Vasile Dragan • Toader Morozan • Adrian-Mihail Stoica

Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems

Springer

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Vasile Dragan
Bucharest, Romania

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Adrian-Mihail Stoica
Bucharest, Romania

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Vasile Dragan
Institute of Mathematics "Simion
Stoilow" of the Romanian Academy
010145 Bucuresti 1
Romania
vdragan@rdslink.ro

Toader Morozan
Institute of Mathematics "Simion
Stoilow" of the Romanian Academy
010145 Bucuresti 1
Romania
toader.morozan@imar.ro

Adrian-Mihail Stoica
Faculty of Aerospace Engineering
University "Politehnica" of Bucharest
Polizu 1
011061 Bucuresti
Romania

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Preface

This monograph contains recent developments in the control theory of linear discrete-time stochastic systems subject both to multiplicative white noise and to Markovian jumping. It provides solutions for various theoretical and practical aspects in connection with this class of stochastic systems including: stability analysis, optimal control, robust stabilization, estimation and filtering, specific numerical algorithms, and computational procedures.

Interest in the topics of the book was generated not only because the considered class of stochastic systems includes as particular cases systems with multiplicative noise and systems with Markov parameters, intensively investigated over the last four decades, but also due to the increasing area of applications in which such dynamic models are used. Engineering domains including communications, fault detection and isolation, robust control, stochastic filtering, navigation, and so on, finance, economics, and biology, are only some of the major fields in which stochastic models using Markov parameters and multiplicative white noise naturally occur.

The monograph can be regarded as a discrete-time counterpart of the book *Mathematical Methods in Robust Control of Linear Stochastic Systems* written by the same authors a few years ago and published by Springer. In fact the discrete-time framework raises many specific aspects both from theoretical and procedural points of view. Therefore when the idea of writing this book was born the authors kept in mind to emphasize even these particularities of the time-domain setting. Another intention was to provide the reader with all prerequisites and analysis tools for a comfortable understanding of the stochastic version of some results firstly derived in the deterministic case, actually belonging to so-called modern control theory. Special attention has been devoted to the numerical aspects determined by the application of these results from stochastic control theory. Thus, for the theoretical results requiring nonstandard numerical procedures, specific algorithms are proposed and illustrated by numerical examples obtained using common commercial software packages. In fact the book does not emphasize the whole potential of the proposed control methods for systems that simultaneously include multiplicative noise components and Markov parameters, but it gives all theoretical and numerical details necessary for the readers to develop their own applications when this is suitable with such stochastic modeling.

The target audience of the book includes researchers in theoretical and applied mathematics, as well as graduate students interested in stochastic modeling and control. Because the authors' intention was to provide a self-contained text, some basic concepts, terminology, and some well-known results are briefly stated in the first chapter where an outline of the book is also presented.

The authors are deeply indebted to Professors O.L.V. Costa, Peng Shi, and I. Yaesh for fruitful discussions on some of the theoretical results and numerical illustrations presented in the book.

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Contents

1

Elements of probability theory

In this introductory chapter we collect several definitions and basic results from probability theory which are used in the developments in the next chapters of the book. Our goal is to present in a unified way some concepts that are presented in different ways in other bibliographic sources. Also we want to establish the basic terminology used in this book. The known results in the field are presented without proofs indicating only the bibliographic source. The less-known results or those which are in less accessible bibliographic sources are presented with their proofs. In the last part of the chapter we describe the classes of stochastic systems under consideration in the book.

1.1 Probability spaces

Let Ω be a nonempty set and 2^{Ω} be the family of its subsets.

Definition 1.1 *A family $\mathcal{F} \subseteq 2^{\Omega}$ is called a σ-*algebra *if the following conditions are simultaneously fulfilled.*

(*i*) $\Omega \in \mathcal{F}$.
(*ii*) *If $A \in \mathcal{F}$ then $\Omega \setminus A \in \mathcal{F}$.*
(*iii*) *For all sequences $\{A_k\}_{k\geq 1} \subset \mathcal{F}$, $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$.*

In this case the pair $(\Omega, \mathcal{F})$ is known as a measurable space.

Usually if Ω is a finite set or a countable set, then $\mathcal{F}$ coincides with 2^{Ω}.

If $\mathcal{M} \subseteq 2^{\Omega}$ then $\sigma[\mathcal{M}]$ stands for the smallest σ-algebra that contains $\mathcal{M}$. It is called the σ-algebra generated by $\mathcal{M}$.

For example, the smallest σ-algebra containing the set $A \subset \Omega$ is $\sigma[A] = \{\emptyset, A, \Omega \setminus A, \Omega\}$.

If $\mathcal{F}, \mathcal{G} \subset 2^{\Omega}$ are two σ-algebras then $\mathcal{F} \vee \mathcal{G}$ stands for the smallest σ-algebra that contains both the σ-algebra $\mathcal{F}$ and the σ-algebra $\mathcal{G}$ (i.e., $\mathcal{F} \vee \mathcal{G} = \sigma[\mathcal{F}, \mathcal{G}]$). The operator $\vee$ may be extended in a natural way to the case of several σ-algebras.

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The Borel σ-algebra $\mathbf{B}(\mathbf{R}^n)$ *is the* σ*-algebra generated by the family of the open subsets of* $\mathbf{R}^n$. *It can be proved that* $\mathbf{B}(\mathbf{R}^n)$ *coincides with the* σ*-algebra generated by the family*

$$\mathcal{M} = \{(-\infty, a_1] \times (-\infty, a_2] \times \cdots (-\infty, a_n] | a_i \in \mathbf{R}, 1 \le i \le n\}. \tag{1.1}$$

In the case $n = 1$ *we write* $\mathbf{B}(\mathbf{R})$ *for the Borel* σ*-algebra on* $\mathbf{R}$.

Definition 1.2 *A collection* $\mathcal{M}$ *of subsets of* Ω *is called a* π-system *if the following conditions are simultaneously fulfilled.*

(*i*) $\emptyset \in \mathcal{M}$.
(*ii*) *If* $A, B \in \mathcal{M}$ *then* $A \bigcap B \in \mathcal{M}$.

If $\mathcal{M}$ *is the family described in* (1.1) *then* $\mathcal{M} \bigcup \{\emptyset\}$ *is a* π*-system.*

The next result is often used in the following.

Theorem 1.1 *Let* $\mathcal{M}$ *be a* π*-system and* $\mathcal{G}$ *be the smallest family of subsets of* Ω *having the properties:*

(*i*) $\mathcal{M} \subseteq \mathcal{G}$.
(*ii*) *If* $A \in \mathcal{G}$ *then* $\Omega \setminus A \in \mathcal{G}$.
(*iii*) *If* $\{A_k\}_{k\ge1} \subseteq \mathcal{G}$ *is a sequence such that* $A_i \bigcap A_j = \emptyset$ *for all* $i \ne j$, *then* $\bigcup_{k=1}^{\infty} A_k \in \mathcal{G}$.

Under these conditions $\mathcal{G} = \sigma[\mathcal{M}]$.

Proof. Because $\sigma[\mathcal{M}]$ verifies $(i), (ii)$, and (iii) in the statement, it follows that $\mathcal{G} \subset \sigma([\mathcal{M}])$. To prove the opposite inclusion we show first that $\mathcal{G}$ is a π-system. Let $A \in \mathcal{G}$ and define $\mathcal{G}(A) = \{B; B \in \mathcal{G}$ and $A \cap B \in \mathcal{G}\}$. Because $A \setminus B = \Omega \setminus [(A \cap B) \cup (\Omega \setminus A)]$, it is easy to check that $\mathcal{G}(A)$ verifies the conditions (ii) and (iii) and if $A \in \mathcal{M}$ then (i) is satisfied. Hence for $A \in \mathcal{M}$ we have $\mathcal{G}(A) = \mathcal{G}$; consequently if $A \in \mathcal{M}$ and $B \in \mathcal{G}$ then $A \cap B \in \mathcal{G}$. But this implies $\mathcal{G}(B) \supset \mathcal{M}$ and therefore $\mathcal{G}(B) = \mathcal{G}$ for any $B \in \mathcal{G}$. Hence $\mathcal{G}$ is a π-system and now, inasmuch as $\mathcal{G}$ verifies (ii) and (iii) it is easy to verify that $\mathcal{G}$ is a σ-algebra and the proof is complete. □

Definition 1.3 *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A function* $\mathcal{P} : \mathcal{F} \to [0, 1]$ *is called a* probability measure *if it has the properties:*

(*i*) $\mathcal{P}(\emptyset) = 0, \mathcal{P}(\Omega) = 1$.
(*ii*) *For all sequences* $\{A_k\}_{k\ge1} \subseteq \mathcal{F}$, *with* $A_i \cap A_j = \emptyset$ *for all* $i \ne j$, *we have* $\mathcal{P}[\bigcup_{k=1}^{\infty} A_k] = \sum_{k=1}^{\infty} \mathcal{P}[A_k]$.

A triple $(\Omega, \mathcal{F}, \mathcal{P})$ *where* Ω *is a nonempty set* $\mathcal{F} \subset 2^{\Omega}$ *is a* σ*-algebra and* $\mathcal{P}\colon \mathcal{F} \to [0, 1]$ *is a probability measure, is called a* probability space. *A set* $A \in \mathcal{F}$ *is called an event and* $\mathcal{P}(A)$ *is the* probability of the event *A*.

Definition 1.4 *We say that the* σ*-algebras* $\mathcal{G}_1, \ldots, \mathcal{G}_m, \mathcal{G}_i \subseteq \mathcal{F}, i = 1, \ldots, m$ *are* mutually independent *if* $\mathcal{P}[\bigcap_{i=1}^{m} A_i] = \Pi_{i=1}^{m} \mathcal{P}(A_i)$ *for all* $A_i \in \mathcal{G}_i$, $1 \le i \le m$.

1.2 Random variables

1.2.1 Definitions and basic results

Let $(\Omega, \mathcal{F}), (\Omega', \mathcal{F}')$ be two measurable spaces.

Definition 1.5

(a) *A function* $\xi : \Omega \to \Omega'$ *is called an* $(\mathcal{F}, \mathcal{F}')$-measurable function *if for all* $A' \in \mathcal{F}'$ *we have* $\xi^{-1}(A') = \{\omega \in \Omega | \xi(\omega) \in A'\} \in \mathcal{F}$.
A function $\xi, (\mathcal{F}, \mathcal{F}')$*-measurable is called an* $(\mathcal{F}, \mathcal{F}')$ random variable *or simply a* random variable *if no confusion is possible.*
(b) *A function* $\xi : \Omega \to \mathbf{R}$ *that is* $(\mathcal{F}, \mathbf{B}[\mathbf{R}])$*-measurable is called a* real random variable *or simply a* random variable *(if no confusions is possible).*
(c) *A function* $\xi : \Omega \to \mathbf{R}^n$ *that is* $(\mathcal{F}, \mathbf{B}[\mathbf{R}])$*-measurable is called a* random vector.

The following results are well known.

Proposition 1.1 *A function* $\xi : \Omega \to \mathbf{R}$ *is a random variable if and only if for all* $a \in \mathbf{R}$ *the sets* $\{\omega \in \Omega | \xi(\omega) \leq a\}$ *are in* $\mathcal{F}$. □

Proposition 1.2 *If* $\xi, \eta, \xi_i, i \geq 1$ *are real random variables and* $\alpha, \beta \in \mathbf{R}$, *then* $\alpha\xi + \beta\eta$, $|\xi|, \xi\eta$ *are also random variables. If the sequence* $\{\xi_i\}_{i \geq 1}$ *is convergent then its limit is also a random variable.* □

Definition 1.6 *A random variable* $\phi : \Omega \to \mathbf{R}$ *that takes a finite number of values is called an* elementary random variable *or a* simple random variable.

If $A \subseteq \Omega, \chi_A$ stands for the *indicator function of the set* A, that is, $\chi_A(\omega) = 1$ if $\omega \in A$ and $\chi_A(\omega) = 0$ if $\omega \in \Omega \setminus A$.
If $A \in \mathcal{F}$ then χ_A is a simple random variable.

Proposition 1.3 *If* $\varphi : \Omega \to \mathbf{R}$ *is a simple random variable taking the values* $a_1, a_2, \ldots, a_n$ *then* $\varphi(\omega) = \sum_{i=1}^{n} a_i \chi_{A_i}(\omega), \omega \in \Omega$, *where* $A_i = \{\omega \in \Omega | \varphi(\omega) = a_i\}, A_i \in \mathcal{F}, 1 \leq i \leq n$. □

At the end of this subsection we remark that each random variable $\xi : \Omega \to \mathbf{R}$ can be written as $\xi = \xi^+ - \xi^-$, where $\xi^+ = \frac{1}{2}(|\xi| + \xi), \xi^- = \frac{1}{2}(|\xi| - \xi), \xi^+ \geq 0, \xi^- \geq 0$.
If ξ and η are two random variables such that $\mathcal{P}\{\omega \in \Omega | \xi(\omega) = \eta(\omega)\} = 1$ then we write $\xi = \eta$ a.s. ($\xi = \eta$ almost surely).

1.2.2 Integrable random variables. Expectation

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a given probability space. Firstly we recall the following.

Theorem 1.2 *If $\xi : \Omega \to \mathbf{R}$ is a nonnegative random variable then there exists an increasing sequence of nonnegative elementary random variables $\{\varphi_j\}_{j\geq 1}$ such that $\lim_{j\to\infty} \varphi_j(\omega) = \xi(\omega), \omega \in \Omega$.*

Proof. It follows immediately taking $\varphi_j(\omega) = \sum_{i=1}^{2^j j+1}((i-1)/2^j)\chi_{A_{ij}}(\omega)$, with $A_{ij} = \{\omega \in \Omega | (i-1/2^j) \leq \xi(\omega) < i/2^j\}, i = 1, 2, \ldots, j2^j, A_{2^j j+1,j} = \{\omega \in \Omega | \xi(\omega) \geq j\}$. □

Definition 1.7

(*a*) *If $\varphi = \sum_{i=1}^n a_i\chi_{A_i}$ is a simple random variable then, by definition, $\int_\Omega \varphi d\mathcal{P} = \sum_{i=1}^n a_i\mathcal{P}(A_i)$.*
(*b*) *If $\xi : \Omega \to \mathbf{R}$ is a nonnegative random variable then, by definition, $\int_\Omega \xi d\mathcal{P} = \lim_{k\to\infty}\int_\Omega \varphi_k d\mathcal{P}$ where $\varphi_k, k \geq 1$ is an increasing sequence of nonnegative simple random variables that converge to ξ.*

It can be shown that $\int_\Omega \xi d\mathcal{P}$ is independent of the increasing sequences of simple random variables $\varphi_k, k \geq 1$, which converge to ξ.

Definition 1.8

(*a*) *We say that a random variable $\xi : \Omega \to \mathbf{R}$ possesses an integral if either*

$$\int_\Omega \xi^+ d\mathcal{P} < \infty \tag{1.2}$$

or

$$\int_\Omega \xi^- d\mathcal{P} < \infty. \tag{1.3}$$

In this case we write

$$\int_\Omega \xi d\mathcal{P} = \int_\Omega \xi^+ d\mathcal{P} - \int_\Omega \xi^- d\mathcal{P}. \tag{1.4}$$

(*b*) *A random variable $\xi : \Omega \to \mathbf{R}$ is called an* integrable random variable *if* (1.2) *and* (1.3) *are simultaneously true.*
(*c*) *We say that a random vector $\xi : \Omega \to \mathbf{R}^m, \xi = (\xi_1, \ldots, \xi_m)^T$ is* integrable *if its components $\{\xi_j\}_{j=\overline{1,m}}$ are integrable random variables.*

As usual the superscript T stands for the transpose of a vector or a matrix.

Because $|\xi| = \xi^+ + \xi^-$ it follows that a random variable $\xi : \Omega \to \mathbf{R}$ is integrable iff $\int_\Omega |\xi| d\mathcal{P} < \infty$.

If $\xi : \Omega \to \mathbf{R}$ is an integrable random variable one denotes $E[\xi] = \int_\Omega \xi d\mathcal{P}$; $E[\xi]$ is called the *expectation of the random variable* ξ. In the case of the random vector $\xi = (\xi_1, \ldots, \xi_m)^T$ one sets $E[\xi] = (E[\xi_1], \ldots, E[\xi_m])^T$. The definition of the expectation can be extended in a natural manner to matrix-valued random variables.

For each $p \geq 1, \mathcal{L}^p(\Omega, \mathcal{F}, \mathbf{R}^m)$ is the set of random vectors ξ with the property $E[|\xi|^p] < \infty$.

1.2.3 Independent random variables

If ξ is a random variable or a random vector then the smallest sub-σ-algebra of $\mathcal{F}$ with respect to which ξ is a measurable function is denoted by $\sigma[\xi]$ and it is called the *σ-algebra generated by ξ*.

If $\{\xi_i\}_{i\in I}$ is a family of random variables or random vectors then $\sigma[\xi_i, i \in I]$ stands for the smallest σ-algebra with respect to which all ξ_i are measurable functions.

Definition 1.9

(a) *We say that the random variables (or random vectors) $\xi_1, \xi_2, \ldots, \xi_m$ are* independent *if $\sigma[\xi_1], \ldots, \sigma[\xi_m]$ are independent σ-algebras.*
(b) *We say that the random vector ξ is* independent of σ-algebra $\mathcal{G} \subset \mathcal{F}$ *if the σ-algebra $\sigma[\xi]$ is independent of the σ-algebra $\mathcal{G}$.*

Proposition 1.4 *Let $x_1, \ldots, x_m$ be independent integrable random variables. Set $x = x_1 x_2 \ldots x_m$. Then*

(i) *x is an integrable random variable.*
(ii) *$E[x] = \Pi_{i=1}^{m} E[x_i]$.* □

1.3 Conditional expectation

Let $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra and x an integrable random variable. By the Radon–Nicodym theorem it follows that there exists a unique (mod $\mathcal{P}$) random variable y with the following properties.

(a) y is measurable with respect to $\mathcal{G}$.
(b) $E[|y|] < \infty$.
(c) $\int_A y d\mathcal{P} = \int_A x d\mathcal{P}$ for all $A \in \mathcal{G}$.

The random variable y with these properties is denoted by $E[x|\mathcal{G}]$ and is called the *conditional expectation of x with respect to the σ-algebra $\mathcal{G}$*.

By definition, for all $A \in \mathcal{F}$,

$$\mathcal{P}(A|\mathcal{G}) := E[\chi_A|\mathcal{G}]$$

and it is called the *conditional probability of the event A with respect to the σ-algebra $\mathcal{G}$*. By definition

$$E[x|y_1, \ldots, y_n] := E[x|\sigma[y_1, \ldots, y_n]].$$

If x is an integrable random variable and $A \in \mathcal{F}$ with $\mathcal{P}(A) > 0$, then by definition

$$E[x|A] := \int_{\Omega} x d\mathcal{P}_A,$$

where $\mathcal{P}_A : \mathcal{F} \to [0,1]$ by $\mathcal{P}_A(B) = ((\mathcal{P}(A \cap B))/(\mathcal{P}(A)))$ for all $B \in \mathcal{F}$.

$E[x|A]$ is called the *conditional expectation of x with respect to the event A.* Because

$$\mathcal{P}_A(B) = \frac{1}{\mathcal{P}(A)} \int_B \chi_A d\mathcal{P},$$

we have

$$E[x|A] = \frac{1}{\mathcal{P}(A)} \int_\Omega (x\chi_A) d\mathcal{P} = \frac{1}{\mathcal{P}(A)} \int_A x d\mathcal{P}.$$

By definition

$$\mathcal{P}(B|A) := \mathcal{P}_A(B), A \in \mathcal{F}, B \in \mathcal{F}, \mathcal{P}(\mathrm{A}) > 0.$$

Obviously, $\mathcal{P}[B|A] = E[\chi_B|A]$.

Theorem 1.3 *Let x, y be integrable random variables and $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$, σ-algebras; then the following assertions hold.*

(i) $E(E[x|\mathcal{G}]) = E[x]$.
(ii) $E[E[x|\mathcal{G}]|\mathcal{H}] = E[x|\mathcal{H}]$ *a.s.if* $\mathcal{G} \supset \mathcal{H}$.
(iii) $E[(\alpha x + \beta y)|\mathcal{G}] = \alpha E[x|\mathcal{G}] + \beta E[y|\mathcal{G}]$ *a.s.* $\alpha, \beta \in \mathbf{R}$.
(iv) $E[xy|\mathcal{G}] = yE[x|\mathcal{G}]$ *a.s. if y is measurable with respect to $\mathcal{G}$ and xy is integrable.*
(v) *If x is independent of $\mathcal{G}$ then* $E[x|\mathcal{G}] = E[x]$ *a.s.*
(vi) $x \geq 0$ *implies* $E[x|\mathcal{G}] \geq 0$ *a.s.*
(vii) *Let $x, x_k, k \geq 1$ be integrable random variables. If* $\lim_{k\to\infty} x_k(\omega) = x(\omega)$ *a.s. and there exists a positive and integrable random variable ψ such that* $|x_k(\omega)| \leq \psi(\omega)$ *a.s., $k \geq 1$, then* $E[x|\mathcal{G}] = \lim_{k\to\infty} E[x_k|\mathcal{G}]$ *a.s.* □

Remark 1.1 It is easy to verify the following.

(a) If x is an integrable random variable and y is a simple random variable with values $c_1, \ldots, c_n$ then

$$E[x|y] = \sum_{j\in M} \chi_{y=c_j} E[x|y = c_j] a.s.,$$

where $M = \{j \in \{1, 2, \ldots, n\}, \mathcal{P}\{y = c_j\} > 0\}$.

(b) If $A \in \mathcal{F}$, $\mathcal{G}_A = \{\emptyset, \Omega, A, \Omega \setminus A\}$, and x is an integrable random variable then

$$E[x|\mathcal{G}_A] = \begin{cases} \chi_A E[x|A] + \chi_{\Omega\setminus A} E[x|\Omega \setminus A] & \text{if } \ 0 < P(A) < 1 \\ Ex & \text{if } \ \mathcal{P}(A) = 0 \ \text{ or } \ \mathcal{P}(A) = 1 \end{cases}$$

Therefore $E[x|\mathcal{G}_A]$ takes at most two values.

(c) If $A_1, A_2, \ldots, A_n \in \mathcal{F}$ are such that $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i=1}^n A_i = \Omega$ then we have

$$\mathcal{P}(B) = \sum_{i=1}^{n} \mathcal{P}(B|A_i)\mathcal{P}(A_i) \tag{1.5}$$

for all $B \in \mathcal{F}$ (with the convention that $\mathcal{P}(B|A_i) = 0$ if $\mathcal{P}(A_i) = 0$).

1.4 Markov chains

1.4.1 Stochastic matrices

In this subsection we recall several issues concerning stochastic matrices. This kind of matrices plays a crucial role in the definition and characterization of Markov chains. More details concerning stochastic matrices as well as sequences of stochastic matrices may be found in [32].

Definition 1.10

(*a*) *A matrix* $P \in \mathbf{R}^{N\times N}, P = \{p(i,j)\}_{i,j\in\overline{1,N}}$ *is called* a stochastic matrix *if* $p(i,j) \geq 0, i,j \in \overline{1,N}$ *and* $\sum_{j=1}^{N} p(i,j) = 1,\ i = 1,\dots,N.$

(*b*) *A stochastic matrix* P *is a* nondegenerate stochastic matrix *if all its columns are not identically zero.*

Proposition 1.5 ([32]) *If* $P \in \mathbf{R}^{N\times N}$ *is a stochastic matrix then the following Cesaro-type limit* $\lim_{k\to\infty} 1/k \sum_{i=0}^{k} P^i$ *exists. Moreover if* $Q = \lim_{k\to\infty}(1/k)\sum_{i=0}^{k} P^i$ *then* Q *is also a stochastic matrix that satisfies* $PQ = QP = Q$. □

1.4.2 Markov chains

Throughout this book, $\mathcal{D}$ stands for the following finite set $\mathcal{D} = \{1, 2, \dots, N\}$, where $N \geq 1$ is a fixed integer. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a given probability space.

Definition 1.11 *A* Markov chain *is a triple* $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$*, where for each* $t \geq 0, \eta_t : \Omega \to \mathcal{D}$ *is a random variable,* $\mathbf{P} = \{P_t\}_{t\geq 0}$ *is a sequence of stochastic matrices* $P_t \in \mathbf{R}^{N\times N}$ *with the property:*

$$\mathcal{P}\{\eta_{t+1} = j|\mathcal{G}_t\} = p_t(\eta_t, j) \ a.s. \tag{1.6}$$

for all $t \geq 0$*, and* $j \in \mathcal{D}$*, where* $\mathcal{G}_t = \sigma[\eta_0, \eta_1, \dots, \eta_t], t \geq 0$.

$\{P_t\}_{t\geq 0}$ *is called the* sequence of transition probability matrices *and* $\mathcal{D}$ *is the* set of the states *of the Markov chain.*

If the sequence P *is constant, that is,* $P_t = P$ *for all* $t \geq 0$*, then* $(\{\eta_t\}_{t\geq 0}, P, \mathcal{D})$ *is called an* homogeneous Markov chain*.*

The following properties of the Markov chains are repeatedly used in this book.

Theorem 1.4 ([32]) *If* $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$ *is a Markov chain then the following hold.*

(*i*) $\mathcal{P}\{\eta_s = i_s, \eta_{s+1} = i_{s+1}, \dots, \eta_t = i_t\}$
$= \mathcal{P}\{\eta_s = i_s\} p_s(i_s, i_{s+1}) p_{s+1}(i_{s+1}, i_{s+2}) \cdots p_{t-1}(i_{t-1}, i_t)$ *for all* $t \geq s+1$, $s \geq 0$.

(ii) $\mathcal{P}(\eta_s = i_s, \eta_{s+1} = i_{s+1}, \dots, \eta_t = i_t | \eta_{s-1} = i)$
$= p_{s-1}(i, i_s)p_s(i_s, i_{s+1}) \cdots p_{t-1}(i_{t-1}, i_t)$ *for all* $t \geq s \geq 1, i \in \mathcal{D}$, *with the property* $\mathcal{P}(\eta_{s-1} = i) > 0$.

(iii) *If* φ *is a bounded random variable that is* $\sigma[\eta_s, s \geq t]$*-measurable, then* $E[\varphi|\mathcal{G}_t] = E[\varphi|\eta_t]$ *a.s.* □

For each $t \geq 0$ we set $\pi_t = (\pi_t(1), \dots, \pi_t(N))$ the distribution of the random variable η_t; that is, $\pi_t(i) = \mathcal{P}(\eta_t = i), i \in \mathcal{D}$. Clearly $\pi_t(i) \geq 0, 1 \leq i \leq N$ and $\sum_{i=1}^N \pi_t(i) = 1, t \geq 0$.

Consider the set $\mathcal{M}_N = \{\pi = (\pi(1), \dots, \pi(N)) | \pi(i) \geq 0, 1 \leq i \leq N, \sum_{i=1}^N \pi(i) = 1\}$. Hence $\pi_t \in \mathcal{M}_N$ for all $t \in \mathbf{Z}_+$.

Based on (i) in Theorem 1.4 for $t = s + 1$ we may write

$$\pi_{s+1}(i) = \sum_{j=1}^N \mathcal{P}\{\eta_s = j, \eta_{s+1} = i\} = \sum_{j=1}^N \pi_s(j)p_s(j, i).$$

This shows that the sequence $\{\pi_t\}_{t\geq 0}$ of the distribution of random variables $\{\eta_t\}_{t\geq 0}$ solves the linear equation:

$$\pi_{t+1} = \pi_t P_t, \qquad t \geq 0. \tag{1.7}$$

Hence the sequence $\{\pi_t\}_{t\geq 0}$ of the distribution of the random variables η_t, $t \geq 0$, is completely determined by the sequence $\mathbf{P} = \{P_t\}_{t\geq 0}$ of the transition probability matrices and by the initial distribution $\pi_0 \in \mathcal{M}_N$. It is known that in the case of a standard continuous-time Markov process, $\pi_t(i) > 0$ for all $i \in \mathcal{D}, t > 0$ if $\pi_0(j) > 0$ for all $j \in \mathcal{D}$. From (1.7) we see that in the discrete-time case it is possible to have $\pi_t(i) = 0$ for some $t \geq 1$ and $i \in \mathcal{D}$ even if $\pi_0(j) > 0$, for all $j \in \mathcal{D}$. The consequences of such aspects specific to the discrete-time case become clearer in Chapter 3 where the difficulties arising in connection with characterization of exponential stability in the mean square for discrete-time linear stochastic systems with matrix switching are investigated.

The next obvious result can be proved by mathematical induction.

Proposition 1.6 *Let* $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$ *by a Markov chain. Then:*

(i) *If for each* $t \geq 0$, P_t *is a nondegenerate stochastic matrix, then* $\pi_t(j) > 0$ *for all* $t \geq 1, 1 \leq j \leq N$, *provided that* $\pi_0(i) > 0$ *for all* $1 \leq i \leq N$.

(ii) *Conversely, if* $\pi_t(i) > 0$ *for all* $t \geq 0$ *and all* $i \in \mathcal{D}$ *then for every* $t \geq 0$, P_t *is a nondegenerate stochastic matrix.* □

Throughout the book the following notation is often used.

$$\mathcal{D}_s = \{i \in \mathcal{D} | \pi_s(i) > 0\}. \tag{1.8}$$

With this notation the statement of Proposition 1.6 becomes: "The matrices P_t are nondegenerate stochastic matrices for all $t \geq 0$, iff $\mathcal{D}_t = \mathcal{D}$ for all $t \geq 1$, if $\mathcal{D}_0 = \mathcal{D}$."

In many control problems for discrete-time linear stochastic systems with Markovian jumping over an infinite time horizon we are interested in knowing if there exist $\lim_{t\to\infty} \pi_t(i)$. Also we are interested in knowing under which conditions these limits are independent of initial distributions π_0.

The next result provides an answer to this issue in the case of an homogeneous Markov chain. If $P_t = P$ for all $t \geq 0$ then from (1.7) we have

$$\pi_t = \pi_0 P^t. \tag{1.9}$$

Setting $p^t(i,j), 1 \leq i \leq N, 1 \leq j \leq N$ the elements of the matrix P^t, from (1.9) one obtains that

$$\pi_t(i) = \sum_{j=1}^{N} \pi_0(j) p^t(j,i), \qquad t \geq 1, \quad 1 \leq i \leq N \tag{1.10}$$

for all $\pi_0 = (\pi_0(1), \ldots, \pi_0(N)) \in \mathcal{M}_N$.

Theorem 1.5 *If* $(\{\eta_t\}_{t\geq 0}, P, \mathcal{D})$ *is an homogeneous Markov chain then the following are equivalent.*

(i) *There exists* $\theta = (\theta(1), \ldots, \theta(N)) \in \mathcal{M}_N$, *such that* $\lim_{t\to\infty} \pi_t(i, \pi_0) = \theta(i), 1 \leq i \leq N$, *for all* $\pi_0 = (\pi_0(1) \cdots \pi_0(N)) \in \mathcal{M}_N$, *where* $\pi_t(i, \pi_0)$ *are given by* (1.10).
(ii) *There exists* $\theta = (\theta(1), \ldots, \theta(N)) \in \mathcal{M}_N$ *such that* $\lim_{t\to\infty} p^t(i,j) = \theta(j)$ *for all* $i, j \in \mathcal{D}$.
(iii) *There exist* $t_0 \geq 1, j \in \mathcal{D}$ *such that* $p^{t_0}(i,j) > 0$ *for all* $i \in \mathcal{D}$.

Proof. To prove the implication $(i) \to (ii)$ we chose

$$\tilde{\pi}^i = (\tilde{\pi}^i(1), \ldots, \tilde{\pi}^i(N)) \in \mathcal{M}_N, \qquad i \in \mathcal{D}$$

defined as follows,

$$\tilde{\pi}^i(i) = 1, \qquad \tilde{\pi}^i(j) = 0, \qquad j \neq i.$$

From (1.10) one sees that $\pi_t(j, \tilde{\pi}^i) = P^t(i,j)$. This shows that (ii) holds if (i) is true. We now prove implication $(ii) \to (iii)$. Let $\theta \in \mathcal{M}_N$ be such that the equality from (ii) takes place.

Because $\sum_{j=1}^{N} \theta(j) = 1$ we deduce that there exists $j \in \mathcal{D}$ such that $\theta(j) > 0$. From the definition of the limit it follows that for each $i \in \mathcal{D}$ there exists $t_0(i) \geq 1$ such that $p^t(i,j) > 0$ for all $t \geq t_0(i)$. Take $t_0 = \max_{i\in\mathcal{D}}\{t_0(i)\}$. One obtains that $p^{t_0}(i,j) > 0$ for all $i \in \mathcal{D}$. This means that (iii) is fulfilled if (ii) holds.

It remains to prove that $(iii) \to (i)$. If (iii) holds, then from Chapter V in ([32]) it follows that there exists $\theta \in \mathcal{M}_N$, such that $\lim_{t\to\infty} p^t(i,j) = \theta(j), i, j \in \mathcal{D}$.

Based on (1.10) we may write

$$\lim_{t\to\infty} \pi_t(j,\pi_0) = \lim_{t\to\infty} \sum_{i=1}^{N} \pi_0(i)p^t(i,j) = \sum_{i=1}^{N} \pi_0(i)\theta(j) = \theta(j)$$

for all $\pi_0 = (\pi_0(1), \dots, \pi_0(N)) \in \mathcal{M}_N$ and thus the proof ends. □

At the end of this section we provide a sufficient condition that guarantees the existence of a positive constant $\delta > 0$ such that $\pi_t(i) \geq \delta$ for all $t \geq 0, i \in \mathcal{D}$.

In Chapter 3 we show how such a condition is involved in the investigation of the exponential stability in the mean square.

Proposition 1.7 *Let* $\{\{\eta_t\}_{t\geq 0}, P, \mathcal{D}\}$ *be an homogeneous Markov chain. If there exists*

$$Q = \lim_{t\to\infty} P^t \tag{1.11}$$

and additionally

$$q_{ii} > 0, \qquad \forall i \in \mathcal{D}, \tag{1.12}$$

q_{ii} *being the diagonal elements of* Q, *then there exists* $\delta > 0$ *such that*

$$\pi_t(i) = \mathcal{P}\{\eta_t = i\} \geq \delta \tag{1.13}$$

for all $t \geq 0$, *if* $\pi_0(i) > 0$ *for all* $i \in \mathcal{D}$.

Proof. One can see that if (1.11)–(1.12) are valid then the matrix P is a nondegenerate stochastic matrix.

Set $\delta_1 = \frac{1}{2}\min\{q_{ii}, i \in \mathcal{D}\}$. If $p^t(i,j)$ are elements of the matrix P^t, then from (1.12) it follows that there exists $t_0 \geq 0$ such that $p^t(i,i) \geq \delta_1$ for all $t \geq t_0$ and $i \in \mathcal{D}$. From (1.10) one obtains that $\pi_t(i) \geq \pi_0(i)\delta_1$.

Setting $\delta_2 = \min\{\pi_0(i)\delta_1, i \in \mathcal{D}\}$ we deduce that $\pi_t(i) \geq \delta_2$, for all $t \geq t_0$. Because P is a nondegenerate stochastic matrix by Proposition 1.6 we have $\pi_t(i) > 0$ for all $t \geq 0$ and $i \in \mathcal{D}$. If $t_0 \geq 1$, set $\delta_3 = \min\{\pi_t(i); 0 \leq t \leq t_0 - 1, i \in \mathcal{D}\}$. Thus one obtains that (1.13) is valid for $\delta = \min\{\delta_2, \delta_3\}$. □

1.5 Some remarkable sequences of random variables

In the developments of this book we assume that the controlled systems under consideration are affected by two classes of stochastic perturbations:

(a) A Markov chain $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$
(b) A sequence of independent random vectors $\{w(t)\}_{t\geq 0}$, where $w(t) : \Omega \to \mathbf{R}^r$

Throughout this book, by a Markov chain we understand any triple $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$ where $\mathbf{P} = \{P_t\}_{t\geq 0}$ is a given sequence of stochastic matrices, $\mathcal{D} = \{1, 2, \ldots, N\}$ is a fixed finite set, and $\{\eta_t\}_{t\geq 0}$ is an arbitrary sequence of random variables $\eta_t : \Omega \to \mathcal{D}$ that verifies (1.6).

Concerning the sequences $\{\eta_t\}_{t\geq 0}, \{w(t)\}_{t\geq 0}$ the following assumptions are made.

$\mathbf{H}_1$. $E[w(t)] = 0, E[|w(t)|^2] < \infty, E[w(t)w^T(t)] = I_r, t \geq 0$, with I_r the identity matrix of size r.

$\mathbf{H}_2$. For each $t \geq 0$, the σ-algebra $\mathcal{F}_t$ is independent of the σ-algebra $\mathcal{G}_t$, where $\mathcal{F}_t = \sigma[w(0), \ldots, w(t)]$, and $\mathcal{G}_t = \sigma[\eta_0, \eta_1, \ldots, \eta_t]$.

Remark 1.2 If $w(t)$ is a sequence of random vectors $w(t) : \Omega \to \mathbf{R}^r$ such that $E[w(t)] = m_t \in \mathbf{R}^r$, and $E[w(t)w^T(t)] = \sum_t > 0$, then the sequence $\{\tilde{w}(t)\}_{t\geq 0}$, where $\tilde{w}(t) = \sum_t^{(-1/2)}(w(t) - m_t)$, will satisfy $E[\tilde{w}(t)] = 0$ and $E[\tilde{w}(t)\tilde{w}^T(t)] = I_r$.

This shows that without loss of generality we may assume that the sequence of independent random vectors $\{w(t)\}_{t\geq 0}$ satisfies $\mathbf{H}_1$.

Together with the σ-algebras $\mathcal{F}_t$ and $\mathcal{G}_t$ defined above we also introduce the following σ-algebras generated by the stochastic processes $\{\eta_t\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$.

- For each $(u, v) \in \mathbf{Z}_+ \times \mathbf{Z}_+$ we set $\mathcal{H}_{u,v} = \sigma[\eta_t, w(s), 0 \leq t \leq u, 0 \leq s \leq v]$. Hence $\mathcal{H}_{u,v}$ is the smallest σ-algebra containing the σ-algebras $\mathcal{G}_u$ and $\mathcal{F}_v$. If $u = v$ we write $\mathcal{H}_u$ instead of $\mathcal{H}_{u,u}$.
- For each $u \in \mathbf{Z}_+$ we set $\check{\mathcal{G}}_u = \sigma[\eta_t, t \geq u]$.
- If $v \in \mathbf{Z}_+$, $\check{\mathcal{F}}_v$ is defined by $\check{\mathcal{F}}_v = \sigma[w(s), s \geq v + 1]$. Obviously, for all $v \in Z_+$, σ-algebra $\check{\mathcal{F}}_v$ is independent of the σ-algebra $\mathcal{F}_v$.
- For each $(u, v) \in \mathbf{Z}_+ \times \mathbf{Z}_+$ we set $\check{\mathcal{H}}_{u,v} = \sigma[\eta_t, w(s), t \geq u, s \geq v + 1]$. Hence $\check{\mathcal{H}}_{u,v}$ is the smallest σ-algebra containing the σ-algebras $\check{\mathcal{F}}_v$ and $\check{\mathcal{G}}_u$.
- $\tilde{H}_u = \sigma[\eta_s, w(t), 0 \leq s \leq u, 0 \leq t \leq u - 1]$ if $u \geq 1$ and $\tilde{H}_0 = \sigma[\eta_0]$.

The next result plays an important role in many proofs in the next chapters. More precisely in our developments the result proved in the next theorem has the role played by property (*iii*) of Theorem 1.4, in the case of systems subject only to Markov perturbations.

Theorem 1.6 *Under the assumptions* $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *if* $\Psi : \Omega \to \mathbf{R}$ *is an integrable random variable that is* $\check{\mathcal{H}}_{u,v}$*-measurable then*

$$E[\Psi|\mathcal{H}_{u,v}] = E[\Psi|\eta_u] \quad a.s. \tag{1.14}$$

Proof. Combining the definition of an integrable random variable, Theorem 1.2, and property (*vii*) from Theorem 1.3 one obtains that it is sufficient to prove (1.14) in the case when Ψ is a bounded random variable $\check{\mathcal{H}}_{u,v}$-measurable. The definition of conditional expectation shows that (1.14) is equivalent to

$$E[\Psi \chi_M] = E[z \chi_M] \tag{1.15}$$

for all $M \in \mathcal{H}_{u,v}$, where $z = E[\Psi | \eta_u]$. To prove (1.15) we show that

$$E[\Psi f g] = E[z f g] \tag{1.16}$$

for all f, g bounded functions such that f is $\mathcal{F}_v$-measurable and g is $\mathcal{G}_u$-measurable. As a first step we show that (1.16) holds for $\Psi = \Psi_1 \Psi_2$ with Ψ_1 and Ψ_2 bounded random variables such that Ψ_1 is $\check{\mathcal{G}}_u$-measurable and Ψ_2 is $\check{\mathcal{F}}_v$-measurable. Because f and Ψ_2 are independent random variables and $g\Psi_1$ is independent of $f\Psi_2$ we have:

$$E[\Psi f g] = E[\Psi_1 \Psi_2 f g] = E[\Psi_2] E[f] E[g \Psi_1]. \tag{1.17}$$

On the other hand $E[\Psi_1 g | \mathcal{G}_u] = g E[\Psi_1 | \mathcal{G}_u] = g E[\Psi_1 | \eta_u]$. For the last equality we have used (iii) from Theorem 1.4 for $\varphi = \Psi_1$.

Furthermore, from (1.17) one obtains that

$$E[\Psi_1 \Psi_2 f g] = E[\Psi_2] E[f] E[g E[\Psi_1 | \eta_u]]. \tag{1.18}$$

We have $z = E[\Psi_2] E[\Psi_1 | \eta_u]$. This leads to

$$E[f g z] = E[\Psi_2] E[f g E[\Psi_1 | \eta_u]] = E[\Psi_2] E[f] E[g E[\Psi_1 | \eta_u]]. \tag{1.19}$$

Combining (1.18) with (1.19) one obtains that (1.16) is fulfilled for $\Psi = \Psi_1 \Psi_2$. Particularly (1.16) holds for $\Psi = \chi_{U_1 \cap U_2}$ for any $U_1 \in \check{\mathcal{G}}_u$ and $U_2 \in \check{\mathcal{F}}_v$.

Applying Theorem 1.1 we conclude that (1.16) is fulfilled for $\Psi = \chi_U$ for any $U \in \check{\mathcal{H}}_{u,v}$. By a standard procedure in measure theory one obtains (via Theorem 1.2) that (1.16) is fulfilled for Ψ-integrable and $\check{\mathcal{H}}_{u,v}$-measurable. Taking now in (1.16) $f = \chi_A, g = \chi_B$, for any $A \in \mathcal{F}_v, B \in \mathcal{G}_u$ one obtains that (1.15) is verified for $\chi_{A \cap B}$. Applying again Theorem 1.1 we deduce that (1.15) is fulfilled and thus the proof ends. □

Taking $u = v = t, \Psi = \chi_{\{\eta_{t+1} = j\}}$ we obtain the following from Theorem 1.6.

Corollary 1.1 *Under the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *we have*

$$\mathcal{P}\{\eta_{t+1} = j | \mathcal{H}_t\} = p_t(\eta_t, j) \quad a.s.$$

for all $t \geq 0, j \in \mathcal{D}$.

1.6 Discrete-time controlled stochastic linear systems

The discrete-time linear systems have been intensively considered in the control literature both in the deterministic framework and in stochastic cases. This interest is wholly motivated by the wide area of applications including engineering, economics, and biology. Most of the results available in the control of discrete-time stochastic systems consider either the case of systems corrupted with white noise perturbations or the case of systems with Markovian jumps.

The class of controlled stochastic linear systems considered in this book is described by:

$$x(t+1) = A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t) + G_0(t,\eta_t)v(t) + \sum_{k=1}^{r}[A_k(t,\eta_t)x(t) + B_k(t,\eta_t)u(t) + G_k(t,\eta_t)v(t)]w_k(t) \quad (1.20)$$

$$y(t) = C_0(t,\eta_t)x(t) + D_0(t,\eta_t)v(t) + \sum_{k=1}^{r}[C_k(t,\eta_t)x(t) + D_k(t,\eta_t)v(t)]w_k(t), \quad (1.21)$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the vector of control parameters, $y(t) \in \mathbf{R}^p$ is a measured signal, and while $v(t) \in \mathbf{R}^{m_v}$ is the exogenous perturbation.

In (1.20) and (1.21), $\{\eta_t\}_{t\geq 0}$ and $w(t) = (w_1(t),\dots,w_r(t))^T$, $t \geq 0$ are stochastic processes whose properties are given in the preceding section.

If $\eta_t = i \in \mathcal{D}$ we set $A_k(t,i), B_k(t,i), G_k(t,i), C_k(t,i), D_k(t,i), k \in \{0,1,\dots,r\}$ for the coefficient matrices of the system (1.20)–(1.21).

If $A_k(t,i) = 0, B_k(t,i) = 0, G_k(t,i) = 0, C_k(t,i) = 0, D_k(t,i) = 0, 1 \leq k \leq r, t \geq 0, i \in \mathcal{D}$ then the system (1.20)–(1.21) becomes

$$x(t+1) = A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t) + G_0(t,\eta_t)v(t) \quad (1.22)$$

$$y(t) = C_0(t,\eta_t)x(t) + D_0(t,\eta_t)v(t), \quad (1.23)$$

$t \in \mathbf{Z}_+$. If $N = 1$, the system (1.20)–(1.21) is:

$$x(t+1) = A_0(t)x(t) + B_0(t)u(t) + G_0(t)v(t) + \sum_{k=1}^{r}[A_k(t)x(t) + B_k(t)u(t) + G_k(t)v(t)]w_k(t) \quad (1.24)$$

$$y(t) = C_0(t)x(t) + D_0(t)v(t) + \sum_{k=1}^{r}[C_k(t)x(t) + D_k(t)v(t)]w_k(t), \quad (1.25)$$

$t \in \mathbf{Z}_+$, with $A_k(t) = A_k(t,1), B_k(t) = B_k(t,1), G_k(t) = G_k(t,1), C_k(t) = C_k(t,1), D_k(t) = D_k(t,1), 0 \leq k \leq r, t \in \mathbf{Z}_+$.

The equations (1.22)–(1.23) are the mathematical model of a discrete-time time-varying controlled stochastic linear system affected by a Markov-type perturbation, whereas (1.24)–(1.25) is the mathematical model of a discrete-time time-varying controlled stochastic linear system affected by a sequence of independent random perturbations (often named *discrete-time white noise*).

In short, a control problem associated with the system (1.20)–(1.21) asks us to construct a control law $u(t) = \mathcal{K}(t, y(t)), t \geq 0$, with the property that in the absence of the exogenous perturbations $v(t)$, the trajectories of the closed-loop system tend towards the equilibrium $x \equiv 0$, when $t \to \infty$.

Additionally it is desired that in the presence of exogenous perturbations $v(t)$ the trajectories of the corresponding closed-loop system satisfy some prescribed performances (other than stable behavior). Usually the prescribed performances imposed on the closed-loop system are expressed in terms of minimization of several norms of some adequately chosen outputs.

Such outputs are of the form

$$z(t) = C_z(t, \eta_t)x(t) + D_z(t, \eta_t)u(t) \tag{1.26}$$

and are known as controlled outputs or quality outputs.

The control laws considered are in the form of dynamic controllers or static controllers (memoryless controllers).

The class of dynamic controllers is described by

$$\begin{aligned} x_c(t+1) &= A_c(t, \eta_t)x_c(t) + B_c(t, \eta_t)u_c(t) \\ y_c(t) &= C_c(t, \eta_t)x_c(t), \qquad t \geq 0, \end{aligned} \tag{1.27}$$

where $x_c(t) \in \mathbf{R}^{n_c}$ is the state vector of the controller, $u_c(t) \in \mathbf{R}^p$ is the input signal of the controller, and $y_c \in \mathbf{R}^m$ stands for the output signal of the controller.

The integer $n_c \geq 1$ is the order of the controller.

In some control problems n_c is prefixed, whereas in other problems it must be chosen together with the matrix coefficients $A_c(t,i), B_c(t,i), C_c(t,i)$.

Let us couple a controller (1.27) to a system (1.20)–(1.21) taking $u_c(t) = y(t)$ and $u(t) = y_c(t)$.

The corresponding closed-loop system is

$$\begin{aligned} x_{cl}(t+1) &= A_{0cl}(t, \eta_t)x_{cl}(t) + G_{0cl}(t, \eta_t)v(t) \\ &\quad + \sum_{k=1}^{r} [A_{kcl}(t\,\eta_t)x_{cl}(t) + G_{kcl}(t, \eta_t)v(t)]w_k(t) \\ z_{cl}(t) &= C_{cl}(t, \eta_t)x_{cl}(t), \end{aligned} \tag{1.28}$$

where

$$A_{0cl}(t,i) = \begin{pmatrix} A_0(t,i) & B_0(t,i)C_c(t,i) \\ B_c(t,i)C_0(t,i) & A_c(t,i) \end{pmatrix}$$

$$A_{kcl}(t,i) = \begin{pmatrix} A_k(t,i) & B_k(t,i)C_c(t,i) \\ B_c(t,i)C_k(t,i) & 0 \end{pmatrix}, \qquad 1 \leq k \leq r$$

$$G_{kcl}(t,i) = \begin{pmatrix} G_k(t,i) \\ B_c(t,i)D_k(t,i) \end{pmatrix}, \qquad 0 \leq k \leq r$$

$$C_{cl}(t,i) = (C_z(t,i) \quad D_z(t,i)C_c(t,i))$$
$$x_{cl} = (x^T(t) \quad x_c^T(t))^T.$$

The closed-loop system (1.28) is obtained under the assumption that the measurements y_t are transferred instantaneously to the controller and the output of y_c of the controller is transmitted instantaneously to the actuators.

If on the channel from the controller to actuators a delay appears in transmission of the dates then $u(t)$ is given by $u(t) = y_c(t-1)$. In this case the corresponding closed-loop system is given by

$$\begin{aligned}
x(t+1) &= A_0(t,\eta_t)x(t) + B_0(t,\eta_t)C_c(t-1,\eta_{t-1})x_c(t-1) + G_0(t,\eta_t)v(t) \\
&\quad + \sum_{k=1}^{r}[A_k(t,\eta_t)x(t) + B_k(t,\eta_t)C_c(t-1,\eta_{t-1})x_c(t-1) \\
&\quad + G_k(t,\eta_t)v(t)]w_k(t) \\
x_c(t+1) &= B_c(t,\eta_t)C_0(t,\eta_t)x(t) + A_c(t,\eta_t)x_c(t) + B_c(t,\eta_t)D_0(t,\eta_t)v(t) \\
&\quad + \sum_{k=1}^{r}[B_c(t,\eta_t)C_k(t,\eta_t)x(t) + B_c(t,\eta_t)D_k(t,\eta_t)v(t)]w_k(t) \quad (1.29) \\
z(t) &= C_z(t,\eta_t)x(t) + D_z(t,\eta_t)C_c(t-1,\eta_{t-1})x_c(t-1).
\end{aligned}$$

Setting $\xi_c(t) = x_c(t-1)$ we rewrite (1.29) in a compact form as follows.

$$\begin{aligned}
x_{cl}(t+1) &= A_{0cl}(t,\eta_t,\eta_{t-1})x_{cl}(t) + G_{0cl}(t,\eta_t)v(t) \\
&\quad + \sum_{k=1}^{r}[A_{kcl}(t,\eta_t,\eta_{t-1})x_{cl}(t) + G_k(t,\eta_t)v(t)]w_k(t) \qquad (1.30) \\
z_{cl}(t) &= C_{cl}(t,\eta_t,\eta_{t-1})x_{cl}(t),
\end{aligned}$$

where

$$x_{cl}(t) = (x^T(t) \quad \xi_c^T(t) \quad x_c^T(t))^T$$

$$A_{0cl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} A_0(t,\eta_t) & B_0(t,\eta_t)C_c(t-1,\eta_{t-1}) & 0 \\ 0 & 0 & I_{n_c} \\ B_c(t,\eta_t)C_0(t,\eta_t) & A_c(t,\eta_t) & 0 \end{pmatrix}$$

$$A_{kcl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} A_k(t,\eta_t) & B_k(t,\eta_t)C_c(t-1,\eta_{t-1}) & 0 \\ 0 & 0 & 0 \\ B_c(t,\eta_t)C_k(t,\eta_t) & 0 & 0 \end{pmatrix}, \quad 1 \le k \le r$$

$$G_{kcl}(t,\eta_t) = \begin{pmatrix} G_k(t,\eta_t) \\ 0 \\ B_c(t,\eta_t)D_k(t,\eta_t) \end{pmatrix}, \quad 0 \le k \le r$$

$$C_{cl}(t,\eta_t,\eta_{t-1}) = \big(C_z(t,\eta_t)\ D_z(t,\eta_t)C_c(t-1,\eta_{t-1})\ 0\big)$$
$$z_{cl}(t) = z(t).$$

If $u_c(t) = y(t-1), u(t) = y_c(t)$ the corresponding closed-loop system becomes:

$$\begin{aligned}
x(t+1) &= A_0(t,\eta_t)x(t) + B_0(t,\eta_t)C_c(t,\eta_t)x_c(t) + G_0(t,\eta_t)v(t)\\
&\quad + \sum_{k=1}^{r}[A_k(t,\eta_t)x(t) + B_k(t,\eta_t)C_c(t,\eta_t)x_c(t) + G_k(t,\eta_t)v(t)]w_k(t)\\
x_c(t+1) &= B_c(t,\eta_t)C_0(t-1,\eta_{t-1})x(t-1) + A_c(t,\eta_t)x_c(t)\\
&\quad + B_c(t,\eta_t)D_0(t-1,\eta_{t-1})v(t-1)\\
&\quad + \sum_{k=1}^{r}[B_c(t,\eta_t)C_k(t-1,\eta_{t-1})x(t-1)\\
&\quad + B_c(t,\eta_t)D_k(t-1,\eta_{t-1})v(t-1)]w_k(t-1), \qquad t \geq 1,\\
z(t) &= C_z(t,\eta_t)x(t) + D_z(t,\eta_t)C_c(t,\eta_t)x_c(t).
\end{aligned}$$

Setting $\xi(t) = x(t-1), \hat{v}(t) = (v^T(t)\ v^T(t-1))^T, \hat{w}(t) = (\hat{w}_1(t)\ \dots\ \hat{w}_{2r}(t))^T$ with

$$\hat{w}_k(t) = \begin{cases} w_k(t), & 1 \leq k \leq r;\\ w_{k-r}(t-1), & r+1 \leq k \leq 2r \end{cases}$$

one obtains the following compact form of the closed-loop system,

$$\begin{aligned}
x_{cl}(t+1) &= A_{0cl}(t,\eta_t,\eta_{t-1})x_{cl}(t) + G_{0cl}(t,\eta_t,\eta_{t-1})\hat{v}(t)\\
&\quad + \sum_{k=1}^{2r}[A_{kcl}(t,\eta_t,\eta_{t-1})x_{cl}(t) + G_{kcl}(t,\eta_t,\eta_{t-1})\hat{v}(t)]\hat{w}_k(t) \qquad (1.31)\\
z_{cl}(t) &= C_{cl}(t,\eta_t,\eta_{t-1})x_{cl}(t), t \geq 1,
\end{aligned}$$

where

$$A_{0cl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} 0 & I_n & 0\\ 0 & A_0(t,\eta_t) & B_0(t,\eta_t)C_c(t,\eta_t)\\ B_c(t,\eta_t)C_0(t-1,\eta_{t-1}) & 0 & A_c(t,\eta_t) \end{pmatrix}$$

$$A_{kcl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} 0 & 0 & 0\\ 0 & A_k(t,\eta_t) & B_k(t,\eta_t)C_c(t,\eta_t)\\ 0 & 0 & 0 \end{pmatrix}, \qquad 1 \leq k \leq r$$

$$A_{kcl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} 0 & 0\ 0\\ 0 & 0\ 0\\ B_c(t,\eta_t)C_{k-r}(t-1,\eta_{t-1}) & 0\ 0 \end{pmatrix}, \qquad r+1 \leq k \leq 2r$$

$$G_{0cl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} 0 & 0 \\ G_0(t,\eta_t) & 0 \\ 0 & B_c(t,\eta_t)D_0(t-1,\eta_{t-1}) \end{pmatrix}$$

$$G_{kcl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} 0 & 0 \\ G_k(t,\eta_t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad 1 \leq k \leq r$$

$$G_{kcl}(t,\eta_t,\eta_{t-1}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & B_c(t,\eta_t)D_{k-r}(t-1,\eta_{t-1}) \end{pmatrix}, \qquad r+1 \leq k \leq 2r$$

$$C_{cl} = (t,\eta_t,\eta_{t-1}) = \begin{pmatrix} 0 & C_z(t,\eta_t) & D_z(t,\eta_t)C_c(t,\eta_t) \end{pmatrix}.$$

In the case when there are delays in transmission of the data both from the sensors to the controller and from the controller to the actuators, the coupling may be done as follows, $u_c(t) = y(t-1), u(t) = y_c(t-1), t \geq 1$.

In this case the corresponding closed-loop system is of the form (1.31) with

$$x_{cl}(t) = \begin{pmatrix} x^T(t-1) & x^T(t) & x_c^T(t-1) & x_c^T(t) \end{pmatrix}^T.$$

The structure of the matrix coefficients of the closed-loop system may be easily derived.

Remark 1.3 If in the system (1.20)–(1.21) either $B_k(t,i) = 0, 1 \leq k \leq r$ or $C_k(t,i) = 0, 1 \leq k \leq r, i \in \mathcal{D}$ then a wider class of admissible controllers can be considered. We refer to dynamic controllers of the form:

$$\begin{aligned} x_c(t+1) &= A_c(t,\eta_t)x_c(t) + B_c(t,\eta_t)u_c(t) \\ y_c(t) &= C_c(t,\eta_t)x_c(t) + D_c(t,\eta_t)u_c(t). \end{aligned} \tag{1.32}$$

According to the terminology used in the case of control of deterministic linear systems the class of admissible controllers (1.27) is called strictly proper controllers, and controllers of type (1.32) are called proper controllers.

In each control problem the class of admissible controllers is taken either among strictly proper controllers (1.27) or among proper controllers (1.32). The type of considered controllers depends upon the well-posedness of the closed-loop system.

1.7 The outline of the book

The material contained in this book could be divided in two main parts. Thus Chapters 2–5 have a strong theoretic character. In these chapters we offer the readers the mathematical machinery necessary for a good understanding of the robust control problems investigated in the second part of the book. In Chapters 6–8 we solve several problems of robust control for discrete-time linear stochastic systems of type (1.20)–(1.21). In the remainder of this section we briefly present the contents of each chapter.

Chapter 2 deals with discrete-time linear equations defined by positive operators. The main goal of the developments in this chapter is to provide a characterization of exponential stability of the equilibrium $x = 0$ in the case of discrete-time linear equations defined by positive operators on an ordered Hilbert space. The criteria for exponential stability derived in this chapter may be viewed as an alternative approach to the problem of exponential stability other than the one based on Lyapunov functions or by Hurwitz criteria.

The results derived in Chapter 2 are used in Chapter 3 in order to characterize exponential stability in the mean square of discrete-time linear stochastic systems. Also in Chapter 5 they are involved in the proof of the existence of some special solutions of Riccati-type equations (maximal solution, stabilizing solution, minimal solution).

Chapter 3 deals with the problem of exponential stability in the mean square of a class of discrete-time stochastic linear systems subject both to independent random perturbations and Markovian switching. We have in mind discrete-time stochastic linear systems that contain as particular cases systems of type (1.28), (1.31) with $G_{kcl}(t, i) = 0$. We show that in the case of discrete-time stochastic linear systems subject to Markovian jumping there are several ways to define the exponential stability in the mean square. We show that the different types of exponential stability in the mean square are not always equivalent. We emphasize important classes of discrete-time stochastic linear systems with Markovian switching for which the concepts of exponential stability in the mean square become equivalent.

In Chapter 4 we investigate several structural properties of stochastic controlled systems such as stabilizability, detectability, and observability. We also provide useful criteria that allow us to check such properties.

In Chapter 5 we consider a general class of nonlinear difference equations containing as particular cases several types of discrete-time Riccati equations involved in many control problems studied in the following chapters. The results developed in Chapter 5 can also be used in the case of Riccati-type equations arising in connection with digital control of deterministic systems and stochastic systems.

In Chapter 6 we deal with several linear quadratic control problems. In short, by a linear quadratic optimization problem we understand the problem of minimization of a quadratic cost functional along the trajectories of a linear controlled system.

Chapter 7 deals with the problem of H_2 optimal control for discrete-time linear stochastic systems subject to sequences of independent random perturbations and Markovian switching. Several kinds of H_2-type performance criteria (often called H_2 norms) are introduced and characterized via solutions of some suitable linear equations on the spaces of symmetric matrices.

The purpose of such performance criteria is to provide a measure of the effect of additive white noise perturbation over an output of the controlled system. The H_2 optimal control is solved both in the case of full access of the measurements and in the case of partial access to the measurements. Many aspects specific to the discrete-time linear stochastic systems perturbed by a Markov chain are emphasized. The chapter ends with an H_2 optimal filtering problem.

In the first part of Chapter 8 we prove a stochastic version of the so-called bounded real lemma. As is known, such a result provides a necessary and sufficient condition that guarantees the norm of an input–output operator from exogenous perturbations $v(t)$ to the controlled output $z(t)$ is less than a prescribed level $\gamma > 0$. Furthermore we prove the small gain theorem and we introduce the notion of stability radius (several estimates of stability radius are derived). In the second part of Chapter 8 the problem of attenuation of the exogenous perturbations under the assumption that the full state is accessible for measurements is solved.

All theoretical developments are illustrated by numerical case studies.

1.8 Notes and references

The results stated in this chapter for which the proofs are omitted are well known and can be found in most monographs about stochastic process theory (see, for instance, [32]). Theorem 1.1 is proved in [114]. Interesting applications of discrete-time linear systems corrupted by sequences of independent random perturbations (often known as discrete-time white noise perturbations) and/or Markovian jumping to model different real-life problems can be seen in [8–12, 27, 63, 83, 84] and their references.

2

Discrete-time linear equations defined by positive operators

In this chapter we study a class of discrete-time deterministic linear equations, namely discrete-time equations defined by sequences of positive linear operators acting on ordered Hilbert spaces. As we show in Chapter 3 such equations play a crucial role in the derivation of some useful criteria for exponential stability in the mean square of the stochastic systems considered in this book.

The results proved in this chapter also provide some powerful devices that help us to prove the existence of some global solutions, maximal solutions, minimal solutions, and stabilizing solutions of a large class of nonlinear equations including Riccati-type equations.

We want to mention that the results of this chapter may be successfully used to derive the solution of some control problems for deterministic positive systems with applications in economy, finances, biology, and so on. Such applications exceed the purpose of this monograph.

2.1 Some preliminaries

In the first part of this section we recall several definitions concerning convex cones and ordered linear spaces, and provide some basic results.

In the second part of this section we investigate the properties of the Minkovski functional and we provide conditions under which such a functional becomes a norm. The Minkovski norm plays a crucial role in our developments in this chapter.

2.1.1 Convex cones

Let $(\mathcal{X}, \|\cdot\|)$ be a real-normed linear space. As usual, if $\mathcal{X}$ is a Hilbert space we use $|\cdot|_2$ instead of $\|\cdot\|$.

Definition 2.1

(a) *A subset $\mathcal{C} \subset \mathcal{X}$ is called a cone if:*

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(i) $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$.
(ii) $\alpha\mathcal{C} \subset \mathcal{C}$ *for all* $\alpha \in \mathbf{R}, \alpha \geq 0$.
(b) A cone $\mathcal{C}$ *is called a* pointed cone *if* $\mathcal{C} \bigcap (-\mathcal{C}) = \{0\}$.
(c) A cone $\mathcal{C}$ *is called a* solid cone *if its interior* Int$\mathcal{C}$ *is not empty.*

We recall that if A, B are two subsets of $\mathcal{X}$ and $\alpha \in \mathbf{R}$, then $A + B = \{x + y | x \in A, y \in B\}$ and $\alpha A = \{\alpha x | x \in A\}$.

It is easy to see that a cone $\mathcal{C}$ is a convex subset and thus we often say *convex cone* when we refer to a cone. A cone $\mathcal{C} \subset \mathcal{X}$ induces an ordering "$\leq$" on $\mathcal{X}$, by $x \leq y$ (or equivalently $y \geq x$) if and only if $y - x \in \mathcal{C}$. If $\mathcal{C}$ is a solid cone then $x < y$ (or equivalently $y > x$) if and only if $y - x \in \text{Int}\mathcal{C}$. Hence $\mathcal{C} = \{x \in \mathcal{X} | x \geq 0\}$ and Int$\mathcal{C} = \{x \in \mathcal{X} | x > 0\}$.

Definition 2.2 *If* $\mathcal{C} \subset \mathcal{X}$ *is a cone then* $\mathcal{C}^* \subset \mathcal{X}^*$ *is called the* dual cone *of* $\mathcal{C}$ *if* $\mathcal{C}^*$ *consists of all bounded and linear functionals* $f \in \mathcal{X}^*$ *with the property that* $f(x) \geq 0$ *for all* $x \in \mathcal{C}$.

Based on the Ritz theorem for representation of a bounded linear functional on a Hilbert space one sees that if $\mathcal{X}$ is a real Hilbert space then the dual cone $\mathcal{C}^*$ of a convex cone $\mathcal{C}$ may be defined as $\mathcal{C}^* = \{y \in \mathcal{X} | \langle y, x \rangle \geq 0, \forall x \in \mathcal{C}\}$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{X}$.

If $\mathcal{X}$ is a real Hilbert space a cone $\mathcal{C}$ is called *selfdual* if $\mathcal{C}^* = \mathcal{C}$.

Lemma 2.1 *Let* $\mathcal{X}$ *be a real Banach space and* $\mathcal{C} \subset \mathcal{X}$ *a solid convex cone. Then* $\mathcal{C}^*$ *is a closed and pointed cone.*

Proof. Let $\varphi \in \bar{\mathcal{C}}^*$. Therefore there exists a sequence $\{\varphi_k\}_{k \geq 1} \subset \mathcal{C}^*$ such that $\lim_{k \to \infty} \varphi_k(x) = \varphi(x)$ for all $x \in \mathcal{X}$. For $x \in \mathcal{C}$ we have $\varphi(x) = \lim_{k \to} \varphi_k(x) \geq 0$. Hence $\varphi \in \mathcal{C}^*$ and $\mathcal{C}^*$ is a closed set. To show that $\mathcal{C}^*$ is a pointed cone, we choose $\varphi \in \mathcal{C}^* \bigcap (-\mathcal{C}^*)$. This leads to $\varphi(x) = 0$ for all $x \in \mathcal{C}$. We have to show that $\varphi(x) = 0$ for all $x \in \mathcal{X}$.

Let $x_0 \in \mathcal{X}$ be arbitrary. Let $\xi \in \text{Int}\mathcal{C}$ be fixed. For $\varepsilon > 0$ small enough we have that $\xi + \varepsilon x_0 \in \mathcal{C}$. Hence $\varphi(\xi + \varepsilon x_0) = 0$. Because $\varphi(\xi) = 0$ we conclude that $\varphi(x_0) = 0$. Thus we obtain that $\mathcal{C}^* \bigcap (-\mathcal{C}^*) = \{0\}$ and the proof is complete. □

In the finite-dimensional case we have the following.

Proposition 2.1 *Let* $\mathcal{X}$ *be a finite-dimensional real Banach space and* $\mathcal{C} \subset \mathcal{X}$ *be a closed, pointed, solid, convex cone. Then the dual cone* $\mathcal{C}^*$ *is a closed, pointed, and solid convex cone.*

Proof. The fact that $\mathcal{C}^*$ is a closed and pointed convex cone follows from the previous lemma. It remains to show that Int$\mathcal{C}^*$ is not empty. Applying Theorem 2.1 [79] one deduces that there exists $\varphi_0 \in \mathcal{C}^*$, such that $\varphi_0(x) > 0$ for all $x \in \mathcal{C}\{0\}$. Because $\mathcal{X}$ is a finite-dimensional linear space it follows that $S_1 = \{x \in \mathcal{C} | \|x\| = 1\}$ is a compact set. Hence there exists $\delta > 0$, such that

$\varphi_0(x) \geq 2\delta, \forall x \in S_1$. Consider the closed ball $B(\varphi_0, \delta) = \{\varphi \in \mathcal{X}^* | \|\varphi - \varphi_0\| \leq \delta\}$. We show that $B(\varphi_0, \delta) \subset \mathcal{C}^*$. If $\varphi \in B(\varphi_0, \delta)$ then $|\varphi(x) - \varphi_0(x)| \leq \delta$, $\forall x \in \mathcal{X}$, with $\|x\| = 1$. Particularly, for $x \in S_1$ we have $\varphi(x) - \varphi_0(x) \geq -\delta$. Hence $\varphi(x) = \varphi_0(x) + (\varphi(x) - \varphi_0(x)) \geq 2\delta - \delta = \delta$.

Thus we have proved that $\varphi(x) \geq \delta\|x\|$ for all $x \in \mathcal{C}\{0\}$ and for all $\varphi \in B(\varphi_0, \delta)$. Hence $B(\varphi_0, \delta) \subset \mathcal{C}^*$ that means $\varphi_0 \in \mathrm{Int}\mathcal{C}^*$ and thus the proof is complete. □

In the last part of this subsection we recall (see [77]) the following.

Definition 2.3 *A cone $\mathcal{C} \subset \mathcal{X}$ is a* regular cone *if for all sequences $\{x_t\}_{t\geq 1} \subseteq \mathcal{X}$ that satisfy $x_1 \geq x_2 \geq \cdots \geq x_t \geq \cdots \geq y$ for some $y \in \mathcal{X}$ not depending upon t, then there exists $x \in \mathcal{X}$ such that* $\lim_{t\to\infty} \|x_t - x\| = 0$.

Example 2.1 Let $\mathcal{X} = \mathbf{R}^n$ and $\mathcal{C} = \mathbf{R}^n_+$ where $\mathbf{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n | x_i \geq 0, 1 \leq i \leq n\}$. In this case $\mathcal{C}$ is a closed, solid, pointed, selfdual, regular convex cone. The ordering induced on $\mathbf{R}^n$ by this cone is known as componentwise ordering.

The next result follows immediately from Lemma 1.8 and Theorem 1.12 in [77].

Proposition 2.2 *If $\mathcal{X}$ is a real Banach space and $\mathcal{C} \in \mathcal{X}$ is a closed, pointed, and solid convex cone then the dual cone $\mathcal{C}^*$ is regular.*

Corollary 2.1 *If $\mathcal{X}$ is a real Hilbert space and $\mathcal{C} \subset \mathcal{X}$ is a closed, pointed, solid, and selfdual convex cone then $\mathcal{C}$ is a regular cone.*

2.1.2 Minkovski seminorms and Minkovski norms

Let $\mathcal{X}$ be a real normed linear space. Let $\mathcal{C} \subset \mathcal{X}$ be a solid convex cone. Assume that $\mathcal{C} \neq \mathcal{X}$. This means that $0 \notin \mathrm{Int}\mathcal{C}$. Let $\xi \in \mathrm{Int}\mathcal{C}$ be fixed. Denote by B_ξ the set defined by $B_\xi = \{x \in \mathcal{X} | -\xi < x < \xi\}$. It is easy to see that

$$B_\xi = (\xi - \mathrm{Int}\mathcal{C}) \bigcap (-\xi + \mathrm{Int}\mathcal{C}). \tag{2.1}$$

From (2.1) one deduces that B_ξ is an open and convex set. For each $x \in \mathcal{X}$, we denote $\mathcal{T}(x) = \{t \in \mathbf{R} | t > 0, (1/t)x \in B_\xi\}$. Because B_ξ is an open set and $0 \in B_\xi$ it follows that $\mathcal{T}(x)$ is not empty for all $x \in \mathcal{X}$. The Minkovski functional associated with the set B_ξ is defined by

$$|x|_\xi = \inf \mathcal{T}(x) \tag{2.2}$$

for every $x \in \mathcal{X}$.

The next theorem collects several important properties of the Minkovski functional.

Theorem 2.1 *The Minkovski functional introduced in* (2.2) *has the properties:*

(*i*) $|x|_\xi \geq 0$ *and* $|0|_\xi = 0$.
(*ii*) $|\alpha x|_\xi = |\alpha||x|_\xi$ *for all* $\alpha \in \mathbf{R}, x \in \mathcal{X}$.
(*iii*) $|x|_\xi < 1$ *if and only if* $x \in B_\xi$.
(*iv*) $|x+y|_\xi \leq |x|_\xi + |y|_\xi$ *for all* $x, y \in \mathcal{X}$.
(*v*) *There exists* $\beta(\xi) > 0$ *such that* $|x|_\xi \leq \beta(\xi)\|x\|, \forall x \in \mathcal{X}$.
(*vi*) $|x|_\xi = 1$ *if and only if* $x \in \partial B_\xi$, ∂B_ξ *being the border of the set* B_ξ.
(*vii*) $|x|_\xi \leq 1$ *iff* $x \in \bar{B}_\xi$ *where* $\bar{B}_\xi = B_\xi \bigcup \partial B_\xi$.
(*viii*) *If* $\mathcal{C}$ *is a closed, solid, convex cone then* $\bar{B}_\xi = \{x \in \mathcal{X} | -\xi \leq x \leq \xi\}$.
(*ix*) $|\xi|_\xi = 1$.
(*x*) *The set* $\mathcal{T}(x)$ *coincides with* $(|x|_\xi, \infty)$.
(*xi*) *If* $x, y, z \in \mathcal{X}$ *are such that* $y \leq x \leq z$ *then* $|x|_\xi \leq \max\{|y|_\xi, |z|_\xi\}$.

Proof. (*i*) follows immediately from the definition of $|\cdot|_\xi$.

(*ii*) Let $\alpha > 0$ be fixed. It is easy to see that $t \in \mathcal{T}(\alpha x)$ iff $\alpha^{-1}t \in \mathcal{T}(x)$. This leads to $\mathcal{T}(\alpha x) = \alpha\mathcal{T}(x)$. Taking the infimum, one concludes that $|\alpha x|_\xi = \alpha|x|_\xi = |\alpha||x|_\xi$.

On the other hand $x \in B_\xi$ iff $-x \in B_\xi$. This allows us to deduce $|-x|_\xi = |x|_\xi$. Let $\alpha < 0$ be fixed. We have $|\alpha x|_\xi = |-|\alpha|x|_\xi = ||\alpha|x|_\xi = |\alpha||x|_\xi$ and thus one obtains that (*ii*) holds.

(*iii*) Let $x \in \mathcal{X}$ be such that $|x|_\xi < 1$. This means that there exists $t \in (0,1)$ such that $-t\xi < x < t\xi$, hence, $-\xi < x < \xi$, therefore $x \in B_\xi$. Conversely let $x \in B_\xi$. Based on the continuity at $t = 1$ of the function $t \to (1/t)x$ we obtain that there exist $t_1 \in (0,1)$ such that $(1/t_1)x \in B_\xi$. Hence, in this case we have $\mathcal{T}(x) \bigcap (0,1) \neq \phi$. This leads to $|x|_\xi < 1$ and thus (*iii*) is true.

To prove (*iv*) it is enough to show that if $\tau > |x|_\xi + |y|_\xi$ then $\tau > |x+y|_\xi$. Let $\tau > |x|_\xi + |y|_\xi$ and define $\varepsilon = \frac{1}{2}(\tau - |x|_\xi - |y|_\xi)$. Let $\tau_1 = \varepsilon + |x|_\xi$, $\tau_2 = \varepsilon + |y|_\xi$. We have

$$\tau_1 + \tau_2 = \tau, \qquad \tau_1 > |x|_\xi, \tau_2 > |y|_\xi \tag{2.3}$$

$$\frac{1}{\tau}(x+y) = \frac{\tau_1}{\tau}x_1 + \frac{\tau_2}{\tau}y_1, \tag{2.4}$$

where $x_1 = (1/\tau_1)x$, $y_1 = (1/\tau_2)y$. From (2.3) and (*iii*) we deduce that $x_1, y_1 \in B_\xi$. Because B_ξ is a convex set and $(\tau_1/\tau) + (\tau_2/\tau) = 1$ we get, using (2.4), that $(1/\tau)(x+y) \in B_\xi$.

Invoking (*iii*) again we have $|(1/\tau)(x+y)|_\xi < 1$. Applying (*iii*) we have $|x+y|_\xi < \tau$ and thus (*iv*) is proved.

(*v*) $\xi \in \mathrm{Int}\mathcal{C}$, thus it follows that there exists $\delta(\xi) > 0$ such that the ball $B(\xi, \delta(\xi)) \subset \mathrm{Int}\mathcal{C}$, with $B(\xi, \delta(\xi)) = \{x \in \mathcal{X} | \|x - \xi\| \leq \delta(\xi)\}$.

Let $x \in \mathcal{X}, x \neq 0$. Because $\xi \pm \delta(\xi)x/\|x\| > 0$, one obtains that $(\delta(\xi)x)/\|x\| \in B_\xi$. From (*iii*) we have $|\delta(\xi)(x/\|x\|)|_\xi < 1$. Using (*ii*) we deduce $|x|_\xi < \beta(\xi)\|x\|$ for all $x \in \mathcal{X}$, with $\beta(\xi) = \delta^{-1}(\xi)$; thus (*v*) is proved.

(vi) Let $x \in \mathcal{X}$ with $|x|_\xi = 1$. This means that there exists a sequence $\{t_k\}_{k\geq 1}$ with $t_k > 1$, $\lim_{k\to\infty} t_k = 1$, and $(1/t_k)x \in B_\xi$. From $x = \lim_{k\to\infty}(1/t_k)x$ one obtains that $x \in \bar{B}_\xi$. Based on (iii) $x \notin B_\xi$. It follows that $x \in \partial B_\xi$.

To prove the converse inclusion we choose $x \in \partial B_\xi$ and assume that $|x|_\xi > 1$. Set $\varepsilon = |x|_\xi - 1$. Let V be the open ball; $V = \{y \in \mathcal{X} | \|y - x\| < \varepsilon/(\beta(\xi))\}$ where $\beta(\xi)$ is the constant from (v). Because $x \in \partial B_\xi$ it follows that there exist $y \in B_\xi \bigcap V$; this means that $\|y - x\| < \varepsilon/(\beta(\xi))$. From (iii) one gets that $|y|_\xi < 1$. Combining $(ii), (iv)$, and (v) one obtains successively $\varepsilon < |x|_\xi - |y|_\xi \leq |x - y|_\xi \leq \beta(\xi)\|x - y\| < \varepsilon$ which is a contradiction. Hence $|x|_\xi \leq 1$. On the other hand, because B_ξ is an open set we have that $B_\xi \bigcap \partial B_\xi = \emptyset$. This means that $x \notin B_\xi$. Hence $|x|_\xi \geq 1$. Therefore $|x|_\xi = 1$ and (vi) is proved.

(vii) It follows from (iii) and (vi).

$(viii)$ Let $x \in \bar{B}_\xi$. From (vii) we have $|x|_\xi \leq 1$. If $|x|_\xi < 1$ then $x \in B_\xi$ which is equivalent to $-\xi < x < \xi$, implying $-\xi \leq x \leq \xi$. If $|x|_\xi = 1$ then there exists a sequence $\{t_k\}_{k\geq 1}, t_k > 1, \lim_{k\to\infty} t_k = 1$ and $(1/t_k)x \in B_\xi$. This means that $\xi \pm (1/t_k)x \in \mathcal{C}$.

Taking the limit for $k \to \infty$ and taking into account that $\mathcal{C}$ is a closed set we conclude that $\xi \pm x \in \mathcal{C}$; this is equivalent to $-\xi \leq x \leq \xi$. Conversely, if $x \in \mathcal{X}$, is such that $-\xi \leq x \leq \xi$, then for all $t > 1$ we have $-t\xi < x < t\xi$.

This means that $(1/t)x \in B_\xi$. Taking the limit for $t \to 1$ we get $x \in \bar{B}_\xi$.

(ix) It follows from (vi).

(x) Let $t \in (|x|_\xi, \infty)$. This is equivalent to $|(1/t)x|_\xi < 1$. Hence $(1/t)x \in B_\xi$. This means that $t \in \mathcal{T}(x)$. Thus we have proved that $(|x|_\xi, \infty) \subset \mathcal{T}(x)$. To prove the converse inclusion we choose $t \in \mathcal{T}(x)$; that is, $(1/t)x \in B_\xi$. From (ii) and (iii) we have $|x|_\xi < t$; that is, $t \in (|x|_\xi, \infty)$.

(xi) Let $x, y, z \in \mathcal{X}$ be such that $y \leq x \leq z$. Let us assume that $\max\{|y|_\xi, |z|_\xi\} < |x|_\xi$. Let t be such that:

$$\max\{|y|_\xi, |z|_\xi\} < t < |x|_\xi. \tag{2.5}$$

Based on (x), it follows that $t \in \mathcal{T}(y) \bigcap \mathcal{T}(z)$. This means that $-\xi < (1/t)y < \xi$ and $-\xi < (1/t)z < \xi$. This leads to

$$-\xi < \frac{1}{t}y \leq \frac{1}{t}x \leq \frac{1}{t}z < \xi$$

and it follows that $t \in \mathcal{T}(x)$. Invoking (x) again one deduces that $t > |x|_\xi$ which is a contradiction to (2.5) and thus the proof ends. □

From $(i), (ii)$, and (iv) of the previous theorem it follows that the Minkovski functional defined by (2.2) is a seminorm.

The next theorem provides a condition which ensures that the seminorm (2.2) is just a norm.

Theorem 2.2 *If B_ξ is a bounded set then the Minkovski seminorm $|\cdot|_\xi$ defined by (2.2) is a norm. Moreover there exists $\alpha_\xi > 0$ such that $\|x\| \leq \alpha_\xi |x|_\xi$ for all $x \in \mathcal{X}$.*

Proof. To prove that $|\cdot|_\xi$ is a norm we have to prove that if $|x|_\xi = 0$ then $x = 0$. If $|x|_\xi = 0$ then, from (x) of Theorem 2.1, we have that $\mathcal{T}(x) = (0, \infty)$. Hence, for all $t > 0$ we have $(1/t)x \in B_\xi$. Because B_ξ is a bounded set it follows that $\|(1/t)x\| \leq \alpha$, with $\alpha > 0$ not depending upon x and t but possibly depending upon ξ. This leads to $\|x\| \leq \alpha t$. Taking the limit for $t \to 0$ one obtains that $\|x\| = 0$, hence $x = 0$. To check the last assertion in the statement we choose $x \in \mathcal{X}, x \neq 0$. Invoking (x) again, we obtain that for all

$$t \in (1, \infty), \qquad \frac{1}{t} \frac{x}{|x|_\xi} \in B_\xi.$$

From the boundedness of B_ξ one obtains that $\|x\| \leq \alpha |x|_\xi$ and thus the proof is complete. □

Corollary 2.2 *Assume that $\mathcal{X}$ is a finite-dimensional real Banach space. Assume also that $\mathcal{C} \subset \mathcal{X}$ is a solid cone, $\mathcal{C} \neq \mathcal{X}$. If $\xi \in \mathrm{Int}\mathcal{C}$ then the following are equivalent.*

(i) The Minkovski seminorm $|x|_\xi$ is a norm.
(ii) B_ξ is a bounded set.

Proof. $(ii) \to (i)$ follows from Theorem 2.2. Suppose (i) holds. Because $\mathcal{X}$ is a finite-dimensional Banach space there exists $\alpha > 0$ (depending on ξ) such that $\|x\| \leq \alpha |x|_\xi$ for all $x \in \mathcal{X}$. Now if $x \in B_\xi$ then $|x|_\xi < 1$, hence $\|x\| < \alpha$; that is, B_ξ is a bounded set. □

In the sequel we provide a sufficient condition which guarantees that $|\cdot|_\xi$ is a norm for all $\xi \in \mathrm{Int}\mathcal{C}$.

To this end we introduce the following.

Definition 2.4 *We say that the $\|\cdot\|$ is* monotone *with respect to the cone $\mathcal{C}$ if from $0 \leq x \leq y$ it follows that $\|x\| \leq \|y\|$.*

Proposition 2.3 *If $\|\cdot\|$ is monotone with respect to the cone $\mathcal{C}$ then for all $\xi \in \mathrm{Int}\mathcal{C}$ the set B_ξ is bounded.*

Proof. Let $\xi \in \mathrm{Int}\mathcal{C}$ be fixed. If $x \in B_\xi$ we have $-\xi < x < \xi$ or equivalently $0 < \xi + x < 2\xi$. Hence $\|x + \xi\| \leq 2\|\xi\|$. This leads to $\|x\| \leq 3\|\xi\|$ and thus the proof is complete. □

Furthermore we prove the following proposition.

Proposition 2.4 *If $\mathcal{X}$ is a real Hilbert space and $\mathcal{C} \subset \mathcal{X}$ is a cone then the following are equivalent.*

(i) The norm $|\cdot|_2$ *is monotone with respect to* $\mathcal{C}$.
(ii) $\mathcal{C} \subset \mathcal{C}^*$.

Proof. $(i) \to (ii)$. Let $x \in \mathcal{C}$. It is easy to see that $0 \leq x \leq x + (1/k)y$, for all $y \in \mathcal{C}$ and $k \geq 1$. If (i) is fulfilled then $|x|_2^2 \leq |x + (1/k)y|_2^2$ which is equivalent to $2\langle x, y\rangle \geq -(1/k)|y|_2^2$.

Taking the limit for $k \to \infty$ we obtain that $\langle x, y\rangle \geq 0$ for all $y \in \mathcal{C}$. This means that $x \in \mathcal{C}^*$, hence $\mathcal{C} \subset \mathcal{C}^*$.

To prove the converse implication let $x, y \in \mathcal{X}$ be such that $0 \leq x \leq y$. This means that both $y-x$ and $y+x$ are in $\mathcal{C}$. If (ii) is fulfilled then $\langle y-x, y+x\rangle \geq 0$. This is equivalent to $(|y|_2)^2 \geq (|x|_2)^2$ which shows that $(ii) \to (i)$ holds. Thus the proof is complete. □

Remark 2.1 If $\mathcal{X}$ is a real Hilbert space such that the norm $|x|_2$ is monotone with respect to the cone $\mathcal{C} \subset \mathcal{X}$ then $\mathcal{C}$ is a pointed cone. Indeed if both $x \in \mathcal{C}$ and $-x \in \mathcal{C}$ then based on $(i) \to (ii)$ of Proposition 2.4 one obtains that $\langle -x, x\rangle \geq 0$ which leads to $\langle x, x\rangle = 0$. Hence $x = 0$. This shows that $\mathcal{C} \bigcap (-\mathcal{C}) = \{0\}$.

The next two examples show that the monotonicity of the norm $\|\cdot\|$ is only a sufficient condition for B_ξ to be a bounded set for all $\xi \in \text{Int}\mathcal{C}$.

Example 2.2 Let $\mathcal{X} = \mathbf{R}^2$ and the cone

$$\mathcal{C} = \{(x, y)^T \in \mathbf{R}^2 | x \geq 0, y \leq x\}. \tag{2.6}$$

It is easy to verify that for all $\xi \in \text{Int}\mathcal{C}$, B_ξ is a bounded set. On the other hand, the dual cone is given by $\mathcal{C}^* = \{(u, v) \in \mathbf{R}^2 | u \geq 0, -u \leq v \leq 0\}$.

Hence $\mathcal{C}^* \subset \mathcal{C}$. From Proposition 2.4 we deduce that the Euclidian norm on $\mathbf{R}^2$ is not monotone with respect to the cone $\mathcal{C}$ defined by (2.6).

Example 2.3 Let $\mathcal{X} = \mathbf{R}^3$ and the cone $\mathcal{C}$ be defined by:

$$\mathcal{C} = \{(x, y, z)^T \in \mathbf{R}^3 | x \geq 0, |y| \leq x, |z| \leq x\}. \tag{2.7}$$

It can be verified that for each $\xi \in \text{Int}\mathcal{C}$, B_ξ is a bounded set. On the other hand the dual cone $\mathcal{C}^*$ is described by $\mathcal{C}^* = \{(u, v, w)^T \in \mathbf{R}^3 | u \geq 0, |v| + |w| \leq u\}$. Obviously $\mathcal{C}^* \subset \mathcal{C}$. Again applying Proposition 2.4 we deduce that the Euclidian norm on $\mathbf{R}^3$ is not monotone with respect to the cone (2.7).

2.2 Discrete-time equations defined by positive linear operators on ordered Hilbert spaces

2.2.1 Positive linear operators on ordered Hilbert spaces

In this subsection as well as in the following $\mathcal{X}$ is a real Hilbert space ordered by the ordering relation "$\leq$" induced by the closed, solid, selfdual, convex cone $\mathcal{X}^+$. Because $\mathcal{X}^+$ is a selfdual convex cone from Lemma 1.8 and

Theorem 1.12 [77] it follows that $\mathcal{X}^+$ is a regular cone and from Remark 2.1 and Proposition 2.4 it follows that $\mathcal{X}^+$ is a pointed cone.

Based on Proposition 2.2 one deduces that if $\mathcal{X}$ is a finite-dimensional real Hilbert space it is enough to assume that $\mathcal{X}^+$ is a closed, pointed, solid, convex cone in order to be sure that it is also a regular cone.

From Proposition 2.4 one obtains that $|\cdot|_2$ defined by

$$|x|_2 = (\langle x, x \rangle)^{1/2} \tag{2.8}$$

is a monotone with respect to $\mathcal{X}^+$.

An example of infinite-dimensional real Hilbert space ordered by a closed, pointed, solid, selfdual, convex cone is given by the following.

Example 2.4 Let $\mathcal{X} = \ell^2(\mathbf{Z}_+, \mathbf{R})$, where $\ell^2(\mathbf{Z}_+, \mathbf{R}) = \{\mathbf{x} = (x_0, x_1, \ldots x_n, \ldots) | x_i \in \mathbf{R}, \sum_{i=0}^{\infty} x_i^2 < \infty\}$. On $\mathcal{X}$ we consider the usual inner product $\langle \mathbf{x}, \mathbf{y} \rangle_{\ell^2} = \sum_{i=0}^{\infty} x_i y_i$ for all $\mathbf{x} = \{x_i\}_{i \geq 0}, \mathbf{y} = \{y_i\}_{i \geq 0}$. We set $\mathcal{X}^+ = \{\mathbf{x} = \{x_i\}_{i \geq 0} | x_0 \geq 0, \sum_{i=1}^{\infty} x_i^2 \leq x_0^2\}$. It is easy to see that $\mathcal{X}^+$ is a closed, pointed, convex cone. In the finite-dimensional case the analogue of this cone is known as a circular cone.

The interior Int$\mathcal{X}^+ = \{\mathbf{x} = \{x_i\}_{i \geq 0} | x_0 > 0, \sum_{i=1}^{\infty} x_i^2 < x_0^2\}$. It remains to prove that $\mathcal{X}^+$ is selfdual.

Let $\mathbf{y} \in (\mathcal{X}^+)^*$. Hence

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\ell^2} \geq 0 \tag{2.9}$$

for all $\mathbf{x} = \{x_i\}_{i \geq 0} \in \mathcal{X}^+$. In particular, taking in (2.9) $\mathbf{x} = \{1, 0, 0, 0\}$ one obtains $y \geq 0$. It is easy to verify that if $y_0 = 0$ then $y_t = 0$ for all $t \geq 1$. Because $y_0 \geq 0$ it is obvious that if $y_t = 0$ for all $t \geq 1$ we have $\mathbf{y} \in \mathcal{X}^+$. Suppose now $\sum_{t=1}^{\infty} y_t^2 > 0$. We take $\tilde{\mathbf{x}} = \{\tilde{x}_i\}_{i \geq 0}$ defined by

$$\tilde{\mathbf{x}}_0 = y_0, \qquad \tilde{x}_i = -\gamma y_i y_0 \tag{2.10}$$

with $\gamma = (\sum_{k=1}^{\infty} y_k^2)^{-1/2}$. Obviously $\tilde{\mathbf{x}} \in \mathcal{X}^+$.

Replacing (2.10) in (2.9) one gets $\sum_{k=1}^{\infty} y_k^2 \leq y_0^2$ which shows that $\mathbf{y} \in \mathcal{X}^+$. Thus it was shown that $(\mathcal{X}^+)^* \subset \mathcal{X}^+$.

Let now $\mathbf{y} = \{y_i\}_{i \geq 0} \in \mathcal{X}^+$. We have to show that (2.9) holds for all $\mathbf{x} \in \mathcal{X}^+$. Indeed for $\mathbf{x} \in \mathcal{X}^+$ we have $(\sum_{k=1}^{\infty} x_k y_k)^2 \leq \sum_{k=1}^{\infty} x_k^2 \sum_{k=1}^{\infty} y_k^2 \leq x_0^2 y_0^2$ which leads to $|\sum_{k=1}^{\infty} x_k y_k| \leq x_0 y_0$. This is equivalent to $-x_0 y_0 \leq \sum_{k=1}^{\infty} x_k y_k \leq x_0 y_0$ which shows that (2.9) is fullfiled. Thus it was proved that $\mathcal{X}^+ \subset (\mathcal{X}^+)^*$. This shows that $\mathcal{X}^+$ is selfdual.

Let $\xi \in$ Int$\mathcal{X}^+$ be fixed; we associate the Minkovski functional $|\cdot|_\xi$ defined by (2.2). Based on Theorem 2.2 and Proposition 2.3 it follows that $|\cdot|_\xi$ is a norm on $\mathcal{X}$. Moreover, from Theorem 2.1(v) and Theorem (2.2) we deduce that $|\cdot|_\xi$ is equivalent to $|\cdot|_2$ defined by (2.8). Hence $(\mathcal{X}, |\cdot|_\xi)$ is a Banach space. Moreover $|\cdot|_\xi$ has the properties:

$\mathbf{P_1}$. If $x, y, x \in \mathcal{X}$ are such that $y \leq x \leq z$ then

$$|x|_\xi \leq \max\{|y|_\xi, |z|_\xi\}. \tag{2.11}$$

$\mathbf{P_2}$. For arbitrary $x \in \mathcal{X}$ with $|x|_\xi \leq 1$ it holds that

$$-\xi \leq x \leq \xi \tag{2.12}$$

and $|\xi|_\xi = 1$.

If $\mathcal{Y}$ is a Banach space, $T : \mathcal{Y} \to \mathcal{Y}$ is a linear bounded operator, and $|\cdot|$ is a norm on $\mathcal{Y}$, then $\|T\| = \sup_{|x|\leq 1} |Tx|$ is the corresponding operator norm.

Remark 2.2

(*a*) Because $|\cdot|_\xi$ and $|\cdot|_2$ are equivalent, then $\|\cdot\|_\xi$ and $\|\cdot\|_2$ are also equivalent. This means that there are two positive constants c_1 and c_2 such that $c_1\|T\|_\xi \leq \|T\|_2 \leq c_2\|T\|_\xi$ for all linear bounded operators $T : \mathcal{X} \to \mathcal{X}$.

(*b*) If $T^* : \mathcal{X} \to \mathcal{X}$ is the adjoint operator of T with respect to the inner product on $\mathcal{X}$, then $\|T\|_2 = \|T^*\|_2$. In general the equality $\|T\|_\xi = \|T^*\|_\xi$ is not true. However, based on (a) it follows that there are two positive constants $\tilde{c}_1, \tilde{c}_2$ such that

$$\tilde{c}_1\|T\|_\xi \leq \|T^*\|_\xi \leq \tilde{c}_2\|T\|_\xi. \tag{2.13}$$

Definition 2.5 *Let* $(\mathcal{X}, \mathcal{X}^+)$ *and* $(\mathcal{Y}, \mathcal{Y}^+)$ *be ordered vector spaces. An operator* $T : \mathcal{X} \to \mathcal{Y}$ *is called* positive*, if* $T(\mathcal{X}^+) \subset \mathcal{Y}^+$*. In this case we write* $T \geq 0$*. If* $T(\mathrm{Int}\mathcal{X}^+) \subset \mathrm{Int}\mathcal{Y}^+$ *we write* $T > 0$*.*

Proposition 2.5 *If* $T : \mathcal{X} \to \mathcal{X}$ *is a linear bounded operator then the following hold.*

(*i*) $T \geq 0$ *if and only if* $T^* \geq 0$*.*
(*ii*) *If* $T \geq 0$ *then* $\|T\|_\xi = |T\xi|_\xi$*.*

Proof.

(*i*) is a direct consequence of the fact that $\mathcal{X}^+$ is a selfdual cone.
(*ii*) If $T \geq 0$ then from (2.12) we have $-T\xi \leq Tx \leq T\xi$. From (2.11) it follows that $|Tx|_\xi \leq |T\xi|_\xi$ for all $x \in \mathcal{X}$ with $|x|_\xi \leq 1$ which leads to

$$\sup_{|x|_\xi \leq 1} |Tx|_\xi \leq |T\xi|_\xi \leq \sup_{|x|_\xi \leq 1} |Tx|_\xi$$

hence $\|T\|_\xi = |T\xi|_\xi$ and thus the proof is complete. □

From (*ii*) of the previous proposition we obtain the following.

Corollary 2.3 *Let* $T_k : \mathcal{X} \to \mathcal{X}, k = 1, 2$ *be linear bounded and positive operators. If* $T_1 \leq T_2$ *then* $\|T_1\|_\xi \leq \|T_2\|_\xi$*.*

Example 2.5 (*i*) As in Example 2.1 we take $\mathcal{X} = \mathbf{R}^n$ ordered by the order relation induced by the cone $\mathbf{R}^n_+$. If $T : \mathbf{R}^n \to \mathbf{R}^n$ is a linear operator then $T \geq 0$ iff its corresponding matrix A with respect to the canonical basis on $\mathbf{R}^n$ has nonnegative entries. For $\xi = (1, 1, 1, \dots, 1)^T \in \text{Int}(\mathbf{R}^n_+)$ the norm $|\cdot|_\xi$ is defined by

$$|x|_\xi = \max_{1 \leq i \leq n} |x_i|. \tag{2.14}$$

The properties $\mathbf{P_1}$ and $\mathbf{P_2}$ are fulfilled for the norm defined by (2.14). The ordered space $(\mathbf{R}^n, \mathbf{R}^n_+)$ is considered in connection with the Perron–Frobenius theorem.

(*ii*) Let $\mathcal{X} = \mathbf{R}^{m \times n}$ be the space of $m \times n$ real matrices, endowed with the inner product

$$\langle A, B \rangle = Tr(B^T A), \tag{2.15}$$

$\forall A, B \in \mathbf{R}^{m \times n}, Tr(M)$ denoting as usual the trace of a matrix M.

On $\mathbf{R}^{m \times n}$ we consider the order relation induced by the cone $\mathcal{X}^+ = \mathbf{R}^{m \times n}_+$, where

$$\mathbf{R}^{m \times n}_+ = \{A \in \mathbf{R}^{m \times n} | A = \{a_{ij}\}, a_{ij} \geq 0, 1 \leq i \leq m, 1 \leq j \leq n\}. \tag{2.16}$$

The interior of the cone $\mathbf{R}^{m \times n}_+$ is not empty. Let A be an element of the dual cone $(\mathbf{R}^{m \times n}_+)^*$. This means that $\langle A, B \rangle \geq 0$ for arbitrary $B \in \mathbf{R}^{m \times n}_+$. Let $E^{ij} \in \mathbf{R}^{m \times n}_+$ be such that $E^{ij} = \{e^{ij}_{lk}\}_{l,k}$, with $e^{ij}_{lk} = 0$ if $(l, k) \neq (i, j)$, $e^{ij}_{lk} = 1$ if $(l, k) = (i, j)$. We have $0 \leq \langle A, E^{ij} \rangle = a_{ij}$ which shows that $A \in \mathbf{R}^{m \times n}_+$ and it follows that the cone (2.16) is selfdual. On $\mathbf{R}^{m \times n}$ we also consider the norm $|\cdot|_{1\xi}$ defined by

$$|A|_\xi = \max_{i,j} |a_{ij}|. \tag{2.17}$$

Properties $\mathbf{P_1}$ and $\mathbf{P_2}$ are fulfilled for norm (2.17) with

$$\xi = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \in \text{Int}\mathbf{R}^{m \times n}_+.$$

An important class of linear operators on $\mathbf{R}^{m \times n}$ is that of the form $\mathcal{L}_{A,B} : \mathbf{R}^{m \times n} \to \mathbf{R}^{m \times n}$ by $\mathcal{L}_{A,B} Y = AYB$, for all $Y \in \mathbf{R}^{m \times n}$ where $A \in \mathbf{R}^{m \times m}, B \in \mathbf{R}^{n \times n}$ are given fixed matrices. These operators are often called "nonsymmetric Stein operators". It can be checked that $\mathcal{L}_{A,B} \geq 0$ iff $a_{ij} b_{lk} \geq 0, \forall i, j \in \{1, \dots, m\}, l, k \in \{1, \dots, n\}$. Hence $\mathcal{L}_{A,B} \geq 0$ iff the matrix $A \otimes B$ defines a positive operator on the ordered space $(\mathbf{R}^{mn}, \mathbf{R}^{mn}_+)$ where $\otimes$ is the Kronecker product.

(*iii*) Let $\mathcal{S}_n \subset \mathbf{R}^{n \times n}$ be the subspace of $n \times n$ symmetric matrices. Let $\mathcal{X} = \mathcal{S}_n \oplus \mathcal{S}_n \oplus \dots \oplus \mathcal{S}_n = \mathcal{S}^N_n$ with $N \geq 1$ fixed. On $\mathcal{S}^N_n$ we consider the inner product

$$\langle X, Y\rangle = \sum_{i=1}^{N} Tr(Y_i X_i) \tag{2.18}$$

for arbitrary $X = (X_1, X_2, \ldots, X_N)$ and $Y = (Y_1, Y_2, \ldots, Y_N)$ in $\mathcal{S}_n^N$. The space $\mathcal{S}_n^N$ is ordered by the convex cone

$$\mathcal{S}_n^{N,+} = \{X = (X_1, X_2, \ldots, X_N) | X_i \geq 0, 1 \leq i \leq N\}. \tag{2.19}$$

The cone $\mathcal{S}_n^{N,+}$ has the interior nonempty.

$$\mathrm{Int}\mathcal{S}_n^{N,+} = \{X \in \mathcal{S}_n^N | X_i > 0, 1 \leq i \leq N\}.$$

Here $X_i \geq 0, X_i > 0$, respectively, means that X_i is a positive semidefinite matrix, positive definite matrix. One may show that $\mathcal{S}_n^{N,+}$ is a selfdual cone.

Together with the norm $|\cdot|_2$ induced by the inner product (2.18), on $\mathcal{S}_n^N$ we consider the norm $|\cdot|_\xi$ defined by

$$|X|_\xi = \max_{1\leq i\leq N} |X_i|, \qquad \forall X = (X_1, \ldots, X_N) \in \mathcal{S}_n^N, \tag{2.20}$$

where $|X_i| = \max_{\lambda\in\sigma(X_i)} |\lambda|, \sigma(X_i)$ being the set of eigenvalues of the matrix X_i. For the norm defined by (2.20) the properties $\mathbf{P_1}$ and $\mathbf{P_2}$ are fulfilled with $\xi = (I_n, I_n, \ldots, I_n) = J \in \mathcal{S}_n^N$.

An important class of positive linear operators on $\mathcal{S}_n^N$ is thoroughly investigated in Section 2.5. The operators considered in Section 2.5 contain as a particular case the symmetric Stein operators.

2.2.2 Discrete-time affine equations

Let $\mathbf{L} = \{\mathcal{L}_k\}_{k\geq k_0}$ be a sequence of linear bounded operators $\mathcal{L}_k : \mathcal{X} \to \mathcal{X}$ and $f = \{f_k\}_{k\geq k_0}$ be a sequence of elements $f_k \in \mathcal{X}$. These two sequences define two affine equations on $\mathcal{X}$:

$$x_{k+1} = \mathcal{L}_k x_k + f_k \tag{2.21}$$

which is called the "forward" affine equation or "causal affine equation" defined by $(\mathbf{L}, f)$ and

$$x_k = \mathcal{L}_k x_{k+1} + f_k, \tag{2.22}$$

which is called the "backward affine equation" or "anticausal affine equation" defined by $(\mathbf{L}, f)$. For each $k \geq l \geq k_0$ let $T(k,l)^c : \mathcal{X} \to \mathcal{X}$ be the causal evolution operator defined by the sequence $\mathbf{L}$, $T(k,l)^c = \mathcal{L}_{k-1}\mathcal{L}_{k-2}\cdots\mathcal{L}_l$ if $k > l$ and $T(k,l)^c = I_\mathcal{X}$ if $k = l, I_\mathcal{X}$ being the identity operator on $\mathcal{X}$.

For all $k_0 \leq k \leq l, T(k,l)^a : \mathcal{X} \to \mathcal{X}$ stands for the anticausal evolution operator on $\mathcal{X}$ defined by the sequence $\mathbf{L}$; that is,

$$T(k,l)^a = \mathcal{L}_k\mathcal{L}_{k+1}\cdots\mathcal{L}_{l-1}$$

if $k < l$ and $T(k,l)^a = I_\mathcal{X}$ if $k = l$.

Often the superscripts a and c are omitted if there will be no confusion.

Let $\tilde{x}_k = T(k,l)^c x, k \geq l, l \geq k_0$ be fixed. One obtains that $\{\tilde{x}_k\}_{k\geq l}$ verifies the forward linear equation

$$x_{k+1} = \mathcal{L}_k x_k \tag{2.23}$$

with initial value $x_l = x$. Also, if $y_k = T(k,l)^a y, k_0 \leq k \leq l$ then from the definition of T^a_{kl} one obtains that $\{y_k\}_{k_0\leq k\leq l}$ is the solution of the backward linear equation

$$y_k = \mathcal{L}_k y_{k+1} \tag{2.24}$$

with given terminal value $y_l = y$.

It must be remarked that, in contrast to the continuous-time case, a solution $\{x_k\}_{k\geq l}$ of the forward linear equation (2.23) with given initial values $x_l = x$ is well defined for $k \geq l$ whereas a solution $\{y_k\}_{k\leq l}$ of the backward linear equation (2.24) with given terminal condition $y_l = y$ is well defined for $k_0 \leq k \leq l$.

If for each k, the operators $\mathcal{L}_k$ are invertible, then all solutions of the equations (2.23), (2.24) are well defined for all $k \geq k_0$.

If $(T(k,l)^c)^*$ is the adjoint operator of the causal evolution operator $T(k,l)^c$ we define

$$z_l = (T(k,l)^c)^* z, \qquad \forall k_0 \leq l \leq k.$$

By direct calculation one obtains that $z_l = \mathcal{L}_l^* z_{l+1}$. This shows that the adjoint of the causal evolution operator associated with the sequence $\mathbf{L}$ generates anticausal evolution.

Definition 2.6 *We say that the sequence* $\mathbf{L} = \{\mathcal{L}_k\}_{k\geq k_0}$ *defines a* positive evolution *if for all* $k \geq l \geq k_0$ *the causal linear evolution operator* $T(kl)^c \geq 0$.

Because $T(l+1,l)^c = \mathcal{L}_l$ *it follows that the sequence* $\{\mathcal{L}_k\}_{k\geq k_0}$ *generates a positive evolution if and only if for each* $k \geq k_0, \mathcal{L}_k$ *is a positive operator. Hence, in contrast to the continuous-time case, in the discrete-time case only sequences of positive operators define equations that generate positive evolutions (see [37]).*

The following result is straightforward and is used in the next sections.

Corollary 2.4 *Let* $\mathbf{L} = \{\mathcal{L}^i_k\}_{k\geq k_0}, i = 1,2$ *be two sequences of linear bounded operators and* $T(k,l)_i$ *be the corresponding causal linear evolution operators. Assume that* $0 \leq \mathcal{L}^1_k \leq \mathcal{L}^2_k$ *for all* $k \geq k_0$. *Under this assumption we have* $T(k,l)_2 \geq T(k,l)_1$ *for all* $k \geq l \geq k_0$.

At the end of this subsection we recall the representation formulae of the solutions of affine equations (2.21), (2.22).

Each solution of the forward affine equation (2.21) has the representation:

$$x_k = T_(k,l)^c x_l + \sum_{i=l}^{k-1} T(k,i+1)^c f_i \tag{2.25}$$

for all $k \geq l+1$. Also, any solution of the backward affine equation (2.22) has a representation

$$y_k = T(k,l)^a y_l + \sum_{i=k}^{l-1} T(k,i)^a f_i, \qquad k_0 \leq k \leq l-1.$$

2.3 Exponential stability

In this section we deal with the exponential stability of the zero solution of a discrete-time linear equation defined by a sequence of linear bounded and positive operators.

Definition 2.7 *We say that the zero solution of the equation*

$$x_{k+1} = \mathcal{L}_k x_k \tag{2.26}$$

is exponentially stable*, or equivalently that the sequence* $\mathbf{L} = \{\mathcal{L}_k\}_{k\geq k_0}$ *generates an* exponentially stable evolution *(E.S. evolution) if there are* $\beta > 0, q \in (0,1)$ *such that*

$$\|T(k,l)\|_\xi \leq \beta q^{k-l}, \qquad k \geq l \geq k_0, \tag{2.27}$$

$T(k,l)$ being the causal linear evolution operator defined by the sequence $\mathbf{L}$*. Based on Remark* 2.2*, in* (2.27) *we may also consider the norm* $\|\cdot\|_2$*. In the case when* $\mathcal{L}_k = \mathcal{L}$ *for all* k*, if* (2.27) *is satisfied we say that the operator* $\mathcal{L}$ *generates a discrete-time exponentially stable evolution. It is well known that* $\mathcal{L}$ *generates a discrete-time exponentially stable evolution if and only if* $\rho[\mathcal{L}] < 1, \rho[\cdot]$ *being the spectral radius. It must be remarked that if the sequence* $\{\mathcal{L}_k\}_{k\geq k_0}$ *generates an exponentially stable evolution then it is a bounded sequence.*

In this section we derive several conditions that are equivalent to the exponential stability of the zero solution of equation (2.26) *in the case* $\{\mathcal{L}_k\}_{k\geq k_0}$*. Such results can be viewed as an alternative characterization of exponential stability to the one in terms of Lyapunov functions.*

Firstly from Proposition 2.5, Corollary 2.3, and Corollary 2.4 we obtain the following result specific to the case of operators that generate positive evolution.

Proposition 2.6 *Let* $\mathbf{L} = \{\mathcal{L}_k\}_{k\geq k_0}$*, and* $\mathbf{L}^1 = \{\mathcal{L}_k^1\}_{k\geq k_0}$ *be two sequences of linear bounded and positive operators on* $\mathcal{X}$*.*

(i) The following are equivalent.
(a) $\mathbf{L}(\cdot)$ *defines E.S. evolution.*
(b) There exist $\beta \geq 1, q \in (0,1)$ *such that* $|T(k,l)\xi|_\xi \leq \beta q^{k-l}$ *for all* $k \geq l \geq k_0$.
(ii) If $\mathcal{L}_k^1 \leq \mathcal{L}_k$ *for all* $k \geq k_0$ *and* $\mathbf{L}$ *generates an E.S. evolution, then* $\mathbf{L}^1$ *generates an E.S. evolution.* □

We further prove the following.

Theorem 2.3 *Let* $\{\mathcal{L}_k\}_{k\geq 0}$ *be a sequence of linear bounded and positive operators* $\mathcal{L}_k : \mathcal{X} \to \mathcal{X}$. *Then the following are equivalent.*

(i) The sequence $\{\mathcal{L}_k\}_{k\geq 0}$ *generates an exponentially stable evolution.*
(ii) There exists $\delta > 0$ *such that* $\sum_{l=k_0}^{k} \|T_{k,l}\|_\xi \leq \delta$ *for arbitrary* $k \geq k_0 \geq 0$.
(iii) There exists $\delta > 0$, *such that* $\sum_{l=k_1}^{k} T(k,l)\xi \leq \delta\xi$ *for arbitrary* $k \geq k_1 \geq 0, \delta > 0$ *being independent of* k, k_1.
(iv) For arbitrary bounded sequence $\{f_k\}_{k\geq 0} \subset \mathcal{X}$ *the solution with zero initial value of the forward affine equation*

$$x_{k+1} = \mathcal{L}_k x_k + f_k, \qquad k \geq 0$$

is bounded.

Proof. The implication $(iv) \to (i)$ is the discrete-time counterpart of Perron's theorem (see [97].) It remains to prove the implications $(i) \to (ii) \to (iii) \to (iv)$.

If (i) is true then (ii) follows immediately from (2.27) with $\delta = \beta/(1-q)$.

Let us prove that

$$0 \leq T(k,l)\xi \leq \|T(k,l)\|_\xi \xi \tag{2.28}$$

for arbitrary $k \geq l \geq 0$. If $T_{k,l}\xi = 0$ then from Proposition 2.5(ii) it follows that $||T_{k,l}||_\xi = 0$ and (2.28) is fulfilled. If $T(k,l)\xi \neq 0$ then from (2.12) applied to $x = (1/|T(k,l)\xi|_\xi)T(k,l)\xi$ one gets $0 \leq T(k,l)\xi \leq |T(k,l)\xi|_\xi \xi$ and (2.28) follows based on Proposition 2.5(ii).

If (ii) holds then (iii) follows from (2.28). We have to prove that $(iii) \to (iv)$. Let $\{f_k\}_{k\geq 0} \subset \mathcal{X}$ be a bounded sequence; that is, $|f_k|_\xi \leq \mu, k \geq 0$. Based on (2.12) we obtain that $-|f_l|_\xi \xi \leq f_l \leq |f_l|_\xi \xi$ which leads to $-\mu\xi \leq f_l \leq \mu\xi$ for all $l \geq 0$.

Because for each $k \geq l+1 \geq 0, T(k,l+1)$ is a positive operator we have:

$$-\mu T(k,l+1)\xi \leq T(k,l+1)f_l \leq \mu T(k,l+1)\xi$$

and

$$-\mu \sum_{l=0}^{k-1} T(k,l+1)\xi \leq \sum_{l=0}^{k-1} T(k,l+1)f_l \leq \mu \sum_{l=0}^{k-1} T(k,l+1)\xi.$$

Applying (2.11) we deduce that

$$\left|\sum_{l=0}^{k-1} T(k,l+1)f_l\right|_\xi \leq \mu \left|\sum_{l=0}^{k-1} T(k,l+1)\xi\right|_\xi .$$

If (iii) is valid we conclude by again using (2.11) that

$$\left|\sum_{l=0}^{k-1} T(k,l+1)f_l\right|_\xi \leq \mu\delta, \qquad k \geq 1$$

which shows that (iv) is fulfilled using (2.25) and thus the proof ends. □

We note that the proof of the above theorem shows that in the case of a discrete-time linear equation (2.26) defined by a sequence of linear bounded and positive operators the exponential stability is equivalent to the boundedness of the solution with the zero initial value of the forward affine equation $x_{k+1} = \mathcal{L}_k x_k + \xi$.

We recall that in the general case of a discrete-time linear equation if we want to use Perron's theorem to characterize the exponential stability we have to check the boundedness of the solution with zero initial value of the forward affine equation $x_{k+1} = \mathcal{L}_k x_k + f_k$ for arbitrary bounded sequence $\{f_k\}_{k\geq 0} \subset \mathcal{X}$.

Let us now introduce the concept of uniform positivity.

Definition 2.8 *We say that a sequence $\{f_k\}_{k\geq k_0} \subset \mathcal{X}^+$ is* uniformly positive *if there exists $c > 0$ such that $f_k > c\xi$ for all $k \geq k_0$. If $\{f_k\}_{k\geq k_0} \subset \mathcal{X}^+$ is uniformly positive we write $f_k \gg 0, k \geq k_0$. If $-f_k \gg 0, k \geq k_0$ then we write $f_k \ll 0, k \geq k_0$.*

The next result provides a characterization of the exponential stability, using solutions of some suitable backward affine equations.

Theorem 2.4 *Let $\{\mathcal{L}_k\}_{k\geq k_0}$ be a sequence of linear bounded and positive operators $\mathcal{L}_k : \mathcal{X} \to \mathcal{X}$. Then the following are equivalent.*

(i) *The sequence $\{\mathcal{L}_k\}_{k\geq k_0}$ generates an exponentially stable evolution.*
(ii) *There exist $\beta_1 > 0, q \in (0,1)$ such that $\|T^*(k,l)\|_\xi \leq \beta_1 q^{k-l}, \forall k \geq l \geq k_0$.*
(iii) *There exists $\delta > 0$, independent of k, such that $\sum_{l=k}^{\infty} T^*(l,k)\xi \leq \delta\xi$.*
(iv) *The discrete-time backward affine equation*

$$x_k = \mathcal{L}_k^* x_{k+1} + \xi \tag{2.29}$$

has a bounded and uniformly positive solution.

(*v*) *For arbitrary bounded and uniformly positive sequence* $\{f_k\}_{k\geq k_0} \subset \mathrm{Int}\mathcal{X}^+$ *the backward affine equation*

$$x_k = \mathcal{L}_k^* x_{k+1} + f_k, \qquad k \geq k_0 \tag{2.30}$$

has a bounded and uniformly positive solution.

(*vi*) *There exists a bounded and uniformly positive sequence* $\{f_k\}_{k\geq k_0} \subset \mathrm{Int}\mathcal{X}^+$ *such that the corresponding backward affine equation* (2.30) *has a bounded solution* $\{\tilde{x}_k\}_{k\geq k_0} \subset \mathcal{X}^+$.

(*vii*) *There exists a bounded and uniformly positive sequence* $\{y_k\}_{k\geq k_0} \subset \mathrm{Int}\mathcal{X}^+$ *that verifies*

$$\mathcal{L}_k^* y_{k+1} - y_k \ll 0, \qquad k \geq k_0. \tag{2.31}$$

Proof. The equivalence $(i) \leftrightarrow (ii)$ follows immediately from (2.13). In a similar way as in the proof of inequality (2.28) one obtains:

$$0 \leq T^*(l,k)\xi \leq \|T^*(l,k)\|_\xi \xi \tag{2.32}$$

for all $l \geq k \geq k_0$.

If (ii) holds, then (iii) follows immediately from (2.32) together with the property that $\mathcal{X}^+$ is a regular cone. To show that $(iii) \to (iv)$ we define $y_k = \sum_{l=k}^{\infty} T^*(l,k)\xi k \geq k_0$. If (iii) holds it follows that $\{y_k\}_{k\geq 0}$ is well defined. Because $y_k = \xi + \mathcal{L}_k^* \sum_{l=k+1}^{\infty} T^*(l,k+1)\xi$ one obtains that $y_k \gg 0, k \geq k_0$ and $\{y_k\}_{k\geq k_0}$ solves (2.29) and thus (iv) is true.

Let us prove now that $(iv) \to (iii)$. Let $\{x_k\}_{k\geq k_0} \subset \mathrm{Int}\mathcal{X}^+$ be a bounded and uniform positive solution of (2.29); that is,

$$0 < \mu_1\xi \leq x_k \leq \mu_2\xi \tag{2.33}$$

for some positive constants μ_i independent of k. The solution $\{x_k\}_{k\geq k_0}$ has the representation formula

$$x_k = T^*(j,k)x_j + \sum_{l=k}^{j-1} T^*(l,k)\xi$$

for all $j \geq k+1 \geq k_0$. Because $T^*(l,k) \geq 0$ we obtain

$$\sum_{l=k}^{j-1} T^*(l,k)\xi \leq x_k. \tag{2.34}$$

For each fixed $k \geq k_0$ we define $z_j = \sum_{l=k}^{j-1} T^*(l,k)\xi$ for all $j \geq k+1$. The sequence $\{z_j\}_{j\geq k+1}$ is monotone increasing. From (2.33) and (2.34) we obtain that

$$\xi \leq z_j \leq \mu_2\xi.$$

Because $\mathcal{X}^+$ is a regular cone we may conclude that there exists

$$\lim_{j\to\infty} z_j = \sum_{l=k}^{\infty} T^*(l,k)\xi \leq \mu_2\xi$$

and thus (iii) is valid.

Now we prove $(iii) \to (v)$. Let $\{f_k\}_{k\geq k_0} \subset \mathrm{Int}\mathcal{X}^+$ be a bounded and uniformly positive sequence. This means that there exist $\nu_i > 0$ such that

$$\nu_1\xi \leq f_l \leq \nu_2\xi, \qquad \forall l \geq k_0.$$

Because $T^*(l,k) \geq 0$ one obtains $\nu_1 T^*(l,k)\xi \leq T^*(l,k)f_l \leq \nu_2 T^*(l,k)\xi, \forall l \geq k \geq k_0$.

Furthermore we may write the inequalities: $\nu_1\xi \leq \nu_1\sum_{l=k}^{j} T^*(l,k)\xi \leq \sum_{l=k}^{j} T^*(l,k)f_l \leq \nu_2\sum_{l=k}^{j} T^*(l,k)\xi \leq \nu_2\delta\xi, j \geq k \geq k_0$. Because $\mathcal{X}^+$ is a regular cone one concludes that the sequence $\{\sum_{l=k}^{j} T^*(l,k)f_l\}_{j\geq k}$ is convergent.

We define $\tilde{x}_k = \sum_{l=k}^{\infty} T^*(l,k)f_l, k \geq k_0$. One obtains that $\tilde{x}_k = f_k + \mathcal{L}_k^*\sum_{l=k+1}^{\infty} T^*(l,k+1)f_l$ which shows that $\{\tilde{x}_k\}_{k\geq k_0}$ is a solution with the desired properties of equation (2.30) and thus (v) holds.

$(v) \to (vi)$ is obvious.

We prove now $(vi) \to (ii)$. Let us assume that there exists a bounded and uniformly positive sequence $\{f_k\}_{k\geq k_0} \subset \mathrm{Int}\mathcal{X}^+$ such that the discrete-time backward affine equation (2.30) has a bounded solution $\{\hat{x}_k\}_{k\geq k_0} \subset \mathcal{X}^+$. Therefore there exist positive constants γ_i such that

$$\begin{aligned} 0 < \gamma_1\xi \leq f_l \leq \gamma_2\xi \\ 0 < \gamma_1\xi \leq \hat{x}_l \leq \gamma_3\xi \end{aligned} \tag{2.35}$$

for all $l \geq k_0$. Writing the representation formula

$$\hat{x}_k = T^*(j,k)\hat{x}_j + \sum_{l=k}^{j-1} T^*(l,k)f_l$$

and taking into account that $T^*(j,k) \geq 0$ if $j \geq k$ one obtains

$$f_k \leq \sum_{l=k}^{j-1} T^*(l,k)f_l \leq \hat{x}_k, \qquad j-1 \geq k \geq k_0. \tag{2.36}$$

Set $y_k = \sum_{l=k}^{\infty} T^*(l,k)f_l, k \geq k_0$, $\mathcal{X}^+$ being a regular cone together with (2.35), (2.36) guarantee that y_k is well defined, and

$$\gamma_1\xi \leq y_k \leq \gamma_3\xi \tag{2.37}$$

for all $k \geq k_0$. Let $k_1 \geq k_0$ be fixed. We define $\tilde{y}_k = T^*(k,k_1)y_k, k \geq k_1$. Because $T^*(k,k_1) \geq 0$ one obtains that

$$\gamma_1 T^*(k,k_1)\xi \leq \tilde{y}_k \leq \gamma_3 T^*(k,k_1)\xi \tag{2.38}$$

for all $k \geq k_1$. On the other hand we have $\tilde{y}_k = \sum_{l=k}^{\infty} T^*(l, k_1) f_l$. This allows us to write

$$\tilde{y}_{k+1} - \tilde{y}_k = -T^*(k, k_1) f_k.$$

From (2.35) we get

$$\tilde{y}_{k+1} - \tilde{y}_k \leq -\gamma_1 T^*(k, k_1)\xi.$$

Furthermore (2.38) leads to

$$\tilde{y}_{k+1} \leq \left(1 - \frac{\gamma_1}{\gamma_3}\right) \tilde{y}_k, \qquad k \geq k_1.$$

Inductively we deduce

$$\tilde{y}_k \leq q^{k-k_1} \tilde{y}_{k_1}, \qquad \forall k \geq k_1, \tag{2.39}$$

where $q = 1 - (\gamma_1/\gamma_3), q \in (0, 1)$ (in (2.38) γ_3 may be chosen large enough so that $\gamma_3 > \gamma_1$). Again invoking (2.38) we may write

$$T^*(k, k_1)\xi \leq \frac{\gamma_3}{\gamma_1} q^{k-k_1} \xi,$$

which by (2.11) leads to $|T^*(k, k_1)\xi|_\xi \leq (\gamma_3/\gamma_1) q^{k-k_1}, k \geq k_1$. Based on Proposition 2.5(ii) we have

$$\|T^*(k, k_1)\|_\xi \leq \frac{\gamma_3}{\gamma_1} q^{k-k_1},$$

which means that (ii) is fulfilled.

The implication $(iv) \to (vii)$ follows immediately because a bounded and uniform positive solution of (2.29) is a solution with the desired properties of (2.31). To end the proof we show that $(vii) \to (vi)$. Let $\{z_k\}_{k \geq k_0} \subset \text{Int}\mathcal{X}^+$ be a bounded and uniformly positive solution of (2.31). Define $\hat{f}_k = z_k - \mathcal{L}_k^* z_{k+1}$. It follows that $\{\hat{f}_k\}_{k \geq k_0}$ is bounded and uniform positive, therefore $\{z_k\}_{k \geq 0}$ will be a bounded and positive solution of (2.30) corresponding to $\{\hat{f}_k\}_{k \geq k_0}$ and thus the proof ends. □

We remark that in the proof of Theorem 2.4 the fact that $\mathcal{X}^+$ is a regular cone was used in order to guarantee the convergence of several series in $\mathcal{X}$. The result proved in Theorem 2.3 does not use that $\mathcal{X}^+$ is a regular cone.

The next result provides more information concerning the bounded solution of the discrete-time backward affine equations.

Theorem 2.5 *Let $\{\mathcal{L}_k\}_{k \geq k_0}$ be a sequence of linear bounded operators that generates an exponentially stable evolution on $\mathcal{X}$. Then the following hold.*

(i) *For each bounded sequence $\{f_k\}_{k \geq k_0} \subset \mathcal{X}$ the discrete-time backward affine equation*

$$x_k = \mathcal{L}_k^* x_{k+1} + f_k \tag{2.40}$$

has an unique bounded solution that is given by

$$\tilde{x}_k = \sum_{l=k}^{\infty} T^*(l,k) f_l, \qquad k \geq k_0. \tag{2.41}$$

(*ii*) *If there exists an integer* $\theta \geq 1$ *such that* $\mathcal{L}_{k+\theta} = \mathcal{L}_k, f_{k+\theta} = f_k$ *for all* k *then the unique bounded solution of equation* (2.40) *is also a periodic sequence with period* θ*.*

(*iii*) *If* $\mathcal{L}_k = \mathcal{L}, f_k = f$ *for all* k *then the unique bounded solution of equation* (2.40) *is constant and it is given by*

$$\tilde{x} = (I_{\mathcal{X}} - \mathcal{L}^*)^{-1} f \tag{2.42}$$

with $I_{\mathcal{X}}$ *the identity operator on* $\mathcal{X}$*.*

(*iv*) *If* $\mathcal{L}_k$ *are positive operators and* $\{f_k\}_{k \geq k_0} \subset \mathcal{X}^+$ *is a bounded sequence then the unique bounded solution of equation* (2.40) *satisfies* $\tilde{x}_k \geq 0$ *for all* $k \geq k_0$*.*

Moreover if $\{f_k\}_{k \geq k_0} \subset \mathrm{Int}\mathcal{X}^+$ *is a bounded and uniformly positive sequence then the unique bounded solution* $\{\tilde{x}_k\}_{k \geq k_0}$ *of equation* (2.40) *is also uniformly positive.*

Proof. (*i*) Based on (*i*) → (*ii*) of Theorem 2.4 we deduce that for all $k \geq k_0$ the series $\{\sum_{l=k}^{j} T^*(l,k) f_l\}_{j \geq k}$ is absolutely convergent and there exists $\delta > 0$ independent of k and j such that

$$\left| \sum_{l=k}^{j} T^*(l,k) f_l \right|_{\xi} \leq \delta. \tag{2.43}$$

Set $\tilde{x}_k = \lim_{j \to \infty} \sum_{l=k}^{j} T^*_{l,k} f_l = \sum_{l=k}^{\infty} T^*(l,k) f_l$. Taking into account the definition of $T(l,k)^*$ we obtain $\tilde{x}_k = f_k + \mathcal{L}_k^* \sum_{l=k+1}^{\infty} T^*(l,k+1) f_l = f_k + \mathcal{L}_k^* \tilde{x}_{k+1}$ which shows that $\{\tilde{x}_k\}_{k \geq k_0}$ solves (2.40).

From (2.43) it follows that $\{\tilde{x}_k\}$ is a bounded solution of (2.40). Let $\{\hat{x}_k\}_{k \geq k_0}$ be another bounded solution of equation (2.40). For each $0 \leq k < j$ we may write

$$\hat{x}_k = T^*(j+1,k)\hat{x}_{j+1} + \sum_{l=k}^{j} T^*(l,k) f_l. \tag{2.44}$$

Because $\{\mathcal{L}_k\}_{k \geq k_0}$ generates an exponentially stable evolution and $\{\hat{x}_k\}_{k \geq k_0}$ is a bounded sequence we have $\lim_{j \to \infty} T^*(j+1,k)\hat{x}_{j+1} = 0$. Taking the limit for $j \to \infty$ in (2.44) we conclude that $\hat{x}_k = \sum_{l=k}^{\infty} T^*(l,k) f_l = \tilde{x}_k$ which proves the uniqueness of the bounded solution of equation (2.40).

(*ii*) If $\{\mathcal{L}_k\}_{k \geq k_0}, \{f_k\}_{k \geq k_0}$ are periodic sequences with period θ then in a standard way using the representation formula (2.41) one shows that the unique bounded solution of equation (2.40) is also periodic with period θ.

In this case we may take that $k_0 = -\infty$.

(iii) If $\mathcal{L}_k = \mathcal{L}, f_k = f$ for all k, then they may be viewed as periodic sequences with period $\theta = 1$. Based on the above result of (ii) one obtains that the unique bounded solution of equation (2.40) is also periodic with period $\theta = 1$, so it is constant. In this case $\tilde{x}$ will verify the equation $\tilde{x} = \mathcal{L}^*\tilde{x} + f$.

The operator $\mathcal{L}$ generates an exponentially stable evolution therefore it follows that $\rho(\mathcal{L}) < 1$. Hence, the operator $I_{\mathcal{X}} - \mathcal{L}^*$ is invertible and one obtains that $\tilde{x}$ is given by (2.42). Finally, if $\mathcal{L}_k$ are positive operators the assertions of (iv) follow immediately from the representation formula (2.41) and thus the proof ends. □

Remark 2.3 From the representation formula (2.25) one obtains that if the sequence $\{\mathcal{L}_k\}_{k \geq k_0}$ generates an exponentially stable evolution and $\{f_k\}_{k \geq k_0}$ is a bounded sequence, then all solutions of the discrete-time forward affine equation (2.21) with given initial values at time $k = k_0$ are bounded on the interval $[k_0, \infty)$. On the other hand from Theorem 2.5(i) it follows that the discrete-time backward equation (2.22) has a unique bounded solution on the interval $[k_0, \infty)$ which is the solution provided by the formula (2.41).

In the case of $k_0 = -\infty$ with the same techniques as in the proof of Theorem 2.5 we may obtain a result concerning the existence and uniqueness of the bounded solution of a forward affine equation similar to the one proved for the case of backward affine equations.

Theorem 2.6 *Assume that $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ is a sequence of linear bounded operators which generates an exponentially stable evolution on $\mathcal{X}$. Then the following assertions hold.*

(i) *For each bounded sequence $\{f_k\}_{k\in Z}$ the discrete-time forward affine equation*

$$x_{k+1} = \mathcal{L}_k x_k + f_k \tag{2.45}$$

has a unique bounded solution $\{\hat{x}_k\}_{k\in\mathbf{Z}}$. Moreover this solution has a representation formula

$$\hat{x}_k = \sum_{l=-\infty}^{k-1} T(k, l+1) f_l, \qquad \forall k \in \mathbf{Z}. \tag{2.46}$$

(ii) *If $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}, \{f_k\}_{k\in\mathbf{Z}}$ are periodic sequences with period θ then the unique bounded solution of equation* (2.45) *is periodic with period θ.*

(iii) *If $\mathcal{L}_k = \mathcal{L}, f_k = f, k \in \mathbf{Z}$ then the unique bounded solution of equation* (2.45) *is constant and it is given by $\hat{x} = (I_{\mathcal{X}} - \mathcal{L})^{-1} f$.*

(iv) *If $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ are positive operators and if $\{f_k\}_{k\in\mathbf{Z}} \subset \mathcal{X}^+$, then the unique bounded solution of equation* (2.45) *satisfies $\hat{x}_k \geq 0$ for all $k \in \mathbf{Z}$. Moreover, if $f_k \gg 0, k \in \mathbf{Z}$ then $\hat{x}_k \gg 0, k \in \mathbf{Z}$.* □

If $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ is a sequence of linear operators on $\mathcal{X}$ we may associate a new sequence of linear operators $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ defined as follows.

$$\mathcal{L}^{\#}{}_k = \mathcal{L}^*_{-k}.$$

Lemma 2.2 *Let $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ be a sequence of linear bounded operators on $\mathcal{X}$. The following assertions hold.*

(i) If $T(k,l)^{\#}$ is the causal linear evolution operator on $\mathcal{X}$ defined by the sequence $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ we have

$$T(k,l)^{\#} = T^*(-l+1,-k+1),$$

where $T(i,j)$ is the causal linear evolution operator defined on $\mathcal{X}$ by the sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$.
(ii) $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ is a sequence of positive linear operators if and only if $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ is a sequence of positive linear operators.
(iii) The sequence $\{\mathcal{L}^{\#}{}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution if and only if the sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution.
(iv) The sequence $\{x_k\}_{k\in\mathbf{Z}}$ is a solution of the discrete-time backward affine equation (2.40) if and only if the sequence $\{y_k\}_{k\in\mathbf{Z}}$ defined by $y_k = x_{-k+1}$ is a solution of the discrete-time forward equation $y_{k+1} = \mathcal{L}^{\#}_k y_k + f_{-k}$, $k \in \mathbf{Z}$.

The proof is straightforward and it is omitted. □

The next result provides a characterization of exponential stability in terms of the existence of the bounded solution of some suitable forward affine equation.

Theorem 2.7 *Let $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ be a sequence of positive linear bounded operators on $\mathcal{X}$. Then the following are equivalent.*

(i) The sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution.
(ii) There exists $\delta > 0$, independent of k such that

$$\sum_{l=-\infty}^{k} T(k,l)\xi \le \delta\xi, \qquad \forall k \in \mathbf{Z}.$$

(iii) The forward affine equation

$$x_{k+1} = \mathcal{L}_k x_k + \xi \tag{2.47}$$

has a bounded and uniformly positive solution.

(*iv*) *For any bounded and uniformly positive sequence* $\{f_k\}_{k\in\mathbf{Z}} \subset \mathrm{Int}\mathcal{X}^+$ *the corresponding forward affine equation*

$$x_{k+1} = \mathcal{L}_k x_k + f_k \tag{2.48}$$

has a bounded and uniformly positive solution.

(*v*) *There exists a bounded and uniformly positive sequence* $\{f_k\}_{k\in\mathbf{Z}} \subset \mathrm{Int}\mathcal{X}^+$ *such that the corresponding forward affine equation* (2.48) *has a bounded solution* $\tilde{x}_k, k \in \mathbf{Z} \subset \mathcal{X}^+$.

(*vi*) *There exists a bounded and uniformly positive sequence* $\{y_k\}_{k\in\mathbf{Z}}$ *that verifies* $y_{k+1} - \mathcal{L}_k y_k \gg 0$.

The proof follows immediately combining the result proved in Theorem 2.4 and Lemma 2.2. □

2.4 Some robustness results

In this section we prove some results that provide a "measure" of the robustness of the exponential stability in the case of positive linear operators. To state and prove this result some preliminary remarks are needed.

So, $\ell^\infty(\mathbf{Z}, \mathcal{X})$ stands for the real Banach space of bounded sequences of elements of $\mathcal{X}$. If $x \in \ell^\infty(\mathbf{Z}, \mathcal{X})$ we denote $|x| = \sup_{k\in\mathbf{Z}} |x_k|_\xi$.

Let $\ell^\infty(\mathbf{Z}, \mathcal{X}^+) \subset \ell^\infty(\mathbf{Z}, \mathcal{X})$ be the subset of bounded sequences $\{x_k\}_{k\in\mathbf{Z}} \subset \mathcal{X}^+$. It can be checked that $\ell^\infty(\mathbf{Z}, \mathcal{X}^+)$ is a solid, closed, convex cone. Therefore, $\ell^\infty(\mathbf{Z}, \mathcal{X})$ is an ordered real Banach space for which the assumptions of Theorem 2.11 in [30] are fulfilled.

Now we are in position to prove the following.

Theorem 2.8 *Let* $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}, \{\mathcal{G}_k\}_{k\in\mathbf{Z}}$ *be sequences of positive linear bounded operators such that* $\{\mathcal{G}_k\}_{k\in\mathbf{Z}}$ *is a bounded sequence. Under these conditions the following are equivalent.*

(*i*) *The sequence* $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ *generates an exponentially stable evolution and* $\rho[\mathcal{T}] < 1$ *where* $\rho[\mathcal{T}]$ *is the spectral radius of the operator* $\mathcal{T} : \ell^\infty(\mathbf{Z}, \mathcal{X}) \to \ell^\infty(\mathbf{Z}, \mathcal{X})$ *by*

$$y = \mathcal{T}x, \qquad y_k = \sum_{l=-\infty}^{k-1} T(k, l+1)\mathcal{G}_l x_l, \tag{2.49}$$

$T(k,l)$ *being the linear evolution operator on* $\mathcal{X}$ *defined by the sequence* $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$.

(*ii*) *The sequence* $\{\mathcal{L}_k + \mathcal{G}_k\}_{k\in\mathbf{Z}}$ *generates an exponentially stable evolution on* $\mathcal{X}$.

Proof. $(i) \to (ii)$ If the sequence $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$ defines an exponentially stable evolution, then we define the sequence $\{f_k\}_{k\in\mathbf{Z}}$ by $\{\mathcal{L}_k\}_{k\in\mathbf{Z}}$,

$$f_k = \sum_{l=-\infty}^{k-1} T(k,l+1)\xi. \tag{2.50}$$

We have $f_k = \xi + \sum_{l=-\infty}^{k-2} T_{k,l+1}\xi$ which leads to $f_k \geq \xi$ thus $f_k \in \mathrm{Int}\mathcal{X}^+$ for all $k \in \mathbf{Z}$. This allows us to conclude that $f = \{f_k\}_{k\in\mathbf{Z}} \in \mathrm{Int}\ell^\infty(\mathbf{Z},\mathcal{X}^+)$.

Applying Theorem 2.11 [30] with $R = -I_{\ell^\infty}$ and $P = \mathcal{T}$ we deduce that there exists $x = \{x_k\}_{k\in\mathbf{Z}} \in \mathrm{Int}\ell^\infty(\mathbf{Z},\mathcal{X}^+)$ which verifies the equation:

$$(I_{\ell^\infty} - \mathcal{T})(x) = f. \tag{2.51}$$

Here I_{ℓ^∞} stands for the identity operator on $\ell^\infty(\mathbf{Z},\mathcal{X})$. Partitioning (2.51) and taking into account (2.49)–(2.50) we obtain that for each $k \in \mathbf{Z}$ we have:

$$x_{k+1} = \sum_{l=-\infty}^{k} T(k+1,l+1)\mathcal{G}_l x_l + \sum_{l=-\infty}^{k} T(k+1,l+1)\xi.$$

Furthermore we may write:

$$\begin{aligned} x_{k+1} &= \mathcal{G}_k x_k + \xi + \mathcal{L}_k \sum_{l=-\infty}^{k-1} T(k,l+1)\mathcal{G}_l x_l + \mathcal{L}_k \sum_{l=-\infty}^{k-1} T(k,l+1)\xi \\ &= \mathcal{G}_k x_k + \xi + \mathcal{L}_k x_k. \end{aligned}$$

This shows that $\{x_k\}_{k\in\mathbf{Z}}$ verifies the equation

$$x_{k+1} = (\mathcal{L}_k + \mathcal{G}_k)x_k + \xi. \tag{2.52}$$

Because $\mathcal{L}_k$ and $\mathcal{G}_k$ are positive operators and $x \geq 0$, (2.52) shows that $x_k \geq \xi$. Thus we get that equation (2.47) associated with the sum operator $\mathcal{L}_k + \mathcal{G}_k$ has a bounded and uniform positive solution. Applying implication $(iii) \to (i)$ of Theorem 2.7 we conclude that the sequence $\{\mathcal{L}_k + \mathcal{G}_k\}_{k\in\mathbf{Z}}$ generates an exponentially stable evolution.

Now we prove the converse implication.

If (ii) holds, then based on the implication $(i) \to (iii)$ of Theorem 2.7 we deduce that equation (2.52) has a bounded and uniform positive solution $\{\tilde{x}_k\}_{k\in\mathbf{Z}} \subset \mathrm{Int}\mathcal{X}^+$. Equation (2.52) verified by $\tilde{x}_k$ may be rewritten as

$$\tilde{x}_{k+1} = \mathcal{L}_k\tilde{x}_k + \tilde{f}_k, \tag{2.53}$$

where $\tilde{f}_k = \mathcal{G}_k\tilde{x}_k + \xi, k \in \mathbf{Z}, \tilde{f}_k \geq \xi, k \in \mathbf{Z}$. Using the implication $(v) \to (i)$ of Theorem 2.7 we deduce that the sequence $\mathcal{L}_k$ generates an exponentially stable evolution. Equation (2.53) has an unique bounded solution which is given by the representation formula (2.46), therefore we have $\tilde{x}_k = \sum_{l=-\infty}^{k-1} T(k,l+1)\tilde{f}_l, k \in \mathbf{Z}$,

$$\tilde{x}_k = \sum_{l=-\infty}^{k-1} T(k,l+1)\mathcal{G}_l\tilde{x}_l + \sum_{l=-\infty}^{k-1} T(k,l+1)\xi. \tag{2.54}$$

Invoking (2.49) the equality (2.54) may be written:

$$\tilde{x} = \mathcal{T}\tilde{x} + \tilde{g}, \tag{2.55}$$

where $\tilde{g} = \{\tilde{g}_k\}_{k\in\mathbf{Z}}, \tilde{g}_k = \sum_{l=-\infty}^{k-1} T(k,l+1)\xi$. It is obvious that $\tilde{g}_k \geq \xi$ for all $k \in \mathbf{Z}$. Hence $\tilde{g} \in \mathrm{Int}\ell^\infty(\mathbf{Z}, \mathcal{X}^+)$.

Applying implication $(v) \to (vi)$ of Theorem 2.11 in [30] for $R = -I_{\ell^\infty}$ and $P = \mathcal{T}$ one obtains that $\rho[\mathcal{T}] < 1$ and thus the proof is complete. □

In the time-invariant case one obtains the following version of Theorem 2.8.

Theorem 2.9 *Let $\mathcal{L} : \mathcal{X} \to \mathcal{X}, \mathcal{G} : \mathcal{X} \to \mathcal{X}$ be linear bounded and positive operators.*

Then the following are equivalent.

(i) $\rho[\mathcal{L}] < 1$ *and* $\rho[(I_\mathcal{X} - \mathcal{L})^{-1}\mathcal{G}] < 1$.
(ii) $\rho[\mathcal{L} + \mathcal{G}] < 1$.

Proof. If (i) holds, then based on $(iii), (iv)$ of Theorem 2.6 we deduce that $(I_\mathcal{X} - \mathcal{L})^{-1}\xi \in \mathrm{Int}\mathcal{X}^+$.

Applying $(vi) \to (iv)$ in Theorem 2.11 [30] for $R = -I_\mathcal{X}$ and $P = I_\mathcal{X} - \mathcal{L}^{-1}\mathcal{G}$ one obtains that there exists $\tilde{x} \in \mathrm{Int}\mathcal{X}^+$ which verifies

$$\tilde{x} = [I_\mathcal{X} - \mathcal{L}]^{-1}\mathcal{G}\tilde{x} + [I_\mathcal{X} - \mathcal{L}]^{-1}\xi,$$

which leads to $I_\mathcal{X} - \mathcal{L}\tilde{x} = \mathcal{G}\tilde{x} + \xi$.

Therefore we obtain that the equation

$$x_{k+1} = [\mathcal{L} + \mathcal{G}]x_k + \xi \tag{2.56}$$

has a bounded and uniform positive solution $\{\tilde{x}_k\}_{k\in Z}$, namely $\tilde{x}_k = \tilde{x}$ for all $k \in Z$.

Applying $(iii) \to (i)$ of Theorem 2.7 one obtains that the operator $\mathcal{L} + \mathcal{G}$ generates a discrete-time exponentially stable evolution which shows that the implication $(i) \to (ii)$ is valid. Let us prove the converse implication. If (ii) holds then based on the implication $(i) \to (ii)$ of Theorem 2.7 we obtain that equation (2.56) has a bounded and uniform positive solution, $\tilde{x}_k, k \in \mathbf{Z}$. Furthermore, from $(iii), (iv)$ of Theorem 2.6 we conclude that $\tilde{x}_k = \tilde{x} \in \mathrm{Int}\mathcal{X}^+$, for all $k \in \mathbf{Z}$. Hence $\tilde{x} = \mathcal{L}\tilde{x} + \tilde{f}$, where $\tilde{f} = \mathcal{G}\tilde{x} + \xi \in \mathrm{Int}\mathcal{X}^+$.

Invoking again $(iii) \to (i)$ of Theorem 2.7 one gets that $\mathcal{L}$ generates a discrete-time exponentially stable evolution. We may write $\tilde{x} = (I_\mathcal{X} - \mathcal{L})^{-1}\tilde{f}$ which leads to

$$\tilde{x} = (I_\mathcal{X} - \mathcal{L})^{-1}\mathcal{G}\tilde{x} + (I_\mathcal{X} - \mathcal{L})\xi.$$

Because $(I_{\mathcal{X}} - \mathcal{L})^{-1}\xi \in \mathrm{Int}\mathcal{X}^+$ then from $(iv) \rightarrow (vi)$ of Theorem 2.11 in [30] we obtain that $\rho[(I_{\mathcal{X}} - \mathcal{L})^{-1}\mathcal{G}] < 1$ which ends the proof of the implication $(ii) \rightarrow (i)$ and the proof is complete. □

In a similar way to that of the proof of Theorem 2.8 we may prove the following result.

Theorem 2.10 *Let $\{\mathcal{L}_k\}_{k\geq k_0}, \{\mathcal{G}_k\}_{k\geq k_0}$ be two sequences of linear and positive operators on $\mathcal{X}$ such that $\{\mathcal{G}_k\}_{k\geq k_0}$ is a bounded sequence.*

Then the following are equivalent.

(i) *The sequence $\{\mathcal{L}_k\}_{k\geq k_0}$ generates an exponentially stable evolution and $\rho[\mathcal{T}^a] < 1$, where $\mathcal{T}^a : \ell^\infty[Z_{k_0}, \mathcal{X}] \rightarrow \ell^\infty[Z_{k_0}, \mathcal{X}]$ by $y = \mathcal{T}^a x$,*

$$y_k = \sum_{l=k}^{\infty} T^*(l,k)\mathcal{G}_l^* x_l, \qquad k \geq k_0, \tag{2.57}$$

$T(l,k)$ being the causal linear evolution operator defined by the sequence $\{\mathcal{L}_k\}_{k\geq k_0}, Z_{k_0}] \subset \mathbf{Z}, Z_{k_0} = \{k \in Z | k \geq k_0\}$.

(ii) *The sequence $\{\mathcal{L}_k + \mathcal{G}_k\}_{k\in Z_{k_0}}$ generates an exponentially stable evolution on $\mathcal{X}$.*

The proof is made combining the results of the above Theorems 2.4 and 2.5, and Theorem 2.11 in [30]. □

2.5 Lyapunov-type operators

2.5.1 Sequences of Lyapunov-type operators

Consider the sequences $\{A_k(t,i)\}_{t\geq 0}$, and $\{p_t(i,j)\}_{t\geq 0}$, where $A_k(t,i) \in \mathbf{R}^{n\times n}$ and $p_t(i,j) \geq 0$, $i,j \in \mathcal{D}, 0 \leq k \leq r$. We define the following operators on $\mathcal{S}_n^N$ ($\mathcal{S}_n^N$ being the linear space introduced in Example 2.5(*iii*)), $\mathcal{L}_t S = (\mathcal{L}_t S(1), \ldots, \mathcal{L}_t S(N))$ by

$$\mathcal{L}_t S(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)A_k(t,j)S(j)A_k^T(t,j) \tag{2.58}$$

for all $i \in \mathcal{D}$, $t \geq 0$ and $\Lambda_t S = (\Lambda_t S(1), \ldots, \Lambda_t S(N))$,

$$\Lambda_t S(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i)S(j)A_k^T(t,i), \tag{2.59}$$

$t \geq 1$ for all $i \in \mathcal{D}$, $S = (S(1), \ldots, S(N)) \in \mathcal{S}_n^N$. Because $p_t(i,j) \geq 0$ it follows easily that $\mathcal{L}_t \geq 0$ for all $t \geq 0$ and $\Lambda_t \geq 0, t \geq 1$. The operators $\mathcal{L}_t$ and Λ_t are termed Lyapunov-type operators defined by the sequences $\{A_k(t,i)\}_{t\geq 0}$ and $\{P_t\}_{t\geq 0}$ (P_t is the matrix with the entries $p_t(i,j), 1 \leq i,j \leq N$).

Remark 2.4 The results concerning exponential stability for discrete-time linear equations defined by the operators $\mathcal{L}_t$ and Λ_t are directly derived from the results presented in Section 2.3. To this end we need that $\mathcal{L}_t$ and Λ_t be positive operators on $\mathcal{S}_n^N$. This happens if $p_t(i,j) \geq 0$. On the other hand there are some results (e.g., Theorem 2.11, Corollary 2.5, Theorem 2.14) asking for $\sum_{j=1}^N p_t(i,j) = 1$. In those statements we assume that P_t are stochastic matrices.

In a standard way one obtains that the adjoint operators with respect to the inner product (2.18) are given by $\mathcal{L}_t^* S = (\mathcal{L}_t^* S(1), \ldots, \mathcal{L}_t^* S(N))$ with

$$\mathcal{L}_t^* S(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_t(i,j) A_k^T(t,i) S(j) A_k(t,i) \tag{2.60}$$

for all $t \geq 0$, and $\Lambda_t^* S = (\Lambda_t^* S(1), \ldots, \Lambda_t^* S(N))$ with

$$\Lambda_t^* S(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j) S(j) A_k(t,j) \tag{2.61}$$

for all $t \geq 1$, $i \in \mathcal{D}$, $S \in \mathcal{S}_n^N$.

For each $t \geq s$ we define the linear operators $T(t,s) : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $S(t,s) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ by

$$T(t,s) = \begin{cases} \mathcal{L}_{t-1}\mathcal{L}_{t-2}\cdots\mathcal{L}_s & \text{if} \quad t > s \geq 0 \\ I_{\mathcal{S}_n^N} & \text{if} \quad t = s \end{cases} \tag{2.62}$$

and

$$S(t,s) = \begin{cases} \Lambda_{t-1}\Lambda_{t-2}\cdots\Lambda_s & \text{if} \quad t > s \geq 1 \\ I_{\mathcal{S}_n^N} & \text{if} \quad t = s \end{cases}. \tag{2.63}$$

The operators $T(t,s)$ and $S(t,s)$ are the linear evolution operators on $\mathcal{S}_n^N$ defined by the sequences $\{\mathcal{L}_t\}_{t \geq 0}$ and $\{\Lambda_t\}_{t \geq 1}$, respectively.

We have:

$$\begin{aligned} T(t+1,s) &= \mathcal{L}_t T(t,s), \qquad t \geq s \geq 0 \\ S(t+1,s) &= \Lambda_t S(t,s), \qquad t \geq s \geq 1 \\ T^*(t,s) &= \mathcal{L}_s^* T^*(t,s+1), \qquad t \geq s+1 \\ S^*(t,s) &= \Lambda_s^* S^*(t,s+1), \qquad t \geq s+1. \end{aligned} \tag{2.64}$$

Remark 2.5

(*a*) If there exists an integer $\theta \geq 2$ such that $A_k(t+\theta,i) = A_k(t,i)$, $0 \leq k \leq r$, $i \in \mathcal{D}$, $P_{t+\theta} = P_t$, $t \in \mathbf{Z}, t \geq 0$ then we have

$$\begin{aligned} T(t+m\theta, s+m\theta) &= T(t,s), \qquad t \geq s \geq 0, m \geq 1 \\ S(t+m\theta, s+m\theta) &= S(t,s), \qquad t \geq s \geq 1, m \geq 1. \end{aligned}$$

(b) If $A_k(t,i) = A_k(i), 0 \leq k \leq r, i \in \mathcal{D}$, $P_t = P, t \in \mathbf{Z}, t \geq 0$ then $\mathcal{L}_t = \mathcal{L}$, $t \geq 0$, $\Lambda_t = \Lambda$, and $T(t,s) = \mathcal{L}^{t-s}$, $S(t,s) = \Lambda^{t-s}$.

The next result establishes a relationship between the linear evolution operators $T^*(t,s)$ and $S^*(t,s)$.

Theorem 2.11 *If for all $t \geq 0$, P_t are stochastic matrices then the following hold.*

(i)

$$[S^*(t,s)J](i) = \sum_{j=1}^{N} p_{s-1}(i,j)[T^*(t,s)J](j) \tag{2.65}$$

for all $t \geq s \geq 1$, $i \in \mathcal{D}$.

(ii)

$$[T^*(t,s)J](i) = \sum_{k=0}^{r} A_k^T(s,i)[S^*(t,s+1)J](i)A_k(s,i), \tag{2.66}$$

$t \geq s+1$, $s \geq 0$, $i \in \mathcal{D}$.

Proof. (i) We prove (2.65) by induction with respect to $s \in \{t, t-1, t-2, \ldots, 1\}$. If $s = t$, (2.65) holds because

$$\sum_{j=1}^{N} p_{s-1}(i,j) = 1. \tag{2.67}$$

Let us assume that (2.65) holds for $s \in \{m+1, m+2, \ldots, t\}$ for some $m < t$ and we prove that (2.65) still holds for $s = m$, $i \in \mathcal{D}$. Indeed, we have

$$\begin{aligned}
&[S^*(t,m)J](i) \\
&\quad = [\Lambda_m^* S^*(t,m+1)J](i) \\
&\quad = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{m-1}(i,j)A_k^T(m,j)[S^*(t,m+1)J](j)A_k(m,j) \\
&\quad = \sum_{j=1}^{N} p_{m-1}(i,j)\sum_{k=0}^{r} A_k^T(m,j)\sum_{l=1}^{N} p_m(j,l)[T^*(t,m+1)J](l)A_k(m,j).
\end{aligned}$$

Considering (2.60) and (2.64) one obtains:

$$\begin{aligned}
[S^*(t,m)J](i) &= \sum_{j=1}^{N} p_{m-1}(i,j)[\mathcal{L}_m^*(T^*(t,m+1)J)](j) \\
&= \sum_{j=1}^{N} p_{m-1}(i,j)[T^*(t,m)J](j)
\end{aligned}$$

which is just (2.65) for $s = m$.

(*ii*) From (2.60) and (2.64) we may write

$$[T^*(t,s)J](i) = [\mathcal{L}_s^* T^*(t,s+1)J](i)$$
$$= \sum_{k=0}^{r} A_k^T(s,i) \sum_{j=1}^{N} p_s(i,j)[T^*(t,s+1)J](j)A_k(s,i).$$

Using (2.65) we deduce

$$[T^*(t,s)J](i) = \sum_{k=0}^{r} A_k^T(s,i)[S^*(t,s+1)J](i)A_k(s,i)$$

and thus the proof ends. □

Corollary 2.5 *If for all $t \geq 0$, P_t are stochastic matrices then we have*

(*i*)

$$\|S^*(t,s)\|_\xi \leq \|T^*(t,s)\|_\xi \tag{2.68}$$

$\forall t \geq s, s \geq 1$.

(*ii*)

$$\|T^*(t,s)\|_\xi \leq \alpha(s)\|S^*(t,s+1)\|_\xi, \tag{2.69}$$

$\forall t \geq s+1, s \geq 0$, *where* $\alpha(s) = \max_{i\in\mathcal{D}}\{\sum_{k=0}^{r} |A_k(s,i)|^2\}$.

Proof. We recall that for any matrix M, $|M|$ stands for the norm defined by $|M| = [\lambda_{\max}(M^T M)]^{1/2}$. Thus from (2.65) we have $|[S^*(t,s)J](i)| \leq \sum_{j=1}^{N} p_{s-1}(i,j)|[T^*(t,s)J](j)| \leq \max_{j\in\mathcal{D}} |[T^*(t,s)J](j)| \sum_{j=1}^{N} p_{s-1}(i,j)$. Invoking (2.67) and (2.20) one gets:

$$|S^*(t,s)J|_\xi \leq |T^*(t,s)J|_\xi \tag{2.70}$$

for all $t \geq s \geq 1$.

Under the considered assumptions we have that $S^*(t,s) \geq 0$ and $T^*(t,s) \geq 0$. Applying Proposition 2.5(*ii*) we conclude that (2.70) is just (2.68). To prove (2.69) we use (2.66) and obtain $|[T^*(t,s)J](i)| \leq [\sum_{k=0}^{r} |A_k(s,i)|^2] \max_{j\in\mathcal{D}} |[S^*(t,s+1)J](j)|$. This leads to $|T^*(t,s)J|_\xi \leq \alpha(s)|S^*(t,s+1)J|_\xi$ for all $t \geq s+1$, $s \geq 0$.

The conclusion follows by again applying Proposition 2.5(*ii*) and thus the proof ends. □

2.5.2 Exponential stability

Consider the discrete-time linear equations

$$X_{t+1} = \mathcal{L}_t X_t, \qquad t \geq 0 \tag{2.71}$$

and

$$X_{t+1} = \Lambda_t X_t, \qquad t \geq 1. \tag{2.72}$$

According to the definition given in Section 2.3 we say that the zero solution of the equation (2.71), ((2.72), respectively) is exponentially stable if there exist $\beta > 0$ and $q \in (0,1)$ such that

$$\|T(t,s)\|_\xi \leq \beta q^{(t-s)} \tag{2.73}$$

for all $t \geq s \geq 0$ and

$$\|S(t,s)\|_\xi \leq \beta q^{(t-s)} \tag{2.74}$$

for all $t \geq s \geq 1$, respectively.

Because $\mathcal{S}_n^N$ is a finite-dimensional linear space in (2.73) and (2.74) any other norm on $\mathcal{S}_n^N$ may be used instead of $\|\cdot\|_\xi$.

If (2.73) ((2.74), respectively) holds we say that the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ (or $\{\Lambda_t\}_{t\geq 1}$, respectively) generates an *exponentially stable evolution*. It is well known that in the time-invariant case, exponential stability is equivalent to the fact that the eigenvalues of the operator $\mathcal{L}$ (or the eigenvalues of the operator Λ) are located inside the unit disk $|z| < 1$.

This may be checked, for example, by applying the Routh–Hurwitz criteria (see [**??**60]) to the polynomials

$$\tilde{f}(z) = f_{\mathcal{L}}\left(\frac{z+1}{z-1}\right)$$

or

$$\tilde{f}(z) = f_{\Lambda}\left(\frac{z+1}{z-1}\right),$$

$f_{\mathcal{L}}$ and f_Λ being the characteristic polynomials of the matrices with respect to the canonical base on $\mathcal{S}_n^N$ associated with the operators $\mathcal{L}$ and Λ. The degree of these two characteristic polynomials is $(Nn(n+1))/2$.

For the time-varying case, criteria for exponential stability of the zero solution of the equations (2.71) and (2.72) (other than Lyapunov criteria) are obtained from Theorem 2.4 in Section 2.3. So, applying Theorem 2.4 to the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ one obtains the following.

Theorem 2.12 *Under the considered assumptions the following are equivalent.*

(*i*) *The sequence $\{\mathcal{L}_t\}_{t\geq 0}$ generates an exponentially stable evolution.*
(*ii*) *There exist $\beta > 0, q \in (0,1)$ such that*

$$\|T^*(t,s)\|_\xi \leq \beta q^{t-s}$$

for all $t \geq s \geq 0$.

(*iii*) *There exists a bounded sequence* $\{X_t\}_{t\geq 0} \subset \text{Int}\mathcal{S}^{N+}$ *solving the following system of equations,*

$$X_t(i) = \sum_{k=0}^{r} A_k^T(t,i) \sum_{j=1}^{N} p_t(i,j) X_{t+1}(j) A_k(t,i) + I_n \tag{2.75}$$

for all $t \geq 0, i \in \mathcal{D}$.

(*iv*) *There exists a bounded sequence* $\{Y_t\}_{t\geq 0} \subset \text{Int}\mathcal{S}_n^{N+}$, $Y_t \gg 0$ *and a positive scalar* α *that verifies the following system of inequalities,*

$$\sum_{k=0}^{r} A_k^T(t,i) \sum_{j=1}^{N} p_t(i,j) Y_{t+1}(j) A_k(t,i) - Y_t(i) \leq -\alpha I_n, \tag{2.76}$$

$\forall i \in \mathcal{D}, t \geq 0$. □

Applying Theorem 2.4 to the sequence $\{\Lambda_t\}_{t\geq 1}$ one obtains the following theorem.

Theorem 2.13 *The following are equivalent.*

(*i*) *The sequence* $\{\Lambda_t\}_{t\geq 1}$ *generates an exponentially stable evolution.*

(*ii*) *There exist* $\beta > 0, q \in (0,1)$ *such that* $\|S^*(t,s)\|_\xi \leq \beta q^{(t-s)}$, $\forall t \geq s \geq 1$.

(*iii*) *There exists a bounded sequence* $\{X_t\}_{t\geq 1} \subset \text{Int}\mathcal{S}_n^{N+}$ *verifying the following system of linear equations,*

$$X_t(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} A_k^T(t,j) p_{t-1}(i,j) X_{t+1}(j) A_k(t,j) + I_n, \tag{2.77}$$

$\forall i \in \mathcal{D}, t \geq 1$.

(*iv*) *There exists a positive scalar* α *and a bounded sequence* $\{Y_t\}_{t\geq 1} \subset \text{Int}\mathcal{S}_n^{N+}, Y_t \gg 0$ *verifying the following system of linear inequalities,*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} A_k^T(t,j) p_{t-1}(i,j) Y_{t+1}(j) A_k(t,j) - Y_t(i) \leq -\alpha I_n, \tag{2.78}$$

$t \geq 1, i \in \mathcal{D}$. □

In the above results, $Y_t \gg 0$ means that there exists $\delta > 0$ such that $Y_t \geq \delta J$ for all t. Based on Theorem 2.5 one obtains that if the sequences $\{A_k(t,i)\}_{t\geq 0}$, $\{p_t(i,j)\}_{t\geq 0}$ are periodic sequences with period θ then the bounded solutions of (2.75) and (2.77), respectively, are periodic sequences with the same period θ.

Moreover if the above sequences are constant then the bounded solutions of (2.75) and (2.77), respectively, are constant.

Thus we have the following.

Corollary 2.6 *Assume that there exists an integer $\theta \geq 2$ such that*

$$A_k(t+\theta, i) = A_k(t, i), \qquad 0 \leq k \leq r$$
$$p_{t+\theta}(i,j) = p_t(i,j) \geq 0, \qquad i,j \in \mathcal{D}, \quad t \geq 0.$$

Then the following are equivalent.

(*i*) *The sequence $\{\mathcal{L}_t\}_{t\geq 0}$ generates an exponentially stable evolution.*
(*ii*) *There exist positive definite matrices $X_t(i)$, $0 \leq t \leq \theta - 1$, $i \in \mathcal{D}$ that verify the following system of linear equations,*

$$X_t(i) = \sum_{k=0}^{r} A_k^T(t,i) \sum_{j=1}^{N} p_t(i,j) X_{t+1}(j) A_k(t,i) + I_n, \tag{2.79}$$
$$0 \leq t \leq \theta - 2,$$
$$X_{\theta-1}(i) = \sum_{k=0}^{r} A_k^T(\theta-1,i) \sum_{j=1}^{N} p_{\theta-1}(i,j) X_0(j) A_k(\theta-1,i) + I_n.$$

Corollary 2.7 *Under the assumptions of Corollary* 2.6 *the following are equivalent.*

(*i*) *The sequence $\{\Lambda_t\}_{t\geq 1}$ generates an exponentially stable evolution.*
(*ii*) *There exist positive definite matrices $X_t(i), 1 \leq t \leq \theta$, $1 \leq i \leq N$ that verify the following system of linear equations,*

$$X_t(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j) X_{t+1}(j) A_k(t,j) + I_n, \tag{2.80}$$
$$1 \leq t \leq \theta - 1,$$
$$X_\theta(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_{\theta-1}(i,j) A_k^T(0,j) X_1(j) A_k(0,j) + I_n, \qquad i \in \mathcal{D}.$$

Corollary 2.8 *Assume that $A_k(t,i) = A_k(i)$, $0 \leq k \leq r$, $p_t(i,j) = p(i,j) \geq 0$, $i,j \in \mathcal{D}, t \geq 0$; then the following are equivalent.*

(*i*) *The eigenvalues of the operator $\mathcal{L}$ are located in the inside of the unit disk $|z| < 1$.*
(*ii*) *There exist positive definite matrices $X(i), 1 \leq i \leq N$ that verify the following system of algebraic matrix equations,*

$$X(i) = \sum_{k=0}^{r} A_k^T(i) \sum_{j=1}^{N} p(i,j) X(j) A_k(i) + I_n, \qquad i \in \mathcal{D}. \tag{2.81}$$

Corollary 2.9 *Under the assumptions of Corollary* 2.8 *the following are equivalent.*

(*i*) *The eigenvalues of the operator* Λ *are located inside the unit disk* $|z| < 1$.
(*ii*) *There exist positive definite matrices* $X(i), 1 \leq i \leq N$ *that verify the following system of algebraic matrix equations,*

$$X(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) A_k^T(j) X(j) A_k(j) + I_n, \qquad i \in \mathcal{D}. \tag{2.82}$$

Remark 2.6

(*a*) It is easy to see that the systems (2.79), (2.80) contain $\hat{n}$ linear scalar equations with $\hat{n}$ scalar unknowns, where $\hat{n} = (N\theta n(n+1))/2$.
(*b*) Each of the system (2.81), (2.82) contains $\tilde{n}$ scalar linear equations with $\tilde{n}$ scalar unknowns with $\tilde{n} = (Nn(n+1))/2$. The solvability of these systems with additional condition $X(i) > 0$, $i \in \mathcal{D}$ may be viewed as an alternative test (those based on Routh–Hurwitz criteria) for stability.

In the last part of this subsection we provide a result that establishes a relationship between the property of generating an exponentially stable evolution of the sequences $\{\mathcal{L}_t\}_{t\geq 0}$ and $\{\Lambda_t\}_{t\geq 1}$.

Theorem 2.14 *Assume that for each* $t \geq 0$, P_t *is a stochastic matrix. Then the following hold.*

(*i*) *If the sequence* $\{\mathcal{L}_t\}_{t\geq 0}$ *generates an exponentially stable evolution then the sequence* $\{\Lambda_t\}_{t\geq 1}$ *generates an exponentially stable evolution.*
(*ii*) *If in addition* $\{A_k(t,i)\}_{t\geq 0}$, $0 \leq k \leq r$, $i \in \mathcal{D}$ *are bounded sequences then the sequence* $\{\mathcal{L}_t\}_{t\geq 0}$ *generates an exponentially stable evolution if and only if the sequence* $\{\Lambda_t\}_{t\geq 1}$ *generates an exponentially stable evolution.*

Proof.

(*i*) It follows immediately from (2.68) and the equivalences $(i) \Longleftrightarrow (ii)$ from Theorem 2.12 and Theorem 2.13.
(*ii*) Under the assumption of (*ii*) it follows that the sequence $\{\alpha(s)\}_{s\geq 0}$ from (2.69) is bounded. The equivalence from (*ii*) is straightforward. □

Remark 2.7

(*a*) If $\sum_{j=1}^{N} p_t(i,j) = 1$, from the second part of Theorem 2.14 it follows that assertion (*ii*) of Corollary 2.6 is equivalent to assertion (*ii*) of Corollary 2.7 and assertion (*ii*) of Corollary 2.7 is equivalent to assertion (*ii*) from Corollary 2.9.
(*b*) If $\{A_k(t,i)\}_{t\geq 0}$, are bounded sequences and P_t are stochastic matrices then assertions (*i*)–(*iv*) of Theorem 2.12 are equivalent to the assertions (*i*)–(*iv*) of Theorem 2.13.

The next two examples show that in the absence of the assumption concerning the boundedness of the sequences $\{A_k(t,i)\}_{t\geq 0}$, it is possible that the sequence $\{\Lambda_t\}_{t\geq 1}$ generates an exponentially stable evolution whereas the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ does not generate an exponentially stable evolution.

Example 2.6 Consider the Lyapunov-type operators (2.58), (2.59) in the particular case

$$N=2, P=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \qquad A_k(t,1)=0, \qquad A_k(t,2)=tI_n.$$

Direct calculation shows that

$$(\mathcal{L}_tH)(1)=\sum_{k=0}^{r}\sum_{j=1}^{2}p_t(j,1)A_k(t,j)H(j)A_k^T(t,j)=t^2(1+r)H(2)$$

$$(\mathcal{L}_tH)(2)=0,$$

$H=(H(1),H(2))\in\mathcal{S}_n^2$. Also (2.59) now gives $(\Lambda_tH)(i)=0$, $i=\overline{1,2}$, $H=(H(1),H(2))\in\mathcal{S}_n^2$. Therefore $\mathcal{S}(t,s)=0$ for all $t\geq s\geq 1$ and thus (2.74) is fulfilled.

On the other hand we deduce that

$$\|T(t+1,t)\|_\xi=|T(t+1,t)J|_\xi=t^2(1+r),$$

which shows that $\lim_{t\to\infty}\|T(t+1,t)\|_\xi=+\infty$ and therefore the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ does not generate an exponentially stable evolution.

Example 2.7 Consider the Lyapunov-type operators (2.58), (2.59) in the particular case $n=2$, $N=2$, $r=1$,

$$P_t=\begin{pmatrix}1-\frac{1}{2^{t+2}} & \frac{1}{2^{t+2}}\\ 1-\frac{1}{2^{t+2}} & \frac{1}{2^{t+2}}\end{pmatrix},$$

$A_k(t,1)=0, A_k(t,2)=2^{(t-1)/4}I_2$. We have $(\Lambda_tH)(1)=0$,

$$(\Lambda_tH)(2)=2(2)^{\frac{t-1}{2}}\frac{1}{2^{(t+1)}}(H(1)+H(2))=\frac{1}{2^{\frac{t+1}{2}}}(H(1)+H(2)),$$

for all $H=(H(1),H(2))$.

It follows that $[S(t,s)J](1)=0$. Furthermore, $[S(s+1,s)J](2)=(1/\sqrt{2})^{s-1}$ I_2 and

$$[S(t,s)J](2)=\left(\frac{1}{\sqrt{2}}\right)^{s-1}\left(\frac{1}{\sqrt{2}}\right)^{s+2+s+3+\cdots+t}I_2=2\left(\frac{1}{2}\right)^{(t+s+1)(t-s)/4}I_2$$

if $t\geq s+1$. Because

$$\frac{t+s+1}{4}\geq\frac{s+1}{2}\geq 1$$

if $t \geq s+1, s \geq 1$ one obtains

$$\frac{1}{2^{((t+s+1)(t-s))/4}} \leq \frac{1}{2^{t-s}}. \tag{2.83}$$

From Proposition 2.5(ii) together with (2.20) and (2.83) we obtain that $\|S(t,s)\|_\xi = |S(t,s)J|_\xi \leq 2(1/2)^{t-s}$ for all $t \geq s \geq 1$ which shows that the sequence $\{\Lambda_t\}_{t \geq 1}$ generates an exponentially stable evolution.

Furthermore, one gets:

$$(\mathcal{L}_t J)(i) = 2(2^{(t-1)/2})p_t(2,i)I_2, \qquad i = 1,2.$$

Therefore

$$\|T(t+1,t)\|_\xi = |T(t+1,t)J|_\xi \geq 2^{(t-1)/2}\left(1 - \frac{1}{2^{t+2}}\right).$$

Hence $\lim_{t\to\infty} \|T(t+1,t)\|_\xi = \infty$ and thus we conclude that the sequence $\{\mathcal{L}_t\}_{t\geq 1}$ does not generate an exponentially stable evolution.

Remark 2.8 It is known that if the sequences $\{\mathcal{L}_t\}_{t\geq 0}$ or $\{\Lambda_t\}_{t\geq 1}$ generate an exponentially stable evolution then those sequences are bounded. Suppose now that the assumptions of Theorem 2.14 are fulfilled. From (2.60) we have

$$(\mathcal{L}_t^* J)(i) = \sum_{k=0}^{r} A_k(t,i)^T A_k(t,i) \stackrel{\text{def}}{=} M(t,i). \tag{2.84}$$

The boundedness of the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ together with (2.13) and Proposition 2.5(ii) lead to the inequality

$$\lambda_{\max}(M(t,i)) \leq \beta_1 \tag{2.85}$$

with some $\beta_1 > 0$ not depending upon (t,i) where $\lambda_{\max}(\cdot)$ stands for the largest eigenvalue of a matrix. From (2.84) and (2.85) we conclude that if the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ generates an exponentially stable evolution then

$$\{A_k(t,i)\}_{t\geq 0}, \qquad i \in \mathcal{D}, \qquad 0 \leq k \leq r \tag{2.86}$$

are bounded sequences. In a similar way we may deduce that if the sequence $\{\Lambda_t\}_{t\geq 1}$ generates an exponentially stable evolution then

$$\{\sqrt{p_{t-1}(i,j)}A_k(t,i)\}_{t\geq 1}, \qquad i,j \in \mathcal{D}, \qquad 0 \leq k \leq r \tag{2.87}$$

are bounded sequences.

This remark shows that the assumption from Theorem 2.14(ii) could be replaced by a weaker one as the boundedness of the sequences (2.87).

2.5.3 Several special cases

In the particular case $N = 1, p_t(1,1) = 1$, the operators (2.58) and (2.59) reduce to

$$\mathcal{L}_t S = \sum_{k=0}^{r} A_k(t) S A_k^T(t) \tag{2.88}$$

for all $S \in \mathcal{S}_n$, where $A_k(t) = A_k(t,1)$.

If $A_k(t,i) = 0, 1 \leq k \leq r, t \geq 0, i \in \mathcal{D}$ the operators (2.58)–(2.59) become:

$$(\mathcal{L}_t^0 S) = ((\mathcal{L}_t^0 S)(1), \ldots, (\mathcal{L}_t^0 S)(N))$$

$$(\mathcal{L}_t^0 S)(i) = \sum_{j=1}^{N} p_t(j,i) A_0(t,j) S(j) A_0^T(t,j) \tag{2.89}$$

$$\Lambda_t^0 S = ((\Lambda_t^0 S)(1), \ldots, (\Lambda_t^0 S)(N))$$

$$(\Lambda_t^0 S)(i) = A_0(t,i) \sum_{j=1}^{N} p_{t-1}(j,i) S(j) A_0^T(t,i) \tag{2.90}$$

for all $S = (S(1), \ldots, S(N)) \in \mathcal{S}_n^N$.

The sequence (2.88) is usually involved in the characterization of the exponential stability in the mean square of discrete-time stochastic linear systems of the form

$$x(t+1) = \left[A_0(t) + \sum_{k=1}^{r} A_k(t) w_k(t) \right] x(t).$$

The operators (2.89), (2.90) are related to the problem of exponential stability in the mean square of discrete-time stochastic linear systems with Markov switching:

$$x(t+1) = A_0(t, \eta_t) x(t).$$

From (2.58) and (2.89), (2.59) and (2.90), respectively, one can see that $\mathcal{L}_t \geq \mathcal{L}_t^0 \geq 0$ for all $t \geq 0$ and $\Lambda_t \geq \Lambda_t^0 \geq 0$ for all $t \geq 1$.

From Corollary 2.4 one obtains that

$$T(t,s) \geq T^0(t,s) \geq 0 \tag{2.91}$$

for all $t \geq s$ and

$$S(t,s) \geq S^0(t,s) \geq 0 \tag{2.92}$$

for all $t \geq s \geq 1$.

$T^0(t,s)$ is the linear evolution operator on $\mathcal{S}_n^N$ defined by the sequence $\{\mathcal{L}_t^0\}_{t \geq 0}$ and $S^0(t,s)$ is the linear evolution operator on $\mathcal{S}_n^N$ defined by $\{\Lambda_t^0\}_{t \geq 1}$.

In Chapter 4 we use the following Lyapunov-type linear operators constructed based on $\{A_k(t,i)\}_{t\geq 0}$ and $\{P_t\}_{t\geq 0}$. Thus for each $i \in \mathcal{D}$ we consider the operator $\mathcal{L}_{it} : \mathcal{S}_n \to \mathcal{S}_n$ by

$$\mathcal{L}_{it} S = \sum_{k=0}^{r} p_t(i,i) A_k(t,i) S A_k^T(t,i) \tag{2.93}$$

for all $S \in \mathcal{S}_n, t \geq 0$.

In Chapters 2 and 4 we show that the operators (2.93) appear in connection with the systems with independent random perturbations

$$x(t+1) = \left[A_0(t,i) + \sum_{k=0}^{r} A_k(t,i) w_k(t) \right] x(t).$$

From (2.58) and (2.93) one obtains that

$$\mathcal{L}_{it} S(i) \leq (\mathcal{L}_t S)(i) \tag{2.94}$$

for all $S = (S(1), \ldots, S(N)) \in \mathcal{S}_n^N, t \geq 0$.

If $T_i(t,s)$ is the linear evolution operator on $\mathcal{S}_n^N$ defined by the sequence (2.93) and $T(t,s)$ is the linear evolution operator on $\mathcal{S}_n^N$ defined by the sequence $\mathcal{L}_t$ one obtains from (2.94) that

$$T_i(t,s) S(i) \leq (T(t,s)S)(i), \tag{2.95}$$

$i \in \mathcal{D}, t \geq s \geq 0, S \in \mathcal{S}_n^N$.

At the end of this subsection we consider the linear operator $L_t : \mathcal{S}_n^N \to \mathcal{S}_n^N$ defined by

$$L_t S = (\tilde{A}_0(t,1) S(1) \tilde{A}_0^T(t,1), \ldots, \tilde{A}_0(t,N) S(N) \tilde{A}_0^T(t,N)) \tag{2.96}$$

for all $S = (S(1), \ldots, S(N)) \in \mathcal{S}_n^N$, where $\tilde{A}_0(t,i) = \sqrt{p_t(i,i)} A_0(t,i), i \in \mathcal{D}$, $t \geq 0$.

In the case $N = 1$, the operator (2.96) is just the symmetric Stein operator defined by $\tilde{A}_0(t,1)$.

Let $T_L(t,s)$ be the linear evolution operator on $\mathcal{S}_n^N$ defined by the sequence $\{L_t\}_{t\geq 0}$. By direct calculation one obtains that

$$T_L(t,s) S = (\Phi_{\tilde{A}}(t,s,1) S(1) \Phi_{\tilde{A}}^T(t,s,1), \ldots, \Phi_{\tilde{A}}(t,s,N) S(N) \Phi_{\tilde{A}}^T(t,s,N)), \tag{2.97}$$

where $\Phi_{\tilde{A}}(t,s,i) = \tilde{A}_0(t-1,i)\tilde{A}_0(t-2,i) \cdots \tilde{A}_0(s,i)$ if $t \geq s+1, s \geq 0$ and $\Phi_{\tilde{A}}(t,s,i) = I_n$ if $t = s$.

If the L_t^* is the adjoint of the operator L_t with respect to the inner product (2.18) then

$$L_t^* S = (\tilde{A}_0^T(t,1) S(1) \tilde{A}_0(t,1), \ldots, \tilde{A}_0^T(t,N) S(N) \tilde{A}_0(t,N))$$

for all $S \in \mathcal{S}_n^N$.

Also the adjoint of the linear evolution operator $T_L(t,s)$ is

$$T_L^*(t,s)S = (\Phi_{\tilde{A}}^T(t,s,1)S(1)\Phi_{\tilde{A}}(t,s,1), \dots, \Phi_{\tilde{A}}^T(t,s,N)S(N)\Phi_{\tilde{A}}(t,s,N)) \tag{2.98}$$

for all $S \in \mathcal{S}_n^N$.

From (2.89) and (2.96) together with Corollary 2.4 one obtains the following.

Corollary 2.10 *The following hold.*

(i) $\mathcal{L}_t^0 \geq L_t \geq 0$ *for all* $t \geq 0$.
(ii) $T^0(t,s) \geq T_L(t,s), t \geq s \geq 0$.

2.5.4 A class of generalized Lyapunov-type operators

Consider the sequences $\{A_k(t,i,j)\}_{t\geq 0}, A_k(t,i,j) \in \mathbf{R}^{n\times n}, 0 \leq k \leq r, i,j \in \mathcal{D}$, $\{P_t\}_{t\geq 0}, P_t = (p_t(i,j)) \in \mathbf{R}^{N\times N}$ with $p_t(i,j) \geq 0$. Based on these sequences we construct the linear operators $\Upsilon_t : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $\Upsilon_t S = (\Upsilon_t S(1), \dots, \Upsilon_t S(N))$ with

$$\Upsilon_t S(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i)A_k(t,i,j)S(j)A_k^T(t,i,j), \qquad t \geq 1, S \in \mathcal{S}_n^N. \tag{2.99}$$

Obviously $\Upsilon_t \geq 0$. By direct calculation based on the definition of the adjoint operator with respect to the inner product (2.18) one obtains that $\Upsilon_t^* S = (\Upsilon_t^* S(1), \dots, \Upsilon_t^* S(N))$ with

$$\Upsilon_t^* S(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(i,j)A_k^T(t,j,i)S(j)A_k(t,j,i), \qquad t \geq 1, \qquad S \in \mathcal{S}_n^N. \tag{2.100}$$

Let $R(t,s)$ be the linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $\{\Upsilon_t\}_{t\geq 1}$. Hence $R(t,s) = \Upsilon_{t-1}\Upsilon_{t-2}\cdots\Upsilon_s$ if $t \geq s+1$ and $R(t,s) = \mathcal{I}_{\mathcal{S}_n^N}$ if $t = s \geq 1$.

The next result shows that the Lyapunov-type operators (2.58)–(2.59) can be viewed as special cases of (2.99).

Proposition 2.7 *We have the following.*

(i) *If* $A_k(t,i,j) = A_k(t,i), 0 \leq k \leq r,\ i,j \in \mathcal{D}, t \geq 1$ *then*

$$\Upsilon_t S = \Lambda_t S, \qquad S \in \mathcal{S}_n^N, \tag{2.101}$$

$$R(t,s)S = S(t,s), \qquad t \geq s \geq 1. \tag{2.102}$$

(ii) If $A_k(t,i,j) = A_k(t-1,j), 0 \leq k \leq r,\ i,j \in \mathcal{D}, t \geq 1$ *then*

$$\Upsilon_t S = \mathcal{L}_{t-1} S \tag{2.103}$$

$$R(t,s) = T(t-1, s-1), t \geq s \geq 1. \tag{2.104}$$

The proof follows immediately based on (2.58)*,* (2.99) *and* (2.59)*,* (2.99)*, respectively.* □

Criteria for exponential stability of the zero solution of the equation $X_{t+1} = \Upsilon_t X_t$ can be derived from Theorem 2.4.

2.6 Notes and references

The Minkovski functional and the Minkovski seminorm associated with some convex sets in topological linear spaces are widely investigated in many monographs on functional analysis (we refer to [51, 76, 100]). The assertions (i)–(iv) and (i), (vii) in Theorem 2.1 follow from general facts in topological linear spaces.

Positive functionals, positive, monotone, quasimonotone linear operators on finite- or infinite-dimensional Banach spaces ordered by cones with different properties are intensively studied in many works (see [5, 30, 79, 78, 77, 101]). The finite-dimensional counterpart of the results in Section 2.3 was published in [38]. The results in this chapter can be found in [46].

3

Mean square exponential stability

The problem of mean square exponential stability for a class of discrete-time linear stochastic systems subject to independent random perturbations and Markovian switching is investigated. Four different definitions of the concept of exponential stability in the mean square are introduced and it is shown that they are not always equivalent. One definition of the concept of mean square exponential stability is done in terms of the exponential stability of the evolution defined by a sequence of linear positive operators on an ordered Hilbert space. The other three definitions are given in terms of different types of exponential behavior of the trajectories of the considered system. In our approach the Markov chain is not prefixed. The only available information about the Markov chain is the sequence of probability transition matrices and the set of its states. In this way one obtains that if the system is affected by Markovian jumping the property of exponential stability is independent of the initial distribution of the Markov chain.

The definition expressed in terms of exponential stability of the evolution generated by a sequence of linear positive operators allows us to characterize the mean square exponential stability based on the existence of some quadratic Lyapunov functions. Unlike the continuous-time framework, for the discrete-time linear stochastic systems with Markovian jumping two types of Lyapunov operators are introduced. Therefore in the case of discrete-time linear stochastic systems subject to Markovian perturbations one obtains characterizations of the mean square exponential stability that do not have an analogue in continuous time.

The results developed in this chapter may be used to derive some procedures for designing stabilizing controllers for the considered class of discrete-time linear stochastic systems.

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3.1 Some representation theorems

Let us consider discrete-time linear stochastic systems of the form:

$$x(t+1) = \left[A_0(t,\eta_t) + \sum_{k=1}^{r} A_k(t,\eta_t)w_k(t)\right] x(t), \tag{3.1}$$

$t \geq 0$, $t \in \mathbf{Z}_+$, where $x(t) \in \mathbf{R}^n, \{\eta_t\}_{t\geq 0}$ and $\{w_k(t)\}_{t\geq 0}, 1 \leq k \leq r$ are sequences of independent random variables with the properties given in Section 1.5.

Remark 3.1 We remark that the only available information concerning the system (3.1) is the set $\mathcal{D}$ and the sequences $\{P_t\}_{t\in\mathbf{Z}_+}$, $\{A_k(t,i)\}_{t\in\mathbf{Z}_+}, 0 \leq k \leq r, i \in \mathcal{D}$. The initial distribution of the Markov chain is not prefixed.

The concept of exponential stability in the mean square introduced in this chapter is a property of the sequences $\{P_t\}_{t\geq 0}, \{A_k(t,i)\}_{t\geq 0}, 0 \leq k \leq r, i \in \mathcal{D}$ and it does not depend upon the initial distribution π_0 of the Markov chain.

If $A_k(t,i) = 0, 1 \leq k \leq r, i \in \mathcal{D}, t \geq 0$ the system (3.1) reduces to

$$x(t+1) = A_0(t,\eta_t)x(t). \tag{3.2}$$

In the particular case $N = 1$, the system (3.1) takes the form

$$x(t+1) = \left[A_0(t) + \sum_{k=1}^{r} A_k(t)w_k(t)\right] x(t), \tag{3.3}$$

$t \geq 0$, where $A_k(t) = A_k(t,1)$, $0 \leq k \leq r$.

In the control literature linear stochastic systems of type (3.2) subject to Markov perturbations and stochastic systems of type (3.3) subject to sequences of independent random perturbations are usually separately investigated.

In this chapter we study the problem of mean square exponential stability for linear stochastic systems of type (3.1). The results derived here contain as particular cases the known results concerning mean square exponential stability for linear stochastic systems with Markov perturbations and for linear stochastic systems with independent random perturbations, respectively.

Set $A(t) = A_0(t,\eta_t) + \sum_{k=1}^{r} A_k(t,\eta_t)w_k(t)$. Then (3.1) becomes:

$$x(t+1) = A(t)x(t), \qquad t \geq 0.$$

For each $t \geq s \geq 0$ we define

$$\Phi(t,s) = A(t-1)A(t-2)\cdots A(s), \quad \text{if } t > s$$

and

$$\Phi(t,s) = I_n, \quad \text{if } t = s.$$

In the sequel $\Phi(t,s)$ is called *the fundamental (random) matrix solution* of (3.1). Each solution of (3.1) verifies $x(t) = \Phi(t,s)x(s),\ t \geq s,\ s \geq 0$.

The next result provides a connection between the adjoint of the linear evolution operators defined by (2.62)–(2.63) and the trajectories of the system (3.1).

Theorem 3.1 *Under the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *the following hold.*

(*i*) $T^*(t,s)H = (T^*(t,s)H(1),\dots,T^*(t,s)H(N))$ *with*

$$T^*(t,s)H(i) = E[\Phi^T(t,s)H(\eta_t)\Phi(t,s)|\eta_s = i] \tag{3.4}$$

for all $t \geq s, i \in \mathcal{D}_s, s \geq 0, H = (H(1),\dots,H(N)) \in \mathcal{S}_n^N, T(t,s)$ *being the linear evolution operator on* $\mathcal{S}_n^N$ *defined by the Lyapunov operator* (2.58) *corresponding to the given matrices* $A_k(t,i)$ *and stochastic matrices* $P_t, t \geq 0$.

(*ii*) $S^*(t,s)H = (S^*(t,s)H(1),\dots,S^*(t,s)H(N))$ *with*

$$S^*(t,s)H(i) = E[\Phi^T(t,s)H(\eta_{t-1})\Phi(t,s)|\eta_{s-1} = i] \tag{3.5}$$

for all $t \geq s, i \in \mathcal{D}_{s-1}, s \geq 1, H \in \mathcal{S}_n^N, S(t,s)$ *being the linear evolution operator on* $\mathcal{S}_n^N$ *defined by the Lyapunov operator* (2.59) *corresponding to the given matrices* $A_k(t,i)$ *and stochastic matrices* $P_t, t \geq 0$*, where* $\mathcal{D}_s$ *are defined in* (1.8).

Proof. (*i*) Consider the linear operators $\mathcal{U}(t,s) : \mathcal{S}_n^N \to \mathcal{S}_n^N, t \geq s \geq 0$ defined by

$$\mathcal{U}(t,s)H = (\mathcal{U}(t,s)H(1),\dots,\mathcal{U}(t,s)H(N)),$$

$\mathcal{U}(t,s)H(i) = E[\Phi^T(t,s)H(\eta_t)\Phi(t,s)|\eta_s = i]$ if $i \in \mathcal{D}_s$ and $\mathcal{U}(t,s)H(i) = T^*(t,s)H(i)$ if $i \in \mathcal{D}\setminus\mathcal{D}_s$, for arbitrary $H = (H(1)\dots H(N)) \in \mathcal{S}_n^N$. From the definition of the fundamental matrix solution we have $\Phi(t+1,s) = A(t)\Phi(t,s)$.

Because $\Phi(t,s)$ and $A(t)$ are $\mathcal{H}_t$-measurable (where $\mathcal{H}_t$ is the σ-algebra introduced in Section 1.5) one obtains, based on Corollary 1.1, that

$$\begin{aligned} &E[\Phi^T(t,s)A^T(t)H(j)A(t)\Phi(t,s)\chi_{\{\eta_{t+1}=j\}}|\mathcal{H}_t] \\ &\quad= \Phi^T(t,s)A^T(t)H(j)A(t)\Phi(t,s)E[\chi_{\{\eta_{t+1}=j\}}|\mathcal{H}_t] \\ &\quad= \Phi^T(t,s)A^T(t)H(j)A(t)\Phi(t,s)p_t(\eta_t,j) \quad \text{a.s.} \end{aligned} \tag{3.6}$$

Consider the σ-algebra $\tilde{\mathcal{H}}_t = \mathcal{H}_{t-1} \vee \sigma[\eta_t]$. Because $\Phi(t,s)$ is $\tilde{\mathcal{H}}_t$-measurable and $\tilde{\mathcal{H}}_t \subset \mathcal{H}_t$ we obtain from (3.6) that

$$\begin{aligned} &E[\Phi^T(t,s)A^T(t)H(j)A(t)\Phi(t,s)\chi_{\{\eta_{t+1}=j\}}|\tilde{\mathcal{H}}_t] \\ &\quad= E[p_t(\eta_t,j)\Phi^T(t,s)A^T(t)H(j)A(t)\Phi(t,s)|\tilde{\mathcal{H}}_t] \\ &\quad= p_t(\eta_t,j)\Phi^T(t,s)E[A^T(t)H(j)A(t)|\tilde{\mathcal{H}}_t]\Phi(t,s). \end{aligned} \tag{3.7}$$

From the assumption $\mathbf{H_1}$ we have that the random vector $w(t)$ is independent of $\tilde{\mathcal{H}}_t$. This shows (via Theorem 1.3(v)) that $E[w(t)|\tilde{\mathcal{H}}_t] = 0$ and $E[w_l(t)w_k(t)|\tilde{\mathcal{H}}_t] = \delta_{lk}$ with $\delta_{lk} = 0$ for $l \neq k$ and $\delta_{lk} = 1$ for $l = k$.

This allows us to write

$$E[A^T(t)H(j)A(t)|\tilde{\mathcal{H}}_t] = \sum_{k=0}^{r} A_k^T(t,\eta_t)H(j)A_k(t,\eta_t). \tag{3.8}$$

Combining (3.7) and (3.8) we get

$$\begin{aligned} &E[\Phi^T(t,s)A^T(t)H(j)A(t)\Phi(t,s)\chi_{\{\eta_{t+1}=j\}}|\tilde{\mathcal{H}}_t] \\ &\quad = p_t(\eta_t,j)\Phi^T(t,s)\left[\sum_{k=0}^{r} A_k^T(t,\eta_t)H(j)A_k(t,\eta_t)\right]\Phi(t,s). \end{aligned} \tag{3.9}$$

On the other hand we have

$$\begin{aligned} &E[\Phi^T(t,s)A^T(t)H(\eta_{t+1})A(t)\Phi(t,s)|\eta_s] \\ &\quad = \sum_{j=1}^{N} E[\Phi^T(t,s)A^T(t)H(j)A(t)\Phi(t,s)\chi_{\{\eta_{t+1}=j\}}|\eta_s]. \end{aligned} \tag{3.10}$$

For $t \geq s$ we have $\sigma[\eta_s] \subset \tilde{\mathcal{H}}_t$.

Thus (3.9) and (3.10) give:

$$\begin{aligned} &E[\Phi^T(t,s)A^T(t)H(\eta_{t+1})A(t)\Phi(t,s)|\eta_s] \\ &\quad = E[\Phi^T(t,s)\left\{\sum_{k=0}^{r} A_k^T(t,\eta_t)\sum_{j=1}^{N} p_t(\eta_t,j)H(j)A_k(t,\eta_t)\right\}\Phi(t,s)|\eta_s]. \end{aligned}$$

Invoking (2.60) we obtain:

$$\begin{aligned} &E[\Phi^T(t,s)A^T(t)H(\eta_{t+1})\Phi(t,s)|\eta_s] \\ &\quad = E[\Phi^T(t,s)(\mathcal{L}_t^* H)(\eta_t)\Phi(t,s)|\eta_s]. \end{aligned}$$

Because $\Phi(t+1,s) = A(t)\Phi(t,s)$ we obtain

$$E[\Phi^T(t+1,s)H(\eta_{t+1})\Phi(t+1,s)|\eta_s] = E[\Phi^T(t,s)(\mathcal{L}_t^* H)(\eta_t)\Phi(t,s)|\eta_s].$$

If $i \in \mathcal{D}_s$ by Remark 1.1 we may write

$$\begin{aligned} &E[\Phi^T(t+1,s)H(\eta_{t+1})\Phi(t+1,s)|\eta_s = i] \\ &\quad = E[\Phi^T(t,s)(\mathcal{L}_t^* H)(\eta_t)\Phi(t,s)|\eta_s = i] \end{aligned} \tag{3.11}$$

for all $H \in \mathcal{S}_n^N, t \geq s \geq 0$.

Taking into account the definition of $\mathcal{U}(t,s)$ (3.11) may be written as

$$\mathcal{U}(t+1,s)H(i) = \mathcal{U}(t,s)(\mathcal{L}_t^* H)(i) \tag{3.12}$$

for all $t \geq s, i \in \mathcal{D}_s, s \geq 0, H \in \mathcal{S}_n^N$.

On the other hand if $i \in \mathcal{D} \setminus \mathcal{D}_s$ we obtain

$$\begin{aligned}(\mathcal{U}(t+1,s)H)(i) &= (T^*(t+1,s)H)(i) = (T^*(t,s)\mathcal{L}_t^* H)(i)\\ &= (\mathcal{U}(t,s)\mathcal{L}_t^* H)(i).\end{aligned}$$

This equality together with (3.12) shows that

$$(\mathcal{U}(t+1,s)H)(i) = (\mathcal{U}(t,s)\mathcal{L}_t^* H)(i)$$

for all $i \in \mathcal{D}, H \in \mathcal{S}_n^N$, which leads to $\mathcal{U}(t+1,s) = \mathcal{U}(t,s)\mathcal{L}_t^*, \forall t \geq s \geq 0$.

Thus we obtain that the sequence of operators $\{\mathcal{U}(t,s)\}_{t\geq s}$ verifies the same discrete-time equation as the sequence of operators $\{T^*(t,s)\}_{t\geq s}$. Because $\mathcal{U}(s,s)H = T^*(s,s)H = H$ we conclude that $\mathcal{U}(t,s)H = T^*(t,s)H$ for arbitrary $t \geq s \geq 0, H \in \mathcal{S}_n^N$ which shows that (3.4) is valid.

(ii) To prove (3.5) we define the operators $\mathcal{V}(t,s) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ by

$$\begin{aligned}\mathcal{V}(t,s)H &= (\mathcal{V}(t,s)H(1),\dots,\mathcal{V}(t,s)H(N)),\\ \mathcal{V}(t,s)H(i) &= E[\Phi^T(t,s)H(\eta_{t-1})\Phi(t,s)|\eta_{s-1} = i]\end{aligned} \tag{3.13}$$

for all $t \geq s, i \in \mathcal{D}_{s-1}, s \geq 1$.

$$\mathcal{V}(t,s)H(i) = S^*(t,s)H(i) \tag{3.14}$$

if $i \in \mathcal{D} \setminus \mathcal{D}_{s-1}, s \geq 1, H \in \mathcal{S}_n^N$.

We write

$$\begin{aligned}&E[\Phi^T(t+1,s)H(\eta_t)\Phi(t+1,s)|\eta_{s-1}]\\ &\quad= E[\Phi^T(t,s)A^T(t)H(\eta_t)A(t)\Phi(t,s)|\eta_{s-1}] \qquad (3.15)\\ &\quad= \sum_{j=1}^N E\left[\Phi^T(t,s)\left\{A_0(t,j) + \sum_{k=1}^r A_k(t,j)w_k(t)\right\}^T\right.\\ &\qquad \left.\times H(j)\left\{A_0(t,j) + \sum_{l=1}^r A_l(t,j)w_l(t)\right\}\Phi(t,s)\chi_{\{\eta_t=j\}}|\eta_{s-1}\right].\end{aligned}$$

On the other hand as in the proof of the first assertion we may write:

$$\begin{aligned}&E\left[\Phi^T(t,s)\left\{A_0(t,j) + \sum_{k=1}^r A_k(t,j)w_k(t)\right\}^T H(j)\right.\\ &\qquad \left.\times\left\{A_0(t,j) + \sum_{l=1}^r A_l(t,j)w_l(t)\right\}\Phi(t,s)\chi_{\{\eta_t=j\}}|\tilde{\mathcal{H}}_t\right]\end{aligned}$$

$$= \chi_{\{\eta_t=j\}}\Phi^T(t,s)E\left[\left\{A_0(t,j)+\sum_{k=1}^r A_k(t,j)w_k(t)\right\}^T \right.$$

$$\left. \times H(j)\left\{A_0(t,j)+\sum_{l=1}^r A_l(t,j)w_l(t)\right\}|\tilde{\mathcal{H}}_t\right]\Phi(t,s)$$

$$= \chi_{\{\eta_t=j\}}\Phi^T(t,s)\sum_{k=0}^r A_k^T(t,j)H(j)A_k(t,j)\Phi(t,s).$$

Again applying Corollary 1.1 we obtain

$$E[\chi_{\{\eta_t=j\}}|\mathcal{H}_{t-1}] = p_{t-1}(\eta_{t-1},j). \tag{3.16}$$

Because $\sigma[\eta_{s-1}] \subset \mathcal{H}_{t-1} \subset \tilde{\mathcal{H}}_t$ we obtain from (3.15) and (3.16) that

$$E[\Phi^T(t+1,s)H(\eta_t)\Phi(t,s)|\eta_{s-1}]$$

$$= E\left[\Phi^T(t,s)\sum_{j=1}^N p_{t-1}(\eta_{t-1},j)\left\{\sum_{k=0}^r A_k^T(t,j)H(j)A_k(t,j)\right\}\Phi(t,s)|\eta_{s-1}\right].$$

Invoking (2.61), we further obtain

$$E[\Phi^T(t+1,s)H(\eta_t)\Phi(t+1,s)|\eta_{s-1}] = E[\Phi^T(t,s)(\Lambda_t^*H)(\eta_{t-1})\Phi(t,s)|\eta_{s-1}].$$

If $i \in \mathcal{D}_{s-1}$ the last equality leads to

$$E[\Phi^T(t+1,s)H(\eta_t)\Phi(t+1,s)|\eta_{s-1}=i]$$
$$= E[\Phi^T(t,s)(\Lambda_t^*H)(\eta_{t-1}\Phi(t,s)|\eta_{s-1}=i].$$

In view of (3.13) this may be written as

$$(\mathcal{V}(t+1,s)H)(i) = (\mathcal{V}(t,s)\Lambda_t^*H)(i) \tag{3.17}$$

for all $i \in \mathcal{D}_{s-1}$.

As in the proof of (i) one checks that (3.17) is also valid for $i \in \mathcal{D}\setminus\mathcal{D}_{s-1}$. Thus we obtain that $\mathcal{V}(t+1,s)H = \mathcal{V}(t,s)\Lambda_t^*H$, for all $t \geq s \geq 1, H \in \mathcal{S}_n^N$.

Thus we conclude that $\mathcal{V}(t,s)H = S^*(t,s)H$ and the proof is complete. □

Corollary 3.1 *Under the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *the following are valid.*

(*i*) $[T^*(t,0)J](i) = E[\Phi_i^T(t,0)\Phi_i(t,0)]$ *for all* $t \geq 0, i \in \mathcal{D}$.
(*ii*) $[S^*(t,1)J](i) = E[\Phi_i^T(t,1)\Phi_i(t,1)]$ *for all* $t \geq 1, i \in \mathcal{D}$, *where* $\Phi_i(t,s)$ *is the fundamental matrix solution of the system* (3.1) *in the particular case of* $\mathcal{D}_0 = \{i\}$*; that is,* $\pi_0(j) = 0$ *if* $j \neq i$ *and* $\pi_0(j) = 1$ *if* $j = i$.

Remark 3.2

(*a*) From (3.4) and (3.5) we conclude that the conditional expectations

$$E[\Phi^T(t,s)H(\eta_t)\Phi(t,s)|\eta_s = i], \qquad i \in \mathcal{D}_s$$

and

$$E[\Phi^T(t,s)H(\eta_{t-1})\Phi(t,s)|\eta_{s-1} = i], \qquad i \in \mathcal{D}_{s-1}$$

do not depend upon the initial distributions $\pi_0(i)$ of the Markov chain.

(*b*) In the case $\mathcal{D} = \{1\}$ (the case of the system of type (3.3)) the two equalities from Theorem 3.1 reduce to

$$T^*(t,s)H = E[\Phi^T(t,s)H\Phi(t,s)]$$

for all $t \geq s \geq 0, H \in \mathcal{S}_n$.

Now we consider the discrete-time time-varying stochastic linear system of the form

$$x(t+1) = \left[A_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} A_k(t,\eta_t,\eta_{t-1})w_k(t)\right] x(t), \tag{3.18}$$

$t \geq 1$, where $\{\eta_t\}_{t\geq 0}, \{w_k(t)\}_{t\geq 0}$ are as in system (3.1).

The investigation of the problem of exponential stability in the mean square for systems of type (3.18) is motivated by the exponential stability in the mean square of the zero state equilibrium of systems of type (1.30) or (1.31), in the absence of exogenous perturbation $v(t)$. Concerning the systems (3.18) we assume that only the sequences $\{A_k(t,i,j)\}_{t\geq 1}, \{P_t\}_{t\geq 0}$, and the set $\mathcal{D}$ are prefixed.

The initial distributions of the Markov chain are arbitrary. Here $A_k(t,i,j) \in \mathbf{R}^{n\times n}$ for all $i,j \in \mathcal{D}, t \geq 1$ and $P_t \in \mathbf{R}^{N\times N}$. Set $\mathcal{A}(t) = A_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} A_k(t,\eta_t,\eta_{t-1})w_k(t), t \geq 1$ and define $\Theta(t,s)$ as follows. $\Theta(t,s) = \mathcal{A}(t-1)\mathcal{A}(t-2)\ldots\mathcal{A}(s)$ if $t \geq s+1$ and $\Theta(t,s) = I_n$ if $t = s, s \geq 1$.

Any solution $x(t)$ of (3.18) verifies

$$x(t) = \Theta(t,s)x(s).$$

$\Theta(t,s)$ is called the fundamental (random) matrix solution of (3.18).

Theorem 3.2 *Under the assumption* $\mathbf{H}_1, \mathbf{H}_2$ *the following equality holds.*

$$(R^*(t,s)H)(i) = E[\Theta^T(t,s)H(\eta_{t-1})\Theta(t,s)|\eta_{s-1} = i] \tag{3.19}$$

for all $H = (H(1),\ldots,H(N)) \in \mathcal{S}_n^N, t \geq s \geq 1, i \in \mathcal{D}_{s-1}$, $R(t,s)$ *being the linear evolution operator on* $\mathcal{S}_n^N$ *generated by the sequence of generalized Lyapunov-type operators* (2.99).

Proof. We consider the family of linear operators $\tilde{\mathcal{V}}(t,s) : \mathcal{S}_n^N \to \mathcal{S}_n^N, t \geq s \geq 1$ defined as follows. $(\tilde{\mathcal{V}}(t,s)H)(i) = E[\Theta^T(t,s)H(\eta_{t-1})\Theta(t,s)|\eta_{s-1} = i]$ if $i \in \mathcal{D}_{s-1}$ and $(\tilde{\mathcal{V}}(t,s)H)(i) = (R^*(t,s)H)(i)$ if $i \in \mathcal{D} \setminus \mathcal{D}_{s-1}$ for all $H \in \mathcal{S}_n^N$.

Firstly we write

$$\begin{aligned}
&\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s) \\
&= \Theta^T(t,s)\mathcal{A}^T(t)H(\eta_t)\mathcal{A}(t)\Theta(t,s) \\
&= \sum_{j=1}^{N} \Theta^T(t,s)\left(A_0(t,j,\eta_{t-1}) + \sum_{k=1}^{r} A_k(t,j,\eta_{t-1})w_k(t)\right)^T H(j) \\
&\quad \times \left(A_0(t,j,\eta_{t-1}) + \sum_{l=1}^{r} A_l(t,j,\eta_{j-1})w_l(t)\right)\Theta(t,s)\chi_{\{\eta_t=j\}}. \qquad (3.20)
\end{aligned}$$

Because $\Theta(t,s)$ and $\chi_{\{\eta_t=j\}}$ are $\tilde{\mathcal{H}}_t$-measurable one obtains:

$$\begin{aligned}
&E\left[\Theta^T(t,s)\left(A_0(t,j,\eta_{t-1}) + \sum_{k=1}^{r} A_k(t,j,\eta_{t-1})w_k(t)\right)^T\right. \\
&\quad \left.\times H(j)\left(A_0(t,j,\eta_{t-1}) + \sum_{l=1}^{r} A_l(t,j,\eta_{t-1})w_l(t)\right)\Theta(t,s)\chi_{\{\eta_t=j\}}|\tilde{\mathcal{H}}_t\right] \\
&= \chi_{\{\eta_t=j\}}\Theta^T(t,s)E\left[\left(A_0(t,j,\eta_{t-1}) + + \sum_{k=1}^{r} A_k(t,j,\eta_{t-1})w_k(t)\right)^T H(j)\right. \\
&\quad \left.\times \left(A_0(t,j,\eta_{t-1}) + \sum_{l=1}^{r} A_l(t,j,\eta_{t-1})w_l(t)\right)|\tilde{\mathcal{H}}_t\right]\Theta(t,s) \\
&= \chi_{\{\eta_t=j\}}\Theta^T(t,s)\sum_{k=0}^{r} A_k^T(t,j,\eta_{t-1})H(j)A_k(t,j,\eta_{t-1})\Theta(t,s).
\end{aligned}$$

For the last equality we take into account that η_{t-1} is $\tilde{\mathcal{H}}_t$-measurable whereas $w_k(t)$ are independent of the σ-algebra $\tilde{\mathcal{H}}_t$.

As we already saw, in this case we have

$$E[w_k(t)|\tilde{\mathcal{H}}_t] = 0$$

$$E[w_k(t)w_l(t)|\tilde{\mathcal{H}}_t] = \delta_{kl},$$

δ_{kl} being introduced in the proof of Theorem 3.1.

Because $\mathcal{H}_{t-1} \subset \tilde{\mathcal{H}}_t$ and $\Theta(t,s), \eta_{t-1}$ are $\mathcal{H}_{t-1}$-measurable we may apply Theorem 1.3(*ii*), (*iv*) to obtain

$$E\left[\chi_{\{\eta_t=j\}}\Theta^T(t,s)\left(A_0(t,j,\eta_{t-1})+\sum_{k=1}^r A_k(t,j,\eta_{t-1})w_k(t)\right)^T \right.$$

$$\left. \times H(j)\left(A_0(t,j,\eta_{t-1})+\sum_{l=1}^r A_l(t,j,\eta_{t-1})w_l(t)\right)\Theta(t,s)|\mathcal{H}_{t-1}\right] \quad (3.21)$$

$$= \Theta^T(t,s)\sum_{k=0}^r (A_k^T(t,j,\eta_{t-1})H(j)A_k(t,j,\eta_{t-1}))\Theta(t,s)E[\chi_{\{\eta_t=j\}}|\mathcal{H}_{t-1}].$$

Based on Corollary 1.1 one gets:

$$\Theta^T(t,s)\sum_{k=0}^r A_k^T(t,j,\eta_{t-1})H(j)A_k(t,j,\eta_{t-1})\Theta(t,s)E[\chi_{\{\eta_t=j\}}|\mathcal{H}_{t-1}]$$

$$= p_{t-1}(\eta_{t-1},j)\Theta^T(t,s)\sum_{k=0}^r (A_k^T(t,j,\eta_{t-1})H(j)A_k(t,j,\eta_{t-1}))\Theta(t,s). \quad (3.22)$$

Combining (3.20)–(3.22) with (2.100) we obtain:

$$E[\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s)|\mathcal{H}_{t-1}]$$
$$= \Theta^T(t,s)(\Upsilon_t^* H)(\eta_{t-1})\Theta(t,s) \quad (3.23)$$

for all $t \geq s \geq 1$.

The inclusion $\sigma(\eta_{s-1}) \subseteq \mathcal{H}_{t-1}$ together with Theorem 1.3*(ii)* allows us to obtain from (3.23) that

$$E[\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s)|\eta_{s-1}]$$
$$= E[\Theta^T(t,s)(\Upsilon_t^* H)(\eta_{t-1})\Theta(t,s)(\eta_{s-1})]. \quad (3.24)$$

If $i \in \mathcal{D}_{s-1}$ then (3.24) leads to

$$E[\Theta^T(t+1,s)H(\eta_t)\Theta(t+1,s)|\eta_{s-1}=i]$$
$$= E[\Theta^T(t,s)(\Upsilon_t^* H)(\eta_{t-1})\Theta(t,s)|\eta_{s-1}=i]. \quad (3.25)$$

Having in mind the definition of $\tilde{\mathcal{V}}(t,s)$ we see that (3.25) may be rewritten as

$$(\tilde{\mathcal{V}}(t+1,s)H)(i) = [\tilde{\mathcal{V}}(t,s)(\Upsilon^* H)](i) \quad (3.26)$$

for all $i \in \mathcal{D}_{s-1}, H \in \mathcal{S}_n^N$.

As in the proof of Theorem 3.1 one establishes that (3.26) still holds for $i \in \mathcal{D} \setminus \mathcal{D}_{s-1}$. Thus we may conclude that

$$\tilde{\mathcal{V}}(t+1,s) = \tilde{\mathcal{V}}(t,s)\Upsilon_t^*, \qquad t \geq s \geq 1. \tag{3.27}$$

On the other hand it is easy to see that $\tilde{\mathcal{V}}(s,s)H = H$ for all $H \in \mathcal{S}_n^N$. Hence $\tilde{\mathcal{V}}(s,s) = \mathcal{I}_{\mathcal{S}_n^N} = R^*(s,s)$.

This equality together with (3.27) shows that $\tilde{\mathcal{V}}(t,s) = R^*(t,s)$ and thus the proof ends. □

Remark 3.3 It is easy to see that in the case $A_k(t,i,j) = A_k(t,i)$, for all $0 \leq k \leq r, t \geq 1, i,j \in \mathcal{D}$ the system (3.18) reduces to (3.1). In this case we have $\Theta(t,s) = \Phi(t,s)$. Based on (2.102) one establishes that (3.19) becomes (3.5).

3.2 Mean square exponential stability. The general case

In this section we introduce four different definitions of the concept of exponential stability in the mean square of the zero solution of system (3.1) and we emphasize that in the general case of time-varying coefficients these definitions are not always equivalent.

Definition 3.1

(*a*) *We say that the zero state equilibrium of system* (3.1) *is* strongly exponentially stable in the mean square of the first kind *(SESMS-I) if there exist* $\beta \geq 1, q \in (0,1)$ *such that* $\|T(t,s)\|_\xi \leq \beta q^{t-s}$ *for all* $t \geq s$, $T(t,s)$ *being the linear evolution operator on* $\mathcal{S}_n^N$ *defined by the corresponding sequence of Lyapunov operators* $\{\mathcal{L}_t\}_{t \geq 0}$.

(*b*) *We say that the zero state equilibrium of system* (3.1) *is* strongly exponentially stable in the mean square of the second kind *(SESMS-II)if there exist* $\beta \geq 1, q \in (0,1)$ *such that* $\|S(t,s)\|_\xi \leq \beta q^{t-s}$ *for all* $t \geq s$, $S(t,s)$ *being the linear solution operator on* $\mathcal{S}_n^N$ *defined by the corresponding sequence of Lyapunov operators* $\{\Lambda_t\}_{t \geq 1}$, *associated with system* (3.1).

(*c*) *We say that the zero state equilibrium of system* (3.1) *is* exponentially stable in the mean square with conditioning of type I *(ESMS-CI) if there exist* $\beta \geq 1, q \in (0,1)$ *such that for any sequence of independent random vectors* $\{w(t)\}_{t \geq 0}$ *and for any Markov chain* $(\{\eta_t\}_{t \geq 0}, \{P_t\}_{t \geq 0}, \mathcal{D})$ *that satisfy* $\mathbf{H_1}, \mathbf{H_2}$ *we have* $E[|\Phi(t,s)x_0|^2 | \eta_s = i] \leq \beta q^{t-s}|x_0|^2$ *for all* $t \geq s$, $i \in \mathcal{D}_s, s \geq 0, x_0 \in \mathbf{R}^n$.

(*d*) *We say that the zero state equilibrium of system* (3.1) *is* exponentially stable in the mean square with conditioning of type II *(ESMS-CII) if there exist* $\beta \geq 1, q \in (0,1)$ *such that for any sequence of independent random vectors* $\{w(t)\}_{t \geq 0}$ *and for any Markov chain that satisfy* $\mathbf{H_1}, \mathbf{H_2}$ *we have* $E[|\Phi(t,s)x_0|^2 | \eta_{s-1} = i] \leq \beta q^{t-s}|x_0|^2$ *for all* $t \geq s, i \in \mathcal{D}_{s-1}$, $s \geq 1$, $x_0 \in \mathbf{R}^n$.

(*e*) *We say that the zero state equilibrium of system* (3.1) *is* exponentially stable in the mean square *(ESMS) if there exist* $\beta \geq 1$, $q \in (0,1)$ *such that for any sequence of independent random vectors* $\{w(t)\}_{t\geq 0}$ *and for any Markov chain* $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *satisfying* $\mathbf{H_1}, \mathbf{H_2}$ *we have* $E[|\Phi(t,s)x_0|^2] \leq \beta q^{t-s}|x_0|^2$ *for all* $t \geq s \geq 0, x_0 \in \mathbf{R}^n$.

If $\mathcal{L}_t^0$ *is defined as in* (2.89) *and* $T^0(t,s)$ *is its corresponding linear evolution operator, then from* (2.91) *and Proposition* 2.6(*ii*) *one can conclude that SESMS of the zero state equilibrium of system* (3.1) *implies SESMS of the zero state equilibrium of system* (3.2).

It can be seen that the concept of strong exponential stability in the mean square introduced by the previous definition does not depend upon the initial distribution of the Markov chain. It depends only on the sequences $\{A_k(t,i)\}_{t\geq 0}$, $\{P_t\}_{t\geq 0}$. *Also it must be remarked that in the definitions of exponential stability in the mean square in terms of the state space trajectories, the sequences* $\{w(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0}$ *are not prefixed. We show later (see Theorems* 3.4 *and* 3.5*) that under some additional assumptions the exponentially stable behavior of the trajectories of the system* (3.1) *for a suitable pair* $(\{w(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0})$ *is enough to guarantee the exponentially stable behavior of the trajectories of the system* (3.1) *for all pairs* $(\{w(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0})$ *that verify* $\mathbf{H_1}, \mathbf{H_2}$.

Now we have the following theorem.

Theorem 3.3 *Under* $\mathbf{H_1}, \mathbf{H_2}$ *the following implications hold.*

(*i*) *If the zero state equilibrium of* (3.1) *is SESMS-I then it is ESMS-CI.*
(*ii*) *If the zero state equilibrium of* (3.1) *is ESMS-CI then it is ESMS-CII.*
(*iii*) *If the zero state equilibrium of* (3.1) *is ESMS-CII then it is ESMS.*
(*iv*) *If the zero state equilibrium of the system* (3.1) *is SESMS-I then it is SESMS-II.*
(*v*) *If the zero state equilibrium of the system* (3.1) *is SESMS-II then it is ESMS-CII.*

Proof. The implication from (*i*) follows immediately combining Theorem 3.1(*i*) and Theorem 2.12. We now prove the implication from (*ii*). Assume that the zero state equilibrium of (3.1) is ESMS-CI. This means that there exist $\beta \geq 1, q \in (0,1)$ such that

$$E[|\Phi(t,s)x_0|^2|\eta_s = i] \leq \beta q^{(t-s)}|x_0|^2 \tag{3.28}$$

for all $t \geq s \geq 0, i \in \mathcal{D}_s, x_0 \in \mathbf{R}^n$. Applying Theorem 1.6 for $u = s, v = s-1, \Psi = |\Phi(t,s)x_0|^2$ we obtain

$$E[|\Phi(t,s)x_0|^2|\mathcal{H}_{s,s-1}] = E[|\Phi(t,s)x_0|^2|\eta_s] \quad \text{a.s.} \tag{3.29}$$

for all $t \geq s, s \geq 1, x_0 \in \mathbf{R}^n$. From Remark 1.1 we have

$$E[|\Phi(t,s)x_0|^2|\eta_s] \leq \sum_{j\in\mathcal{D}_s} E[|\Phi(t,s)x_0|^2|\eta_s = j] \quad \text{a.s.}$$

and using (3.28) we have that

$$E[|\Phi(t,s)x_0|^2|\eta_s] \le N\beta q^{t-s}|x_0|^2. \tag{3.30}$$

Combining (3.29) and (3.30) we may conclude that if $s \ge 1, i \in \mathcal{D}_{s-1}$ we have

$$E[|\Phi(t,s)x_0|^2|\eta_{s-1} = i] \le N\beta q^{t-s}|x_0|^2$$

for all $t \ge s, x_0 \in \mathbf{R}^n$ which shows that the implication from (ii) holds.

Now we prove the implication from (iii). Let us assume that the zero state equilibrium of (3.1) is ESMS-CII. That means that there exist $\beta \ge 1, q \in (0,1)$ such that

$$E[|\Phi(t,s)x_0|^2|\eta_{s-1} = i] \le \beta q^{t-s}|x_0|^2 \tag{3.31}$$

for all $t \ge s \ge 1, i \in \mathcal{D}_{s-1}, x_0 \in \mathbf{R}^n$.

For $t \ge s \ge 1, x_0 \in \mathbf{R}^n$ we have $E[|\Phi(t,s)x_0|^2] \le \sum_{j\in\mathcal{D}_{s-1}} E[|\Phi(t,s)x_0|^2 |\eta_{s-1} = j]$. Invoking (3.31) one gets

$$E[|\Phi(t,s)x_0|^2] \le N\beta q^{t-s}|x_0|^2 \tag{3.32}$$

for all $t \ge s \ge 1, x_0 \in \mathbf{R}^n$. Now we show that there exists $\tilde{\beta} \ge 1$ such that

$$E[|\Phi(t,0)x_0|^2] \le \tilde{\beta} q^t |x_0|^2$$

for all $t \ge 0, x_0 \in \mathbf{R}^n$, where q is the same as in (3.31). For $t \ge 1$ we have $\Phi(t,0)x_0 = \Phi(t,1)\xi$, where $\xi = \Phi(1,0)x_0$ is $\sigma(\eta_0, w(0))$-measurable. This allows us to write

$$\begin{aligned} E[|\Phi(t,0)x_0|^2|(\eta_0, w(0))] &\le E[|\Phi(t,1)|^2|\xi|^2|(\eta_0, w(0))] \\ &\le |\xi|^2 \sum_{k=1}^{n} E[|\Phi(t,1)e_k|^2|(\eta_0, w(0))], \end{aligned}$$

where $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ are vectors of canonical bases in $\mathbf{R}^n$. Because for $t \ge 2, |\Phi(t,1)e_k|^2$ is $\sigma(\eta_0, \eta_1, \dots \eta_{t-1}, w(1), \dots w(t),)$-measurable and $|\Phi(1,1)e_k|^2 = 1$ are obtained via Theorem 1.6 (for $u = v = 0$) that

$$E[|\Phi(t,1)e_k|^2|(\eta_0, w(0))] = E[|\Phi(t,1)e_k|^2|\eta_0] \quad \text{a.s.} \tag{3.33}$$

for all $t \ge 1$. This allows us to write

$$\begin{aligned} E[|\Phi(t,0)x_0|^2|(\eta_0, w(0))] &\le |\xi|^2 \sum_{k=1}^{n} E[|\Phi(t,1)e_k)^2|\eta_0] \\ &\le |\xi|^2 \sum_{k=1}^{n} \sum_{j\in\mathcal{D}_0} E[|\Phi(t,1)e_k|^2|\eta_0 = j] \le nN|\xi|^2\beta q^{t-1}. \end{aligned}$$

Taking the expectation one obtains $E[|\Phi(t,0)x_0|^2] \leq \beta_1 q^t E[|\xi|^2]$. Having in mind the definition of ξ we deduce that

$$E[|\Phi(t,0)x_0|^2] \leq \tilde{\beta} q^t |x_0|^2 \tag{3.34}$$

for all $t \geq 0, x_0 \in \mathbf{R}^n$. From (3.32) and (3.34) we conclude that the zero state equilibrium of (3.1) is ESMS. The implication from (iv) follows immediately from the Theorem 2.14(i) and the implication from (v) follows combining Theorem 3.1(ii) and Theorem 2.13. Thus the proof ends. □

Remark 3.4

(a) In the case $\mathcal{D} = \{1\}$ the concepts introduced by Definition 3.1(c) and (d) are the same as the concept introduced in Definition 3.1(e). Also, in this case, the concepts of strong exponential stability introduced by Definition 3.1(a) and (b) coincide. Therefore in the case of systems of type (3.3) we can talk only about the concepts of SESMS and ESMS. On the other hand the equality from Remark 3.2(b) shows that the zero state equilibrium of (3.3) is SESMS iff it is ESMS.

(b) Examples 3.1 and 3.2 show that a part of converse implications in Theorem 3.3 are not always true. Hence in the absence of some additional assumptions the five types of mean square exponential stability introduced by Definition 3.1 are not equivalent. In Theorems 3.4 and 3.5 below, one shows that under some additional assumptions the converse implications from (i) and (v) of Theorem 3.3 hold.

Now we prove the following.

Theorem 3.4 *Assume that $P_t, t \geq 0$ are nondegenerate stochastic matrices. Then the following are equivalent.*

(i) *The zero state equilibrium of the system* (3.1) *is SESMS-I.*

(ii) *The zero state equilibrium of the system* (3.1) *is ESMS-CI.*

(iii) *There exist a sequence of independent random vectors $\{w(t)\}_{t\geq 0}$ and a Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ satisfying $\mathbf{H_1}, \mathbf{H_2}$ and $\mathcal{P}\{\eta_0 = i\} > 0$ for all $i \in \mathcal{D}$ such that*

$$E[|\Phi(t,s)x_0|^2 | \eta_s = i] \leq \beta q^{t-s} |x_0|^2 \tag{3.35}$$

for all $t \geq s \geq 0, i \in \mathcal{D}, x_0 \in \mathbf{R}^n$, where $\beta > 0$ and $q \in (0,1)$.

Proof. $(i) \to (ii)$ follows from Theorem 3.3 and $(ii) \to (iii)$ is obvious. It remains to prove implication $(iii) \to (i)$. Applying Theorem 3.1 (i) for $H = J = (I_n, \ldots, I_n)$, from (3.35) and Proposition 1.6 we obtain that $x_0^T [T^*(t,s)J](i) x_0 \leq \beta q^{t-s} |x_0|^2$ for all $t \geq s \geq 0, i \in \mathcal{D}, x_0 \in \mathbf{R}^n$.

This allows us to conclude that $|[T^*(t,s)J](i)| \leq \beta q^{t-s}$ for all $t \geq s \geq 0, i \in \mathcal{D}$. Based on (2.20) one gets $|T^*(t,s)J|_\xi \leq \beta q^{t-s}$ for all $t \geq s$.

Furthermore, from Proposition 2.5(ii) we have $\|T^*(t,s)\|_\xi \leq \beta q^{t-s}$ for all $t \geq s \geq 0$. The conclusion follows now from (2.13). □

Theorem 3.5 *Assume that for each $t \geq 0$, P_t is a nondegenerate stochastic matrix.*

Then the following are equivalent.

(i) The zero state equilibrium of the system (3.1) *is SESMS-II.*
(ii) The zero state equilibrium of the system (3.1) *is ESMS-CII.*
(iii) There exist a sequence of independent random vectors $\{w(t)\}_{t\geq 0}$ and a Markov chain $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ that satisfy $\mathbf{H_1}, \mathbf{H_2}$ and $\mathcal{P}\{\eta_0 = i\} > 0$ for all $i \in \mathcal{D}$, such that we have $E[|\Phi(t,s)x_0|^2|\eta_{s-1} = i] \leq \beta q^{t-s}|x_0|^2$ for all $t \geq s, s \geq 1, i \in \mathcal{D}, x_0 \in \mathbf{R}^n$, where $\beta > 1, q \in (0,1)$.

Proof. It is similar to that of Theorem 3.4 and is based on Theorems 3.3, 3.1(ii), and 2.13(ii). □

The next result follows immediately from Theorems 2.14(ii), 3.4, and 3.5.

Corollary 3.2 *Assume that $\{A_k(t,i)\}_{t\geq 0}, 0 \leq k \leq r, i \in \mathcal{D}$ are bounded sequences; then*

(i) The zero state equilibrium of the system (3.1) *is SESMS-I iff it is SESMS-II.*
(ii) If additionally for each $t \in \mathbf{Z}_+, P_t$ is a nondegenerate stochastic matrix, then the zero state equilibrium of the system (3.1) *is ESMS-CI iff it is ESMS-CII.*

Example 3.1 (i) Consider the system (3.1) in the particular case $n = 2, N = 2$, $r = 1$,

$$A_k(t,1) = 0, \qquad A_k(t,2) = 2^{t-1/4}I_2, \qquad k \in \{0,1\},$$

$$P_t = \begin{pmatrix} 1 - \frac{1}{2^{t+2}} & \frac{1}{2^{t+2}} \\ 1 - \frac{1}{2^{t+2}} & \frac{1}{2^{t+2}} \end{pmatrix}, \qquad t \geq 0. \tag{3.36}$$

For each $t \geq 0, P_t$ is a nondegenerate matrix, thus via Theorems 3.4 and 3.5, one obtains that for the considered system the SESMS-I is equivalent to ESMS-C I and SESMS-II is equivalent to ESMS-CII.

From Example 2.7 we deduce that the zero state equilibrium of the system under consideration is SESMS-II but it is not SESMS-I. Hence the zero state equilibrium of this system is ESMS-CII but it is not ESMS-CI.

(ii) The following example shows that ESMS-CII does not always imply SESMS-II, SESMS-I, and ESMS-CI. Consider system (3.1) in the particular case $n = 1, N = 2$, and having the coefficients described by

$$A_0(t,1) = 0 \forall\, t \geq 0; \qquad A_0(t,2) = 2^{t/2} \quad \text{if t is even,}$$

$$A_0(t,2) = 0 \quad \text{if t is odd} \tag{3.37}$$

$$A_k(t,i) = 0, i \in \{1,2\}, \qquad 1 \leq k \leq r, t \in \mathbf{Z}_+ \tag{3.38}$$

The transition probability matrix is

$$P_t = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad t = 2m \tag{3.39}$$

$$P_t = \begin{pmatrix} 1 - \frac{1}{4^{t+1}} & \frac{1}{4^{t+1}} \\ 1-\alpha & \alpha \end{pmatrix}, \qquad t = 2m+1, m \in \mathbf{Z}, \tag{3.40}$$

where $0 < \alpha \leq 1$. Because $A_k(t,i) = 0, 1 \leq k \leq r, i \in \{1,2\}, t \geq 0$ it follows that $\Phi(t,s) = A_0(t-1,\eta_{t-1})\dots A_0(s+1,\eta_{s+1})A_0(s,\eta_s)$ if $t > s \geq 0$.

Moreover if $t \geq s+2, s \geq 1$ we deduce that $\Phi(t,s) = 0$ a.s. because $A_0(\tau,\eta_\tau) = 0$, if τ is odd. Hence

$$E[|\Phi(t,s)x|^2|\eta_{s-1} = i] = 0 \tag{3.41}$$

for all $t \geq s+2, s \geq 1, i \in \mathcal{D}_{s-1}$. For s odd $\Phi(s+1,s) = A_0(s,\eta_s) = 0$ a.s. This shows that (3.41) holds also for $t \geq s+1$, s odd. It remains to estimate $E[|\Phi(s+1,s)x|^2|\eta_{s-1} = i]$ for s even, $s \geq 2$. Firstly we show that if s is even then $\mathcal{D}_{s-1} = \{1\}$. To this end, we write the equality:

$$\begin{aligned} \mathcal{P}\{\eta_t = 2\} = \mathcal{P}\{\eta_{t-1} = 1\}p_{t-1}(1,2) \\ + \mathcal{P}\{\eta_{t-1} = 2\}p_{t-1}(2,2), \qquad t \geq 1, t \in \mathbf{Z}. \end{aligned} \tag{3.42}$$

Writing (3.40) for $t = s-1$, s even, $s \geq 2$ we obtain that $\mathcal{P}\{\eta_{s-1} = 2\} = 0$ for all s even, $s \geq 2$. Therefore $\mathcal{D}_{s-1} = \{1\}$. we have

$$\begin{aligned} E[|\Phi(s+1,s)x|^2|\eta_{s-1} = 1] &= x^2 \sum_{j=1}^{2} A_0^2(s,j)p_{s-1}(1,j) = x^2 A_0^2(s,2)p_{s-1}(1,2) \\ &= x^2 2^s \frac{1}{4^s} = x^2 \frac{1}{2^s} \leq \frac{1}{2}x^2, \qquad \forall x \in \mathbf{R}. \end{aligned}$$

Thus we obtain

$$E[|\Phi(s+1,s)x|^2|\eta_{s-1} = 1] \leq \frac{1}{2^{s+1-s}}x^2$$

for all $s \geq 2$, even. The last inequality together with (3.39) shows that the property of ESMS-CII takes place for system (3.1) with the coefficients given by (3.37)–(3.38). On the other hand if $s \geq 2$ is even, we have

$$\begin{aligned} (S(s+1,s)J)(2) &= A_0^2(s,2) \sum_{j=1}^{2} p_{s-1}(j,2) \\ &= 2^s[p_{s-1}(1,2) + p_{s-1}(2,2)] = 2^s\left(\frac{1}{4^s} + \alpha\right) > \alpha 2^s. \end{aligned}$$

Because $\|S(s+1,s)\|_\xi = |S(s+1,s)J| \geq |(S(s+1,s)J)(2)|$ one obtains that $\lim_{s\to\infty} \|S(s+1,s)\|_\xi \geq \lim_{s\to\infty} 2^s = +\infty$. Thus we conclude that the property of SESMS-II cannot take place for system (3.1) with the coefficients given by (3.37)–(3.38). Based on Theorem 2.14(i) we deduce that the property SESMS-I cannot take place for this system, too.

Now we show that the zero state equilibrium of the system (3.1) with the coefficients given by (3.37)–(3.38) cannot be ESMS-CI. First we show that if $s \geq 2$, even, then the state $i = 2$ is in $\mathcal{D}_s$. To this end we write (3.42) for $t = s, s$ even, and obtain: $\mathcal{P}\{\eta_s = 2\} = p_{s-1}(1,2) = (1/4^s) > 0$. Therefore we may compute

$$E[|\Phi(s+1,s)x|^2|\eta_s = 2] = E[|A_0(s,\eta_s)x|^2|\eta_s = 2] = x^2 A_0^2(s,2).$$

Hence for s even, $s \geq 2$, we have

$$E[|\Phi(s+1,s)x|^2|\eta_s = 2] = 2^s|x|^2.$$

Thus one gets $\lim_{s\to\infty} E[|\Phi(s+1,s)x|^2|\eta_s = 2] = \infty$. This shows that the zero state equilibrium of the system under consideration cannot be ESMS-CI.

The previous computations show that the zero state equilibrium of system (3.1) with the coefficients given by (3.37)–(3.38) is only ESMS-CII and ESMS and it is not SESMS-I, SESMS-II, and ESMS-CI.

The next simple example shows that even in the case of discrete-time linear stochastic systems subject to an homogeneous Markov chain, ESMS does not always imply SESMS.

Example 3.2 Let the system (3.1) in the particular case when $\{\eta_t\}_{t\geq 0}$ is a Markov chain with two states, having the transition probability matrix

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

and the coefficients matrices be $A_0(t,1) = 0, A_0(t,2) = tI_n, A_k(t,i) = 0, 1 \leq k \leq r, i \in \{1,2\}, t \geq 0$.

We have $\mathcal{P}\{\eta_t = 2\} = 0$ a.s. for all $t \geq 1$. This leads to $\eta_t = 1$ a.s, $t \geq 1$. Hence $\Phi(t,s) = 0$ a.s. if $t \geq s+1, s \geq 0$ for any Markov chain $(\{\eta_t\}_{t\geq 0}, P, \{1,2\})$. This shows that in this particular case the zero state equilibrium of the considered system is both ESMS-CI and ESMS-CII, as well as ESMS.

On the other hand we see that $(T^*(t+1,t)J)(2) = t^2 I_n$. From Proposition 2.5($ii$) and (2.20) it follows that $\|T^*(t+1,t)\|_\xi \geq t^2$. This shows that $\lim_{t\to\infty} \|T^*(t+1,t)\|_\xi = +\infty$. This allows us to conclude that the zero state equilibrium of the considered system is not SESMS-I. Furthermore, one sees that for the system under consideration we have $(S(t,s)H)(i) = 0$, $i \in \{1,2\}, H = (H(1), H(2)) \in \mathcal{S}_n^2$. hence the zero state equilibrium of the system under consideration is SESMS-II.

Remark 3.5 The validity of the converse implication in (iii) of Theorem 3.3 in the absence of some additional assumptions still remains an open problem. Our conjecture is that this converse implication is not true.

The next result provides a sufficient condition which guarantees that in the special case of system (3.2) four types of exponential stability introduced in Definition 3.1 become equivalent.

Theorem 3.6 *Assume that for a Markov chain* $(\{\tilde{\eta}_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *there exists* $\delta > 0$ *such that*

$$\tilde{\pi}_t(i) = \mathcal{P}\{\tilde{\eta}_t = i\} \geq \delta \tag{3.43}$$

for all $t \geq 0, i \in \mathcal{D}$*. Then the following are equivalent.*

(i) *There exist* $\beta \geq 0, q \in (0,1)$ *such that*

$$E[|\tilde{\Phi}(t,s)x_0|^2 | \tilde{\eta}(s) = i] \leq \beta q^{t-s}|x_0|^2 \tag{3.44}$$

for all $t \geq s \geq 0, i \in \mathcal{D}, x_0 \in \mathbf{R}^n, \tilde{\Phi}(t,s)$ *being the fundamental matrix solution of the system* (3.2) *corresponding to the Markov chain* $\tilde{\eta}_t$*.*

(ii) *The zero state equilibrium of* (3.2) *is SESMS-I.*

(iii) *The zero state equilibrium of* (3.2) *is ESMS-CI.*

(iv) *The zero state equilibrium of* (3.2) *is ESMS-CII.*

(v) *The zero state equilibrium of* (3.2) *is ESMS.*

Proof. To prove $(i) \to (ii)$ we remark firstly that (3.43) implies that for the Markov chain $\{\tilde{\eta}_t\}_{t\geq 0}$ we have $\mathcal{D}_s = \mathcal{D}$ for all $s \geq 0$. Combining Proposition 1.6(ii), Theorem 3.1(i) and the inequality (3.44) one gets

$$\|T^*(t,s)\|_\xi \leq \beta q^{t-s},$$

for all $t \geq s \geq 0$. Now, Theorem 2.12 allows us to conclude that the zero state equilibrium of (3.2) is SESMS-I, hence $(i) \to (ii)$ is true. The implications $(ii) \to (iii) \to (iv) \to (v)$ follow from Theorem 3.3.

It remains to prove $(v) \to (i)$.

If (v) holds then there exist $\beta \geq 1, q \in (0,1)$ such that

$$E[|\tilde{\Phi}(t,s)x_0|^2] \leq \beta q^{t-s}|x_0|^2$$

for all $t \geq s \geq 0, x_0 \in \mathbf{R}^n$. From

$$\tilde{\pi}_s(i)E[|\tilde{\Phi}(t,s)x_0|^2 | \tilde{\eta}_s = i] \leq E[|\tilde{\Phi}(t,s)x_0|^2]$$

we obtain

$$E[|\tilde{\Phi}(t,s)x_0|^2 | \tilde{\eta}_s = i] \leq \delta^{-1}\beta q^{t-s}|x_0|^2$$

for all $t \geq s \geq 0, i \in \mathcal{D}, x_0 \in \mathbf{R}^n$ and thus the proof ends. □

Remark 3.6 Proposition 1.7 provides a sufficient condition assuring the validity of condition (3.43) in the case of an homogeneous Markov chain $(\{\eta_t\}_{t\geq 0}, P, \mathcal{D})$ with $\mathcal{P}\{\eta_0 = i\} > 0$ for all $i \in \mathcal{D}$.

3.3 Lyapunov-type criteria

Based on Theorem 2.12, Theorem 2.13, and Definition 3.1(a) one immediately obtains two results that allow us to characterize the SESMS-I in terms of solvability of some systems of linear matrix equations or linear matrix inequalities.

We have the following.

Theorem 3.7 (Lyapunov-type criteria derived via operators $\mathcal{L}_t$) *The following are equivalent.*

(i) *The zero state equilibrium of the system* (3.1) *is SESMS-I.*

(ii) *There exists a bounded sequence* $\{X_t\}_{t\geq 0}$, $X_t \in \mathcal{S}^N$, $X_t > 0, t \geq 0$ *solving the following system of equations,*

$$X_t(i) = \sum_{k=0}^{r} A_k^T(t,i) \sum_{j=1}^{N} p_t(i,j) X_{t+1}(j) A_k(t,i) + I_n \tag{3.45}$$

for all $t \geq 0, i \in \mathcal{D}$.

(iii) *There exists a bounded sequence* $\{Y_t\}_{t\geq 0}$, $Y_t \gg 0, t \geq 0$ *and a positive scalar* α *that verify the following system of inequalities,*

$$\sum_{k=0}^{r} A_k^T(t,i) \sum_{j=1}^{N} p_t(i,j) Y_{t+1}(j) A_k(t,i) - Y_t(i) \leq -\alpha I_n \tag{3.46}$$

for all $i \in \mathcal{D}, t \geq 0$.

Theorem 3.8 (Lyapunov-type criteria derived via operators Λ_t) *The following are equivalent.*

(i) *The zero state equilibrium of* (3.1) *is SESMS-II.*

(ii) *There exists a bounded sequence* $\{X_t\}_{t\geq 1} \subset \mathcal{S}_n^N, X_t > 0, t \geq 1$ *verifying the following system of linear equations,*

$$X_t(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} A_k^T(t,j) p_{t-1}(i,j) X_{t+1}(j) A_k(t,j) + I_n \tag{3.47}$$

for all $i \in \mathcal{D}, t \geq 1$.

(iii) *There exists a positive scalar* α *and a bounded sequence* $\{Y_t\}_{t\geq 1} \subset \mathcal{S}_n^N, Y_t \gg 0, t \geq 1$ *verifying the following system of linear inequalities,*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} A_k^T(t,j) p_{t-1}(i,j) Y_{t+1}(j) A_k(t,j) - Y_t(i) \leq -\alpha I_n \tag{3.48}$$

$t \geq 1, i \in \mathcal{D}$.

Proof. From Theorem 2.13 it follows that the assertions (*ii*) and (*iii*) are equivalent to the fact that the sequence $\{\Lambda_t\}_{t\geq 1}$ generates an exponentially stable evolution. □

It is worth mentioning that under the assumption of the boundedness of the sequences $\{A_k(t)\}_{t\geq 0}, 0 \leq k \leq r, i \in \mathcal{D}$, Theorem 3.8 provides criteria for SESMS-I for the system (3.1).

Even if the criteria provided by Theorems 3.7 and 3.8 are expressed in terms of solvability of some systems of linear equations or linear inequations with an infinite number of unknowns, they may be useful in the derivation of some sufficient conditions for SESMS-I and ESMS. This is illustrated by the following simple example.

Example 3.3 Consider the system (3.1) in the particular case $n = 1$. That is,

$$x(t+1) = (a_0(t,\eta_t) + \sum_{k=1}^{r} a_k(t,\eta_t)w_k(t))x(t), \tag{3.49}$$

$t \geq 0, x(t) \in \mathbf{R}, a_k(t,i) \in \mathbf{R}$.

If

$$\max_{i\in\mathcal{D}} \sup_{t\geq 0} \left\{ \sum_{k=0}^{r} a_k^2(t,i) \right\} < 1, \tag{3.50}$$

then the zero state equilibrium of (3.49) is SESMS-I. Indeed if (3.50) holds then the system (3.46) associated with (3.49) is verified for $Y_t(i) = 1$ for all $i \in \mathcal{D}, t \geq 0$.

We remark that in the particular case of (3.49) given by $x(t+1) = a_0(\eta_t)x(t)$, the condition (3.50) reduces to $|a_0(i)| < 1, i \in \mathcal{D}$. This shows that in the one-dimensional case the zero state equilibrium of a system with jump Markov perturbations is SESMS-I if for each mode i the deterministic system $y(t+1) = a_0(i)y(t)$ is exponentially stable. We show later that for $n \geq 2$ the fact that for each mode i the deterministic system is exponentially stable does not always imply exponential stability in the mean square of the stochastic system.

3.4 The case of homogeneous Markov chain

The next result shows that in the case of homogeneous Markov chain strong mean square exponential stability of the zero state equilibrium of the system (3.2) is always equivalent with SESMS of the zero solution of a system of the form:

$$x(t+1) = A_0(t,\tilde{\eta}_t)x(t),$$

where $\{\{\tilde{\eta}_t\}_{t\geq 0}, \tilde{P}, \tilde{\mathcal{D}}\}$ is an homogeneous Markov chain with the transition probability matrix $\tilde{P}$ which is a nondegenerate stochastic matrix.

If the stochastic matrix P has some null columns then we set $\hat{\mathcal{D}} = \mathcal{D} - \{j_1, j_2, \ldots, j_p\}$ where $j_l, 1 \leq l \leq p$ are such that $P(i, j_l) = 0$ for all $i \in \mathcal{D}$. Let $\hat{P}$ be the $(N-p) \times (N-p)$ matrix obtained by canceling the columns $j_1, \ldots, j_p$ and the corresponding rows in P. Obviously, $\hat{P}$ is still a stochastic matrix. Let $\{\{\hat{\eta}_t\}_{t \geq 0}, \hat{P}, \hat{\mathcal{D}}\}$ be the homogeneous Markov chain, having the state space $\hat{\mathcal{D}}$ and the transition probability matrix $\hat{P}$ defined above.

We have the following.

Theorem 3.9 *If the sequences $\{A_0(t,i)\}_{t \geq 0}, i \in \mathcal{D}$ are bounded and if the matrix P has some null columns the following are equivalent.*

(i) *The zero state equilibrium of the system* (3.2) *is SESMS-I.*
(ii) *The zero state equilibrium of the system*

$$y(t+1) = \hat{A}_0(t, \hat{\eta}_t) y(t) \tag{3.51}$$

is SESMS-I, where $(\hat{\eta}_t, \hat{P}, \hat{\mathcal{D}})$ is a Markov chain defined above from the given Markov chain $\{\{\eta_t\}_{t \geq 0}, P, \mathcal{D}\}$ and $\hat{A}_0(t,i) = A_0(t,i)$, $i \in \hat{\mathcal{D}}, t \geq 0$.

Proof. If (i) holds then based on Definition 3.1(a) and the equivalence $(i) \iff (ii)$ in Theorem 3.7 (in the particular case $A_k(t,i) = 0, 1 \leq k \leq r, t \geq 0, i \in \mathcal{D}$) one deduces that there exists the bounded sequence $\{X_t\}_{t \geq 0}, X_t = \{X_t(1), \ldots, X_t(N)\}$, $X_t(i) > 0$ that solves (3.45).

It is immediate that in this particular case (3.45) may be written as

$$X_t(i) = A_0^T(t,i) \sum_{j \in \hat{\mathcal{D}}} p(i,j) X_{t+1}(j) A_0(t,i) + I_n, \tag{3.52}$$

for all $i \in \mathcal{D}$.

Considering (3.52) for $i \in \hat{\mathcal{D}}$ one obtains based on the implication $(ii) \Longrightarrow (i)$ of Theorem 3.7 that the sequence of Lyapunov operators $\{\hat{\mathcal{L}}_t\}_{t \geq 0}$ associated with the system (3.41) generates an exponentially stable evolution. This means that (ii) is fulfilled. To prove the converse implication, we remark that if (ii) holds then based on the implication $(i) \Longrightarrow (ii)$ of Theorem 3.7 and Definition 3.1(a) there exist the bounded sequences $\{X_t(i)\}_{t \geq 0}, i \in \hat{\mathcal{D}}$ that verify $X_t(i) > 0, t \geq 0, i \in \hat{\mathcal{D}}$ and

$$X_t(i) = \hat{A}_0^T(t,i) \sum_{j \in \hat{\mathcal{D}}} p(i,j) X_{t+1} \hat{A}_0(t,i) + I_n. \tag{3.53}$$

Let $i \in \mathcal{D}$ and define $\hat{X}_t(i) = X_t(i)$ if $i \in \hat{\mathcal{D}}$ and

$$\hat{X}_t(i) = A_0^T(t,i) \sum_{j \in \hat{\mathcal{D}}} p(i,j) X_{t+1}(j) A_0(t,i) + I_n$$

if $i \in \mathcal{D} \setminus \hat{\mathcal{D}}, t \geq 0$. Obviously $\hat{X}_t(i) > 0, t \geq 0, i \in \mathcal{D}$, $\{\hat{X}_t(i)\}_{t \geq 0}, i \in \mathcal{D}$ are bounded sequences and $\hat{X}_t(i)$ verifies (3.45) for $t \geq 0, i \in \mathcal{D}$. Therefore by again using implication $(ii) \to (i)$ of Theorem 3.7 one concludes that the zero state equilibrium of system (3.2) is SESMS-I and thus the proof ends. □

If we repeatedly apply the reduction of number of states of the Markov chain described above we finally obtain a system of the form

$$x(t+1) = \tilde{A}_0(t, \tilde{\eta}_t)x(t)$$

driven by a Markov chain $(\{\tilde{\eta}_t\}_{t\geq 0}, \tilde{P}, \tilde{\mathcal{D}})$ such that $\tilde{P}$ is a nondegenerate stochastic matrix.

3.5 Some special cases

3.5.1 The periodic case

In this subsection we consider the case of systems (3.1) with periodic coefficients. For such a class of stochastic systems we prove that the five types of mean square exponential stability introduced by Definition 3.1 are equivalent. First we remark that if the coefficients $\{A_k(t,i)\}_{t\geq 0}$ of the system (3.1) are periodic sequences then they are bounded sequences. Therefore from Corollary 3.2 we deduce that in this case the two types of strong exponential stability introduced by Definition 3.1(a) and (b) coincide.

Firstly we introduce the following definition.

Definition 3.2 *We say that the zero state equilibrium of system* (3.1) *is* asymptotically stable in mean square *(ASMS) if for any sequence of independent random vectors* $\{w(t)\}_{t\geq 0}$ *and for any Markov chain* $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *that satisfy* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *we have* $\lim_{t\to\infty} E[|\Phi(t,0)x_0|^2] = 0$ *for all* $x_0 \in \mathbf{R}^n$.

Now we prove the following.

Theorem 3.10 *Assume that there exists an integer* $\theta \geq 1$ *such that* $A_k(t+\theta, i) = A_k(t,i), 0 \leq k \leq r, i \in \mathcal{D}$, $p_{t+\theta}(i,j) = p_t(i,j), i,j \in \mathcal{D}, t \geq 0$. *Under these conditions the following are equivalent.*

- (i) *The zero state equilibrium of system* (3.1) *is SESMS-I.*
- (ii) *The zero state equilibrium of system* (3.1) *is ESMS-CI.*
- (iii) *The zero state equilibrium of system* (3.1) *is ESMS-CII.*
- (iv) *The zero state equilibrium of system* (3.4) *is ESMS.*
- (v) *The zero state equilibrium of system* (3.4) *is ASMS.*
- (vi) *There exists a sequence of independent random vectors* $\{w(t)\}_{t\geq 0}$ *and a Markov chain* $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *with* $\mathcal{P}\{\eta_0 = i\} > 0$ *for all* $i \in \mathcal{D}$ *which satisfy* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *such that*

$$\lim_{m\to\infty} E[|\Phi(m\theta, 0)x_0|^2] = 0$$

for all $x_0 \in \mathbf{R}^n$.

(*vii*) $\lim_{m\to\infty} E[|\Phi_i(m\theta,0)x_0|^2] = 0$ *for all* $i \in \mathcal{D}$ *and* $x_0 \in \mathbf{R}^n$, *where* $\Phi_i(t,s)$ *is the fundamental matrix solution of the system* (3.1) *for a fixed sequence of independent random vectors* $\{w(t)\}_{t\geq 0}$ *and a Markov chain* $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *with* $\mathcal{D}_0 = \{i\}$ *that satisfy* $\mathbf{H_1}$ *and* $\mathbf{H_2}$.

(*viii*) $\lim_{m\to\infty} E[|\Phi_i(m\theta,1)x_0|^2] = 0$ *for all* $i \in \mathcal{D}, x_0 \in \mathbf{R}^n, \Phi_i(t,s)$ *being as before.*

Proof. The implications $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$ follow from Theorem 3.3; $(i) \Longrightarrow (iv) \Longrightarrow (v) \Longrightarrow (vii), (iii) \Longrightarrow (viii)$, and $(v) \Longrightarrow (vi)$ are obvious. Now we prove $(vi) \Longrightarrow (i)$. Because $\mathcal{D}_0 = \mathcal{D}$ we may write $E[|\Phi(m\theta,0)x_0|^2] = \sum_{i\in\mathcal{D}} \mathcal{P}\{\eta_0 = i\} E[|\Phi(m\theta,0)x_0|^2 | \eta_0 = i]$.

Hence (vi) is equivalent to

$$\lim_{m\to\infty} E[|\Phi(m\theta,0)x_0|^2 | \eta_0 = i] = 0 \tag{3.54}$$

for all $i \in \mathcal{D}$ and $x_0 \in \mathbf{R}^n$.

Based on (3.4) one obtains that

$$\lim_{m\to\infty} x_0^T (T^*(m\theta,0)J)(i)x_0 = \lim_{m\to\infty} E[|\Phi(m\theta,0)x_0|^2 | \eta_0 = i] = 0. \tag{3.55}$$

This allows us to write

$$\lim_{m\to\infty} |(T^*(m\theta,0)J)(i)| = 0.$$

Based on (2.20) and Proposition 2.5(ii) one gets:

$$\lim_{m\to\infty} \|T^*(m\theta,0)\|_\xi = 0. \tag{3.56}$$

From the periodicity assumption it follows that $\mathcal{L}^*_{t+\theta} = \mathcal{L}^*_t$ for all $t \geq 0$.

By induction one obtains that

$$T^*(m\theta,0) = (T^*(\theta,0))^m,$$

for all $m \geq 0$. Therefore (3.54) may be written as

$$\lim_{m\to\infty} \|[T^*(\theta,0)]^m\|_\xi = 0. \tag{3.57}$$

Let $m_0 \geq 1$ be such that $\|(T^*(\theta,0))^{m_0}\|_\xi < \frac{1}{2}$. It follows that

$$\|T^*(lm_0\theta,0)\|_\xi < \left(\frac{1}{2}\right)^l$$

for all $l \geq 1$.

In a standard way, by using Remark 2.5(a) and the equality

$$T^*(t,s) = T^*(u,s)T^*(t,u), \qquad t \geq u \geq s \geq 0 \tag{3.58}$$

one obtains that there exist $\beta \geq 1$ and $q \in 0, 1, (q = (\frac{1}{2})^{1/m_0})$ such that $\|T^*(t,s)\|_\xi \leq \beta q^{t-s}$, for all $t \geq s \geq 0$.

Invoking Theorem 2.12 we deduce that the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ generates an exponentially stable evolution and thus, in view of Definition 3.1(a) it follows that (i) holds.

We prove now $(vii) \Longrightarrow (i)$. The representation formula (3.4) together with (vii) shows that (3.55) is fulfilled for all $i \in \mathcal{D}$ and $x_0 \in \mathbf{R}^n$. The remainder of the proof of this implication is similar to that of the implication $(vi) \Longrightarrow (i)$.

It remains to prove $(viii) \Longrightarrow (i)$. If $(viii)$ holds then from the representation formula (3.5) one obtains that

$$\lim_{m\to\infty} x_0^T(S^*(m\theta, 1)J)(i)x_0 = 0$$

for all $i \in \mathcal{D}, x_0 \in \mathbf{R}^n$. This leads to

$$\lim_{m\to\infty} \|S^*(m\theta, 1)\|_\xi = 0.$$

Based on Corollary 2.5(ii) one gets

$$\|T^*(m\theta, 0)\|_\xi \leq \alpha(0)\|S^*(m\theta, 1)\|_\xi.$$

This shows that $\lim_{m\to\infty} \|T^*(m\theta, 0)\|_\xi = 0$. The conclusion is obtained now as in the proof of the implication $(vi) \to (i)$. Thus the proof is complete. □

Remark 3.7 Part of the equivalences in the above theorem are known from the case of discrete-time time-invariant linear systems affected by Markov perturbations (see [52, 86, 87]). Here we have shown that these equivalences are still valid for the periodic case without the additional assumption that P is a nondegenerate stochastic matrix.

Based on Theorem 2.5 one obtains that if the sequences $\{A_k(t,i)\}_{t\geq 0}$, $\{p_t(i,j)\}_{t\geq 0}$ are periodic sequences with period θ then the bounded solutions of (3.45) and (3.47), respectively, are periodic sequences with the same period θ.

Combining Theorems 2.5, 2.7, and 2.4 we obtain the following specialized version of Theorems 3.7 and 3.8.

Theorem 3.11 (Lyapunov-type criteria derived via operators $\mathcal{L}_t$) *Under the periodicity assumption of Theorem* 3.10 *with* $\theta \geq 2$, *the following are equivalent.*

- (i) *The zero state equilibrium of* (3.1) *is ESMS.*
- (ii) *There exist positive definite matrices* $X_t(i)$, $0 \leq t \leq \theta - 1$, $i \in \mathcal{D}$ *that verify the following system of linear equations,*

$$X_t(i) = \sum_{k=0}^{r} A_k^T(t,i) \sum_{j=1}^{N} p_t(i,j) X_{t+1}(j) A_k(t,i) + I_n$$

$$0 \leq t \leq \theta - 2, \tag{3.59}$$

$$X_{\theta-1}(i) = \sum_{k=0}^{r} A_k^T(\theta-1,i) \sum_{j=1}^{N} p_{\theta-1}(i,j) X_0(j) A_k(\theta-1,i) + I_n.$$

(*iii*) *There exist positive definite matrices* $X_t(i), 0 \leq t \leq \theta - 1, i \in \mathcal{D}$ *that solve the following system of linear equations,*

$$X_{t+1}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i) A_k(t,j) X_t(j) A_k^T(t,j) + I_n,$$

$$0 \leq t \leq \theta - 2$$

$$X_0(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{\theta-1}(j,i) A_k(\theta-1,j) X_{\theta-1}(j) A_k^T(\theta-1,j) + I_n, \quad (3.60)$$

$$i \in \mathcal{D}.$$

(*iv*) *There exist positive definite matrices* $Y_t(i), 0 \leq t \leq \theta-1, i \in \mathcal{D}$ *that solve the following system of LMIs,*

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i) A_k(t,j) Y_t(j) A_k^T(t,j) - Y_{t+1}(i) < 0,$$

$$0 \leq t \leq \theta - 2$$

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p_{\theta-1}(j,i) A_k(\theta-1,j) Y_{\theta-1}(j) A_k^T(\theta-1,j) - Y_0(i) < 0 \qquad (3.61)$$

$$i \in \mathcal{D}.$$

Proof. The equivalence $(i) \leftrightarrow (ii)$ follows from Theorems 2.4 and 2.5 applied to the sequence of operators $\{\mathcal{L}_t^*\}_{t\in\mathbf{Z}}$. The equivalences $(i) \leftrightarrow (iii) \leftrightarrow (iv)$ follow from Theorems 2.6 and 2.7 applied to the sequence $\{\mathcal{L}_t\}_{t\in\mathbf{Z}}$.

In fact $(i) \leftrightarrow (ii)$ is just the equivalence $(i) \leftrightarrow (ii)$ of Theorem 3.7 above specialized for the periodic case. It is worth mentioning that conditions for the existence of a Lyapunov function expressed in terms of the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ instead of the sequence $\{\mathcal{L}_t^*\}_{t\geq 0}$ can be derived if $\mathcal{L}_t$ are defined for all $t \in \mathbf{Z}$. This is the case if $\{\mathcal{L}_t\}_t$ is a periodic sequence or a constant sequence. □

Theorem 3.12 (Lyapunov-type criteria derived via operators Λ_t) *Under the assumptions of Theorem* 3.10 *with* $\theta \geq 2$ *the following are equivalent.*

(*i*) *The zero state equilibrium of* (3.1) *is ESMS.*
(*ii*) *There exist positive definite matrices* $X_t(i), 1 \leq t \leq \theta,\ 1 \leq i \leq N$ *that verify the following system of linear equations,*

$$X_t(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j) X_{t+1}(j) A_k(t,j) + I_n, \tag{3.62}$$

$$1 \le t \le \theta - 1,$$

$$X_\theta(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_{\theta-1}(i,j) A_k^T(0,j) X_1(j) A_k(0,j) + I_n,$$

$$i \in \mathcal{D}.$$

(*iii*) *There exist positive definite matrices* $X_t(i)$, $1 \le t \le \theta, i \in \mathcal{D}$ *that solve the following system of linear equations,*

$$X_{t+1}(i) = \sum_{k=0}^{r} A_k(t,i) \sum_{j=1}^{N} p_{t-1}(j,i) X_t(j) A_k^T(t,i) + I_n, \tag{3.63}$$

$$1 \le t \le \theta - 1, i \in \mathcal{D}$$

$$X_1(i) = \sum_{k=0}^{r} A_k(\theta,i) \sum_{j=1}^{N} p_{\theta-1}(j,i) X_\theta(j) A_k^T(\theta,i) + I_n,$$

$$i \in \mathcal{D}.$$

(*iv*) *There exist positive definite matrices* $Y_t(i), 1 \le t \le \theta, i \in \mathcal{D}$ *that solve the following system of LMIs,*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j) Y_{t+1}(j) A_k(t,j) - Y_t(i) < 0,$$

$$1 \le t \le \theta - 1, i \in \mathcal{D}$$

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p_{\theta-1}(i,j) A_k^T(\theta,j) Y_1(j) A_k(\theta,j) - Y_\theta(i) < 0, \qquad i \in \mathcal{D}.$$

Proof. The equivalence $(i) \leftrightarrow (ii) \leftrightarrow (iv)$ follows from combining Theorem 2.14(ii) with Theorems 2.4 and 2.5 applied to the sequence $\{\Lambda_t^*\}_{t\in\mathbf{Z}}$. The equivalence $(i) \leftrightarrow (iii)$ follows from Theorem 2.14(ii) together with Theorems 2.6 and 2.7 applied to the sequence $\{\Lambda_t\}_{t\in\mathbf{Z}}$. □

Remark 3.8 It is easy to see that the systems (3.57), (3.58) and (3.60), (3.61), respectively, contain $\hat{n}$ linear scalar equations with $\hat{n}$ scalar unknowns, where $\hat{n} = (N\theta n(n+1))/2$.

3.5.2 The time-invariant case

Let us consider the following version of the system (3.1),

$$x(t+1) = \left[A_0(\eta_t) + \sum_{k=1}^{r} A_k(\eta_t) w_k(t) \right] x(t). \tag{3.64}$$

It corresponds to the case $A_k(t,i) = A_k(i), t \geq 0, i \in \mathcal{D}, 0 \leq k \leq r$.

Definition 3.3 *We say that the system* (3.1) *is* in the time-invariant case *if it takes the form* (3.64)*, and* $\eta_t, t \geq 0$ *is an homogeneous Markov chain.*

We note that the time-invariant case may be viewed as a periodic case with period $\theta = 1$*. Hence the equivalences proved in Theorem* 3.10 *are still valid in the time-invariant case.*

The next example shows that if there exists $i_0 \in \mathcal{D}$ *such that* $\mathcal{P}\{\eta_0 = i_0\} = 0$ *it is possible that the zero solution of the considered stochastic system is not mean square exponentially stable even if*

$$\lim_{t \to \infty} E[|\Phi(t,s)x_0|^2] = 0$$

for all $x_0 \in \mathbf{R}^n$*.*

Example 3.4 Consider the system (3.2) in the particular case $n = 1, N = 2$, $A(1) = 0, A(2) = 1$, $\mathcal{P}\{\eta_0 = 2\} = 0$, $\mathcal{P}\{\eta_0 = 1\} = 1$,

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 0 & 1 \end{pmatrix},$$

$\alpha \in (0,1)$, $x_{t+1} = A(\eta_t)x_t, t \geq 0$. Because $\eta_0 = 1$ a.s. it follows that $x_1 = A(1)x_0 = 0$ a.s. By induction one obtains $x_t = 0$ a.s. for all $t \geq 1$.

Therefore $\lim_{t\to\infty} E[|x(t)|^2] = 0$. On the other hand $(\mathcal{L}_t^* J)(1) = (\mathcal{L}^*)^t J(1) = 0, t \geq 1$ and $(\mathcal{L}_t^* J)(2) = ((\mathcal{L}^*)^t J)(2) = 1, t \geq 0$ hence $\|(\mathcal{L}^*)^t\|_\xi = 1$, $\lim_{t\to\infty} \|(\mathcal{L}^*)^t\|_\xi = 1 \neq 0$.

Therefore in order to have exponential stability in the mean square, in (vi) of Theorem 3.10 we asked $\mathcal{P}\{\eta_0 = i\} > 0$ for all $i \in \mathcal{D}$.

In the time-invariant case, the results of Theorems 3.11 and 3.12 become the following.

Corollary 3.3 (Lyapunov-type criteria derived via the operator $\mathcal{L}$) *Assume that* $A_k(t,i) = A_k(i)$*,* $0 \leq k \leq r$*,* $p_t(i,j) = p(i,j)$*,* $i,j \in \mathcal{D}, t \geq 0$*; then the following are equivalent.*

(i) *The zero state equilibrium of* (3.1) *is ESMS.*

(ii) *There exist positive definite matrices* $X(i), 1 \leq i \leq N$ *that verify the following system of algebraic matrix equations,*

$$X(i) = \sum_{k=0}^{r} A_k^T(i) \sum_{j=1}^{N} p(i,j) X(j) A_k(i) + I_n, \tag{3.65}$$

$i \in \mathcal{D}$.

(*iii*) *There exist positive definite matrices* $X(i), i \in \mathcal{D}$ *that solve the following system of linear equations,*

$$X(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(j,i) A_k(j) X(j) A_k^T(j) + I_n, \tag{3.66}$$

$i \in \mathcal{D}$.

(*iv*) *There exist positive definite matrices* $Y(i), i \in \mathcal{D}$ *that solve the following system of LMIs,*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p(j,i) A_k(j) Y(j) A_k^T(j) - Y(i) < 0, \tag{3.67}$$

$i \in \mathcal{D}$.

Corollary 3.4 (Lyapunov-type criteria derived via the operator Λ) *Under the assumptions of Corollary* 3.2 *the following are equivalent.*

(*i*) *The zero state equilibrium of* (3.1) *is ESMS.*

(*ii*) *There exist positive definite matrices* $X(i), 1 \le i \le N$ *that verify the following system of algebraic matrix equations,*

$$X(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) A_k^T(j) X(j) A_k(j) + I_n, \tag{3.68}$$

$i \in \mathcal{D}$.

(*iii*) *There exist positive definite matrices* $X(i), 1 \le i \le N$ *that verify the following system of algebraic matrix equations,*

$$X(i) = \sum_{k=0}^{r} A_k(i) \sum_{j=1}^{N} p(j,i) X(j) A_k^T(i) + I_n, \tag{3.69}$$

$i \in \mathcal{D}$.

(*iv*) *There exist positive definite matrices* $Y(i)$, $i \in \mathcal{D}$ *that solve the following system of LMIs*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) A_k^T(j) Y(j) A_k(j) - Y(i) < 0,$$

$i \in \mathcal{D}$.

For Markovian systems (i.e., $A_k(i) = 0, 1 \le k \le r$) the equivalence $(i) \leftrightarrow (ii)$ in Corollary 3.2 has been proved in [21, 22, 53, 54, 75, 86, 88]. The equivalence $(i) \leftrightarrow (ii)$ in Corollary 3.3 has been proved in a different way in [52, 86].

3.5.3 Another particular case

In this subsection we focus our attention on a particular case of system (3.1) which may be interesting in applications. We refer to the case $p_t(i,j) = p_t(j)$ for all $i, j \in \mathcal{D}, t \geq 0$.

This means that $\mathcal{P}\{\eta_{t+1} = j | \eta_t = i\}$ does not depend upon i. It is easy to see that in this case the solution of the system (3.62) does not depend upon i.

This remark together with the Theorem 3.12 leads to the following.

Corollary 3.5 *Assume that:*

(*a*) *There exists an integer* $\theta \geq 2$ *such that* $A_k(t+\theta, i) = A_k(t,i), 0 \leq k \leq r, i \in \mathcal{D}$, $P_{t+\theta} = P_t$, *for all* $t \geq 0$.
(*b*) $p_t(i,j) \equiv p_t(j), \ \forall i, j \in \mathcal{D}, t \geq 0$.

Then the following are equivalent.

(*i*) *The zero state equilibrium of the system* (3.1) *is ESMS.*
(*ii*) *There exist positive definite matrices* $X_t, 1 \leq t \leq \theta$, *that satisfy*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p_{s-1}(j) A_k^T(s,j) X_{s+1} A_k(s,j) - X_s = -I_n, \tag{3.70}$$

$$1 \leq s \leq \theta - 1,$$

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p_{\theta-1}(j) A_k^T(0,j) X_1 A_k(0,j) - X_\theta = -I_n.$$

Also from Corollary 3.3 we obtain the following.

Corollary 3.6 *Assume that:*

(*a*) $A_k(t,i) = A_k(i), 0 \leq k \leq r, i \in \mathcal{D}$, $P_t = P, t \geq 0$.
(*b*) $p(i,j) = p(j), i, j \in \mathcal{D}$.

Then the following are equivalent.

(*i*) *The zero state equilibrium of system* (3.1) *is ESMS.*
(*ii*) *There exists a positive definite matrix* $X \in \mathcal{S}_n$ *that solves*

$$X = \sum_{k=0}^{r} \sum_{j=1}^{N} p(j) A_k^T(j) X A_k(j) + I_n. \tag{3.71}$$

We remark that under the assumption of Corollary 3.4 *to check the exponential stability we need to solve a system of* n_1 *scalar linear equations with* n_1 *scalar unknowns, with* $n_1 = (\theta n(n+1))/2$.

In the case of system (3.1) which verifies the assumptions of Corollary 3.5 to check the mean square exponential stability means to solve a system of n_2 scalar linear equations with n_2 scalar unknowns with $(n_2 = n(n+1))/2$.

Remark 3.9 In applications (see the next examples), Theorem 3.12 and Corollaries 3.4–3.6 often are used.

These results provide characterizations of the mean square exponential stability in terms of the operators $\Lambda_t, \Lambda_t^*, t \geq 1$. Such operators do not have an analogue in the continuous-time case. We consider that the usefulness of the aforementioned criteria for SESMS represents a good motivation to introduce the sequences $\Lambda_t, \Lambda_t^*, S(t,s), S^*(t,s)$.

At the end of this section we present several examples that illustrate the applicability of the criteria for exponential stability in the mean square derived in this section.

Example 3.5 Consider the time-invariant system with $n = 1, N = 2$ described by

$$x(t+1) = \left(a_0(\eta_t) + \sum_{k=1}^{r} a_k(\eta_t) w_k(t)\right) x(t), \tag{3.72}$$

where the probability transition matrix is

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix}, \quad \alpha \in [0,1].$$

Applying Corollary 3.6 we deduce that the zero state equilibrium of (3.72) is ESMS if and only if

$$\alpha\delta_1 + (1-\alpha)\delta_2 - 1 < 0, \tag{3.73}$$

where $\delta_j = \sum_{k=0}^{r} a_k^2(j), j = 1,2$. The inequality (3.71) is verified if and only if

$$\delta_1 \leq \delta_2 < 1 \quad \text{and} \quad \alpha \in [0,1]$$

or

$$\delta_1 < 1, \qquad \delta_2 \geq 1, \qquad \alpha \in \left(\frac{\delta_2 - 1}{\delta_2 - \delta_1}, 1\right]$$

or

$$\delta_2 < \min\{1, \delta_1\}, \qquad \alpha < \min\left\{1, \frac{1-\delta_2}{\delta_1 - \delta_2}\right\}.$$

Example 3.6 Consider the discrete-time time-invariant system with jump Markov perturbations, in the particular case $n = 1$, $N = 2$,

$$x(t+1) = a_0(\eta_t)x(t) \tag{3.74}$$

and the probability transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix}.$$

Considering the discussions from Example 3.5 in the particular case $a_k(i) = 0, 1 \leq k \leq r, i \in \{1, 2\}$ we conclude that the zero state equilibrium of (3.74) is ESMS if and only if

$$a_0^2(1) \leq a_0^2(2) < 1, \qquad \alpha \in [0, 1]$$

or

$$a_0^2(1) < 1, \qquad a_0^2(2) \geq 1, \qquad \alpha \in \left(\frac{a_0^2(2) - 1}{a_0^2(2) - a_0^2(1)}, 1 \right]$$

or

$$a_0^2(2) < \min\{1, a_0^2(1)\}, \qquad 0 \leq \alpha < \min\left\{ 1, \frac{1 - a_0^2(2)}{a_0^2(1) - a_0^2(2)} \right\}.$$

Particularly, if $a_0^2(1) = \frac{1}{2}, a_0^2(2) = 2, \alpha = \frac{4}{5}$ the zero state equilibrium of the corresponding equation (3.74) is ESMS.

This example shows that it is possible for the zero state equilibrium of a linear stochastic system with Markov perturbations to be ESMS even if for some mode $i \in \mathcal{D}$ the deterministic system is not exponentially stable.

Example 3.7 Consider the particular case of (3.2) with

$$n = 2, \qquad N = 2, \qquad P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

described by

$$x(t+1) = A_0(\eta_t)x(t), \tag{3.75}$$

where

$$A_0(1) = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}, \qquad A_0(2) = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \qquad a \in \mathbf{R}.$$

Applying Corollary 3.6 we deduce that the zero state equilibrium of (3.75) is ESMS if and only if there exists

$$X = \begin{pmatrix} x & y \\ y & z \end{pmatrix} > 0$$

such that

$$\frac{1}{2} \sum_{j=1}^{2} A_0^T(j) X A_0(j) - X = -I_2.$$

One obtains

$$x = z = \frac{2(1 - a^2)}{2a^4 - 5a^2 + 1}, \qquad y = \frac{2a}{2a^4 - 5a^2 + 1}$$

therefore $X > 0$ if and only if $a^2 \in [0, (5 - \sqrt{17}/4))$. We see that if $a^2 \in [(5 - \sqrt{17}/4), 1)$ the spectral radii of the matrices $A_0(1)$ and $A_0(2)$ are less than 1, but the corresponding system (3.75) does not have a mean square exponentially stable evolution. This shows that for $n \geq 2$ it is possible that for each mode $i \in \mathcal{D}$ the deterministic system $y(t+1) = A_0(i)y(t)$ is exponentially stable, whereas the corresponding stochastic system of type (3.2) is not exponentially stable in the mean square.

Example 3.8 Consider the periodic case of (3.1) for $r = 1, N = 2, \theta = 2$; that is,

$$x(t+1) = (A_0(t,\eta_t) + A_1(t,\eta_t)w_1(t))x(t), \tag{3.76}$$

where the probability transition matrices

$$P_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad P_1 = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix},$$

$$A_k(t,i) = a_k(i)I_n, k \in \{0,1\}, i \in \mathcal{D}, t \geq 0.$$

Applying Corollary 3.5 we deduce that the zero state equilibrium of system (3.76) is ESMS iff there exist $X_1 > 0, X_2 > 0$ such that

$$\frac{1}{2}[a_0^2(1) + a_0^2(2) + a_1^2(1) + a_1^2(2)]X_2 - X_1 = -I_n$$

$$\frac{1}{3}[(a_0^2(1) + a_1^2(1)) + 2(a_0^2(2) + a_1^2(2))]X_1 - X_2 = -I_n.$$

This is equivalent to

$$[a_0^2(1) + a_0^2(2) + a_1^2(1) + a_1^2(2)][a_0^2(1) + a_1^2(1) + 2a_0^2(2) + 2a_1^2(2)] < 6.$$

3.6 The case of the systems with coefficients depending upon η_t and η_{t-1}

In this section we study the problem of mean square exponential stability of the linear stochastic systems (3.18).

As in the case of the systems of type (3.1), we define the exponential stability in the mean square both in terms of exponential stability of the evolution generated by the Lyapunov operators associated with the system (3.18) as well as in terms of exponentially stable behavior of the state space trajectories of the system.

Definition 3.4

(*a*) *We say that the zero state equilibrium of the system* (3.18) *is* strongly exponentially stable in the mean square *(SESMS) if there exist $\beta \geq 1, q \in (0,1)$ such that $\|R(t,s)\|_\xi \leq \beta q^{t-s}$ for all $t \geq s \geq 1$, $R(t,s)$ being the linear evolution operator on $\mathcal{S}_n^N$ defined by the sequence Υ_t associated with the system* (3.18).

(*b*) *We say that the zero state equilibrium of the system* (3.18) *is* exponentially stable in the mean square with conditioning *(ESMS-C) if there exist* $\beta \geq 1$, $q \in (0,1)$ *such that for any sequence of independent random vectors* $\{w(t)\}_{t\geq 1}$ *and for any Markov chain* $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$ *that satisfy* $\mathbf{H}_1, \mathbf{H}_2$ *we have:*

$$E[|\Theta(t,s)x|^2|\eta_{s-1} = i] \leq \beta q^{t-s}|x|^2$$

for all $t \geq s \geq 1, x \in \mathbf{R}^n, i \in \mathcal{D}_{s-1}$.

(*c*) *We say that the zero state equilibrium of the system* (3.18) *is* exponentially stable in the mean square *(ESMS) if there exist* $\beta \geq 1, q \in (0,1)$ *such that for any sequence of independent random vectors* $\{w(t)\}_{t\geq 1}$ *and for any Markov chain* $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$ *that satisfy* $\mathbf{H}_1, \mathbf{H}_2$ *we have:*

$$E[|\Theta(t,s)x|^2] \leq \beta q^{t-s}|x|^2$$

for all $t \geq s \geq 1, x \in \mathbf{R}^n$.

Remark 3.10 As we have seen in Remark 3.3 the system (3.1) may be viewed as a special case of the system (3.18). It is natural to want to see if the types of exponential stability in the mean square introduced by Definition 3.4 reduce to the ones introduced by Definition 3.1 if the systems (3.18) take the particular form (3.1).

It is obvious that if the system (3.1) is regarded as a system of type (3.18) then the concepts of ESMS introduced by Definition 3.1(*e*) and Definition 3.4(*c*) coincide. The concept of ESMS-C introduced by Definition 3.4(*b*) coincides with the concept ESMS-CII introduced by Definition 3.1(*d*). The concept of SESMS introduced by Definition 3.4(*a*) coincides with the concept of SESMS-II introduced by Definition 3.1(*b*).

Concerning the relations between the concepts of exponential stability in the mean square introduced by Definition 3.4 we have the following theorem.

Theorem 3.13 *Under the assumptions* $\mathbf{H}_1, \mathbf{H}_2$ *we have:*

(*i*) *If the zero state equilibrium of the system* (3.18) *is SESMS then it is ESMS-C.*

(*ii*) *If the zero state equilibrium of the system* (3.18) *is ESMS-C then it is ESMS.*

Proof. (*i*) follows immediately from (3.19); (*ii*) follows from the inequality

$$E[|\Theta(t,s)x|^2] \leq \sum_{i\in\mathcal{D}_{s-1}} E[|\Theta(t,s)x|^2|\eta_{s-1} = i].$$

The analogue of Theorem 3.5 for the systems (3.18) is as follows. □

Theorem 3.14 *If for all* $t \geq 0$, P_t *are nondegenerate stochastic matrices then the following are equivalent.*

(i) *The zero state equilibrium of the system* (3.18) *is SESMS.*
(ii) *The zero state equilibrium of the system* (3.18) *is ESMS-C.*
(iii) *There exist a sequence of independent random vectors* $\{w(t)\}_{t\geq 1}$ *and a Markov chain* $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$ *with* $\mathcal{P}\{(\eta_0 = i)\} > 0, i \in \mathcal{D}$ *satisfying* $\mathbf{H}_1, \mathbf{H}_2$ *such that*

$$E[|\Theta(t,s)x|^2|\eta_{s-1} = i] \leq \beta q^{t-s}|x|^2$$

for all $t \geq s \geq 1, i \in \mathcal{D}, x \in \mathbf{R}^n$, *where* $\beta \geq 1, q \in (0,1)$.

Proof. It is similar to the proof of Theorem 3.4 and is based on Theorem 3.2. The details are omitted. □

As in the case of the systems of type (3.1), it is expected that the types of exponential stability in the mean square introduced for systems of type (3.18) will not be equivalent in the absence of some additional assumptions.

In the sequel we show that under the periodicity assumption the concepts of exponential stability introduced by Definition 3.4 become equivalent.

Firstly we introduce the following.

Definition 3.5 *We say that the zero state equilibrium of the system* (3.18) *is* asymptotically stable in the mean square *(ASMS) if for any sequence of independent random vectors* $\{w(t)\}_{t\geq 1}$ *and for any Markov chain* $(\{\eta_t\}_{t\geq 0}, \mathbf{P}, \mathcal{D})$ *that satisfy* $\mathbf{H}_1, \mathbf{H}_2$ *we have*

$$\lim_{t\to\infty} E[|\Theta(t,1)x|^2] = 0, \forall x \in \mathbf{R}^n.$$

Now we are in position to state the following.

Theorem 3.15 *Assume that there exists an integer* $\theta \geq 1$ *such that* $A_k(t+\theta, i, j) = A_k(t,i,j), 0 \leq k \leq r, i,j \in \mathcal{D}, P_{t+\theta} = P_t, t \geq 0$. *Under these conditions the following are equivalent.*

(i) *The zero state equilibrium of the system* (3.18) *is (SESMS).*
(ii) *The zero state equilibrium of the system* (3.18) *is (ESMS-C).*
(iii) *The zero state equilibrium of the system* (3.18) *is (ESMS).*
(iv) *The zero state equilibrium of the system* (3.18) *is (ASMS).*
(v) *There exist a sequence of independent random vectors* $\{w(t)\}_{t\geq 1}$ *and a Markov chain* $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *with* $\mathcal{P}\{(\eta_0 = i)\} > 0, i \in \mathcal{D}$ *satisfying* $\mathbf{H}_1, \mathbf{H}_2$ *such that*

$$\lim_{t\to\infty} E[|\Theta(\theta t, 1)x|^2] = 0 \tag{3.77}$$

for all $x \in \mathbf{R}^n$.
(vi) $\rho[R(\theta+1,1)] < 1$ *where* $\rho[\cdot]$ *is the spectral radius.*

Proof. The implications $(i) \to (ii) \to (iii)$ follow from Theorem 3.13. The implications $(iii) \to (iv) \to (v)$ are straightforward. Now we prove the implication $(v) \to (vi)$.

From (3.77) together with the equality:

$$E[|\Theta(\theta t, 1)x|^2] = \sum_{i=1}^{N} \pi_0(i) E[|\Theta(\theta t, 1)x|^2 | \eta_0 = i]$$

we deduce that

$$\lim_{t\to\infty} E[|\Theta(\theta t, 1)x|^2 | \eta_0 = i] = 0, \qquad i \in \mathcal{D}. \tag{3.78}$$

Based on (3.19) and (3.78) one gets

$$\lim_{t\to\infty} x^T [(R^*(\theta t, 1)J)(i)]x = 0. \tag{3.79}$$

If we take into account the definition of the norm $|\cdot|_\xi$ (see (2.20)), we may write

$$\lim_{t\to\infty} |R^*(\theta t, 1)J|_\xi = 0$$

or equivalently

$$\lim_{t\to\infty} \|R^*(\theta t, 1)\|_\xi = 0. \tag{3.80}$$

Because $\|\cdot\|_\xi, \|\cdot\|_2$ are equivalent norms on $\mathcal{S}_n^N$ and $\|R^*(\theta t, 1)\|_2 = \|R(\theta t, 1)\|_2$ one obtains from (3.80) that $\lim_{t\to\infty} \|R(\theta t, 1)\|_\xi = 0$. Using the fact that $R(\theta t + 1, 1) = \Upsilon_{\theta t} R(\theta t, 1) = \Upsilon_\theta R(\theta t, 1)$ we deduce that

$$\lim_{t\to\infty} \|R(\theta t + 1, 1)\|_1 = 0. \tag{3.81}$$

Based on periodicity of the coefficients, one shows inductively that $R(\theta t + 1, 1) = (R(\theta + 1, 1))^t$ for all $t \geq 1$. Thus (3.81) may be rewritten $\lim_{t\to\infty} \|(R(\theta + 1), 1)^t\|_\xi = 0$.

From the definition of the spectral radius we conclude that (3.81) is equivalent to $\rho[R(\theta + 1, 1)]_\xi < 1$. This shows that (vi) holds. If (vi) is true then there exist $\beta \geq 1, q \in (0, 1)$ such that $\|(R(\theta + 1, 1))^t\|_\xi \leq \beta q^t$. Furthermore one shows in a standard way that there exists $\beta_1 \geq \beta$ such that $\|R(t, s)\|\xi \leq \beta_1 q^{(t-s)}$ for all $t \geq s \geq 1$ (for more details one can see the proof of implication $(vi) \to (i)$ in Theorem 3.10). Thus we obtain that the implication $(vi) \to (i)$ holds and the proof is complete. □

Definition 3.6 *We say that the system* (3.18) is in the time-invariant case *if* $A_k(t, i, j) = A_k(i, j)$ *for all* $t \geq 1$, $i, j \in \mathcal{D}$, $0 \leq k \leq r$ *and* $P_t = P$ *for all* $t \geq 0$.

In this case we have $\Upsilon_t = \Upsilon$, for all $t \geq 1$. One sees that the system (3.18) is in the time-invariant case if and only if it is periodic with period $\theta = 1$. Hence, the equivalences from the above theorem hold in the time-invariant case too. In this case, the statement (vi) becomes $\rho(\Upsilon) < 1$.

Based on Theorems 2.4 and 2.5 applied to the sequence (2.99) we obtain a set of Lyapunov-type criteria based on the properties of the sequence Υ_t^*.

Theorem 3.16 *Under the assumption of Theorem* 3.15 *with* $\theta \geq 2$ *the following are equivalent.*

(i) *The zero state equilibrium of the system* (3.18) *is ESMS.*
(ii) *There exist the positive definite matrices* $X_t(i) \in \mathcal{S}_n^N, 1 \leq t \leq \theta, i \in \mathcal{D}$ *that solve the following system of linear matrix equations,*

$$X_t(i) = \sum_{k=1}^{r} \sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j,i) X_{t+1}(j) A_k(t,j,i) + I_n$$

for all $1 \leq t \leq \theta - 1$,

$$X_\theta(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p_{\theta-1}(i,j) A_k^T(\theta,j,i) X_1(j) A_k(\theta,j,i) + I_n$$

for $i \in \mathcal{D}$.
(iii) *There exist positive definite matrices* $Y_t(i) \in \mathcal{S}_n, 1 \leq t \leq \theta, i \in \mathcal{D}$ *that solve the following system of LMIs,*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j,i) Y_{t+1}(j) A_k(t,j,i) - Y_t(i) < 0$$

for $1 \leq t \leq \theta - 1$,

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p_{\theta-1}(i,j) A_k^T(\theta,j,i) Y_1(j) A_k(\theta,j,i) - Y_\theta(i) < 0$$

for $i \in \mathcal{D}$.

Proof. From Theorems 2.4 and 2.5 applied to the sequence Υ_t^* it follows that (ii) and (iii) in the statement are equivalent to the fact that the sequence $\{\Upsilon_t\}_{t \geq 1}$ generates an exponentially stable evolution. Definition 3.4(a) and Theorem 3.15 show that this is equivalent to ESMS of the zero state equilibrium of the system (3.18). □

The next result provides a set of Lyapunov-type criteria based on exponential stability of the evolution defined by the sequence $\{\Upsilon_t\}_{t \in \mathbf{Z}}$.

Theorem 3.17 *Under the assumptions of Theorem* 3.16 *the following are equivalent.*

(*i*) *The zero state equilibrium of the system* (3.18) *is ESMS.*
(*ii*) *The system of linear matrix equations*

$$X_{t+1}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i)A_k(t,i,j) \times X_t(j)A_k^T(t,i,j) + I_n \qquad 1 \le t \le \theta - 1 \tag{3.82}$$

$$X_1(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{\theta-1}(j,i)A_k(\theta,i,j)X_\theta(j)A_k^T(\theta,i,j) + I_n,$$

$i \in \mathcal{D}$, has a solution $X_t = (X_t(1),\ldots,X_t(N))$ such that $X_t(i) > 0$, $i \in \mathcal{D}$.
(*iii*) *There exist positive definite matrices $Y_t(i), 1 \le t \le \theta, i \in \mathcal{D}$, that solve the following system of LMIs,*

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i)A_k(t,i,j)Y_t(j)A_k^T(t,i,j) - Y_{t+1}(i) < 0 \qquad 1 \le t \le \theta - 1 \tag{3.83}$$

$$\sum_{k-0}^{r}\sum_{j=1}^{N} p_{\theta-1}(j,i)A_k(\theta,i,j)Y_\theta(j)A_k^T(\theta,i,j) - Y_1(i) < 0,$$

$i \in \mathcal{D}$.

In the time-invariant case the results of Theorems 3.16 and 3.17 become as follows.

Corollary 3.7 *If the system is in the time-invariant case, the following are equivalent.*

(*i*) *The zero state equilibrium of the system* (3.18) *is ESMS.*
(*ii*) *The system of linear equations*

$$X(i) = \sum_{k=0}^{R}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)X(j)A_k(j,i) + I_n,$$

$i \in \mathcal{D}$, has a solution $X = (X(1),\ldots,X(N))$ with $X(i) > 0, i \in \mathcal{D}$.
(*iii*) *There exist positive definite matrices $Y(i), i \in \mathcal{D}$, that solve the following system of LMIs,*

$$\sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)Y(j)A_k(j,i) - Y(i) < 0,$$

$i \in \mathcal{D}$.

Corollary 3.8 *Under the conditions of Corollary* 3.6 *the following are equivalent.*

(*i*) *The zero state equilibrium of the system* (3.18) *is ESMS.*
(*ii*) *The system of linear matrix equations*

$$X(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(j,i) A_k(i,j) X(j) A_k^T(i,j) + I_n,$$

$i \in \mathcal{D}$, has a solution $X = (X(1), \ldots, X(N))$ with $X(i) > 0, i \in \mathcal{D}$.
(*iii*) *There exist positive definite matrices $Y(i), i \in \mathcal{D}$, that solve the following system of LMIs,*

$$\sum_{k=0}^{r} \sum_{j=1}^{N} p(j,i) A_k(i,j) Y(j) A_k^T(i,j) - Y(i) < 0, \qquad i \in \mathcal{D}.$$

3.7 Discrete-time affine systems

In this section we deal with discrete-time affine stochastic systems obtained from (3.1) adding some forcing terms. More precisely, the systems under consideration are of the form

$$\begin{aligned} x(t+1) = {} & A_0(t,\eta_t)x(t) + g_0(t) \\ & + \sum_{k=1}^{r} w_k(t)[A_k(t,\eta_t)x(t) + g_k(t)], t \geq 0, \end{aligned} \tag{3.84}$$

where $\{g_k(t)\}_{t \geq 0}, 0 \leq k \leq r$ are stochastic processes with the property that for each $t \geq 0$, the vectors $g_k(t)$ are $\tilde{\mathcal{H}}_t$-measurable and $E|g_k(t)|^2 < \infty$, $\tilde{\mathcal{H}}_t$ being the σ-algebra defined in Section 1.5.

The following result is used repeatedly in the developments of the next chapters.

Lemma 3.1 *Let $V(t,x,i) = x^T X(t,i) x + 2x^T \kappa(t,i) + \mu(t,i)$, where $X(t,i) = X^T(t,i) \in \mathbf{R}^{n \times n}$, $\kappa(t,i) \in \mathbf{R}^n$, $\mu(t,i) \in \mathbf{R}$, $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$. Then we have:*

$$\begin{aligned} & E[V(t+1, x(t+1), \eta_{t+1}) | \eta_{t_0}] \\ & = \sum_{k=0}^{r} E[x^T(t) A_k^T(t,\eta_t) \mathcal{E}_{\eta_t}(t, X(t+1)) A_k(t,\eta_t) x(t) + 2x^T(t) A_k^T(t,\eta_t) \\ & \quad \times \mathcal{E}_{\eta_t}(t, X(t+1)) g_k(t) + g_k^T(t) \mathcal{E}_{\eta_t}(t, X(t+1)) g_k(t) | \eta_{t_0}] \\ & \quad + 2E[(A_0(t,\eta_t)x(t) + g_0(t))^T \mathcal{E}_{\eta_t}(t, \kappa(t{+}1)) | \eta_{t_0}] \\ & \quad + E[\mathcal{E}_{\eta_t}(t, \mu(t+1)) | \eta_{t_0}] \end{aligned}$$

for all $t \geq t_0 \geq 0$ *and any trajectories* $\{x(t)\}_{t \geq t_0}$ *of the system* (3.84) *with* $x(t_0)$ $\tilde{\mathcal{H}}_{t_0}$*-measurable and* $E[|x(t_0)|^2] < \infty$*, where*

$$\mathcal{E}_i(t, Z) = \sum_{j=1}^{N} p_t(i, j) Z(j), \tag{3.85}$$

$1 \leq i \leq N$*, for all* $Z = (Z(1), Z(2), \dots, Z(N))$*, where either* $Z(i) \in \mathbf{R}^{n \times n}$*,* $Z(i) \in \mathbf{R}^n$*, or* $Z(i) \in \mathbf{R}$*, respectively.*

Proof. Because $V(t+1, x(t+1), j)$ is $\mathcal{H}_t$-measurable we obtain via Corollary 1.1 the equalities:

$$\begin{aligned} E[V(t+1, x(t+1), \eta_{t+1})|\mathcal{H}_t] &= \sum_{j=1}^{N} V(t+1, x(t+1), j) E[\chi_{\{\eta_{t+1}=j\}}|\mathcal{H}_t] \\ &= \sum_{j=1}^{N} V(t+1, x(t+1), j) p_t(\eta_t, j), \quad \text{a.s.} \end{aligned} \tag{3.86}$$

Taking into account that $\tilde{\mathcal{H}}_t \subset \mathcal{H}_t$, $g_k(t), x(t)$, and η_t are $\tilde{\mathcal{H}}_t$-measurable and $w_k(t)$ are independent of $\tilde{\mathcal{H}}_t$, we obtain from (3.86) the equality

$$\begin{aligned} E[V(t+1, x(t+1), \eta_{t+1})|\tilde{\mathcal{H}}_t] &= (A_0(t, \eta_t)x(t) + g_0(t))^T \mathcal{E}_{\eta_t}(t, X(t+1)) \\ &\quad \times (A_0(t, \eta_t)x(t) + g_0(t)) \\ &\quad + 2(A_0(t, \eta_t)x(t) + g_0(t))^T \mathcal{E}_{\eta_t}(t, X(t+1)) \\ &\quad \times \sum_{k=1}^{r} (A_k(t, \eta_t)x(t) + g_k(t)) E[w_k(t)|\tilde{\mathcal{H}}_t] \\ &\quad + \sum_{k,l=1}^{r} (A_k(t, \eta_t)x(t) + g_k(t))^T \mathcal{E}_{\eta_t}(t, X(t+1)) \\ &\quad \times (A_l(t, \eta_t)x(t) + g_l(t)) E[w_k(t)w_l(t)|\tilde{\mathcal{H}}_t] \\ &\quad + (A_0(t, \eta_t)x(t) + g_0(t))^T \mathcal{E}_{\eta_t}(t, \kappa(t+1)) \\ &\quad + 2\sum_{k=1}^{r} (A_k(t, \eta_t)x(t) + g_k(t))^T \mathcal{E}_{\eta_t}(t, \kappa(t+1)) \\ &\quad \times E[w_k(t)|\tilde{\mathcal{H}}_t] + \mathcal{E}_{\eta_t}(t, \mu_{t+1}). \end{aligned}$$

From assumption H_1 and Theorem 1.3(v) we deduce: $E[w_k(t)|\tilde{\mathcal{H}}_t] = E[w_k(t)] = 0$ and $E[w_k(t)w_l(t)|\tilde{\mathcal{H}}_t] = E[w_k(t)w_l(t)] = \delta_{kl}$ where as usual

$$\delta_{kl} = \begin{cases} 0, & k \neq l; \\ 1, & k = l. \end{cases}$$

This allows us to write:

$$\begin{aligned} E[V(t+1, x(t+1), \eta_{t+1}) | \tilde{\mathcal{H}}_t] = & \sum_{k=0}^{r} (A_k(t, \eta_t) x(t) + g_k(t))^T \\ & \times \mathcal{E}_{\eta_t}(t, X(t+1))(A_k(t, \eta_t) x(t) + g_k(t)) \\ & + 2(A_0(t, \eta_t) x(t) + g_0(t))^T \\ & \times \mathcal{E}_{\eta_t}(t, \kappa(t+1)) + \mathcal{E}_{\eta_t}(t, \mu(t+1)). \quad (3.87) \end{aligned}$$

Finally, taking the conditional expectation with respect to $\sigma(\eta_{t_0}) \subset \tilde{\mathcal{H}}_t$ in (3.87) one obtains the equality in the statement and thus the proof ends. □

Concerning the trajectories of the affine system (3.84) we prove the following.

Theorem 3.18 *Under the assumptions H_1, H_2 if the zero state equilibrium of the linear system* (3.1) *is SESMS-I then the trajectories $x(t), t \geq t_0$ of the affine system* (3.84) *such that $x(t_0)$ is $\tilde{\mathcal{H}}_{t_0}$-measurable and $E[|x(t_0)|^2] < \infty$ satisfy the estimates*

$$E[|x(t)|^2 | \eta_{t_0}] \leq c \left\{ q^{t-t_0} E[|x(t_0)|^2 | \eta_{t_0}] + \sum_{k=0}^{r} \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2 | \eta_{t_0}] \right\} \quad (3.88)$$

for all $t > t_0$, where $c > 0$, $q \in (0,1)$, are independent of $t, t_0 \in Z_+$.

Proof. Based on the property of SESMS-I one obtains via Theorem 3.7 that there exists the sequence $\{X(t)\}_{t \geq 0}$, $X(t) = (X(t,1), X(t,2), \ldots, X(t,N))$ that satisfies

$$X(t,i) = \sum_{k=0}^{r} A_k^T(t,i) \mathcal{E}_i(t, X(t+1)) A_k((t,i) + I_n, \quad (3.89)$$

$t \geq 0, i \in \mathcal{D}$ and

$$I_n \leq X(t,i) \leq \tilde{c} I_n \quad (3.90)$$

for all $t \geq 0$, $i \in \mathcal{D}$, $\tilde{c} \geq 1$ not depending upon t, i.

Applying Lemma 3.1 to the function $V(t,x,i) = x^T X(t,i) x$ and to the system (3.84) we obtain, via (3.89) that

$$E[V(t+1,x(t+1),\eta_{t+1})|\eta_{t_0}] - E[V(t,x(t),\eta_t)|\eta_{t_0}] = -E[|x(t)|^2|\eta_{t_0}]$$
$$+ 2\sum_{k=0}^{r} E[x^T(t)A_k^T(t,\eta_t)\mathcal{E}_{\eta_t}(t,X(t+1))g_k(t)|\eta_{t_0}]$$
$$+ \sum_{k=0}^{r} E[g_k^T(t)\mathcal{E}_{\eta_t}(t,X(t+1))g_k(t)|\eta_{t_0}]. \quad (3.91)$$

Denoting $\varphi(t)$ the right-hand side of (3.91) we may write:

$$\varphi(t) = -\frac{1}{2}E[|x(t)|^2|\eta_{t_0}] - \frac{1}{2}\sum_{k=0}^{r} E\left[\left|\frac{1}{\sqrt{r+1}}x(t)\right.\right.$$
$$\left.\left. -2\sqrt{r+1}A_k^T(t,\eta_t)\mathcal{E}_{\eta_t}(t,X(t+1))g_k(t)\right|^2 |\eta_{t_0}\right]$$
$$+ \sum_{k=0}^{r} E[g_k^T(t)\mathcal{M}_k(t,\eta_t)g_k(t)|\eta_{t_0}], \quad (3.92)$$

where $\mathcal{M}_k(t,i) = \mathcal{E}_i(t,X(t+1)) + 2(r+1)\mathcal{E}_i(t,X(t+1))A_k(t,i)A_k^T(t,i)$ $\mathcal{E}_i(t,X(t+1))$.

Because the zero state equilibrium of the system (3.1) is SESMS-I it follows that the sequences $\{A_k(t,i)\}_{t\geq 0}$, $0 \leq k \leq r$, $i \in \mathcal{D}$ are bounded. This guarantees that we have

$$0 \leq \mathcal{M}_k(t,i) \leq \nu I_n, \quad (3.93)$$

$0 \leq k \leq r$, $i \in \mathcal{D}$, $t \in Z_+$, where $\nu > 0$ does not depend upon t,k,i. Setting $v(t) = E[V(t,x(t),\eta_t)|\eta_{t_0}]$ we obtain from (3.93), (3.92), and (3.91) that

$$v(t+1) - v(t) \leq -\frac{1}{2}E[|x(t)|^2|\eta_{t_0}] + \nu\sum_{k=0}^{r} E[|g_k(t)|^2|\eta_{t_0}].$$

Using (3.90) one gets:

$$v(t+1) \leq qv(t) + \tilde{g}(t), \quad (3.94)$$

where $q = 1 - \tilde{c}^{-1}$ and $\tilde{g}(t) = \nu\sum_{k=0}^{r} E[|g_k(t)|^2|\eta_{t_0}]$, $q \in (0,1)$.

Let $\tilde{v}(t)$, $t \geq t_0$ be the solution of $\tilde{v}(t+1) = q\tilde{v}(t) + \tilde{g}(t)$, $\tilde{v}(t_0) = v(t_0)$; hence

$$\tilde{v}(t) = q^{t-t_0}v(t_0) + \sum_{s=t_0}^{t-1} q^{t-s-1}\tilde{g}(s) \quad (3.95)$$

for all $t > t_0$. Furthermore, one obtains inductively via (3.94) that $v(t) \leq \tilde{v}(t)$, $t \geq t_0$.

Thus from (3.95) we deduce that $v(t) \le q^{t-t_0} v(t_0) + \nu \sum_{k=0}^{r} \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2 | \eta_{t_0}]$. Invoking again (3.90) one obtains

$$E[|x(t)|^2|\eta_{t_0}] \le v(t) \le \tilde{c} q^{t-t_0} E[|x(t_0)|^2|\eta_{t_0}] + \nu \sum_{k=0}^{r} \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0}].$$

Thus the proof is complete. □

Corollary 3.9 *Under the assumptions in Theorem* 3.18 *the following hold.*

(*i*) *If the series* $\sum_{t=0}^{\infty} E[|g_k(t)|^2|\eta_{t_0} = i]$, $0 \le k \le r$, $i \in \mathcal{D}_{t_0}$ *are convergent then the series* $\sum_{t=t_0}^{\infty} E[|x(t)|^2|\eta_{t_0} = i]$ *are convergent for all* $i \in \mathcal{D}_{t_0}$ *and for all trajectories of the system* (3.84) *with* $x(t_0)$, $\tilde{\mathcal{H}}_{t_0}$*-measurable, and* $E[|x(t_0)|^2] < \infty$. *Moreover, we have*

$$\sum_{t=t_0}^{\infty} E[|x(t)|^2|\eta_{t_0} = i]$$
$$\le c_1 \left(E[|x(t_0)|^2|\eta_{t_0} = i] + \sum_{k=0}^{r} \sum_{t=t_0}^{\infty} E[|g_k(t)|^2|\eta_{t_0} = i] \right).$$

(*ii*) *If* $\sup_{t \ge 0} E[|g_k(t)|^2|\eta_{t_0} = i] < +\infty$, $0 \le k \le r$, $i \in \mathcal{D}_{t_0}$ *then*

$$\sup_{t \ge t_0} E[|x(t)|^2|\eta_{t_0} = i] < c < +\infty$$

for all $t \ge 0$ *and all the trajectories of the affine system* (3.84) *with the properties as in (i).*

(*iii*) *If* $\lim_{t\to\infty} E[|g_k(t)|^2|\eta_{t_0} = i] = 0$, $0 \le k \le r$, *and* $i \in \mathcal{D}_{t_0}$, *then* $\lim_{t\to\infty} E[|x(t)|^2|\eta_{t_0} = i] = 0$ *for any trajectories of the affine system* (3.84) *with the additional properties as in (i).*

Proof. (*i*) If $i \in \mathcal{D}_{t_0}$, then taking the conditional expectation with respect to the event $\{\eta_{t_0} = i\}$ in (3.88) one gets

$$E[|x(t)|^2|\eta_{t_0} = i] \le c \Bigg(q^{t-t_0} E[|x(t_0)|^2|\eta_{t_0} = i] + \sum_{k=0}^{r} \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0} = i] \Bigg). \quad (3.96)$$

This leads to

$$\sum_{t=t_0+1}^{\tau} E[|x(t)|^2|\eta_{t_0} = i] \le c \Bigg(\sum_{t=t_0+1}^{\tau} q^{t-t_0} E[|x(t_0)|^2|\eta_{t_0} = i] + \sum_{k=0}^{r} \sum_{t=t_0+1}^{\tau} \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0} = i] \Bigg) \quad (3.97)$$

for all $\tau > t_0 \geq 0$, $i \in \mathcal{D}_{t_0}$. Performing a change of the order of summation we obtain:

$$\sum_{t=t_0+1}^{\tau} \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0} = i] \sum_{s=t_0}^{\tau-1} \sum_{t=s+1}^{\tau} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0} = i].$$

Because $q \in (0, 1)$ we deduce

$$\sum_{t=t_0+1}^{\tau} \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0} = i] \leq \frac{1}{1-q} \sum_{s=t_0}^{\tau-1} E[|g_k(s)|^2|\eta_{t_0} = i]. \quad (3.98)$$

From (3.97) and (3.98) we obtain

$$\sum_{t=t_0+1}^{\tau} E[|x(t)|^2|\eta_{t_0} = i]$$

$$\leq \frac{c}{1-q} \left(E[|x(t_0)|^2|\eta_{t_0} = i] + \sum_{k=0}^{r} \sum_{s=t_0}^{\tau-1} E[|g_k(s)|^2|\eta_{t_0} = i] \right) \quad (3.99)$$

for all $\tau > t_0 \geq 0$, $i \in \mathcal{D}_{t_0}$.

The conclusion from (i) follows by taking the limit for $\tau \to \infty$ in (3.99).

(ii) This follows immediately from (3.96).

(iii) From (3.96) it follows that we have to prove

$$\lim_{t\to\infty} \sum_{t=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0} = i] = 0, \qquad i \in \mathcal{D}_{t_0}, \quad (3.100)$$

$0 \leq k \leq r$. To this end we use Stolz–Cesaro criteria for the convergence of a sequence of real numbers. Denoting

$$\xi_k(t) = \sum_{s=t_0}^{t-1} q^{t-s-1} E[|g_k(s)|^2|\eta_{t_0} = i], \qquad t > t_0 \geq 0$$

one sees that $\xi_k(t) = (\tilde{\xi}_k(t))/q^{-t}$ with $\tilde{\xi}_k(t) = \sum_{s=t_0}^{t-1} q^{-s-1} E[|g_k(s)|^2|\eta_{t_0} = i]$. We have

$$\frac{\tilde{\xi}_k(t+1) - \tilde{\xi}_k(t)}{q^{-t-1} - q^{-t}} = \frac{q^{-1}}{q^{-1} - 1} E[|g_k(t)|^2|\eta_{t_0} = i].$$

Hence

$$\lim_{t\to\infty} \frac{\tilde{\xi}_k(t+1) - \tilde{\xi}_k(t)}{q^{-t-1} - q^{-t}} = 0.$$

This implies that $\lim_{t\to\infty} \xi_k(t) = 0$ and thus the proof is complete. □

At the end of this section it should be noted that if $x(t)$ is a trajectory of the affine system (3.84) starting from $x_0 \in \mathbf{R}^n$ at time $t = \hat{t}$, then under the assumptions $\mathbf{H_1}$, $\mathbf{H_2}$ one deduces that $x(t_0)$ is $\tilde{\mathcal{H}}_{t_0}$-measurable and $E[|x(t_0)|^2] < \infty$ for any $t_0 \geq \hat{t}$.

3.8 Notes and references

The exponential stability in the mean square for discrete-time stochastic systems affected by multiplicative white noise perturbations is discussed in [7, 13, 34, 58, 65, 88, 92, 93, 96, 113].

The exponential stability in the mean square for discrete-time systems with Markovian perturbations is studied in a great number of papers; see [6, 15, 16, 18, 21, 22, 27, 38, 52–54, 75, 80, 84, 86–90, 95]. Theorem 3.6 is proved in [41]. The results from Section 3.6 are proved in [38, 43] and those from Section 3.7 can be found in [45]. Example 3.1(ii) is presented for the first time in this work. All other results from this chapter can be found in [42].

4

Structural properties of linear stochastic systems

In this chapter we present the stochastic version of some basic concepts in control theory, namely stabilizability, detectability, and observability. All these concepts are defined both in Lyapunov operator terms and in stochastic system terms. The definitions given in this chapter extend the corresponding definitions from the deterministic time-varying systems. Some examples show that stochastic observability does not always imply stochastic detectability and stochastic controllability does not necessarily imply stochastic stabilizability. As in the deterministic case the concepts of stochastic detectability and observability are used in some criteria of exponential stability in the mean square.

4.1 Stochastic stabilizability and stochastic detectability

4.1.1 Definitions and criteria for stochastic stabilizability and stochastic detectability

In this subsection we introduce the stochastic version of concept of stabilizability and detectability. These two properties of controlled systems play an important role in the developments in the next chapters of the book.

Consider the discrete-time controlled system

$$\begin{aligned} x(t+1) = A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t) \\ + \sum_{k=1}^{r}[A_k(t,\eta_t)x(t) + B_k(t,\eta_t)u(t)]w_k(t), \end{aligned} \tag{4.1}$$

where $u \in \mathbf{R}^m$ is the control input, $\{w(t)\}_{t\geq 0}$ is a sequence of independent random vectors with $w(t) = (w_1(t),\dots,w_r(t))^T$, and $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ is a Markov chain such that assumptions $\mathbf{H_1}$ and $\mathbf{H_2}$ are fulfilled.

V. Dragan et al., *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, DOI 10.1007/978-1-4419-0630-4_4,

Definition 4.1 *We say that the system* (4.1) *is* stochastic stabilizable *if there exist the bounded sequences* $\{F(t,i)\}_{t\geq 0}, i \in \mathcal{D}, F(t,i) \in \mathbf{R}^{m\times n}$ *such that the zero state equilibrium of the closed-loop system*

$$x(t+1) = \Bigg[A_0(t,\eta_t) + B_0(t,\eta_t)F(t,\eta_t) + \sum_{k=1}^{r}(A_k(t,\eta_t) + B_k(t,\eta_t)F(t,\eta_t))w_k(t)\Bigg] x(t) \tag{4.2}$$

is SESMS-I. The sequences $\{F(t)\}_{t\geq 0}, F(t) = (F(t,1), F(t,2), \dots, F(t,N))$ *involved in the above definition are called stabilizing feedback gains.*

Let us consider the discrete-time system:

$$x(t+1) = (A_0(t,\eta_t) + \sum_{k=1}^{r} A_k(t,\eta_t)w_k(t))x(t) \tag{4.3}$$

$$y(t) = (C_0(t,\eta_t) + \sum_{k=1}^{r} C_k(t,\eta_t)w_k(t))x(t),$$

$x(t) \in \mathbf{R}^n$ *and* $y(t) \in \mathbf{R}^p$.

Definition 4.2 *We say that the system* (4.3) *is* stochastic detectable *if there exist sequences* $\{K(t,i)\}_{t\geq 0}, i \in \mathcal{D}, K(t,i) \in \mathbf{R}^{n\times p}$ *such that the zero state equilibrium of the following system,*

$$x(t+1) = \Bigg[A_0(t,\eta_t) + K(t,\eta_t)C_0(t,\eta_t) + \sum_{k=1}^{r}(A_k(t,\eta_t) + K(t,\eta_t)C_k(t,\eta_t))w_k(t)\Bigg] x(t) \tag{4.4}$$

is SESMS-I.

The sequences $\{K(t,i)\}_{t\geq 0}$*, involved in the previous definition, are called stabilizing injections.*

In the case of periodic coefficients in the definition of stochastic stabilizability and stochastic detectability we restrict our attention only to stabilizing feedback gains and stabilizing injections that are periodic sequences. In the time-invariant case in the definition of stochastic stabilizability and stochastic detectability we consider only stabilizing feedback gains $F = (F(1), \dots, F(N))$ *and stabilizing injections of the form* $K = (K(1), \dots, K(N))$*. In the next chapter we show that this can be done without loss of generality.*

The following notations are often used in this chapter.

$$A_k(t) = (A_k(t,1), \dots, A_k(t,N)) \in \mathbf{R}^{n\times n} \oplus \cdots \oplus \mathbf{R}^{n\times n} := \mathcal{M}_n^N,$$

$$B_k(t) = (B_k(t,1), \dots, B_k(t,N)) \in \mathbf{R}^{n\times m} \oplus \cdots \oplus \mathbf{R}^{n\times m} := \mathcal{M}_{nm}^N,$$

$$C_k(t) = (C_k(t,1), \ldots, C_k(t,n)) \in \mathbf{R}^{p\times n} \oplus \cdots \oplus \mathbf{R}^{p\times n} := \mathcal{M}_{pn}^N,$$

$$A(t) = (A_0(t), A_1(t), \ldots, A_r(t)) \in \mathcal{M}_n^N \oplus \cdots \oplus \mathcal{M}_n^N,$$

$$B(t) = (B_0(t), B_1(t), \ldots, B_r(t)) \in \mathcal{M}_{nm}^N \oplus \cdots \oplus \mathcal{M}_{nm}^N,$$

$$C(t) = (C_0(t), C_1(t), \ldots, C_r(t)) \in \mathcal{M}_{pn}^N \oplus \cdots \oplus \mathcal{M}_{pn}^N,$$

$$\mathbf{A} = \{A(t)\}_{t\geq 0}, \qquad \mathbf{B} = \{B(t)\}_{t\geq 0}, \qquad \mathbf{C} = \{C(t)\}_{t\geq 0}, \qquad \mathbf{P} = \{P_t\}_{t\geq 0}.$$

We remark that in the time-invariant case $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{P}$ *are constant sequences; that is,* $\mathbf{A} = (A_0, A_1, \ldots, A_r) \in \mathcal{M}_n^N \oplus \cdots \oplus \mathcal{M}_n^N, \mathbf{B} = (B_0, B_1, \ldots, B_r) \in \mathcal{M}_{nm}^N \oplus \cdots \oplus \mathcal{M}_{nm}^N, \mathbf{C} = (C_0, C_1, \ldots, C_r) \in \mathcal{M}_{pn}^N \oplus \cdots \oplus \mathcal{M}_{pn}^N, \mathbf{P} = P \in \mathbf{R}^{n\times n}$.

Now we introduce a definition of stabilizability and detectability expressed in terms of Lyapunov operators.

Definition 4.3 *Let* $\{\mathcal{L}_t\}_{t\geq 0}$ *be a sequence of operators of type* (2.58) *and* $\{B(t)\}_{t\geq 0}$ *and* $\{C(t)\}_{t\geq 0}$ *be as before.*

(a) *We say that the pair* $(\mathcal{L}_t, B(t))$ *is* stabilizable, *or equivalently the triple* $(\mathbf{A}, \mathbf{B}, \mathbf{P})$ *is* stabilizable *if there exists a bounded sequence* $\{F(t)\}_{t\geq 0}$, $F(t) = (F(t,1), \ldots, F(t,N)), F(t,i) \in \mathbf{R}^{m\times n}$ *such that the sequence* $\mathcal{L}_{F,t}, t \geq 0$ *generates an exponentially stable evolution, where* $\mathcal{L}_{F,t} : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $\mathcal{L}_{F,t}X = (\mathcal{L}_{F,t}X(1), \ldots, \mathcal{L}_{F,t}X(N))$ *with*

$$\mathcal{L}_{F,t}X(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)(A_k(t,j) + B_k(t,j)F(t,j))X(j)(A_k(t,j) + B_k(t,j)F(t,j))^T \tag{4.5}$$

for all $X \in \mathcal{S}_n^N$.

(b) *We say that the pair* $(C(t), \mathcal{L}_t)$ *is* detectable *or equivalently the triple* $(\mathbf{C}, \mathbf{A}, \mathbf{P})$ *is* detectable *if there exists a bounded sequence* $\{H(t)\}_{t\geq 0}$, *where* $H(t) = (H(t,1), \ldots, H(t,N)), H(t,i) \in \mathbf{R}^{n\times p}$ *such that the sequence* $\{\mathcal{L}_t^H\}_{t\geq 0}$ *generates an exponentially stable evolution, where* $\mathcal{L}_t^H$ *is defined by* $\mathcal{L}_t^H X = (\mathcal{L}_t^H X(1), \ldots, \mathcal{L}_t^H X(N))$ *with*

$$(\mathcal{L}_t^H X)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)(A_k(t,j) + H(t,j)C_k(t,j))X(j)(A_k(t,j) + H(t,j)C_k(t,j))^T \tag{4.6}$$

for all $X \in \mathcal{S}_n^N$.

Remark 4.1 The concepts of stabilizability and detectability introduced by Definition 4.3 do not require that $p_t(i,j)$ be necessarily the elements of a stochastic matrix. Only the condition $p_t(i,j) \geq 0, i,j \in \mathcal{D}, t \geq 0$ is assumed.

The next simple result provides the relation between the concept of stabilizability and detectability defined above.

Corollary 4.1 *Assume that the scalars* $p_t(i,j)$ *satisfy the additional condition* $\sum_{j=1}^{N} p_t(i,j) = 1, i \in \mathcal{D}, t \geq 0.$

Then the following two equivalences are true.

(a) *The system* (4.1) *is stochastic stabilizable iff the pair* $(\mathcal{L}_t, B(t))$ *is detectable.*

(b) *The system* (4.3) *is stochastic detectable iff the pair* $(C(t), \mathcal{L}_t)$ *is detectable.*

Based on Lyapunov criteria for SESMS derived in Chapter 3, one obtains some criteria for stochastic stabilizability and stochastic detectability as well as for stabilizability of the pair $(\mathcal{L}_t, B(t))$ and detectability of the pair $(C(t), \mathcal{L}_t)$. Applying Corollary 3.3 to the time-invariant version of system (4.3) we have the following.

Corollary 4.2 *Assume that* $A_k(t,i) = A_k(i), B_k(t,i) = B_k(i),\ k \in \{0,1,\dots,r\},\ i \in \mathcal{D}, P_t = P, t \in \mathbf{Z}_+$*. Then the following are equivalent.*

(i) *The system* (4.1) *is stochastic stabilizable.*

(ii) *There exist* $Y = (Y(1),\dots,Y(N)) \in \mathcal{S}_n^N, Z = (Z(1),\dots,Z(N)), Z(i) \in \mathbf{R}^{m\times n}, Y(i) > 0, i \in \mathcal{D}$*, that solve the following system of LMIs,*

$$\begin{pmatrix} -Y(i) & \Psi_{0i}(Y,Z) & \Psi_{1i}(Y,Z) & \dots & \Psi_{ri}(Y,Z) \\ \Psi_{0i}^T(Y,Z) & -\mathcal{Y} & 0 & \dots & 0 \\ \Psi_{1i}^T(Y,Z) & 0 & -\mathcal{Y} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \Psi_{ri}^T(Y,Z) & 0 & 0 & \dots & -\mathcal{Y} \end{pmatrix} < 0, \qquad (4.7)$$

where $\Psi_{ki}(Y,Z) = (\sqrt{p(1,i)}(A_k(1)Y(1) + B_k(1)Z(1)) \dots \sqrt{p(N,i)}(A_k(N)$ $Y(N) + B_k(N)Z(N))), k \in \{0,1,\dots,r\},$

$$\mathcal{Y} = \mathrm{diag}(Y(1),\dots,Y(N)).$$

If (4.7) is solvable then a stabilizing feedback gain is given by $F(i) = Z(i)Y^{-1}(i), i \in \mathcal{D}$.

Applying Corollary 3.4 to the time-invariant case of the closed-loop system (4.4) one obtains the following.

Corollary 4.3 *Under the assumptions of Corollary* 4.2 *the following are equivalent.*

(i) *The system* (4.3) *is stochastic detectable.*

(ii) *There exist* $Y = (Y(1),\dots,Y(N)) \in \mathcal{S}_n^N$ *and* $Z = (Z(1),\dots,Z(N)), Z(i) \in \mathbf{R}^{n\times p}, Y(i) > 0, i \in \mathcal{D}$ *that solve the following system of LMIs,*

$$\begin{pmatrix} -Y(i) & \tilde{\Psi}_{0i}(Y,Z) & \tilde{\Psi}_{1i}(Y,Z) & \dots & \tilde{Psi}_{ri}(Y,Z) \\ \tilde{\Psi}_{0i}^T(Y,Z) & -\mathcal{Y} & 0 & \dots & 0 \\ \tilde{\Psi}_{1i}^T(Y,Z) & 0 & -\mathcal{Y} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \tilde{\Psi}_{ri}^T(Y,Z) & 0 & 0 & \dots & -\mathcal{Y} \end{pmatrix} < 0,$$

$i \in \mathcal{D}$, where $\tilde{\Psi}_{ki}(Y,Z) = (\sqrt{p(i,1)}(A_k^T(1)Y(1) + C_k^T(1)Z^T(1)) \ \dots \ \sqrt{p(i,N)}(A_k^T(N)Y(N) + C_k^T(N)Z^T(N)))$, $k \in \{0,1,\dots,r\}$ and $\mathcal{Y}$ is as before.

If the previous system of LMIs is feasible then a stabilizing injection is given by $K(i) = Y^{-1}(i)Z(i), i \in \mathcal{D}$.

Similar results as in Corollary 4.2 and Corollary 4.3 can be easily derived for the periodic case applying Theorem 3.11 and Theorem 3.12, respectively.

4.1.2 A stability criterion

In this subsection we prove a necessary and sufficient condition for the exponential stability of the evolution generated by the operators (2.58). That condition may not be directly derived from the result proved in Section 2.3.

Theorem 4.1 *Let $\{\mathcal{L}_t\}_{t\geq 0}$ be a sequence defined by* (2.58) *with the additional property that $\{p_t(i,j)\}_{t\geq 0}$ and $\{A_k(t,i)\}_{t\geq 0}$ are bounded sequences.*

Consider the discrete-time backward affine equation

$$Y_t = \mathcal{L}_t^* Y_{t+1} + \tilde{C}(t), \quad t \geq 0, \tag{4.8}$$

where $\tilde{C}(t) = (\tilde{C}(t,1), \tilde{C}(t,2), \dots, \tilde{C}(t,N)), \tilde{C}(t,i) = \sum_{k=0}^{r} C_k^T(t,i)C_k(t,i)$. Assume that $\{C_k(t,i)\}_{t\geq 0}$ are bounded sequences and the pair $(C(t), \mathcal{L}_t)$ is detectable.

Under these conditions the following are equivalent.

(i) *The sequence $\{\mathcal{L}_t\}_{t\geq 0}$ generates an exponentially stable evolution.*
(ii) *The equation* (4.8) *has a bounded solution $\{\tilde{Y}_t\}_{t\geq 0} \subset \mathcal{S}_n^{N+}$.*

Proof. The implication $(i) \to (ii)$ follows immediately from Theorem 2.5(iv). It remains to prove the converse implication.

Let $\{X_t\}_{t\geq t_0}$ be a solution of the problem with given initial values:

$$X_{t+1} = \mathcal{L}_t X_t, \quad t \geq t_0 \tag{4.9}$$

$$X_{t_0} = H, \quad H \in \mathcal{S}_n^{N+}. \tag{4.10}$$

We show that there exists $\gamma > 0$ not depending upon t_0 and H such that

$$\sum_{t=t_0}^{\infty} |X_t|_\xi \leq \gamma |H|_\xi \tag{4.11}$$

for all $t_0 \geq 0, H \in \mathcal{S}_n^{N+}$.

Let $\{H_t\}_{t\geq 0}$ be a stabilizing injection. This means that there exist $\beta_1 > 0$, $q_1 \in (0,1)$ such that

$$\|T(t,s)^H\|_\xi \leq \beta_1 q_1^{t-s}$$

for all $t \geq s \geq 0, T(t,s)^H$ being the causal linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $\{\mathcal{L}_t^H\}_{t\geq 0}$ where $\mathcal{L}_t^H$ is defined as in (4.6).

The equation (4.9) may be rewritten as

$$X_{t+1} = \mathcal{L}_t^H X_t + \mathcal{G}_t X_t, \tag{4.12}$$

where $\mathcal{G}_t X_t = (\mathcal{G}_t X_t(1), \ldots, \mathcal{G}_t X_t(N))$,

$$\begin{aligned}\mathcal{G}_t X_t(i) = -\sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)[&H(t,j)C_k(t,j)X_t(j)A_k^T(t,j)\\ &+ A_k(t,j)X_t(j)C_k^T(t,j)H^T(t,j)\\ &+ H(t,j)C_k(t,j)X_t(j)C_k^T(t,j)H^T(t,j)].\end{aligned}$$

Furthermore we define the perturbed operators

$$\mathcal{L}_t^\varepsilon = \mathcal{L}_t^H + \varepsilon^2 \hat{\mathcal{G}}_t, \tag{4.13}$$

where $\hat{\mathcal{G}}_t X = (\hat{\mathcal{G}}_t X(1), \ldots, \hat{\mathcal{G}}_t(N))$ with

$$\hat{\mathcal{G}}_t X(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i) A_k(t,j) X(j) A_k^T(t,j)$$

for all $X = (X(1), \ldots, X(N)) \in \mathcal{S}_n^N$.

If $q \in (q_1, 1)$ one shows in a standard way using a discrete-time version of the Belman–Gronwall lemma that there exists $\varepsilon_0 > 0$ such that

$$\|T^\varepsilon(t,s)\|_\xi \leq \beta q^{t-s}, \tag{4.14}$$

for all $t \geq s \geq 0, 0 < \varepsilon \leq \varepsilon_0, T^\varepsilon(t,s)$ being the causal linear evolution operator defined on $\mathcal{S}_n^N$ by the sequence $(\mathcal{L}_t^\varepsilon)_{t\geq 0}$.

Let $\varepsilon \in (0, \varepsilon_0)$ be fixed and $\{Z_t\}_{t\geq t_0}$ be the solution of the problem with given initial condition:

$$Z_{t+1} = \mathcal{L}_t^\varepsilon Z_t + \frac{1}{\varepsilon^2}\Psi_t, \qquad Z_{t_0} = H, \tag{4.15}$$

where $\Psi_t = (\Psi_t(1), \dots, \Psi_t(N))$,

$$\Psi_t(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)H(t,j)C_k(t,j)X_t(j)C_k^T(t,j)H(t,j)^T. \tag{4.16}$$

If we set $\tilde{Z}_t = Z_t - X_t$ then by direct calculations based on (4.12) and (4.15) one obtains that $\tilde{Z}_t$ solves

$$\tilde{Z}_{t+1} = \mathcal{L}_t^\varepsilon \tilde{Z}_t + \tilde{\Psi}_t, \qquad \tilde{Z}_{t_0} = 0, \tag{4.17}$$

where $\tilde{\Psi}_t = (\tilde{\Psi}_t(1), \dots, \tilde{\Psi}_t(N))$,

$$\begin{aligned}\tilde{\Psi}_t(i) = {} & \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)\left(\varepsilon A_k(t,j) + \frac{1}{\varepsilon}H(t,j)C_k(t,j)\right) \\ & \times X_t(j)\left(\varepsilon A_k(t,j) + \frac{1}{\varepsilon}H(t,j)C_k(t,j)\right)^T \\ & + \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)H(t,j)C_k(t,j)X_t(j)C_k^T(t,j)H^T(t,j).\end{aligned}$$

The solution of (4.9) is in $\mathcal{S}_n^{N+}$, thus it follows that $\tilde{\Psi}_t(i) \geq 0$ for all $t \geq t_0$ and $i \in \mathcal{D}$; that is, $\tilde{\Psi}_t \in \mathcal{S}_n^{N+}$.

Because $\mathcal{L}_t^\varepsilon$ are positive operators, then based on (4.17) one obtains inductively that $\tilde{Z}_t \geq 0$ for all $t \geq t_0$, which is equivalent to $X_t \leq Z_t$ for all $t \geq t_0$.

The last inequality allows us to write

$$|X_t|_\xi \leq |Z_t|_\xi, \quad t \geq t_0. \tag{4.18}$$

From (4.15) we obtain the representation formula

$$Z_t = T^\varepsilon(t,t_0)H + \frac{1}{\varepsilon}\sum_{l=t_0}^{t-1} T^\varepsilon(t,l+1)\Psi_l, \qquad t \geq t_0 + 1.$$

Based on (4.14) we get:

$$|Z_t|_\xi \leq \beta q^{t-t_0}|H|_\xi + \frac{\beta}{\varepsilon^2}\sum_{l=t_0}^{t-1} q^{t-l-1}|\Psi_l|_\xi. \tag{4.19}$$

Taking into account the definition of the norm $|\cdot|_\xi$ on $\mathcal{S}_n^N$ one obtains (see also (4.16)):

$$|\Psi_l|_\xi = \max_{i\in\mathcal{D}} |\Psi_l(i)| \leq \max_{i\in\mathcal{D}} \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(j,i)|H(l,j)C_k(l,j)X_l(j)C_k^T(l,j)H^T(l,j)|,$$

which leads to

$$|\Psi_l|_\xi \le \rho_1\rho_2 \sum_{k=0}^{r}\sum_{j=1}^{N} |C_k(l,j)X_l(j)C_k(l,j)^T|, \tag{4.20}$$

where $\rho_1 \ge p_t(j,i), \rho_2 \ge |H(l,j)|^2$ for all $l \ge 0, i,j \in \mathcal{D}$. Because $|C_k(l,j)X_l(j)$ $C_k^T(l,j)| = \lambda_{\max}[C_k(l,j)X_l(j)C_k^T(l,j)]$ we may write

$$|\Psi_l|_\xi \le \rho_1\rho_2 \sum_{k=0}^{r}\sum_{j=1}^{N} Tr(C_k(l,j)X_l(j)C_k^T(l,j))$$

$$= \rho_1\rho_2 \sum_{j=1}^{N} Tr \sum_{k=0}^{r}(C_k^T(l,j)C_k(l,j)X_l(j)).$$

In view of the definition of the inner product on $\mathcal{S}_n^N$ we get:

$$|\Psi_l|_\xi \le \rho_1\rho_2 \langle \tilde{C}(l), X_l\rangle. \tag{4.21}$$

Based on equation (4.8) verified by $\{\tilde{Y}_l\}_{l\ge 0}$ we may write

$$\langle \tilde{C}_l, X_l\rangle = \langle \tilde{Y}_l, X_l\rangle - \langle \mathcal{L}_l^* \tilde{Y}_{l+1}, X_l\rangle = \langle \tilde{Y}_l, X_l\rangle - \langle \tilde{Y}_{l+1}, X_{l+1}\rangle. \tag{4.22}$$

Because $(\tilde{Y}_l)_{l\ge 0}$ is a bounded sequence and $\langle \tilde{Y}_i, X_i\rangle \ge 0$ for arbitrary $i \ge 0$, we obtain from (4.21) and (4.22) that

$$\sum_{l=t_0}^{t_1} |\Psi_l|_\xi \le \rho_3 |H|_\xi, \qquad \forall\, t_1 > t_0 \tag{4.23}$$

with $\rho_3 > 0$ independent of t_0, t_1 and H. Using (4.19) we may write:

$$\sum_{t=t_0}^{t_2} |Z_t|_\xi \le \left(1 + \beta \sum_{t=t_0+1}^{t_2} q^{t-t_0}\right)|H|_\xi + \frac{\beta}{\varepsilon^2} \sum_{t=t_0+1}^{t_2}\sum_{l=t_0}^{t-1} q^{t-l-1}|\Psi_l|_\xi.$$

Changing the order of summation and taking into account (4.23) we obtain finally

$$\sum_{t=t_0}^{t_2} |Z_t|_\xi \le \gamma |H|_\xi, \qquad \forall\, t_2 > t_0$$

and

$$\gamma = 1 + \frac{\beta q}{1-q} + \frac{\beta\varepsilon^{-2}\rho_3}{1-q}$$

does not depend upon t_0, t_2, H. Taking the limit for $t_2 \to \infty$ one gets:

$$\sum_{t=t_0}^{\infty} |Z_t|_\xi \le \gamma |H|_\xi.$$

Invoking (4.18) we conclude that (4.11) is valid. Taking $H = J = (I_n, I_n, \ldots, I_n)$, (4.11) becomes $\sum_{t=t_0}^{\infty} |T(t,t_0)J|_\xi \leq \gamma$ for all $t_0 \geq 0$, or equivalently

$$\sum_{t=t_0}^{\infty} \|T(t,t_0)\|_\xi \leq \gamma. \tag{4.24}$$

Based on (2.13), (4.24) leads to $\sum_{t=t_0}^{\infty} \|T^*(t,t_0)\|_\xi \leq \gamma_1$ for all $t_0 \geq 0$, $\gamma_1 > 0$ being independent of t_0.

Because $T^*(t,t_0)J \leq \|T^*(t,t_0)\|_\xi J$ one obtains $0 \leq \sum_{t=t_0}^{\infty} T^*(t,t_0)J \leq \delta J$.

Applying now the implication $(iii) \to (i)$ of Theorem 2.4 we conclude that the sequence $\mathcal{L}_t$ generates an exponentially stable evolution and thus the proof ends. □

The result proved in the above theorem may be viewed as an alternative of the equivalence $(i) \leftrightarrow (vi)$ of Theorem 2.4 for the case when the forced term of the corresponding equation (2.30) is not uniform positive.

Remark 4.2 In Theorem 4.1 we do not assume that P_t are stochastic matrices. In this way, we may apply the result proved in Theorem 4.1 for Lyapunov-type operators $\mathcal{L}_t$ constructed with P_t^T instead of P_t. The transposed P_t is not always a stochastic matrix even if P_t is a stochastic matrix.

4.2 Stochastic observability

Let us consider discrete-time linear stochastic systems of the form

$$x(t+1) = [A_0(t,\eta_t) + \sum_{k=1}^{r} A_k(t,\eta_t)w_k(t)]x(t) \tag{4.25}$$

$$y(t) = C(t,\eta_t)x(t),$$

$t \in \mathbf{Z}_+$, where $\{\eta_t\}_{t\geq 0}$ and $\{w(t)\}_{t\geq 0}$ are as in the case of system (4.1), $x(t) \in \mathbf{R}^n$, $y(t) \in \mathbf{R}^p$.

If $A_k(t,i) = 0, 1 \leq k \leq r, i \in \mathcal{D}, t \geq 0$ (4.25) reduces to a system with Markovian perturbations:

$$x(t+1) = A_0(t,\eta_t)x(t) \tag{4.26}$$

$$y(t) = C(t,\eta_t)x(t), \qquad t \in \mathbf{Z}_+.$$

In the particular case $N = 1$, system (4.25) takes the form

$$x(t+1) = [A_0(t) + \sum_{k=1}^{r} A_k(t)w_k(t)]x(t) \tag{4.27}$$

$$y(t) = C(t)x(t),$$

$t \geq 0$, where $A_k(t) = A_k(t,1)$, $0 \leq k \leq r, C(t) = C(t,1)$.

The time-invariant version of the systems (4.25), (4.26), and (4.27), respectively, is described by

$$x(t+1) = \left[A_0(\eta_t) + \sum_{k=1}^{r} A_k(\eta_t)w_k(t)\right] x(t) \tag{4.28}$$

$$y(t) = C(\eta_t)x(t)$$

$$x(t+1) = A_0(\eta_t)x(t) \tag{4.29}$$

$$y(t) = C(\eta_t)x(t)$$

and

$$x(t+1) = \left[A_0 + \sum_{k=1}^{r} A_k w_k(t)\right] x(t) \tag{4.30}$$

$$y(t) = Cx(t).$$

In the case of systems (4.28), (4.29) $\{\eta_t\}_{t\geq 0}$ is an homogeneous Markov chain. It must be remarked that the coefficients of the systems (4.28)–(4.30) are not constants as in the time-invariant case of a deterministic framework. They are still time-dependent via the stochastic processes $w(t)$ and η_t.

Setting C for the sequences $\{C(t,i)\}_{t\geq 0}, i \in \mathcal{D}$ we introduce the following definition.

Definition 4.4 *We say that the system* (4.25) *is* stochastic uniformly observable *or equivalently the triple* $(C, \mathbf{A}, \mathbf{P})$ *is uniformly observable if there exist* $\tau_0 \in Z_+$ *and* $\gamma > 0$ *such that*

$$\sum_{t=s}^{s+\tau_0} T^*(t,s)\tilde{C}(t) \geq \gamma J \tag{4.31}$$

for all $s \geq 0$ *where* $\tilde{C}(t) = (\tilde{C}(t,1), \ldots, \tilde{C}(t,N))$ *with* $\tilde{C}(t,i) = C^T(t,i)C(t,i)$ *and* $J = (I_n, I_n, \ldots, I_n) \in \mathcal{S}_n^N$, $T(t,s)$ *being the linear evolution operator defined by the sequence of Lyapunov-type operators* $\{\mathcal{L}_t\}_{t\geq 0}$ *introduced in* (2.58)*.*

In the time-invariant case we say that the system (4.28) *is stochastic observable or that* $(C, \mathbf{A}, P)$ *is observable if* (4.31) *is fulfilled.*

Remark 4.3

(a) In the case of the discrete-time linear stochastic systems affected only by Markov perturbations we say that the system (4.26) is stochastic uniformly observable or equivalently the triple (C, A_0, P) is uniformly observable if (4.31) is satisfied with $T^0(t,s)$ instead of $T(t,s)$.

(b) In the case of discrete-time linear stochastic systems subject to independent random perturbations we say that the system (4.27) is stochastic uniformly observable or equivalently that the pair $(C, \mathbf{A})$ is uniformly observable if

$$\sum_{t=s}^{s+\tau_0} \hat{T}^*(t,s)(C^T(t)C(t)) \geq \gamma I_n \tag{4.32}$$

for all $s \geq 0$, $\hat{T}(t,s)$ being the linear evolution operator on $\mathcal{S}_n$ defined by the sequence $\{\hat{\mathcal{L}}_t\}_{t\geq 0}$, $\hat{\mathcal{L}}_t$ being defined by (2.88).

(c) If we take into account the expression of $T^*(t,s)$ for $N = 1$ it is easy to see that the condition (4.31) is a natural extension of the definition of uniform observability in a deterministic framework (see [68] and [81]).

(d) We remark that the concept of stochastic observability introduced above is completely characterized by the sequences A_k, C, and $\mathbf{P}$ where the matrices P_t have only the property that $p_t(i,j) \geq 0, i, j \in \mathcal{D}$.
The fact that P_t are stochastic matrices, that is, $\sum_j p_t(i,j) = 1$, is used only in Theorem 3.1 and in the results based on this theorem, namely Theorem 4.4 below.

(e) Based on Remark 2.5(b) one obtains that in the time-invariant case the triple $(C, \mathbf{A}, P)$ is observable if and only if there exists $\tau_0 \in Z_+$ such that

$$\sum_{t=0}^{\tau_0} (\mathcal{L}^*)^t \tilde{C} > 0, \tag{4.33}$$

where $\tilde{C} = (\tilde{C}(1), \ldots, \tilde{C}(N)), \tilde{C}(i) = C^T(i)C(i), i \in \mathcal{D}$.

In the case of the system (4.30) the pair $(C, \mathbf{A})$ is observable if and only if there exists $\tau_0 \in Z_+$ such that

$$\sum_{t=0}^{\tau_0} (\hat{\mathcal{L}}^*)^t (C^T C) > 0. \tag{4.34}$$

For each $0 \leq k \leq r$ we set

$$\tilde{A}_k(t,i) = \sqrt{p_t(i,i)} A_k(t,i), \qquad t \geq 0, i \in \mathcal{D}. \tag{4.35}$$

Now we prove the following.

Theorem 4.2 *We have:*

(i) *If for each* $1 \leq i \leq N$ *the pair* $(C(\cdot,i), \tilde{A}_0(\cdot,i))$ *is uniformly observable (in the deterministic sense) then the system* (4.26) *is stochastic uniformly observable.*

(ii) *If the system* (4.26) *is stochastic uniformly observable then the system* (4.25) *is stochastic uniformly observable,* (*iii*) *If for each* $1 \leq i \leq N$ *the discrete-time linear stochastic system with independent random perturbations*

$$x(t+1) = \left(\tilde{A}_0(t,i) + \sum_{k=1}^{r} \tilde{A}_k(t,i)w_k(t)\right)x(t)$$

$$y(t) = C(t,i)x(t) \tag{4.36}$$

is stochastic uniformly observable then the system (4.25) *is stochastic uniformly observable.*

(*iv*) *If there exists* $k_0 \in \{0, \ldots, r\}$ *such that the pair* $(C(\cdot), A_{k_0}(\cdot))$ *is uniformly observable (in the deterministic sense) then the system* (4.27) *is stochastic uniformly observable.*

Proof. (*i*) If $(C(\cdot,i), \tilde{A}_0(\cdot,i))$ is uniformly observable then there exist $\tilde{\tau}_0 \in \mathbf{Z}_+, \tilde{\gamma} > 0$ (depending upon i) such that

$$\sum_{t=s}^{s+\tilde{\tau}_0} \tilde{\Phi}_i^T(t,s)C^T(t,i)C(t,i)\tilde{\Phi}_i(t,s) \geq \tilde{\gamma} I_n \tag{4.37}$$

for all $s \geq 0$ where $\tilde{\Phi}(t,s) = \Phi_{\tilde{A}}(t,s,i)$ defined by (2.97).

Consider the operators $L_t : \mathcal{S}_n^N \to \mathcal{S}_n^N$ defined by (2.96). From (4.37) we deduce that there exist $\tilde{\tau}_0 \in Z_+, \tilde{\gamma} > 0$ such that

$$\sum_{t=s}^{s+\tilde{\tau}_0} T_L^*(t,s)\tilde{C}(t) \geq \tilde{\gamma} J \tag{4.38}$$

for all $s \geq 0$.

Applying Corollary 2.10 one gets that $T^0(t,s) \geq T_L(t,s)$ for all $t \geq s \geq 0$. This inequality together with (4.38) leads to

$$\sum_{t=s}^{s+\tilde{\tau}_0} \left(T^0\right)^*(t,s)\tilde{C}(t) \geq \tilde{\gamma} J \tag{4.39}$$

and thus (*i*) is fulfilled.

(*ii*) If the triple (C, A_0, P) is uniformly observable then there exist $\tau_0 \in Z_+$ and $\gamma > 0$ such that (4.39) is fulfilled. On the other hand from (2.91) we have that $T(t,s) \geq T^0(t,s)$ for all $t \geq s \geq 0$. This last inequality together with (4.39) shows that (4.31) is fulfilled and thus (*ii*) is proved.

To prove (*iii*) we consider the operators $\mathcal{L}_{it}$ and $T_i(t,s)$ defined in Section 2.5.3. If the system (4.36) is uniformly observable then there exist $\tau_i \in Z_+, \gamma_i > 0$ such that

$$\sum_{t=s}^{s+\tau_i} T_i^*(t,s)(C^T(t,i)C(t,i)) > \gamma_i I_n \tag{4.40}$$

for all $s \geq 0$.

Taking into account (4.40) and (2.95) and by using Proposition 2.5 we obtain that (4.31) is verified and thus the proof of (*iii*) is complete.

It remains to prove (*iv*). To this end we define the operators

$$\bar{\mathcal{L}}_{k_0 t} Y = A_{k_0}(t) Y A_{k_0}^T(t) \tag{4.41}$$

for all $Y \in \mathcal{S}_n$.

Let $\bar{T}_{k_0}(t,s)$ be the linear evolution operator on $\mathcal{S}_n$ defined by the sequence $\{\bar{\mathcal{L}}_{k_0 t}\}_{t \geq 0}$. If the pair $(C(\cdot), A_{k_0}(\cdot))$ is uniformly observable it follows that there exist $\tau_0 \geq 0, \gamma > 0$ such that

$$\sum_{t=s}^{s+\tau_0} \bar{T}_{k_0}^*(t,s)(C^T(t)C(t)) \geq \gamma I_n \tag{4.42}$$

for all $s \geq 0$.

Because $\hat{\mathcal{L}}_t \geq \bar{\mathcal{L}}_{k_0,t}$ for all $t \geq 0$ we deduce that $\hat{T}(t,s) \geq \bar{T}_{k_0}(t,s)$ for all $t \geq s \geq 0$. Using again Proposition 2.5 we obtain that (4.32) is verified and thus the proof ends. □

From (*i*) and (*iv*) one obtains the following result concerning the systems (4.29) and (4.30).

Corollary 4.4

(*i*) *If for each* $i \in \mathcal{D}, p(i,i) > 0$ *and the pair* $(C(i), A_0(i))$ *is observable (in the deterministic sense) then the system* (4.29) *is stochastic observable.*

(*ii*) *If there exists* $0 \leq k_0 \leq r$ *such that the pair* (C, A_{k_0}) *is observable (in the deterministic sense) then the system* (4.30) *is stochastic observable.*

Remark 4.4

(*a*) Theorem 4.2 and Corollary 4.4 provide a set of sufficient conditions for stochastic observability. We show later by some examples that the converse implications are not always true. This means that these sufficient conditions are not also necessary conditions for stochastic observability.

(*b*) In [87, 88] the following concept of stochastic observability is introduced. The system (2.6) is stochastic observable if for each $i \in \mathcal{D}, p(i,i) > 0$ and the pair $(C(i), A_0(i))$ is observable. The result of Corollary 4.4(i) shows that the concept of stochastic observability introduced here is more general then the one defined in the above-cited paper.

By Remark 3.2(*b*) we have:

$$\hat{T}^*(t,t_0)Y = E[\Phi^T(t,t_0) Y \Phi(t,t_0)], \tag{4.43}$$

$t \geq t_0 \geq 0, Y \in \mathcal{S}_n$.

Combining (4.32) with (4.43) we obtain the following directly.

Theorem 4.3 *The following are equivalent.*

(i) *The pair* $(C, \mathbf{A})$ *is uniformly observable.*
(ii) *There exist* $\tau_0 \in Z_+, \gamma > 0$, *such that*

$$\sum_{t=s}^{s+\tau_0} E[\Phi^T(t,s)C^T(t)C(t)\Phi(t,s)] \geq \gamma I_n \tag{4.44}$$

for all $s \in Z_+$.

In the case when the system is subject to Markov perturbations an additional assumption is required in order to obtain some significant results.

$\mathbf{H_3}$. Assume that for each $t \geq 0$ the stochastic matrices P_t is nondegenerate.

The next result follows directly from Definition 4.3 and Theorem 3.1.

Theorem 4.4 *Under the assumption* $\mathbf{H_3}$ *the following are equivalent.*

(i) *The triple* $(C, \mathbf{A}, P)$ *is uniformly observable.*
(ii) *There exist* $\tau_0 \in Z_+$ *and* $\gamma > 0$ *such that for an arbitrary sequence of independent random vectors* $\{w(t)\}_{t\geq 0}$ *and for an arbitrary Markov chain* $(\{\eta_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *which verify* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *we have:*

$$\sum_{t=s}^{s+\tau_0} E[\Phi^T(t,s)C^T(t,\eta_t)C(t,\eta_t)\Phi(t,s)|\eta_s = i] \geq \gamma I_n$$

for all $s \geq 0, i \in \mathcal{D}_s$.
(iii) *There exist a sequence of independent random vectors* $\{\tilde{w}(t)\}_{t\geq 0}$ *and a Markov chain* $(\{\tilde{\eta}_t\}_{t\geq 0}, \{P_t\}_{t\geq 0}, \mathcal{D})$ *with* $\mathcal{P}\{\tilde{\eta}_0 = i\} > 0$ *for all* $i \in \mathcal{D}$, *that verify* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *with the property that there exist* $\tau_0 \geq 0, \gamma > 0$ *such that*

$$\sum_{t=s}^{s+\tau_0} E[\Phi^T(t,s)C^T(t,\tilde{\eta}_t)C(t,\tilde{\eta}_t)\Phi(t,s)|\tilde{\eta}_s = i] \geq \gamma I_n$$

for all $s \geq 0, i \in \mathcal{D}$.

Proof. The implication $(i) \to (ii)$ follows from (4.31) and Theorem 3.1(i) and the implication $(ii) \to (iii)$ is obvious. The implication $(iii) \to (i)$ follows by using assumption $\mathbf{H_3}$, Theorem 3.1, and Proposition 1.6. □

In the time-invariant case we do not need the assumption $\mathbf{H_3}$ and we prove the following.

Theorem 4.5 *The following are equivalent.*

(i) *The system* (4.28) *is stochastic observable.*

(*ii*) *There exist* $\tau_0 \in Z_+, \gamma > 0$ *such that for an arbitrary sequence of independent random vectors* $\{w(t)\}_{t\geq 0}$ *and for an arbitrary homogeneous Markov chain* $(\{\eta_t\}_{t\geq 0}, P, \mathcal{D})$ *which verify* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *we have:*

$$\sum_{t=0}^{\tau_0} E[|Y(t,x)|^2 | \eta_0 = i] \geq \gamma |x|^2$$

for all $i \in \mathcal{D}_0, 0 \neq x \in \mathbf{R}^n$ *where* $Y(t,x) = C(\eta_t)\Phi(t,0)x$.

(*iii*) *There exist a sequence of independent random vectors* $\{\tilde{w}(t)\}_{t\geq 0}$ *and an homogeneous Markov chain* $(\{\tilde{\eta}_t\}_{t\geq 0}, P, \mathcal{D})$ *with* $\mathcal{P}\{\tilde{\eta}_0 = i\} > 0$ *for all* $i \in \mathcal{D}$ *verifying* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *with the property that there exist* $\tau_0 \in Z_+, \gamma > 0$ *such that*

$$\sum_{t=0}^{\tau_0} E[|\tilde{Y}(t,x)|^2 | \tilde{\eta}_0 = i] \geq \gamma |x|^2$$

for all $x \in \mathbf{R}^n, x \neq 0, i \in \mathcal{D}$, *where* $\tilde{Y}(t,x) = C(\tilde{\eta}_t)\Phi(t,0)x$.

(*iv*) *There exists* $\tau_0 \in Z_+$ *such that for an arbitrary sequence of independent random vectors* $\{w(t)\}_{t\geq 0}$ *and for an arbitrary homogeneous Markov chain* $(\{\eta_t\}_{t\geq 0}, P, \mathcal{D})$ *which verify* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *there exists* $\delta > 0$ *such that we have*

$$\sum_{t=0}^{\tau_0} E[\Phi^T(t,0)C^T(\eta_t)C(\eta_t)\Phi(t,0)] \geq \delta I_n. \tag{4.45}$$

(*v*) *For each* $i \in \mathcal{D}$ *there exist* $\tau_i \in Z_+, \gamma_i > 0$ *such that*

$$\sum_{t=0}^{\tau_i} E[\Phi_i(t,0)C^T(\eta_t^i)C(\eta_t^i)\Phi_i(t,0)] \geq \gamma_i I_n,$$

where $\Phi_i(t,s)$ *is the fundamental matrix solution of the system* (4.28) *for a pair* $(\{w(t)\}_{t\geq 0}, \{\eta_t^i\}_{t\geq 0})$, *where* $\{w(t)\}_{t\geq 0}$ *is a sequence of independent random vectors and* $\{\eta_t^i\}_{t\geq 0}$ *is an homogeneous Markov chain with* $\mathcal{P}\{\eta_0^i = i\} = 1$ *verifying* $\mathbf{H_1}$ *and* $\mathbf{H_2}$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$ follow from Theorem 3.1 and Remark 4.3(e). We now prove $(ii) \Rightarrow (iv)$.

We write:

$$\sum_{t=0}^{\tau_0} E[\Phi^T(t,0)C^T(\eta_t)C(\eta_t)\Phi(t,0)]$$

$$= \sum_{i\in\mathcal{D}_0} \sum_{t=0}^{\tau_0} \pi_0(i) E[\Phi^T(t,0)C^T(\eta_t)\Phi(t,0) | \eta_0 = i].$$

If (ii) holds we obtain that

$$\sum_{t=0}^{\tau_0} E[\Phi^T(t,0)C^T(\eta_t)C(\eta_t)\Phi(t,0)] \geq \delta I_n,$$

where $\delta = \sum_{i \in \mathcal{D}_0} \pi_0(i)\gamma$ and we see that δ depends upon the Markov chain. The implication $(iv) \Rightarrow (v)$ is obvious.

It remains to prove $(v) \rightarrow (i)$. To this end, let $\{w(t)\}_{t\geq 0}$ be a sequence of independent random vectors and $(\{\eta_t^i\}_{t\geq 0}, P, \mathcal{D})$ an homogeneous Markov chain with $\mathcal{P}\{\eta_0^i = i\} = 1$ that verify $\mathbf{H_1}$ and $\mathbf{H_2}$. We have

$$E[\Phi^T(t,0)C^T(\eta_t^i)C(\eta_t^i)\Phi(t,0)] = E[\Phi^T(t,0)C^T(\eta_t^i)C(\eta_t^i)\Phi(t,0)|\eta_0^i = i].$$

Applying Theorem 3.1 one obtains that

$$E[\Phi^T(t,0)C^T(\eta_t^i)C(\eta_t^i)\Phi(t,0)] = [T^*(t,0)\tilde{C}](i).$$

If (v) holds then for each $i \in \mathcal{D}$ we have:

$$\sum_{t=0}^{\tau_i} (T^*(t,0)\tilde{C})(i) \geq \gamma_i I_n \tag{4.46}$$

for all $i \in \mathcal{D}$.

Taking $\tau_0 = \max_i \tau_i, \gamma = \min_i \gamma_i$, and taking into account Remark 2.5(b) we obtain from (4.46) that $\sum_{t=0}^{\tau_0} (\mathcal{L}^*)^t \tilde{C} > \gamma J$ which shows that the system (4.28) is stochastic observable and thus the proof is complete. □

Remark 4.5 If (4.45) is fulfilled only for a pair $(\{w(t)\}_{t\geq 0}, \{\eta_t\}_{t\geq 0})$ that verifies $\mathbf{H_1}$ and $\mathbf{H_2}$ it is not sure that the corresponding system (4.28) is stochastic observable. This may be illustrated by the following particular case of the system (4.28) with $n = 2, N = 2, p = 1$,

$$A_0(1) = A_0(2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$A_k(i) = 0, k \geq 1, i \in \{1,2\}, C(1) = (1 \quad 0), C(2) = (0 \quad 1), \{\eta_t\}_{t\geq 0}$ is a homogeneous Markov chain with two states and the probability transition matrix

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the initial distribution $\mathcal{P}\{\eta_0 = 1\} = \mathcal{P}\{\eta_0 = 2\} = \frac{1}{2}$. It is shown later (see Example 4.2) that the triple $(C, \mathbf{A}_0, P)$ is not stochastic observable. On the other hand we have

$$E[C^T(\eta_0)C(\eta_0)] = \frac{1}{2}C^T(1)C(1) + \frac{1}{2}C^T(2)C(2)$$

$$\frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}I_2.$$

This shows that in this particular case (4.45) is fulfilled with $\tau_0 = 0$ and $\delta = \frac{1}{2}$.

In the time-invariant case we also prove the following result which provides some useful necessary and sufficient conditions for stochastic observability.

Theorem 4.6 *The following are equivalent.*

(*i*) *The system* (4.28) *is stochastic observable.*
(*ii*) *There exists* $\tau_0 \in Z_+$ *such that*

$$\sum_{t=0}^{\tau_0} (\mathcal{L}^*)^t \tilde{C} > 0.$$

(*iii*) *There exists* $\tau_0 \geq 1$ *such that* $\tilde{K}_{\tau_0}(i) > 0$ *for all* $i \in \mathcal{D}$*, where* $t \to \tilde{K}_t(i), i \in \mathcal{D}$ *verifies the system of affine forward equations:*

$$K_{t+1}(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) A_k^T(i) K_t(j) A_k(i) + C^T(i)C(i), \tag{4.47}$$

$\tilde{K}_0(i) = 0, i \in \mathcal{D}$.

(*iv*) $\lim_{t\to\infty} \det \tilde{K}_t(i) > 0$ *for all* $i \in \mathcal{D}$*, where* $\tilde{K}_t(i)$ *is the solution of* (4.47).

Proof. $(i) \iff (ii)$ follows immediately from Remark 4.3(e). We now prove the equivalence $(ii) \iff (iii)$. To this end, we define $\tilde{K}_t = (\tilde{K}_t(1), \ldots, \tilde{K}_t(N))$ by $\tilde{K}_0 = (0, \ldots, 0)$, $\tilde{K}_t = \sum_{s=0}^{t-1} (\mathcal{L}^*)^s \tilde{C}$ if $t \geq 1$. We have

$$\tilde{K}_{t+1} = \sum_{s=0}^{t} (\mathcal{L}^*)^s \tilde{C} = \mathcal{L}^* \sum_{s=0}^{t-1} (\mathcal{L}^*)^s \tilde{C} + \tilde{C} \tag{4.48}$$

hence $\tilde{K}_{t+1} = \mathcal{L}^* \tilde{K}_t + \tilde{C}$.

The ith component of this equation is

$$\tilde{K}_{t+1}(i) = (\mathcal{L}^* \tilde{K}_t)(i) + C^T(i)C(i). \tag{4.49}$$

If we consider the formula for $\mathcal{L}^*$ we see that (4.49) is just (4.47).

If (ii) is valid then from (4.48) we deduce that there exists $\tau_0 \geq 1$ such that $\tilde{K}_{\tau_0} > 0$ and (iii) is fulfilled. The implication $(iii) \to (ii)$ follows in the same way.

To prove $(iii) \to (iv)$ we rewrite that from (4.48) it follows that $0 \leq \tilde{K}_t(i) \leq \tilde{K}_{t+1}(i)$ for all $t \geq 0, i \in \mathcal{D}$. Applying Theorem 3, Chapter 7 in [4] we obtain that $\det \tilde{K}_t(i) \leq \det \tilde{K}_{t+1}(i)$ for all $t \geq 0, i \in \mathcal{D}$. Hence the sequences $\{\det \tilde{K}_t(i)\}_{t\geq 0}$ are monotone increasing. From (iii) it follows that there exists $\tau_0 \geq 1$ such that $\det \tilde{K}_{\tau_0}(i) > 0$. Because $\lim_{t\to\infty} \det \tilde{K}_t(i) \geq \det \tilde{K}_{\tau_0}(i)$ we conclude that (iv) holds.

Finally we prove $(iv) \to (iii)$. From (4.47) one sees that $\tilde{K}_t(i) = \tilde{K}_t^T(i) \geq 0$. If (iv) holds then for each $i \in \mathcal{D}$ there exist $\tau_i \geq 1$ such that $\det \tilde{K}_{\tau_i}(i) > 0$ which shows that $\tilde{K}_{\tau_i}(i) > 0$. Taking $\tau_0 = \max_i \tau_i$ one obtains that (iii) holds and thus the proof is complete. □

With the same proof one obtains the following.

Theorem 4.7 *The following are equivalent.*

(*i*) *The system* (4.30) *is stochastic observable.*
(*ii*) *There exists* $\tau_0 \in Z_+$ *such that*

$$\sum_{s=0}^{\tau_0} (\hat{\mathcal{L}}^*)^s (C^T C) > 0.$$

(*iii*) *There exists* $\tau_0 \geq 1$ *such that* $\tilde{K}_{\tau_0} > 0$, *where* $t \to \tilde{K}_t$ *is the solution of the following problem with initial value,*

$$\tilde{K}_{t+1} = \sum_{k=0}^{r} A_k^T \tilde{K}_t A_k + C^T C$$
$$\tilde{K}_0 = 0. \qquad (4.50)$$

(*iv*) $\lim_{t\to\infty} \det \tilde{K}_t > 0$, *where* $\tilde{K}_t$ *is the solution of the problem* (4.23).

Now we show that in a particular case the converse implication from Corollary 4.3(*ii*) is true.

Proposition 4.1 *The following assertions are equivalent.*

(*i*) *The system* (4.30) *in the particular case* $n = 2, r = 1, p = 1$ *is stochastic observable.*
(*ii*) *There exists* $k_0 \in \{0, 1\}$ *such that the pair* (C, A_{k_0}) *is observable in the deterministic sense.*

Proof. (*ii*) $\to$ (*i*) follows from Corollary 4.3(*ii*). It remains to prove that (*i*) $\to$ (*ii*). Let

$$A_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad C = (c_1 \quad c_2).$$

Assume that the pairs $(C, A_0), (C, A_1)$ are not observable and we prove that in this case $\det \tilde{K}_t = 0$ for all $t \geq 0$, where $\tilde{K}_t$ is the solution of (4.50) for the considered case. The fact that $(C, A_0), (C, A_1)$ are not observable is equivalent to the following equalities,

$$bc_1^2 - cc_2^2 + c_1 c_2 (d - a) = 0 \qquad (4.51)$$

$$\beta c_1^2 - \gamma c_2^2 + c_1 c_2 (\delta - \alpha) = 0. \qquad (4.52)$$

Let

$$\tilde{K}_t = \begin{pmatrix} x_t & y_t \\ y_t & z_t \end{pmatrix}.$$

The equation (4.50) in this particular case becomes:

$$\begin{aligned} x_{t+1} &= (a^2+\alpha^2)x_t + 2(ac+\alpha\gamma)y_t + (c^2+\gamma^2)z_t + c_1^2 \\ y_{t+1} &= (ab+\alpha\beta)x_t + (ad+bc+\alpha\delta+\beta\gamma)y_t + (cd+\gamma\delta)z_t + c_1c_2 \\ z_{t+1} &= (b^2+\beta^2)x_t + 2(bd+\beta\delta)y_t + (d^2+\delta^2)z_t + c_2^2, \end{aligned} \tag{4.53}$$

$x_0 = y_0 = z_0 = 0$. If $c_1 = 0$ or $c_2 = 0$ from (4.51)–(4.53) we obtain that $x_t = y_t = 0, t \geq 0$, or $y_t = z_t = 0$ for all $t \geq 0$ hence $\det\hat{K}_t = 0, t \geq 0$.

Assume that $c_1 \neq 0$ and $c_2 \neq 0$. Let $\tilde{y}_{t+1} = \{(ab+\alpha\beta)(c_1/c_2) + 2(ad + bc + \alpha\delta + \beta\gamma) + (cd+\gamma\delta)(c_2/c_1)\}\tilde{y}_t + c_1c_2$, $\tilde{y}_0 = 0$ and let $\tilde{x}_t = (c_1/c_2)\tilde{y}_t$, $\tilde{z}_t = (c_2/c_1)\tilde{y}_t$. Using (4.51) and (4.52) it can be verified by direct computation that $\tilde{x}_t, \tilde{y}_t, \tilde{z}_t$ verify (4.53). Because $\tilde{x}_0 = \tilde{y}_0 = \tilde{z}_0 = 0$ one obtains from uniqueness that $x_t = \tilde{x}_t, y_t = \tilde{y}_t, z_t = \tilde{z}_t$. Hence $\det\tilde{K}_t = 0$ for all $t \geq 0$ and the proof is complete. □

4.3 Some illustrative examples

The next example shows that the converse implication from Corollary 4.3(*ii*) is not true if $n \geq 3$.

Example 4.1 Consider the system (4.30) in the particular case $n = 3$, $r = 1, p = 1$, described by

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad C = (1 \quad 1 \quad 1).$$

By direct calculations one verifies that $(C, A_0), (C, A_1)$ are not observable but the corresponding system (4.30) is stochastic observable. This is due to the fact that

$$\tilde{K}_2 = \begin{pmatrix} 11 & 12 & 0 \\ 12 & 14 & 2 \\ 0 & 2 & 6 \end{pmatrix} > 0.$$

The next example shows that the condition $p(i,i) > 0, i \in \mathcal{D}$ is essential for Corollary 4.3(*i*) in order for the assertion of that corollary to be valid.

Example 4.2 Consider the system (4.29) in the particular case $n = 2$, $N = 2, p = 1$,

$$A_0(1) = A_0(2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad C(1) = (1 \quad 0),$$

$$C(2) = (0 \quad 1), \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The pairs $(C(1), A_0(1)), (C(2), A_0(2))$ are observable in the deterministic sense. However, the corresponding system (4.29) is not stochastic observable because by direct calculations one obtains that if

$$K_{t+1}(i) = A_0^*(i) \sum_{j=1}^{2} p(i,j) K_t(j) A_0(i) + C^*(i)C(i), \tag{4.54}$$

$t \geq 0, K_0(i) = 0$ it follows that

$$K_t(1) = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad K_t(2) = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}, \qquad t \geq 0.$$

Using Theorem 4.6 we conclude that the triple (C, A_0, P) is not observable.

The next example shows that it is possible that $(C(i), A_0(i))$ are not observable for each $i \in \mathcal{D}$ but the corresponding system (4.29) is stochastic observable.

Example 4.3 Consider the system (4.29) in the particular case $n = 2, N = 2, p = 1 A_0(1) = A_0(2) = \alpha I_2$,

$$C(1) = (1 \qquad 0), \qquad C(2) = (0 \qquad 1), \qquad P = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}, \qquad q \in (0,1).$$

It is easy to see that the pairs $(C(1), A_0(1))$ and $(C(2), A_0(2))$ are not observable. On the other hand if K_t solves (4.54) for this choice of the matrix coefficients then

$$K_2(1) = \begin{pmatrix} \alpha^2 q + 1 & 0 \\ 0 & \alpha^2(1-q) \end{pmatrix}, \qquad K_2(2) = \begin{pmatrix} \alpha^2(1-q) & 0 \\ 0 & \alpha^2 q + 1 \end{pmatrix}.$$

Hence $K_2(1) > 0, K_2(2) > 0$ for all $q \in (0,1)$ and $\alpha \in \mathbf{R}, \alpha \neq 0$ thus (C, A_0, P) is observable due to Theorem 4.6.

Example 4.4 Consider the following particular form of (4.29), namely $n = 1, N = 2$,

$$P = \begin{pmatrix} q & 1-q \\ q & 1-q \end{pmatrix},$$

$q \in (0,1), C(1) = 0, C(2) = 1, A_0(1) = a_1, A_0(2) = a_2, a_i \neq 0$. The equations (4.47) lead to

$$K_2(1) = a_1^2(1-q) > 0, \qquad K_2(2) = a_2^2(1-q) + 1 > 0$$

hence (C, A_0, P) is observable. Because in our case $p(i,j)$ do not depend upon i, by using Corollary 3.5 (C, A_0, P) is detectable iff there exist δ_j such that

$$\sum_{j=1}^{2}(a_j + \delta_j c_j)^2 p(i,j) - 1 < 0.$$

If $i = 1$ we have $(a_2 + \delta_2)^2(1-q) + a_1^2 q - 1 < 0$. Hence (C, A_0, P) is not detectable for $a_1^2 q - 1 \geq 0, q \in (0,1)$.

Example 4.5 Consider the following scalar version of equation (4.30),

$$\begin{aligned} x(t+1) &= a_0 x(t) + a_1 w(t) x(t), \\ y(t) &= c x(t), \qquad a_i \neq 0, \qquad c \neq 0. \end{aligned} \tag{4.55}$$

Applying Theorem 4.7(iii) one verifies that the system (4.55) is stochastic observable. On the other hand if we use Corollary 3.3 for $\mathcal{D} = \{1\}$ we obtain that the system (4.55) is not stochastic detectable if $1 - a_1^2 < 0$.

Example 4.6 Consider again Example 4.3. Hence the triple (C, A_0, P) is observable, but (C, A_0, P) cannot be detectable if $q\alpha^2 - 1 \geq 0$. Indeed from Corollary 3.2(ii) it follows that if (C, A_0, P) is detectable then the pair $(\sqrt{q}C(1), \sqrt{q}A(1))$ is detectable. But this pair is detectable iff $q\alpha^2 - 1 < 0$.

4.4 A generalization of the concept of uniform observability

In this section we show how we can extend the concept of uniform observability to an abstract framework of ordered Hilbert spaces.

Let $\mathcal{X}$ be a real Hilbert space. We assume that $\mathcal{X}$ is ordered by a order relation "$\leq$" induced by a solid, closed, pointed, selfdual, and convex cone $\mathcal{X}^+$. Let $\xi \in \operatorname{int} \mathcal{X}^+$ be fixed.

Definition 4.5 *Let $\{\mathcal{L}_t\}_{t \geq 0}$ be a sequence of linear bounded positive operators on $\mathcal{X}$ and $\{g_t\}_{t \geq 0} \subset \mathcal{X}^+$. We say that $\{g_t, \mathcal{L}_t\}$ is uniformly observable if there exist $\tau_0 \in Z_+$ and $\gamma > 0$ such that*

$$\sum_{t=s}^{s+\tau_0} T^*(t,s) g_t \geq \gamma \xi \tag{4.56}$$

for all $s \geq 0$.

Remark 4.6 If $\mathcal{X} = \mathcal{S}_n^N, \mathcal{X}^+ = \mathcal{S}_n^{N,+}$. The triple $(C, \mathbf{A}, \mathbf{P})$ is uniformly observable (in the sense of the Definition 4.4) if and only if $(g_t, \mathcal{L}_t)$ is uniformly observable, where

$$g_t = \tilde{C}(t) = (C^T(t,1)C(t,1), \ldots, C^T(t,N)C(t,N))$$

and $\mathcal{L}_t : \mathcal{S}_n^N \to \mathcal{S}_n^N$ is the Lyapunov operator defined by (2.58).

We need the next result.

Lemma 4.1 *Let* $\{\mathcal{L}_t\}_{t\geq 0}$ *be a bounded sequence of linear bounded and positive operators on* $\mathcal{X}$. *Also let* $\{h_t\}_{t\geq 0}$ *be a bounded and uniform positive sequence. If there exist* $\tau_0 \geq 1, q \in (0,1)$ *such that*

$$T^*(t+\tau_0, t)h_{t+\tau_0} \leq qh_t \tag{4.57}$$

for all $t \geq 0$, *then* $\{\mathcal{L}_t\}_{t\geq 0}$ *generates an exponentially stable evolution.*

Proof. It is obvious that

$$T^*(\tau, s)T^*(t, \tau) = T^*(t, s) \tag{4.58}$$

for all $t \geq \tau \geq s \geq 0$. Combining (4.58) and (4.57) we obtain inductively that

$$T^*(s + m\tau_0, s)h_{s+m\tau_0} \leq q^m h_s \tag{4.59}$$

for all $s \geq 0, m \geq 1$. Because $\{h_t\}_{t\geq 0}$ is a bounded and uniform positive sequence one obtains that

$$0 \leq T^*(s + m\tau_0, s)\xi \leq \beta_1 q^m \xi \tag{4.60}$$

for all $s \geq 0, m \geq 1$, for some $\beta_1 \geq 1$. From Proposition 2.5(ii) one obtains that (4.60) leads to

$$\|T^*(s + m\tau_0, s)\|_\xi \leq \beta_1 q^m \tag{4.61}$$

for all $s \geq 0, m \geq 1$. On the other hand from the boundedness of the sequence $\{\mathcal{L}_t\}_{t\geq 0}$ we deduce that there exist $\beta_2 \geq 1$ such that

$$\|T^*(t, s)\|_\xi \leq \beta_2^{(t-s)} \tag{4.62}$$

for all $t \geq s \geq 0$. If $t \geq s \geq 0$, we have $t - s = m\tau_0 + n_0$ with $0 \leq n_0 \leq \tau_0 - 1$. Invoking again (4.58) we obtain from (4.61) that

$$\|T^*(t, s)\|_\xi \leq q^m \beta_1 \|T^*(t, s + m\tau_0)\|_\xi.$$

Also, using (4.62) we deduce that

$$\|T^*(t, s)\|_\xi \leq \beta_1 \beta_2^{\tau_0 - 1} q^m.$$

Because $m = (t - s/\tau_0) - (n_0/\tau_0)$ we finally obtain

$$\|T^*(t, s)\|_\xi \leq \beta_3 q_1^{t-s}, \tag{4.63}$$

where $\beta_3 = \beta_1 \beta_2^{\tau_0 - 1} q^{-(n_0/\tau_0)}$ and $q_1 = q^{(1/\tau_0)} \in (0,1)$. Combining (2.13) with (4.63) we conclude that $\|T(t, s)\|_\xi \leq \beta q_1^{t-s}, \forall t \geq s \geq 0$ and thus the proof ends. $\square$

The next result may be viewed as a criterion for exponential stability in the case of sequences of linear and positive operators.

Theorem 4.8 *Let* $\{\mathcal{L}_t\}_{t\geq 0}$ *be a bounded sequence of linear bounded and positive operators on* $\mathcal{X}$ *and* $\{g_t\}_{t\geq 0} \subset \mathcal{X}^+$ *be a bounded sequence.*

Assume that $(g_t, \mathcal{L}_t)$ *is uniformly observable. Under these conditions the following are equivalent.*

(i) $\{\mathcal{L}_t\}_{t\geq 0}$ *defines an exponentially stable evolution.*
(ii) The backward affine equation

$$x_t = \mathcal{L}_t^* x_{t+1} + g_t \tag{4.64}$$

has a bounded solution $\{\tilde{x}_t\}_{t\geq 0} \subset \mathcal{X}^+$.

Proof. The implication $(i) \rightarrow (ii)$ follows from Theorem 2.5. Now we prove $(ii) \rightarrow (i)$. It is easy to see that for each $0 \leq t \leq m$ we have

$$\tilde{x}_t = T^*(m+1,t)\tilde{x}_{m+1} + \sum_{l=t}^{m} T^*(l,t)g_l. \tag{4.65}$$

Because $\tilde{x}_{m+1} \geq 0, T^*(m+1,t) \geq 0$ and $\{\tilde{x}_t\}_{t\geq 0}$ is a bounded sequence we deduce from (4.65) that

$$0 \leq \sum_{l=t}^{m} T^*(l,t)g_l \leq \tilde{x}_t \leq \mu\xi \tag{4.66}$$

for all $0 \leq t \leq m$ and $\mu > 0$ not depending upon t and m. Because $\mathcal{X}^+$ is a regular cone (see Proposition 2.2) we conclude that the sequence $\{\sum_{l=t}^{m} T^*(l,t)g_l\}_{m\geq t}$ is convergent.

Set $h_t = \sum_{l=t}^{\infty} T^*(l,t)g_l$. From (4.66) we get that

$$h_t \leq \mu\xi \tag{4.67}$$

for all $t \geq 0$.

On the other hand from the uniform observability condition one obtains that there exist $\tau_0 \geq 1, \gamma > 0$ such that (4.56) is valid. This allows us to write

$$h_t \geq \gamma\xi \tag{4.68}$$

for all $t \geq 0$. From (4.67) and (4.68) we have that $\{h_t\}_{t\geq 0}$ is a bounded and uniformly positive sequence.

For τ_0 from (4.56) we may write:

$$T^*(t+\tau_0,t)h_{t+\tau_0} = \sum_{l=t+\tau_0}^{\infty} T^*(l,t)g_l = h_t - \sum_{l=t}^{t+\tau_0-1} T^*(l,t)g_l.$$

Based on (4.56) and (4.67) we obtain

$$T^*(t+\tau_0, t)h_{t+\tau_0} \leq qh_t$$

for all $t \geq 0$ where $q = (1 - (\gamma/\mu)) \in (0,1)$. The conclusion now follows from Lemma 4.1 and the proof ends. □

Remark 4.7 From Theorem 2.5 one obtains that under the assumptions of Theorem 4.8, if the backward affine equation (4.64) has a bounded solution $\{\tilde{x}_t\}_{t\geq 0} \subset \mathcal{X}^+$ then $\{\tilde{x}_t\}_{t\geq 0}$ is a unique bounded solution of (4.64) and $\tilde{x}_t \gg 0, t \geq 0$.

Specializing the above to $\mathcal{X} = \mathcal{S}_n^N$ we have the following.

Corollary 4.5 *Assume that:*

(*a*) $\{A_k(t,i)\}_{t\geq 0}, 0 \leq k \leq r$, $\{C(t,i)\}_{t\geq 0}, i \in \mathcal{D}$, *and* $\{P_t\}_{t\geq 0}$ *are bounded sequences.*
(*b*) $(C, \mathbf{A}, \mathbf{P})$ *is uniform observable.*

Then the following are equivalent.

(*i*) *The sequence* $\{\mathcal{L}_t\}_{t\geq 0}$ *defined by* (2.58) *generates an exponentially stable evolution.*
(*ii*) *The system of linear backward equations*

$$X_t(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(i,j)A_k^T(t,i)X_{t+1}(j)A_k(t,i) + C^T(t,i)C(t,i) \quad (4.69)$$

has a bounded solution $\tilde{X}_t = (\tilde{X}_t(1)\cdots\tilde{X}_t(N))$, *with* $\tilde{X}_t(i) \geq 0$, *for all* $t \geq 0, i \in \mathcal{D}$.
Moreover the bounded solution of (4.69) *if it exists is unique and uniform positive definite.*

If $P_t, t \geq 0$ are stochastic matrices, then under the assumptions of Corollary 4.5, (*ii*) holds iff the zero solution of the system (4.25) is SESMS-I.

4.5 The case of the systems with coefficients depending upon η_t, η_{t-1}

Consider the system

$$x(t+1) = \left[A_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t,\eta_{t-1})\right]x(t)$$
$$+ \left[B_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)B_k(t,\eta_t,\eta_{t-1})\right]u(t). \quad (4.70)$$

Definition 4.6 *We say that the system* (4.70) *is* stochastic stabilizable *if there exist bounded sequences* $\{F(t,i)\}_{t \geq 1}, 1 \leq i \leq N$*, such that the zero state equilibrium of the closed-loop system*

$$x(t+1) = \Bigg[A_0(t,\eta_t,\eta_{t-1}) + B_0(t,\eta_t,\eta_{t-1})F(t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t,\eta_{t-1}) + B_k(t,\eta_t,\eta_{t-1})F(t,\eta_{t-1}))\Bigg] x(t) \quad (4.71)$$

$t \geq 1$ *is SESMS.*

Let $\Upsilon_F(t) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ be the Lyapunov-type operator associated with (4.71). $\Upsilon_F(t)H = (\Upsilon_F(t)H(1), \Upsilon_F(t)H(2), \dots, \Upsilon_F(t)H(N))$,

$$\Upsilon_F(t)H(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(j,i)[A_k(t,i,j) + B_k(t,i,j)F(t,j)]H(j)[A_k(t,i,j) + B_k(t,i,j)F(t,j)]^T, \qquad \forall\, H \in \mathcal{S}_n^N, \qquad i \in \mathcal{D}. \quad (4.72)$$

Using Corollary 3.7 and some Schur complement techniques one obtains the following criteria for stochastic stabilizability of the systems (4.70) in the time-invariant case.

Corollary 4.6 *Assume that the system* (4.70) *is in the time-invariant case. Under the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *the following are equivalent.*

(*i*) *The system* (4.70) *is stochastic stabilizable.*

(*ii*) *There exist* $F = (F(1), F(2), \dots, F(N)), F(i) \in \mathbf{R}^{m \times n}, i \in \mathcal{D}, X = (X(1), X(2), \dots, X(N)) \in \mathcal{S}_n^N, X(i) > 0, i \in \mathcal{D}$ *that solve*

$$\Upsilon_F X(i) - X(i) < 0, \qquad i \in \mathcal{D}. \quad (4.73)$$

(*iii*) *There exist* $X = (X(1), X(2), \dots, X(N)) \in \mathcal{S}_n^N, \Gamma = (\Gamma(1), \Gamma(2), \dots, \Gamma(N)), \Gamma(i) \in \mathbf{R}^{m \times n}, i \in \mathcal{D}$ *that solve the following system of LMIs,*

$$\begin{pmatrix} -X(i) & \mathcal{M}_0(i) & \mathcal{M}_1(i) & \dots & \mathcal{M}_r(i) \\ \mathcal{M}_0^T(i) & -\mathbf{X} & 0 & \dots & 0 \\ \mathcal{M}_1^T(i) & 0 & -\mathbf{X} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \mathcal{M}_r^T(i) & 0 & 0 & \dots & -\mathbf{X} \end{pmatrix} < 0, \quad (4.74)$$

$i \in \mathcal{D}$*, where* $\mathcal{M}_k(i) = (\sqrt{p(1,i)}(A_k(i,1)X(1) + B_k(i,1)\Gamma(1))\sqrt{p(2,i)}(A_k(i,2)X(2) + B_k(i,2)\Gamma(2)) \cdots \sqrt{p(N,i)}(A_k(i,N)X(N) + B_k(i,N)\Gamma(N)))$, $0 \leq k \leq r$,

$$\mathbf{X} = \mathrm{diag}\ (X(1), X(2), \dots, X(N)) \in \mathbf{R}^{n \times N}.$$

Moreover, if (X, Γ) is a solution of (4.73) then $F(i) = \Gamma(i)X^{-1}(i), i \in \mathcal{D}$ provide a stabilizing feedback gain for (4.70).

4.6 A generalization of the concept of stabilizability

In this section we introduce a general definition of stabilizability that contains the concepts of stabilizability introduced by Definitions 4.1 and 4.6 as special cases. Let $\{\Pi(t)\}_{t\in\mathcal{I}}$ be a sequence of linear operators and $\Pi(t) : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$, $\mathcal{I} \subset \mathbf{Z}$ is a right unbounded set of consecutive integers. Hence $\Pi(t)X = (\Pi_1(t)X, \Pi_2(t)X, \ldots, \Pi_N X)$ for all $X \in \mathcal{S}_n^N$. Let $F(t) = (F(t,1), F(t,2), \ldots, F(t,N)), F(t,i) \in \mathbf{R}^{m\times n}$ be given. Then the pair $(\Pi(t), F(t))$ defines a linear operator $\Pi_F(t) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ as follows,

$$\Pi_F(t)X = (\Pi_{F1}(t)X, \Pi_{F2}(t)X, \ldots, \Pi_{FN}(t)X),$$

where

$$\Pi_{Fi}(t)X = \begin{pmatrix} I_n & F^T(t,i) \end{pmatrix} \Pi_i(t)X \begin{pmatrix} I_n & F^T(t,i) \end{pmatrix}^T \tag{4.75}$$

for all $X \in \mathcal{S}_n^N$. It should be remarked that $\Pi_F(t) \geq 0$ if $\Pi(t) \geq 0$.

Definition 4.7 *We say that the sequence of linear and positive operators* $\{\Pi(t)\}_{t\in\mathcal{I}}$ *is stabilizable if there exist bounded sequences* $\{F(t,i)\}_{t\in\mathcal{I}} \subset \mathbf{R}^{m\times n}, 1 \leq i \leq N$, *such that the zero state equilibrium of the discrete-time linear equation*

$$X(t+1) = (\Pi_F(t))^* X(t)$$

is exponentially stable, where $(\Pi_F(t))^*$ *is the adjoint operator of* $\Pi_F(t)$ *with respect to the inner product* (2.18), $\Pi_F(t)$ *being defined as in* (4.75).

The sequence $\{F(t)\}_{t\in\mathcal{I}}, F(t) = (F(t,1), F(t,2), \ldots, F(t,N))$ *involved in the above definition is termed the* stabilizing feedback gain.

Remark 4.8

(*a*) In the special case of $\Pi(t) = \Pi, \Pi : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$ being a linear operator, we say that the operator Π is stabilizable if the conditions of the previous definition are fulfilled for the constant sequence $\{F\}$; that is, $F(t,i) = F(i), 1 \leq i \leq N$.

(*b*) If $\mathcal{L}_F(t)$ and $\Upsilon_F(t)$ are the linear operators defined by (4.5) and (4.72), respectively, then the equalities

$$\mathcal{L}_F^*(t) = \hat{\Pi}_F(t) \tag{4.76}$$

and

$$\Upsilon_F^*(t) = \check{\Pi}_F(t) \tag{4.77}$$

show that the system (4.1) is stochastic stabilizable if and only if the sequence $\{\hat{\Pi}(t)\}_{t\geq 0}$ is stabilizable, and the system (4.70) is stochastic stabilizable if and only if the sequence $\{\check{\Pi}(t)\}_{t\geq 1}$ is stabilizable. In (4.76)

$$\hat{\Pi}(t)X = \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(i,j)(A_k(t,i) \quad B_k(t,i))^T X(j)(A_k(t,i) \quad B_k(t,i))$$

whereas in (4.77), $\check{\Pi}(t)X = \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(i,j)(A_k(t,j,i) \quad B_k(t,j,i))^T X(j)(A_k(t,j,i) \quad B_k(t,j,i))$. Therefore the operators $\hat{\Pi}$ and $\check{\Pi}$ are positive.

4.7 Notes and references

Different concepts of stochastic detectability and observability for discrete-time linear stochastic systems are defined and studied in [25–27, 38, 44, 58, 86–91, 107, 108, 113]. The results from Section 4.1.1 may be found in [42]. The notions from Section 4.6 are in [48]. Theorem 4.1 is proved in [38]. All other results presented in this chapter are proved in [39]. Here the loss of the uniform positivity is compensated by the detectability property. The continuous-time time-invariant version of the result proved in Theorem 4.1 may be found in [57], and the continuous-time time-varying counterpart of this result may be found in [35].

Such a result is sometimes useful to derive the existence of the stabilizing solutions for generalized Riccati equations; see [44].

5

Discrete-time Riccati equations of stochastic control

In this chapter a class of discrete-time backward nonlinear equations defined on the ordered Hilbert space $\mathcal{S}_n^N$ is considered. The problem of the existence of some global solutions is investigated. The class of considered discrete-time nonlinear equations contains, as special cases, a great number of difference Riccati equations both from the deterministic and the stochastic framework. The results proved in Sections 5.3–5.6 provide sets of necessary and sufficient conditions that guarantee the existence of some special solutions of the considered equations such as the maximal solution, the stabilizing solution, and the minimal positive semidefinite solution. These conditions are expressed in terms of the feasibility of some suitable systems of linear matrix inequalities, LMIs. One shows that in the case of the equations with periodic coefficients to verify the conditions that guarantee the existence of the maximal or the stabilizing solution we have to check the solvability of some systems of LMI with a finite number of inequations. The proofs are based on some suitable properties of discrete-time linear equations defined by positive operators on some ordered Hilbert spaces developed in Chapter 2. In Section 5.7 an iterative procedure is proposed for the computation of the maximal and stabilizing solution of the discrete-time backward nonlinear equations under consideration. In the last part of this chapter, one shows how the obtained results can be specialized to derive useful conditions that guarantee the existence of the maximal solution or the stabilizing solution for different classes of difference matrix Riccati equations involved in many problems of robust control in the stochastic framework.

5.1 An overview on discrete-time Riccati-type equations of stochastic control

In the literature related to the topic of control of discrete-time linear stochastic systems two types of discrete-time Riccati equations are usually involved.

V. Dragan et al., *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, DOI 10.1007/978-1-4419-0630-4_5,

$$X_t = \sum_{k=0}^{r} A_k^T(t) X_{t+1} A_k(t) + C^T(t)C(t) - \left(\sum_{k=0}^{r} A_k^T(t) X_{t+1} B_k(t) + C^T(t)D(t) \right)$$
$$\times \left(D^T(t)D(t) + \sum_{k=0}^{r} B_k^T(t) X_{t+1} B_k(t) \right)^{-1}$$
$$\times \left(\sum_{k=0}^{r} B_k^T(t) X_{t+1} A_k(t) + D^T(t)C(t) \right), \qquad t \in Z, \qquad t \geq 0 \tag{5.1}$$

and

$$X(t,i) = \sum_{j=1}^{N} p_t(i,j) A^T(t,i) X(t+1,j) A(t,i) + C^T(t,i)C(t,i)$$
$$- \left(\sum_{j=1}^{N} p_t(i,j) A(t,i) X(t+1,j) B(t,i) + C^T(t,i)D(t,i) \right)$$
$$\times \left(D^T(t,i)D(t,i) + \sum_{j=1}^{N} p_t(i,j) B(t,i) X(t+1,j) B(t,i) \right)^{-1}$$
$$\times \left(\sum_{j=1}^{N} p_t(i,j) B^T(t,i) X(t+1,j) A(t,i) + D^T(t,i)C(t,i) \right),$$
$$1 \leq i \leq N, \qquad t \in Z, \qquad t \geq 0. \tag{5.2}$$

The equation (5.1) occurs in connection with the linear quadratic optimization problem associated with a discrete-time linear system with independent random perturbations described by

$$x(t+1) = \left[A_0(t) + \sum_{k=1}^{r} w_k(t) A_k(t) \right] x(t) + \left[B_0(t) + \sum_{k=1}^{r} w_k(t) B_k(t) \right] u(t)$$

and the output

$$y(t) = C(t)x(t) + D(t)u(t),$$

where $\{w_k(t)\}_{t \geq 0}, 1 \leq k \leq r$, are independent random perturbations on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with zero mean and finite second moments.

Systems of coupled discrete-time Riccati equations (5.2) occur in connection with the linear quadratic optimization problems associated with discrete-time linear systems subject to Markovian switching:

$$x(t+1) = A(t, \eta_t)x(t) + B(t, \eta_t)u(t)$$

and the output

$$y(t) = C(t, \eta_t)x(t) + D(t, \eta_t)u(t),$$

where $\{\eta_t\}_{t\geq 0}$ is a Markov chain with the set of states $\mathcal{D} = \{1, 2, \ldots, N\}$ and the sequence of transition probability matrix $\{P_t\}_{t\geq 0}, P_t = (p_t(i,j))_{1\leq i,j\leq N}$.

Lately, there exists an increasing interest in considering discrete-time linear stochastic systems subject to both independent random perturbations and Markovian jumping.

In this case the following type of discrete-time Riccati equations is involved.

$$\begin{aligned}
X(t,i) = &\sum_{k=0}^{r}\sum_{j=1}^{N} p_t(i,j)A_k^T(t,i)X(t+1,j)A_k(t,i) + C^T(t,i)C(t,i) \\
&- \left(\sum_{k=0}^{r}\sum_{j=1}^{N} p_t(i,j)A_k(t,i)X(t+1,j)B_k(t,i) + C^T(t,i)D(t,i)\right) \\
&\times \left(D^T(t,i)D(t,i) + \sum_{k=0}^{r}\sum_{j=1}^{N} p_t(i,j)B_k^T(t,i)X(t+1,j)B_k(t,i)\right)^{-1} \\
&\times \left(\sum_{k=0}^{r}\sum_{j=1}^{N} p_t(i,j)B_k(t,i)X(t+1,j)A_k(t,i) + D^T(t,i)C(t,i)\right), \\
&t \in Z, \qquad t \geq 0.
\end{aligned} \tag{5.3}$$

A more general case is the one of the discrete-time linear stochastic systems of the form:

$$\begin{aligned}
x(t+1) = &\left[A_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t,\eta_{t-1})\right]x(t) \\
&+ \left[B_0(t,\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)B_k(t,\eta_t,\eta_{t-1})\right]u(t) \\
z(t) = &\; C(t,\eta_t,\eta_{t-1})x(t) + D(t,\eta_t,\eta_{t-1})u(t),
\end{aligned}$$

where $\{\eta_t\}_{t\geq 0}, \{w_k(t)\}_{t\geq 0}, k \in \{1, 2, \ldots, r\}$ are as before.

Such a system occurs when in the control process for a discrete-time linear stochastic system subject to Markovian jumping and independent random perturbations, some delays in the transmission of the data are possible (see Chapter 1, Section 1.6).

To solve the LQG problem and the H_2 control problem for this type of stochastic system the following system of coupled discrete-time Riccati equations is involved.

$$
\begin{aligned}
&X(t,i)\\
&= \sum_{k=0}^{r}\sum_{j=1}^{N} p_{t-1}(i,j) A_k^T(t,j,i) X(t+1,j) A_k(t,j,i)\\
&\quad + \sum_{j=1}^{N} p_{t-1}(i,j) C^T(t,j,i) C(t,j,i) - \left[\sum_{j=1}^{N} p_{t-1}(i,j)\left(C^T(t,j,i)D(t,j,i)\right.\right.\\
&\quad \left.\left. + \sum_{k=0}^{r} A_k^T(t,j,i) X(t+1,j) B_k(t,j,i)\right)\right]\left[\sum_{j=1}^{N} p_{t-1}(i,j)\left(D^T(t,j,i)D(t,j,i)\right.\right.\\
&\quad \left.\left. + \sum_{k=0}^{r} B_k^T(t,j,i) X(t+1,j) B_k(t,j,i)\right)\right]^{-1}\\
&\quad \times \left[\sum_{j=1}^{N} p_{t-1}(i,j)\left(D^T(t,j,i)C(t,j,i)\right.\right.\\
&\quad \left.\left. + \sum_{k=0}^{r} B_k^T(t,j,i) X(t+1,j) A_k(t,j,i)\right)\right] \qquad (5.4)
\end{aligned}
$$

Let us consider the following linear stochastic system with finite jumps,

$$
\begin{aligned}
dx(\tau) &= A_0 x(\tau) d\tau + A_1 x(\tau) dv(\tau), \qquad \tau \neq th\\
x(th_+) &= A_{0d} x(th) + B_{0d} u(t) + [A_{1d} x(th) + B_{1d} u(t)] v_d(t)
\end{aligned}
$$

with the continuous time output $z(\tau) = Cx(\tau)$ and discrete-time output $z_d(t) = C_d x(th) + D_d u(t), t \in \{0,1,2,\dots\}; h > 0$ is the sampling period, where $\{v(\tau)\}_{\tau \geq 0}$ is a standard scalar Wiener process; and $\{v_d(t)\}_{t\geq 0}$ is a sequence of independent random variables.

Such systems occur in connection with the control by piecewise constant controls of linear stochastic systems described by Ito differential equations. To solve the LQG problem or the H_2 optimal control problem for this type of controlled system the following system of Riccati equations with jumps

$$
\begin{aligned}
\frac{dX(\tau)}{d\tau} &= A_0^T X(\tau) + X(\tau) A_0 + A_1^T X(\tau) A_1 + C^T C, \qquad \tau \neq th\\
X(th_-) &= A_{0d}^T X(th) A_{0d} + A_{1d}^T X(th) A_{1d} + C_d^T C_d\\
&\quad - (A_{0d}^T X(th) B_{0d} + A_{1d}^T X(th) B_{1d} + C_d^T D_d)
\end{aligned}
$$

$$\times (D_d^T D_d + B_{0d}^T X(th) B_{0d} + B_{1d}^T X(th) B_{1d})^{-1}$$
$$\times (B_{0d}^T X(th) A_{0d} + B_{1d}^T X(th) A_{1d} + D_d^T C_d), \tag{5.5}$$

$t \in \{0, 1, 2, \dots\}$ is used.

From the first equation of (5.5) one obtains:

$$X(th) = e^{\mathcal{L}h} X((t+1)h_-) + \int_0^h e^{\mathcal{L}s} C^T C ds, \tag{5.6}$$

where $\mathcal{L}$ is the perturbed Lyapunov operator, $\mathcal{L}S = A_0^T S + SA_0 + A_1^T SA_1$ for any symmetric matrices S, and

$$e^{\mathcal{L}h} = \sum_{k=0}^{\infty} \frac{\mathcal{L}^k h^k}{k!}.$$

We remark that in the special case $A_1 = 0$ we have $e^{\mathcal{L}h} S = e^{A_0^T h} S e^{A_0 h}$. By replacing (5.6) in the second equation of (5.5) one obtains a discrete-time backward nonlinear equation for the sequence $X(th_-), t \geq 0$. It is shown that this discrete-time nonlinear equation is of type (5.8) below.

Our goal is to find a large enough class of discrete-time nonlinear equations containing as special cases all discrete-time Riccati equations described above. For this class of discrete-time nonlinear equation which we call a discrete-time system of generalized Riccati equations (DTSGRE) we provide necessary and sufficient conditions for the existence of some global special solutions such as maximal solution, stabilizing solution, minimal solution, and so on.

The results derived in that general framework can be applied to each of the Riccati-type equations described above as well as to other Riccati equations connected with different control problems in both a stochastic framework and deterministic framework.

The conditions that guarantee the existence of the maximal solution, stabilizing solution, and minimal solution of DTSGRE considered in this chapter are expressed in terms of solvability of some suitable systems of LMIs.

5.2 A class of discrete-time backward nonlinear equations

5.2.1 Several notations

Together with the notations introduced in Example 2.5(iii) in this chapter we use new conventions of notations displayed in this subsection. For a fixed integer $N \geq 1$, $\mathcal{M}_{nm}^N = \mathbf{R}^{n \times m} \oplus \mathbf{R}^{n \times m} \oplus \cdots \oplus \mathbf{R}^{n \times m}$. Hence $B \in \mathcal{M}_{nm}^N$ if and only if $B = (B(1), B(2), \dots, B(N))$ with $B(i) \in \mathbf{R}^{n \times m}, i \in \{1, 2, \dots, N\}$.

In the sequel we write $\mathcal{M}_n^N$ instead of $\mathcal{M}_{nn}^N$. Obviously $\mathcal{S}_n^N \subset \mathcal{M}_n^N$. $\mathcal{M}_{nm}^N$ is a Hilbert space with respect to the following inner product,

$$\langle X, Y \rangle = \sum_{i=1}^{N} Tr[Y^T(i)X(i)] \tag{5.7}$$

for all $X = (X(1), X(2), \dots, X(N))$, $Y = (Y(1), Y(2), \dots, Y(N))$ from $\mathcal{M}_{nm}^N$. If $m = n$ the inner product (5.7) induces a Hilbert space structure on $\mathcal{M}_n^N$ and on its subspace $\mathcal{S}_n^N$. The restriction of (5.7) to the subspace $\mathcal{S}_n^N$ is just (2.18).

Throughout this chapter we use the following calculus convention.

(α) If $C = (C(1), C(2), \dots, C(N)) \in \mathcal{M}_{nm}^N$ then $C^T \in \mathcal{M}_{mn}^N$ and it is defined by $C^T = (C^T(1), C^T(2), \dots, C^T(N))$.

(β) If $A = (A(1), A(2), \dots, A(N)) \in \mathcal{M}_n^N$ with $\det A(i) \neq 0$, $1 \leq i \leq N$, then $A^{-1} = (A^{-1}(1), A^{-1}(2), \dots, A^{-1}(N)) \in \mathcal{M}_n^N$ and $\det A \neq 0$ means that $\det A(i) \neq 0$ for all $1 \leq i \leq N$.

(γ) If $B \in \mathcal{M}_{nm}^N, C \in \mathcal{M}_{pn}^N, B = (B(1), B(2), \dots, B(N)), C = (C(1), C(2), \dots, C(N))$, then $D = CB \in \mathcal{M}_{pm}^N$, $D = (D(1), D(2), \dots, D(N))$, $D(i) = C(i)B(i), 1 \leq i \leq N$.

(δ) If $C = (C(1), C(2), \dots, C(N)) \in \mathcal{M}_{pn}^N$ and $F = (F(1), F(2), \dots, F(N)) \in \mathcal{M}_{mn}^N$ then

$$D = \begin{pmatrix} C \\ F \end{pmatrix} \in \mathcal{M}_{p+m,n}^N$$

is defined as $D = (D(1), D(2), \dots, D(N))$,

$$D(i) = \begin{pmatrix} C(i) \\ F(i) \end{pmatrix}, \qquad 1 \leq i \leq N.$$

(ϵ) If $K = (K(1), K(2), \dots, K(N)) \in \mathcal{M}_{np}^N$ and $B = (B(1), B(2), \dots, B(N)) \in \mathcal{M}_{nm}^N$ then $Y = (K \quad B) \in \mathcal{M}_{n,p+m}^N$ is defined by $Y = (Y(1), Y(2), \dots, Y(N)), Y(i) = (K(i) \quad B(i)), i \in \{1, 2, \dots, N\}$.

We often use the special element of $\mathcal{S}_n^N$, $J_n = (I_n \quad I_n \quad \cdots \quad I_n)$, I_n being the identity matrix. Throughout the chapter $\mathcal{I} \subset \mathbf{Z}$ is a subset of consecutive integers. That is, $\mathcal{I} = \mathbf{Z}$ or $\mathcal{I} = \{s, s+1, \dots\}$. Often $\mathcal{I}$ is called the interval of integers.

$\mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ denotes the set of linear operators $\Pi : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$.

5.2.2 A class of discrete-time generalized Riccati equations

In this chapter we deal with nonlinear discrete-time backward equations on the space $\mathcal{S}_n^N$, of the form:

$$X(t) = \Pi_1(t)X(t+1) + M(t) - (L(t) + \Pi_2(t)X(t+1))(R(t) + \Pi_3(t)X(t+1))^{-1}(L(t) + \Pi_2(t)X(t+1))^T, \tag{5.8}$$

$t \in \mathcal{I}$, where $\mathcal{I} \subset \mathbf{Z}$ is a right unbounded interval of integers.

For each $t \in \mathcal{I}, M(t) \in \mathcal{S}_n^N, L(t) \in \mathcal{M}_{nm}^N, R(t) \in \mathcal{S}_m^N, \Pi_1(t) : \mathcal{S}_n^N \to \mathcal{S}_n^N, \Pi_2(t) : \mathcal{S}_n^N \to \mathcal{M}_{nm}^N, \Pi_3 : \mathcal{S}_n^N \to \mathcal{S}_m^N$ are linear operators, where n, m, N are fixed positive integers.

The equation (5.8) contains as special cases the Riccati equations (5.1)–(5.5) that appear in stochastic control. Therefore the results proved in this chapter concerning the properties of the solutions of equation (5.8) allow us to obtain useful information about the solutions of a wide class of discrete-time Riccati-type equations, involved in both deterministic and stochastic control.

Equation (5.8) can be written in a compact form as

$$X(t) = \mathcal{R}(t, X(t+1)), \tag{5.9}$$

where $\mathcal{R} : \mathrm{Dom}\mathcal{R} \to \mathcal{S}_n^N$ is given by

$$\mathcal{R}(t, X) = \Pi_1(t)X - (L(t) + \Pi_2(t)X)(R(t) + \Pi_3(t)X)^{-1} \times (L(t) + \Pi_2(t)X)^T + M(t) \tag{5.10}$$

and

$$\mathrm{Dom}\mathcal{R} = \{(t, X) \in \mathcal{I} \times \mathcal{S}_n^N \,|\, \det(R(t) + \Pi_3(t)X) \neq 0\}. \tag{5.11}$$

We set

$$\Pi(t)X = \begin{pmatrix} \Pi_1(t)X & \Pi_2(t)X \\ (\Pi_2(t)X)^T & \Pi_3(t)X \end{pmatrix} \tag{5.12}$$

$$\mathcal{Q}(t) = \begin{pmatrix} M(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix}. \tag{5.13}$$

We have $\Pi(t) \in \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ and $\mathcal{Q}(t) \in \mathcal{S}_n^N$. It is clear that the equations (5.8) and (5.9) are associated with the pair $\Sigma = (\{\Pi(t)\}_{t \in I}, \{\mathcal{Q}(t)\}_{t \in I})$. We often write $\Sigma = (\Pi, \mathcal{Q})$ for simplicity.

Throughout the chapter we make the following assumption.

A.5.1

(i) The sequences $\{\Pi(t)\}_{t \in \mathcal{I}} \subset \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ and $\{\mathcal{Q}(t)\}_{t \in \mathcal{I}} \subset \mathcal{S}_{n+m}^N$ are bounded.

(ii) For each $t \in \mathcal{I}, \Pi(t)$ is a positive operator; that is, $\Pi(t)X \geq 0$ if $X \geq 0$.

Let $\ell^\infty\{\mathcal{I}, \mathcal{S}_n^N\}$ be the set of bounded sequences $\mathbf{X} = \{X(t)\}_{t\in\mathcal{I}} \subset \mathcal{S}_n^N$. With a pair $\Sigma = (\Pi, \mathcal{Q})$ we associate the so-called dissipation operator $\mathcal{D}^\Sigma : \ell^\infty(\mathcal{I}, \mathcal{S}_n^N) \to \ell^\infty(\mathcal{I}, \mathcal{S}_{n+m}^N)$ by

$$(\mathcal{D}^\Sigma \mathbf{X})(t) = \begin{pmatrix} \Pi_1(t)X(t+1) + M(t) - X(t) & L(t) + \Pi_2(t)X(t+1) \\ (L(t) + \Pi_2(t)X(t+1))^T & R(t) + \Pi_3(t)X(t+1) \end{pmatrix} \tag{5.14}$$

for arbitrary $\mathbf{X} = \{X(t)\}_{t\in\mathcal{I}} \in \ell^\infty(\mathcal{I}, \mathcal{S}_n^N)$.

Remark 5.1 It is easy to see that

$$\mathcal{D}^\Sigma(\mathbf{X}(t)) = \begin{pmatrix} -X(t) & 0 \\ 0 & 0 \end{pmatrix}$$

$+\Pi(t)X(t+1) + \mathcal{Q}(t)$ for any $\mathbf{X} = \{X(t)\}_{t\in\mathcal{I}}$.

The following two subsets of $\ell^\infty(\mathcal{I}, \mathcal{S}_n^N)$ play an important role in our further developments.

$$\begin{aligned} \Gamma^\Sigma = \{\mathbf{X} = \{X(t)\}_{t\in\mathcal{I}} \in \ell^\infty(\mathcal{I}, \mathcal{S}_n^N) | \mathcal{D}^\Sigma(\mathbf{X}(t)) \geq 0, \\ R(t) + \Pi_3(t)X(t+1) \gg 0, t \in \mathcal{I}\} \end{aligned} \tag{5.15}$$

$$\tilde{\Gamma}^\Sigma = \{\mathbf{X} = \{X(t)\}_{t\in\mathcal{I}} \in \ell^\infty(\mathcal{I}, \mathcal{S}_n^N) | \mathcal{D}^\Sigma(\mathbf{X}(t)) \gg 0, \quad t \in \mathcal{I}\}. \tag{5.16}$$

We mention that for a sequence $\{X(t)\}_{t\in\mathcal{I}}$ the notation $X(t) \gg 0, t \in \mathcal{I}$ is equivalent to $X(t,i) \geq \varepsilon I_n > 0$ for all $t \in \mathcal{I}, 1 \leq i \leq N$. Such a sequence is called uniformly positive.

Remark 5.2

(*a*) From (5.15) and (5.16) it follows that $\tilde{\Gamma}^\Sigma \subset \Gamma^\Sigma$.
(*b*) Based on the Schur complement technique, one deduces that Γ^Σ contains all global and bounded solutions $\{X(t)\}_{t\in\mathcal{I}}$ of (5.8) that verify the additional condition $R(t) + \Pi_3(t)X(t+1) \gg 0, t \in \mathcal{I}$.

Definition 5.1 *We say that* $\{X(t)\}_{t\in\mathcal{I}}$ *is a* maximal solution *of DTSGRE* (5.8) *if* $X(t) \geq \hat{X}(t), t \in \mathcal{I}$ *for arbitrary* $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^\Sigma$.

If $\{X(t)\}_{t\in\mathcal{I}}$ is a solution of (5.8) then $\mathcal{R}'(t, X(t+1)) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ stands for the Frechet derivative of the operator $\mathcal{R}$ given by (5.10)–(5.11).

A new kind of global solution of (5.8) that plays an important role in different applications is introduced by the following.

Definition 5.2 *A solution* $\{X_s(t)\}_{t\in\mathcal{I}} \subset \mathcal{S}_n^N$ *is called a* stabilizing solution *of DTSGRE* (5.8) *if the zero solution of the discrete-time linear equation on* $\mathcal{S}_n^N$,

$$Z(t+1) = (\mathcal{R}'(t, X_s(t+1)))^* Z(t), \tag{5.17}$$

is exponentially stable, where $(\mathcal{R}'(t, X_s(t+1)))^*$ *is the adjoint operator of* $\mathcal{R}'(t, X_s(t+1))$ *with respect to the inner product* (5.7).

The next result is used repeatedly in the next developments in this chapter.

Lemma 5.1 *Let* $\{X(t)\}_{t\in\mathcal{I}_1}$ *be a solution of* (5.8) *and* $\{W(t)\}_{t\in\mathcal{I}_1} \subset \mathcal{M}_{mn}^N$ *be a given sequence;* $\mathcal{I}_1 \subset \mathcal{I}$ *is a subinterval of integers. Under these conditions* $\{X(t)\}_{t\in\mathcal{I}_1}$ *also verifies the following modified backward affine equation,*

$$X(t) = \Pi_W(t)X(t+1) + \mathcal{Q}_W(t) - (W(t) - F^X(t))^T$$

$$\times (R(t) + \Pi_3(t)X(t+1))(W(t) - F^X(t)),$$

where for each $t \in \mathcal{I}_1, \Pi_W(t) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ *is given by*

$$\Pi_W(t)X = \begin{pmatrix} J_n & W^T(t) \end{pmatrix} \begin{pmatrix} \Pi_1(t)X & \Pi_2(t)X \\ (\Pi_2(t)X)^T & \Pi_3(t)X \end{pmatrix} \begin{pmatrix} J_n \\ W(t) \end{pmatrix}, \qquad \forall X \in \mathcal{S}_n^N, \tag{5.18}$$

$$\mathcal{Q}_W(t) = \begin{pmatrix} J_n & W^T(t) \end{pmatrix} \begin{pmatrix} M(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix} \begin{pmatrix} J_n \\ W(t) \end{pmatrix}, \tag{5.19}$$

and

$$F^X(t) = -(R(t) + \Pi_3(t)X(t+1))^{-1}(L(t) + \Pi_2(t)X(t+1))^T. \tag{5.20}$$

Proof. It can be done by direct computations and can be a useful exercise for the reader.

At the end of this subsection we rewrite the Frechet derivative of the operator $\mathcal{R}$ in a easier form which is used in the developments of the next sections.

If we take into account that

$$\mathcal{R}'(t, X)U = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}(\mathcal{R}(t, X + \varepsilon U) - \mathcal{R}(t, X))$$

for arbitrary $(t, X) \in \text{Dom}\mathcal{R}$ and $U \in \mathcal{S}_n^N$ we obtain

$$\mathcal{R}'(t, X)U = \Pi_{F^X}(t)U, \tag{5.21}$$

where $\Pi_{F^X}(t)$ is defined as in (5.18) with W replaced by F^X and

$$F^X = -(R(t) + \Pi_3(t)X)^{-1}(L(t) + \Pi_2(t)X)^T. \tag{5.22}$$

Thus we see that the discrete-time linear equation (5.17) may be written as

$$Z(t+1) = \Pi_{F^{X_s}}^*(t)Z(t) \tag{5.23}$$

with

$$F^{X_s}(t) = -(R(t) + \Pi_3(t)X_s(t+1))^{-1}(L(t) + \Pi_2(t)X_s(t+1))^T. \tag{5.24}$$

5.3 A comparison theorem and several consequences

In this section we prove a result that mainly shows the monotonicity of the solutions of DTSGRE (5.8) with respect to $\mathcal{Q}(t)$. So we extend to this general framework a similar result in [58].

Theorem 5.1 *Let* $\{X_k(t)\}_{t\in\mathcal{I}_1}, k = 1,2, \mathcal{I}_1 \subset \mathcal{I}$ *be the solutions of the equations*

$$X_k(t) = \mathcal{R}_k(t, X_k(t+1)), \tag{5.25}$$

where $\mathcal{R}_k : \mathrm{Dom}\mathcal{R}_k \to \mathcal{S}_n^N$ *are operators of the form* (5.10) *defined by the pairs* $\Sigma_k = (\Pi, \mathcal{Q}_k)$*, where* $\Pi = \{\Pi(t)\}_{t\in\mathcal{I}}$ *is as in the previous section and* $\mathcal{Q}_k = \{\mathcal{Q}_k(t)\}_{t\in\mathcal{I}}$*, where*

$$\mathcal{Q}_k(t) = \begin{pmatrix} M_k(t) & L_k(t) \\ L_k^T(t) & R_k(t) \end{pmatrix}, k = 1, 2.$$

Assume:

(a) $\mathcal{Q}_1(t) \geq \mathcal{Q}_2(t), t \in \mathcal{I}$.
(b) $R_2(t) + \Pi_3(t)X_2(t+1) > 0, t, t+1 \in \mathcal{I}_1$.
(c) *There exists* $\tau \in \mathcal{I}_1$ *such that* $X_1(\tau) \geq X_2(\tau)$.

Under these conditions $X_1(t) \geq X_2(t)$ *for all* $t \in \mathcal{I}_1, t \leq \tau$.

Proof. Let $F_k(t) = F^{X_k}(t), k = 1,2$ be defined as in (5.22) with $X_k(t+1)$ instead of X.

Applying Lemma 5.1 to equations (5.25) with $W(t) = F_1(t)$ we obtain that $X_1(t) - X_2(t)$ verifies the discrete-time backward affine equation:

$$X_1(t) - X_2(t) = \Pi_{F_1}(t)(X_1(t+1) - X_2(t+1)) + \tilde{M}(t), \qquad t \in \mathcal{I}_1 \tag{5.26}$$

where

$$\tilde{M}(t) = (F_1(t) - F_2(t))^T (R_2(t) + \Pi_3(t)X_2(t+1))(F_1(t) - F_2(t))$$
$$+ \begin{pmatrix} J_n \\ F_1(t) \end{pmatrix}^T (\mathcal{Q}_1(t) - \mathcal{Q}_2(t)) \begin{pmatrix} J_n \\ F_1(t) \end{pmatrix}.$$

Based on assumptions (a), and (b) in the statement one obtains that $\tilde{M}(t) \geq 0, t \in \mathcal{I}_1$. Furthermore, from (5.18) and assumption **A5.1**(ii) we conclude that $\Pi_{F_1}(t) \geq 0$ for all $t \in \mathcal{I}_1$. The conclusion now follows inductively from (5.26) and assumption (c) in the statement. Thus the proof ends. □

Based on the above theorem we may provide a sufficient condition that guarantees the existence of a solution of equation (5.8) with the given terminal values. Before stating this result we introduce a notation. So, for each $\tau \in \mathcal{I}, X(t, \tau, H)$ denotes the solution of DTSGRE (5.8) that verifies $X(\tau, \tau, H) = H$.

Theorem 5.2 *Let* $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$ *and* $\tau \in \mathcal{I}$ *be given. If* $H \geq \hat{X}(\tau)$ *then* $X(t,\tau,H)$ *is well defined for all* $t \in \mathcal{I}, t \leq \tau$.

Proof. Suppose that $\tau - 1 \in \mathcal{I}$. From $H \geq \hat{X}(\tau)$ and (5.15) we obtain

$$R(\tau-1) + \Pi_3(\tau-1)H \geq R(\tau-1) + \Pi_3(\tau-1)\hat{X}(\tau) > 0.$$

This shows that $(\tau-1, H) \in \mathrm{Dom}\mathcal{R}$ and $X(t,\tau,H)$ is well defined for $t = \tau-1$.

Let $\mathcal{I}_\tau = \{\hat{t}, \hat{t}+1, \hat{t}+2, \ldots, \tau\} \subseteq \mathcal{I}$ be the maximal interval of integers where $X(t,\tau,H)$ is defined. If $\mathcal{I}_\tau = (-\infty; \tau] \cap \mathcal{I}$ then the proof is complete. In contrast, we prove that the solution $X(t,\tau,H)$ may be computed for $t = \hat{t} - 1$.

Let $\hat{M}(t) = \mathcal{R}(t, \hat{X}(t+1)) - \hat{X}(t), t \in \mathcal{I}$. Using the Schur complement technique one obtains from (5.15) that $\hat{M}(t) \geq 0, t \in \mathcal{I}$. Hence $\hat{X}(t)$ verifies the equation

$$\hat{X}(t) = \hat{\mathcal{R}}(t, \hat{X}(t+1)), \tag{5.27}$$

where $\hat{\mathcal{R}} : \mathrm{Dom}\hat{\mathcal{R}} \to \mathcal{S}_n^N$ is an operator of type (5.10) defined by the pair $\hat{\Sigma} = (\Pi, \hat{\mathcal{Q}})$, where Π is as before and

$$\hat{\mathcal{Q}} = \{\hat{\mathcal{Q}}(t)\}_{t\in\mathcal{I}}, \qquad \hat{\mathcal{Q}}(t) = \begin{pmatrix} M(t) - \hat{M}(t) & L(t) \\ L^T(t) & R(t) \end{pmatrix}.$$

It is obvious that $\mathcal{Q}(t) \geq \hat{\mathcal{Q}}(t), t \in \mathcal{I}$. Applying Theorem 5.1 to equations (5.8) and (5.27) one obtains that $X(t,\tau,H) \geq \hat{X}(t)$ for all $t \in \mathcal{I}_\tau$. Hence $R(\hat{t}-1) + \Pi_3(\hat{t}-1)X(\hat{t},\tau,H) \geq R(\hat{t}-1) + \Pi_3(\hat{t}-1)\hat{X}(\hat{t}) > 0$. This shows that $(\hat{t}-1, X(\hat{t},\tau,H)) \in \mathrm{Dom}\mathcal{R}$. Therefore $X(t,\tau,H)$ is well defined for $t = \hat{t}-1$ and thus the proof is complete. □

Corollary 5.1 *Assume that* $0 \in \Gamma^{\Sigma}$. *Then for arbitrary* $H \in \mathcal{S}_n^{N+}$ *and* $\tau \in \mathcal{I}$ *the solution* $X(t,\tau,H)$ *of* (5.8) *is defined for all* $t \in \mathcal{I}, t \leq \tau$. *Moreover* $X(t,\tau,H) \geq 0, t \in \mathcal{I}, t \leq \tau$.

5.4 The maximal solution

In this section we study the problem of the existence of the maximal solution of equation (5.8). We also display the monotonicity of the maximal solution with respect to $\mathcal{Q}(t)$. The next auxiliary result is used repeatedly in the proofs of this section.

Lemma 5.2 *Let* $W(t) \in \mathcal{M}_{mn}^N$ *be given and* $X(t), t \in \mathcal{I}_1$ *be a solution of the following discrete-time backward affine equation,*

$$X(t) = \Pi_W(t)X(t+1) + \mathcal{Q}_W(t)$$

with the additional property $[R(t) + \Pi_3(t)X(t+1)]^{-1}$ *is well defined for all* $t, t+1 \in \mathcal{I}_1$, *where* $\Pi_W(t)$ *and* $\mathcal{Q}_W(t)$ *are as in* (5.18) *and* (5.19), *respectively.*

Under these conditions $\{X(t)\}_{t\in\mathcal{I}_1}$ *also solves the following modified equation,* $X(t) = \Pi_{F^X}(t)X(t+1) + \mathcal{Q}_{F^X}(t) + (F^X(t) - W(t))^T(R(t) + \Pi_3(t) X(t+1))(F^X(t) - W(t))$, *where* $F^X(t)$ *is defined as in* (5.22).

Proof. It is based on standard algebraic calculations. □

The next result provides a necessary and sufficient condition for the existence of the maximal solution of the DTSGRE (5.8).

Theorem 5.3 *Assume that the sequence of linear operators* $\{\Pi(t)\}_{t\in\mathcal{I}} \subset \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ *is stabilizable. Then the following are equivalent.*

(*i*) *The set* Γ^Σ *is not empty.*
(*ii*) *The equation* (5.8) *has a maximal and bounded solution* $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ *that satisfies the condition:*

$$R(t) + \Pi_3(t)X_{\max}(t+1) \gg 0.$$

Moreover if there exists an integer $\theta \geq 1$ *such that* $\Pi(t+\theta) = \Pi(t)$, $\mathcal{Q}(t+\theta) = \mathcal{Q}(t)$ *for all* $t \in \mathcal{I}$, *then the maximal solution* $X_{\max}(\cdot)$ *is periodic with period* θ.

Proof. The implication $(ii) \to (i)$ is obvious because $\{X_{\max}(t)\}_{t\in\mathcal{I}}$, if it exists, belongs to Γ^Σ. It remains to prove the converse implication. Based on the stabilizability property we may choose a bounded sequence $\{F_0(t)\}_{t\in\mathcal{I}} \subset \mathcal{M}_{mn}^N$ such that the zero state equilibrium of the discrete-time linear equation:

$$X(t+1) = \Pi_{F_0}^*(t)X(t) \tag{5.28}$$

is exponentially stable.

Applying Theorem 2.5 one obtains that the discrete-time affine equation

$$X_1(t) = \Pi_{F_0}(t)X_1(t+1) + \mathcal{Q}_{F_0}(t) + \varepsilon J_n \tag{5.29}$$

has a unique bounded solution $\{X_1(t)\}_{t\in\mathcal{I}} \subset \mathcal{S}_n^N$, where $\varepsilon > 0$ is fixed.

Taking $\{X_1(t)\}_{t\in\mathcal{I}}$ as a first step, we iteratively construct sequences $\{X_k(t)\}_{t\in\mathcal{I}}$, $\{F_k(t)\}_{t\in\mathcal{I}}, k \geq 1$ as follows. At each step k, $\{X_k(t)\}_{t\in\mathcal{I}}$ is the unique bounded solution of the discrete-time backward affine equation:

$$X_k(t) = \Pi_{F_{k-1}}(t)X_k(t+1) + \mathcal{Q}_{F_{k-1}}(t) + \frac{\varepsilon}{k}J_n \tag{5.30}$$

and

$$F_k(t) = -(R(t) + \Pi_3(t)X_k(t+1))^{-1}(L(t) + \Pi_2(t)X_k(t+1))^T. \tag{5.31}$$

The following items are proved inductively.

(a_k) $X_k(t) - \hat{X}(t) \geq \mu_k J_n$ for any $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$, where $\mu_k > 0$ do not depend upon $\hat{X}(t)$.
(b_k) The sequence $\{\mathcal{L}_k(t)\}_{t\in\mathcal{I}}$ defines an exponentially stable evolution, where $\mathcal{L}_k(t) = (\Pi_{F_k}(t))^*$.
(c_k) $X_k(t) \geq X_{k+1}(t), \forall t \in \mathcal{I}$.

Let $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$. Applying Lemma 5.1 to equation (5.27) verified by $\hat{X}(t)$ and $W(t) = F_0(t)$ one obtains

$$\hat{X}(t) = \Pi_{F_0}(t)\hat{X}(t+1) + Q_{F_0}(t) - \hat{M}(t) - (F_0(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_0(t) - \hat{F}(t)), \tag{5.32}$$

where

$$\hat{F}(t) = -(R(t) + \Pi_3(t)\hat{X}(t+1))^{-1}(L(t) + \Pi_2(t)\hat{X}(t+1))^T. \tag{5.33}$$

Subtracting (5.32) from (5.29) one obtains that $X_1(t) - \hat{X}(t)$ is a bounded solution of the backward affine equation:

$$X_1(t) - \hat{X}(t) = \Pi_{F_0}(t)(X_1(t+1) - \hat{X}(t+1)) + \Delta_1(t), \tag{5.34}$$

where $\Delta_1(t) = \varepsilon J_n + \hat{M}(t) + (F_0(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_0(t) - \hat{F}(t))$. Because $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$ one gets that $\Delta_1(t) \geq \varepsilon J_n \gg 0, t \in \mathcal{I}$.

Invoking Theorem 2.5(*iv*), we deduce that equation (5.34) has a unique bounded solution which is also uniformly positive. Moreover from (5.34) one obtains that $X_1(t) - \hat{X}(t) \geq \varepsilon J_n, t \in \mathcal{I}$. This confirms the validity of item (a_k) for $k = 1$.

If (a_1) is fulfilled, then $R(t) + \Pi_3(t)X_1(t+1) \gg 0$ and thus $F_1(t)$ is well defined by (5.31) for $k = 1$.

Applying Lemma 5.1 again to the equation (5.27) with $W(t) = F_1(t)$ we obtain:

$$\hat{X}(t) = \mathcal{L}_1^*(t)\hat{X}(t+1) + \mathcal{Q}_{F_1}(t) - \hat{M}(t) - (F_1(t) - \hat{F}(t))^T \times (R(t) + \Pi_3(t)\hat{X}(t+1))(F_1(t) - \hat{F}(t)). \tag{5.35}$$

On the other hand, applying Lemma 5.2 to equation (5.29) we obtain:

$$X_1(t) = \mathcal{L}_1^*(t)X_1(t+1) + \mathcal{Q}_{F_1}(t) + \varepsilon J_n + (F_1(t) - F_0(t))^T \times (R(t) + \Pi_3(t)X_1(t+1))(F_1(t) - F_0(t)). \tag{5.36}$$

Subtracting (5.35) from (5.36) we deduce that $X_1(t) - \hat{X}(t)$ solves the following discrete-time backward affine equation,

$$X_1(t) - \hat{X}(t) = \mathcal{L}_1^*(t)(X_1(t+1) - \hat{X}(t+1)) + H_1(t), \tag{5.37}$$

$t \in \mathcal{I}$, where

$$H_1(t) = \varepsilon J_n + \hat{M}(t) + (F_1(t) - F_0(t))^T (R(t) + \Pi_3(t) X_1(t+1))(F_1(t) - F_0(t))$$
$$+ (F_1(t) - \hat{F}(t))^T (R(t) + \Pi_3(t)\hat{X}(t+1))(F_1(t) - \hat{F}(t)).$$

Because $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$ and (a_1) is fulfilled, we may conclude that $H_1(t) \geq \varepsilon J_n \gg 0, t \in \mathcal{I}$.

Using implication $(vi) \to (i)$ in Theorem 2.4 with equation (5.37), we deduce that the sequence $\{\mathcal{L}_1(t)\}_{t\in\mathcal{I}}$ generates an exponentially stable evolution and thus we obtain that item (b_k) for $k = 1$ holds.

Furthermore, based on (b_1) together with Theorem 2.5, we deduce that $\{X_2(t)\}_{t\in\mathcal{I}}$ is well defined as a unique bounded solution of (5.30) for $k = 2$.

To check the validity of item (c_1) we subtract equation (5.30) written for $k = 2$ from equation (5.36) and obtain

$$X_1(t) - X_2(t) = \mathcal{L}_1^*(t)(X_1(t+1) - X_2(t+1)) + \tilde{H}_1(t) \tag{5.38}$$

with $\tilde{H}(t) = (\varepsilon/2)J_n + (F_1(t) - F_0(t))^T (R(t) + \Pi_3(t)X_1(t+1))(F_1(t) - F_0(t))$. We have $\tilde{H}_1(t) \geq (\varepsilon/2)J_n \gg 0, t \in \mathcal{I}$.

Using Theorem 2.5(iv) again in the case of equation (5.38), we conclude that

$$X_1(t) - X_2(t) \geq 0$$

which confirms the validity of item (c_k) for $k = 1$.

Let us assume that the items $(a_i), (b_i), (c_i)$ are fulfilled for $1 \leq i \leq k-1$ and prove their validity for $i = k$.

Using Lemma 5.1 with $W(t) = F_{k-1}(t)$ we rewrite equation (5.27) as

$$\hat{X}(t) = \Pi_{F_{k-1}}(t)\hat{X}(t+1) + \mathcal{Q}_{F_{k-1}}(t) - \hat{M}(t)$$
$$- (F_{k-1}(t) - \hat{F}(t))^T (R(t) + \Pi_3(t)\hat{X}(t+1))(F_{k-1}(t) - \hat{F}(t)), \tag{5.39}$$

$t \in \mathcal{I}$ with $\hat{F}(t)$ as in (5.33).

Subtracting (5.39) from (5.30) we deduce that $X_k(t) - \hat{X}(t)$ solves the discrete-time affine equation

$$X_k(t) - \hat{X}(t) = \Pi_{F_{k-1}}(t)(X_k(t) - \hat{X}(t+1)) + \Delta_k(t) \tag{5.40}$$

with $\Delta_k(t) = (\varepsilon/k)J_n + \hat{M}(t) + (F_{k-1}(t) - \hat{F}(t))^T (R(t) + \Pi_3(t)\hat{X}(t+1))$ $(F_{k-1}(t) - \hat{F}(t))$.

Because $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$ it follows that $\Delta_k(t) \geq (\varepsilon/k)J_n \gg 0, t \in \mathcal{I}$. Again applying Theorem 2.5(iv) we deduce that $X_k(t) - \hat{X}(t) \geq 0$; furthermore from (5.40) we get that

$$X_k(t) - \hat{X}(t) \geq \frac{\varepsilon}{k} J_n \tag{5.41}$$

which confirms the validity of (a_k) with $\mu_k = \varepsilon/k$.

From (5.41) it follows that $R(t) + \Pi_3(t)X_k(t+1) \gg 0$. This allows us to construct $F_k(t)$ as in (5.31).

To check the validity of (b_k) we apply Lemma 5.2 to rewrite equation (5.30) in the form:

$$\begin{aligned} X_k(t) = \mathcal{L}_k^*(t)X_k(t+1) + \mathcal{Q}_{F_k}(t) + \frac{\varepsilon}{k}J_n \\ + (F_k(t) - F_{k-1}(t))^T(R(t) + \Pi_3(t)X_k(t+1))(F_k(t) - F_{k-1}(t)). \end{aligned} \tag{5.42}$$

On the other hand applying Lemma 5.1 with $W(t) = F_k(t)$ we rewrite (5.27) in the form:

$$\begin{aligned} \hat{X}(t) = \mathcal{L}_k^*(t)\hat{X}(t+1) + \mathcal{Q}_{F_k}(t) - \hat{M}(t) \\ - (F_k(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_k(t) - \hat{F}(t)) \end{aligned} \tag{5.43}$$

with $\hat{F}(t)$ as in (5.32). Subtracting (5.43) from (5.42) and taking into account (5.41) we obtain that $X_k(t) - \hat{X}(t), t \in \mathcal{I}$ is a bounded and uniform positive solution of the following discrete-time backward affine equation,

$$Y(t) = \mathcal{L}_k^*(t)Y(t+1) + H_k(t), \tag{5.44}$$

where $H_k(t) = (\varepsilon/k)J_n + \hat{M}(t) + (F_k(t) - F_{k-1}(t))^T(R(t) + \Pi_3(t)X_k(t+1))(F_k(t) - F_{k-1}(t)) + (F_k(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_k(t) - \hat{F}(t)), t \in \mathcal{I}$.

Because (a_k) is fulfilled it follows that $H_k(t) \geq (\varepsilon/k)J_n > 0, t \in \mathcal{I}$. Applying the implication $(vi) \to (i)$ of Theorem 2.4 to equation (5.44) we may conclude that $\{\mathcal{L}_k(t)\}_{t\in\mathcal{I}}$ generates an exponentially stable evolution. This confirms the validity of (b_k). To check that (c_k) holds we subtract equation (5.30) (with k replaced by $k+1$) from (5.42) and obtain:

$$X_k(t) - X_{k+1}(t) = \mathcal{L}_k^*(t)(X_k(t+1) - X_{k+1}(t+1)) + \tilde{H}_k(t), \qquad t \in \mathcal{I}, \tag{5.45}$$

where $\tilde{H}_k(t) = (\varepsilon/(k(k+1)))J_n + (F_k(t) - F_{k-1}(t))^T(R(t) + \Pi_3(t)X_k(t+1))(F_k(t) - F_{k-1}(t))$. We have $\tilde{H}_k(t) \geq (\varepsilon/(k(k+1)))J_n > 0, t \in \mathcal{I}$.

Applying Theorem 2.5(iv) to equation (5.45) we deduce that $X_k(t) - X_{k+1}(t) \geq 0$ and thus (c_k) is fulfilled.

From (a_k) and (c_k) we conclude that for each $t \in \mathcal{I}$ the sequence $\{X_k(t)\}_{k\geq 1}$ is convergent in $\mathcal{S}_n^N$.

Let

$$X_{\max}(t) = \lim_{k\to\infty} X_k(t), \qquad t \in \mathcal{I}. \tag{5.46}$$

Taking the limit for $k \to \infty$ in (5.30) and (5.31) one obtains that $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ defined by (5.46) solves (5.8).

On the other hand from (a_k) one obtains that $X_{\max}(t) \geq \hat{X}(t), t \in \mathcal{I}$ for the arbitrary sequence $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$. This shows that $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ defined by (5.46) is the maximal and bounded solution of (5.8). If $\{\Pi(t)\}_{t\in\mathcal{I}}, \{\mathcal{Q}(t)\}_{t\in\mathcal{I}}$ are periodic sequences with period $\theta \geq 1$, then, based on Theorem 5.7, we may choose a stabilizing feedback gain $\{F_0(t)\}_{t\in\mathcal{I}}$ which is a periodic sequence with the same period θ. Using Theorem 2.5(ii), we deduce that the unique bounded solutions $\{X_k(t)\}_{t\in\mathcal{I}}$ of (5.29), (5.30) are periodic with period θ. Also the feedback gains $\{F_k(t)\}_{t\in\mathcal{I}}$ defined by (5.31) will be periodic with the same period θ.

Thus from (5.46) we conclude that $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ is periodic with the same period θ and the proof is complete. □

The following result can be viewed as an extension of the comparison theorem to the case of maximal solutions of (5.8).

Theorem 5.4 *Let $\Sigma_j = (\Pi, \mathcal{Q}_j)_{j\geq 1}, \Sigma = (\Pi, \mathcal{Q})$, where for each $t \in \mathcal{I}$, $\Pi(t), \mathcal{Q}(t)$ are as in* (5.12) *and* (5.13), *respectively, and*

$$\mathcal{Q}_j(t) = \begin{pmatrix} M_j(t) & L_j(t) \\ L_j^T(t) & R_j(t) \end{pmatrix}.$$

For each $j \geq 1$ let $X(t) = \mathcal{R}_j(t, X(t+1))$ be the solution of the equation of type (5.9) *associated with the pair $\Sigma_j = (\Pi, \mathcal{Q}_j)$. Assume that:*

(a) *The sequence of linear positive operators $\{\Pi(t)\}_{t\in\mathcal{I}}$ is stabilizable.*
(b) $\mathcal{Q}_j(t) \geq \mathcal{Q}_{j+1}(t) \geq \mathcal{Q}(t), t \in \mathcal{I}, j \geq 1.$
(c) $\lim\limits_{j\to\infty} \mathcal{Q}_j(t) = \mathcal{Q}(t), t \in \mathcal{I}.$
(d) *Γ^{Σ} is not empty.*

Then:

(i) *For each $j \geq 1$ the corresponding equation* (5.8) *associated with the pair Σ_j has a maximal and bounded solution $\{X^j_{\max}(t)\}_{t\in\mathcal{I}}$ that satisfies $R(t) + \Pi_j(t)X^j_{\max}(t+1) \gg 0, t \in \mathcal{I}$.*
(ii) *$X^j_{\max}(t) \geq X^{j+1}_{\max}(t) \geq X_{\max}(t), t \in \mathcal{I}$, $X_{\max}(t)$ being the maximal solution of* (5.8).
(iii) $\lim_{j\to\infty} X^j_{\max}(t) = X_{\max}(t), t \in \mathcal{I}.$

Proof. Based on Remark 5.1 one obtains that $\Gamma^{\Sigma} \subset \Gamma^{\Sigma_{j+1}} \subset \Gamma^{\Sigma_j}$. Applying $(i) \to (ii)$ in Theorem 5.3 we deduce the existence of the maximal and bounded solutions $\{X^j_{\max}(t)\}_{t\in\mathcal{I}}, j \geq 1$. Thus ($i$) is proved.

For each fixed $j \geq 1$, we consider the corresponding iterations $\{X^j_k(t)\}_{t\in\mathcal{I}}$ and $\{F_{j,k}(t)\}_{t\in\mathcal{I}}$, $k \geq 1$ defined by (5.30) and (5.31) specialized for the pairs $\Sigma_j = (\Pi, \mathcal{Q}_j)$. More precisely $\{X^j_k(t)\}_{t\in\mathcal{I}}$ is the unique and bounded solution of:

$$X^j_k(t) = \Pi_{F_{j,k-1}}(t)X^j_k(t+1) + \frac{\varepsilon}{k}J_n + \mathcal{Q}^j_{F_{j,k-1}}(t)$$

and

$$F_{j,k}(t) = -(R_j(t) + \Pi_3(t)X_k^j(t+1))^{-1}(L_j(t) + \Pi_2(t)X_k^j(t+1))^T,$$

where

$$\mathcal{Q}^j_{F_{j,k-1}}(t) = \begin{pmatrix} J_n \\ F_{j,k-1}(t) \end{pmatrix}^T \mathcal{Q}_j(t) \begin{pmatrix} J_n \\ F_{j,k-1}(t) \end{pmatrix}.$$

From item (b_k) in the proof of Theorem 5.3 we deduce that the sequences $\{\mathcal{L}_{j,k}(t)\}_{t\in\mathcal{I}}$ generate exponentially stable evolutions, where $\mathcal{L}_{j,k}(t) = \Pi^*_{F_{j,k}}(t)$. On the other hand applying Lemma 5.1 with $W(t) = F_{j,k-1}(t)$ one obtains that $\{X_k^j(t) - X_{\max}^{j+1}(t)\}_{t\in\mathcal{I}}$ is a bounded solution of

$$X_k^j(t) - X_{\max}^{j+1}(t) = \mathcal{L}^*_{j,k-1}(t)(X_k^{j(t+1)} - X_{\max}^{j+1}(t+1)) + G_k^j(t), \qquad t \in \mathcal{I} \tag{5.47}$$

with

$$\begin{aligned} G_k^j(t) = {} & \frac{\varepsilon}{k}J_n + \begin{pmatrix} J_n \\ F_{j,k-1}(t) \end{pmatrix}^T (\mathcal{Q}_j(t) - \mathcal{Q}_{j+1}(t)) \begin{pmatrix} J_n \\ F_{j,k-1}(t) \end{pmatrix} \\ & + (F_{j+1}(t) - F_{j,k-1}(t))^T (R_{j+1}(t) + \Pi_3(t)X_{\max}^{j+1}(t+1)) \\ & \times (F_{j+1}(t) - F_{j,k-1}(t)). \end{aligned}$$

We have $G_k^j(t) \geq (\varepsilon/k)J_n > 0, t \in \mathcal{I}$.

Invoking Theorem 2.5(iv) one deduces that (5.47) has a unique bounded solution that is positive, hence

$$X_k^j(t) - X_{\max}^{j+1}(t) \geq 0, \qquad \forall\, t \in \mathcal{I}, k \geq 1. \tag{5.48}$$

Taking the limit for $k \to \infty$ in (5.48) one gets that $X_{\max}^j(t) \geq X_{\max}^{j+1}(t), t \in \mathcal{I}, j \geq 1$. Thus the first inequality from (ii) is confirmed.

To obtain the second inequality of (ii) in the statement one repeats the above reasoning replacing $\{X_{\max}^{j+1}(t)\}_{t\in\mathcal{I}}$ with an arbitrary $\{\hat{X}_t\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$. Thus we deduce that

$$X_{\max}^j(t) \geq \hat{X}(t) \tag{5.49}$$

for all $j \geq 1, t \in \mathcal{I}$.

The second inequality of (ii) follows from (5.49) because $\{X_{\max}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$. Furthermore, from (ii) one obtains that for each $t \in \mathcal{I}$, the sequence $\{X_{\max}^j(t)\}_{j\geq 1}$ is convergent. Set

$$Y(t) = \lim_{j\to\infty} X_{\max}^j(t), \qquad t \in \mathcal{I}.$$

Taking the limit for $j \to \infty$ in equation (5.48) verified by $X_{\max}^j(t)$ one obtains that $\{Y(t)\}_{t\in\mathcal{I}}$ is a solution of (5.8). On the other hand taking the limit for $j \to \infty$ in (5.49) one obtains that $Y(t) \geq \hat{X}(t), t \in \mathcal{I}$ for arbitrary $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$. This shows that $Y(t) = X_{\max}(t)$ and thus the proof is complete. □

5.5 The stabilizing solution

In this section we deal with the problem of the existence of the stabilizing solution of DTSGRE (5.8). Also we display several useful properties of the stabilizing solution.

First, we prove the following.

Proposition 5.1 *If the set Γ^{Σ} is not empty then the stabilizing solution of the equation* (5.8) *if it exists coincides with the maximal solution of the equation* (5.8).

Proof. Let $\mathbf{X}_s = \{X_s(t)\}_{t\in\mathcal{I}}$ be a stabilizing solution of (5.8) and $F_s(t)$ the stabilizing feedback gain constructed as in (5.24). Let $\hat{\mathbf{X}} = \{\hat{X}(t)\}_{t\in\mathcal{I}}$ be an arbitrary sequence in Γ^{Σ}. Applying Lemma 5.1 with $W(t) = F_s(t)$, equation (5.8) verified by $\mathbf{X}_s$ may be written as

$$X_s(t) = \Pi_{F_s}(t)X_s(t+1) + \mathcal{Q}_{F_s}(t). \tag{5.50}$$

On the other hand equation (5.27) verified by $\hat{\mathbf{X}}$ may be rewritten in the form:

$$\begin{aligned}\hat{X}(t) = \Pi_{F_s}(t)\hat{X}(t+1) + \mathcal{Q}_{F_s}(t) - \hat{M}(t) - (F_s(t) - \hat{F}(t))^T \\ \times (R(t) + \Pi_3(t)X_s(t+1))(F_s(t) - \hat{F}(t)),\end{aligned} \tag{5.51}$$

where $\hat{M}(t) \geq 0$ and $\hat{F}(t)$ is defined as in (5.33). Subtracting (5.51) from (5.50) we obtain that $X_s(t) - \hat{X}(t)$ is a bounded solution of the discrete-time backward affine equation:

$$Z(t) = \mathcal{L}_s^*(t)Z(t+1) + H_s(t), \tag{5.52}$$

where $\mathcal{L}_s(t) = \Pi_{F_s}^*(t)$ and $H_s(t) = (F_s(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)$ $\hat{X}(t+1))(F_s(t) - \hat{F}(t)) + \hat{M}(t)$.

From (5.15) one obtains that $H_s(t) \geq 0, t \in \mathcal{I}$.

On the other hand, the fact that $\mathbf{X}_s$ is a stabilizing solution guarantees that the sequence $\{\mathcal{L}_s(t)\}_{t\in\mathcal{I}}$ generates an exponentially stable evolution. Applying Theorem 2.5 we conclude that (5.52) has a unique bounded solution that belongs to $\mathcal{S}_n^{N+}$. Hence $X_s(t) - \hat{X}(t) \geq 0, t \in \mathcal{I}$ for arbitrary $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$. This means that $\mathbf{X}_s$ is just the maximal solution of (5.8) and the proof is complete. □

From the above proposition one obtains the following corollary.

Corollary 5.2 *If Γ^{Σ} is not empty then equation* (5.8) *has at most one stabilizing solution.*

The next result provides an interesting property of the stabilizing solution of equation (5.8).

Theorem 5.5 *Assume:*

(a) *There exists an integer* $\theta \geq 1$, *such that* $\Pi(t+\theta) = \Pi(t), \mathcal{Q}(t+\theta) = \mathcal{Q}(t)$ *for all* $t \in \mathcal{I}$.
(b) *The set* Γ^{Σ} *is not empty.*

Under these conditions the stabilizing solution of equation (5.8), *if it exists, is periodic with the same period* θ.

Proof. Let $\mathbf{X}_s = \{X_s(t)\}_{t\in\mathcal{I}}$ be a stabilizing solution of (5.8). This means that if $T_s(t,t_0)$ is the linear evolution operator on $\mathcal{S}_n^N$ defined by the corresponding equation (5.23) then

$$\|T_s(t,t_0)\| \leq \beta q^{t-t_0}, \qquad t \geq t_0, \qquad t, t_0 \in \mathcal{I} \tag{5.53}$$

for some $\beta \geq 1, q \in (0,1)$ independent of t and t_0.

We define the sequence $\{X(t)^{\theta}\}_{t\in\mathcal{I}}$ by $X(t)^{\theta} = X_s(t+\theta), t \in \mathcal{I}$. It is easy to check that the sequence $\{X(t)^{\theta}\}_{t\in\mathcal{I}}$ is a solution of (5.8). We show that $\{X(t)^{\theta}\}_{t\in\mathcal{I}}$ is a stabilizing solution of (5.8) too.

Let $F^{\theta}(t)$ be defined as in (5.24) with $X(t)^{\theta}$ instead of $X_s(t)$. From (a) in the statement one obtains that $F^{\theta}(t) = F_s(t+\theta), t \in \mathcal{I}$. In the same way one can see that

$$\Pi_{F^{\theta}}(t) = \Pi_{F_s}(t+\theta), \qquad t \in \mathcal{I}. \tag{5.54}$$

Let $T_{\theta}(t,t_0)$ be the linear evolution operator defined by the discrete-time linear equation

$$Z(t+1) = \Pi^*_{F^{\theta}}(t)Z(t).$$

This means that $T_{\theta}(t,t_0) = \Pi^*_{F^{\theta}}(t-1)\Pi^*_{F^{\theta}}(t-2)\cdots\Pi^*_{F^{\theta}}(t_0)$ if $t > t_0$ and $T_{\theta}(t,t_0) = I_{\mathcal{S}_n^N}$ if $t = t_0$, where $I_{\mathcal{S}_n^N}$ is the identity operator on $\mathcal{S}_n^N$.

From (5.54) one gets $T_{\theta}(t,t_0) = T_s(t+\theta, t_0+\theta)$ for all $t \geq t_0, t, t_0 \in \mathcal{I}$. Hence $\|T_{\theta}(t,t_0)\| = \|T_s(t+\theta, t_0+\theta)\| \leq \beta q^{t-t_0}$. This shows that $\{X(t)^{\theta}\}_{t\in\mathcal{I}}$ is also a stabilizing solution of (5.8). Furthermore, from Corollary 5.2 we obtain that equation (5.8) has at most one bounded and stabilizing solution. Thus $X(t)^{\theta} = X_s(t), t \in \mathcal{I}$, which is equivalent to $X_s(t+\theta) = X_s(t), t \in \mathcal{I}$ and thus the proof is complete. □

Remark 5.3 In the special case $\theta = 1$, the coefficients of equation (5.8) are constant sequences. This corresponds to the so-called time-invariant case. The above theorem shows that in the time-invariant case the stabilizing solution of (5.8), if it exists, is constant and solves the following nonlinear algebraic equation,

$$X = \Pi_1 X + M - (L + \Pi_2 X)(R + \Pi_3 X)^{-1}(L + \Pi_2 X)^T. \tag{5.55}$$

The main result of this section is the following.

Theorem 5.6 *Under the considered assumptions the following are equivalent.*

(*i*) *The sequence of linear and positive operators* $\{\Pi(t)\}_{t\in\mathcal{I}} \subset \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ *is stabilizable and the set* $\tilde{\Gamma}^\Sigma$ *is not empty.*

(*ii*) *The equation* (5.8) *has a bounded and stabilizing solution* $\{X_s(t)\}_{t\in\mathcal{I}}$ *that satisfies*

$$R(t) + \Pi_3(t)X_s(t+1) \gg 0, \qquad t \in \mathcal{I}. \tag{5.56}$$

Proof. To prove $(i) \to (ii)$ we remark that if (i) holds then Γ^Σ is not empty. We apply the implication $(i) \to (ii)$ in Theorem 5.3 and deduce that equation (5.8) has a maximal solution $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ which satisfies

$$R(t) + \Pi_3(t)X_{\max}(t+1) \gg 0, \qquad t \in \mathcal{I}. \tag{5.57}$$

We show that $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ is just a stabilizing solution. Set

$$F(t) = -(R(t) + \Pi_3(t)X_{\max}(t+1))^{-1}(L(t) + \Pi_2(t)X_{\max}(t+1))^T. \tag{5.58}$$

Let $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \tilde{\Gamma}^\Sigma$ be fixed. We define $\hat{M}(t) = \mathcal{R}(t, \hat{X}(t+1)) - \hat{X}(t), t \in \mathcal{I}$. Using the Schur complement technique one obtains via (5.16) that $\hat{M}(t) \gg 0, t \in \mathcal{I}$. Applying Lemma 5.1 with $W(t) = F(t)$ to the equation of type (5.27) verified by $\hat{X}_t$ one obtains the following modified equation,

$$\begin{aligned}\hat{X}(t) = \Pi_F(t)\hat{X}(t+1) + \mathcal{Q}_F(t) - \hat{M}(t) - (F(t) - \hat{F}(t))^T \\ \times (R(t) + \Pi_3(t)\hat{X}(t+1))(F(t) - \hat{F}(t)),\end{aligned} \tag{5.59}$$

where $\hat{F}(t)$ is defined as in (5.33).

On the other hand, the DTSGRE (5.8) verified by $X_{\max}(t)$ can be written as

$$X_{\max}(t) = \Pi_F(t)X_{\max}(t+1) + \mathcal{Q}_F(t). \tag{5.60}$$

Subtracting (5.59) from (5.60) one obtains that $X_{\max}(t) - \hat{X}(t)$ is a bounded and positive semidefinite solution of the following backward affine equation,

$$Z(t) = \Pi_F(t)Z(t+1) + \tilde{G}(t), \tag{5.61}$$

where $\tilde{G}(t) = \hat{M}(t) + (F(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F(t) - \hat{F}(t)), t \in \mathcal{I}$.

We have $\tilde{G}(t) \geq \hat{M}(t) \gg 0, t \in \mathcal{I}$.

Using the implication $(vi) \to (i)$ in Theorem 2.4 in the case of equation (5.61) we conclude that the sequence $\{\Pi_F^*(t)\}_{t\in\mathcal{I}}$ generates an exponentially stable evolution on $\mathcal{S}_n^N$. This shows that $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ is just the stabilizing solution of (5.8) and thus we obtain the validity of the implication $(i) \Rightarrow (ii)$.

Conversely, if (ii) is fulfilled, we denote $\{X_s(t)\}_{t\in\mathcal{I}}$ the stabilizing solution of DTSGRE (5.8) that satisfies (5.56).

Let $\{F_s(t)\}_{t\in\mathcal{I}}$ be the corresponding stabilizing feedback gain associated as in (5.24). This means that the sequence $\{\Pi^*_{F_s}(t)\}_{t\in\mathcal{I}}$ generates an exponentially stable evolution on $\mathcal{S}_n^N$.

Thus we obtained that the sequence of linear and positive operators $\{\Pi(t)\}_{t\in\mathcal{I}}$ is stabilizable.

It remains to show that $\tilde{\Gamma}^{\Sigma}$ is not empty. To this end let us consider $\ell^\infty(\mathcal{I},\mathcal{S}_n^N)$ equipped with the norm:

$$\|\mathbf{X}\| = \sup_{t\in\mathcal{I}} |X(t)|_2$$

for all $\mathbf{X} = \{X(t)\}_{t\in\mathcal{I}} \in \ell^\infty(\mathcal{I},\mathcal{S}_n^N)$, $|\cdot|_2$ being the norm induced by the inner product (2.7).

$$(\ell^\infty(\mathcal{I},\mathcal{S}_n^N), \|\cdot\|)$$

is a real Banach space.

In this Banach space we consider the subset $\mathcal{U} = \{\mathbf{X} \in \ell^\infty(\mathcal{I},\mathcal{S}_n^N) | \mathbf{X} = \{X(t)\}_{t\in\mathcal{I}}$ and $R(t) + \Pi_3(t)X(t+1) \gg 0, t\in\mathcal{I}\}$.

One sees that $\{X_s(t)\}_{t\in\mathcal{I}} \in \mathcal{U}$ and $\mathcal{U}$ is an open subset. Let $\Psi : \mathcal{U}\times\mathbf{R} \to \ell^\infty(\mathcal{I},\mathcal{S}_n^N)$ be defined by

$$\Psi(\mathbf{X},\delta) = \mathbf{Y}, \tag{5.62}$$

where $\mathbf{Y} = \{Y(t)\}_{t\in\mathcal{I}}$, and

$$Y(t) = \mathcal{R}(t, X(t+1)) - X(t) + \delta J_n, \tag{5.63}$$

$\forall \mathbf{X}\in\mathcal{U}, \delta\in\mathbf{R}$.

We apply the implicit function theorem (see [102]) to the equation

$$\Psi(\mathbf{X},\delta) = 0. \tag{5.64}$$

We have $\Psi(\mathbf{X}_s, 0) = 0$. Let $(\partial\Psi/\partial\mathbf{X}) : \ell^\infty(\mathcal{I},\mathcal{S}_n^N) \to \ell^\infty(\mathcal{I},\mathcal{S}_n^N)$ be the partial derivative of the operator Ψ with respect to $\mathbf{X}$. We have to show that $(\partial\Psi/\partial\mathbf{X})(\mathbf{X}_s,0)$ is an isomorphism. To check that $(\partial\Psi/\partial\mathbf{X})(\mathbf{X}_s,0)$ is injective, we have to show that the linear equation on $\ell^\infty(\mathcal{I},\mathcal{S}_n^N)$,

$$\frac{\partial\Psi}{\partial\mathbf{X}}(\mathbf{X}_s,0)\mathbf{U} = 0, \tag{5.65}$$

has only the solution $\mathbf{U} = 0$. The equation (5.15) may be written as

$$\mathcal{R}'(t, X_s(t+1))U_{t+1} - U_t = 0, \qquad t\in\mathcal{I}. \tag{5.66}$$

Based on (5.21) one obtains that (5.65) becomes

$$U_t = \Pi_{F_s}(t)U_{t+1}. \tag{5.67}$$

The fact that X_s is the stabilizing solution and (5.8) together with Theorem 2.5 allows us to conclude that equation (5.67) has only the solution

$U_t = 0, t \in \mathcal{I}$. Thus we have that the equation (5.65) has only the solution $\mathbf{U} = 0$ which is equivalent to the injectivity of $(\partial \Psi / \partial \mathbf{X})(\mathbf{X}_s, 0)$.

To show the surjectivity of the partial derivative, we have to check that for any $\mathbf{Z} \in \ell^\infty(\mathcal{I}, \mathcal{S}_n^N)$ the equation

$$\frac{\partial \Psi}{\partial \mathbf{X}}(\mathbf{X}_s, 0)\mathbf{U} = \mathbf{Z} \tag{5.68}$$

has a solution. Based on (5.21), one obtains that (5.68) may be written as

$$U_t = \Pi_{F_s}(t) U_{t+1} - Z(t). \tag{5.69}$$

Applying Theorem 2.5 one obtains that the equation (5.69) has a solution $\{U_t\}_{t \in \mathcal{I}} \in \ell^\infty(\mathcal{I}, \mathcal{S}_n^N)$. This confirms the surjectivity of the partial derivative $(\partial \Psi / \partial \mathbf{X})(\mathbf{X}_s, 0)$. Also the continuity of $(\mathbf{X}, \delta) \to (\partial \Psi / \partial \mathbf{X})(\mathbf{X}, \delta)$ at $(\mathbf{X}, \delta) = (\mathbf{X}_s, 0)$ is obvious. Thus we obtained that the assumptions of the implicit function theorem are fulfilled for equation (5.64). Hence we deduce that there exist $\tilde{\delta} > 0$ and a smooth function $\delta \to \mathbf{X}_\delta : (-\tilde{\delta}; \tilde{\delta}) \to \mathcal{U}$ that satisfy

$$\Psi(\mathbf{X}_\delta, \delta) = 0 \tag{5.70}$$

for all $\delta \in (-\tilde{\delta}, \tilde{\delta})$ and $\lim_{\delta \to 0} \mathbf{X}_\delta = \mathbf{X}_s$.

From (5.63) one obtains that (5.70) becomes

$$X_\delta(t) = \mathcal{R}(t, X_\delta(t+1)) + \delta J_n \tag{5.71}$$

for all $t \in \mathcal{I}, \delta \in (-\tilde{\delta}, \tilde{\delta})$.

Because $\mathbf{X}_\delta \in \mathcal{U}$ it follows that

$$R(t) + \Pi_3(t) X_\delta(t+1) \gg 0, \qquad t \in \mathcal{I}. \tag{5.72}$$

From (5.71) and (5.72) it follows that $\mathbf{X}_\delta \in \tilde{\Gamma}^\Sigma$ if $\delta \in (-\tilde{\delta}, 0)$ and thus the proof ends. □

Furthermore we have the following.

Theorem 5.7 *Let $\{\Pi(t)\}_{t \in \mathcal{I}} \subset \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ be a sequence of linear and positive operators. If there exists an integer $\theta \geq 1$ such that $\Pi(t+\theta) = \Pi(t)$ for all $t \in \mathcal{I}$, then the following are equivalent.*

- *(i) The sequence $\{\Pi(t)\}_{t \in \mathcal{I}}$ is stabilizable.*
- *(ii) There exists a stabilizing feedback gain $\{F(t)\}_{t \in \mathcal{I}}$ with the property $F(t+\theta) = F(t), t \in \mathcal{I}$.*

Proof. The implication $(ii) \to (i)$ is obvious. We have to prove $(i) \to (ii)$. To this end we apply Theorem 5.6 to the DTSGRE defined by the pair $\Sigma_0 = (\Pi, \mathcal{Q}_0)$, where

$$\mathcal{Q}_0(t) = \begin{pmatrix} J_n & 0 \\ 0 & J_m \end{pmatrix}, \qquad t \in \mathcal{I}.$$

It is obvious that $0 \in \tilde{\Gamma}^{\Sigma_0}$. Thus from Theorem 5.6 we deduce that the equation

$$X(t) = \Pi_1(t)X(t+1) + J_n - (\Pi_2(t)X(t+1)) \times (J_m + \Pi_3(t)X(t+1))^{-1}(\Pi_2(t)X(t+1))^T \tag{5.73}$$

has a bounded and stabilizing solution $\{X_s(t)\}_{t\in\mathcal{I}}$.

This means that the sequence of linear operators $\{\Pi^*_{F_s}(t)\}_{t\in\mathcal{I}}$ generates an exponentially stable evolution on $\mathcal{S}_n^N$, with $F_s(t)$ defined as in (5.24). On the other hand from Theorem 5.5 applied to equation (5.73) we deduce that the stabilizing solution is periodic with period θ. Therefore the stabilizing feedback gain $F_s(t)$ is periodic. The proof is complete. □

Remark 5.4 The result proved in the above theorem shows that in the case of periodic sequences of linear and positive operators $\{\Pi(t)\}_{t\in\mathcal{I}}$, the definition of the concept of stabilizability may be done using only stabilizing feedback gains $\{F(t)\}_{t\in\mathcal{I}} \subset \mathcal{M}^N_{m,s}$ that are periodic sequences.

Finally one can conclude that from Theorem 5.5 and the proof of Theorem 5.6 (see (5.71), (5.72)) the next result directly follows.

Proposition 5.2 *Assume that the pair $\Sigma = (\Pi, \mathcal{Q})$ that defines the DTSGRE* (5.8) *satisfies the condition $\Pi(t+\theta) = \Pi(t)$ and $\mathcal{Q}(t+\theta) = \mathcal{Q}(t), t \in \mathcal{I}, \theta \geq 1$.*

Under these assumptions the following are equivalent.

(*i*) *The sequence $\{\Pi(t)\}_{t\in\mathcal{I}}$ is stabilizable and the set $\tilde{\Gamma}^{\Sigma}$ is not empty.*
(*ii*) *The DTSGRE* (5.8) *has a bounded and stabilizing solution $\{X_s(t)\}_{t\in\mathcal{I}}$ that satisfies the condition $R(t) + \Pi_3(t)X_s(t+1) \gg 0, t \in \mathcal{I}$.*
(*iii*) *The sequence $\{\Pi(t)\}_{t\in\mathcal{I}}$ is stabilizable and there exists a periodic sequence with period θ $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \tilde{\Gamma}^{\Sigma}$.*

Remark 5.5 From the above proposition we deduce that in the periodic case, with period $\theta \geq 1$, to check the fact that $\tilde{\Gamma}^{\Sigma}$ is not empty we have to verify the solvability of the system of LMIs $\mathcal{D}^{\Sigma}[X](t) > 0, t_0 \leq t \leq t_0 + \theta - 1$ for some $t_0 \in \mathcal{I}$ where $\mathcal{D}^{\Sigma}$ is the dissipation operator introduced by (5.14). Combining Corollary 5.2, Theorem 5.5, and Proposition 5.2 one obtains the following time-invariant counterpart of Theorem 5.6.

Theorem 5.8 *If $\Pi(t) = \Pi \in \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N), \mathcal{Q}(t) = \mathcal{Q} \in \mathcal{S}_{n+m}^N$, $t \in \mathcal{I}$, then the following are equivalent.*

(*i*) *The linear and positive operator Π is stabilizable and there exist $\hat{X} \in \mathcal{S}_n^N, \hat{X} = (\hat{X}(1), \hat{X}(2), \ldots, \hat{X}(N))$ such that*

$$\mathcal{D}^{\Sigma}\hat{X} > 0. \tag{5.74}$$

(*ii*) *The algebraic equation*

$$X = \Pi_1 X + M - (\Pi_2 X + L)(R + \Pi_3 X)^{-1}(\Pi_2 X + L)^T \tag{5.75}$$

has a stabilizing solution $X_s = (X_s(1), X_s(2), \dots, X_s(N))$ *that satisfies* $R + \Pi_3 X_s > 0$.

Remark 5.6

(*a*) Based on (5.14) it follows that the inequality (5.74) becomes

$$\begin{pmatrix} \Pi_1 \hat{X} - \hat{X} + M & \Pi_2 \hat{X} + L \\ (\Pi_2 \hat{X} + L)^T & R + \Pi_3 \hat{X} \end{pmatrix} > 0, \tag{5.76}$$

where

$$\begin{pmatrix} M & L \\ L^T & R \end{pmatrix}$$

is the corresponding partition of $\mathcal{Q}$.

(*b*) In the time-invariant case the stabilizing solution of (5.8) is a constant sequence and it solves (5.75), therefore one obtains that $X_s = (X_s(1), X_s(2), \dots, X_s(N))$ is a stabilizing solution of (5.75) if and only if the eigenvalues of the operators $\mathcal{R}'(X_s)$ are located in the inside of the disk $|\lambda| < 1$.

5.6 The Minimal Solution

In this section we focus our attention on the case $0 \in \Gamma^{\Sigma}$. From (5.15) it follows that $0 \in \Gamma^{\Sigma}$ if and only if

$$R(t) \gg 0, \qquad t \in \mathcal{I} \tag{5.77}$$

and

$$M(t) - L(t)R^{-1}(t)L^T(t) \geq 0, \qquad t \in \mathcal{I}. \tag{5.78}$$

In this case we prove the following.

Theorem 5.9 *Assume:*

(*a*) *the sequence of positive operators* $\{\Pi(t)\}_{t \in \mathcal{I}} \subset \mathcal{B}(\mathcal{S}_n^N, \mathcal{S}_{n+m}^N)$ *is stabilizable.*

(*b*) $0 \in \Gamma^{\Sigma}$.

Under these conditions the DTSGRE defined by $\Sigma = (\Pi, \mathcal{Q})$ *has two global and bounded solutions* $\{X_{\max}(t)\}_{t \in \mathcal{I}}$ *and* $\{X_{\min}(t)\}_{t \in \mathcal{I}}$ *with the property:*

$$0 \leq X_{\min}(t) \leq X(t) \leq X_{\max}(t), \qquad t \in \mathcal{I} \tag{5.79}$$

for arbitrary bounded solution $\{X(t)\}_{t \in \mathcal{I}}$ *of* (5.8) *with* $X(t) \geq 0, t \in \mathcal{I}$. *Furthermore, if there exists an integer* $\theta \geq 1$ *such that* $\Pi(t + \theta) = \Pi(t)$, $\mathcal{Q}(t + \theta) = \mathcal{Q}(t), t \in \mathcal{I}$, *then* $X_{\max}(t + \theta) = X_{\max}(t), X_{\min}(t + \theta) = X_{\min}(t)$, $t \in \mathcal{I}$.

Proof. The assumptions (a) and (b) in the statement guarantee (via Theorem 5.3) the existence of the maximal solution $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ with the required properties. We have to prove the existence of the minimal solution $\{X_{\min}(t)\}_{t\in\mathcal{I}}$. For each integer $\tau \in \mathcal{I}$, let $X_\tau(t) = X(t,\tau,0)$ be the solution of (5.8) with the terminal value $X(\tau,\tau,0) = 0$.

From Corollary 5.1 we deduce that $X_\tau(t)$ is well defined for all $t \in \mathcal{I}_\tau = (-\infty;\tau]\bigcap\mathcal{I}$ and

$$X_\tau(t) \geq 0 \tag{5.80}$$

for all $t \in I_\tau$.

If $\tau_1 < \tau_2, \tau_1, \tau_2 \in \mathcal{I}$ we have $X_{\tau_2}(\tau_1) \geq 0 = X_{\tau_1}(\tau_1)$.

Applying Theorem 5.1 we conclude that

$$X_{\tau_2}(t) \geq X_{\tau_1}(t) \tag{5.81}$$

for all $t \in \mathcal{I}_{\tau_1}$. Based on assumption (a) in the statement we deduce that there exists a stabilizing feedback gain $\{F_0(t)\}_{t\in\mathcal{I}} \subset \mathcal{M}_{mn}^N$ such that the sequence of linear and positive operators $\{\Pi_{F_0}^*(t)\}_{t\in\mathcal{I}}$ generates an exponentially stable evolution.

Applying Theorem 2.5(i) we deduce that the discrete-time backward affine equation

$$Y_0(t) = \Pi_{F_0}(t)Y_0(t+1) + Q_{F_0}(t) \tag{5.82}$$

has a bounded solution $\{Y_0(t)\}_{t\in\mathcal{I}} \subset \mathcal{S}_n^{N+}, \Pi_{F_0}(t), Q_{F_0}(t)$ being defined as in (5.18) and (5.19), respectively, with F_0 instead of W. On the other hand, Lemma 5.1 applied to equation (5.8) verified by $X_\tau(t)$ leads to

$$\begin{aligned} X_\tau(t) = {} & \Pi_{F_0}(t)X_\tau(t+1) + \mathcal{Q}_{F_0}(t) - (F_0(t) - F_\tau(t))^T \\ & \times (R(t) + \Pi_3(t)X_\tau(t+1))(F_0(t) - F_\tau(t)), \qquad t \in \mathcal{I}, \end{aligned} \tag{5.83}$$

where $F_\tau(t) = -(R(t) + \Pi_3(t)X_\tau(t+1))^{-1}(\Pi_2(t)X_\tau(t+1) + L(t))^T$.

Subtracting (5.83) from (5.82) we obtain

$$Y_0(t) - X_\tau(t) = \Pi_{F_0}(t)(Y_0(t+1) - X_\tau(t+1)) + H_\tau(t), \tag{5.84}$$

where $H_\tau(t) = (F_0(t) - F_\tau(t))^T(R(t) + \Pi_3(t)X_\tau(t+1))(F_0(t) - F_\tau(t))$. From (5.77) and (5.80) we conclude that $H_\tau(t) \geq 0, t \in \mathcal{I}$.

Because $Y_0(\tau) - X_\tau(\tau) = Y_0(\tau) \geq 0$ we deduce inductively from (5.84) that

$$X_\tau(t) \leq Y_0(t) \tag{5.85}$$

for all $t \in \mathcal{I}_\tau$ and all $\tau \in \mathcal{I}$. From (5.80), (5.81), and (5.85)

$$0 \leq X_{\tau_1}(t) \leq X_{\tau_2}(t) \leq Y_0(t) \leq cJ_n \tag{5.86}$$

for all $t \in \mathcal{I}_{\tau_1}, \tau_1 < \tau_2, \tau_1, \tau_2 \in \mathcal{I}$, where $c > 0$ is independent of t and τ_i. From (5.86) one deduces that for each $t \in \mathcal{I}$ the sequence $\{X_\tau(t)\}_{\tau \geq t}$ is convergent.

Set

$$X_{\min}(t) = \lim_{\tau \to \infty} X_\tau(t), \qquad t \in \mathcal{I}. \tag{5.87}$$

Replacing $X(t)$ by $X_\tau(t)$ in (5.8) and taking the limit for $\tau \to \infty$ one obtains that $X_{\min}(t)$ solves DTSGRE (5.8).

Moreover from (5.86) and (5.87) we have $X_{\min}(t) \geq 0, t \in \mathcal{I}$. Let $\{X(t)\}_{t \in \mathcal{I}} \subset \mathcal{S}_n^{N+}$ be another bounded solution of DTSGRE (5.8). From $X(\tau) \geq 0 = X_\tau(\tau)$ together with Theorem 5.1 we deduce that

$$X(t) \geq X_\tau(t) \tag{5.88}$$

for all $t \in \mathcal{I}_\tau$.

From (5.87) and (5.88) we obtain that $X_{\min}(t) \leq X(t)$ for all $t \in \mathcal{I}$. This confirms the validity of (5.79).

It remains to prove that if $\{\Pi(t)\}_{t \in \mathcal{I}}$ and $\{\mathcal{Q}(t)\}_{t \in \mathcal{I}}$ are periodic sequences with period $\theta \geq 1$, then $\{X_{\min}(t)\}_{t \in \mathcal{I}}$ is also a periodic sequence with the same period θ.

To this end we define

$$\check{X}_\tau(t) = X_{\tau+\theta}(t + \theta), \qquad t \in \mathcal{I}_\tau. \tag{5.89}$$

Based on the periodicity assumption one obtains that $\{\check{X}_\tau(t)\}_{t \in \mathcal{I}_\tau}$ is a solution of DTSGRE (5.8).

Also we have $\check{X}_\tau(\tau) = 0 = X_\tau(\tau)$. This allows us to obtain inductively that

$$\check{X}_\tau(t) = X_\tau(t), \qquad t \in \mathcal{I}_\tau \tag{5.90}$$

for all $\tau \in \mathcal{I}$. From (5.87) and (5.90) one obtains

$$\lim_{\tau \to \infty} \check{X}_\tau(t) = X_{\min}(t), \qquad t \in \mathcal{I}. \tag{5.91}$$

On the other hand from (5.87) and (5.89) we deduce that

$$\lim_{\tau \to \infty} \check{X}_\tau(t) = X_{\min}(t + \theta). \tag{5.92}$$

From (5.91) and (5.92) one concludes that $X_{\min}(t + \theta) = X_{\min}(t)$ for all $t \in \mathcal{I}$ and thus the proof is complete. □

The result proved in the previous theorem suggests the following definition.

Definition 5.3 *We say that a solution* $\{X_{\min}(t)\}_{t \in \mathcal{I}}$ *of DTSGRE* (5.8) *is minimal in the class of positive semidefinite solutions of* (5.8) *if* $0 \leq X_{\min}(t) \leq X(t)$ *for any solution* $\{X(t)\}_{t \in \mathcal{I}} \subset \mathcal{S}_n^{N+}$ *of DTSGRE* (5.8).

Remark 5.7 It is known (see [27, 89, 90]) that under the assumption of stochastic detectability, any positive semidefinite solution of difference Riccati equations (5.1)–(5.3) is a stabilizing solution. On the other hand from the uniqueness of the stabilizing solution, in the presence of stochastic detectability the discrete-time Riccati equations (5.1)–(5.3) have at most one positive semidefinite solution. In this case $X_{\min}(t) = X_{\max}(t), t \in \mathcal{I}$.

The following example shows that in the absence of stochastic detectability the maximal solution does not coincide with the minimal solution.

Example 5.1 Consider equation (5.1) in the special case $n = 2, r = 1$:

$$\begin{aligned} X(t) = {} & A_0^T X(t+1)A_0 + A_1^T X(t+1)A_1 + C^T C \\ & - (A_0^T X(t+1)B_0 + A_1^T X(t+1)B_1 + C^T D) \\ & \times (D^T D + B_0^T X(t+1)B_0 + B_1^T X(t+1)B_1)^{-1} \\ & \times (B_0^T X(t+1)A_0 + B_1^T X(t+1)A_1 + D^T C), \end{aligned} \tag{5.93}$$

where

$$A_0 = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{8}{5} \end{pmatrix}; \qquad A_1 = \frac{3}{5} I_2; \qquad B_0 = (2 \quad 1)^T; \qquad B_1 = (0 \quad 0)^T;$$

$$C = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & 0 \end{pmatrix}; \qquad D = (0 \quad 1)^T.$$

It can be seen that the pair $(C, (A_0, A_1))$ is not stochastic detectable. Let

$$X_s = \begin{pmatrix} \frac{7+\sqrt{2}}{2} & -3(3+\sqrt{2}) \\ -3(3+\sqrt{2}) & 3(11+6\sqrt{2}) \end{pmatrix}. \tag{5.94}$$

By direct calculations one verifies that X_s is a solution of (5.93). We prove that X_s is the stabilizing solution of (5.93). Indeed, by the special case of Theorem 3.7 for $N = 1$ one obtains that X_s is the stabilizing solution of (5.93), iff there exists $\tilde{X} > 0$ such that $\tilde{A}^T \tilde{X} \tilde{A} + (9/25)\tilde{X} - \tilde{X} < 0$, where

$$\tilde{A} = \begin{pmatrix} \frac{4\sqrt{2}}{5} & -\frac{12}{5}(7-6\sqrt{2}) \\ \frac{2}{5}(-1+\sqrt{2} & \frac{2}{5}(7-6\sqrt{2}) \end{pmatrix}.$$

The above inequality is equivalent to $\hat{A}^T \tilde{X} \hat{A} - \tilde{X} < 0$, where $\hat{A} = (5/4)\tilde{A}$. By the well-known discrete-time Lyapunov theorem from the deterministic case the preceding inequality is equivalent to $\rho(\hat{A}) < 1$. This inequality is verified easily by simple calculations. Thus we conclude that X_s is a stabilizing solution of (5.93).

Because $0 \in \Gamma^{\Sigma}$ one obtains via Proposition 5.1 that $X_s = X_{\max}$. By direct calculations one obtains that (5.93) has only two positive semidefinite solutions, namely X_s and

$$\hat{X} = \begin{pmatrix} \frac{1+\sqrt{2}}{2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.95}$$

Because $X_s \geq \hat{X}$ it follows that $\hat{X}$ is the minimal solution of (5.93).

From (5.94) and (5.95) we conclude that in the case of equation (5.93) the maximal solution does not coincide with the minimal solution.

5.7 An iterative procedure to compute the maximal solution and the stabilizing solution of DTSGRE

The iterative procedure described by (5.30)–(5.31) could be used to compute numerically the maximal solution of (5.8). However, at each step of the above procedure, we have to compute the unique bounded solution of a linear equation on $\mathcal{S}_n^N$. Unfortunately, in the case of discrete-time Riccati equations, corresponding to the stochastic framework, the equations (5.30) have a complicated structure and consequently it is not easy to solve them efficiently. That is why it is desirable to have numerical procedures to compute the maximal solution of DTSGRE (5.8), simpler than the ones based on the Newton–Kantorovich method. Our goal is to provide such an iterative procedure to compute the maximal (stabilizing) solution of (5.8).

Assume that $\{\Pi(t)\}_{t\in\mathcal{I}}$ is a stabilizable sequence of linear operators. Let $\{\tilde{F}_0(t)\}_{t\in\mathcal{I}}$ be a stabilizing feedback gain. This means that the zero state equilibrium of the linear equation $Z(t+1) = \Pi^*_{\tilde{F}_0}(t)Z(t)$ is exponentially stable.

Let $\{X_0(t)\}_{t\in\mathcal{I}}$ be a bounded solution of the following inequality,

$$X_0(t) \geq \Pi_{\tilde{F}_0}(t)X_0(t+1) + \mathcal{Q}_{\tilde{F}_0}(t) + \varepsilon J_n, \tag{5.96}$$

and $t \in \mathcal{I}$, $\varepsilon > 0$ are fixed.

We prove the following.

Lemma 5.3 *Assume:*

(*a*) *The sequence* $\{\Pi(t)\}_{t\in\mathcal{I}}$ *is stabilizable.*
(*b*) *The set* Γ^{Σ} *is not empty.*

Let $\{\tilde{F}_0(t)\}_{t\in\mathcal{I}}$ *be a stabilizing feedback gain. Then any bounded solution* $\{X_0(t)\}_{t\in\mathcal{I}}$ *of* (5.96) *has the properties:*

(i) $X_0(t) \geq \hat{X}(t)$ *for any* $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$.

(ii) If F_0 *is defined by*

$$F_0(t) = -(R(t) + \Pi_3(t)X_0(t+1))^{-1}(L(t) + \Pi_2(t)X_0(t+1))^T \quad (5.97)$$

then the zero state equilibrium of the linear equation $Z(t+1) = \Pi_{F_0}(t)^*$ $Z(t)$ *is exponentially stable.*

(iii) Let $\{X_1(t)\}_{t\in\mathcal{I}}$ *be defined by*

$$X_1(t) = \Pi_{F_0}(t)X_0(t+1) + Q_{F_0}(t) + \varepsilon J_n \quad (5.98)$$

$t \in \mathcal{I}$. *Then we have* $X_0(t) \geq X_1(t)$, $t \in \mathcal{I}$.

Proof. At the beginning we remark that (5.96) can be written as a linear equation:

$$X_0(t) = \Pi_{\tilde{F}_0}(t)X_0(t+1) + \mathcal{Q}_{\tilde{F}_0}(t) + \varepsilon J_n + \check{M}(t), \quad (5.99)$$

where $\check{M}(t) \in \mathcal{S}_n^{N+}$, $t \in \mathcal{I}$.

If $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$ then one obtains via the Schur complement technique that $\{\hat{X}(t)\}_{t\in\mathcal{I}}$ solves

$$\hat{X}(t) = \mathcal{R}(t, \hat{X}(t+1)) - \hat{M}(t) \quad (5.100)$$

for some $\hat{M}(t) \in \mathcal{S}_n^{N+}$. Applying Lemma 5.1$(i)$ to the operator $\hat{\mathcal{R}}(t,X) = \mathcal{R}(t,X) - \hat{M}(t)$ with $W(t) = \tilde{F}_0(t)$ one obtains that equation (5.100) may be rewritten:

$$\begin{aligned}\hat{X}(t) = \Pi_{\tilde{F}_0}(t)\hat{X}(t+1) + \mathcal{Q}_{\tilde{F}_0}(t) - \hat{M}(t) - (\tilde{F}_0(t) - \hat{F}(t))^T \\ \times (R(t) + \Pi_3(t)\hat{X}(t+1))(\tilde{F}_0(t) - \hat{F}(t)),\end{aligned} \quad (5.101)$$

$t \in \mathcal{I}$, where $\hat{F}(t) = F^{\hat{X}}(t)$, being as in (5.20) with $\hat{X}(t)$ instead of $X(t)$.

From (5.101) and (5.99) one deduces that $\{X_0(t) - \hat{X}(t)\}_{t\in\mathcal{I}}$ is a bounded solution of

$$X_0(t) - \hat{X}(t) = \Pi_{\tilde{F}_0}(t)[X_0(t+1) - \hat{X}(t+1)] + H_0(t), \quad (5.102)$$

where $H_0(t) = \varepsilon J_n + (\tilde{F}_0(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(\tilde{F}_0(t) - \hat{F}(t)) + \check{M}(t) + \hat{M}(t)$.

Based on (5.15) we may conclude that $H_0(t) \geq \varepsilon J_n, \forall t \in \mathcal{I}$. This allows us to deduce via Theorem 2.5 that (5.102) has a unique bounded solution that is uniformly positive. More precisely, we have

$$X_0(t) - \hat{X}(t) \geq \varepsilon J_n, \qquad t \in \mathcal{I}. \quad (5.103)$$

Thus (i) is proved. Furthermore, from (5.103) we obtain

$$R(t) + \varPi_3(t)X_0(t+1) \geq R(t) + \varPi_3(t)\hat{X}(t+1) \gg 0, \qquad t \in \mathcal{I}.$$

Hence $F_0(t)$ is well defined by (5.97).

To check that (ii) holds we rewrite (5.99) (via Lemma 5.2) as

$$\begin{aligned} X_0(t) = \varPi_{F_0}(t)X_0(t+1) + \mathcal{Q}_{F_0}(t) + \varepsilon J_n + (F_0(t) - \tilde{F}_0(t))^T \\ \times (R(t) + \varPi_3(t)X_0(t+1))(F_0(t) - \tilde{F}_0(t)) + \check{M}(t). \end{aligned} \tag{5.104}$$

On the other hand, applying Lemma 5.1 with $W(t) = F_0(t)$ in the case of equation (5.100) we obtain that $\{\hat{X}(t)\}_{t\in\mathcal{I}}$ verifies:

$$\begin{aligned} \hat{X}(t) = \varPi_{F_0}(t)\hat{X}(t+1) + \mathcal{Q}_{F_0}(t) - (F_0(t) - \hat{F}(t))^T \\ \times (R(t) + \varPi_3(t)\hat{X}(t+1))(F_0(t) - \hat{F}(t)) - \hat{M}(t). \end{aligned} \tag{5.105}$$

Subtracting (5.105) from (5.104) we deduce that $\{X_0(t) - \hat{X}(t)\}_{t\in\mathcal{I}}$ is a bounded and uniform positive solution of the following backward affine equation,

$$Y(t) = \varPi_{F_0}(t)Y(t+1) + \tilde{H}_0(t), \qquad t \in \mathcal{I}, \tag{5.106}$$

where $\tilde{H}_0(t) = \varepsilon J_n + \check{M}(t) + \hat{M}(t) + (F_0(t) - \tilde{F}_0(t))^T(R(t) + \varPi_3(t) X_0(t+1))(F_0(t) - \tilde{F}_0(t)) + (F_0(t) - \hat{F}(t))^T(R(t) + \varPi_3(t)\hat{X}(t+1))(F_0(t) - \hat{F}(t))$.

One sees that $\tilde{H}_0(t) \geq \varepsilon J_n, t \in \mathcal{I}$.

Using the implication $(vi) \to (i)$ in Theorem 2.4 together with (5.103) to equation (5.106) we conclude that the zero state equilibrium of the equation $Z(t+1) = \varPi^*_{F_0}(t)Z(t)$ is exponentially stable. This shows that (ii) is valid.

To check the validity of (iii) one subtracts (5.98) from (5.104) and we obtain that

$$X_0(t) - X_1(t) = \varDelta_0(t), \tag{5.107}$$

where $\varDelta_0(t) = \check{M}(t) + (F_0(t) - \tilde{F}_0(t))^T(R(t) + \varPi_3(t)X_0(t+1))(F_0(t) - \tilde{F}_0(t))$.

We have $\varDelta_0(t) \geq \check{M}(t) \geq 0, t \in \mathcal{I}$. Therefore (5.107) shows that $X_0(t) - X_1(t) \geq 0, t \in \mathcal{I}$. Thus the proof is complete. □

Taking $\{X_0(t)\}_{t\in\mathcal{I}}$, $\{F_0(t)\}_{t\in\mathcal{I}}$ as the first step we construct inductively the sequences $\{X_k(t)\}_{k\geq 1}$, $\{F_k(t)\}_{k\geq 1}$, $t \in \mathcal{I}$ as follows,

$$X_k(t) = \varPi_{F_{k-1}}(t)X_{k-1}(t+1) + \mathcal{Q}_{F_{k-1}}(t) + \frac{\varepsilon}{k}J_n, \qquad t \in \mathcal{I}, \qquad k \geq 1, \tag{5.108}$$

$$\begin{aligned} F_k(t) = -(R(t) + \varPi_3(t)X_k(t+1))^{-1}(L(t) + \varPi_2(t)X_k(t+1))^T, \\ t \in \mathcal{I}, \qquad k \geq 1. \end{aligned} \tag{5.109}$$

Remark 5.8 Applying Theorem 5.7, we deduce that if $\{\Pi(t)\}_{t\in\mathcal{I}}$ is a periodic sequence with period $\theta \geq 1$, then it is stabilizable if and only if there exists a stabilizing feedback gain $\{\tilde{F}_0(t)\}_{t\in\mathcal{I}}$ that is a periodic sequence with the same period θ. This allows us to conclude that if the coefficients of the DTSGRE (5.8) are periodic sequences with period $\theta \geq 1$, then one can find a solution $\{X_0(t)\}_{t\in\mathcal{I}}$ of (5.96) that is a periodic sequence with the same period θ. Furthermore, from (5.108) and (5.109) it follows that at each step k, $\{X_k(t)\}_{t\in\mathcal{I}}$ will be periodic sequences with the same period θ.

To prove the convergence of the sequences defined above, the following auxiliary result is helpful.

Lemma 5.4 *Assume that for a* $k \geq 1$, *the sequences* $\{X_q(t)\}_{t\in\mathcal{I}}$, $1 \leq q \leq k$, *are well defined via* (5.108) *and* $(t, X_q(t+1)) \in \mathrm{Dom}(\mathcal{R})$, $1 \leq q \leq k, t \in \mathcal{I}$. *Then the following equality holds.*

$$\begin{aligned} X_k(t) = {} & \Pi_{F_k}(t)X_k(t+1) + \Pi_{F_{k-1}}(t)(X_{k-1}(t+1) - X_k(t+1)) + \mathcal{Q}_{F_k}(t) \\ & + (F_k(t) - F_{k-1}(t))^T(R(t) + \Pi_3(t)X_k(t+1)) \\ & \times (F_k(t) - F_{k-1}(t)) + \frac{\varepsilon}{k}J_n. \end{aligned} \tag{5.110}$$

Proof. From the assumption in the statement of the lemma it follows that $\{F_q(t)\}_{t\in\mathcal{I}}$ are well defined via (5.109). Applying Lemma 5.1 with $W(t) = F_{k-1}(t)$ we obtain:

$$\begin{aligned} \mathcal{R}(t, X_k(t+1)) = {} & \Pi_{F_{k-1}}(t)X_k(t+1) + \mathcal{Q}_{F_{k-1}}(t) - (F_k(t) - F_{k-1}(t))^T \\ & \times (R(t) + \Pi_3(t)X_k(t+1))(F_k(t) - F_{k-1}(t)), \qquad t \in \mathcal{I}. \end{aligned} \tag{5.111}$$

Furthermore, using (5.108) we have

$$\begin{aligned} \mathcal{R}(t, X_k(t+1)) = {} & X_k(t) - \Pi_{F_{k-1}}(t)(X_{k-1}(t+1) - X_k(t+1)) \\ & - n(F_k(t) - F_{k-1}(t))^T(R(t) + \Pi_3(t)X_k(t+1)) \\ & \times (F_k(t) - F_{k-1}(t)) - \frac{\varepsilon}{k}J_n. \end{aligned} \tag{5.112}$$

Rewriting the left-hand side of (5.112) via Lemma 5.1 with $W(t) = F_k(t)$ one obtains (5.110) and thus the proof ends. □

Now we are in position to prove the main result of this section.

Theorem 5.10 *Assume:*

(a) The sequence of linear and positive operators $\{\Pi(t)\}_{t\in\mathcal{I}}$ *is stabilizable.*
(b) The set Γ^{Σ} *is not empty.*

Let $\varepsilon > 0$ be fixed and $\{\tilde{F}_0(t)\}_{t\in\mathcal{I}}$ be a stabilizing feedback gain. Then for any bounded solution $\{X_0(t)\}_{t\in\mathcal{I}}$ of (5.96) *the sequences $\{X_k(t)\}_{k\geq 1}$, $\{F_k(t)\}_{k\geq 1}$, $t \in \mathcal{I}$ introduced by* (5.108)*, and* (5.109)*, respectively, are well defined and convergent. Moreover, if $X_{\max}(t) = \lim_{k\to\infty} X_k(t)$, then $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ is just the maximal solution of DTSGRE* (5.8) *satisfying $R(t)+\Pi_3(t)X_{\max}(t+1) \gg 0$, $t \in \mathcal{I}$.*

Proof. We prove inductively for $k \geq 1$ the following items.

(a_k) $X_k(t) - \hat{X}(t) \geq \mu_k J_n$ for arbitrary $\{\hat{X}(t)\}_{t\in\mathcal{I}} \in \Gamma^{\Sigma}$, where $\mu_k > 0$ do not depend upon $\hat{X}(t)$.
(b_k) The zero state equilibrium of the linear equation $Z(t+1) = \Pi^*_{F_k}(t)Z(t)$ is exponentially stable.
(c_k) $X_k(t) \geq X_{k+1}(t)$, $t \in \mathcal{I}$.

Subtracting (5.105) from (5.98) one obtains that $\{X_1(t) - \hat{X}(t)\}_{t\in\mathcal{I}}$ is a bounded solution of the discrete-time backward affine equation:

$$Z_1(t) = \Pi_{F_0}(t)Z_1(t+1) + H_1(t), \qquad t \in \mathcal{I} \tag{5.113}$$

with $H_1(t) = \varepsilon J_n + \hat{M}(t) + (F_0(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_0(t) - \hat{F}(t))$.
It is obvious that $H_1(t) \geq \varepsilon J_n > 0, t \in \mathcal{I}$.

Because $\{F_0(t)\}_{t\in\mathcal{I}}$ is a stabilizing feedback gain, one obtains, via Theorem 2.5, that equation (5.113) has a unique bounded solution and that solution is uniformly positive. Finally from (5.113) follows

$$X_1(t) - \hat{X}(t) \geq \varepsilon J_n, \qquad t \in \mathcal{I}. \tag{5.114}$$

This shows that (a_1) is valid with $\mu_1 = \varepsilon > 0$.

Furthermore, from (5.114) one obtains that $F_1(t)$ is well defined via (5.109) for $k = 1$. To check that (b_1) is fulfilled we rewrite equation (5.100) in the form:

$$\begin{aligned}\hat{X}(t) = \Pi_{F_1}(t)\hat{X}(t) + \mathcal{Q}_{F_1}(t) - \hat{M}(t) - (F_1(t) - \hat{F}(t))^T \\ \times (R(t) + \Pi_3(t)\hat{X}(t+1))(F_1(t) - \hat{F}(t)).\end{aligned} \tag{5.115}$$

Subtracting (5.115) from (5.110) (written for $k = 1$), we obtain that $\{X_1(t) - \hat{X}(t)\}_{t\in\mathcal{I}}$ is a bounded and uniformly positive solution of

$$Y_1(t) = \Pi_{F_1}(t)Y_1(t+1) + \tilde{H}_1(t), \qquad t \in \mathcal{I}, \tag{5.116}$$

where $\tilde{H}_1(t) = \Pi_{F_0}(t)[X_0(t+1) - X_1(t+1)] + \varepsilon J_n + \hat{M}(t) + (F_1(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_1(t) - \hat{F}(t)) + (F_1(t) - F_0(t))^T(R(t) + \Pi_3(t) X_1(t+1))(F_1(t) - F_0(t))$, $t \in \mathcal{I}$.

Based on Lemma 5.3(*iii*) we conclude that $\tilde{H}_1(t) \geq \varepsilon J_n > 0$, $t \in \mathcal{I}$. Now, the implication (*vi*) $\to$ (*i*) of Theorem 2.4 together with (5.114) applied to equation (5.116) leads to the exponential stability of the zero state equilibrium of the equation

$$Z(t+1) = \Pi^*_{F_1}(t)Z(t).$$

Thus the validity of item (b_1) is confirmed.

Furthermore, subtracting (5.108) (written for $k = 2$) from (5.110) (written for $k = 1$) we obtain:

$$X_1(t) - X_2(t) = \Delta_2(t), \tag{5.117}$$

where $\Delta_2(t) = (\varepsilon/2)J_n + (F_0(t) - F_1(t))^T(R(t) + \Pi_3(t)X_1(t+1))(F_0(t) - F_1(t)) + \Pi_{F_1}(t)(X_0(t+1) - X_1(t+1))$.

From Lemma 5.3(iii) we conclude that $\Delta_2(t) \geq (\varepsilon/2)J_n > 0$, $t \in \mathcal{I}$. Therefore (5.117) leads to $X_1(t) - X_2(t) \geq 0$, $t \in \mathcal{I}$, and thus the item (c_1) is fulfilled. Let us assume that $(a_q), (b_q), (c_q)$ are fulfilled for $1 \leq q \leq k-1$ and we prove that they are fulfilled for $q = k$. If (a_{k-1}) holds then $R(t) + \Pi_3(t)X_{k-1}(t+1) \gg 0$, $t \in \mathcal{I}$. Hence, $F_{k-1}(t)$ is well defined by (5.109) written for $k-1$ instead of k. This allows us to construct $X_k(t)$ via (5.108).

Applying Lemma 5.1 with $W(t) = F_{k-1}(t)$ we rewrite equation (5.100) in the form:

$$\begin{aligned}\hat{X}(t) &= \Pi_{F_{k-1}}(t)\hat{X}(t+1) + Q_{F_{k-1}}(t) - (F_{k-1}(t) - \hat{F}(t))^T \\ &\quad \times (R(t) + \Pi_3(t)\hat{X}(t+1))(F_{k-1}(t) - \hat{F}(t)) - \hat{M}(t)\end{aligned} \tag{5.118}$$

with $\hat{M}(t) \geq 0$. Subtracting (5.118) from (5.108) one obtains that $\{X_k(t) - \hat{X}(t)\}_{t\in\mathcal{I}}$ verifies

$$X_k(t) - \hat{X}(t) = \Pi_{F_{k-1}}(t)(X_{k-1}(t+1) - \hat{X}(t+1)) + H_k(t), \qquad t \in \mathcal{I}, \tag{5.119}$$

where $H_k(t) = (\varepsilon/k)J_n + \hat{M}(t) + (F_{k-1}(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_{k-1}(t) - \hat{F}(t))$. From (5.15) we have that $H_k(t) \geq (\varepsilon/k)J_n > 0$, $t \in \mathcal{I}$. Invoking again (a_{k-1}) one obtains that $\Pi_{F_{k-1}}(t)(X_{k-1}(t+1) - \hat{X}(t+1)) \geq 0$, $t \in \mathcal{I}$.

Thus we obtain via (5.119) that

$$X_k(t) - \hat{X}(t) \geq \frac{\varepsilon}{k}J_n \tag{5.120}$$

which confirms the validity of (a_k) with $\mu_k = (\varepsilon/k) > 0$.

From (5.120) it follows that $R(t) + \Pi_3(t)X_k(t+1) \gg 0$, $t \in \mathcal{I}$. This shows that $F_k(t)$ is well defined via (5.109). Moreover $\{F_k(t)\}_{t\in\mathcal{I}}$ is a bounded sequence.

Applying again Lemma 5.1 with $W(t) = F_k(t)$ we obtain:

$$\begin{aligned}\hat{X}(t) &= \Pi_{F_k}(t)\hat{X}(t+1) + \mathcal{Q}_{F_k}(t) - (F_k(t) - \hat{F}(t))^T \\ &\quad \times (R(t) + \Pi_3(t)\hat{X}(t+1))(F_k(t) - \hat{F}(t)) - \hat{M}(t), \qquad t \in \mathcal{I}.\end{aligned} \tag{5.121}$$

Subtracting (5.121) from (5.110) we obtain that $\{X_k(t) - \hat{X}(t)\}_{t\in\mathcal{I}}$ is a bounded and uniformly positive solution of the discrete-time backward affine equation:

$$Z_k(t) = \Pi_{F_k}(t)Z_k(t+1) + \tilde{H}_k(t), \tag{5.122}$$

where $\tilde{H}_k(t) = (\varepsilon/k)J_n + \hat{M}(t) + \Pi_{F_{k-1}}(t)(X_{k-1}(t) - X_k(t+1)) + (F_k(t) - F_{k-1}(t))^T(R(t) + \Pi_3(t)X_k(t+1))(F_k(t) - F_{k-1}(t)) + (F_k(t) - \hat{F}(t))^T(R(t) + \Pi_3(t)\hat{X}(t+1))(F_k(t) - \hat{F}(t))$.

Because (c_{k-1}) is fulfilled we deduce that $\tilde{H}_k(t) \geq (\varepsilon/k)J_n > 0$, $t \in \mathcal{I}$. Using the implication $(vi) \to (i)$ from Theorem 2.4 with equation (5.122) we conclude that the zero state equilibrium of the linear equation

$$Z(t+1) = \Pi^*_{F_k}(t)Z(t),$$

$t \in \mathcal{I}$ is exponentially stable. Thus we have shown that (b_k) is true. Finally subtracting equation (5.108), written for $k+1$, from equation (5.110) we obtain:

$$X_k(t) - X_{k+1}(t) = \Delta_k(t), \qquad t \in \mathcal{I}, \tag{5.123}$$

where $\Delta_k(t) = (\varepsilon/(k(k+1)))J_n + \Pi_{F_{k-1}}(t)(X_{k-1}(t+1) - X_k(t+1)) + (F_k(t) - F_{k-1}(t))^T(R(t) + \Pi_3(t)X_k(t+1))(F_k(t) - F_{k-1}(t))$.

If we take into account that $(a_k), (c_{k-1})$ are fulfilled, we deduce that $\Delta_k(t) \geq (\varepsilon/(k(k+1)))J_n$. Thus from (5.123) we deduce that $X_k(t) - X_{k+1}(t) \geq 0$ which means that (c_k) is true. Furthermore, from (a_k) and (c_k) one deduces that the sequences $\{X_k(t)\}_{k\geq 1}$, $t \in \mathcal{I}$ are convergent. From (5.109) one gets that $\{F_k(t)\}_{k\geq 1}$, $t \in \mathcal{I}$ are also convergent. Let

$$X_{\max}(t) = \lim_{k\to\infty} X_k(t), \qquad t \in \mathcal{I} \tag{5.124}$$

$$F_{\max}(t) = \lim_{k\to\infty} F_k(t), \qquad t \in \mathcal{I}.$$

Taking the limit for $k \to \infty$ in (5.108) and (5.109) one obtains that $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ is a bounded solution of DTSGRE (5.8). On the other hand taking the limit in (a_k) one obtains that $\{X_{\max}(t)\}_{t\in\mathcal{I}}$ is just the maximal solution of (5.8). Thus the proof is complete. □

Remark 5.9

(a) As we have already pointed out in Remark 5.8, if $\{\Pi(t)\}_{t\in\mathcal{I}}$ and $\{\mathcal{Q}(t)\}_{t\in\mathcal{I}}$ are periodic sequences with period $\theta \geq 1$, then we may construct a periodic solution $X_0(t)$ of (5.96). Furthermore, if $X_0(t)$ is a periodic sequence with period $\theta \geq 1$, then the iterations $\{X_k(t)\}_{t\in\mathcal{I}}$, $\{F_k(t)\}_{t\in\mathcal{I}}$, $k \geq 1$ are also periodic sequences with the same period $\theta \geq 1$. Hence, in the periodic case at each iteration we need to compute finite sequences $\{X_k(t)\}$ and $\{F_k(t)\}$, $t_0 \leq t \leq t_0 + \theta - 1$ $k \geq 1$, where $t_0 \in \mathcal{I}$ is fixed.

(*b*) In the special case $\theta = 1$ the coefficients of DTSGRE (5.8) are constant. In this case, the stabilizing feedback gain $\tilde{F}_0$ will be constant and (5.96) will become a system of LMIs. Thus $X_0(t)$ and consequently all iterations X_k and F_k are also constant.

(*c*) If $\tilde{\Gamma}^{\Sigma}$ is not empty then the maximal solution of (5.8) coincides with the stabilizing solution of (5.8). Thus the iterative procedure described in Theorem 5.10 can also be used to compute the stabilizing solution of DTSGRE (5.8).

The following result may be viewed as an alternative procedure for finding a periodic solution of (5.96).

Proposition 5.3 *Assume that the assumptions from Theorem* 5.10 *are fulfilled. Let* $\{\tilde{F}_0(t)\}_{t\in\mathcal{I}}$ *be a stabilizing feedback gain and* $\varepsilon > 0$ *be given. Let* $\{Y_k(t)\}_{k\geq 0}, t \in \mathcal{I}$ *be defined iteratively as follows,*

$$Y_k(t) = \Pi_{\tilde{F}_0}(t)Y_{k-1}(t+1) + \mathcal{Q}_{\tilde{F}_0}(t) + 2\varepsilon J_n, \tag{5.125}$$

$t \in \mathcal{I}$, $k \geq 1$, $Y_0(t) \equiv 0$. *Then we have*

$$0 \leq Y_k(t) \leq Y_{k+1}(t) \leq \tilde{Y}(t), \qquad \forall k \geq 1, \qquad t \in \mathcal{I}, \tag{5.126}$$

where $\tilde{Y}(t)$ *is the unique bounded solution of the linear equation*

$$\tilde{Y}(t) = \Pi_{\tilde{F}_0}(t)\tilde{Y}(t+1) + \mathcal{Q}_{\tilde{F}_0}(t) + 2\varepsilon J_n, \qquad t \in \mathcal{I}. \tag{5.127}$$

Additionally, for each $t \in \mathcal{I}$ *we have* $\lim_{k\to\infty} Y_k(t) = \tilde{Y}(t)$.

Proof. The inequalities (5.126) can be proved inductively. Furthermore, from (5.126) one deduces that the sequences $\{Y_k(t)\}_{k\geq 1}$, $t \in \mathcal{I}$, are convergent. Set $Z(t) = \lim_{k\to\infty} Y_k(t)$, $t \in \mathcal{I}$. Taking the limit for $k \to \infty$ in (5.125) one obtains that $\{Z(t)\}_{t\in\mathcal{I}}$ is a bounded solution of (5.127). Because (5.127) has a unique bounded solution we conclude that $Z(t)$ coincides with $\tilde{Y}(t)$ and thus the proof ends. □

Corollary 5.3 *Assume that the sequence of linear and positive operators* $\{\Pi(t)\}_{t\in\mathcal{I}}$ *and the sequence of symmetric matrices* $\{\mathcal{Q}(t)\}_{t\in\mathcal{I}}$ *are periodic with period* $\theta \geq 1$. *Assume that the stabilizing feedback gain* $\{\tilde{F}_0(t)\}_{t\in\mathcal{I}}$ *is also a periodic sequence with period* θ. *Then* $\{Y_k(t)\}_{t\in\mathcal{I}}$ *constructed by* (5.125) *are also periodic sequences with the same period* θ. *Moreover, if* k_0 *is such that* $\Pi_{\tilde{F}_0}(t)Y_{k_0-1}(t+1) - Y_{k_0}(t+1) + \varepsilon J_n \geq 0$, $t_0 \leq t \leq t_0 + \theta - 1$ *(for a fixed* $t_0 \in \mathcal{I}$*), then* $\{Y_{k_0}(t)\}_{t\in\mathcal{I}}$ *is a solution of* (5.96).

Proof. It follows immediately writing (5.125) in the form:

$$\begin{aligned} Y_{k_0}(t) = {} & \Pi_{\tilde{F}_0}(t)Y_{k_0}(t+1) + \mathcal{Q}_{\tilde{F}_0}(t) + \varepsilon J_n \\ & + (\Pi_{\tilde{F}_0}(t)(Y_{k_0-1}(t+1) - Y_{k_0}(t+1)) + \varepsilon J_n). \end{aligned}$$

Thus the proof is complete. □

5.8 Discrete-time Riccati equations of stochastic control

In this section we apply the general results proved in the previous sections to derive necessary and sufficient conditions for the existence of the stabilizing solution of discrete-time Riccati-type equations arising in connection with several kinds of linear quadratic optimization problems associated with discrete-time stochastic systems with Markovian jumping and independent random perturbations.

5.8.1 The maximal solution and the stabilizing solution of DTSRE-C

Let us consider the following system of discrete-time Riccati-type equations of stochastic control (DTSRE-C).

$$\begin{aligned} X(t,i) &= \sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i) + M(t,i) \\ &\quad - \left(\sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i) + L(t,i)\right) \\ &\quad \times \left(R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))\right)^{-1} \\ &\quad \times \left(\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i) + L^T(t,i)\right), \qquad 1 \le i \le N, \end{aligned} \tag{5.128}$$

where $\mathcal{E}_i(t,X(t+1)) = \sum_{j=1}^{N} p_t(i,j)X(t+1,j)$, $M(t,i) = M^T(t,i)$, $R(t,i) = R^T(t,i)$ for all $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$.

We remark that no assumption concerning the sign of the matrices $M(t,i)$ and $R(t,i)$ is made.

Remark 5.10

(*a*) The DTSGRE-C (5.128) arises in connection with an optimization problem described by a system of type (4.1) and a quadratic cost functional

$$J(u) = \sum_{t=t_0}^{\infty} E[x^T(t)M(t,\eta_t)x(t) + 2x^T(t)L(t,\eta_t)u(t) + u^T(t)R(t,\eta_t)u(t)].$$

Details concerning the classes of the admissible controls are fully discussed in the next chapter.

(*b*) Motivated by the definition of $\mathcal{E}_i(t,\cdot)$ from above we may define a linear operator $X \to \mathcal{E}(t,X); \mathcal{S}_n^N \to \mathcal{S}_n^N$ by $\mathcal{E}(t,X) = (\mathcal{E}_1(t,X),\ldots,\mathcal{E}_N(t,X))$;

$$\mathcal{E}_i(t,X) = \sum_{j=1}^{N} p_t(i,j)X(j), \qquad \forall\, X = (X(1),\ldots,X(N)) \in \mathcal{S}_n^N. \quad (5.129)$$

It is clear that for each $t, \mathcal{E}(t,\cdot)$ is a linear and positive operator on $\mathcal{S}_n^N$.

(*c*) In the special case $M(t,i) = C^T(t,i)C(t,i)$, $L(t,i) = C^T(t,i)D(t,i)$, $R(t,i) = D^T(t,i)D(t,i)$ the DTSRE-C (5.128) reduces to (5.4).

Set $\Pi_l(t)X = (\Pi_{l1}(t)X, \Pi_{l2}(t)X, \ldots, \Pi_{lN}(t)X)$, $l = 1,2,3$, with

$$\Pi_{1i}(t)X = \sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X)A_k(t,i) \qquad (5.130)$$

$$\Pi_{2i}(t)X = \sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X)B_k(t,i)$$

$$\Pi_{3i}(t)X = \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X)B_k(t,i),$$

$1 \le i \le N$. With these notations the DTSRE-C (5.128) takes the form of (5.8). The analogue of the dissipation operator (5.14) is: $\mathcal{D}\mathbf{X}(t) = (\mathcal{D}_1\mathbf{X}(t),\ldots, \mathcal{D}_N\mathbf{X}(t))$ with

$$\mathcal{D}_i\mathbf{X}(t) = \begin{pmatrix} \sum\limits_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i) - X(t,i) + M(t,i) & \sum\limits_{k=0}^{r} A_k(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i) + L(t,i) \\ \sum\limits_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i) + L^T(t,i) & \sum\limits_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i) + R(t,i) \end{pmatrix}$$

for all $\mathbf{X} = \{X(t)\}_{t\ge 0} \in l^\infty(\mathbf{Z}_+, \mathcal{S}_n^N)$.

Thus we can associate the sets Γ^Σ and $\tilde{\Gamma}^\Sigma$ as in (5.15) and (5.16), respectively.

In order to be sure that the assumption **A.5.1** is fulfilled for the operators (5.130) we make the following new assumption.

A.5.2 The sequences $\{A_k(t,i)\}_{t\ge0}$, $\{B_k(t,i)\}_{t\ge0}$, $0 \le k \le r$, $\{M(t,i)\}_{t\ge0}$, $\{L(t,i)\}_{t\ge0}$, $\{R(t,i)\}_{t\ge0}$, $1 \le i \le N$, $\{P_t\}_{t\ge0}$ are bounded and $p_t(i,j) \ge 0$.

It is worth mentioning that the results in this section could be derived without the assumption that P_t is a stochastic matrix. We need only the fact that P_t is a matrix with nonnegative elements. Thus the assumption that $\{P_t\}_{t\geq 0}$ is a bounded sequence is not redundant.

Theorem 5.3 together with Remark 4.8(b) leads to the following.

Theorem 5.11 *Assume:*

(a) *The triple* $\mathbf{A}, \mathbf{B}, \mathbf{P}$ *is stabilizable.*
(b) *There exists a sequence* $\hat{\mathbf{X}} = \{\hat{X}(t)\}_{t\geq 0} \in l^\infty(\mathbf{Z}_+, \mathcal{S}_n^N)$ *with the properties* $\mathcal{D}_i\hat{\mathbf{X}}(t) \geq 0$, $R(t,i) + \sum_{k=0}^r B_k^T(t,i)\mathcal{E}_i(t, \hat{X}(t+1))B_k(t,i) \geq \nu I_n$ *for all* $1 \leq i \leq N$.

Under these conditions the DTSRE-C (5.128) *has a bounded and maximal solution* $\{X_{\max}(t)\}_{t\geq 0}$ *with the additional property*

$$R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, X_{\max}(t,i))B_k(t,i) \geq \nu I_n, \tag{5.131}$$

$\forall\, t \geq 0,\ 1 \leq i \leq N.$ □

Definition 5.4 *We say that a solution* $\{X_s(t)\}_{t\geq 0}$ *is a stabilizing solution of* (5.128) *if the sequence of perturbed Lyapunov operators* $\{\mathcal{L}_{F_s}(t)\}_{t\geq 0}$ *generates an exponentially stable evolution where* $\mathcal{L}_{F_s}(t) : \mathcal{S}_n^N \to \mathcal{S}_n^N$ *is defined as in* (4.5) *with* $F_s(t,i)$ *instead of* $F(t,i)$, *where*

$$F_s(t,i) = -\left(R(t,i) + \sum_{k=0}^{r} B_k(t,i)\mathcal{E}_i(t, X_s(t+1))B_k(t,i)\right)^{-1}$$
$$\times \left(\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, X_s(t+1))A_k(t,i) + L^T(t,i)\right). \tag{5.132}$$

It should be noted that any time we refer to a solution $\{X(t)\}_{t\geq 0}$ *of DTSRE-C* (5.128) *we assume tacitly that the matrices* $R(t,i) + \sum_{k=0}^r B_k^T(t,i)$ $\mathcal{E}_i(t, X(t+1))B_k(t,i)$ *are invertible.*

Remark 5.11 If P_t are stochastic matrices then one obtains via Corollary 4.1 that $\{X_s\}_{t\geq 0}$ is a stabilizing solution of (5.128) if and only if the zero state equilibrium of the closed-loop system

$$x(t+1) = \Bigg[A_0(t,\eta_t) + B_0(t,\eta_t)F_s(t,\eta_t)$$
$$+ \sum_{k=1}^{r} w_k(t)(A_k(t,\eta_t) + B_k(t,\eta_t)F_s(t,\eta_t))\Bigg] x(t), t \geq 0 \tag{5.133}$$

is SESMS-I.

Let $\{\mathcal{L}_{F_s}(t)\}_{t\geq 0}$ be the sequence of Lyapunov operators introduced by (4.5) and $\Pi_{F_s}(t)$ be associated with the operators (5.130) via (5.18) with $W(t) = F_s(t)$, $F_s(t)$ as in (5.132). Then the equality $\mathcal{L}_{F_s}(t) = \Pi^*_{F_s}(t)$ shows that $\{X_s(t)\}_{t\geq 0}$ is a stabilizing solution of DTSRE-C (5.128) in the sense of Definition 5.4 if and only if it is a stabilizing solution of the corresponding equation (5.8).

Therefore from Corollary 5.2 we deduce that (5.128) has at most one bounded and stabilizing solution.

From Theorem 5.6 one obtains the following.

Theorem 5.12 *The following are equivalent.*

(i) *The triple* $\mathbf{A}, \mathbf{B}, \mathbf{P}$ *is stabilizable and there exists a sequence* $\hat{\mathbf{X}} = \{\hat{X}(t)\}_{t\geq 0} \in l^\infty(\mathbf{Z}_+, \mathcal{S}_n^N)$ *that satisfies* $(\mathcal{D}_i\hat{\mathbf{X}})(t) \geq \nu I_{n+m}$, *for all* $t \in \mathbf{Z}_+$, $1 \leq i \leq N$, *with* $\nu > 0$ *independent of* t *and* i.

(ii) *The DTSRE-C* (5.128) *has a bounded and stabilizing solution* $\{X_s(t)\}_{t\geq 0}$ *such that*

$$R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, X_s(t+1))B_k(t,i) \geq \nu I_m \tag{5.134}$$

for all $t \in \mathbf{Z}_+$, $1 \leq i \leq N$.

In the previous developments we considered the time-varying case in order to cover the systems with periodic coefficients with period $\theta \geq 1$.

As we already remarked in Section 5.5 to check the validity of the conditions of (i) in Theorem 5.12 we have to verify the feasability of some system of LMIs with a finite number of inequalities.

The case $\theta = 1$ is more frequent in applications therefore we present a version of Theorem 5.12 for this special case. From Theorem 5.5 it follows that in this case the stabilizing solution of DTSRE-C (5.128) if it exists, is constant and it solves:

$$\begin{aligned} X(i) = \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i) + M(i) - \left(\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)B_k(i) + L(i)\right) \\ \times \left(R(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i)\right)^{-1} \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)A_k(i) + L^T(i)\right). \end{aligned} \tag{5.135}$$

We also set $\mathcal{D}X = (\mathcal{D}_1 X, \ldots, \mathcal{D}_N X)$,

$$\mathcal{D}_i X = \begin{pmatrix} \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i) - X(i) + M(i) & \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)B_k(i) + L(i) \\ \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i) + L^T(i) & R(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i) \end{pmatrix}$$

for all $X = (X(1), \ldots, X(N)) \in \mathcal{S}_n^N$.

Applying Theorem 5.8 we have the following.

Theorem 5.13 *Assume that* $P_t = P, A_k(t,i) = A_k(i)$, $B_k(t,i) = B_k(i)$, $0 \leq k \leq r$, $M(t,i) = M(i)$, $L(t,i) = L(i)$, $R(t,i) = R(i)$, $t \in \mathbf{Z}$, $1 \leq i \leq N$. *Then the following are equivalent.*

(i) *The triple* (A, B, P) *is stabilizable and there exists* $\hat{X} \in \mathcal{S}_n^N$, *such that* $\mathcal{D}_i \hat{X} > 0, 1 \leq i \leq N$.
(ii) *DTSRE-C* (5.135) *has a stabilizing solution* $X_s = (X_s(1), \ldots, X_s(N))$ *that satisfies*

$$R(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i) > 0, \qquad 1 \leq i \leq N. \tag{5.136}$$

5.8.2 The case of DTSRE-C with definite sign of weighting matrices

In the developments in the previous subsection no assumptions about the sign of the weighting matrices $M(t,i), R(t,i)$ were made. The absence of information about the sign of those matrices was supplied by conditions (5.131), (5.134), and (5.136) verified by the maximal solution and the stabilizing solution, respectively.

In this subsection we consider the case of DTSRE-C (5.128) for which conditions (5.77) and (5.78) are verified.

By direct calculation one obtains that (5.128) can be rewritten as

$$\begin{aligned} X(t,i) = &\sum_{k=0}^{r}(A_k(t,i) - B_k(t,i)R^{-1}(t,i)L^T(t,i))^T\mathcal{E}_i(t, X(t+1)) \\ &\times (A_k(t,i) - B_k(t,i)R^{-1}(t,i)L(t,i)) \\ &+ M(t,i) - L(t,i)R^{-1}(t,i)L^T(t,i) \\ &- \left(\sum_{k=0}^{r}(A_k(t,i) - B_k(t,i)R^{-1}(t,i)L^T(t,i))^T\right)\mathcal{E}_i(t, X(t+1))B_k(t,i) \\ &\times \left(R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, X(t+1))B_k(t,i)\right)^{-1} \\ &\times \left(\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, X(t+1))(A_k(t,i) - B_k(t,i)R^{-1}(t,i)L^T(t,i)\right). \end{aligned} \tag{5.137}$$

To simplify the presentation we write $A_k(t,i)$ instead of $A_k(t,i) - B_k(t,i)$ $R^{-1}(t,i)L^T(t,i)$ in (5.137). Thus we obtain:

$$
\begin{aligned}
X(t,i) = {} & \sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i) + C^T(t,i)C(t,i) \\
& - \left(\sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i)\right) \\
& \times \left(R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i)\right)^{-1} \\
& \times \left(\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i)\right), 1 \le i \le N,
\end{aligned} \tag{5.138}
$$

where $C^T(t,i)C(t,i) = M(t,i) - L(t,i)R^{-1}(t,i)L^T(t,i) \ge 0$.

Applying Theorem 5.9 to the special case of DTSRE-C (5.138) we deduce that under the assumption of stabilizability of the triple $(\mathbf{A}, \mathbf{B}, \mathbf{P})$, the DTSRE-C (5.138) has a bounded and maximal solution $X_{\max}(t)$, $t \ge 0$ and a bounded and minimal solution $X_{\min}(t)$, $t \ge 0$. As we already remarked in Section 5.6, these two global solutions do not always coincide.

In this section we show that under the additional assumption of detectability these two solutions coincide.

Setting $C_0(t,i) = C(t,i), C_k(t,i) = 0, 1 \le k \le r, (t,i) \in \mathbf{Z}_+ \times \mathcal{D}$, we may define the sequences $\mathbf{C}, \mathbf{A}, \mathbf{P}$ as in Section 4.1. We prove the following lemma.

Lemma 5.5 *Assume that* **A.5.2** *is fulfilled and the triple* $(\mathbf{C}, \mathbf{A}, \mathbf{P})$ *is detectable. Under these assumptions any bounded solution* $\tilde{X}(t) = (\tilde{X}(t,1), \dots, \tilde{X}(t,N))$ *of DTSRE* (5.138) *with* $\tilde{X}(t,i) \ge 0$ *for all* $t \ge 0, i \in \mathcal{D}$ *is a stabilizing solution.*

Proof. Let $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), \dots, \tilde{X}(t,N))$ be a bounded and positive semidefinite solution of (5.138). Set $\tilde{F}(t,i) = F^{\tilde{X}}(t,i)$, $(t,i) \in Z_+ \times \mathcal{D}$. Applying Lemma 5.1 with $W(t,i) = \tilde{F}(t,i)$ one obtains that $\tilde{X}(t)$ solves:

$$
\begin{aligned}
\tilde{X}(t,i) = {} & \sum_{k=0}^{r} (A_k(t,i) + B_k(t,i)\tilde{F}(t,i))^T \mathcal{E}_i(t, \tilde{X}(t+1)) \\
& \times (A_k(t,i) + B_k(t,i)\tilde{F}(t,i)) + C^T(t,i)C(t,i) \\
& + \tilde{F}^T(t,i)R(t,i)\tilde{F}(t,i).
\end{aligned} \tag{5.139}
$$

Also, (5.139) may be written in a compact form as

$$
\tilde{X}(t) = \mathcal{L}^*_{\tilde{F}}(t)\tilde{X}(t+1) + \hat{C}(t), \tag{5.140}
$$

where $\hat{C}(t) = (\hat{C}(t,1), \dots, \hat{C}(t,N)), \hat{C}(t,i) = \sum_{k=0}^{r} \hat{C}_k(t,i)^T \hat{C}_k(t,i)$ with $\hat{C}_k(t,i) \in \mathbf{R}^{\hat{p} \times n}, \hat{p} = p + (r+1)m$ given by

$$\hat{C}_0(t,i) = \begin{pmatrix} C(t,i) \\ (r+1)^{-(1/2)} R(t,i)^{1/2} \tilde{F}(t,i) \\ 0_{mn} \\ \dots \\ 0_{mn} \end{pmatrix}$$

$$\hat{C}_1(t,i) = \begin{pmatrix} 0_{p+m,m} \\ (r+1)^{-(1/2)} R^{1/2}(t,i) \tilde{F}(t,i) \\ 0_{mn} \\ \dots \\ 0_{mn} \end{pmatrix} \quad \dots$$

$$\hat{C}_r(t,i) = \begin{pmatrix} 0_{p+m,n} \\ 0_{mn} \\ \dots \\ 0_{mn} \\ (r+1)^{-(1/2)} R^{1/2}(t,i) \tilde{F}(t,i) \end{pmatrix}.$$

Setting $\hat{C}_k(t) = (\hat{C}_k(t,i),\dots,\hat{C}_k(t,N)), \hat{C}(t) = (\hat{C}_0(t),\dots,\hat{C}_r(t))$ we show that $(\hat{C}(t), \mathcal{L}_{\tilde{F}}(t))$ is detectable. To this end, we show that there exist some bounded sequences $\{\hat{K}(t,i)\}_{t\geq 0}, 1 \leq i \leq N$ such that the sequence of Lyapunov operators $\{\hat{\mathcal{L}}^{\hat{K}}(t)\}_{t\geq 0}$ generates an exponentially stable evolution, where $\hat{\mathcal{L}}^{\hat{K}}(t)X = ((\hat{\mathcal{L}}^{\hat{K}}(t)X)(1),\dots,(\hat{\mathcal{L}}^{\hat{K}}(t)X)(N))$ with

$$((\hat{\mathcal{L}}^{\hat{K}}(t))^*X)(i) = \sum_{k=0}^{r} (A_k(t,i) + B_k(t,i)\tilde{F}(t,i) + \hat{K}(t,i)\hat{C}_k(t,i))^T \times \mathcal{E}_i(t,X)(A_k(t,i) + B_k(t,i)\tilde{F}(t,i) + \hat{K}(t,i)\hat{C}_k(t,i)). \tag{5.141}$$

The assumption concerning the detectability of $\mathbf{C}, \mathbf{A}, \mathbf{P}$ guarantees the existence of the bounded sequences $\{K(t,i)\}_{t\geq 0}$ such that the sequence of Lyapunov operators $\{\mathcal{L}^K(t)\}_{t\geq 0}$ generates an exponentially stable evolution, where $\mathcal{L}^K(t)X = ((\mathcal{L}^K(t)X)(1),\dots,(\mathcal{L}^K(t)X)(N))$ with

$$((\mathcal{L}^K(t))^*X)(i) = (A_0(t,i) + K(t,i)C(t,i))^T \mathcal{E}_i(t,X)(A_0(t,i) + K(t,i)C(t,i)) + \sum_{k=1}^{r} A_k(t,i)\mathcal{E}_i(t,X)A_k^T(t,i). \tag{5.142}$$

We take $\hat{K}(t,i) = (\hat{K}_0(t,i)\hat{K}_1(t,i)\dots\hat{K}_r(t,i)) \in \mathbf{R}^{n\times\hat{p}}$ as follows.

$$\hat{K}_0(t,i) = (K(t,i) - (r+1)^{1/2}B_0(t,i)R^{-(1/2)}(t,i),$$

$$\hat{K}_k(t,i) = -(r+1)^{1/2}B_k(t,i)R^{-(1/2)}(t,i), 1 \le k \le r.$$

By direct calculation one obtains that $A_0(t,i) + B_0(t,i)\tilde{F}(t,i) + \hat{K}(t,i)$ $\hat{C}_0(t,i) = A_0(t,i)+K(t,i)C(t,i)$ and $A_k(t,i)+B_k(t,i)\tilde{F}(t,i)+\hat{K}(t,i)\hat{C}_k(t,i) = A_k(t,i), 1 \le k \le r$. Thus we obtain that (5.141) is just (5.142). This allows us to conclude that $(\hat{C}(t), \mathcal{L}_{\tilde{F}}(t)$ is detectable. Now, applying Theorem 4.1 to equation (5.140) we deduce that the sequence $\{\mathcal{L}_{\tilde{F}}(t)\}_{t\ge 0}$ generates an exponentially stable evolution. Hence $\{\tilde{X}(t)\}_{t\ge 0}$ is a stabilizing solution of (5.138) and thus the proof is complete. □

At the end of this subsection we prove the following.

Theorem 5.14 *Assume:*

(*a*) *The assumption* **A 5.2** *is fulfilled.*
(*b*) *The triple* $(\mathbf{A}, \mathbf{B}, \mathbf{P})$ *is stabilizable.*
(*c*) *The triple* $(\mathbf{C}, \mathbf{A}, \mathbf{P})$ *is detectable.*
(*d*) *There exists* $\delta \ge 0$ *such that* $R(t,i) \ge \delta I_m$ *for all* $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$.

Under these conditions DTSRE-C (5.138) *has a unique bounded and positive semidefinite solution* $\tilde{X}(t) = (\tilde{X}(t,1), \tilde{X}(t,2), \dots, \tilde{X}(t,N))$. *This solution is also the stabilizing solution.*

Moreover if the coefficients of the DTSRE-C (5.138) *are periodic sequences with period* $\theta \ge 1$ *then* $\tilde{X}(t)$ *is a periodic solution with the same period* θ.

Proof. It follows immediately from Theorem 5.9, Lemma 5.5, and Corollary 5.2.

5.8.3 The case of the systems with coefficients depending upon η_t and η_{t-1}

Let us consider the following discrete-time system of generalized Riccati equations DTSRE-C,

$$X(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_k^T(j,i)X(j)A_k(j,i) + \sum_{j=1}^{N} p(i,j)C_z^T(j,i)C_z(j,i)$$
$$- \left[\sum_{j=1}^{N} p(i,j)\left(C_z^T(j,i)D_z(j,i) + \sum_{k=0}^{r} A_k^T(j,i)X(j)B_k(j,i)\right)\right]$$

$$\times \left[\sum_{j=1}^{N} p(i,j)\left(D_z^T(j,i)D_z(j,i)+\sum_{k=0}^{r} B_k^T(j,i)X(j)B_k(j,i)\right)\right]^{-1}$$

$$\times \left[\sum_{j=1}^{N} p(i,j)\left(D_z^T(j,i)C_z(j,i)+\sum_{k=0}^{r} B_k^T(j,i)X(j)A_k(j,i)\right)\right]. \quad (5.143)$$

This kind of DTSRE-C arises in connection with the solution of the H_2 control problem for discrete-time linear stochastic systems of the form:

$$\begin{cases} x(t+1) = \left(A_0(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t,\eta_{t-1})\right)x(t) \\ \qquad + \left(B_0(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t,\eta_{t-1})\right)u(t) + B_v(\eta_t,\eta_{t-1})v(t) \\ y(t) = x(t) \\ z(t) = C_z(\eta_t,\eta_{t-1})x(t) + D_z(\eta_t,\eta_{t-1})u(t), \qquad t \geq 1. \end{cases} \quad (5.144)$$

A solution $X_s = (X_s(1), X_s(2), \dots, X_s(N))$ of DTSRE-C (5.143) is called a stabilizing solution if the zero state equilibrium of the corresponding closed-loop system

$$x_s(t+1) = \Bigg[A_0(\eta_t,\eta_{t-1}) + B_0(\eta_t,\eta_{t-1})F_s(\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t,\eta_{t-1}) + B_k(\eta_t,\eta_{t-1})F_s(\eta_{t-1}))\Bigg] x_s(t) \quad (5.145)$$

is ESMS where

$$F_s(i) = -\left[\sum_{j=1}^{N} p(i,j)\left(D_z^T(j,i)D_z(j,i)+\sum_{k=0}^{r} B_k^T(j,i)X(j)B_k(j,i)\right)\right]^{-1} \times \left[\sum_{j=1}^{N} p(i,j)\left(D_z^T(j,i)C_z(j,i)+\sum_{k=0}^{r} B_k^T(j,i)X(j)A_k(j,i)\right)\right]. \quad (5.146)$$

In this subsection we briefly show how we can use the result proved in the previous sections of the chapter to obtain a set of conditions that guarantee the existence of a stabilizing solution of DTSRE-C (5.143).

Let $F = (F(1), F(2), \dots, F(N)), F(i) \in \mathbf{R}^{m\times n}, i \in \mathcal{D}$ be a stabilizing feedback gain for (5.144); this means that the zero state equilibrium of the closed-loop system,

$$x(t+1) = \Bigg[A_0(\eta_t, \eta_{t-1}) + B_0(\eta_t, \eta_{t-1})F(\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t, \eta_{t-1}) + B_k(\eta_t, \eta_{t-1})F(\eta_{t-1}))\Bigg] x(t), \quad (5.147)$$

$t \geq 1$ is ESMS.

Let $\Upsilon_F : \mathcal{S}_n^N \to \mathcal{S}_n^N$ be the Lyapunov-type operator associated with (5.147). We have $\Upsilon_F H = (\Upsilon_F H(1), \Upsilon_F H(2), \ldots, \Upsilon_F H(N))$,

$$\Upsilon_F H(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(j,i)[A_k(i,j) + B_k(i,j)F(j)]H(j)[A_k(i,j) + B_k(i,j)F(j)]^T \quad (5.148)$$

for all $H \in \mathcal{S}_n^N, i \in \mathcal{D}$.

The adjoint operator of Υ_F with respect to the inner product (5.7) is given by $\Upsilon_F^* H = ((\Upsilon_F^* H)(1), (\Upsilon_F^* H)(2), \ldots, (\Upsilon_F^* H)(N))$,

$$(\Upsilon_F^* H)(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j)(A_k(j,i) + B_k(j,i)F(i))^T H(j)(A_k(j,i) + B_k(j,i)F(i)). \quad (5.149)$$

One sees that

$$(\Upsilon_F^* H)(i) = \begin{pmatrix} I_n \\ F(i) \end{pmatrix}^T \begin{pmatrix} (\Pi_1 H)(i) & (\Pi_2 H)(i) \\ ((\Pi_2 H)(i))^T & (\Pi_3 H)(i) \end{pmatrix} \begin{pmatrix} I_n \\ F(i) \end{pmatrix}, \quad (5.150)$$

where

$$(\Pi_1 H)(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) A_k^T(j,i) H(j) A_k(j,i),$$

$$(\Pi_2 H)(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) A_k^T(j,i) H(j) B_k(j,i)$$

$$(\Pi_3 H)(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) B_k^T(j,i) H(j) B_k(j,i),$$

$i \in \mathcal{D}, H \in \mathcal{S}_n^N$. Setting $\Pi_k H = ((\Pi_k H)(1), (\Pi_k H)(2), \ldots, (\Pi_k H)(N))$ we may define the operator $\Pi : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$ by

$$\Pi H = \begin{pmatrix} \Pi_1 H & \Pi_2 H \\ (\Pi_2 H)^T & \Pi_3 H \end{pmatrix}. \tag{5.151}$$

Here we use the notation convention introduced in Section 5.2.1 Using the above operators equation (5.143) can be rewritten in a compact form as

$$X = \Pi_1 X + M - (\Pi_2 X + L)(R + \Pi_3 X)^{-1}(\Pi_2 X + L)^T, \tag{5.152}$$

where $M = (M(1), M(2), \dots, M(N))$,

$$M(i) = \sum_{j=1}^{N} p(i,j) C_z^T(j,i) C_z(j,i),$$

$L = (L(1), L(2), \dots, L(N))$,

$$L(i) = \sum_{j=1}^{N} p(i,j) C_z^T(j,i) D_z(j,i),$$

$R = (R(1), R(2), \dots, R(N))$,

$$R(i) = \sum_{j=1}^{N} p(i,j) D_z^T(j,i) D_z(j,i), \qquad i \in \mathcal{D}.$$

Also the equalities (5.150), (5.151) show that the system (5.144) is stochastic stabilizable iff the linear positive operator Π is stabilizable in the sense of Definition 4.7.

With the above notations we may introduce the so-called dissipation operator associated with (5.152), $\mathbf{D} : \mathcal{S}_n^N \to \mathcal{S}_{n+m}^N$ by

$$(\mathbf{D}X)(i) = \begin{pmatrix} (\Pi_1 X)(i) + M(i) - X(i) & (\Pi_2 X)(i) + L(i) \\ ((\Pi_2 X)(i) + L(i))^T & (\Pi_3 X)(i) + R(i) \end{pmatrix}, \tag{5.153}$$

$i \in \mathcal{D}, X \in \mathcal{S}_n^N$.

Applying Theorem 5.8 to equation (5.152) we obtain the following.

Theorem 5.15 *Under the assumptions* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *the following are equivalent.*

(i) *The system* (5.144) *is stochastic stabilizable and there exist* $\hat{X} = (\hat{X}(1), \hat{X}(2), \dots, \hat{X}(N)) \in \mathcal{S}_n^N$ *such that*

$$(\mathbf{D}\hat{X})(i) > 0, \qquad i \in \mathcal{D}. \tag{5.154}$$

(ii) *The DTSRE-C* (5.143) *has a stabilizing solution* X_s *that satisfies*

$$\sum_{j=1}^{N} p(i,j)\Big(D_z^T(j,i) D_z(j,i) + \sum_{k=1}^{r} B_k^T(j,i) X_s(j) B_k(j,i)\Big) > 0, \qquad i \in \mathcal{D}. \tag{5.155}$$

We remark that a set of conditions equivalent to the existence of a stabilizing solution of (5.143) that verify condition (5.155) consists of the solvability of the systems of LMIs (4.74) and (5.154).

5.9 Discrete-time Riccati filtering equations

To derive the optimal controller in the H_2 optimization problem, together with the stabilizing solution of DTSRE-C (5.135) we need the stabilizing solution of the following system of so-called discrete-time Riccati filtering equations (DTSRE-F):

$$\begin{aligned} Y(i) = \sum_{j=1}^{N} p(j,i) \Bigg[& \sum_{k=0}^{r} A_k(j)Y(j)A_k^T(j) + \varepsilon_\mu(j)B_v(j)B_v^T(j) \\ & - \left(\sum_{k=0}^{r} A_k(j)Y(j)C_k^T(j) + \varepsilon_\mu(j)B_v(j)D_v^T(j) \right) \\ & \times \left(\varepsilon_\mu(j)D_v(j)D_v^T(j) + \sum_{k=0}^{r} C_k(j)Y(j)C_k^T(j) \right)^{-1} \\ & \times \left(\sum_{k=0}^{r} C_k(j)Y(j)A_k^T(j) + \varepsilon_\mu(j)D_v(j)B_v^T(j) \right) \Bigg], \end{aligned} \tag{5.156}$$

where $\varepsilon_\mu(j)$ are some given nonnegative real numbers. Anytime we refer to a solution Y of (5.156) we assume tacitly that

$$\det[\varepsilon_\mu(j)D_v(j)D_v^T(j) + \sum_{k=0}^{r} C_k(j)Y(j)C_k^T(j)] \neq 0$$

for all $j \in \mathcal{D}$.

We say that a solution $Y_s = (Y_s(1), Y_s(2), \dots, Y_s(N))$ is a stabilizing solution of DTSRE-F (5.156) if the zero state equilibrium of the closed-loop system

$$x(t+1) = \left[A_0(\eta_t) + K_s(\eta_t)C_0(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + K_s(\eta_t)C_k(\eta_t)) \right] x(t) \tag{5.157}$$

is ESMS, where

$$K_s(i) = -\left[\sum_{k=0}^{r} A_k(i)Y_s(i)C_k^T(i) + \varepsilon_\mu(i)B_v(i)D_v^T(i)\right] \times \left[\varepsilon_\mu(i)D_v^T(i)D_v(i) + \sum_{k=0}^{r} C_k(i)Y_s(i)C_k^T(i)\right]^{-1}, \qquad 1 \le i \le N. \tag{5.158}$$

One sees that (5.156) cannot be written in the form (5.152) in order to apply the general result from Section 5.5 to obtain the existence of the stabilizing solution Y_s. Following the ideas from [27] for the special case $A_k(i) = 0, 1 \le k \le r, 1 \le i \le N$, we associate a new system of discrete-time Riccati equations, called a "dual Riccati equation." To this dual equation one may apply Theorem 5.8. Thus, if $Y = (Y(1), Y(2), \dots, Y(N))$ is a solution of (5.156), we define

$$\begin{aligned} X(j) = \mathcal{R}_j(Y(j)) = &\sum_{k=0}^{r} A_k(j)Y(j)A_k^T(j) + \varepsilon_\mu(j)B_v(j)B_v^T(j) \\ &- \left(\sum_{k=0}^{r} A_k(j)Y(j)C_k^T(j) + \varepsilon_\mu(j)B_v(j)D_v^T(j)\right) \\ &\times \left(\varepsilon_\mu(j)D_v(j)D_v^T(j) + \sum_{k=0}^{r} C_k(j)Y(j)C_k^T(j)\right)^{-1} \\ &\times \left(\sum_{k=0}^{r} C_k(j)Y(j)A_k^T(j) + \varepsilon_\mu(j)D_v(j)B_v^T(j)\right). \end{aligned} \tag{5.159}$$

Hence

$$Y(i) = \sum_{j=1}^{N} p(j,i)X(j). \tag{5.160}$$

From (5.159), (5.160) one deduces that $X = (X(1), X(2), \dots, X(N))$ is a solution of the following DTSRE-C,

$$\begin{aligned} X(i) = &\sum_{k=0}^{r} A_k(i)\mathcal{E}_i^*(X)A_k^T(i) + \varepsilon_\mu(i)B_v(i)B_v^T(i) \\ &- \left(\sum_{k=0}^{r} A_k(i)\mathcal{E}_i^*(X)C_k^T(i) + \varepsilon_\mu(i)B_v(i)D_v^T(i)\right) \end{aligned}$$

$$\times \left(\varepsilon_\mu(i) D_v(i) D_v^T(i) + \sum_{k=0}^{r} C_k(i) \mathcal{E}_i^*(X) C_k^T(i) \right)^{-1}$$

$$\times \left(\sum_{k=0}^{r} C_k(i) \mathcal{E}_i^*(X) A_k^T(i) + \varepsilon_\mu(i) D_v(i) B_v^T(i) \right), \qquad 1 \le i \le N, \quad (5.161)$$

where

$$\mathcal{E}_i^*(X) = \sum_{j=1}^{N} p(j,i) X(j), \qquad 1 \le i \le N. \quad (5.162)$$

One can see that $\mathcal{E}^* : \mathcal{S}_n^N \to \mathcal{S}_n^N$ defined by (5.162) is the adjoint operator of the defined (5.129). Conversely, if $X = (X(1), X(2), \ldots, X(N))$ is a solution of (5.161) then $Y = (Y(1), Y(2), \ldots, Y(N))$ defined by $Y(i) = \mathcal{E}_i^*(X)$ is a solution of (5.156). The system (5.161) can be written in a compact form as a nonlinear equation on $\mathcal{S}_n^N$:

$$X = \tilde{\Pi}_1 X + \tilde{M} - (\tilde{L} + \tilde{\Pi}_2 X)(\tilde{R} + \tilde{\Pi}_3 X)^{-1}(\tilde{L} + \tilde{\Pi}_2 X)^T. \quad (5.163)$$

The i-th component of (5.163) is

$$X(i) = \tilde{\Pi}_{1i} X + \tilde{M}(i) - (\tilde{L}(i) + \tilde{\Pi}_{2i} X)(\tilde{R}(i) + \tilde{\Pi}_{3i})^{-1}(\tilde{L}(i) + \tilde{\Pi}_{2i} X)^T,$$

where

$$\tilde{\Pi}_{1i} X = \sum_{k=0}^{r} A_k(i) \mathcal{E}_i^*(X) A_k^T(i),$$

$$\tilde{\Pi}_{2i} X = \sum_{k=0}^{r} A_k(i) \mathcal{E}_i^*(X) C_k^T(i), \quad (5.164)$$

$$\tilde{\Pi}_{3i} X = \sum_{k=0}^{r} C_k(i) \mathcal{E}_i^*(X) C_k^T(i),$$

$$\begin{pmatrix} \tilde{M}(i) & \tilde{L}(i) \\ \tilde{L}^T(i) & \tilde{R}(i) \end{pmatrix} = \varepsilon_\mu(i) \begin{pmatrix} B_v(i) \\ D_v(i) \end{pmatrix} \begin{pmatrix} B_v(i) \\ D_v(i) \end{pmatrix}^T. \quad (5.165)$$

Lemma 5.6 *The following hold.*

(*i*) *The system*

$$x(t+1) = \left(A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t) \right) x(t) \quad (5.166)$$

$$y(t) = \left(C_0(\eta_t) + \sum_{k=1}^{r} w_k(t) C_k(\eta_t) \right) x(t)$$

is stochastic detectable if and only if the linear operator $\tilde{\Pi}$ *defined by* (5.164) *is stabilizable.*

(*ii*) *If* $Y_s = (Y_s(1), Y_s(2), \ldots, Y_s(N))$ *is the stabilizing solution of DTSRE-F* (5.156) *then* $X_s = (X_s(1), X_s(2), \ldots, X_s(N))$ *defined by* $X_s(i) = \mathcal{R}_i(Y_s(i))$*, is the stabilizing solution of DTSRE-C* (5.161).

(*iii*) *If* $X_s = (X_s(1), X_s(2), \ldots, X_s(N))$ *is the stabilizing solution of DTSRE-C* (5.161) *then* $Y_s = (Y_s(1), Y_s(2), \ldots, Y_s(N))$*defined by* $Y_s(i) = \sum_{j=1}^{N} p(j,i)X_s(j), 1 \leq i \leq N$ *is the stabilizing solution of DTSRE-F* (5.156).

Proof. (*i*) Let $K = (K(1), K(2), \ldots, K(N))$ be a stabilizing injection for the system (5.166). Based on Theorem 2.14 and Corollary 3.3 one obtains that the zero state equilibrium of

$$x(t+1) = \left[A_0(\eta_t) + K(\eta_t)C_0(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + K(\eta_t)C_k(\eta_t))\right] x(t) \tag{5.167}$$

is ESMS if and only if the eigenvalues of the operator $\Lambda_K : \mathcal{S}_n^N \to \mathcal{S}_n^N$ are located in the inside of the disk $|\lambda| < 1$, where

$$(\Lambda_K X)(i) = \sum_{k=0}^{r} (A_k(i) + K(i)C_k(i))\mathcal{E}_i^*(X)(A_k(i) + K(i)C_k(i))^T, \qquad 1 \leq i \leq N.$$

By direct calculation one sees that $\Lambda_K = \tilde{\Pi}_F$ with $F = (F(1), F(2), \ldots, F(N)), F(i) = K^T(i)$. Thus we obtained that the system (5.166) is stochastic detectable iff there exists a feedback gain $F = (F(1), F(2), \ldots, F(N)), F(i) \in \mathbf{R}^{n_y \times n}$ such that the eigenvalues of the operator $\tilde{\Pi}_F$ are in the inside of the disk $|\lambda| < 1$, which means that the operator $\tilde{\Pi}$ is stabilizable. Thus the proof of (*i*) is complete.

(*ii*) Let $Y_s = (Y_s(1), Y_s(2), \ldots, Y_s(N))$ be the stabilizing solution of DTSRE-F (5.156). Invoking again Theorem 2.14 and Corollary 3.3, we deduce that the zero state equilibrium of the closed-loop system (5.157) is ESMS if and only if the eigenvalues of the linear operator $\Lambda_{K_s} : \mathcal{S}_n^N \to \mathcal{S}_n^N$ are located in the inside of the disk $|\lambda| < 1$, where

$$\Lambda_{K_s} Y(i) = \sum_{k=0}^{r} (A_k(i) + K_s(i)C_k(i))\mathcal{E}_i^*(Y_s)(A_k(i) + K_s(i)C_k(i))^T, \qquad 1 \leq i \leq N \tag{5.168}$$

for all $Y \in \mathcal{S}_n^N$. Set

$$X_s(i) = \mathcal{R}_i(Y_s(i)), \qquad 1 \leq i \leq N \tag{5.169}$$

and define

$$F_s(i) = -(\tilde{R}(i) + \tilde{\Pi}_{3i}X_s)^{-1}(\tilde{L}(i) + \tilde{\Pi}_{2i}X_s)^T, \qquad 1 \le i \le N. \tag{5.170}$$

Then $X_s = (X_s(1), X_s(2), \dots, X_s(N))$ is the solution of (5.161). By direct calculation one obtains that $\Lambda_{K_s} = \tilde{\Pi}_{F_s}$ where $\tilde{\Pi}_{F_s}$ is defined as in (5.18) with $\tilde{\Pi}$ instead of Π and $F_s(i)$ instead of $W(i)$. The equality $\Lambda_{K_s} = \tilde{\Pi}_{F_s}$ shows that X_s is the stabilizing solution of (3.40) if Y_s is the stabilizing solution of (5.156). Thus (ii) holds.

The implication of (iii) can be proved in a similar manner as the one in (ii) and it is based on the equality $\tilde{\Pi}_{F_s} = \Lambda_{K_s}$.Thus the proof is complete.

□

The generalized dissipation operator corresponding to equation (5.163) is $\tilde{\mathbf{D}} : \mathcal{S}_n^N \to \mathcal{S}_{n+n_y}^N$:

$$\tilde{\mathbf{D}}Y(i) = \begin{pmatrix} \tilde{\Pi}_{1i}Y - Y(i) + \tilde{M}(i) & \tilde{L}(i) + \tilde{\Pi}_{2i}Y \\ (\tilde{L}(i) + \Pi_{2i}Y)^T & \tilde{R}(i) + \tilde{\Pi}_{3i}Y \end{pmatrix}. \tag{5.171}$$

Combining Theorem 5.8 and Lemma 5.6 from above, one obtains the following.

Theorem 5.16 *The following are equivalent.*

(i) *The system* (5.166) *is stochastic detectable and there exists* $\hat{Y} = (\hat{Y}(1), \hat{Y}(2), \dots, \hat{Y}(N)) \in \mathcal{S}_n^N$ *such that* $(\tilde{\mathbf{D}}\hat{Y})(i) > 0, 1 \le i \le N$.

(ii) *The DTSRE-F* (5.156) *has a stabilizing solution* $Y_s = (Y_s(1), Y_s(2), \dots, Y_s(N))$ *with the additional property:*

$$\varepsilon_\mu(i)D_v(i)D_v^T(i) + \sum_{k=0}^{r} C_k(i)Y_s(i)C_k^T(i) > 0, \qquad 1 \le i \le N. \tag{5.172}$$

5.10 A numerical example

In order to illustrate the iterative procedure given by Theorem 5.10 the following Riccati-type system was considered,

$$\begin{aligned} X(i) = {} & \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i) + C^T(i)C(i) - \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)B_k(i) \\ & \times \left(R(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i)\right)^{-1} B_k^T(i)\mathcal{E}_i(X)A_k(i), \quad i = 1, \dots, N, \end{aligned} \tag{5.173}$$

where $R(i) \geq 0, i = 1, \dots, N$ and $\mathcal{E} : \mathcal{S}_n^N \to \mathcal{S}_n^N$, $\mathcal{E}(X) = (\mathcal{E}_1(X), \dots, \mathcal{E}_N(X))$,

$$\mathcal{E}_i(X) = \sum_{j=1}^{N} p(i,j)X(j), i = 1, \dots, N. \tag{5.174}$$

Such systems are used to determine the solutions of linear quadratic optimization problems for discrete-time stochastic systems with state-dependent noise and Markovian jumps (see Chapter 6). In this numerical example, $n = 3$, $N = 2$, $r = 1$, and

$$A_0(1) = \begin{pmatrix} 0.5 & -0.5 & 1 \\ 1.5 & 1 & -0.5 \\ 1 & 0.5 & 2 \end{pmatrix}, \qquad A_0(2) = \begin{pmatrix} -0.25 & 0 & 0.5 \\ 0.5 & -0.75 & 0.25 \\ 0.25 & -0.25 & -0.75 \end{pmatrix},$$

$$A_1(1) = \begin{pmatrix} 0.2 & 0.4 & -0.2 \\ 0.4 & 0.2 & 0.6 \\ -0.2 & 0.4 & 0.2 \end{pmatrix}, \qquad A_1(2) = \begin{pmatrix} 0.2 & 0.3 & -0.1 \\ 0.1 & 0.1 & 0.2 \\ -0.1 & 0.1 & 0.1 \end{pmatrix},$$

$$B_0(1) = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \qquad B_0(2) = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix},$$

$$B_1(1) = \begin{pmatrix} -0.1 \\ 0.1 \\ 0.2 \end{pmatrix}, \qquad B_1(2) = \begin{pmatrix} -1 \\ 0.5 \\ 1.5 \end{pmatrix},$$

$$C(1) = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}, \qquad C(2) = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}, \qquad R(1) = R(2) = 1,$$

$$p(1,1) = p(2,1) = 0.1, \qquad p(1,2) = p(2,2) = 0.9.$$

With the notations introduced in Section 5.8, one can directly see that

$$\Pi_{1i}X := \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i)$$

$$\Pi_{2i}X := \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)B_k(i)$$

$$\Pi_{3i}X := \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i)$$

$$M(i) := C^T(i)C(i), \qquad L(i) := 0, i = 1, \dots, N.$$

Before starting the iterative procedure one must determine the initial values $F_0(i)$ and $X_0(i), \imath = 1, \dots, N$. A stabilizing gain $F_0(i), \imath = 1, \dots, N$ can be determined using Corollary 4.3. Thus for $\varepsilon = 10^{-2}$ one obtains the stabilizing gain

$$F_0(1) = \begin{pmatrix} 0.0074 & 0.1585 & -0.8732 \end{pmatrix},$$
$$F_0(2) = \begin{pmatrix} -0.1432 & 0.1440 & 0.1898 \end{pmatrix}. \tag{5.175}$$

Then substituting the above values for F_0 in (5.96) and solving this system of LMIs with respect to X_0 one obtains

$$X_0(1) = \begin{pmatrix} 10.1414 & 0.0119 & 3.8104 \\ 0.0119 & 8.4770 & -3.0157 \\ 3.8104 & -3.0157 & 9.3539 \end{pmatrix},$$

$$X_0(2) = \begin{pmatrix} 8.1611 & -3.4041 & 2.4325 \\ -3.4041 & 4.2300 & -3.4394 \\ 2.4325 & -3.4394 & 4.5159 \end{pmatrix}.$$

Further applying the iterative procedure given in Theorem 5.10 with the stop condition

$$\max\left(X_k(1) - X_{k+1}(1), X_k(2) - X_{k+1}(2)\right) < 10^{-4},$$

after 53 iterations one obtains the following stabilizing solution of the system (5.173),

$$X(1) = \begin{pmatrix} 2.2913 & -1.0861 & 2.6794 \\ -1.0861 & 2.7089 & -2.9888 \\ 2.6794 & -2.9888 & 5.5858 \end{pmatrix},$$

$$X(2) = \begin{pmatrix} 4.6750 & -2.3108 & 2.2413 \\ -2.3108 & 1.8165 & -1.6733 \\ 2.2413 & -1.6733 & 1.7533 \end{pmatrix}.$$

5.11 Notes and references

Different aspects concerning the existence and properties of the solutions of discrete-time Riccati-type equations arising in different stochastic control problems are investigated in numerous works. Here we refer only to the monographs [1, 27, 84, 85, 63, 64]. Monotonicity properties of the solutions of such equations may be found in [1, 58, 104]. The main part of the results contained in this chapter are presented in [48]. The results from Section 5.7 were published for the first time in [44], and the ones from Section 5.9 appear for the first time in [50]. For the continuous-time case, similar results were proved in [35] and [40]. Numerical aspects concerning the computation of the stabilizing solution of the systems of coupled algebraic Riccati equations arising in connection with the linear quadratic problem for discrete-time linear systems with Markovian jumping can be found in [72–74].

6

Linear quadratic optimization problems

In this chapter several problems of the optimization of a quadratic cost functional along the trajectories of a discrete-time linear stochastic system affected by jumping Markov perturbations are independent random perturbations are investigated. In Section 6.2 we deal with the classical problem of the linear quadratic optimal regulator which means the minimization of a quadratic cost functional with definite sign along the trajectories of a controlled linear system. Also in Section 6.3 the general case of a linear quadratic optimization problem with a cost functional without sign is treated. It is shown that in the case of a linear quadratic optimal regulator, the optimal control is constructed via the minimal solution of a system of discrete-time Riccati-type equations, whereas in the general case of the linear quadratic optimization problem without sign, the optimal control, if it exists, is constructed based on the stabilizing solution of a system of discrete-time Riccati-type equations. In Section 6.4 we deal with the problem of the optimization of a quadratic cost functional of a discrete-time affine stochastic system affected by jumping Markov perturbations and independent random perturbations. Both the case of finite time horizon as well as the infinite time horizon are considered. Optimal control is constructed using the stabilizing solution for a system of discrete-time Riccati-type equations. A set of necessary and sufficient conditions ensuring the existence of the desired solutions of the discrete-time Riccati equations involved in this chapter were given in Chapter 5. A tracking problem is also solved.

6.1 Some preliminaries

6.1.1 A brief discussion on the linear quadratic optimization problems

Let us consider the discrete-time linear controlled system with the state space representation given by:

V. Dragan et al., *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, DOI 10.1007/978-1-4419-0630-4_6,

$$
\begin{aligned}
x(t+1) &= \left(A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(t,\eta_t)\right) x(t) \\
&\quad + \left(B_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t) B_k(t,\eta_t)\right) u(t) + B_v(t,\eta_t) v(t) \\
y(t) &= \left(C_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t) C_k(t,\eta_t)\right) x(t) + D_v(t,\eta_t) v(t) \qquad (6.1) \\
z(t) &= C(t,\eta_t) x(t) + D(t,\eta_t) u(t),
\end{aligned}
$$

where $x(t) \in \mathbf{R}^n$ is the state vector; $u(t) \in \mathbf{R}^m$ is the vector of the control parameters; $y(t) \in \mathbf{R}^{n_y}$ is the vector of the measurements; $z(t) \in \mathbf{R}^{n_z}$ is the regulated output; $\{w_k(t)\}_{t \geq 0}$, $1 \leq k \leq r$ are sequences of random variables; $\{v(t)\}_{t \geq 0}$ is a sequence of m_v-dimensional independent random vectors; and $\{\eta_t\}_{t \geq 0}$ is a Markov chain on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

The class of admissible controls consists of static or dynamic controllers,

$$u(t) = (G_c(\eta) y)(t), \qquad (6.2)$$

where $G_c(\eta)$ is a linear and causal operator. This means that at each time t, the control parameters $u(t)$ are computed via the measurements $y(s)$, $0 \leq s \leq t$. It should be remarked that together with the measurements $y(t)$ it is assumed that we have access to the current mode i of the Markov chain.

By coupling the state space realization of the controller (6.2) to (6.1) one obtains a closed-loop system of the form:

$$
\begin{aligned}
x_{cl}(t+1) &= \left(A_{0cl}(t,\eta_t) + \sum_{k=1}^{r} w_k(t) A_{kcl}(t,\eta_t)\right) x(t) + B_{vcl}(t,\eta_t) v(t) \\
z_{cl}(t) &= C_{cl}(t,\eta_t) x_{cl}(t). \qquad (6.3)
\end{aligned}
$$

The coefficient matrices $A_{kcl}(t,i)$, $B_{vcl}(t,i)$, $C_{cl}(t,i)$ of the closed-loop system (6.3) are detailed later.

If $\Phi_{cl}(t,t_0)$, $t \geq t_0 \geq 0$ is the fundamental matrix solution of the linear system obtained from (6.3) by taking $B_{vcl}(t,i) = 0$, then we may write the following representation of the trajectories of the closed-loop system,

$$x_{cl}(t) = \Phi_{cl}(t,t_0) x_{cl}(t_0) + \sum_{s=t_0}^{t-1} \Phi_{cl}(t,s+1) B_{vcl}(s,\eta_s) v(s).$$

This allows us to obtain the following decomposition of the output $z_{cl}(t)$,

$$z_{cl}(t) = z^1(t,t_0) + z^2(t,t_0,v).$$

The component $z^1(t,t_0)$ depends only upon the initial state $x_{cl}(t_0)$ of the closed-loop system, whereas the component $z^2(t,t_0,v)$ depends upon the additive noises $v(s)$, $0 \leq s \leq t-1$.

In short, the linear quadratic regulator problem denotes the optimization problem that is required to find a controller in the class of the controllers of type (6.2) minimizing a suitable norm of the component $z^1(t,t_0)$ of the output of the closed-loop system.

An optimization problem asking for the construction of a controller of type (6.2) minimizing a suitable norm of the component $z^2(t,t_0,v)$ of the output of the closed-loop system is usually known as an H_2 optimal control problem.

In the next sections of this chapter we investigate in detail the linear quadratic regulator problem (LQRP). We show that depending upon the class of admissible controls the optimal regulator is constructed either via the minimal solution of DTSRE-C (5.128) or the maximal and stabilizing solution of that Riccati equation, respectively.

In the developments of this chapter we consider systems with time-varying coefficients in order to cover the case of systems with periodic coefficients. The problem of H_2 optimal control associated with a system of type (6.1) is discussed in detail in the next chapter. The H_2 control problem is addressed to the time-invariant case of systems (6.1) because in this case there are more possibilities to introduce the H_2 performances, than in the case of systems with time-varying coefficients.

6.1.2 A usual class of stochastic processes

In this subsection we describe a class of stochastic processes widely involved in the construction of the class of admissible controls in the developments of this chapter. First we recall that $\ell^2\{t_0,\infty;\mathbf{R}^d\}$ stands for the space of d-dimensional stochastic processes $\{u(t)\}_{t\geq t_0}$ with the property that $\sum_{t=t_0}^{\infty} E[|u(t)|^2]<\infty$.

If $\hat{\mathcal{F}}=\{\hat{\mathcal{F}}_t, t_0\leq t\leq t_1\}$ is an increasing sequence of σ-algebras $\hat{\mathcal{F}}_t\subset\hat{\mathcal{F}}_{t+1}\subset\mathcal{F}$, then $\ell^2_{\hat{\mathcal{F}}}\{t_0,t_1;\mathbf{R}^d\}$ stands for the d-dimensional stochastic processes $f=\{f(t)\}_{t_0\leq t\leq t_1}$, $f(t):\Omega\to\mathbf{R}^d$, $E[|f(t)|^2]<\infty$, and $f(t)$ is $\hat{\mathcal{F}}_t$-measurable for all $t_0\leq t\leq t_1$. Also $\ell^2_{\hat{\mathcal{F}}}\{t_0,\infty;\mathbf{R}^d\}$ denotes the subspace of stochastic processes $f=\{f(t)\}_{t\geq t_0}$, $f(t):\Omega\to\mathbf{R}^d$, $\sum_{t=t_0}^{\infty}E[|f(t)|^2]<+\infty$, and $f(t)$ is $\hat{\mathcal{F}}_t$-measurable for all $t\geq t_0$. In the developments in this chapter an important role is played by $\ell^2_{\tilde{\mathcal{H}}}\{t_0,\infty;\mathbf{R}^m\}$ and $\ell^2_{\tilde{\mathcal{H}}}\{t_0,t_1;\mathbf{R}^m\}$, where $\tilde{\mathcal{H}}_t$ are the σ-algebras defined in Section 1.5.

6.1.3 Several auxiliary results

In this subsection we deduce several useful equalities used in the solution of the linear quadratic optimization problems discussed in this chapter.

Consider the discrete-time controlled system described by:

$$\begin{aligned} x(t+1) &= \left[A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t)\right]x(t) \\ &\quad + \left[B_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(t,\eta_t)\right]u(t) \\ y(t) &= x(t) \\ z(t) &= C(t,\eta_t)x(t) + D(t,\eta_t)u(t) \end{aligned} \tag{6.4}$$

obtained from (6.1) by taking $B_v(t,i) = 0$, $C_0(t,i) = I_n$, $C_k(t,i) = 0$ $1 \le k \le r$, $D_v(t,i) = 0$, $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$.

If $0 \le t_0 < t_1 \in \mathbf{Z}$, $x_0 \in \mathbf{R}^n$, $i \in \mathcal{D}_{t_0}$ we define $J(t_0,t_1,x_0,i,\cdot) : \ell^2_{\tilde{\mathcal{H}}}(t_0, t_1-1;\mathbf{R}^m) \to \mathbf{R}$ by

$$J(t_0,t_1,x_0,i,u) = \sum_{t=t_0}^{t_1-1} E\left[(x^T(t) \quad u^T(t))\mathcal{Q}(t,\eta_t)\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} |\eta_{t_0} = i\right], \tag{6.5}$$

where

$$\mathcal{Q}(t,i) = \begin{pmatrix} M(t,i) & L(t,i) \\ L^T(t,i) & R(t,i) \end{pmatrix} \in \mathcal{S}_{n+m}, \qquad (t,i) \in \mathbf{Z}_+ \times \mathcal{D}_{t_0}.$$

The following simple result provides the so-called "squares completion technique" and is used repeatedly in the next developments.

Lemma 6.1 *Let*

$$\mathbf{Q}(\xi,u) = \begin{pmatrix} \xi^T & u^T \end{pmatrix} \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \\ \mathbf{Q}_2^T & \mathbf{Q}_3 \end{pmatrix} \begin{pmatrix} \xi \\ u \end{pmatrix}$$

be a quadratic form on $\mathbf{R}^{n+m}$. *If* $\mathbf{Q}_3$ *is an invertible matrix then we have:*

$$\mathbf{Q}(\xi,u) = \xi^T(\mathbf{Q}_1 - \mathbf{Q}_2\mathbf{Q}_3^{-1}\mathbf{Q}_2^T)\xi + (u + \mathbf{Q}_3^{-1}\mathbf{Q}_2^T\xi)^T\mathbf{Q}_3(u + \mathbf{Q}_3^{-1}\mathbf{Q}_2^T\xi)$$

for all $\xi \in \mathbf{R}^n$, $u \in \mathbf{R}^m$.

Proof. It consists of simple calculations and is omitted. □

Now we prove the following.

Proposition 6.1 *Assume* $\mathbf{H_1}$ *and* $\mathbf{H_2}$. *Let* $0 \le t_0 < t_1 \in \mathbf{Z}$, *and* $X(t) = (X(t,1), X(t,2), \ldots, X(t,N))$ *be a solution of DTSRE-C* (5.128) *defined for all* $t_0 \le t \le t_1$. *Then we have the representation:*

$$J(t_0,t_1,x_0,i,u) = x_0^T X(t_0,i)x_0 - E[x^T(t_1)X(t_1,\eta_{t_1})x(t_1)|\eta_{t_0}=i]$$
$$+ \sum_{t=t_0}^{t_1-1} E\Bigg[(u(t) - F^X(t,\eta_t)x(t))^T(R(t,\eta_t)$$
$$+ \sum_{k=0}^{r} B_k^T(t,\eta_t)\mathcal{E}_{\eta_t}(t,X(t+1))B_k(t,\eta_t))$$
$$\times (u(t) - F^X(t,\eta_t)x(t))|\eta_{t_0}=i\Bigg] \tag{6.6}$$

for all $i \in \mathcal{D}_{t_0}$*,* $u = \{u(t), t_0 \le t \le t_1 - 1\} \in \ell^2_{\tilde{\mathcal{H}}}\{t_0, t_1-1, \mathbf{R}^m\}$*,* $x(t)$ *being the solution of* (6.4) *corresponding to the input* u *starting from* x_0 *at* $t = t_0$*,*

$$F^X(t,i) = -\left[R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i)\right]^{-1}$$
$$\times \left[\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i) + L^T(t,i)\right]. \tag{6.7}$$

Proof. Setting $g_k(t) = B_k(t,\eta_t)u(t)$, $0 \le k \le r$, we obtain that $g_k(t)$ is $\tilde{\mathcal{H}}_t$-measurable. Hence the system (6.4) is of type (3.84). Thus we may apply Lemma 3.1 to system (6.4) and the function $V(t,x,i) = x^T X(t,i)x$ to obtain:

$$E[V(t_1,x(t_1),\eta_{t_1})|\eta_{t_0}] - V(t_0,x(t_0),\eta_{t_0})$$
$$= \sum_{t=t_0}^{t_1-1} E\left[\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} \Pi_{1\eta_t}(t)X(t+1) - X(t,\eta_t) & \Pi_{2\eta_t}(t)X(t+1) \\ (\Pi_{2\eta_t}(t)X(t+1))^T & \Pi_{3\eta_t}(t)X(t+1) \end{pmatrix}\right.$$
$$\left.\times \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} |\eta_{t_0}\right], \tag{6.8}$$

where $\Pi_{li}(t)$ are the operators introduced in (5.130). Taking the expectation with respect to the event $\{\eta_{t_0} = i\}$, $i \in \mathcal{D}_{t_0}$, in (6.8) and adding (6.5) we obtain:

$$J(t_0,t_1,x_0,i,u)$$
$$= x_0^T X(t_0,i)x_0 - E[x^T(t_1)X(t_1,\eta_{t_1})x(t_1)|\eta_{t_0}=i]$$

$$+\sum_{t=t_0}^{t_1-1} E\left[\begin{pmatrix} x(t)\\ u(t)\end{pmatrix}^T\right.$$

$$\times\begin{pmatrix} \Pi_{1\eta_t}(t)X(t+1)-X(t,\eta_t)+M(t,\eta_t) & \Pi_{2\eta_t}(t)X(t+1)+L(t,\eta_t)\\ (\Pi_{2\eta_t}(t)X(t+1)+L(t,\eta_t))^T & R(t,\eta_t)+\Pi_{3\eta_t}(t)X(t+1)\end{pmatrix}$$

$$\left.\times\begin{pmatrix} x(t)\\ u(t)\end{pmatrix}|\eta_{t_0}=i\right] \tag{6.9}$$

for all $u\in\ell^2_{\tilde{\mathcal{H}}}\{t_0,t_1-1;\mathbf{R}^m\}$, $i\in\mathcal{D}_{t_0}$. Applying Lemma 6.1 with $\mathbf{Q}_3=R(t,\eta_t)+\Pi_{3\eta_t}(t)X(t+1)$ and taking into account that $X(t)$ solves the DTSRE-C (5.128) one obtains (6.6). Thus the proof is complete. □

Let us consider the discrete-time affine controlled system of the form

$$x(t+1)=\left(A_0(t,\eta_t)+\sum_{k=1}^{r}w_k(t)A_k(t,\eta_t)\right)x(t)+\left(B_0(t,\eta_t)\right. \tag{6.10}$$
$$\left.+\sum_{k=1}^{r}w_k(t)B_k(t,\eta_t)\right)u(t)+f_0(t,\eta_t)+\sum_{k=1}^{r}w_k(t)f_k(t,\eta_t)$$

obtained from (6.4) by adding the forcing terms $f_k(t,\eta_t)$. For each solution $X(t)=(X(t,1),X(t,2),\dots,X(t,N))$ of DTSRE-C (5.128) defined for $t_0\le t\le t_1$ we construct the following discrete-time backward affine equations on $\mathbf{R}^n\oplus\mathbf{R}^n\oplus\cdots\oplus\mathbf{R}^n$ and $\mathbf{R}^N$, respectively:

$$\kappa(t,i)=[A_0(t,i)+B_0(t,i)F^X(t,i)]^T\mathcal{E}_i(t,\kappa(t+1))+g(t,i),\qquad 1\le i\le N \tag{6.11}$$

$$\mu(t)=P_t\mu(t+1)+h(t) \tag{6.12}$$

with the unknowns $\kappa(t)=(\kappa(t,1),\dots,\kappa(t,N))\in\mathbf{R}^n\oplus\mathbf{R}^n\oplus\cdots\oplus\mathbf{R}^n$, $\mu(t)=(\mu(t,1),\dots,\mu(t,N))^T\in\mathbf{R}^N$, where

$$g(t,i)=\sum_{k=0}^{r}[A_k(t,i)+B_k(t,i)F^X(t,i)]^T\mathcal{E}_i(t,X(t+1))f_k(t,i);$$

$h(t)=(h(t,1),\dots,h(t,N))^T$ with

$$h(t,i)=\sum_{k=0}^{r}f_k^T(t,i)\mathcal{E}_i(t,X(t+1))f_k(t,i)-(\mathcal{E}_i^T(t,\kappa(t+1))B_0(t,i)$$
$$+\sum_{k=0}^{r}f_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i))(R(t,i)$$

$$+\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i))^{-1}(B_0^T(t,i)\mathcal{E}_i(t,\kappa(t+1))$$

$$+\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))f_k(t,i)) + 2f_0^T(t,i)\mathcal{E}_i(t,\kappa(t+1)). \qquad (6.13)$$

If we denote $\hat{J}(t_0,t_1,x_0,i,u)$ the value of (6.5) along the trajectories of the affine system (6.10) we obtain the following.

Proposition 6.2 *Assume* $\mathbf{H_1}$ *and* $\mathbf{H_2}$. *Let* $0 \le t_0 < t_1 \in \mathbf{Z}$ *be given. Then for any solution* $X(t) = (X(t,1),\ldots,X(t,N))$ *of DTSRE-C* (5.128) *defined for* $t_0 \le t \le t_1$ *and for any solution* $\kappa^X(t) = (\kappa^X(t,1),\ldots,\kappa^X(t,N))$ *of* (6.11) *defined for* $t_0 \le t \le t_1$, *we have:*

$$\begin{aligned}\hat{J}(t_0,t_1,x_0,i,u) &= x_0^T X(t_0,i)x_0 + 2x_0^T\kappa^X(t_0,i) + \mu^X(t_0,i)\\ &\quad - E[x^T(t_1)X(t_1,\eta_{t_1})x(t_1) + 2x^T(t_1)\kappa^X(t_1,\eta_{t_1})|\eta_{t_0}=i]\\ &\quad + \sum_{t=t_0}^{t_1-1} E\Bigg[(u(t) - F^X(t,\eta_t)x(t) - \psi^X(t,\eta_t))^T\\ &\quad \times \left(R(t,\eta_t) + \sum_{k=0}^{r} B_k^T(t,\eta_t)\mathcal{E}_{\eta_t}(t,X(t+1))B_k(t,\eta_t)\right)\\ &\quad \times (u(t) - F^X(t,\eta_t)x(t) - \psi^X(t,\eta_t))|\eta_{t_0}=i\Bigg]\end{aligned}$$

for all $u \in \ell^2_{\tilde{\mathcal{H}}}\{t_0,t_1-1;\mathbf{R}^m\}$, $i \in \mathcal{D}_{t_0}$, $x_0 \in \mathbf{R}^n$, *where* $\mu^X(t) = (\mu^X(t,1),\ \ldots,\mu^X(t,N))^T$ *is the solution of* (6.12), *with* $\mu^X(t_1,i)=0$, $1 \le i \le N$, *and*

$$\begin{aligned}\psi^X(t,i) = &-\left(R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i)\right)^{-1}\\ &\times \Bigg(B_0^T(t,i)\mathcal{E}_i(t,\kappa^X(t+1))\\ &+\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))f_k(t,i)\Bigg). \qquad (6.14)\end{aligned}$$

Proof. Applying Lemma 3.1 to the system (6.10) and the function

$$V(t,x,i) = x^T X(t,i)x + 2x^T\kappa^X(t,i) + \mu^X(t,i),$$

taking the conditional expectation with respect to the event $\{\eta_{t_0}=i\}$, $i \in \mathcal{D}_{t_0}$, and summing for t from t_0 to t_1-1, one gets:

$$\hat{J}(t_0,t_1,x_0,i,u) = V(t_0,x_0,i) - E[V(t_1,x(t_1),\eta_{t_1})|\eta_{t_0}=i]$$
$$+\sum_{t=t_0}^{t_1-1} E\left[\begin{pmatrix} x(t)\\ 1\\ 1\\ u(t)\end{pmatrix}^T \mathcal{W}(t,\eta_t)\begin{pmatrix} x(t)\\ 1\\ 1\\ u(t)\end{pmatrix}|\eta_{t_0}=i\right] \tag{6.15}$$

for all $u \in \ell^2_{\tilde{\mathcal{H}}}\{t_0,t_1-1;\mathbf{R}^m\}$ where

$$\mathcal{W}(t,i) = \begin{pmatrix} \mathcal{W}_{11}(t,i) & \mathcal{W}_{12}(t,i) & \mathcal{W}_{13}(t,i) & \mathcal{W}_{14}(t,i)\\ \mathcal{W}_{12}^T(t,i) & \mathcal{W}_{22}(t,i) & \mathcal{W}_{23}(t,i) & \mathcal{W}_{24}(t,i)\\ \mathcal{W}_{13}^T(t,i) & \mathcal{W}_{23}^T(t,i) & \mathcal{W}_{33}(t,i) & \mathcal{W}_{34}(t,i)\\ \mathcal{W}_{14}^T(t,i) & \mathcal{W}_{24}^T(t,i) & \mathcal{W}_{34}^T(t,i) & \mathcal{W}_{44}(t,i)\end{pmatrix}$$

with

$$\mathcal{W}_{11}(t,i) = \Pi_{1i}(t)X(t+1) - X(t,i) + M(t,i),$$

$$\mathcal{W}_{12}(t,i) = \sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1))f_k(t,i),$$

$$\mathcal{W}_{13}(t,i) = A_0^T(t,i)\mathcal{E}_i(t,\kappa^X(t+1)) - \kappa^X(t,i),$$

$$\mathcal{W}_{14}(t,i) = \Pi_{2i}(t)X(t+1) + L(t,i),$$

$$\mathcal{W}_{22}(t,i) = \sum_{k=0}^{r} f_k^T(t,i)\mathcal{E}_i(t,X(t+1))f_k(t,i),$$

$$\mathcal{W}_{23}(t,i) = f_0^T(t,i)\mathcal{E}_i(t,\kappa^X(t+1)),$$

$$\mathcal{W}_{24}(t,i) = \sum_{k=0}^{r} f_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i),$$

$$\mathcal{W}_{33}(t,i) = \mathcal{E}_i(t,\mu^X(t+1)) - \mu^X(t,i),$$

$$\mathcal{W}_{34}(t,i) = \mathcal{E}_i^T(t,\kappa^X(t+1))B_0(t,i),$$

$$\mathcal{W}_{44}(t,i) = \Pi_{3i}(t)X(t+1) + R(t,i).$$

Finally applying Lemma 6.1 in (6.15) with $\mathbf{Q}_3 = R(t,i) + \Pi_{3i}(t)X(t+1)$ and taking into account that $X(t), \kappa^X(t), \mu^X(t)$ are solutions of (5.128), (6.11) and (6.12), respectively, one obtains the equality in the statement and thus the proof ends. □

6.2 The problem of the linear quadratic regulator

Considering the controlled system (6.4) we associate the cost function

$$J_1(t_0, x_0, u) = \sum_{t=t_0}^{\infty} E[|z_u(t, t_0, x_0)|^2] \tag{6.16}$$

with

$$z_u(t, t_0, x_0) = C(t, \eta_t)x_u(t, t_0, x_0) + D(t, \eta_t)u(t),$$

$x_u(t, t_0, x_0)$ being the solution of (6.4) corresponding to the control $u(t)$ and having the initial condition $x_u(t_0, t_0, x_0) = x_0$.

The class of admissible controls $\mathcal{U}_1(t_0, x_0)$ consists of the stochastic processes $u = \{u(t)\}_{t \geq t_0} \in \ell^2_{\tilde{\mathcal{H}}}\{t_0, t_1; \mathbf{R}^m\}$ for all $t_1 > t_0$, with the property that the series in (6.16) is convergent. The optimization problem we want to solve asks us to find the control $u_{\text{opt}} \in \mathcal{U}_1(t_0, x_0)$ such that $J_1(t_0, x_0, u_{\text{opt}}) \leq J_1(t_0, x_0, u)$ for any $u \in \mathcal{U}_1(t_0, x_0)$. We solve this optimization problem under the following assumption.

A.6.1 The coefficient matrices of the system (6.4) have the properties:

(*i*) $\{A_k(t, i)\}_{t \geq 0}, \{B_k(t, i)\}_{t \geq 0}, \{C(t, i)\}_{t \geq 0}, \{D(t, i)\}_{t \geq 0}, (t, i) \in \mathbf{Z}_+ \times \mathcal{D}$ are bounded sequences.
(*ii*) $C^T(t, i)D(t, i) = 0, (t, i) \in \mathbf{Z}_+ \times \mathcal{D}$.
(*iii*) There exists $\delta > 0$, such that

$$R(t, i) := D^T(t, i)D(t, i) \geq \delta I_n \tag{6.17}$$

for all $(t, i) \in \mathbf{Z}_+ \times \mathcal{D}$.

It should be noted that if (6.17) is fulfilled then the condition (*ii*) can be obtained without losing generality with the following change of the control variable $u(t) = \tilde{u}(t) - R^{-1}(t, \eta_t)D^T(t, \eta_t)C(t, \eta_t)x(t)$, where $\tilde{u}(t)$ denotes the new input variable.

Let $x^X(t), t \geq t_0$ be the solution of the following problem with given initial values

$$x(t+1) = \Big(A_0(t, \eta_t) + B_0(t, \eta_t)F^X(t, \eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(t, \eta_t) + B_k(t, \eta_t)F^X(t, \eta_t))\Big)x(t) \tag{6.18}$$

$$x^X(t_0) = x_0,$$

F^X being defined as in (6.7).

Let $u^X = F^X(t, \eta_t)x^X(t),\ t \geq t_0$. Then we have the following useful result.

Lemma 6.2 *Under the assumptions* H_1, H_2, *and* **A6.1,** *for any bounded and positive semidefinite solution* $X(t)$ *of DTSRE-C* (5.138) *and for any* $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ *the controls* $u^X = \{u^X(t)\}_{t \geq t_0}$ *are in* $\mathcal{U}_1(t_0, x_0)$.

Proof. Because $x^X(t)$ is $\mathcal{H}_{t-1}$-measurable and $\mathcal{H}_{t-1} \subset \tilde{\mathcal{H}}_t$ one obtains that $u^X(t) \in \ell^2_{\tilde{H}}\{t_0, t_1; \mathbf{R}^m\}$ for arbitrary $t_1 > t_0$. Thus it remains to show that $J_1(t_0, x_0, u^X)$ is well defined. Applying Proposition 6.1 in the special case $M(t,i) = C^T(t,i)C(t,i), L(t,i) = 0, R(t,i) = D^T(t,i)D(t,i), u(t) = u^X(t)$, one deduces that

$$\sum_{t=t_0}^{\tau} E[|C(t,\eta_t)x^X(t)|^2 + |D(t,\eta_t)u^X(t)|^2 | \eta_{t_0} = i]$$
$$= x_0^T X(t_0, i)x_0 - E[(x^X(\tau+1))^T X(\tau+1, \eta_{\tau+1})x^X(\tau+1) | \eta_{t_0} = i] \tag{6.19}$$

for all $i \in \mathcal{D}_{t_0}$. Under the considered assumptions it follows that there exists a positive constant c such that $0 \leq X(t,i) \leq cI_n$ for all $(t,i) \in \mathbf{Z}_+ \times \mathcal{D}$. Thus from (6.19) it follows that

$$\sum_{t=t_0}^{\tau} E[|C(t,\eta_t)x^X(t)|^2 + |D(t,\eta_t)u^X(t)|^2 | \eta_{t_0} = i] \leq x_0^T X(t_0,i)x_0 \leq c|x_0|^2$$

for all $\tau > t_0$. Therefore we may conclude that

$$\sum_{t=t_0}^{\tau} E[|C(t,\eta_t)x^X(t)|^2 + |D(t,\eta_t)u^X(t)|^2 | \eta_{t_0} = i]$$

is convergent. Moreover, we have

$$\sum_{t=t_0}^{\infty} E[|C(t,\eta_t)x^X(t)|^2 + |D(t,\eta_t)u^X(t)|^2 | \eta_{t_0} = i]$$
$$\leq x_0^T X(t_0, i)x_0, \qquad i \in \mathcal{D}_{t_0}, \qquad x_0 \in \mathbf{R}^n. \tag{6.20}$$

Furthermore, (6.20) shows that $J_1(t_0, x_0, u^X)$ is well defined and

$$J_1(t_0, x_0, u^X) \leq \sum_{i \in \mathcal{D}_{t_0}} \pi_{t_0}(i) x_0^T X(t_0, i)x_0 \tag{6.21}$$

for all $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ with $\pi_{t_0}(i) = \mathcal{P}\{\eta_{t_0} = i\}$; thus the proof is complete. □

The main result of this section is given in the following theorem.

Theorem 6.1 *Assume that the system* (6.4) *is stochastic stabilizable. Then the optimal solution of the problem of the linear quadratic regulator described* *by the controlled system* (6.4)*, the cost functional* (6.16)*, and the class of admissible controls* $\mathcal{U}_1(t_0, x_0)$ *is given by*

$$u_{\text{opt}}(t) = F_{\text{opt}}(t, \eta_t)x_{\text{opt}}(t), \tag{6.22}$$

where $F_{\text{opt}}(t,i) = F^{X_{\min}}(t,i)$ *is associated by* (6.7) *with the minimal solution* $X_{\min}(t) = (X_{\min}(t,1), \dots, X_{\min}(t,N))$ *of the DTSRE-C* (5.138) *and* $\{x_{\text{opt}}(t)\}_{t\geq t_0}$ *is the solution of the closed-loop system* (6.18) *written for* $F_{\text{opt}}(t,i)$ *instead of* $F^X(t,i)$*. The minimal value of the cost functional is*

$$J_1(t_0, x_0, u_{\text{opt}}) = \sum_{i\in\mathcal{D}_{t_0}} \pi_{t_0}(i) x_0^T X_{\min}(t_0, i)x_0. \tag{6.23}$$

Proof. For each integer $\tau \geq t_0 + 1$ we consider $X_\tau(t) = (X(t,1), \dots, X(t,N))$ the solution of DTSRE-C (5.138) with the terminal value $X_\tau(\tau, i) = 0$, $1 \leq i \leq N$. From Corollary 5.1 it follows that $X_\tau(t)$ is well defined for $0 \leq t \leq \tau$; we also deduce from the proof of Theorem 5.9 that $X_{\min}(t,i) = \lim_{\tau\to\infty} X_\tau(t,i), (t,i) \in \mathbf{Z}_+ \times \mathcal{D}$. Further choose $u = \{u(t)\}_{t\geq t_0} \in \mathcal{U}_1(t_0, x_0)$. Applying Proposition 6.1 for $X(t,i) = X_\tau(t,i)$ one obtains

$$\sum_{t=t_0}^{\tau-1} E[|C(t,\eta_t)x(t)|^2 + |D(t,\eta_t)u(t)|^2 | \eta_{t_0} = i]$$

$$= x_0^T X_\tau(t_0,i)x_0 + \sum_{t=t_0}^{\tau-1} E\left[(u(t) - F_\tau(t,\eta_t)x(t))^T\right.$$

$$\times \left(R(t,\eta_t) + \sum_{k=0}^{r} B_k^T(t,\eta_t)\mathcal{E}_{\eta_t}(t, X_\tau(t+1))B_k(t,\eta_t)\right)$$

$$\times\ (u(t) - F_\tau(t,\eta_t)x(t)) | \eta_{t_0} = i]$$

for all $i \in \mathcal{D}_{t_0}$, where $F_\tau(t,i) = F^{X_\tau}(t,i)$ is constructed as in (6.7) with

$$R(t,i) = D^T(t,i)D(t,i), L(t,i) = 0$$

and $X_\tau(t)$ instead of $X(t)$. The fact that $X_\tau(t,i) \geq 0$ together with (6.17) allows us to deduce that

$$\sum_{t=t_0}^{\tau-1} E[|C(t,\eta_t)x(t)|^2 + |D(t,\eta_t)u(t)|^2 | \eta_{t_0} = i] \geq x_0^T X_\tau(t_0,i)x_0 \tag{6.24}$$

for all $\tau \geq t_0 + 1$, $u \in \mathcal{U}_1(t_0, x_0)$, $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$. Taking the limit $\tau \to \infty$ in (6.24) it follows that

$$\sum_{t=t_0}^{\infty} E[|C(t,\eta_t)x(t)|^2 + |D(t,\eta_t)u(t)|^2 | \eta_{t_0} = i] \geq x_0^T X_{\min}(t_0,i)x_0$$

for all $i \in \mathcal{D}_{t_0}$, $u \in \mathcal{U}_1(t_0, x_0)$, $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$. This allows us to conclude that

$$J_1(t_0, x_0, u) \geq \sum_{i \in \mathcal{D}_{t_0}} \pi_{t_0}(i) x_0^T X_{\min}(t_0, i) x_0 \tag{6.25}$$

for all $u \in \mathcal{U}_1(t_0, x_0)$. However, from (6.22) and from Lemma 6.2 it follows that $u_{\text{opt}} = \{u_{\text{opt}}(t)\}_{t \geq 0}$ is in $\mathcal{U}_1(t_0, x_0)$. From (6.21) and (6.25) we deduce that $J_1(t_0, x_0, u_{\text{opt}}) = \sum_{i \in \mathcal{D}_{t_0}} \pi_{t_0}(i) x_0^T X_{\min}(t_0, i) x_0$, respectively. This shows that the equality (6.23) holds. Now (6.25) becomes $J_1(t_0, x_0, u) \geq J_1(t_0, x_0, u_{\text{opt}})$ for all $u \in \mathcal{U}(t_0, x_0)$. This confirms the optimality of u_{opt} and thus the proof is complete. □

Remark 6.1 In the definition of the class of admissible controls $\mathcal{U}_1(t_0, x_0)$ no assumption about the asymptotic behavior for $t \to \infty$ of the trajectories $x_u(t, t_0, x_0)$ is made. However, from the convergence of the series in (6.16) one deduces that

$$\lim_{t \to \infty} E[|C(t, \eta_t) x_u(t, t_0, x_0)|^2] = 0.$$

On the other hand from (6.19) with $u = u_{\text{opt}}$ and taking into account (6.23) we deduce that

$$\lim_{\tau \to \infty} \sum_{i \in \mathcal{D}_{t_0}} \pi_{t_0}(i) E[x_{\text{opt}}^T(\tau) X_{\min}(\tau, \eta_\tau) x_{\text{opt}}(\tau) | \eta_{t_0} = i] = 0,$$

or equivalently

$$\lim_{\tau \to \infty} E[x_{\text{opt}}^T(\tau) X_{\min}(\tau, \eta_\tau) x_{\text{opt}}(\tau) | \eta_{t_0} = i] = 0, \qquad i \in \mathcal{D}_{t_0}.$$

This is additional information concerning the asymptotic behavior for $t \to \infty$ for the optimal trajectory $x_{\text{opt}}(t)$.

6.3 The linear quadratic optimization problem

In this section we deal with the optimization problem described by the system (6.4) and the cost functional

$$\begin{aligned} J_2(t_0, x_0, u) = \sum_{t=t_0}^{\infty} E[&x_u^T(t, t_0, x_0) M(t, \eta_t) x_u(t, t_0, x_0) \\ &+ 2x_u^T(t, t_0, x_0) L(t, \eta_t) u(t) + u^T(t) R(t, \eta_t) u(t)], \end{aligned} \tag{6.26}$$

where $x_u(t, t_0, x_0)$ is the solution of (6.4) corresponding to the input u having the initial value $x_u(t_0, t_0, x_0) = x_0$. The class of admissible controls $\mathcal{U}_2(t_0, x_0)$ consists of the stochastic processes $u = \{u(t)\}_{t \geq t_0} \in \ell^2_{\tilde{\mathcal{H}}}\{t_0, t_1; \mathbf{R}^m\}$ for all $t_1 > t_0$, with the property that the series (6.26) is convergent and

$$\lim_{t\to\infty} E[|x_u(t,t_0,x_0)|^2|\eta_{t_0}=i]=0 \tag{6.27}$$

for all $i \in \mathcal{D}_{t_0}$.

Throughout this section we assume that the following hypothesis is true.

A.6.2 The sequences $\{A_k(t,i)\}_{t\geq 0} \subset \mathbf{R}^{n\times n}$, $\{B_k(t,i)\}_{t\geq 0} \subset \mathbf{R}^{n\times m}$, $0 \leq k \leq r$, $\{M(t,i)\}_{t\geq 0} \subset \mathcal{S}_n$, $\{L(t,i)\}_{t\geq 0} \subset \mathbf{R}^{n\times m}$, $\{R(t,i)\}_{t\geq 0} \subset \mathcal{S}_m$ are bounded.

The optimization problem investigated in this section requires us to find a control $\tilde{u} = \{u(t)\}_{t\geq t_0} \in \mathcal{U}_2(t_0,x_0)$ with the property that $J_2(t_0,x_0,\tilde{u}) \leq J_2(t_0,x_0,u)$ for all $u \in \mathcal{U}_2(t_0,x_0)$.

Remark 6.2 It can be seen that the cost functional $J_1(t_0,x_0,u)$ from (6.16) is a special case of $J_2(t_0,x_0,u)$ from (6.26). Hence in the case of the cost functional $J_1(t_0,x_0,\cdot)$ we may construct two classes of admissible controls $\mathcal{U}_1(t_0,x_0)$ and $\mathcal{U}_2(t_0,x_0)$. We have $\mathcal{U}_2(t_0,x_0) \subseteq \mathcal{U}_1(t_0,x_0)$. This leads to

$$\inf_{u\in\mathcal{U}_1(t_0,x_0)} J_1(t_0,x_0,u) \leq \inf_{u\in\mathcal{U}_2(t_0,x_0)} J_1(t_0,x_0,u). \tag{6.28}$$

Because in (6.26) no assumptions concerning the sign of the weighting matrices $M(\cdot,\cdot), L(\cdot,\cdot)$, and $R(\cdot,\cdot)$ were made, it is possible that $u \mapsto J_2(t_0,x_0,u)$: $\mathcal{U}_2(t_0,x_0) \to \mathbf{R}$ is unbounded from below.

Set

$$\mathbf{V}(t_0,x_0) = \inf_{u\in\mathcal{U}_2(t_0,x_0)} J_1(t_0,x_0,u), \qquad (t_0,x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n \tag{6.29}$$

to be the value function associated with the optimization problem under consideration.

Definition 6.1 *We say that the linear quadratic optimization problem described by the controlled system* (6.4), *the cost functional* (6.26), *and the class of admissible controls* $\mathcal{U}_2(t_0,x_0)$ *is well posed if* $-\infty < \mathbf{V}(t_0,x_0) < +\infty$ *for all* $(t_0,x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$.

To make the statement of the next results clearer we adopt the following notations.

$$\mathcal{Q}(t) = (\mathcal{Q}(t,1),\ldots,\mathcal{Q}(t,N))$$

$$\mathcal{Q}(t,i) = \begin{pmatrix} M(t,i) & L(t,i) \\ L^T(t,i) & R(t,i) \end{pmatrix}$$

$$\varPi(t)X = ((\varPi(t)X)(1),\ldots,(\varPi(t)X)(N))$$

$$(\varPi(t)X)(i) = \begin{pmatrix} \varPi_{1i}(t)X & \varPi_{2i}(t)X \\ (\varPi_{2i}(t)X)^T & \varPi_{3i}(t)X \end{pmatrix},$$

$\Pi_{li}(t)X$ being constructed as in (5.130) *using the coefficients of the system* (6.4) *and the elements $p_t(i,j)$ of P_t.*

In the sequel, Γ^{Σ} is the set associated with the DTSRE-C (5.128) *via* (5.15).

Theorem 6.2 *Assume:*

(*a*) *The system* (6.4) *is stochastic stabilizable.*
(*b*) *The set Γ^{Σ} is not empty.*

Under these conditions the linear quadratic optimization problem described by the system (6.4), *the cost functional* (6.26), *and the class of admissible controls $\mathcal{U}_2(t_0,x_0)$ is well posed. Moreover, we have*

$$\mathbf{V}(t_0,x_0)=\sum_{i\in\mathcal{D}}\pi_{t_0}(i)x_0^T X_{\max}(t_0,i)x_0 \tag{6.30}$$

for all $(t_0,x_0)\in\mathbf{Z}_+\times\mathbf{R}^n$, where

$$X_{\max}(t)=(X_{\max}(t,1),\dots,X_{\max}(t,N))$$

is the maximal bounded solution of the DTSRE-C (5.128) *that verifies* (5.131).

Proof. First we remark that under the assumptions (a) and (b) in the statement, the DTSRE-C (5.128) has a bounded and maximal solution $X_{\max}(t)$ that verifies (5.131). Also from assumption (a) it follows that $\mathcal{U}_2(t_0,x_0)\neq\Phi$ for each $t_0\geq 0$ and $x_0\in\mathbf{R}^n$. Applying Proposition 6.1 for $X(t,i)=X_{\max}(t,i)$ we obtain

$$\begin{aligned}\sum_{t=t_0}^{\tau-1}&E\left[\begin{pmatrix}x_u(t)\\u(t)\end{pmatrix}^T\mathcal{Q}(t,\eta_t)\begin{pmatrix}x_u(t)\\u(t)\end{pmatrix}|\eta_{t_0}=i\right]\\&=x_0^TX_{\max}(t_0,i)x_0-E[x_u^T(\tau)X_{\max}(\tau,\eta_\tau)x_u(\tau)|\eta_{t_0}=i]\\&\quad+\sum_{t=t_0}^{\tau-1}E[(u(t)-\tilde F(t,\eta_t)x_u(t))^T(R(t,\eta_t)+\Pi_{3\eta_t}(t)X_{\max}(t+1))\\&\quad\times(u(t)-\tilde F(t,\eta_t)x_u(t))|\eta_{t_0}=i]\end{aligned} \tag{6.31}$$

for all $\tau\geq t_0+1$, $i\in\mathcal{D}_{t_0}$, $u\in\mathcal{U}_2(t_0,x_0),(t_0,x_0)\in\mathbf{Z}_+\times\mathbf{R}^n$, and $\tilde F(t,i)=F^{X_{\max}}(t,i)$. Because $\{X_{\max}(t)\}_{t\geq 0}$ is a bounded sequence it follows that there exists a positive constant c_1 such that $|X_{\max}(t,i)|\leq c_1$ for all $(t,i)\in\mathbf{Z}_+\times\mathcal{D}$ ($|\cdot|$ being the Euclidian norm of a matrix). This allows us to write

$$|E[x_u^T(\tau)X_{\max}(\tau,\eta_\tau)x_u(\tau)|\eta_{t_0}=i]|\leq c_1E[|x_u(\tau)|^2|\eta_{t_0}=i]$$

for all $\tau\geq t_0+1$, $i\in\mathcal{D}_{t_0}$. Taking into account (6.27) we deduce that

$$\lim_{\tau\to\infty}E[x_u^T(\tau)X_{\max}(\tau,\eta_\tau)x_u(\tau)|\eta_{t_0}=i]=0,\qquad\forall i\in\mathcal{D}_{t_0}. \tag{6.32}$$

From (6.31) we get

$$\sum_{t=t_0}^{\tau-1} E\left[\begin{pmatrix} x_u(t) \\ u(t) \end{pmatrix}^T Q(t,\eta_t)\begin{pmatrix} x_u(t) \\ u(t) \end{pmatrix}\right] + E[x_u^T(\tau)X_{\max}(\tau,\eta_\tau)x_u(\tau)]$$
$$= \sum_{i\in\mathcal{D}} \pi_{t_0}(i)x_0^T X_{\max}(t_0,i)x_0 + \sum_{t=t_0}^{\tau-1} E[(u(t)-\tilde{F}(t,\eta_t)x_u(t))^T$$
$$\times (R(t,\eta_t)+\Pi_{3\eta_t}(t)X_{\max}(t+1))(u(t)-\tilde{F}(t,\eta_t)x_u(t))]. \quad (6.33)$$

Because the left-hand side of (6.33) converges for $\tau \to \infty$ it follows that the right-hand side is also convergent. Taking the limit for $\tau \to \infty$ in (6.33) and taking into account (6.32) we obtain

$$J_2(t_0,x_0,u) = \sum_{i\in\mathcal{D}} \pi_{t_0}(i)x_0^T X_{\max}(t_0,i)x_0 + \sum_{t=t_0}^{\infty} E[(u(t)-\tilde{F}(t,\eta_t)x_u(t))^T$$
$$\times (R(t,\eta_t)+\Pi_{3\eta_t}(t)X_{\max}(t+1))(u(t)-\tilde{F}(t,\eta_t)x_u(t))] \quad (6.34)$$

for all $u \in \mathcal{U}_2(t_0,x_0); (t_0,x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$. Furthermore, (6.34) together with (5.131) leads to

$$J_2(t_0,x_0,u) \geq \sum_{i\in\mathcal{D}} \pi_{t_0}(i)x_0^T X_{\max}(t_0,i)x_0$$

for all $u \in \mathcal{U}_2(t_0,x_0); (t_0,x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$. Hence

$$\mathbf{V}(t_0x_0) \geq \sum_{i\in\mathcal{D}} \pi_{t_0}(i)x_0^T X_{\max}(t_0,i)x_0. \quad (6.35)$$

Thus we obtain that the linear quadratic optimization problem under consideration is well posed. It remains to show that in (6.35) we have equality. To this end we choose a decreasing sequence of real and positive numbers $\{\varepsilon_j\}_{j\geq 0}$ such that $\lim_{j\to\infty}\varepsilon_j = 0$. We associate the cost functionals

$$J^{\varepsilon_j}(t_0,x_0,u) = J_2(t_0,x_0,u) + \varepsilon_j \sum_{t=t_0}^{\infty} E[|x_u(t)|^2] \quad (6.36)$$

$u \in \tilde{\mathcal{U}}_2(t_0,x_0)$, where $x_u(t) = x_u(t,t_0,x_0)$ and $\tilde{\mathcal{U}}_2(t_0,x_0) = \{u = \{u(t)\}_{t\geq t_0} \in \mathcal{U}_2(t_0,x_0) | x_u \in \ell^2_{\tilde{\mathcal{H}}}(t_0,\infty,\mathbf{R}^n)\}$.

Let $\mathbf{V}^j(t_0,x_0) = \inf_{u\in\tilde{\mathcal{U}}_2(t_0,x_0)} J^{\varepsilon_j}(t_0,x_0)$. Because $\tilde{\mathcal{U}}_2(t_0,x_0) \subset \mathcal{U}_2(t_0,x_0)$ and $J_2(t_0,x_0,u) \leq J^{\varepsilon_j}(t_0,x_0), u \in \tilde{\mathcal{U}}_2(t_0,x_0)$ we deduce that $\mathbf{V}^j(t_0,x_0) \geq \mathbf{V}(t_0,x_0)$ for all $j \geq 0$. We remark that under the assumptions $(a), \tilde{\mathcal{U}}_2(t_0,x_0) \neq \Phi$, $t_0 \geq 0, x_0 \in \mathbf{R}^n$.

Consider the DTSRE-C

$$X(t,i) = \Pi_{1i}(t)X(t+1) + M(t,i) + \varepsilon_j I_n - [L(t,i) + \Pi_{2i}(t)X(t+1)] \\ \times [R(t,i) + \Pi_{3i}(t)X(t+1)]^{-1}[L(t,i) + \Pi_{2i}(t)X(t+1)]^T, \quad (6.37)$$

where $\Pi_{li}(t), l \in \{1,2,3\}$ are defined as in (5.130). The DTSRE-C (6.37) is defined by the pair $\Sigma_j = (\Pi(t), \mathcal{Q}^j(t))$, where $\Pi(t)$ is given by (5.130) and where

$$\mathcal{Q}^j(t) = (\mathcal{Q}^j(t,1), \dots, \mathcal{Q}^j(t,N)),$$

$$\mathcal{Q}^j(t,i) = \begin{pmatrix} M(t,i) + \varepsilon_j I_n & L(t,i) \\ L^T(t,i) & R(t,i) \end{pmatrix}.$$

For each $j \geq 0, \tilde{\Gamma}^{\Sigma_j}$ is not empty because $\tilde{\Gamma}^{\Sigma_j} \supset \Gamma^{\Sigma}$. Applying Theorem 5.12 we deduce that for each $j \geq 0$, DTSRE-C (6.37) has a bounded and stabilizing solution $X_s^j(t) = (X_s^j(t,1), \dots, X_s^j(t,N)), t \geq 0$. Based on Corollary 5.2 it follows that $X_s^j(t)$ coincides with the maximal solution of (6.37). Furthermore, from Theorem 5.4 one deduces that $X_s^j(t,i) \geq X_s^{j+1}(t,i) \geq X_{\max}(t,i)$ for all $j \geq 0$ and $\lim_{j\to\infty} X_s^j(t,i) = X_{\max}(t,i)$ for all $t \geq 0, i \in \mathcal{D}$. Proceeding as in the first part of the proof we deduce that

$$J_2^{\varepsilon_j}(t_0,x_0,u) = \sum_{i\in\mathcal{D}} \pi_{t_0}(i) x_0^T X_s^j(t_0,i)x_0 + \sum_{t=t_0}^{\infty} E[(u(t) - F_s^j(t,\eta_t)x_u(t))^T \\ \times (R(t,\eta_t) + \Pi_{3\eta_t}(t)X_s^j(t+1))(u(t) - F_s^j(t,\eta_t)x_u(t))] \quad (6.38)$$

for all $u \in \tilde{\mathcal{U}}_2(t_0,x_0)$, where $F_s^j(t,i) = F^{X_s^j}(t,i)$ is a stabilizing feedback associated with $X_s^j(t)$. Take the control $u_s^j(t) = F_s^j(t,\eta_t)x_s^j(t)$, $\{x_s^j(t)\}_{t\geq t_0}$ being the solution of the system (6.18) when $F^X(t,i)$ is replaced by $F_s^j(t,i)$. Because $X_s^j(t)$ is the stabilizing solution of (6.37) it follows that $u_s^j = \{u_s^j(t)\}_{t\geq t_0} \in \tilde{\mathcal{U}}_2(t_0,x_0)$. Taking $u = u_s^j$ in (6.38) we obtain

$$J_2^{\varepsilon_j}(t_0,x_0,u_s^j) = \sum_{i\in\mathcal{D}} \pi_{t_0}(i)x_0^T X_s^j(t_0,i)x_0.$$

This leads to $\mathbf{V}(t_0,x_0) \leq \mathbf{V}^j(t_0,x_0) \leq \sum_{i\in\mathcal{D}} \pi_{t_0}(i)x_0^T X_s^j(t_0,i)x_0$ for all $j \geq 0$. Taking the limit for $j \to \infty$, we obtain

$$\mathbf{V}(t_0,x_0) \leq \sum_{i\in\mathcal{D}} \pi_{t_0}(i)x_0^T X_{\max}(t_0,i)x_0, \qquad \forall (t_0,x_0) \in \mathbf{Z}_t \times \mathbf{R}^n. \quad (6.39)$$

Thus from (6.39) and (6.35) one obtains (6.30) and the proof is complete. □

The previous theorem provides a lower bound of the cost functional $J_2(t_0,x_0,u)$ on $\mathcal{U}_2(t_0,x_0)$. However, it cannot provide any information about the existence of an optimal control.

Definition 6.2 *We say that a control* $u_{\text{opt}} = \{u_{\text{opt}}(t)\}_{t \geq t_0} \in \mathcal{U}_2(t_0, x_0)$ *is called the optimal control for the linear quadratic optimization problem under consideration if* $\mathbf{V}(t_0, x_0) = J_2(t_0, x_0, u_{\text{opt}}) \leq J_2(t_0, x_0, u)$ *for all* $u \in \mathcal{U}_2(t_0, x_0)$.

The following result provides a sufficient condition for the existence of an optimal control for the optimization problem described by the cost functional (6.26)*, the controlled system* (6.4)*, and the set of admissible controls* $\mathcal{U}_2(t_0, x_0)$.

Proposition 6.3 *If DTSRE-C* (5.128) *has a bounded and stabilizing solution* $\{X_s(t)\}_{t \geq 0}$ *that satisfies* (5.134) *then the linear quadratic optimization problem under consideration has an optimal control given by* $u_{\text{opt}}(t) = F_s(t, \eta_t)x_s(t)$ *where* $F_s(t, i)$ *is defined in* (5.132) *and* $\{x_s(t)\}_{t \geq t_0}$ *is the solution of the system* (5.133) *with the initial condition* $x_s(t_0) = x_0$.

Proof. Because $\{X_s(t)\}_{t \geq 0}$ is the bounded and stabilizing solution of (5.128) one obtains that the control $u_{\text{opt}} = F_s(t, \eta_t)x_s(t)$ is admissible. The conclusion follows immediately from (6.34) written for $u = u_{\text{opt}}$ and taking into account (5.134). □

Now we prove a result that provides a necessary and sufficient condition for the existence of an optimal control.

Theorem 6.3 *Assume that*

(*a*) *The assumptions of Theorem* 6.2 *are fulfilled.*
(*b*) *For each* $t \geq 0$, P_t *is a nondegenerate stochastic matrix.*
(*c*) $\pi_0(i) = \mathcal{P}\{\eta_0 = i\} \geq 0$ *for* $1 \leq i \leq N$.

Then the following are equivalent.

(*i*) *For any* $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ *the optimization problem described by the system* (6.4)*, the cost functional* (6.26)*, and the class of admissible controls* $\mathcal{U}_2(t_0, x_0)$ *admits an optimal control* $\hat{u}_{t_0,x_0}(t), t \geq t_0$*; that is,* $\mathbf{V}(t_0, x_0) = J_2(t_0, x_0, \hat{u}_{t_0,x_0})$.

(*ii*)
$$\lim_{t \to \infty} \|T_{\tilde{F}}(t, t_0)\|_\xi = 0, \qquad \forall t_0 \in \mathbf{Z}_+, \tag{6.40}$$

where $T_{\tilde{F}}(t, t_0)$ *is the linear evolution operator on* $\mathcal{S}_n^N$ *defined by the sequence of Lyapunov operators* $\{\mathcal{L}_{\tilde{F}}(t)\}_{t \geq 0}, \mathcal{L}_{\tilde{F}}$ *being defined by* (4.5) *with* $\tilde{F}(t, i)$ *instead of* $F(t, i)$ *and* $\tilde{F}(t, i) = F^{X_{\max}}(t, i)$.

If (i) or (ii) are fulfilled then the optimal control of the problem under consideration is given by $u_{\text{opt}}(t) = \tilde{F}(t, \eta_t)\hat{x}(t)$*, where* $\hat{x}(t)$ *is the solution of the system* (6.43).

Proof. Let us assume that (*i*) is fulfilled. Let $(t_0, x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ and $\hat{u} = \{\hat{u}(t)\}_{t \geq t_0} \in \mathcal{U}_2(t_0, x_0)$ be such that $\mathbf{V}(t_0, x_0) = J_2(t_0, x_0, \hat{u})$. From (6.34) one obtains

$$\mathbf{V}(t_0,x_0)=\sum_{i\in\mathcal{D}}\pi_{t_0}(i)x_0^TX_{\max}(t_0,i)x_0+\sum_{t=t_0}^{\infty}E[(\hat{u}(t)-\tilde{F}(t,\eta_t)\hat{x}(t))^T$$
$$\times\,(R(t,\eta_t)+\Pi_{3\eta_t}(t)X_{\max}(t+1))(\hat{u}(t)-\tilde{F}(t,\eta_t)\hat{x}(t))],\quad (6.41)$$

where $\hat{x}=x_{\hat{u}}(t)$ is the optimal trajectory. Combining (6.30) and (6.41) we deduce

$$\sum_{t=t_0}^{\infty}E[(\hat{u}(t)-\tilde{F}(t,\eta_t)\hat{x}(t))^T$$
$$\times\,(R(t,\eta_t)+\Pi_{3\eta_t}(t)X_{\max}(t+1))(\hat{u}(t)-\tilde{F}(t,\eta_t)\hat{x}(t))]=0.\qquad (6.42)$$

If we take into account (5.131) then (6.42) leads to $\hat{u}(t)=\tilde{F}(t,\eta_t)\hat{x}(t)$ a.s. $t\geq t_0$. Substituting this equality in (6.4) one obtains that $\hat{x}(t)$ is the solution of the following problem with given initial value,

$$\hat{x}(t+1)=\Big[A_0(t,\eta_t)+B_0(t,\eta_t)\tilde{F}(t,\eta_t)$$
$$+\sum_{k=1}^{r}w_k(t)(A_k(t,\eta_t)+B_k(t,\eta_t)\tilde{F}(t,\eta_t))\Big]\hat{x}(t),\qquad \hat{x}(t_0)=x_0.$$
$$(6.43)$$

Because $\hat{u}\in\mathcal{U}_2(t_0,x_0)$ it follows from (6.27) that

$$\lim_{t\to\infty}E[|\hat{x}(t)|^2|\eta_{t_0}=i]=0,\qquad i\in\mathcal{D}.\qquad (6.44)$$

If $\Phi_{\tilde{F}}(t,t_0)$ is the fundamental matrix solution of (6.43) then (6.44) may be rewritten

$$\lim_{t\to\infty}E[x_0^T\Phi_{\tilde{F}}^T(t,t_0)\Phi_{\tilde{F}}(t,t_0)x_0|\eta_{t_0}=i]=0,\qquad \forall i\in\mathcal{D},\qquad (t_0,x_0)\in\mathbf{Z}_+\times\mathbf{R}^n.$$

Based on Theorem 3.1(i) we deduce that the last equality is equivalent to $\lim_{t\to\infty}x_0^T(T_{\tilde{F}}^*(t,t_0)J)(i)x_0=0$, for all $(t_0,x_0)\in\mathbf{Z}_+\times\mathbf{R}^n$, $i\in\mathcal{D}$. Recalling that

$$\|T_{\tilde{F}}^*(t,t_0)\|_\xi=|T_{\tilde{F}}^*(t,t_0)J|_\xi=\max_{i\in\mathcal{D}}\sup_{|x_0|\leq 1}x_0^T[T_{\tilde{F}}^*(t,t_0)J](i)x_0$$

we deduce that

$$\lim_{t\to\infty}\|T_{\tilde{F}}^*(t,t_0)\|_\xi=0.\qquad (6.45)$$

Finally invoking (2.13) we deduce that (6.45) is equivalent to (6.40). Thus we obtain that the implication (i) $\Rightarrow$ (ii) is true.

To prove the converse implication, one obtains via Theorem 3.1(i) that if (6.40) is true then (6.44) holds. This means by (6.33) that the control $\hat{u}(t)=\tilde{F}(t,\eta_t)\hat{x}(t)$ is admissible. Furthermore, from (6.34) and (6.30) one obtains that $\hat{u}$ is an optimal control and thus the proof is complete. □

Remark 6.3 Combining Definition 3.1(a) and Definition 5.4 one obtains that the maximal solution $X_{\max}(t)$ of DTSRE-C (5.128) is a stabilizing solution if and only if there exist $\beta \geq 1$, $q \in (0,1)$ such that

$$\|T_{\tilde{F}}(t,t_0)\|_\xi \leq \beta q^{t-t_0} \tag{6.46}$$

for all $t \geq t_0 \geq 0$.

From Theorem 6.3 we deduce that the condition verified by the maximal solution of (5.128) which is equivalent to the existence of an optimal control of the problem under consideration is weaker than (6.46). This can explain why the result proved in Proposition 6.3 provides only a sufficient condition for the existence of an optimal control.

Theorem 6.4 *Assume that*

(a) *The coefficients of the system* (6.4) *and the weights of the cost functional* (6.26) *are periodic sequences with the period* $\theta \geq 1$ *and* $P_{t+\theta} = P_t, t \geq 0$*;*
(b) *The assumptions of Theorem* 6.2 *are fulfilled.*

Under these conditions the following are equivalent.

(i) *For every* $(t_0,x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ *the optimization problem described by the controlled system* (6.4)*, the cost functional* (6.26)*, and the class of admissible controls* $\mathcal{U}_2(t_0,x_0)$ *has the optimal control* $\hat{u}_{t_0x_0} = \{u_{t_0x_0}(t)\}_{t\geq t_0}$ *(i.e.,* $\mathbf{V}(t_0,x_0) = J_2(t_0,x_0,\hat{u}_{t_0x_0})$*).*
(ii) *The DTSRE-C* (5.128) *has a bounded stabilizing solution* $\{X_s(t)\}_{t\geq 0}$ *that satisfies* (5.134)*.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from Proposition 6.3. If (i) is fulfilled, reasoning as in the proof of Theorem 6.3 one deduces by Theorem 3.10 that the zero state equilibrium of the closed-loop system (6.43) is SESMS. This allows us to conclude that the maximal solution $\{X_{\max}(t)\}$ coincides with the stabilizing solution of (5.128). Thus the proof ends. □

Remark 6.4 As we have already seen in Remark 6.2 for the cost functional $J_1(t_0,x_0,\cdot)$ we may consider two optimization problems. Based on Theorem 6.1 one obtains that the optimal control u_{opt} of the optimization problem described by the cost functional $J_1(t_0,x_0,\cdot)$, the controlled system (6.4), and the class of admissible controls $\mathcal{U}_1(t_0,x_0)$ is constructed via the minimal solution of DTSRE-C (5.138), whereas from Theorem 6.3 we have that the optimal control $\tilde{u}_{\text{opt}}$ of the optimization problem described by the cost functional $J_1(t_0,x_0,\cdot)$, the controlled system (6.4), and the class of admissible controls $\mathcal{U}_2(t_0,x_0)$ is constructed via the maximal solution of DTSRE-C (5.138) (viewed as a special form of (5.128)).

From (6.28) one obtains that

$$J_1(t_0,x_0,u_{\text{opt}}) \leq J_1(t_0,x_0,\tilde{u}_{\text{opt}}). \tag{6.47}$$

Invoking Theorem 5.14 one obtains that under the assumption of stochastic detectability, the minimal solution of (5.138) coincides with its stabilizing solution. In this case (6.47) becomes $J_1(t_0, x_0, u_{\mathrm{opt}}) = J_1(t_0, x_0, \tilde{u}_{\mathrm{opt}})$. By the next example we show that in the absence of the assumption of stochastic detectability u_{opt} may not coincide with $\tilde{u}_{\mathrm{opt}}$ and in this case $J_1(t_0, x_0, u_{\mathrm{opt}}) < J_1(t_0, x_0, \tilde{u}_{\mathrm{opt}})$.

Example 6.1 Consider the system (6.4) in the particular case $N=1, n=2r=1$,

$$x(t+1) = (A_0 + w_1(t)A_1)x(t) + B_0 u(t) \tag{6.48}$$

with

$$A_0 = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{8}{5} \end{pmatrix}, \quad A_1 = \frac{3}{5} I_2, \; B_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbf{R}^2, \quad u(t) \in \mathbf{R}.$$

The cost functional is

$$J_1(0, x_0, u) = \sum_{t=0}^{\infty} E\left[\frac{4}{5} x_1^2(t) + u^2(t)\right]. \tag{6.49}$$

In this case the DTSRE-C (5.138) reduces to (5.93). Hence the maximal solution coincides with the stabilizing solution and it is given by (5.94). The minimal solution is given by (5.95). From Theorem 6.1 we have

$$J_1(0, x_0, u_{\mathrm{opt}}) = \begin{pmatrix} x_{10} & x_{20} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{2}}{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}.$$

Using Theorem 6.3 or Theorem 6.4 one obtains that

$$J_1(0, x_0, \tilde{u}_{\mathrm{opt}}) = \begin{pmatrix} x_{10} & x_{20} \end{pmatrix} \begin{pmatrix} \frac{7+\sqrt{2}}{2} & -3(3+\sqrt{2}) \\ -3(3+\sqrt{2}) & 3(11+6\sqrt{2}) \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

for all $x_0 = \begin{pmatrix} x_{10} & x_{20} \end{pmatrix}^T$.

6.4 The linear quadratic problem. The affine case

In this section we consider the linear quadratic optimization problem associated with the affine system (6.10). We consider both the finite time horizon and infinite time horizon. The results developed in this section are used in the next section to solve a tracking problem for discrete-time linear stochastic systems subject to independent random perturbations and Markovian jumping.

6.4.1 The problem setting

Let us consider the controlled system (6.10) where $w(t) = (w_1(t), \dots, w_r(t))^T$ and η_t are stochastic processes that verify the general assumptions $\mathbf{H_1}$ and $\mathbf{H_2}$. Throughout this section $\{\eta_t\}_{t \geq 0}$ is assumed to be a nonhomogeneous Markov chain with the transition probability matrices $P_t, t \geq 0$.

The following two classes of admissible controls are involved in this section.

(a) If $0 \leq t_0 < t_f \in \mathbf{Z}$, $\mathcal{U}_{t_0,t_f}$ consists of the stochastic processes $u = \{u(t), t_0 \leq t \leq t_f\}$, where $u(t)$ is an m-dimensional random vector with finite second moments and $\tilde{\mathcal{H}}_t$-measurable; that is, $\mathcal{U}_{t_0,t_f} = \ell^2_{\tilde{\mathcal{H}}}\{t_0, t_f - 1; \mathbf{R}^m\}$

(b) If $t_f = \infty$ and $x_0 \in \mathbf{R}^n$, $\mathcal{U}_{t_0,\infty}(x_0)$ consists of all stochastic processes $u = u(t), t_0 \leq t < \infty$ where for each t, $u(t)$ is an m-dimensional random vector that is $\tilde{\mathcal{H}}_t$-measurable, having the following two additional properties,

$$
\begin{aligned}
&(\alpha) \quad E[|u(t)|^2] < \infty, \qquad t \geq t_0 \\
&(\beta) \quad \sup_{t \geq t_0} E[|x_u(t, t_0, x_0)|^2] < \infty,
\end{aligned}
\tag{6.50}
$$

$x_u(\cdot, t_0, x_0)$ being the solution of (6.10) determined by the control u and starting from x_0 at $t = t_0$

It must be remarked that in the case $t_f < +\infty$ the initial value x_0 does not play any role in the definition of the admissible controls $\mathcal{U}_{t_0 t_f}$. On the other hand in the infinite time horizon case $(t_f = +\infty)$ it is expected that the set of admissible controls will be dependent upon the initial state x_0. This could happened due to condition (6.50).

That is why the dependence with respect to the initial state x_0 is emphasized, writing $\mathcal{U}_{t_0,\infty}(x_0)$.

We associate the following two cost functionals with the system (6.10): $J_3(t_0, t_f, x_0, \cdot) : \mathcal{U}_{t_0,t_f} \to \mathbf{R}$ and $J_3(t_0, \infty, x_0, \cdot) : \mathcal{U}_{t_0,\infty}(x_0) \to \bar{\mathbf{R}}$ by

$$
J_3(t_0, t_f, x_0, u) = E\left[x^T(t_f) K_f(\eta_{t_f}) x(t_f) + \sum_{t=t_0}^{t_f - 1} |y(t)|^2\right] \tag{6.51}
$$

$$
J_3(t_0, \infty, x_0, u) = \overline{\lim_{T \to \infty}} \frac{1}{T - t_0} \sum_{t=t_0}^{T} E[|y(t)|^2], \tag{6.52}
$$

where

$$
y(t) = C(t, \eta_t) x_u(t, t_0, x_0) + D(t, \eta_t) u(t) \tag{6.53}
$$

and $x_u(t, t_0, x_0)$ is as before.

Now we are in position to formulate the two optimization problems that are solved in this section.

OP 1. *Given* $0 \le t_0 < t_f \in \mathbf{Z}$ *and* $x_0 \in \mathbf{R}^n$, *find an admissible control* $\tilde{u} \in \mathcal{U}_{t_0,t_f}$ *that satisfies* $J_3(t_0,t_f,x_0,\tilde{u}) \le J_3(t_0,t_f,x_0,u)$ *for all* $u \in \mathcal{U}_{t_0,t_f}$.

OP 2. *Given* $t_0 \ge 0, x_0 \in \mathbf{R}^n$ *find a control* $\tilde{u} \in \mathcal{U}_{t_0,\infty}(x_0)$ *such that* $J_3(t_0,\infty,x_0,\tilde{u}) < \infty$ *and* $J_3(t_0,\infty,x_0,\tilde{u}) \le J(t_0,\infty,x_0,u)$ *for all* $u \in \mathcal{U}_{t_0,\infty}(x_0)$.

In the case of the cost functional (6.52) it is not known that there exists $u \in \mathcal{U}_{t_0,\infty}(x_0)$ such that

$$J_3(t_0,\infty,x_0,u) < +\infty. \tag{6.54}$$

That is why it is natural to introduce the following definition.

Definition 6.3 *We say that the optimization problem* **OP 2** *is* well posed *if for every* $x_0 \in \mathbf{R}^n$ *and* $t_0 \in \mathbf{Z}_+$ *there exists* $u \in \mathcal{U}_{t_0,\infty}(x_0)$ *such that* (6.54) *is fulfilled.*

In the construction of the optimal control $\tilde{u}$ in the above optimization problems a crucial role is played by the solutions of the following system of discrete-time stochastic generalized Riccati equations (DTSRE-C),

$$\begin{aligned} X(t,i) = {} & \sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1)) - \left[\sum_{k=0}^{r} A_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i)\right. \\ & \left. +C^T(t,i)D(t,i)\right]\left[D^T(t,i)D(t,i)+\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t.i)\right]^{-1} \\ & \times\left[\sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i)+D^T(t,i)C(t,i)\right] + C^T(t,i)C(t,i), \end{aligned} \tag{6.55}$$

where $\mathcal{E}_i(t,Y) = \sum_{j=1}^{N} p_t(i,j)Y(j)$ (see also (5.129)).

For the problem **OP 1** we need the solution of (6.55) with the terminal condition $X(t_f,i) = K_f(i)$, whereas in the case of problem **OP 2** the bounded stabilizing solution of (6.55) is involved. It should be remarked that the DTSRE-C (6.55) is a special case of (5.128). It follows that the existence of the solutions of (6.55) involved in this section can be obtained via the results derived in Chapter 5.

6.4.2 Solution of the problem OP 1

Concerning the optimization problem **OP 1** we prove the following.

Theorem 6.5 *Assume that in the cost functional* (6.51) *we have:*

(a) $K_f(i) \ge 0, i \in \mathcal{D}$.
(b) $D^T(t,i)D(t,i) > 0, t_0 \le t \le t_f - 1, i \in \mathcal{D}$.

Let $\hat{X}(t) = (\hat{X}(t,1), \dots, \hat{X}(t,N))$ *be the solution of the system* (6.55) *that verifies the terminal condition* $\hat{X}(t_f, i) = K_f(i), i \in \mathcal{D}$.

Let $\hat{\kappa}(t) = (\hat{\kappa}(t,1), \dots, \hat{\kappa}(t,N))$ *be the solution of the corresponding backward affine equation* (6.11), *with the terminal condition* $\hat{\kappa}(t_f, i) = 0, i \in \mathcal{D}$. *Under these conditions the optimal control in the optimization problem* **OP 1** *is given by*

$$\hat{u}(t) = \hat{F}(t, \eta_t)\hat{x}(t) + \hat{\psi}(t, \eta_t), \tag{6.56}$$

where $\hat{F}(t,i) = F^{\hat{X}}(t,i)$ *and* $\hat{\psi}(t,i)$ *is as in* (6.14) *with* $(\hat{X}(t,i), \hat{\kappa}(t,i))$ *instead of* $(X(t,i), \kappa(t,i))$ *and* $\hat{x}(t)$ *is a solution of the problem with given initial values:*

$$\begin{aligned} x(t+1) = {} & [A_0(t,\eta_t) + B_0(t,\eta_t)\hat{F}(t,\eta_t)]x(t) + \hat{f}_0(t,\eta_t) \\ & + \sum_{k=1}^{r} w_k(t)[(A_k(t,\eta_t) + B_k(t,\eta_t)\hat{F}(t,\eta_t))x(t) + \hat{f}_k(t,\eta_t)], \end{aligned} \tag{6.57}$$

$x(t_0) = x_0$ *and* $\hat{f}_k(t,i) = B_k(t,i)\hat{\psi}(t,i) + f_k(t,i), 0 \le k \le r$.

The optimal value is

$$J_3(t_0, t_f, x_0, \hat{u}) = \sum_{l=1}^{N} \pi_{t_0}(l)\{x_0^T \hat{X}(t_0,l)x_0 + 2x_0^T \hat{\kappa}(t_0,l) + \hat{\mu}(t_0,l)\}, \tag{6.58}$$

where $\pi_{t_0}(l) = \mathcal{P}\{\eta_{t_0} = l\}$ *is the distribution of the Markov chain and* $\hat{\mu}(t) = (\hat{\mu}(t,1) \cdots \hat{\mu}(t,N))^T$ *is the solution of* (6.12) *written for* $\hat{X}(t,i), \hat{\kappa}(t,i)$ *and having the terminal value* $\hat{\mu}(t_f, i) = 0$.

Proof. The assumptions (a) and (b) guarantee (via Corollary 5.1) that the solution $\hat{X}(t)$ of (6.55) is well defined for $t_0 \le t \le t_f - 1$ and $\hat{X}(t,i) \ge 0$. On the other hand $\hat{\kappa}(t,i)$ is well defined as the solution of (6.11). Applying Proposition 6.2 for the triple $(\hat{X}(t,i), \hat{\kappa}(t,i), \hat{\mu}(t,i))$ and taking into account that if α is a random integrable variable then $E\alpha = \sum_{i \in \mathcal{D}_{t_0}} \pi_{t_0}(i)E[\alpha|\eta_{t_0} = i]$, one gets:

$$\begin{aligned} J_3(t_0, t_f, x_0, u) = {} & E[x_0^T \hat{X}(t_0, \eta_{t_0})x_0 + 2x_0^T \hat{\kappa}(t_0, \eta_{t_0}) + \hat{\mu}(t_0, \eta_{t_0})] \\ & + \sum_{t=t_0}^{t_f - 1} E[(u(t) - \hat{u}(t))^T \mathcal{R}_{\eta_t}(t, \hat{X}(t+1))(u(t) - \hat{u}(t))], \end{aligned} \tag{6.59}$$

where $\hat{u}(t)$ is given by (6.56) and

$$\mathcal{R}_i(t, \hat{X}(t+1)) = R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, \hat{X}(t+1))B_k(t,i). \tag{6.60}$$

The fact that $\hat{u} \in \mathcal{U}_{t_0,t_f}$ follows from its formula. From (6.59) we deduce that

$$\begin{aligned} J_3(t_0, t_f, x_0, u) &\geq J_3(t_0, t_f, x_0, \hat{u}) \\ &= E[x_0^T \hat{X}(t_0, \eta_{t_0}) x_0 + 2x_0^T \hat{\kappa}(t_0, \eta_{t_0}) + \hat{\mu}(t_0, \eta_{t_0})]. \end{aligned} \tag{6.61}$$

The fact that $\mathcal{R}_{\eta_t}(t, \hat{X}(t+1)) > 0$ if $\hat{X}(t,i) \geq 0$ was used. Equation (6.59) also shows that $\hat{u}$ is the unique optimal control. Using

$$\begin{aligned} &E[x_0^T \hat{X}(t_0, \eta_{t_0}) x_0 + 2x_0^T \hat{\kappa}(t_0, \eta_{t_0}) + \hat{\mu}(t_0, \eta_{t_0})] \\ &\quad = \sum_{l=1}^{N} \pi_{t_0}(l)(x_0^T \hat{X}(t_0, l) x_0 + 2x_0^T \hat{\kappa}(t_0, l) = \hat{\mu}(t_0, l)) \end{aligned}$$

in (6.61) one obtains (6.58) and thus the proof ends. □

6.4.3 On the global bounded solution of (6.11)

To derive the solution of the problem **OP 2** we need the global bounded solution of the backward affine equation of type (6.11), written for $X_s(t)$ and $F_s(t,i)$ instead of $X(t)$ and $F^X(t,i)$, where $X_s(t) = (X_s(t,1), \ldots, X_s(t,N))$ is the stabilizing bounded solution of DTSRE-C (6.55) and $F_s(t,i)$ is the corresponding stabilizing feedback gain. To obtain conditions that guarantee the existence of such a global solution we need some auxiliary results.

Let $\mathbf{R}^{n \cdot N} = \mathbf{R}^n \oplus \mathbf{R}^n \oplus \cdots \oplus \mathbf{R}^n$(N times). If $x \in \mathbf{R}^{n \cdot N}$ then $x = (x(1), \ldots, x(N))$ with $x(i) \in \mathbf{R}^n$, $x(i) = (x_1(i), x_2(i), \ldots, x_n(i))^T$. $\mathbf{R}^{n \cdot N}$ is a Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{N} x^T(i) y(i) \tag{6.62}$$

for all $x, y \in \mathbf{R}^{n \cdot N}$. Together with the norm $|\cdot|_2$ induced on $\mathbf{R}^{n \cdot M}$ by the inner product (6.62) we consider also the norm

$$|x|_1 = \max_{i \in \mathcal{D}} (x^T(i) x(i))^{1/2}. \tag{6.63}$$

If $L : \mathbf{R}^{n \cdot N} \to \mathbf{R}^{n \cdot N}$ is a linear operator then $\|L\|_k$ is the operator norm induced by $|\cdot|_k$, $k \in \{1, 2\}$. Based on the sequences $\{A_0(t,i)\}_{t \geq 0}$, $\{P_t\}_{t \geq 0}$ we construct the linear operators $\mathcal{A}_t : \mathbf{R}^{n \cdot N} \to \mathbf{R}^{n \cdot N}$ by $\mathcal{A}_t x = ((\mathcal{A}_t x)(1), \ldots, (\mathcal{A}_t x)(N))$ with

$$(\mathcal{A}_t x)(i) = \sum_{j=1}^{N} p_t(j,i) A_0(t,j) x(j). \tag{6.64}$$

It is easy to see that the adjoint operator of $\mathcal{A}_t$ with respect to the inner product (6.62) is given by $\mathcal{A}_t^* x = ((\mathcal{A}_t^* x)(1), \ldots, (\mathcal{A}_t^* x)(N))$ with

$$(\mathcal{A}_t^* x)(i) = A_0^T(t,i)\mathcal{E}_i(t,x), \tag{6.65}$$

where $\mathcal{E}_i(t,x)$ is defined as in (5.129) with $x(i)$ instead of $X(i)$. In the sequel $\Xi(t,s)$ stands for the linear evolution operator on $R^{n\cdot N}$ defined by $\mathcal{A}_t$; that is,

$$\Xi(t,s) = \begin{cases} \mathcal{A}_{t-1}\dots\mathcal{A}_s & \text{if} \quad t > s \geq 0 \\ I_{\mathbf{R}^{n\cdot N}} & \text{if} \quad t = s, \end{cases}$$

where $I_{\mathbf{R}^{n\cdot N}}$ is the identity operator on $\mathbf{R}^{n\cdot N}$.

Consider the linear system derived from (6.10)

$$x(t+1) = \left[A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t)\right] x(t), \qquad t \geq 0. \tag{6.66}$$

We denote $\Phi(t,s)$ the fundamental matrix solution of (6.66). We have

$$\Phi(t+1,s) = \left[A_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(t,\eta_t)\right] \Phi(t,s). \tag{6.67}$$

The next result provides a connection between the adjoint of linear evolution operators defined by (6.64) and the trajectories of the system (6.66). It can be viewed as a counterpart of Theorem 3.1.

Lemma 6.3 *Under the assumptions* $\mathbf{H_1}, \mathbf{H_2}$ *we have*

$$(\Xi^*(t,s)x)(i) = E[\Phi^T(t,s)x(\eta_t)|\eta_s = i] \tag{6.68}$$

for all $i \in \mathcal{D}_s$, $t \geq s > 0$, $x = (x(1),\dots,x(N)) \in \mathbf{R}^{n\cdot N}$.

Proof. We define the linear operators $\mathcal{U}(t,s) : \mathbf{R}^{n\cdot N} \to \mathbf{R}^{n\cdot N}$, $t \geq s \geq 0$ by

$$(\mathcal{U}(t,s)x)(i) = \begin{cases} E[\Phi^T(t,s)x(\eta_t)|\eta_s = i & \text{if} \quad i \in \mathcal{D}_s \\ (\Xi^*(t,s)x)(i) & \text{if} \quad i \in \mathcal{D} - \mathcal{D}_s, \end{cases} \tag{6.69}$$

where $\mathcal{D}_s$ is defined as in (1.8). Taking successively the conditional expectation with respect to $\mathcal{H}_t$, $\tilde{\mathcal{H}}_t$, and $\sigma(\eta_s)$ and taking into account Corollary 1.1, one obtains via (6.67) that

$$E[\Phi^T(t+1,s)x(\eta_{t+1})|\eta_s] = \sum_{j=1}^{N} E[\Phi^T(t,s)A_0^T(t,\eta_t)x(j)p_t(\eta_t,j)|\eta_s]. \tag{6.70}$$

We also used the fact that $E[w_k(t)|\tilde{\mathcal{H}}_t] = E[w_k(t)] = 0$, $1 \leq k \leq r$. If $i \in \mathcal{D}_s$ (6.70) leads to

$$E[\Phi^T(t+1,s)x(\eta_{t+1})|\eta_s = i] = \sum_{j=1}^{N} E[\Phi^T(t,s)A_0^T(t,\eta_t)x(j)p_t(\eta_t,j)|\eta_s = i].$$

Using (6.65) one gets

$$E[\Phi^T(t+1,s)x(\eta_{t+1})|\eta_s = i] = E[\Phi^T(t,s)(\mathcal{A}_t^* x)(\eta_t)|\eta_s = i]. \tag{6.71}$$

Based on (6.69), the equality (6.71) may be written:

$$(\mathcal{U}(t+1,s)x)(i) = (\mathcal{U}(t,s)\mathcal{A}_t^* x)(i), \tag{6.72}$$

$i \in \mathcal{D}_s$. By direct calculation one obtains that (6.72) still holds for $i \in \mathcal{D} \setminus \mathcal{D}_s$. Therefore (6.72) leads to $\mathcal{U}(t+1,s) = \mathcal{U}(t,s)\mathcal{A}_t^*$. This shows that the sequence $\{\mathcal{U}(t,s)\}_{t \geq s}$ solves the same equation as $\Xi^*(t,s)$. Also we have $\mathcal{U}(s,s)x = x = \Xi(s,s)x$ for all $x \in \mathbf{R}^{n \cdot N}$. This allows us to conclude that $\mathcal{U}(t,s) = \Xi^*(t,s)$ for all $t \geq s \geq 0$ and thus the proof ends. □

In the sequel we use the following assumption.

A.6.3 For each $t \geq 0$, P_t is a nondegenerate stochastic matrix.

From the representation formula (6.68) and Theorem 3.4 one obtains the following.

Corollary 6.1 *Under the assumptions* **H1**, **H2**, *and* **A.6.3**, *if the zero solution of* (6.66) *is exponentially stable in the mean square with conditioning of type I (ESMS-CI) then the zero solution of the discrete-time linear equation on* $R^{n \cdot N}$,

$$x_{t+1} = \mathcal{A}_t x_t$$

is exponentially stable.

Let us consider the system of backward affine equations

$$\kappa(t,i) = (A_0(t,i) + B_0(t,i)\tilde{F}_s(t,i))^T \mathcal{E}_i(t, \kappa(t+1)) + \tilde{g}(t,i), \tag{6.73}$$

$i \in \mathcal{D}$, where

$$\tilde{g}(t,i) = \sum_{k=0}^{r} (A_k(t,i) + B_k(t,i)F_s(t,i))^T \mathcal{E}_i(t, X_s(t+1)) f_k(t,i),$$

$F_s(t,i)$ being the stabilizing feedback gain determined by the stabilizing bounded solution of (6.55).

In the sequel we need the following assumption.

A.6.4

(*i*) $\{A_k(t,i)\}_{t \geq 0}, \{B_k(t,i)\}_{t \geq 0}, 0 \leq k \leq r, \{C(t,i)\}_{t \geq 0}, \{D(t,i)\}_{t \geq 0}$ are bounded sequences.

(*ii*) $\{f_k(t,i)\}_{t \geq 0}, 0 \leq k \leq r, i \in \mathcal{D}$ are bounded sequences.

(*iii*) (*a*) There exists $\delta_0 > 0$ not depending upon t such that $R(t,i) := D^T(t,i)D(t,i) \geq \delta_0 I_m, \forall (t,i) \in Z_+ \times \mathcal{D}$.
(*b*) $C^T(t,i)D(t,i) = 0, \forall (t,i) \in Z_+ \times \mathcal{D}$.

It should be noted that under the assumption **A.6.4** the DTSRE-C (6.55) takes the form of (5.138).

Lemma 6.4 *Under the assumptions* **A.6.4** *the system of backward affine equations* (6.73) *has a unique bounded solution on* $\mathbf{Z}_+$, $\tilde{\kappa}(t) = (\tilde{\kappa}(t,1), \ldots, \tilde{\kappa}(t,N))$.

Proof. Let $X_s(t) = (X_s(t,1), \ldots, X_s(t,N))$ be the stabilizing bounded solution of (6.55) and $F_s(t) = (F_s(t,1), \ldots, F_s(t,N))$ be the corresponding stabilizing feedback gain. Let $\tilde{\mathcal{A}}_t : \mathbf{R}^{n \cdot N} \to \mathbf{R}^{n \cdot N}$ defined by

$$(\tilde{\mathcal{A}}_t x)(i) = \sum_{j=1}^{N} p_t(j,i)(A_0(t,j) + B_0(t,j)F_s(t,j))x(j)$$

for all $x = (x(1), \ldots, x(N)) \in \mathbf{R}^{n \cdot N}$. It is easy to see that the backward affine equation (6.73) may be written as

$$\kappa(t) = \tilde{\mathcal{A}}_t^* \kappa(t+1) + \tilde{g}(t) \tag{6.74}$$

with $\tilde{g}(t) = (\tilde{g}(t,1), \ldots, \tilde{g}(t,N))$. Under the considered assumptions it follows that $|\tilde{g}(t)|_1 \leq \mu$, where $\mu > 0$ is independent of t. From Corollary 6.1 it follows that the sequence $\{\tilde{\mathcal{A}}_t\}_{t \geq 0}$ defines an exponentially stable evolution. The conclusion of Lemma 6.4 now follows, applying Theorem 2.5(i) to equation (6.74). □

6.4.4 The solution of the problem OP 2

In this subsection we derive the solution of the optimization problem **OP 2** stated in Section 6.4.1.

Based on the stabilizing bounded solution $\tilde{X}(t)$ of (6.55) and the unique bounded solution $\tilde{\kappa}(t)$ of (6.73) we construct the following control law,

$$\tilde{u}(t) = F_s(t,\eta_t)\tilde{x}(t) + \tilde{\psi}(t,\eta_t), \tag{6.75}$$

where $F_s(t,i)$ is the stabilizing feedback gain,

$$\begin{aligned}\tilde{\psi}(t,i) = &-\left(R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, X_s(t+1))B_k(t,i)\right)^{-1} \\ &\times \left(B_0^T(t,i)\mathcal{E}_i(t,\tilde{\kappa}(t+1)) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, X_s(t+1))f_k(t,i)\right),\end{aligned} \tag{6.76}$$

and $\tilde{x}(t)$ is the solution of the closed-loop system

$$\tilde{x}(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)F_s(t,\eta_t)]\tilde{x}(t) + \tilde{f}_0(t,\eta_t) + \sum_{k=1}^{r} w_k(t)[(A_k(t,\eta_t) + B_k(t,\eta_t)F_s(t,\eta_t))\tilde{x}(t) + \tilde{f}_k(t,\eta_t)] \quad (6.77)$$

$$\tilde{x}(t_0) = x_0,$$

where $\tilde{f}_k(t,i) = B_k(t,i)\tilde{\psi}(t,i) + f_k(t,i), 0 \leq k \leq r, (t,i) \in \mathbf{Z}_+ \times \mathcal{D}$. It is easy to see that if $f_k(t,i) = 0$, $0 \leq k \leq r$, $i \in \mathcal{D}$, $t \in \mathbf{Z}_+$ then $\tilde{\kappa}(t,i) = 0$; this leads to $\tilde{\psi}(t,i) = 0$, $t \geq 0$, $i \in \mathcal{D}$. In this case the control (6.75) reduces to $\tilde{u}(t) = F_s(t,\eta_t)\tilde{x}(t)$, $\tilde{x}(t)$ being the solution of (6.77).

Lemma 6.5 *Under the assumptions* $\mathbf{H_1}, \mathbf{H_2}, \mathbf{A.6.3}$, *and* $\mathbf{A.6.4}$ *the following hold.*

(i) *For each* $x_0 \in \mathbf{R}^n$, $\tilde{u} \in \mathcal{U}_{t_0\infty}(x_0)$.
(ii) $J_3(t_0,\infty,x_0,\tilde{u}) < +\infty$.

Proof. Based on Lemma 6.4 and the assumption **A.6.4** we deduce that

$$\sup_{t\geq 0} |\tilde{f}_k(t,i)| < +\infty, \qquad 0 \leq k \leq r, \qquad 1 \leq i \leq N.$$

Applying Corollary 3.8(ii) to the system (6.77) we conclude that in the case of the control $\tilde{u}$ condition (6.50) is fulfilled. Because $\tilde{x}(t)$ is $\mathcal{H}_{t-1}$-measurable and $\mathcal{H}_{t-1} \subset \tilde{\mathcal{H}}_t$ we obtain from (6.75) that $\tilde{u}(t)$ is $\tilde{\mathcal{H}}_t$-measurable. This allows us to conclude that $\tilde{u}$ is an admissible control. Also from (6.75) it follows that $\sup_{t\geq t_0} E[|\tilde{u}(t)|^2] < +\infty$. Hence $J_3(t_0,x_0,\tilde{u}) < +\infty$. This completes the proof. □

For each $(t_0,x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ we introduce the sets

$$\tilde{\mathcal{U}}_{t_0,\infty}(x_0) = \{u \in \mathcal{U}_{t_0,\infty}(x_0) | J(t_0,\infty,x_0,u) < +\infty\}.$$

Under the conditions of the above lemma it follows that $\tilde{u}$ defined by (6.75–6.77) leads in $\tilde{\mathcal{U}}_{t_0,\infty}(x_0)$.

Moreover based on Corollary 3.8(ii) one obtains that if the linear control system (6.10) (with $f_k(t,i) \equiv 0, 0 \leq k \leq r$) is stochastic stabilizable and if the assumptions $\mathbf{H_1}, \mathbf{H_2}$, and $\mathbf{A.6.4}$ are fulfilled then for each $(t_0,x_0) \in \mathbf{Z}_+ \times \mathbf{R}^n$ the set $\tilde{\mathcal{U}}_{t_0,\infty}(x_0)$ contains the controls of the form

$$u(t) = F(t,\eta_t)\hat{x}(t) + h(t)$$

for arbitrary stabilizing feedback gain $\{F(t,i)\}_{t\geq 0}, i \in \mathcal{D}$ and for arbitrary stochastic process $\{h(t)\}_{t\geq 0}$ with the properties:

(a) For each $t \in \mathbf{Z}_+, h(t)$ is $\tilde{\mathcal{H}}_t$-measurable.
(b) $\sup_{t\geq 0} E[|h(t)|^2] < \infty$.

$\hat{x}(t)$ is the solution of

$$x(t+1) = [A_0(t,\eta_t) + B_0(t,\eta_t)F(t,\eta_t)]x(t) + \check{f}_0(t) \\ + \sum_{k=1}^{r} w_k(t)[(A_k(t,\eta_t)+B_k(t,\eta_t)F(t,\eta_t))x(t)+\check{f}_k(t)], \quad \hat{x}(t_0)=x_0,$$

where

$$\check{f}_k(t) = f_k(t,\eta_t) + B_k(t,\eta_t)h(t).$$

Let us consider the backward affine equation on

$$\mu(t) = P_t\mu(t+1) + \tilde{h}(t), \tag{6.78}$$

where $\tilde{h}(t) = (\tilde{h}(t,1),\ldots,\tilde{h}(t,N))^T$, $\tilde{h}(t,i)$ being constructed as in (6.13) replacing $X(t)$, $F^X(t,i)$, and $\kappa^X(t)$ by $X_s(t), F_s(t,i)$, and $\tilde{\kappa}(t)$.

Lemma 6.6 *Under the assumptions of Lemma* 6.5*, if* $\mu_T(t) = (\mu_T(t,1),\ldots,\mu_T(t,N))^T$ *is the solution of* (6.78) *with the final value* $\mu_T(T,i) = 0, 1 \le i \le N$*, then, for all* $i \in \mathcal{D}$*,*

$$\overline{\lim_{T\to\infty}} \frac{1}{T-t_0}\mu_T(t_0,i)$$

is finite.

Proof. We have:

$$\mu_T(t_0) = \sum_{s=t_0}^{T-1} \mathbf{P}(t_0,s)\tilde{h}(s), \tag{6.79}$$

where $\mathbf{P}(t,s) = P_tP_{t+1}\cdots P_{s-1}$ if $s > t$ and $\mathbf{P}(t,s) = I_N$ if $s = t$. One can verify that $\mathbf{P}(t_0,s)$ is also a stochastic matrix. Based on the assumptions **A.6.3** and **A.6.4** we deduce that $|\tilde{h}(t) \le c_h$ for all $t \ge 0$, where c_h is a positive constant not depending upon t. Therefore, if $\mathbf{P}_i(t_0,s)$ is the ith row of $\mathbf{P}(t,s)$, one obtains that

$$|\mathbf{P}_i(t_0,s)\tilde{h}(s)| \le c_h \tag{6.80}$$

for all $s \ge t_0 \ge 0$. The conclusion follows combining (6.79) and (6.80). □

The main result of this section is the following.

Theorem 6.6 *Assume that:*

(*a*) *The hypotheses* $\mathbf{H_1}, \mathbf{H_2}, \mathbf{A.6.3}$, *and* $\mathbf{A.6.4}$, *are fulfilled.*
(*b*) *The DTSRE-C* (6.55) *has a bounded stabilizing solution.*

Under these conditions the optimal control of the problem **OP 2** *is given by* (6.75)–(6.77).

The optimal value of the cost functional is given by

$$J_3(t_0,\infty,x_0,\tilde{u}) = \overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{l=1}^{N}\pi_0(l)\mu_T(0,l), \tag{6.81}$$

where $\mu_T(t) = (\mu_T(t,1)\cdots\mu_T(t,N))^T$ *is the solution of* (6.78) *with the final value*

$$\mu_T(T,l) = 0.$$

Proof. Let $\tilde{X}(t) = (\tilde{X}(t,1),\dots,\tilde{X}(t,N))$ be the bounded stabilizing solution of DTSRE-C (6.55) and $\tilde{\kappa}(t) = (\tilde{\kappa}(t,1),\dots,\tilde{\kappa}(t,N))$ be the unique bounded solution of the corresponding backward affine equation (6.73). For each $T > t_0$ let $\mu_{T+1}(t) = (\mu_{T+1}(t,1)\cdots\mu_{T+1}(t,N))$ be the solution of the backward affine equation (6.78) with the terminal value $\mu_{T+1}(T+1,i) = 0, 1 \le i \le N$. Applying Proposition 6.2, one obtains:

$$\begin{aligned}\sum_{t=t_0}^{T} E[|y(t)|^2|\eta_{t_0}=i] = {}& x_0^T\tilde{X}(t_0,i)x_0 + 2x_0^T\tilde{\kappa}(t_0,i) + \mu_{T+1}(t_0,i)\\ &- E[x^T(T+1)\tilde{X}(T+1,\eta_{T+1})x(T+1)\\ &+ 2x^T(T+1)\tilde{\kappa}(T+1,\eta_{T+1})|\eta_{t_0}=i]\\ &+ \sum_{t=t_0}^{T} E[(u(t)-\tilde{F}(t,\eta_t)x(t))^T(R(t,\eta_t)\\ &+ \sum_{k=0}^{r} B_k^T(t,\eta_t)\mathcal{E}_{\eta_t}(t,\tilde{X}(t+1))B_k(t,\eta_t))\\ &\times (u(t)-\tilde{F}(t,\eta_t)x(t))|\eta_{t_0}=i]\end{aligned}$$

for all $u \in \mathcal{U}_{t_0\infty}(x_0)$ and $x(t)$ stands for the trajectory $x_u(t,t_0,x_0)$ of (6.10) corresponding to the control u. Invoking (6.50) together with Lemma 6.4 we deduce that

$$\begin{aligned}&\overline{\lim_{T\to\infty}}\frac{1}{T-t_0}\{x_0^T\tilde{X}(t_0,i)x_0 + 2x_0^T\tilde{\kappa}(t_0,i)\\ &\quad - E[x^T(T+1)\tilde{X}(T+1,\eta_{T+1})x(T+1)\\ &\quad + 2x^T(T+1)\tilde{\kappa}(T+1,\eta_{T+1})|\eta_{t_0}=i]\} = 0, \qquad i \in \mathcal{D}_{t_0}.\end{aligned}$$

So we obtain

$$J_3(t_0,\infty,x_0,u) = \overline{\lim_{t\to\infty}}\frac{1}{T-t_0}\Bigg\{\sum_{l=1}^{N}\pi_{t_0}(l)\mu_{T+1}(t_0,l)$$
$$+\sum_{t=t_0}^{T}E\Bigg[(u(t)-\tilde{F}(t,\eta_t)x(t))^T\Bigg(R(t,\eta_t)+\sum_{k=0}^{r}B_k^T(t,\eta_t)$$
$$\times\ \mathcal{E}_{\eta_t}(t,\tilde{X}(t+1))B_k(t,\eta_t)\Bigg)(u(t)-\tilde{F}(t,\eta_t)x(t))\Bigg]\Bigg\}$$
$$\geq \overline{\lim_{T\to\infty}}\frac{1}{T-t_0}\sum_{l=1}^{N}\pi_{t_0}(l)\mu_{T+1}(t_0,l) = J_3(t_0,\infty,x_0,\tilde{u})$$

for all $u \in \mathcal{U}_{t_0\infty}(x_0)$. Here we used the fact that the bounded stabilizing solution of (6.55) verifies

$$R(t,i)+\sum_{k=0}^{r}B_k^T(t,i)\mathcal{E}_i(t,\tilde{X}(t+1))B_k(t,i)\geq \nu I_m > 0$$

for all $(t,i)\in \mathbf{Z}_+\times\mathcal{D}$. Finally we remark that under the considered assumptions by using (1.7) and (6.79) we have

$$\overline{\lim_{T\to\infty}}\frac{1}{T-t_0}\sum_{l=1}^{N}\pi_{t_0}(l)\mu_{T+1}(t_0,l) = \overline{\lim_{T\to\infty}}\frac{1}{T}\sum_{l=1}^{N}\pi_0(l)\mu_T(0,l).$$

This completes the proof. □

Remark 6.5

(a) From (6.81) it follows that under the assumptions of Theorem 6.6 the optimal value of the problem **OP 2** does not depend upon t_0 and x_0.
(b) In the time-invariant case we have $P_t = P, \tilde{h}(t,i) = \tilde{h}(i)$, $t\geq 0$ and therefore by Theorem 6.6 and Proposition 1.5 one obtains that the optimal value of the problem **OP** is $\sum_{\ell=1}^{N}\sum_{j=1}^{N}q(\ell,j)\tilde{h}(j)\pi_0(\ell)$; in the time-invariant case it is not necessary to assume that **P** is a nondegenerate stochastic matrix (see Theorem 3.10).
(c) If the condition (6.50) from the definition of the set of admissible controls $\mathcal{U}_{t_0,\infty}(x_0)$ is replaced by

$$\lim_{t\to\infty}E[|x_u(t,t_0,x_0)|^2]=0 \tag{6.82}$$

one obtains a new class of admissible controls $\hat{\mathcal{U}}_{t_0,\infty}(x_0)$. It is obvious that $\hat{\mathcal{U}}_{t_0,\infty}(x_0)\subset\mathcal{U}_{t_0,\infty}(x_0)$.

Thus we may consider a new optimization problem asking for the minimization of the cost functional (6.52) over the set of admissible controls $\hat{\mathcal{U}}_{t_0,\infty}(x_0)$. To be sure that condition (6.82) is satisfied, the assumption **A.6.4**(ii) should be replaced with a stronger one:

A.6.5 $\lim_{t\to\infty} f_k(t,i) = 0, \qquad i \in \mathcal{D}, 0 \leq k \leq r.$

One proves that the unique bounded solution of (6.73) satisfies $\lim_{t\to\infty} \tilde{\kappa}(t) = 0$.

Furthermore if $\tilde{\psi}(t,i)$ is defined by (6.76) we have $\lim_{t\to\infty} \tilde{\psi}(t,i) = 0$, $i \in \mathcal{D}$. Applying Corollary 3.9(iii), one obtains that the solution of (6.77) satisfies $\lim_{t\to\infty} E[|\tilde{x}(t)|^2] = 0$. This shows that the control $\tilde{u}(t)$ defined by (6.75)–(6.77) belongs to the new class of admissible controls $\hat{\mathcal{U}}_{t_0,\infty}(x_0)$.

Reasoning as in the proof of Theorem 6.6 one obtains that if the assumption **A.6.4**(ii) is replaced by **A.6.5** the control $\tilde{u}(t)$ defined by (6.75)–(6.77) achieves the optimal value of the cost functional (6.52) with respect to both classes of admissible controls $\mathcal{U}_{t_0,\infty}(x_0)$ as well as $\hat{\mathcal{U}}_{t_0,\infty}(x_0)$.

6.5 Tracking problems

Consider the discrete-time controlled system described by

$$x(t+1) = A_0(t,\eta_t)x(t) + B_0(t,\eta_t)u(t) + \sum_{k=1}^{r}[A_k(t,\eta_t)x(t) + B_k(t,\eta_t)u(t)]w_k(t), \tag{6.83}$$

$t \geq t_0$, $x(t_0) = x_0$.

Let $\{r(t)\}_{t\geq 0}$, $r(t) \in \mathbf{R}^n$ be a given signal called the *reference signal*. The control problem we want to solve is to find a control $\tilde{u}(t)$ that minimizes the deviation $x(t) - r(t)$.

For a more rigorous setting of this problem let us introduce the following cost functionals,

$$J_4(t_0,t_f,x_0,u) = E\Big\{(x(t_f) - r(t_f))^T \kappa_f(\eta_t)(x(t_f) - r(t_f)) + \sum_{t=t_0}^{t_f-1}[(x(t) - r(t))^T M(t,\eta_t)(x(t) - r(t)) + u^T(t)R(t,\eta_t)u(t)]\Big\} \tag{6.84}$$

in the case of a finite time horizon and

$$J_4(t_0,\infty,x_0,u)$$
$$= \overline{\lim}_{T\to\infty}\frac{1}{T-t_0}\sum_{t=t_0}^{T} E[(x(t)-r(t))^T M(t,\eta_t)(x(t)-r(t))$$
$$+ u^T(t)R(t,\eta_t)u(t)] \tag{6.85}$$

in the case of an infinite time horizon, where $M(t,i)=M^T(t,i)\geq 0$, $K_f(i)=K_f^T(i)\geq 0$, $R(t,i)=R^T(t,i)>0$, and $x(t)=x_u(t,t_0,x_0)$.

The tracking problems considered in this section ask for finding a control law $u_{\text{opt}}\in\mathcal{U}_{t_0,t_f}$, ($\tilde{u}_{\text{opt}}\in\mathcal{U}_{t_0,\infty}(x_0)$, respectively) in order to minimize the cost (6.84) (the cost (6.85), respectively). If we set $\xi(t)=x(t)-r(t)$ then we obtain $\xi(t+1)=A_0(t,\eta_t)\xi(t)+B_0(t,\eta_t)u(t)+f_0(t,\eta_t)+\sum_{k=1}^{r}[A_k(t,\eta_t)\xi(t)+B_k(t,\eta_t)u(t)+f_k(t,\eta_t)]w_k(t)$ and the cost functionals

$$J_4(t_0,t_f,x_0,u)=E[\xi^T(t_f)K_f(\eta_t)\xi(t_f)$$
$$+\sum_{t=t_0}^{t_f-1}(\xi^T(t)M(t,\eta_t)\xi(t)+u^T(t)R(t,\eta_t)u(t))]$$

and

$$J_4(t_0,\infty,x_0,u)=\overline{\lim}_{T\to\infty}\frac{1}{T-t_0}\sum_{t=t_0}^{T}E[\xi^T(t)M(t,\eta_t)\xi(t)+u(t)^T R(t,\eta_t)u(t)],$$

where

$$f_0(t,i)=A_0(t,i)r(t)-r(t+1)$$
$$f_k(t,i)=A_k(t,i)r(t),\qquad 1\leq k\leq r, t\geq 1. \tag{6.86}$$

Let us consider the following system of Riccati type equations.

$$X(t,i)=\sum_{k=0}^{r}A_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i)$$
$$-\left[\sum_{k=0}^{r}A_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i)\right]$$
$$\times\left[R(t,i)+\sum_{k=0}^{r}B_k^T(t,i)\mathcal{E}_i(t,X(t+1))B_k(t,i)\right]^{-1}$$
$$\times\left[\sum_{k=0}^{r}B_k^T(t,i)\mathcal{E}_i(t,X(t+1))A_k(t,i)\right]+M(t,i), \tag{6.87}$$

where $\mathcal{E}_i(t,\cdot)$ is defined as in (5.129). It is easy to see that (6.87) is a special case of (5.128) with $L(t,i)=0$.

The solutions of the tracking problems are derived directly from Theorem 6.5 and Theorem 6.6.

Corollary 6.2 *Under the considered assumptions, the optimal control of the tracking problem described by the system* (6.83) *and the cost* (6.84) *is given by*

$$\hat{u}_{\mathrm{opt}}(t)=\hat{F}(t,\eta_t)(\hat{x}(t)-r(t))+\hat{\psi}(t,\eta_t), \tag{6.88}$$

where $\hat{F}(t,i)=F^{\hat{X}}(t,i)$ *is constructed as in* (6.7) *based on the solution* $\hat{X}(t,i)$ *of the system* (6.87) *with the terminal condition* $\hat{X}(t_f,i)=K_f(i)$, $i\in\mathcal{D}$, $\hat{\psi}(t,i)$ *is constructed as in* (6.14) *based on* $\hat{X}(t,i)$*; and* $\hat{\kappa}(t)$ *with* ($\hat{\kappa}(t)=(\hat{\kappa}(t,1),\dots,\hat{\kappa}(t,N))$ *is the solution of the system of backward affine equations*

$$\begin{aligned}\kappa(t,i)&=(A_0(t,i)+B_0(t,i)\hat{F}(t,i))^T\mathcal{E}_i(t,\hat{\kappa}(t+1))+\hat{g}(t,i), \qquad (6.89)\\ \hat{\kappa}(t_f,i)&=0,\end{aligned}$$

$i\in\mathcal{D}$, *where* $\hat{g}(t,i)=\sum_{k=0}^{r}(A_k(t,i)+B_k(t,i)\hat{F}(t,i))^T\mathcal{E}_i(t,\hat{X}(t+1)f_k(t,i)$, $f_k(t,i)$ *given by* (6.86), $\hat{x}(t)$ *is the solution of the closed-loop system:*

$$\begin{aligned}\hat{x}(t+1)=&\Bigg[A_0(t,\eta_t)+B_0(t,\eta_t)\hat{F}(t,\eta_t)\\ &+\sum_{k=1}^{r}(A_k(t,\eta_t)+B_k(t,\eta_t)\hat{F}(t,\eta_t))w_k(t)\Bigg]\hat{x}(t)\\ &+f_0(t,\eta_t)+B_0(t,\eta_t)\hat{\psi}(t,\eta_t)+\sum_{k=1}^{r}w_k(t)(f_k(t,\eta_t)+B_k(t,\eta_t)\hat{\psi}(t,\eta_t)),\end{aligned}$$

$t\geq t_0$, $\hat{x}(t_0)=x_0$. *The optimal cost is given by* $J_4(t_0,t_f,x_0,\hat{u}_{\mathrm{opt}})=\sum_{l=1}^{N}[\pi_0(l)[(x_0-r(t_0))^T\hat{X}(t_0,l)(x_0-r(t_0))+2(x_0-r(t_0))^T\kappa(t_0,l)+\hat{\mu}(t_0,l)]$, *where* $\hat{\mu}(t,l)$ *are as in Theorem* 6.5.

For the tracking problem on the infinite time horizon, we have the following.

Corollary 6.3 *Assume:*

(*a*) *The hypotheses* $\mathbf{H_1},\mathbf{H_2},\mathbf{A.6.3}$ *are fulfilled.*
(*b*) *The sequences* $\{A_k(t,i)\}_{t\geq 0},\{B_k(t,i)\}_{t\geq 0},0\leq k\leq r$, $\{M(t,i)\}_{t\geq 0}$, $\{R(t,i)\}_{t\geq 0},i\in\mathcal{D},\{r(t)\}_{t\geq 0}$ *are bounded.*
(*c*) $R(t,i)\geq\delta I_n>0$ *for all* $(t,i)\in\mathbf{Z}_+\times\mathcal{D}$.
(*d*) *The DTSRE-C* (6.87) *has a bounded stabilizing solution.*

Under these conditions the optimal control of the tracking problem described by the system (6.83) *and the cost* (6.85) *is:*

$$\tilde{u}_{\text{opt}}(t) = \tilde{F}(t, \eta_t)(\tilde{x}(t) - r(t)) + \tilde{\psi}(t, \eta_t), \tag{6.90}$$

where $\tilde{F}(t,i) = F^{\tilde{X}}(t,i)$ *is constructed as in* (6.7) *based on the bounded stabilizing solution* $\tilde{X}(t) = ((\tilde{X}(t,1), \dots, \tilde{X}(t,N))$ *of DTSRE-C* (6.87) *and* $\tilde{\psi}(t,i)$ *is given by*

$$\tilde{\psi}(t,i) = -\left(R(t,i) + \sum_{k=0}^{r} B_k(t,i)\mathcal{E}_i(t, \tilde{X}(t+1))B_k(t,i)\right)^{-1}$$
$$\times \left[B_0^T(t,i)\mathcal{E}_i(t, \tilde{\kappa}(t+1)) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, \tilde{X}(t+1)) f_k(t,i)\right],$$

where $\tilde{\kappa}(t) = (\tilde{\kappa}(t,1), \dots, \tilde{\kappa}(t,N))$ *is the unique bounded solution of the system of backward affine equations*

$$\kappa(t,i) = (A_0(t,i) + B_0(t,i)\tilde{F}(t,i))^T \mathcal{E}_i(t, \kappa(t+1)) + \tilde{g}(t,i) \tag{6.91}$$

with

$$\tilde{g}(t,i) = \sum_{k=0}^{r} (A_k(t,i) + B_k(t,i)\tilde{F}(t,i))^T \mathcal{E}_i(t, \tilde{X}(t+1)) f_k(t,i),$$

$f_k(t,i)$ *given by* (6.86)*;* $\tilde{x}(t)$ *is the solution of the closed-loop system*

$$\tilde{x}(t+1) = \Bigg[A_0(t,\eta_t) + B_0(t,\eta_t)\tilde{F}(t,\eta_t)$$
$$+ \sum_{k=1}^{r} (A_k(t,\eta_t) + B_k(t,\eta_t)\tilde{F}(t,\eta_t)) w_k(t)\Bigg] \tilde{x}(t) + f_0(t,\eta_t)$$
$$+ B_0(t,\eta_t)\tilde{\psi}(t,\eta_t) + \sum_{k=1}^{r} w_k(t,\eta_t)(f_k(t,\eta_t) + B_k(t,\eta_t)\tilde{\psi}(t,\eta_t)),$$

$t \geq t_0$*,* $x(t_0) = x_0$*.*

Remark 6.6 Necessary and sufficient conditions that guarantee the existence of the bounded stabilizing solution of (6.87) are obtained via Theorem 5.12. The condition $R(t,i) + \sum_{k=0}^{r} B_k^T(t,i)\mathcal{E}_i(t, \tilde{X}(t+1))B_k(t,i) \geq \nu I_n, \forall (t,i) \in \mathbf{Z}_+ \times \mathcal{D}$ is automatically satisfied, due to assumption (c), together with $M(t,i) = M^T(t,i) \geq 0$.

Remark 6.7 The construction of the optimal control in the case of the tracking problem investigated in this section is based on advance knowledge of the reference signal on the whole time interval. This is a difficulty that is hard to overcome if the reference signal $\{r(t\}_{t\geq 0}$ is a sequence without other additional properties. In this case an infinite is required to compute the terms $\tilde{\Psi}(t,i)$ occurring in the optimal control. However, in some important cases, such as the case of periodic reference signals or in the case of a constant reference signal, we need a finite memory to compute the bounded solution on $\mathbf{Z}_+$ of the backward affine equation (6.91). Let us assume that there exists an integer $\theta \geq 2$ such that the coefficients of the system (6.83), the weights of the cost functional (6.85), and the reference signal satisfy: $A_k(t+\theta,i) = A_k(t,i), B_k(t+\theta,i) = B_k(t,i), 0 \leq k \leq r, M(t+\theta,i) = M(t,i), R(t+\theta,i) = R(t,i), i \in \mathcal{D}$, and $r(t+\theta) = r(t)$ for all $t \in \mathbf{Z}_+$. Under these conditions one obtains, via Theorem 5.5, that the stabilizing solution of the DTSRE-C (6.87) is a bounded sequence with the same period θ. Therefore the corresponding stabilizing feedback gain $\tilde{F}(t,i)$ is also periodic with the period θ. Applying Theorem 2.5(ii) to equation (6.91) we deduce that the unique bounded solution $\tilde{\kappa}(t)$ of (6.91) is periodic with period θ. As in the case of the equation (6.73), equation (6.91) can be regarded as a discrete-time backward affine equation on the space $\mathbf{R}^{nN}$:

$$\tilde{\kappa}(t) = \tilde{\mathcal{A}}_t^* \tilde{\kappa}(t+1) + \tilde{g}(t), \tag{6.92}$$

where $\tilde{\mathcal{A}}_t^*$ is the adjoint of the operator $\tilde{\mathcal{A}}_t$ defined in the proof of Lemma 6.4. Setting $\tilde{\Xi}(t,s)$ for the linear evolution operator on $\mathbf{R}^{nN}$ defined by the sequence $\{\tilde{\mathcal{A}}_t\}_{t\geq 0}$ we obtain the representation

$$\tilde{\kappa}(t) = \tilde{\Xi}^*(\theta,t)\tilde{\kappa}(\theta) + \sum_{s=t}^{\theta-1} \tilde{\Xi}^*(s,t)\tilde{g}(s), \tag{6.93}$$

$0 \leq t \leq \theta - 1$. The final value $\tilde{\kappa}(\theta)$ of the periodic solution is obtained solving the following linear equation, on $\mathbf{R}^{nN}$;

$$(\mathcal{I}_{\mathbf{R}^{nN}} - \tilde{\Xi}^*(\theta,0))x = \sum_{s=0}^{\theta-1} \tilde{\Xi}^*(s,0)\tilde{g}(s), \tag{6.94}$$

where $\mathcal{I}_{\mathbf{R}^{nN}}$ is the identity operator on $\mathbf{R}^{nN}$. Because the sequence $\{\tilde{\mathcal{A}}_t\}_{t\geq 0}$ defines an exponentially stable evolution, it follows that $\lambda = 1$ is not in the spectrum of the operator $\tilde{\Xi}^*(\theta,0)$. Hence equation (6.94) has a unique solution $\tilde{\kappa}(\theta) = (\mathcal{I}_{\mathbf{R}^{nN}} - \tilde{\Xi}^*(\theta,0))^{-1}(\sum_{s=0}^{\theta-1} \tilde{\Xi}^*(s,0)\tilde{g}(s))$. Plugging this in (6.93) we obtain the values of the solution $\tilde{\kappa}(t)$ required for the construction of the optimal control.

In the time-invariant case ($\theta = 1$), the bounded solution of (6.91) is constant. It is given by $\tilde{\kappa} = (\mathcal{I}_{\mathbf{R}^{nN}} - \tilde{\mathcal{A}})^{-1}\tilde{g}$. In this case $\tilde{\psi}(t,i) - \tilde{\psi}(i)$.

6.6 Notes and references

In the discrete-time stochastic framework the linear quadratic optimization problem was separately investigated for systems with independent random perturbations and systems with Markov perturbations, respectively. Thus, for the case of discrete-time stochastic systems with independent random perturbations we refer to [88, 89, 113], and for discrete-time systems with Markovian switching we mention [2, 6, 15, 16, 18, 27, 55, 59, 75, 84, 86, 87, 90, 89].

In this chapter we consider different aspects of the linear quadratic optimization problem for a general class of discrete-time time-varying linear stochastic systems subject to multiplicative white noise perturbations and Markov jump perturbations. The results contained in Sections 6.2 and 6.3 are the discrete-time counterparts of those published in Sections 5.1 and 5.2 from [40]. They are presented in detail for the first time in this monograph. The results included in Section 6.4 were presented firstly in [45], and the tracking problem on the infinite time horizon was published in [44].

7

Discrete-time stochastic H_2 optimal control

In this chapter the problem of H_2 control of a discrete-time linear system subject to Markovian jumping and independent random perturbations is considered. Several kinds of H_2-type performance criteria (often called H_2 norms) are introduced and characterized via solutions of some suitable linear equations on the spaces of symmetric matrices. The purpose of such performance criteria is to provide a measure of the effect of additive white noise perturbation over an output of the controlled system. Different aspects specific to the discrete-time framework are emphasized. Firstly, the problem of optimization of H_2 norms is solved under the assumption that a full state vector is available for measurements. One shows that among all stabilizing controllers of higher dimension, the best performance is achieved by a zero-order controller. The corresponding feedback gain of the optimal controller is constructed based on the stabilizing solution of a system of discrete-time generalized Riccati equations. The case of discrete-time linear stochastic systems with coefficients depending upon the states both at time t and at time $t-1$ of the Markov chain is also considered. Secondly, the H_2 optimization problem is solved under the assumption that only an output is available for measurements. The state space realization of the H_2 optimal controller coincides with the stochastic version of the well-known Kalman–Bucy filter. In the construction of the optimal controller the stabilizing solutions of two systems of discrete-time coupled Riccati equations are involved. Because in the case of the systems affected by multiplicative white noise the optimal controller is hard to implement, a procedure for designing a suboptimal controller with the state space realization in a state estimator form is provided. Finally a problem of H_2 filtering in the case of stochastic systems affected by multiplicative and additive white noise and Markovian switching is solved.

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7.1 H_2 norms of discrete-time linear stochastic systems

7.1.1 Model setting

Consider the discrete-time linear system (G) described by:

$$(G): \begin{cases} x(t+1) = \left(A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t) \right) x(t) + B_v(\eta_t) v(t) \\ z(t) \qquad = C(\eta_t) x(t), t \in \mathbf{Z}_+, \end{cases} \tag{7.1}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $z(t) \in \mathbf{R}^{n_z}$ a controlled output, $\{w(t)\}_{t \geq 0}$ (where $w(t) = (w_1(t), \ldots, w_r(t))^T$) is a sequence of independent random vectors, $\{v(t)\}_{t \geq 0}$ is a sequence of m_v-dimensional independent random vectors on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $\{\eta_t\}_{t \geq 0}$ is an homogeneous Markov chain with the set of the states $\mathcal{D} = \{1, 2, \ldots, N\}$ and the transition probability matrix P. In (7.1), $A_k(i) \in \mathbf{R}^{n \times n}$, $B_v(i) \in \mathbf{R}^{n \times m_v}$, and $C(i) \in \mathbf{R}^{n \times n_z}$ are given matrices. As usually, $\mathbf{Z}_+$ stands for the set of nonnegative integers.

Throughout this chapter, we assume that the stochastic processes $\{\eta_t\}_{t \geq 0}$, $\{w(t)\}_{t \geq 0}$ satisfy the hypotheses $\mathbf{H_1}$, and $\mathbf{H_2}$ introduced in Section 1.5, and related to the sequence $\{v(t)\}_{t \geq 0}$, we make the following assumptions.

A.7.1 $\{v(t)\}_{t \geq 0}$ is a sequence of independent random vectors with the properties:

$$E[v(t)] = 0, E[v(t)v^T(t)] = I_{m_v}, \qquad t \geq 0$$

and $\{v(t)\}_{t \geq 0}$ is independent of stochastic processes $\{w(t)\}_{t \geq 0}$ and $\{\eta_t\}_{t \geq 0}$.

Let $A(t) = A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t), t \geq 0$. Set

$$\Phi(t,s) = \begin{cases} A(t-1)A(t-2)\cdots A(s), & \text{if} \quad t \geq s+1 \\ I_n, & \text{if} \quad t = s. \end{cases}$$

If $x(t, t_0, x_0, v)$ is the solution of (7.1) with the initial value $x(t_0, t_0, x_0, v) = x_0$ then we have the following representation formula,

$$x(t, t_0, x_0, v) = \Phi(t, t_0) x_0 + \sum_{l=t_0}^{t-1} \Phi(t, l+1) B_v(\eta_l) v(l) \tag{7.2}$$

for all $t \geq t_0 + 1$.

Due to the linearity of (7.2), we have the decomposition:

$$x(t, t_0, x_0, v) = x(t, t_0, x_0, 0) + x(t, t_0, 0, v) \tag{7.3}$$

with $x(t, t_0, x_0, 0) = \sum_{l=t_0}^{t-1} \Phi(t, l+1) B_v(\eta_l) v(l)$. The corresponding output is

$$z(t, t_0, x_0, v) = C(\eta_t) x(t, t_0, x_0, 0) + z(t, t_0, 0, v), \tag{7.4}$$

where $z(t, t_0, 0, v) = C(\eta_t) x(t, t_0, 0, v)$.

In (7.4) $C(\eta_t)x(t, t_0, x_0, 0)$ is the transitory component of the output signal, whereas $z(t, t_0, 0, v)$ is the answer of the system determined by the exogenous noise $v(t)$.

7.1.2 H_2-type norms

The linear system obtained from (7.1) is:

$$x(t+1) = (A_0(\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t))x(t). \tag{7.5}$$

Under the assumption that the zero state equilibrium of (7.5) is exponentially stable in the mean square (ESMS), we introduce the following performance criteria associated with the system (7.1).

$$\|G\|_2 = \left(\lim_{l\to\infty} \frac{1}{l}\sum_{t=0}^{l} E[|z(t, 0, x_0, v)|]\right)^{1/2} \tag{7.6}$$

$$\tilde{\|}G\tilde{\|}_2 = \left(\lim_{l\to\infty} \frac{1}{l}\sum_{t=0}^{l}\sum_{i\in\mathcal{D}_0} E[|z(t, 0, x_0, v)|^2|\eta_0 = i]\right)^{1/2}, \tag{7.7}$$

where $\mathcal{D}_0 \subseteq \mathcal{D}$ is the subset introduced by (1.8) for $s = 0$,

$$|||G|||_2 = \left(\lim_{t\to\infty} E[|z(t, 0, x_0, v)|^2]\right)^{1/2}. \tag{7.8}$$

Because in the deterministic framework (i.e., $\mathcal{D} = \{1\}$ and $A_k(1) = 0, 1 \leq k \leq r$) the right-hand side of (7.6)–(7.8) provides the state space characterization of the H_2 norm of a linear time-invariant deterministic system, we preserve the same terminology in this general framework of stochastic systems (7.1). That is why we call H_2 norms the cost functionals introduced by (7.6)–(7.8).

Having in mind (7.4) together with the exponential stability in the mean square of the zero state equilibrium of (7.5) one can see that the transitory component of the output $z(t, s, x_0, v)$ does not influence the performances (7.6)–(7.8). Explicit formulae for the performances (7.6)–(7.8) are derived in Section 7.2.

It is worth mentioning that in the literature which deals with the problem of H_2 optimal control for discrete-time linear stochastic systems only the H_2 performances of type (7.8) are considered. In this chapter we consider also H_2 norms defined via Cesaro limits (see (7.6)–(7.7)). In the next section we show that the H_2 norms (7.6)–(7.7) can be computed under some weaker assumptions than (7.8).

7.1.3 Systems with coefficients depending upon η_t and η_{t-1}

The explicit formulae of the H_2 norms (7.6)–(7.8) are derived as special cases of some corresponding H_2 norms defined for a class of discrete-time stochastic systems with coefficients depending upon η_t and η_{t-1}. A motivation for the consideration of the systems with coefficients depending upon η_t and η_{t-1} can be found in Section 1.6. Here we show that regarding systems (7.1) as special cases of systems (7.9) allows us to obtain new formulae for the H_2 performances (7.6)–(7.8) that do not have an analogue in the continuous-time case.

To redefine the H_2 norms of type (7.6)–(7.8) in the case of systems with coefficients depending upon η_t and η_{t-1} we consider the uncontrolled system:

$$(\mathbf{G}): \begin{cases} x(t+1) = \Big(A_0(\eta_t,\eta_{t-1}) \\ \qquad\qquad + \sum_{k=1}^{r} w_k(t)A_k(\eta_t,\eta_{t-1})\Big)x(t) + B_v(\eta_t,\eta_{t-1})v(t) \\ z(t) \qquad = C(\eta_t,\eta_{t-1})x(t), t \geq 1. \end{cases} \tag{7.9}$$

As in the case of system (7.1), $x(t,t_0,x_0,v), t \geq t_0 \geq 1, x_0 \in \mathbf{R}^n$ stands for the trajectory of (7.9) with the initial value $x(t_0,t_0,x_0,v) = x_0$ and $z(t,t_0,x_0,v) = C(\eta_t,\eta_{t-1})x(t,t_0,x_0,v)$ is a corresponding output.

The analogues of norms (7.6)–(7.8) defined for the system (7.9) are:

$$\|\mathbf{G}\|_2 = \left[\lim_{l\to\infty}\frac{1}{l}\sum_{t=1}^{l} E[|z(t,1,x_0)|^2]\right]^{1/2} \tag{7.10}$$

$$\tilde{\|}\mathbf{G}\tilde{\|}_2 = \left[\lim_{l\to\infty}\frac{1}{l}\sum_{t=1}^{l}\sum_{i\in\mathcal{D}_0} E[|z(t,1,x_0)|^2|\eta_0 = i]\right]^{1/2} \tag{7.11}$$

$$|||\mathbf{G}|||_2 = \left[\lim_{t\to\infty} E[|z(t,s,x_0)|^2]\right]^{1/2}. \tag{7.12}$$

In the next section we show how we can express the right-hand side of (7.10)–(7.12) in terms of the solution of some suitable linear equations. Such linear equations extend to this framework the well-known equations of observability Gramian and controllability Gramian from the deterministic framework.

7.2 The computation of H_2-type norms

Consider the discrete-time linear system

$$x(t+1) = \left[A_0(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t,\eta_{t-1})\right]x(t) \tag{7.13}$$

obtained from (7.9) taking $B_v(i,j) = 0$. Using the matrices $A_k(i,j)$ and the transition probability matrix P we construct the linear operator (see also (2.99)) $\Upsilon : \mathcal{S}_n^N \to \mathcal{S}_n^N$ as $\Upsilon H = (\Upsilon H(1), \Upsilon H(2), \ldots, \Upsilon H(N))$ with

$$\Upsilon H(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(j,i) A_k(i,j) H(j) A_k^T(i,j), \tag{7.14}$$

$i \in \mathcal{D}, H \in \mathcal{S}_n^N$. By direct computation one obtains that the adjoint operator Υ^* with respect to the inner product (2.18) is given by $\Upsilon^* H = (\Upsilon^* H(1), \Upsilon^* H(2), \ldots, \Upsilon^* H(N))$,

$$\Upsilon^* H(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j) A_k^T(j,i) H(j) A_k(j,i), \qquad i \in \mathcal{D}, \tag{7.15}$$

$H \in \mathcal{S}_n^N$. From Theorem 3.15 for $\theta = 1$ one obtains that the zero state equilibrium of (7.13) is ESMS if and only if the eigenvalues of the operator Υ are in the inside of the disk $|\lambda| < 1$.

For the proofs of the results in this section we need the following auxiliary results derived from Theorem 1.6. For each $(t,s) \in \mathbf{Z}_+ \times \mathbf{Z}_+$, we denote

$$\check{\mathcal{H}}_{t,s} = \sigma[\eta_\mu, \check{w}(\nu); 0 \le \mu \le t, 0 \le \nu \le s],$$

where either $\check{w}(\nu) = w(\nu)$ or $\check{w}(\nu) = (w(\nu), v(\nu)), \nu \ge 0$.

In the special case $t = s$ we write $\check{\mathcal{H}}_t$ instead of $\check{\mathcal{H}}_{tt}$. It is obvious that $\check{\mathcal{H}}_t = \mathcal{H}_t$ if $\check{w}(\nu) = w(\nu)$ and $\check{\mathcal{H}}_t = \hat{\mathcal{H}}_t$ if $\check{w}(\nu) = (w(\nu), v(\nu)), \nu \ge 0$. The next result is derived directly from Theorem 1.6.

Corollary 7.1 *Under the assumptions* $\mathbf{H}_1, \mathbf{H}_2$, *and* **A.7.1** *the following equality holds:*

$$E[\chi_{\{\eta_{t+1}=j\}} | \check{\mathcal{H}}_t] = E[\chi_{\{\eta_{t+1}=j\}} | \eta_t] = p(\eta_t, j) \qquad a.s.$$

for all $j \in \mathcal{D}, t \ge 0$.

It must be remarked that equality in the previous corollary extends (1.6) to the joint process $\{\eta_t, w(t)\}_{t\ge 0}$ or $\{\eta_t, w(t), v(t)\}_{t \ge 0}$.

7.2.1 The computations of the norm $\|\mathbf{G}\|_2$ and the norm $\tilde{\|}\mathbf{G}\tilde{\|}_2$

We start with the following auxiliary result.

Lemma 7.1 *Under the assumptions* $\mathbf{H}_1, \mathbf{H}_2$, *and* **A.7.1** *we have*

$$\begin{aligned} &E[x^T(t+1) H(\eta_t) x(t+1) | \eta_{s-1}] \\ &\quad = E[x^T(t)(\Upsilon^* H)(\eta_{t-1}) x(t) | \eta_{s-1}] \\ &\qquad + \sum_{j=1}^{N} E[Tr[H(j) B_v(j, \eta_{t-1}) B_v^T(j, \eta_{t-1})] p(\eta_{t-1}, j) | \eta_{s-1}], \end{aligned}$$

$\forall\, t \geq s \geq 1$, $H \in \mathcal{S}_n^N$, *where* $x(t) = x(t, s, x_0, v)$ *is a trajectory of the system* (7.9) *starting from* x_0 *at* $t = s$.

Proof. First we write

$$\begin{aligned}
&x^T(t+1)H(\eta_t)x(t+1)\\
&\quad = x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_0(\eta_t,\eta_{t-1})x(t)\\
&\qquad + \sum_{k,l=1}^r w_k(t)w_l(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)\\
&\qquad + v^T(t)B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)\\
&\qquad + 2\sum_{k=1}^r w_k(t)x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)\\
&\qquad + 2x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)\\
&\qquad + 2\sum_{k=1}^r w_k(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t).
\end{aligned} \tag{7.16}$$

If we take into account that $x(t)$ is $\hat{\mathcal{H}}_{t-1}$-measurable, $\hat{\mathcal{H}}_{t-1} \subset \tilde{\mathcal{H}}_t$, and $w_k(t)$, $v(t)$ are independent of $\tilde{\mathcal{H}}_t$ one obtains

$$\begin{aligned}
&E[x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_0(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t]\\
&\quad = x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_0(\eta_t,\eta_{t-1})x(t)
\end{aligned} \tag{7.17}$$

$$\begin{aligned}
&E\left[\sum_{k,l=1}^r w_k(t)w_l(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t\right]\\
&\quad = \sum_{k,l=1}^r x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)E[w_k(t)w_l(t)|\tilde{\mathcal{H}}_t]\\
&\quad = \sum_{k,l=1}^r x^T(t)A_k^T(\eta_t\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)E[w_k(t)w_l(t)].
\end{aligned}$$

Based on $\mathbf{H}_1$ one gets:

$$\begin{aligned}
&E\left[\sum_{k,l=1}^r w_k(t)w_l(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_l(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t\right]\\
&\quad = \sum_{k=1}^r x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)
\end{aligned} \tag{7.18}$$

$$E\left[\sum_{k=1}^{r} w_k(t)x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t\right]$$
$$=\sum_{k=1}^{r} x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)E[w_k(t)|\tilde{\mathcal{H}}_t]$$
$$=\sum_{k=1}^{r} x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)E[w_k(t)].$$

Invoking again the assumption $\mathbf{H}_1$ we conclude:

$$E\left[\sum_{k=1}^{r} w_k(t)x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t)|\tilde{\mathcal{H}}_t\right]=0 \tag{7.19}$$

$$E\left[\sum_{k=1}^{r} w_k(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t\right]$$
$$=\sum_{k=1}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[w_k(t)v(t)|\tilde{\mathcal{H}}_t]$$
$$=\sum_{k=1}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[w_k(t)v(t)].$$

Based on the assumptions $\mathbf{H}_1, \mathbf{H}_2$, and **A.7.1** we deduce:

$$E\left[\sum_{k=1}^{r} w_k(t)x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t\right]=0. \tag{7.20}$$

Similarly

$$E[x^T(t)A_0^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t]=0. \tag{7.21}$$

Invoking again **A.7.1** we write:

$$E[v^T(t)B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)|\tilde{\mathcal{H}}_t]$$
$$=E[Tr(B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})v(t)v^T(t))|\tilde{\mathcal{H}}_t]$$
$$=Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[v(t)v^T(t)|\tilde{\mathcal{H}}_t]]$$
$$=Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})E[v(t)v^T(t)]]$$
$$=Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})]. \tag{7.22}$$

Combining (7.16)–(7.22) one obtains

$$\begin{aligned}
&E[x^T(t+1)H(\eta_t)x(t+1)|\tilde{\mathcal{H}}_t] \\
&= \sum_{k=0}^{r} x^T(t)A_k^T(\eta_t,\eta_{t-1})H(\eta_t)A_k(\eta_t,\eta_{t-1})x(t) \\
&\quad + Tr[B_v^T(\eta_t,\eta_{t-1})H(\eta_t)B_v(\eta_t,\eta_{t-1})]. \qquad (7.23)
\end{aligned}$$

Further taking the conditional expectation with respect to $\hat{\mathcal{H}}_{t-1}$ in (7.23) one obtains:

$$\begin{aligned}
&E[x^T(t+1)H(\eta_t)x(t+1)|\hat{\mathcal{H}}_{t-1}] \\
&= \sum_{k=0}^{r}\sum_{j=1}^{N} A_k^T(j,\eta_{t-1})H(j)A_k(j,\eta_{t-1})x(t)E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}] \\
&\quad + Tr\left[\sum_{j=1}^{N} H(j)B_v(j,\eta_{t-1})B_v^T(j,\eta_{t-1})E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}]\right]. \qquad (7.24)
\end{aligned}$$

Applying Corollary 7.1 one deduces

$$E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}] = E[\chi_{\{\eta_t=j\}}|\eta_{t-1}]p(\eta_{t-1},j) \qquad \text{a.s.} \qquad (7.25)$$

Combining (7.24) and (7.25) and taking the conditional expectation with respect to $\sigma[\eta_{s-1}] \subset \hat{\mathcal{H}}_{t-1}$ we obtain the equality in the statement and thus the proof is complete. □

Let $\mathcal{A}(t) = A_0(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t,\eta_{t-1})$. We define $\Theta(t,s) = \mathcal{A}(t-1)\mathcal{A}(t-2)\cdots\mathcal{A}(s)$ if $t > s \geq 1$ and $\Theta(t,s) = I_n$ if $t = s$. $\Theta(t,s)$ is the fundamental matrix solution of the system (7.13).

The solutions of the affine system (7.9) have the representation

$$x(t,s,x_0,v) = \Theta(t,s)x_0 + \sum_{l=s}^{t-1} \Theta(t,l+1)B_v(\eta_l,\eta_{l-1})v(l) \qquad (7.26)$$

for all $t \geq s+1, s \geq 1, x_0 \in \mathbf{R}^n$. We often write $x_0(t,s,v)$ instead of $x(t,s,0,v)$.

Lemma 7.2 *Under the assumptions* $\mathbf{H}_1, \mathbf{H}_2$, *and* **A.7.1** *the following hold.*

(*i*) $E[x_0(t,s,v)x_0^T(t,s,v)] = \sum_{l=s}^{t-1} E[\Theta(t,l+1)B_v(\eta_l,\eta_{l-1})B_v^T(\eta_l,\eta_{l-1})\Theta^T(t,l+1)]$.

(*ii*)

$$\begin{aligned}
&E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=j\}}] \\
&= \sum_{l=s}^{t-1} E[\Theta(t,l+1)B_v(\eta_l,\eta_{l-1})B_v^T(\eta_l,\eta_{l-1})\Theta^T(t,l+1)\chi_{\{\eta_{t-1}=j\}}]
\end{aligned}$$

for all $t > s \geq 1$.

Proof. Using (7.26) for $x_0 = 0$ one first computes the conditional expectations

$$E[x_0(t,s,v)x_0^T(t,s,v)|\mathcal{H}_{t-1}]$$

and

$$E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=j\}}|\mathcal{H}_{t-1}].$$

To this end one takes into account that $\Theta(t,l+1), B_v(\eta_l,\eta_{l-1})$ are $\mathcal{H}_{t-1}$-measurable, and $v(l)$ are independent of $\mathcal{H}_{t-1}$. Details are omitted. □

Remark 7.1 If together with assumptions $\mathbf{H}_1, \mathbf{H}_2$, and **A.7.1** we assume that the zero state equilibrium of (7.13) is ESMS, then from Lemma 7.2 one obtains that:

$$\sup_{t\geq s\geq 1} E[|x_0(t,s,v)|^2] \leq \gamma < \infty. \tag{7.27}$$

On the other hand from the representation formula (7.26) one deduces that

$$E[|x(t,s,x_0,v) - x_0(t,s,v)|^2] \leq \beta q^{t-s}|x_0|^2 \tag{7.28}$$

for all $t \geq s \geq 1, x_0 \in \mathbf{R}^n, \beta \geq 1, q \in (0,1)$. Combining (7.27) and (7.28) we may conclude that

$$\sup_{t\geq s\geq 1} E[|x(t,s,x_0,v)|^2] \leq \gamma_1(1+|x_0|^2), \qquad \forall\, x_0 \in \mathbf{R}^n. \tag{7.29}$$

Lemma 7.3 *Assume:*

(*a*) *The assumptions* $\mathbf{H}_1, \mathbf{H}_2$, *and* **A.7.1** *are fulfilled.*
(*b*) *The zero state equilibrium of* (7.13) *is ESMS.*

Under these conditions we have:

$$\lim_{l\to\infty}\frac{1}{l}\sum_{t=1}^{l} E[|C(\eta_t,\eta_{t-1})x(t,1,x_0,v)|^2|\eta_0 = i]$$

$$= \sum_{i_1,i_2=1}^{N} Tr[B_v^T(i_1,i_2)\tilde{X}(i_1)B_v(i_1,i_2)]p(i_2,i_1)q(i,i_2)$$

for all $x_0 \in \mathbf{R}^n, i \in \mathcal{D}_0, x(t,1,x_0,v)$ *is the trajectory of* (7.9) *starting from* x_0 *at* $t=1$, $\tilde{X} = (\tilde{X}(1),\tilde{X}(2),\ldots,\tilde{X}(N))$ *is the unique solution of the affine equation on* $\mathcal{S}_n^N$

$$X = \Upsilon^* X + \tilde{C}, \tag{7.30}$$

where $\tilde{C} = (\tilde{C}(1),\tilde{C}(2),\ldots,\tilde{C}(N))$ *with*

$$\tilde{C}(i) = \sum_{j=1}^{N} p(i,j)C^T(j,i)C(j,i), \tag{7.31}$$

and $q(i,i_2)$ *are the entries of the matrix* Q *introduced by the Cesaro limit from Proposition* 1.5.

Proof. Under the considered assumptions the eigenvalues of the operator Υ introduced by (7.14) are located in the inside of the disk $|\lambda| < 1$. Then applying Theorem 2.5(*iii*) we deduce that the linear equation (7.30)–(7.31) has a unique solution $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), \dots, \tilde{X}(N)) \in \mathcal{S}_n^{N+}$. Applying Lemma 7.1 for $H(i) = \tilde{X}(i)$ one obtains for $i \in \mathcal{D}_0$,

$$E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i] = E[x^T(t)(\Upsilon^*\tilde{X})(\eta_{t-1})x(t)|\eta_0 = i)]$$

$$+\sum_{j=1}^{N} E[Tr[\tilde{X}(j)B_v(j,\eta_{t-1})B_v^T(j,\eta_{t-1})]p(\eta_{t-1},j)|\eta_0 = i],$$

$\forall\, x(t) = x(t,1,x_0,v)$ the solution of (7.9) with the initial value x_0 at $t = 1$.

Based on (7.30) we deduce

$$E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i] - E[x^T(t)\tilde{X}(\eta_{t-1})x(t)|\eta_0 = i]$$
$$= -E[x^T(t)\tilde{C}(\eta_{t-1})x(t)|\eta_0 = i]$$
$$+\sum_{j=1}^{N} E[Tr[\tilde{X}(j)B_v(j,\eta_{t-1})B_v^T(j,\eta_{t-1})]p(\eta_{t-1},j)|\eta_0 = i],$$

where $\tilde{C}(i)$ is defined in (7.31).

Furthermore we have

$$E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i] - E[x^T(t)\tilde{X}(\eta_{t-1})x(t)|\eta_0 = i]$$
$$= -E[x^T(t)\tilde{C}(\eta_{t-1})x(t)|\eta_0 = i]$$
$$+\sum_{j,i_2=1}^{N} Tr[\tilde{X}(j)B_v(j,i_2)B_v^T(j,i_2)]p(i_2,j)p^{t-1}(i,i_2), \tag{7.32}$$

where $p^{t-1}(i,i_2)$ is an element of P^{t-1}. On the other hand,

$$E[|C(\eta_t,\eta_{t-1})x(t)|^2|\hat{\mathcal{H}}_{t-1}] = \sum_{j=1}^{N} E[|C(j,\eta_{t-1})x(t)\chi_{\{\eta_t=j\}}|^2|\hat{\mathcal{H}}_{t-1}]$$
$$= \sum_{j=1}^{N} |C(j,\eta_{t-1})x(t)|^2 E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}]. \tag{7.33}$$

Using Corollary 7.1 again, one deduces that

$$E[\chi_{\{\eta_t=j\}}|\hat{\mathcal{H}}_{t-1}] = p(\eta_{t-1},j). \tag{7.34}$$

Combining (7.33) and (7.34) together with (7.31) we may write

$$E[|C(\eta_t,\eta_{t-1})x(t)|^2|\hat{\mathcal{H}}_{t-1}] = x^T(t)\tilde{C}(\eta_{t-1})x(t).$$

Taking the conditional expectation with respect to the event $\{\eta_0 = i\}$ in the last equality and replacing the obtained result in (7.32) we have

$$E[|C(\eta_t,\eta_{t-1})x(t)|^2|\eta_0 = i]$$

$$= \sum_{j,i_2=1}^{N} Tr[\tilde{X}(j)B_v(j,i_2)B_v^T(j,i_2)]p(i_2,j)p^{t-1}(i,i_2)$$

$$+ E[x^T(t)\tilde{X}(\eta_{t-1})x(t)|\eta_0 = i] - E[x^T(t+1)\tilde{X}(\eta_t)x(t+1)|\eta_0 = i].$$

Thus

$$\frac{1}{l}\sum_{t=1}^{l} E[|C(\eta_t,\eta_{t-1})x(t)|^2|\eta_0 = i]$$

$$= \sum_{i_1,i_2=1}^{N}\left[Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)]p(i_2,i_1)\frac{1}{l}\sum_{t=1}^{l}p^{t-1}(i,i_2)\right] \quad (7.35)$$

$$+ \frac{1}{l}[x_0^T\tilde{X}(i)x_0 - E[x^T(l+1)\tilde{X}(\eta_l)x(l+1)|\eta_0 = i]].$$

Based on Remark 7.1 we obtain $E[x^T(l+1)\tilde{X}(\eta_l)x(l+1)|\eta_0 = i] \leq \hat{\gamma}(1/\pi_0(i))(1+|x_0|^2)$. Therefore

$$\lim_{l\to\infty}\frac{1}{l}[x_0^T\tilde{X}(i)x_0 - E[x^T(l+1)\tilde{X}(\eta_l)x(l+1)|\eta_0 = i]] = 0. \quad (7.36)$$

On the other hand from Proposition 1.5 we obtain

$$\lim_{l\to\infty}\frac{1}{l}\sum_{t=1}^{l}p^{t-1}(i,i_2) = q(i,i_2). \quad (7.37)$$

Taking the limit for $l \to \infty$ in (7.35) and taking into account (7.36)–(7.37) one obtains the equality in the statement and thus the proof is complete. □

Now we are in position to prove the result that provides the explicit formulae of the H_2 norms (7.10)–(7.11).

Theorem 7.1 *Assume:*

(*a*) *The assumptions* $\mathbf{H}_1$,$\mathbf{H}_2$, *and* **A.7.1** *are fulfilled.*
(*b*) *The zero state equilibrium of* (7.13) *is ESMS.*
Then:
(*i*)

$$(\|\mathbf{G}\|_2)^2 = \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)]$$

$$= \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2,i_1)Y^{\pi_0}(i_1)C(i_2,i_1)]$$

(ii)

$$(\tilde{\|}\mathbf{G}\tilde{\|}_2)^2 = \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)]$$
$$= \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2,i_1)Y^{\mathcal{D}_0}(i_1)C(i_2,i_1)],$$

where $\tilde{X} \in \mathcal{S}_n^{N+}$ *is the unique solution of the linear equation* (7.30)–(7.31) *and* $Y^{\pi_0} = (Y^{\pi_0}(1), Y^{\pi_0}(2), \ldots, Y^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$*, and* $Y^{\mathcal{D}_0} = (Y^{\mathcal{D}_0}(1), Y^{\mathcal{D}_0}(2), \ldots, Y^{\mathcal{D}_0}(N)) \in \mathcal{S}_n^{N+}$*, respectively, are the unique solutions of the linear equations:*

$$Y = \Upsilon Y + B^{\pi_0} \tag{7.38}$$

and

$$Y = \Upsilon Y + B^{\mathcal{D}_0}, \tag{7.39}$$

respectively, with $B^{\pi_0} = (B^{\pi_0}(1), B^{\pi_0}(2), \ldots, B^{\pi_0}(N))$*,*

$$B^{\pi_0}(i) = \sum_{j=1}^{N} q^{\pi_0}(j)p(j,i)B_v(i,j)B_v^T(i,j), \tag{7.40}$$

and $B^{\mathcal{D}_0} = (B^{\mathcal{D}_0}(1), B^{\mathcal{D}_0}(2), \ldots, B^{\mathcal{D}_0}(N))$*,*

$$B^{\mathcal{D}_0}(i) = \sum_{j=1}^{N} q^{\mathcal{D}_0}(j)p(j,i)B_v(i,j)B_v^T(i,j), \qquad 1 \le i \le N, \tag{7.41}$$

$q^{\pi_0}(i) = \sum_{j=1}^{N} \pi_0(j)q(j,i)$ *and* $q^{\mathcal{D}_0}(i) = \sum_{j\in\mathcal{D}_0} q(j,i), 1 \le i \le N.$

Proof. We start with the proof of *(ii)*. Directly from the equalities in Lemma 7.3 one obtains that

$$(\tilde{\|}\mathbf{G}\tilde{\|}_2)^2 = \lim_{l\to\infty} \frac{1}{l} \sum_{t=1}^{l} \sum_{i\in\mathcal{D}_0} E[|z(t,1,x_0)|^2|\eta_0 = i]$$
$$= \sum_{i_1,i_2=1}^{N} \sum_{i\in\mathcal{D}_0} q(i,i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)]$$
$$= \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)] \tag{7.42}$$

which confirms the validity of the first equality of (*ii*).

Furthermore (2.18) and (7.41) allow us to write

$$\sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2)]$$

$$= \sum_{i_1=1}^{N} Tr[\tilde{X}(i_1)B^{\mathcal{D}_0}(i_1)] = \langle \tilde{X}, B^{\mathcal{D}_0}\rangle.$$

Using the equation verified by $Y^{\mathcal{D}_0}$ and equality (7.42) we have:

$$(\tilde{\|}\mathbf{G}\tilde{\|}_2)^2 = \langle \tilde{X}, Y^{\mathcal{D}_0}\rangle - \langle \tilde{X}, \Upsilon Y^{\mathcal{D}_0}\rangle = \langle \tilde{X} - \Upsilon^* \tilde{X}, Y^{\mathcal{D}_0}\rangle = \langle \tilde{C}, Y^{\mathcal{D}_0}\rangle.$$

Taking into account (2.18) and (7.31) we may write finally

$$\tilde{\|}\mathbf{G}\tilde{\|}_2)^2 = \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C^T(i_2,i_1)C(i_2,i_1)Y^{\mathcal{D}_0}(i_1)],$$

which confirms the second equality of (ii).

To prove (i) we take into account that $E[|z(t,1,x_0)|^2] = \sum_{i\in\mathcal{D}_0}\pi_0(i)$ $E[|z(t,1,x_0)^2|\eta_0 = i]$. Thus, multiplying by $\pi_0(i)$ the equalities proved in Lemma 7.3 and proceeding as in the first part of the proof one obtains that (i) holds and the proof is complete. □

Using Lemma 7.1 for $H = \tilde{X}$ one can prove the following.

Proposition 7.1 *Assume:*

(a) *Assumptions* $\mathbf{H}_1, \mathbf{H}_2$, *and* **A.7.1** *are fulfilled.*
(b) *The zero state equilibrium of* (7.13) *is ESMS.*
(c) *P is a nondegenerate stochastic matrix.*
(d) $\pi_0(i) > 0, \quad 1 \le i \le N.$

Under these conditions, the following hold.

(i)

$$\lim_{l\to\infty}\frac{1}{l}\sum_{t=s}^{s+l-1}\sum_{i=1}^{N} E[|C(\eta_t,\eta_{t-1})x(t,s,x_0)|^2|\eta_{s-1} = i]$$

$$= \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2).$$

(ii)

$$\lim_{l\to\infty}\frac{1}{l}\sum_{t=s}^{s+l-1} E[|C(\eta_t,\eta_{t-1})x(t,s,x_0)|^2]$$

$$\times \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{X}(i_1)B_v(i_1,i_2)B_v^T(i_1,i_2),$$

for every solution $x(t,s,x_0,v)$ *of the system* (7.9) *starting from* x_0 *at* $t=s, q^{\pi_0}(i_2)$ *is defined as in Theorem* 7.1 *and* $\tilde{q}(i_2)=\sum_{i=1}^N q(i,i_2)$.

To prove the equality in (ii) one uses the fact that $\pi_{s-1}(i)=\sum_{j=1}^N \pi_0(j)$ $p^{s-1}(j,i)$, where $p^{s-1}(j,i)$ are the entries of P^{s-1}. The details are omitted.

From Theorem 7.1 one sees that the H_2 norms introduced by (7.10) and (7.11) do not depend upon the initial values x_0 of the solutions $x(t,1,x_0,v)$ of the system (7.9). The result stated in Proposition 7.1 shows that under some additional assumptions the norms (7.10) and (7.11) do not depend upon the initial time $t=s$, too.

7.2.2 The computation of the norm $|||\mathbf{G}|||_2$

We start with the following.

Lemma 7.4 *Under the assumptions* $\mathbf{H}_1, \mathbf{H}_2$, *and* **A.7.1** *the following hold.*

$$E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=j\}}]=\sum_{l=s}^{t-1}(\Upsilon^{t-l-1}H_l)(j),$$

where $H_l=(H_l(1),H_l(2),\dots,H_l(N))$,

$$H_l(i)=\sum_{i_1,i_2=1}^{N}\pi_0(i_1)p^{l-1}(i_1,i_2)p(i_2,i)B_v(i,i_2)B_v^T(i,i_2),$$

with $p^{l-1}(i_1,i_2)$ *as in Proposition* 7.1.

Proof. For each $1\leq j\leq N$ we choose $H^j=(H^j(1),H^j(2),\dots,H^J(N))\in\mathcal{S}_n^N$ defined as follows: $H^j(i)=0$ if $i\not=j$ and $H^j(i)=\xi\xi^T$ if $i=j$, where $\xi\in\mathbf{R}^n$ is arbitrary but fixed. Applying Lemma 7.1 for $H=H^j$, one obtains

$$\begin{aligned}
&E[x_0^T(t+1,s,v)\xi\xi^T x_0(t+1,s,v)\chi_{\{\eta_t=j\}}|\eta_{s-1}]\\
&\quad=E[x_0^T(t,s,v)(\Upsilon^*H^j)(\eta_{t-1})x_0(t,s,v)|\eta_{s-1}]\\
&\qquad+E[p(\eta_{t-1},j)Tr[B_v^T(j,\eta_{t-1})\xi\xi^T B_v(j,\eta_{t-1})]|\eta_{s-1}]. \qquad (7.43)
\end{aligned}$$

Using (7.15) we obtain

$$(\Upsilon^*H^j)(\eta_{t-1})=p(\eta_{t-1},j)\sum_{k=0}^{r}A_k^T(j,\eta_{t-1})\xi\xi^T A_k(j,\eta_{t-1}).$$

This leads to

$$x_0^T(t,s,v)(\Upsilon^* H^j)(\eta_{t-1})x_0(t,s,v)$$
$$= Tr\left[\sum_{k=0}^{r} A_k^T(j,\eta_{t-1})\xi\xi^T A_k(j,\eta_{t-1})x_0(t,s,v)x_0^T(t,s,v)\right]p(\eta_{t-1},j)$$
$$= \sum_{k=0}^{r}\sum_{i=1}^{N} p(i,j)\xi^T A_k(j,i)[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=i\}}]A_k^T(j,i)\xi.$$

Now (7.43) becomes:

$$E[x_0^T(t+1,s,v)\xi\xi^T x_0(t+1,s,v)\chi_{\{\eta_t=j\}}|\eta_{s-1}]$$
$$= \sum_{k=0}^{r}\sum_{i=1}^{N} p(i,j)\xi^T A_k(j,i)E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=i\}}|\eta_{s-1}]A_k^T(j,i)\xi$$
$$+ \sum_{i=1}^{N} p(i,j)\xi^T B_v(j,i)B_v^T(j,i)\xi E[\chi_{\{\eta_{t-1}=i\}}|\eta_{s-1}]. \tag{7.44}$$

Taking the expectation in (7.44) and having in mind that

$$E[\chi_{\{\eta_{t-1}=i\}}] = \sum_{i_1=1}^{N} \pi_0(i_1)p^{t-1}(i_1,i)$$

we get:

$$\xi^T E[x_0(t+1,s,v)x_0^T(t+1,s,v)\chi_{\{\eta_t=j\}}]\xi$$
$$= \xi^T\left(\sum_{k=0}^{r}\sum_{i=1}^{N} p(i,j)A_k(j,i)E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=i\}}]A_k^T(j,i)\right)\xi$$
$$+ \xi^T\left(\sum_{i,i_1=1}^{N} \pi_0(i_1)p^{t-1}(i_1,i)p(i,j)B_v(j,i)B_v^T(j,i)\right)\xi. \tag{7.45}$$

Because ξ is arbitrarily chosen, (7.45) becomes:

$$E[x_0(t+1,s,v)x_0^T(t+1,s,v)\chi_{\{\eta_t=j\}}]$$
$$= \sum_{k=0}^{r}\sum_{i=1}^{N} p(i,j)A_k(j,i)E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=i\}}]A_k^T(j,i) + H_t(j). \tag{7.46}$$

Let $Y(t) = (Y(t,1), Y(t,2) \dots, Y(t,N))$, where

$$Y(t,i) = E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=i\}}], \qquad t \geq s.$$

With this notation (7.46) may be rewritten in a compact form:

$$Y(t+1) = \Upsilon Y(t) + H_t, \quad t \geq s. \tag{7.47}$$

Because $Y(s) = 0$ one obtains via (7.47) that

$$Y(t) = \sum_{l=s}^{t-1} \Upsilon^{t-l-1} H_l. \tag{7.48}$$

But the jth component of (7.48) coincides with the equality from the statement. Thus the proof ends. □

Before stating the next result we introduce an additional assumption.

A.7.2 The transition probability matrix P has the following property: $\lim_{l\to\infty} P^l$ exists.

Remark 7.2 Under the assumption **A.7.2** if $Q = \lim_{l\to\infty} P^l$ then the matrix Q is the same as the one given by the Cesaro limit in Proposition 1.5.

Lemma 7.5 *Assume:*

(*a*) *The assumptions* $\mathbf{H}_1$,$\mathbf{H}_2$, **A.7.1** *and* **A.7.2** *are fulfilled.*
(*b*) *The zero state equilibrium of the system* (7.13) *is ESMS.*

Under these conditions we have:

$$\lim_{t\to\infty} E[x(t,s,x_0,v)x^T(t,s,x_0,v)\chi_{\{\eta_{t-1}=j\}}]Y^{\pi_0}(j)$$

for all $j \in \mathcal{D}$, *where* $Y^{\pi_0} = (Y^{\pi_0}(1), Y^{\pi_0}(2), \dots, Y^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$ *is a unique solution of the linear equation* (7.38), (7.40).

Proof. If the assumption **A.7.2** is fulfilled it follows that $\lim_{l\to\infty} H_l(i_1) = B^{\pi_0}(i_1), \forall i_1 \in \mathcal{D}$.

Using the equality proved in Lemma 7.4 we may write successively

$$\begin{aligned}
&E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=j\}}] \\
&= \sum_{l=s}^{t-1}[(\Upsilon^{t-l-1}H_l)(j) \\
&= \sum_{l=s}^{t-1}[(\Upsilon^{t-l-1}B^{\pi_0})(j) + \sum_{l=s}^{t-1}(\Upsilon^{t-l-1}(H_l - B^{\pi_0}))(j) \\
&= \sum_{l=0}^{t-s-1}(\Upsilon^l B^{\pi_0})(j) + \sum_{l=s}^{t-1}(\Upsilon^{t-l-1}(H_l - B^{\pi_0}))(j).
\end{aligned} \tag{7.49}$$

From assumption (b) in the statement we deduce firstly that

$$\lim_{t\to\infty}\sum_{l=0}^{t-s-1}(\Upsilon^l B^{\pi_0})(j)=\sum_{l=0}^{\infty}(\Upsilon^l B^{\pi_0})(j)=Y^{\pi_0}(j). \tag{7.50}$$

Also from assumption (b) we deduce that there exist $\beta\geq 1, q\in(0,1)$ such that

$$\|\Upsilon^l\|_\xi\leq \beta q^l, \qquad \forall l\geq 0, \tag{7.51}$$

where $\|\cdot\|_\xi$ is the Minkovsky norm of the operator Υ induced by (2.20).

If $|M|$ is the spectral norm of a symmetric matrix then based on the definition of (2.20) we deduce

$$\begin{aligned}\left|\sum_{l=s}^{t-1}[\Upsilon^{t-l-1}(H_l-B^{\pi_0})](j)\right| &\leq \left|\sum_{l=s}^{t-l-1}\Upsilon^{t-l-1}(H_l-B^{\pi_0})\right|_\xi\\ &\leq \sum_{l=s}^{t-1}\|\Upsilon^{t-l-1}\|_\xi|H_l-B^{\pi_0}|_\xi.\end{aligned}$$

Furthermore, (7.51) allows us to write

$$\left|\sum_{l=s}^{t-1}[\Upsilon^{t-l-1}(H_l-B^{\pi_0})](j)\right|\leq\sum_{l=s}^{t-1}\beta q^{t-l-1}|H_l-B^{\pi_0}|_\xi. \tag{7.52}$$

Because $\lim_{l\to\infty}|H_l-B^{\pi_0}|_\xi=0$ and $q\in(0,1)$ one obtains from (7.52) that

$$\lim_{t\to\infty}\sum_{l=s}^{t-1}[\Upsilon^{t-l-1}(H_l-B^{\pi_0})](j)=0. \tag{7.53}$$

Taking the limit for $t\to\infty$ in (7.49) and using (7.50) and (7.53) one obtains

$$\lim_{t\to\infty}E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=j\}}]Y^{\pi_0}(j), \qquad \forall\, j\in\mathcal{D},\ s\geq 1. \tag{7.54}$$

Furthermore, the representation formula (7.26) together with assumption (b) in the statement allows us to write

$$E[|x(t,s,x_0,v)-x_0(t,s,v)|^2]\leq\beta q^{t-s}|x_0|^2,$$

$\forall\, t\geq s\geq 1, x_0\in\mathbf{R}^n$, where $\beta\geq 1, q\in(0,1)$.

Hence

$$\begin{aligned}&\lim_{t\to\infty}E[x(t,s,x_0,v)x^T(t,s,x_0,v)\chi_{\{\eta_{t-1}=j\}}]\\ &\quad=\lim_{t\to\infty}E[x_0(t,s,v)x_0^T(t,s,v)\chi_{\{\eta_{t-1}=j\}}]\end{aligned} \tag{7.55}$$

for all $t\geq s\geq 1, x_0\in\mathbf{R}^n$. The equality in the statement now follows from (7.54) and (7.55) and thus the proof ends. □

The main result of this subsection is the following.

Theorem 7.2 *Under the assumptions of Lemma* 7.5 *we have the following formula for the H_2 norm* (7.12)*.*

$$(|||\mathbf{G}|||_{\mathbf{2}}{}^2 = \sum_{i_1,i_2=1}^{N} Tr[C(i_1,i_2)Y^{\pi_0}(i_2)C^T(i_1,i_2)]p(i_2,i_1)$$

$$= \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[B_v^T(i_1,i_2)\tilde{X}(i_1)B_v(i_1,i_2)],$$

where $\tilde{X} = (\tilde{X}(1), \tilde{X}(2), \ldots, \tilde{X}(N)) \in \mathcal{S}_n^{N+}$ is the unique solution of the linear equation (7.30) *and* (7.31)*, $Y^{\pi_0} = (Y^{\pi_0}(1), Y^{\pi_0}(2), \ldots, Y^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$ is the unique solution of the linear equation* (7.38)–(7.40)*, and q^{π_0} is defined as in Theorem* 7.1*.*

Proof. Set $x(t) = x(t,s,x_0,v)$ and $z(t) = z(t,s,x_0,v), t \geq s \geq 1, x_0 \in \mathbf{R}^n$. Because $x(t)$ is $\hat{\mathcal{H}}_{t-1}$-measurable we may write successively

$$\begin{aligned} &E[|z(t)|^2|\hat{\mathcal{H}}_{t-1}] \\ &= E[Tr(C(\eta_t,\eta_{t-1})x(t)x^T(t)C^T(\eta_t,\eta_{t-1}))|\hat{\mathcal{H}}_{t-1}] \\ &= \sum_{i_1,i_2=1}^{N} E[Tr(C(i_1,i_2)x(t)x^T(t)C^T(i_1,i_2))\chi_{\{\eta_t=i_1\}}\chi_{\{\eta_{t-1}=i_2\}}|\hat{\mathcal{H}}_{t-1}] \\ &= \sum_{i_1,i_2=1}^{N} Tr[C(i_1,i_2)x(t)x^T(t)C^T(i_1,i_2)]\chi_{\{\eta_{t-1}=i_2\}}E[\chi_{\{\eta_t=i_1\}}|\hat{\mathcal{H}}_{t-1}]. \end{aligned}$$

Using Corollary 7.1 with $\hat{\mathcal{H}}_{t-1}$ instead of $\mathcal{H}_t$ we obtain $E[\chi_{\{\eta_t=i_1\}}|\hat{\mathcal{H}}_{t-1}] = p(\eta_{t-1},i_1)$.

Thus we have

$$E[|z(t)|^2|\hat{\mathcal{H}}_{t-1}] = \sum_{i_1,i_2=1}^{N} p(i_2,i_1)Tr[C(i_1,i_2)x(t)x^T(t)C^T(i_1,i_2)]\chi_{\{\eta_{t-1}=i_2\}}.$$

Taking the expectation in the last equality one gets:

$$E[|z(t)|^2] = \sum_{i_1,i_2=1}^{N} p(i_2,i_1)Tr\{C(i_1,i_2)E[x(t)x^T(t)\chi_{\{\eta_{t-1}=i_2\}}]C^T(i_1,i_2)\},$$

$t \geq s \geq 1, x_0 \in \mathbf{R}^n$. Based on Lemma 7.5 we may conclude

$$\begin{aligned} &\lim_{t\to\infty} E[|z(t,s,x_0,v)|^2] \\ &= \sum_{i_1,i_2=1}^{N} p(i_2,i_1)Tr[C(i_1,i_2)Y^{\pi_0}(i_2)C^T(i_1,i_2)], s \geq 1, \qquad x_0 \in \mathbf{R}^n. \end{aligned}$$

This confirms the validity of the first equality in the statement. The second equality may be proved in the same way as in Theorem 7.1. Thus the proof ends. □

7.2.3 The computation of the H_2 norms for the system of type (7.1)

The systems described by (7.1) can be regarded as systems of type (7.9) in two ways.

First we may transform the system (7.1) as

$$(\tilde{\mathbf{G}}) : \begin{cases} \tilde{x}(t+1) = \left(\tilde{A}_0(\eta_t, \eta_{t-1}) + \sum\limits_{k=1}^{r} w_k(t)\tilde{A}_k(\eta_t, \eta_{t-1})\right)\tilde{x}(t) + \tilde{B}_v(\eta_t, \eta_{t-1})v(t) \\ \tilde{z}(t) \qquad = \tilde{C}(\eta_t, \eta_{t-1})\tilde{x}(t), \end{cases} \tag{7.56}$$

$t \geq 1$, where

$$\begin{aligned} &\tilde{A}_k(i,j) = A_k(i), \qquad 0 \leq k \leq r, \qquad \tilde{B}_v(i,j) = B_v(i), \\ &\tilde{C}(i,j) = C(i), \qquad i, j \in \mathcal{D}. \end{aligned} \tag{7.57}$$

Also, (7.1) could be viewed as a system of type (7.9) as follows:

$$(\hat{\mathbf{G}}) : \begin{cases} \hat{x}(t+1) = \left[\hat{A}_0(\eta_t, \eta_{t-1}) + \sum\limits_{k=1}^{r} \hat{w}_k(t)\hat{A}_k(\eta_t, \eta_{t-1})\right]\hat{x}(t) + \hat{B}_v(\eta_t, \eta_{t-1})\hat{v}(t) \\ \hat{z}(t) \qquad = \hat{C}(\eta_t, \eta_{t-1})\hat{x}(t), \qquad t \geq 1, \end{cases} \tag{7.58}$$

where

$$\begin{aligned} &\hat{A}_k(i,j) = A_k(j), \qquad 0 \leq k \leq r, \qquad \hat{B}_v(i,j) = B_v(j), \\ &\hat{C}(i,j) = C(j), \qquad i, j \in \mathcal{D} \qquad \hat{x}(t) = x(t-1), \\ &\hat{w}_k(t) = w_k(t-1), \qquad \hat{v}(t) = v(t-1), \qquad t \geq 1. \end{aligned} \tag{7.59}$$

For each $s \geq 1, x_0 \in \mathbf{R}^n$, let $\tilde{x}(t,s,x_0,v), \hat{x}(t,s,x_0,\hat{v}), x(t,s,x_0,v)$ be the solutions of (7.56), (7.58), and (7.1), respectively, starting from x_0, at $t = s$. It is easy to see that

$$\tilde{x}(t,s,x_0,v) = x(t,s,x_0,v), \qquad t \geq s \geq 1, \qquad x_0 \in \mathbf{R}^n \tag{7.60}$$

$$\hat{x}(t,s,x_0,\hat{v}) = x(t-1,s-1,x_0,v), \qquad t \geq s \geq 1, \qquad x_0 \in \mathbf{R}^n. \tag{7.61}$$

Furthermore, if

$$\tilde{z}(t,s,x_0,v) = \tilde{C}(\eta_t, \eta_{t-1})\tilde{x}(t,s,x_0,v),$$

$$\hat{z}(t,s,x_0,\hat{v}) = \hat{C}(\eta_t,\eta_{t-1})\hat{x}(t,s,x_0,\hat{v}),$$
$$z(t,s,x_0,v) = C(\eta_t)x(t,s,x_0,v), \quad t \geq s \geq 1,$$

then from (7.57) and (7.59)–(7.61) we have

$$\tilde{z}(t,s,x_0,v) = z(t,s,x_0,v), \qquad t \geq s \geq 1, \qquad x_0 \in \mathbf{R}^n \tag{7.62}$$

$$\hat{z}(t,s,x_0,\hat{v}) = z(t-1,s-1,x_0,v), \qquad t \geq s \geq 1, \qquad x_0 \in \mathbf{R}^n. \tag{7.63}$$

If $\tilde{\Upsilon} : \mathcal{S}_n^N \to \mathcal{S}_n^N, \hat{\Upsilon} : \mathcal{S}_n^N \to \mathcal{S}_n^N$ are the Lyapunov operators associated with system (7.56), (7.58), respectively, then from (7.14), (7.57), and (7.59) we have:

$$(\tilde{\Upsilon}H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)A_k(i)H(j)A_k^T(i) = (\Lambda H)(i) \tag{7.64}$$

$$(\hat{\Upsilon}H)(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)A_k(j)H(j)A_k^T(j) = (\mathcal{L}H)(i) \tag{7.65}$$

for all $i \in \mathcal{D}, H \in \mathcal{S}_n^N$ where $\mathcal{L}$ and Λ are the Lyapunov-type operators associated with the linear system (7.5) by (2.58) and (2.59), respectively.

Using the equality (7.63) and Theorem 7.1 specialized to the system ($\hat{\mathbf{G}}$) we obtain the following.

Theorem 7.3 *Assume:*

(*a*) *The assumptions* $\mathbf{H}_1, \mathbf{H}_2$, *and* **A.7.1** *are fulfilled.*
(*b*) *The zero state equilibrium of the system* (7.5) *is ESMS.*

Under these conditions the H_2 *norms of the system* (7.1) *defined by* (7.6) *and* (7.7) *are given by*

(*i*)

$$\|G\|_2^2 = \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}(i_1)B_v(i_2)B_v^T(i_2)]$$
$$= \sum_{i=1}^{N} Tr[C(i)\mathcal{Y}^{\pi_0}(i)C^T(i)]$$

(*ii*)

$$\tilde{\|G\|}_2^2 = \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}(i_1)B_v(i_2)B_v^T(i_2)]$$
$$= \sum_{i=1}^{N} Tr[C(i)\mathcal{Y}^{\mathcal{D}_0}(i)C^T(i)],$$

where $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}(1), \tilde{\mathcal{X}}(2), \dots, \tilde{\mathcal{X}}(N)) \in \mathcal{S}_n^{N+}$ *is the unique solution of the algebraic equation*

$$\mathcal{X} = \mathcal{L}^* \mathcal{X} + \tilde{\mathcal{C}}, \tag{7.66}$$

where $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}(1), \tilde{\mathcal{C}}(2), \dots, \tilde{\mathcal{C}}(N))$,

$$\tilde{\mathcal{C}}(i) = C^T(i)C(i), \qquad i \in \mathcal{D}, \tag{7.67}$$

$\mathcal{Y}^{\pi_0} = (\mathcal{Y}^{\pi_0}(1), \mathcal{Y}^{\pi_0}(2), \dots, \mathcal{Y}^{\pi_0}(N)) \in \mathcal{S}_n^{N+}$ *and* $\mathcal{Y}^{\mathcal{D}_0} = (\mathcal{Y}^{\mathcal{D}_0}(1), \mathcal{Y}^{\mathcal{D}_0}(2), \dots, \mathcal{Y}^{\mathcal{D}_0}(N)) \in \mathcal{S}_n^{N+}$ *are the unique solutions of the algebraic equations*

$$\mathcal{Y} = \mathcal{L}\mathcal{Y} + \mathcal{B}^{\pi_0} \tag{7.68}$$

$$\mathcal{Y} = \mathcal{L}\mathcal{Y} + \mathcal{B}^{\mathcal{D}_0}, \tag{7.69}$$

where $\mathcal{B}^{\pi_0} = (\mathcal{B}^{\pi_0}(1), \mathcal{B}^{\pi_0}(2), \dots, \mathcal{B}^{\pi_0}(N))$,

$$\mathcal{B}^{\pi_0}(i) = \sum_{j=1}^{N} q^{\pi_0}(j)p(j,i)B_v(j)B_v^T(j) \tag{7.70}$$

and $\mathcal{B}^{\mathcal{D}_0} = (\mathcal{B}^{\mathcal{D}_0}(1), \mathcal{B}^{\mathcal{D}_0}(2), \dots, \mathcal{B}^{\mathcal{D}_0}(N))$,

$$\mathcal{B}^{\mathcal{D}_0}(i) = \sum_{j=1}^{N} q^{\mathcal{D}_0}(j)p(j,i)B_v(j)B_v^T(j), \qquad i \in \mathcal{D}, \tag{7.71}$$

$q^{\pi_0}(j) = \sum_{i=1}^{N} \pi_0(i)q(i,j)$ *and* $q^{\mathcal{D}_0}(j) = \sum_{i \in \mathcal{D}_0} q(i,j)$.

It must be remarked that if $\mathcal{D}_0 = \mathcal{D}$ then the H_2 norm defined by (7.7) does not depend upon the initial distribution of the Markov chain.

From Theorem 7.2 specialized for the system $\hat{\mathbf{G}}$, defined by (7.58), we obtain the following

Theorem 7.4 *Assume:*

(*a*) *Assumptions* $\mathbf{H}_1, \mathbf{H}_2$, **A.7.1**, *and* **A.7.2** *are fulfilled.*
(*b*) *The zero state equilibrium of the system* (7.5) *is ESMS.*

Under these conditions the H_2 *norm of the system* (7.1) *defined by* (7.8) *is given by*

$$|||G|||_2^2 = \sum_{j=1}^{N} Tr[C(j)\mathcal{Y}^{\pi_0}(j)C^T(j)]$$

$$\times \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}(i_1)B_v(i_2)B_v^T(i_2)],$$

where $\mathcal{Y}^{\pi_0}$ *is the unique solution of the equation* (7.68), (7.70) *and* $\tilde{\mathcal{X}}$ *is the unique solution of the equation* (7.66), (7.67) *and* q^{π_0} *is defined as before.*

At the end of this subsection we remark that the equality (7.62) together with Theorems 7.1 and 7.2 lead to some expressions of the H_2 norms (7.6)–(7.8) that do not have a correspondent in the continuous time framework.

Thus we have the following.

Theorem 7.5 *Under the assumptions of Theorem* 7.4 *the following hold.*

(*i*)

$$(\|G\|_2)^2 = (|||G|||_2)^2 = \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[B_v^T(i_1)\overline{\mathcal{X}}(i_1)B_v(i_1)]$$
$$= \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2)\mathcal{Z}^{\pi_0}(i_1)C^T(i_2)].$$

(*ii*)

$$(\tilde{\|}G\tilde{\|}_2)^2 = \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[B_v^T(i_1)\overline{\mathcal{X}}(i_1)B_v(i_1)]$$
$$= \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2)\mathcal{Z}^{\mathcal{D}_0}(i_1)C^T(i_2)],$$

where $\overline{\mathcal{X}} = (\overline{\mathcal{X}}(1), \overline{\mathcal{X}}(2), \dots, \overline{\mathcal{X}}(N)) \in \mathcal{S}_n^{N+}$ *is the unique solution of the algebraic equation*

$$\mathcal{X} = \Lambda^* \mathcal{X} + \overline{\mathcal{C}}, \tag{7.72}$$

where $\overline{\mathcal{C}} = (\overline{\mathcal{C}}(1), \overline{\mathcal{C}}(2), \dots, \overline{\mathcal{C}}(N))$,

$$\overline{\mathcal{C}}(i) = \sum_{j=1}^{N} p(i,j)C^T(j)C(j), \qquad i \in \mathcal{D}, \tag{7.73}$$

and $\mathcal{Z}^{\pi_0} = (\mathcal{Z}^{\pi_0}(1), \mathcal{Z}^{\pi_0}(2), \dots, \mathcal{Z}^{\pi_0}(N))$ *and* $\mathcal{Z}^{\mathcal{D}_0} = (\mathcal{Z}^{\mathcal{D}_0}(1), \mathcal{Z}^{\mathcal{D}_0}(2), \dots, \mathcal{Z}^{\mathcal{D}_0}(N))$ *are the unique solutions of the algebraic equations*

$$Z = \Lambda \mathcal{Z} + \mathcal{B}^{\pi_0} \tag{7.74}$$

$$Z = \Lambda \mathcal{Z} + \mathcal{B}^{\mathcal{D}_0}, \tag{7.75}$$

respectively, with $\mathcal{B}^{\pi_0}$ *and* $\mathcal{B}^{\mathcal{D}_0}$ *given by* (7.70) *and* (7.71).

Taking into account Theorem 7.1, assertion (*ii*) in the above theorem holds if the assumptions of Theorem 7.3 are fulfilled.

7.3 Some robustness issues

As we can see from Theorems 7.1, 7.2, 7.3, and 7.4, respectively, if $N \geq 2$ the H_2 norms associated with the stochastic systems (7.9) and (7.1), respectively, are strongly dependent upon the initial distributions π_0 of the Markov chain, or upon the subset $\mathcal{D}_0$ of the states i, such that $\mathcal{P}\{\eta_0 = i\} > 0$. Unfortunately, the initial distributions of the Markov chain are not known a priori. To avoid such inconvenience specific to the stochastic systems subject to Markovian jumping, one could make the additional assumption: for each $i \in \mathcal{D}$, $\lim_{t\to\infty} \mathcal{P}\{\eta_t = i\}$ exists and it does not depend upon the initial distribution $\mathcal{P}\{\eta_0 = j\}, j \in \mathcal{D}$.

One can obtain, via Theorem 1.5, that the above hypothesis is equivalent to the fact that assumption **A.7.2** is fulfilled and additionally the matrix $Q = \lim_{t\to\infty} P^t$ has the property $q(i,j) = q(j), i, j \in \mathcal{D}$.

Another idea to overcome the problems due to the presence of the initial distribution of the Markov chain in the formula of the H_2 norms is to introduce a suitable upper bound of these norms.

Thus in the case of the system (7.9) we define

$$(\hat{\|}\mathbf{G}\hat{\|})_2^2 = \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[B_v^T(i_1,i_2)\tilde{X}(i_1)B_v(i_1,i_2)], \tag{7.76}$$

where $\tilde{q}(i_2) = \sum_{i_1=1}^{N} q(i_1,i_2)$. We have

$$q^{\pi_0}(i_2) \leq \tilde{q}(i_2)$$
$$q^{\mathcal{D}_0}(i_2) \leq \tilde{q}(i_2) \tag{7.77}$$

for every initial distribution π_0 and for all subsets $\mathcal{D}_0 \subset \mathcal{D}$. So, under the assumptions of Theorem 7.1 we have:

$$\|\mathbf{G}\|_2 \leq \hat{\|}\mathbf{G}\hat{\|}_2, \tilde{\|}\mathbf{G}\tilde{\|}_2 \leq \hat{\|}\mathbf{G}\hat{\|}_2. \tag{7.78}$$

Under the assumptions of Theorem 7.2 we also have

$$|||\mathbf{G}|||_2 \leq \hat{\|}\mathbf{G}\hat{\|}_2. \tag{7.79}$$

Reasoning as in the proof of Theorem 7.1 we may obtain

$$(\hat{\|}\mathbf{G}\hat{\|}_2)^2 = \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[B_v^T(i_1,i_2)\tilde{X}(i_1)B_v(i_1,i_2)]$$
$$= \sum_{i_1,i_2=1}^{N} p(i_1,i_2)Tr[C(i_2,i_1)\tilde{Y}(i_1)C^T(i_2,i_1)], \tag{7.80}$$

where $\tilde{X}$ is the solution of (7.30) and (7.31), whereas $\tilde{Y} = (\tilde{Y}(1), \tilde{Y}(2), \dots, \tilde{Y}(N)) \in \mathcal{S}_n^{N+}$ is the unique solution of the algebraic equation

$$Y = \Upsilon Y + \tilde{B} \tag{7.81}$$

with $\tilde{B} = (\tilde{B}(1), \tilde{B}(2), \dots, \tilde{B}(N))$,

$$\tilde{B}(i) = \sum_{j=1}^{N} \tilde{q}(j) p(j,i) B_v(i,j) B_v^T(i,j). \tag{7.82}$$

Using Lemma 7.3 we may prove the following.

Proposition 7.2 *Under the assumptions in Theorem* 7.1

$$(\hat{\|}\mathbf{G}\hat{\|}_2)^2 = \lim_{l \to \infty} \frac{1}{l} \sum_{t=1}^{l} \sum_{i=1}^{N} E[|z^i(t,1,x_0)|^2],$$

where $z^i(t,1,x_0) = C(\eta_t, \eta_{t-1}), x^i(t,1,x_0), x^i(t,1,x_0), t \geq 1$ *being the solution of the system* (7.9) *corresponding to the Markov chain with the initial distribution* $\mathcal{P}\{\eta_0 = i\} = 1$ *and* $\mathcal{P}\{\eta_0 = j\} = 0$ *if* $j \neq i$.

In the case of system (7.1) the equality (7.80) becomes

$$\begin{aligned}(\hat{\|}G\hat{\|}_2)^2 &= \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2) p(i_2,i_1) Tr[B_v^T(i_2) \tilde{\mathcal{X}}(i_1) B_v(i_2)] \\ &= \sum_{i=1}^{N} Tr[C(i) \tilde{\mathcal{Y}}(i) C^T(i)],\end{aligned} \tag{7.83}$$

where $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}(1), \tilde{\mathcal{X}}(2), \dots, \tilde{\mathcal{X}}(N))$ is the unique solution of the equations (7.66) and (7.67) and $\tilde{\mathcal{Y}} = (\tilde{\mathcal{Y}}(1), \tilde{\mathcal{Y}}(2), \dots, \tilde{\mathcal{Y}}(N)) \in \mathcal{S}_n^{N+}$ is the unique solution of the algebraic equation

$$\mathcal{Y} = \mathcal{L}\mathcal{Y} + \tilde{\mathcal{B}}, \tag{7.84}$$

where $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}(1), \tilde{\mathcal{B}}(2), \dots, \tilde{\mathcal{B}}(N))$,

$$\tilde{\mathcal{B}}(i) = \sum_{j=1}^{N} \tilde{q}(j) p(j,i) B_v(j) B_v^T(j), \tag{7.85}$$

$\tilde{q}(j)$ being as before.

In the process of designing an H_2 optimal controller one may add to the list of H_2 performance criteria another one which asks the minimization of $\hat{\|} \cdot \hat{\|}_2$ of the closed-loop system.

7.4 H_2 optimal controllers. The case with full access to measurements

In this section we illustrate how the results proved in the previous sections can be used to solve several H_2 optimization problems for discrete-time linear stochastic systems subject to independent random perturbations and Markovian jumping. We consider the case of full access to the measurements. This means that at each time t, the pair $(x(t), \eta_t)$ are available and can be used to compute the control. The general case when only an output is available for measurements is considered in the next section.

7.4.1 H_2 optimization

Consider the discrete-time controlled stochastic system (G) described by

$$(G): \begin{cases} x(t+1) = \left[A_0(\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t)\right] x(t) \\ \qquad\qquad + \left[B_0(\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t)\right] u(t) + B_v(\eta_t)v(t) \\ y(t) \qquad = x(t) \\ z(t) \qquad = C_z(\eta_t)x(t) + D_z(\eta_t)u(t), \end{cases} \tag{7.86}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input, $y(t) \in \mathbf{R}^n$ is the vector of the measurements, $z(t) \in \mathbf{R}^{n_z}$ is the controlled output, and $\{w_k(t)\}_{t\geq 0}, 1 \leq k \leq r$, $\{\eta_t\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$ are as before and verify $\mathbf{H_1}, \mathbf{H_2}$, and **A.7.1**. It is assumed that the whole state vector $x(t)$ and the system mode i are available for measurements. The coefficients $A_k(i), B_k(i), 0 \leq k \leq r$, $B_v(i), C_z(i), D_z(i), i \in \mathcal{D}$ are constant matrices of appropriate dimensions.

To control the systems of type (7.86) we consider dynamic controllers of the form:

$$(G_c): \begin{cases} x_c(t+1) = \left[A_{c0}(\eta_t) + \sum_{k=1}^{r} w_k(t)A_{ck}(\eta_t)\right] x_c(t) \\ \qquad\qquad + \left(B_{c0}(\eta_t) + \sum_{k=1}^{r} w_k(t)B_{ck}(\eta_t)\right) u_c(t) \\ y_c(t) \qquad = C_c(\eta_t)x_c(t) + F_c(\eta_t)u_c(t), \end{cases} \tag{7.87}$$

$t \geq 0$, where $x_c \in \mathbf{R}^{n_c}$ is the vector of the states of the controller, $u_c(t) \in \mathbf{R}^n$ is the vector of the inputs of the controller, and $y_c(t) \in \mathbf{R}^m$ is the output of the controller. The integer n_c, often known as the order of the controller, is not prefixed. It is determined together with the matrices $A_{ck}(i), B_{ck}(i), C_c(i), F_c(i)$. If $n_c = 0$ the controller (G_c) reduces to a feedback gain $y_c(t) = F_c(\eta_t)u_c(t)$.

Coupling a controller (G_c) of type (7.87) to a system (G) of type (7.86) taking $u_c(t) = y(t), u(t) = y_c(t)$ one obtains the following closed-loop system.

$$(G_{cl}) : \begin{cases} x_{cl}(t+1) = \left[A_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t)\right] x_{cl}(t) + B_{vcl}(\eta_t)v(t) \\ z_{cl}(t) \qquad = C_{cl}(\eta_t)x_{cl}(t), t \geq 0, \end{cases} \tag{7.88}$$

where $x_{cl}(t) = \begin{pmatrix} x^T(t) & x_c^T(t) \end{pmatrix}^T$,

$$A_{kcl}(i) = \begin{pmatrix} A_k(i) + B_k(i)F_c(i) & B_k(i)C_c(i) \\ B_{ck}(i) & A_{ck}(i) \end{pmatrix}, \qquad 0 \leq k \leq r,$$

$$B_{vcl}(i) = \begin{pmatrix} B_v(i) \\ 0 \end{pmatrix}$$

$$C_{cl}(i) = \begin{pmatrix} C_z(i) + D_z(i)F_c(i) & D_z(i)C_c(i) \end{pmatrix}. \tag{7.89}$$

Definition 7.1 *We say that a controller* (G_c) *of type* (7.87) *is a* stabilizing controller *for the system* G *of type* (7.86) *if the zero state equilibrium of the linear system*

$$x_{cl}(t+1) = \left(A_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t)\right) x_{cl}(t)$$

is exponentially stable in the mean square.

In the sequel we denote $\mathcal{K}_s(G)$ the class of stabilizing controllers for a given system (G) of type (7.86). Now we are in position to state the optimization problems associated with a system (7.86).

$\mathbf{OP_1}$. Find an admissible controller $\tilde{G}_c$ such that the corresponding closed-loop system $\tilde{G}_{cl}$ satisfies

$$\|\tilde{G}_{cl}\|_2 = \min_{G_c \in \mathcal{K}_s(G)} \|G_{cl}\|_2.$$

$\mathbf{OP_2}$. Find an admissible controller $\tilde{G}_c$ such that the corresponding closed-loop system $\tilde{G}_{cl}$ satisfies $\tilde{\|}\tilde{G}_{cl}\tilde{\|}_2 = \min_{G_c \in \mathcal{K}_s(G)} \tilde{\|}G_{cl}\tilde{\|}_2$.

$\mathbf{OP_3}$. Find an admissible controller $\tilde{G}_c$ such that the corresponding closed-loop system $\tilde{G}_{cl}$ satisfies $|||\tilde{G}_{cl}|||_2 = \min_{G_c \in \mathcal{K}_s(G)} |||G_{cl}|||$.

$\mathbf{OP_4}$. Find an admissible controller $\tilde{G}_c$ such that the corresponding closed-loop system $\tilde{G}_{cl}$ satisfies $\hat{\|}\tilde{G}_{cl}\hat{\|}_2 = \min_{G_c \in \mathcal{K}_s(G)} \hat{\|}G_{cl}\hat{\|}$.

In the case $N = 1$ the norms (7.6), (7.7), and (7.83) coincide, therefore it follows that for the system subject to independent random perturbations we have only two H_2 optimization problems, $\mathbf{OP}_1$ and $\mathbf{OP}_3$, respectively.

To have a unified approach to the four optimization problems stated before, we introduce the notation $\|\cdot\|_{2.\mu}, \mu \in \{1,2,3,4\}$ as $\|\cdot\|_{21}$ instead of $\|\cdot\|_2$ defined by (7.6), $\|\cdot\|_{22}$ instead of $\tilde{\|}\cdot\tilde{\|}_2$, defined by (7.7), $\|\cdot\|_{23}$ instead of $|||\cdot|||_2$ defined by (7.8), and $\|\cdot\|_{24}$ instead of $\hat{\|}\cdot\hat{\|}_2$ defined by (7.83).

From Theorem 7.3, Theorem 7.4, and (7.83)–(7.85) applied to the closed-loop system (7.88) we have

$$\|G_{cl}\|^2_{2,\mu} = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B^T_{vcl}(i_2)\mathcal{X}_{cl}(i_1)B_{vcl}(i_2)], \tag{7.90}$$

where $\mathcal{X}_{cl} = (\mathcal{X}_{cl}(1), \mathcal{X}_{cl}(2), \ldots, \mathcal{X}_{cl}(N)) \in \mathcal{S}^{N+}_{n+n_c}$ is the unique solution of the linear equation:

$$\mathcal{X}_{cl}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A^T_{kcl}(i)\mathcal{X}_{cl}(j)A_{kcl}(i) + C^T_{cl}(i)C_{cl}(i), \qquad i \in \mathcal{D} \tag{7.91}$$

with

$$\varepsilon_\mu(i_2) = \begin{cases} q^{\pi_0}(i_2), & \text{for } \mu \in \{1,3\}; \\ q^{\mathcal{D}_0}(i_2), & \text{for } \mu = 2; \\ \tilde{q}(i_2), & \text{for } \mu = 4. \end{cases}$$

Consider the following system of discrete-time Riccati equations of stochastic control DTSRE-C:

$$\begin{aligned} X(i) &= \sum_{k=0}^{r} A^T_k(i)\mathcal{E}_i(X)A_k(i) + C^T_z(i)C_z(i) \\ &\quad - \left(\sum_{k=0}^{r} A^T_k(i)\mathcal{E}_i(X)B_k(i) + C^T_z(i)D_z(i)\right) \\ &\quad \times \left(D^T_z(i)D_z(i) + \sum_{k=0}^{r} B^T_k(i)\mathcal{E}_i(X)B_k(i)\right)^{-1} \\ &\quad \times \left(\sum_{k=0}^{r} B^T_k(i)\mathcal{E}_i(X)A_k(i) + D^T_z(i)C_z(i)\right), \qquad i \in \mathcal{D}, \end{aligned} \tag{7.92}$$

where

$$\mathcal{E}_i(X) = \sum_{j=1}^{N} p(i,j)X(j) \tag{7.93}$$

for all $X = (X(1), X(2), \ldots, X(N)) \in \mathcal{S}^N_n$.

Such systems are special cases of DTSRE-C (5.135). We recall that a solution $X_s = (X_s(1), X_s(2), \ldots, X_s(N))$ of DTSRE-C (7.92) is called a stabilizing solution if the zero state equilibrium of the closed-loop system

$$x_s(t+1) = \left[A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F_s(\eta_t))\right] x_s(t), \qquad t \geq 0 \tag{7.94}$$

is ESMS, where

$$F_s(i) = -\left(D_z^T(i)D_z(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i)\right)^{-1} \times \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)A_k(i) + D_z^T(i)C_z(i)\right). \tag{7.95}$$

In Theorem 5.13 a set of necessary and sufficient conditions for the existence of a stabilizing solution of (7.92) that satisfies

$$D_z^T(i)D_z(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i) > 0, \qquad i \in \mathcal{D} \tag{7.96}$$

is given. For each stabilizing controller G_c of type (7.87) we introduce the following performances,

$$J_\mu(G_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\mathcal{X}_{cl}(i_1)B_{vcl}(i_2)], \qquad \mu \in \{1,2,3,4\}. \tag{7.97}$$

Furthermore, under some additional assumptions which are as in Theorem 7.3 and in Theorem 7.4, respectively, $J_\mu(G_c)$ is just the H_2 norm $\|\cdot\|_{2\mu}$ of the corresponding closed-loop system. In the process of designing an H_2 optimal controller we try to minimize $J_\mu(G_c)$ for some $\mu \in \{1,2,3,4\}$.

Now we are in position to prove the following.

Theorem 7.6 *Assume that* (7.92) *has a stabilizing solution* $X_s = (X_s(1), X_s(2), \ldots, X_s(N))$ *that satisfies* (7.96). *Then*

$$\min_{G_c \in \mathcal{K}_s(G)} J_\mu(G_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_v^T(i_2)X_s(i_1)B_v(i_2)], \quad \mu \in \{1,2,3,4\}.$$

The optimal value is achieved for the zero-order controller

$$\tilde{G} : u_s(t) = F_s(\eta_t)x_s(t), \tag{7.98}$$

where $F_s(i), i \in \mathcal{D}$ *are as in* (7.95) *and* $x_s(t)$ *is the solution of* (7.94).

Proof. Let us remark that in the case of the zero-order controller (7.98) the corresponding closed-loop system is:

$$x_{cl}(t+1) = \left[A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F_s(\eta_t))\right]x_{cl}(t) \\ + B_v(\eta_t)v(t) \tag{7.99}$$
$$z_{cl}(t) = (C_z(\eta_t) + D_z(\eta_t)F_s(\eta_t))x_{cl}(t), \qquad t \geq 0.$$

On the other hand, by direct calculation one obtains that DTSRE-C (7.92) verified by X_s can be rewritten as

$$X_s(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)(A_k(i) + B_k(i)F_s(i))^T X_s(j)(A_k(i) + B_k(i)F_s(i)) \\ + (C_z(i) + D_z(i)F_s(i))^T(C_z(i) + D_z(i)F_s(i)), \qquad i \in \mathcal{D}. \tag{7.100}$$

One sees that the linear equation (7.91) corresponding to the closed-loop system (7.99) is just (7.100). Therefore the value of the corresponding performance is

$$J_\mu(\tilde{G}_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_v^T(i_2)X_s(i_1)B_v(i_2)]. \tag{7.101}$$

Let G_c be an arbitrary stabilizing controller of type (7.87). Let

$$X_{cl}(i) = \begin{pmatrix} X_{11}(i) & X_{12}(i) \\ X_{12}^T(i) & X_{22}(i) \end{pmatrix}$$

be a partition of the solution of (7.91) according to the partition (7.89) of the coefficients of the closed-loop system.

Using (7.89) we obtain the following partition of (7.91).

$$X_{11}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)[(A_k(i) + B_k(i)F_c(i))^T X_{11}(j)(A_k(i) + B_k(i)F_c(i)) \\ + B_{ck}^T(i)X_{12}^T(j)(A_k(i) + B_k(i)F_c(i)) \\ + (A_k(i) + B_k(i)F_c(i))^T X_{12}(j)B_{ck}(i) \\ + B_{ck}^T(i)X_{22}(j)B_{ck}(i)] + [C_z(i) + D_z(i)F_c(i)]^T[C_z(i) + D_z(i)F_c(i)]$$

$$X_{12}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)[(A_k(i) + B_k(i)F_c(i))^T X_{11}(j)B_k(i)C_c(i) \\ + B_{ck}^T(i)X_{12}^T(j)B_k(i)C_c(i) + (A_k(i) + B_k(i)F_c(i))^T X_{12}(j)A_{ck}(i) \\ + B_{ck}^T(i)X_{22}(j)A_{ck}(i)] + (C_z(i) + D_z(i)F_c(i))^T D_z(i)C_c(i)$$

$$X_{22}(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j)[C_c^T(i)B_k^T(i)X_{11}(j)B_k(i)C_c(i)$$
$$+ A_{ck}^T(i)X_{12}^T(j)B_k(i)C_c(i) + C_c^T(i)B_k^T(i)X_{12}(j)A_{ck}(i)$$
$$+ A_{ck}^T(i)X_{22}(j)A_{ck}(i)] + C_c^T(i)D_z^T(i)D_z(i)C_c(i). \qquad (7.102)$$

On the other hand the DTSRE-C (7.92) verified by the stabilizing solution X_s can be rewritten as

$$X_s(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j)[A_k(i) + B_k(i)F_c(i)]^T X_s(j)[A_k(i) + B_k(i)F_c(i)]$$
$$+ [C_z(i) + D_z(i)F_c(i)]^T [C_z(i) + D_z(i)F_c(i)]$$
$$- [F_s(i) - F_c(i)]^T \Delta(i)[F_s(i) - F_c(i)], \qquad (7.103)$$

where

$$\Delta(i) = D_z^T(i)D_z(i) + \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j)B_k^T(i)X_s(j)B_k(i) > 0. \qquad (7.104)$$

Set

$$\hat{\mathcal{X}}_{cl}(i) = \mathcal{X}_{cl}(i) - \begin{pmatrix} X_s(i) & 0 \\ 0 & 0 \end{pmatrix}, \qquad i \in \mathcal{D}.$$

Subtracting (7.103) from (7.102) and taking into account (7.95) and (7.104) one obtains that $\hat{\mathcal{X}}_{cl} = (\hat{\mathcal{X}}_{cl}(1), \hat{\mathcal{X}}_{cl}(2), \ldots, \hat{\mathcal{X}}_{cl}(N))$ is the solution of the following linear equation,

$$\hat{\mathcal{X}}_{cl}(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(i,j)A_{kcl}^T(i)\hat{\mathcal{X}}_{cl}(j)A_{kcl}(i) + \Psi^T(i)\Delta(i)\Psi(i), \qquad i \in \mathcal{D}, \qquad (7.105)$$

where $\Psi(i) = \begin{pmatrix} F_s(i) - F_c(i) & -C_c(i) \end{pmatrix}$. Because G_c is a stabilizing controller and $\Delta(i) > 0$, it follows that the unique solution of (7.105) satisfies

$$\hat{\mathcal{X}}_{cl}(i) \geq 0, \qquad i \in \mathcal{D}. \qquad (7.106)$$

The value of the performance $J_\mu(G_c)$ from (7.97) can be rewritten as

$$J_\mu(G_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_v(i_2)X_s(i_1)B_v(i_2)]$$
$$+ \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\hat{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)]. \qquad (7.107)$$

Based on (7.101), (7.106), and (7.107) one obtains that $J_\mu(G_c) \geq J_\mu(\tilde{G}_c)$ and thus the proof is complete. □

Remark 7.3 The result proved in the above theorem shows that in the case of full access to the measurements of the states, the best performance with respect to all four H_2 performance criteria is provided by the same zero-order controller. In fact it is the same state feedback that provides the optimal control in the linear quadratic optimization problem.

7.4.2 The case of systems with coefficients depending upon η_t and η_{t-1}

Let us consider the controlled system:

$$(\mathbf{G}) : \begin{cases} x(t+1) = \left(A_0(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t,\eta_{t-1})\right) x(t) + \Big(B_0(\eta_t,\eta_{t-1}) \\ \qquad\qquad + \sum_{k=1}^{r} w_k(t) B_k(\eta_t,\eta_{t-1})\Big) u(t) + B_v(\eta_t,\eta_{t-1}) v(t) \\ y(t) \qquad = x(t) \\ z(t) \qquad = C_z(\eta_t,\eta_{t-1}) x(t) + D_z(\eta_t,\eta_{t-1}) u(t), \qquad t \geq 1. \end{cases} \tag{7.108}$$

The class of admissible controllers consists of the family of dynamic compensators of the form:

$$(\mathbf{G}_c) \begin{cases} x_c(t+1) = \left(A_{c0}(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t) A_{ck}(\eta_t,\eta_{t-1})\right) x_c(t) \\ \qquad\qquad + \left(B_{c0}(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t) B_{ck}(\eta_t,\eta_{t-1})\right) u_c(t) \\ y_c(t) \qquad = C_c(\eta_{t-1}) x_c(t) + F_c(\eta_{t-1}) u_c(t) \\ \qquad\qquad x_c \in \mathbf{R}^{n_c}, \qquad u_c \in \mathbf{R}^{n}, \qquad y_c \in \mathbf{R}^{m}. \end{cases} \tag{7.109}$$

Coupling (7.109) with (7.108), taking $u_c(t) = y(t), u(t) = y_c(t)$, one obtains the following closed-loop system.

$$(\mathbf{G}_{cl}) : \begin{cases} x_{cl}(t+1) = \left(A_{0cl}(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t) A_{kcl}(\eta_t,\eta_{t-1})\right) x_{cl}(t) \\ \qquad\qquad + B_{vcl}(\eta_t,\eta_{t-1}) v(t) \\ z_{cl}(t) \qquad = C_{cl}(\eta_t,\eta_{t-1}) x_{cl}(t), \qquad t \geq 1, \end{cases} \tag{7.110}$$

where $x_{cl}(t) = \begin{pmatrix} x^T(t) & x_c^T(t) \end{pmatrix}^T \in \mathbf{R}^{n+n_c}$,

$$A_{kcl}(i,j) = \begin{pmatrix} A_k(i,j) + B_k(i,j)F_c(j) & B_k(i,j)C_c(j) \\ B_{ck}(i,j) & A_{ck}(i,j) \end{pmatrix}, \qquad 0 \leq k \leq r,$$

$$B_{vcl}(i,j) = \begin{pmatrix} B_v(i,j) \\ 0 \end{pmatrix},$$

$$C_{cl}(i,j) = \begin{pmatrix} C_z(i,j) + D_z(i,j)F_c(j) & D_z(i,j)C_c(j) \end{pmatrix}, \qquad i,j \in \mathcal{D}.$$

Let us remark that if we consider the system (7.86) with a controller of type (7.87) and a delay occurs on the channel between controllers and actuators (i.e., $y_c(t-1)$ is used instead of $y_c(t)$), then the closed-loop system is of the form (7.110). Hence it is natural to consider an H_2 control problem for the systems with coefficients depending upon η_t, η_{t-1}. Such a problem is specific to the discrete-time framework. It has no analogue in the continuous-time case.

As in the first part of this section we denote $\|\mathbf{G}_{cl}\|_{2\mu}, \mu \in \{1,2,3,4\}$ the four types of H_2 norms defined for the closed-loop system by (7.10), (7.11), (7.12), and (7.80). Based on Theorem 7.1, Theorem 7.2, and equality (7.80) one deduces that

$$\|\mathbf{G}_{cl}\|_{2\mu} = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_1,i_2)X_{cl}(i_1)B_{vcl}(i_1,i_2),$$
$$\mu \in \{1,2,3,4\}, \tag{7.111}$$

where $\varepsilon_\mu(i_2)$ are defined as before and $X_{cl} = (X_{cl}(1), X_{cl}(2), \ldots, X_{cl}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ is the unique solution of

$$X_{cl}(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(i,j)A_{kcl}^T(j,i)X_{cl}(j)A_{kcl}(j,i) + \sum_{j=1}^{N} p(i,j)C_{cl}^T(j,i)C_{cl}(j,i),$$
$$i \in \mathcal{D}. \tag{7.112}$$

As before we introduce the performances of an admissible controller (7.109) by

$$J_\mu(\mathbf{G}_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_1,i_2)X_{cl}(i_1)B_{vcl}(i_1,i_2)]. \tag{7.113}$$

It must be remarked that to be sure (7.113) is well defined we need to know that the assumptions $\mathbf{H}_1, \mathbf{H}_2$, and **A.7.1** are fulfilled and the zero state equilibrium of the linear closed-loop system

$$x_{cl}(t+1) = \left[A_{0cl}(\eta_t,\eta_{t-1}) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t,\eta_{t-1})\right] x_{cl}(t)$$

is ESMS. As we proceed in the first part of this section we minimize $J_\mu(\mathbf{G}_{cl})$ in order to obtain the solution of the H_2 optimization problem for systems of type (7.108).

To construct the solution of the H_2 optimization problems associated with the system (7.108) we use the stabilizing solution of the DTSRE-C (5.143).

The main result of this subsection is the following theorem.

Theorem 7.7 *Assume that the DTSRE-C* (5.143) *has a stabilizing solution* $X_s = (X_s(1), X_s(2), \dots, X_s(N))$ *which satisfies the condition* (5.155). *Then*

$$\min_{\mathbf{G}_c \in \mathcal{K}_s(\mathbf{G})} J_\mu(\mathbf{G}_c) \sum_{i_1, i_2 = 1}^{N} \varepsilon_\mu(i_2) p(i_2, i_1) Tr(B_v^T(i_1, i_2) X_s(i_1) B_v(i_1, i_2)).$$

The optimal value is achieved by the zero-order controller

$$\tilde{\mathbf{G}}_c : u_s(t) = F_s(\eta_{t-1}) x_s(t),$$

where $F_s(i), i \in \mathcal{D}$ *are constructed in* (5.146) *and* $x_s(t)$ *is the solution of the closed-loop system* (5.145).

Proof. It is similar to the one of Theorem 7.6 and is omitted. □

7.5 The H_2 optimal control. The case with partial access to measurements

7.5.1 Problem formulation

Consider a discrete-time controlled system (G) described by

$$(G): \begin{cases} x(t+1) = \left[A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)\right] x(t) + \left[B_0(\eta_t)\right. \\ \qquad\qquad \left. + \sum_{k=1}^{r} w_k(t) B_k(\eta_t)\right] u(t) + B_v(\eta_t) v(t) \\ y(t) \quad = \left[C_0(\eta_t) + \sum_{k=1}^{r} w_k(t) C_k(\eta_t)\right] x(t) + D_v(\eta_t) v(t) \\ z(t) \quad = C_z(\eta_t) x(t) + D_z(\eta_t) u(t), \end{cases} \tag{7.114}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^{m_u}$ is the vector of control inputs, and $y(t) \in \mathbf{R}^{n_y}$ is the measured output, whereas $z(t) \in \mathbf{R}^{n_z}$ is the controlled output, $\{w_k(t)\}_{t \geq 0}, 1 \leq k \leq r$ are sequences of random variables, $\{v(t)\}_{t \geq 0}$ is a sequence of m_v-dimensional random vectors, and $(\{\eta_t\}_{t \geq 0}, P, \mathcal{D})$ is an homogeneous Markov chain with the set of states $\mathcal{D} = \{1, 2, \dots, N\}$ and the transition probability matrix P. We assume that the assumptions $\mathbf{H_1}, \mathbf{H_2}$, and **A.7.1** are fulfilled.

It should be remarked that (7.114) is a hybrid system with the states $(x(t), i)$ where $x(t)$ is the vector of the state parameters of the system and i is the mode of the system, which evolves according to the Markov chain. In this section we assume that the pair $(y(t), i)$ is available at each time t.

In (7.114), $A_k(i) \in \mathbf{R}^{n\times n}, B_k(i) \in \mathbf{R}^{n\times m_u}, C_k(i) \in \mathbf{R}^{n_y\times n}$, $0 \le k \le r$, $B_v(i) \in \mathbf{R}^{n\times m_v}$, $C_z(i) \in \mathbf{R}^{n_z\times n}, D_v(i) \in \mathbf{R}^{n_y\times m_v}, D_z(i) \in \mathbf{R}^{n_z\times m_u}$, $1 \le i \le N$ are given matrices.

The family of admissible controllers consists of the dynamic compensators (G_c) of the form:

$$(G_c) : \begin{cases} x_c(t+1) = \left[A_{c0}(\eta_t) + \sum\limits_{k=1}^{r} w_k(t)A_{ck}(\eta_t)\right] x_c(t) + B_c(\eta_t)u_c(t) \\ y_c(t) \qquad\quad = C_c(\eta_t)x_c(t), \end{cases} \tag{7.115}$$

where $x_c(t) \in \mathbf{R}^{n_c}$ is the state of the controller, $u_c(t) \in \mathbf{R}^{n_y}$ is the input of the controller, and $y_c \in \mathbf{R}^{m_u}$ is the output of the controller. As in the case discussed in the previous section the order n_c of the controllers (7.115) is not prefixed. It is determined in the process of designing the optimal controller. Thus a controller (7.115) is described by the set of parameters $\{n_c, A_{ck}(i), 0 \le k \le r, B_c(i), C_c(i), i \in \mathcal{D}\}$. When coupling a controller (G_c) of type (7.115) to the system (G) by taking $u_c(t) = y(t), u(t) = y_c(t)$ one obtains the closed-loop system (G_{cl}):

$$(G_{cl}) : \begin{cases} x_{cl}(t+1) = \left[A_{0cl}(\eta_t) + \sum\limits_{k=1}^{r} w_k(t)A_{kcl}(\eta_t)\right] x_{cl}(t) + B_{vcl}(\eta_t)v(t) \\ z_{cl}(t) \qquad\quad = C_{cl}(\eta_t)x_{cl}(t), \end{cases} \tag{7.116}$$

where

$$x_{cl} = (x^T(t)x_c^T(t))^T, \qquad A_{kcl}(i) = \begin{pmatrix} A_k(i) & B_k(i)C_c(i) \\ B_c(i)C_k(i) & A_{ck}(i) \end{pmatrix},$$
$$0 \le k \le r$$
$$B_{vcl}(i) = \begin{pmatrix} B_v(i) \\ B_c(i)D_v(i) \end{pmatrix}, \qquad C_{cl}(i) = (C_z(i) \qquad D_z(i)C_c(i)). \tag{7.117}$$

A controller (G_c) is called a *stabilizing controller* if the zero state equilibrium of the linear system

$$x_{cl}(t+1) = \left[A_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t)\right] x_{cl}(t) \tag{7.118}$$

is exponentially stable in the mean square.

In the sequel $\mathcal{K}_s(G)$ stands for the family of stabilizing controllers (G_c) of type (7.115).

For a closed-loop system (G_{cl}), the H_2 norms (7.6)–(7.8) are:

$$\|G_{cl}\|_{2,1}^2 = \lim_{l\to\infty} \frac{1}{l} \sum_{t=0}^{l} E[|z_{cl}(t)|^2] \tag{7.119}$$

$$\|G_{cl}\|_{2,2}^2 = \lim_{l\to\infty} \frac{1}{l} \sum_{t=0}^{l} \sum_{i\in\mathcal{D}_0} E[|z_{cl}(t)|^2|\eta_0 = i] \tag{7.120}$$

$$\|G_{cl}\|_{2,3}^2 = \lim_{t\to\infty} E[|z_{cl}(t)|^2]. \tag{7.121}$$

In (7.119)–(7.121) $z_{cl}(t), t \geq 0$ is the output of the closed-loop system (G_{cl}). Also, based on (7.83) applied to the closed-loop system (7.116), one may introduce the norm $\|(G_{cl})\|_{24}$.

The aim of this section is to solve the following optimization problems.

$\mathbf{OP}_\mu : \mu \in \{1, 2, 3, 4\}$. Find a controller $(\tilde{G}_c(\mu)) \in \mathcal{K}_s(G)$ such that the corresponding closed-loop system $(\tilde{G}_{cl})$ satisfies

$$\|\tilde{G}_{cl}\|_{2,\mu} = \min_{G_c\in\mathcal{K}_s(G)} \|G_{cl}\|_{2,\mu}. \tag{7.122}$$

7.5.2 Some preliminaries

Based on the matrix coefficients $A_{kcl}(i)$ of the closed-loop system and the elements $p(i, j)$ of the stochastic matrix P we define the linear operator $\mathcal{L}_{cl} : \mathcal{S}_{n+n_c}^N \to \mathcal{S}_{n+n_c}^N$ by

$$(\mathcal{L}_{cl}X)(i) = \sum_{k=0}^{r} \sum_{j=1}^{N} p(j, i)A_{kcl}(j)X(j)A_{kcl}^T(j), \qquad 1 \leq i \leq N, \tag{7.123}$$

for all $X = (X(1), X(2), \ldots, X(N)) \in \mathcal{S}_{n+n_c}^N$.

Applying Theorems 7.3 and 7.4 to the H_2 norms (7.119)–(7.121) one obtains the following.

Corollary 7.2 *Assume:*

(a) *The assumptions* $\mathbf{H}_1, \mathbf{H}_2$ *and* **A.7.1** *are fulfilled.*
(b) *The zero state equilibrium of the system* (7.118) *is ESMS.*

Under these conditions the H_2 norms defined by (7.119) *and* (7.120) *are given by*

(i)

$$\begin{aligned}\|G_{cl}\|_{2,1}^2 &= \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2, i_1)Tr[\tilde{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)B_{vcl}^T(i_2)] \\ &= \sum_{i=1}^{N} Tr[C_{cl}(i)\mathcal{Y}_{cl}^{\pi_0}(i)C_{cl}^T(i)],\end{aligned}$$

(*ii*)

$$\|G_{cl}\|_{2,2}^2 = \sum_{i_1,i_2=1}^{N} q^{\mathcal{D}_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)B_{vcl}^T(i_2)]$$
$$= \sum_{i=1}^{N} Tr[C_{cl}(i)\mathcal{Y}_{cl}^{\mathcal{D}_0}(i)C_{cl}^T(i)],$$

where $\tilde{\mathcal{X}}_{cl} = (\tilde{\mathcal{X}}_{cl}(1), \tilde{\mathcal{X}}_{cl}(2), \ldots, \tilde{\mathcal{X}}_{cl}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ *is the unique solution of the algebraic equation*

$$\mathcal{X}_{cl} = \mathcal{L}_{cl}^* \mathcal{X}_{cl} + \tilde{\mathcal{C}}_{cl}, \tag{7.124}$$

where $\tilde{\mathcal{C}}_{cl} = (\tilde{\mathcal{C}}_{cl}(1), \tilde{\mathcal{C}}_{cl}(2), \ldots, \tilde{\mathcal{C}}_{cl}(N)$,

$$\tilde{\mathcal{C}}_{cl}(i) = C_{cl}^T(i)C_{cl}(i), \qquad i \in \mathcal{D}, \tag{7.125}$$

$\mathcal{Y}_{cl}^{\pi_0} = (\mathcal{Y}_{cl}^{\pi_0}(1), \mathcal{Y}_{cl}^{\pi_0}(2), \ldots, \mathcal{Y}_{cl}^{\pi_0}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ *and* $\mathcal{Y}_{cl}^{\mathcal{D}_0} = (\mathcal{Y}_{cl}^{\mathcal{D}_0}(1), \mathcal{Y}_{cl}^{\mathcal{D}_0}(2), \ldots, \mathcal{Y}_{cl}^{\mathcal{D}_0}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ *are the unique solutions of the algebraic equations*

$$\mathcal{Y}_{cl} = \mathcal{L}_{cl}\mathcal{Y}_{cl} + \mathcal{B}_{cl}^{\pi_0} \tag{7.126}$$

$$\mathcal{Y}_{cl} = \mathcal{L}_{cl}\mathcal{Y}_{cl} + \mathcal{B}_{cl}^{\mathcal{D}_0}, \tag{7.127}$$

where $\mathcal{B}_{cl}^{\pi_0} = (\mathcal{B}_{cl}^{\pi_0}(1), \mathcal{B}_{cl}^{\pi_0}(2), \ldots, \mathcal{B}_{cl}^{\pi_0}(N))$,

$$\mathcal{B}_{cl}^{\pi_0}(i) = \sum_{j=1}^{N} q^{\pi_0}(j)p(j,i)B_{vcl}(j)B_{vcl}^T(j), \qquad i \in \mathcal{D} \tag{7.128}$$

and $\mathcal{B}_{cl}^{\mathcal{D}_0} = (\mathcal{B}_{cl}^{\mathcal{D}_0}(1), \mathcal{B}_{cl}^{\mathcal{D}_0}(2), \ldots, \mathcal{B}_{cl}^{\mathcal{D}_0}(N)$,

$$\mathcal{B}_{cl}^{\mathcal{D}_0}(i) = \sum_{j=1}^{N} q^{\mathcal{D}_0}(j)p(j,i)B_{vcl}(j)B_{vcl}^T(j), \qquad i \in \mathcal{D}, \tag{7.129}$$

$q^{\pi_0}(j) = \sum_{i=1}^{N} \pi_0(i)q(i,j)$ *and* $q^{\mathcal{D}_0}(j) = \sum_{i \in \mathcal{D}_0} q(i,j)$.

Corollary 7.3 *Assume:*

(*a*) *Assumptions* $\mathbf{H}_1, \mathbf{H}_2$, **A.7.1** *and* **A.7.2** *are fulfilled.*
(*b*) *The zero state equilibrium of the system* (7.118) *is ESMS.*

Under these conditions the H_2 *norm of the system* (7.116) *defined by* (7.121) *is given by*

$$\|G_{cl}\|_{2,3}^2 = \sum_{j=1}^{N} Tr[C_{cl}(j)\mathcal{Y}_{cl}^{\pi_0}(j)C_{cl}^T(j)]$$
$$\times \sum_{i_1,i_2=1}^{N} q^{\pi_0}(i_2)p(i_2,i_1)Tr[\tilde{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)B_{vcl}^T(i_2)],$$

where $\mathcal{Y}_{cl}^{\pi_0}$ *is the unique solution of the equation* (7.126), (7.128) *and* $\tilde{\mathcal{X}}$ *is the unique solution of the equation* (7.124)–(7.125) *and* q^{π_0} *is defined as before.*

One sees that the values of the H_2 norms of the closed-loop system (7.116) are strongly dependent either upon the initial distributions π_0 of the Markov chain or upon the subset $\mathcal{D}_0$ of the states $i \in \mathcal{D}$, such that $\mathcal{P}\{\eta_0 = i\} > 0$. To overcome the difficulties due to the dependence of the coefficients of the system upon a Markov chain, we consider a new performance:

$$\|G_{cl}\|_{2,4}^2 = \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\tilde{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)], \tag{7.130}$$

where

$$\tilde{q}(i_2) = \sum_{i_1=1}^{N} q(i_1,i_2), \tag{7.131}$$

$q(i_1,i_2)$ being the entries of the matrix Q introduced by the Proposition 1.5.

Because $q^{\pi_0}(i_2) \le \tilde{q}(i_2)$ and $q^{\mathcal{D}_0}(i_2) \le \tilde{q}(i_2)$ we deduce that $\|G_{cl}\|_{2,\mu} \le \|G_{cl}\|_{2,4}, \mu \in \{1,2,3\}$. Hence the minimization of $\|G_{cl}\|_{2,4}$ may lead to satisfactory suboptimal values for $\|G_{cl}\|_{2,\mu}, \mu \in \{1,2,3\}$. Reasoning as in the proof of Theorem 7.1, one obtains the following new expression for $\|G_{cl}\|_{2,4}$ introduced by (7.130),

$$\begin{aligned}\|G_{cl}\|_{2,4}^2 &= \sum_{i_1,i_2=1}^{N} \tilde{q}(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\tilde{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)] \\ &= \sum_{i=1}^{N} Tr[C_{cl}(i)\tilde{\mathcal{Y}}_{cl}(i)C_{cl}^T(i)]\end{aligned} \tag{7.132}$$

with $\tilde{\mathcal{Y}}_{cl} = (\tilde{\mathcal{Y}}_{cl}(1),\dots,\tilde{\mathcal{Y}}_{cl}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ is the unique solution of the linear equation

$$\mathcal{Y} = \mathcal{L}_{cl}\mathcal{Y} + \tilde{\mathcal{B}} \tag{7.133}$$

with $\tilde{\mathcal{B}} = (\tilde{\mathcal{B}}(1), \tilde{\mathcal{B}}(2), \dots, \tilde{\mathcal{B}}(N))$,

$$\tilde{\mathcal{B}}(i) = \sum_{j=1}^{N} \tilde{q}(j)p(j,i)B_{vcl}(j)B_{vcl}^T(j). \tag{7.134}$$

For a controller $G_c \in \mathcal{K}_s(G)$ we introduce the performances $J_\mu(G_c), \mu \in \{1,2,3,4\}$, as follows.

$$J_\mu(G_c) = \sum_{i_1,i_2=1}^{N} \varepsilon_\mu(i_2)p(i_2,i_1)Tr[B_{vcl}^T(i_2)\tilde{\mathcal{X}}_{cl}(i_1)B_{vcl}(i_2)], \qquad \mu \in \{1,2,3,4\}, \tag{7.135}$$

where

$$\varepsilon_\mu(i) = q^{\pi_0}(i), \quad \text{if } \mu \in \{1,3\},$$
$$\varepsilon_\mu(i) = q^{\mathcal{D}_0}(i), \quad \text{if } \mu = 2,$$
$$\varepsilon_\mu(i) = \tilde{q}(i), \quad \text{if } \mu = 4, \tag{7.136}$$

$\tilde{\mathcal{X}}_{cl} = (\tilde{\mathcal{X}}_{cl}(1), \tilde{\mathcal{X}}_{cl}(2), \dots, \tilde{\mathcal{X}}_{cl}(N)) \in \mathcal{S}_{n+n_c}^{N+}$ is the unique solution of equation (7.124)–(7.125). Based on Corollaries 7.2 and 7.3 one sees that under some additional assumptions $J_\mu(G_c)$ coincides with the H_2 norm $\|G_{cl}\|_{2,\mu}^2$. Therefore, to solve the H_2 optimization problems stated before we design stabilizing controllers $\tilde{G}_c \in \mathcal{K}_s(G)$ that minimize the performance $J_\mu(G_c)$. Finally it is worth pointing out the following equality that can be derived from the above results,

$$\sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_{vcl}^T(j)\tilde{\mathcal{X}}_{cl}(i)B_{vcl}(j)] = \sum_{i=1}^{N} Tr[C_{cl}(i)\mathcal{Y}_{cl}^\mu(i)C_{cl}(i)] \tag{7.137}$$

with $\mu \in \{1,2,3,4\}$, where $\mathcal{Y}_{cl}^\mu(i) = \mathcal{Y}_{cl}^{\pi_0}(i)$ for $i \in \{1,3\}$, $\mathcal{Y}_{cl}^\mu(i) = \mathcal{Y}_{cl}^{\mathcal{D}_0}$ for $\mu = 2$, and $\mathcal{Y}_{cl}^\mu(i) = \tilde{\mathcal{Y}}_{cl}(i)$ for $\mu = 4$.

7.5.3 The solution of the H_2 optimization problems

In this subsection we construct the state space realization of a controller $\tilde{G}_c(\mu) \in \mathcal{K}_s(G)$ that minimizes the cost $J_\mu(G_c)$ in the class of stabilizing controllers $\mathcal{K}_s(G)$.

In the applications, one may choose one of the performances $J_\mu(\cdot)$, $\mu \in \{1,2,3,4\}$ depending upon the available information.

To begin we prove the following.

Lemma 7.6 *If $X_s = (X_s(1), X_s(2), \dots, X_s(N))$ is the stabilizing solution of DTSRE-C* (7.92) *then for any controller $G_c \in \mathcal{K}_s(G)$ and any $\mu \in \{1,2,3,4\}$ we have:*

$$J_\mu(G_c) = \sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_v^T(j)X_s(i)B_v(j)] + \sum_{i=1}^{N} Tr[\check{C}_{cl}(i)\mathcal{Y}_{cl}^\mu(i)\check{C}_{cl}^T(i)],$$

where

$$\check{C}_{cl}(i) = V(i)(F_s(i) - C_c(i)). \tag{7.138}$$

$F_s(i)$ is the stabilizing feedback gain defined in (7.95)*,*

$$V(i) = \left[D_z^T(i)D_z(i) + \sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i)\right]^{1/2}, \qquad 1 \le i \le N; \tag{7.139}$$

$Y_{cl}^\mu(i)$ is as in (7.138)*, $\mu \in \{1,2,3,4\}$.*

Proof. Let $X_{cl} = (X_{cl}(1), X_{cl}(2), \ldots, X_{cl}(N))$ be the solution of (7.124)–(7.125). Define

$$U(i) = X_{cl}(i) - \begin{pmatrix} X_s(i) & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.140}$$

By direct calculations based on (7.92) and (7.95) one obtains that $U = (U(1), \ldots, U(N))$ solves the linear equation on $\mathcal{S}^N_{n+n_c}$:

$$U = \mathcal{L}^*_{cl} U + \check{\mathcal{M}} \tag{7.141}$$

with $\check{\mathcal{M}} = (\check{\mathcal{M}}(1), \check{\mathcal{M}}(2), \ldots, \check{\mathcal{M}}(N))$, $\check{\mathcal{M}}(i) = \check{C}^T_{cl}(i)\check{C}_{cl}(i)$, $\check{C}_{cl}(i)$ being defined by (7.138).

Because G_c is a stabilizing controller, one obtains, via Theorem 2.5(iii) and (iv) that the equation (7.141) has a unique solution and that solution is in $\mathcal{S}^{N+}_{n+n_c}$.

From (7.135) one gets:

$$\begin{aligned} J_\mu(G_c) = &\sum_{i,j=1}^{N} \varepsilon_\mu(j) p(j,i) Tr[B_v^T(j) X_s(i) B_v(j)] \\ &+ \sum_{i,j=1}^{N} \varepsilon_\mu(j) p(j,i) Tr[B_{vcl}^T(j) U(i) B_{vcl}(j)]. \end{aligned} \tag{7.142}$$

The conclusion now follows from (7.142) applying the equality (7.137) with $U(i)$ instead of $\mathcal{X}_{cl}(i)$, $\check{C}_{cl}(i)$ instead of $C_{cl}(i)$. Thus the proof is complete. □

The main result of this section is as follows.

Theorem 7.8 *Assume that for some* $\mu \in \{1,2,3,4\}$ *assertion (i) in Theorem* 5.13 *applied to the DTSRE-C* (7.92) *and assertion (i) in Theorem* 5.16 *hold. Let* $X_s = (X_s(1), X_s(2), \ldots, X_s(N))$ *and* $Y_s = (Y_s(1), Y_s(2), \ldots, Y_s(N))$ *be the stabilizing solution of DTSRE-C* (7.92) *and DTSRE-F* (5.156), *respectively.*

We construct the controller $\tilde{G}_c(\mu)$ *described by*

$$\begin{aligned} x_c(t+1) = &\Bigg[A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + K_s(\eta_t)C_0(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) \\ &+ B_k(\eta_t)F_s(\eta_t) + K_s(\eta_t)C_k(\eta_t))\Bigg] x_c(t) - K_s(\eta_t)u_c(t) \end{aligned} \tag{7.143}$$

$$y_c(t) = F_s(\eta_t)x_c(t),$$

where $F_s(i)$ and $K_s(i)$ are as in (7.95) *and* (5.158)*, respectively. Under the considered assumptions $\tilde{G}_c(\mu) \in \mathcal{K}_s(G)$ and $J_\mu(\tilde{G}_c(\mu)) \le J_\mu(G_c)$ for any $G_c \in \mathcal{K}_s(G)$.*

Moreover, the optimal value of the performance is:

$$J_\mu(\tilde{G}_c(\mu)) = \sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_v^T(j)X_s(i)B_v(j)] + \sum_{i=1}^{N} Tr[V(i)F_s(i)Y_s(i)F_s^T(i)V(i)]. \tag{7.144}$$

Proof. Let us consider the closed-loop system obtained by coupling (7.114) with (7.143) taking $u(t) = y_c(t)$, $u_c(t) = y(t)$:

$$x_{cl}(t+1) = \left[\tilde{A}_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t)\tilde{A}_{kcl}(\eta_t)\right] x_{cl}(t) + \tilde{B}_{vcl}(\eta_t)v(t)$$

$$z_{cl}(t) = \tilde{C}_{cl}(\eta_t)x_{cl}(t).$$

To check that $\tilde{G}_c(\mu) \in \mathcal{K}_s(G)$ we have to show that the zero state equilibrium of the linear system

$$x_{cl}(t+1) = \left[\tilde{A}_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t)\tilde{A}_{kcl}(\eta_t)\right] x_{cl}(t) \tag{7.145}$$

is ESMS. To this end, in (7.145) we make the transformation:

$$\zeta(t) = \begin{pmatrix} \zeta_1(t) \\ \zeta_2(t) \end{pmatrix} = \mathcal{T} x_{cl}(t), \qquad \text{where } \mathcal{T} = \begin{pmatrix} I_n & 0 \\ I_n & -I_n \end{pmatrix}. \tag{7.146}$$

It is easy to see that $\mathcal{T}^{-1} = \mathcal{T}$. By direct calculation one obtains that

$$\hat{A}_{kcl}(i) = \mathcal{T}\tilde{A}_{kcl}(i)\mathcal{T}^{-1} = \begin{pmatrix} A_k(i) + B_k(i)F_s(i) & -B_k(i)F_s(i) \\ 0 & A_k(i) + K_s(i)C_k(i) \end{pmatrix}, \tag{7.147}$$

$1 \le i \le N$, $0 \le k \le r$.

Thus (7.145) becomes:

$$\zeta_1(t+1) = \left[A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F_s(\eta_t))\right] \zeta_1(t) - \left[B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t)F_s(\eta_t)\right] \zeta_2(t), \tag{7.148}$$

$$\zeta_2(t) = \left[A_0(\eta_t)+K_s(\eta_t)C_0(\eta_t)+\sum_{k=1}^{r} w_k(t)(A_k(\eta_t)+K_s(\eta_t)C_k(\eta_t))\right]\zeta_2(t).$$

Let $\Phi_{F_s}(t,t_0)$ and $\Phi_{K_s}(t,t_0)$ be the fundamental matrix solution of (7.94) and (5.157), respectively. Because X_s and Y_s are stabilizing solutions of (7.92) and (5.156), respectively, it follows that there exist $\beta \geq 1$, $q \in (0,1)$ such that

$$E[|\Phi_{F_s}(t,0)\zeta_1(0)|^2] \leq \beta q^t |\zeta_1(0)|^2 \tag{7.149}$$

$$E[|\Phi_{K_s}(t,0)\zeta_2(0)|^2] \leq \beta q^t |\zeta_2(0)|^2 \tag{7.150}$$

for all $\zeta_1(0), \zeta_2(0) \in \mathbf{R}^n$, $t \geq 0$.

From the second equation of (7.148) one obtains $\zeta_2(t) = \Phi_{K_s}(t,0)\zeta_2(0)$, $t \geq 0$. Hence

$$E[|\zeta_2(t)|^2] \leq \beta q^t |\zeta_2(0)|^2 \tag{7.151}$$

for all $t \geq 0$, $\zeta_2(0) \in \mathbf{R}^n$. Furthermore, using (7.149) and (7.151) one obtains via Corollary 3.9(iii), that $\lim_{t\to\infty} E[|\zeta_1(t)|^2] = 0$ for $\zeta_1(0) \in \mathbf{R}^n, \zeta_2(0) \in \mathbf{R}^n$. From

$$x_{cl}(t) = \mathcal{T}^{-1}\begin{pmatrix}\zeta_1(t)\\ \zeta_2(t)\end{pmatrix}$$

we deduce $\lim_{t\to\infty} E[|x_{cl}(t)|^2] = 0$. Finally invoking the implication $(v) \to (i)$ in Theorem 3.10 in the special case $\theta = 1$, we conclude that the zero state equilibrium of (7.145) is ESMS. This confirms that $\tilde{G}_c(\mu) \in \mathcal{K}_s(G)$.

Let $\mathcal{Y}^{\mu}_{cl} = (\mathcal{Y}^{\mu}_{cl}(1), \mathcal{Y}^{\mu}_{cl}(2), \ldots, \mathcal{Y}^{\mu}_{cl}(N))$ be as in (7.137) corresponding to an arbitrary controller $G_c \in \mathcal{K}_s(G)$. Set

$$\hat{U}(i) = \mathcal{Y}^{\mu}_{cl}(i) - \begin{pmatrix} Y_s(i) & 0\\ 0 & 0\end{pmatrix}, \qquad 1 \leq i \leq N.$$

By direct calculations based on (5.156) and (5.158) together with the equations verified by $\mathcal{Y}^{\mu}_{cl}$, one gets that $\hat{U} = (\hat{U}(1), \hat{U}(2), \ldots, \hat{U}(N))$ solves the following linear equation on $\mathcal{S}^N_{n+n_c}$,

$$\hat{U} = \mathcal{L}_{cl}\hat{U} + \check{\mathcal{B}}^{\mu}_{cl}, \tag{7.152}$$

where $\check{\mathcal{B}}^{\mu}_{cl} = (\check{\mathcal{B}}^{\mu}_{cl}(1), \check{\mathcal{B}}^{\mu}_{cl}(2), \ldots, \check{\mathcal{B}}^{\mu}_{cl}(N))$ with $\check{\mathcal{B}}^{\mu}_{cl}(i) = \mathcal{E}^*_i(\Delta^{\mu}_{cl})$ and

$$\Delta^{\mu}_{cl}(i)\begin{pmatrix} K_s(i)\\ -B_c(i)\end{pmatrix}\hat{V}^2(i)\begin{pmatrix} K_s(i)\\ -B_c(i)\end{pmatrix}^T;$$

$$\hat{V}(i) = (\varepsilon_{\mu}(i)D_v(i)D_v^T(i) + \sum_{k=0}^{r} C_k(i)Y_s(i)C_k^T(i))^{1/2}.$$

The eigenvalues of the operator $\mathcal{L}_{cl}$ are located in the inside of the disk $|\lambda| < 1$, therefore we deduce that the unique solution of equation (7.152) lies in $\mathcal{S}_{n+n_c}^{N+}$. Furthermore, the equality from Lemma 7.6 may be rewritten:

$$\begin{aligned} J_\mu(G_c) = &\sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_v^T(j)X_s(i)B_v(j)] \\ &+ \sum_{i=1}^{N} Tr[V(i)F_s(i)Y_s(i)F_s(i)^T V(i)] \\ &+ \sum_{i=1}^{N} Tr[\check{C}_{cl}(i)\hat{U}(i)\check{C}_{cl}(i)^T]. \end{aligned}$$

Because $\hat{U}(i) \geq 0$ we deduce that

$$\begin{aligned} J_\mu(G_c) \geq &\sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_v^T(j)X_s(i)B_v(j)] \\ &+ \sum_{i=1}^{N} Tr[V(i)F_s(i)Y_s(i)F_s(i)^T V(i)]. \end{aligned} \tag{7.153}$$

We remark that the right-hand side of (7.153) does not depend upon the choice of the stabilizing controller. It remains to show that for the controller $\tilde{G}_c$ with the state space realization given by (7.143), we have equality in (7.153). To this end we remark that in the case of the controller (7.143) we have:

$$\check{C}_{cl}(i)\hat{U}(i)\check{C}_{cl}^T(i) = V(i)F_s(i)(\, I_n \; -I_n \,)\hat{U}(i)\begin{pmatrix} I_n \\ -I_n \end{pmatrix} F_s^T(i)V(i). \tag{7.154}$$

It can be remarked that

$$(I_n \quad -I_n)\hat{U}(i)\begin{pmatrix} I_n \\ -I_n \end{pmatrix}$$

is the 2×2 block of the matrix $Z(i) = \mathcal{T}\hat{U}(i)\mathcal{T}^T$, where $\mathcal{T}$ is as in (7.146).

Based on (7.123) and (7.152) we obtain the following equation verified by $Z = (Z(1), Z(2), \dots, Z(N))$,

$$Z(i) = \sum_{k=0}^{r}\sum_{j=1}^{N} p(j,i)[\hat{A}_{kcl}(j)Z(j)\hat{A}_{kcl}^T(j) + \overline{\mathcal{B}}_{cl}^{\mu}(i)], \tag{7.155}$$

where $\hat{A}_{kcl}(j)$ are as in (7.147), and

$$\overline{\mathcal{B}}_{cl}^{\mu}(i) = \mathcal{T}\check{\mathcal{B}}_{cl}^{\mu}(i)\mathcal{T}^T = \begin{pmatrix} \mathcal{E}_i^*(K_s\hat{V}^2K_s^T) & 0 \\ 0 & 0 \end{pmatrix}.$$

Setting

$$Z(i) = \begin{pmatrix} Z_{11}(i) & Z_{12}(i) \\ Z_{12}^T(i) & Z_{22}(i) \end{pmatrix}$$

one obtains from (7.155) the following equation,

$$Z_{22} = \mathcal{L}_{K_s} Z_{22}, \tag{7.156}$$

where $\mathcal{L}_{K_s}$ is the linear Lyapunov operator associated with the system (5.157).

The eigenvalues of the operator $\mathcal{L}_{K_s}$ are in the inside of the disk $|\lambda| < 1$, therefore we deduce that (7.156) has only the solution $Z_{22} = 0$. Thus we prove that in the case of the controller (7.143) we have $\sum_{i=1}^{N} Tr[\check{C}_{cl}(i)\hat{U}(i)\check{C}_{cl}^T(i)] = 0$. This shows that in the case of the controller (7.143), the inequality (7.153) becomes equality. This also confirms the validity of (7.144). Thus the proof is complete. □

Remark 7.4 In the special case $N = 1, A_k(1) = 0, B_k(1) = 0, C_k(1) = 0, 1 \leq k \leq r$, the controller (7.143) reduces to the well-known Kalman–Bucy filter, which is the solution of the H_2 control problem for deterministic systems. In the stochastic framework, if $N = 1$ the fact that the controller (7.143) is optimal is proved in [65].

If $N \geq 2$ but $A_k(i) = 0, B_k(i) = 0, C_k(i) = 0, 1 \leq k \leq r, i \in \mathcal{D}$, the result proved in Theorem 7.8 reduces to the one derived in [26]; see also [27]. Unfortunately, if $A_k(i) \neq 0$, for some $k \in \{1, 2, \ldots, r\}$, $i \in \{1, 2, \ldots, N\}$ then the controllers of type (7.143) are hard to implement due to the presence of white noise type perturbation in their coefficients. However, we consider that the result proved in Theorem 7.8 is useful even if multiplicative white noise perturbations are presented in the coefficients of the controlled system. Then equality (7.144) provides the best H_2 performance that can be achieved by a stabilizing controller. The right-hand side of (7.144) can be used to evaluate the performances of some suboptimal controllers with prescribed structure. This is illustrated in the next section where we design a suboptimal controller in the state estimator form.

7.6 H_2 suboptimal controllers in a state estimator form

In this section we focus our attention on the class of stabilizing controllers of the form:

$$x_c(t+1) = [A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + L(\eta_t)C_0(\eta_t)]x_c(t) - L(\eta_t)u_c(t) \tag{7.157}$$

$$y_c(t) = F_s(\eta_t)x_c(t),$$

where the matrix gains $L(i) \in \mathbf{R}^{n \times n_y}$ are unknown and they have to be chosen in order to obtain a suboptimal H_2 performance; $F_s(i)$ are given by (7.95).

It is clear that the controllers of type (7.157) are special cases of the controllers (7.115). Hence, for a controller (7.157) the equality given by Lemma 7.6 holds. Our aim is to describe a procedure that allows us to choose the matrix gains $L(i)$, $1 \le i \le N$, such that the deviation of $J_\mu(G_c)$ from the right-hand side of (7.144) is smaller than a prescribed level $\gamma > 0$.

Throughout this section we suppose that DTSRE-C (7.92) has a stabilizing solution X_s.

The coefficients (7.117) of the closed-loop system obtained by coupling a controller (7.157) to the system (7.114) by taking $u_c(t) = y(t)$ and $u(t) = y_c(t)$ are given by

$$
\begin{aligned}
A_{0cl}(i) &= \begin{pmatrix} A_0(i) & B_0(i)F_s(i) \\ -L(i)C_0(i) & A_0(i) + B_0(i)F_s(i) + L(i)C_0(i) \end{pmatrix} \\
A_{kcl}(i) &= \begin{pmatrix} A_k(i) & B_k(i)F_s(i) \\ -L(i)C_k(i) & 0 \end{pmatrix}, \qquad 1 \le k \le r, \qquad (7.158) \\
B_{vcl}(i) &= \begin{pmatrix} B_v(i) \\ -L(i)D_v(i) \end{pmatrix}.
\end{aligned}
$$

Applying Lemma 7.6 for a stabilizing controller of type (7.157) one obtains:

$$
\begin{aligned}
J_\mu(G_c) = &\sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_v(j)X_s(i)B_v(j)] \\
&+ \sum_{i=1}^{N} Tr[V(i)F_s(i)\mathcal{J}\mathcal{Y}_{cl}^\mu(i)\mathcal{J}^T F_s^T(i)V(i)], \qquad (7.159)
\end{aligned}
$$

where $\mathcal{J} = (\, I_n \quad -I_n \,)$, $\mathcal{Y}_{cl}^\mu = (\mathcal{Y}_{cl}^\mu(1), \mathcal{Y}_{cl}^\mu(2), \dots, \mathcal{Y}_{cl}^\mu(N)) \in \mathcal{S}_{2n}^{N+}$ is the unique solution of the linear equation

$$
\mathcal{Y}_{cl}^\mu = \mathcal{L}_{cl}\mathcal{Y}_{cl}^\mu + \mathcal{B}_{cl}^\mu \qquad (7.160)
$$

with

$$
\mathcal{B}_{cl}^\mu(i) = \sum_{j=1}^{N} \varepsilon_\mu(j)p(j,i)B_{vcl}(j)B_{vcl}^T(j), \qquad 1 \le i \le N, \qquad (7.161)
$$

$\mathcal{L}_{cl}$ being the linear operator defined by (7.123) for the coefficients $A_{kcl}(i)$ given by (7.158).

Let us consider the following linear inequality on $\mathcal{S}_{2n}^N$,

$$
\mathcal{L}_{cl}W - W + \mathcal{B}_{cl}^\mu < 0. \qquad (7.162)
$$

Because the eigenvalues of the operator $\mathcal{L}_{cl}$ are in the inside of the disk $|\lambda| < 1$ then any solution $W = (W(1), W(2), \ldots, W(N))$ of (7.162) with $W(i) \geq 0$ will satisfy

$$W(i) \geq \mathcal{Y}_{cl}^{\mu}(i), \qquad 1 \leq i \leq N. \tag{7.163}$$

This shows that

$$J_\mu(G_c) \leq \sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_v^T(j)X_s(i)B_v(j)] + \sum_{i=1}^{N} Tr[V(i)F_s(i)\mathcal{J}W(i)\mathcal{J}^T F_s^T(i)V(i)]. \tag{7.164}$$

Thus, to find a suboptimal controller of the form (7.157) we may proceed in two ways.

1. Given a prescribed level

$$\gamma > J_\mu^{\text{opt}} := \sum_{i,j=1}^{N} \varepsilon_\mu(j)p(j,i)Tr[B_v^T(j)X_s(i)B_v(j)] + \sum_{i=1}^{N} Tr[V(i)F_s(i)Y_s(i)F_s^T(i)V(i)] \tag{7.165}$$

find the matrix gains $L(i)$, $1 \leq i \leq N$, such that the corresponding LMI (7.162) has a solution $W = (W(1), W(2), \ldots, W(N))$, $W(i) > 0$ satisfying

$$\sum_{i=1}^{N} Tr[V(i)F_s(i)\mathcal{J}W(i)\mathcal{J}^T F_s^T(i)V(i)] - \sum_{i=1}^{N} Tr[V(i)F_s(i)Y_s(i)F_s^T(i)V(i)] < \gamma. \tag{7.166}$$

It is worth mentioning that if the gains $L(i)$ are such that the linear inequality (7.162) has a solution $W > 0$, then one obtains via Corollary 3.3 that (7.157) is a stabilizing controller.

2. Solve the following minimization problem,

$$\min_{L(i),W(i)} \sum_{i=1}^{N} Tr[V(i)F_s(i)\mathcal{J}W(i)\mathcal{J}^T F_s^T(i)V(i)] \tag{7.167}$$

subject to (7.162) and $W(i) > 0, 1 \leq i \leq N$.

Furthermore we show how we can separate the computation of the variables $W(i)$ and $L(i)$ in (7.162) in connection with (7.166) and (7.167), respectively.

To this end, let us remark that $\mathcal{J}W(i)\mathcal{J}^T = Z_{11}(i)$ is the (1,1) block of the matrix $Z(i) = \check{\mathcal{T}}W(i)\check{\mathcal{T}}^T$, where

$$\check{\mathcal{T}} = \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix}. \tag{7.168}$$

If $A_{kcl}(i)$, $B_{vcl}(i)$ are as in (7.158) and $\check{A}_{kcl}(i) = \check{\mathcal{T}}A_{kcl}(i)\check{\mathcal{T}}^{-1}$, $\check{B}_{vcl}(i) = \check{\mathcal{T}}B_{vcl}$, $0 \le k \le r$, $1 \le i \le N$, then one obtains that $Z = (Z(1), Z(2), \ldots, Z(N))$ solves the following system of linear inequalities,

$$\sum_{j=1}^{N} p(j,i)\left\{\sum_{k=0}^{r} \check{A}_{kcl}(j)Z(j)\check{A}_{kcl}^T(j) + \varepsilon_\mu(j)\check{B}_{vcl}(j)\check{B}_{vcl}^T(j)\right\}$$
$$- Z(i) < 0, \qquad 1 \le i \le N. \tag{7.169}$$

Lemma 7.7 *Assume that P is a nondegenerate stochastic matrix and DTSRE-C* (7.92) *has a stabilizing solution X_s. Then for any matrix gains $L(i)$, $1 \le i \le N$, the following are equivalent.*

(*i*) *The system of LMI* (7.169) *has a solution $Z = (Z(1), Z(2), \ldots, Z(N)) \in \mathcal{S}_{2n}^N$, $Z(i) > 0$, $1 \le i \le N$.*
(*ii*) *The system of LMIs on $\mathcal{S}_{2n}^N$,*

$$\sum_{k=0}^{r} \check{A}_{kcl}(i)\mathcal{E}_i^*(R)\check{A}_{kcl}^T(i) + \varepsilon_\mu(i)\check{B}_{vcl}(i)\check{B}_{vcl}^T)(i) - R(i) < 0, \qquad 1 \le i \le N, \tag{7.170}$$

has a solution $R = (R(1), R(2), \ldots, R(N))$ with $R(i) > 0$, $1 \le i \le N$. Moreover if $R = (R(1), R(2), \ldots, R(N))$ is a solution of (7.170) *then $Z = (Z(1), Z(2), \ldots, Z(N))$ with*

$$Z(i) = \sum_{j=1}^{N} p(j,i)R(j) \tag{7.171}$$

is a solution of (7.169).

Proof. Let $Z = (Z(1), Z(2), \ldots, Z(N)) \in \mathcal{S}_{2n}^N$ be a solution of (7.169) with $Z(i) > 0$, $1 \le i \le N$. Because P is a nondegenerate stochastic matrix there exist $\delta(i) > 0$, $1 \le i \le N$, small enough such that we have:

$$Z(i) \ge \sum_{j=1}^{N} p(j,i)\left\{\sum_{k=0}^{r} \check{A}_{kcl}(j)Z(j)\check{A}_{kcl}^T(j)\right.$$
$$\left. + \varepsilon_\mu(j)\check{B}_{vcl}(j)\check{B}_{vcl}^T(j) + \delta(j)I_{2n}\right\}, \qquad 1 \le i \le N. \tag{7.172}$$

Let

$$R(j) = \sum_{k=0}^{r} \check{A}_{kcl}(j) Z(j) \check{A}_{kcl}^T(j) + \varepsilon_\mu(j) \check{B}_{vcl}(j) \check{B}_{vcl}^T(j) + \delta(j) I_{2n}, \qquad 1 \leq j \leq N. \tag{7.173}$$

From (7.172) and (7.173) one obtains

$$Z(i) \geq \sum_{j=1}^{N} p(j,i) R(j), \qquad 1 \leq i \leq N. \tag{7.174}$$

Furthermore, from (7.173) and (7.174) one gets that $R = (R(1), R(2), \ldots, R(N))$ solves (7.170), $R(j) \geq \delta(j) I_{2n}$ and thus we have shown that the implication $(i) \to (ii)$ holds.

To check the implication $(ii) \to (i)$ one multiplies (7.170) (written with j instead of i) by $p(j,i)$ and taking the sum with respect to j from 1 to N. Thus we obtain that $Z = (Z(1), Z(2), \ldots, Z(N))$ defined by (7.171) is a solution of (7.169). Because P is a nondegenerate stochastic matrix from $R(i) > 0, 1 \leq i \leq N$ it follows that $Z(i) > 0, 1 \leq i \leq N$. Thus the proof is complete. □

From (7.158) and (7.168) one obtains

$$\check{A}_{0cl}(i) = \begin{pmatrix} A_0(i) + L(i)C_0(i) & 0 \\ -L(i)C_0(i) & A_0(i) + B_0(i)F_s(i) \end{pmatrix}$$

$$\check{A}_{kcl}(i) = \begin{pmatrix} A_k(i) + L(i)C_k(i) & A_k(i) + B_k(i)F_s(i) + L(i)C_k(i) \\ -L(i)C_k(i) & -L(i)C_k(i) \end{pmatrix}, \quad 1 \leq k \leq r$$

$$\check{B}_{vcl}(i) = \begin{pmatrix} D_v(i) + L(i)D_v(i) \\ -L(i)D_v(i) \end{pmatrix}, \qquad 1 \leq i \leq N. \tag{7.175}$$

From (7.175) one obtains the decomposition:

$$\check{A}_{kcl}(i) = \mathcal{A}_k(i) + \mathcal{B} L(i) \Gamma_k(i), \qquad 0 \leq k \leq r$$
$$\check{B}_{vcl}(i) = \mathcal{B}_v(i) + \mathcal{B} L(i) D_v(i), \tag{7.176}$$

where

$$\mathcal{A}_0(i) = \begin{pmatrix} A_0(i) & 0 \\ 0 & A_0(i) + B_0(i)F_s(i) \end{pmatrix}$$

$$\mathcal{A}_k(i) = \begin{pmatrix} A_k(i) & A_k(i) + B_k(i)F_s(i) \\ 0 & 0 \end{pmatrix}, \qquad 1 \leq k \leq r$$

$$\mathcal{B} = \begin{pmatrix} I_n \\ -I_n \end{pmatrix}, \qquad \Gamma_0(i) = \begin{pmatrix} C_0(i) & 0 \end{pmatrix}, \qquad \Gamma_k(i) = \begin{pmatrix} C_k(i) & C_k(i) \end{pmatrix},$$

$$1 \le k \le r, \qquad 1 \le i \le N$$

$$\mathcal{B}_v(i) = \begin{pmatrix} B_v(i) \\ 0 \end{pmatrix}.$$

Thus (7.170) becomes:

$$\sum_{k=0}^{r}[\mathcal{A}_k(i) + \mathcal{B}L(i)\Gamma_k(i)]\mathcal{E}_i^*(R)[\mathcal{A}_k(i) + \mathcal{B}L(i)\Gamma_k(i)]^T$$
$$+ \varepsilon_\mu(i)[\mathcal{B}_v(i) + \mathcal{B}L(i)D_v(i)][\mathcal{B}_v(i) + \mathcal{B}L(i)D_v(i)]^T$$
$$- R(i) < 0, \qquad 1 \le i \le N. \tag{7.177}$$

Lemma 7.8 *The system of LMI* (7.177) *with the unknowns $L(i)$, $R(i)$, $1 \le i \le N$, can be rewritten as*

$$\mathcal{Z}(i) + \mathcal{U}^T(i)L(i)\mathcal{V}(i) + \mathcal{V}^T(i)L^T(i)\mathcal{U}(i) < 0, \tag{7.178}$$

where

$$\mathcal{Z}(i) = \begin{pmatrix} -R(i) & \mathcal{Z}_{12}(i) \\ \mathcal{Z}_{12}^T(i) & \mathcal{Z}_{22}(i) \end{pmatrix} \in \mathcal{S}_{\tilde{n}}$$

with $\tilde{n} = 2n[N(r+1)+1] + m_v$.

$$\mathcal{Z}_{12}(i) = \begin{pmatrix} \mathcal{Z}_{12}^0(i) & \mathcal{Z}_{12}^1(i) & \dots & \mathcal{Z}_{12}^r(i) & \sqrt{\varepsilon_\mu(i)}\mathcal{B}_v(i) \end{pmatrix} \in \mathbf{R}^{2n \times \tilde{n}_1},$$

$$\tilde{n}_1 = 2nN(r+1) + m_v.$$

$$\mathcal{Z}_{12}^k(i) = \begin{pmatrix} \sqrt{p(1,i)}\mathcal{A}_k(i)R(1) & \dots & \sqrt{p(N,i)}\mathcal{A}_k(i)R(N) \end{pmatrix}.$$

$$\mathcal{Z}_{22}(i) = \text{diag}\{\mathcal{Z}_{22}^0, \mathcal{Z}_{22}^1, \dots, \mathcal{Z}_{22}^r, I_{m_v}\}.$$

$$\mathcal{Z}_{22}^k(i) = \text{diag}\{R(1), R(2), \dots, R(N)\}.$$

$$\mathcal{U}(i) = (\mathcal{B}^T \quad 0_{n \times \hat{n}} \quad 0_{n \times \hat{n}} \quad \dots \quad 0_{n \times m_v}) \in \mathbf{R}^{n \times \tilde{n}}, \quad \hat{n} = 2nN(r+1).$$

$$\mathcal{V}(i) = \begin{pmatrix} 0_{n_y \times 2n} & \mathcal{V}_0(i) & \mathcal{V}_1(i) & \dots & \mathcal{V}_r(i) & \sqrt{\varepsilon_\mu(i)}D_v(i) \end{pmatrix} \in \mathbf{R}^{n_y \times \tilde{n}}.$$

$$\mathcal{V}_k(i) = \big(\sqrt{p(1,i)}\Gamma_k(i)R(1) \quad \sqrt{p(2,i)}\Gamma_k(i)R(2)$$
$$\dots \quad \sqrt{p(1,N)}\Gamma_k(i)R(N) \big), \qquad 0 \le k \le r.$$

Proof. It is standard and is based on the Schur complement technique. It is omitted. □

In addition (7.178) allows us to separate the computation of $R(i)$ and $L(i)$, $1 \leq i \leq N$. To this end we apply the projection lemma and Finsler's lemma.

Lemma 7.9 (The projection lemma [103]) *Let $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n\times n}, \mathcal{U} \in \mathbf{R}^{m\times n}, \mathcal{V} \in \mathbf{R}^{p\times n}$ be given matrices, $n \geq \max\{m, p\}$. Let $\mathcal{U}^\perp, \mathcal{V}^\perp$ be full column rank matrices such that $\mathcal{U}\mathcal{U}^\perp = 0$ and $\mathcal{V}\mathcal{V}^\perp = 0$. Then the following are equivalent.*

(*i*) *The linear matrix inequation:*

$$\mathcal{Z} + \mathcal{U}^T L \mathcal{V} + \mathcal{V}^T L^T \mathcal{U} < 0$$

with the unknown matrix $L \in \mathbf{R}^{m\times p}$ is solvable.

(*ii*)

$$(\mathcal{U}^\perp)^T \mathcal{Z} \mathcal{U}^\perp < 0,$$
$$(\mathcal{V}^\perp)^T \mathcal{Z} \mathcal{V}^{\perp|} < 0.$$

Lemma 7.10 (Finsler's lemma [103]) *Let $\mathcal{Z} = \mathcal{Z}^T \in \mathbf{R}^{n\times n}, \mathcal{C} \in \mathbf{R}^{p\times n}$, $n > p$ be given. Take $\mathcal{C}^\perp$ a full column rank matrix such that $\mathcal{C}\mathcal{C}^\perp = 0$. Then the following are equivalent.*

(*i*) *There exist a scalar μ such that $\mathcal{Z} + \mu \mathcal{C}^T \mathcal{C} < 0$.*
(*ii*) $(\mathcal{C}^\perp)^T \mathcal{Z} \mathcal{C}^\perp < 0$.

From Lemma 7.9 one deduces that for each $1 \leq i \leq N$, there exist $L(i) \in \mathbf{R}^{n\times n_y}$, which solve (7.178), if and only if

$$\mathcal{U}_\perp^T(i)\mathcal{Z}(i)\mathcal{U}_\perp(i) < 0 \tag{7.179}$$

$$\mathcal{V}_\perp^T(i)\mathcal{Z}(i)\mathcal{V}_\perp(i) < 0, \tag{7.180}$$

where $\mathcal{U}_\perp(i)$ and $\mathcal{V}_\perp(i)$ are full column rank matrices with the properties $\mathcal{U}(i)\mathcal{U}_\perp(i) = 0$ and $\mathcal{V}(i)\mathcal{V}_\perp(i) = 0$, respectively.

One can see that we may choose

$$\mathcal{U}_\perp(i) = \begin{pmatrix} \mathcal{B}_\perp & 0 \\ 0 & I_{\tilde{n}_1} \end{pmatrix}$$

where

$$\mathcal{B}_\perp = \begin{pmatrix} I_n \\ I_n \end{pmatrix}.$$

Thus, using the Schur complement techniques one obtains that (7.179) is equivalent to:

$$\mathcal{B}_\perp^T \left(\sum_{k=0}^{r} \mathcal{A}_k \mathcal{E}_i^*(R) \mathcal{A}_k(i) - R(i) + \varepsilon_\mu(i) \mathcal{B}_v(i) \mathcal{B}_v^T(i) \right) \mathcal{B}_\perp < 0, \qquad 1 \leq i \leq N, \tag{7.181}$$

where $\mathcal{E}_i^*(\cdot)$ is defined as in (5.162).

On the other hand, it can be seen that we have the decomposition

$$\mathcal{V}(i) = \begin{pmatrix} 0_{n_y \times 2n} & \hat{\mathcal{V}}(i) \end{pmatrix} \begin{pmatrix} R(i) & 0 \\ 0 & \mathcal{Z}_{22}(i) \end{pmatrix},$$

where $\hat{\mathcal{V}}(i) = (\hat{\mathcal{V}}_0(i) \quad \hat{\mathcal{V}}_1(i) \quad \dots \quad \hat{\mathcal{V}}_r(i) \quad \sqrt{\varepsilon_\mu(i)} D_v(i))$ with

$$\hat{\mathcal{V}}_k(i) = (\sqrt{p(1,i)}\Gamma_k(i) \quad \dots \quad \sqrt{p(N,i)}\Gamma_k(i)).$$

Thus one obtains that a base for the null subspace of $\mathcal{V}(i)$ is given by the columns of the matrix

$$\mathcal{V}_\perp(i) = \begin{pmatrix} R^{-1}(i) & 0 \\ 0 & \mathcal{Z}_{22}^{-1}(i) \end{pmatrix} \begin{pmatrix} I_{2n} & 0 \\ 0 & \hat{\mathcal{V}}_\perp(i) \end{pmatrix},$$

where $\hat{\mathcal{V}}_\perp(i)$ is a full column rank matrix such that $\hat{\mathcal{V}}(i)\hat{\mathcal{V}}_\perp(i) = 0$.

Setting $S(i)$ instead of $R^{-1}(i)$ one obtains that (7.180) is equivalent to the following LMI,

$$\begin{pmatrix} -S(i) & \mathcal{W}_{12}(i,S)\hat{\mathcal{V}}_\perp(i) \\ \hat{\mathcal{V}}_\perp^T(i)\mathcal{W}_{12}^T(i,S) & -\hat{\mathcal{V}}_\perp^T(i)\mathcal{W}_{22}(i,S)\hat{\mathcal{V}}_\perp(i) \end{pmatrix} < 0, \tag{7.182}$$

where

$$\begin{aligned} \mathcal{W}_{12}(i,S) &= (S(i)\mathcal{W}_{12}^0(i) \quad S(i)\mathcal{W}_{12}^1(i) \quad \dots \quad S(i)\mathcal{W}_{12}^r(i) \\ &\qquad \sqrt{\varepsilon_\mu(i)} S(i)\mathcal{B}_v(i) \quad \text{with} \\ \mathcal{W}_{12}^k(i) &= \Big(\sqrt{p(1,i)}\mathcal{A}_k(i) \quad \sqrt{p(2,i)}\mathcal{A}_k(i) \quad \dots \\ &\qquad \sqrt{p(N,i)}\mathcal{A}_k(i)\Big), \qquad 0 \le k \le r, 1 \le i \le N, \\ \mathcal{W}_{22}(i,S) &= \operatorname{diag}(\mathcal{W}_{22}^0(i,S) \quad \mathcal{W}_{22}^1(i,S) \quad \dots \quad \mathcal{W}_{22}^r(i,S) \quad I_{m_v}) \quad \text{with} \\ \mathcal{W}_{22}^k(i,S) &= \operatorname{diag}(S(1), S(2), \dots, S(N)). \end{aligned}$$

From the above developments one obtains the following.

Theorem 7.9 *Under the assumptions of Theorem* 7.8 *suppose that* P *is a nondegenerate stochastic matrix. Then the following are equivalent.*

(i) There exist gain matrices $L(i)$, $1 \le i \le N$, *such that the corresponding controller* (G_c) *of type* (7.157) *satisfies* $J_\mu(G_c) < J_\mu^{\text{opt}} + \gamma$ *for a prescribed level* $\gamma > 0$.

(*ii*) *There exist positive definite matrices* $R(i) \in \mathcal{S}_{2n}$, $S(i) \in \mathcal{S}_{2n}$, $1 \leq i \leq N$ *that solve* (7.181), (7.182) *together with*

$$R(i)S(i) = I_{2n} \tag{7.183}$$

and

$$\sum_{i=1}^{N} Tr[V(i)F_s(i)(\mathcal{E}_i^*(R_{11}) - Y_s(i))F_s^T(i)V(i)] < \gamma, \tag{7.184}$$

where $R_{11}(i) \in \mathcal{S}_n$ *is the* 1×1 *block of* $R(i)$, $F_s(i)$ *is given by* (7.95), *and* J_μ^{opt} *is defined in* (7.165).

To avoid the conditions (7.182) and (7.183) a sufficient criterion could be derived via Finsler's lemma (Lemma 7.10).

Theorem 7.10 *Assume that the assumptions in Theorem* 7.9 *are fulfilled. Then there exist gain matrices* $L(i)$, $1 \leq i \leq N$ *such that the corresponding controller* G_c *of type* (7.157) *satisfies* $J_\mu(G_c) \leq J_\mu^{\text{opt}} + \gamma$ *(for a prescribed level* $\gamma > 0$ *and some* $\mu \in \{1, 2, 3, 4\}$*) if there exist positive definite matrices* $R(i) \in \mathcal{S}_{2n}$, *and the scalars* $\nu(i) < 0$, $1 \leq i \leq N$, *that solve* (7.181), (7.184) *together with*

$$\begin{pmatrix} \mathcal{Z}(i) & \mathcal{V}^T(i) \\ \mathcal{V}(i) & \nu(i)I_{n_y} \end{pmatrix} < 0, \qquad 1 \leq i \leq N. \tag{7.185}$$

Proof. Based on Finsler's lemma (7.180) is equivalent to the existence of the scalars $\hat{\nu}(i) \in \mathbf{R}, \hat{\nu}(i) \neq 0$ such that

$$\mathcal{Z}(i) - \hat{\nu}\mathcal{V}^T(i)\mathcal{V}(i) < 0. \tag{7.186}$$

If $\hat{\nu}(i) < 0$ then by the Schur complement technique (7.186) is equivalent to (7.185) with $\nu(i) = \hat{\nu}^{-1}(i)$. Thus the proof is complete. □

Remark 7.5 To obtain the best level of suboptimality achieved by a controller of type (7.157) one may solve the following optimization problem,

$$\min_{R(i),\nu(i)} \sum_{i=1}^{N} Tr[V(i)F_s(i) \begin{pmatrix} I_n \\ 0 \end{pmatrix} \mathcal{E}_i^*(R) \begin{pmatrix} I_n \\ 0 \end{pmatrix}^T F_s^T(i)V(i)] \tag{7.187}$$

subject to (7.181), (7.185), $R(i) > 0, \nu(i) < 0, 1 \leq i \leq N$. If $R = (R(1), R(2), \ldots, R(N))$ is obtained solving the minimization problem (7.187) then it is introduced in (7.178) for obtaining the gain matrices $L(i)$.

7.7 An H_2 filtering problem

Let us consider the system:

$$G : \begin{cases} x(t+1) & = \left(A_0(\eta_t) + \sum\limits_{k=1}^{r} w_k(t) A_k(\eta_t)\right) x(t) + B_v(\eta_t) v(t) \\ y(t) & = \left(C_0(\eta_t) + \sum\limits_{k=1}^{r} w_k(t) C_k(\eta_t)\right) x(t) + D_v(\eta_t) v(t), \qquad t \geq 0, \end{cases} \tag{7.188}$$

where $x(t) \in \mathbf{R}^n$ is the state, $y(t) \in \mathbf{R}^{n_y}$ is the vector of the measurements, $\{\eta_t\}_{t\geq 0}$, $\{w(t)\}_{t\geq 0}$, $\{v(t)\}_{t\geq 0}$ are as in (7.1) and satisfy the assumptions $\mathbf{H_1}, \mathbf{H_2}$, and $\mathbf{A.7.1}$. Throughout this section we assume that the zero state equilibrium of the linear system

$$x(t+1) = \left(A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)\right) x(t) \tag{7.189}$$

is ESMS. Our goal is to construct a discrete-time linear system $y_F(t) = (\mathcal{G}_F(\eta)y)(t)$, called a linear filter, activated by the measurements $y(s)$, $0 \leq s \leq t$ such that the output $y_F(t)$ is a "good estimation" of the state $x(t)$ of the given system. Because the given system is affected by additive white noise perturbations, we consider that a good estimation could be expressed in terms of an H_2 norm.

We remark that (7.188) is the special case of (7.114) for $B_k(i) = 0$, $0 \leq k \leq r$, $C_z(i) = I_n$, $D_z(i) = -I_n$, $1 \leq i \leq N$. It follows that the best estimation of $x(t)$ in terms of an H_2 norm can be obtained applying the results in Section 7.5. In this special case the DTSRE-C (7.92) reduces to

$$X(i) = \sum_{k=0}^{r} A_k^T(i) \mathcal{E}_i(X) A_k(i). \tag{7.190}$$

Under the assumption of mean square exponential stability of the zero state equilibrium of (7.189) it follows that the stabilizing solution of (7.190) is $X_s = (X_s(1), \ldots, X_s(N))$, $X_s(i) = 0$, $1 \leq i \leq N$. The stabilizing feedback gain (7.95) is $F_s(i) = I_n$, $1 \leq i \leq N$.

Assuming that for some $\mu \in \{1, 2, 3, 4\}$ the DTSRE-F (5.156) has a stabilizing solution $Y_s = (Y_s(1), \ldots, Y_s(N))$ then applying Theorem 7.8 one obtains that the optimal filter has the state space realization given by

$$\begin{aligned} x_F(t+1) &= (A_0(\eta_t) + K_s(\eta_t) C_0(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) \\ &\qquad + K_s(\eta_t) C_k(\eta_t))) x_F(t) - K_s(\eta_t) y(t) \end{aligned} \tag{7.191}$$

$$y_F(t) = x_F(t), \quad t \geq 0,$$

where $K_s(i)$ is the stabilizing injection constructed as in (5.158). Moreover, the minimal value of the corresponding H_2 performance is

$$J_{\text{opt}} = \sum_{i=1}^{N} Tr(Y_s(i)). \tag{7.192}$$

It should be noted that as in the general case investigated in Section 7.5, the optimal filter (7.191) is hard to implement, due to the presence of the white noise $w(t)$, $t \geq 0$ in its state space realization. To avoid this inconvenience we try to find a filter with a simpler structure:

$$\begin{aligned} x_L(t+1) &= (A_0(\eta_t) + L(\eta_t)C_0(\eta_t))x_L(t) - L(\eta_t)y(t) \\ y_L(t) &= x_L(t), \end{aligned} \tag{7.193}$$

where the gain matrices $L(i)$, $1 \leq i \leq N$ are arbitrary but have the property that the zero state equilibrium of the system

$$x(t+1) = [A_0(\eta_t) + L(\eta_t)C_0(\eta_t)]x(t) \tag{7.194}$$

is ESMS. Let $\hat{\mathcal{K}}_s(G)$ be the set of all gains $L = (L(1), \dots, L(N))$, $L(i) \in \mathbf{R}^{n \times n_y}$ such that the zero state equilibrium of the corresponding system (7.194) is ESMS.

Our aim is to provide conditions that guarantee the existence of a gain $\tilde{L} \in \hat{\mathcal{K}}_s(G)$ with the property that

$$\lim_{t\to\infty} E[|x(t) - x_{\tilde{L}}(t)|^2] \leq \lim_{t\to\infty} E[|x(t) - x_L(t)|^2] \tag{7.195}$$

for any $L \in \hat{\mathcal{K}}_s(G)$. Let $e_L(t) = x(t) - x_L(t)$ be the estimation error corresponding to a filter (7.193). Combining (7.188) and (7.193) one obtains

$$\begin{aligned} e_L(t+1) &= (A_0(\eta_t) + L(\eta_t)C_0(\eta_t))e_L(t) \\ &\quad + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + L(\eta_t)C_k(\eta_t))x(t) + (B_v(\eta_t) + L(\eta_t)D_v(\eta_t))v(t). \end{aligned}$$

Setting $\xi_L(t) = (\, e_L^T(t) \quad x^T(t)\,)^T$ one obtains the system:

$$G_L : \begin{cases} \xi_L(t+1) &= A_{0cl}(\eta_t)\xi_L(t) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t)\xi_L(t) + B_{vcl}(\eta_t)v(t) \\ z_L(t) &= C_{cl}(\eta_t)\xi_L(t), \end{cases} \tag{7.196}$$

where

$$A_{0cl}(i) = \begin{pmatrix} A_0(i) + L(i)C_0(i) & 0 \\ 0 & A_0(i) \end{pmatrix},$$

$$A_{kcl} = \begin{pmatrix} 0 & A_k(i) + L(i)C_k(i) \\ 0 & A_k(i) \end{pmatrix}, \qquad 1 \leq k \leq N, \tag{7.197}$$

$$B_{vcl}(i) = \begin{pmatrix} B_v(i) + L(i)D_v(i) \\ B_v(i) \end{pmatrix}, \qquad C_{cl}(i) = \begin{pmatrix} I_n & 0 \end{pmatrix}.$$

Specializing (7.8) to the case of the system (7.196) we may rewrite (7.195) as

$$|||G_{\tilde{L}}|||_2 \leq |||G_L|||_2. \tag{7.198}$$

In (7.198) we choose the norm $|||\cdot|||_2$ to measure the quality of the estimation achieved by a filter of type (7.193) because this type of H_2 norm is more frequently used in the literature. It is worth mentioning that the use of the norms $||\cdot||_2$ or $\tilde{||}\cdot\tilde{||}_2$ instead of $|||\cdot|||_2$ is possible without major changes. They are pointed out at the end of this section.

Assuming that the hypotheses $\mathbf{H_1}, \mathbf{H_2}, \mathbf{A.7.1}$, and $\mathbf{A.7.2}$ are fulfilled, we may obtain via Theorem 7.4 applied to the system (7.196) that:

$$|||G_L|||_2^2 = \sum_{i=1}^{N} Tr[C_{cl}(i)Y^{\pi_0}(i)C_{cl}^T(i)], \tag{7.199}$$

where $Y^{\pi_0} = (Y^{\pi_0}(1), \ldots, Y^{\pi_0}(N))$ is the unique solution of the following linear equation on $\mathcal{S}_n^N$,

$$Y^{\pi_0}(i) = \sum_{j=1}^{N} p(j,i) \left[\sum_{k=0}^{r} A_{kcl}(j)Y^{\pi_0}(j)A_{kcl}^T(j) + q^{\pi_0}(j)B_{vcl}(j)B_{vcl}^T(j) \right], \tag{7.200}$$

where $q^{\pi_0}(j)$ are defined as in Theorem 7.1.

Let

$$\begin{pmatrix} Y_1(i) & Y_2(i) \\ Y_2^T(i) & Y_3(i) \end{pmatrix}$$

be the partition of the matrix $Y^{\pi_0}(i)$ according to the partition of the coefficients of (7.196). Thus (7.199) becomes:

$$|||G_L|||_2^2 = \sum_{i=1}^{N} Tr[Y_1(i)]. \tag{7.201}$$

Using (7.197) one obtains the following partition of (7.200).

$$Y_1(i) = \sum_{j=1}^{N} p(j,i)\Bigg\{[A_0(j)+L(j)C_0(j)]Y_1(j)[A_0(j)+L(j)C_0(j)]^T$$
$$+\sum_{k=1}^{r}[A_k(j)+L(j)C_k(j)]Y_3(j)[A_k(j)+L(j)C_k(j)]^T$$
$$+q^{\pi_0}(j)[B_v(j)+L(j)D_v(j)][B_v(j)+L(j)D_v(j)]^T\Bigg\} \tag{7.202}$$

$$Y_2(i) = \sum_{j=1}^{N} p(j,i)\Bigg\{[A_0(j)+L(j)C_0(j)]Y_2(j)A_0^T(j)$$
$$+\sum_{k=1}^{r}[A_k(j)+L(j)C_k(j)]Y_3(j)A_k^T(j)$$
$$+q^{\pi_0}(j)[B_v(j)+L(j)D_v(j)]B_v^T(j)\Bigg\} \tag{7.203}$$

$$Y_3(i) = \sum_{j=1}^{N} p(j,i)\left[\sum_{k=0}^{r} A_k(j)Y_3(j)A_k^T(j) + q^{\pi_0}(j)B_v(j)B_v^T(j)\right]. \tag{7.204}$$

Having in mind (7.201) it follows that only the equation (7.202) would be of interest for us. However, one sees that it is dependent upon $Y_3(j)$ which solves (7.204). Hence the equation (7.203) is not involved in our developments. On the other hand if the zero state equilibrium of (7.189) is ESMS one obtains that (7.204) has a unique solution $\tilde{Y}_3 = (\tilde{Y}_3(1),\ldots,\tilde{Y}_3(N)) \in \mathcal{S}_n^{N+}$. Plugging $\tilde{Y}(j)$ in (7.202) we look for the gain $\tilde{L} \in \hat{\mathcal{K}}_s(G)$ such that the right-hand side of (7.201) minimized.

Let us consider the following system of nonlinear equations on $\mathcal{S}_n^N$.

$$Y_1(i) = \sum_{j=1}^{N} p(j,i)\Bigg[A_0(j)Y_1(j)A_0^T(j) - (A_0(j)Y_1(j)C_0^T(j)$$
$$+\sum_{k=1}^{r} A_k(j)\tilde{Y}_3(j)C_k^T(j) + q^{\pi_0}(j)B_v(j)D_v^T(j))(C_0(j)Y_1(j)C_0^T(j)$$
$$+\sum_{k=1}^{r} C_k(j)\tilde{Y}_3(j)C_k^T(j) + q^{\pi_0}(j)D_v(j)D_v^T(j))^{-1}(C_0(j)Y_1(j)A_0^T(j)$$
$$+\sum_{k=1}^{r} C_k(j)\tilde{Y}_3(j)A_k^T(j) + q^{\pi_0}(j)D_v(j)B_v^T(j))$$
$$+\sum_{k=1}^{r} A_k(j)\tilde{Y}_3(j)A_k^T(j) + q^{\pi_0}(j)B_v(j)B_v^T(j)\Bigg] \tag{7.205}$$

with the unknown $Y_1 = (Y_1(1), \dots, Y_1(N))$. In (7.205) $\tilde{Y}_3(j)$ is the jth component of the unique solution of (7.204). Let us define $\overline{B}(j)$, $\overline{D}(j)$ via the following factorization,

$$\begin{pmatrix} \overline{B}(j) \\ \overline{D}(j) \end{pmatrix} \begin{pmatrix} \overline{B}(j) \\ \overline{D}(j) \end{pmatrix}^T = \sum_{k=1}^{r} \begin{pmatrix} A_k(j) \\ C_k(j) \end{pmatrix} \tilde{Y}_3(j) \begin{pmatrix} A_k(j) \\ C_k(j) \end{pmatrix}^T + q^{\pi_0}(j) \begin{pmatrix} B_v(j) \\ D_v(j) \end{pmatrix} \begin{pmatrix} B_v(j) \\ D_v(j) \end{pmatrix}^T .$$

Thus, (7.205) can be rewritten:

$$\begin{aligned} Y_1(i) = \sum_{j=1}^{N} p(j,i)[&A_0(j)Y_1(j)A_0^T(j) - (A_0(j)Y_1(j)C_0^T(j) \\ &+ \overline{B}(j)\overline{D}^T(j))(C_0(j)Y_1(j)C_0^T(j) \\ &+ \overline{D}(j)\overline{D}^T(j))^{-1}(C_0(j)Y_1(j)A_0^T(j) + \overline{D}(j)\overline{B}^T(j)) + \overline{B}(j)\overline{B}^T(j)]. \end{aligned} \tag{7.206}$$

Hence (7.206) is the special form of the DTSRE-F (5.156) associated with the following system subject to Markovian switching,

$$\begin{aligned} \overline{x}(t+1) &= A_0(\eta_t)\overline{x}(t) + \overline{B}(\eta_t)\overline{u}(t) \\ \overline{y}(t) &= C_0(\eta_t)\overline{x}(t) + \overline{D}(\eta_t)\overline{u}(t). \end{aligned}$$

Based on this analogy we may introduce the concept of a stabilizing solution of (7.205). Thus, a solution $\tilde{Y}_1 = (\tilde{Y}_1(1), \dots, \tilde{Y}_1(N))$ of (7.205) is a stabilizing solution if the zero state equilibrium of the closed-loop system

$$\tilde{x}(t+1) = [A_0(\eta_t) + \tilde{L}(\eta_t)C_0(\eta_t))\tilde{x}(t) \tag{7.207}$$

is ESMS, where

$$\begin{aligned} \tilde{L}(j) = -\Bigg(&A_0(j)\tilde{Y}_1(j)C_0^T(j) + \sum_{k=1}^{r} A_k(j)\tilde{Y}_3(j)C_k^T(j) \\ &+ q^{\pi_0}(j)B_v(j)D_v^T(j))(C_0(j)\tilde{Y}_1(j)C_0^T(j) \\ &+ \sum_{k=1}^{r} C_k(j)\tilde{Y}_3(j)C_k^T(j) + q^{\pi_0}(j)D_v(j)D_v^T(j)\Bigg)^{-1}. \end{aligned} \tag{7.208}$$

If we take $\tilde{L} = (\tilde{L}(1), \dots \tilde{L}(N))$, then $\tilde{L} \in \hat{\mathcal{K}}_s(G)$.

Specializing the result of Theorem 5.16 to the DTSRE-F (7.206) one obtains the following.

Corollary 7.4 *Assume:*

(*a*) *The hypotheses* $\mathbf{H_1}$ *and* $\mathbf{H_2}$ *are fulfilled.*
(*b*) *The zero state equilibrium of* (7.189) *is ESMS.*

Then the following are equivalent.

(*i*) *The DTSRE-F* (7.205) *has a stabilizing solution* $\tilde{Y}_1 = (\tilde{Y}_1(1), \dots, \tilde{Y}_1(N)) \in \mathcal{S}_n^{N+}$ *verifying the condition:*

$$C_0(j)\tilde{Y}_1(j)C_0^T(j) + \sum_{k=1}^{r} C_k(j)\tilde{Y}_3(j)C_k^T(j) + q^{\pi_0}(j)D_v(j)D_v^T(j) > 0, \tag{7.209}$$

$1 \le j \le N$.

(*ii*) *There exists* $Z = (Z(1), \dots, Z(N)) \in \mathcal{S}_n^N$ *that solves the following system of LMIs,*

$$\begin{pmatrix} \mathbf{\Psi}_{1i}(Z) - Z(i) & \mathbf{\Psi}_{2i}(Z) \\ \mathbf{\Psi}_{2i}^T(Z) & \mathbf{\Psi}_{3i}(Z) \end{pmatrix} > 0, \tag{7.210}$$

$1 \le i \le N$, *where*

$$\mathbf{\Psi}_{1i}(Z) = A_0(i)\mathcal{E}_i^*(Z)A_0^T(i) + \sum_{k=1}^{r} A_k(i)\tilde{Y}_3(i)A_k^T(i) + q^{\pi_0}(i)B_v(i)B_v^T(i)$$

$$\mathbf{\Psi}_{2i}(Z) = A_0(i)\mathcal{E}_i^*(Z)C_0^T(i) + \sum_{k=1}^{r} A_k(i)\tilde{Y}_3(i)C_k^T(i) + q^{\pi_0}(i)B_v(i)D_v^T(i)$$

$$\mathbf{\Psi}_{3i}(Z) = C_0(i)\mathcal{E}_i^*(Z)C_0^T(i) + \sum_{k=1}^{r} C_k(i)\tilde{Y}_3(i)C_k^T(i) + q^{\pi_0}(i)D_v(i)D_v^T(i).$$

It must be remarked that if assumption (b) in the statement is fulfilled then the zero state equilibrium of $x(t+1) = A_0(\eta_t)x(t)$ *is ESMS. Therefore, in this case, the stochastic detectability involved in Theorem* 5.16 *is automatically satisfied in the case of the DTSRE-F* (7.205).

The next result can be proved by direct calculations.

Lemma 7.11 *The stabilizing solution* $\tilde{Y}_1$ *of the DTSRE-F* (7.205) *also solves the following modified equation.*

$$\tilde{Y}_1(i) = \sum_{j=1}^{N} p(j,i)\Big\{ [A_0(j) + L(j)C_0(j)]\tilde{Y}_1(j)[A_0(j) + L(j)C_0(j)]^T$$

$$+ \sum_{k=1}^{r}[A_k(j) + L(j)C_k(j)]\tilde{Y}_3(j)[A_k(j) + L(j)C_k(j)]^T$$

$$+ q^{\pi_0}(j)[B_v(j) + L(j)D_v(j)][B_v(j) + L(j)D_v(j)]^T$$
$$- [L(j) - \tilde{L}(j)][C_0(j)\tilde{Y}_1(j)C_0^T(j)$$
$$\left. + \sum_{k=1}^{r} C_k(j)\tilde{Y}_3(j)C_k^T(j) + q^{\pi_0}(j)D_v(j)D_v^T(j)][L(j) - \tilde{L}(j)]^T \right\}, \quad (7.211)$$

where $L(j) \in \mathbf{R}^{n \times n_y}$, $1 \le j \le N$ *are arbitrary but fixed and* $\tilde{L}(j)$, $1 \le j \le N$ *are given by* (7.208).

Taking in (7.211) $L(j) = \tilde{L}(j)$ we obtain the following version of (7.205), verified by $\tilde{Y}_1$.

$$\tilde{Y}_1(i) = \sum_{j=1}^{N} p(j,i) \left\{ [A_0(j) + \tilde{L}(j)C_0(j)]\tilde{Y}_1(j)[A_0(j) + \tilde{L}(j)C_0(j)]^T \right.$$
$$+ \sum_{k=1}^{r} [A_k(j) + \tilde{L}(j)C_k(j)]\tilde{Y}_3(j)[A_k(j) + \tilde{L}(j)C_k(j)]^T$$
$$\left. + q^{\pi_0}(j)[B_v(j) + \tilde{L}(j)D_v(j)][B_v(j) + \tilde{L}(j)D_v(j)]^T \right\}. \quad (7.212)$$

It is easy to see that (7.212) coincides with equation (7.202) associated with the gains $L(j) = \tilde{L}(j)$, $1 \le j \le N$. So, (7.201) becomes in this case:

$$|||G_{\tilde{L}}|||_2^2 = \sum_{j=1}^{N} Tr[\tilde{Y}_1(j)]. \quad (7.213)$$

Now we are in position to prove the main result of this section.

Theorem 7.11 *Assume:*

(*a*) *The hypotheses* $\mathbf{H_1}, \mathbf{H_2}, \mathbf{A.7.1}$, *and* $\mathbf{A.7.2}$ *are fulfilled.*
(*b*) *The zero state equilibrium of* (7.189) *is ESMS.*
(*c*) *There exists* $Z = (Z(1), \ldots, Z(N)) \in \mathcal{S}_n^N$ *that solves* (7.210). *Under these conditions the linear equation* (7.204) *has a unique solution* $\tilde{Y}_3 = (\tilde{Y}_3(1), \tilde{Y}_3(2), \ldots, \tilde{Y}_3(N)) \in \mathcal{S}_n^{N+}$ *and the DTSRE-F* (7.205) *has a stabilizing solution* $\tilde{Y}_1 = (\tilde{Y}_1(1), \tilde{Y}_2, \ldots, \tilde{Y}_1(N)) \in \mathcal{S}_n^N$ *that verifies* (7.209). *Let* $\tilde{L}(i)$ *be the stabilizing injection defined by* (7.208) *and*

$$x_{\tilde{L}}(t+1) = (A_0(\eta_t) + \tilde{L}(\eta_t)C_0(\eta_t))x_{\tilde{L}}(t) - \tilde{L}(\eta_t)y(t) \quad (7.214)$$
$$y_{\tilde{L}}(t) = x_{\tilde{L}}(t)$$

be the corresponding filter of type (7.193).

Under these conditions we have:

$$\lim_{t\to\infty} E[|x(t) - x_{\tilde{L}}(t)|^2] \le \lim_{t\to\infty} E[|x(t) - x_L(t)|^2]$$

for all $L = (L(1), \ldots, L(N)) \in \hat{\mathcal{K}}_s(G)$.

Proof. The fact that (7.204) has a unique solution follows from assumption (b), Theorem 2.5, and Theorem 3.10 for $\theta = 1$. Furthermore, Corollary 7.4 guarantees the existence of the solution of (7.205) with the property (7.209). Also, the fact that the zero state equilibrium of (7.194) is ESMS guarantees the existence of the solution $Y_1 = (Y_1(1), \ldots, Y_1(N))$ of (7.202), for each $L \in \hat{\mathcal{K}}_s(G)$.

Based on (7.201) and (7.213) it follows that it is sufficient to verify that $Y_1(i) \ge \tilde{Y}_1(i)$ for all $1 \le i \le N$ and for all $L \in \hat{\mathcal{K}}_s(G)$.

Let $\Delta = (\Delta(1), \ldots, \Delta(N))$ be defined by $\Delta(i) = (Y_1(i) - \tilde{Y}_1(i))$, $1 \le i \le N$. Subtracting (7.211) from (7.202) one obtains that Δ solves the equation:

$$\Delta(i) = \sum_{j=1}^{N} p(j,i)[(A_0(j) + L(j)C_0(j))\Delta(j)(A_0(j) + L(j)C_0(j))^T + H(j)], \tag{7.215}$$

where $H(j) = (L(j) - \tilde{L}(j))(C_0(j)\tilde{Y}_1(j)C_0^T(j) + \sum_{k=1}^{r} C_k(j)\tilde{Y}_3(j)C_k^T(j) + q^{\pi_0}(j)D_v(j)D_v^T(j))(L(j) - \tilde{L}(j))^T$. Because (7.209) is fulfilled we deduce that $H(j) \ge 0$, $1 \le j \le N$.

From the definition of the admissible gains it is known that the zero state equilibrium of (7.194) is ESMS. This allows us to deduce, via Theorems 2.5 and 3.10 in the special case $\theta = 1$, that equation (7.215) has a unique solution $\Delta = (\Delta(1), \ldots, \Delta(N))$. Moreover this solution has the property $\Delta(i) \ge 0$, for all $1 \le i \le N$. This means that $Y_1(i) \ge \tilde{Y}_1(i)$, $1 \le i \le N$, and this completes the proof. □

Remark 7.6

(a) The result proved in the previous theorem shows that the filter (7.214) provides the best estimation of the states $x(t)$ of the system (7.188) with respect to the H_2 performance $|||\cdot|||_2$. Based on Theorem 7.3(i) one obtains that the filter (7.214) also provides the best estimation of the states $x(t)$ of (7.188) with respect to the H_2 performance $||\cdot||_2$ defined by (7.6).

(b) If in (7.205) and (7.209) the numbers $q^{\pi_0}(j)$ are replaced by $q^{\mathcal{D}_0}(j)$, $1 \le j \le N$, then one can show that the filter (7.214) constructed using the new version of the gain $\tilde{L}(i)$ given by (7.209) provides the best estimation of the states $x(t)$ of (7.188) with respect to the H_2 performance $\tilde{||} \cdot \tilde{||}_2$ introduced by (7.7).

(c) In light of the discussion in Section 7.3, if in (7.205) and (7.209) the numbers $q^{\pi_0}(j)$ and $q^{\mathcal{D}_0}(j)$ are replaced by $\tilde{q}(j) = \sum_{i=1}^{N} q_{ij}$ then the corresponding filter (7.214) has some robustness properties in the sense that it does not depend upon the initial distribution π_0 of the Markov chain.

7.8 A case study

We next consider a numerical example that demonstrates the advantages of using the new estimator derived in the present chapter, for a simplified navigation problem where a vehicle's attitude is estimated from its noisy position and velocity measurements (e.g., from GPS) utilizing inaccurate inertial sensors. The example is a modified version of the three-axis simplified navigation model of [3].

The three axis model of [3] is given by

$$\ddot{x} = \beta_x - g(-\phi_y), \qquad -\dot{\phi_y} = \dot{x}/R_e + (\omega_x + \epsilon_x)\phi_z - \epsilon_y$$

$$\ddot{y} = \beta_y - g\phi_x, \qquad \dot{\phi_x} = \dot{y}/R_e + (\omega_y + \epsilon_y)\phi_z + \epsilon_x$$

$$\ddot{z} = \beta_z, \qquad \dot{\phi_z} = \epsilon_z,$$

where x, y, z are the components of the vehicle position, ϕ_x, ϕ_y, ϕ_z are the tilt errors, and $\omega_x, \omega_y, \omega_z$ are the angular rates. In [3] it was assumed that the constant bias and drifts of the accelerometers and rate sensors have been compensated via calibration and, therefore, the driving terms $\beta_x, \beta_y, \beta_z$ and $\epsilon_x, \epsilon_y, \epsilon_z$ are white noise processes. In our case, we assume low-cost noisy measurement devices for the angular rates (e.g., by differentiation of angles computed from magnetometers) and we have, therefore, added the noise terms β_x to ω_x and β_y to ω_y. Defining the state vector to be $x = (x \quad \dot{x} \quad -\phi_y \quad y \quad \dot{y} \quad \phi_x \quad z \quad \dot{x} \quad \phi_z)^T$ and the measurements to be $y = (x \quad y \quad z \quad \dot{x} \quad \dot{y} \quad \dot{z})^T$ and considering the discrete-time version of the above system with a sampling time of $h = 0.1\,\text{sec}$ we obtain the system of (7.188) with [3], where

$$A_0 \approx I + h \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/R_e & 0 & 0 & 0 & 0 & 0 & 0 & \omega_x \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/R_e & 0 & 0 & 0 & \omega_y \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_v v = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\epsilon_d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon_d \end{pmatrix} \begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \\ \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{pmatrix},$$

where we slightly abused notations when writing the discrete-time version. We also define the terms with state-dependent noise:

$$A_1 x(t) w_1(t) = \begin{pmatrix} 0_{2\times 8} & 0_{2\times 1} \\ 0_{1\times 8} & \epsilon_d \\ 0_{6\times 8} & 0_{6\times 1} \end{pmatrix} x(t)\epsilon_x(t),$$

$$A_2 x(t) w_2(t) = \begin{pmatrix} 0_{5\times 8} & 0_{5\times 1} \\ 0_{1\times 8} & \epsilon_d \\ 0_{3\times 8} & 0_{3\times 1} \end{pmatrix} x(t)\epsilon_y(t)$$

and we note that the apparent correlation between v and $w := [\epsilon_x \ \epsilon_y \ \epsilon_z]^T$ does not affect the results of Theorem 7.1 above. The covariance of the measurement noise is taken as $D_v D_v^T = \text{diag}\{100, 100, 100, 0.01, 0.01, 0.01\}$. We took $\epsilon_d = 0.0483 \,\text{radian}/\sqrt{\text{sec}}$ which amounts to $500 \,\text{deg}/\sqrt{\text{hour}}$. Although this level of random walk is very high and beyond commonly encountered practical values, it may represent cases where angular rates are obtained with very cheap and noisy components or under severe environmental conditions.

We compare two estimators. One is the classical Kalman filter (KF) which ignores the state-multiplicative noise and is derived by solving the discrete-time recursive Riccati equation obtained from Theorem 7.11 above by nulling A_1, A_2, C_1, C_2, and $\tilde{Y}_3$ and by replacing $\tilde{Y}_1$ in the left-hand side of (7.205) by $\tilde{Y}_3(t+1)$ and by replacing $\tilde{Y}_3$ in the left-hand side of (7.205) by $\tilde{Y}_3(t+1)$. All values in the right-hand sides of (7.205) and (7.204) correspond to the moment t. Similarly, the new filter of the present paper, is referred to as the state-multiplicative Kalman filter (MKF).

The vehicle maneuvers are assumed to behave according to $\omega_x = \omega_y = \omega_z = 0.5 \sin(0.5t \cdot h)$. The results of 50 Monte Carlo simulation runs depicting the standard deviations of the tilt errors ϕ_x, ϕ_y, ϕ_z for the KF and MKF are given in Figures 7.1 and 7.2, respectively. The solid lines are the actual ensemble based standard deviations whereas the dashed lines are the standard deviations predicted by the filter, namely the square roots of elements (3,3), (6,6), and (9,9) respectively, of $\tilde{Y}_1$. Clearly the prediction by the MKF is considerably more accurate and tighter. Moreover, the standard deviations of the tilt errors are smaller with the MKF, where the benefit of using MKF over using KF is best observed in the estimation of ϕ_z where the errors are smaller by a factor of two with the MKF with respect to the KF. Note also that with both filters, the errors in ϕ_z are larger than those in ϕ_x and ϕ_y due to the weaker observability in ϕ_z due to the lack of direct relation between ϕ_z and the measured velocities that exists, in contrast, between $\ddot{x}$ and ϕ_y and $\ddot{y}$ and ϕ_x.

7.9 Notes and references

The H_2 optimal control problems refer to the minimization of a quadratic cost functional in the solutions of a linear control system affected by additive

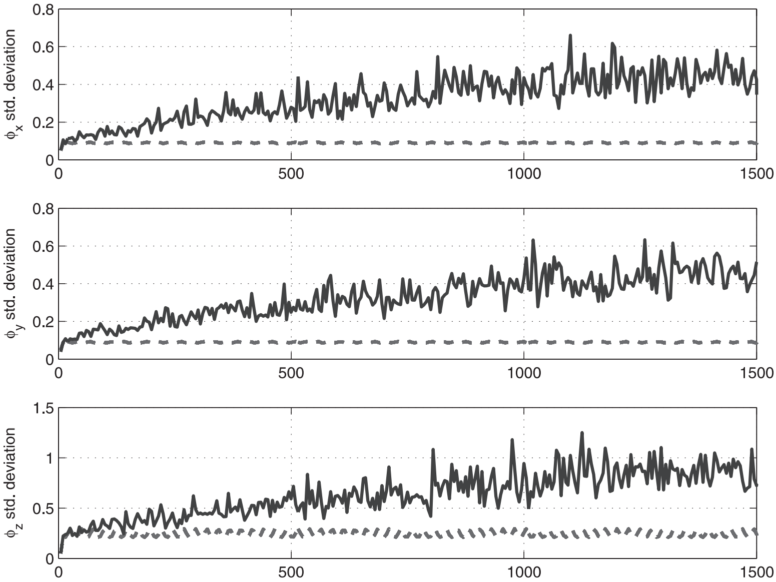

Figure 7.1. Kalman filter, 50 Monte Carlo runs.

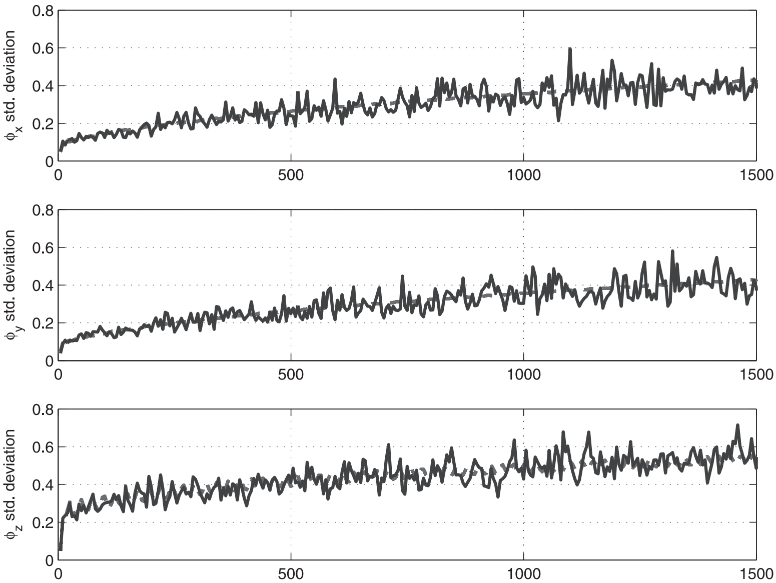

Figure 7.2. State-multiplicative Kalman filter, 50 Monte Carlo runs.

white noise. As is known from the deterministic framework (see, e.g., [33]), the value of such a performance criterion does not depend upon the initial conditions of the trajectories of the controlled system and it coincides with the H_2 norm of a linear time-invariant system. That is why such criteria are often called H_2 norms. As in the case of other controlled problems for discrete-time linear stochastic systems, the H_2 optimal control problems were investigated separately for the case of discrete-time controlled systems subject to multiplicative and additive white noise and for systems subject to jump Markov perturbations and additive white noise. Thus for the case of discrete-time linear systems subject to multiplicative and additive white noise the H_2 optimal control problem was considered in [63, 65]. For the case of discrete-time linear stochastic systems with Markovian switching and additive white noise, the H_2 optimal control problem was considered in [26, 27] and their references. In this chapter we have considered the H_2 optimization problem for a general class of discrete-time linear stochastic systems affected simultaneously by multiplicative and additive white noise as well as Markovian switching. The results included in this chapter may be found in [47, 49]. The results from Section 7.6, concerning the H_2 filtering problem, were published for the first time in this book. The counterpart of this H_2 filtering problem for the case of discrete-time linear stochastic systems was considered in [106].

8

Robust stability and robust stabilization of discrete-time linear stochastic systems

The main goal of this chapter is to investigate several aspects of the problem of robust stability and robust stabilization for a class of discrete-time linear stochastic systems subject to sequences of independent random perturbations and Markov jump perturbations. As a measure of the robustness of the stability of an equilibrium of a nominal system a concept of stability radius is introduced. A crucial role in determining the lower bound of the stability radius is played by the norm of a linear bounded operator associated with the given plant. This operator is called the input–output operator and it is introduced in Section 8.2. In Section 8.3 a stochastic version of the so-called bounded real lemma is proved. This result provides an estimation of the norm of the input–output operator in terms of the feasibility of some linear matrix inequalities (LMIs) or in terms of the existence of stabilizing solutions of a discrete-time generalized algebraic Riccati-type equation. In Section 8.4 the stochastic version of the so-called small gain theorem is proved. Then this result is used to derive a lower bound of robustness with respect to linear structured uncertainties. In the second part of this chapter we consider the robust stabilization problem of systems subject to both multiplicative white noise and Markovian jumps with respect to some classes of parametric uncertainty. Based on the bounded real lemma we obtain a set of necessary and sufficient conditions for the existence of a stabilizing feedback gain that ensures a prescribed level of attenuation of the exogenous disturbance. We also show that in the case of full state measurement if the disturbance attenuation problem has a solution in a dynamic controller form then the same problem is solvable via a control in a state feedback form. Finally a problem of H_∞ filtering is solved.

8.1 A brief motivation

In many applications the mathematical model of the controlled process is not completely known. Even if the multiplicative white noise perturbations are

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introduced in order to model the stochastic environmental perturbations that are hard to quantify, it is also possible that some parametric uncertainties occur in the coefficients of the stochastic system. Thus a robust stabilization problem, is given asking us to construct a control law in a static or dynamic feedback form that stabilizes all discrete-time linear stochastic systems into a neighborhood of a given system often called the nominal system.

To be more specific, let us consider the controlled system:

$$x(t+1) = \left(A_0(\eta_t) + \Delta_A(t,\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t) \right) x(t) + B(\eta_t)u(t), \quad (8.1)$$

where $A_k(i)$, $0 \le k \le r$, $B(i)$, $1 \le i \le N$, are known matrices of appropriate dimensions, and $\Delta_A(t,i)$, $t \ge 0$ are unknown matrices. A robust stabilization problem, via state feedback control law, asks us to construct a control $u(t) = F(\eta_t)x(t)$ such that the zero state equilibrium of the nominal system

$$x(t+1) = \left(A_0(\eta_t) + B(\eta_t)F(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t) \right) x(t) \qquad (8.2)$$

and the zero state equilibrium of the perturbed system

$$x(t+1) = \left(A_0(\eta_t) + B(\eta_t)F(\eta_t) + \Delta_A(t,\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t) \right) x(t) \quad (8.3)$$

are exponentially stable in the mean square (ESMS) for all uncertainties $\Delta_A(t,i)$ in a neighborhood of the origin in $\mathbf{R}^{n \times n}$.

It is known that if the zero state equilibrium of the nominal system (8.2) is ESMS then the zero state equilibrium of the perturbed system (8.3) is still ESMS for some "small perturbations" $\Delta_A(t,i)$. In a robust stability problem, as well as in a robust stabilization problem, the goal is to preserve the stability of the nominal system for the perturbed systems in the case of the variation of coefficients of the system that are not necessarily small.

In this chapter we investigate different aspects of the problem of robust stability and robust stabilization of discrete-time linear stochastic systems (8.1) with structured parametric uncertainties of the form:

$$\Delta_A(t,\eta_t) = \left(G_0(\eta_t) + \sum_{k=1}^{r} w_k(t) G_k(\eta_t) \right) \Delta(\eta_t) C(\eta_t),$$

where the matrices $G_k(i)$, $0 \le k \le r$, $C(i)$, $1 \le i \le N$ are assumed to be known, and $\Delta(i)$, $1 \le i \le N$ are unknown matrices of appropriate dimensions. We show that in the definition of the set of uncertainties $\Delta = (\Delta(1), \dots, \Delta(N))$ for which the exponential stability in the mean square is preserved, an important role is played by the norm of the linear operator adequately chosen, named the input–output operator.

For this reason we start with the proof of the stochastic version of the bounded real lemma. This result allows us to obtain information about the norm of an input–output operator.

We also prove a stochastic version of the small gain theorem which is a powerful tool in the estimation of the stability radius of a perturbed system, with structured parametric uncertainties.

8.2 Input–output operators

Let us consider the system (G) with the state space representation:

$$\begin{aligned} x(t+1) &= \left(A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)\right) x(t) \\ &\quad + \left(B_0(\eta_t) + \sum_{k=1}^{r} w_k(t) B_k(\eta_t)\right) v(t) \\ z(t) &= C(\eta_t)x(t) + D(\eta_t)v(t), \end{aligned} \tag{8.4}$$

where $x(t) \in \mathbf{R}^n$ is the state of the system; $v(t) \in \mathbf{R}^{m_v}$ is the external input and $z(t) \in \mathbf{R}^{n_z}$ is the output; $\{w(t)\}_{t\geq 0}, (w(t) = (w_1(t), w_2(t), \dots, w_r(t))^T)$ is a sequence of independent random vectors; and the triple $(\{\eta_t\}_{t\geq 0}, P, \mathcal{D})$ is an homogeneous Markov chain, on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the set of the states $\mathcal{D} = \{1, 2, \dots, N\}$ and the transition probability matrix P.

Throughout this chapter we assume that together with the hypotheses H_1 and H_2 (introduced in Section 1.5), the Markov chain verifies the following additional assumption.

A.8.1

(i) The transition probability matrix P is a nondegenerate stochastic matrix.
(ii) $\pi_0(i) = \mathcal{P}\{\eta_0 = i\} > 0$, $1 \leq i \leq N$.

It should be remarked that in the developments of this chapter the Markov chain is not prefixed, but it is assumed that the initial distributions $\pi_0 = (\pi_0(1), \dots, \pi_0(N))$ lie in the subset $\mathcal{M}_N = \{\pi_0 = (\pi_0(1), \dots, \pi_0(N)) | \pi_0(i) > 0, 1 \leq i \leq N, \sum_{i=1}^{N} \pi_0(i) = 1\}$.

In (8.4) $A_k(i), B_k(i)$, $0 \leq k \leq r$, $C(i)$, $D(i)$, $1 \leq i \leq N$ are given matrices of appropriate dimensions.

In this chapter, the inputs $v = \{v(t)\}_{t\geq 0}$ are stochastic processes either in $\ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ for $\tau > 0$ or in $\ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$. Both $\ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ and $\ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$ are real Hilbert spaces.

The norms induced by the usual inner product on each of these Hilbert spaces are:

$$\|v\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}} = \left(\sum_{t=0}^{\tau} E[|v(t)|^2]\right)^{1/2}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}$ and

$$\|\overline{v}\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}} = \left(\sum_{t=0}^{\infty} E[|\overline{v}(t)|^2]\right)^{1/2},$$

respectively, for all $\overline{v} \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$.

Let $x(t,0,v)$ be the solution of the system (8.4) corresponding to the input $v = \{v(t)\}_{t\geq 0}$ with the initial condition $x(0,0,v) = 0$. Let

$$z(t,0,v) = C(\eta_t)x(t,0,v) + D(\eta_t)v(t) \tag{8.5}$$

be the corresponding output. One can see that if $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}$ for some $\tau \geq 1$, then $x(t,0,v)$ is $\mathcal{H}_{t-1}$-measurable and $E[|x(t,0,v)|^2] < \infty$. Hence from (8.5) it follows that $\{z(t,0,v)\}_{0\leq t\leq\tau} \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\}$.

We have $E[|z(t,0,v)|^2] \leq \delta_1 E[|x(t,0,v)|^2] + \delta_2 E[|v(t)|^2]$, where $\delta_1 = 2\max_{i\in\mathcal{D}} |C(i)|^2$ and $\delta_2 = 2\max_{i\in\mathcal{D}} |D(i)|^2$.

Applying Corollary 3.8(i) we deduce that if the zero state equilibrium of the system

$$x(t+1) = \left(A_0(\eta_t) + \sum_{k=1}^{r} w_k(t)A_k(\eta_t)\right)x(t) \tag{8.6}$$

is ESMS, then there exists $\gamma > 0$ such that

$$\sum_{t=0}^{\infty} E[|z(t,0,v)|^2] \leq \gamma^2 \sum_{t=0}^{\infty} E[|v(t)|^2] \tag{8.7}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$.

It can be remarked that in the absence of the property of exponential stability in the mean square of the linear system (8.6) one can prove that for each $\tau \geq 1$ there exists $\gamma(\tau) > 0$ such that

$$\sum_{t=0}^{\tau} E[|z(t,0,v)|^2] \leq \gamma^2(\tau) \sum_{t=0}^{\tau} E[|v(t)|^2] \tag{8.8}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}$.

Because $v \to z(t,0,v)$ is a linear dependence, we deduce that if the state equilibrium of (8.6) is ESMS, we may define a linear operator $\mathcal{T}$: $\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}$ by

$$(\mathcal{T}v)(t) = z(t,0,v) \tag{8.9}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$.

In the absence of the assumption of exponential stability for each $\tau \geq 1$, the equality (8.9) defines a linear operator $\mathcal{T}_\tau : \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{n_z}\}$.

From (8.7) and (8.8) one obtains that $\mathcal{T}$ and $\mathcal{T}_\tau$ are bounded operators.

The linear operator $\mathcal{T}$ introduced by (8.9) is called the input–output operator defined by the system (8.4) and the system (8.4) is known as a state space representation of the operator $\mathcal{T}$. From the definition of the input–output operator one sees that such an operator maps only finite-energy disturbance signal v into the corresponding finite energy output signal z of the considered system.

Concerning the product and the inversion of the input–output operators we have the following proposition.

Proposition 8.3

(*i*) *Let* $\mathcal{T}^1_\tau : \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^m\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^p\}$ *and* $\mathcal{T}^2_\tau : \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_1}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^m\}$ *be the input–output operators with the state space representations*

$$
\begin{aligned}
x_1(t+1) &= \left(A_{10}(\eta_t) + \sum_{k=1}^{r} w_k(t) A_{1k}(\eta_t)\right) x(t) \\
&\quad + \left(B_{10}(\eta_t) + \sum_{k=1}^{r} w_k(t) B_{1k}(\eta_t)\right) v_1(t) \\
y_1(t) &= C_1(\eta_t) x(t) + D_1(\eta_t) v_1(t)
\end{aligned}
$$

and

$$
\begin{aligned}
x_2(t+1) &= \left(A_{20}(\eta_t) + \sum_{k=1}^{r} w_k(t) A_{2k}(\eta_t)\right) x(t) \\
&\quad + \left(B_{20}(\eta_t) + \sum_{k=1}^{r} w_k(t) B_{2k}(\eta_t)\right) v_2(t) \\
y_2(t) &= C_2(\eta_t) x(t) + D_2(\eta_t) v_2(t);
\end{aligned}
$$

then the product $\mathcal{T}^1_\tau \mathcal{T}^2_\tau : \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_1}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^p\}$ *has the state space realization of the form* (8.4) *where*

$$
A_k(i) = \begin{pmatrix} A_{1k}(i) & B_{1k}(i) C_2(i) \\ 0 & A_{2k}(i) \end{pmatrix}
$$

$$
B_k(i) = \begin{pmatrix} B_{1k}(i) D_2(i) \\ B_{2k}(i) \end{pmatrix}, \qquad 0 \leq k \leq r
$$

$$
C(i) = \begin{pmatrix} C_1(i) & D_1(i) C_2(i) \end{pmatrix}, \qquad D(i) = D_1(i) D_2(i), \qquad i \in \mathcal{D}.
$$

(*ii*) *Assume that in* (8.4) $m_v = n_z$ *and* $\det D(i) \neq 0$, $i \in \mathcal{D}$. *Then for every* $\tau > 0$ *the input–output operator* $\mathcal{T}_\tau : \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\}$ *is invertible and its inverse* $\mathcal{T}_\tau^{-1}$ *has the state space realization*

$$\begin{aligned}\xi(t+1) &= \left(\tilde{A}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)\tilde{A}_k(\eta_t)\right)\xi(t)\\ &\quad + \left(\tilde{B}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)\tilde{B}_k(\eta_t)\right)z(t)\\ v(t) &= \tilde{C}(\eta_t)\xi(t) + \tilde{D}(\eta_t)z(t),\end{aligned} \tag{8.10}$$

where $\tilde{A}_k(i) = A_k(i) - B_k(i)D^{-1}(i)C(i), \tilde{B}_k(i) = B_k(i)D^{-1}(i), \tilde{C}(i) = -D^{-1}(i)C(i), \tilde{D}(i) = D^{-1}(i), i \in \mathcal{D}$. *Moreover, if the zero state equilibrium of the system* (8.6) *and the zero state equilibrium of the system obtained from* (8.10) *with* $\tilde{B}_k(i) = 0$ *are ESMS then the input–output operator* $\mathcal{T}$ *associated with the system* (8.4) *is invertible and its inverse* $\mathcal{T}^{-1}$ *has a state space representation given by* (8.10).

To obtain an estimate of a robustness radius of the stabilization achieved by a control law, an important role is played by the norm of an input–output operator. It is well known, from the deterministic context, that the norm of an input–output operator cannot be explicitly computed as in the case of H_2 norms. That is why we are looking for necessary and sufficient conditions which guarantee that the norm of an input–output operator is smaller than a prescribed level $\gamma > 0$.

Such conditions are provided by the well-known bounded real lemma. The discrete-time stochastic version of the bounded real lemma is proved in the next section.

In the last part of this section we present several auxiliary results useful in the developments of the next sections.

Remark 8.7 Each stochastic process $v = \{v(t)\}_{0 \le t \le \tau} \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}$ can be extended in a natural way to a process $\overline{v} = \{\overline{v}(t)\}_{t \ge 0} \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$ by taking $\overline{v}(t) = v(t)$ for $0 \le t \le \tau$ and $\overline{v}(t) = 0$ for $t \ge \tau + 1$. We have $\|v\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}} = \|\overline{v}\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}}$ and also $\|\mathcal{T}_\tau v\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\}} \le \|\mathcal{T}\overline{v}\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}} \le \|\mathcal{T}\| \cdot \|\overline{v}\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}} = \|\mathcal{T}\| \cdot \|v\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}}$ for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}$. Therefore this leads to

$$\|\mathcal{T}_\tau\| \le \|\mathcal{T}\| \tag{8.11}$$

for all $\tau \ge 1$.

Let $\gamma > 0$, $0 < \tau \in \mathbf{Z} \cup \{\infty\}$, and $x_0 \in \mathbf{R}^n$ be arbitrary but fixed. We consider the following cost functionals

$$J_\gamma(\tau, x_0, i, v) = \sum_{t=0}^{\tau} E[|z(t, x_0, v)|^2 - \gamma^2|v(t)|^2|\eta_0 = i], \tag{8.12}$$

$i \in \mathcal{D}$ and

$$\tilde{J}_\gamma(\tau, x_0, v) = \sum_{t=0}^{\tau} E[|z(t, x_0, v)|^2 - \gamma^2|v(t)|^2] \tag{8.13}$$

for all $v = \{v(t)\}_{0\le t\le\tau} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

It should be noted that if (8.12) and (8.13) are written for $\tau = +\infty$ we assume tacitly that the zero state equilibrium of the system (8.6) is ESMS. It is clear that $\|\mathcal{T}_\tau\| \le \gamma$ if and only if $\tilde{J}_\gamma(\tau, 0, v) \le 0$ for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ and $\|\mathcal{T}\| \le \gamma$ if and only if $\tilde{J}_\gamma(\infty, 0, v) \le 0$ for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$.

Applying Lemma 3.1 we obtain the following.

Corollary 8.5 *For all $\tau \ge 1$, $x_0 \in \mathbf{R}^n$, and for all $X(t) = (X(t, 1), \dots, X(t, N)) \in \mathcal{S}_n^N$, $0 \le t \le \tau + 1$, we have*

$$J_\gamma(\tau, x_0, i, v) x_0^T X(0, i) x_0 - E[x^T(\tau+1) X(\tau+1, \eta_{\tau+1}) x(\tau+1)|\eta_0 = i]$$

$$+ \sum_{t=0}^{\tau} E\left[\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}^T \mathbf{Q}(t, \eta_t) \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} |\eta_0 = i\right]$$

for all $v = \{v(t)\}_{0\le t\le\tau} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$, $i \in \mathcal{D}$, where $x(t) = x(t, x_0, v)$ and

$$\mathbf{Q}(t, i) = \begin{pmatrix} \Pi_{1i} X(t+1) - X(t, i) + C^T(i) C(i) & \Pi_{2i} X(t+1) + C^T(i) D(i) \\ (\Pi_{2i} X(t+1) + C^T(i) D(i))^T & \Pi_{3i} X(t+1) + D^T(i) D(i) - \gamma^2 I_{m_v} \end{pmatrix} \tag{8.14}$$

with

$$\begin{pmatrix} \Pi_{1i} X(t+1) & \Pi_{2i} X(t+1) \\ (\Pi_{2i} X(t+1))^T & \Pi_{3i} X(t+1) \end{pmatrix}$$

$$= \sum_{k=0}^{r} \begin{pmatrix} A_k(i) & B_k(i) \end{pmatrix}^T \mathcal{E}_i(X(t+1)) \begin{pmatrix} A_k(i) & B_k(i) \end{pmatrix} \tag{8.15}$$

with $\mathcal{E}_i(X(t+1)) = \sum_{j=1}^{N} p(i, j) X(t+1, j)$, $1 \le i \le N$.

Proof. One applies Lemma 3.1 to the function $V(t, x, i) = x^T X(t, i) x$ and to the system (8.4). Details are omitted. □

Let $F(t) = (F(t,1),\ldots,F(t,N))$, $F(t,i) \in \mathbf{R}^{m_v \times n}$,$0 \le t \le \tau$, $\tau \ge 1$. Let $X_F^\gamma(t) = (X_F^\gamma(t,1),\ldots,X_F^\gamma(t,N))$ be the solution of the following problem with the given final value

$$X(t,i) = \sum_{k=0}^{r}(A_k(i)+B_k(i)F(t,i))^T\mathcal{E}_i(X(t+1))(A_k(i)+B_k(i)F(t,i))$$
$$+ (C(i)+D(i)F(t,i))^T(C(i)+D(i)F(t,i)) - \gamma^2F^T(t,i)F(t,i) \tag{8.16}$$

$$X(\tau+1,i) = 0,$$

$1 \le i \le N$. Because (8.16) is a backward affine equation, its solution $X_F(t)$ is well defined for all $0 \le t \le \tau+1$. Let us remark that (8.16) may be written as

$$\begin{pmatrix} I_n \\ F(t,i) \end{pmatrix}^T \mathbf{Q}(X_F^\gamma,i,t)\begin{pmatrix} I_n \\ F(t,i) \end{pmatrix} = 0, \tag{8.17}$$

$0 \le t \le \tau+1$, $i \in \mathcal{D}$, where $\mathbf{Q}(X_F^\gamma,i,t)$ is obtained from (8.14) when $X(t)$ is replaced by $X_F^\gamma(t)$. Let $x_F = \{x_F(t)\}_{0 \le t \le \tau+1}$ be the solution of the following problem with the initial given value,

$$x(t+1) = \Bigg[A_0(\eta_t) + B_0(\eta_t)F(t,\eta_t)$$
$$+ \sum_{k=1}^{r} w_k(t)(A_k(\eta_t)+B_k(\eta_t)F(t,\eta_t))\Bigg]x(t)$$
$$+ \left[B_0(\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t)\right]v(t) \tag{8.18}$$

$$x(0) = x_0.$$

Lemma 8.12 *Let* $F = \{F(t)\}_{0 \le t \le \tau}$*,* $F(t) = (F(t,1),\ldots,F(t,N))$*,* $F(t,i) \in \mathbf{R}^{m_v \times n}$ *be a sequence of gain matrices. If* $\{X_F^\gamma(t)\}_{0 \le t \le \tau+1}$ *is the solution of the problem* (8.16)*, then we have:*

$$J_\gamma(\tau,x_0,i,v+Fx_F) = x_0^TX_F^\gamma(0,i)x_0$$
$$+ \sum_{t=0}^{\tau} E[v^T(t)\mathfrak{H}_\gamma(X_F^\gamma(t+1),\eta_t)v(t)$$
$$+ 2v^T(t)\mathfrak{N}(X_F^\gamma(t+1),\eta_t)x_F(t)|\eta_0 = i]$$

for all $i \in \mathcal{D}$*,* $x_0 \in \mathbf{R}^n$*,* $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}$*,* $x_F(t)$ *being the solution of the problem* (8.18) *corresponding to the input* v *and*

$$\mathfrak{H}_\gamma(X_F^\gamma(t+1), i) = \Pi_{3i} X_F^\gamma(t+1) + D^T(i)D(i) - \gamma^2 I_{m_v} \tag{8.19}$$

$$\mathfrak{N}(X_F^\gamma(t+1), i) = (\Pi_{2i} X_F^\gamma(t+1) + C^T(i)D(i))^T + \mathfrak{H}_\gamma(X_F^\gamma(t+1), i)F(t, i). \tag{8.20}$$

Proof. Applying Corollary 8.1 with $X(t,i)$ replaced by $X_F^\gamma(t,i)$ and $v(t)$ replaced by $v(t) + F(t,\eta_t)x_F(t)$ we obtain

$$\begin{aligned} &J_\gamma(\tau, x_0, i, v + Fx_F) \\ &\quad = x_0^T X_F^\gamma(0,i)x_0 + \sum_{k=0}^{\tau} E\left[\begin{pmatrix} x_F(t) \\ v(t) + F(t,\eta_t)x_F(t) \end{pmatrix}^T \right. \\ &\qquad \left. \times \mathbf{Q}(X_F^\gamma, \eta_t, t) \begin{pmatrix} x_F(t) \\ v(t) + F(t,\eta_t)x_F(t) \end{pmatrix} |\eta_0 = i \right] \end{aligned} \tag{8.21}$$

for all $x_0 \in \mathbf{R}^n$, $i \in \mathcal{D}$, $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

Furthermore, we may write

$$\begin{aligned} &\begin{pmatrix} x_F(t) \\ v(t) + F(t,\eta_t)x_F(t) \end{pmatrix}^T \mathbf{Q}(X_F^\gamma, \eta_t, t) \begin{pmatrix} x_F(t) \\ v(t) + F(t,\eta_t)x_F(t) \end{pmatrix} \\ &= x_F^T(t) \begin{pmatrix} I_n \\ F(t,\eta_t) \end{pmatrix}^T \mathbf{Q}(X_F^\gamma, \eta_t, t) \begin{pmatrix} I_n \\ F(t,\eta_t) \end{pmatrix} x_F(t) \\ &\quad + 2x_F^T(t) \begin{pmatrix} I_n \\ F(t,\eta_t) \end{pmatrix}^T \mathbf{Q}(X_F^\gamma, \eta_t, t) \begin{pmatrix} 0 \\ I_{m_v} \end{pmatrix} v(t) \\ &\quad + v^T(t) \begin{pmatrix} 0 \\ I_{m_v} \end{pmatrix}^T \mathbf{Q}(X_F^\gamma, \eta_t, t) \begin{pmatrix} 0 \\ I_{m_v} \end{pmatrix} v(t). \end{aligned}$$

Using (8.17) we deduce

$$\begin{aligned} &\begin{pmatrix} x_F(t) \\ v(t) + F(t,\eta_t)x_F(t) \end{pmatrix}^T \mathbf{Q}(X_F^\gamma, \eta_t, t) \begin{pmatrix} x_F(t) \\ = v(t) + F(t,\eta_t)x_F(t) \end{pmatrix} \\ &2v^T(t)\mathfrak{N}(X_F^\gamma(t+1), \eta_t)x_F(t) + v^T(t)\mathfrak{H}_\gamma(X_F^\gamma(t+1), \eta_t)v(t). \end{aligned}$$

The conclusion follows plugging the last equality in (8.21). This completes the proof. □

Now we prove the following.

Proposition 8.4 *If for an integer $\tau \geq 1$ and a real number $\gamma > 0$, $\|\mathcal{T}_\tau\| < \gamma$, then*

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_F^\gamma(t+1))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} \leq -\varepsilon_0 I_{m_v} \tag{8.22}$$

for all $0 \leq t \leq \tau$, *with* $\varepsilon_0 \in (0, \gamma^2 - \|\mathcal{T}_\tau\|^2)$.

Proof. Let us remark that (8.22) can be rewritten

$$\mathfrak{H}_\gamma(X_F^\gamma(t+1), i) \leq -\varepsilon_0 I_{m_v}, \qquad 0 \leq t \leq \tau. \tag{8.23}$$

We prove (8.23) in two steps. First we show that

$$\mathfrak{H}_\gamma(X_F^\gamma(t+1), i) \leq 0 \tag{8.24}$$

for all $0 \leq t \leq \tau$, $i \in \mathcal{D}$.

In the second step, using (8.24), we show the validity of (8.23). Let us assume to the contrary that (8.24) is not true. This implies that there exist $0 \leq t_0 \leq \tau$, $i_0 \in \mathcal{D}$, and $v \in \mathbf{R}^{m_v}$ with $|v| = 1$, such that

$$v^T \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), i_0)v = \nu_0 > 0 \tag{8.25}$$

for a $\nu_0 > 0$.

Let $\hat{v} = \{\hat{v}(t)\}_{0 \leq t \leq \tau}$ be defined as follows,

$$\hat{v}(t) = \begin{cases} \chi_{\{\eta_{t_0} = i_0\}} v, & \text{if } \ t = t_0 \\ 0, & \text{if } \ t \neq t_0. \end{cases}$$

It is clear that $\hat{v} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

Also let $\hat{x} = \{\hat{x}(t)\}_{0 \leq t \leq \tau+1}$ be the solution of the problem with zero initial value:

$$\begin{aligned} x(t+1) = &\left[A_0(\eta_t) + B_0(\eta_t)F(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F(t,\eta_t))\right] x(t) \\ &+ \left[B_0(\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t)\right] \hat{v}(t) \end{aligned} \tag{8.26}$$

$$x(0) = 0.$$

Let $\check{v}(t) = \hat{v}(t) + F(t,\eta_t)\hat{x}(t)$. It is clear that $\check{v} = \{\check{v}(t)\}_{0 \leq t \leq \tau}$ lies in $\ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$.

Hence

$$\tilde{J}_\gamma(\tau, 0, \check{v}) = \|\mathcal{T}_\tau \check{v}\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\}} - \gamma^2 \|\check{v}\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}} \leq 0. \tag{8.27}$$

Because $\tilde{J}_\gamma(\tau, x_0, v) = \sum_{i=1}^{N} \pi_0(i) J_\gamma(\tau, x_0, i, v)$ we obtain via Lemma 8.1 and the inequality (8.27) that

$$0 \geq \sum_{i=1}^{N} \pi_0(i) J_\gamma(\tau, 0, i, \hat{v}) = \sum_{i=1}^{N} \pi_0(i) \left\{ \sum_{t=0}^{\tau} E[\hat{v}^T(t) \mathfrak{H}_\gamma(X_F^\gamma(t+1), \eta_t) \hat{v}(t) \right.$$

$$\left. + 2\hat{v}^T(t) \mathfrak{N}(X_F^\gamma(t+1), \eta_t) \hat{x}(t) | \eta_0 = i] \right\}$$

$$= \sum_{i=1}^{N} \pi_0(i) E[\hat{v}(t_0) \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), \eta_{t_0}) \hat{v}(t_0) | \eta_0 = i]. \tag{8.28}$$

For the last equality we used the fact that $\hat{v}(t) = 0$ for $t \neq t_0$ and $\hat{x}(t) = 0$ for $t \leq t_0$.

On the other hand

$$\hat{v}^T(t_0) \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), \eta_{t_0}) \hat{v}(t_0) = \sum_{l=1}^{N} \hat{v}^T(t_0) \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), l) \hat{v}(t_0) \chi_{\{\eta_{t_0} = l\}}$$

$$= v^T \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), i_0) v \chi_{\{\eta_{t_0} = i_0\}}. \tag{8.29}$$

From (8.25), (8.28), and (8.29) we deduce

$$0 \geq \sum_{i=1}^{N} \pi_0(i) E[v^T \mathfrak{H}_\gamma(X_F^\gamma(t_0+1), i_0) v \chi_{\{\eta_{t_0} = i_0\}} | \eta_0 = i]$$

$$= \nu_0 \sum_{i=1}^{N} \pi_0(i) E[\chi_{\{\eta_{t_0} = i_0\}} | \eta_0 = i] = \nu_0 \pi_{t_0}(i_0) > 0.$$

This is a contradiction, hence (8.24) is correct. Note that $\pi_{t_0}(i_0) > 0$ is a consequence of the assumption $A.8.1$.

Let $0 < \varepsilon_0 < \gamma^2 - \|\mathcal{T}_\tau\|^2$. Set $\tilde{\gamma} = (\gamma^2 - \varepsilon_0)^{1/2}$. We have $\|\mathcal{T}_\tau\| < \tilde{\gamma}$. Hence (8.24) is fulfilled for γ replaced by $\tilde{\gamma}$. This means that

$$\mathfrak{H}_{\tilde{\gamma}}(X_F^{\tilde{\gamma}}(t+1), i) \leq 0, \qquad i \in \mathcal{D}, \qquad 0 \leq t \leq \tau, \tag{8.30}$$

where $X_F^{\tilde{\gamma}}(\cdot)$ is the solution of the problem with given final value

$$X_F^{\tilde{\gamma}}(t, i) = \sum_{k=0}^{r} (A_k(i) + B_k(i) F(t, i))^T \mathcal{E}_i(X_F^{\tilde{\gamma}}(t+1))$$

$$\times (A_k(i) + B_k(i) F(t, i)) + (C(i) + D(i) F(t, i))^T (C(i)$$

$$+ D(i) F(t, i)) - \tilde{\gamma}^2 F^T(t, i) F(t, i)$$

$$X_F^{\tilde{\gamma}}(\tau+1, i) = 0, \qquad i \in \mathcal{D}. \tag{8.31}$$

Subtracting (8.16) from (8.31) one gets

$$X_F^{\tilde{\gamma}}(t,i) - X_F^{\gamma}(t,i) = \sum_{k=0}^{r} (A_k(i) + B_k(i)F(t,i))^T \mathcal{E}_i(X_F^{\tilde{\gamma}}(t+1) - X_F^{\gamma}(t+1))(A_k(i) + B_k(i)F(t,i)) + \varepsilon_0 F^T(t,i)F(t,i).$$

The last equality allows us to deduce recursively that

$$X_F^{\tilde{\gamma}}(t,i) \geq X_F^{\gamma}(t,i), \qquad 0 \leq t \leq \tau, \qquad i \in \mathcal{D}. \tag{8.32}$$

From (8.19), (8.30), and (8.32) we deduce that

$$\mathfrak{H}_{\tilde{\gamma}}(X_F^{\gamma}(t+1), i) \leq \mathfrak{H}_{\tilde{\gamma}}(X_F^{\tilde{\gamma}}(t+1), i) \leq 0.$$

Having in mind the definition of $\tilde{\gamma}$ we obtain that

$$\mathfrak{H}_{\gamma}(X_F^{\gamma}(t+1), i) \leq -\varepsilon_0 I_{m_v}, \qquad 0 \leq t \leq \tau, \qquad i \in \mathcal{D}$$

which completes the proof. □

Let $X^{\gamma}(t) = (X^{\gamma}(t,1), \ldots, X^{\gamma}(t,N))$ be the solution of the problem (8.16) in the special case $F(t) = 0$. One obtains recursively for $t \in \{\tau+1, \tau, \ldots, 0\}, i \in \mathcal{D}$ that $X^{\gamma}(t,i) \geq 0$. Applying Proposition 8.2 for $X^{\gamma}(t)$ instead of $X_F^{\gamma}(t)$ one obtains the following.

Corollary 8.6 *If there exists an integer $\tau \geq 1$ such that $\|\mathcal{T}_\tau\| < \gamma$, then $\gamma^2 I_{m_v} - D^T(i)D(i) > 0, i \in \mathcal{D}$.*

8.3 Stochastic version of bounded real lemma

In the developments of this section an important role is played by the following backward discrete-time nonlinear equation.

$$\begin{aligned} X(t,i) = &\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X(t+1))A_k(i) + C^T(i)C(i) \\ &- \left(\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X(t+1))B_k(i) + C^T(i)D(i)\right) \\ &\times \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X(t+1))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v}\right)^{-1} \\ &\times \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X(t+1))A_k(i) + D^T(i)C(i)\right), \qquad 1 \leq i \leq N. \end{aligned} \tag{8.33}$$

Using the notation introduced in (8.15) we may rewrite (8.33) in the following compact form,

$$X(t) = \Pi_1 X(t+1) + M - (\Pi_2 X(t+1) + L) \times (\Pi_3 X(t+1) + R)^{-1}(\Pi_2 X(t+1) + L)^T, \tag{8.34}$$

where

$$M = (M(1), M(2), \dots, M(N)) \in \mathcal{S}_n^N, \qquad M(i) = C^T(i)C(i),$$

$$L = (L(1), L(2), \dots, L(N)) \in \mathcal{M}_{n,m_v}^N, \qquad L(i) = C^T(i)D(i),$$

$$R = (R(1), R(2), \dots, R(N)) \in \mathcal{S}_{m_v}^N, \qquad R(i) = D^T(i)D(i) - \gamma^2 I_{m_v}.$$

We recall that as in Chapter 5, $\mathcal{M}_{n,m_v}^N$ stands for $\mathbf{R}^{n\times m_v} \oplus \mathbf{R}^{n\times m_v} \oplus \cdots \oplus \mathbf{R}^{n\times m_v}$. Hence (8.34) is a time-invariant version of (5.8). That is why often we refer to (8.33) and (8.34) as a discrete-time stochastic generalized Riccati equation (DTSGRE).

8.3.1 Stochastic bounded real lemma. The finite horizon time case

For each integer $\tau \geq 1$, let $X_\tau(t) = (X_\tau(t,1), \dots, X_\tau(t,N))$ be the solution of DTSGRE (8.33) with the final value

$$X_\tau(\tau+1, i) = 0, \qquad i \in \mathcal{D}. \tag{8.35}$$

Concerning the well-definedness of the solution $X_\tau(t)$ of (8.33)–(8.35) we prove the following.

Lemma 8.13 *If for an integer $\tau \geq 1$ and a real number $\gamma > 0$ we have $\|\mathcal{T}_\tau\| < \gamma$, then the solution $X_\tau(t)$ of the problem* (8.33)–(8.35) *is well defined for all $0 \leq t \leq \tau$ and it has the properties:*
$X_\tau(t,i) \geq 0$ and

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_\tau(t+1))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} \leq -\varepsilon_0 I_{m_v} \tag{8.36}$$

for all $0 \leq t \leq \tau$, $i \in \mathcal{D}$, where $\varepsilon_0 \in (0, \gamma^2 - \|\mathcal{T}_\tau\|^2)$.

Proof. Based on Corollary 8.2 we deduce that $X_\tau(\tau, i)$ can be computed via (8.33) for all $i \in \{1, 2, \dots, N\}$. Also (8.36) is fulfilled for $t = \tau$. Let us assume that for an integer $0 < t_0 < \tau$ the solution $X_\tau(t)$ is well defined for $t \in \{t_0, t_0+1, \dots, \tau\}$ and has the properties in the statement. Because (8.36) is fulfilled for $t = t_0$ it follows that $X_\tau(t_0, i)$ can be computed via (8.33) and $X_\tau(t_0, i) \geq 0$ for all $i \in \mathcal{D}$. Now we show that (8.36) is fulfilled for $t = t_0 - 1$. For $t_0 \leq t \leq \tau$ we define

$$F_\tau(t,i) = -\left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_\tau(t+1))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v}\right)^{-1}$$
$$\times \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_\tau(t+1))A_k(i) + D^T(i)C(i)\right), \qquad i \in \mathcal{D}. \tag{8.37}$$

By direct calculation one obtains that $X_\tau(t)$ verifies

$$X_\tau(t,i) = \sum_{k=0}^{r}(A_k(i) + B_k(i)F_\tau(t,i))^T\mathcal{E}_i(X_\tau(t+1))(A_k(i) + B_k(i)F_\tau(t,i))$$
$$+ (C(i) + D(i)F_\tau(t,i))^T(C(i) + D(i)F_\tau(t,i)) - \gamma^2 F_\tau^T(t,i)F_\tau(t,i). \tag{8.38}$$

Let $F = \{F(t)\}_{0\le t\le\tau}$ be the sequence of the gain matrices defined as follows.

$$F(t) = (F(t,1),\dots,F(t,N)), \qquad F(t,i) = \begin{cases} 0, & \text{if } 0 \le t \le t_0 - 1, \\ F_\tau(t,i), & \text{if } t_0 \le t \le \tau. \end{cases}$$

If $X_F^\gamma = \{X_F^\gamma(t)\}_{0\le t\le\tau}$ is the solution of the problem (8.16) corresponding to this choice of F, from the uniqueness of the solution of this problem with given final value we conclude that $X_F^\gamma(t) = X_\tau(t)$, $t_0 \le t \le \tau$.

Applying Proposition 8.2 we deduce that

$$\mathfrak{H}_\gamma(X_F^\gamma(t+1), i) \le -\varepsilon_0 I_{m_v}$$

for all $0 \le t \le \tau$. Particularly for $t = t_0 - 1$ this inequality becomes:

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_\tau(t_0))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} \le -\varepsilon_0 I_{m_v} < 0.$$

This shows that (8.36) is fulfilled for $t = t_0 - 1$. This allows us to deduce that $X_\tau(t_0-1) = (X_\tau(t_0-1,1), X_\tau(t_0-1,2),\dots,X_\tau(t_0-1,N))$ can be computed via (8.33). Using again (8.33) and (8.36) for $t = t_0 - 1$ we deduce that $X_\tau(t_0-1,i) \ge 0$, $i \in \mathcal{D}$. Thus the proof ends. □

Let us remark that if $x(t) = x(t,x_0,v)$ is a solution of the system (8.4) determined by the input $v(t), t \ge 0$ then it verifies

$$x(t+1) = \Bigg[A_0(\eta_t) + B_0(\eta_t)F_\tau(t,\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t)$$
$$+B_k(\eta_t)F_\tau(t,\eta_t))\Bigg]x(t) + \left(B_0(\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t)\right)v_\tau(t), \tag{8.39}$$

where $v_\tau(t) = v(t) - F_\tau(t, \eta_t)x(t) \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$. Applying Lemma 8.1 for $F(t, i) = F_\tau(t, i)$ we obtain the following.

Corollary 8.7 *Assume that for an integer $\tau \geq 1$, $\|\mathcal{T}_\tau\| < \gamma$. Then we have*

$$J_\gamma(\tau, x_0, i, v) = x_0^T X_\tau(0, i)x_0 + \sum_{t=0}^{\tau} E[(v(t) - F_\tau(t, \eta_t)x(t))^T \mathfrak{H}_\gamma$$

$$\times (X_\tau(t+1), \eta_t)(v(t) - F_\tau(t, \eta_t)x(t))|\eta_0 = i]$$

for all $x_0 \in \mathbf{R}^n$, $i \in \mathcal{D}$, $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$, $x(t) = x(t, x_0, v)$, where $X_\tau = \{X_\tau(t)\}_{0 \leq t \leq \tau}$ is a solution of the problem (8.33)–(8.35) *and $F_\tau(t, i)$ is given by* (8.37).

Proof. From the previous lemma it follows that $X_\tau(t)$ is well defined for all $0 \leq t \leq \tau$ and(8.36) is fulfilled. Therefore $F_\tau(t, i)$ can be constructed via (8.37) for all t, i. The problem with given final value (8.33)–(8.35) is just the problem with given final value (8.16) corresponding to the matrix gains $F(t, i) = F_\tau(t, i)$. Applying Lemma 8.1 for $F(t, i) = F_\tau(t, i)$ and taking into account that from (8.37) and (8.20) we have $\mathfrak{N}(X_\tau(t+1), i) = 0$ for all t, i, we obtain the equality in the statement. Thus the proof ends. □

The next result is the finite horizon time version of the bounded real lemma.

Theorem 8.12 *Under the considered assumptions, for an integer $\tau \geq 1$ and a given scalar $\gamma > 0$ the following are equivalent.*

(*i*) $\|\mathcal{T}_\tau\| < \gamma$.
(*ii*) *The solution $X_\tau = \{X_\tau(t)\}_{0 \leq t \leq \tau}$ of the DTSGRE* (8.33) *with the final value $X_\tau(\tau + 1, i) = 0$, $i \in \mathcal{D}$, is well defined for $0 \leq t \leq \tau$ and verifies* (8.36) *with $\varepsilon_0 \in (0, \gamma^2 - \|\mathcal{T}_\tau\|^2)$.*

Proof. The implication $(i) \to (ii)$ follows from Lemma 8.2.

We have to prove the implication $(ii) \to (i)$. Let us assume that (ii) is fulfilled. Then the result from Corollary 8.3 allows us to write:

$$\|\mathcal{T}_\tau v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\}} - \gamma^2 \|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}}$$

$$= \tilde{J}_\gamma(\tau, 0, v) = \sum_{t=0}^{\tau} E[(v(t) - F_\tau(t, \eta_t)x(t))^T$$

$$\times \mathfrak{H}_\gamma(X_\tau(t+1, \eta_t))(v(t) - F_\tau(t, \eta_t)x(t))] \leq 0 \qquad (8.40)$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$. Hence $\|\mathcal{T}_\tau\| \leq \gamma$. Let us assume that $\|\mathcal{T}_\tau\| = \gamma$. This means that there exists a sequence $\{v_j\}_{j \geq 1} \subset \ell^2_{\tilde{\mathcal{H}}}\{0, \tau; \mathbf{R}^{m_v}\}$ such that

$$\|v_j\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{m_v}\}} = 1$$

$$\lim_{j\to\infty} \|\mathcal{T}_\tau v_j\|_{\ell^2_{\tilde{\mathcal{H}}}\{0,\tau;\mathbf{R}^{n_z}\}} = \gamma. \tag{8.41}$$

Let $x_j = \{x_j(t)\}_{0\le t\le \tau+1}$, be the solution of (8.4) corresponding to the input v_j and $x_j(0) = 0$. Writing (8.40) for v_j and taking the limit for $j \to \infty$ one obtains

$$\lim_{j\to\infty} E[|v_j(t) - F_\tau(t,\eta_t)x_j(t)|^2] = 0$$

for all $0 \le t \le \tau$. If we take into account that $x_j(t)$ verifies

$$\begin{aligned} x_j(t+1) = &\left[A_0(\eta_t) + B_0(\eta_t)F_\tau(t,\eta_t) + \sum_{k=1}^{r} w_k(t)\right.\\ &\left.\times(A_k(\eta_t) + B_k(\eta_t)F_\tau(t,\eta_t))\right] x_j(t) + \left[B_0(\eta_t) + \sum_{k=1}^{r} w_k(t)B_k(\eta_t)\right]\\ &\times (v_j(t) - F_\tau(t,\eta_t)x_j(t)), \end{aligned}$$

one obtains inductively that $\lim_{j\to\infty} E[|x_j(t)|^2] = 0$. Furthermore, one obtains

$$\begin{aligned} \lim_{j\to\infty} E[|v_j(t)|^2] \le 2 \lim_{j\to\infty} E[|F_\tau(t,\eta_t)x_j(t)|^2] \\ + 2 \lim_{j\to\infty} E[|v_j(t) - F_\tau(t,\eta_t)x_j(t)|^2] = 0. \end{aligned}$$

This is in contradiction to (8.41) hence $\|\mathcal{T}_\tau\| < \gamma$. This completes the proof. □

8.3.2 The bounded real lemma. The infinite time horizon case

First we prove the following.

Lemma 8.14 *Assume:*

(*a*) *The zero state equilibrium of the system* (8.6) *is ESMS.*
(*b*) *The input–output operator $\mathcal{T}$ associated with the system* (8.4) *satisfies $\|\mathcal{T}\| < \gamma$. Then there exists $\rho > 0$ such that $\tilde{J}_\gamma(\infty, x_0, v) \le \rho|x_0|^2$ for all $x_0 \in \mathbf{R}^n$ and $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$.*

Proof. Let $Z = (Z(1), Z(2), \ldots, Z(N)) \in \mathcal{S}_n^{N+}$ be the unique solution of the linear equation

$$Z(i) = \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(Z)A_k(i) + C^T(i)C(i), \qquad 1 \le i \le N. \tag{8.42}$$

We recall that under the assumption (a), if $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$ then, based on Corollary 3.8, $\lim_{t\to\infty} E[|x(t,x_0,v)|^2] = 0$.

Applying Corollary 8.1 in the special case $X(t,i) = Z(i)$ and taking the limit for $\tau \to \infty$, one gets:

$$\tilde{J}_\gamma(\infty,x_0,v) = \sum_{i=1}^{N} \pi_0(i) x_0^T Z(i) x_0 \\ + \sum_{t=0}^{\infty} E[v^T(t)\mathfrak{H}_\gamma(Z,\eta_t)v(t) + 2x^T(t,x_0,v)\mathfrak{N}^T(Z,\eta_t)v(t)] \tag{8.43}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$ and all $x_0 \in \mathbf{R}^n$, where $\mathfrak{N}(Z,i)$ and $\mathfrak{H}_\gamma(Z,i)$ are as in (8.19) and (8.20) with $Z(i)$ instead of $X_F^\gamma(t,i)$. Writing (8.43) for $x_0 = 0$ one obtains:

$$\tilde{J}_\gamma(\infty;x_0,v) - \tilde{J}_\gamma(\infty;0,v) \sum_{i=1}^{N} \pi_0(i) x_0^T Z(i) x_0 \\ + 2\sum_{t=0}^{\infty} E[x^T(t,x_0,0)\mathfrak{N}^T(Z,\eta_t)v(t)] \tag{8.44}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$, $x_0 \in \mathbf{R}^n$.

Let ε be such that $\|\mathcal{T}\|^2 < \gamma^2 - \varepsilon^2$. Thus we may write:

$$\tilde{J}_\gamma(\infty,0,v) = \|\mathcal{T}v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}} - \gamma^2\|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}} \le -\varepsilon^2\|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}}.$$

Plugging the last inequality in (8.44) we deduce

$$\tilde{J}_\gamma(\infty;x_0,v) \le \sum_{i=1}^{N} \pi_0(i) x_0^T Z(i) x_0 \\ + \sum_{t=0}^{\infty} E[2x^T(t,x_0,0)\mathfrak{N}^T(Z,\eta_t)v(t) - \varepsilon^2|v(t)|^2]$$

or

$$\tilde{J}_\gamma(\infty;x_0,v) \le \sum_{i=1}^{N} \lambda_{\max}(Z(i))|x_0|^2 + \frac{1}{\varepsilon^2}\sum_{t=0}^{\infty} E[|\mathfrak{N}(Z,\eta_t)x(t,x_0,0)|^2] \\ - \sum_{t=0}^{\infty} E[|\varepsilon v(t) - \frac{1}{\varepsilon}\mathfrak{N}(Z,\eta_t)x(t,x_0,0)|^2].$$

Let $\nu > 0$ such that $\max |\mathfrak{N}(Z,i)| \le \nu$. Thus we have

$$\tilde{J}_\gamma(\infty; x_0, v) \le \sum_{i=1}^{N} \lambda_{\max}(Z(i))|x_0|^2 + \frac{\nu^2}{\varepsilon^2} \sum_{t=0}^{\infty} E[|x(t,x_0,0)|^2]. \tag{8.45}$$

From assumption (a) we deduce that there exists $\rho_1 > 0$ not depending upon x_0 such that $\sum_{t=0}^{\infty} E[|x(t,x_0,0)|^2] \le \rho_1 |x_0|^2$. Introducing the last inequality in (8.45) one obtains the inequality from the statement with $\rho = \sum_{i=1}^{N} \lambda_{\max} Z(i) + \rho_1(\nu^2/\varepsilon^2)$. Thus the proof is complete. □

If $X_\tau(t)$, $0 \le t \le \tau + 1$ is the solution of the problem with given final value (8.33)–(8.35) we define $K(t) = (K(t,1), \dots, K(t,N))$ by

$$K(t,i) = X_\tau(\tau + 1 - t, i). \tag{8.46}$$

We see that $K(0,i) = X_\tau(\tau + 1, i) = 0$, $1 \le i \le N$. Also, by direct calculation one obtains that $K = \{K(t)\}_{t \ge 0}$ solves the following forward nonlinear equation on $\mathcal{S}_n^N$,

$$\begin{aligned} K(t+1,i) = {} & \Pi_{1i}K(t) + C^T(i)C(i) - (\Pi_{2i}K(t) + C^T(i)D(i))(\Pi_{3i}K(t) \\ & + D^T(i)D(i) - \gamma^2 I_{m_v})^{-1}(\Pi_{2i}K(t) + C^T(i)D(i))^T. \end{aligned} \tag{8.47}$$

Let us denote $K_0(t) = (K_0(t,1), \dots, K_0(t,N))$ the solution of (8.47) with given initial value $K_0(0,i) = 0$, $1 \le i \le N$.

Several properties of the solution $K_0(t)$ are summarized in the next result.

Proposition 8.5 *Assume:*

(a) *the zero state equilibrium of* (8.6) *is ESMS.*
(b) $\|\mathcal{T}\| < \gamma$.

Then the solution $K_0(t)$ *of the forward equation* (8.47) *with the given initial value* $K_0(0,i) = 0$ *is defined for all* $t \ge 0$. *It has the properties:*

(i)
$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(K_0(t))B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} \le -\varepsilon_0 I_{m_v}, \tag{8.48}$$

where $\varepsilon_0 \in (0, \gamma^2 - \|\mathcal{T}\|^2)$.
(ii) $0 \le K_0(\tau,i) \le K_0(\tau+1,i) \le cI_n, \forall\ t,i \in \mathbf{Z}_+ \times \mathcal{D}$, *where* $c > 0$ *is a constant not depending upon* t, i.

Proof. Based on (8.11) we obtain that $\|\mathcal{T}_\tau\| \le \|\mathcal{T}\| < \gamma$ for all $\tau \ge 1$. Therefore, we deduce, via Lemma 8.2, that for any integer $\tau \ge 1$ the solution $X_\tau(t)$ of the problem with given final value (8.33), (8.35) is well defined for $0 \le t \le \tau + 1$ and it verifies (8.36). Thus we deduce via (8.46) that $K_0(t)$ is well defined for all $t \ge 0$. If $0 < \varepsilon_0 < \gamma^2 - \|\mathcal{T}\|^2$ it follows that $\varepsilon_0 < \gamma^2 - \|\mathcal{T}_\tau\|^2$ for all $\tau \ge 1$.

Hence in (8.36) we may choose ε_0 independent of τ. Writing (8.36) for $t = 0$ and taking into account that $K_0(\tau, i) = X_\tau(1, i)$ we obtain that (i) is fulfilled. Furthermore, from (8.48) and (8.47) we deduce that $K_0(t, i) \geq 0$ for all $(t, i) \in \mathbf{Z}_+ \times \mathcal{D}$.

Let $X_\tau(t)$ and $X_{\tau+1}(t)$ be the solutions of the DTSGRE (8.33) with the final value $X_\tau(\tau + 1) = 0$ and $X_{\tau+1}(\tau + 2) = 0$ in $\mathcal{S}_n^N$. Under the considered assumptions we know that these two solutions are well defined for $0 \leq t \leq \tau+1$ and $0 \leq t \leq \tau + 2$, respectively.

If $F_\tau(t, i)$ is defined as in (8.37) we rewrite the equation (8.33) verified by $X_{\tau+1}(t)$ in the form

$$\begin{aligned} X_{\tau+1}(t, i) = \sum_{k=0}^{r} &(A_k(i) + B_k(i)F_\tau(t, i))^T \mathcal{E}_i(X_{\tau+1}(t + 1))(A_k(i)) \\ &+ B_k(i)F_\tau(t, i)) + (C(i) + D(i)F_\tau(t, i))^T (C(i) + D(i)F_\tau(t, i)) \\ &- \gamma^2 F_\tau^T(t, i)F_\tau(t, i) - (F_\tau(t, i) - F_{\tau+1}(t, i))^T \mathfrak{H}_\gamma \\ &\times (X_{\tau+1}(t + 1), i)(F_\tau(t, i) - F_{\tau+1}(t, i), \qquad 1 \leq i \leq N, \quad (8.49) \end{aligned}$$

where $F_{\tau+1}(t, i)$ is constructed as in (8.37) with $X_{\tau+1}(t)$ instead of $X_\tau(t)$.

Let $Z_\tau(t, i) = X_{\tau+1}(t, i) - X_\tau(t, i)$, $0 \leq t \leq \tau + 1$, $1 \leq i \leq N$.

Subtracting (8.38) from (8.49) we obtain that $Z_\tau(t) = (Z_\tau(t, 1), \ldots, Z_\tau(t, N))$ is the solution of the problem:

$$\begin{aligned} Z_\tau(t, i) &= \sum_{k=0}^{r} (A_k(i) + B_k(i)F_\tau(t, i))^T \mathcal{E}_i(Z_\tau(t + 1)) \\ &\qquad \times (A_k(i) + B_k(i)F_\tau(t, i)) + H_\tau(t, i) \\ Z_\tau(\tau + 1, i) &= X_{\tau+1}(\tau + 1, i) \geq 0, \qquad (8.50) \end{aligned}$$

where $H_\tau(t, i) = -(F_\tau(t, i) - F_{\tau+1}(t, i))^T \mathfrak{H}_\gamma(X_{\tau+1}(t + 1), i)(F_\tau(t, i) - F_{\tau+1}(t, i))$.

Having in mind (8.36) written for $X_{\tau+1}(t)$ instead of $X_\tau(t)$ we deduce that $H_\tau(t, i) \geq 0$ for all $0 \leq t \leq \tau + 1$, $i \in \mathcal{D}$. Thus (8.50) allows us to deduce recursively that $Z_\tau(t, i) \geq 0$ for $0 \leq t \leq \tau + 1$.

This means that $X_\tau(t, i) \leq X_{\tau+1}(t, i)$, $0 \leq t \leq \tau + 1$, $i \in \mathcal{D}$. Particularly $X_\tau(1, i) \leq X_{\tau+1}(1, i)$, $i \in \mathcal{D}$.

Using (8.46) we see that the above inequality is equivalent to $K_0(\tau, i) \leq K_0(\tau + 1, i)$, $i \in \mathcal{D}$, $\tau \geq 1$. We further consider $v_\tau = \{v_\tau(t)\}_{0 \leq t \leq \tau}$ defined by $v_\tau(t) = F_\tau(t, \eta_t)x_\tau(t)$, where $x_\tau(t)$ is the solution of

$$\begin{aligned} x_\tau(t + 1) &= \left[A_0(\eta_t) + B_0(\eta_t)F_\tau(t, \eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(i) + B_k(i)F_\tau(t, i)) \right] x_\tau(t) \\ x_\tau(0) &= x_0. \qquad (8.51) \end{aligned}$$

Let $\overline{v}_\tau = \{\overline{v}_\tau(t)\}_{t\geq 0} \in \ell^2_{\mathcal{H}}\{0,\infty;\mathbf{R}^{m_v}\}$ be the natural extension of v_τ taking $\overline{v}_\tau(t) = 0$ for $t \geq \tau+1$.

Applying Corollary 8.3 and Lemma 8.3 we may write successively

$$\pi_0(i)x_0^T X_\tau(0,i)x_0 \leq E[x_0^T X_\tau(0,\eta_0)x_0]$$
$$= \tilde{J}_\gamma(\tau,x_0,v_\tau) \leq \tilde{J}_\gamma(\infty;x_0,\overline{v}_\tau) \leq \rho|x_0|^2$$

for all $x_0 \in \mathbf{R}^n$, $i \in \mathcal{D}$. Hence

$$\pi_0(i)x_0^T X_\tau(0,i)x_0 \leq \rho|x_0|^2 \tag{8.52}$$

for all $x_0 \in \mathbf{R}^n$, $i \in \mathcal{D}$, and for all initial distribution $\pi_0 = (\pi_0(1),\dots,\pi_0(N))$ with $\pi_0(i) > 0$. Particularly, (8.52) is valid for the special case $\pi_0(i) = 1/N$.

This leads to $x_0^T X_\tau(0,i)x_0 \leq N\rho|x_0|^2$ for all $i \in \mathcal{D}$. Thus $x_0^T K_0(\tau+1,i)x_0 \leq c|x_0|^2 \forall\ \tau \geq 1$, $i \in \mathcal{D}$, $x_0 \in \mathbf{R}^n$, where $c = N\rho$. Thus the proof is complete. □

Let us consider the following system of discrete-time coupled algebraic Riccati equations (DTSARE),

$$\begin{aligned} X(i) &= \sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)A_k(i) + C^T(i)C(i) \\ &\quad - \left(\sum_{k=0}^{r} A_k^T(i)\mathcal{E}_i(X)B_k(i) + C^T(i)D(i)\right) \\ &\quad \times \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v}\right)^{-1} \\ &\quad \times \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)A_k(i) + D^T(i)C(i)\right). \end{aligned} \tag{8.53}$$

We have the following corollary.

Corollary 8.8 *Under the assumptions of Proposition* 8.3 *the DTSARE* (8.53) *has a solution* $\tilde{X} = (\tilde{X}(1),\dots,\tilde{X}(N)) \in \mathcal{S}_n^{N+}$ *with the additional property*

$$\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X)B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v} < 0, \tag{8.54}$$

$1 \leq i \leq N$.

Proof. From Proposition 8.3 one obtains that the sequence $\{K_0(\tau,i)\}_{\tau\geq 1}$, $1 \leq i \leq N$ is convergent. Let $\tilde{X}(i) = \lim_{\tau\to\infty} K_0(\tau,i)$. Taking the limit for $t \to \infty$ in (8.47) one obtains that $\tilde{X} = (\tilde{X}(1),\dots,\tilde{X}(N))$ is a solution of DTSARE (8.53). Finally, taking the limit for $t \to \infty$ in (8.48) we deduce that (8.54) is fulfilled. The proof ends. □

We recall that a solution $X_s = (X_s(1), \dots, X_s(N))$ of the DTSARE (8.53) is a stabilizing solution if the zero state equilibrium of the closed-loop system $x_s(t+1) = [A_0(\eta_t) + B_0(\eta_t)F_s(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F_s(\eta_t))]x_s(t)$ is ESMS, where

$$F_s(i) = -\left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)B_k(i) + D^T(i)D(i) - \gamma^2 I_{m_v}\right)^{-1} \times \left(\sum_{k=0}^{r} B_k^T(i)\mathcal{E}_i(X_s)A_k(i) + D^T(i)C(i)\right), \tag{8.55}$$

$1 \le i \le N$.

The main result of this section is the following.

Theorem 8.13 (Bounded real lemma) *Under the considered assumptions, for a given scalar $\gamma > 0$, the following are equivalent.*

(*i*) *The zero state equilibrium of* (8.6) *is ESMS and the input–output operator $\mathcal{T}$ defined by the system* (8.4) *satisfies $\|\mathcal{T}\| < \gamma$.*

(*ii*) *There exists $X = (X(1), \dots, X(N)) \in \mathcal{S}_n^N$, $X(i) > 0$, $1 \le i \le N$, which solves the following systems of LMIs,*

$$\begin{pmatrix} \Pi_{1i}X - X(i) + C^T(i)C(i) & \Pi_{2i}X + C^T(i)D(i) \\ (\Pi_{2i}X + C^T(i)D(i))^T & \Pi_{3i}X + D^T(i)D(i) - \gamma^2 I_{m_v} \end{pmatrix} < 0, \quad 1 \le i \le N, \tag{8.56}$$

where the operators Π_{li} are introduced by (8.15).

(*iii*) *The DTSARE* (8.53) *has a stabilizing solution $\tilde{X} = (\tilde{X}(1), \dots, \tilde{X}(N)) \in \mathcal{S}_n^{N+}$ that satisfies* (8.54).

(*iv*) *There exists $X = (X(1), \dots, X(N)) \in \mathcal{S}_n^N$, $X(i) > 0$, $1 \le i \le N$ that satisfies the following system of LMIs.*

$$\begin{pmatrix} -X(i) & O_{n,m_v} & \mathcal{M}_{0i}(X) & \mathcal{M}_{1i}(X) & \dots & \mathcal{M}_{ri}(X) & C^T(i) \\ O_{m_v,n} & -\gamma^2 I_{m_v} & \tilde{\mathcal{M}}_{0i}(X) & \tilde{\mathcal{M}}_{1i}(X) & \dots & \tilde{\mathcal{M}}_{ri}(X) & D^T(i) \\ \mathcal{M}_{0i}^T(X) & \tilde{\mathcal{M}}_{0i}^T(X) & -\mathcal{X} & O_{nN,nN} & \dots & O_{nN,nN} & O_{nN,n_z} \\ \mathcal{M}_{1i}^T(X) & \tilde{\mathcal{M}}_{1i}^T(X) & O_{nN,nN} & -\mathcal{X} & \dots & O_{nN,nN} & O_{nN,n_z} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathcal{M}_{ri}^T(X) & \tilde{\mathcal{M}}_{ri}^T(X) & O_{nN,nN} & O_{nN,nN} & \dots & -\mathcal{X} & O_{nN,n_z} \\ C(i) & D(i) & O_{n_z,nN} & O_{n_z,nN} & \dots & O_{n_z,nN} & -I_{n_z} \end{pmatrix} < 0, \tag{8.57}$$

$1 \leq i \leq N$, *where*

$$\mathcal{M}_{ki}(X) = \big(\sqrt{p(i,1)}A_k^T(i)X(1) \quad \sqrt{p(i,2)}A_k^T(i)X(2) \quad \dots \quad \sqrt{p(i,N)}A_k^T(i)X(N) \big)$$

$$\tilde{\mathcal{M}}_{ki}(X) = \big(\sqrt{p(i,1)}B_k^T(i)X(1) \quad \sqrt{p(i,2)}B_k^T(i)X(2) \big) \quad \dots \quad \sqrt{p(i,N)}B_k^T(i)X(N) \big), \qquad 0 \leq k \leq r,$$

$$\mathcal{X} = \mathrm{diag}(X(1), \dots, X(N)) \in \mathcal{S}_{nN}.$$

(v) *There exists* $Y = (Y(1), \dots, Y(N))$, $Y(i) > 0$, $1 \leq i \leq N$ *that solves the following system of LMIs.*

$$\begin{pmatrix} -Y(i) & O_{n,m_v} & \Psi_{0i}(Y) & \Psi_{1i}(Y) & \dots & \Psi_{ri}(Y) & Y(i)C^T(i) \\ O_{m_v,n} & -\gamma^2 I_{m_v} & \tilde{\Psi}_{0i} & \tilde{\Psi}_{1i} & \dots & \tilde{\Psi}_{ri} & D^T(i) \\ \Psi_{0i}^T(Y) & \tilde{\Psi}_{0i}^T & -\mathcal{Y} & O_{nN,nN} & \dots & O_{nN,nN} & O_{nN,n_z} \\ \Psi_{1i}^T(Y) & \tilde{\Psi}_{1i}^T & O_{nN,nN} & -\mathcal{Y} & \dots & O_{nN,nN} & O_{nN,n_z} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Psi_{ri}^T(Y) & \tilde{\Psi}_{ri}^T & O_{nN,nN} & O_{nN,nN} & \dots & -\mathcal{Y} & O_{nN,n_z} \\ C(i)Y(i) & D(i) & O_{n_z,nN} & O_{n_z,nN} & \dots & O_{n_z,nN} & -I_{n_z} \end{pmatrix} < 0, \tag{8.58}$$

$1 \leq i \leq N$, *where*

$$\Psi_{ki}(Y) = \big(\sqrt{p(i,1)}Y(i)A_k^T(i) \quad \sqrt{p(i,2)}Y(i)A_k^T(i) \quad \dots \quad \sqrt{p(i,N)}Y(i)A_k^T(i) \big),$$

$$\tilde{\Psi}_{ki} = \big(\sqrt{p(i,1)}B_k^T(i) \quad \sqrt{p(i,2)}B_k^T(i) \quad \dots \quad \sqrt{p(i,N)}B_k^T(i) \big),$$

$$0 \leq k \leq r, \qquad 1 \leq i \leq N,$$

$\mathcal{Y} = \mathrm{diag}(Y(1), \dots, Y(N)) \in \mathcal{S}_{nN}$

(vi) *There exists* $Y = (Y(1), Y(2), \dots, Y(N)) \in \mathcal{S}_n^N$, $Y(i) > 0$, $1 \leq i \leq N$, *that solves the following system of LMIs.*

$$\begin{pmatrix} -Y(i) & \Psi_{0i}(Y) & \Psi_{1i}(Y) & \dots & \Psi_{ri}(Y) & Y(i)C^T(i) \\ \Psi_{0i}^T(Y) & \mathfrak{G}_{00}(i) - \mathcal{Y} & \mathfrak{G}_{01}(i) & \dots & \mathfrak{G}_{0r}(i) & \mathfrak{G}_{0r+1}(i) \\ \Psi_{1i}^T(Y) & \mathfrak{G}_{01}^T(i) & \mathfrak{G}_{11}(i) - \mathcal{Y} & \dots & \mathfrak{G}_{1r}(i) & \mathfrak{G}_{1r+1}(i) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \Psi_{ri}^T(Y) & \mathfrak{G}_{0r}^T(i) & \mathfrak{G}_{1r}^T(i) & \dots & \mathfrak{G}_{rr}(i) - \mathcal{Y} & \mathfrak{G}_{rr+1}(i) \\ C(i)Y(i) & \mathfrak{G}_{0r+1}^T(i) & \mathfrak{G}_{1r+1}^T(i) & \dots & \mathfrak{G}_{rr+1}^T(i) & D(i)D^T(i) - \gamma^2 I_{n_z} \end{pmatrix} < 0, \tag{8.59}$$

where $\Psi_{li}(Y)$ *and* $\mathcal{Y}$ *are as before, whereas*

$$\mathfrak{G}_{lk}(i) = \mathfrak{I}^T(i)B_l(i)B_k^T(i)\mathfrak{I}(i), \qquad 0 \le l \le k \le r,$$

$$\mathfrak{G}_{lr+1}(i) = \mathfrak{I}^T(i)B_l(i)D^T(i)$$

and

$$\mathfrak{I}(i) = \left(\sqrt{p(i,1)}I_n \quad \sqrt{p(i,2)}I_n \quad \dots \quad \sqrt{p(i,N)}I_n \right).$$

Proof. Let us assume that (i) holds. If $\delta > 0$ denotes $\mathcal{T}_\delta : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n+n_z}\}$ the linear operator defined by $v \to (\mathcal{T}_\delta v)(t) = C_\delta(\eta_t) x(t,0,v) + D_\delta(\eta_t)v(t)$ where $x(t,0,v)$ is the zero initial value solution of (8.4) corresponding to the input v and

$$C_\delta(i) = \begin{pmatrix} C(i) \\ \delta I_n \end{pmatrix}, D_\delta(i) = \begin{pmatrix} D(i) \\ 0 \end{pmatrix}.$$

Based on (8.7) we deduce that for $\delta > 0$ sufficiently small we have $\|\mathcal{T}_\delta\| < \gamma$. Applying Corollary 8.4 we deduce that there exists $X_\delta = (X_\delta(1),\dots,X_\delta(N))$, $X_\delta(i) \ge 0$ solving the DTSARE:

$$\begin{aligned} X_\delta(i) = \Pi_{1i}X_\delta - (\Pi_{2i}X_\delta + C^T(i)D(i))(\Pi_{3i}X_\delta + D^T(i)D(i) - \gamma^2 I_{m_v})^{-1} \\ \times (\Pi_{2i}X_\delta + C^T(i)D(i))^T + C^T(i)C(i) + \delta^2 I_n, \qquad 1 \le 1 \le N, \end{aligned} \tag{8.60}$$

with the additional property

$$\Pi_{3i}X_\delta + D^T(i)D(i) - \gamma^2 I_{m_v} < 0, \qquad 1 \le 1 \le N. \tag{8.61}$$

Because the right-hand side of (8.60) is positive definite it follows that $X_\delta(i) > 0$, $1 \le i \le N$. Also (8.60) implies

$$\begin{aligned} \Pi_{1i}X_\delta - X_\delta(i) + C^T(i)C(i) - (\Pi_{2i}X_\delta + C^T(i)D(i)) \\ \times (\Pi_{3i}X_\delta + D^T(i)D(i) - \gamma^2 I_{m_v})^{-1} \\ \times (\Pi_{2i}X_\delta + C^T(i)D(i))^T < 0, \qquad 1 \le 1 \le N. \end{aligned} \tag{8.62}$$

By a Schur complement technique one obtains that (8.61) and (8.62) are equivalent to (8.56) and thus the proof of the implication $(i) \to (ii)$ is complete. To prove the converse implication, $(ii) \to (i)$, we remark that if (ii) is fulfilled then the (1; 1) block of (8.56) is negative definite. Thus we obtained that there exists $X = (X(1),\dots,X(N)) \in \mathcal{S}_n^N$ with $X(i) > 0$, such that $X(i) > \sum_{k=0}^r A_k^T(i)\mathcal{E}_i(X)A_k(i)$, $1 \le i \le N$. Applying Corollary 3.6 we deduce that the zero state equilibrium of the system (8.6) is ESMS. Furthermore, applying Corollary 8.1 for $X(t,i) = X(i)$, $0 \le t \le \tau$, $\tau \ge 1$, $1 \le i \le N$, and taking the limit for $\tau \to \infty$ we have:

$$\tilde{J}_\gamma(\infty; 0, v) = \sum_{t=0}^{\infty} E\left[\begin{pmatrix} x(t,0,v) \\ v(t) \end{pmatrix}^T \mathbf{Q}(X, \eta_t) \begin{pmatrix} x(t,0,v) \\ v(t) \end{pmatrix}\right], \tag{8.63}$$

where $\mathbf{Q}(X, i)$ is the left-hand side of (8.56). If $X = (X(1), \ldots, X(N))$ verifies (8.56) then for $\varepsilon > 0$ small enough we have

$$\mathbf{Q}(X, i) \leq -\varepsilon^2 I_{n+m_v}, \qquad 1 \leq 1 \leq N. \tag{8.64}$$

Combining (8.63) and (8.64) we deduce

$$\tilde{J}_\gamma(\infty; 0, v) \leq -\varepsilon^2 \sum_{t=0}^{\infty} E[|x(t,0,v)|^2] + E[|v(t)|^2]$$

or equivalently

$$\tilde{J}_{\tilde{\gamma}}(\infty; 0, v) \leq -\varepsilon^2 \sum_{t=0}^{\infty} E[|x(t,0,v)|^2] < 0$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$ where $\tilde{\gamma} = (\gamma^2 - \varepsilon^2)^{1/2}$.

The last inequality may be written:

$$\|\mathcal{T} v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}} \leq \tilde{\gamma}^2 \|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}}$$

for all $v \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$. This leads to $\|\mathcal{T}\|^2 \leq \gamma^2 - \varepsilon^2$ and thus the implication $(ii) \to (i)$ is proved.

To prove the equivalence $(ii) \leftrightarrow (iii)$ let us consider the DTSGRE:

$$\begin{aligned} X(t) = &\Pi_1 X(t+1) - M - (\Pi_2 X(t+1) - L) \\ &\times (\Pi_3 X(t+1) - R)^{-1} (\Pi_2 X(t+1) - L)^T, \end{aligned} \tag{8.65}$$

where M, L, R are defined as in the case of (8.34). One can see that (8.65) is a nonlinear equation of type (5.8) defined by the pair $\Sigma = (\Pi, \mathcal{Q})$ with

$$\mathcal{Q} = \begin{pmatrix} -M & -L \\ -L^T & -R \end{pmatrix} \in \mathcal{S}^N_{n+m_v}.$$

One can check that if $X = (X(1), X(2), \ldots, X(N))$ solves (8.56) then $\hat{X} = (\hat{X}(1), \ldots, \hat{X}(N))$ with $\hat{X}(i) = -X(i)$, $i \in \mathcal{D}$ belongs to the set $\tilde{\Gamma}^\Sigma$ associated by (5.16) with equation (8.65). Also if (ii) is fulfilled then from the (1, 1) bolock of (8.56) one obtains that $\Pi_{1i} X - X(i) < 0$, $1 \leq i \leq N$. Using the implication $(vii) \to (i)$ of Theorem 2.4 in the special case of the positive operator Π_1 we deduce that the eigenvalues of this operator are located in the inside of the disk $|\lambda| < 1$. This means that the operator Π defined by

$$\Pi X = \begin{pmatrix} \Pi_1 X & \Pi_2 X \\ (\Pi_2 X)^T & \Pi_3 X \end{pmatrix}$$

is stabilizable. Thus we obtain that if (ii) is fulfilled then in case of DTSGRE (8.65) assertion (i) in Theorem 5.6 is fulfilled. Hence, (8.65) has a bounded and stabilizing solution $\{X_s(t)\}_{t\geq 0}$ that satisfies $\Pi_3 X_s(t+1) - R(i) > 0$, $1 \leq i \leq N$, $t \geq 0$.

On the other hand, because the coefficients of (8.65) are constant we obtain via Theorem 5.5, for $\theta = 1$ that $X_s(t)$ is constant.

A simple computation shows that $\tilde{X} = (\tilde{X}(1), \dots, \tilde{X}(N))$ defined by $\tilde{X}(i) = -X_s(i)$ is the stabilizing solution of DTSARE (8.53) that satisfies (8.54). The eigenvalues of the positive operator Π_1 are located in the inside of the disk $|\lambda| < 1$ from Theorem 2.5 thus it follows that $\tilde{X}(i) \geq 0, i \in \mathcal{D}$ and then $(ii) \to (iii)$ is true.

Conversely, if (iii) is fulfilled we deduce via Proposition 5.2 for $\theta = 1$, that there exists $\hat{X} = (\hat{X}(1), \dots, \hat{X}(N)) \in \tilde{\Gamma}^{\Sigma}$. This means that the constant sequence $\hat{X}$ verifies a condition of type (5.16) associated with the special case of DTSGRE (8.65). One obtains that $X = (X(1), \dots, X(N))$ defined by $X(i) = -\hat{X}(i)$, $i \in \mathcal{D}$, solves (8.56). Hence $(iii) \to (ii)$ is true.

The equivalence $(ii) \leftrightarrow (iv)$ follows immediately by a Schur complement technique.

If (iv) is true, then multiplying at the left and at the right of (8.57) with $\text{diag}(X^{-1}(i), I_{m_v}, \mathcal{X}^{-1}, \dots, \mathcal{X}^{-1}, I_{n_z})$, one obtains that (8.58) is feasible with $Y(i) = X^{-1}(i)$. Conversely, if (v) is fulfilled then multiplying (8.58) at the left and at the right by $\text{diag}(Y^{-1}(i), I_{m_v}, \mathcal{Y}^{-1}, \mathcal{Y}^{-1}, \dots, \mathcal{Y}^{-1}, I_{n_z})$ we deduce that (8.57) is feasible with $X(i) = Y^{-1}(i)$. This shows that $(iv) \longleftrightarrow (v)$. If (v) holds then taking the Schur complement of the block $-\gamma^2 I_{m_v}$ one obtains that (8.58) is equivalent to

$$\left(\begin{array}{cccc} -Y(i) & \Psi_{0i}(Y) & \Psi_{1i}(Y) & \dots \\ \Psi_{0i}^T(Y) & \gamma^{-2}\mathfrak{G}_{00}(i) - \mathcal{Y} & \gamma^{-2}\mathfrak{G}_{01}(i) & \dots \\ \Psi_{1i}^T(Y) & \gamma^{-2}\mathfrak{G}_{01}^T(i) & \gamma^{-2}\mathfrak{G}_{11}(i) - \mathcal{Y} & \dots \\ \dots & \dots & \dots & \dots \\ \Psi_{ri}^T(Y) & \gamma^{-2}\mathfrak{G}_{0r}^T(i) & \gamma^{-2}\mathfrak{G}_{1r}^T(i) & \dots \\ C(i)Y(i) & \gamma^{-2}\mathfrak{G}_{0r+1}^T(i) & \gamma^{-2}\mathfrak{G}_{1r+1}^T(i) & \dots \end{array}\right.$$

$$\left.\begin{array}{cc} \Psi_{ri}(Y) & Y(i)C^T(i) \\ \gamma^{-2}\mathfrak{G}_{0r}(i) & \gamma^{-2}\mathfrak{G}_{0r+1}(i) \\ \gamma^{-2}\mathcal{G}_{1r}(i) & \gamma^{-2}\mathfrak{G}_{1r+1}(i) \\ \dots & \dots \\ \gamma^{-2}\mathfrak{G}_{rr}(i) - \mathcal{Y} & \gamma^{-2}\mathfrak{G}_{rr+1}(i) \\ \gamma^{-2}\mathfrak{G}_{rr+1}^T(i) & \gamma^{-2}D(i)D^T(i) - I_{n_z} \end{array}\right) < 0.$$

Pre- and postmultiplying this inequality by $\gamma I_{\tilde{n}}$ ($\tilde{n} = n[1+(r+1)N]+n_z$) and taking $\gamma^2 Y(i)$ as new variables one obtains to (8.58) is equivalent to (8.59). This completes the proof. □

If the system (8.4) is either in the case $N = 1$ or $N \geq 2$, with $A_k(i) = 0$, $B_k(i) = 0$, $1 \leq k \leq r$, $i \in \mathcal{D}$, the result proved in Theorem 8.2 recovers as special cases the stochastic version of the bounded real lemma for discrete-time linear stochastic systems perturbed by independent random perturbations and the discrete-time linear stochastic systems with Markovian switching, respectively.

Let us remark that if the zero state equilibrium of (8.6) is ESMS from Theorem 8.2, it follows that

$$\begin{aligned}\|\mathcal{T}\| &= \inf\{\gamma > 0, \text{ for which exists } X \in \mathcal{S}_n^N, X > 0,\\ &\qquad \text{such that (8.56) holds}\}\\ &= \inf\{\gamma > 0, \text{ DTSARE (8.53) has a positive semidefinite}\\ &\qquad \text{solution verifying (8.54)}\}.\end{aligned}$$

8.3.3 An H_∞-type filtering problem

Consider the ESMS discrete-time system with state-dependent noise

$$\begin{aligned}x_{k+1} &= A_0 x_k + A_1 x_k \xi_k + B u_k, \quad k = 0, 1, \dots\\ y_k &= C_0 x_k + C_1 x_k \xi_k + D u_k,\end{aligned} \tag{8.66}$$

where $x_k \in \mathbf{R}^n$ is the state vector at moment k, $y_k \in \mathbf{R}^p$ represents the measured output and $\xi_k \in \mathbf{R}$, $k = 0, 1, \dots$ are independent random variables with zero mean and unit covariance. It is assumed that the exogenous signals are energy bounded, namely $u_k \in \ell^2\{0, \infty; \mathbf{R}^m\}$. The problem considered in this section has the following statement. Given the ESMS system (8.66) determine, if possible, a stable system of the form

$$\begin{aligned}x_{f,k+1} &= A_f x_{f,k} + B_f y_k, \qquad k = 0, 1, \dots\\ y_{f,k} &= C_f x_k\end{aligned} \tag{8.67}$$

with the specified order $n_f \geq 1$, such that

$$J_\gamma := \sum_{k=0}^{\infty} E[|z_k|^2 - \gamma^2 |u_k|^2] < 0, \tag{8.68}$$

where

$$z_k := y_{f,k} - H x_k \tag{8.69}$$

denotes a quality output, $H \in \mathbf{R}^{r \times n}$, $\gamma > 0$ is a given level of attenuation, and x_k is the solution of (8.66) with zero initial condition.

The solution of the above filtering problem is derived using the following result which is a direct consequence of Theorem 8.2(i), (ii) in the particular case when the Markovian jumps are missing.

Corollary 8.9 *Assume that the system with multiplicative noise*

$$\begin{aligned} x_{k+1} &= A_0 x_k + A_1 x_k \xi_k + B u_k, \qquad k = 0, 1, \ldots \\ y_k &= C x_k \end{aligned}$$

is ESMS. Then $\sum_{k=0}^{\infty} E[|y_k|^2 - \gamma^2 |u_k|^2] < 0$ *for a given* $\gamma > 0$*, where* x_k *is the solution of the above system with zero initial condition, if and only if there exists a symmetric matrix* $X > 0$ *such that*

$$-X + A_0^T X A_0 + A_1^T X A_1 + A_0^T X B(\gamma^2 I - B^T X B)^{-1} B^T X A_0 + C^T C < 0.$$

Remark 8.8 The filter (8.67) has a deterministic structure. A more complex structure including state-dependent noise terms in (8.67) may provide better filtering performance but in such a situation implementation problems occur because these noises cannot be directly measured.

From (8.66), (8.67), and (8.69) one obtains the resulting system

$$\begin{aligned} \tilde{x}_{k+1} &= \mathcal{A}_0 \tilde{x}_k + \mathcal{A}_1 \tilde{x}_k \xi_k + \mathcal{B} u_k, \qquad k = 0, 1, \ldots \\ z_k &= \mathcal{C} \tilde{x}_k, \end{aligned} \tag{8.70}$$

where $\tilde{x}_k := [x_k^T \quad x_{f,k}^T]^T$ and

$$\mathcal{A}_0 = \begin{pmatrix} A_0 & 0 \\ B_f C_0 & A_f \end{pmatrix}, \qquad \mathcal{A}_1 = \begin{pmatrix} A_1 & 0 \\ B_f C_1 & 0 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} B \\ B_f D \end{pmatrix}, \qquad \mathcal{C} = \begin{pmatrix} -H & C_f \end{pmatrix}. \tag{8.71}$$

Applying Corollary 8.5 for the resulting system (8.70) it follows that the considered H_∞-type filtering problem is feasible if and only if there exists a symmetric matrix $X \in \mathbf{R}^{(n+n_f)\times(n+n_f)}$, $X > 0$ such that

$$\begin{aligned} &- X + \mathcal{A}_0^T X \mathcal{A}_0 + \mathcal{A}_1^T X \mathcal{A}_1 \\ &\quad + \mathcal{A}_0^T X \mathcal{B}(\gamma^2 I - \mathcal{B}^T X \mathcal{B})^{-1} \mathcal{B}^T X \mathcal{A}_0 + \mathcal{C}^T \mathcal{C} < 0. \end{aligned} \tag{8.72}$$

Based on the Schur complement formula, the above condition is equivalent to

$$\begin{pmatrix} -X + \mathcal{C}^T \mathcal{C} & \mathcal{A}_0^T & \mathcal{A}_1^T & \mathcal{A}_0^T X \mathcal{B} \\ \mathcal{A}_0 & -X^{-1} & 0 & 0 \\ \mathcal{A}_1 & 0 & -X^{-1} & 0 \\ \mathcal{B}^T X \mathcal{A}_0 & 0 & 0 & -(\gamma^2 I - \mathcal{B}^T X \mathcal{B}) \end{pmatrix} < 0. \tag{8.73}$$

In the above inequality the unknown variables are X, A_f, B_f, and C_f. Without reducing the generality of the problem one can chose B_f as an arbitrary full rank matrix. Indeed if the filtering problem has a solution with B_f having not full rank then there always exists a small enough perturbation of B_f for which it has full rank and (8.73) is fulfilled. Therefore it remains to solve (8.73) with respect to X, A_f, and C_f.

Denoting

$$\Omega := \begin{pmatrix} A_f \\ C_f \end{pmatrix}, \qquad X := \begin{pmatrix} R & M \\ M^T & S \end{pmatrix}, \tag{8.74}$$

direct algebraic computations together with the Schur complement formula show that condition (8.73) can be written as

$$Z + P^T \Omega^T Q + Q^T \Omega P < 0, \tag{8.75}$$

where the following notations have been used.

$$Z := \begin{pmatrix} Z_{11} & -M & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & 0 \\ -M^T & -S & 0 & 0 & 0 & 0 & 0 & 0 \\ Z_{13}^T & 0 & -R & -M & 0 & 0 & 0 & 0 \\ Z_{14}^T & 0 & -M^T & -S & 0 & 0 & 0 & 0 \\ Z_{15}^T & 0 & 0 & 0 & -R & -M & 0 & 0 \\ Z_{16}^T & 0 & 0 & 0 & -M^T & -S & 0 & 0 \\ Z_{17}^T & 0 & 0 & 0 & 0 & 0 & Z_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -I \end{pmatrix}$$

with

$$Z_{11} := -R + H^T H$$

$$Z_{13} := A_0^T R + C_0^T B_f^T M^T$$

$$Z_{14} := A_0^T M + C_0^T B_f^T S$$

$$Z_{15} := A_1^T R + C_1^T B_f^T M^T$$

$$Z_{16} := A_1^T M + C_1^T B_f^T S$$

$$Z_{17} := (A_0^T R + C_0^T B_f^T M^T)B + (A_0^T M + C_0^T B_f^T S)B_f D$$

$$Z_{77} := -(\gamma^2 I - B^T R B - D^T B_f^T M^T B - B^T M B_f D - D^T B_f^T S B_f D)$$

and where

$$P := \begin{pmatrix} 0 & 0 & M^T & S & 0 & 0 & M^T B + SB_f D & 0 \\ -H & 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \tag{8.76}$$

and

$$Q := \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{8.77}$$

According to the so-called projection lemma (see, e.g., [14]), there exists Ω satisfying (8.75) if and only if the following two conditions are accomplished,

$$W_P^T Z W_P < 0 \tag{8.78}$$

and

$$W_Q^T Z W_Q < 0, \tag{8.79}$$

where W_P and W_Q denote bases of the null spaces of P and Q, respectively. Furthermore, conditions (8.78) and (8.79) are explicit. Firstly, taking into account (8.76) it results that a basis of the null space of P is

$$W_P = \begin{pmatrix} 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 & 0 & I \\ -B_f D & 0 & 0 & 0 & 0 & -S^{-1}M^T \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H & 0 \end{pmatrix}.$$

Then direct algebraic computations together with Schur complement arguments give that the inequality (8.78) is equivalent to

$$\begin{pmatrix} -\hat{R} + \gamma^{-2} B B^T & 0 & 0 & A_0 \hat{R} \\ 0 & -\hat{R} & -N & A_1 \hat{R} \\ 0 & -N^T & -\hat{S} & B_f C_1 \hat{R} \\ \hat{R} A_0^T & \hat{R} A_1^T & \hat{R} C_1^T B_f^T & -\hat{R} \end{pmatrix} < 0, \tag{8.80}$$

where $\hat{R} \in \mathbf{R}^{n \times n}$, $N \in \mathbf{R}^{n \times n_f}$, and $\hat{S} \in \mathbf{R}^{n_f \times n_f}$ are the block elements of X^{-1}, namely

$$X^{-1} := \begin{pmatrix} \hat{R} & N \\ N^T & \hat{S} \end{pmatrix}. \tag{8.81}$$

In addition, because

$$W_Q = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix},$$

one obtains that condition (8.79) is equivalent to

$$\begin{pmatrix} -R + H^T H & 0 & A_0^T & C_0^T B_f^T & A_1^T & C_1^T B_f^T \\ 0 & -\gamma^2 I & B^T & D^T B_f^T & 0 & 0 \\ A_0 & B & -\hat{R} & -N & 0 & 0 \\ B_f C_0 & B_f D & -N^T & -\hat{S} & 0 & 0 \\ A_1 & 0 & 0 & 0 & -\hat{R} & -N \\ B_f C_1 & 0 & 0 & 0 & -N^T & -\hat{S} \end{pmatrix} < 0. \qquad (8.82)$$

The above developments are concluded in the following result.

Theorem 8.14 *The H_∞-type filtering problem formulated in Section* 8.2 *has an n_f-order solution if and only if there exists a symmetric matrix $X > 0, X \in \mathbf{R}^{(n+n_f)\times(n+n_f)}$ such that its block element R from partition* (8.74) *and the block elements $\hat{R}, N$, and $\hat{S}$ of X^{-1} defined by* (8.81)*, verify the inequalities* (8.80) *and* (8.82)*.* □

Remark 8.9 As in standard design methodology based on linear matrix inequalities (LMIs), one firstly must solve the system (8.80), (8.82) with respect with $R, \hat{R}, N$, and $\hat{S}$, and then determine Ω from the basic LMI (8.75). The system (8.80), (8.82) cannot be solved using the usual semidefinite programming-based methods because R and $\hat{R}, N, \hat{S}$ are related by the condition that X^{-1} is the inverse of X.

A simple method to avoid the computational problem mentioned in Remark 8.3 is to take into account that $R \geq \hat{R}^{-1}$. Indeed this follows using the Schur complement formula in the obvious inequality

$$\begin{pmatrix} X & I \\ I & X^{-1} \end{pmatrix} \geq 0.$$

Then one can state the following corollary.

Corollary 8.10 *If the system obtained from* (8.80), (8.82) *replacing* R *from the block* (1, 1) *of* (8.82) *by* $\hat{R}^{-1}$ *is feasible, then the filtering problem has an* n_f*-order solution.* □

The inequality (8.82) in which R is replaced by $\hat{R}^{-1}$ can be pre- and postmultiplied by diag($\hat{R}$, I, I, I, I, I) and using again a Schur complement argument, one obtains

$$\begin{pmatrix} -\hat{R} & 0 & \hat{R}A_0^T & \hat{R}C_0^TB_f^T & \hat{R}A_1^T & \hat{R}C_1^TB_f^T & \hat{R}H^T \\ 0 & -\gamma^2 I & B^T & D^TB_f^T & 0 & 0 & 0 \\ A_0\hat{R} & B & -\hat{R} & -N & 0 & 0 & 0 \\ B_fC_0\hat{R} & B_fD & -N^T & -\hat{S} & 0 & 0 & 0 \\ A_1\hat{R} & 0 & 0 & 0 & -\hat{R} & -N & 0 \\ B_fC_1\hat{R} & 0 & 0 & 0 & -N^T & -\hat{S} & 0 \\ H\hat{R} & 0 & 0 & 0 & 0 & 0 & -I \end{pmatrix} < 0. \quad (8.83)$$

The system (8.80), (8.83) can then be solved to respect to $\hat{R}, N$, and $\hat{S}$ such that $X^{-1} > 0$ using semidefinite programming-based algorithms.

Remark 8.10 Corollary 8.6 gives sufficient feasibility conditions for the considered H_∞ filtering problem and therefore the results obtained solving the system (8.80), (8.83) instead of (8.80), (8.82) may be conservative.

In order to illustrate some of the above developments consider a navigation problem consisting in determining an estimation of an airplane altitude using measurements from a barometric altimeter and from a RADAR altimeter. Due to some inherent sources of error, the barometric altimeter indication is altered by a bias error and by a small additive white noise ([63]). Its continuous-time dynamics may be approximated by the following state space equations

$$\begin{aligned} \dot{h} &= -\frac{1}{\tau}(h - v) \\ h_{\text{baro}} &= h + b + \eta, \end{aligned} \quad (8.84)$$

where v represents the commanded altitude, h is the true altitude, b denotes the bias, and η is a standard zero-mean white noise with known intensity R_1. On the other hand, the RADAR altimeter determines the altitude without bias but that a measurement noise that intensity increases with the altitude as follows,

$$h_{\text{radar}} = h(1 + \xi) + \nu \quad (8.85)$$

with $E[\nu(t)\nu(\tau)] = R_2\delta(t - \tau)$ and $E[\xi(t)\xi(\tau)] = R_3\delta(t - \tau)$.

If the bias b in the second equation (8.84) is approximated as the solution of the differential stochastic equation $\dot{b} = \sqrt{2}\bar{w}$, where $\bar{w}$ is a standard white noise independent of ξ, ν, and η, the following model is obtained.

$$\begin{pmatrix} \dot{h} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h \\ b \end{pmatrix} + \begin{pmatrix} -\frac{1}{\tau} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \bar{w} \\ \eta \\ \nu \end{pmatrix}$$

$$y = \begin{pmatrix} h_{\text{baro}} \\ h_{\text{radar}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h \\ b \end{pmatrix} + \begin{pmatrix} 0 & 0 & \sqrt{R_1} & 0 \\ 0 & 0 & 0 & \sqrt{R_2} \end{pmatrix} \begin{pmatrix} v \\ \bar{w} \\ \eta \\ \nu \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ \sqrt{R_3} & 0 \end{pmatrix} \begin{pmatrix} h \\ b \end{pmatrix} \xi. \tag{8.86}$$

The estimated state is the true altitude h. Sampling the above continuous-time system one obtains a discrete-time system of form (8.66). Because the above system is not stable, a small negative perturbation of the zero $(2,2)$ element in the state matrix has been introduced.

For a sampling period $T = 0.25$ sec and for the time constant $\tau = 30$ sec, $R_1 = R_2 = 0.4$, and $R_3 = 0.0016$ one obtains using Corollary 8.1 with the attenuation level $\gamma = 1.07$, an H_∞ filter the response of which is depicted in Figure 8.1.

8.4 Robust stability. An estimate of the stability radius

8.4.1 The small gain theorems

One of the important consequences of the bounded real lemma is the so-called small gain theorem. It is known that this result is a powerful tool in the derivation of some estimates of the stability radius with respect to several classes of parametric uncertainties. We start with an auxiliary result which is interesting in itself.

Theorem 8.15 *Regarding the system* (8.4) *we assume that the following assumptions are fulfilled.*

(*a*) *The number of inputs equals the number of outputs (i.e., $m_v = n_z = n$).*
(*b*) *The zero state equilibrium of the corresponding linear system* (8.6) *is ESMS.*
(*c*) *The input–output operator $\mathcal{T}$ associated with the system* (8.4) *satisfies* $\|\mathcal{T}\| < 1$.

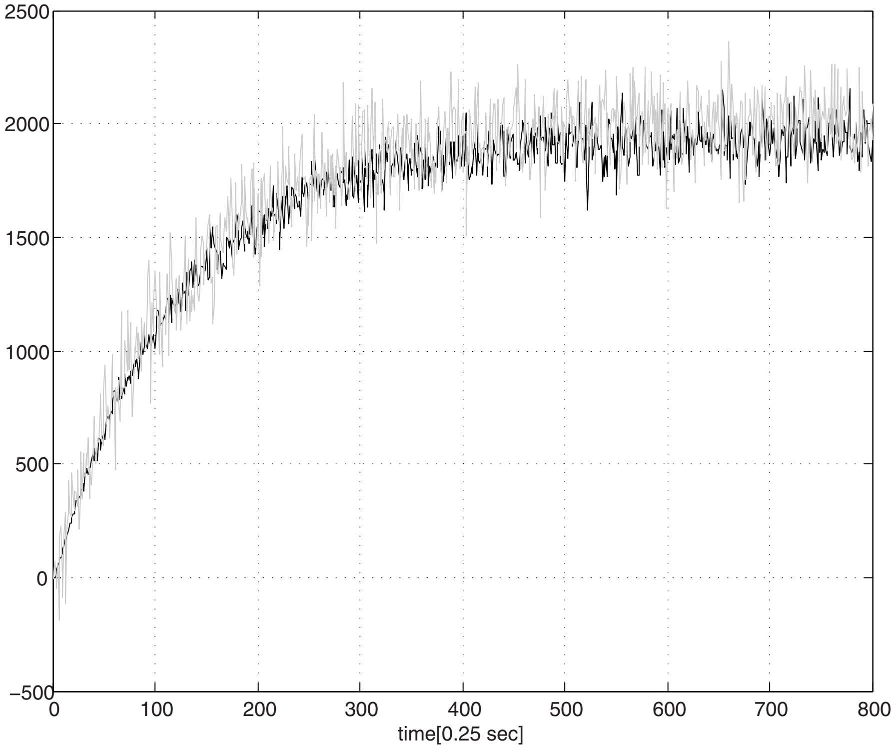

Figure 8.1. Filtered signal (black) and unfiltered (grey).

Under these assumptions we have:

(i) *The matrices* $I_m \pm D(i)$, $i \in \{1, 2, \dots, N\}$ *are invertible.*
(ii) *The zero state equilibrium of the system*

$$x(t+1) = \left(\overline{A}(\eta_t) + \sum_{k=1}^{r} w_k(t) \overline{A}_k(\eta_t) \right) x(t) \tag{8.87}$$

is ESMS, where either $\overline{A}_k(i) = A_k(i) - B_k(i)(I_m + D(i))^{-1} C(i)$ *or* $\overline{A}_k(i) = A_k(i) + B_k(i)(I_m - D(i))^{-1} C(i)$.

Proof. Based on (8.11) and assumption (c) we deduce that $\|\mathcal{T}_\tau\| < 1$ for any integer $\tau \geq 1$. Thus applying Corollary 8.2 one obtains that $I_m - D^T(i)D(i) > 0$, $i \in \{1, 2, \dots, N\}$. Therefore for each i the eigenvalues of the matrix $D(i)$ are located in the inside of the disk $|\lambda| < 1$. Hence $\det(I_m \pm D(i)) \neq 0$, $1 \leq i \leq N$. Thus we obtain that (i) is true. To prove (ii) we use the implication (i) $\to$ (ii) of Theorem 8.2. Thus if the assumptions (b) and (c) are fulfilled, then there exist $X = (X(1), \dots, X(N)) \in \mathcal{S}_n^N$, $X(i) > 0$ such that

$$Q_1(X,i) < 0, \tag{8.88}$$

$1 \leq i \leq N$, where $Q_1(X,i)$ is the left-hand side of (8.56) written for $\gamma = 1$.

Taking

$$F(i) = \pm(I_m \mp D(i))^{-1}C(i) \tag{8.89}$$

we obtain that (8.88) is equivalent to the inequalities

$$\begin{pmatrix} I_n & F^T(i) \\ 0 & I_m \end{pmatrix} Q_1(X,i) \begin{pmatrix} I_n & 0 \\ F(i) & I_m \end{pmatrix} < 0.$$

Displaying the $(1,1)$-block of this LMI one gets

$$\begin{pmatrix} I_n & F^T(i) \end{pmatrix} Q_1(X,i) \begin{pmatrix} I_n \\ F(i) \end{pmatrix} < 0.$$

By direct calculation one obtains via (8.15) and (8.56) that

$$\sum_{k=0}^{r}[A_k(i) + B_k(i)F(i)]^T \mathcal{E}_i(X)[A_k(i) + B_k(i)F(i)] \\ - X(i) + (C(i) + D(i)F(i))^T(C(i) + D(i)F(i)) - F^T(i)F(i) < 0, \tag{8.90}$$

$1 \leq i \leq N$.

If we take into account (8.89) we obtain $C(i) + D(i)F(i) = (I_m \mp D(i))^{-1}C(i)$. Thus we have $(C(i)+D(i)F(i))^T(C(i)+D(i)F(i))-F^T(i)F(i) = 0$. Hence (8.90) becomes:

$$\sum_{k=0}^{r} \overline{A}_k^T(i)\mathcal{E}_i(X)\overline{A}_k(i) - X(i) < 0, \qquad X(i) > 0, \qquad 1 \leq i \leq N. \tag{8.91}$$

Applying Corollary 3.6 one deduces that the zero state equilibrium of the system (8.87) is ESMS. This completes the proof. □

Theorem 8.16 (The first small gain theorem) *Assume that the assumptions of Theorem* 8.3 *are fulfilled. Then the operators* $I \mp \mathcal{T} : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^m\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^m\}$ *are invertible and the operators* $(I \mp \mathcal{T})^{-1} : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^m\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^m\}$ *have the following state space representation,*

$$x(t+1) = \left[\overline{A}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)\overline{A}_k(\eta_t)\right] x(t) + \left[\overline{B}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)\overline{B}_k(\eta_t)\right] z(t) \\ u(t) = \overline{C}(\eta_t)x(t) + \overline{D}(\eta_t)z(t), \tag{8.92}$$

$\overline{A}_k(i)$ *being defined in Theorem* 8.3 *and* $\overline{B}_k(i) = B_k(i)(I_m \mp D(i))^{-1}$, $\overline{C}_k(i) = \pm(I_m \mp D(i))^{-1}C(i)$, $\overline{D}(i) = (I_m \mp D(i))^{-1}$, $0 \leq k \leq r$, $i \in \mathcal{D}$.

Proof. It follows immediately using Theorem 8.4 and part (*ii*) of Proposition 8.1. □

Let us consider the systems:

$$\begin{aligned} x_1(t+1) &= \left(A_{10}(\eta_t) + \sum_{k=1}^{r} w_k(t) A_{1k}(\eta_t)\right) x_1(t) \\ &\quad + \left(B_{10}(\eta_t) + \sum_{k=1}^{r} w_k(t) B_{1k}(\eta_t)\right) v_1(t) \\ z_1(t) &= C_1(\eta_t) x_1(t), \end{aligned} \tag{8.93}$$

and

$$\begin{aligned} x_2(t+1) &= \left(A_{20}(\eta_t) + \sum_{k=1}^{r} w_k(t) A_{2k}(\eta_t)\right) x_2(t) \\ &\quad + \left(B_{20}(\eta_t) + \sum_{k=1}^{r} w_k(t) B_{2k}(\eta_t)\right) v_2(t) \\ z_2(t) &= C_2(\eta_t) x_2(t) + D_2(\eta_t) v_2(t), \end{aligned} \tag{8.94}$$

where $x_i(t) \in \mathbf{R}^{n_i}$, $i \in \{1, 2\}$, $z_1(t), v_2(t) \in \mathbf{R}^{n_z}$, $z_2(t), v_1(t) \in \mathbf{R}^{m_v}$. When interconnecting these two systems taking $v_2(t) = z_1(t)$ and $v_1(t) = z_2(t)$ one obtains:

$$\begin{aligned} x_1(t+1) &= (A_{10}(\eta_t) + B_{10}(\eta_t) D_2(\eta_t) C_1(\eta_t)) x_1(t) + B_{10}(\eta_t) C_2(\eta_t) x_2(t) \\ &\quad + \sum_{k=1}^{r} w_k(t) [(A_{1k}(\eta_t) + B_{1k}(\eta_t) D_2(\eta_t) C_1(\eta_t)) x_1(t) \\ &\quad + B_{1k}(\eta_t) C_2(\eta_t) x_2(t)] \\ x_2(t+1) &= B_{20}(\eta_t) C_1(\eta_t) x_1(t) + A_{20}(\eta_t) x_2(t) \\ &\quad + \sum_{k=1}^{r} w_k(t) (B_{2k}(\eta_t) C_1(\eta_t) x_1(t) + A_{2k}(\eta_t) x_2(t)). \end{aligned} \tag{8.95}$$

It is natural to ask if the zero state equilibrium of the system (8.95) is ESMS in the case when the zero state equilibrium of the linear systems

$$x_l(t+1) = \left(A_{l0}(\eta_t) + \sum_{k=1}^{r} w_k(t) A_{lk}(\eta_t)\right) x_l(t), \tag{8.96}$$

$l \in \{1, 2\}$ are ESMS.

An answer to this question is given by the following.

Theorem 8.17 (The second small gain theorem) *Assume:*

(*a*) *the zero state equilibria of the linear systems* (8.96) *are ESMS.*
(*b*) $\|\mathcal{T}_1\| < \gamma$ *and* $\|\mathcal{T}_2\| < \gamma^{-1}$ *for* $\gamma > 0$, *where* $\mathcal{T}_1 : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}$, $\mathcal{T}_2 : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}$ *are the input–output operators associated with the systems* (8.93), (8.94), *respectively.*

Under these conditions the zero state equilibrium of the interconnected system (8.95) *is ESMS.*

Proof. Setting $x(t) = (\, x_1^T(t) \quad x_2^T(t)\,)^T$ the system (8.95) may be written in a compact form on $\mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}$:

$$x(t+1) = \left(\overline{A}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)\overline{A}_k(\eta_t)\right) x(t), \tag{8.97}$$

where

$$\overline{A}_k(i) = \begin{pmatrix} A_{1k}(i) + B_{1k}(i)D_2(i)C_1(i) & B_{1k}(i)C_2(i) \\ B_{2k}(i)C_1(i) & A_{2k}(i) \end{pmatrix},$$

$0 \le k \le r$, $i \in \mathcal{D}$. On the other hand one obtains, via Proposition 8.1, that a state space representation of the product operator $\mathcal{T}_1\mathcal{T}_2$ is:

$$\begin{aligned} \begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} &= \left(\tilde{A}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)\tilde{A}_k(\eta_t)\right) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ &\quad + \left(\tilde{B}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)\tilde{B}_k(\eta_t)\right) v_2(t) \\ z_1(t) &= \tilde{C}(\eta_t) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \end{aligned} \tag{8.98}$$

where

$$\tilde{A}_k(i) = \begin{pmatrix} A_{1k}(i) & B_{1k}(i)C_2(i) \\ 0 & A_{2k}(i) \end{pmatrix},$$

$$\tilde{B}_k(i) = \begin{pmatrix} B_{1k}(i)D_2(i) \\ B_{2k}(i) \end{pmatrix}, \qquad \tilde{C}(i) = \begin{pmatrix} C_1(i) & 0 \end{pmatrix}.$$

One sees that $\overline{A}_k(i) = \tilde{A}_k(i) + \tilde{B}_k(i)\tilde{C}(i)$, $0 \le k \le r$.

The conclusions may be obtained applying Theorem 8.4 to the system (8.98). To this end we have to check that the assumptions of Theorem 8.4 are verified in the case of the system (8.98). First, we remark that in the case of this system the number of inputs equals the number of outputs, because $v_2(t), z_1(t) \in \mathbf{R}^{n_z}$.

The linear system obtained from (8.98) is:

$$\begin{aligned} x_1(t+1) &= A_{10}(\eta_t)x_1(t) + B_{10}(\eta_t)C_2(\eta_t)x_2(t) + \sum_{k=1}^{r} w_k(t)(A_{1k}(\eta_t)x_1(t) \\ &\quad + B_{1k}(\eta_t)C_2(\eta_t)x_2(t)) \\ x_2(t+1) &= A_{20}(\eta_t)x_2(t) + \sum_{k=1}^{r} w_k(t)A_{2k}(\eta_t)x_2(t). \end{aligned} \tag{8.99}$$

Let $\Phi_l(t,s)$, $l \in \{1,2\}$ be the fundamental matrix solutions of the linear systems (8.96). By assumption (a) we know that there exist $\beta \geq 1$, $q \in (0,1)$ such that

$$E[|\Phi_l(t,s)x_l|^2] \leq \beta q^{t-s}|x_l|^2 \tag{8.100}$$

for all $t \geq s \geq 0$, $x_l \in \mathbf{R}^{n_l}$, $l \in \{1,2\}$.

Proceeding as in the first part of the proof of Theorem 7.8 one obtains, via (8.100), that

$$\lim_{t\to\infty} E[|x_1(t)|^2 + |x_2(t)|^2] = 0 \tag{8.101}$$

for any solution $\begin{pmatrix} x_1^T(t) & x_2^T(t) \end{pmatrix}^T$ of the linear system (8.99). Invoking Theorem 3.10 for $\theta = 1$ we deduce that (8.101) is equivalent to the exponential stability in the mean square of the zero state equilibrium of the system (8.99). Finally, let us remark that $\|\mathcal{T}_1\mathcal{T}_2\| \leq \|\mathcal{T}_1\| \cdot \|\mathcal{T}_2\| < 1$. This completes the proof. □

Let us consider the special case $C_2(i) = 0$, $1 \leq i \leq N$ in (8.94). In this case the interconnected system (8.95) is partially decoupled. Thus, the exponential stability in the mean square of the zero state equilibrium of the interconnected system is strongly dependent upon the exponential stability in the mean square of the system:

$$\begin{aligned} x_1(t+1) &= (A_{10}(\eta_t) + B_{10}(\eta_t)D_2(\eta_t)C_1(\eta_t))x_1(t) \\ &\quad + \sum_{k=1}^{r} w_k(t)(A_{1k}(\eta_t) + B_{1k}(\eta_t)D_2(\eta_t)C_1(\eta_t))x_1(t). \end{aligned} \tag{8.102}$$

Moreover, in this special case, the input–output operator $\mathcal{T}_2$ defined by the system (8.94) reduces to:

$$(\mathcal{T}_2 v_2)(t) = D_2(\eta_t)v_2(t), \qquad t \in \mathbf{Z}_+.$$

One obtains that $\|\mathcal{T}_2 v_2\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\}} \leq |D_2|^2 \|v\|^2_{\ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}}$, where

$$|D_2|^2 = \max_{i\in\mathcal{D}} |D_2(i)|^2 = \max_{i\in\mathcal{D}} \lambda_{\max}[D_2^T(i) D_2(i)]^{1/2}.$$

From the proof of Theorem 8.6, we obtain the following.

Corollary 8.11 *Assume:*

(a) *The zero state equilibrium of the system* (8.96) *for* $l = 1$, *is ESMS.*
(b) $\|\mathcal{T}_1\| < \gamma$ *where* $\mathcal{T}_1 : \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0,\infty;\mathbf{R}^{n_z}\}$ *is the input–output operator defined by the system* (8.93).
(c) $|D_2| < \gamma^{-1}$.

Under these conditions the zero state equilibrium of the system (8.102) *is ESMS.*

Moreover, if the zero state equilibrium of the system (8.96) *for* $l = 2$ *is ESMS, too, then the zero state equilibrium of the interconnected system* (8.95) *for* $C_2(i) = 0$, $i \in \mathcal{D}$, *is also ESMS.* □

8.4.2 An estimate of the stability radius

When analyzing the robust stability of a solution of a discrete-time linear stochastic system we refer to the preservation of the stability property when the system is subject to some variation of the coefficients that is not necessarily small. Such variations of the system parameters (often known as parametric uncertainties) are due to inaccurate knowledge of the coefficients of the system or due to some simplifications of the mathematical model. It should be taken into consideration that a stabilizing controller designed for a simplified model must work in the real model which is subject to uncertainties.

In this section the problem of robust stability is investigated for a class of discrete-time linear stochastic systems subject to linear parametric uncertainties. Let us consider the discrete-time linear stochastic system described by

$$x(t+1) = \Bigg[A_0(\eta_t) + B_0(\eta_t)\Delta(\eta_t)C(\eta_t) + \sum_{k=1}^{r} w_k(t) \times (A_k(\eta_t) + B_k(\eta_t)\Delta(\eta_t)C(\eta_t))\Bigg] x(t), \tag{8.103}$$

where $A_k(i) \in \mathbf{R}^{n\times n}$, $B_k(i) \in \mathbf{R}^{n\times m}$, $0 \leq k \leq r$, $C(i) \in \mathbf{R}^{p\times n}$ are assumed to be known matrices; $\Delta(i) \in \mathbf{R}^{m\times p}$ are unknown matrices. The system (8.103) is a perturbed model of the nominal system

$$x(t+1) = \Bigg[A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)\Bigg] x(t). \tag{8.104}$$

The matrices $B_k(i)$, $C(i)$ occurring in (8.103) determine the structure of the parametric uncertainties presented in the perturbed model.

If the zero state equilibrium of the nominal system (8.104) is ESMS we analyze whether the zero state equilibrium of the perturbed model (8.103) remains ESMS for $\Delta(i) \neq 0$. This would be, in short, the formulation of the problem of robust stability. For a more precise formulation of the robust stability problem we introduce a norm in the set of uncertainties.

If $\Delta = (\Delta(1), \Delta(2), \dots, \Delta(N)) \in \mathcal{M}_{mp}^N$ we set

$$|\Delta| = \max_{i \in \mathcal{D}} |\Delta(i)| = \max_{i \in \mathcal{D}} (\lambda_{\max}(\Delta^T(i)\Delta(i)))^{1/2}. \tag{8.105}$$

Based on the known matrices $A_k(i)$, $B_k(i)$, $C(i)$ occurring in (8.103), we introduce the notations:

$$\begin{aligned}
A_k &= (A_k(1), A_k(2), \dots, A_k(N)) \in \mathcal{M}_n^N, \\
B_k &= (B_k(1), B_k(2), \dots, B_k(N)) \in \mathcal{M}_{n,m}^N \\
C &= (C(1), C(2), \dots, C(N)) \in \mathcal{M}_{pn}^N, \\
\mathbf{A} &= (A_0, A_1, \dots, A_r) \in \mathcal{M}_n^N \oplus \mathcal{M}_n^N \oplus \cdots \oplus \mathcal{M}_n^N, \\
\mathbf{B} &= (B_1, B_2, \dots, B_r) \in \mathcal{M}_{n,m}^N \oplus \mathcal{M}_{n,m}^N \oplus \cdots \oplus \mathcal{M}_{n,m}^N.
\end{aligned}$$

We recall that according to the notations introduced in Chapter 5, $\mathcal{M}_{n,m}^N$ stands for $\mathbf{R}^{n\times m} \oplus \mathbf{R}^{n\times m} \oplus \cdots \oplus \mathbf{R}^{n\times m}$ and $\mathcal{M}_n^N$ stands for $\mathcal{M}_{n,n}^N$.

As a measure of the robustness of the stability we introduce the concept of stability radius.

Definition 8.2 *The stability radius of the nominal system* (8.104)*, or equivalently, the stability radius of the pair* $(\mathbf{A}, P)$ *with respect to the structured parametric uncertainties with the structure determined by the pair* $(\mathbf{B}, C)$ *is the number* $\rho_L[\mathbf{A}, P|\mathbf{B}, C] = \inf\{\rho > 0 | (\exists)\Delta = (\Delta(1), \dots, \Delta(N)) \in \mathcal{M}_{mp}^N$ *with* $|\Delta| \leq \rho$ *that the zero state equilibrium of the corresponding system (8.103) is not ESMS*$\}$.

The next result provides a lower bound of the stability radius introduced in the above definition. To this end, let us consider the fictitious system constructed based on the known matrices occurring in the perturbed model (8.103)*:*

$$\begin{aligned}
x(t+1) &= \left(A_0(\eta_t) + \sum_{k=1}^{r} w_k(t) A_k(\eta_t)\right) x(t) \\
&\quad + \left(B_0(\eta_t) + \sum_{k=1}^{r} w_k(t) B_k(\eta_t)\right) v(t) \\
z(t) &= C(\eta_t) x(t).
\end{aligned} \tag{8.106}$$

Theorem 8.18 *Assume that the zero state equilibrium of the nominal system* (8.104) *is ESMS. Let* $\mathcal{T} : \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^m\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^p\}$ *be the input–output operator defined by the fictitious system* (8.106). *Then we have:*

$$\rho_L[\mathbf{A}, P|\mathbf{B}, C] \geq \|\mathcal{T}\|^{-1}. \tag{8.107}$$

Proof. Let $\rho < \|\mathcal{T}\|^{-1}$ be arbitrary but fixed. We show that for any perturbation $\Delta = (\Delta(1), \Delta(2), \dots, \Delta(N)) \in \mathcal{M}^N_{mp}$ with $|\Delta| < \rho$, the zero state equilibrium of the perturbed system (8.103) is ESMS. Let $\Delta \in \mathcal{M}^N_{mp}$ be a perturbation with $|\Delta| < \rho$. Setting $\gamma = \rho^{-1}$, we have $\|\mathcal{T}\| < \gamma$ and $|\Delta| < \gamma^{-1}$. Hence the fictitious system (8.106) and the perturbation Δ are in the conditions of Corollary 8.4.

Thus applying the result stated in that corollary we deduce that the zero state equilibrium of the corresponding system (8.103) is ESMS.

Therefore $\rho_L[\mathbf{A}, P|\mathbf{B}, C] \geq \rho$ for all $\rho \leq \|\mathcal{T}\|^{-1}$. Thus we may conclude that (8.107) is fulfilled and the proof is complete. □

At the end of this subsection we show that several structures of the parametric uncertainties frequently used in the literature are embedded in the general form of (8.103).

First we consider the following perturbed system,

$$x(t+1) = \Bigg[A_0(\eta_t) + \hat{B}_0(\eta_t)\Delta_0(\eta_t)C(\eta_t)$$
$$+ \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + \hat{B}_k(\eta_t)\Delta_k(\eta_t)C(\eta_t))\Bigg] x(t), \tag{8.108}$$

where $A_k(i) \in \mathbf{R}^{n\times n}$, $\hat{B}_k(i) \in \mathbf{R}^{n\times m_k}$, $C(i) \in \mathbf{R}^{p\times n}$, $0 \leq k \leq r$, $i \in \mathcal{D}$, are assumed to be known matrices and $\Delta_k(i) \in \mathcal{M}_{kp}$ are unknown matrices.

Let us define $B_k(i) \in \mathbf{R}^{n\times m}$, $m = \sum_{k=0}^{r} m_k$ by

$$B_0(i) = \begin{pmatrix} \hat{B}_0(i) & 0 & \dots & 0 \end{pmatrix}$$
$$B_k(i) = \begin{pmatrix} 0 & \dots & 0 & \hat{B}_k(i) & 0 & \dots & 0 \end{pmatrix}. \tag{8.109}$$

Set $\Delta = (\Delta(1), \Delta(2), \dots, \Delta(N))$ with

$$\Delta(i) = \begin{pmatrix} \Delta_0(i) \\ \Delta_1(i) \\ \vdots \\ \Delta_r(i) \end{pmatrix} \in \mathbf{R}^{m\times p}. \tag{8.110}$$

With these notations one obtains that the system (8.108) is a special case of the system (8.103). From (8.105) and (8.110) we have $|\Delta|^2 = \max_{i\in\mathcal{D}} \{\lambda_{\max}[\Delta^T(i)\Delta(i)]\} = \max_{i\in\mathcal{D}}\{\lambda_{\max}[\sum_{k=0}^{r} \Delta_k^T(i)\Delta_k(i)]\}$.

Another interesting case is that of the perturbed system of the form:

$$x(t+1) = \Bigg[A_0(\eta_t) + \hat{B}_0(\eta_t)\Delta_0(\eta_t)\hat{C}_0(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + \hat{B}_k(\eta_t)\Delta_k(\eta_t)\hat{C}_k(\eta_t))\Bigg] x(t), \qquad (8.111)$$

where $A_k(i) \in \mathbf{R}^{n\times n}$, $\hat{B}_k(i) \in \mathbf{R}^{n\times m_k}$, $\hat{C}_k(i) \in \mathbf{R}^{p_k\times n}$, $0 \le k \le r$, $i \in \mathcal{D}$ are assumed known matrices and $\Delta_k(i) \in \mathcal{M}^{m_k\times p_k}$ are unknown matrices.

We define $B_k(i) \in \mathbf{R}^{n\times m}$, $m = \sum_{k=0}^{r} m_k$ as in (8.109) and $C(i) \in \mathbf{R}^{p\times n}$, $p = \sum_{k=0}^{r} p_k$ by

$$C(i) = \begin{pmatrix} C_0(i) \\ C_1(i) \\ \vdots \\ C_r(i) \end{pmatrix}.$$

Also we set

$$\Delta(i) = \mathrm{diag}(\Delta_0(i), \Delta_1(i), \dots, \Delta_r(i)) \in \mathbf{R}^{m\times p}. \qquad (8.112)$$

With these notations the system (8.111) can be regarded as a special case of the system (8.103). Therefore a lower bound of the stability radius can be obtained using Theorem 8.7.

We remark that combining (8.105) and (8.112) one may obtain:

$$|\Delta(i)|^2 = \lambda_{\max}[\Delta^T(i)\Delta(i)] = \max_{0\le k\le r} \{\lambda_{\max}[\Delta_k^T(i)\Delta_k(i)]\}.$$

8.5 The disturbance attenuation problem

8.5.1 The problem formulation

Let us consider the system with the following state space representation.

$$G : \begin{cases} x(t+1) = A_0(\eta_t)x(t) + B_0(\eta_t)u(t) + G_0(\eta_t)v(t) \\ \qquad + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t)x(t) + B_k(\eta_t)u(t) + G_k(\eta_t)v(t)) \\ y(t) = C_0(\eta_t)x(t) + D(\eta_t)v(t) \\ z(t) = C_z(\eta_t)x(t) + D_{zu}(\eta_t)u(t) + D_{zv}(\eta_t)v(t), \end{cases} \qquad (8.113)$$

where $x(t) \in \mathbf{R}^n$ is a vector of the state parameters, $u(t) \in \mathbf{R}^{m_u}$ is the vector of control parameters, $v(t) \in \mathbf{R}^{m_v}$ is the vector of the exogenous disturbances, $y(t) \in R^{n_y}$ is the vector of the measurements, and $z(t) \in \mathbf{R}^{n_z}$ is the controlled output. In the sequel, the exogenous disturbances are modeled by stochastic processes $v = \{v(t)\}_{t\geq 0} \in \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\}$. The class of admissible controllers consists of systems of the form,

$$G_c : \begin{cases} x_c(t+1) = A_c(\eta_t)x_c(t) + B_c(\eta_t)u_c(t) \\ y_c(t) = C_c(\eta_t)x_c(t) + D_c(\eta_t)u_c(t), \end{cases} \tag{8.114}$$

$x_c(t) \in \mathbf{R}^{n_c}$ is the vector of the state parameters of the controller, $u_c(t) \in \mathbf{R}^{n_y}$ is the input of the controller, and $y_c(t) \in \mathbf{R}^{m_u}$ is the output of the controller. When coupling a controller (G_c) of type (8.114) to the system (8.113) by taking $u_c(t) = y(t)$ and $u(t) = y_c(t)$ one obtains the following closed-loop system,

$$\begin{aligned} x_{cl}(t+1) &= \left(A_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t) A_{kcl}(\eta_t) \right) x_{cl}(t) \\ &\quad + \left(G_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t) G_{kcl}(\eta_t) \right) v(t) \\ z_{cl}(t+1) &= C_{cl}(\eta_t) x_{cl}(t) + D_{cl}(\eta_t) v(t), \end{aligned} \tag{8.115}$$

where

$$\begin{aligned} A_{0cl}(i) &= \begin{pmatrix} A_0(i) + B_0(i)D_c(i)C_0(i) & B_0(i)C_c(i) \\ B_c(i)C_0(i) & A_c(i) \end{pmatrix} \\ A_{kcl}(i) &= \begin{pmatrix} A_k(i) + B_k(i)D_c(i)C_0(i) & B_k(i)C_c(i) \\ 0 & 0 \end{pmatrix} \\ G_{0cl}(i) &= \begin{pmatrix} G_0(i) + B_0(i)D_c(i)D(i) \\ B_c(i)D(i) \end{pmatrix} \\ G_{kcl}(i) &= \begin{pmatrix} G_k(i) + B_k(i)D_c(i)D(i) \\ 0 \end{pmatrix} \\ C_{cl}(i) &= \begin{pmatrix} C_z(i) + D_{z_u}(i)D_c(i)C_0(i) & D_{zu}(i)C_c(i) \end{pmatrix} \\ D_{cl}(i) &= D_{zv}(i) + D_{zu}(i)D_c(i)D(i), \quad x_{cl} = \begin{pmatrix} x^T(t) & x_c^T(t) \end{pmatrix}^T. \end{aligned} \tag{8.116}$$

As usual a controller G_c is a stabilizing controller if the zero state equilibrium of the closed-loop linear system $x_{cl}(t+1) = (A_{0cl}(\eta_t) + \sum_{k=1}^{r} w_k(t)A_{kcl}(\eta_t))$ $x_{cl}(t)$ is ESMS.

Because the closed-loop system (8.115) corresponding to a stabilizing controller takes the form of a system of type (8.4), it follows that we may associate a corresponding input–output operator $\mathcal{T}_{cl} : \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{n_z}\}$ with $(\mathcal{T}_{cl}v)(t) = C_{cl}(\eta_t)x_{cl}(t, 0, v) + D_{cl}(\eta_t)v(t)$, $t \in \mathbf{Z}_+$, where $x_{cl}(t, 0, v)$ is the solution of the closed-loop system (8.105) determined by the input v and taking the initial value $x_{cl}(0, 0, v) = 0$.

The disturbance attenuation problem with attenuation level $\gamma, (DAP_\gamma)$ requires the construction of a stabilizing controller G_c of type (8.114) with additional property: $\|\mathcal{T}_{cl}\| < \gamma$.

Remark 8.11

(a) The disturbance attenuation problem stated before extends to this general framework the H_∞ control problem from the deterministic context. Therefore, this problem is often named the stochastic H_∞ problem.

(b) To implement a controller (8.114) we need to know at each time t both the measurements $y(t)$ as well as the system mode i.

(c) In the case $n_c = 0$ the controller (8.114) is a memoryless one; that is, $u(t) = D_c(\eta_t)y(t)$.

(d) In the special case of the plant (8.113) with $B_k(i) = 0$, $0 \leq k \leq r$, $D_{zu}(i) = -I_{n_z}$ the disturbance attenuation problem stated before becomes an H_∞ filtering problem. It requires the construction of a filter G_c of type (8.114) whose output approximates the output $z(t)$ of the given plant G with the accuracy given by $\|\mathcal{T}_{cl}\|$ of the input–output operator.

In the sequel, the disturbance attenuation problem is solved under the assumption that the whole state vector is accessible for measurements. This means that in (8.113), $C_0(i) = I_n$ and $D(i) = 0$, $1 \leq i \leq N$.

The case when only an output is available for measurements is illustrated for an H_∞ filtering problem.

8.5.2 The solution of the disturbance attenuation problem. The case of full state measurements

In this subsection we present the solution of the disturbance attenuation problem with level of attenuation $\gamma > 0$ under the assumption that the whole state vector $x(t)$ and the mode i are available. In this case, the controlled system (8.113) becomes:

$$x(t+1) = A_0(\eta_t)x(t) + G_0(\eta_t)v(t) + B_0(\eta_t)u(t)$$
$$+ \sum_{k=1}^{r} w_k(t)[A_k(\eta_t)x(t) + G_k(\eta_t)v(t) + B_k(\eta_t)u(t)] \qquad (8.117)$$
$$y(t) = x(t)$$
$$z(t) = C_z(\eta_t)x(t) + D_{zv}(\eta_t)v(t) + D_{zu}(\eta_t)u(t).$$

As we have already shown in Remark 8.2(c), if $n_c = 0$, a controller (8.114) reduces now to $u(t) = D_c(\eta_t)x(t)$, or, using a traditional notation, to

$$u(t) = F(\eta_t)x(t). \qquad (8.118)$$

The closed-loop system obtained when coupling (8.118) and (8.117) is:

$$x(t+1) = \left[A_0(\eta_t) + B_0(\eta_t)F(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F(\eta_t))\right] x(t)$$
$$+ \left(G_0(\eta_t) + \sum_{k=1}^{r} w_k(t)G_k(\eta_t)\right) v(t) \qquad (8.119)$$
$$z(t) = (C_z(\eta_t) + D_{zu}(\eta_t)F(\eta_t))x(t) + D_{zv}(\eta_t)v(t).$$

If $F = (F(1), F(2), \ldots, F(N))$ is a stabilizing feedback gain, then the system (8.119) defines an input–output operator, $\mathcal{T}_F : \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{m_v}\} \to \ell^2_{\tilde{\mathcal{H}}}\{0, \infty; \mathbf{R}^{n_z}\}$ by $(\mathcal{T}_F v)(t) = (C_z(\eta_t) + D_{zu}(\eta_t)F(\eta_t))x(t, 0, v) + D_{zv}(\eta_t)v(t)$, $t \in \mathbf{Z}_+$.

The disturbance attenuation problem with level of attenuation $\gamma > 0$ asks for the construction of a stabilizing feedback gain F, such that $\|\mathcal{T}_F\| < \gamma$. Our aim is to find conditions guarantee the existence of a control in a state feedback from (8.118) which solves this problem.

Thus we prove the following.

Theorem 8.19 *For the system* (8.117) *and a given scalar* $\gamma > 0$*, the following are equivalent.*

(i) *There exists a control law* $u(t) = F(\eta_t)x(t)$ *such that the zero state equilibrium of the linear system* $x(t+1) = [A_0(\eta_t) + B_0(\eta_t)F(\eta_t) + \sum_{k=1}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F(\eta_t))]x(t)$ *is ESMS and* $\|\mathcal{T}_F\| < \gamma$.

(ii) *There exist* $Y = (Y(1), Y(2), \ldots, Y(N)) \in \mathcal{S}_n^N$ *and* $\Gamma = (\Gamma(1), \Gamma(2), \ldots, \Gamma(N)) \in \mathcal{M}_{mn}^N$, $Y(i) > 0$, $1 \le i \le N$, *that solve the following system of LMIs.*

$$\left(\begin{array}{cccccc} -Y(i) & \mathcal{W}_{0i}(Y,\Gamma) & \mathcal{W}_{1i}(Y,\Gamma) & \dots & \mathcal{W}_{ri}(Y,\Gamma) & Y(i)C_z^T(i)+\Gamma^T(i)D_{zu}^T(i) \\ \mathcal{W}_{0i}^T(Y,\Gamma) & \mathfrak{G}_{00}-\mathcal{Y} & \mathfrak{G}_{01}(i) & \dots & \mathfrak{G}_{0r}(i) & \mathfrak{G}_{0r+1}(i) \\ \mathcal{W}_{1i}^T(Y,\Gamma) & \mathfrak{G}_{01}^T(i) & \mathfrak{G}_{11}(i)-\mathcal{Y} & \dots & \mathfrak{G}_{1r}(i) & \mathfrak{G}_{1r+1}(i) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \mathcal{W}_{ri}^T(Y,\Gamma) & \mathfrak{G}_{0r}^T(i) & \mathfrak{G}_{1r}^T(i) & \dots & \mathfrak{G}_{rr}(i)-\mathcal{Y} & \mathfrak{G}_{rr+1}(i) \\ C_z(i)Y(i)+D_{zu}(i)\Gamma(i) & \mathfrak{G}_{0r+1}^T(i) & \mathfrak{G}_{1r+1}^T(i) & \dots & \mathfrak{G}_{rr+1}^T(i) & D_{zv}^T(i)D_{zv}^T(i)-\gamma^2 I_{n_z} \end{array}\right) < 0, \tag{8.120}$$

where $\mathcal{W}_{ki}(Y,\Gamma)=(Y(i)A_k^T(i)+\Gamma^T(i)B_k^T(i))\mathfrak{I}(i)$, $0\le k\le r$,

$$\mathfrak{I}(i)=\left(\sqrt{p(i,1)}I_n \quad \sqrt{p(i,2)}I_n \quad \dots \quad \sqrt{p(i,N)}I_n\right)$$

$$\mathfrak{G}_{lk}(i)=\mathfrak{I}^T(i)G_l(i)G_k^T(i)\mathfrak{I}(i), \qquad 0\le l\le k\le r,$$

$$\mathfrak{G}_{lr+1}(i)=\mathfrak{I}^T(i)G_l(i)D_{zv}^T(i), \qquad 0\le l\le r \tag{8.121}$$

$$\mathcal{Y}=\mathrm{diag}(Y(1),Y(2),\dots,Y(N)).$$

Moreover, if (Y,Γ) *is a solution of the above LMI* (8.120)*, then a solution of the disturbance attenuation problem under consideration is given by* $F=(F(1),F(2),\dots,F(N))$, $F(i)=\Gamma(i)Y^{-1}(i)$, $1\le i\le N$.

Proof. Applying the implication $(i)\to(vi)$ of Theorem 8.2 to the system (8.119), we obtain that the assertion (i) in the statement is equivalent to the existence of $Y=(Y(1),Y(2),\dots,Y(N))\in\mathcal{S}_n^N$, with $Y(i)>0$ which solves the following system of LMIs.

$$\left(\begin{array}{cccc} -Y(i) & \Psi_{0i}(Y,F) & \Psi_{1i}(Y,F) & \dots \\ \Psi_{0i}^T(Y,F) & \mathfrak{G}_{00}(i)-\mathcal{Y} & \mathfrak{G}_{01}(i) & \dots \\ \Psi_{1i}^T(Y,F) & \mathfrak{G}_{01}^T(i) & \mathfrak{G}_{11}(i)-\mathcal{Y} & \dots \\ \dots & \dots & \dots & \dots \\ \Psi_{ri}^T(Y,F) & \mathfrak{G}_{0r}^T(i) & \mathfrak{G}_{1r}^T(i) & \dots \\ (C_z(i)+D_{zu}(i)F(i))Y(i) & \mathfrak{G}_{0r+1}^T(i) & \mathfrak{G}_{1r+1}^T(i) & \dots \end{array}\right.$$

$$\left.\begin{array}{cc} \Psi_{ri}(Y,F) & Y(i)(C_z(i)+D_{zu}(i)F(i))^T \\ \mathfrak{G}_{0r}(i) & \mathfrak{G}_{0r+1}(i) \\ \mathfrak{G}_{1r}(i) & \mathfrak{G}_{1r+1}(i) \\ \dots & \dots \\ \mathfrak{G}_{rr}(i)-\mathcal{Y} & \mathfrak{G}_{rr+1}(i) \\ \mathfrak{G}_{rr+1}^T(i) & D_{zv}(i)D_{zv}^T(i)-\gamma^2 I_{n_z} \end{array}\right) < 0, \tag{8.122}$$

where $\mathfrak{G}_{lk}(i), \mathcal{Y}$ are as in (8.121) and $\Psi_{ki}(Y,F) = Y(i)(A_k(i)+B_k(i)F(i))^T\mathfrak{I})(i)$. Setting $\Gamma(i) = F(i)Y(i)$ one obtains that (8.122) is equivalent to (8.120) and thus the proof is complete. □

Furthermore, we consider a dynamic controller of type (8.114) with $n_c \geq 1$ and $n_y = n$. In this case a closed-loop system of type (8.115) obtained when coupling a controller (8.114) to the system (8.117) has a coefficient with the following structure.

$$A_{0cl}(i) = \begin{pmatrix} A_0(i)+B_0(i)D_c(i) & B_0(i)C_c(i) \\ B_c(i) & A_c(i) \end{pmatrix}$$

$$A_{kcl}(i) = \begin{pmatrix} A_k(i)+B_k(i)D_c(i) & B_k(i)C_c(i) \\ 0 & 0 \end{pmatrix}$$

$$G_{kcl}(i) = \begin{pmatrix} G_k(i) \\ 0 \end{pmatrix},$$

$$C_{cl}(i) = \begin{pmatrix} C_z(i)+D_{zu}(i)D_c(i) & D_{zu}(i)C_c(i) \end{pmatrix}, \qquad D_{cl}(i) = D_{zv}(i). \tag{8.123}$$

Theorem 8.20 *For the system* (8.117) *and a given attenuation level* $\gamma > 0$*, the following are equivalent.*

(*i*) *There exists a dynamic controller of order* $n_c \geq 1$ *that solves the disturbance attenuation problem with level of attenuation* γ*.*

(ii) There exists a zero-order controller $u(t) = F(\eta_t)x(t)$ that solves the disturbance attenuation problem with the level of attenuation γ.

Proof. If (i) is true then, applying Theorem 8.2 to the corresponding closed-loop system one obtains that there exists $Y = (Y(1), Y(2), \dots, Y(N)) \in \mathcal{S}_{n+n_c}^N$, $Y(i) > 0$, $1 \le i \le N$ which solves the following system of LMIs,

$$\begin{pmatrix} -Y(i) & \hat{\Psi}_{0i}(Y) & \hat{\Psi}_{1i}(Y) & \dots \\ \hat{\Psi}_{0i}^T(Y) & \hat{\mathfrak{G}}_{00}(i) - \mathcal{Y} & \hat{\mathfrak{G}}_{01}(i) & \dots \\ \hat{\Psi}_{1i}^T(Y) & \hat{\mathfrak{G}}_{01}^T(i) & \hat{\mathfrak{G}}_{11}(i) - \mathcal{Y} & \dots \\ \dots & \dots & \dots & \dots \\ \hat{\Psi}_{ri}^T(Y) & \hat{\mathfrak{G}}_{0r}^T(i) & \hat{\mathfrak{G}}_{1r}^T(i) & \dots \\ C_{cl}(i)Y(i) & \hat{\mathfrak{G}}_{0r+1}^T(i) & \hat{\mathfrak{G}}_{1r+1}^T(i) & \dots \end{pmatrix.$$

$$\begin{matrix} \hat{\Psi}_{ri}(Y) & Y(i)C_{cl}^T(i) \\ \hat{\mathfrak{G}}_{0r}(i) & \hat{\mathfrak{G}}_{0r+1}(i) \\ \hat{\mathfrak{G}}_{1r}(i) & \hat{\mathfrak{G}}_{1r+1}(i) \\ \dots & \dots \\ \hat{\mathfrak{G}}_{rr}(i) - \mathcal{Y} & \hat{\mathfrak{G}}_{rr+1}(i) \\ \hat{\mathfrak{G}}_{rr+1}^T(i) & D_{zv}(i)D_{zv}^T(i) - \gamma^2 I_{n_z} \end{pmatrix} < 0, \tag{8.124}$$

where

$$\hat{\Psi}_{ki}(i) = \left(\sqrt{p(i,1)}Y(i)A_{kcl}^T(i) \quad \dots \quad \sqrt{p(i,N)}Y(i)A_{kcl}^T(i) \right),$$

$$\hat{\mathfrak{G}}_{lk}(i) = \hat{\mathfrak{I}}^T(i)G_{lcl}(i)G_{kcl}^T(i)\hat{\mathfrak{I}}(i), \qquad 0 \le l \le k \le r,$$

$$\hat{\Psi}_{lr+1}(i) = \hat{\mathfrak{I}}^T(i)G_{lcl}(i)D_{zv}^T(i), \qquad 0 \le l \le r,$$

$$\hat{\mathfrak{I}}(i) = \left(\sqrt{p(i,1)}I_{n+n_c} \quad \dots \quad \sqrt{p(i,N)}I_{n+n_c} \right).$$

Take $\mathfrak{T} \in \mathbf{R}^{\tilde{n} \times \hat{n}}$, $(\tilde{n} = (n+n_c)[1+(r+1)N] + n_z, \hat{n} = n[1+(r+1)N] + n_z)$,

$$\mathfrak{T} = \mathrm{diag}(\mathfrak{T}_0, \mathfrak{T}_0, \dots, \mathfrak{T}_0, I_{n_z}), \qquad \mathfrak{T}_0 = \begin{pmatrix} I_n \\ O_{n_c,n} \end{pmatrix}.$$

One sees that $\mathfrak{T}$ has full column rank; that is, rank$\mathfrak{T} = \hat{n}$.

Pre- and postmultiplying (8.124) by $\mathfrak{T}^T$ and $\mathfrak{T}$, respectively, we obtain the following LMIs,

$$\begin{pmatrix} -Y_{11}(i) & \mathcal{V}_{0i}(Y) & \mathcal{V}_{1i}(Y) & \dots \\ \mathcal{V}_{0i}^T(Y) & \mathfrak{G}_{00} - \mathcal{Y}_{11} & \mathfrak{G}_{01}(i) & \dots \\ \mathcal{V}_{1i}^T(Y) & \mathfrak{G}_{01}^T(i) & \mathfrak{G}_{11}(i) - \mathcal{Y}_{11} & \dots \\ \dots & \dots & \dots & \dots \\ \mathcal{V}_{ri}^T(Y) & \mathfrak{G}_{0r}^T(i) & \mathfrak{G}_{1r}^T(i) & \dots \\ \mathcal{V}_{r+1,i}^T(Y) & \mathfrak{G}_{0r+1}^T(i) & \mathfrak{G}_{1r+1}^T(i) & \dots \end{pmatrix.$$

$$\left.\begin{matrix} \mathcal{V}_{ri}(Y) & \mathcal{V}_{r+1,i}(Y) \\ \mathfrak{G}_{0r}(i) & \mathfrak{G}_{0r+1}(i) \\ \mathfrak{G}_{1r}(i) & \mathfrak{G}_{1r+1}(i) \\ \dots & \dots \\ \mathfrak{G}_{rr}(i) - \mathcal{Y}_{11} & \mathfrak{G}_{rr+1}(i) \\ \mathfrak{G}_{rr+1}^T(i) & D_{zv}(i)D_{zv}^T(i) - \gamma^2 I_{n_z} \end{matrix}\right) < 0 \tag{8.125}$$

where $\mathfrak{G}_{lk}(i)$ are as in (8.121), $\mathcal{V}_{ki}(Y) = \mathfrak{T}_0^T Y(i) A_{kcl}^T(i)\mathfrak{T}_0(i)\mathfrak{I}(i)$, $0 \le k \le r$, $\mathcal{V}_{r+1,i}(Y) = \mathfrak{T}_0^T Y(i) C_{cl}^T(i)$, $\mathcal{Y}_{11} = \mathrm{diag}(Y_{11}(i), Y_{11}(2), \dots, Y_{11}(N))$, and $Y_{11}(i)$ is obtained from the partition

$$Y(i) = \begin{pmatrix} Y_{11}(i) & Y_{12}(i) \\ Y_{12}^T(i) & Y_{22}(i) \end{pmatrix}$$

according to the partition of the coefficients of the closed-loop system. By direct calculation one obtains that $\mathcal{V}_{ki}(Y) = \mathcal{W}_{ki}(Y_{11}, \Gamma)$, where $\Gamma = (\Gamma(1), \Gamma(2), \dots, \Gamma(N))$,

$$\Gamma(i) = D_c(i) Y_{11}(i) + C_c(i) Y_{12}^T(i). \tag{8.126}$$

This means that (8.125) is equivalent to (8.120) written for $Y_{11}(i)$ instead of $Y(i)$ and $\Gamma(i)$ given by (8.127). Hence, the state feedback given by $F(i) = D_c(i) + C_c(i) Y_{12}^T(i) Y_{11}^{-1}(i)$ is a solution of the disturbance attenuation problem with the level of attenuation $\gamma > 0$. Thus we proved that $(i) \to (ii)$.

If (ii) is fulfilled, then there exists a stabilizing feedback control law $u(t) = F(\eta_t)x(t)$ such that $\|\mathcal{T}_F\| < \gamma$. Take $n_c \ge 1$ and matrices $A_c(i) \in \mathbf{R}^{n_c \times n_c}$, $1 \le i \le N$, such that the zero state equilibrium of the linear system $x_c(t+1) = A_c(\eta_t)x_c$ is ESMS.

Taking $B_c(i) = 0$, $C_c(i) = 0$, $D_c(i) = F(i)$, $1 \le i \le N$, we obtain a stabilizing controller of type (8.114) of order $n_c \ge 1$, that stabilizes the system (8.117). One can see that the corresponding input–output operator $\mathcal{T}_{cl}$ is just $\mathcal{T}_F$. Hence $\|\mathcal{T}_{cl}\| < \gamma$. Thus we have shown that $(ii) \to (i)$ holds and the proof is complete. □

8.5.3 Solution of a robust stabilization problem

We apply Theorem 8.8 in order to solve a robust stabilization problem.

Consider the system described by

$$\begin{aligned} x(t+1) = & [A_0(\eta_t) + \hat{G}_0(\eta_t)\Delta_1(\eta_t)\hat{C}(\eta_t)]x(t) \\ & + [B_0(\eta_t) + \hat{B}_0(\eta_t)\Delta_2(\eta_t)\hat{D}(\eta_t)]u(t) \\ & + \sum_{k=1}^{r} w_k(t)\{[A_k(\eta_t) + \hat{G}_k(\eta_t)\Delta_1(\eta_t)\hat{C}(\eta_t)]x(t) \\ & + [B_k(\eta_t)\Delta_2(\eta_t)\hat{D}(\eta_t)]u(t)\}, \end{aligned} \tag{8.127}$$

where $A_k(\eta_t), \hat{G}_k(i), B_k(i), \hat{C}(i), \hat{D}(i), 0 \le k \le r, i \in \mathcal{D}$ are known matrices of appropriate dimensions and $\Delta_1 = (\Delta_1(1), \ldots, \Delta_1(N))$ and $\Delta_2 = (\Delta_2(1), \ldots, \Delta_2(N))$ are unknown matrices and they describe the magnitude of the uncertainties of the system (8.127). It is assumed that the whole state vector is accessible for measurements.

The robust stabilization problem considered here can be stated as follows.

For a given $\rho > 0$ find a control $u(t) = F(\eta_t)x(t)$ stabilizing (8.127) for any Δ_1 and Δ_2 such that $\max(|\Delta_1|, |\Delta_2|) < \rho$.

The closed-loop system obtained with $u(t) = F(\eta_t)x(t)$ is given by

$$\begin{aligned} x(t+1) = & \{A_0(\eta_t) + B_0(\eta_t)F(\eta_t) + G_0(\eta_t)\Delta(\eta_t)[C(\eta_t) + D(\eta_t)F(\eta_t)]\}x(t) \\ & + \sum_{k=1}^{r} w_k(t)\{A_k(\eta_t) + B_k(\eta_t)F(\eta_t) \\ & + G_k(\eta_t)\Delta(\eta_t)[C(\eta_t) + D(\eta_t)F(\eta_t)]\}x(t), \end{aligned} \tag{8.128}$$

where

$$G_k(i) = (\hat{G}_k(i) \quad \hat{B}_k(i)), \qquad C(i) = \begin{pmatrix} \hat{C}(i) \\ 0 \end{pmatrix},$$

$$D(i) = \begin{pmatrix} 0 \\ \hat{D}(i) \end{pmatrix}, \qquad \Delta(i) = \begin{pmatrix} \Delta_1(i) & 0 \\ 0 & \Delta_2(i) \end{pmatrix}.$$

If the zero state equilibrium of the linear system obtained from (8.128) taking $\Delta = 0$ is ESMS, then from Theorem 8.6 it follows that the zero state equilibrium of (8.128) is ESMS for all Δ with $|\Delta| < \rho$, $|\Delta| = \max(|\Delta_1|, |\Delta_2|)$, if the input–output operator $\mathcal{T}_F$ associated with the system (8.119) with $z(t) = [C(\eta_t) + D(\eta_t)F(\eta_t)]x(t)$ satisfies the condition $\|\mathcal{T}_F\| < \frac{1}{\rho}$.

Therefore, F is a robust stabilizing feedback with the robustness radius ρ if it is a solution of the DAP with level of attenuation $\gamma = 1/\rho$ for the system

(8.117) with $z(t) = C(\eta_t)x(t) + D(\eta_t)u(t)$, where the matrices $C(i)$ and $D(i)$ were defined above.

The next result follows directly from Theorem 8.7.

Theorem 8.21 *Suppose that there exist* $Y \in \mathcal{S}_n^N$ *and* $\Gamma \in \mathcal{M}_{mn}^N$, $Y > 0$ *verifying the system of LMIs* (8.120), *where* $C_z(i) = C(i), D_{zu}(i) = D(i), D_{zv}(i) = 0, \gamma = 1/\rho$. *Then the state feedback gain* $F(i) = \Gamma(i)Y^{-1}(i)$ *is a solution of the robust stabilization problem.*

8.6 Notes and references

The bounded real lemma and other H_∞ control problems for discrete-time linear systems affected by independent random perturbations were considered in [7, 33, 34, 62–64, 93, 105, 112] and in the Markovian case in [23, 27, 69, 94, 110, 115, 116]. The proof of the bounded real lemma in this chapter follows the ideas in [7, 94]. The small gain theorems given in Section 8.4 are proved for the first time in this book.

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Abbreviations

ESMS–exponentially stable in the mean square, 61
SEMS-I–strongly exponentially stable in the mean square of the first kind, 61
SEMS-II–strongly exponentially stable in the mean square of the second kind, 61
ESMS-CI–exponentially stable in the mean square with conditioning of type I, 61
ESMS-CII–exponentially stable in the mean square with conditioning of type II, 61
ASMS–asymptotically stable in the mean square, 71
LMI–linear matrix inequality, 73
LQG–linear quadratic Gaussian,117
DTSGRE–discrete-time system of generalized Riccati equation, 118
DTSRE-C–discrete-time Riccati equation of stochastic control, 145
DTSRE-F–discrete-time Riccati equation of filtering, 154
DTSARE–discrete-time coupled algebraic Riccati equation, 263
DAP–disturbance attenuation problem, 281

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