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Progress in Nonlinear Differential Equations and Their Applications

Advances in Phase Space Analysis of Partial Differential Equations

In Honor of Ferruccio Colombini's 60th Birthday

Antonio Bove
Daniele Del Santo
M.K. Venkatesha Murthy
Editors

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Volume 78

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Advances in Phase Space Analysis of Partial Differential Equations

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ISBN 978-0-8176-4860-2 e-ISBN 978-0-8176-4861-9
DOI 10.1007/978-0-8176-4861-9

Library of Congress Control Number: 2009934006

Printed on acid-free paper

springer.com

Ferruccio Colombini

Se vòi d'Amor o d'altro bene stare,
magistra sit tibi vita aliena,
disse Cato in su' versificare.

Cecco Angiolieri, Rime, CVIII

Preface

The present volume is a collection of papers mainly concerning *Phase Space Analysis*, also known as *Microlocal Analysis*, and its applications to the theory of *Partial Differential Equations* (PDEs).

The basic idea behind this theory, at the crossing of harmonic analysis, functional analysis, quantum mechanics and algebraic analysis, is that many phenomena depend on both position and frequency (or wave numbers, or momentum) and therefore must be understood and described in the phase space. Including time and its dual variable, the energy, leads to the space-time phase space. From this perspective major progress has been achieved in the analysis of PDEs over the last forty years, based on the development of powerful tools of microlocal analysis.

A number of the following papers, all written by leading experts in their respective fields, are expanded versions of talks given at a meeting held in October 2007 at the Certosa di Pontignano, a former 1400 cloister sprawling on the hills surrounding Siena.

The Siena workshop was in honor of Ferruccio Colombini on the occasion of his 60th birthday and it is our pleasure to dedicate to him this volume, to which a number of friends and collaborators promptly manifested their willingness to contribute.

In this sense the present volume can be seen as a scientific portrait of Ferruccio.

Many people deserve our gratitude. We would like to thank all the contributors as well as the people who took part in the workshop, who made a lively mathematical attendance.

A number of institutions made possible to hold the Siena workshop through their financial support. They are the Italian Ministero dell'Istruzione, dell'Università e della Ricerca, Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, Università di Bologna, Università di Pisa and the scientific cooperation agreement between the universities of Pisa and Paris VI. We thank all of them for their generosity.

Bologna, Trieste, Pisa
August 2008

Antonio Bove
Daniele Del Santo
M. K. Venkatesha Murthy†

Note added in Proofs – During the preparation of this volume, on November 22, 2008, Prof. M. K. Venkatesha Murthy passed away after a sudden and brief illness. He was an outstanding mathematician, a friend and an example for us.
[A. B., D. D. S.]

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Tangent Halfspaces to Sets of Finite Perimeter in Carnot Groups

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Summary. We consider sets of locally finite perimeter in Carnot groups. We show that if E is a set of locally finite perimeter in a Carnot group G then, for almost every $x \in G$ with respect to the perimeter measure of E, some tangent of E at x is a vertical halfspace. This is a partial extension of a theorem of Franchi-Serapioni-Serra Cassano in step 2 Carnot groups.

2000 AMS Subject Classification: 53C17, 49Q15.

Key words: Rectifiability, Carnot groups, sets of finite perimeter.

1 Introduction

The content of this paper reflects, with additional comments and extensions, the talk given in the meeting in Pontignano. I will describe a recent joint work [5] with B. Kleiner and E. Le Donne devoted to the rectifiability of sets of finite perimeter in Carnot groups. I will spend some time in the description of the basic results in this subject, and only in the end I will illustrate our results, still not conclusive, and the open problems.

It is a pleasure to dedicate this paper to my friend Ferruccio Colombini, on the occasion of his 60th birthday.

2 Differentiability and rectifiability

Let us start from two classical results:

Theorem 2.1 (Rademacher, Math. Ann., 1919) *Any (locally) Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathscr{L}^n$-a.e. point x.*

Theorem 2.2 (De Giorgi, Ricerche Mat., 1955, [14]) *If $E \subseteq \mathbb{R}^n$ is a set of (locally) finite perimeter, then $(E - x)/r$ converges locally in measure as $r \downarrow 0$ to a halfspace H with $0 \in \partial H$ for $|D\chi_E|$-a.e. x.*

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_1,

Postponing to the next section the precise technical meaning of the "surface measure" $|D\chi_E|$, we just underline the links between the two results: in both cases, a blow-up procedure produces at a.e. point a simpler object (a linear function in the former case, a halfspace in the latter). In some sense, the second result is the geometric counterpart of the first one. The analogy becomes also more clear if we write differentiability in the form

$$\lim_{r\downarrow 0} f_{x,r}(y) = \nabla f(x)y \qquad \text{locally uniformly, with } f_{x,r}(y) := \frac{f(x+ry)-f(x)}{r}.$$

On the other hand, the "exceptional" sets in both statements are related by the coarea formula:

$$|\nabla f|\mathscr{L}^n = \int_{-\infty}^{+\infty} |D\chi_{\{f>t\}}|\, dt.$$

In this sense, the second result is more precise than the first (because a collection of $|D\chi_{\{f>t\}}|$-negligible sets gives rise to an $\mathscr{L}^n$-negligible set, at least in the region where $|\nabla f| > 0$). Indeed, in the framework of Carnot groups that soon we are going to describe, the analog of Rademacher's theorem is known, while only partial results are available on the analog of De Giorgi's theorem.

2.1 Sets of finite perimeter in Euclidean spaces

Here we explain the notation and the terminology used in the statement of De Giorgi's theorem.

A Borel set $E \subseteq \mathbb{R}^n$ is said to be a set of *finite perimeter* if the derivative in the sense $D\chi_E$ of the characteristic function χ_E is an $\mathbb{R}^n$-valued measure with finite total variation. Equivalently, there exists signed measure with finite total variation $D_i\chi_E$, $i = 1, \ldots, n$, such that

$$\int_E \frac{\partial f}{\partial x_i}\, dx = -\int_{\mathbb{R}^n} f\, D_i\chi_E \qquad \forall f \in C^1_c(\mathbb{R}^n).$$

Setting $D\chi_E = \nu_E|D\chi_E|$ with $\nu_E : \mathbb{R}^n \to \mathbf{S}^{n-1}$ (this is the so-called polar representation of vector-valued measures, the existence of ν_E and its uniqueness up to $|D\chi_E|$-negligible sets being ensured by the Radon–Nikodym theorem), we have a weak formulation of Green's formula:

$$\int_E \mathrm{div} g\, dx = -\int_{\mathbb{R}^n} \langle g, \nu_E\rangle\, d|D\chi_E| \qquad \forall g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n).$$

Sets E whose boundary is sufficiently regular and of finite surface measure have finite perimeter; in addition $|D\chi_E|(B) = \mathscr{H}^{n-1}(B \cap \partial E)$ for all Borel sets $B \subset \mathbb{R}^n$ and $\nu_E(x)$ is the inner normal. See [3, 16] for a comprehensive treatment of sets of finite perimeter (and BV functions) in Euclidean spaces.

3 Motivations

The study of the differentiability properties of maps (between manifolds, metric spaces, etc.) is motivated, for instance, by rigidity results. As an example, the question of nonexistence of Lipschitz onto maps between $\mathbb{R}^3$ and the first Heisenberg group $\mathbb{H}^1$ can be reduced, via a differentiability result, to the much simpler question of nonexistence of onto (even *homogeneous*) homeomorphisms between $\mathbb{R}^3$ and $\mathbb{H}^1$. We will precisely define the group $\mathbb{H}^1$ later on, here it suffices to say that $\mathbb{H}^1$ is a noncommutative group, hence such homeomorphism can exist.

On the other hand, the asymptotic behavior of the sets $(E-x)/r$ leads to a representation of the perimeter measure $|D\chi_E|$ as a Hausdorff $(n-1)$-dimensional measure, precisely

$$|D\chi_E|(B) = \mathscr{H}^{n-1}(B \cap \partial^* E) \qquad \text{for all Borel sets } B \subset \mathbb{R}^n.$$

Here $\partial^* E$ is the *essential boundary* of E, i.e., the set of points where the density of E is neither 0 nor 1.

As shown by De Giorgi in [14], this analysis leads also to the *rectifiability* of the essential boundary of E (i.e., there exist countably many hypersurfaces Γ_i such that $\mathscr{H}^{n-1}(\partial^* E \setminus \cup_i \Gamma_i) = 0$), and it is one of the most basic tools in Geometric Measure Theory and in the regularity theory of minimal surfaces.

3.1 Generalized differentiability

Some very recent work by Cheeger and Kleiner shows that the “geometric” viewpoint, exemplified by De Giorgi’s theorem, can be used as a replacement of the “analytic” one in some cases when the former fails.

A typical case is the one of L^1-valued maps, for which the usual concept of differentiability fails: for instance the map $t \mapsto \chi_{(0,t)}$ is nowhere differentiable.

A possible way to circumvent this difficulty is to look at the *metric derivative*: [27] this concept works well for Lipschitz maps from Euclidean spaces $\mathbb{R}^n$ into any metric space E, and provides a “metric differential,” i.e., a local norm χ_x such that

$$\lim_{r\downarrow 0} \frac{d_E(f(x+ry), f(x)) - \pi\chi_x(y)}{r} = 0 \qquad \text{locally uniformly.}$$

However, here I shall describe also another approach that works well, instead, when $f : X \to L^1$: according to [15], the pull-back distance on E given by $d_f(x,y) := \|f(x) - f(y)\|_1$ can be represented as a superposition of the so-called cut distance:

$$d_f(u,v) = \int d_C(u,v)\, d\Sigma_f(C).$$

Here the cut distance d_C induced by C is defined by $d_C(u,v) := |\chi_C(u) - \chi_C(v)|$, and Σ_f is a suitable measure in the space of subsets C of X.

Using this result, the asymptotic behavior of d_f on small scales is reduced to the behavior of the sets $C \in \operatorname{supp} \Sigma_f$ on small scales. Cheeger and Kleiner went further, proving in [9] that, under quite natural assumption on the metric measure space (X, d, μ), Σ_f is concentrated on the class of sets *of finite perimeter*. Here finiteness of perimeter must be understood in the generalized sense introduced by M. Miranda in [33].

Using this fact, together with the analog of De Giorgi's theorem in the Heisenberg groups $\mathbb{H}^n$ endowed with the Carnot–Carathéodory distance d_{cc} [19] (I am going to state this result and the definition of d_{cc} more precisely later on), they obtain the following rigidity result:

Theorem 3.1 *There exists no bi-Lipschitz embedding of $\mathbb{H}^1$ into L^1. More precisely, if $f : \mathbb{H}^1 \to L^1$ is Lipschitz, then*

$$\lim_{r \downarrow 0} \frac{\|f(x \exp(r^2 T)) - f(x)\|_1}{r} = 0 \qquad \text{for } \mathrm{vol}_{\mathbb{H}^1}\text{-a.e. } x \in \mathbb{H}^1.$$

Here, in exponential coordinates (x_1, x_2, t), $T = \partial_t$ and

$$d_{cc}(x \exp(r^2 T), x) = d_{cc}(\exp(r^2 T), 0) = cr.$$

So, it turns out that, at vol_{H^1}-a.e. point, f contracts too much the distance (and hence is not bi-Lipschitz) in the direction T. The geometric counterpart of this, as we will see, is the fact tangent sets to sets of finite perimeter in $\mathbb{H}^1$ are "vertical halfspaces," i.e., they are invariant along the T direction.

4 Carnot groups, differentiability of Lipschitz functions and sets of finite perimeter

Carnot groups are a natural object of study in Subelliptic PDEs, Harmonic Analysis, Control Theory, Geometry [17, 18, 24, 26, 41, 42]. They arise, for instance, as "tangent" spaces to Carnot–Carathéodory spaces [6].

A Carnot group $\mathbb{G}$ is a connected, simply connected and nilpotent Lie group whose Lie algebra $\mathfrak{g}$ of left-invariant vector fields admits a stratification:

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_s.$$

Here $V_{i+1} = [V_i, V_1]$ for $i \geq 1$, $V_s \neq \{0\}$ and $V_{s+1} = \{0\}$.

It is convenient to introduce a notation for some relevant parameters depending on $\mathfrak{g}$ and its stratification:

- s is the step of the group;
- V_1 is the space of *horizontal* vector fields;
- $Q := \sum_1^s i \dim(V_i)$ is the *homogeneous* dimension of $\mathbb{G}$.

Notice that $Q > \dim(\mathbb{G}) = \dim(\mathfrak{g})$, unless $s = 1$.

4.1 The Heisenberg groups

The Heisenberg group $\mathbb{H}^1$ is the step 2 group whose Lie algebra $\mathfrak{g}$ is spanned by X, Y, T satisfying

$$[X,Y]=T,\qquad [X,T]=0,\qquad [Y,T]=0.$$

Here $n=3$, $Q=4$. In exponential coordinates $(x,y,t)\mapsto \exp(xX+yY+tT)$ the vector fields are representable by

$$X=\partial_x+2y\partial_t,\qquad Y=\partial_y-2x\partial_t,\qquad T=-4\partial_t$$

and the group law is

$$(x,y,t)(x',y',t')=(x+x',y+y',t+t'-2xy'+2x'y).$$

More generally, in exponential coordinates, for any Carnot group the vector fields of the Lie algebra are divergence-free and have polynomial coefficients with respect to the canonical basis $\partial_{x_1},\ldots,\partial_{x_n}$ (this is a consequence of the Baker–Campbell–Hausdorff formula [30], which provides a formula for the polynomials depending only on the commutator relations in $\mathfrak{g}$).

4.2 Dilations and Carnot–Carathéodory distance

We may define dilations δ_r in $\mathfrak{g}$ setting $\delta_r X = r^i X$ for $X\in V_i$, and then extending δ_r linearly. Via the exponential map $\exp:\mathfrak{g}\to\mathbb{G}$ ($\exp\circ\delta_r=\delta_r\circ\exp$), the dilations can be defined on $\mathbb{G}$, and are well-behaved with respect to the group operations:

$$\delta_r(xy)=\delta_r(x)\delta_r(y)\qquad \forall x,\ y\in\mathbb{G},\ \forall r\geq 0.$$

The Carnot–Carathéodory distance $d_{\rm cc}$ is defined [34] by

$$d^2_{\rm cc}(x,y):=\inf\left\{\int_0^1|\dot\gamma|^2(t)\,dt:\ \gamma(0)=x,\ \gamma(1)=y\right\},$$

where the infimum is constrained to *horizontal* curves, i.e., those such that $\dot\gamma(t)\in(V_1)_{\gamma(t)}$ for $\mathscr{L}^1$-a.e. t.

This distance is compatible with the group law and with the dilations: we have $\mathrm{vol}_{\mathbb{G}}(B_r(x))=c\,r^Q$ for some $c>0$ and

$$d_{\rm cc}(zx,zy)=d_{\rm cc}(x,y),\qquad d_{\rm cc}(\delta_r x,\delta_r y)=r\,d_{\rm cc}(x,y).$$

4.3 Pansu differentiability theorem

The following result provides the natural extension of Rademacher's theorem to the context of Lie groups.

Theorem 4.1 (Pansu, Ann. Mat. 1989, [38]) *Let $f:\mathbb{G}\to\mathbb{M}$ be a Lipschitz map. Then, for $\mathrm{vol}_{\mathbb{G}}$-a.e. x, the limit*

$$df_x(Y) := \lim_{t\downarrow 0} \delta_{1/t}\big[(f(x))^{-1} f(x\exp(\delta_t Y))\big] \qquad Y\in\mathfrak{g}$$

exists and $\ln(df_x(Y))$ defines a homogeneous homeomorphism between the Lie algebras of $\mathbb{G}$ and $\mathbb{M}$.

Notice that the formula for the difference quotients is dictated by the following two requirements: first, intrinsic dilations (both in $\mathbb{G}$ and in $\mathbb{M}$) should be used; second, the property of being differentiable should be left invariant both with respect to translations in the domain ($f(x)$ replaced by $f(gx)$, with $g\in\mathbb{G}$) and in the target ($f(x)$ replaced by $mf(x)$, with $m\in\mathbb{M}$).

If $\mathbb{M}=\mathbb{R}$, the formula becomes simpler, and says that the limit

$$\lim_{t\downarrow 0}\frac{f(x\exp(\delta_t Y))-f(x)}{t}$$

exists. Moreover, the homogeneity implies that $df_x(Y)=0$ for all $Y\in V_i$ with $i>1$: indeed

$$2^i df_x(Y) = df_x(2^i Y) = df_x(\delta_2 Y) = \delta_2 df_x(Y) = 2df_x(Y).$$

Notice that, at *all* points, the Lipschitz continuity guarantees only $|f(x\exp(\delta_t Y))-f(x)|\le Ct$. The underlying reason for this phenomenon can be explained as follows: assume $Y=[X_1,X_2]$ with X_1, $X_2\in V_1$. Then

$$d_{\mathrm{cc}}\big(\bar x\exp(t^2Y), \bar x\exp(tX_1)\exp(tX_2)\exp(-tX_1)\exp(-tX_2)\big)=o(t)$$

yields

$$f\big(\bar x\exp(\delta_t Y)\big) - f\big(\bar x\exp(tX_1)\exp(tX_2)\exp(-tX_1)\exp(-tX_2)\big)=o(t).$$

But, since the target space is commutative, cancellations at Lebesgue points $\bar x$ of the horizontal gradient yield

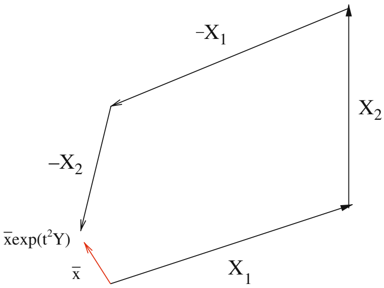
$$f(\bar x\exp(tX_1)\exp(tX_2)\exp(-tX_1)\exp(-tX_2)) - f(\bar x) = o(t).$$

Combining these two bits of information we get

$$f\big(\bar x\exp(\delta_t Y)\big) - f(\bar x) = o(t).$$

4.4 *X*-derivative and sets of finite perimeter

If X is a (smooth) vector field in $\mathbb{G}$, we can define the divergence $\div X$ by duality:



$$\int_{\mathbb{G}} Xu \, d\mathrm{vol}_{\mathbb{G}} = -\int_{\mathbb{G}} u \, \mathrm{div} X \, d\mathrm{vol}_{\mathbb{G}} \qquad \forall u \in C^1_c(\mathbb{G}).$$

Accordingly, if X is divergence-free, we define X-derivative in the sense of distributions by

$$\langle Xu, v \rangle := -\int_{\mathbb{G}} uXv \, d\mathrm{vol}_{\mathbb{G}} \qquad v \in C^\infty_c(\mathbb{G}).$$

As usual, we will write $Xu = f$ (resp. $Xu = \mu$) whenever this distribution is representable by integration with respect to $f\mathrm{vol}_{\mathbb{G}}$ (resp. μ).

Given a Borel set $E \subseteq \mathbb{G}$ we define

$$\mathrm{Reg}(E) := \{X \in \mathfrak{g} : \; X\chi_E \text{ is a Radon measure in } \mathbb{G}\}.$$

We say that E has finite perimeter if $\mathrm{Reg}(E) \supseteq V_1$.

It will also be useful to consider the subspace of *invariant* directions:

$$\mathrm{Inv}(E) := \{X \in \mathrm{Reg}(E) : \; X\chi_E = 0\}.$$

The set $\mathrm{Inv}(E)$ is easily seen to be a Lie subalgebra of $\mathfrak{g}$, because the identity $[X,Y]\chi_E = X(Y\chi_E) - Y(X\chi_E)$ still holds in the sense of distributions. While the whole theory is (conventionally) *left* invariant, the property $X\chi_E = 0$ ensures *right* invariance! Indeed

$$X\chi_E = 0 \qquad \Longleftrightarrow \qquad E\exp(tX) = E \quad \forall t \in \mathbb{R}.$$

4.5 Generalized inner normal to sets of finite perimeter

Keeping in mind the analogy with the Euclidean case, we would like to define a generalized inner normal. Obviously this is a metric-dependent concept, so we fix an orthonormal basis $X_1, \ldots, X_m$ of V_1 and we define the $\mathbb{R}^m$-valued measure

$$D\chi_E := (X_1\chi_E, \ldots, X_m\chi_E).$$

We have, by the Radon–Nikodym theorem,

$$D\chi_E = \nu_E |D\chi_E|, \text{ with } \nu_E : \mathbb{G} \to \mathbf{S}^{m-1}$$

and we call $\nu_E = (\nu_{E,1}, \ldots, \nu_{E,m})$ generalized (horizontal) inner normal.

5 Rectifiability in Euclidean spaces and in step 2 groups

Let us call *vertical halfspace* a Borel set $E \subseteq \mathbb{G}$ satisfying:

(i) E is invariant along all directions in V_i, $i > 1$, and along a codimension 1 subspace of V_1;
(ii) For some $X \in V_1$, $X\chi_E$ is a nonnegative Radon measure, not equal to 0.

In exponential coordinates, these sets correspond to the usual halfpsaces (up to a left translation)

$$\left\{x \in \mathbb{R}^n : \sum_{i=1}^{m} x_i \nu_i \geq 0\right\} \qquad \nu \in \mathbf{S}^{m-1},$$

with the only difference that only the first m coordinates of $x \in \mathbb{R}^n$ are involved. For this reason we use the adjective "vertical," because all these sets are invariant along the directions in V_i with $i > 1$.

Now, if we define

$$\mathrm{Tan}(E, x) := \left\{\lim_{i \to \infty} \delta_{1/r_i}(x^{-1}E) : \ (r_i) \downarrow 0\right\}$$

(the limit being understood with respect to local convergence in measure, i.e., L^1_{loc} convergence of characteristic functions), the rectifiability problem can be stated as follows: *show that, for $|D\chi_E|$-a.e. $x \in \mathbb{G}$, $\mathrm{Tan}(E, x)$ contains only a vertical halfspace H with $e \in \partial H$.*

5.1 Measure-theoretic properties of $|D\chi_E|$

Let us define the essential boundary of E as the set of points where the density is neither 0 nor 1:

$$\partial^* E := \left\{x \in \mathbb{G} : \ \liminf_{r \downarrow 0} \min\Big(\frac{\mathrm{vol}_{\mathbb{G}}(B_r(x) \cap E)}{\mathrm{vol}_{\mathbb{G}}(B_r(x))}, \frac{\mathrm{vol}_{\mathbb{G}}(B_r(x) \setminus E)}{\mathrm{vol}_{\mathbb{G}}(B_r(x))}\Big) > 0\right\}.$$

The following result could be considered as a very weak version of De Giorgi's theorem (it holds, however, in all doubling metric measure spaces supporting a Poincaré inequality, see [1]) and provides some information on the "perimeter" measure $|D\chi_E|$:

Theorem 5.1 (A., Adv. Math. '01) *Let $E \subseteq \mathbb{G}$ be a set with finite perimeter. Then, for $|D\chi_E|$-a.e. x, for $r > 0$ sufficiently small we have*

$$c_{\mathbb{G}}\omega_{Q-1}r^{Q-1} \leq |D\chi_E|(B_r(x)) \leq C_{\mathbb{G}}\omega_{Q-1}r^{Q-1},$$

with $0 < c_{\mathbb{G}} \leq C_{\mathbb{G}}$. As a consequence, $|D\chi_E| = \theta \mathscr{S}^{Q-1} \llcorner \partial^* E$ for some function $\theta \in [c_{\mathbb{G}}, C_{\mathbb{G}}]$.

Here $\mathscr{S}^{Q-1}$ is the spherical Hausdorff measure. The missing piece of information is the *explicit* characterization of the function θ (whose existence is provided by a nonconstructive result, the Radon–Nikodym theorem): this requires a more precise blow-up analysis, which is one of the motivations of studying the rectifiability problem.

5.2 De Giorgi's rectifiability proof

In essence (and omitting some deep preliminary volume and perimeter estimates), De Giorgi's proof [14] is based on the following steps:

- (Choice of the blow-up point)

A Lebesgue point $\bar{x}$ of ν_E, relative to $|D\chi_E|$; this means that $\int_{B_r(\bar{x})} |\nu_E(y) - \nu_E(\bar{x})|^2 \, d|D\chi_E|(y)$ is an infinitesimal faster than $|D\chi_E|(B_r(\bar{x}))$ as $r \downarrow 0$.

- (Normals of blow-ups are constant)

Since $\nu_{(E-\bar{x})/r}(y)$ equals $\nu_E(\bar{x}+ry)$, it turns out that all $F \in \mathrm{Tan}(E,\bar{x})$ have *constant* normal, equal to $\nu_E(\bar{x})$.

- (Classification of blow-ups)

By the previous step, any tangent set F at $\bar{x}$ satisfies $X\chi_F = 0$ if $X \perp \nu_E(\bar{x})$, and $X\chi_F \geq 0$ if $X = \nu_E(\bar{x})$.

These properties imply, by a classical and elementary smoothing argument, that F is a halfspace orthogonal to $\nu_E(\bar{x})$. Here the fact that we are dealing with constant coefficient operators plays an important role.

5.3 De Giorgi's argument in Carnot groups

Franchi, Serapioni and Serra Cassano reproduced (first in the Heisenberg groups [19] and then in all step 2 groups [20]) this argument in Carnot groups:

- (Choice of the blow-up point)

A Lebesgue point $\bar{x}$ of ν_E, relative to $|D\chi_E|$. Here no difference appears with the Euclidean case: thanks to the density estimates of Theorem 5.1 Lebesgue points are a set of full $|D\chi_E|$ measure.

- (Normals of blow-ups are constant)

Since $\nu_{\delta_{1/r}(\bar{x}^{-1}E)}(y)$ equals $\nu_E(\bar{x}\delta_r y)$, it turns out that all $F \in \mathrm{Tan}(E,\bar{x})$ have *constant horizontal* normal, equal to $\nu_E(\bar{x})$. Also here no essential difference appears, the only one is that we gain constancy of the horizontal normal and not of the Euclidean normal (which makes no sense, in this setting).

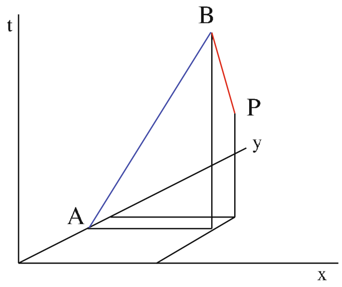
- (Classification of blow-ups)

By the previous step, any tangent set F at $\bar{x}$ satisfies $X\chi_F = 0$ if $X = \sum_i \xi_i X_i$ with $\xi \perp \nu_E(\bar{x})$, and $X\chi_F \geq 0$ if $X = \sum_i \nu_{E,i}(\bar{x}) X_i$.

Now the question is: do these properties imply that F is a vertical halfspace? Franchi, Serapioni and Serra Cassano proved that the answer is affirmative in step 2 Carnot groups. This leads to a complete proof of the rectifiability in this class of groups.

The proof, in $\mathbb{G} = \mathbb{H}^1$, is based on the following key geometric observation: if $x > x'$, it is possible to move from (x,y,t) to (x',y',t) using integral lines of $Y = \partial_y - 2x\partial_t$, and integral lines of $X = \partial_x + 2y\partial_t$ *only* in the positive direction.

Let us illustrate this with an example: in order to reach $P = (1, 1, 1)$ from $(0, 0, 0)$, we reach $A = (0, 3/4, 0)$ following $Y = (\partial_y - 2x\partial_t)$ for $T_1 = 3/4$, then we reach $B = (1, 3/4, 3/2)$ following $X = (\partial_x + 2y\partial_t)$ for $T_2 = 1$, and eventually we reach $P = (1, 1, 1)$ following Y for $T_3 = 1/4$.



In general step 2 groups $\mathbb{G}$ one uses the fact that, given any two horizontal directions X and Y, $\mathfrak{g}' := \mathrm{span}(X, Y, [X, Y])$ is a Lie subalgebra of $\mathfrak{g}$, so that $\mathbb{H}^1$ "embeds" into $\mathbb{G}$.

However, as shown in [20], this program fails in groups of step 3, or higher, because there exist sets with a constant horizontal normal which are not halfspaces.

Let e be the Carnot group (called Engel group, or group of Engel type) whose Lie algebra is $\mathfrak{g} = V_1 \oplus V_2 \oplus V_3$ with $V_1 = \mathrm{span}\{X_1, X_2\}$, $V_2 = \{\mathbb{R}X_3\}$ and $V_3 = \{\mathbb{R}X_4\}$, the only nonzero commutation relations being

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = -X_4.$$

In this case $n = 4$, $s = 3$, $Q = 2\cdot 1 + 1\cdot 2 + 1\cdot 3 = 7$. In exponential coordinates, an explicit representation of the vector fields is

$$\begin{cases} X_1 = \partial_1 + \frac{x_2}{2}\partial_3 + (\frac{x_3}{2} - \frac{x_1 x_2}{12})\partial_4, \\ X_2 = \partial_2 - \frac{x_1}{2}\partial_3 + \frac{x_1^2}{12}\partial_4, \\ X_3 = \partial_3 - \frac{x_1}{2}\partial_4, \\ X_4 = \partial_4. \end{cases}$$

Now, let $P : \mathbb{R}^4 \to \mathbb{R}$ be the polynomial

$$P(x) = \frac{1}{6}x_2(x_1^2 + x_2^2) - \frac{1}{2}x_1 x_3 + x_4,$$

whose gradient is

$$\nabla P(x) = \left(\frac{x_1 x_3}{3} - \frac{x_3}{2}, \frac{x_2^2}{2} + \frac{x_1^2}{6}, -\frac{x_1}{2}, 1 \right).$$

All level sets $\{P = c\}$ of P are obviously graphs of smooth functions depending on (x_1, x_2, x_3). We have

$$X_1 P(x) = 0, \quad X_2 P(x) = \frac{1}{2}(x_1^2 + x_2^2) \qquad \forall x \in \mathbb{R}^4. \tag{5.1}$$

We define

$$C := \{x \in \mathbb{R}^4 : \ P(x) \leq 0\},$$

whose boundary ∂C is the set $\{P = 0\}$. Notice that, due to the (intrinsic) homogeneity of degree 3 of the polynomial, the set C is a cone, i.e., $\delta_r C = C$ for all $r > 0$. Thus, C is a set with constant normal that is not a vertical halfspace (it is not invariant with respect to the X_3 and X_4 directions).

This example is reminiscent of the celebrated Simons' cone

$$\left\{x \in \mathbb{R}^8 : \ x_1^2 + x_2^2 + x_3^2 + x_4^2 < x_5^2 + x_6^2 + x_7^2 + x_8^2\right\},$$

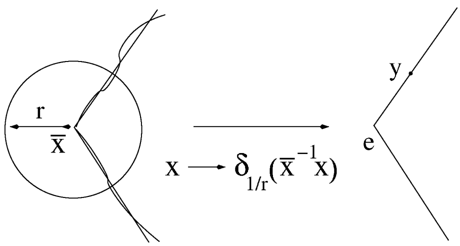
whose minimality has been proved by Bombieri, De Giorgi and Giusti. As a matter of fact, one can show that all sets with a constant horizontal normal, and C in particular, are *minimal surfaces* with respect to the Carnot–Carathéodory distance: they minimize the intrinsic perimeter $|D\chi_E|$ with respect to compactly supported variations.

6 Our main results, and open problems

In order to get a halfspace, the Engel cone example suggests to consider *iterated* tangents: we define $\mathrm{Tan}^{(1)}(E, x) = \mathrm{Tan}(E, x)$ and

$$\mathrm{Tan}^{(k+1)}(E, x) := \left\{\lim_{i \to \infty} \delta_{1/r_i}(y^{-1}F) : \ F \in \mathrm{Tan}^{(k)}(E, x), \ (r_i) \downarrow 0\right\}.$$

Here we need to iterate the blow-up procedure even at points $y \neq e$: if we do not do this, any conical tangent set (as the cone we considered before) would stop the process.



Theorem 6.1 *For $|D\chi_E|$-a.e. $\bar{x}$ and $k \geq 1 + 2(n - m)$, $\mathrm{Tan}^{(k)}(E, \bar{x})$ contains the vertical halfspace H with $e \in \partial H$ and with inner normal $\nu_E(\bar{x})$.*

The idea of the proof is to provide a mechanism for the growth, in higher and higher tangents, of regular and invariant directions. I shall describe this mechanism later on, but first I want to discuss the relation between this result and the original rectifiability problem. The connection is provided by this result:

Theorem 6.2 *For $|D\chi_E|$-a.e. $\bar{x}$ we have $\bigcup_k \mathrm{Tan}^{(k)}(E,\bar{x}) = \mathrm{Tan}(E,\bar{x})$. As a consequence, for $|D\chi_E|$-a.e. $\bar{x}$ there exists $(r_i) \downarrow 0$ satisfying*

$$\lim_{i\to\infty} \delta_{1/r_i}(\bar{x}^{-1}E) = H,$$

where H is the vertical halfspace with $e \in \partial H$ and inner normal $\nu_E(\bar{x})$.

This result is not yet a complete solution of the rectifiability problem. Indeed, we know that $(\bar{x}^{-1}E)$, when seen at *some small* scales r_i, looks like a vertical halfspace, but we would like to show that this happens on *all sufficiently small* scales r. What is still missing is some monotonicity/stability argument that singles out halfspaces among all possible tangent sets.

The principle "iterated tangents are tangent" was discovered by Preiss [39], in his complete proof (after partial solutions by Besicovitch, Marstrand, Mattila) of Federer's conjecture:

$$\exists\ \Theta(\bar{x}) := \lim_{r\downarrow 0} \frac{\mu(B_r(\bar{x}))}{\omega_k r^k} \in (0,\infty)\ \mu\text{-a.e.} \quad \Longrightarrow \quad \mu\ \mathscr{H}^k\text{-rectifiable.}$$

In the proof of Preiss's theorem, blow-up measures ν at $\bar{x}$ are uniform:

$$\nu(B_r(x)) = \Theta(\bar{x}) r^k \qquad \forall r > 0,\ x \in \operatorname{supp}\nu.$$

For instance, $\mathscr{H}^3$ restricted to the cone $\{x_4^2 = x_1^2 + x_2^2 + x_3^2\} \subseteq \mathbb{R}^4$ has this property. Flat measures are singled out, among all possible blow-ups, by an asymptotic stability property.

7 Some ideas from the proof

Let F be a tangent set at $\bar{x}$. We know that the Lie algebra $\mathrm{Inv}(F)$ contains a codimension 1 subspace of V_1, and that F has constant horizontal normal. We set

$$X := \sum_{i=1}^m \nu_{E,i}(\bar{x}) X_i \in V_1.$$

Our proof is based on the following two principles:

(1) *Regular nonhorizontal directions become, for the tangent sets, invariant* (at least at $\mathscr{H}^{Q-1}$-a.e. point);

(2) $\mathrm{Reg}(F) \supsetneq \mathrm{Inv}(F) + \{\mathbb{R}X\}$, *unless* $\mathrm{Inv}(F)$ *has codimension 1.*

The first principle is natural, because the group dilations δ_r shrink more in the nonhorizontal directions. Roughly speaking, it corresponds to saying that if f is a BV function, then

$$\lim_{r\downarrow 0} \frac{f(x+r^i y)-f(x)}{r} = 0 \qquad \text{in } L^1_{\rm loc}(\mathbb{R}^n_y)$$

for $\mathscr{L}^n$-a.e. x, provided $i>1$.

7.1 Regular directions become invariant

If $Y\in V_i\cap \mathrm{Reg}(E)$ for some $i>1$ and $\mu=Y\chi_E$, we have indeed

$$\begin{aligned}\int_{\delta_{1/r}(\bar x^{-1}E)} Y\phi\,dx &= r^{-Q}\int_{\bar x^{-1}E}(Y\phi)\circ\delta_{1/r}\,dy = r^{i-Q}\int_{\bar x^{-1}E} Y(\phi\circ\delta_{1/r})\,dy\\ &= -r^{i-Q}\int \phi(\delta_{1/r}(\bar x^{-1}y))\,d\mu(y).\end{aligned}$$

Now, if $\mathrm{supp}\,\phi\subseteq B_R$, we have $\mathrm{supp}\,\phi(\delta_{1/r}(\bar x^{-1}y))\subseteq B_{Rr}(\bar x)$ and therefore (for $\mathscr{H}^{Q-1}$-a.e. $\bar x$)

$$r^{i-Q}\int \phi(\delta_{1/r}(\bar x^{-1}y))\,d\mu(y) = r^{i-Q}O\big(|\mu|(B_{Rr}(\bar x))\big) = O(r^{i-1}).$$

Passing to the limit as $r\downarrow 0$ we conclude that $Y\chi_F=0$ for all $F\in \mathrm{Tan}(E,\bar x)$.

7.2 New regular directions

Recall that $\mathrm{Ad}_k:\mathfrak{g}\to\mathfrak{g}$ is defined by $Ad_k(Y):=dC_k(Y)$, where $C_k:\mathbb{G}\to\mathbb{G}$ is the conjugation by k:

$$C_k(g):=kgk^{-1}, \qquad g\in\mathbb{G}.$$

The first remark is that Ad_k maps $\mathrm{Reg}(F)$ into $\mathrm{Reg}(F)$ whenever F is right k-invariant. This is natural and not difficult to prove: indeed, on the one hand, the theory is left invariant, so that left multiplication by k preserves the regularity property; on the other hand, because of the right k-invariance of F, the potentially dangerous right multiplication by k (a map that is only Hölder continuous with respect to the Carnot–Carathéodory distance) has no effect. A simple computation shows that $Ad_k(X)\chi_F$ is the push-forward of the measure $X\chi_F$ under the right multiplication with k^{-1}.

The following lemma provides the basic mechanism for the generation of new regular directions, starting from the monotonicity direction and the invariant directions.

Lemma 7.1 $\{\mathrm{Ad}_k(X):\ k \in \exp(\mathrm{Inv}(F))\}$ *is not contained in* $\mathrm{Inv}(F) + \{\mathbb{R}X\}$*, unless* $\mathrm{Inv}(F)$ *has codimension* 1.

A sketchy proof goes as follows: if this does not happen, setting $\mathbb{K} = \exp(\mathrm{Inv}(F))$ and denoting by $\pi : \mathbb{G} \to \mathbb{G}/\mathbb{K}$ the projection map, we can "project" $\mathrm{Ad}_k(X)$ to a smooth 1-dimensional distribution W on $\mathbb{G}/\mathbb{K}$. This would be integrable, and the inverse image $\pi^{-1}(\mathcal{F})$ of a foliation $\mathcal{F}$ of $\mathbb{G}/\mathbb{K}$ tangent to W would give a foliation of $\mathbb{G}$ tangent to W. This contradicts the fact that the Lie algebra generated by W is the whole of $\mathfrak{g}$.

Denoting for simplicity by $\mathfrak{g}'$ the Lie subalgebra $\mathrm{Inv}(F)$, a closer analysis of the operator $\mathrm{Ad}_{\exp(Y)}$, written as an exponential of the map $Y \mapsto [X,Y]$, reveals that

$$\mathrm{span}\big(\{\mathrm{Ad}_k(X):\ k \in \exp(\mathfrak{g}')\}\big) = X + [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] + \cdots .$$

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The Heat Kernel and Frequency Localized Functions on the Heisenberg Group

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Summary. The goal of this paper is to study the action of the heat operator on the Heisenberg group $\mathbb{H}^d$, and in particular to characterize Besov spaces of negative index on $\mathbb{H}^d$ in terms of the heat kernel. That characterization can be extended to positive indexes using Bernstein inequalities. As a corollary we obtain a proof of refined Sobolev inequalities in $\dot{W}^{s,p}$ spaces.

Mathematics Subject Classification (2000): 43A85, 43A30.

Key words: Heat kernel, Besov space, Heisenberg group, frequency localization.

1 Introduction

This paper is concerned mainly with a characterization of Besov spaces on the Heisenberg group using the heat kernel. In [1], a Littlewood–Paley decomposition on the Heisenberg group is constructed, and Besov spaces are defined using that decomposition. It is classical that in $\mathbb{R}^d$ there is an equivalent definition, for negative regularity indexes, in terms of the heat kernel. This characterization in $\mathbb{R}^d$ can be extended to positive regularity indexes thanks to Bernstein's inequalities which express that derivatives act almost as homotheties on distributions, the Fourier transform of which is supported in a ring of $\mathbb{R}^d$ centered at zero.

The aim of this text is to present a similar characterization of Besov spaces on $\mathbb{H}^d$ using the heat flow. One of the main steps of the procedure in $\mathbb{R}^d$ consists in studying frequency localized functions and the action of derivatives, and more generally Fourier multipliers, on such functions (the corresponding

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_2,

inequalities for derivatives are known as Bernstein inequalities). In the Heisenberg group there is a priori no simple notion of frequency localization, since the Fourier transform is a family of operators on a Hilbert space; however, frequencies may be understood by studying the action of the Laplacian on a Hilbertian basis of that space, which allows one to define a notion of frequency localization (see Definition 2.10 below). One can then try to investigate the action of the semigroup of the heat equation on the Heisenberg group on such frequency localized functions. That is achieved in this paper; we also prove a similar characterization of Besov spaces in terms of the heat flow, as in the classical $\mathbb{R}^d$ case. This allows us to prove refined Sobolev inequalities, for $\dot{W}^{s,p}$ spaces. Finally we are able by similar techniques to recover the fact that the heat semigroup is the convolution by a function in the Schwartz class (as in previous works by Gaveau in [6] and Hulanicki in [8]).

Let us mention that by a different method, Furioli, Melzi and Veneruso obtained in [5] a characterization of Besov spaces in terms of the heat kernel for Lie groups of polynomial growth.

1.1 The Heisenberg group $\mathbb{H}^d$

In this introductory section, let us recall some basic facts on the Heisenberg group $\mathbb{H}^d$. The Heisenberg group $\mathbb{H}^d$ is the Lie group with underlying $\mathbb{C}^d \times \mathbb{R}$ endowed with the following product law:

$$\forall\,((z,s),(z',s')) \in \mathbb{H}^d \times \mathbb{H}^d, \quad (z,s)\cdot(z',s') = (z+z', s+s'+2\mathrm{Im}(z\cdot\overline{z}')),$$

where $z\cdot\overline{z}' = \sum_{j=1}^d z_j\overline{z}'_j$. It follows that $\mathbb{H}^d$ is a noncommutative group, the identity of which is $(0,0)$; the inverse of the element (z,s) is given by $(z,s)^{-1} = (-z,-s)$. The Lie algebra of left invariant vector fields on the Heisenberg group $\mathbb{H}^d$ is spanned by the vector fields

$$Z_j = \partial_{z_j} + i\overline{z}_j\partial_s, \quad \overline{Z}_j = \partial_{\overline{z}_j} - iz_j\partial_s \quad \text{and} \quad S = \partial_s = \frac{1}{2i}[\overline{Z}_j, Z_j],$$

with $j \in \{1,\dots,d\}$. In all that follows, we shall denote by $\mathcal{Z}$ the family of vector fields defined by Z_j for $j \in \{1,\dots,d\}$ and $Z_j = \overline{Z}_{j-d}$ for $j \in \{d+1,\dots,2d\}$ and for any multi-index $\alpha \in \{1,\dots,2d\}^k$, we will write

$$\mathcal{Z}^\alpha \stackrel{\text{def}}{=} Z_{\alpha_1}\dots Z_{\alpha_k}. \tag{1.1}$$

The space $\mathbb{H}^d$ is endowed with a smooth left invariant measure, the Haar measure, which in the coordinate system (x,y,s) is simply the Lebesgue measure $dxdyds$.

Let us point out that on the Heisenberg group $\mathbb{H}^d$, there is a notion of dilation defined for $a>0$ by $\delta_a(z,s) = (az, a^2 s)$. The homogeneous dimension of $\mathbb{H}^d$ is therefore $N \stackrel{\text{def}}{=} 2d+2$, noticing that the Jacobian of the dilation δ_a is a^N.

The Schwartz space $\mathcal{S}(\mathbb{H}^d)$ on the Heisenberg group is defined as follows.

Definition 1.1 *The Schwartz space $\mathcal{S}(\mathbb{H}^d)$ is the set of smooth functions u on $\mathbb{H}^d$ such that, for any $k \in \mathbb{N}$, we have*

$$\|u\|_{k,\mathcal{S}} \stackrel{def}{=} \sup_{\substack{|\alpha|\leq k \\ (z,s)\in\mathbb{H}^d}} \left| \mathcal{Z}^\alpha \left((|z|^2 - is) \right)^{2k} u(z,s) \right| < \infty.$$

Remark 1.2 *The Schwartz space on the Heisenberg group $\mathcal{S}(\mathbb{H}^d)$ coincides with the classical Schwartz space $\mathcal{S}(\mathbb{R}^{2d+1})$. The weight in (z,s) appearing in the definition above is related to the fact that the Heisenberg distance to the origin is defined by $\rho(z,s) \stackrel{def}{=} \left(|z|^4 + s^2\right)^{\frac{1}{4}}$.*

Finally, let us present the Laplacian–Kohn operator, which is central in the study of partial differential equations on $\mathbb{H}^d$, and is defined by

$$\Delta_{\mathbb{H}^d} \stackrel{\text{def}}{=} 2\sum_{j=1}^{d} (Z_j \overline{Z}_j + \overline{Z}_j Z_j).$$

Powers of that operator allow us to construct positive order Sobolev spaces: for example we define the homogeneous space $\dot{W}^{s,p}(\mathbb{H}^d)$, for $0 < s < N/p$, as the completion of $\mathcal{S}(\mathbb{H}^d)$ for the norm

$$\|f\|_{\dot{W}^{s,p}(\mathbb{H}^d)} \stackrel{\text{def}}{=} \left\|(-\Delta_{\mathbb{H}^d})^{\frac{s}{2}} f\right\|_{L^p(\mathbb{H}^d)}.$$

1.2 Statement of the results

In [1] and [3] a dyadic unity partition is built on the Heisenberg group $\mathbb{H}^d$, similar to the one defined in the classical $\mathbb{R}^d$ case. A significant application of this decomposition is the definition of Besov spaces on the Heisenberg group in the same way as in the classical case (see [1],[3]). In Section 2, we shall give a full account of this theory.

The main result of this paper describes the action of the semigroup associated with the heat equation on the Heisenberg group, on a frequency localized function. We refer to Definition 2.10 below for the notion of a frequency localized function, which requires the definition of the Fourier transform on $\mathbb{H}^d$, and is therefore slightly technical.

Lemma 1.3 *Let (r_1, r_2) be two positive real numbers, and define $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0, r_1, r_2)$ the ring centered at the origin, of small and large radius respectively r_1 and r_2. Two positive constants c and C exist such that, for any real number $p \in [1,\infty]$, any couple (t,β) of positive real numbers and any function u frequency localized in the ring $\beta\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, we have*

$$\|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p(\mathbb{H}^d)} \leq C e^{-ct\beta^2} \|u\|_{L^p(\mathbb{H}^d)}. \tag{1.2}$$

That lemma is the key argument in the proof of the following theorem which is well known in $\mathbb{R}^d$ and proved by a different method in [5] for Lie groups of polynomial growth. The definition of Besov spaces is provided in the next section.

Theorem 1.4 *Let s be a positive real number and $(p,r) \in [1,\infty]^2$. A constant C exists which satisfies the following property. For $u \in \dot{B}^{-2s}_{p,r}(\mathbb{H}^d)$, we have*

$$C^{-1}\|u\|_{\dot{B}^{-2s}_{p,r}(\mathbb{H}^d)} \leq \Big\| \|t^s e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p(\mathbb{H}^d)} \Big\|_{L^r(\mathbb{R}^+, \frac{dt}{t})} \leq C\|u\|_{\dot{B}^{-2s}_{p,r}(\mathbb{H}^d)}. \tag{1.3}$$

Remark 1.5 *Thanks to Bernstein's inequalities (see Proposition 2.12 below), we have*

$$\|u\|_{\dot{B}^{\sigma}_{p,r}(\mathbb{H}^d)} \equiv \sup_{|\alpha|=k} \big\|(-\Delta_{\mathbb{H}^d})^{\frac{\alpha}{2}} u\big\|_{\dot{B}^{\sigma-k}_{p,r}(\mathbb{H}^d)}\,.$$

We deduce that the characterization of Besov spaces on the Heisenberg group in terms of the heat kernel can be extended to any positive regularity index.

This characterization is useful for instance to prove refined Sobolev inequalities. In this paper we will prove the following result.

Theorem 1.6 *Let $p \in [1,\infty]$ and $0 < s < N/p$ be given. There exists a positive constant C such that for any function f in $\dot{W}^{s,p}(\mathbb{H}^d)$ we have*

$$\|f\|_{L^q(\mathbb{H}^d)} \leq C\|f\|^{1-\frac{sp}{N}}_{\dot{W}^{s,p}(\mathbb{H}^d)}\|f\|^{\frac{sp}{N}}_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}},$$

with $q = pN/(N-ps)$.

Remark 1.7 *This is a refined Sobolev inequality since it is easy to see that $\dot{W}^{s,p}(\mathbb{H}^d)$ is continuously embedded in $\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}$, so that Theorem 1.6 is a refined version of the classical inequality*

$$\|f\|_{L^q(\mathbb{H}^d)} \leq C\|f\|_{\dot{W}^{s,p}(\mathbb{H}^d)}.$$

The above continuous embedding is simply due to the following estimate, applied to $u = (-\Delta_{\mathbb{H}^d})^{\frac{s}{2}} f$:

$$\|u\|_{\dot{B}^{-\frac{N}{p}}_{\infty,\infty}} = \sup_{t>0} t^{\frac{N}{2p}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^\infty(\mathbb{H}^d)} \leq C\|u\|_{L^p(\mathbb{H}^d)}.$$

Note that in the special case when $p = 2$, such an inequality was proved in [3], using the method developed in the classical case in [7].

It turns out that the techniques involved in the proof of Lemma 1.3 enable us to recover the following theorem, which was proved (by different methods) by Gaveau in [6] and Hulanicki in [8].

Theorem 1.8 *There exists a function $h \in \mathcal{S}(\mathbb{H}^d)$ such that if u denotes the solution of the free heat equation on the Heisenberg group*

$$\begin{cases} \partial_t u - \Delta_{\mathbb{H}^d} u = 0 \quad in \quad \mathbb{R}^+ \times \mathbb{H}^d, \\ u_{|t=0} = u_0, \end{cases} \tag{1.4}$$

then we have

$$u(t, \cdot) = u_0 \star h_t,$$

where $\star$ denotes the convolution on the Heisenberg group defined in Section 2 below, while h_t is defined by

$$h_t(x, y, s) = \frac{1}{t^{d+1}} h\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, \frac{s}{t}\right).$$

The rest of this paper is devoted to the proof of Theorems 1.4 to 1.8, as well as Lemma 1.3.

The structure of the paper is the following. First, in Section 2, we present a short review of the Littlewood–Paley theory on the Heisenberg group, giving the notation and results that will be used in the proofs, as well as the main references of the theory. Section 3 is devoted to the proof of Theorem 1.4, assuming Lemma 1.3, and finally the proof of Lemma 1.3 can be found in Section 4. In Section 4 we also give the proofs of Theorems 1.6 and 1.8.

2 Elements of Littlewood–Paley theory on the Heisenberg group

2.1 The Fourier transform on the Heisenberg group

To introduce the Littlewood–Paley theory on the Heisenberg group, we need to recall the definition of the Fourier transform in that framework. We refer for instance to [10], [11] or [12] for more details. The Heisenberg group being noncommutative, the Fourier transform on $\mathbb{H}^d$ is defined using irreducible unitary representations of $\mathbb{H}^d$. As explained for instance in [12, Chapter 2], all irreducible representations of $\mathbb{H}^d$ are unitarily equivalent to one of two representations: the Bargmann representation or the L^2-representation. The representations on $L^2(\mathbb{R}^d)$ can be deduced from Bargmann representations thanks to interlacing operators. The reader can consult J. Faraut and K. Harzallah [4] for more details. We shall choose here the Bargmann representations described by $(u^\lambda, \mathcal{H}_\lambda)$, with $\lambda \in \mathbb{R} \setminus \{0\}$, where $\mathcal{H}_\lambda$ are the spaces defined by

$$\mathcal{H}_\lambda = \{F \text{ holomorphic on } \mathbb{C}^d, \|F\|_{\mathcal{H}_\lambda} < \infty\},$$

while we define

$$\|F\|^2_{\mathcal{H}_\lambda} \stackrel{\text{def}}{=} \left(\frac{2|\lambda|}{\pi}\right)^d \int_{\mathbb{C}^d} e^{-2|\lambda||\xi|^2} |F(\xi)|^2 d\xi, \tag{2.1}$$

and u^λ is the map from $\mathbb{H}^d$ into the group of unitary operators of $\mathcal{H}_\lambda$ defined by

$$\begin{aligned} u^\lambda_{z,s} F(\xi) &= F(\xi - \overline{z}) e^{i\lambda s + 2\lambda(\xi\cdot z - |z|^2/2)} \quad \text{for} \quad \lambda > 0, \\ u^\lambda_{z,s} F(\xi) &= F(\xi - z) e^{i\lambda s - 2\lambda(\xi\cdot \overline{z} - |z|^2/2)} \quad \text{for} \quad \lambda < 0. \end{aligned}$$

Let us notice that $\mathcal{H}_\lambda$ equipped with the norm (2.1) is a Hilbert space and that the monomials

$$F_{\alpha,\lambda}(\xi) = \frac{(\sqrt{2|\lambda|}\,\xi)^\alpha}{\sqrt{\alpha!}}, \quad \alpha \in \mathbb{N}^d,$$

constitute an orthonormal basis.

If f belongs to $L^1(\mathbb{H}^d)$, its Fourier transform is given by

$$\mathcal{F}(f)(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{H}^d} f(z,s) u^\lambda_{z,s} dz ds.$$

Note that the function $\mathcal{F}(f)$ takes its values in the bounded operators on $\mathcal{H}_\lambda$. As in the $\mathbb{R}^d$ case, one has a Plancherel theorem and an inversion formula. More precisely, let $\mathcal{A}$ denote the Hilbert space of one-parameter families $A = \{A(\lambda)\}_{\lambda \in \mathbb{R}\setminus\{0\}}$ of operators on $\mathcal{H}_\lambda$ which are Hilbert–Schmidt for almost every $\lambda \in \mathbb{R}$ with norm

$$\|A\| = \left(\frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \|A(\lambda)\|^2_{HS(\mathcal{H}_\lambda)} |\lambda|^d d\lambda\right)^{\frac{1}{2}} < \infty,$$

where $\|A(\lambda)\|_{HS(\mathcal{H}_\lambda)}$ denotes the Hilbert–Schmidt norm of the operator $A(\lambda)$. Then the Fourier transform can be extended to an isometry from $L^2(\mathbb{H}^d)$ onto $\mathcal{A}$ and we have the Plancherel formula:

$$\|f\|^2_{L^2(\mathbb{H}^d)} = \frac{2^{d-1}}{\pi^{d+1}} \sum_{\alpha \in \mathbb{N}^d} \int_{-\infty}^{\infty} \|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda} |\lambda|^d d\lambda.$$

On the other hand, if

$$\sum_{\alpha \in \mathbb{N}^d} \int_{-\infty}^{\infty} \|\mathcal{F}(f)(\lambda) F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda} |\lambda|^d d\lambda < \infty, \tag{2.2}$$

then we have for almost every w,

$$f(w) = \frac{2^{d-1}}{\pi^{d+1}} \int_{-\infty}^{\infty} \operatorname{tr}\left(u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda)\right) |\lambda|^d d\lambda, \tag{2.3}$$

where

$$\operatorname{tr}\left(u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda)\right) = \sum_{\alpha \in \mathbb{N}^d} (u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda) F_{\alpha,\lambda}, F_{\alpha,\lambda})_{\mathcal{H}_\lambda}$$

denotes the trace of the operator $u^\lambda_{w^{-1}} \mathcal{F}(f)(\lambda)$.

Remark 2.1 *The above hypothesis* (2.2) *is satisfied in* $\mathcal{S}(\mathbb{H}^d)$*, where* $\mathcal{S}(\mathbb{H}^d)$ *is defined in Definition* 1.1*. This follows from Proposition* 2.2 *which is proved for the sake of completeness, directly below its statement.*

Let us moreover point out that we have the following useful formulas, for any $k \in \{1, ..., d\}$.

Denoting by $1_k = (0, \dots, 1, \dots)$ the vector whose k-component is one and all the others are zero, one has

$$\mathcal{F}(Z_k f)(\lambda)F_{\alpha,\lambda} = -\sqrt{2|\lambda|}\sqrt{\alpha_k + 1}\mathcal{F}(f)(\lambda)F_{\alpha+1_k,\lambda} \tag{2.4}$$

if $\lambda > 0$, and similarly

$$\mathcal{F}(Z_k f)(\lambda)F_{\alpha,\lambda} = \sqrt{2|\lambda|}\sqrt{\alpha_k}\mathcal{F}(f)(\lambda)F_{\alpha-1_k,\lambda} \tag{2.5}$$

if $\lambda < 0$. Furthermore,

$$\mathcal{F}(\overline{Z_k} f)(\lambda)F_{\alpha,\lambda} = \sqrt{2|\lambda|}\sqrt{\alpha_k}\mathcal{F}(f)(\lambda)F_{\alpha-1_k,\lambda} \tag{2.6}$$

if $\lambda > 0$, and

$$\mathcal{F}(\overline{Z_k} f)(\lambda)F_{\alpha,\lambda} = -\sqrt{2|\lambda|}\sqrt{\alpha_k + 1}\mathcal{F}(f)(\lambda)F_{\alpha+1_k,\lambda} \tag{2.7}$$

if $\lambda < 0$. Therefore, we have easily, for any $\rho \in \mathbb{R}$,

$$\mathcal{F}((-\Delta_{\mathbb{H}^d})^\rho f)(\lambda)F_{\alpha,\lambda} = (4|\lambda|(2|\alpha| + d))^\rho\, \mathcal{F}(f)(\lambda)F_{\alpha,\lambda} \tag{2.8}$$

and

$$\mathcal{F}\left(e^{t\Delta_{\mathbb{H}^d}} f\right)(\lambda)F_{\alpha,\lambda} = e^{-t(4|\lambda|(2|\alpha|+d))}\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}.$$

Using those formulas, we can prove the following proposition, which justifies Remark 2.1 stated above. The proof of this proposition is new to our knowledge.

Proposition 2.2 *For any function* $f \in \mathcal{S}(\mathbb{H}^d)$, (2.2) *is satisfied. More precisely, for any* $\rho > \frac{N}{2}$*, there exists a positive constant* C *such that*

$$\sum_{\alpha\in\mathbb{N}^d} \int_{-\infty}^{\infty} \|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda} |\lambda|^d \, d\lambda \leq C\Big(\|f\|_{L^1(\mathbb{H}^d)} + \|(-\Delta_{\mathbb{H}^d})^\rho f\|_{L^1(\mathbb{H}^d)}\Big).$$

Let us prove that result. By definition of $\mathcal{S}(\mathbb{H}^d)$, for any $\rho \in \mathbb{R}$, the function $(-\Delta_{\mathbb{H}^d})^\rho f$ belongs to $\mathcal{S}(\mathbb{H}^d)$. Therefore, we can write, using (2.8),

$$\begin{aligned}\mathcal{F}(f)(\lambda)F_{\alpha,\lambda} &= \mathcal{F}((-\Delta_{\mathbb{H}^d})^{-\rho}(-\Delta_{\mathbb{H}^d})^\rho f)(\lambda)F_{\alpha,\lambda}\\ &= (4|\lambda|(2|\alpha| + d))^{-\rho}\mathcal{F}((-\Delta_{\mathbb{H}^d})^\rho f)(\lambda)F_{\alpha,\lambda}.\end{aligned}$$

But that implies that

$$\|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda} = (4|\lambda|(2|\alpha|+d))^{-2\rho}\Big(\frac{2|\lambda|}{\pi}\Big)^d \int_{\mathbb{C}^d} e^{-2|\lambda||\xi|^2}\Big|\mathcal{F}((-\Delta_{\mathbb{H}^d})^\rho f)(\lambda)F_{\alpha,\lambda}(\xi)\Big|^2 d\xi.$$

According to the definition of the Fourier transform on the Heisenberg group, we thus have

$$\|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda} = (4|\lambda|(2|\alpha|+d))^{-2\rho}\Big(\frac{2|\lambda|}{\pi}\Big)^d \int_{\mathbb{C}^d} e^{-2|\lambda||\xi|^2} \times\Big(\int_{\mathbb{H}^d}((-\Delta_{\mathbb{H}^d})^\rho f(z,s))u^\lambda_{z,s}F_{\alpha,\lambda}dzds\overline{\int_{\mathbb{H}^d}((-\Delta_{\mathbb{H}^d})^\rho f(z',s'))u^\lambda_{z',s'}F_{\alpha,\lambda}dz'ds'}\Big)d\xi.$$

Fubini's theorem allows us to write

$$\|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|^2_{\mathcal{H}_\lambda} = (4|\lambda|(2|\alpha|+d))^{-2\rho} \times\int_{\mathbb{H}^d}\int_{\mathbb{H}^d}(-\Delta_{\mathbb{H}^d})^\rho f(z,s)\overline{(-\Delta_{\mathbb{H}^d})^\rho f(z',s')}(u^\lambda_{z,s}F_{\alpha,\lambda}\,|\,u^\lambda_{z',s'}F_{\alpha,\lambda})_{\mathcal{H}_\lambda}dzdsdz'ds'.$$

Since the operators $u^\lambda_{z,s}$ and $u^\lambda_{z',s'}$ are unitary on $\mathcal{H}_\lambda$ and the family $(F_{\alpha,\lambda})$ is a Hilbert basis of $\mathcal{H}_\lambda$, we deduce that

$$\|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda} \leq (4|\lambda|(2|\alpha|+d))^{-\rho}\|(-\Delta_{\mathbb{H}^d})^\rho f\|_{L^1(\mathbb{H}^d)}.$$

To conclude we decompose the integral on λ into two parts, corresponding to "high and low" frequencies (the parameter $|\lambda|^{\frac{1}{2}}$ may be identified as a frequency, as will be clear in the next section—it is in fact already apparent in (2.8) above). Thus denoting $\lambda_m = (2m+d)\lambda$, we write

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}^d}\int_{-\infty}^{\infty}\|\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}\|_{\mathcal{H}_\lambda}|\lambda|^d d\lambda &\leq \sum_{m\in\mathbb{N}}\binom{m+d-1}{m}\Big(\|f\|_{L^1(\mathbb{H}^d)}\int_{|\lambda_m|\leq 1}|\lambda|^d d\lambda \\ &\quad + (4(2m+d))^{-\rho}\|(-\Delta_{\mathbb{H}^d})^\rho f\|_{L^1(\mathbb{H}^d)}\int_{|\lambda_m|\geq 1}|\lambda|^{-\rho}|\lambda|^d d\lambda\Big).\end{aligned}$$

This gives the announced result for $\rho > N/2$. The proposition is proved. $\square$

Finally the convolution product of two functions f and g on $\mathbb{H}^d$ is defined by

$$f\star g(w) = \int_{\mathbb{H}^d} f(wv^{-1})g(v)dv = \int_{\mathbb{H}^d} f(v)g(v^{-1}w)dv.$$

It should be emphasized that the convolution on the Heisenberg group is not commutative. Moreover, if P is a left invariant vector field on $\mathbb{H}^d$, then one sees easily that

$$P(f \star g) = f \star Pg, \tag{2.9}$$

whereas in general $P(f \star g) \neq Pf \star g$. Nevertheless, the usual Young inequalities are valid on the Heisenberg group, and one has moreover

$$\mathcal{F}(f \star g)(\lambda) = \mathcal{F}(f)(\lambda) \circ \mathcal{F}(g)(\lambda). \tag{2.10}$$

It turns out that for radial functions on the Heisenberg group, the Fourier transform becomes simplified and puts into light the quantity that will play the role of the frequency size. Let us first recall the concept of radial functions on the Heisenberg group.

Definition 2.3 *A function f defined on the Heisenberg group $\mathbb{H}^d$ is said to be radial if it is invariant under the action of the unitary group $U(d)$ of $\mathbb{C}^d$, which means that for any $u \in U(d)$, we have*

$$f(z,s) = f(u(z),s), \quad \forall (z,s) \in \mathbb{H}^d.$$

A radial function on the Heisenberg group can then be written under the form

$$f(z,s) = g(|z|,s).$$

It can be shown (see for instance [10]) that the Fourier transform of radial functions of $L^2(\mathbb{H}^d)$ satisfies the following formulas:

$$\mathcal{F}(f)(\lambda)F_{\alpha,\lambda} = R_{|\alpha|}(\lambda)F_{\alpha,\lambda},$$

where

$$R_m(\lambda) = \binom{m+d-1}{m}^{-1} \int e^{i\lambda s} f(z,s) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} dz ds,$$

and where $L_m^{(p)}$ are Laguerre polynomials defined by

$$L_m^{(p)}(t) = \sum_{k=0}^{m} (-1)^k \binom{m+p}{m-k} \frac{t^k}{k!}, \quad t \geq 0, \quad m,p \in \mathbb{N}.$$

Note that in that case

$$\|f\|_{L^2(\mathbb{H}^d)} = \|(R_m)\|_{L^2_d(\mathbb{N}\times\mathbb{R})}$$
$$\stackrel{\text{def}}{=} \left(\frac{2^{d-1}}{\pi^{d+1}} \sum_m \binom{m+d-1}{m} \int_{-\infty}^{\infty} |Q_m(\lambda)|^2 |\lambda|^d d\lambda \right)^{\frac{1}{2}}, \tag{2.11}$$

which corresponds to the Plancherel formula recalled above, in the radial case. We also have the following inversion formula: if R_m belongs to $L^2_d(\mathbb{N} \times \mathbb{R})$ defined in (2.11), then the function

$$f(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} R_m(\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda \tag{2.12}$$

is a radial function in $L^2(\mathbb{H}^d)$ and satisfies

$$\mathcal{F}(f)(\lambda)F_{\alpha,\lambda} = R_{|\alpha|}(\lambda)F_{\alpha,\lambda}.$$

2.2 Littlewood–Paley theory on the Heisenberg group

Now we are ready to define the Littlewood–Paley decomposition on $\mathbb{H}^d$. We will not give any proof but refer to the construction in [1] and [3] for all the details. We simply recall that the key point in the construction of the Littlewood–Paley decomposition on $\mathbb{H}^d$ lies in the following proposition proved in [1]. Note that Proposition 2.4 enables one to show in particular that functions of $-\Delta_{\mathbb{H}^d}$ may be seen as convolution operators by Schwartz class functions (a result proved by Hulanicki [8] in the case of general nilpotent Lie groups).

Proposition 2.4 *For any* $Q \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, *the series*

$$g(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} Q((2m+d)\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda$$

converges in $\mathcal{S}(\mathbb{H}^d)$.

The Littlewood–Paley operators are then constructed using the following proposition (see [1] and [3]).

Proposition 2.5 *Define the ring* $\mathcal{C}_0 = \left\{\tau \in \mathbb{R},\ \frac{3}{4} \leq |\tau| \leq \frac{8}{3}\right\}$ *and the ball* $\mathcal{B}_0 = \left\{\tau \in \mathbb{R},\ |\tau| \leq \frac{4}{3}\right\}$. *Then there exist two radial functions* $\widetilde{R}^*$ *and* R^* *the values of which are in the interval* $[0,1]$, *belonging respectively to* $\mathcal{D}(\mathcal{B}_0)$ *and to* $\mathcal{D}(\mathcal{C}_0)$ *such that*

$$\forall \tau \in \mathbb{R}, \quad \widetilde{R}^*(\tau) + \sum_{j \geq 0} R^*(2^{-2j}\tau) = 1 \quad \textit{and} \quad \forall \tau \in \mathbb{R}^*, \quad \sum_{j \in \mathbb{Z}} R^*(2^{-2j}\tau) = 1,$$

and satisfying as well the support properties

$$|p-q| \geq 1 \Rightarrow \textit{supp } R^*(2^{-2q}\cdot) \cap \textit{supp } R^*(2^{-2p}\cdot) = \emptyset$$

$$\textit{and} \quad q \geq 1 \Rightarrow \textit{supp } \widetilde{R}^* \cap \textit{supp } R^*(2^{-2q}\cdot) = \emptyset.$$

Moreover, there are radial functions of $\mathcal{S}(\mathbb{H}^d)$, *denoted* ψ *and* φ, *such that*

$$\mathcal{F}(\psi)(\lambda)F_{\alpha,\lambda} = \widetilde{R}^*_{|\alpha|}(\lambda)F_{\alpha,\lambda} \quad \textit{and} \quad \mathcal{F}(\varphi)(\lambda)F_{\alpha,\lambda} = R^*_{|\alpha|}(\lambda)F_{\alpha,\lambda},$$

where we have noted $\widetilde{R}^*_m(\tau) = \widetilde{R}^*((2m+d)\tau)$ *and* $R^*_m(\tau) = R^*((2m+d)\tau)$.

Now as in the $\mathbb{R}^d$ case, we define Littlewood–Paley operators in the following way.

Definition 2.6 *The Littlewood–Paley operators* Δ_j *and* S_j, *for* $j \in \mathbb{Z}$, *are defined by*

$$\mathcal{F}(\Delta_j f)(\lambda)F_{\alpha,\lambda} = R^*_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(f)(\lambda)F_{\alpha,\lambda},$$

$$\mathcal{F}(S_j f)(\lambda)F_{\alpha,\lambda} = \widetilde{R}^*_{|\alpha|}(2^{-2j}\lambda)\mathcal{F}(f)(\lambda)F_{\alpha,\lambda}.$$

Remark 2.7 *It is easy to see that*

$$\Delta_j u = u \star 2^{Nj}\varphi(\delta_{2^j}\cdot) \quad \text{and} \quad S_j u = u \star 2^{Nj}\psi(\delta_{2^j}\cdot),$$

which implies that those operators map L^p into L^p for all $p \in [1,\infty]$ with norms which do not depend on j.

Along the same lines as in the $\mathbb{R}^d$ case, we can define homogeneous Besov spaces on the Heisenberg group (see [1]).

Definition 2.8 *Let $s \in \mathbb{R}$ be given, as well as p and r, two real numbers in the interval $[1,\infty]$. The Besov space $\dot{B}^s_{p,r}(\mathbb{H}^d)$ is the space of tempered distributions u such that*

- *The series $\sum_{-m}^{m} \Delta_q u$ converges to u in $\mathcal{S}'(\mathbb{H}^d)$.*
- $\|u\|_{\dot{B}^s_{p,r}(\mathbb{H}^d)} \stackrel{def}{=} \left\| 2^{qs}\|\Delta_q u\|_{L^p(\mathbb{H}^d)}\right\|_{\ell^r(\mathbb{Z})} < \infty.$

Remark 2.9 *Sobolev spaces $\dot{H}^s(\mathbb{H}^d)$ have a characterization using Littlewood–Paley operators, as well as noninteger Hölder spaces (see* [1],[3]*). More precisely, one has $\dot{H}^s(\mathbb{H}^d) = \dot{B}^s_{2,2}(\mathbb{H}^d)$ for any $s \in \mathbb{R}$, and for any $\rho \in \mathbb{R}\setminus\mathbb{N}$, $\dot{C}^\rho(\mathbb{H}^d) = \dot{B}^\rho_{\infty,\infty}(\mathbb{H}^d)$.*

2.3 Frequency localized functions and Bernstein inequalities on the Heisenberg group

Let us first define the concept of localization procedure in the frequency space in the framework of the Heisenberg group. We will only state the definition in the case of smooth functions—otherwise one proceeds by regularizing by convolution (see [1] or [3]).

Definition 2.10 *Let $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0,r_1,r_2)$ be a ring of $\mathbb{R}$ centered at the origin. A function u in $\mathcal{S}(\mathbb{H}^d)$ is said to be frequency localized in the ring 2^j $\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, if*

$$\mathcal{F}(u)(\lambda)F_{\alpha,\lambda} = \mathbf{1}_{(2|\alpha|+d)^{-1}2^{2j}\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}}(\lambda)\mathcal{F}(u)(\lambda)F_{\alpha,\lambda}.$$

Remark 2.11 *Equivalently, a frequency localized function in the sense of Definition* 2.10 *satisfies*

$$u = u \star \phi_j,$$

where $\phi_j = 2^{Nj}\phi(\delta_{2^j}\cdot)$, and ϕ is a radial function in $\mathcal{S}(\mathbb{H}^d)$ such that

$$\mathcal{F}(\phi)(\lambda)F_{\alpha,\lambda} = R((2|\alpha|+d)\lambda)F_{\alpha,\lambda},$$

with R compactly supported in a ring of $\mathbb{R}$ centered at zero.

In order to estimate the cost of applying powers of the Laplacian on a frequency localized function, we shall need the following proposition, which ensures that the action powers of the Laplacian act as homotheties on such frequency localized functions. The proof of that proposition may be found in [3].

Proposition 2.12 ([3]) *Let p be an element of $[1,\infty]$ and let (r_1,r_2) be two positive real numbers. Define $\mathcal{C}_{(r_1,r_2)} = \mathcal{C}(0,r_1,r_2)$ the ring centered at the origin, of small and large radius respectively r_1 and r_2. Then for any real number ρ, there is a constant C_ρ such that if u is a function defined on $\mathbb{H}^d$, frequency localized in the ring $2^j\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$, then*

$$C_\rho^{-1}2^{-j\rho}\|(-\Delta_{\mathbb{H}^d})^{\frac{\rho}{2}}u\|_{L^p(\mathbb{H}^d)} \leq \|u\|_{L^p(\mathbb{H}^d)} \leq C_\rho 2^{-j\rho}\|(-\Delta_{\mathbb{H}^d})^{\frac{\rho}{2}}u\|_{L^p(\mathbb{H}^d)}.$$

3 Proof of Theorem 1.4

In this section we shall prove Theorem 1.4, assuming Lemma 1.3. It turns out that the proof is very similar to the $\mathbb{R}^d$ case, and we sketch it here for the convenience of the reader.

Let us start by estimating $\|t^s e^{t\Delta_{\mathbb{H}^d}}u\|_{L^p}$. Using Lemma 1.3 and the fact that the operator Δ_j commutes with the operator $e^{t\Delta_{\mathbf{H}^d}}$, we can write

$$\|t^s\Delta_j e^{t\Delta_{\mathbb{H}^d}}u\|_{L^p} \leq Ct^s 2^{2js}e^{-ct2^{2j}}2^{-2js}\|\Delta_j u\|_{L^p}.$$

Using the definition of the homogeneous Besov (semi) norm, we get

$$\|t^s e^{t\Delta_{\mathbb{H}^d}}u\|_{L^p} \leq C\|u\|_{\dot{B}^{-2s}_{p,r}}\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c_{r,j},$$

where $(c_{r,j})_{j\in\mathbb{Z}}$ denotes, as in all this proof, a generic element of the unit sphere of $\ell^r(\mathbb{Z})$. In the case when $r=\infty$, the required inequality comes immediately from the following easy result: for any positive s, we have

$$\sup_{t>0}\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}} < \infty. \tag{3.1}$$

In the case when $r<\infty$, using the Hölder inequality with the weight $2^{2js}e^{-ct2^{2j}}$ and Inequality (3.1) we obtain

$$\begin{aligned}
&\int_0^\infty t^{rs}\|e^{t\Delta_{\mathbb{H}^d}}u\|^r_{L^p}\frac{dt}{t}\\
&\quad\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}\Big)^{r-1}\Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c^r_{r,j}\Big)\frac{dt}{t}\\
&\quad\leq C\|u\|^r_{\dot{B}^{-2s}_{p,r}}\int_0^\infty\sum_{j\in\mathbb{Z}} t^s 2^{2js}e^{-ct2^{2j}}c^r_{r,j}\frac{dt}{t}\cdot
\end{aligned}$$

This gives directly the result by Fubini's theorem.

In order to prove the other inequality, let us observe that for any s greater than -1, we have

$$\int_0^\infty \tau^s e^{-\tau} d\tau \stackrel{\text{def}}{=} C_s.$$

Using the fact that the Fourier transform on the Heisenberg group is injective, we deduce the following identity (which may be easily proved by taking the Fourier transform of both sides):

$$\Delta_j u = C_s^{-1} \int_0^\infty t^s (-\Delta_{\mathbb{H}^d})^{s+1} e^{t\Delta_{\mathbb{H}^d}} \Delta_j u dt.$$

Then Lemma 1.3, the obvious identity $e^{t\Delta_{\mathbb{H}^d}} u = e^{\frac{t}{2}\Delta_{\mathbf{H}^d}} e^{\frac{t}{2}\Delta_{\mathbb{H}^d}} u$ and the fact that the operator Δ_j commutes with the operator $e^{t\Delta_{\mathbf{H}^d}}$, lead to

$$\|\Delta_j u\|_{L^p} \leq C \int_0^\infty t^s 2^{2j(s+1)} e^{-ct2^{2j}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p} dt. \tag{3.2}$$

In the case $r = \infty$, we simply write

$$\begin{aligned}
\|\Delta_j u\|_{L^p} &\leq C \Big(\sup_{t>0} t^s \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}\Big) \int_0^\infty 2^{2j(s+1)} e^{-ct2^{2j}} dt \\
&\leq C 2^{2js} \Big(\sup_{t>0} t^s \|e^{t\Delta_{\mathbf{H}^d}} u\|_{L^p}\Big).
\end{aligned}$$

In the case $r < \infty$, Hölder's inequality with the weight $e^{-ct2^{2j}}$ gives

$$\begin{aligned}
&\Big(\int_0^\infty t^s e^{-ct2^{2j}} \|e^{t\Delta_{\mathbf{H}^d}} u\|_{L^p} dt\Big)^r \\
&\qquad \leq \Big(\int_0^\infty e^{-ct2^{2j}} dt\Big)^{r-1} \int_0^\infty t^{rs} e^{-ct2^{2j}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r dt \\
&\qquad \leq C 2^{-2j(r-1)} \int_0^\infty t^{rs} e^{-ct2^{2j}} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r dt.
\end{aligned}$$

Thanks to (3.1) and Fubini's theorem, we infer from (3.2) that

$$\begin{aligned}
\sum_j 2^{-2jsr} \|\Delta_j u\|_{L^p}^r &\leq C \int_0^\infty \Big(\sum_{j\in\mathbb{Z}} t 2^{2j} e^{-ct2^{2j}}\Big) t^{rs} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r \frac{dt}{t} \\
&\leq C \int_0^\infty t^{rs} \|e^{t\Delta_{\mathbb{H}^d}} u\|_{L^p}^r \frac{dt}{t}.
\end{aligned}$$

The theorem is proved. □

4 Proofs of Lemma 1.3 and Theorems 1.6 and 1.8

Now we are left with the proof of Lemma 1.3 and Theorems 1.6 and 1.8. Lemma 1.3 is proved in Section 4.1, while the proofs of Theorems 1.6 and 1.8 can be found in Sections 4.2 and 4.3, respectively.

4.1 Proof of Lemma 1.3

By density, it suffices to suppose that the function u is an element of $\mathcal{S}(\mathbb{H}^d)$. Now the frequency localization of u in the ring $\beta\mathcal{C}_{(\sqrt{r_1},\sqrt{r_2})}$ allows us to write

$$\mathcal{F}(e^{t\Delta_{\mathbb{H}^d}}u)(\lambda)F_{\alpha,\lambda} = e^{-t\beta^2(4|\beta^{-2}\lambda|(2|\alpha|+d))}R_{|\alpha|}(\beta^{-2}\lambda)\mathcal{F}(u)(\lambda)F_{\alpha,\lambda}, \quad (4.1)$$

with $R_{|\alpha|}(\lambda) = R((2|\alpha|+d)\lambda)$ and $R \in \mathcal{D}(\mathbb{R}\setminus\{0\})$ is equal to 1 near the ring $\mathcal{C}_{(r_1,r_2)}$. We can then assume in what follows that $\beta = 1$.

Since R belongs to $\mathcal{D}(\mathbb{R}\setminus\{0\})$, Proposition 2.4 ensures the existence of a radial function $g^t \in \mathcal{S}(\mathbb{H}^d)$ such that

$$\mathcal{F}(g^t)(\lambda)F_{\alpha,\lambda} = e^{-t(4|\lambda|(2|\alpha|+d))}R_{|\alpha|}(\lambda)F_{\alpha,\lambda}.$$

We deduce that

$$e^{t\Delta_{\mathbb{H}^d}}u = u\star g^t.$$

If we prove that two positive real numbers c and C exist such that, for all positive t, we have

$$\|g^t\|_{L^1(\mathbb{H}^d)} \leq Ce^{-ct}, \quad (4.2)$$

then the lemma is proved. To prove (4.2), let us first recall that thanks to Proposition 2.4,

$$g^t(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \times \sum_m \int e^{-i\lambda s}e^{-t(4|\lambda|(2m+d))}R((2m+d)\lambda)L_m^{(d-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda.$$

Now, we shall follow the idea of the proof of Proposition 2.4 established in [1] to obtain Estimate (4.2). Let us denote by $\mathcal{Q}$ the subspace of $L^2_d(\mathbb{N}\times\mathbb{R})$ (defined in (2.11)) generated by the sequences (Q_m) of the type

$$Q_m(\lambda) = \int_{\mathbb{R}^n} Q((2m+f(\sigma))\lambda)P(\lambda)d\mu(\sigma), \quad (4.3)$$

where μ is a bounded measure compactly supported on $\mathbb{R}^n$, f is a bounded function on the support of μ, P is a polynomial function and Q is a function of $\mathcal{D}(\mathbb{R}\setminus\{0\})$ under the form

$$Q(\tau) = e^{-4t|\tau|}\mathcal{P}(t\tau)R(\tau), \quad (4.4)$$

with $\mathcal{P}$ a polynomial and R a function of $\mathcal{D}(\mathbb{R}\setminus\{0\})$.

Now, let us recall the following useful formulas (proved for instance in [1] and [9]).

Lemma 4.1 *For any radial function $f \in \mathcal{S}(\mathbb{H}^d)$, we have for any $m \geq 1$,*

$$\mathcal{F}((is-|z|^2)f)(m,\lambda) = \frac{d}{d\lambda}\mathcal{F}f(m,\lambda) - \frac{m}{\lambda}\Big(\mathcal{F}f(m,\lambda) - \mathcal{F}f(m-1,\lambda)\Big) \quad \text{for } \lambda > 0 \text{ and}$$

$$\mathcal{F}((is-|z|^2)f)(m,\lambda) = \frac{d}{d\lambda}\mathcal{F}f(m,\lambda) + \frac{m+d}{|\lambda|}\Big(\mathcal{F}f(m,\lambda) - \mathcal{F}f(m+1,\lambda)\Big) \quad \text{for } \lambda < 0.$$

Moreover, we have the following classical property on Laguerre polynomials:

$$|L_m^{(p)}(y)e^{-y/2}| \leq C_p(m+1)^p, \quad \forall y \geq 0. \tag{4.5}$$

Let us start by proving that for any integer k, one has the following formula:

$$(is-|z|^2)^k g^t(z,s) = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} Q_m^{(k)}(\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda, \tag{4.6}$$

where $(Q_m^{(k)})$ is an element of the space $\mathcal{Q}$. By induction the problem is reduced to proving that for (Q_m) element of $\mathcal{Q}$, the sequence $(Q_m^\star)$ defined as follows is still an element of $\mathcal{Q}$: for all $m \geq 1$,

$$Q_m^\star(\lambda) = \frac{d}{d\lambda}Q_m(\lambda) - \frac{m}{\lambda}(Q_m(\lambda) - Q_{m-1}(\lambda)), \quad \lambda > 0,$$
$$Q_m^\star(\lambda) = \frac{d}{d\lambda}Q_m(\lambda) + \frac{m+d}{|\lambda|}(Q_m(\lambda) - Q_{m+1}(\lambda)), \quad \lambda < 0.$$

Let us for instance compute $Q_m^\star(\lambda)$ for $\lambda > 0$ and $m \geq 1$. Considering (4.3), the Taylor formula implies that

$$\frac{m}{\lambda}(Q_m(\lambda) - Q_{m-1}(\lambda)) = 2m \int_{\mathbb{R}^n} \int_0^1 Q'((2m+f(\sigma)-2u)\lambda)P(\lambda)du d\mu(\sigma).$$

Therefore,

$$\begin{aligned} Q_m^\star(\lambda) = & \int_{\mathbb{R}^n} Q((2m+f(\sigma))\lambda)P'(\lambda)d\mu(\sigma) \\ & + \int_{\mathbb{R}^n} Q'((2m+f(\sigma))\lambda)P(\lambda)f(\sigma)d\mu(\sigma) \\ & + 2\int_{\mathbb{R}^n}\int_0^1\int_0^1 (2m+f(\sigma)-2us)\lambda Q''((2m+f(\sigma)-2us)\lambda) \\ & \qquad \times \; P(\lambda)\; u \; du \; ds \; d\mu(\sigma) \\ & - 2\int_{\mathbb{R}^n}\int_0^1\int_0^1 Q''((2m+f(\sigma)-2us)\lambda)\lambda P(\lambda) u du ds f(\sigma) d\mu(\sigma) \\ & + 4\int_{\mathbb{R}^n}\int_0^1\int_0^1 Q''((2m+f(\sigma)-2us)\lambda)\lambda P(\lambda) u^2 du s ds d\mu(\sigma). \end{aligned}$$

This proves that the sequence $(Q^\star_m)$ belongs to the space $\mathcal{Q}$.

Now let us end the proof of Lemma 1.3: defining

$$f^t_m(z,s) = \int e^{-i\lambda s} Q_m(\lambda) L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda,$$

with (Q_m) element of $\mathcal{Q}$, and in view of (4.6) it is enough to prove that there exist two constants c and C which do not depend on m, such that

$$|f^t_m(z,s)| \leq C e^{-ct} \frac{1}{m^2} \cdot \tag{4.7}$$

Due to the condition on the support of the function R appearing in (4.4), there exist two fixed constants which only depend on R, denoted c_1 and c_2, such that

$$\begin{aligned} f^t_m(z,s) & \\ &= \int_{\mathbb{R}^n} \int_{c_1 \leq |(2m+f(\sigma))\lambda| \leq c_2} e^{-i\lambda s} Q((2m+f(\sigma))\lambda) P(\lambda) \\ &\quad \times L_m^{(d-1)}(2|\lambda||z|^2) e^{-|\lambda||z|^2} d\mu(\sigma) |\lambda|^d d\lambda. \end{aligned}$$

In view of (4.4) and (4.5), we obtain

$$|f^t_m(z,s)| \leq c_{d-1} \int_{\mathbb{R}^n} \int_{c_1 \leq |(2m+f(\sigma))\lambda| \leq c_2} e^{-ct} m^{d-1} d\mu(\sigma) |\lambda|^d d\lambda,$$

which leads easily to (4.7) and ends the proof of the lemma. □

4.2 Proof of Theorem 1.6

The proof of Theorem 1.6 presented here relies on the maximal function on the Heisenberg group; before starting the proof let us collect a few useful results on this function, starting with the definition of the maximal function (the interested reader can consult [11] for details and proofs).

Definition 4.2 *Let f be in $L^1_{loc}(\mathbb{H}^d)$. The maximal function of f is defined by*

$$Mf(z,s) \stackrel{def}{=} \sup_{R>0} \frac{1}{m(B((z,s),R))} \int_{B((z,s),R)} |f(z',s')| dz' ds',$$

where $m(B((z,s),R))$ denotes the measure of the Heisenberg ball $B((z,s),R)$ of center (z,s) and radius R.

The key propoerties we will use on the maximal function are collected in the following proposition.

Proposition 4.3 *The maximal function satisfies the following properties.*

1. *If f is a function in $L^p(\mathbb{H}^d)$, with $1 < p \leq \infty$, then Mf belongs to $L^p(\mathbb{H}^d)$ and we have*
$$\|Mf\|_{L^p(\mathbb{H}^d)} \leq A_p \|f\|_{L^p(\mathbb{H}^d)},$$
where A_p is a constant which depends only on p and d.
2. *Let φ be a function in $L^1(\mathbb{H}^d)$ and suppose that the function $\psi(w) \stackrel{def}{=} \sup_{\rho(w') \geq \rho(w)} \varphi(w')$ belongs to $L^1(\mathbb{H}^d)$, where ρ denotes the Heisenberg distance to the origin defined in Remark* 1.2*. Then for any measurable function f, we have*
$$\Big|(f \star \varphi)(w)\Big| \leq \|\psi\|_{L^1(\mathbb{H}^d)} Mf(w).$$

Now we are ready to prove Theorem 1.6. By density we can suppose that f belongs to $\mathcal{S}(\mathbb{H}^d)$. Let us write

$$f = \int_0^\infty e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt$$

and decompose the integral in two parts:

$$f = \int_0^A e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt + \int_A^\infty e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt,$$

where A is a constant to be fixed later.

On the one hand, by Theorem 1.4, we have

$$\|e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f\|_{L^\infty} \leq \frac{C}{t^{1+\frac{1}{2}(\frac{N}{p}-s)}} \|f\|_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}(\mathbb{H}^d)}.$$

Therefore, after integration we get

$$\int_A^\infty \|e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f\|_{L^\infty} \leq A^{\frac{1}{2}(s-\frac{N}{p})} \|f\|_{\dot{B}^{s-\frac{N}{p}}_{\infty,\infty}(\mathbf{H}^d)}.$$

On the other hand, denoting by $g = (-\Delta_{\mathbf{H}^d})^{\frac{s}{2}} f$, we have

$$e^{t\Delta_{\mathbf{H}^d}} \Delta_{\mathbb{H}^d} f = \frac{1}{(-t)^{1-\frac{s}{2}}} e^{t\Delta_{\mathbf{H}^d}} (-t\Delta_{\mathbb{H}^d})^{1-\frac{s}{2}} g.$$

It is well-known that the heat kernel on the Heisenberg group satisfies the second assumption of Proposition 4.3 (the reader can consult [2], [5] or [6]), so we deduce that

$$\Big|e^{t\Delta_{\mathbb{H}^d}} (-t\Delta_{\mathbb{H}^d})^{1-\frac{s}{2}} g(x)\Big| \leq C_s M_g(x),$$

where $M_g(x)$ denotes the maximal function of the function g. This leads to

$$\Big|\int_0^A e^{t\Delta_{\mathbb{H}^d}} \Delta_{\mathbb{H}^d} f dt\Big| \leq CA^{\frac{s}{2}} M_g(x).$$

In conclusion, we get

$$\Big|\int_0^\infty e^{t\Delta_{\mathbb{H}^d}}\Delta_{\mathbb{H}^d}f(x)dt\Big| \leq C\Big(A^{\frac{s}{2}}M_g(x)+A^{\frac12(s-\frac{N}{p})}\|f\|_{\dot B^{s-\frac{N}{p}}_{\infty,\infty}(\mathbb{H}^d)}\Big),$$

and the choice of A such that $A^{\frac{N}{2p}}M_g(x)=\|f\|_{\dot B^{s-\frac{N}{p}}_{\infty,\infty}}$ ensures that

$$\Big|\int_0^\infty e^{t\Delta_{\mathbb{H}^d}}\Delta_{\mathbb{H}^d}f(x)dt\Big| \leq CM_g(x)^{1-\frac{ps}{N}}\|f\|^{\frac{ps}{N}}_{\dot B^{s-\frac{N}{p}}_{\infty,\infty}(\mathbb{H}^d)}.$$

Finally, taking the L^q norm with $q=\frac{pN}{N-ps}$ ends the proof of Theorem 1.6 thanks to Proposition 4.3. □

4.3 Proof of Theorem 1.8

The proof of Theorem 1.8 is similar to the proof of Lemma 1.3 and relies on the following result.

Lemma 4.4 *The series*

$$h(z,s)=\frac{2^{d-1}}{\pi^{d+1}}\sum_m\int e^{-i\lambda s}e^{-4|\lambda|(2m+d)}L_m^{(d-1)}(2|\lambda||z|^2)e^{-|\lambda||z|^2}|\lambda|^d d\lambda \quad (4.8)$$

converges in $\mathcal{S}(\mathbb{H}^d)$.

Notice that Lemma 4.4 implies directly the theorem, as by a rescaling, it is easy to see that the heat kernel on the Heisenberg group is given by

$$h_t(x,y,s)=\frac{1}{t^{d+1}}h(\frac{x}{\sqrt t},\frac{y}{\sqrt t},\frac{s}{t}).$$

□

Proof of Lemma 4.4 Due to the subellipticity of $-\Delta_{\mathbb{H}^d}$ (see for instance [1]), it suffices to prove that for any integers k and ℓ,

$$\big\|(-\Delta_{\mathbb{H}^d})^\ell(|z|^2-is)^k h\big\|_{L^2(\mathbb{H}^d)}<\infty.$$

In order to do so, let us introduce the set $\widetilde{\mathcal{Q}}$ of sequences (Q_m) of the type

$$Q_m(\lambda)=\int_{\mathbb{R}^n}Q((2m+\theta(\sigma))\lambda)P(\lambda)d\mu(\sigma), \quad (4.9)$$

where μ is a bounded measure compactly supported on $\mathbb{R}^n$, θ is a bounded function on the support of μ, P is a polynomial function and Q is a function of $\mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$ under the form

$$Q(\tau)=e^{-4|\tau|}\mathcal{P}(\tau), \quad (4.10)$$

where $\mathcal{P}$ is a polynomial function. As in the proof of Lemma 1.3 and thanks to (2.8) and Lemma 4.1, we obtain

$$\Delta^{\ell}_{\mathbb{H}^d}(|z|^2 - is)^k h(z,s) \\ = \frac{2^{d-1}}{\pi^{d+1}} \sum_m \int e^{-i\lambda s} Q^{\ell,k}_m(\lambda) L^{(d-1)}_m(2|\lambda||z|^2) e^{-|\lambda||z|^2} |\lambda|^d d\lambda,$$

with $(Q^{\ell,k}_m)$ an element of $\widetilde{\mathcal{Q}}$ which ends the proof of the lemma thanks to (2.11). □

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A Generalization of the Rudin–Carleson Theorem

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Summary. We prove a generalization of the Rudin–Carleson theorem for homogeneous solutions of locally solvable real analytic vector fields.

2000 AMS Subject Classification: Primary: 35F15, 35B30, Secondary: 42A38, 30E25

Key words: Rudin–Carleson property, Sussmann's orbits, condition $(\mathcal{P})$.

1 Introduction

It is well known that if a continuous function f on the closure of the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ is holomorphic on D and vanishes on a subset E of the boundary ∂D of positive Lebesgue measure, then $f \equiv 0$. Conversely, Rudin [R] and Carleson [C] independently proved that if $E \subset \partial D$ is a closed set of Lebesgue measure zero, and if g is a continuous function on E, then there is $f \in C(\overline{D})$, holomorphic on D such that f agrees with g on E. Refinements, new proofs, and some generalizations of the Rudin–Carleson theorem were given in the works [B], [D], [G], and [O]. For applications to peak interpolation manifolds for holomorphic functions on domains in $\mathbb{C}^n$ see [Bh], [Na], [R2] and the references in these works. In a recent paper [BH1], we proved the following generalization of the Rudin–Carleson theorem for a class of real analytic complex vector fields:

Theorem A *Let D be the unit disk and let L be a nonvanishing real analytic vector field defined on a neighborhood U of $\overline{D}$ satisfying the Nirenberg–Treves condition $(\mathcal{P})$. Assume that L does not have a relatively compact orbit in U.*

*Work supported in part by NSF DMS 0714696, CNPq, and FAPESP.

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_3,

Let $E \subset \partial D$ be a closed set with Lebesgue measure zero and assume that $g \in C(\partial D)$ is constant on the intersection $\gamma \cap \partial D$ whenever γ is a one-dimensional orbit of L. Then there is $h \in C(\overline{D})$ satisfying

$$\begin{aligned} Lh &= 0 \quad \text{in } D, \\ h(z) &= g(z), \quad z \in E, \\ \sup_{z \in \partial D} |h(z)| &\leq \sup_{z \in E} |g(z)|. \end{aligned}$$

We recall that condition $(\mathcal{P})$ is a geometric condition that characterizes the class of locally solvable vector fields. This condition and the notion of orbits will be reviewed in Section 2. Observe that a continuous solution u of $Lu = 0$ is constant on any one-dimensional orbit of L and this explains why g is assumed to be constant on the sets $\gamma \cap \partial D$ in Theorem A.

In [BH1] we gave examples that showed that in general, if a real analytic, locally solvable vector field has a compact orbit, it may not have the Rudin–Carleson property. The main goal of this article is to characterize those locally solvable, real analytic vector fields L with compact orbits which exhibit the Rudin–Carleson property. This characterization is given in terms of the conformal type of a one-sided tubular neighborhood of each closed orbit endowed with the natural holomorphic structure induced by L (see Section 2 and Theorem 2.1 for the precise formulation). Section 3 is devoted to the proof of this result and Section 4 presents various examples. In Section 5, we show that for any smooth vector field, local solvability is a necessary and sufficient condition for the validity of the Rudin–Carleson property in arbitrary small neighborhoods of a point in an open set. In the final section we briefly discuss the relationship between the Rudin–Carleson theorem and the F. and M. Riesz theorem in the spirit of [B].

2 Preliminaries and statement of the main result

Let $L = X + iY$ be a smooth vector field on an open set Ω in $\mathbb{C}$ where X and Y are real vector fields. Assume that L is nonzero at each point of Ω. It is well known that condition $(\mathcal{P})$ can be expressed in terms of the orbits of the pair of vector fields $\{X, Y\}$ in the sense of Sussmann ([S]). Two points belong to the same orbit of $\{X, Y\}$ in Ω if they can be joined by a continuous, piecewise differentiable curve such that each piece is an integral curve of X or Y. Since X and Y are assumed to have no common zeros, the orbits of L in Ω are immersed submanifolds of Ω of dimension one or two; moreover, the two-dimensional orbits are open subsets of Ω. Let $\mathcal{O} \subset \Omega$ be a two-dimensional orbit of L in Ω and consider $X \wedge Y \in C^\infty(\Omega; \bigwedge^2(T(\Omega)))$. Since $\bigwedge^2(T(\Omega))$ has a global nonvanishing section $e_1 \wedge e_2$, $X \wedge Y$ is a real multiple of $e_1 \wedge e_2$ and this gives a meaning to the requirement that $X \wedge Y$ does not change sign on any two-dimensional orbit $\mathcal{O}$ of $\{X, Y\}$ in Ω. The vector field L satisfies condition $(\mathcal{P})$ at $p \in \Omega$ if there is a disk $U \subset \Omega$ centered at p such that $X \wedge Y$ does not change sign on any two-dimensional orbit of L in U.

Suppose now $L = X + iY$ is a locally solvable, real analytic vector field in a neighborhood U of $\overline{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. If L is a multiple of a real vector field, then by Lemma 2.1 in [BH1], it has the Rudin–Carleson property, and so we will assume throughout that L is not a multiple of a real vector field. If ∂D is an orbit of L, then any solution in D that is continuous on $\overline{D}$ will be constant on ∂D and so we will assume that

$$\partial D \quad \text{is not an orbit.} \tag{2.1}$$

By real analyticity and the assumption that it is not a multiple of a real vector field, L has a finite number of one-dimensional orbits in D. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k$ denote all the closed orbits of L in D. Observe that these orbits are real analytic, Jordan curves. We divide these closed orbits into two types. A closed orbit $\mathcal{C}_j$ is a type I orbit if it is not enclosed in any other closed orbit. There may, however, be closed orbits in the precompact component of $D \backslash \mathcal{C}_j$ in D, i.e., closed orbits enclosed by $\mathcal{C}_j$. The remaining closed orbits will be referred to as type II orbits. Suppose now Ω is a connected open subset of a two-dimensional orbit of L in D. Since L satisfies condition $(\mathcal{P})$ and is real analytic, for each $p \in \Omega$, there is a neighborhood U_p of p, a real analytic function $Z : U_p \to \mathbb{C}$ such that $LZ = 0$, $dZ \neq 0$, and Z is a homeomorphism. If $Z' : U'_p :\to \mathbb{C}$ is also another such function on a neighborhood U'_p of p, then $Z' \circ Z^{-1} : Z(U_p \cap U'_p) \to \mathbb{C}$ is holomorphic. In other words, L induces a Riemann surface structure on Ω which we will denote by (Ω, L).

For each type I closed orbit $\mathcal{C}_j$, we fix a one-sided tubular neighborhood Ω_j of $\mathcal{C}_j$ that lies in the non-precompact component of $D \backslash \mathcal{C}_j$ such that Ω_j does not intersect any one-dimensional orbit of L. We will also assume that the Ω_j are pairwise disjoint. The following theorem provides a necessary and sufficient condition for L to have the Rudin–Carleson property.

Theorem 2.1 *Let D be the unit disk and let L be a nonvanishing real analytic vector field defined on a neighborhood U of $\overline{D}$ satisfying the Nirenberg–Treves condition $(\mathcal{P})$ and (2.1). Let $E \subset \partial D$ be a closed set with Lebesgue measure zero. Then the following are equivalent:*

(1) *For each type I closed orbit $\mathcal{C}_j$, the Riemann surface (Ω_j, L) is conformal to the punctured disk with the standard structure.*
(2) *For every $g \in C(\partial D)$ that is constant on the intersection $\gamma \cap \partial D$ whenever γ is a one-dimensional orbit of L, there is $h \in C(\overline{D})$ satisfying*

$$\begin{aligned} Lh &= 0 \quad \text{in } D, \\ h(z) &= g(z), \quad z \in E, \\ \sup_{z \in \partial D} |h(z)| &\leq \sup_{z \in E} |g(z)|. \end{aligned}$$

Remark 2.2 In Section 4 we will give examples of real analytic, locally solvable vector fields with compact orbits with and without the Rudin–Carleson property.

3 Proof of Theorem 2.1

We will consider two cases.

3.1 Case 1

Assume that L has a single type I compact orbit $\mathcal{C}$ contained in D. Let γ_j, $j = 1, \ldots, n$, denote the one-dimensional, noncompact orbits of L in D. Consider the complement in D of the noncompact one-dimensional orbits, $D^\sharp = D \setminus \bigcup_{j=1}^n \gamma_j$, and consider the connected components D_k, $k = 0, \ldots, N$, of $D^\sharp$. We now fix k and study the boundary of D_k. Note that if $p \in D \cap \partial D_k$, then $p \in \gamma_j$ for some $j \geq 1$. Therefore, if V is a sufficiently small disk centered at p, $V \setminus \gamma_j$ is a disjoint union of two domains V_1 and V_2 with $V \cap D_k = V_1$. It follows that D_k has a real analytic boundary near p. We suppose from now on that $p \in \partial D_k \cap \partial D$. We will consider different possibilities. If the orbit of L at p is two-dimensional, by the real analyticity of L, we can find a disk B centered at p such that $B \cap D \subset D_k$. This means that near p, ∂D_k consists of $\partial D \cap B$ in this case. Assume next that the orbit at p is one-dimensional. If L is transversal to ∂D at p, then the orbit γ_j through p divides a disk W centered at p into two connected pieces W_1 and W_2 with $W_1 \cap D = W \cap D_k$. Thus near p, D_k has a piecewise real analytic boundary consisting of two curves that intersect at p. Suppose now that L is tangent to ∂D at p. Let γ_j continue to denote the one-dimensional orbit through p. By the real analyticity, in a small disk V centered at p, if $\gamma_j^\flat = \gamma_j \cap V$, there are three possibilities:

(a) Assume $\gamma_j^\flat \subset D \cup \{p\}$. Then since $p \in \partial D_k$, either $\gamma_j^\flat \subset \partial D_k$ or $\gamma_j^\flat \setminus \{p\}$ contains a subarc with an endpoint at $\{p\}$ that bounds ∂D_k (in the second case, replace $\gamma_j^\flat$ by this subarc and call it $\gamma_j^\flat$). Near each $q \in \gamma_j^\flat$, D_k lies on one side of $\gamma_j^\flat$. Hence near p, either $\partial D_k = \gamma_j^\flat$, or ∂D_k consists of $\gamma_j^\flat$ and a subarc of ∂D with one endpoint at p.

(b) Suppose $\gamma_j^\flat \cap \overline{D} = \{p\}$. Near p, each side of $\gamma_j^\flat$ is contained in distinct two-dimensional orbits. It follows that for some neighborhood V' of p, $V' \cap D \subset D_k$ and so ∂D_k near p equals $V' \cap \partial D$.

(c) Assume $\gamma_j^\flat = \gamma^+ \cup \gamma^-$, where $\gamma^- \subset D$, and $\gamma^+ \cap D = \emptyset$. Again each side of $\gamma_j^\flat$ is contained in a two-dimensional orbit and so near p, ∂D_k consists of γ^- and an arc in ∂D with p as an endpoint.

We thus see that ∂D_k is piecewise real analytic consisting of a finite number of curves each of which is either an arc of some γ_j, $1 \leq j \leq n$, and the endpoints of this arc belong to the intersection of γ_j with ∂D, or an arc in ∂D with endpoints contained in the intersection of ∂D with a pair of one-dimensional orbits $\gamma_j \cup \gamma_{j'}$, $1 \leq j < j' \leq n$. Note also that, whatever $0 \leq k \leq N$, $\partial D_k \cap \partial D$ contains an open arc. Let D_0 be the connected component that contains the closed orbit $\mathcal{C}$. Notice that $D_0 \setminus \mathcal{C}$ has two connected components, a simply connected one that we may call the interior of $\mathcal{C}$ and denote it by Ω_0, and an annular one that we will denote by Ω, so $\Omega = D_0 \setminus \overline{\Omega}_0$.

Assume (1) in the theorem holds for the Riemann surface (Ω_1, L) where Ω_1 is the one-sided annular neighborhood of $\mathcal{C}$ in Ω. We may assume that Ω_1 is bounded by $\mathcal{C}$ and a real analytic, simple closed Jordan curve Σ. Consider the Riemann surface (Ω, L). Because the fundamental group of Ω is the integers, by Theorem IV.6.1 in [FK], this Riemann surface is conformal to either the punctured disk, an annulus of the form $\{z : a < |z| < b\}$ for some $a, b > 0$, or the punctured plane where each one is equipped with the standard structure. Since $\partial\Omega$ intersects ∂D on an arc where the orbit of L is two-dimensional, there is an open set Ω_2 such that $\Omega_2 \setminus \Omega$ has nonempty interior and Ω_2 is contained in an orbit of L in U of dimension two. This shows that (Ω, L) is a prolongable Riemann surface (we recall that a Riemann surface S is called prolongable if there exists a Riemann surface S' and an injective holomorphic map $f : S \longrightarrow S'$ with $f(S)$ not dense in S'). In particular, (Ω, L) cannot be conformal to the punctured plane.

Next, if (Ω, L) is conformal to an annulus of the form $\{z : a < |z| < b\}$ for some $a, b > 0$, let $F : (\Omega, L) \longrightarrow \{z : a < |z| < b\}$ be a conformal map. Since Σ divides Ω into two components, it follows that $F(\Omega_1)$ equals one of the components of $\{z : a < |z| < b\} \setminus F(\Sigma)$. This contradicts the assumption that (Ω_1, L) is conformal to the punctured disk. It follows that (Ω, L) is conformal to the punctured disk. Let $F : (\Omega, L) \longrightarrow \Delta \setminus \{0\}$ be a conformal map. We will now use the arguments in [BH1] to show that F extends continuously to $\partial\Omega \setminus \mathcal{C}$, and this extension is injective and preserves sets of null Lebesgue measure on the part of $\partial\Omega \setminus \mathcal{C}$ that is disjoint from the one-dimensional orbits. We will also show that F has a continuous extension to $\overline{\Omega}$ which is injective away from the one-dimensional orbits, and that maps distinct one-dimensional orbits to distinct single points.

Let $z_0 \in \partial\Omega$. Assume first that $z_0 \in \partial D$ and that z_0 is not contained in a one-dimensional orbit of L. Suppose z_k is a sequence in Ω that converges to z_0. If the sequence $F(z_k)$ does not have a limit, then it clusters at least at two points on $\partial\Delta \setminus \{0\}$. Without loss of generality we may assume $p_k = F(z_{2k})$ converges to v and $q_k = F(z_{2k+1})$ converges to w where v and w are two points on the boundary of $\Delta \setminus \{0\}$. Let T_1 and T_2 be two continuous arcs in Δ such that T_1 contains the p_k and ends at v while T_2 contains the q_k and ends at w. We may assume that $\operatorname{dist}(T_1, T_2) > c$ for some $c > 0$. Let Z be a first integral which is a homeomorphism from a disk U' about z_0 to a neighborhood of the origin and mapping $U' \cap \Omega$ onto V. Let $G = F \circ Z^{-1}$. Let $S_j = F^{-1}(T_j),\ j = 1, 2$. For $r > 0$ small, let C_r be the intersection of the circle of radius r centered at 0 with the region V. Observe that if r is small enough, say $r \leq r_0$ for some $r_0 > 0$, $Z^{-1}(C_r)$ intersects both S_1 and S_2 since these sets are connected and both accumulate at z_0. Let $C_r = \{re^{i\theta} : \theta_1(r) < \theta < \theta_2(r)\}$. Let $C'_r = G(C_r)$. Observe that C'_r contains points of both T_1 and T_2 since $Z^{-1}(C_r)$ intersects both S_1 and S_2. Since G is holomorphic, it follows that

$$c < \ell(C'_r) = \int_{\theta_1(r)}^{\theta_2(r)} |G'(re^{i\theta})| r \, d\theta.$$

Applying the Schwarz inequality we get

$$\frac{c^2}{r} < 2\pi \int_{\theta_1(r)}^{\theta_2(r)} |G'(re^{i\theta})|^2 r \, d\theta,$$

which in turn leads to the contradiction that

$$\infty = c^2 \int_0^{r_0} \frac{dr}{r} \, dr < 2\pi \int_0^{r_0} \int_{\theta_1(r)}^{\theta_2(r)} |G'(re^{i\theta})|^2 r \, d\theta \, dr < \pi.$$

It follows that $F(z_k)$ has a limit and therefore F extends continuously up to the point z_0. If $F(z_0) = 0$, then F will map a neighborhood of z_0 in ∂D to 0 which would force F to be constant since by the analyticity of L and (2.1), except possibly at a finite number of points, ∂D is noncharacteristic for L. It follows that $F(z_0) \in \partial \Delta$.

We wish to prove the same property for F^{-1} away from the finite number of points which as we will see later are the images of the one-dimensional orbits under F. This would be equivalent to showing that F is locally one-to-one at the boundary points z_0 as above which we will next show.

We claim that at each hypocomplex point $z_0 \in \partial \Omega$ as above, the function F extends to be a homeomorphism up to z_0. To see this, first assume that L is transversal to ∂D at z_0. In this case, after contracting U about z_0, $Z(U \cap \partial D)$ is a real analytic piece of the boundary of V through the origin. The function G is holomorphic on V, continuous up to the boundary near the origin, and sends a real analytic boundary piece of V through 0 into the boundary of the disk Δ. By the Schwarz reflection principle, G extends as a holomorphic function in a neighborhood of the origin which in turn leads to a real analytic extension of F past z_0. Suppose now $z_1 \in \partial \Omega$ is another hypocomplex point where L is transversal to ∂D and assume that $F(z_0) = F(z_1) = w$. Then since F extends as a solution past both z_0 and z_1, and L is hypocomplex at these points, the extended F is an open map and hence there are neighborhoods U_0, U_1, W of z_0, z_1, and w, respectively, such that $F(U_0) = F(U_1) = W$. Moreover, because F is extended using the reflection principle, we may assume that $F(U_0 \cap \Omega) = W \cap \Delta = F(U_1 \cap \Omega)$. But this contradicts the injectivity of F on Ω. Hence $F(z_0) \neq F(z_1)$. Recall that since ∂D is not an orbit of L, there are only a finite number of points on ∂D where L is not transversal to ∂D. Suppose now z_2, z_3 are two points in $\partial \Omega \cap \partial D$ where $F(z_2) = F(z_3)$ and assume that L is transversal to ∂D at z_2 and tangent to ∂D at z_3. Assume that L is hypocomplex at z_2 and z_3. We have seen that there is a neighborhood

U_2 of z_2 in $\overline{D}$ where F is one-to-one and such that $F(U_2)$ is a neighborhood of $F(z_2)$ in $\overline{\Delta}$. But then, if $z \in \Omega$ and is sufficiently close to z_3, $F(z) \notin F(U_2)$, contradicting the continuity of F at z_3. Therefore, $F(z_2) \neq F(z_3)$. Finally, suppose z_4 and z_5 are two points in $\partial\Omega \cap \partial D$ where L is hypocomplex and assume L is tangent to ∂D at both points and $F(z_4) = F(z_5) = w_0$. Since there are only a finite number of such points in ∂D, there is an open arc I in ∂D containing z_4 in its interior and consisting of hypocomplex points such that z_4 is the only point where L is not transversal to ∂D. Since F is one-to-one on $I \backslash \{z_4\}$, $F(I)$ is an open arc in $\partial\Delta$ containing w_0 in its interior. There is also a similar arc J with z_5 in its interior and we may assume that I and J are disjoint. But then this would contradict the injectivity of F on $I \cup J \backslash \{z_4, z_5\}$ and so we must have $F(z_4) \neq F(z_5)$. Hence F can be extended as a homeomorphism up to the part of the boundary of Ω that is disjoint from the one-dimensional orbits. It is also real analytic past all but a finite number of the points that do not lie in the one-dimensional orbits.

Assume next that $z_0 \in \partial\Omega \cap D \cap \gamma_j$ for some $j \geq 1$ and write, for simplicity of notation, $\gamma_j = \Gamma$. Write $L = X + iY$ with X and Y real vector fields. Replacing L, if necessary, by a convenient nonvanishing multiple of L we may assume that Γ is a closed integral curve of X joining two points A and B that belong to ∂D. Since Y vanishes on Γ, it vanishes identically on any integral curve of X that contains Γ (by analyticity). We may consider an integral curve Γ_1 of X that extends Γ past both endpoints A and B, so that Γ_1 is a one-dimensional orbit of L in a neighborhood U of $\overline{D}$ with endpoints in $U \setminus \overline{D}$. In a tubular neighborhood V of Γ_1 we may choose coordinates that rectify the flow of X and in which L has a canonical form. More precisely, we may choose local coordinates (x, t), so that V is expressed as $|x| \leq 1$, $|t| \leq 2$, $x(z_0) = t(z_0) = 0$, $x(A) = x(B) = 0$, $t(A) = 1$, $t(B) = -1$ and L has the form

$$L = \frac{\partial}{\partial t} + ib(x,t)\frac{\partial}{\partial t},$$

with $t \mapsto b(x,t) \geq 0$ and not identically zero for $0 < x \leq 1$, and $b(0,t) \equiv 0$, $-2 \leq t \leq 2$. The intersection of Ω with V is described by

$$\Omega \cap V = \{(x,t) : \ 0 < x \leq 1, \ \beta(x) < t < \alpha(x)\}.$$

Here, $\alpha(x)$, $\beta(x)$ are continuous on $[0,1]$ and analytic on $(0,1]$, $\alpha(0) = 1$, $\beta(0) = -1$ and their graphs are contained in $\partial D \cap \partial\Omega$. By restricting F to $\Omega \cap V$, we obtain an injective map $F(x,t)$ from $\Omega \cap V$ into Δ. We may assume that F has already been extended as a homeomorphism from

$$\{(x,t) : \ 0 < x \leq 1, \ \beta(x) \leq t \leq \alpha(x)\}$$

into $\overline{\Delta}$. Hence, $F(x,t)$ maps the graphs $t = \alpha(x)$, $t = \beta(x)$, $0 < x < 1$, into some open arcs $\widehat{A'C'}$, $\widehat{B'D'} \subset \partial\Delta$. Consider the vertical segment $T_\varepsilon = \{\varepsilon\} \times [\beta(\varepsilon), \alpha(\varepsilon)]$, $0 < \varepsilon < 1$, that is mapped by F into a curve $F(T_\varepsilon)$ contained in

Δ that joins two boundary points $A'_\varepsilon \doteq F(\varepsilon, \alpha(\varepsilon)) \in \widehat{A'C'}$, $B'_\varepsilon \doteq F(\varepsilon, \beta(\varepsilon)) \in \widehat{B'D'}$. Notice that $A'_\varepsilon \to A'$ and $B'_\varepsilon \to B'$ as $\varepsilon \to 0$. The length of $F(T_\varepsilon)$ is

$$\ell(F(T_\varepsilon)) = \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} |F_t(\varepsilon,t)|\,dt = \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \big(U_t^2(\varepsilon,t) + V_t^2(\varepsilon,t)\big)^{1/2}\,dt$$

with $F = \Re F + i\Im F = U + iV$. Since $LF = 0$, i.e., $U_t = bV_x$, $V_t = -bU_x$, we have

$$\begin{aligned}\ell(F(T_\varepsilon))^2 &= \left(\int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\big(U_x^2(\varepsilon,t) + V_x^2(\varepsilon,t)\big)^{1/2}\,dt\right)^2 \\ &\le \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\,dt \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\big(U_x^2(\varepsilon,t) + V_x^2(\varepsilon,t)\big)\,dt \\ &\le \int_{-2}^{2} b(\varepsilon,t)\,dt\, I(\varepsilon) \le C\varepsilon\, I(\varepsilon). \end{aligned} \tag{3.1}$$

On the other hand,

$$\begin{aligned}\int_0^1 I(\varepsilon)\,d\varepsilon &= \int_0^1 \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} b(\varepsilon,t)\big(U_x^2(\varepsilon,t) + V_x^2(\varepsilon,t)\big)\,dtd\varepsilon \\ &= \int_0^1 \int_{\beta(\varepsilon)}^{\alpha(\varepsilon)} \det \frac{\partial(U,V)}{\partial(\varepsilon,t)}\,dtd\varepsilon = \operatorname{area}(F(\Omega \cap V)) < \pi.\end{aligned}$$

Since the integral on the left-hand side is finite, we see that the product $\varepsilon I(\varepsilon)$ cannot remain bounded below by a positive constant in any neighborhood of the origin. In other words, there is a sequence $\varepsilon_j \searrow 0$ such that $\varepsilon_j I(\varepsilon_j) \searrow 0$ and (3.1) shows that $\ell(F(T_{\varepsilon_j})) \to 0$. Hence, $|A'_{\varepsilon_j} - B'_{\varepsilon_j}| \to 0$ and we conclude that $A' = B'$. Notice that the region

$$F\big(\{(x,t):\ 0 < x < \varepsilon_j,\ \beta(x) < t < \alpha(x)\}\big)$$

is bounded by the closed curve made of three arcs, to wit, the circular arc from A'_{ε_j} to A', the circular arc from $B' = A'$ to B'_{ε_j}, and the curve $F(T_{\varepsilon_j})$ that joins B'_{ε_j} to A'_{ε_j}. It is therefore easy to see that the diameter of that region tends to zero as $j \to \infty$, so given $r > 0$ we may find j_0 such that

$$F\big(\{(x,t):\ 0 < x < \varepsilon_j,\ \beta(x) < t < \alpha(x)\}\big) \subset \Delta(A', r), \quad j \ge j_0.$$

This shows that if we extend F to $\{0\} \times [-1,1]$ by setting $F(0,t) = A'$, $-1 \le t \le 1$, we obtain a continuous extension.

Finally, we need to consider the continuous extendability up to points z_0 that are in $\partial\Omega \cap \partial D \cap \gamma_j$, for some $j \ge 1$. Such a point will be in $D_r = \{z : |z| < r\}$, $r > 1$, so it must belong to Ω_r (the analog of Ω). Reasoning as above we see that a conformal map $F_r : \Omega_r \longrightarrow \Delta \setminus \{0\}$ extends continuously up to z_0. This in turn leads to the continuity of F since we could have taken it to be the restriction of F_r to Ω. Observe next that $F(\Sigma)$ is a simple closed curve that disconnects Δ and with 0 in its interior. Since Ω_1 is conformal to $\Delta\setminus\{0\}$, we conclude that as $p \to \mathcal{C}$, $F(p) \to 0$, so F has a unique continuous extension up to $\mathcal{C}$ and $F(\mathcal{C}) = \{0\}$. Summing up, F can be continuously extended up to $\Omega \cup \partial D_0$, this extension is injective on the pieces of ∂D_0 made up of subarcs of ∂D and sends each of the pieces made up of one-dimensional orbits γ_j, $j \geq 1$, to points $A_j \in \partial\Delta$, in particular $F(\partial D_0) \subset \partial\Delta$. Moreover, it takes sets $X \subset \partial\mathcal{D}_0$ of Lebesgue measure zero into subsets of $\partial\Delta$ of Lebesgue measure zero.

To consider the Rudin–Carleson problem, we are given a closed set $E \subset \partial D$ with $|E| = 0$ ($|\cdot|$ denoting the Lebesgue measure in ∂D) and we may assume without loss of generality that the intersections $\gamma_j \cap \partial D \subset E$, $1 \leq j \leq n$. We are also given a continuous function $g \in C(E)$ which is constant on each $\gamma_j \cap \partial D \subset E$, $j = 1, \ldots, n$. Since $F : \partial D_0 \longrightarrow \partial\Delta$ is continuous and preserves sets of Lebesgue measure zero, $\tilde{E} \doteq F(E \cap \partial D_0)$ is a closed set in $\partial\Delta$ of measure zero. The fact that F maps the γ_j to single points implies that there is $\tilde{g} \in C(\tilde{E})$ such that $\tilde{g} \circ F = g$ on the set $E \cap \partial D_0$. By the Rudin–Carleson theorem, there is a holomorphic function $\tilde{h}$ on Δ which is continuous on $\overline{\Delta}$, agrees with $\tilde{g}$ on $\tilde{E}$ and

$$\sup_{z\in\partial\Delta} |\tilde{h}(z)| \leq \sup_{z\in\tilde{E}} |\tilde{g}(z)|.$$

Set $h_0 = \tilde{h} \circ F$. Then h_0 is continuous on $\overline{\Omega} \cup \partial D_0$ and $Lh_0 = 0$ in Ω. If γ_j, $j \geq 1$, is a one-dimensional orbit of L that is a piece of ∂D_0, we know that $F(\gamma_j) = A_j$, so $h_0(\gamma_j) = \tilde{h}(A_j) = \tilde{g}(A_j)$. This shows that h_0 agrees with g on the intersection

$$\overline{D}_0 \cap E.$$

Finally, we may extend h_0 continuously to all of $\overline{D_0}$ by declaring that h_0 assumes the value $\tilde{h}(0)$ on the closed orbit $\mathcal{C}$ as well as in the interior of $\mathcal{C}$. Thus h_0 is continuous on $\overline{D_0}$ and it is easy to see that $Lh_0 = 0$ on D_0. The construction of h_0 also shows that

$$\sup_{z\in\partial D_0} |h_0(z)| \leq \sup_{z\in E} |g(z)|.$$

For the components D_j, $j \geq 1$, using the method of proof of the main theorem in [BH1], we may find continuous functions $F_{D_j} \in C(\overline{D}_j)$ such that $LF_{D_j} = 0$ in D_j, $F_{D_j}(\overline{D}_j) = \overline{\Delta}$ (where Δ is a copy of the unit disk in $\mathbb{C}$), and F_{D_j} is injective on D_j and on the portion of ∂D_j that lies in ∂D and is disjoint from the one-dimensional orbits. The remaining part of ∂D_j is made up of subarcs C_ℓ of some γ_j, $1 \leq j \leq n$, and each subarc C_ℓ is mapped by F_{D_j} into a single point $A_\ell \in \partial\Delta$, with $A_\ell \neq A_{\ell'}$ for $\ell \neq \ell'$. Moreover, we may use these functions F_{D_j} $j \geq 1$, and h_0, to obtain a continuous function $h \in C(\overline{D})$ such that $Lh = 0$ in D, and that agrees with g on E. Furthermore,

$$\sup_{\overline{D}} |h(z)| \leq \sup_{E} |g(z)|.$$

Thus (2) in the theorem is satisfied.

Suppose now (2) is satisfied. Consider the structure (Ω_1, L). This structure cannot be conformal to the punctured plane since it is prolongable. Assume $F : (\Omega_1, L) \longrightarrow \{z : a < |z| < b\}$ is conformal for some $a, b > 0$. The methods in [BH1] show that F extends as a homeomorphism up to the boundary piece Σ and we may assume that it maps Σ onto $\{w : |w| = b\}$. It then follows that as $z \to \mathcal{C}$, $F(z) \longrightarrow \{w : |w| = a\}$. Let $h \in C(\overline{D})$ be a solution of L in D. There exists $\tilde{h}$ holomorphic on $\{z : a < |z| < b\}$ such that $h = \tilde{h} \circ F$ on Ω_1. Since h is continuous up to $\mathcal{C}$ and is constant on $\mathcal{C}$, $\tilde{h}$ extends continuously up to $\{w : |w| = a\}$ and is constant on this circle. It follows that h is constant on Ω. But then h would be constant on arcs of ∂D that intersect $\partial \Omega$. This contradicts the validity of (2). Hence (1) holds.

3.2 Case 2

Assume that L has at least two compact orbits of type I. Let $\gamma_1, \ldots, \gamma_m$ be the one-dimensional noncompact orbits in D and write $D \setminus \cup_{j=1}^{m} \gamma_j$ as a union $\bigcup_{i=1}^{N} W_i$ of components. Assume that (1) in the theorem holds. As before for each i, ∂W_i is piecewise real analytic consisting of arcs of the γ_j or ∂D. For each i, let $\mathcal{C}_1^i, \ldots, \mathcal{C}_{n_i}^i$ be all the type I compact orbits in W_i. For each $t = 1, \ldots, n_i$, let D_t^i be the relatively compact region in D bounded by $\mathcal{C}_t^i$ and let Ω_t^i denote the one-sided tubular neighborhood of $\mathcal{C}_t^i$ that is disjoint from $\overline{D_t^i}$. We may assume that the Ω_t^i are pairwise disjoint and $\partial \Omega_t^i = \mathcal{C}_t^i \cup \Sigma_t^i$ for some analytic, closed Jordan curves Σ_t^i. When (1) in the theorem holds, for each i and t, there is a conformal map $Z_t^i : \Omega_t^i \to \Delta \setminus \{0\}$. Moreover, as we saw in Case 1, we can extend each Z_t^i continuously to $\overline{D_t^i}$ by setting it to be zero there. Fix i and consider the component W_i. Consider the equivalence relation on W_i such that the equivalence classes $[z]$ are

- single points $[z] = \{z\}$ if $z \notin \bigcup_{t=2}^{n_i} \overline{D_t^i}$,
- $[z] = \overline{D_t^i}$ if $z \in \overline{D}_t^i$ and $2 \leq t \leq n_i$.

In other words, for $t \geq 2$, we collapse $\overline{D_t^i}$ to a single class. For each $t \geq 2$, fix once for all a point $z_t \in \overline{D_t^i}$. We denote by $\widehat{W_i} = W_i/\sim$ the quotient space with its natural topology and we will define a conformal structure on $\widehat{W_i} \setminus \overline{D_1^i}$. We need to define an atlas of local holomorphic charts. If $[z] = \{z\}$, $z \notin \bigcup_{t=1}^{n_i} \overline{D_t^i}$, is a class of type (1), we may take a local first integral of L that is a homeomorphism on a neighborhood of z that does not intersect $\bigcup_{t=1}^{n_i} \overline{D_t^i}$. In the case of the $[z_t]$ $(2 \leq t \leq n_i)$, we choose the neighborhood as $[z_t] \cup [\Omega_t^i] = [z_t] \cup \Omega_t^i$ and the holomorphic coordinate will be given by Z_t^i on Ω_t^i and maps $[z_t]$ to zero. These charts are holomorphically related on their overlaps and turn $\widehat{W_i} \setminus \overline{D_1^i}$ into a Riemann surface. Observe that $\widehat{W_i} \setminus \overline{D_1^i}$ has the integers as its fundamental group and it is conformal to the punctured disk, due to the assumption that this is so for the structure (Ω_1^i, L). Given g as in (2) in the theorem, we may now reason as in case (1) to solve the Rudin–Carleson problem in $\widehat{W_i}$ which also solves the same problem in W_i by composition with the quotient map $W_i \longrightarrow \widehat{W_i}$. The solutions on the W_i can then be glued together to lead to a solution on D and hence (2) holds. Conversely, if (2) in the theorem holds, the arguments used in case (1) show that (1) has to hold.

4 Some examples

The motivation for Examples 1 and 2 below comes from [BM].

Example 1. Consider a one-form ω expressed in polar coordinates as

$$\omega = e^{i\theta}\left(dr + ir(1-4r^2)h(r^2)d\theta\right)$$

where $h(t)$ is real analytic on $\mathbb{R}$, and $h(0) = 1$. We can express ω as

$$\omega = A(z,\overline{z})dz + B(z,\overline{z})d\overline{z}$$

where

$$2A(z,\overline{z}) = 1 + (1-4r^2)h(r^2) \text{ and } 2B(z,\overline{z}) = e^{2i\theta}\left(1-(1-4r^2)h(r^2)\right).$$

The condition that $h(0) = 1$ ensures that ω is a real analytic form in the plane. Let L be a real analytic, nonvanishing vector field in the plane such that $\langle \omega, L\rangle = 0$. Observe that the only one-dimensional orbit of L is given by

$$\gamma = \left\{(r,\theta) : r = \frac{1}{2}\right\}.$$

It is easy to see that L is elliptic away from γ and hence L is locally solvable everywhere in the plane. Let $\frac{-1}{2h(\frac{1}{4})} = a + ib$. By separation of variables, in the region $\Omega = \{z : \frac{1}{2} < |z| < 1\}$, one gets a solution of L of the form

$$Z(r,\theta) = \left(r - \frac{1}{2}\right)^{a+ib} e^{E(r)+i\theta},$$

where $E(r)$ is a real analytic function. If h is chosen so that $a > 0$, then the structure induced by L on Ω is conformal to the standard one on a punctured disk as can be seen by using the injective solution $Z(r,\theta)$. By Theorem 2.1, such an L will have the Rudin–Carleson property. On the other hand, if h is chosen so that $a = 0$, then L will not have the Rudin–Carleson property.

Example 2. Example 1 can be modified to get a locally solvable, real analytic vector field with several orbits that are concentric circles (see [BM]). We will now give an example with two compact orbits which are not contained in each other.

Set

$$X = (3-x^2)(x^2-1)\frac{\partial}{\partial x} - 2x(x^2-2)y\frac{\partial}{\partial y},$$
$$Y = -x(x^2-2)y\frac{\partial}{\partial x} + 2x^2(1-y^2)\frac{\partial}{\partial y}, \qquad (x,y)\in\mathbb{R}^2.$$

The vector field X has four critical points, $(\pm 1,0)$, $(\pm\sqrt{3},0)$ on which Y does not vanish, so the complex vector field $L = X + iY$ has no zeros. We have

$$\begin{aligned} X\wedge Y &= 2(3-x^2)(x^2-1)x^2(1-y^2) - 2x^2(x^2-2)^2y^2\partial_x\wedge\partial_y \\ &= 2x^2\big(1-y^2-(2-x^2)^2\big)\partial_x\wedge\partial_y \end{aligned}$$

which means that X and Y are linearly dependent if and only if

$$x = 0 \quad \text{or} \quad (x^2-2)^2 + y^2 = 1.$$

Thus the analytic set $X\wedge Y = 0$ has three connected components that are analytic curves, two Jordan curves $\mathcal{C}_1$ and $\mathcal{C}_2$ (each one of them is the mirror image of the other with respect to the y-axis) plus the y-axis. Call Ω_j the interior domain bounded by $\mathcal{C}_j$, $j = 1, 2$. Notice that X and Y are tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ and since L never vanishes, $\mathcal{C}_1$ and $\mathcal{C}_2$ are orbits of L. For instance, to check that X is tangent to $\mathcal{C}_j$ observe that the gradient of $(x^2-2)^2+y^2$ is proportional to $Z = (x^2-2)2x\partial_x + y\partial_y$ while

$$\begin{aligned} X\cdot Z &= (3-x^2)(x^2-1)(x^2-2)2x - 2x(x^2-2)y^2 \\ &= -2x(x^2-2)\big(-4x^2+3+y^2+x^4\big) \\ &= -2x(x^2-2)\big((x^2-2)^2+y^2-1\big) \end{aligned}$$

and the last factor of the right-hand side vanishes on C_j, $j = 1, 2$. Since X and Y are linearly independent on Ω_1 and Ω_2, both are 2-orbits. Set $\Omega_3 = \mathbb{R}^2\setminus\big(\overline{\Omega}_1\cup\overline{\Omega}_2\big)$. Then X and Y are linearly independent on $\Omega_3^+ \doteq \Omega_3\cap\{x>0\}$ and $\Omega_3^- \doteq \Omega_3\cap\{x<0\}$ and since X is transversal to the y-axis it is easy to see that Ω_3 is an orbit. Thus, $X\wedge Y$ vanishes neither on Ω_1 nor on Ω_2 and vanishes but does not change sign on Ω_3, showing that L satisfies condition $(\mathcal{P})$.

Example 3. We will now describe a method to produce locally solvable vector fields L with a large number of closed one-dimensional orbits. Suppose we are given a locally solvable vector field L_0 defined on $\mathbb{R}^2$ with closed one-dimensional orbits C_j bounding disjoint two-dimensional orbits Ω_j, $j = 1, \dots k$ on which L_0 is elliptic. Assume that these orbits are contained in the half plane $x>0$ and no noncompact one-dimensional orbit intersects $x>0$ (what happens for $x<0$ is irrelevant). We will also assume that L_0 is elliptic at any point of the y-axis, that X_0 is transversal to the y-axis, and that $\{x>0\}\setminus \bigcup_{j=1}^{k}\overline{\Omega}_j$ is a two-dimensional orbit of L_0 on $\{x>0\}$. We may write $L_0 = X_0 + iY_0$ where

$$X_0 = a(x,y)\tfrac{\partial}{\partial x} + b(x,y)\tfrac{\partial}{\partial y},$$
$$Y_0 = c(x,y)\tfrac{\partial}{\partial x} + d(x,y)\tfrac{\partial}{\partial y}$$

are real vector fields. Mimicking the way one obtains the Mizohata vector field out of the Cauchy–Riemann vector field, consider the twofold transformation $\Phi(x,y)=(x^2,y)$. This leads us to define new vector fields

$$X_1 = a(x^2,y)\tfrac{\partial}{\partial x} + 2xb(x^2,y)\tfrac{\partial}{\partial y},$$
$$Y_1 = c(x^2,y)\tfrac{\partial}{\partial x} + 2xd(x^2,y)\tfrac{\partial}{\partial y}.$$

The restriction of Φ to the half-planes $x>0$ and $x<0$ is a diffeomorphism that pulls back $\big(X_0+iY_0\big)|_{\{x>0\}}$ to a multiple of $L_1 = X_1+iY_1$. Then each one of the k 1-orbits of L_0 contained in $x>0$ is mapped by Φ^{-1} into a couple of 1-orbits of L_1, generating $2k$ 1-orbits for L_1. Furthermore, the restriction of L_1 to either $x>0$ or $x<0$, satisfies $(\mathcal{P})$. By the hypothesis made on L_0, $X_0\wedge Y_0$ is, say, positive for $x=0$ and this implies—writing $X_1\wedge Y_1 = \beta(x,y)\partial_x\wedge\partial_y$—that $\beta(x,y) = x\gamma(x,y)$ with $\gamma(0,y)>0$ and since β does not change sign on the complement of the closure of the bounded 2-orbits, it follows that $\gamma>0$ everywhere on the complement of the closure of the bounded 2-orbits. Hence $\beta(x,y)$ will change sign across the y-axis which is contained in a 2-orbit. We now consider the vector field

$$L_2 = X_1 + ixY_1 \doteq X_1 + iY_2.$$

Since Y_2 is a nonvanishing multiple of Y_1 for $x\neq 0$, the 1-orbits of L_2 and L_1 are the same, in particular L_2 has exactly $2k$ closed 1-orbits. On the other hand, $X_1\wedge Y_2 = x^2\gamma(x,y)\partial_x\wedge\partial_y$ will not change sign on any 2-orbit, because γ does not change sign on any 2-orbit.

For instance, by a translation to the right of the vector field $X+iY$ defined in Example 2, we may obtain an L_0 satisfying the required hypothesis with $k=2$ and duplicate the number of closed 1-orbits to 4. This process can be continued.

Example 4. We will next describe a locally solvable, real analytic vector field with one compact and one noncompact orbit. Let $\rho_1(x,y) = x^2+y^2-1$ and $\rho_2(x,y) = x-3$. Define the real vector fields X and Y by

$$X = 4(1-x^2)\rho_2(x,y)^2\frac{\partial}{\partial x} - 2y\rho_2(x,y)(2x\rho_2(x,y)+\rho_1(x,y))\frac{\partial}{\partial y}$$

and

$$Y = -2y\rho_2(x,y)(2x\rho_2(x,y) + \rho_1(x,y))\frac{\partial}{\partial x} + (2x\rho_2(x,y) + \rho_1(x,y))^2\frac{\partial}{\partial y}.$$

Let $L = X + iY$. If $\rho_2(a,b) = 0$, then $Y(a,b) = \rho_1(a,b)^2\frac{\partial}{\partial y} = (8+b^2)\frac{\partial}{\partial y} \neq 0$. Suppose $\rho_2(a,b) \neq 0$. Then if $X(a,b) = 0$, either $|a| = 1$ and $b = 0$ or $|a| = 1$ and $2a\rho_2(a,b) + \rho_1(a,b) = 0$. Suppose first $|a| = 1$ and $b = 0$. Then

$$Y(a,b) = (2a\rho_2(a,b) + \rho_1(a,b))^2\frac{\partial}{\partial y} = (2-6a)^2\frac{\partial}{\partial y} \neq 0.$$

On the other hand, if $|a| = 1$ and $2a\rho_2(a,b) + \rho_1(a,b) = 0$, then

$$0 = 2a\rho_2(a,b) + \rho_1(a,b) = 2 - 6a + b^2$$

and the latter equals zero when $|a| = 1$ only if $(a,b) = (1,-2)$ or $(a,b) = (1,2)$. Thus we see that the vector field L is nonzero away from these two points. We have: $X(\rho_1) = 0 = Y(\rho_1)$ on the set where $\rho_1 = 0$ and $X(\rho_2) = 0 = Y(\rho_2)$ on the set where $\rho_2 = 0$. It follows that the circle $\gamma_1 = \{(x,y) : \rho_1(x,y) = 0\}$ and the line $\gamma_2 = \{(x,y) : \rho_2(x,y) = 0\}$ are one-dimensional orbits of $L = X + iY$. Let Ω be a bounded, simply connected region containing $\{(x,y) : x^2 + y^2 \leq 1\} \cup \{(3,y) : -3 \leq y \leq 3\}$ and such that $\Omega \cap \gamma_2$ is connected. We choose Ω so that the two points $(1,-2)$ and $(1,2)$ are not in Ω. We have

$$X \wedge Y = 4\rho_2(x,y)^2(2x\rho_2(x,y) + \rho_1(x,y))^2(1 - x^2 - y^2)\partial_x \wedge \partial_y.$$

Observe that the set $\sigma = \{(x,y) : 2x\rho_2(x,y) + \rho_1(x,y) = 0\}$ is a circle which intersects γ_1 at two points and is disjoint from γ_2. Since

$$X(2x\rho_2 + \rho_1) = 24(1-x^2)\rho_2^2(x-1) - 4y^2\rho_2(2x\rho_2 + \rho_1),$$

we see that X is transversal to σ except at the points $(1,2)$, $(1,-2)$ which are not in Ω. It follows that in Ω, the vector field L has two one-dimensional orbits, namely, γ_1 and $\gamma_2 \cap \Omega$, and three two-dimensional orbits: $\{(x,y) : x^2 + y^2 < 1\}$, $\{(x,y) \in \Omega : x^2 + y^2 > 1,\ x < 3\}$, and $\{(x,y) \in \Omega : x^2 + y^2 > 1,\ x > 3\}$. Observe also that L is real analytic and locally solvable in Ω.

5 A local version of the Rudin–Carleson property

The next result characterizes those locally integrable, smooth vector fields which satisfy a local version of the Rudin–Carleson theorem.

Theorem 5.1 *Let L be a smooth vector field satisfying condition $(\mathcal{P})$ in an open set D. For each point $p \in D$ there is a neighborhood U_p such that if $Q \subset U_p$ is a rectangle, $E \subset \partial Q$ is a closed set with Lebesgue measure zero,* *and $g \in C(\partial Q)$ is constant on the fibers of a first integral Z, then there is $h \in C(\overline{Q})$ satisfying*

$$Lh = 0 \text{ in } Q, \quad h(z) = g(z) \;\; \forall z \in E, \;\; \text{and } \sup|h| \leq 2 \sup|g|.$$

Conversely, given a locally integrable smooth vector field L on D, if there is a neighborhood W_p of each point $p \in D$ such that for every rectangle $Q \subset W_p$, and every closed set $E \subset \partial Q$ of Lebesgue measure zero and $g \in C(\partial Q)$ constant on the fibers of a first integral Z, there is $h \in C(\overline{Q})$ satisfying $Lh = 0$ in Q, $h(z) = g(z)$ $\forall z \in E$, then L satisfies condition $(\mathcal{P})$ in D.

Proof Suppose L is a smooth vector field satisfying condition $(\mathcal{P})$. Then it is well known that it is locally integrable (see Theorem 3.2 in [T]). We may assume that in a rectangle $Q = (-A, A) \times (0, T)$, $L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$ and $Z(x,t) = x + i\varphi(x,t)$ is a first integral of L, with φ real-valued. The local solvability of L implies that for each $x \in (-A, A)$,

$$\text{the function} \quad t \longmapsto \varphi(x,t) \quad \text{is monotonic on } (0,T).$$

Suppose the set E and the function $g \in C(\partial Q)$ are as in the theorem. Let Ω be the union of the two-dimensional orbits of L in Q. We can write

$$\Omega = \bigcup_{j=1}^{N} (a_j, b_j) \times (0, T)$$

where $N \leq \infty$ and the union is a disjoint union. For each j, let $Q_j = (a_j, b_j) \times (0,T)$. For each j, set $E_j = ([a_j, b_j] \times \{0,T\} \cap E) \cup (\{a_j, b_j\} \times \{0,T\})$. Since L satisfies condition $(\mathcal{P})$, the set $Z(\overline{Q_j})$ is a simply connected set whose boundary is a rectifiable, simple closed curve. In particular, by a version of the Riemann mapping theorem, the classical Rudin–Carleson theorem applies to $Z(\overline{Q_j})$. Fix j. We can find points $p, q \in [a_j, b_j] \times \{0, T\}$ such that the oscillation of g on the set $[a_j, b_j] \times \{0, T\}$,

$$\text{osc}_{[a_j,b_j]\times\{0,T\}}(g) = |g(p) - g(q)|.$$

Let G_j be continuous on $\overline{Z(Q_j)}$, holomorphic on the interior $Z(Q_j)$ such that $G_j(Z(x,t)) = g(x,t) - g(p)$ for $(x,t) \in E_j$ and

$$\sup_{Z(Q_j)}|G_j| \leq \sup_{[a_j,b_j]\times\{0,T\}}|g - g(p)|.$$

This is possible by the Rudin–Carleson theorem since g is constant on the fibers of Z. Observe that

$$\text{osc}_{Z(Q_j)} G_j \leq 2\, \text{osc}_{[a_j,b_j]\times\{0,T\}}(g).$$

Define $F_j(z) = G_j(z) + g(p)$. Then F_j is continuous on $\overline{Z(Q_j)}$, holomorphic on the interior $Z(Q_j)$, $F_j(Z(x,t)) = g(x,t)$ for $(x,t) \in E_j$ and

$$\text{osc}_{Z(Q_j)} F_j \leq 2\, \text{osc}_{[a_j,b_j]\times\{0,T\}}(g).$$

Define now

$$h(x,t)=\begin{cases}F_k(Z(x,t)), & \text{if } (x,t)\in\overline{Q_k}\\ g(x,0), & \text{if } (x,t)\notin\cup_i[a_i,b_i]\times[0,T].\end{cases}$$

Observe that $h(x,t)=g(x,t)$ for $(x,t)\in E$. We will show next that h is continuous on $\overline{Q}$. Clearly h is continuous on Ω. Suppose $(x_0,t_0)\in\overline{Q}$ and $x_0\notin\cup_i(a_i,b_i)$. Let $(x_k,t_k)\to(x_0,t_0)$. Suppose (x_{k_i},t_{k_i}) is any subsequence. If there is an infinite subset (y_m,t_m) of this subsequence which is disjoint from Ω, then $h(y_m,t_m)=g(y_m,0)$ and so by the continuity of g, $h(y_m,t_m)\to g(x_0,0)=h(x_0,t_0)$. If there is no such subsequence, without loss of generality, we may assume that for each k_i, there is k_i' such that $x_{k_i}\in[a_{k_i'},b_{k_i'}]$. Assume first that $x_0\notin\cup_{j=1}^N\{a_j,b_j\}$. Then $h(x_{k_i},t_{k_i})=F_{k_i'}(Z(x_{k_i},t_{k_i}))$ and

$$\begin{aligned}|h(x_{k_i},t_{k_i})-h(a_{k_i'},0)|&=|F_{k_i'}(Z(x_{k_i},t_{k_i}))-F_{k_i'}(Z(a_{k_i'},0))|\\&\le 2\operatorname{osc}_{[a_{k_i'},b_{k_i'}]\times\{0,T\}}(g).\end{aligned}$$

Observe that $|a_{k_i'}-b_{k_i'}|\to 0$ because we are assuming that $x_0\notin\cup_{j=1}^N[a_j,b_j]$. Since $g(a_{k_i'},0)=g(a_{k_i'},T), g(b_{k_i'},0)=g(b_{k_i'},T)$, and $|a_{k_i'}-b_{k_i'}|\to 0$, the oscillation $\operatorname{osc}_{[a_{k_i'},b_{k_i'}]\times\{0,T\}}(g)$ goes to zero as $k_i\to\infty$, and hence since

$$h(a_{k_i'},0)=g(a_{k_i'},0)\to g(x_0,0),$$

it follows that $h(x_{k_i},t_{k_i})\to h(x_0,0)=h(x_0,t_0)$. We have shown that if $x_0\notin\cup_{j=1}^N\{a_j,b_j\}$, and $(x_k,t_k)\to(x_0,t_0)$, then every subsequence of $h(x_k,t_k)$ has a further subsequence that converges to $h(x_0,t_0)$. It follows that h is continuous at (x_0,t_0) whenever $x_0\notin\cup_{j=1}^N\{a_j,b_j\}$. Suppose now $x_0\in\cup_{j=1}^N\{a_j,b_j\}$. Without loss of generality, assume $x_0=a_i$ for some i. Then clearly h is continuous from the right (in x) at (x_0,t_0). If $(x_j,t_j)\to(x_0,t_0)$ with each $x_j<x_0$, we can consider subsequences of $h(x_j,t_j)$ as before to conclude that $h(x_j,t_j)\to h(x_0,t_0)$. We have thus shown that h is continuous on $\overline{Q}$.

We will next show that $Lh=0$ in Q. Let $\psi(x,t)\in C_0^\infty(Q)$. Fix a two-dimensional orbit $Q_j=(a_j,b_j)\times(0,T)$. For each sufficiently small $\epsilon>0$, let $\psi_\epsilon(x)\in C_0^\infty(a_j,b_j)$ such that $\psi_\epsilon(x)\equiv 1$ on $(a_j+\epsilon,b_j-\epsilon)$ and for some constant C independent of ϵ, $|\psi_\epsilon'(x)|\le C\,\epsilon^{-1}$. From the definition of h, it is clear that $Lh=0$ in Q_j. We therefore have

$$\begin{aligned}0&=\int_{Q_j}hL^t(\psi_\epsilon(x)\psi(x,t))\,dxdt\quad(\text{since } \psi_\epsilon(x)\psi(x,t)\in C_0^\infty(Q_j))\\&=\int_{Q_j}h(x,t)\psi_\epsilon(x)L^t\psi(x,t)\,dxdt\\&\quad-\int_{Q_j}h(x,t)a(x,t)\psi_\epsilon'(x)\psi(x,t)\,dxdt.\end{aligned}\tag{5.1}$$

Clearly, as $\epsilon \to 0$,

$$\int_{Q_j} h(x,t)\psi_\epsilon(x)L^t\psi(x,t)\,dxdt \to \int_{Q_j} h(x,t)L^t\psi(x,t)\,dxdt. \tag{5.2}$$

The function $\psi'_\epsilon(x)$ is supported on a set of measure at most 2ϵ and on the support of this function, $a(x,t) = O(\epsilon)$. Since $\psi'_\epsilon(x) = O(\epsilon^{-1})$, it follows that when $\epsilon \to 0$,

$$\int_{Q_j} h(x,t)a(x,t)\psi'_\epsilon(x)\psi(x,t)\,dxdt \to 0. \tag{5.3}$$

From (5.1)–(5.3), we conclude that

$$\int_{Q_j} hL^t\psi(x,t)\,dxdt = 0$$

and hence

$$\int_{\Omega} hL^t\psi(x,t)\,dxdt = 0 \tag{5.4}$$

where by definition, Ω was the union of the two-dimensional orbits of L in Q. Recall that $L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x}$. Let

$$\mathcal{N} = \{x \in (-A,A) : a(x,t) \equiv 0,\ 0 \le t \le T\}$$

and set

$$\tilde{\mathcal{N}} = \{x \in \mathcal{N} : \frac{\partial a}{\partial x}(x,t) \equiv 0,\ 0 \le t \le T\}.$$

The implicit function theorem implies that the set $\mathcal{N}\backslash\tilde{\mathcal{N}}$ is a countable set. Therefore, using this and (5.4), we have

$$\begin{aligned}
\int_Q hL^t\psi(x,t)\,dxdt &= \int_0^T \int_{\mathcal{N}} h(x,t)L^t\psi(x,t)\,dxdt \\
&= \int_0^T \int_{\tilde{\mathcal{N}}} h(x,t)L^t\psi(x,t)\,dxdt \\
&= \int_{\tilde{\mathcal{N}}} \int_0^T h(x,t)L^t\psi(x,t)\,dtdx \\
&= -\int_{\tilde{\mathcal{N}}} h(x,0)\left(\int_0^T \frac{\partial\psi}{\partial t}(x,t)\,dt\right)dx \\
&\quad \text{(since } h(x,t) \equiv h(x,0) \text{ for } x \in \mathcal{N}) \\
&= 0.
\end{aligned}$$

It follows that $Lh = 0$ in Q.

Conversely, suppose the locally integrable vector field L satisfies the Rudin–Carleson property for every smooth subdomain of a neighborhood of the origin. Let $Z(x,t) = x + i\varphi(x,t)$ be a first integral of L near the origin such that

$$\left|\frac{\partial\varphi}{\partial x}(x,t)\right| \leq \frac{1}{2}. \tag{5.5}$$

Assume L does not satisfy property $(\mathcal{P})$. Then we may assume that for some A, $T > 0$, $x_0 \in (-A, A)$, and $0 < t_0 < T$,

$$\varphi(x_0, 0) < \varphi(x_0, t_0) \quad \text{and} \quad \varphi(x_0, t_0) > \varphi(x_0, T). \tag{5.6}$$

By changing t_0 if necessary, and choosing T close enough to t_0, we may also assume that

$$\varphi(x_0, t) \leq \varphi(x_0, t_0) \quad \forall t \in [0, T] \quad \text{and } \varphi(x_0, 0) < \varphi(x_0, T) < \varphi(x_0, t_0). \tag{5.7}$$

Let $\delta > 0$ such that

$$\varphi(x, 0) < \varphi(x, T) < \varphi(x, t_0) \quad \text{whenever } |x - x_0| \leq \delta. \tag{5.8}$$

We will reason in the rectangle $Q = [x_0 - \delta, x_0 + \delta] \times [0, T]$. Let $x_k \in [x_0 - \delta, x_0 + \delta]$ be a sequence converging to x_0. Let $E \subset \partial Q$ be a closed set with measure zero containing the sequence $\{(x_k, T)\}$. Choose $g \in C(E)$ such that

$$g(x_k, T) = 0 \ \forall k \quad \text{and } g(p) \neq 0 \text{ for some } p \in E \cap \{(x, T) : |x - x_0| < \delta\}. \tag{5.9}$$

Suppose now $h(x,t) \in C(Q)$, $Lh = 0$ in the interior of Q, and $h = g$ on the set E. Estimate (5.5) allows us to use the Baouendi–Treves approximation theorem ([BT], [BCH, p. 53]) to produce a sequence of entire functions H_k such that $H_k(Z(x,t)) \to h(x,t)$ uniformly on Q. In particular, by (5.8), there is a connected open neighborhood V of the set $\{Z(x,T) : |x - x_0| \leq \delta\}$ on which the sequence $H_k(z)$ will converge to a holomorphic function H. Since $H(Z(x_k, T)) = 0$ for every k and $Z(x_k, T) \to Z(x_0, T) \in V$, we must have $H \equiv 0$ on V. But this contradicts the assumption that $h(p) = g(p) \neq 0$. Therefore, L does not have the Rudin–Carleson property on Q.

6 A link with the F. and M. Riesz theorem

In [B] Bishop proved an abstract theorem which permits a generalization of the Rudin–Carleson theorem to some situations where a version of the F. and M. Riesz theorem is valid. Bishop's theorem has been a key tool in the study of peak-interpolation sets for $A(\Omega)$ where Ω is a bounded domain in $\mathbb{C}^n$ (typically strictly pseudoconvex) and $A(\Omega)$ is the algebra of holomorphic functions on Ω that are continuous up to the boundary (see [Bh], [R2], [Na] and the references therein). We state here a strengthened version from [G] of the theorem proved in [B]:

Theorem 6.1 (Theorem 12.5 in [G]) *Let $C(X)$ be the uniformly normed Banach space of all continuous complex-valued functions on a compact Hausdorff space X. Let B be a closed subspace of $C(X)$. Let $B^{\perp}$ consist of all (finite, complex-valued, Baire) measures μ on X such that $\int f\, d\mu = 0$ for all f in B. Let $\hat{\mu}$ be the regular Borel extension of the Baire measure μ. Let S be a closed subset of X with the property that $\hat{\mu}(T) = 0$ for every Borel subset T of S and every μ in $B^{\perp}$. Let f be a continuous complex-valued function on S and Δ a positive continuous function on X such that $|f(x)| \leq \Delta(x)$ for all x in S. Then there exists F in B with $|F(x)| \leq \Delta(x)$ for all x in X and $F(x) = f(x)$ for all x in S.*

If $X = \mathbb{T}$ equals the unit circle and B denotes the space of continuous functions on $\mathbb{T}$ which are restrictions of functions holomorphic on the unit disk D and continuous on the closure $\overline{D}$, then a measure μ on $\mathbb{T}$ is in $B^{\perp}$ if and only if it is the boundary value of a holomorphic function on D. By the F. and M. Riesz theorem, it follows that any $\mu \in B^{\perp}$ is absolutely continuous with respect to Lebesgue measure and so the preceding theorem implies the Rudin–Carleson theorem. The classical F. and M. Riesz theorem was generalized for solutions of locally integrable vector fields in the paper [BH2]. However, unlike the holomorphic case, there are two reasons why we cannot use the F. and M. Riesz property of a vector field together with Theorem 6.1 to deduce the Rudin–Carleson property. Given a vector field L in a neighborhood of $\overline{D}$, let $\mathcal{A}$ denote the subspace of $C(\partial D)$ which are restrictions of functions $u \in C(\overline{D})$ that satisfy $Lu = 0$ in D. In general, $\mathcal{A}$ is not a closed subspace of $C(\partial D)$. Moreover, if $\mu \in \mathcal{A}^{\perp}$, it may not be the boundary value of a solution of L in D. For example, if $M = \frac{\partial}{\partial y} + iy\frac{\partial}{\partial x}$ is the Mizohata vector field and μ is a measure on $\mathbb{T}$ which is of the form $\mu = \delta_{(0,1)} - \delta_{(0,-1)}$ where δ_p denotes the Dirac mass at p, then $\int_{\mathbb{T}} h\, d\mu = 0$ for every $h \in C(\overline{D})$ that satisfies $Mh = 0$ on D. Such a measure cannot be the boundary value of a solution of M. Thus for a general vector field, a measure that is orthogonal to the boundary values of continuous solutions may not be a boundary value of a solution and in fact, it may not be absolutely continuous with respect to Lebesgue measure. If a vector field L satisfies the hypotheses of Theorem 2.1, we have the following:

Corollary 6.2 *Suppose L is a vector field as in Theorem 2.1 defined on a neighborhood U of $\overline{D}$ and satisfying the equivalent conditions (1), (2) in the theorem. Let $\mathcal{A}$ denote the algebra of continuous functions h on $\overline{D}$ satisfying the equation $Lh = 0$ in D. Let μ be a complex Baire measure defined on ∂D with the property that*

$$\int_{\partial D} h\, d\mu = 0$$

for every $h \in \mathcal{A}$. If a closed set $E \subseteq \partial D$ has Lebesgue measure zero and it is disjoint from the one-dimensional orbits of L in U, then $\mu(E) = 0$.

Proof Let F be a closed subset of E. Let P be a positive continuous function on ∂D such that:

- $P \equiv 1$ on F.
- For any $y \notin F$, $P(y) < 1$.

An application of Theorem 6.1 in the proof of Theorem 2.1 shows that there is $h \in \mathcal{A}$ that equals 1 on F and satisfies $|h(p)| < 1$ for $p \notin F$. By hypothesis, for each positive integer n, we have $\int h^n \, d\mu = 0$. Letting $n \to \infty$, we are led to conclude that $\mu(F) = 0$. By the regularity of the measure μ, it follows that $\mu(E) = 0$.

Let $L = \frac{\partial}{\partial y} + ix\frac{\partial}{\partial x}$. This vector field is locally solvable and the y-axis is a one-dimensional orbit. Therefore, if $u \in C(\overline{D})$ satisfies $Lu = 0$ in D, then it is constant on the y-axis. It follows that if $\mu = \delta_{(0,1)} - \delta_{(0,-1)}$, then $\int_{\partial D} u \, d\mu = 0$ for all such solutions. Note that L satisfies the hypotheses of Theorem 2.1 since it has no compact orbits. This example shows that in Corollary 6.2, the set E has to be disjoint from the one-dimensional orbits.

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Evolution Equations and Generalized Fourier Integral Operators

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Dedicato a Ferruccio Colombini in occasione del suo sessantesimo compleanno

Summary. We consider evolution equations $\partial u/\partial t = ia^w(x,D)u$ where a is the (real valued) Weyl symbol of the operator $A = a^w$. For instance, Schrödinger-like equations. After recalling what are generalized Fourier integral operators in the framework of the Weyl-Hörmander calculus, we give conditions on a and on the dynamics of its hamiltonian flow which imply: 1. The operator a^w is essentially self-adjoint and the propagators e^{itA} are bounded between (conveniently related) generalized Sobolev spaces. 2. The propagators e^{itA} are generalized Fourier integral operators.

2000 AMS Subject Classification: 35G10, 35S05, 35S30.

Key words: Evolution equations, Weyl-Hörmander calculus, Fourier integral operators, Schrödinger equations.

Introduction

It is well known that, given a classical selfadjoint pseudodifferential operator of order 1, one can define the strongly continuous group (P_t) of unitary operators, such that $u_t = P_t u_0$ gives the solution of the hyperbolic equation $\frac{\partial}{\partial t} + iAu = 0$ with Cauchy data u_0. Moreover, the operators $P_t = e^{-itA}$ are classical Fourier integral operators associated to the canonical transformations F_t, where $(t,x) \mapsto F_t(x)$ is the flow of the Hamiltonian field $H_a = (\partial a/\partial \xi_j; -\partial a/\partial x_j)$ and a is the principal symbol of A. In particular, we have the following two properties:

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_4,

- The operators P_t are bounded from the Sobolev space H^s into itself.
- The conjugate $P_{-t}BP_t$ of a classical pseudodifferential operator B (with principal symbol b) is a classical pseudodifferential, whose principal symbol is $b \circ F_t^{-1}$.

It turns out that to extend this theory to a more general evolution equation such as Schrödinger-type equations, one has just to modify the properties above. Let us consider for instance the harmonic oscillator $A = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2\right)$. The group of unitary operators $P_t = e^{-itA}$ is well known and, in particular, for $t = \pi/2$, P_t is, up to some factor, the Fourier transformation while the canonical transformation (still associated to the Hamiltonian flow of the principal symbol) becomes $F_t(x, \xi) = (\xi, -x)$. One has the corresponding properties:

- P_t maps the Sobolev spaces H^s into weighted L^2 spaces.
- The symbols of B and of its conjugate $\widetilde{B} = P_{-t}BP_t$ are still related by $\widetilde{b} = b \circ F_t^{-1}$, but if b is a symbol of order m satisfying the standard estimates

$$\left|\partial_\xi^\alpha \partial_x^\beta b\right| \leq C^{\mathrm{st}}(1+|\xi|)^{m-|\alpha|},$$

the symbol $\widetilde{b}$ satisfies the exotic ones

$$\left|\partial_\xi^\alpha \partial_x^\beta \widetilde{b}\right| \leq C^{\mathrm{st}}(1+|x|)^{m-|\beta|}.$$

Such $\widetilde{b}$ can be considered as symbols of (generalized) pseudodifferential symbols if we use the Weyl–Hörmander calculus.

In this theory, many different pseudodifferential calculi are defined, each of which is associated to a "good" Riemannian metric g on the phase space $\mathcal{X} = \mathbb{R}^n \times (\mathbb{R}^n)^*$. Moreover, to any "good" positive function M on $\mathcal{X}$, one can associate a generalized "Sobolev space" $H(M, g)$. In the example above, the P_t maps the usual Sobolev spaces into unusual ones (the weighted L^2 spaces), and conjugates of usual pseudodifferential operators are unusual ones.

We will use systematically the Weyl–Hörmander calculus and, in order to generalize Fourier integral operators and hyperbolic equations, we have to study two problems.

1. We are given a canonical transformation F (symplectic diffeomorphism) of $\mathcal{X}$ onto itself, an initial calculus (defined by a Riemannian metric g) and a final calculus (defined by $\widetilde{g}$). One can then define (under convenient assumptions) a class $\mathrm{FIO}(F, g, \widetilde{g})$ of operators whose main property is the following: conjugates of g-pseudodifferential operators are $\widetilde{g}$-pseudodifferential ones. These generalized Fourier integral operators have good properties (composition, boundedness in generalized Sobolev spaces) and enjoy a symbolic calculus. This has been developped in [Bo3] and is recalled in Section 2.

2. We are given an evolution equation $\frac{\partial}{\partial t} + iAu = 0$ and an initial calculus (defined by a metric g_0). Then, one can expect that the propagators P_t exist

and belong to $\mathrm{FIO}(F_t, g_0, g_t)$. The calculus at time t depends on t and is actually forced by the Hamiltonian flow. Theorems 3.1 and 3.2 give sufficient conditions (on the symbol a and its Hamiltonian flow F_t) for getting such results. Proofs will be sketched in Sections 4 and 5.

Our assumptions are exclusively expressed in terms of differential geometry, starting from the symbol a of A. In particular, no selfadjoint extension in L^2 is a priori given and an important part of the task is to deduce from the dynamic assumption on a that A is essentially selfadjoint. One can see easily that these assumptions are grosso modo necessary if one wants to fulfill the program above. However, they are not so easy to check: they require estimates which may be touchy, not only on a but also on its Hamiltonian flow.

1 Weyl–Hörmander calculus of pseudodifferential operators

We refer to [Hö, Sections 18.5, 18.6] but we will need some results from [BL], [BC], [Bo1] and [Bo2].

1.1 Quantization

We will denote by $X = (x, \xi)$ a point of the *phase space* $\mathcal{X} = \mathbb{R}^n \times (\mathbb{R}^n)^*$. The *symplectic form* σ on $\mathcal{X}$ is defined by

$$\sigma(X, Y) = \langle \xi, y \rangle - \langle \eta, x \rangle\,; \qquad X = (x, \xi),\ Y = (y, \eta).$$

For $a(x, \xi)$ belonging to the Schwartz space $\mathcal{S}(\mathcal{X})$, the operator $a^w(x, D)$, or a^w for short, is defined by

$$a^w(x, D)u(x) = \iint e^{i(x-y)\cdot\xi}\, a\left(\tfrac{x+y}{2}, \xi\right) u(y) \tfrac{dy d\xi}{(2\pi)^n}. \tag{1.1}$$

Such an operator maps $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. If now a belongs to the space $\mathcal{S}'(\mathcal{X})$ of tempered distributions on $\mathcal{X}$, the same formula, taken in the weak sense, defines an operator mapping $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$. One says that a is the Weyl symbol of a^w.

The product of composition of two symbols a and b (belonging say to $\mathcal{S}(\mathcal{X})$, but this will be widely extended) is defined by $(a\#b)^w = a^w \circ b^w$ and is given by the formula

$$a\#b(X) = \iint e^{-2i\sigma(X-S, X-T)} a(S) b(T) \tfrac{dS\, dT}{\pi^{2n}}. \tag{1.2}$$

The following expansion is given here with a remainder of order 3, which is sufficient for our purpose, but it exists at any order:

$$a\#b = ab + \tfrac{1}{2i}\,\{a\,,\, b\} + \tfrac{1}{2}\left(\tfrac{1}{2i}\sigma(\partial_Y, \partial_Z)\right)^2 a(Y) b(Z)|_{Y=Z=X} + R_3(a, b). \tag{1.3}$$

Here, $\{a\,,\,b\}$ is the usual Poisson bracket in $\mathcal{X}$. There is an integral formula, more or less similar to (1.2) and for which we refer to [Bo2], giving the value of $R_3(a,b)$. An important point is that it depends only on the derivatives of order 3 of a and b.

1.2 Admissible metrics

A Riemannian metric g on the phase space is identified to a family $Y \mapsto g_Y$ of positive definite quadratic forms on $\mathcal{X}$. For each Y, one can choose symplectic coordinates (depending on Y but still denoted by (x,ξ)) such that g_Y is diagonalized:

$$g_Y(dx,d\xi) = \sum_1^n \frac{dx_j^2}{a_j^2} + \sum_1^n \frac{d\xi_j^2}{\alpha_j^2}. \tag{1.4}$$

The a_j and α_j depend on the choice of the coordinates, but the products $a_j\alpha_j$ depend just on Y.

Such a metric g is said to be *admissible* if the following five properties are satisfied.

A1. Simplifying assumption. The products $a_j\alpha_j$ above are equal and their common value is denoted by $\lambda(Y)$. This means that there is a (linear) symplectic transformation mapping the unit ball $B_Y = \{X |\, g_Y(X-Y) \le 1\}$ onto the euclidean ball of radius $\sqrt{\lambda(Y)}$. One has

$$|\sigma(S,T)| \le \lambda(Y) g_Y(S)^{1/2} g_Y(T)^{1/2}.$$

A2. Fundamental assumption. $\forall Y, \quad \lambda(Y) \ge 1$.
This means that localizing in unit balls is not a violation of the uncertainty principle.

A3. Slowness. There exists $C > 0$ such that

$$g_Y(Y-Z) \le C^{-1} \Longrightarrow \big(g_Y(T)/g_Z(T)\big)^{\pm 1} \le C$$

uniformly with respect to Y, Z, T.

A4. Temperance. There exist C and N such that

$$\forall Y, \forall Z, \forall T, \quad \big(g_Y(T)/g_Z(T)\big)^{\pm 1} \le C\big(1 + \lambda(Y)^2 g_Y(Y-Z)\big)^N.$$

A5. Geodesic temperance. The geodesic distance $D(Y,Z)$ *for the Riemannian metric* $\lambda(Y)^2 g_Y(dx,d\xi)$ is equivalent to $\lambda(Y) g_Y(Y-Z)^{1/2}$ in the following sense:

$$\exists C, \exists N, \forall Y, \forall Z$$
$$C^{-1}(1+D(Y,Z))^{1/N} \le 1+\lambda(Y) g_Y(Y-Z)^{1/2} \le C(1+D(Y,Z))^N.$$

In view of A4, this property is equivalent to

$$\exists C, \exists N, \forall Y, \forall Z, \forall T, \quad \big(g_Y(T)/g_Z(T)\big)^{\pm 1} \le C\big(1 + D(Y,Z)\big)^N.$$

Remark 1.1 The first assumption A1 makes things simpler, for instance it is not necessary to introduce the inverse metric g^σ which in this case is just $\lambda^2 g$, but it is not necessary. On the contrary, the geodesic temperance plays an important rôle: thanks to A5, one has a simple characterization of pseudodifferential operators (see Section 1.4), one can define very easily the Fourier integral operators and thus prove in a few lines our Theorem 3.2.

It could be possible to define the Fourier integral operators without A5, using localized twisted commutators (as in [BC, th. 5.5]), but the proofs are much more complicated. Moreover, there is no known example of a metric satisfying A4 and not A5.

1.3 Weights and symbols

A positive function M defined on $\mathcal{X}$ is a *g-weight* if it satisfies the following conditions (slowness and temperance), for convenient constants C' andN':

$$g_Y(Y-Z) \le C'^{-1} \Longrightarrow \big(M(Y)/M(Z)\big)^{\pm 1} \le C',$$
$$\big(M(Y)/M(Z)\big)^{\pm 1} \le C'\big(1 + \lambda(Y)^2 g_Y(Y-Z)\big)^{N'}.$$

Modifying the constants if necessary, $\big(1 + \lambda(Y)^2 g_Y(Y-Z)\big)$ can be replaced above by $\big(1 + D(Y,Z)\big)$.

The *classes of symbols* $S(M,g)$ (for admissible metrics and g-weights) are defined as the set of functions $a \in C^\infty(\mathcal{X})$ such that

$$|\partial_{T_1} \dots \partial_{T_k} a(X)| \le C_k M(X) \quad \text{for } g_X(T_j) \le 1. \tag{1.5}$$

Here, $\partial_T a = \langle T, da\rangle$ denotes the directional derivative along T. The space of operators a^w for $a \in S(M,g)$ (the pseudodifferential operators of weight M) is denoted by $\Psi(M,g)$. The following properties are now well known:

- $\Psi(M,g) \subset \mathcal{L}(\mathcal{S},\mathcal{S})$ and $\Psi(M,g) \subset \mathcal{L}(\mathcal{S}',\mathcal{S}')$.
- $\Psi(1,g) \subset \mathcal{L}(L^2,L^2)$.
- In the expansion (1.3), for $a \in S(M_1,g)$ and $b \in S(M_2,g)$, one has

$$a\#b \text{ and } ab \in S(M_1M_2,g),$$
$$\{a\,,\,b\}\,,\ (a\#b - ab) \text{ and } (a\#b - b\#a) \in S(M_1M_2\lambda^{-1},g), \tag{1.6}$$
$$R_3(a,b) \in S(M_1M_2\lambda^{-3},g). \tag{1.7}$$

Let us recall some complements which are proved in [Bo2].

Proposition 1.1 *The classes of symbols* $\hat{S}\ (M,g)$ *[resp.* $\hat{\hat{S}}\ (M,g)$, $\hat{\hat{\hat{S}}}$ (M,g)*] are defined as the spaces of functions satisfying* (1.5) *for* $k \geq 1$ *[resp.* $k \geq 2$, $k \geq 3$*].*

(a) *There exists a weight* M' *depending on* M *such that* $\hat{\hat{\hat{S}}}\ (M,g) \subset S(M',g)$.

(b) *The properties* (1.6) *are still valid for* $a \in \hat{S}\ (M_1,g)$ *and* $b \in \hat{S}\ (M_2,g)$.

(c) *The property* (1.7) *is still valid for* $a \in \hat{\hat{\hat{S}}}\ (M_1,g)$ *and* $b \in \hat{\hat{\hat{S}}}\ (M_2,g)$.

The seminorms of the spaces $S(M,g)$, $\hat{S}\ (M,g)$, ... are the best constants C_k in (1.5). The $S(M,g)$ are Fréchet spaces, the $\hat{S}\ (M,g)$, ... are complete but not Hausdorff.

1.4 Characterization of pseudodifferential operators

Theorem 1.1 (a) *Given* $b \in \hat{S}\ (\lambda,g)$ *and* $A \in \Psi(M,g)$, *one has*

$$\operatorname{ad} b^w \cdot A \stackrel{\text{def}}{=} b^w A - A\, b^w \in \Psi(M,g).$$

When $M = 1$, *this operator is thus bounded on* L^2.

(b) *Conversely, let* A *be an operator which is bounded on* L^2 *as well as its iterated commutators*

$$\operatorname{ad} b_1^w \dots \operatorname{ad} b_k^w \cdot A \quad \textit{for } b_j \in \hat{S}\ (\lambda,g).$$

Then A *belongs to* $\Psi(1,g)$.

The first part is an immediate consequence of Proposition 1.1 (b). For the converse, we refer to [Bo1] where the geodesic temperance plays a decisive role.

Generalized Sobolev spaces $H(M,g)$**.** — We refer to [BC] for equivalent definitions; the following properties will be sufficient:

- For any g-weight M, there exist $A \in \Psi(M,g)$ and $B \in \Psi(M^{-1},g)$ such that $AB = BA = I$.
- The *Sobolev space* $H(M,g)$ (sometimes denoted $H(M)$ for short) is the set of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $Au \in L^2$ for any $A \in \Psi(M,g)$. It is sufficient that $Au \in L^2$ for one invertible A as above, and one can choose $\|u\|_{H(M)} = \|Au\|_{L^2}$.
- For any g-weights M and M_1, any $A \in \Psi(M,g)$ maps continuously $H(M_1)$ into $H(M_1/M)$.
- If $A \in \Psi(M,g)$ is bijective from $H(M_1)$ onto $H(M_1/M)$ for some g-weight M_1, then A^{-1} belongs to $\Psi(M^{-1},g)$.

2 Generalized Fourier integral operators

We recall here some of the definitions and results of [Bo3]. We consider only Fourier integral operators P of weight 1 (or of order 0, i.e., bounded on L^2), Fourier integral operators of weight M being just products PA of such P with $A \in \Psi(M, g)$.

An *admissible triple* $(F, g, \widetilde{g})$ is made of a diffeomorphism F of $\mathcal{X}$ onto itself, and of two Riemannian metrics g and $\widetilde{g}$, such that the four following conditions are satisfied.

B1. F is a canonical transformation (or symplectomorphism), which means that $F_*\sigma = \sigma$. For any $Y \in \mathcal{X}$, the differential $F'(Y)$ belongs to the symplectic group $\mathrm{Sp}(n)$.

B2. F is an isometry of $(\mathcal{X}, g)$ onto $(\mathcal{X}, \widetilde{g})$. This means that $\widetilde{g}$ is the direct image F_*g of g, i.e., the Riemannian metric defined by

$$\widetilde{g}_{F(Y)}(T) = g_Y(F'(Y)^{-1} \cdot T).$$

B3. g and $\widetilde{g}$ are admissible metrics, satisfying conditions A1 to A5.

B4. One has the following estimates on the derivatives of F, for convenient constants C_k:

$$\widetilde{g}_{F(X)}\big(\partial_{T_1} \dots \partial_{T_k} F(X)\big) \le C_k \quad \text{for } g_X(T_j) \le 1. \tag{2.1}$$

Remark 2.1 In most applications, the canonical transformation F and an admissible metric g are given and $\widetilde{g}$ is thus determined by B2. The problem is to know and to prove that $\widetilde{g}$ is also admissible. It is easy to see that A1 and A2 are satisfied and that the slowness A3 is a consequence of B4 for $k = 2$, but the temperance is touchy.

It cannot be expressed simply in terms of F and g because it mixes up the symplectic and Riemannian structures (which are preserved by F) and the affine structure (which is not). One has to compare the values of the quadratic forms $\widetilde{g}_Y$ and $\widetilde{g}_Z$ for *the same vector* T in two points which can be very far, and this requires a good knowledge of the behavior of F at infinity.

For g and $\widetilde{g}$, the functions defined in Section 1.2 are denoted by λ and $\widetilde{\lambda}$. The quadratic forms g_Y and $\widetilde{g}_{F(Y)}$ being symplectically equivalent, one has $\widetilde{\lambda}(F(Y)) = \lambda(Y)$.

The condition B4 for $k = 1$ is automatically satisfied (with $C_1 = 1$) for F and F^{-1}. A simple computation shows that the conditions B4 (for $k > 1$) are also valid for F^{-1} which imply that the triple $(F^{-1}, \widetilde{g}, g)$ is also admissible.

The condition B4 is actually equivalent to the following properties, which are of course essential for our purpose.

Proposition 2.1 *If m is a g-weight, then $\widetilde{m} = m \circ F^{-1}$ is a $\widetilde{g}$-weight and one has*

$$a \circ F^{-1} \in S(\widetilde{m}, \widetilde{g}) \iff a \in S(m, g),$$
$$a \circ F^{-1} \in \hat{S}\,(\widetilde{m}, \widetilde{g}) \iff a \in \hat{S}\,(m, g).$$

There is no analogous result for $a \in \hat{\hat{S}}\,(m, g)$: an estimate of the second derivatives of $a \circ F^{-1}$ requires an estimate of the first derivatives of a.

Definition 2.1 (Fourier integral operators and twisted commutators) *The space* $\mathrm{FIO}(F, g, \widetilde{g})$ *of Fourier integral operators associated to the admissible triple* $(F, g, \widetilde{g})$ *is the set of operators P such that*

$$\widetilde{\mathrm{ad}}(b_1) \ldots \widetilde{\mathrm{ad}}(b_k) \cdot P \in \mathcal{L}(L^2) \quad \textit{for } b_j \in \hat{S}\,(\lambda, g), \tag{2.2}$$

where $\widetilde{\mathrm{ad}}(b) \cdot P$ *is a notation for the twisted commutator:*

$$\widetilde{\mathrm{ad}}(b) \cdot P = (b \circ F^{-1})^w P - P b^w. \tag{2.3}$$

This definition is of course modeled on the characteristic property of pseudodifferential operators given in Theorem 1.1. It implies easily the following properties:

- $\mathrm{FIO}(I, g, g) = S(1, g)$.
- For $P \in \mathrm{FIO}(F, g, \widetilde{g})$, its adjoint P^* belongs to $\mathrm{FIO}(F^{-1}, \widetilde{g}, g)$.
- For $P \in \mathrm{FIO}(F, g, \widetilde{g})$ and $Q \in \mathrm{FIO}(G, \widetilde{g}, \overline{g})$, where $(F, g, \widetilde{g})$ and $(G, \widetilde{g}, \overline{g})$ are two admissible triples, one has $QP \in \mathrm{FIO}(G \circ F, g, \overline{g})$.

To prove the existence of nontrivial Fourier integral operators, a more concrete definition is necessary.

2.1 Principal symbol of Fourier integral operators

Let Γ be the graph of F and for each point $(Y, F(Y)) \in \Gamma$, let χ_Y be the affine tangent map, defined by $\chi_Y(X) = Y + F'(Y) \cdot (X - Y)$. One can define a fiber bundle $\widetilde{\Gamma} \to \Gamma$ such that its fiber at $(Y, F(Y))$ is made of the metaplectic operators V associated to χ_Y, i.e., such that $a^w\, V = V\, (a \circ \chi_Y)^w$ for any symbol a. Such a V is determined by χ_Y up to multiplication by a complex number $\omega \in U(1)$ and the fiber is thus a circle.

We refer to [Bo3] for the definition of the *horizontal sections* $Y \mapsto V_Y$ of $\widetilde{\Gamma}$ as well as for the construction of a refined partition of unity $Y \mapsto \psi_Y$. The following result is Theorem 6.6 of [Bo3].

Theorem 2.1 (i) *For such V_Y and ψ_Y and for $p \in S(1, g)$, the integral*

$$P_1 = \int p(Y)\; V_Y \circ \psi_Y^w \,\tfrac{dY}{\pi^n} \tag{2.4}$$

defines an element of $\mathrm{FIO}(F, g, \widetilde{g})$.
(ii) *Conversely, any* $P \in \mathrm{FIO}(F, g, \widetilde{g})$ *can be written* $P = P_1 + R$, *with* P_1 *as above and* R *a regularizing Fourier integral operator, i.e., such that*

$$\forall N, \quad (\widetilde{\lambda}^w)^N \circ R \circ (\lambda^w)^N \in \mathrm{FIO}(F, g, \widetilde{g}). \tag{2.5}$$

(iii) *The section* $(Y, F(Y)) \mapsto p(Y)V_Y$ *of the line bundle* $\widetilde{\Gamma} \otimes_{U(1)} \mathbb{C}$ *is said to be a principal symbol of* P. *The principal symbol of* P *is unique, up to a symbol* $(Y, F(Y)) \mapsto q(Y)V_Y$ *with* $q \in S(\lambda^{-1}, g)$.

A principal symbol for the adjoint P^* is $(F(Y), Y) \mapsto \overline{p(Y)}V_Y^*$. With evident notations, for $Q \in \mathrm{FIO}(G, \widetilde{g}, \overline{g})$, a principal symbol of $Q \circ P$ is the section $(Y, G \circ F(Y)) \mapsto p(Y)q(F(Y))\, W_{F(Y)} \circ V_Y$. Thanks to part (i) of the theorem, there exist almost invertible Fourier integral operators.

3 Evolution equations

Let a be a real-valued and C^∞ function on $\mathcal{X}$, belonging to a class of symbols which will be specified later, let g_0 be an admissible metric and $T > 0$. We make the following assumptions:

C1. The flow F_t of the hamiltonian field of a is global: it is defined for all $t \in \mathbb{R}$ par $\frac{d}{dt}F_t(X) = H_a(F_t(X))\,;\ F_0(X) = X$. Set $g_t = F_{t*}g_0$.

C2. The metrics g_t satisfy A1 to A5 for any $t \in [-T, T]$, with uniform constants.

C3. The triples (F_t, g_0, g_t) satisfy B1 to B4 for any $t \in [-T, T]$, with uniform constants.

The group law $F_{t+s} = F_t \circ F_s$ implies that the triples (F_t, g_s, g_{s+t}) are admissible when s and $s{+}t$ belong to $[-T, T]$.

The "function λ" defined in Section 1.2 corresponding to g_t will be denoted by λ_t. One has $\lambda_t{=}\lambda_0 \circ F_t^{-1}$. For any g_0-weight μ_0, we will denote by μ_* the family of g_t-weight $\mu_t = \mu_0 \circ F_t^{-1};\ t \in [-T, T]$.

Theorem 3.1 *Assume that* a *belongs uniformly to* $\hat{\hat{\hat{S}}}\,(\lambda_t^3, g_t)$ *(i.e., the kth semi-norm of* a *in these spaces is bounded by a constant* C_k *independent of* t*). Then*
(i) *The operator* a^w *with domain* $\mathcal{S}(\mathbb{R}^n)$ *is essentially selfadjoint on* L^2. *The domain of its closure* A *is* $\left\{u \in L^2 \middle| a^w u \in L^2\right\}$, *which means that weak and strong extension coincide.*
(ii) A *is thus the infinitesimal generator of a one-parameter strongly continuous group* $P_t = e^{-itA}$. *For any* g_0*-weight* μ_0 *and for* $|t| \leq T$, *the operator* P_t *is bounded from* $H(\mu_0, g_0)$ *onto* $H(\mu_t, g_t)$.

The assumption on a is satisfied when $a \in \hat{S}\,(\lambda_0^3, g_0)$ but it is not sufficient in general that $a \in \overset{\wedge\wedge\wedge}{S}\,(\lambda_0^3, g_0)$. For the same reason, it is sufficient to assume $a \in \hat{S}\,(\lambda_0^2, g_0)$ in the next theorem.

Theorem 3.2 *Assume now that a belongs uniformly to $\overset{\wedge\wedge\wedge}{S}\,(\lambda_t^2, g_t)$. Then P_t belongs to* $\mathrm{FIO}(F, g_0, g_t)$ *for $|t| \leq T$.*

Remark 3.1 The meaning of the condition $a \in \overset{\wedge\wedge\wedge}{S}\,(\lambda_0^2, g_0)$ depends strongly on the choice of the initial metric g_0. For instance, for the standard metric $dx^2 + \frac{d\xi^2}{1+|\xi|^2}$, terms like $|\xi|^2 \log |\xi|$ or x^3 are allowed. If g_0 is the euclidean metric, any polynomial of total degree 3 (in x and ξ) belongs to $\overset{\wedge\wedge\wedge}{S}\,(1, g_0)$.

It is clear from these examples that the assumption on the class of a cannot imply the global character of the flow nor the essential selfadjointness of a^w. The dynamic assumption C1 is crucial.

4 Proof of Theorem 3.1

Let us write $A = a^w$. If we think of the equation $\frac{d}{dt}u_t + iAu_t = f_t$ as a Schrödinger equation, the associated "Heisenberg equation" is $\frac{d}{dt}B_t = i(B_tA - AB_t)$. It turns out that our dynamic assumptions give immediately approximate solutions of this last equation, which will give a priori estimates.

Let b_0 be a symbol for the metric g_0 whose weight will be specified later, and $b_t = b_0 \circ F_t^{-1}$. We have $\frac{\partial}{\partial t} b_t = \{b_t\,,\, a\}$ and thus, according to (1.3),

$$b_t \# a = ab_t + \tfrac{1}{2i}\,\{b_t\,,\, a\} + \text{order } 2 + R_3(b_t, a),$$
$$a \# b_t = b_t a + \tfrac{1}{2i}\,\{a\,,\, b_t\} + \text{order } 2 + R_3(a, b_t).$$

Terms of order 0 and 2 are symmetric in a and b, and thus

$$\tfrac{d}{dt} B_t = i(B_tA - AB_t) + R_t \tag{4.1}$$

where the symbol of R_t belongs to the same class as $R_3(a, b_t)$. As a consequence of Proposition 1.1 (c), under the assumptions of Theorem 3.1, and for $b_0 \in S(\mu_0, g_0)$ (or $b_0 \in \hat{S}\,(\mu_0, g_0)$), one has $R_t \in \Psi(\mu_t, g_t)$.

We have to define the spaces $L^p([-T, T]; H(\mu_*))$ made of (classes of) measurable functions $u : t \mapsto u_t$ (the weak measurability with values in $\mathcal{S}'$ is sufficient) such that

$$\|u\|_{L^p(H(\mu_*))} = \left(\int_{-T}^{T} \|u_t\|^p_{H(\mu_t)}\, dt \right)^{1/p} < \infty, \tag{4.2}$$

with the usual convention for $p = \infty$. This definition is meaningful if we define the norms of the spaces $H(\mu_t, g_t)$ in a coherent way. This can be specified thanks to the following proposition.

Proposition 4.1 *Let μ_0 be a g_0-weight. There exist $\delta > 0$ and for each $\theta \in [-T, T]$ a $b_\theta \in S(\mu_\theta, g_\theta)$ such that, for $|s| \le \delta$, the operators $(b_\theta \circ F_s^{-1})^w$ have an inverse belonging to $S(\mu_{\theta+s}, g_{\theta+s})$.*

Choosing a finite number of points θ, each t can be written $\theta + s$ and we can choose $\|u\|_{H(\mu_t)} = \left\|(b_\theta \circ F_s^{-1})^w u\right\|_{L^2}$ in (4.2). Changing the points θ and the b_θ would replace the norm in $L^p(H(\mu_*))$ by an equivalent one.

We should verify, in the proof of [BC, th. 6.4], that we can choose $b_\theta \in S(\mu_\theta, g_\theta)$ and $c_\theta \in S(\mu_\theta^{-1}, g_\theta)$ whose seminorms are independent of θ such that $b_\theta \# c_\theta = 1$. We are thus reduced to prove the result for $\theta = 0$. With evident notations for c_s and C_s, we get

$$\tfrac{d}{ds} B_s C_s = i(B_s C_s A - A B_s C_s) + R_s$$

where R_s belongs to $\Psi(1, g_s)$ (with uniform seminorms). Setting $e_s = (b_s \# c_s) \circ F_s$, we get an equation $\frac{d}{ds} e_s = r'_s$ with a right-hand side bounded uniformly in $S(1, g_0)$. For s small, the seminorms of $(1 - e_s)$ in $S(1, g_0)$ and thus those of $(1 - b_s \# c_s)$ in $S(1, g_s)$ are small. As a consequence, $B_s C_s$ is invertible in $\mathcal{L}(L^2)$, its inverse belongs to $\Psi(1, g_s)$ (see Section 1.4), and B_s itself is invertible.

The functions of class C^∞ are dense in $L^1(H(\mu_*))$ and the dual of this space is $L^\infty(H(\mu_*^{-1}))$. The space $C(H(\mu_*))$ ("continuous" functions with values in a variable space!) is defined as the closure, in $L^\infty(H(\mu_*))$, of the set of continuous functions with value in $\mathcal{S}$.

Proposition 4.2 *Let μ_0 be a g_0-weight.*
(a) *There exists C such that for any $u \in C^1([-T, T], \mathcal{S})$ solution of the equation*

$$\tfrac{d}{dt} u_t + i A u_t = f_t, \tag{4.3}$$

one has

$$\|u\|_{L^\infty(H(\mu_*))} \le C \left(\|u_0\|_{H(\mu_0)} + \|f\|_{L^1(H(\mu_*))} \right). \tag{4.4}$$

(b) *There exist $\overline{\mu}_0 > \mu_0$ such that any solution u of* (4.3) *which belongs to $L^\infty(H(\overline{\mu}_*))$ belongs to $C(H(\mu_*))$ and satisfies the estimate* (4.4).

It suffices to prove the result on an interval of size δ centered at 0. Keeping the notations above, we have

$$\begin{aligned} \tfrac{d}{dt}(B_t u_t) &= i B_t A u_t - i A B_t u_t + R_t u_t - i B_t A u_t + B_t f_t \\ &= -i A (B_t u_t) + R_t u_t + B_t f_t \end{aligned}$$

and thus

$$\tfrac{d}{dt} \|u_t\|^2_{H(\mu_t)} = \tfrac{d}{dt} \|B_t u_t\|^2_{L^2} \le C \left(\|u_t\|^2_{H(\mu_t)} + \|u_t\|_{H(\mu_t)} \|f_t\|_{H(\mu_t)} \right)$$

which proves part (a) of the theorem for δ small.

To prove part (b), we need the following lemma, where μ_0 and $\overline{\mu}_0$ will be g_0-weights, and where $\mathcal{H}^N$ is the classical weighted Sobolev space $\{u|\, x^\alpha D^\beta u \in L^2\,;\, |\alpha{+}\beta| \le N\}$ for $N \ge 0$, and is the dual of $\mathcal{H}^{-N}$ for $N < 0$.

Lemma 4.1

$$\forall \mu_0,\ \exists N,\ \exists C,\ \forall t \in [-T,T], \qquad \|u\|_{H(\mu_t)} \le C\,\|u\|_{\mathcal{H}^N}\,.$$
$$\forall N,\ \exists \overline{\mu}_0,\ \exists C,\ \forall t \in [-T,T], \qquad \|u\|_{\mathcal{H}^N} \le C\,\|u\|_{H(\overline{\mu}_t)}\,.$$

For a fixed t, the first estimate says that pseudodifferential operators are bounded from $\mathcal{S}$ into L^2, while the second one is a consequence of the fact that any linear form on $\mathcal{X}$ belongs to a class of symbols for a convenient weight. One has just to make uniform these arguments.

Let us go back to part (b) of Proposition 4.2. We know (Proposition 1.1 (a)) that there exists a weight m_0 such that $a \in S(m_0, g_0)$. We have also $a \in S(m_t, g_t)$ because $a \circ F_t^{-1} = a$. We can choose $\overline{\mu}_0$ sufficiently large, such that $H(m_t\mu_t) \supset \mathcal{H}^N \supset H(\overline{\mu}_t/m_t)$. We know then that $du/dt \in L^\infty(\mathcal{H}^N)$ and u is continuous with values in $\mathcal{H}^N$. It is then possible to find a sequence $u^\nu \in C^1(\mathcal{S})$ such that $u_\nu \to u$ in $C^0(\mathcal{H}^N)$. The estimate (4.4) is valid for u_ν, we have $u^\nu(0) \to u(0)$ in $H(\mu_0)$ while $u^\nu \to u$ and $Au^\nu \to Au$ in $L^\infty(H(\mu_*))$, which ends the proof.

Theorem 4.1 *Let μ_0 be a g_0-weight, let u_0 and f belong to $H(\mu_0, g_0)$ and $L^1([0,T]; H(\mu_*))$, respectively. Then there exists a unique solution $u \in C([0,T]; H(\mu_*))$ of the Cauchy problem*

$$\frac{du(t)}{dt} + iAu(t) = f(t); \qquad u(0) = u_0.$$

We use a classical duality argument. Let $v \in \mathcal{S}(\mathbb{R}^{n+1})$ vanishing near $t = T$ and let $g = \frac{\partial v}{\partial t} + iAv$. From (4.4) (with the time going from T to 0), we know that one has

$$\|v\|_{L^\infty(H(\mu_*^{-1}))} \le C\,\|g\|_{L^1(H(\mu_*^{-1}))}$$

and thus that v is uniquely determined by g. The linear form $g \mapsto (u_0\,|\,v(0)) + \int_0^T (f(t)\,|\,v(t))\,dt$ is defined and continuous on the subspace of $L^1(H(\mu_*^{-1}))$ made of such g. From the Hahn–Banach theorem, we get the existence of $u \in L^\infty(H(\mu_*))$ such that

$$\forall v \in \mathcal{S},\ (u_0\,|\,v(0)) + \int_0^T (f(t)\,|\,v(t))\,dt = -\int_0^T \big(u(t)\,\big|\, \tfrac{\partial v}{\partial t} + iAv\big)\,dt. \qquad (4.5)$$

Using functions v vanishing also near $t = 0$, this proves that u, in the sense of distributions, is a solution of $\frac{\partial u}{\partial t} + iAu = f$ in $]0,T[\times\mathbb{R}^n$. Let us choose a weight $\underline{\mu}_0 \ll \mu_0$ such that part (b) of Proposition 4.2 applies to this couple of weights. One has $u \in C(H(\underline{\mu}_*))$ and $u(0)$ is now well defined. Integrating by parts in (4.5) we get that $u(0) = u_0$. The estimate (4.4), with μ replaced by $\underline{\mu}$, shows the uniqueness of u.

It remains to prove that $u \in C([0,T]; H(\mu_*))$. Let us introduce a weight $\overline{\mu}_0 \gg \mu_0$ such that part (b) of Proposition 4.2 applies. Let us approximate u_0 and f, in $H(\mu_0, g_0)$ and $L^1([0,T]; H(\mu_*))$, respectively, by regular functions u_0^ν and f^ν. From the analysis above, one gets solutions u^ν belonging to $L^\infty(H(\overline{\mu}_*))$ and thus to $C([0,T]; H(\mu_*))$. Using again (4.4), the sequence u^ν is a Cauchy sequence in $L^\infty(H(\mu_*))$, its limit u should belong to $C([0,T]; H(\mu_*))$ which ends the proof of Theorem 4.1.

Proof of Theorem 3.1 (end). Taking $\mu_0 = 1$, the last theorem shows that for any $u_0 \in L^2$, there exists a unique solution $t \mapsto u_t = P_t u_0$, continuous from $[-T, T]$ into L^2, of the equation $\frac{\partial u}{\partial t} + i a^w u = 0$. The group law and the relation $P_t^* = P_t^{-1}$ are valid in the interval in view of the uniqueness. One can thus extend P_t to $\mathbb{R}$ and get a strongly continuous group of unitary operators. Its infinitesimal generator will be denoted by $-i\mathcal{A}$, where $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$, is selfadjoint. Moreover, the P_t, for $|t| \leq T$, are continuous from $H(\mu_0)$ into $H(\mu_t)$.

We know that $\frac{1}{t}(u_0 - P_t u_0)$ converges always toward $-i a^w u_0$ in the sense of distributions, and this limit belongs thus to L^2 when $u_0 \in \mathcal{D}(\mathcal{A})$. This proves that

$$\mathcal{D}(\mathcal{A}) \subset \left\{ u_0 \in L^2 \middle| \, a^w u_0 \in L^2 \right\}. \tag{4.6}$$

Conversely, assume that u_0 and $a^w u_0$ belong to L^2. For $|t| \leq T$, one has

$$\tfrac{d}{dt} P_t u_0 = -i a^w P_t u_0 = -i P_t (a^w u_0).$$

The right-hand side is continuous from $[-T, T]$ into L^2, and $P_t u_0$ has a derivative in L^2. This proves that $\mathcal{D}(\mathcal{A})$ is exactly the right-hand side of (4.6). It is well known that a^w, with that domain, is the adjoint of the closure of a^w defined on $\mathcal{S}$. The selfadjointness of $\mathcal{A}$ shows that the weak and strong extensions coincide, which ends the proof of Theorem 3.1.

5 Proof of Theorem 3.2

We assume now that $a \in \hat{\hat{\hat{S}}}(\lambda_t^2, g_t)$ and we have to prove that the iterated twisted commutators of P_t are bounded on L^2.

Let $b_0 \in \hat{S}(\lambda_0, g_0)$, let $b_t = b_0 \circ F_t^{-1}$ and, using capital letters for the corresponding operators, set

$$K_t = P_{-t} \, \widetilde{\mathrm{ad}}(b) \cdot P_t = P_{-t} B_t P_t - B_0.$$

One has

$$\frac{d}{dt} K_t = P_{-t} \left\{ i A B_t - i B_t A + \{b_t \,,\, a\}^w \right\} P_t = P_{-t} R_t P_t.$$

Proposition 1.1 (c) shows that R_t belongs to $\Psi(1, g_t)$ (its seminorms being controlled) and is thus uniformly bounded on L^2. We get

$$\widetilde{\mathrm{ad}}(b) \cdot P_t = \int_0^t P_{t-s} R_s P_s \, ds \in \mathcal{L}(L^2).$$

By induction, it is possible to write the iterated twisted commutators as sums of terms of the following type:

$$\int \cdots \int_{0 \leq s_1 \leq \ldots \leq s_N} P_{t-s_N} R_{s_N} \ldots P_{s_2 - s_1} R_{s_1} P_{s_1} \, ds_1 \ldots ds_N \in \mathcal{L}(L^2),$$

which ends the proof.

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The Solvability and Subellipticity of Systems of Pseudodifferential Operators

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Dedicated to Ferruccio Colombini on his sixtieth birthday

Summary. The paper studies the local solvability and subellipticity for square systems of principal type. These are the systems for which the principal symbol vanishes of first order on its kernel. For systems of principal type having constant characteristics, local solvability is equivalent to condition (Ψ) on the eigenvalues. This is a condition on the sign changes of the imaginary part along the oriented bicharacteristics of the real part of the eigenvalue. In the generic case when the principal symbol does not have constant characteristics, condition (Ψ) is not sufficient and in general not well defined. Instead we study systems which are quasi-symmetrizable, these systems have natural invariance properties and are of principal type. We prove that quasi-symmetrizable systems are locally solvable. We also study the subellipticity of quasi-symmetrizable systems in the case when principal symbol vanishes of finite order along the bicharacteristics. In order to prove subellipticity, we assume that the principal symbol has the approximation property, which implies that there are no transversal bicharacteristics.

2000 AMS Subject Classification: Primary: 35S05, Secondary: 35A07, 35H20, 47G30, 58J40

Key words: solvability, subelliptic, pseudodifferential, principal type, system

1 Introduction

In this paper we shall study the question of solvability and subellipticity of square systems of classical pseudodifferential operators of principal type on a C^∞ manifold X. These are the pseudodifferential operators which have an asymptotic expansion in homogeneous terms, where the highest order term, the principal symbol, vanishes of first order on the kernel. Local solvability for

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_5,

an $N \times N$ system of pseudodifferential operators P at a compact set $K \subseteq X$ means that the equations

$$Pu = v \tag{1.1}$$

have a local weak solution $u \in \mathcal{D}'(X, \mathbf{C}^N)$ in a neighborhood of K for all $v \in C^\infty(X, \mathbf{C}^N)$ in a subset of finite codimension. We can also define microlocal solvability at any compactly based cone $K \subset T^*X$, see [5, Definition 26.4.3]. Hans Lewy's famous counterexample [6] from 1957 showed that not all smooth linear partial differential operators are solvable.

In the scalar case, Nirenberg and Treves conjectured in [7] that local solvability of scalar classical pseudodifferential operators of principal type is equivalent to condition (Ψ) on the principal symbol p. Condition (Ψ) means that

$$\begin{aligned}&\operatorname{Im}(ap) \text{ does not change sign from } - \text{ to } + \\ &\qquad\qquad \text{along the oriented bicharacteristics of } \operatorname{Re}(ap)\end{aligned} \tag{1.2}$$

for any $0 \neq a \in C^\infty(T^*X)$. These oriented bicharacteristics are the positive flow-outs of the Hamilton vector field

$$H_{\operatorname{Re}(ap)} = \sum_j \partial_{\xi_j} \operatorname{Re}(ap) \partial_{x_j} - \partial_{x_j} \operatorname{Re}(ap) \partial_{\xi_j}$$

on $\operatorname{Re}(ap) = 0$, and are called semibicharacteristics of p. The Nirenberg–Treves conjecture was recently proved by the author, see [2].

Condition (1.2) is obviously invariant under symplectic changes of coordinates and multiplication with nonvanishing factors. Thus the condition is invariant under conjugation of P with elliptic Fourier integral operators. We say that p satisfies condition $(\overline{\Psi})$ if $\overline{p}$ satisfies condition (Ψ), which means that only sign changes from $-$ to $+$ are allowed in (1.2). We also say that p satisfies condition (P) if there are no sign changes on the semibicharacteristics, that is, p satisfies both conditions (Ψ) and $(\overline{\Psi})$. For partial differential operators, condition (Ψ) and (P) are equivalent, since the principal symbol is either odd or even in ξ.

For systems there is no corresponding conjecture for solvability. We shall consider systems of principal type, so that the principal symbol vanishes of first order on the kernel, see Definition 2.1. By looking at diagonal operators, one finds that condition (Ψ) for the eigenvalues of the principal symbol is necessary for solvability. A special case is when we have constant characteristics, so that the eigenvalue close to the origin has constant multiplicity, see Definition 2.6. Then, the eigenvalue is a C^∞ function and condition (Ψ) is well-defined. For classical systems of pseudodifferential operators of principal type having eigenvalues of the principal symbol with constant multiplicity, the generalization of the Nirenberg–Treves conjecture is that local solvability is equivalent to condition (Ψ) on the eigenvalues. This has recently been proved by the author, see Theorem 2.7 in [4].

But when the principal symbol is not diagonalizable, condition (Ψ) is not sufficient for local solvability, see Example 2.7 below. In fact, it is not even known if condition (Ψ) is sufficient in the case when the principal system is C^∞ diagonalizable. Instead, we shall study the *quasi-symmetrizable* systems introduced in [3], see Definition 2.8. These are of principal type, are invariant under taking adjoints and multiplication with invertible systems. A scalar quasi-symmetrizable symbol is of principal type and satisfies condition (P). Our main result is that quasi-symmetrizable systems are locally solvable, see Theorem 2.17.

We shall also study the subellipticity of square systems. An $N \times N$ system of pseudodifferential operators $P \in \Psi^m_{cl}(X)$ is *subelliptic* with a loss of $\gamma < 1$ derivatives if $Pu \in H_{(s)}$ implies that $u \in H_{(s+m-\gamma)}$ locally for $u \in \mathcal{D}'(X, \mathbf{C}^N)$. Here $H_{(s)}$ are the standard L^2 Sobolev spaces, thus ellipticity corresponds to $\gamma = 0$ so we may assume $\gamma > 0$. For scalar operators, subellipticity is equivalent to condition $(\overline{\Psi})$ and the bracket condition on the principal symbol p, i.e., that some repeated Poisson bracket of $\operatorname{Re} p$ and $\operatorname{Im} p$ is nonvanishing. This is not true for systems, and there seems to be no general results on the subellipticity for systems of pseudodifferential operators. In fact, the real and imaginary parts do not commute in general, making the bracket condition meaningless. Even when they do, the bracket condition is not invariant and not sufficient for subellipticity, see Example 3.2.

Instead we shall study quasi-symmetrizable symbols, for which we introduce invariant conditions on the order of vanishing of the symbol along the semibicharacteristics of the eigenvalues. Observe that for systems, there could be several (limit) semibicharacteristics of the eigenvalues going through a characteristic point, see Example 3.10. Therefore we introduce the *approximation property* in Definition 3.11 which gives that the all (limit) semibicharacteristics of the eigenvalues are parallel at the characteristics, see Remark 3.12. We shall study systems of *finite type* introduced in [3], these are quasi-symmetrizable systems satisfying the approximation property, for which the imaginary part on the kernel vanishes of finite order along the bicharacteristics of the real part of the eigenvalues. This definition is invariant under multiplication with invertible systems and taking adjoints. For scalar symbols this corresponds to the case when the operator satisfies condition (P) and the bracket condition. For systems of finite type we obtain subellipticity with a loss of $2k/2k+1$ derivatives as in the scalar case, where $2k$ is the order of vanishing, see Theorem 3.21. For the proof, we shall use the estimates developed in [3]. The results in this paper are formulated for operators acting on the trivial bundle. But since our results are mainly local, they can be applied to operators on sections of fiber bundles.

2 Solvability of systems

Recall that a scalar symbol $p(x, \xi) \in C^\infty(T^*X)$ is of *principal type* if $dp \neq 0$ when $p = 0$. We shall generalize this definition to systems $P \in C^\infty(T^*X)$. For $\nu \in T_w(T^*X)$, $w = (x, \xi)$, we let $\partial_\nu P(w) = \langle \nu, dP(w) \rangle$. We shall denote $\operatorname{Ker} P$ the kernel and $\operatorname{Ran} P$ the range of the matrix P.

Definition 2.1 *The $N \times N$ system $P(w) \in C^\infty(T^*X)$ is of* principal type *at w_0 if*

$$\operatorname{Ker} P(w_0) \ni u \mapsto \partial_\nu P(w_0)u \in \operatorname{Coker} P(w_0) = \mathbf{C}^N / \operatorname{Ran} P(w_0) \qquad (2.1)$$

is bijective for some $\nu \in T_{w_0}(T^*X)$. The operator $P \in \Psi^m_{cl}(X)$ is of principal type if the homogeneous principal symbol $\sigma(P)$ is of principal type.

Observe that if P is homogeneous in ξ, then the direction ν cannot be radial. In fact, if ν has the radial direction and P is homogeneous, then $\partial_\nu P = cP$ which vanishes on $\operatorname{Ker} P$.

Remark 2.2 *If $P(w) \in C^\infty$ is of principal type and $A(w)$, $B(w) \in C^\infty$ are invertible, then APB is of principal type. We have that P is of principal type if and only if the adjoint P^* is of principal type.*

In fact, by Leibniz's rule we have

$$\partial(APB) = (\partial A)PB + A(\partial P)B + AP\partial B \qquad (2.2)$$

and $\operatorname{Ran}(APB) = A(\operatorname{Ran} P)$ and $\operatorname{Ker}(APB) = B^{-1}(\operatorname{Ker} P)$ when A and B are invertible, which gives invariance under left and right multiplication. Since $\operatorname{Ker} P^*(w_0) = \operatorname{Ran} P(w_0)^\perp$ we find that P satisfies (2.1) if and only if

$$\operatorname{Ker} P(w_0) \times \operatorname{Ker} P^*(w_0) \ni (u, v) \mapsto \langle \partial_\nu P(w_0)u, v\rangle \qquad (2.3)$$

is a nondegenerate bilinear form. Since $\langle \partial_\nu P^* v, u\rangle = \overline{\langle \partial_\nu P u, v\rangle}$ we then obtain that P^* is of principal type.

Observe that if P only has one vanishing eigenvalue λ (with multiplicity one), then the condition that P is of principal type reduces to the condition in the scalar case: $d\lambda \neq 0$ when $\lambda = 0$. In fact, by using the spectral projection one can find invertible systems A and B so that

$$APB = \begin{pmatrix} \lambda & 0 \\ 0 & E \end{pmatrix} \in C^\infty$$

where E is an invertible $(N-1) \times (N-1)$ system. Since this system is of principal type we obtain the result by the invariance.

Example 2.3 *Consider the system*

$$P(w) = \begin{pmatrix} \lambda_1(w) & 1 \\ 0 & \lambda_2(w) \end{pmatrix}$$

where $\lambda_j(w) \in C^\infty$, $j = 1, 2$. Then $P(w)$ is not of principal type when $\lambda_1(w) = \lambda_2(w) = 0$ since then $\operatorname{Ker} P(w) = \operatorname{Ran} P(w) = \mathbf{C} \times \{0\}$, which is preserved by ∂P.

Observe that the property of being of principal type is not stable under C^1 perturbation, not even when $P = P^*$ is symmetric by the following example.

Example 2.4 *The system*

$$P(w) = \begin{pmatrix} w_1 - w_2 & w_2 \\ w_2 & -w_1 - w_2 \end{pmatrix} = P^*(w) \qquad w = (w_1, w_2)$$

is of principal type when $w_1 = w_2 = 0$*, but* not *of principal type when* $w_2 \neq 0$ *and* $w_1 = 0$*. In fact,*

$$\partial_{w_1} P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is invertible, and when $w_2 \neq 0$ *we have that*

$$\operatorname{Ker} P(0, w_2) = \operatorname{Ker} \partial_{w_2} P(0, w_2) = \{ z(1,1) : z \in \mathbf{C} \}$$

which is mapped to $\operatorname{Ran} P(0, w_2) = \{ z(1,-1) : z \in \mathbf{C} \}$ *by* $\partial_{w_1} P$*. The eigenvalues of* $P(w)$ *are* $-w_2 \pm \sqrt{w_1^2 + w_2^2}$ *which are equal if and only if* $w_1 = w_2 = 0$*. When* $w_2 \neq 0$ *the eigenvalue close to zero is* $w_1^2/2w_2 + \mathcal{O}(w_1^4)$ *which has vanishing differential at* $w_1 = 0$*.*

Recall that the multiplicity of λ as a root of the characteristic equation $|P(w) - \lambda \operatorname{Id}_N| = 0$ is the *algebraic* multiplicity of the eigenvalue, and the dimension of $\operatorname{Ker}(P(w) - \lambda \operatorname{Id}_N)$ is the *geometric* multiplicity. Observe the geometric multiplicity is lower or equal to the algebraic, and for symmetric systems they are equal.

Remark 2.5 *If the eigenvalue* $\lambda(w)$ *has constant* algebraic *multiplicity, then it is a* C^∞ *function.*

In fact, if k is the multiplicity, then $\lambda = \lambda(w)$ solves $\partial_\lambda^{k-1} |P(w) - \lambda \operatorname{Id}_N| = 0$ so we obtain this from the Implicit Function Theorem. This is not true when we have constant geometric multiplicity, for example $P(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$, $t \in \mathbf{R}$, has geometric multiplicity equal to one for the eigenvalues $\pm\sqrt{t}$.

Definition 2.6 *The* $N \times N$ *system* $P(w) \in C^\infty$ *has* constant characteristics *near* w_0 *if there exists an* $\varepsilon > 0$ *such that an eigenvalue* $\lambda(w)$ *of* $P(w)$ *with* $|\lambda(w)| < \varepsilon$ *has both constant algebraic and constant geometric multiplicity in a neighborhood of* w_0*.*

If P has constant characteristics, then the eigenvalue close to zero has constant algebraic multiplicity, thus it is a C^∞ function close to zero. We obtain from Proposition 2.10 in [4] that if $P(w) \in C^\infty$ is an $N \times N$ system of constant characteristics near w_0, then $P(w)$ is of principal type at w_0 if and only if the algebraic and geometric multiplicities of P agree at w_0 and $d\lambda(w_0) \neq 0$ for the C^∞ eigenvalues for P at w_0 satisfying $\lambda(w_0) = 0$, thus there are no nontrivial Jordan boxes in the normal form.

For classical systems of pseudodifferential operators of principal type and constant characteristics, the eigenvalues are homogeneous C^∞ functions when the values are close to zero, so the condition (Ψ) given by (1.2) is well-defined on the eigenvalues. Then, the natural generalization of the Nirenberg–Treves conjecture is that local solvability is equivalent to condition (Ψ) on the eigenvalues. This has recently been proved by the author, see Theorem 2.7 in [4].

When the multiplicity of the eigenvalues of the principal symbol is not constant the situation is much more complicated. The following example shows that then it is not sufficient to have conditions only on the eigenvalues in order to obtain solvability, not even in the principal type case.

Example 2.7 *Let $x \in \mathbf{R}^2$, $D_x = \frac{1}{i}\partial_x$ and*

$$P(x, D_x) = \begin{pmatrix} D_{x_1} & x_1 D_{x_2} \\ x_1 D_{x_2} & -D_{x_1} \end{pmatrix} = P^*(x, D_x).$$

This system is symmetric of principal type and $\sigma(P)$ has real eigenvalues $\pm\sqrt{\xi_1^2 + x_1^2\xi_2^2}$ but

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} P \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} D_{x_1} - ix_1 D_{x_2} & 0 \\ 0 & D_{x_1} + ix_1 D_{x_2} \end{pmatrix}$$

which is not solvable at $(0,0)$ because condition (Ψ) is not satisfied. The eigenvalues of the principal symbol are now $\xi_1 \pm ix_1\xi_2$.

Of course, the problem is that the eigenvalues are not invariant under multiplication with elliptic systems. We shall instead study *quasi-symmetrizable* systems, which generalize the normal forms of the scalar symbol at the boundary of the numerical range of the principal symbol, see Example 2.9.

Definition 2.8 *The $N \times N$ system $P(w) \in C^\infty(T^*X)$ is* quasi-symmetrizable *with respect to a real C^∞ vector field V in $\Omega \subseteq T^*X$ if $\exists$ $N \times N$ system $M(w) \in C^\infty(T^*X)$ so that*

$$\operatorname{Re}\langle M(VP)u, u\rangle \geq c\|u\|^2 - C\|Pu\|^2 \qquad c > 0 \qquad \forall\, u \in \mathbf{C}^N, \tag{2.4}$$

$$\operatorname{Im}\langle MPu, u\rangle \geq -C\|Pu\|^2 \qquad \forall\, u \in \mathbf{C}^N \tag{2.5}$$

on Ω, the system M is called a symmetrizer *for P. If $P \in \Psi^m_{cl}(X)$, then it is quasi-symmetrizable if the homogeneous principal symbol $\sigma(P)$ is quasi-symmetrizable when $|\xi| = 1$, one can then choose a homogeneous symmetrizer M.*

The definition is clearly independent of the choice of coordinates in T^*X and choice of basis in $\mathbf{C}^N$. When P is elliptic, we find that P is quasi-symmetrizable with respect to any vector field since $\|Pu\| \cong \|u\|$. Observe that the set of symmetrizers M satisfying (2.4)–(2.5) is a convex cone, a sum of two multipliers is also a multiplier. Thus for a given vector field V it suffices to make a local choice of symmetrizer and then use a partition of unity to get a global one.

Example 2.9 *A scalar function* $p \in C^\infty$ *is quasi-symmetrizable if and only*

$$p(w) = e(w)(w_1 + if(w')) \qquad w = (w_1, w') \tag{2.6}$$

for some choice of coordinates, where $f \geq 0$*. Then* 0 *is at the boundary of the numerical range of* p*.*

In fact, it is obvious that p in (2.6) is quasi-symmetrizable. On the other hand, if p is quasi-symmetrizable, then there exists $m \in C^\infty$ such that $mp = p_1 + ip_2$ where p_j are real satisfying $\partial_\nu p_1 > 0$ and $p_2 \geq 0$. Thus 0 is at the boundary of the numerical range of p. By using Malgrange preparation theorem and changing coordinates as in the proof of Lemma 4.1 in [1], we obtain the normal form (2.6) with $\pm f \geq 0$.

Taylor has studied *symmetrizable* systems of the type $D_t \operatorname{Id} + iK$, for which there exists $R > 0$ making RK symmetric (see Definition 4.3.2 in [8]). These systems are quasi-symmetrizable with respect to ∂_τ with symmetrizer R. We shall denote $\operatorname{Re} A = \frac{1}{2}(A + A^*)$ and $i\operatorname{Im} A = \frac{1}{2}(A - A^*)$ the symmetric and antisymmetric parts of the matrix A. Next, we recall the following result from Proposition 4.7 in [3].

Remark 2.10 *If the* $N \times N$ *system* $P(w) \in C^\infty$ *is quasi-symmetrizable, then it is of principal type. Also, the symmetrizer* M *is invertible if* $\operatorname{Im} MP \geq cP^*P$ *for some* $c > 0$*.*

Observe that by adding $i\varrho P^*$ to M we may assume that $Q = MP$ satisfies

$$\operatorname{Im} Q \geq (\varrho - C)P^*P \geq P^*P \geq cQ^*Q \qquad c > 0 \tag{2.7}$$

for $\varrho \geq C + 1$, and then the symmetrizer is invertible by Remark 2.10.

Remark 2.11 *The system* $P \in C^\infty$ *is quasi-symmetrizable with respect to* V *if and only if there exists an invertible symmetrizer* M *such that* $Q = MP$ *satisfies*

$$\operatorname{Re} \langle (VQ)u, u\rangle \geq c\|u\|^2 - C\|Qu\|^2 \qquad c > 0, \tag{2.8}$$
$$\operatorname{Im} \langle Qu, u\rangle \geq 0 \tag{2.9}$$

for any $u \in \mathbf{C}^N$*.*

In fact, by the Cauchy–Schwarz inequality we find

$$|\langle (VM)Pu, u\rangle| \leq \varepsilon\|u\|^2 + C_\varepsilon\|Pu\|^2 \qquad \forall\, \varepsilon > 0 \quad \forall\, u \in \mathbf{C}^N.$$

Since M is invertible, we also have that $\|Pu\| \cong \|Qu\|$.

Definition 2.12 *If* $Q \in C^\infty(T^*X)$ *satisfies* (2.8)–(2.9)*, then* Q *is* quasi-symmetric *with respect to the real* C^∞ *vector field* V*.*

The invariance properties of quasi-symmetrizable systems are partly due to the following properties of semibounded matrices. Let $U+V = \{ u+v : u \in U \wedge v \in V \}$ for linear subspaces U and V of $\mathbf{C}^N$.

Lemma 2.13 *Assume that* Q *is an* $N \times N$ *matrix such that* $\operatorname{Im} zQ \geq 0$ *for some* $0 \neq z \in \mathbf{C}$. *Then we find*

$$\operatorname{Ker} Q = \operatorname{Ker} Q^* = \operatorname{Ker}(\operatorname{Re} Q) \bigcap \operatorname{Ker}(\operatorname{Im} Q) \tag{2.10}$$

and $\operatorname{Ran} Q = \operatorname{Ran}(\operatorname{Re} Q) + \operatorname{Ran}(\operatorname{Im} Q) \perp \operatorname{Ker} Q$.

Proof By multiplying with z we may assume that $\operatorname{Im} Q \geq 0$, clearly the conclusions are invariant under multiplication with complex numbers. If $u \in \operatorname{Ker} Q$, then we have $\langle \operatorname{Im} Qu, u\rangle = \operatorname{Im} \langle Qu, u\rangle = 0$. By using the Cauchy–Schwarz inequality on $\operatorname{Im} Q \geq 0$ we find that $\langle \operatorname{Im} Qu, v\rangle = 0$ for any v. Thus $u \in \operatorname{Ker}(\operatorname{Im} Q)$ so $\operatorname{Ker} Q \subseteq \operatorname{Ker} Q^*$. We get equality and (2.10) by the rank theorem, since $\operatorname{Ker} Q^* = \operatorname{Ran} Q^{\perp}$.

For the last statement we observe that $\operatorname{Ran} Q \subseteq \operatorname{Ran}(\operatorname{Re} Q) + \operatorname{Ran}(\operatorname{Im} Q) = (\operatorname{Ker} Q)^{\perp}$ by (2.10) where we also get equality by the rank theorem.

Proposition 2.14 *If* $Q \in C^\infty(T^*X)$ *is quasi-symmetric and* $E \in C^\infty(T^*X)$ *is invertible, then* E^*QE *and* $-Q^*$ *are quasi-symmetric.*

Proof First we note that (2.8) holds if and only if

$$\operatorname{Re} \langle (VQ)u, u\rangle \geq c\|u\|^2 \qquad \forall\, u \in \operatorname{Ker} Q \tag{2.11}$$

for some $c > 0$. In fact, Q^*Q has a positive lower bound on the orthogonal complement $\operatorname{Ker} Q^{\perp}$ so that

$$\|u\| \leq C\|Qu\| \qquad \text{for } u \in \operatorname{Ker} Q^{\perp}.$$

Thus, if $u = u' + u''$ with $u' \in \operatorname{Ker} Q$ and $u'' \in \operatorname{Ker} Q^{\perp}$, we find that $Qu = Qu''$,

$$\operatorname{Re} \langle (VQ)u', u''\rangle \geq -\varepsilon\|u'\|^2 - C_\varepsilon\|u''\|^2 \geq -\varepsilon\|u'\|^2 - C'_\varepsilon\|Qu\|^2 \qquad \forall\, \varepsilon > 0$$

and $\operatorname{Re} \langle (VQ)u'', u''\rangle \geq -C\|u''\|^2 \geq -C'\|Qu\|^2$. By choosing ε small enough we obtain (2.8) by using (2.11) on u'.

Next, we note that $\operatorname{Im} Q^* = -\operatorname{Im} Q$ and $\operatorname{Re} Q^* = \operatorname{Re} Q$, so $-Q^*$ satisfies (2.9) and (2.11) with V replaced by $-V$, and thus it is quasi-symmetric. Finally, we shall show that $Q_E = E^*QE$ is quasi-symmetric when E is invertible. We obtain from (2.9) that

$$\operatorname{Im} \langle Q_E u, u\rangle = \operatorname{Im} \langle QEu, Eu\rangle \geq 0 \qquad \forall\, u \in \mathbf{C}^N.$$

Next, we shall show that Q_E satisfies (2.11) on $\operatorname{Ker} Q_E = E^{-1} \operatorname{Ker} Q$, which will give (2.8). We find from Leibniz's rule that $VQ_E = (VE^*)QE + E^*(VQ)E + E^*Q(VE)$ where (2.11) gives

$$\operatorname{Re}\langle E^*(VQ)Eu, u\rangle \ge c\|Eu\|^2 \ge c'\|u\|^2 \qquad u \in \operatorname{Ker} Q_E \quad c' > 0$$

since then $Eu \in \operatorname{Ker} Q$. Similarly we obtain that $\langle (VE^*)QEu, u\rangle = 0$ when $u \in \operatorname{Ker} Q_E$. Now since $\operatorname{Im} Q_E \ge 0$ we find from Lemma 2.13 that

$$\operatorname{Ker} Q_E^* = \operatorname{Ker} Q_E \tag{2.12}$$

which gives $\langle E^*Q(VE)u, u\rangle = \langle E^{-1}(VE)u, Q_E^*u\rangle = 0$ when $u \in \operatorname{Ker} Q_E = \operatorname{Ker} Q_E^*$. Thus Q_E satisfies (2.11) so it is quasi-symmetric, which finishes the proof.

Proposition 2.15 *Let $P(w) \in C^\infty(T^*X)$ be a quasi-symmetrizable $N \times N$ system, then P^* is quasi-symmetrizable. If $A(w)$ and $B(w) \in C^\infty(T^*X)$ are invertible $N \times N$ systems, then BPA is quasi-symmetrizable.*

Proof Clearly (2.8)–(2.9) are invariant under *left* multiplication of P with invertible systems E, just replace M with ME^{-1}. Since we may write $BPA = B(A^*)^{-1}A^*PA$ it suffices to show that E^*PE is quasi-symmetrizable if E is invertible. By Remark 2.11 there exists a symmetrizer M so that $Q = MP$ is quasi-symmetric, i.e., satisfies (2.8)–(2.9). It then follows from Proposition 2.14 that

$$Q_E = E^*QE = E^*M(E^*)^{-1}E^*PE$$

is quasi-symmetric, thus E^*PE is quasi-symmetrizable.

Finally, we shall prove that P^* is quasi-symmetrizable if P is. Since $Q = MP$ is quasi-symmetric, we find from Proposition 2.14 that $Q^* = P^*M^*$ is quasi-symmetric. By multiplying with $(M^*)^{-1}$ from the right, we find from the first part of the proof that P^* is quasi-symmetrizable.

For scalar symbols of principal type, we find from the normal form in Example 2.9 that 0 is on the boundary of the local numerical range of the principal symbol. This need not be the case for systems by the following example.

Example 2.16 *Let*

$$P(w) = \begin{pmatrix} w_2 + iw_3 & w_1 \\ w_1 & w_2 - iw_3 \end{pmatrix}$$

which is quasi-symmetrizable with respect to ∂_{w_1} with symmetrizer $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. In fact, $\partial_{w_1} MP = \mathrm{Id}_2$ and

$$MP(w) = \begin{pmatrix} w_1 & w_2 - iw_3 \\ w_2 + iw_3 & w_1 \end{pmatrix} = (MP(w))^*$$

so $\operatorname{Im} MP \equiv 0$. Since eigenvalues of $P(w)$ are $w_2 \pm \sqrt{w_1^2 - w_3^2}$ we find that 0 is not a boundary point of the local numerical range of the eigenvalues.

For quasi-symmetrizable systems we have the following semiglobal solvability result.

Theorem 2.17 *Assume that $P \in \Psi^m_{cl}(X)$ is an $N \times N$ system and that there exists a real-valued function $T(w) \in C^\infty(T^*X)$ such that P is quasi-symmetrizable with respect to the Hamilton vector field $H_T(w)$ in a neighborhood of a compactly based cone $K \subset T^*X$. Then P is locally solvable at K.*

The cone $K \subset T^*X$ is compactly based if $K \bigcap \{\, (x, \xi) : \ |\xi| = 1 \,\}$ is compact. We also get the following local result:

Corollary 2.18 *Let $P \in \Psi^m_{cl}(X)$ be an $N \times N$ system that is quasi-symmetrizable at $w_0 \in T^*X$. Then P is locally solvable at w_0.*

This follows since we can always choose a function T such that $V = H_T$ at w_0. Recall that a semibicharacteristic of $\lambda \in C^\infty$ is a bicharacteristic of $\operatorname{Re}(a\lambda)$ for some $0 \neq a \in C^\infty$.

Remark 2.19 *If Q is quasi-symmetric with respect to H_T, then the limit set at the characteristics of the nontrivial semibicharacteristics of the eigenvalues close to zero of Q is a union of curves on which T is strictly monotone, thus they cannot form closed orbits.*

In fact, we have that an eigenvalue $\lambda(w)$ is C^∞ almost everywhere. The Hamilton vector field $H_{\operatorname{Re} z\lambda}$ then gives the semibicharacteristics of λ, and that is determined by $\langle dQu, u\rangle$ with $0 \neq u \in \operatorname{Ker}(P - \lambda \operatorname{Id}_N)$ by the invariance property given by (2.2). Now $\operatorname{Re}\langle (H_T Q)u, u\rangle > 0$ and $\operatorname{Im} d\langle Qu, u\rangle = 0$ for $u \in \operatorname{Ker} P$ by (2.8)–(2.9). Thus by picking subsequences when $\lambda \to 0$ we find that the limits of nontrivial semibicharacteristics of the eigenvalues close to zero give curves on which T is strictly monotone, since $H_T \lambda \neq 0$.

Example 2.20 *Let*

$$P(t, x; \tau, \xi) = \tau M(t, x, \xi) + iF(t, x, \xi) \in S^1_{cl}$$

where $M \geq c_0 > 0$ and $F \geq 0$. Then P is quasi-symmetrizable with respect to ∂_τ with symmetrizer Id_N, so Theorem 2.17 gives that $P(t, x, D_t, D_x)$ is locally solvable.

Proof (Proof of Theorem 2.17) We shall modify the proof of Theorem 4.15 in [3], and derive estimates for the L^2 adjoint P^* which will give solvability. By Proposition 2.15 we find that P^* is quasi-symmetrizable in K. By the invariance of the conditions, we may multiply with an elliptic scalar operator to obtain that $P^* \in \Psi^1_{cl}$. By the assumptions, Definition 2.8 and (2.7), we find that there exists a real valued function $T(w) \in C^\infty$ and a symmetrizer $M(w) \in C^\infty$ so that $Q = MP^*$ satisfies

$$\operatorname{Re} H_T Q \geq c - C_0 Q^* Q \geq c - C_1 \operatorname{Im} Q, \tag{2.13}$$

$$\operatorname{Im} Q \geq c\, Q^* Q \geq 0 \tag{2.14}$$

when $|\xi| = 1$ near K for some $c > 0$, and we find that M is invertible by Remark 2.10. Extending by homogeneity, we may assume that M and T are homogeneous of degree 0 in ξ, then $T \in S^0_{1,0}$ and $Q \in S^1_{1,0}$. Let

$$M(x,D)P^*(x,D) = Q(x,D) \in \Psi^1_{cl} \tag{2.15}$$

which has principal symbol $Q(x,\xi)$. Leibniz's rule gives that $\exp(\pm\gamma T) \in S^0_{1,0}$ for any $\gamma > 0$, so we can define

$$Q_\gamma(x,D) = \exp(-\gamma T)(x,D)Q(x,D)\exp(\gamma T)(x,D) \in \Psi^1_{cl}.$$

Since T is a scalar function, we obtain that the symbol of

$$\operatorname{Im} Q_\gamma = Q_1 + \gamma Q_0 \qquad \text{modulo } S^{-1} \text{ near } K \tag{2.16}$$

where $0 \le Q_1 = \operatorname{Im} Q \in S^1$ and $Q_0 \in S^0$ satisfies

$$Q_0 = \operatorname{Re} H_T Q \ge c - C|\xi|^{-1}Q_1 \qquad \text{near } K \tag{2.17}$$

by (2.13), (2.14) and homogeneity.

Now take $0 \le \varphi \in S^0_{1,0}$ such that $\varphi = 1$ near K and φ is supported where (2.13) and (2.14) hold. If $\chi = \varphi^2$, then we obtain from (2.17) and the sharp Gårding inequality [5, Theorem 18.6.14] that

$$Q_0(x,D) \ge c_0\chi(x,D) - C\langle D\rangle^{-1}Q_1(x,D) + R(x,D) + S(x,D)$$

where $c_0 > 0$, $R \in S^{-1}$ and $S \in S^0$ with $\operatorname{supp} S \bigcap K = \emptyset$. Thus we obtain

$$\operatorname{Im} Q_\gamma(x,D) \ge c_0\gamma\chi(x,D) + (1+\varrho_\gamma)Q_1(x,D) + R_\gamma(x,D) + S_\gamma(x,D) \tag{2.18}$$

where $R_\gamma \in S^{-1}$, $\varrho_\gamma = -\gamma C\langle D\rangle^{-1} \in \Psi^{-1}$ and $S_\gamma \in S^0$ with $\operatorname{supp} S_\gamma \bigcap K = \emptyset$. The calculus gives that $\chi(x,D) \cong \varphi(x,D)\varphi(x,D)$ modulo Ψ^{-1} and

$$(1+\varrho_\gamma)Q_1(x,D) = (1+\varrho_\gamma/2)Q_1(x,D)(1+\varrho_\gamma/2) \qquad \text{modulo } \Psi^{-1}$$

By using the sharp Gårding inequality we obtain that $Q_1(x,D) \ge R_0(x,D)$ for some $R_0 \in S^0_{1,0}$. Thus we find

$$(1+\varrho_\gamma)Q_1(x,D) \ge (1+\varrho_\gamma/2)R_0(x,D)(1+\varrho_\gamma/2) = R_0(x,D) \ge -C_0$$

modulo terms in Ψ^{-1} (depending on γ). Combining this with (2.18) and using that $\operatorname{supp}(1-\varphi)\bigcap K = \emptyset$, we find for large enough γ that

$$\begin{aligned} &c_1\gamma\|\varphi(x,D)u\|^2 \\ &\le \operatorname{Im}\langle Q_\gamma(x,D)u,u\rangle + \langle A_\gamma(x,D)u,u\rangle + \langle B_\gamma(x,D)u,u\rangle \qquad u \in C_0^\infty \end{aligned} \tag{2.19}$$

where $c_1 > 0$, $A_\gamma \in S^{-1}$ and $B_\gamma \in S^0$ with $\operatorname{supp} B_\gamma \bigcap K = \emptyset$. Next, we fix γ and apply this to $\exp(-\gamma T)(x,D)u$. We find by the calculus that

$$\|\varphi(x,D)u\| \le C(\|\varphi(x,D)\exp(-\gamma T)(x,D)u\| + \|u\|_{(-1)}) \qquad u \in C_0^\infty.$$

We also obtain from the calculus that

$$\exp(\gamma T)(x,D)\exp(-\gamma T)(x,D) = 1 + r(x,D)$$

with $r \in S^{-1}$, which gives

$$Q_\gamma(x,D)\exp(-\gamma T)(x,D) = \exp(-\gamma T)(x,D)(1+r(x,D))Q(x,D) \\ + \exp(-\gamma T)(x,D)[Q(x,D), r(x,D)]$$

where $[Q(x,D), r(x,D)] \in \Psi^{-1}$. Since $Q(x,D) = M(x,D)P^*(x,D)$ we find

$$|\langle \exp(-\gamma T)(x,D)(1+r(x,D))Q(x,D)v, \exp(-\gamma T)(x,D)u\rangle| \\ \le C\|P^*(x,D)u\|\|u\|.$$

Since $\|u\| \le \|\varphi(x,D)u\| + \|(1-\varphi(x,D))u\|$ and $\varphi = 1$ near K we obtain that

$$\|u\| \le C\left(\|P^*(x,D)u\| + \|Q(x,D)u\| + \|u\|_{(-1)}\right) \qquad u \in C_0^\infty$$

where $Q \in S^0$ with $\operatorname{supp} Q \bigcap K = \emptyset$. We then obtain the local solvability by standard arguments.

3 Subellipticity of systems

We shall consider the question when a quasi-symmetrizable system is subelliptic. Recall that an $N \times N$ system of operators $P \in \Psi^m_{cl}(X)$ is *(micro)subelliptic* with a loss of $\gamma < 1$ derivatives at w_0 if

$$Pu \in H_{(s)} \text{ at } w_0 \implies u \in H_{(s+m-\gamma)} \text{ at } w_0$$

for $u \in \mathcal{D}'(X, \mathbf{C}^N)$. Here $H_{(s)}$ is the standard Sobolev space of distributions u such that $\langle D\rangle^s u \in L^2$. We say that $u \in H_{(s)}$ microlocally at w_0 if there exists $a \in S^0_{1,0}$ such that $a \ne 0$ in a conical neighborhood of w_0 and $a(x,D)u \in H_{(s)}$. Of course, ellipticity corresponds to $\gamma = 0$ so we shall assume $\gamma > 0$.

Example 3.1 *Consider the scalar operator*

$$D_t + if(t,x,D_x)$$

with $0 \le f \in C^\infty(\mathbf{R}, S^1_{cl})$, $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, *then we obtain from Proposition* 27.3.1 *in* [5] *that this operator is subelliptic with a loss of* $k/k+1$ *derivatives microlocally near* $\{\, \tau = 0 \,\}$ *if and only if*

$$\sum_{j \le k} |\partial_t^j f(t,x,\xi)| \ne 0 \qquad \forall\, x\ \xi \tag{3.1}$$

where we can choose k *even.*

The following example shows that condition (3.1) is not sufficient for systems.

Example 3.2 *Let* $P = D_t \operatorname{Id}_2 + iF(t)|D_x|$ *where*

$$F(t) = \begin{pmatrix} t^2 & t^3 \\ t^3 & t^4 \end{pmatrix} \ge 0.$$

Then we have $F^{(3)}(0) = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$ *which gives that*

$$\bigcap_{j\le 3} \operatorname{Ker} F^{(j)}(0) = \{\, 0 \,\}\,. \tag{3.2}$$

But

$$F(t) = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}$$

so we find

$$P = (1+t^2)^{-1} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} \begin{pmatrix} D_t + i(t^2+t^4)|D_x| & 0 \\ 0 & D_t \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \qquad \text{modulo } \Psi^0$$

which is not subelliptic near $\{\, \tau = 0 \,\}$*, since* D_t *is not by Example* 3.1*.*

Example 3.3 *Let* $P = hD_t \operatorname{Id}_2 + iF(t)|D_x|$ *where*

$$F(t) = \begin{pmatrix} t^2 + t^8 & t^3 - t^7 \\ t^3 - t^7 & t^4 + t^6 \end{pmatrix} = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} \begin{pmatrix} t^2 & 0 \\ 0 & t^6 \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}.$$

Then we have

$$P = (1+t^2)^{-1} \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} \begin{pmatrix} D_t + i(t^2+t^4)|D_x| & 0 \\ 0 & D_t + i(t^6+t^8)|D_x| \end{pmatrix} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}$$

modulo Ψ^0*, which is subelliptic near* $\{\, \tau = 0 \,\}$ *with a loss of* $6/7$ *derivatives by Example* 3.1*. This operator is, element for element, a higher order perturbation of the operator of Example* 3.2*.*

The problem is that condition (3.2) in *not* invariant in the systems case. Instead, we shall consider the following invariant generalization of (3.1).

Definition 3.4 *Let* $0 \le F(t) \in L^\infty_{loc}(\mathbf{R})$ *be an* $N \times N$ *system, then we define*

$$\Omega_\delta(F) = \left\{\, t : \min_{\|u\|=1} \langle F(t)u, u\rangle \le \delta \,\right\} \qquad \delta > 0 \tag{3.3}$$

which is well-defined almost everywhere and contains $|F|^{-1}(0)$*.*

Observe that one may also use this definition in the scalar case, then $\Omega_\delta(f) = f^{-1}([0,\delta])$ for nonnegative functions f.

Remark 3.5 *Observe that if $F \geq 0$ and E is invertible, then we find that*

$$\Omega_\delta(E^*FE) \subseteq \Omega_{C\delta}(F) \tag{3.4}$$

where $C = \|E^{-1}\|^2$.

Example 3.6 *For the matrix $F(t)$ in Example 3.3 we find that $|\Omega_\delta(F)| \leq C\delta^{1/6}$ for $0 < \delta \leq 1$, and for the matrix in Example 3.2 we find that $|\Omega_\delta(F)| = \infty$, $\forall\,\delta$.*

We also have examples when the semidefinite imaginary part vanishes of infinite order.

Example 3.7 *Let $0 \leq f(t,x) \leq Ce^{-1/|t|^\sigma}$, $\sigma > 0$, then we obtain that*

$$|\Omega_\delta(f_x)| \leq C_0|\log\delta|^{-1/\sigma} \qquad \forall\,\delta > 0 \quad \forall\,x$$

where $f_x(t) = f(t,x)$. (We owe this example to Y. Morimoto.)

We shall study systems where the imaginary part F vanishes of finite order, so that $|\Omega_\delta(F)| \leq C\delta^\mu$ for $\mu > 0$. In general, the largest exponent could be any $\mu > 0$, for example when $F(t) = |t|^{1/\mu}\,\mathrm{Id}_N$. But for C^∞ systems the best exponent is $\mu = 1/k$ for an even k, by the following result, which is Proposition A.2 in [3].

Remark 3.8 *Assume that $0 \leq F(t) \in C^\infty(\mathbf{R})$ is an $N \times N$ system such that $F(t) \geq c > 0$ when $|t| \gg 1$. Then we find that*

$$|\Omega_\delta(F)| \leq C\delta^\mu \qquad 0 < \delta \leq 1$$

if and only if $\mu \leq 1/k$ for an even $k \geq 0$ so that

$$\sum_{j \leq k} |\partial_t^j \langle F(t)u(t), u(t)\rangle| / \|u(t)\|^2 > 0 \qquad \forall\,t \tag{3.5}$$

for any $0 \neq u(t) \in C^\infty(\mathbf{R})$.

Example 3.9 *For the scalar symbols $\tau + if(t,x,\xi)$ in Example 3.1 we find from Remark 3.8 that (3.1) is equivalent to*

$$|\,\{\,t : f(t,x,\xi) \leq \delta\,\}\,| = |\Omega_\delta(f_{x,\xi})| \leq C\delta^{1/k} \qquad 0 < \delta \leq 1 \qquad |\xi| = 1$$

where $f_{x,\xi}(t) = f(t,x,\xi)$.

The following example shows that for subelliptic type of estimates it is not sufficient to have conditions only on the vanishing of the symbol, we also need conditions on the semibicharacteristics of the eigenvalues.

Example 3.10 *Let*

$$P = D_t \operatorname{Id}_2 + \alpha \begin{pmatrix} D_x & 0 \\ 0 & -D_x \end{pmatrix} + i(t - \beta x)^2 |D_x| \operatorname{Id}_2 \qquad (t, x) \in \mathbf{R}^2$$

with α, $\beta \in \mathbf{R}$, *then we see from the scalar case in Example* 3.1 *that* P *is subelliptic near* $\{\, \tau = 0 \,\}$ *with a loss of* $2/3$ *derivatives if and only either* $\alpha = 0$ *or* $\alpha \neq 0$ *and* $\beta \neq \pm 1/\alpha$.

Definition 3.11 *Let* $Q \in C^\infty(T^*X)$ *be an* $N \times N$ *system and let* $w_0 \in \Sigma \subset T^*X$, *then* Q *satisfies the* approximation property *on* Σ *near* w_0 *if there exists a* Q *invariant* C^∞ *subbundle* $\mathcal{V}$ *of* $\mathbf{C}^N$ *over* T^*X *such that* $\mathcal{V}(w_0) = \operatorname{Ker} Q^N(w_0)$ *and*

$$\operatorname{Re} \langle Q(w)v, v \rangle = 0 \qquad v \in \mathcal{V}(w) \qquad w \in \Sigma \tag{3.6}$$

near w_0. *That* $\mathcal{V}$ *is* Q *invariant means that* $Q(w)v \in \mathcal{V}(w)$ *for* $v \in \mathcal{V}(w)$.

Here $\operatorname{Ker} Q^N(w_0)$ is the space of the generalized eigenvectors corresponding to the zero eigenvalue. The symbol of the system in Example 3.10 satisfies the approximation property on $\Sigma = \{\, \tau = 0 \,\}$ if and only if $\alpha = 0$.

Let $\widetilde{Q} = Q\big|_{\mathcal{V}}$ then $\operatorname{Im} i\widetilde{Q} = \operatorname{Re} \widetilde{Q} = 0$ so Lemma 2.13 gives that $\operatorname{Ran} \widetilde{Q} \perp \operatorname{Ker} \widetilde{Q}$ on Σ. Thus $\operatorname{Ker} \widetilde{Q}^N = \operatorname{Ker} \widetilde{Q}$ on Σ, and since $\operatorname{Ker} \widetilde{Q}^N(w_0) = \mathcal{V}(w_0)$ we find that $\operatorname{Ker} Q^N(w_0) = \mathcal{V}(w_0) = \operatorname{Ker} Q(w_0)$.

Remark 3.12 *Assume that* Q *satisfies the approximation property on the* C^∞ *hypersurface* Σ *and is quasi-symmetric with respect to* $V \notin T\Sigma$. *Then the limits of the nontrivial semibicharacteristics of the eigenvalues of* Q *close to zero coincide with the bicharacteristics of* Σ.

In fact, the approximation property in Definition 3.11 gives that $\langle \operatorname{Re} Qu, u \rangle = 0$ for $u \in \operatorname{Ker} Q$ when $\tau = 0$. Since $\operatorname{Im} Q \geq 0$ we find that

$$\langle dQu, u \rangle = 0 \qquad \forall\, u \in \operatorname{Ker} Q \qquad \text{on } T\Sigma. \tag{3.7}$$

By Remark 2.19 the limits of the nontrivial semibicharacteristics of the eigenvalues close to zero of Q are curves with tangetnts determined by $\langle dQu, u \rangle$ for $u \in \operatorname{Ker} Q$. Since $V \operatorname{Re} Q \neq 0$ on $\operatorname{Ker} Q$ we find from (3.7) that the limit curves coincide with the bicharacteristics of Σ, which are the flow-outs of the Hamilton vector field.

Example 3.13 *Observe that Definition* 3.11 *is empty if* $\operatorname{Dim} \operatorname{Ker} Q^N(w_0) = 0$. *If* $\operatorname{Dim} \operatorname{Ker} Q^N(w_0) > 0$, *then there exists* $\varepsilon > 0$ *and a neighborhood* ω *to* w_0 *so that*

$$\Pi(w) = \frac{1}{2\pi i} \int_{|z| = \varepsilon} (z \operatorname{Id}_N - Q(w))^{-1}\, dz \in C^\infty(\omega) \tag{3.8}$$

is the spectral projection on the (generalized) eigenvectors with eigenvalues having absolute value less than ε*. Then* $\operatorname{Ran} \Pi$ *is a* Q *invariant bundle over* ω *so that* $\operatorname{Ran} \Pi(w_0) = \operatorname{Ker} Q^N(w_0)$*. Condition* (3.6) *with* $\mathcal{V} = \operatorname{Ran} \Pi$ *means that* $\Pi^* \operatorname{Re} Q \Pi \equiv 0$ *in* ω*. When* $\operatorname{Im} Q(w_0) \geq 0$ *we find that* $\Pi^* Q \Pi(w_0) = 0$*, then* Q *satisfies the approximation property on* Σ *near* w_0 *with* $\mathcal{V} = \operatorname{Ran} \Pi$ *if and only if*

$$d(\Pi^* (\operatorname{Re} Q) \Pi)\big|_{T\Sigma} \equiv 0 \qquad \text{near } w_0.$$

Example 3.14 *If* Q *satisfies the approximation property on* Σ*, then by choosing an orthonormal basis for* $\mathcal{V}$ *and extending it to an orthonormal basis for* $\mathbf{C}^N$ *we obtain the system on the form*

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix}$$

where Q_{11} *is a* $K \times K$ *system such that* $Q_{11}^N(w_0) = 0$*,* $\operatorname{Re} Q_{11} = 0$ *on* Σ *and* $|Q_{22}| \neq 0$*. By multiplying from the left with*

$$\begin{pmatrix} \mathrm{Id}_K & -Q_{12} Q_{22}^{-1} \\ 0 & \mathrm{Id}_{N-K} \end{pmatrix}$$

we obtain that $Q_{12} \equiv 0$ *without changing* Q_{11} *or* Q_{22}*.*

In fact, the eigenvalues of Q are then eigenvalues of either Q_{11} or Q_{22}. Since $\mathcal{V}(w_0)$ are the (generalized) eigenvectors corresponding to the zero eigenvalue of $Q(w_0)$ we find that all eigenvalues of $Q_{22}(w_0)$ are nonvanishing, thus Q_{22} is invertible near w_0.

Remark 3.15 *If* Q *satisfies the approximation property on* Σ *near* w_0*, then it satisfies the approximation property on* Σ *near* w_1*, for* w_1 *sufficiently close to* w_0*.*

In fact, let Q_{11} be the restriction of Q to $\mathcal{V}$ as in Example 3.14, then since $\operatorname{Re} Q_{11} = \operatorname{Im} i Q_{11} = 0$ on Σ we find from Lemma 2.13 that $\operatorname{Ran} Q_{11} \perp \operatorname{Ker} Q_{11}$ and $\operatorname{Ker} Q_{11} = \operatorname{Ker} Q_{11}^N$ on Σ. Since Q_{22} is invertible in (3.14), we find that $\operatorname{Ker} Q \subseteq \mathcal{V}$. Thus, by using the spectral projection (3.8) of Q_{11} near $w_1 \in \Sigma$ for small enough ε we obtain a Q invariant subbundle $\widetilde{\mathcal{V}} \subseteq \mathcal{V}$ so that $\widetilde{\mathcal{V}}(w_1) = \operatorname{Ker} Q_{11}(w_1) = \operatorname{Ker} Q^N(w_1)$.

If $Q \in C^\infty$ satisfies the approximation property and $Q_E = E^* Q E$ with invertible $E \in C^\infty$, then it follows from the proof of Proposition 3.20 below that there exist invertible $A, B \in C^\infty$ so that AQ_E and $Q^* B$ satisfy the approximation property.

Definition 3.16 *Let* $P(w) \in C^\infty(T^*X)$ *be an* $N \times N$ *system and* $\mu \in \mathbf{R}_+$*. Then* P *is of* finite type μ *at* $w_0 \in T^*X$ *if there exists a neighborhood* ω *of* w_0*, a* C^∞ *hypersurface* $\Sigma \ni w_0$*, a real* C^∞ *vector field* $V \notin T\Sigma$ *and an invertible symmetrizer* $M \in C^\infty$ *so that* $Q = MP$ *is quasi-symmetric with* *respect to* V *in* ω *and satisfies the approximation property on* $\Sigma \bigcap \omega$. *Also, for every bicharacteristic* γ *of* Σ *the arc length*

$$\left|\gamma \cap \Omega_\delta(\operatorname{Im} Q) \cap \omega\right| \le C\delta^\mu \qquad 0 < \delta \le 1. \tag{3.9}$$

The operator $P \in \Psi^m_{cl}$ *is of finite type* μ *at* w_0 *if the principal symbol* $\sigma(P)$ *is of finite type when* $|\xi| = 1$.

Recall that the bicharacteristics of a hypersurface in T^*X are the flow-outs of the Hamilton vector field of Σ. Of course, if P is elliptic, then it is trivially of finite type 0, just choose $M = iP^{-1}$ to obtain $Q = i\,\mathrm{Id}_N$. If P is of finite type, then it is quasi-symmetrizable by definition and thus of principal type.

Remark 3.17 *Observe that since* $0 \le \operatorname{Im} Q \in C^\infty$ *we obtain from Remark* 3.8 *that the largest exponent in* (3.9) *is* $\mu = 1/k$ *for an even* $k \ge 0$. *Also, we may assume that*

$$\operatorname{Im}\langle Qu, u\rangle \ge c\|Qu\|^2 \qquad \forall\, u \in \mathbf{C}^N. \tag{3.10}$$

In fact, by adding $i\varrho P^*$ to M we obtain (3.10) for large enough ϱ by (2.7), and this does not change $\operatorname{Re} Q$.

Example 3.18 *Assume that* Q *is quasi-symmetric with respect to the real vector field* V, *satisfying* (3.9) *and the approximation property on* Σ. *Then by choosing an orthonormal basis and changing the symmetrizer as in Example* 3.14 *we obtain the system on the form*

$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix}$$

where Q_{11} *is a* $K \times K$ *system such that* $Q_{11}^N(w_0) = 0$, $\operatorname{Re} Q_{11} = 0$ *on* Σ *and* $|Q_{22}| \ne 0$. *Since* Q *is quasi-symmetric with respect to* V *we also obtain that* $Q_{11}(w_0) = 0$, $\operatorname{Re} VQ_{11} > 0$, $\operatorname{Im} Q \ge 0$ *and* Q *satisfies* (3.9). *In fact, then we find from Lemma* 2.13 *that* $\operatorname{Im} Q \perp \operatorname{Ker} Q$ *which gives* $\operatorname{Ker} Q^N = \operatorname{Ker} Q$. *Note that* $\Omega_\delta(\operatorname{Im} Q_{11}) \subseteq \Omega_\delta(\operatorname{Im} Q)$, *so* Q_{11} *satisfies* (3.9).

Example 3.19 *In the scalar case, we find from Example 2.9 that* $p \in C^\infty(T^*X)$ *is quasi-symmetrizable with respect to* $H_t = \partial_\tau$ *if and only if*

$$p(t, x; \tau, \xi) = q(t, x; \tau, \xi)(\tau + if(t, x, \xi)) \tag{3.11}$$

with $f \ge 0$ *and* $q \ne 0$. *If* $f(t, x, \xi) \ge c > 0$ *when* $|(t, x, \xi)| \gg 1$ *we find by taking* q^{-1} *as symmetrizer that* p *is of finite type* μ *if and only if* $\mu = 1/k$ *for an even* k *such that*

$$\sum_{j \le k} |\partial_t^k f(t, x, \xi)| > 0 \qquad \forall\, x\ \xi$$

by Remark 3.8. *In fact, the approximation property on* $\Sigma = \{\, \tau = 0 \,\}$ *is trivial since* f *is real.*

Proposition 3.20 *If $P(w) \in C^\infty(T^*X)$ is of finite type μ at w, then P^* is of finite type μ at w. If $A(w)$ and $B(w) \in C^\infty(T^*X)$ are invertible, then APB is of finite type μ at w.*

Proof Let M be the symmetrizer in Definition 3.16 so that $Q = MP$ is quasi-symmetric with respect to V. By choosing a suitable basis and changing the symmetrizer as in Example 3.18, we may write

$$Q = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{pmatrix} \tag{3.12}$$

where Q_{11} is a $K \times K$ system such that $Q_{11}(w_0) = 0$, $V \operatorname{Re} Q_{11} > 0$, $\operatorname{Re} Q_{11} = 0$ on Σ and Q_{22} is invertible. We also have $\operatorname{Im} Q \geq 0$ and Q satisfies (3.9). Let $\mathcal{V}_1 = \left\{ u \in \mathbf{C}^N : \ u_j = 0 \text{ for } j > K \right\}$ and $\mathcal{V}_2 = \left\{ u \in \mathbf{C}^N : \ u_j = 0 \text{ for } j \leq K \right\}$, these are Q invariant bundles such that $\mathcal{V}_1 \oplus \mathcal{V}_2 = \mathbf{C}^N$.

First we are going to show that $\widetilde{P} = APB$ is of finite type. By taking $\widetilde{M} = B^{-1} M A^{-1}$ we find that

$$\widetilde{M}\widetilde{P} = \widetilde{Q} = B^{-1} Q B \tag{3.13}$$

and it is clear that $B^{-1}\mathcal{V}_j$ are $\widetilde{Q}$ invariant bundles, $j = 1, 2$. By choosing bases in $B^{-1}\mathcal{V}_j$ for $j = 1, 2$, we obtain a basis for $\mathbf{C}^N$ in which $\widetilde{Q}$ has a block form:

$$\widetilde{Q} = \begin{pmatrix} \widetilde{Q}_{11} & 0 \\ 0 & \widetilde{Q}_{22} \end{pmatrix} \tag{3.14}$$

Here $\widetilde{Q}_{jj} : B^{-1}\mathcal{V}_j \mapsto B^{-1}\mathcal{V}_j$ is given by $\widetilde{Q}_{jj} = B_j^{-1} Q_{jj} B_j$ with

$$B_j : B^{-1}\mathcal{V}_j \ni u \mapsto Bu \in \mathcal{V}_j \qquad j = 1,\ 2.$$

By multiplying $\widetilde{Q}$ from the left with

$$\mathcal{B} = \begin{pmatrix} B_1^* B_1 & 0 \\ 0 & B_2^* B_2 \end{pmatrix}$$

we obtain that

$$\overline{Q} = \mathcal{B}\widetilde{Q} = \mathcal{B}\widetilde{M}\widetilde{P} = \begin{pmatrix} B_1^* Q_{11} B_1 & 0 \\ 0 & B_2^* Q_{22} B_2 \end{pmatrix} = \begin{pmatrix} \overline{Q}_{11} & 0 \\ 0 & \overline{Q}_{22} \end{pmatrix}.$$

It is clear that $\operatorname{Im} \overline{Q} \geq 0$, $Q_{11}(w_0) = 0$, $\operatorname{Re} \overline{Q}_{11} = 0$ on Σ, $|\overline{Q}_{22}| \neq 0$ and $V \operatorname{Re} \overline{Q}_{11} > 0$ by Proposition 2.14. Finally, we obtain from Remark 3.5 that

$$\Omega_\delta(\operatorname{Im} \overline{Q}) \subseteq \Omega_{C\delta}(\operatorname{Im} Q) \tag{3.15}$$

for some $C > 0$, which proves that $\widetilde{P} = APB$ is of finite type. Observe that $\overline{Q} = \mathcal{A} Q_B$, where $Q_B = B^* Q B$ and $\mathcal{A} = \mathcal{B} B^{-1} (B^*)^{-1}$.

To show that P^* also is of finite type, we may assume as before that $Q = MP$ is on the form (3.12) with $Q_{11}(w_0) = 0$, $V\operatorname{Re} Q_{11} > 0$, $\operatorname{Re} Q_{11} = 0$ on Σ, Q_{22} is invertible, $\operatorname{Im} Q \geq 0$ and Q satisfies (3.9). Then we find that

$$-P^*M^* = -Q^* = \begin{pmatrix} -Q_{11}^* & 0 \\ 0 & -Q_{22}^* \end{pmatrix}$$

satisfies the same conditions with respect to $-V$, so it is of finite type with multiplier Id_N. By the first part of the proof we obtain that P^* is of finite type, which finishes the proof.

Theorem 3.21 *Assume that* $P \in \Psi_{cl}^m(X)$ *is an* $N \times N$ *system of finite type* $\mu > 0$ *near* $w_0 \in T^*X \setminus 0$, *then* P *is subelliptic at* w_0 *with a loss of* $1/\mu + 1$ *derivatives:*

$$Pu \in H_{(s)} \text{ at } w_0 \implies u \in H_{(s+m-1/\mu+1)} \text{ at } w_0 \qquad (3.16)$$

for $u \in \mathcal{D}'(X, \mathbf{C}^N)$.

Observe that the largest exponent is $\mu = 1/k$ for an even k by Remark 3.17, and then $1/\mu + 1 = k/k + 1$. Thus Theorem 3.21 generalizes Proposition 27.3.1 in [5] by Example 3.19.

Example 3.22 *Let*

$$P(t, x; \tau, \xi) = \tau M(t, x, \xi) + iF(t, x, \xi) \in S_{cl}^1$$

where $M \geq c_0 > 0$ *and* $F \geq 0$ *satisfies*

$$\left| \left\{ t : \inf_{|u|=1} \langle F(t, x, \xi)u, u \rangle \leq \delta \right\} \right| \leq C\delta^\mu \qquad |\xi| = 1 \qquad (3.17)$$

for some $\mu > 0$. *Then* P *is quasi-symmetrizable with respect to* ∂_τ *with symmetrizer* Id_N. *When* $\tau = 0$ *we obtain that* $\operatorname{Re} P = 0$, *so by taking* $\mathcal{V} = \operatorname{Ran} \Pi$ *for the spectral projection* Π *given by* (3.8) *for* F, *we find that* P *satisfies the approximation property with respect to* $\Sigma = \{ \tau = 0 \}$. *Since* $\Omega_\delta(\operatorname{Im} P) = \Omega_\delta(F)$ *we find from* (3.17) *that* P *is of finite type* μ. *Observe that if* $F(t, x, \xi) \geq c > 0$ *when* $|(t, x, \xi)| \gg 1$, *we find from Remark* 3.8 *that* (3.17) *is satisfied if and only if* $\mu \leq 1/k$ *for an even* $k \geq 0$ *so that*

$$\sum_{j \leq k} |\partial_t^j \langle F(t, x, \xi)u(t), u(t) \rangle| > 0 \qquad \forall\, t, x, \xi$$

for any $0 \neq u(t) \in C^\infty(\mathbf{R})$. *Theorem* 3.21 *gives that* $P(t, x, D_t, D_x)$ *is subelliptic near* $\{ \tau = 0 \}$ *with a loss of* $k/k + 1$ *derivatives.*

Proof (Proof of Theorem 3.21*)* First, we may reduce to the case $m = s = 0$ by replacing u and P by $\langle D \rangle^{s+m} u$ and $\langle D \rangle^s P \langle D \rangle^{-s-m} \in \Psi_{cl}^0$. Now $u \in H_{(-K)}$

for some K near w_0, and it is no restriction to assume $K = 1$. In fact, if $K > 1$, then by using that $Pu \in H_{(1-K)}$ near w_0, we obtain that $u \in H_{(-K+\mu/\mu+1)}$ near w_0 and we may iterate this argument until $u \in H_{(-1)}$ near w_0. By cutting off with $\varphi \in S^0_{1,0}$ we may assume that $v = \varphi(x,D)u \in H_{(-1)}$ and $Pv = [P, \varphi(x,D)]u + \varphi(x,D)Pu \in H_{(0)}$ since $[P, \varphi(x,D)] \in \Psi^{-1}$. If $\varphi \neq 0$ in a conical neighborhood of w_0, it suffices to prove that $v \in H_{(-1/\mu+1)}$.

By Definition 3.16 and Remark 3.17 there exist a C^∞ hypersurface Σ, a real C^∞ vector field $V \notin T\Sigma$, an invertible symmetrizer $M \in C^\infty$ so that $Q = MP$ satisfies (3.9), the approximation property on Σ, and

$$V\operatorname{Re} Q \geq c - C\operatorname{Im} Q \qquad c > 0, \tag{3.18}$$

$$\operatorname{Im} Q \geq c\, Q^* Q \tag{3.19}$$

in a neighborhood ω of w_0. By extending by homogeneity, we can assume that V, M and Q are homogeneous of degree 0.

Since (3.18) is stable under small perturbations in V we can replace V with H_t for some real $t \in C^\infty$. By solving the initial value problem $H_t \tau \equiv -1$, $\tau\big|_\Sigma = 0$, and completing to a symplectic C^∞ coordinate system (t, τ, x, ξ), we obtain that $\Sigma = \{\, \tau = 0 \,\}$ in a neighborhood of $w_0 = (0, 0, x_0, \xi_0)$, $\xi_0 \neq 0$. We obtain from Definition 3.11 that

$$\operatorname{Re} \langle Qu, u \rangle = 0 \qquad \text{when } u \in \mathcal{V} \text{ and } \tau = 0 \tag{3.20}$$

near w_0. Here $\mathcal{V}$ is a Q invariant C^∞ subbundle of $\mathbf{C}^N$ such that $\mathcal{V}(w_0) = \operatorname{Ker} Q^N(w_0) = \operatorname{Ker} Q(w_0)$ by Lemma 2.13. By condition (3.9) we have that

$$\big|\Omega_\delta(\operatorname{Im} Q_{x,\xi}) \cap \{\, |t| < c \,\}\big| \leq C\delta^\mu \tag{3.21}$$

when $|(x, \xi) - (x_0, \xi_0)| < c$, here $Q_{x,\xi}(t) = Q(t, 0, x, \xi)$.

Next, we shall localize the estimate. Choose $\{\, \varphi_j \,\}_j \in S^0_{1,0}$ and $\{\, \psi_j \,\}_j \in S^0_{1,0}$ with values in ℓ^2, such that $\varphi_j \geq 0$, $\psi_j \geq 0$, $\sum_j \varphi_j^2 = 1$, $\psi_j \varphi_j \equiv \varphi_j$ and ψ_j is supported where $|(\tau, \xi)| \cong 2^j$. Since these are Fourier multipliers we find that $\sum_j \varphi_j(D_{t,x})^2 = 1$ and

$$\|u\|^2_{(s)} \cong \sum_j 2^{2sj} \|\varphi_j(D_{t,x})u\|^2 \qquad u \in \mathcal{S}.$$

Let $Q_j = \psi_j Q$ be the localized symbol, and let $h_j = 2^{-j} \leq 1$. Since $Q_j \in S^0_{1,0}$ is supported where $|(\tau, \xi)| \cong 2^j$, we find that $Q_j(t, x, \tau, \xi) = \widetilde{Q}_j(t, x, h_j\tau, h_j\xi)$ where $\widetilde{Q}_j \in C_0^\infty(T^*\mathbf{R}^n)$ uniformly. We shall obtain Theorem 3.21 from the following result, which is Proposition 6.1 in [3].

Proposition 3.23 *Assume that $Q \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$ is an $N \times N$ system satisfying (3.18)–(3.21) in a neighborhood of $w_0 = (0, 0, x_0, \xi_0)$ with $V = \partial_\tau$ and*

$\mu > 0$. Then there exists $h_0 > 0$ and $R \in C_{\mathrm{b}}^\infty(T^*\mathbf{R}^n)$ so that $w_0 \notin \operatorname{supp} R$ and

$$h^{1/\mu+1}\|u\| \le C(\|Q(t,x,hD_{t,x})u\|+\|R^w(t,x,hD_{t,x})u\|+h\|u\|) \qquad 0 < h \le h_0 \tag{3.22}$$

for any $u \in C_0^\infty(\mathbf{R}^n, \mathbf{C}^N)$.

Here C_{b}^∞ are C^∞ functions with L^∞ bounds on any derivative, and the result is uniform in the usual sense. Observe that this estimate can be extended to a semiglobal estimate. In fact, let ω be a neighborhood of w_0 such that $\operatorname{supp} R \bigcap \omega = \emptyset$, where R is given by Proposition 3.23. Take $\varphi \in C_0^\infty(\omega)$ such that $0 \le \varphi \le 1$ and $\varphi = 1$ in a neighborhood of w_0. By substituting $\varphi(t,x,hD_{t,x})u$ in (3.22) we obtain from the calculus

$$\begin{aligned} &h^{1/\mu+1}\|\varphi(t,x,hD_{t,x})u\| \\ &\quad \le C_N(\|\varphi(t,x,hD_{t,x})Q(t,x,hD_{t,x})u\| + h\|u\|) \qquad \forall\, u \in C_0^\infty \end{aligned} \tag{3.23}$$

for small enough h since $R\varphi \equiv 0$ and $\|[Q(t,x,hD_{t,x}),\varphi(t,x,hD_{t,x})]u\| \le Ch\|u\|$. Thus, if Q satisfies conditions (3.18)–(3.21) near any $w \in K \Subset T^*\mathbf{R}^n$, then by using Bolzano–Weierstrass we obtain the estimate (3.22) with $\operatorname{supp} R \bigcap K = \emptyset$.

Now, by using that $\widetilde{Q}_j$ satisfies (3.18)–(3.21) in a neighborhood of $\operatorname{supp} \varphi_j$, we obtain the estimate (3.22) for $\widetilde{Q}_j(t,x,hD_{t,x})$ with $h = h_j = 2^{-j} \ll 1$ and $R = R_j \in S_{1,0}^0$ such that $\operatorname{supp} \varphi_j \bigcap \operatorname{supp} R_j = \emptyset$. Substituting $\varphi_j(D_{t,x})u$ we obtain for $j \gg 1$ that

$$\begin{aligned} &2^{-j/\mu+1}\|\varphi_j(D_{t,x})u\| \\ &\quad \le C_N(\|Q_j(t,x,D_{t,x})\varphi_j(D_{t,x})u\| + \|\widetilde{R}_j u\| + 2^{-j}\|\varphi_j(D_{t,x})u\|) \quad \forall\, u \in \mathcal{S}' \end{aligned}$$

where $\widetilde{R}_j = R_j(t,x,D_{t,x})\varphi_j(D_{t,x}) \in \Psi^{-N}$ with values in ℓ^2. Now since Q_j and Q are uniformly bounded in $S_{1,0}^0$ the calculus gives that

$$Q_j(t,x,D_{t,x})\varphi_j(D_{t,x}) = \varphi_j(D_{t,x})Q(t,x,D_{t,x}) + \varrho_j(t,x,D_{t,x})$$

where $\left\{ \varrho_j \right\}_j \in \Psi^{-1}$ with values in ℓ^2. Thus, by squaring and summing up, we obtain by continuity that

$$\|u\|_{(-1/\mu+1)}^2 \le C(\|Q(t,x,D_{t,x})u\|^2 + \|u\|_{(-1)}^2) \qquad u \in H_{(-1)}. \tag{3.24}$$

Since $Q(t,x,D_{t,x}) = M(t,x,D_{t,x})P(t,x,D_{t,x})$ modulo Ψ^{-1} where $M \in \Psi^0$, the calculus gives

$$\begin{aligned} \|Q(t,x,D_{t,x})u\| &\le C(\|M(t,x,D_{t,x})P(t,x,D_{t,x})u\| + \|u\|_{(-1)}) \\ &\le C'(\|P(t,x,D_{t,x})u\| + \|u\|_{(-1)}) \quad u \in H_{(-1)} \end{aligned} \tag{3.25}$$

which together with (3.24) proves Theorem 3.21.

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Uniform Exponential Decay for Viscous Damped Systems*

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Dedicated to Ferruccio Colombini with friendship

Summary. We consider a class of viscous damped vibrating systems. We prove that, under the assumption that the damping term ensures the exponential decay for the corresponding inviscid system, then the exponential decay rate is uniform for the viscous one, regardless what the value of the viscosity parameter is. Our method is mainly based on a decoupling argument of low and high frequencies. Low frequencies can be dealt with because of the effectiveness of the damping term in the inviscid case while the dissipativity of the viscous term guarantees the decay of the high-frequency components. This method is inspired in previous work by the authors on time-discretization schemes for damped systems in which a numerical viscosity term needs to be added to ensure the uniform exponential decay with respect to the time-step parameter.

2000 AMS Subject Classification: Primary: 35B35, 93D15, Secondary: 74D05, 35L90

Key words: dissipative semigroups, exponential decay, elasticity, viscoelasticity.

* This work has been partially supported by grant MTM 2008-03541 of the MICINN (Spain).

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_6,

1 Introduction

Let X and Y be Hilbert spaces endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $A : \mathcal{D}(A) \subset X \to X$ be a skew-adjoint operator with compact resolvent and $B \in \mathfrak{L}(X, Y)$.

We consider the system described by

$$\dot{z} = Az + \varepsilon A^2 z - B^* B z, \quad t \geq 0, \qquad z(0) = z_0 \in X. \tag{1.1}$$

Here and henceforth, a dot $(\dot{})$ denotes differentiation with respect to time t. The element $z_0 \in X$ is the initial state, and $z(t)$ is the state of the system. Most of the linear equations modeling the damped viscous vibrations of elastic structures (strings, beams, plates,...) can be written in the form (1.1) or some variants that we shall also discuss, in which the viscosity term has a more general form, namely,

$$\dot{z} = Az + \varepsilon \mathcal{V}_\varepsilon z - B^* B z, \quad t \geq 0, \qquad z(0) = z_0 \in X, \tag{1.2}$$

for a suitable viscosity operator $\mathcal{V}_\varepsilon$, which might depend on ε.

We define the energy of the solutions of system (1.1) by

$$E(t) = \frac{1}{2} \|z(t)\|_X^2, \quad t \geq 0, \tag{1.3}$$

which satisfies

$$\frac{dE}{dt}(t) = -\|Bz(t)\|_Y^2 - \varepsilon ||Az||_X^2, \quad t \geq 0. \tag{1.4}$$

In this paper, we assume that system (1.1) is exponentially stable when $\varepsilon = 0$. For the sake of completeness and clarity we distinguish the case in which the viscosity parameter vanishes:

$$\dot{z} = Az - B^* B z, \quad t \geq 0, \qquad z(0) = z_0 \in X. \tag{1.5}$$

This model corresponds to a conservative system in which a bounded damping term has been added. The damped wave and Schrödinger equations enter in this class, for instance.

Thus, we assume that there exist positive constants μ and ν such that any solution of (1.5) satisfies

$$E(t) \leq \mu \, E(0) \exp(-\nu t), \quad t \geq 0. \tag{1.6}$$

Our goal is to prove that the exponential decay property (1.6) for (1.5) implies the uniform exponential decay of solutions of (1.1) with respect to the viscosity parameter $\varepsilon > 0$.

This result might seem immediate a priori since the viscous term that (1.1) adds to (1.5) should in principle increase the decay rate of the solutions of the latter. But, this is far from being trivial because of the possible presence of overdamping phenomena. Indeed, in the context of the damped wave equation, for instance, it is well known that the decay rate does not necessarily behave monotonically with respect to the size of the damping operator (see, for instance, [6, 7, 15]). In our case, however, the viscous damping operator is such that the decay rate is kept uniformly on ε. This is so because it adds dissipativity to the high-frequency components, while it does not deteriorate the low-frequency damping that the bounded feedback operator $-B^*B$ introduces.

The main result of this paper is that system (1.1) enjoys a uniform stabilization property. It reads as follows:

Theorem 1.1 *Assume that system* (1.5) *is exponentially stable and satisfies* (1.6) *for some positive constants* μ *and* ν, *and that* $B \in \mathfrak{L}(X, Y)$.

Then there exist two positive constants μ_0 *and* ν_0 *depending only on* $\|B\|_{\mathfrak{L}(X,Y)}$, ν *and* μ *such that any solution of* (1.1) *satisfies* (1.6) *with constants* μ_0 *and* ν_0 *uniformly with respect to the viscosity parameter* $\varepsilon > 0$.

Our strategy is based on the fact that the uniform exponential decay properties of the energy for systems (1.5) and (1.1), respectively, are equivalent to observability properties for the conservative system

$$\dot{y} = Ay, \quad t \in \mathbb{R}, \qquad y(0) = y_0 \in X, \tag{1.7}$$

and its viscous counterpart

$$\dot{u} = Au + \varepsilon A^2 u, \quad t \in \mathbb{R}, \qquad u(0) = u_0 \in X. \tag{1.8}$$

For (1.7) the observability property consists in the existence of a time $T^* > 0$ and a positive constant $k_* > 0$ such that

$$k_* \|y_0\|_X^2 \leq \int_0^{T^*} \|By(t)\|_Y^2 \, dt, \tag{1.9}$$

for every solution of (1.7) (see [11]).

A similar argument can be applied to the viscous system (1.8). In this case the relevant inequality is the following: There exist a time $T > 0$ and a positive constant $k_T > 0$ such that any solution of (1.8) satisfies

$$k_T \|u_0\|_X^2 \leq \int_0^T \|Bu(t)\|_Y^2 \, dt + \varepsilon \int_0^T \|Au(t)\|_X^2 \, dt. \tag{1.10}$$

Note, however, that, for the uniform exponential decay property of the solutions of (1.1) to be independent of ε, we also need the time T and the observability constant k_T in (1.10) to be uniform. Actually we will prove the observability property (1.10) for the time $T = T^*$ given in (1.9).

The observability inequality (1.10) cannot be obtained directly from (1.9) by a perturbation argument since the viscosity operator εA^2 is an unbounded perturbation of the dynamics associated to the conservative system (1.7). Therefore, we decompose the solution u of (1.8) into its low- and high-frequency parts, which we handle separately. We first use the observability of (1.7) to prove (1.10), uniformly on ε, for the low-frequency components. Second, we use the dissipativity of (1.8) to obtain a similar estimate for the high-frequency components.

In this way, we derive observability properties of the low- and high-frequency components separately, which, together, yield the needed observability property (1.10) leading to the uniform exponential decay result.

Our arguments do not apply when the damping operator B is not bounded, as it happens when the damping acts on the boundary for the wave equation, see for instance [7]. Dealing with unbounded damping operators B needs further work.

As we mentioned above, the results in this paper are related with the literature on the uniform stabilization of numerical approximation schemes for damped equations of the form (1.5) and in particular with [21, 20, 18, 19, 9]. Similar techniques have also been employed to obtain uniform dispersive estimates for numerical approximation schemes to Schrödinger equations in [12].

The recent work [8] is also worth mentioning. There, observability issues were discussed for time and fully discrete approximation schemes of (1.7) and served as one of the sources of motivation for this work.

The outline of this paper is as follows. In Section 2, we recall the results of [8] and prove Theorem 1.1. In Section 3, we present a generalization of Theorem 1.1 to other viscosity operators. We also specify an application of our technique for viscous second order in time evolution equations which fit (1.2). In Section 4, we present some applications to viscous approximations of damped Schrödinger and wave equations. Finally, some further comments and open problems are collected in Section 5.

2 Proof of Theorem 1.1

We first need to introduce some notations.

Since A is a skew-adjoint operator with compact resolvent, its spectrum is discrete and $\sigma(A) = \{i\mu_j \ : \ j \in \mathbb{N}\}$, where $(\mu_j)_{j\in\mathbb{N}}$ is a sequence of real numbers such that $|\mu_j| \to \infty$ when $j \to \infty$. Set $(\Phi_j)_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j\in\mathbb{N}}$, that is,

$$A\Phi_j = i\mu_j\Phi_j. \tag{2.1}$$

Moreover, define

$$\mathcal{C}_s = \mathrm{span}\,\{\Phi_j : \text{ the corresponding } i\mu_j \text{ satisfies } |\mu_j| \leq s\}. \tag{2.2}$$

In the sequel, we assume that system (1.5) is exponentially stable and that $B \in \mathfrak{L}(X, Y)$, i.e., there exists a constant K_B such that

$$\|Bz\|_Y \leq K_B \|z\|_X, \qquad \forall z \in X. \tag{2.3}$$

The proof of Theorem 1.1 is divided into several steps.

First, we write carefully the energy identity for z solution of (1.1).

Consider z a solution of (1.1). Its norm $\|z(t)\|_X^2$ satisfies

$$\|z(T)\|_X^2 + 2\int_0^T \|Bz(t)\|_Y^2 \; dt + 2\int_0^T \varepsilon \|Az(t)\|_Y^2 \; dt = \|z(0)\|_X^2 . \tag{2.4}$$

Therefore our goal is to prove that, with T^* as in (1.9), there exists a constant $c > 0$ such that any solution of (1.1) satisfies

$$c\|z(0)\|_X^2 \leq \int_0^{T^*} \|Bz(t)\|_Y^2 \; dt + \varepsilon \int_0^{T^*} \|Az(t)\|_X^2 \; dt. \tag{2.5}$$

It is indeed easy to see that, combining (2.4) and (2.5), the semigroup S_ε generated by (1.1) satisfies

$$\|S_\varepsilon(T^*)\| \leq \gamma = 1 - c, \tag{2.6}$$

for a constant $0 < \gamma < 1$ independent of $\varepsilon > 0$. This, by the semigroup property, yields the uniform exponential decay result.

We also claim that, for (2.5) to hold for the solutions of (1.1), it is sufficient to show (1.10) for solutions of (1.8). To do that, it is sufficient to follow the argument in [11] developed in the context of system (1.5).

We decompose z as $z = u + w$ where u is the solution of the system (1.8) with initial data $u(0) = z_0$ and w satisfies

$$\dot{w} = Aw + \varepsilon A^2 w - B^*Bz, \quad t \geq 0, \qquad w(0) = 0. \tag{2.7}$$

Indeed, multiplying (2.7) by w and integrating in time, we get

$$\|w(t)\|_X^2 + 2\varepsilon \int_0^t \|Aw(s)\|_X^2 \; ds + 2\int_0^t < Bz(s), Bw(s) >_Y \; ds = 0.$$

Using that B is bounded, this gives

$$\|w(t)\|_X^2 + 2\varepsilon \int_0^t \|Aw(s)\|_X^2 \; ds \leq \int_0^t \|Bz(s)\|_Y^2 + K_B^2 \int_0^t \|w(s)\|_X^2 \; ds. \tag{2.8}$$

Grönwall's inequality then gives a constant G, which depends only on K_B and T^*, such that

$$\sup_{t\in[0,T^*]} \left\{ \|w(t)\|_X^2 \right\} + \varepsilon \int_0^{T^*} \|Aw(s)\|_X^2 \; ds \leq G \int_0^{T^*} \|Bz(s)\|_Y^2 \; ds. \tag{2.9}$$

Therefore in the sequel we deal with solutions u of (1.8), for which we prove (1.10) for $T = T^*$.

As stated in the introduction, we decompose the solution u of (1.8) into its low- and high-frequency parts. To be more precise, we consider

$$u_l = \pi_{1/\sqrt{\varepsilon}} u, \quad u_h = (I - \pi_{1/\sqrt{\varepsilon}})u, \tag{2.10}$$

where $\pi_{1/\sqrt{\varepsilon}}$ is the orthogonal projection on $\mathcal{C}_{1/\sqrt{\varepsilon}}$ defined in (2.2). Here the notation u_l and u_h stands for the low- and high-frequency components, respectively.

Note that both u_l and u_h are solutions of (1.8) since the projection $\pi_{1/\sqrt{\varepsilon}}$ and the viscosity operator A^2 commute.

Besides, u_h lies in the space $\mathcal{C}^{\perp}_{1/\sqrt{\varepsilon}}$, in which the following property holds:

$$\sqrt{\varepsilon}\,\|Ay\|_X \geq \|y\|_X\,, \quad \forall y \in \mathcal{C}^{\perp}_{1/\sqrt{\varepsilon}}. \tag{2.11}$$

In a first step, we compare u_l with y_l solution of (1.7) with initial data $y_l(0) = u_l(0)$. Now, set $w_l = u_l - y_l$. From (1.9), which is valid for solutions of (1.7), we get

$$k_* \|u_l(0)\|_X^2 = k_* \|y_l(0)\|_X^2 \leq 2\int_0^{T^*} \|Bu_l(t)\|_Y^2 \;dt + 2\int_0^{T^*} \|Bw_l(t)\|_Y^2 \;dt. \tag{2.12}$$

In the sequel, to simplify the notation, $c > 0$ will denote a positive constant that may change from line to line, but which does not depend on ε.

Let us therefore estimate the last term on the right-hand side of (2.12). To this end, we write the equation satisfied by w_l, which can be deduced from (1.7) and (1.8):

$$\dot{w}_l = Aw_l + \varepsilon A^2 u_l, \quad t \geq 0, \qquad w_l(0) = 0.$$

Note that $w_l \in \mathcal{C}_{1/\sqrt{\varepsilon}}$, since u_l and y_l both belong to $\mathcal{C}_{1/\sqrt{\varepsilon}}$. Therefore, the energy estimate for w_l leads, for $t \geq 0$, to

$$\begin{aligned}\|w_l(t)\|_X^2 &= -2\varepsilon \int_0^t < Au_l(s), Aw_l(s) >_X ds \\ &\leq \varepsilon \int_0^t \|Au_l(s)\|_X^2\, ds + \int_0^t \|w_l(s)\|_X^2\, ds.\end{aligned}$$

Grönwall's Lemma applies and allows us to deduce from (2.12) and the fact that the operator B is bounded, the existence of a positive c independent of ε, such that

$$c\,\|u_l(0)\|_X^2 \leq \int_0^{T^*} \|Bu_l(t)\|_Y^2 \;dt + \varepsilon \int_0^{T^*} \|Au_l(s)\|_X^2\, ds.$$

Besides,

$$\int_0^{T^*} \|Bu_l(t)\|_Y^2 \;dt \leq 2\int_0^{T^*} \|Bu(t)\|_Y^2 \;dt + 2\int_0^{T^*} \|Bu_h(t)\|_Y^2 \;dt$$

and, since $u_h(t) \in \mathcal{C}^{\perp}_{1/\sqrt{\varepsilon}}$ for all t,

$$\int_0^{T^*} \|Bu_h(t)\|_Y^2 \, dt \le K_B^2 \int_0^{T^*} \|u_h(t)\|_X^2 \, dt \le K_B^2 \varepsilon \int_0^{T^*} \|Au_h(t)\|_X^2 \, dt.$$

It follows that there exists $c > 0$ independent of ε such that

$$c \|u_l(0)\|_X^2 \le \int_0^{T^*} \|Bu(t)\|_Y^2 \, dt + \varepsilon \int_0^{T^*} \|Au(t)\|_X^2 \, dt. \tag{2.13}$$

Let us now consider the high-frequency component u_h. Since $u_h(t)$ is a solution of (1.8) and belongs to $\mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$ for all time $t \ge 0$, the energy dissipation law for u_h solution of (1.8) reads

$$\|u_h(t)\|_X^2 + 2\varepsilon \int_0^t \|Au_h(s)\|_X^2 \, ds = \|u_h(0)\|_X^2, \quad t \ge 0, \tag{2.14}$$

and

$$\|u_h(t)\|_X^2 \le \exp(-2t) \|u_h(0)\|_X^2, \quad \forall t \ge 0.$$

In particular, these last two inequalities imply the existence of a constant $c > 0$ independent of ε such that any solution u_h of (1.8) with initial data $u_h(0) \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$ satisfies

$$c \|u_h(0)\|_X^2 \le \varepsilon \int_0^{T^*} \|Au_h(s)\|_X^2 \, ds. \tag{2.15}$$

Combining (2.13) and (2.15) leads to the observability inequality (1.10). This, combined with the arguments of [11] and (2.9), allows us to prove that any solution z of (1.1) satisfies (2.5), and proves (2.6), from which Theorem 1.1 follows.

3 Variants of Theorem 1.1

3.1 General viscosity operators

Other viscosity operators could have been chosen. In our approach, we used the viscosity operator εA^2, which is unbounded, but we could have considered the viscosity operator

$$\varepsilon \mathcal{V}_\varepsilon = \frac{\varepsilon A^2}{I - \varepsilon A^2}, \tag{3.1}$$

which is well defined, since A^2 is a definite negative operator, and commutes with A. This choice presents the advantage that the viscosity operator now is bounded, keeping the properties of being small at frequencies of order less than $1/\sqrt{\varepsilon}$ and of order 1 on frequencies of order $1/\sqrt{\varepsilon}$ and more. Again, the same proof as the one presented above works.

The following result constitutes a generalization of Theorem 1.1, which applies to a wide range of viscosity operators, and, in particular, to (3.1).

Theorem 3.1 *Assume that system* (1.5) *is exponentially stable and satisfies* (1.6)*, and that* $B \in \mathfrak{L}(X, Y)$*.*

Consider a viscosity operator $\mathcal{V}_\varepsilon$ *such that*

1. $\mathcal{V}_\varepsilon$ *defines a self-adjoint definite negative operator.*
2. *The projection* $\pi_{1/\sqrt{\varepsilon}}$ *and the viscosity operator* $\mathcal{V}_\varepsilon$ *commute.*
3. *There exist positive constants* c *and* C *such that for all* $\varepsilon > 0$*,*

$$\begin{cases} \sqrt{\varepsilon}\left\| \left(\sqrt{-\mathcal{V}_\varepsilon}\right) z \right\|_X \leq C \left\| z \right\|_X, \ \forall z \in \mathcal{C}_{1/\sqrt{\varepsilon}}, \\ \sqrt{\varepsilon}\left\| \left(\sqrt{-\mathcal{V}_\varepsilon}\right) z \right\|_X \geq c \left\| z \right\|_X, \ \forall z \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}. \end{cases}$$

Then the solutions of (1.2) *are exponentially decaying in the sense of* (1.6)*, uniformly with respect to the viscosity parameter* $\varepsilon \geq 0$*.*

The proof of Theorem 3.1 can be easily deduced from that of Theorem 1.1 and is left to the reader.

Especially, note that the second item implies that both spaces $\mathcal{C}_{1/\sqrt{\varepsilon}}$ and $\mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$ are left globally invariant by the viscosity operator $\mathcal{V}_\varepsilon$. Therefore, if $u_l \in \mathcal{C}_{1/\sqrt{\varepsilon}}$ and $u_h \in \mathcal{C}_{1/\sqrt{\varepsilon}}^{\perp}$, we have

$$< \mathcal{V}_\varepsilon(u_l + u_h), (u_l + u_h) >_X = < \mathcal{V}_\varepsilon u_l, u_l >_X + < \mathcal{V}_\varepsilon u_h, u_h >_X .$$

Also remark that the second item is always satisfied when the operators $\mathcal{V}_\varepsilon$ and A commute.

3.2 Wave-type systems

In this subsection we investigate the exponential decay properties for viscous approximations of second order in time evolution equation.

Let H be a Hilbert space endowed with the norm $\|\cdot\|_H$. Let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint positive definite operator with compact resolvent and $C \in \mathfrak{L}(H, Y)$.

We then consider the initial value problem

$$\begin{cases} \ddot{v} + A_0 v + \varepsilon A_0 \dot{v} + C^* C \dot{v} = 0, \quad t \geq 0, \\ v(0) = v_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{v}(0) = v_1 \in H. \end{cases} \tag{3.2}$$

System (3.2) can be seen as a particular instance of (1.2) modeling wave and beam equations.

The energy of solutions of (3.2) is given by

$$E(t) = \frac{1}{2} \|\dot{v}(t)\|_H^2 + \frac{1}{2} \left\| A_0^{1/2} v(t) \right\|_H^2, \tag{3.3}$$

and satisfies

$$\frac{dE}{dt}(t) = -\|C\dot{v}(t)\|_Y^2 - \varepsilon \left\|A_0^{1/2}\dot{v}(t)\right\|_H^2. \tag{3.4}$$

As before, we assume that, for $\varepsilon = 0$, the system

$$\ddot{v} + A_0 v + C^*C\dot{v} = 0, \quad t \geq 0, \qquad v(0) = v_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{v}(0) = v_1 \in H \tag{3.5}$$

is exponentially stable, i.e., (1.6) holds.

We are indeed in the setting of (1.2), since (3.2) can be written as

$$\dot{Z} = AZ + \varepsilon \mathcal{V}_\varepsilon Z - B^*BZ, \tag{3.6}$$

with

$$Z = \begin{pmatrix} v \\ \dot{v} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \quad \mathcal{V}_\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & -A_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & C \end{pmatrix}. \tag{3.7}$$

Note that the viscosity operator $\mathcal{V}_\varepsilon$ introduced in (3.7) does not satisfy Condition 1 in Theorem 3.1. However, we can prove the following theorem:

Theorem 3.2 *Assume that system* (3.5) *is exponentially stable and satisfies* (1.6) *for some positive constants* μ *and* ν*, and that* $C \in \mathfrak{L}(H, Y)$*. Set* $K < \infty$*.*

Then there exist two positive constants μ_K *and* ν_K *depending only on* $\|C\|_{\mathfrak{L}(H,Y)}$*,* K*,* ν *and* μ *such that any solution of* (3.2) *satisfies* (1.6) *with constants* μ_0 *and* ν_0 *uniformly with respect to the viscosity parameter* $\varepsilon \in [0, K]$*.*

Before going into the proof, we introduce the spectrum of A_0. Since A_0 is self-adjoint positive definite with compact resolvent, its spectrum is discrete and $\sigma(A_0) = \{\lambda_j^2 : \ j \in \mathbb{N}\}$, where λ_j is an increasing sequence of real positive numbers such that $\lambda_j \to \infty$ when $j \to \infty$. Set $(\Psi_j)_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors of A_0 associated to the eigenvalues $(\lambda_j^2)_{j\in\mathbb{N}}$.

These notations are consistent with the ones introduced in Section 2, by setting A as in (3.7), and

$$\mu_{\pm j} = \pm\lambda_j, \qquad \Phi_j = \begin{pmatrix} \dfrac{1}{i\mu_j}\Psi_j \\ \Psi_j \end{pmatrix}.$$

For convenience, similarly as in (2.2), we define

$$\mathfrak{C}_s = \operatorname{span}\{\Psi_j : \text{ the corresponding } \lambda_j \text{ satisfies } |\lambda_j| \leq s\}, \tag{3.8}$$

which satisfies $\mathcal{C}_s = (\mathfrak{C}_s)^2$.

Proof (Sketch of the proof) The proof of Theorem 3.2 closely follows that of Theorem 1.1.

As before, we read the exponential stability of (3.5) into the following observability inequality: There exist a time T^* and a positive constant k_* such that any solution of

$$\ddot{y} + A_0 y = 0, \quad t \geq 0, \qquad y(0) = y_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{y}(0) = y_1 \in H \tag{3.9}$$

satisfies

$$k_* \Big(\|y_1\|_H^2 + \left\| A_0^{1/2} y_0 \right\|_H^2 \Big) \leq \int_0^{T^*} \|C\dot{y}(t)\|_Y^2 \, dt. \tag{3.10}$$

Due to (3.4), as in (2.5), the exponential decay of the energy for solutions of (3.2) is equivalent to the following observability inequality: There exist a time $\tilde{T}$ and a positive constant c such that for any $\varepsilon \in [0, K]$,

$$c\Big(\|v_1\|_H^2 + \left\| A_0^{1/2} v_0 \right\|_H^2 \Big) \leq \int_0^{\tilde{T}} \|C\dot{v}(t)\|_Y^2 \, dt + \varepsilon \int_0^{\tilde{T}} \left\| A_0^{1/2} \dot{v}(t) \right\|_H^2 \, dt \tag{3.11}$$

holds for any solution v of (3.2).

Using the same perturbative arguments as in [11] or (2.7)–(2.9), the observability inequality (3.11) holds if and only if there exist a time T and a positive constant $k_T > 0$ such that, for any $\varepsilon \in [0, K]$, the observability inequality

$$k_T \Big(\|u_1\|_H^2 + \left\| A_0^{1/2} u_0 \right\|_H^2 \Big) \leq \int_0^{T} \|C\dot{u}(t)\|_Y^2 \, dt + \varepsilon \int_0^{T} \left\| A_0^{1/2} \dot{u}(t) \right\|_H^2 \, dt \tag{3.12}$$

holds for any solution u of

$$\ddot{u} + A_0 u + \varepsilon A_0 \dot{u} = 0, \quad t \geq 0, \qquad u(0) = u_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{u}(0) = u_1 \in H. \tag{3.13}$$

As before, we then focus on the observability inequality (3.12) for solutions of (3.13). As in the proof of Theorem 1.1, we now decompose the solution of (3.13) into its low- and high-frequency parts, which we handle separately. To be more precise, we consider

$$u_l = P_{1/\sqrt{\varepsilon}}\, u, \qquad u_h = (I - P_{1/\sqrt{\varepsilon}})u.,$$

where $P_{1/\sqrt{\varepsilon}}$ is the orthogonal projection in H on $\mathfrak{C}_{1/\sqrt{\varepsilon}}$ as defined in (3.8). Again, both u_l and u_h are solutions of (3.13) since $P_{1/\sqrt{\varepsilon}}$ commutes with A_0.

Arguing as before, the low-frequency component u_l can be compared to y_l solution of (3.9) with initial data $(y_0, y_1) = (P_{1/\sqrt{\varepsilon}} u_0, P_{1/\sqrt{\varepsilon}} u_1)$, and using (3.10) for solutions of (3.9), we obtain the existence of a positive constant c_1 such that

$$c_1\Big(\left\|P_{1/\sqrt{\varepsilon}}u_1\right\|_H^2 + \left\|A_0^{1/2}P_{1/\sqrt{\varepsilon}}u_0\right\|_H^2 \Big)$$
$$\leq \int_0^{T^*} \|C\dot{u}(t)\|_Y^2 \, dt + \varepsilon \int_0^{T^*} \left\|A_0^{1/2}\dot{u}(t)\right\|_H^2 \, dt. \quad (3.14)$$

For the high-frequency component u_h, the situation is slightly more intricate than in Theorem 1.1. The energy of the solution u_h satisfies the dissipation law

$$\frac{1}{2}\frac{d}{dt}\Big(\|\dot{u}_h(t)\|_H^2 + \left\|A_0^{1/2}u_h(t)\right\|_H^2 \Big) = -\varepsilon \left\|A_0^{1/2}\dot{u}_h\right\|_H^2 \leq - \|\dot{u}_h\|_H^2 , \quad (3.15)$$

where the last inequality comes from $\dot{u}_h \in \mathfrak{C}_{1/\sqrt{\varepsilon}}^{\perp}$.

Setting

$$E_h(t) = \frac{1}{2}\|\dot{u}_h(t)\|_H^2 + \frac{1}{2}\left\|A_0^{1/2}u_h(t)\right\|_H^2 ,$$

we thus obtain that

$$E_h(t) + \int_0^t \|\dot{u}_h(s)\|_H^2 \, ds \leq E_h(0). \quad (3.16)$$

We now prove the so-called equirepartition of the energy for the solutions u of (3.13). Multiplying (3.13) by u and integrating by parts between 0 and t, we obtain

$$< \dot{u}(t), u(t) >_H - < \dot{u}(0), u(0) >_H - \int_0^t \|\dot{u}(s)\|_H^2 \, ds + \int_0^t \left\|A_0^{1/2}u(s)\right\|_H^2 \, ds$$
$$+ \varepsilon \int_0^t < A_0^{1/2}\dot{u}(s), A_0^{1/2}u(s) >_H \, ds = 0.$$

In particular,

$$\int_0^t \|\dot{u}(s)\|_H^2 \, ds = \int_0^t \left\|A_0^{1/2}u(s)\right\|_H^2 \, ds + \frac{\varepsilon}{2}\Big(\left\|A_0^{1/2}u(t)\right\|_H^2 - \left\|A_0^{1/2}u_0\right\|_H^2 \Big)$$
$$+ < \dot{u}(t), u(t) >_H - < \dot{u}(0), u(0) >_H . \quad (3.17)$$

Now, for u_h, which is a solution of (3.13), for all $t \geq 0$, $u_h(t) \in \mathfrak{C}_{1/\sqrt{\varepsilon}}^{\perp}$. In particular, for all $t \geq 0$, we have

$$\Big| < \dot{u}_h(t), u_h(t) >_H \Big| \leq \frac{\sqrt{\varepsilon}}{2}\|\dot{u}_h\|_H^2 + \frac{1}{2\sqrt{\varepsilon}}\|u_h(t)\|_H^2 \leq \sqrt{\varepsilon}E_h(t), \quad (3.18)$$

where we used that for $\varphi \in \mathfrak{C}_{1/\sqrt{\varepsilon}}^{\perp}$,

$$\|\varphi\|_H^2 \leq \varepsilon \left\|A_0^{1/2}\varphi\right\|_H^2 .$$

Combining (3.18) with identity (3.17) for u_h, we obtain

$$\int_0^t \|\dot{u}_h(s)\|_H^2 \, ds \geq \int_0^t \left\| A_0^{1/2} u_h(s) \right\|_H^2 \, ds - \left(\sqrt{\varepsilon} + \varepsilon\right)(E_h(t) + E_h(0)). \quad (3.19)$$

This yields

$$\int_0^t \|\dot{u}_h(s)\|_H^2 \, ds \geq \int_0^t E_h(s) \, ds - \frac{1}{2}\left(\sqrt{\varepsilon} + \varepsilon\right)(E_h(t) + E_h(0)). \quad (3.20)$$

Combined with (3.16), we obtain

$$\left(1 - \frac{1}{2}(\sqrt{\varepsilon} + \varepsilon)\right) E_h(t) + \int_0^t E_h(s) \, ds \leq E_h(0)\left(1 + \frac{1}{2}(\sqrt{\varepsilon} + \varepsilon)\right). \quad (3.21)$$

Assuming, without loss of generality, that $K \geq 1$, for $\varepsilon \in [0, K]$, we thus have

$$(1 - K)E_h(t) + \int_0^t E_h(s) \, ds \leq (1 + K)E_h(0).$$

The decay of $E_h(t)$, guaranteed by the dissipation law (3.15), then proves that

$$(t + 1 - K)E_h(t) \leq (1 + K)E_h(0).$$

For $t = 1 + 3K$, we thus have $E_h(1 + 3K) \leq E_h(0)/2$. We then deduce from the dissipation law (3.15) the existence of a positive constant c_K such that

$$c_K E_h(0) \leq \varepsilon \int_0^{1+3K} \left\| A_0^{1/2} \dot{u}_h(s) \right\|_H^2 \, ds. \quad (3.22)$$

We finally conclude Theorem 3.2 by combining (3.14) and (3.22) as before.

Remark 3.3 *One cannot expect the results of Theorem* 3.2 *to hold uniformly with respect to* $\varepsilon \in [0, \infty)$. *Indeed, an overdamping phenomenon appears when* $\varepsilon \to \infty$. *This can indeed be deduced from the existence of the following solutions of* (3.13)*:*

$$u_j(t) = \exp(t\tau_j^\varepsilon)\Psi_j, \quad t \geq 0, \quad \textit{where } \tau_j^\varepsilon = \frac{\varepsilon\lambda_j^2}{2}\left(\sqrt{1 - \frac{4}{(\varepsilon\lambda_j)^2}} - 1\right) \underset{\varepsilon\lambda_j \to \infty}{\sim} -\frac{1}{\varepsilon}.$$

Plugging these solutions in (3.12)*, one can check that the observability inequality* (3.12) *cannot hold uniformly with respect to* $\varepsilon \in [0, \infty)$. *Finally, using the equivalence between the observability inequality* (3.12) *for solutions of* (3.13) *and the observability inequality* (3.11) *for solutions of* (3.2)*, this proves that the results of Theorem* 3.2 *do not hold uniformly with respect to* $\varepsilon \in [0, \infty)$.

Remark 3.4 *To avoid the overdamping phenomenon when $\varepsilon \to \infty$, one can for instance add a dispersive term in* (3.2)*, and consider the initial value problem*

$$\begin{cases} \ddot{v} + A_0 v + \varepsilon A_0 \dot{v} + \varepsilon A_0 v + C^* C \dot{v} = 0, \quad t \geq 0, \\ v(0) = v_0 \in \mathcal{D}(A_0^{1/2}), \quad \dot{v}(0) = v_1 \in H. \end{cases} \tag{3.23}$$

The energy of solutions of (3.23) *is now given by*

$$E_\varepsilon(t) = \frac{1}{2} \|\dot{v}(t)\|_H^2 + \left(\frac{1+\varepsilon}{2}\right) \left\| A_0^{1/2} v(t) \right\|_H^2. \tag{3.24}$$

One can then prove that if system (3.5) *is exponentially stable, then the energy E_ε of solutions of systems* (3.23) *is exponentially stable, uniformly with respect to the viscosity parameter $\varepsilon \in [0, \infty)$. The proof can be done similarly as that of Theorem* 3.2 *and is left to the reader. The main difference that the dispersive term introduces is that the high-frequency solutions u_h of*

$$\ddot{u}_h + A_0 u_h + \varepsilon A_0 \dot{u}_h + \varepsilon A_0 u_h = 0, \quad t \geq 0, \tag{3.25}$$

with initial data $(u_h(0), \dot{u}_h(0)) \in (\mathfrak{C}^\perp_{1/\sqrt{\varepsilon}})^2 \cap (\mathcal{D}(A_0^{1/2}) \times H)$ now satisfy, instead of (3.19)*, which deteriorates when $\varepsilon \to \infty$, the following property of equirepartition of the energy:*

$$\left| \int_0^t \|\dot{u}_h\|_H^2 \, ds - (1+\varepsilon) \int_0^t \left\| A_0^{1/2} u(s) \right\|_H^2 \, ds \right| \leq 2E_{h,\varepsilon}(t) + 2E_{h,\varepsilon}(0), \tag{3.26}$$

where $E_{h,\varepsilon}$ is the energy of the solutions u_h of (3.25).

4 Applications

This section is devoted to present some precise examples.

4.1 The viscous Schrödinger equation

Let Ω be a smooth bounded domain of $\mathbb{R}^N$.

Let us now consider the following damped Schrödinger equation:

$$\begin{cases} i\dot{z} + \Delta_x z + ia(x)\, z = 0, \text{ in } \Omega \times (0, \infty), \\ z = 0, \qquad \text{on } \partial\Omega \times (0, \infty), \\ z(0) = z_0, \qquad \text{in } \Omega, \end{cases} \tag{4.1}$$

where $a = a(x)$ is a nonnegative damping function in $L^\infty(\Omega)$, which we assume to be positive in some open subdomain ω of Ω, that is, there exists $a_0 > 0$ such that

$$a(x) \geq a_0, \quad \forall x \in \omega. \tag{4.2}$$

The energy of solutions of (4.1), given by

$$E(t) = \frac{1}{2} \|z(t)\|^2_{L^2(\Omega)}, \tag{4.3}$$

satisfies

$$\frac{dE}{dt}(t) = -\int_\Omega a(x)|z(t,x)|^2 \, dx. \tag{4.4}$$

The stabilization problem for (4.1) has been studied in recent years. Let us briefly present some known results. Some of them concern the problem of exact controllability but, as explained for instance in [16], it is equivalent to the observability and the stabilization ones addressed in this article in the case where the damping operator B is bounded.

For instance, in [14], it is proved that the Geometric Control Condition (GCC) is sufficient to guarantee the stabilization property (1.6) for the damped Schrödinger equation (4.1). The GCC can be, roughly, formulated as follows (see [2] for the precise setting): The subdomain ω of Ω is said to satisfy the GCC if there exists a time $T > 0$ such that all rays of Geometric Optics that propagate inside the domain Ω at velocity one reach the set ω in time less than T. This condition is necessary and sufficient for the stabilization property to hold for the wave equation.

But, in fact, the Schrödinger equation behaves slightly better than a wave equation from the stabilization point of view because of the infinite velocity of propagation and, in this case, the GCC is sufficient but not always necessary. For instance, in [13], it has been proved that when the domain Ω is a square, for any nonempty bounded open subset ω, the stabilization property (1.6) holds for system (4.1). Other geometries have also been dealt with: We refer to the articles [4, 1].

Now, we assume that ω satisfies the GCC and, consequently, that we are in a situation where the stabilization property (1.6) for (4.1) holds, and we consider the viscous approximations

$$\begin{cases} i\dot{z} + \Delta_x z + ia(x)\, z - i\sqrt{\varepsilon}\Delta_x z = 0, & \text{in } \Omega \times (0,\infty), \\ z = 0, & \text{on } \partial\Omega \times (0,\infty), \\ z(0) = z_0, & \text{in } \Omega, \end{cases} \tag{4.5}$$

where $\varepsilon \geq 0$.

System (4.1) can be seen as a Ginzburg–Landau-type approximation. More precisely, system (4.1) is the inviscid limit of (4.5). We refer to the works [17, 3] where inviscid limits were analyzed in a nonlinear context.

For the stabilization problem, Theorem 3.1 applies and provides the following result:

Theorem 4.1 *Assume that system* (4.1) *is exponentially stable, i.e., it satisfies* (1.6).

Then the solutions of (4.5) *are exponentially decaying in the sense of* (1.6), *uniformly with respect to the viscosity parameter* $\varepsilon \geq 0$.

Proof Let us check the hypotheses of Theorem 3.1.

This example enters in the abstract setting given in the introduction: The operator $A = i\Delta_x$ with the Dirichlet boundary conditions is indeed skew-adjoint in $L^2(\Omega)$ with compact resolvent and domain $\mathcal{D}(A) = H^2 \cap H_0^1(\Omega) \subset L^2(\Omega)$. Since a is a nonnegative function, the damping term in (4.1) takes the form B^*Bz where B is defined as the multiplication by $\sqrt{a(x)}$, which is obviously bounded from $L^2(\Omega)$ to $L^2(\Omega)$.

The viscosity operator is

$$\varepsilon \mathcal{V}_\varepsilon = \sqrt{\varepsilon}\Delta_x = -i\sqrt{\varepsilon}A = -\sqrt{\varepsilon}|A|.$$

Obviously, this viscosity operator $\mathcal{V}_\varepsilon$ satisfies the assumptions 1, 2 and 3, and therefore Theorem 3.1 applies.

4.2 The viscous damped wave equation

Again, let Ω be a smooth bounded domain of $\mathbb{R}^N$.

We now consider the damped wave equation

$$\begin{cases} \ddot{v} - \Delta_x v + a(x)\,\dot{v} = 0, \text{ in } \Omega \times (0,\infty), \\ v = 0, \qquad \text{on } \partial\Omega \times (0,\infty), \\ v(0) = v_0, \quad \dot{v}(0) = v_1, \text{ in } \Omega, \end{cases} \tag{4.6}$$

where a is a nonnegative function as before, and satisfies (4.2) for some nonempty open subset ω of Ω.

The energy of solutions of (4.6), given by

$$E(t) = \frac{1}{2}\ \|\dot{v}\|^2_{L^2(\Omega)} + \frac{1}{2}\ \|\nabla v\|^2_{L^2(\Omega)}\,, \tag{4.7}$$

satisfies the dissipation law

$$\frac{dE}{dt}(t) = -\int_\Omega a(x)|\dot{v}|^2\ dx. \tag{4.8}$$

We assume that system (4.6) is exponentially stable. From the works [2, 5], this is the case if and only if ω satisfies the Geometric Control Condition given above.

We now consider viscous approximations of (4.6) given, for $\varepsilon > 0$, by

$$\begin{cases} \ddot{v} - \Delta_x v + a(x)\dot{v} - \varepsilon\Delta_x\dot{v} = 0, & \text{in } \Omega \times (0,\infty), \\ v = 0, & \text{on } \partial\Omega \times (0,\infty), \\ v(0) = v_0 \in H_0^1(\Omega), \quad \dot{v}(0) = v_1 \in L^2(\Omega). \end{cases} \tag{4.9}$$

Setting $A_0 = -\Delta_x$ with Dirichlet boundary conditions and $C = \sqrt{a(x)}$, Theorem 3.2 applies:

Theorem 4.2 *Assume that ω satisfies the Geometric Control Condition.*

Then the solutions of (4.9) *decay exponentially, i.e., satisfy* (1.6) *uniformly with respect to the viscosity parameter $\varepsilon \in [0, 1]$. To be more precise, there exist positive constants μ_0 and ν_0 such that for all $\varepsilon \in [0, 1]$, for any initial data in $H_0^1(\Omega) \times L^2(\Omega)$, the solution of* (4.9) *satisfies*

$$E(t) \leq \mu_0 \, E(0) \exp(-\nu_0 t), \quad t \geq 0. \tag{4.10}$$

5 Further comments

1. In this article, we have identified a class of damped systems, with added viscosity term, in which overdamping does not occur. This is to be compared with the existing literature on the overdamping phenomenon for the damped wave equation ([6, 7]).

2. As we mentioned in the introduction, our methods and results require the assumption that the damping operator B is bounded. This is due to the method we employ, which is based on the equivalence between the exponential decay of the energy and the observability properties of the conservative system, that requires the damping operator to be bounded. However, in several relevant applications, as for instance when dealing with the problem of boundary stabilization of the wave equation (see [16]), the feedback law is unbounded, and our method does not apply. This issue requires further work.

3. The same methods allow obtaining numerical approximation schemes with uniform decay properties.

The discrete analogue of the viscosity term added above for the stabilization of the wave equation has already been discussed in the works [21, 20, 18, 9] for space semidiscrete approximation schemes of damped wave equations. In those articles, though, the viscosity term is needed due to the presence of high-frequency spurious solutions that do not propagate and therefore are not efficiently damped by the damping operator B^*B when it is localized in space as in the examples considered above.

Following the same ideas as in [21, 20, 18, 9], if observability properties such as (1.9) hold for fully discrete approximation schemes of the conservative linear system (1.7) in a filtered space (see [8]), then adding a suitable viscosity term to the corresponding fully discrete version of the dissipative system (1.5) suffices to obtain uniform (with respect to space time discretization parameters) stabilization properties. This issue is currently investigated by the authors and will be published in [10].

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The Hyperbolic Symmetrizer: Theory and Applications

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Summary. The hyperbolic symmetrizer is a matrix which symmetrizes in a standard way any Sylvester hyperbolic matrix. This paper deals with the theory of the hyperbolic symmetrizer, its relationships with the concept of Bezout matrix, its perturbations which originate the so–called quasi-symmetrizer and its applications to Cauchy problems for linear weakly hyperbolic equations.

2000 AMS Subject Classification: Primary: 35L30, 35L80; Secondary: 15A21.

Key words: Hyperbolic Symmetrizer, Hyperbolic equation, Cauchy problem well posed, Bezout Matrix, Sylvester Matrix.

1 Introduction

Let us consider the system

$$U_t = \sum_{h=1}^{n} A_h(t) U_{x_h} \qquad \text{in } \mathbb{R}^n_x \times \mathbb{R}_t \,. \tag{1.1}$$

Let V be the Fourier transform with respect to x of U, so that

$$V'(t,\xi) = i|\xi| A(t,\xi) V(t,\xi) \tag{1.2}$$

with $A(t,\xi) = \sum_h A_h(t)\xi_h/|\xi|$ 0–homogeneous in ξ. Hyperbolicity means that $A(t,\xi)$ has real eigenvalues $\forall t,\xi \in \mathbb{R}_t \times \mathbb{R}^n_\xi$.

There are two favorable cases for the symbol A: A is hermitian, or A is uniformly and regularly diagonalizable. In the first case, if we set $E = \langle V, V\rangle$, we get

$$E' = 2\mathrm{Re}\,\langle V', V\rangle = 2\mathrm{Re}\,\langle i|\xi|A(t,\xi)V, V\rangle = 0, \tag{1.3}$$

i.e., E (which may be called *energy* of the system) is an invariant; in the second case, there exists $N(t,\xi)$, regular with respect to t, bounded together with its inverse, and such that $NA = DN$, D being the diagonal matrix bearing the eigenvalues (real by hypothesis) of A. Setting $Q = N^*N$, Q is regular and positive defined, and $QA = A^*Q$. We will call Q a *symmetrizer* for the symbol A.

In this case we define the energy of the system as $E = \langle QV, V\rangle$. We get

$$\begin{aligned} E' &= \langle Q'V, V\rangle + 2\mathrm{Re}\,\langle QV', V\rangle \\ &= \langle Q'V, V\rangle + 2\mathrm{Re}\,\langle i|\xi|(QA)\,V, V\rangle = \langle Q'V, V\rangle \le CE, \end{aligned} \tag{1.4}$$

from which we deduce the so–called *energy estimate* $E(t,\xi) \le C(t)E(0,\xi)$. The energy estimate, together with the Fourier transform, allow us to solve the hyperbolic system, and to determine the functional spaces in which the Cauchy problem for the system is well–posed.

Therefore, it is clear that the core of the problem consists in symmetrizing, uniformly and regularly, the symbol $A(t,\xi)$, if it is not already hermitian. It is easy to see that we may find a good symmetrizer in a standard way if the system is strictly hyperbolic, i.e., A is regular in t and it has real and distinct eigenvalues. In this case, the matrix N is made up by eigenvectors of A, and the symmetrizer is $Q = N^*N$.

The problem is much more delicate when only weak hyperbolicity (i.e., the symbol A has real but possibly coincident eigenvalues) is assumed. Throughout this paper, we will be concerned with scalar equations of order N, which may be regarded as a particular (and more favorable) case among first-order systems.

Let us consider the N–order homogeneous Kovalevskian operator

$$L = \partial_t^N - \sum_{\substack{1\le j\le N\\ |\nu|=j}} a_{\nu,j}(t)\,\partial_x^\nu\,\partial_t^{N-j}\,. \tag{1.5}$$

Let u be any solution to $Lu = 0$. Fourier transform with respect to the x variable plus standard transformation into a first–order system leads to consider a problem like (1.2), where A is the Sylvester matrix

$$A(t,\xi) = \begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & 0 & \dots \\ & & \ddots & & \\ & & & 0 & 1 \\ h_N & \dots & \dots & \dots & h_1 \end{pmatrix} \tag{1.6}$$

and any h_j is a (fixed) polynomial expression in the coefficients $a_{\nu,h}$ of the symbol of L, i.e.,

$$P(\tau,\xi;t) = \tau^N - \sum_{\substack{1\le j\le N \\ |\nu|=j}} a_{\nu,j}(t)\,\xi^\nu\,\tau^{N-j} := \tau^N - \sum_{j=1}^{N} h_j(t,\xi)\,|\xi|^j \tau^{N-j} \quad (1.7)$$

whose roots $\tau_1(\xi),\ldots,\tau_N(\xi)$ coincide with the eigenvalues of A multiplied by $|\xi|$. Let us remark that $A(t,\xi)$ is 0–homogeneous with respect to ξ.

Hyperbolicity of L means that A is a real–valued Sylvester matrix with only real eigenvalues, i.e., what we call a *hyperbolic Sylvester matrix*.

Then the following theorem holds:

Theorem 1 (see [6]) *Let A be an $N\times N$ hyperbolic Sylvester matrix. Then there exists a hermitian $N\times N$ matrix Q such that:*

i) the entries of Q are polynomials in the N–tuple $h_1 \ldots h_N$ whose coefficients depend only on N;

ii) Q is strictly (resp. weakly) positive defined iff L is strictly (resp. weakly) hyperbolic;

iii) $QA = A^*Q$.

From now on we will refer to Q as the *standard symmetrizer*, or simply the symmetrizer.

In particular, Theorem 1-i) states that it is always possible to *regularly* symmetrize a symbol under Sylvester form. This is important because, if only weak hyperbolicity is assumed, the characteristic roots $\tau_i(t,\xi)$ of the principal symbol P may be much less regular – with respect to t – than the coefficients $a_{\nu,h}(t)$, while Theorem 1, essentially, states that the symmetrizer Q inherits the t–regularity of L.

Unfortunately, the symmetrizer Q is only weakly positive defined under the assumption of weak hyperbolicity; hence, it is not possible to estimate the energy defined by means of Q. Moreover, an estimate on the energy $E = \langle QV, V\rangle$ does not automatically imply an estimate on the solution V; therefore, a perturbation argument is needed.

This leads us to the concept of *quasi–symmetrizer*. The construction of a quasi–symmetrizer Q_ε is the object of the following theorem, due to P. D'Ancona and S. Spagnolo:

Theorem 2 (see [5]) *Let $A(t,\xi)$ be an $N\times N$ Sylvester hyperbolic matrix, where $t\in[0,T]$. Let A be continuous in t,ξ, 0–homogeneous in ξ. Then, for* *any $\varepsilon > 0$ there exists a quasi–symmetrizer $Q_\varepsilon(t,\xi)$, such that (C denotes a positive constant):*

i) the entries of Q_ε are polynomials in the entries of A whose coefficients depend only on N and ε;

ii) $\frac{1}{C}\varepsilon^{2(N-1)} I \le Q_\varepsilon = Q_\varepsilon^* \le CI$;

iii) $Q_\varepsilon A - A^* Q_\varepsilon \le C\varepsilon Q_\varepsilon$.

As a consequence of Theorem 2, D'Ancona and Spagnolo obtain in their cited work the following result:

Theorem 3 (see [5]) *Let us consider the following semilinear system of order N:*

$$\partial_t u + A(t,D)u = f(t,x,u)$$

where $A(t,\xi)$ is a weakly hyperbolic symbol belonging to C^N class in t, and let u be a solution which is uniformly Gevrey of order s, with

$$1 \le s < N/(N-1)\,. \tag{1.8}$$

Let us suppose that f is C^N in t, real analytic in x and entire holomorphic in u.

Then, if u is analytic at $t=0$, it remains analytic for all the time in which solution exists.

As regards Theorem 3, let us only remark that A is not, in general, a Sylvester symbol, but the proof is based on quasi–symmetrizers for Sylvester symbols and on a transformation of the system, by means of the cofactor matrix of $\tau I + A(t,\xi)$, into a system whose symbol is N–block Sylvester type.

Another interesting result, based on quasi–symmetrizer technique, has been recently obtained by T. Kinoshita and S. Spagnolo (see [9]). Namely, they consider a homogeneous weakly hyperbolic equation of order N with time-dependent coefficients, of the form

$$\begin{cases} \partial_t^N u = \left[a_1(t)\partial_x\partial_t^{N-1} + a_2(t)\partial_x^2\partial_t^{N-2} + \cdots + a_N(t)\partial_x^N\right] u, \\ \partial_t^h u\,|_{t=0} = u_h(x), \qquad h = 0,\ldots,N-1. \end{cases} \tag{1.9}$$

Here weak hyperbolicity means that

$$z^m - \sum_{j=1}^{N} a_j(t) z^{N-j} = \prod_{j=1}^{N} (z - \lambda_j(t)) \qquad \text{with } \lambda_1(t) \le \cdots \le \lambda_N(t)\,.$$

Kinoshita and Spagnolo, moreover, assume a condition on characteristic roots, i.e., they suppose that

$$\lambda_i^2(t) + \lambda_j^2(t) \le M(\lambda_i(t) - \lambda_j(t))^2, \quad 1 \le i < j \le N,\ t \in [0,T]. \tag{1.10}$$

Then they are able to prove the following:

Theorem 4 (see [9]) *Let us consider the weakly hyperbolic Cauchy problem* (1.9) *and let us suppose that* (1.10) *holds. If $a_j \in C^k([0,T])$ for some $k \ge 2$, then* (1.9) *is well-posed in Gevrey spaces of order s for*

$$1 \le s < 1 + \frac{k}{2(N-1)}\,. \tag{1.11}$$

When the a_j's are real analytic, (1.9) *is C^∞ well-posed.*

In the present work a systematic theory of hyperbolic standard symmetrizer and quasi–symmetrizer is developed. In particular we will show that:

1) The symmetrizer Q is nothing but the Bezout matrix associated to the couple of polynomials $(P, \frac{\partial P}{\partial \tau})$; this allows us to get an explicit formula for Q in terms of the coefficients $a_{\nu,h}$. Moreover, we provide analogous explicit formula for the quasi–symmetrizer Q_ε.

2) A refinement of Theorem 2 is possible. In particular, in Section 3 below we will prove the following:

Theorem 5 *Let $A(t,\xi)$ be an $N \times N$ Sylvester hyperbolic matrix, where $t \in [0,T]$. Let A be continuous in t, ξ, 0–homogeneous in ξ, and let m be the maximum multiplicity of the eigenvalues of A. Then, for any $\varepsilon > 0$ there exists a quasi–symmetrizer $Q_\varepsilon(t,\xi)$, such that (C denotes a positive constant):*

- *i) the entries of Q_ε are polynomials in the entries of A whose coefficients depend only on N and ε;*
- *ii) $\frac{1}{C}\varepsilon^{2(m-1)} I \le Q_\varepsilon = Q_\varepsilon^* \le CI$;*
- *iii) $Q_\varepsilon A - A^* Q_\varepsilon \le C\varepsilon Q_\varepsilon$.*

Theorem 5, in turn, allows a refinement of Theorems 3 and 4 as follows:

Theorem 3 holds verbatim but the assumption (1.8), which is replaced, denoting by m the maximum of the variable multiplicity of the characteristic roots of symbol $A(t,\xi)$, by the weaker assumption

$$1 \le s < m/(m-1).$$

Theorem 4 holds verbatim but the result (1.11), which is replaced, denoting by m the maximum of the variable multiplicity of the characteristic roots λ_i, by the stronger result

$$1 \le s < 1 + \frac{k}{2(m-1)}$$

(for further improvements of Theorem 4, see Section 4 below).

3) The standard symmetrizer may be used to get well-posedness results for linear weakly hyperbolic equation *without assuming any hypotheses about characteristic roots*: the hypotheses concern only coefficients and symmetrizer, which in turn (see above) may be explicitly written in terms of the coefficients. As an example, in Section 4 a result (joint with G. Taglialatela) of C^∞ well–posedness for linear homogeneous weakly hyperbolic equations with time-dependent coefficients is announced (for the proof see [7]).

The present paper is organized as follows: Section 2 is devoted to the standard symmetrizer: definition, properties, examples, relationship with the Bezout matrix. In Section 3 we study the D'Ancona and Spagnolo quasi–symmetrizer, giving a sketch proof of Theorem 2 and a detailed proof of Theorem 5; moreover, a general formula to get an explicit expression for the quasi–symmetrizer is given. In Section 4 we show the main ideas and techniques which make the symmetrizer useful in getting energy estimates for weakly hyperbolic equations; starting from this, a result of C^∞ well–posedness is announced (see Theorem 6) and conclusions are drawn.

2 The standard symmetrizer

2.1 Definition and elementary properties

Let $\varphi(x) = \sum\limits_{h=0}^{r} a_h x^{r-h}$ be a monic real hyperbolic polynomial with (eventually coincident) roots $x_1, \dots x_r$. If $1 \le h \le r$ is the number of the distinct roots of $\varphi(x)$, we will denote (after eventual renaming) by $x_1, \dots, x_h$ the distinct roots, with multiplicity $m_1, \dots, m_h$ resp., so that

$$\varphi(x) = a_0 \prod_{j=1}^{r} (x - x_j) = a_0 \prod_{j=1}^{h} (x - x_j)^{m_j}, \qquad m_1 + \dots + m_h = r. \tag{2.1}$$

If we set

$$\begin{aligned} \operatorname{coeff}(\varphi) &= (a_r, a_{r-1}, \dots, a_1, a_0), \\ V_p^0[x] &= (1, x, x^2, \dots, x^{p-2}, x^{p-1}), \\ V_p^h[x] &= D_x^h V_p^0[x], \end{aligned} \tag{2.2}$$

then we obviously get

$$\varphi(x) = \text{coeff}(\varphi) \cdot V^0_{r+1}[x]; \quad D^h_x\varphi(x) = \text{coeff}(\varphi) \cdot V^h_{r+1}[x]\,. \tag{2.3}$$

For the sake of commodity, from now on we will identify D^0_x with the identity operator, so that we will indifferently write $\varphi(x)$ or $D^0_x\varphi(x)$. Moreover, with the same symbol $V^q_p[x]$ we will denote the row vector or the column vector defined by (2.2). Now let us define

$$\varphi^{(i)}(x) = \frac{\varphi(x)}{(x - x_i)} \qquad 1 \leq i \leq r\,. \tag{2.4}$$

Then a straightforward verification shows that

$$\text{coeff}(\varphi^{(i)}) = \Big(\sum_{j=0}^{r-1} a_j x_i^{r-1-j}, \sum_{j=0}^{r-2} a_j x_i^{r-2-j}, \cdots \sum_{j=0}^{1} a_j x_i^{1-j}, a_0\Big). \tag{2.5}$$

Throughout this paper we will reserve the symbol $W(\varphi)$ to the $r \times r$ matrix

$$W(\varphi) = \begin{pmatrix} \text{coeff}(\varphi^{(1)}) \\ \dots \\ \text{coeff}(\varphi^{(r)}) \end{pmatrix}. \tag{2.6}$$

The following proposition points out some useful properties of the matrices $W(\varphi)$:

Proposition 1 *Let* $\varphi(x)$ *as in* (2.1), *and let* $W(\varphi)$ *be defined by* (2.6). *Then:*

a) $$\det W(\varphi) = a_0^r \det \begin{pmatrix} x_1^{r-1} & \dots & x_1 & 1 \\ \dots & \dots & \dots & \\ x_r^{r-1} & \dots & x_r & 1 \end{pmatrix} = a_0^r \prod_{1 \leq i < j \leq r} (x_i - x_j)\,;$$

b) If $h < r$, *where* h *is the number of distinct roots of* $\varphi(x)$, *and* Σ *is the set of multiple roots of* φ, *then the* $n - h$ *vectors*

$$V^i_r[x_j] \qquad\qquad x_j \in \Sigma, \quad 0 \leq i \leq m_j - 2, \tag{2.7}$$

constitute a basis for $\operatorname{Ker} W(\varphi)$.

Proof a) If w^j denotes the jth column of $W(\varphi)$ and v^j denotes the jth column of the Vandermonde matrix $\begin{pmatrix} x_1^{r-1} & \dots & x_1 & 1 \\ \dots & \dots & \dots & \\ x_r^{r-1} & \dots & x_r & 1 \end{pmatrix}$, then by (2.5) we have

$$w^j = \sum_{k=0}^{r-j} a_k v^{k+j}, \tag{2.8}$$

hence claim a) follows.

b) From a) we know that $W(\varphi)$ has trivial kernel iff φ is strictly hyperbolic, i.e., $h = r$. Now let us suppose that multiple roots occur, so that $h < r$. By means of (2.8) we see that the first h rows of $W(\varphi)$ are independent, as the determinant of the uppermost rightmost $h \times h$ minor of $W(\varphi)$ is equal to $a_0^h \prod_{1 \le i < j \le h} (x_i - x_j)$; therefore $\operatorname{rank} W(\varphi) \ge h$.

Let us consider the $r \times r$ matrix B whose columns are

$$B = \left(V_r^0[x_1] \dots V_r^{m_1-1}[x_1] \dots V_r^0[x_j] \dots V_r^{m_j-1}[x_j] \dots V_r^0[x_h] \dots V_r^{m_h-1}[x_h] \right).$$

The matrix B is a generalized Vandermonde matrix (it would be a classical Vandermonde iff $m_1 = \cdots = m_h = 1$, i.e., in the strict hyperbolic case). It is known (see for instance [8], Theorem 20) that the determinant of such a generalization of the classical Vandermonde matrix is given by

$$\det B = \left(\prod_{i=1}^{h} \prod_{j=1}^{m_i-1} j! \right) \prod_{1 \le i < j \le h} (x_j - x_i)^{m_i m_j}. \tag{2.9}$$

In particular, the columns of B are independent, and the vectors in (2.7) are some of these columns. Moreover, being (see (2.3))

$$W(\varphi) \cdot V_r^j[x] = \begin{pmatrix} D_x^j \varphi^{(1)}(x) \\ \cdots \\ D_x^j \varphi^{(r)}(x) \end{pmatrix} \qquad j = 0, 1, \dots,$$

we get that

$$W(\varphi) \cdot V_r^i[x_j] = \vec{0} \qquad \forall\, x_j \in \Sigma, \quad 0 \le i \le m_j - 2;$$

hence, remembering that $\operatorname{rank} W(\varphi) \ge h$, claim b) follows (and $\operatorname{rank} W(\varphi) = h$ as a matter of fact). □

Now let us come to $A(t,\xi)$ and its related symbol P. Being $A(t,\xi)$ 0–homogeneous with respect to ξ, most of the theory will be developed for $\xi \in S^N$.

So, let $|\xi| = 1$ and let P be as in (1.7). Let us define $h_0 \equiv -1$, so that

$$P(\tau) = -\sum_{j=0}^{N} h_j \tau^{N-j}. \tag{2.10}$$

A direct verification immediately shows that the ith row of $W = W(P)$ is a left eigenvector of A with eigenvalue τ_i. Hence, if $D = \begin{pmatrix} \tau_1 & & \\ & \ddots & \\ & & \tau_N \end{pmatrix}$, we get $WA = DW$. Denoting by W^* the transposed of W, we obviously have that both W^*W and $W^*WA = W^*DW$ are symmetric.

Definition 1 *The matrix $Q = W^*W$ is said to be the* standard symmetrizer *of the Sylvester matrix A.*

Of course, there is a one–to–one correspondence between symbols P and Sylvester matrices A; therefore, we can refer to Q as the symmetrizer related to A or the symmetrizer related to P as well, and we may denote Q also by $Q(A)$ or $Q(P)$ depending on what relationship we want to emphasize.

We remark that, by construction, the entries of Q are certain (fixed) symmetric polynomials in $\tau_1 \dots \tau_N$; hence, by the fundamental theorem of symmetric functions, they are polynomials in $h_1 \dots h_N$.

Moreover, by Proposition 1 we get

$$\det Q = (\det W(P))^2 = \prod_{1 \le i < j \le N} (\tau_j - \tau_i)^2, \tag{2.11}$$

hence Q is strictly (resp. weakly) positive defined iff L is strictly (resp. weakly) hyperbolic.

2.2 The standard symmetrizer and the Bezout matrix

Up to now we have reobtained Theorem 1 (see [6]); now let us show the link between the standard symmetrizer and the concept of Bezout matrix.

Let us define the elementary symmetric functions of k variables $\tau_1, \dots, \tau_k$ as

$$e_{k,0}(\tau) = 1, \quad e_{k,r}(\tau) = \sum_{i_1 < i_2 \dots < i_r} \tau_{i_1}\tau_{i_2}\dots\tau_{i_r}$$

$$i_1, \dots, i_r = 1, \dots, k, \quad r = 1, \dots, k\,;$$

if $\tau_1, \dots, \tau_N$ are the roots of the symbol P defined by (2.1), then $h_r = (-1)^{(r+1)} e_{N,r}(\tau)$, $r = 0, \dots, N$. Moreover, let $e^{(i_1 \dots i_r)}_{N,p}(\tau)$ be the pth elementary symmetric function formed from $(\tau_1, \dots, \tau_N)$ omitting $\tau_{i_1}, \dots, \tau_{i_r}$. Then a direct verification shows that

$$W_{ij} = (-1)^{N-i} e^{(j)}_{N,N-i}(\tau)$$

and therefore

$$Q_{ij} = (-1)^{i+j} \sum_{k=1}^{N} e^{(k)}_{N,N-i}(\tau) e^{(k)}_{N,N-j}(\tau). \tag{2.12}$$

Now, in [1] it is proved (see formula (33)) that

$$\sum_{k=1}^{N} e^{(k)}_{N,N-i}(\tau)e^{(k)}_{N,N-j}(\tau)$$
$$= i\, e_{N,N-i}(\tau)e_{N,N-j}(\tau) - \sum_{r=1}^{N-j}(j-i+2r)e_{N,N-i+r}(\tau)e_{N,N-j-r}(\tau),$$

hence

$$Q_{ij} = i\, h_{N-i}h_{N-j} - \sum_{r=1}^{N-j}(j-i+2r)h_{N-i+r}h_{N-j-r}. \tag{2.13}$$

As is well–known, if

$$f(z) = \sum_{i=0}^{N} u_i z^{N-i}, \qquad g(z) = \sum_{i=0}^{N} v_i z^{N-i}$$

are two polynomials of degree at most N, then the Bezout matrix $\mathrm{Bez}(f,g)$ of order N associated to f and g is a symmetric $N\times N$ matrix implicitly defined by means of the quadratic form

$$B(f,g;x,y) = \frac{f(x)g(y)-f(y)g(x)}{(x-y)} = \sum_{i,j=1}^{N}\{\mathrm{Bez}(f,g)\}_{ij}\, x^{i-1}y^{j-1}.$$

It turns out that if $m_{ij} = \min(N-i, j+1)$, then

$$\{\mathrm{Bez}(f,g)\}_{ij} = \sum_{k=1}^{m_{ij}} u_{N-1-j-k}v_{N-i-k} - u_{N-i+k}v_{N-1-j-k}. \tag{2.14}$$

By comparing (2.13) and (2.14) we get that

$$Q = \mathrm{Bez}(P, \frac{\partial P}{\partial \tau}) \tag{2.15}$$

and we reobtain (2.11) from (2.15), on account of the following facts:

- the determinant of $\mathrm{Bez}(f,g)$ is the resultant of f and g;
- the resultant of f, f' is the discriminant of f.

Moreover, there is another way to write (2.13), which will be useful in the next section. Let us define, for any n, a transformation Φ from the space of $n\times n$ real symmetric matrices to the space of $(n-1)\times(n-1)$ real symmetric matrices: if X is any $n\times n$ real symmetric matrix, then

$$\Phi(X)_{i,j} = \begin{cases} iX_{i+1,j+1} - \displaystyle\sum_{k=1}^{\min(i,n-j-1)} (j-i+2k)X_{i+1-k,\,j+1+k} & j \ge i\,; \\ jX_{j+1,i+1} - \displaystyle\sum_{k=1}^{\min(j,n-i-1)} (i-j+2k)X_{j+1-k,\,i+1+k} & j < i\,. \end{cases} \tag{2.16}$$

$\Phi(X)$, by construction, is an $(n-1) \times (n-1)$ real symmetric matrix.

Now, if P is defined by (2.10), and if we consider the $(N+1) \times (N+1)$ real symmetric matrix

$$\Pi = \begin{pmatrix} h_m h_m & h_m h_{m-1} & \dots & h_m h_1 & h_m h_0 \\ h_{m-1} h_m & h_{m-1} h_{m-1} & \dots & h_{m-1} h_1 & h_{m-1} h_0 \\ \dots & \dots & \dots & \dots & \dots \\ h_0 h_m & h_0 h_{m-1} & \dots & h_0 h_1 & h_0 h_0 \end{pmatrix},$$

then

$$Q = \Phi(\Pi). \tag{2.17}$$

2.3 How symmetrizer checks hyperbolicity

Let P be a real monic polynomial of degree N. Then it is well-known (see for instance [11]) that it is possible to count the real roots of P by means of a suitable sequence of principal minors of $\mathrm{Bez}(P, P')$, i.e., by means of its standard symmetrizer Q.

Namely, for $j = 1 \dots N-1$ let

$$Q_j = \begin{pmatrix} Q_{j,j} & \dots & Q_{j,N} \\ \vdots & \vdots & \vdots \\ Q_{N,j} & \dots & Q_{N,N} \end{pmatrix} \tag{2.18}$$

be the principal $(N-j+1) \times (N-j+1)$ minor of Q obtained by removing the first $j-1$ rows and the first $j-1$ columns of Q. We remark that $Q_1 = Q$ and $Q_N = \{N\}$. Let

$$\Delta_1 = \det Q,\ \Delta_2 = \det Q_2,\ \dots\ \Delta_j = \det Q_j,\ \dots\ \Delta_N = \det Q_N = N \tag{2.19}$$

(observe that $\Delta_1 = \det Q$ is the discriminant of P). Then:

P is strictly hyperbolic $\Longleftrightarrow \Delta_1 > 0, \Delta_2 > 0, \dots, \Delta_N > 0;$
P is weakly hyperbolic $\Longleftrightarrow$
$$\exists\, h : \Delta_1 = \dots = \Delta_{N-h} = 0, \Delta_{N-h+1} > 0, \dots, \Delta_N > 0;$$
in this case P has exactly h distinct roots.

Let us remark that weak hyperbolicity of a real polynomial P is not only a matter of weak positivity of the sequence $\Delta_1 \dots \Delta_N$; the (eventual) zeros must all be confined at the leftmost part of the sequence.

2.4 A few examples

Let $N = 2$, so that

$$L = \partial_t^2 - \sum_{\substack{1\le j\le 2\\ |\nu|=j}} a_{\nu,j}(t)\,\partial_x^\nu\,\partial_t^{2-j}\,;$$

$$\begin{aligned} P(\tau,\xi;t) &= \tau^2 - \sum_{\substack{1\le j\le 2\\ |\nu|=j}} a_{\nu,j}(t)\,\xi^\nu\,\tau^{2-j} = \tau^2 - h_1(t,\xi)|\xi|\tau - h_2(t,\xi)|\xi|^2 \\ &= (\tau-\tau_1(t,\xi)|\xi|)(\tau-\tau_2(t,\xi)|\xi|)\,; \end{aligned}$$

$$A = \begin{pmatrix} 0 & 1 \\ h_2 & h_1 \end{pmatrix}, \quad W = \begin{pmatrix} -\tau_1 & 1 \\ -\tau_2 & 1 \end{pmatrix}.$$

Hence

$$\begin{cases} h_0 = -1 \\ h_1 = \tau_1 + \tau_2 \\ h_2 = -\tau_1\tau_2 \end{cases} ; \qquad \Pi = \begin{pmatrix} h_2h_2 & h_2h_1 & h_2h_0 \\ h_1h_2 & h_1h_1 & h_1h_0 \\ h_0h_2 & h_0h_1 & h_0h_0 \end{pmatrix}.$$

Then

$$Q = \Phi(\Pi) = \begin{pmatrix} 2h_2 + h_1^2 & -h_1 \\ -h_1 & 2 \end{pmatrix} = \begin{pmatrix} \tau_1^2 + \tau_2^2 & -\tau_1 - \tau_2 \\ -\tau_1 - \tau_2 & 2 \end{pmatrix} = W^*W.$$

Let $N = 3$, so that

$$L = \partial_t^3 - \sum_{\substack{1\le j\le 3\\ |\nu|=j}} a_{\nu,j}(t)\,\partial_x^\nu\,\partial_t^{3-j}\,;$$

$$\begin{aligned} P(\tau,\xi;t) &= \tau^3 - \sum_{\substack{1\le j\le 3\\ |\nu|=j}} a_{\nu,j}(t)\,\xi^\nu\,\tau^{3-j} \\ &= \tau^3 - h_1(t,\xi)|\xi|\tau - h_2(t,\xi)|\xi|^2 - h_3(t,\xi)|\xi|^3 \\ &= (\tau-\tau_1(t,\xi)|\xi|)(\tau-\tau_2(t,\xi)|\xi|)(\tau-\tau_3(t,\xi)|\xi|)\,; \end{aligned}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ h_3 & h_2 & h_1 \end{pmatrix}, \quad W = \begin{pmatrix} \tau_1\tau_2 & -(\tau_1+\tau_2) & 1 \\ \tau_1\tau_3 & -(\tau_1+\tau_3) & 1 \\ \tau_2\tau_3 & -(\tau_2+\tau_3) & 1 \end{pmatrix}.$$

Hence

$$\begin{cases} h_0 = -1 \\ h_1 = \tau_1 + \tau_2 + \tau_3 \\ h_2 = -\tau_1\tau_2 - \tau_1\tau_3 - \tau_2\tau_3 \\ h_3 = \tau_1\tau_2\tau_3 \end{cases} \quad ; \qquad \Pi = \begin{pmatrix} h_3h_3 & h_3h_2 & h_3h_1 & h_3h_0 \\ h_2h_3 & h_2h_2 & h_2h_1 & h_2h_0 \\ h_1h_3 & h_1h_2 & h_1h_1 & h_1h_0 \\ h_0h_3 & h_0h_2 & h_0h_1 & h_0h_0 \end{pmatrix}.$$

Then

$$\begin{aligned} Q = \Phi(\Pi) &= \begin{pmatrix} h_2^2 - 2h_1h_3 & h_1h_2 + 3h_3 & -h_2 \\ h_1h_2 + 3h_3 & 2h_1^2 + 2h_2 & -2h_1 \\ -h_2 & -2h_1 & 3 \end{pmatrix} \\ &= \sum_{1\le i<j\le 3} \begin{pmatrix} \tau_i^2\tau_j^2 & -\tau_i^2\tau_j - \tau_i\tau_j^2 & \tau_i\tau_j \\ -\tau_i^2\tau_j - \tau_i\tau_j^2 & (\tau_i + \tau_j)^2 & -2\tau_i \\ \tau_i\tau_j & -2\tau_i & 3 \end{pmatrix} = W^*W. \end{aligned} \tag{2.20}$$

3 The quasi–symmetrizer

3.1 Sketch proof of Theorem 2

Let T be the $N \times N$ lower triangular matrix defined as

$$T_{ij} = \begin{cases} (-1)^{i+j} e_{i-1,i-j}(\tau) & \text{if } j \le i, \\ 0 & \text{if } j > i, \end{cases} \tag{3.1}$$

i.e.,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -\tau_1 & 1 & 0 & 0 & 0 & \dots & 0 \\ \tau_1\tau_2 & -(\tau_1 + \tau_2) & 1 & 0 & 0 & \dots & 0 \\ -\tau_1\tau_2\tau_3 & \tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1 & -(\tau_1 + \tau_2 + \tau_3) & 1 & 0 & \dots & 0 \\ & & \dots & & & & \end{pmatrix},$$

then we see, by direct inspection, that T triangulates A, in the sense that

$$TAT^{-1} = \begin{pmatrix} \tau_1 & 1 & & & \\ & \tau_2 & 1 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & \tau_{N-1} & 1 \\ & & & & \tau_N \end{pmatrix} = D + K \tag{3.2}$$

with obvious meaning of the symbols. Now, defining

$$H_\varepsilon = \begin{pmatrix} \varepsilon^{N-1} & & & 0 \\ & \varepsilon^{N-2} & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \qquad \text{and} \qquad S_\varepsilon = T^* H_\varepsilon^2 T \tag{3.3}$$

we see that S_ε is a (nonsmooth, when unfreezing coefficients) *quasi–symmetrizer* for A, which satisfies ii) and iii) (in place of Q_ε). It satisfies ii) by its very definition; as regards iii), we have

$$H_\varepsilon K = \varepsilon K H_\varepsilon \tag{3.4}$$

and

$$\begin{aligned} S_\varepsilon A - A^* S_\varepsilon &= T^* H_\varepsilon^2 T A - (T^* H_\varepsilon^2 T A)^* \\ &= T^* H_\varepsilon^2 (D+K) T - (T^* H_\varepsilon^2 (D+K) T)^* \\ &= T^* (H_\varepsilon^2 K - K^* H_\varepsilon^2) T = \varepsilon T^* (H_\varepsilon K H_\varepsilon - H_\varepsilon K^* H_\varepsilon) T\,; \end{aligned} \tag{3.5}$$

hence, on account of

$$|\langle Kz, z\rangle| \le |z|^2 \quad \forall z \tag{3.6}$$

we deduce

$$\begin{aligned} |\langle (S_\varepsilon A - A^* S_\varepsilon) z, z\rangle| &= 2\varepsilon |\mathrm{Im}\langle (K H_\varepsilon T) z, H_\varepsilon T z\rangle \\ &\le 2\varepsilon |H_\varepsilon T z|^2 = 2\varepsilon \langle S_\varepsilon z, z\rangle\,. \end{aligned} \tag{3.7}$$

Now we must manage to fulfill i), in order to get a *smooth* quasi–symmetrizer, whose entries are polynomials in the entries of A. To this aim, if σ is any permutation on $\tau_1 \dots \tau_N$ and S_{ε_σ} is the matrix obtained by S_ε under the action of σ, let us define

$$Q_\varepsilon = \sum_\sigma S_{\varepsilon_\sigma} = \sum_{k=0}^{N-1} \varepsilon^{2k} Q^{(k)} \tag{3.8}$$

where the sum runs over all possible permutations σ. Then again Q_ε fulfills ii) and iii), but now any of its entries is a symmetric polynomial in the N–tuple $\tau_1 \dots \tau_N$, hence it is a polynomial in the entries of A. Standard continuity and compactness arguments allow us to manage variable coefficients, so that thesis easily follows. $\square$

3.2 Going deep into quasi–symmetrizer

Let us go back to the definition of S_ε in (3.3). The matrix S_ε is a polynomial expression in ε with $N \times N$ matrices as coefficients, i.e.,

$$S_\varepsilon = \sum_{h=0}^{N-1} \varepsilon^{2h} M^{(h)}$$

where $M_{ij}^{(N-h)} = T_{hi}T_{hj}$; hence, by (3.1),

$$M_{ij}^{(N-h)} = \begin{cases} (-1)^{i+j} e_{h-1,h-i}(\tau) e_{h-1,h-j}(\tau) & \text{if } 1 \le i,j \le h \\ 0 & \text{otherwise} \end{cases} \qquad h = 1,\dots,N. \tag{3.9}$$

By (3.9) we immediately learn the following facts about the $M^{(h)}$:

- The last h rows and h columns of $M^{(h)}$ are identically zero;
- $M^{(N-h)}$ depends only on $\tau_1,\dots,\tau_{h-1}$, $h = 2\dots N$, while $M^{(N-1)}$ is constant;
- $M_{hh}^{(N-h)} = 1, \quad h = 1,\dots,N.$

To get a smooth quasi–symmetrizer, in the proof of Theorem 2, following [5], a sum is performed over *all* possible permutations σ. This is redundant; for instance, $M^{(N-1)}$ is constant, so that it does not need any sum. Let us be more precise about any $M^{(h)}$.

The matrix $M^{(0)}$ depends on $\tau_1,\dots,\tau_{N-1}$, so for $M^{(0)}$ we may restrict the sum to the subgroup of permutations $\sigma \in \Sigma_1$ which choose any subset of $N-1$ roots τ_j keeping indexes j in increasing order. We have $\#(\Sigma_1) = N$ (here $\#$ stands for cardinality).

Analogously, for $h = 1\dots N-2$, the matrix $M^{(h)}$ depends only on $\tau_1,\dots,$ τ_{N-h-1}, so we may restrict the sum to the subgroup of permutations $\sigma \in \Sigma_{h+1}$ which choose any subset of $N-h-1$ roots τ_j keeping indexes j in increasing order. We have $\#(\Sigma_{h+1}) = \binom{N}{N-h-1}$.

Let us define

$$Q^{(h)} = \sum_{\sigma \in \Sigma_{h+1}} M_\sigma^{(h)}, \quad h = 0,\dots,N-2; \quad Q^{(N-1)} = M^{(N-1)} \tag{3.10}$$

where obviously $M_\sigma^{(h)}$ denotes $M^{(h)}$ under the action of σ. Then the matrices $Q^{(j)}$ defined in (3.8) coincide with the $Q^{(j)}$ defined in (3.10) apart from multiplicative constants due to the redundance of useless permutations of roots; from now on we will neglect these constants and adopt (3.10) for the $Q^{(j)}$.

Let us remark that, just as for the $M^{(j)}$, the last j rows and last j columns of the matrices $Q^{(j)}$ are identically zero. As we need to study the properties of the nonzero part of matrices $Q^{(j)}$, we introduce the symbol $\widetilde{Q}^{(j)}$ which stands for the $(N-j)\times(N-j)$ matrix obtained taking the first $N-j$ rows and first $N-j$ columns of $Q^{(j)}$.

From (3.10), (3.9) and (2.12) we immediately get

$$Q_{ij}^{(0)} = (-1)^{i+j}\sum_{k=1}^{N} e^k_{N,N-i}(\tau)e^k_{N,N-j}(\tau) = Q_{ij},$$

i.e., the quasi–symmetrizer is a perturbation of the standard symmetrizer; Q_e and Q coincide when $\varepsilon = 0$.

The following Proposition 2 will establish the relevant properties of the matrices $\widetilde{Q}^{(h)}$, with $h \geq 1$. Let us introduce the notation

$$P^{(i_1\dots i_r)}(\tau) = \frac{P(\tau)}{(\tau-\tau_{i_1})\dots(\tau-\tau_{i_r})}, \qquad 1 \leq i_1 < \dots < i_r \leq N.$$

Proposition 2

a) The following relations hold:

$$\begin{aligned}\widetilde{Q}^{(h)} &= \frac{1}{h+1}\sum_{1\leq i_1<\dots<i_h\leq N} Q(P^{(i_1\dots i_h)}) \qquad h = 1,\dots,N-2\,;\\ \widetilde{Q}^{(h)}_{N-h,N-h} &= \binom{N}{N-h-1}.\end{aligned} \tag{3.11}$$

b) Let us fix τ and $\xi : |\xi| = 1$. Let m_1 be the maximum multiplicity of the roots τ_j of $P = (\tau-\tau_1)^{m_1}\dots(\tau-\tau_h)^{m_h}$. Then

$$\widetilde{Q}^{(j)} \text{ is strictly positive} \Longleftrightarrow j \geq m_1 - 1. \tag{3.12}$$

Proof a) Let us carry out the proof for $h = 1$, the general case being quite analogous. From (3.9) we have

$$\begin{aligned}M_{ij}^{(1)} &= (-1)^{i+j}e_{N-2,N-1-i}(\tau)e_{N-2,N-1-j}(\tau)\\ &= (-1)^{i+j}e^{(N-1,N)}_{N,N-1-i}(\tau)e^{(N-1,N)}_{N,N-1-j}(\tau),\end{aligned}$$

hence, by (3.10),

$$\widetilde{Q}^{(1)}_{ij} = \sum_{\sigma\in\Sigma_2} M^{(1)}_{ij\ \sigma}$$

$$= \frac{1}{2}(-1)^{(i+j)} \sum_{k=1}^{N}\sum_{r\neq k} e^{(k,r)}_{N,N-1-i}(\tau)e^{(k,r)}_{N,N-1-j}(\tau) = \frac{1}{2}\sum_{k=1}^{N} Q(P^{(k)})_{ij}$$

(the factor $\frac{1}{2}$ above depends on the definition of subgroup of permutations Σ_2, which was chosen as small as possible, keeping indexes of the roots in increasing order; of course, from the viewpoint of ε estimates about the quasi–symmetrizer, any multiplicative constant is negligible).

Analogously

$$\widetilde{Q}^{(h)}_{ij}$$

$$= \sum_{\sigma\in\Sigma_{h+1}} M^{(h)}_{ij\ \sigma} = \cdots = \frac{1}{h+1}\sum_{1\leq i_1<\cdots<i_h\leq N} Q(P^{(i_1\ldots i_h)}), \quad h=1,\ldots,N-2,$$

and, being $M^{(h)}_{N-h,N-h}=1$, from the above formula we get

$$\widetilde{Q}^{(h)}_{N-h,N-h} = \#(\Sigma_{h+1}) = \binom{N}{N-h-1}.$$

b) From a) and from the definition of standard symmetrizer, we know that

$$\langle\widetilde{Q}^{(h)}y,y\rangle = \frac{1}{h+1}\sum_{1\leq i_1<\cdots<i_h\leq N}\left|W(P^{(i_1\ldots i_h)})\cdot y\right|^2, \qquad y\in\ \mathbb{R}^{N-h}. \quad (3.13)$$

Therefore the quadratic form $\langle\widetilde{Q}^{(h)}y,y\rangle$ is not strictly positive defined, but only weakly positive, if and only if the intersection of the kernels of $W(P^{(i_1\ldots i_h)})$, when $i_1,\ldots,i_h$ vary, is nontrivial.

But Proposition 1–b) gives a basis of $\operatorname{Ker} W(\varphi)$ for any real hyperbolic polynomial φ, and it is quite easy to see that if $\{\varphi_\alpha\}_\alpha$ is a family of r–degree monic hyperbolic polynomials, then

$$\bigcap_\alpha \operatorname{Ker} W(\varphi_\alpha)\neq\emptyset \Longleftrightarrow \exists\,\overline{x} : \overline{x} \text{ is a multiple root for any } \varphi_\alpha. \qquad (3.14)$$

Being $P=(\tau-\tau_1)^{m_1}\ldots(\tau-\tau_h)^{m_h}$, we have that

$$\tau_j \text{ is a multiple root for any } P^{(i_1\ldots i_r)} \Longleftrightarrow r\leq m_j-2 \qquad (3.15)$$

and the highest value of r for which (3.15) may occur is $r=m_1-2$; therefore, (3.12) follows from (3.13)–(3.15).

3.3 The proof of Theorem 5

Let

$$Q_\varepsilon = \sum_{k=0}^{m-1} \varepsilon^{2k} Q^{(k)} . \tag{3.16}$$

On account of Theorem 2, all we have to prove is the existence of $\delta > 0$ such that

$$Q_\varepsilon \geq \delta \varepsilon^{2(m-1)} I . \tag{3.17}$$

We have

$$\langle Q_\varepsilon y, y\rangle = \sum_{k=0}^{m-1} \varepsilon^{2k} \langle Q^{(k)} y, y\rangle \qquad y \in \mathbb{R}^N \tag{3.18}$$

and any term of the sum is not negative. We claim that

$$\forall\, y : |y| = 1 \quad \exists\, j,\, 0 \leq j \leq m-1 \,:\, \langle Q^{(j)} y, y\rangle > 0 \tag{3.19}$$

from which (3.17) follows. Indeed, by absurd, if this is not the case, then there exists $\overline{y} = (\overline{y}_1 \dots \overline{y}_N) : |\overline{y}| = 1$ and

$$\langle Q_\varepsilon \overline{y}, \overline{y}\rangle = 0 \Longrightarrow \langle Q^{(h)} y, y\rangle = 0, \quad h = 0 \dots m-1 . \tag{3.20}$$

Now, by Proposition 2 we know that $\widetilde{Q}^{(m-1)} > 0$, hence

$$\overline{y}_1 = \cdots = \overline{y}_{N-m+1} = 0 ; \tag{3.21}$$

therefore, remembering that $Q^{(j)}_{N-j,N-j} > 0$ (see (3.11)) and that the last j rows and last j columns of $Q^{(j)}$ are identically zero, we obtain

$$\begin{aligned} 0 = \langle Q^{(m-2)} \overline{y}, \overline{y}\rangle &= Q^{(m-2)}_{N-m+2,N-m+2} \overline{y}^2_{N-m+2} \\ \Longrightarrow \overline{y}_{N-m+2} &= 0 ; \end{aligned} \tag{3.22}$$

backward induction brings us to the conclusion $\overline{y} = \vec{0}$, an absurd. □

At the end of this section, we want to point out that, analogously to what happens for the standard symmetrizer, there is a closed formula to write down the quasi–symmetrizer. Namely, if $\varPhi$ is defined by (2.16), then it may be proved that

$$\widetilde{Q}^{(1)} = \frac{1}{2}\varPhi(Q), \quad \widetilde{Q}^{(2)} = \frac{1}{3}\varPhi(\widetilde{Q}^{(1)}) \quad \dots \quad \widetilde{Q}^{(j+1)} = \frac{1}{j+2}\varPhi(\widetilde{Q}^{(j)}) \quad \dots . \tag{3.23}$$

3.4 An example

Let $N = 3$. Then (see (2.20))

$$Q \quad = \begin{pmatrix} h_2^2 - 2h_1h_3 & h_1h_2 + 3h_3 & -h_2 \\ h_1h_2 + 3h_3 & 2h_1^2 + 2h_2 & -2h_1 \\ -h_2 & -2h_1 & 3 \end{pmatrix},$$

$$\widetilde{Q}^{(1)} = \frac{1}{2}\Phi(Q) = \begin{pmatrix} 2h_2 + h_1^2 & -h_1 \\ -h_1 & 3 \end{pmatrix}, \quad \widetilde{Q}^{(2)} = \frac{1}{3}\Phi(\widetilde{Q}^{(1)}) = (1),$$

hence

$$Q_\varepsilon =$$

$$\begin{pmatrix} h_2^2 - 2h_1h_3 & h_1h_2 + 3h_3 & -h_2 \\ h_1h_2 + 3h_3 & 2h_1^2 + 2h_2 & -2h_1 \\ -h_2 & -2h_1 & 3 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 2h_2 + h_1^2 & -h_1 & 0 \\ -h_1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4 Hyperbolic symmetrizer and weakly hyperbolic equations

In this section we want to show that the hyperbolic symmetrizer is a useful tool for getting energy estimates and proving well–posedness of weakly hyperbolic Cauchy problems. Moreover, the symmetrizer allows us to get well–posedness results assuming hypotheses *on the coefficients* of a scalar hyperbolic operator, instead of considering the characteristic roots, as most of the literature about the subject does. This is meaningful for high-order scalar hyperbolic operators, in which an explicit expression for characteristic roots is often not available. For instance, let L as in (1.5), and let us suppose that the coefficients $a_{\nu,j}$ are analytic in $[0, T]$. We want to state a result of C^∞ well–posedness for the equation $L[u] = 0$ in the strip $\mathbb{R}^n_x \times [0, T]$. We know from [2] that if $a(t)$ is a nonnegative analytic function, then the Cauchy problem at $t = 0$ for the model equation

$$u_{tt} = a(t)u_{xx}\,, \qquad (x, t) \in \mathbb{R}_x \times [0, T] \tag{4.1}$$

is C^∞ well–posed, and indeed [2] is essentially concerned with weakly hyperbolic wave–type homogeneous equations with regular coefficients.

On the other hand, both the equations

$$\begin{aligned} &u_{tt} - u_x = 0, \\ &u_{tt} - 2tu_{tx} + t^2 u_{xx} = 0 \end{aligned} \tag{4.2}$$

are C^∞ ill–posed. The first equation in (4.2) is not homogeneous, and its lower order term does not fulfill the Levi condition, which is necessary for C^∞ well–posedness; but the second equation is homogeneous, and nevertheless it is C^∞ ill–posed, as it may be transformed into the first equation by means of a change of variables. Therefore, any extension of [2] results must assume some algebraic hypotheses about the operator L. Here we show how it is possible to formulate these hypotheses by means of the symmetrizer.

In order to better understand the assumptions we are going to make about $Q = Q(L)$, let us go back to our model equation (4.1).

4.1 Sketch proof of C^∞ well–posedness of (4.1)

If $f(x)$ is a nonnegative real analytic function on an interval $[a, b]$ with isolated zeros, say $a \leq x_1 < \cdots < x_k \leq b$, we will denote by $z_f(x)$ the function

$$z_f(x) = \prod_{i=1}^{k} |x - x_i|. \tag{4.3}$$

Obviously

$$\exists\, C : |z_f(x) f'(x)| \leq C f(x) \qquad \forall\, x \in [a, b]. \tag{4.4}$$

Now, coming to equation (4.1), if $a(t) \equiv 0$, or if $a(t) > 0\, \forall t \in [0, T]$, there is nothing to prove. Otherwise, being $a(t)$ analytic, its zeros, say $0 \leq t_1 < \cdots < t_l \leq T$, are isolated and have finite order.

Let $v(t, \xi)$ be the Fourier transform of $u(t, x)$ with respect to x, so that

$$v'' + \xi^2 a(t) v = 0\,. \tag{4.5}$$

Following [9], we introduce two kinds of energy for (4.5): a Kovalevskian energy, which we are going to use near the t_j, and a hyperbolic energy, adopted in the rest of the interval $[0, T]$. Namely, let us fix for the moment $\varepsilon > 0$ sufficiently small and let us define

$$\begin{aligned} &\widetilde{E}(t, \xi) = \xi^2 |v|^2 + |v'|^2 && \text{(Kovalevskian energy)}, \\ &E(t, \xi) = (a(t))\xi^2 |v|^2 + |v'|^2 && \text{(hyperbolic energy)}. \end{aligned} \tag{4.6}$$

Being (here and in the following C denotes any suitable positive constant)

$$\widetilde{E}' = 2\xi^2 \mathrm{Re}\,(v, v') + 2\mathrm{Re}\,(v', v'') \leq C|\xi|\widetilde{E}$$

we get

$$\begin{aligned} &\widetilde{E}(t, \xi) \leq e^{C\varepsilon|\xi|} \widetilde{E}(0, \xi) && \forall\, t \in [0, \varepsilon], \\ &\widetilde{E}(t, \xi) \leq e^{C\varepsilon|\xi|} \widetilde{E}(t_1 - \varepsilon, \xi) && \forall\, t \in [t_1 - \varepsilon, t_1], \end{aligned} \tag{4.7}$$

while, on account of (4.4),

$$E' = a'(t)\xi^2|v|^2 + 2(a(t))\xi^2\mathrm{Re}\,(vv') + 2\mathrm{Re}\,(v'v'') = a'(t)\xi^2|v|^2 \le \frac{C}{z_a(t)}E \quad (4.8)$$

from which we have

$$E(t,\xi) \le E(\varepsilon,\xi)e^{C(\log(1/\varepsilon))} \qquad \forall\, t \in [\varepsilon, t_1 - \varepsilon]. \tag{4.9}$$

Moreover, having $a(t)$ only zeros of finite order, say at most ν,

$$|\xi v(t,\xi)|^2 + |v'(t,\xi)|^2 \le \frac{C}{\varepsilon^\nu}E(t,\xi) \qquad \forall\, t \in [\varepsilon, t_1 - \varepsilon]. \tag{4.10}$$

By (4.7), (4.9) and (4.10) we easily get

$$\begin{aligned} &|\xi v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\qquad \le Ce^{C(\log(1/\varepsilon)+\varepsilon|\xi|)}(|\xi v(0,\xi)|^2 + |v'(0,\xi)|^2) \qquad \forall\, t \in [0, t_1]. \end{aligned} \tag{4.11}$$

By iterating this process on the whole interval $[0,T]$ we have

$$\begin{aligned} &|\xi v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\qquad \le Ce^{C(\log(1/\varepsilon)+\varepsilon|\xi|)}(|\xi v(0,\xi)|^2 + |v'(0,\xi)|^2) \qquad \forall\, t \in [0, T] \end{aligned} \tag{4.12}$$

and finally, by choosing $\varepsilon = 1/|\xi|$ for $|\xi|$ sufficiently large, we deduce from (4.12) that there exists a suitable integer k such that

$$\begin{aligned} &|\xi v(t,\xi)|^2 + |v'(t,\xi)|^2 \\ &\qquad \le C(1 + |\xi|^k)(|\xi v(0,\xi)|^2 + |v'(0,\xi)|^2) \qquad \forall\, t \in [0, T] \end{aligned} \tag{4.13}$$

now C^∞ well–posedness of (4.1) follows from the Paley–Wiener theorem. □

4.2 Toward more general homogeneous equations

It is quite clear that the crucial point in the above proof is the estimate

$$|z_a(t)a'(t)| \le Ca(t) \qquad \forall\, t \in [0, T].$$

Now, suppose that we want to adapt our previous argument to a general weakly hyperbolic equation with real analytic coefficients, by means of the symmetrizer. For the sake of simplicity, for the time being let us confine ourselves to one space variable. Moreover, let us suppose that the discriminant $\Delta_1(t)$ of the principal symbol is not identically zero.

Then, as explained in Section 1, after Fourier transform in space and reduction to first-order system, we come to a problem like (1.2) where A is a hyperbolic Sylvester matrix like (1.6), i.e.,

$$V'(t,\xi) = i\xi A(t) V(t,\xi) \tag{4.14}$$

(note that A does not depend on ξ as we are in one space dimension).

If we want to retrace the previous proof for model equation (4.1), we can consider again two types of energy, namely,

$$\begin{aligned} &\widetilde{E}(t,\xi) = |V(t,\xi)|^2 && \text{(Kovalevskian energy)}, \\ &E(t,\xi) = \langle Q(t)V(t,\xi), V(t,\xi)\rangle && \text{(hyperbolic energy)}, \end{aligned} \tag{4.15}$$

where $Q(t)$ is the standard symmetrizer of L. Now it is clear that, instead of $\dfrac{a'(t)}{a(t)}$, we must manage an expression like

$$\frac{\langle Q'(t)V, V\rangle}{\langle Q(t)V, V\rangle}. \tag{4.16}$$

Here the main difficulty pops up: *it does not exist, in general, a constant C such that*

$$z_{\Delta_1}(t)|\langle Q'(t)W, W\rangle| \le C\langle Q(t)W, W\rangle \qquad \forall\, W \in S^N. \tag{4.17}$$

Example Let us consider the equation

$$u_{tt} - 2tu_{tx} + (t^2 - t^4)u_{xx} \qquad \text{on } \ \mathbb{R}_x \times [0,1]. \tag{4.18}$$

Let $v(t,\xi)$ be the Fourier transform of $u(t,x)$ with respect to x, so that

$$v'' - 2i\xi t v' - \xi^2(t^2 - t^4)v = 0. \tag{4.19}$$

Let us reduce (4.19) to a first–order system: if

$$V(t,\xi) = \begin{pmatrix} i\xi v(t,\xi) \\ v'(t,\xi) \end{pmatrix},$$

then

$$V' = i\xi \begin{pmatrix} 0 & 1 \\ t^4 - t^2 & 2t \end{pmatrix} V. \tag{4.20}$$

The symmetrizer is

$$Q(t) = \begin{pmatrix} 2t^2 + 2t^4 & -2t \\ -2t & 2 \end{pmatrix} \tag{4.21}$$

and $\Delta_1(t) = \det Q(t) = 4t^4$ so that $z_{\Delta_1}(t) = t$ in $[0,1]$. So, if $W = (w_1, w_2)$,

$$z_{\Delta_1}(t)\frac{\langle Q'(t)W,W\rangle}{\langle Q(t)W,W\rangle} = \frac{(2t^2+4t^4)w_1^2-2w_1w_2}{(tw_1-w_2)^2+t^4w_1^2} := f(t,w_1,w_2). \tag{4.22}$$

Straightforward calculation shows that

$$f(\varepsilon+\varepsilon^2,\sqrt{1-\varepsilon^2},\varepsilon) = \frac{2\varepsilon^3+o(\varepsilon^3)}{2\varepsilon^4+o(\varepsilon^4)},$$

hence the ratio

$$z_{\Delta_1}(t)\frac{\langle Q'(t)W,W\rangle}{\langle Q(t)W,W\rangle}$$

is unbounded.

Of course, (4.17) would hold if $Q = Q(L)$ would be a diagonal matrix, but this happens iff $L = \partial_t^2 - \sum a_{ij}(t)\partial^2_{x_ix_j}$, i.e., for wave–type equations, which in fact do not need a symmetrizer. Moreover, from a heuristic viewpoint, it is clear the reason why Kinoshita and Spagnolo in [9] introduce the concept of *nearly diagonal* family of matrices, which they can apply, due to hypothesis (1.10), to the quasi–symmetrizer Q_ε, regarded as a family of matrices indexed by ε.

From the preceding discussion it is clear that we must look for algebraic hypotheses on the operator L, hence on Q, to ensure that (4.17) holds.

To this aim, let us remark that:

- If $B_1(t)$, $B_2(t)$ are two real symmetric $N\times N$ matrices, $B_2(t)$ is nonnegative and $\det B_2(t)$ has only isolated zeros, then

$$\frac{|\langle B_1(t)W,W\rangle|}{\langle B_2(t)W,W\rangle} \le C$$

iff the roots of the Hamilton–Cayley polynomial

$$\det(\lambda B_2(t) - B_1(t)) = \sum_{h=0}^{N} c_h(t)\lambda^{N-h}.$$

are bounded functions of t;

- The hyperbolic polynomial $\sum\limits_{h=0}^{N} c_h(t)\lambda^{N-h}$ has bounded roots $\lambda_1(t),\ldots,$ $\lambda_N(t)$ iff the ratios $c_1(t)/c_0(t)$, $c_2(t)/c_0(t)$ are bounded, as

$$\sum_{i=1}^{N}\lambda_i^2(t) = \frac{c_1^2(t)}{c_0^2(t)} - 2\frac{c_2(t)}{c_0(t)}\,;$$

- If $\gamma(t)$ is any scalar function and $\sum_{h=0}^{N} c_h(t)\lambda^{N-h}$ is the Hamilton–Cayley polynomial of $B_1(t), B_2(t)$, then the Hamilton–Cayley polynomial of $\gamma(t)B_1(t)$, $B_2(t)$ is

$$\sum_{h=0}^{N} \gamma^h(t) c_h(t)\lambda^{N-h}.$$

Now, let

$$\begin{aligned} R(\lambda, t) = \det(\lambda Q(t) - Q'(t)) &= \sum_{h=0}^{n} d_h(t)\lambda^{N-h} \\ &= \Delta_1(t)\lambda^N - \Delta_1'(t)\lambda^{N-1} + \sum_{h=2}^{n} d_h(t)\lambda^{N-h}\,; \end{aligned} \tag{4.23}$$

then we get (4.17) iff

$$\exists C : \quad |z_{\Delta_1}(t)\Delta_1'(t)| \le C\Delta_1(t), \quad |z_{\Delta_1}^2(t) d_2(t)| \le C\Delta_1(t) \qquad \forall\, t \in [0, T]. \tag{4.24}$$

But the first inequality in (4.24) is automatically verified, while the second is false in general, unless it is assumed as a hypothesis. This leads us to the following:

Definition 2 *Let $B(t)$ be an $N \times N$ real symmetric C^1 matrix. The coefficient of λ^{N-2} in the polynomial* $\det(\lambda B(t) - B'(t))$ *is termed the* check function *of the matrix $B(t)$.*

Throughout the rest of the paper, we will denote by $\psi_B(t)$ the check function of $B(t)$.

By (4.24) we see that (4.17) holds provided that, at any point $\bar{t}$ at which $\Delta_1(t)$ has a zero of order h, $\psi_Q(t)$ has a zero of order not less than $h-2$. This motivates the following:

Definition 3 *Let f, g be two real analytic functions on an interval $I \subset \mathbb{R}$, let $g \ge 0$. We say that f* is $-k$ dominated by *g, and we denote this by*

$$f \overset{(k)}{\prec} g$$

if, at any point $t \in I$ at which $g(t)$ has a zero of order h, the function $f(t)$ has a zero of order greater than or equal to $h - k$.

Now we may state a C^∞ well–posedness result.

Proposition 3 *Let us consider the equation*

$$L[u] = \partial_t^N u - \sum_{j=1}^{N} a_j(t)\, \partial_x^j \partial_t^{N-j} u = 0 \qquad \text{on } \ \mathbb{R}_x \times [0, T] \tag{4.25}$$

where L is a weakly hyperbolic operator with real analytic coefficients $a_j(t)$. Let $Q = Q(L)$ the symmetrizer of L and $\Delta_1(t)$ its determinant. Let us suppose that:

i) $\Delta_1(t) \not\equiv 0$,

ii) $\psi_Q(t) \overset{(2)}{\prec} \Delta_1(t)$,

then the Cauchy problem at $t = 0$ for (4.25) *is C^∞ well–posed.*

Proof The proof of C^∞ well–posedness of (4.1) we sketched above holds verbatim in this case, provided that (4.25) is tranformed into a system like (4.14) and the definition (4.6) of the energies is replaced by (4.15); therefore, we will not write down this proof again. We want only to remark that now (4.8) is replaced by

$$E' = \langle Q'(t)V(t,\xi), V(t,\xi)\rangle \leq \frac{C}{z_{\Delta_1}(t)} E$$

due to the fact that Q is the symmetrizer, which means that QA is symmetric (see also (1.4)), and (4.17) holds, thanks to hypothesis ii). □

Proposition 3 may be regarded as a provisional result, as we assumed to be in the simpler frame of one space variable and $\Delta_1(t) \neq 0$, but it clarifies the role of the check function of a matrix in our theory.

Of course, it may happen that $\Delta_1(t) \equiv 0$. If we assume hypothesis ii) of Proposition 3 (and indeed we must assume it), then $\psi_Q(t) \equiv 0$, hence $R(\lambda, t) \equiv 0$, where $R(\lambda, t)$ is defined in (4.23). So, if $\Delta_1(t) \equiv 0$, the Hamilton–Cayley polynomial of Q', Q vanishes at all. This is the situation, for instance, for the operators

$$L_1 = \partial_t^4, \qquad L_2 = \partial_t^2(\partial_t^2 - 2t\partial_t\partial_x + t^2\partial_x^2)$$

and obviously L_1 is C^∞ well–posed while L_2 is C^∞ ill–posed (being a trivial variation of the second equation in (4.2)).

When $\Delta_1(t) \equiv \psi_Q \equiv 0$, it turns out that the behavior of the ratio

$$\frac{\langle Q'(t)W, W\rangle}{\langle Q(t)W, W\rangle}$$

is completely described by the minors Q_j defined in (2.18) and their check functions ψ_{Q_j}. Starting from this fact, we may state the general result (for the proof see [7]):

Theorem 6 *Let L as in* (1.5) *be a weakly hyperbolic homogeneous operator with real analytic coefficients in* $[0, T]$. *Let $A(t,\xi)$ as in* (1.6) *be its Sylvester* *matrix, and* $Q(t,\xi) = Q(A)(t,\xi)$ *its standard symmetrizer. Let* Q_j *be as in* (2.18)*.*

For any fixed $\overline{\xi} \in S^N$*, let* $r = r(\overline{\xi})$ *be the minimum integer such that* $\Delta_r(t,\overline{\xi}) \not\equiv 0$ *in* $[0,T]$*. Let us suppose that:*

if $r > 1$*, then* $\psi_{Q_{r-1}}(t,\overline{\xi}) \equiv 0$ *in* $[0,T]$*;*

if $r < N$ *(*$r < N-1$ *when* $N \geq 3$*), then* $\psi_{Q_r}(t,\overline{\xi}) \overset{(2)}{\prec} \Delta_r(t,\overline{\xi})$ *in* $[0,T]$*.*

Then, the Cauchy problem for L *at* $t = 0$ *is* C^∞ *well–posed.*

Remark In the frame of C^∞ well–posedness for homogeneous weakly hyperbolic operators with time-dependent analytic coefficients, Theorem 6 is more general than [9]. Moreover, in this context, operators which satisfy hypotheses of [3] (see also [4]) satisfy also hypotheses of Theorem 6. It is likely that, as a matter of facts, there is an equivalence between our Theorem 6 and its counterpart in [3]. The novelty consists in assuming hypotheses only about the coefficients of the operator, instead of the characteristic roots.

In conclusion, the underlying reason for adopting the hyperbolic symmetrizer as a standard tool in the theory of hyperbolic equations is that any hypothesis about characteristic roots must be invariant under permutation of the roots, and so it may (and definitely *it should*) be expressed in terms of the coefficients of the operator.

Acknowledgments

The author wishes to thank Peter W. Michor for some useful discussions about Bezout matrix and roots of polynomials, and Armin Rainer, who pointed out, in a private communication, some very interesting links among various types of symmetrizers and the so–called *Bezoutiant* (for a definition, see for instance [10]).

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Time Global Solutions to the Cauchy Problem for Multidimensional Kirchhoff Equations

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Summary. The aim of this work is to get the time global solutions to the Cauchy problem in Sobolev spaces for multidimensional Kirchhoff equations.

2000 AMS Subject Classification: Primary: 35L70, Secondary: 35L15, 35P25

Key words: Kirchhoff equations, Scattering, nonlinear hyperbol equations.

1 Introduction

In [3] we showed the existence of time global solutions to the Cauchy problem and the scattering for one-dimensional perturbed Kirchhoff equations. In this paper we shall get the time global solutions to the Cauchy problem for one- and multidimensional Kirchhoff equations. We consider the following equation:

$$\begin{cases} u_{tt}(t,x) = (1 - \varepsilon(Au(t), u(t))_{L^2})Au(t,x), \ t \in \mathbb{R}, \ x \in \mathbb{R}^n, \\ u(0,x) = f(x), \ u_t(0,x) = g(x), \qquad\qquad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $A = \sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k}$. We assume that the coefficients $a_{jk}(x) \in C^\infty(\mathbb{R}^n)$ are real valued, have bounded derivatives in R^n and satisfy that $a_{jk}(x) = a_{kj}(x)$ and

$$a(x,\xi) = \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq 0, \tag{1.2}$$

for $x, \xi \in \mathbb{R}^n$.

Let $\mu \in \mathbb{R}$ and $1 \leq p \leq \infty$ and $L^p = L^p(\mathbb{R}^n)$ the set of integrable functions over $\mathbb{R}^n$ with pth power. We denote by $W_\mu^{l,p}$ the set of functions $u(x)$ defined

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_8,

in $\mathbb{R}^n$ such that $(1+|x|)^\mu \partial_x^\alpha u(x)$ is contained in L^p for $|\alpha| \le l$. For brevity we denote $L^p_\mu = W^{0,p}_\mu, W^{l,p} = W^{l,p}_0$, $H^l_k = W^{l,2}_k$ and $H^l = W^{l,2}$. Denote $H = \sqrt{-A}$, $D(H)$ the definition domain of H and $H_0 = \sqrt{-\Delta}$, $\Delta = \sum_{j=1}^n (\frac{\partial}{\partial x_j})^2$.

We assume that the initial data (f,g) belongs to $D(H^{\frac{3}{2}}) \times D(H^{\frac{1}{2}})$ and satisfies

$$||(f,g)||_{Y(H)} = \int_{-\infty}^{\infty} \{|(e^{itH}H^3 f, f)| + |(e^{itH}H^2 f, g)| + |(e^{itH}Hg, g)|\}dt < \infty, \tag{1.3}$$

where $(\cdot,\cdot)$ stands for an inner product of $L^2(\mathbb{R}^n)$. The first result we mention is the following theorem.

Theorem 1.1 *Assume that* (1.2) *is valid and moreover assume that* $(f,g) \in D(H^{\frac{3}{2}}) \times D(H^{\frac{1}{2}})$ *satisfies* (1.3). *Then there is* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *there exists the unique solution* u *of the Cauchy problem* (1.1) *which belongs to* $\cap_{j=0}^2 C^j(R; D(H_0^{\frac{3}{2}-j}))$ *and satisfies* $\sup_{s\in\mathbb{R}} ||(u(s), u_s(s)||_{Y(H)} < \infty$.

We remark that Greenberg and Hu [2], D'Ancona and Spagnolo [1], and Yamazaki [4], [5], [6] proved the existence of time global solution to the Cauchy problem for Kirchhoff equations under the decay conditions in space variables on the initial data. In [3], Kajitani investigates one-dimensional perturbed Kirchhoff equations.

Next we consider the Kirchhoff equation associated to Δ, that is, the case of $A = \Delta$,

$$\begin{cases} v_{tt}(t,x) = (1+\varepsilon||\nabla v(t)||^2_{L^2})\Delta v(t,x), \ t \in \mathbb{R}, \ x \in \mathbb{R}^n, \\ v(0,x) = f_0(x), \ v_t(0,x) = g_0(x), \qquad x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

Applying Theorem 1.1 we can get the following theorem.

Theorem 1.2 *Assume that the initial data* (f_0, g_0) *belongs to* $D(H^{\frac{3}{2}}) \cap W^{l,1} \times D(H^{\frac{1}{2}}) \cap W^{l-1,1}, l > n+2$. *Then there is* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *we have the unique solution* v *of the Cauchy problem* (1.4) *which belongs to* $\cap_{j=0}^2 C^j(\mathbb{R}; D(H_0^{\frac{3}{2}-j}))$.

We remark that the above theorem is proved essentially by Yamazaki [5], [6] in the case of $n \ge 3$.

2 Proof of Theorem 1.1

First we transform our original equation into a two-by-two system of first-order equations following Greenberg and Hu [2]. We let $A^1(t,x) = u_t + ic(t)Hu$ and

$B^1(t,x) = u_t - ic(t)Hu$, where $c(t)^2 = 1+\varepsilon||a(\cdot)Hu(t)||^2_{L^2}$. We write $c' = \frac{dc(t)}{dt}$. Then equation (1.1) yields

$$A^1_t - ic(t)HA^1 = \frac{c'(t)}{2c(t)}(A^1 - B^1), \quad B^1_t + ic(t)HB^1_x = -\frac{c'(t)}{2c(t)}(A^1 - B^1). \quad (2.1)$$

The initial conditions for A^1 and B^1 are computable in terms of Hf and g. They are

$$A^1(0,x) = A_0(x); = g + ic_0Hf, \quad B^1(0,x) = B_0(x); = g - ic_0Hf, \qquad (2.2)$$

where $c_0 = c(0) = (1 + \varepsilon||Hf||^2_{L^2})^{\frac{1}{2}}$. The defining relation for $c(t)$ becomes

$$c(t)^2 = 1 + \frac{\varepsilon}{4c(t)^2}||A^1(t,\cdot) - B^1(t,\cdot)||^2_{L^2}. \qquad (2.3)$$

We now introduce the change of variable $\tau = \int_0^t c(s)ds$. Clearly, τ is a strictly increasing function of t. We denote its inverse function by $t = T(\tau)$ and regard A^1, B^1, c as functions of τ, that is, we write $A^2(\tau,x) = A^1(T(\tau),x), B^2(\tau,x) = B^1(T(\tau),x), \gamma(\tau) = c(T(\tau))$. Then by applying the change of variable to equations (2.1), we get

$$A^2_\tau - iHA^2 = \frac{\gamma'}{2\gamma}(A^2 - B^2), \quad B^2_\tau + iHB^2 = -\frac{\gamma'}{2\gamma}(A^2 - B^2), \qquad (2.4)$$

and the initial condition is given by (2.2).

We put

$$A(\tau,x) = \frac{1}{\gamma^{\frac{1}{2}}}e^{-i\tau H}A^2(\tau,x), \quad B(\tau,x) = \frac{1}{\gamma^{\frac{1}{2}}}e^{i\tau H}B^2(\tau,x)$$

and

$$q(\tau) = q_\gamma(\tau) = \frac{\gamma'(\tau)}{2\gamma(\tau)}.$$

Then (A, B, γ) satisfies from (2.4) and (2.3)

$$A_\tau = -q(\tau)e^{2i\tau}B(\tau,x), \quad B_\tau = -q(\tau)e^{-2i\tau}A(\tau,x) \qquad (2.5)$$

and

$$\gamma(\tau)^2 = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}||e^{i\tau H}A(\tau,\cdot) - e^{-i\tau H}B(\tau,\cdot)||^2_{L^2}, \qquad (2.6)$$

and the initial condition is given by (2.2). We note that if $(f,g) \in D(H^{\frac{3}{2}}) \times D(H^{\frac{1}{2}})$ satisfies (1.3), then $(A_0, B_0) = (g + ic_0Hf, g - ic_0Hf)$ belongs to $D(H^{\frac{1}{2}}) \times D(H^{\frac{1}{2}})$ and satisfies

$$\int_{-\infty}^{\infty} \{|(e^{2i\tau H}HA_0, A_0)| + |(e^{2i\tau H}HA_0, B_0)| + |(e^{2i\tau H}HB_0, B_0)|\}d\tau < \infty. \tag{2.7}$$

We introduce a functional space as follows:

$$X_{\delta,M} = \{\gamma(\tau) \in C^1(R^1); 1 \le \gamma(\tau) \le M, \int_{-\infty}^{\infty} |\gamma'(\tau)|d\tau < \delta\}$$

with a norm $|\gamma|_X = \sup|\gamma(\tau)| + \int |\gamma'(\tau)|d\tau$. Let γ be in $X_{\delta,M}$ and consider the linear Cauchy problem (2.5) and (2.2). We denote its solution by (A_γ, B_γ). We define for $\gamma \in X_{\delta,M}$

$$\Phi(\gamma)^2(\tau) = 1 + \frac{\varepsilon}{4\gamma(\tau)^2}||e^{i\tau H}A_\gamma(\tau,\cdot) - e^{-i\tau H}B_\gamma(\tau,\cdot)||_{L^2}^2. \tag{2.8}$$

Then we can prove the following theorem.

Theorem 2.1 *Assume that* $(A_0, B_0) \in D(H) \times D(H)$ *satisfies* (2.7) *and* $0 < \delta < \frac{1}{2}$. *Then there is* $\varepsilon_0 > 0$ *such that* Φ *is a contraction mapping in* $X_{\delta,M}$, *that is,*

$$|\Phi(\gamma_1) - \Phi(\gamma_2)|_X \le C\varepsilon|\gamma_1 - \gamma_2|_X, \tag{2.9}$$

for any $\gamma_1, \gamma_2 \in X_{\delta,M}$ *and* $0 < \varepsilon \le \varepsilon_0$.

Define

$$G(A, B : \tau, s) = (e^{2i\tau H}HA(s,\cdot), B(s,\cdot))_{L^2(\mathbb{R}^n)}$$

and

$$Y(A, B, \tau) = \sup_{s\in\mathbb{R}} |G(A, B : \tau, s)|.$$

For the proof of the above theorem the following lemma is convenient.

Lemma 2.2 *Let* $\gamma_k \in X_{\delta,M}, k = 1, 2$ *and* $(A_k, B_k), k = 1, 2$ *be a solution of* (2.2) *and* (2.5) *for* γ_k *and* $0 < \delta < \frac{1}{2}$. *Then if* $(A_0, B_0) \in D(H) \times D(H)$ *satisfies* (2.7), *then* $Y(A_j, B_k : \tau), Y(A_j, A_k : \tau)$ *and* $Y(B_j, B_k : \tau)$ *belong to* $L^1(\mathbb{R})$ *and satisfy*

$$\begin{aligned} \int_{-\infty}^{\infty} \{\sup\nolimits_{j,k=1,2} Y(A_j, A_k : \tau) + \sup\nolimits_{j,k=1,2} Y(A_j, B_k : \tau) \\ + \sup\nolimits_{j,k=1,2} Y(B_j, B_k : \tau)\}d\tau \\ \le \frac{1}{1-2\delta}\int_{-\infty}^{\infty} \{|(e^{2\tau H}HA_0, B_0)| + |(e^{2\tau H}HA_0, A_0)| \\ + |(e^{2\tau H}HB_0, B_0)|\}d\tau. \end{aligned} \tag{2.10}$$

Moreover we have

$$\int_{-\infty}^{\infty}\{\sup_{k=1,2} Y(A_1 - A_2, A_k : \tau) + \sup_{k=1,2} Y(A_1 - A_2, B_k : \tau)$$
$$+ \sup_{k=1,2} Y(B_1 - B_2, B_k : \tau) + \sup_{k=1,2} Y(B_1 - B_2, A_k : \tau)\}d\tau$$
$$\leq C|\gamma_1 - \gamma_2|_X. \tag{2.11}$$

Proof. Differentiating $G(A_j, B_k : \tau, s)$ with respect to s, we get from (2.5)

$$\frac{\partial}{\partial s}G(A_j, B_k : \tau, s) = \frac{\partial}{\partial s}(e^{2i\tau H}HA_j(s,\cdot), B_k(s,\cdot))$$
$$= -q_j(s)(e^{2i(\tau+s)H}HB_j(s,\cdot), B_k(s,\cdot)) - q_k(e^{2i(\tau+s)H}HA_j(s,\cdot), A_k(s,\cdot))\}$$
$$= -q_j(s)G(B_j, B_k : \tau + s, s) - q_k(s)G(A_j, A_k : \tau + s, s),$$

where we denote $q_j = q_{\gamma_j} = \frac{\gamma_j'}{2\gamma_j}$. Similarly

$$\frac{\partial}{\partial s}G(A_j, A_k : \tau, s) = -q_j(s)G(B_j, A_k : \tau + s, s) - q_k(s)G(A_j, B_k : \tau - s, s)$$

and

$$\frac{\partial}{\partial s}G(B_j, B_k : \tau, s) = -q_j(s)G(A_j, B_k : \tau - s, s) - q_k(s)G(B_j, A_k : \tau + s, s).$$

Integrating the above relations from 0 to s and noting that

$$G(A_j, B_k : \tau, 0) = (e^{2i\tau H}HA_0, B_0), \quad G(A_j, A_k : \tau, 0) = (e^{2i\tau H}HA_0, A_0)$$

and

$$G(B_j, B_k : \tau, 0) = (e^{2i\tau H}HB_0, B_0)$$

we get

$$G(A_j, B_k : \tau, s) =$$
$$(e^{2i\tau H}HA_0, B_0) - \int_0^s q_j(t)G(B_j, B_k : \tau + t, t) + q_k(t)G(A_j, A_k : \tau + t, t)dt, \tag{2.12}$$

$$G(A_j, A_k : \tau, s) =$$
$$(e^{2i\tau H}HA_0, A_0) - \int_0^s q_j(t)G(B_j, A_k : \tau + t, t) + q_k(t)G(A_j, B_k : \tau - t, t)dt \tag{2.13}$$

and

$$G(B_j, B_k : \tau, s) =$$
$$(e^{2i\tau H}HB_0, B_0) - \int_0^s q_j(t)G(A_j, B_k : \tau - t, t) + q_k(t)G(B_j, A_k : \tau + t, t)dt. \tag{2.14}$$

From (2.12) we can estimate

$$Y(A_j, B_k, \tau) \leq$$

$$|(e^{2i\tau H} HA_0, B_0)| + \int_{-\infty}^{\infty} \{|q_j(t)| Y(B_j, B_k : \tau + t) + |q_k(t)| Y(A_j, A_k : \tau + t)\} dt. \tag{2.15}$$

Analogously from (2.13)

$$Y(A_j, A_k, \tau) \leq$$

$$|(e^{2i\tau H} HA_0, A_0)| + \int_{-\infty}^{\infty} |q_j(t)| Y(B_j, A_k : \tau + t) + |q_k(t)| Y(A_\gamma, B_\gamma : \tau - t) dt,$$

and from (2.14)

$$Y(B_j, B_k, \tau) \leq$$

$$|(e^{2i\tau H} HB_0, B_0)| + \int_{-\infty}^{\infty} |q_j(t)| Y(A_j, B_k : \tau - t) + |q_k(t)| Y(B_j, A_k : \tau + t)\} dt.$$

Since $\int |q_k(t)| dt \leq \delta$, integrating (2.15) with respect to τ we obtain

$$\int_{-\infty}^{\infty} Y(A_j, B_k, \tau) d\tau \leq$$
$$\int_{-\infty}^{\infty} |(e^{2i\tau H} HA_0, B_0)| d\tau + \delta \int_{-\infty}^{\infty} \{Y(A_j, A_k : \tau) + Y(B_j, B_k : \tau)\} d\tau.$$

Analogously

$$\int_{-\infty}^{\infty} Y(A_j, A_k, \tau) d\tau \leq$$
$$\int_{-\infty}^{\infty} |(e^{2i\tau H} HA_0, A_0)| d\tau + \delta \int_{-\infty}^{\infty} \{Y(B_j, A_k : \tau) + Y(A_j, B_k : \tau)\} d\tau$$

and

$$\int_{-\infty}^{\infty} Y(B_j, B_k, \tau) d\tau \leq$$
$$\int_{-\infty}^{\infty} |(e^{2i\tau H} HB_0, B_0)| d\tau + \delta \int_{\infty}^{\infty} \{Y(A_j, B_k : \tau) + Y(B_j, A_k : \tau)\} d\tau.$$

Therefore the above estimates imply (2.10), if we take $0 < \delta < \frac{1}{2}$. Next we shall show (2.11) similarly. Differentiating $G(A_1 - A_2, B_k : \tau, s)$ with respect to s, we have

$$\begin{aligned} \tfrac{\partial}{\partial s} G(A_1 - A_2, B_k : \tau, s) &= (e^{2i\tau H} \tfrac{\partial}{\partial s}(A_1 - A_2), B_k) + (e^{2i\tau H}(A_1 - A_2), \tfrac{\partial}{\partial s} B_k) \\ &= -(q_1 - q_2)(s) G(B_1, B_k : \tau + s, s) - q_2(s) G(B_1 - B_2, B_k : \tau + s, s) \\ &\quad - q_k(s) G(A_1 - A_2, A_k : \tau + s, s). \end{aligned}$$

Hence noting that $G(A_1 - A_2, A_k : \tau, 0) = 0$, analogously we get

$$Y(A_1 - A_2, B_k : \tau) \leq \int_{-\infty}^{\infty} |(q_1 - q_2)(t)| G(B_1, B_k : \tau - t) dt$$

$$+ \int_{-\infty}^{\infty} |q_2(t)| Y(B_1 - B_2, A_k : \tau - t) + |q_k(t)| Y(A_1 - A_2, A_k : \tau - t) dt.$$

Hence we get

$$\int_{-\infty}^{\infty} Y(A_1 - A_2, B_k : \tau) d\tau \leq \int_{-\infty}^{\infty} |(q_1 - q_2)(t) dt \int_{-\infty}^{\infty} Y(B_1, B_k : \tau) d\tau$$

$$+\delta \int_{-\infty}^{\infty} Y(B_1 - B_2, A_k : \tau) + Y(A_1 - A_2, A_k : \tau) d\tau.$$

Analogously

$$\int_{-\infty}^{\infty} Y(A_1 - A_2, A_k : \tau) d\tau \leq \int_{-\infty}^{\infty} |(q_1 - q_2)(t)| dt \int_{-\infty}^{\infty} Y(B_1, A_k : \tau) d\tau$$

$$+\delta \int_{-\infty}^{\infty} Y(B_1 - B_2, A_k : \tau) + Y(A_1 - A_2, B_k : \tau) d\tau,$$

$$\int_{-\infty}^{\infty} Y(B_1 - B_2, A_k : \tau) d\tau \leq \int_{-\infty}^{\infty} |(q_1 - q_2)(t)| dt \int_{-\infty}^{\infty} Y(A_1, A_k : \tau) d\tau$$

$$+\delta \int_{-\infty}^{\infty} Y(B_1 - B_2, B_k : \tau) + Y(A_1 - A_2, A_k : \tau) d\tau$$

and

$$\int_{-\infty}^{\infty} Y(B_1 - B_2, B_k : \tau) d\tau \leq \int_{-\infty}^{\infty} |(q_1 - q_2)(t)| dt \int_{-\infty}^{\infty} Y(A_1, B_k : \tau) d\tau$$

$$+\delta \int_{-\infty}^{\infty} Y(B_1 - B_2, A_k : \tau) + Y(A_1 - A_2, A_k : \tau) d\tau.$$

Therefore noting that $\int |q_1 - q_2| dt \leq C|\gamma_1 - \gamma_2|_X$ and using (2.10) we get

$$\int_{-\infty}^{\infty} Y(A_1 - A_2, B_k : \tau) + Y(A_1 - A_2, A_k : \tau)$$
$$+Y(B_1 - B_2, A_k : \tau) + Y(B_1 - B_2, B_k : \tau) d\tau$$
$$\leq C(1 - 2\delta)^{-1} |\gamma_1 - \gamma_2|_X \int_{-\infty}^{\infty} \{|(e^{2\tau H} A_0, A_0)| + |(e^{2\tau H} A_0, B_0)|$$
$$+|(e^{2\tau H} B_0, B_0)|\} d\tau$$

which implies (2.11). Q.E.D.

Proof of Theorem 2.1. It follows from (2.5) that

$$||A_\gamma(\tau)|| + ||B_\gamma(\tau)|| \le e^\delta(||A_0|| + ||B_0||),$$

for $\gamma \in X_{\delta,M}$. Hence we can see that $1 \le \Phi(\gamma)(\tau) \le M$ holds if we take $M > O$ suitably for any $\gamma \in X_{\delta,M}$. Next we show

$$\int_{-\infty}^{\infty} |\Phi(\gamma)'(\tau)| d\tau \le \delta. \tag{2.16}$$

Differentiating (2.8) and using (2.5) we have

$$\begin{aligned}
&2\Phi(\gamma)\Phi(\gamma)'(\tau)\\
&= -\frac{\varepsilon\gamma'(\tau)}{4\gamma(\tau)^2}||e^{i\tau H}A_\gamma(\tau,\cdot) - e^{-i\tau H}B_\gamma(\tau,\cdot)||^2\\
&\quad + \frac{\varepsilon}{4\gamma(\tau)}\{2\Re(A'_\gamma(\tau), A_\gamma(\tau)) + 2\Re(B'_\gamma(\tau), B_\gamma(\tau))\\
&\quad -2\Re(2iHe^{2i\tau H}A_\gamma(\tau,\cdot), B_\gamma(\tau,\cdot))\\
&\quad -2\Re(e^{2i\tau H}A'_\gamma(\tau,\cdot), B_\gamma(\tau,\cdot)) - 2\Re(e^{2i\tau H}A_\gamma(\tau,\cdot), B'_\gamma(\tau,\cdot))\}\\
&= \frac{\varepsilon}{\gamma(\tau)}\Re\{(ie^{2i\tau H}HA_\gamma(\tau), B_\gamma(\tau))\} = -\frac{\varepsilon}{\gamma(\tau)}\Im G(A_\gamma, B_\gamma : 2\tau, \tau).
\end{aligned} \tag{2.17}$$

Hence we get by use of (2.10)

$$\int_{-\infty}^{\infty} |\Phi(\gamma)'(\tau)| d\tau \le \varepsilon \int_{-\infty}^{\infty} Y(A_\gamma, B_\gamma, 2\tau) d\tau \le \delta,$$

which implies (2.16), if we take $\varepsilon > 0$ sufficiently small. Thus we proved that $\Phi(\gamma)(\tau)$ belongs to $X_{\delta,M}$ for $\gamma \in X_{\delta,M}$. Now we shall prove

$$|\Phi(\gamma_1)(\tau) - \Phi(\gamma_2)(\tau)| \le C\varepsilon|\gamma_1 - \gamma_2|_X, \tau \in \mathbb{R} \tag{2.18}$$

for any $\gamma_k \in X_{\delta,M}, k = 1, 2$. In fact, it follows from (2.8) and (2.5) we have

$$\begin{aligned}
&|\Phi(\gamma_1)(\tau)^2 - \Phi(\gamma_2)(\tau)^2|\\
&\le \frac{\varepsilon}{4}(|\frac{1}{\gamma_1(\tau)} - \frac{1}{\gamma_1(\tau)})|||e^{i\tau H}A_{\gamma_1}(\tau,\cdot) - e^{-i\tau H}B_{\gamma_2}(\tau,\cdot)||\\
&+\frac{\varepsilon}{4\gamma_2(\tau)}|||e^{i\tau H}A_{\gamma_1}(\tau,\cdot) - e^{-i\tau H}B_{\gamma_1}(\tau,\cdot)||^2 - ||e^{i\tau H}A_{\gamma_2}(\tau,\cdot) - e^{-i\tau H}B_{\gamma_2}(\tau,\cdot)||^2)|\\
&\le \varepsilon(M|\gamma_1 - \gamma_2|_X + C(||A_{\gamma_2}(\tau) - A_{\gamma_2}(\tau)|| + ||B_{\gamma_1}(\tau) - B_{\gamma_2}(\tau)||)\\
&\le \varepsilon C|\gamma_1 - \gamma_2|_X,
\end{aligned}$$

which proves (2.18). Here we used the inequality

$$||A_{\gamma_2}(\tau) - A_{\gamma_2}(\tau)|| + ||B_{\gamma_1}(\tau) - B_{\gamma_2}(\tau)|| \le C|\gamma_1 - \gamma_2|_X \tag{2.19}$$

which follows from (2.5). In fact, it follows from (2.5)

$$(A_{\gamma_1} - A_{\gamma_2})(\tau) = -\int_0^\tau (q_1 - q_2)(s)e^{2isH}B_{\gamma_1}(s) + q_2(s)e^{2isH}(B_{\gamma_1} - B_{\gamma_2})(s)ds$$

and

$$(B_1 - B_2)(\tau) = -\int_0^\tau (q_1 - q_2)(s)e^{2isH}A_1(s) + q_2(s)e^{2isH}(A_{\gamma_1} - A_{\gamma_2})(s)ds$$

which imply

$$||A_{\gamma_2}(\tau) - A_{\gamma_2}(\tau)|| + ||B_{\gamma_1}(\tau) - B_{\gamma_2}(\tau)||$$

$$\le \int_0^\tau \{|(q_1 - q_2)(s)|(||A_{\gamma_1}(s)|| + ||B_{\gamma_1}(s)||)$$

$$+|q_2(s)|(||A_{\gamma_2}(s) - A_{\gamma_2}(s)|| + ||B_{\gamma_1}(s) - B_{\gamma_2}(s)||)\}ds.$$

Hence we get (2.19) applying Gronwall's inequality.

Finally we shall show

$$\int |\Phi(\gamma_1)'(\tau) - \Phi(\gamma_2)'(\tau)|d\tau \le \varepsilon C|\gamma_1 - \gamma_2|_X \tag{2.20}$$

for any $\gamma_k \in X_{\delta,M}, k = 1, 2$. In fact, (2.17) implies

$$2|\Phi(\gamma_1)(\tau)\Phi(\gamma_1)'(\tau) - \Phi(\gamma_2)(\tau)\Phi(\gamma_2)'(\tau)|$$

$$= |\frac{\varepsilon}{2\gamma_1(\tau)}\Re\{(ie^{-2i\tau H}HA_{\gamma_1}(\tau), B_{\gamma_1}(\tau))\}$$

$$-\frac{\varepsilon}{2\gamma_2(\tau)}\Re\{(ie^{-2i\tau H}HA_{\gamma_2}(\tau), B_{\gamma_2}(\tau))\}|$$

$$\le |\frac{\varepsilon}{2\gamma_1(\tau)} - \frac{\varepsilon}{2\gamma_2(\tau)}||\Re\{(ie^{-2i\tau H}HA_{\gamma_1}(\tau), B_{\gamma_1}(\tau))\}|$$

$$+\frac{\varepsilon}{2\gamma_2(\tau)}|G(A_1, B_1 : -2\tau, \tau) - G(A_2, B_2 : -2\tau, \tau)|.$$

On the other hand,

$$|G(A_{\gamma_1}, B_{\gamma_2}1 : -2\tau, \tau) - G(A_{\gamma_2}, B_{\gamma_2} : -2\tau, \tau)|$$

$$\le |G(A_{\gamma_1} - A_{\gamma_2}, B_{\gamma_1} : -2\tau, \tau)| + |G(A_{\gamma_2}, B_{\gamma_1} - B_{\gamma_2} : -2\tau, \tau)|$$

$$\le Y(A_{\gamma_1}, B_{\gamma_2} : -2\tau) + Y(A_{\gamma_2}, B_{\gamma_1} - B_{\gamma_1} : -2\tau).$$

Therefore integrating the above inequalities with respect to τ, we obtain (2.20) by use of (2.10) and (2.11). Thus we have completed the proof of Theorem 2.1. Q.E.D.

Proof of Theorem 1.1. It follows from Theorem 2.1 that we have the fixed point $\gamma \in X_{\delta,M}$ of Φ, if we choose $\varepsilon > 0$ small. Let $(A, B)(\tau, x)$ be a solution of (2.5) with $q = \frac{\gamma'}{2\gamma}$ satisfying the initial condition (2.2) and put $T(\tau) = \int_0^\tau \gamma(s)^{-1} ds$. Let $S(t)$ be the inverse function of $T(\tau) = t$. Put

$$c(t) = \gamma(S(t))$$

and

$$(A^1, B^1)(t, x) = c(t)^{\frac{1}{2}}(e^{S(t)H} A(S(t), x), e^{-S(t)H} B(S(t), x))$$

which satisfy (2.3) and (2.1), (2.2), respectively. Define

$$u(t, x) = f(x) + \int_0^t \frac{A_1(s, x) + B_1(s, x)}{2} ds.$$

Then it follows from (2.1), (2.2) and (2.3) that we can see that $u(t, x)$ solves (1.1). Thus we complete the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In order to prove Theorem 1.2 we need the following lemma and proposition.

Lemma 3.1 *Let k be a nonnegative number and $l > k + 1$. Then*

$$|\int_0^\infty e^{it\rho} \rho^k (\rho + 1)^{-l} d\rho| \leq C(1 + |t|)^{-k-1}, t \in R. \tag{3.1}$$

Proof. Let $t > 0$ and $\chi \in C^\infty$ such that $\chi(\rho) = 0$ for $\rho \leq 1$ and $\chi(\rho) = 1$ for $\rho \geq 2$. We have

$$\int_0^\infty e^{it\rho} \rho^k (\rho + 1)^{-l} d\rho$$
$$= \int_0^\infty e^{it\rho} \rho^k (\rho + 1)^{-l} \chi(t\rho) d\rho + \int_0^\infty e^{it\rho} \rho^k (\rho + 1)^{-l} (1 - \chi(t\rho)) d\rho =: I_1 + I_2.$$

It is trivial that I_2 satisfies (3.1). We shall show I_1 satisfies (3.1). The transform of variable implies

$$I_1 = t^{k+1} \int_1^\infty e^{i\rho} \rho^k (\frac{\rho}{t} + 1)^{-l} \chi(\rho) d\rho.$$

On the other hand,

$$\int_0^\infty e^{i\rho}\rho^k(\frac{\rho}{t}+1)^{-l}\chi(\rho)d\rho = \int_0^\infty e^{i\rho}(\frac{\partial}{\partial\rho})^N\{\rho^k(\frac{\rho}{t}+1)^{-l}\chi(\rho)\}d\rho,$$

for any nonnegative integer N. Since

$$\left|(\frac{\partial}{\partial\rho})^N\left\{\rho^k(\frac{\rho}{t}+1)^{-l}\chi(\rho)\right\}\right| \le C_N\rho^{k-N}, \rho > 1,$$

where C_N is independent of t. Hence if we take $N > k+1$, we can see easily that I_1 satisfies (3.1). Q.E.D.

Besides we need

Proposition 3.2 *Let $H_0 = \sqrt{-\Delta}$ and k be a nonnegative number. Then there is $C > 0$ such that we have*

$$\begin{aligned}&|e^{itH_0}H_0^k f(x)|\\ &\le C\int_{\mathbb{R}^n}\{(1+|t|)^{-n-k} + (1+|t|)^{-\frac{n-1}{2}}((1+||x-y|+t|)^{-\frac{n+1}{2}-k} \qquad (3.2)\\ &\qquad +(1+||x-y|-t|)^{-\frac{n+1}{2}-k})\}|(1-\Delta_y)^{\frac{l}{2}}f(y)|dy,\end{aligned}$$

for $f \in W^{l,1}$ and for $l > k+n$.

Proof. Assume that k is a nonnegative integer. Let $l > n+k$. We have

$$e^{itH_0}f(x) = \int_{\mathbb{R}^n} K(x-y,t)\langle D_y\rangle^l f(y)dy, \qquad (3.3)$$

where

$$K(z,t) = \int e^{iz\xi+it|\xi|}|\xi|^k\langle\xi\rangle^{-l}d\xi \qquad (3.4)$$

and we use the notation $\langle D\rangle = (1-\Delta)^{\frac{1}{2}}$ and $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. Let $|z| \le \frac{|t|}{2}$. Denote $w = \frac{\xi}{|\xi|}$. Then $|zw+t| \ge \frac{|t|}{2}$. Hence by use of (3.1)

$$\begin{aligned}|K(z,t)| &= \left|\int_{|w|=1}\int_0^\infty e^{i(zw+t)\rho}\rho^{n-1+k}\langle\rho\rangle^{-l}d\rho dw\right|\\ &\le C\int|zw+t|^{-n-k}dw \le C|t|^{-n-k}, \qquad (3.5)\end{aligned}$$

for $|z| \le \frac{|t|}{2}$. For $|z| \ge \frac{|t|}{2}$ and for $|z|\rho \le 1$ we can see easily

$$\left|\int_{0\le\rho\le|z|^{-1}}\int_{|w|=1} e^{i(zw+t)\rho}\rho^{n-1+k}\langle\rho\rangle^{-l}dwd\rho\right| \le C|z|^{-n-k} \le C|t|^{-n-k}. \quad (3.6)$$

On the other hand, when $|z|\rho \ge 1$, by use of the stationary phase method we can get

$$\begin{aligned}&\int_{|w|=1} e^{izw\rho}dw \\ &= (|z|\rho)^{-\frac{n-1}{2}}\left\{\sum_{k=0}^{N-1}(e^{i|z|\rho}q_{+j}+e^{-i|z|\rho}q_{-j})(|z|\rho)^{-j}+q_N(|z|\rho)\right\},\end{aligned} \quad (3.7)$$

where $q_{\pm j}$ are constants and $|q_N(|z|\rho)| \le C_N(|z|\rho)^{-N}$ for any positive integer N. Hence applying again (3.1) we get from (3.7)

$$\begin{aligned}&\left|\int_{|z|\rho\ge1}\left\{\int_{|w|=1} e^{i(zw+t)\rho}\rho^{n-1+k}\langle\rho\rangle^{-l}dwd\rho\right|\right. \\
&= \left|\int(|z|\rho)^{-\frac{n-1}{2}}\left\{\sum_{j=0}^{N-1}(e^{i(t+|z|)\rho}q_{+j}+e^{i(t-|z|)\rho}q_{-j})(|z|\rho)^{-j}\right.\right. \\
&\quad \left.+\,q_N(|z|\rho)\}\rho^{n-1+k}\langle\rho\rangle^{-l}d\rho\right| \\
&\le \sum_{j=0}^{N-1}|z|^{-\frac{n-1}{2}-j}\left|\int(e^{i(t+|z|)\rho}q_{+j}+e^{i(t-|z|)\rho}q_{-j})\rho^{\frac{n-1}{2}+k-j}\langle\rho\rangle^{-l}d\rho\right| \\
&\quad + C|z|^{-\frac{n-1}{2}-N} \\
&\le \sum_{j=0}^{N-1}|z|^{-\frac{n-1}{2}-j}\left\{(1+|t-|z||)^{-\frac{n+1}{2}-k+j}+(1+|t+|z||)^{-\frac{n+1}{2}-k+j}\right\} \\
&\quad + C|z|^{-\frac{n-1}{2}-N} \\
&\le C|t|^{-\frac{n-1}{2}}\left\{(1+|t-|z||)^{-\frac{n+1}{2}-k}+(1+|t+|z||)^{-\frac{n+1}{2}-k}\right\}+C|t|^{-n-k},\end{aligned}$$

for $|z| \ge \frac{|t|}{2}$, which gives (3.2), if $N > \frac{n-1}{2}+k+1$. Q.E.D.

Proof of Theorem 1.2. Now we can prove Theorem 1.2, applying Proposition 3.2. It suffices to show that $(f_0, g_0) \in H^{\frac{3}{2}} \cap W^{l,1} \times H^{\frac{1}{2}} \cap W^{l-1,1}, l > n+\frac{3}{2}$ satisfies (1.3) with $H = H_0$, that is,

$$||(f_0, g_0)||_{Y(H_0)}$$

$$= \int_{-\infty}^{\infty} \left\{ \left| (e^{itH_0} H_0^3 f_0, f_0) \right| + \left| (e^{itH_0} H g_0, g_0) \right| + \left| (e^{itH_0} H_0^2 f_0, g_0) \right| \right\} dt$$

$$\leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \left\{ |e^{itH_0} H_0^{\frac{3}{2}} f_0(x)||H_0^{\frac{3}{2}} f_0(x)| + |e^{itH_0} H_0^{\frac{1}{2}} g_0(x)||H_0^{\frac{1}{2}} g_0(x)| \right.$$

$$\left. + |e^{itH_0} H_0^{\frac{3}{2}} f_0(x)|H_0^{\frac{1}{2}} g_0(x)| \right\} dxdt < \infty.$$

In fact, for example we show the second term of the right side is finite if $l > n + \frac{1}{2}$,

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \left| e^{itH_0} H_0^{\frac{1}{2}} g_0(x) \right| \left| H_0^{\frac{1}{2}} g_0(x) \right| dxdt \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{(1+|t|)^{-n-\frac{1}{2}}$$

$$+ (1+|t|)^{-\frac{n-1}{2}} ((1+||x-y|+t|)^{-\frac{n+1}{2}-\frac{1}{2}}$$

$$+ (1+||x-y|-t|)^{-\frac{n+1}{2}-\frac{1}{2}})\} |(1-\Delta_y)^{\frac{l}{2}} g_0(y)| dy |H_0^{\frac{1}{2}} g_0(x)| dxdt$$

$$\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(1-\Delta_y)^{\frac{l}{2}} H_0^{\frac{1}{2}} g_0(y)||H_0^{\frac{1}{2}} g_0(x)| dydx$$

$$\leq C||(1-\Delta)^{\frac{l}{2}} H_0^{\frac{1}{2}} g_0||_{L^1} ||H_0^{\frac{1}{2}} g_0||_{L^1}.$$

We can estimate the other terms in the same way. Thus we completed the proof of Theorem 1.2.

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The Order of Accuracy of Quadrature Formulae for Periodic Functions

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We dedicate this paper to Ferruccio Colombini on the event of his sixtieth birthday. He has been an inspiring colleague, coauthor and close friend.
We wish him happy and creative years till the next milestone.

Summary. The trapezoidal quadrature rule on a uniform grid has spectral accuracy when integrating C^∞ periodic function over a period. The same holds for quadrature formulae based on piecewise polynomial interpolations. In this paper, we prove that these quadratures applied to $W^{r;p}_{\rm per}$ periodic functions with $r > 2$ and $p \geq 1$ have error $\mathcal{O}((\Delta x)^r)$. The order is independent of p, sharp, and for $p < \infty$ is higher than predicted by best trigonometric approximation. For $p = 1$ it is higher by 1.

2000 AMS Subject Classification: Primary: 65D32, Secondary: 65T40

Key words: Numerical quadrature, periodic functions, order of accuracy, trigonometric polynomial, piecewise polynomial interpolation, trapezoidal rule.

*The research of A. Kurganov was supported in part by the National Science Foundation under grants DMS-0310585 and DMS-0610430. The research of J. Rauch was supported in part by the National Science Foundation under grant DMS-0405899.

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_9,

1 Upper bound on the error

Denote by $W^{r,p}_{\rm per}$ the Banach space of periodic functions on $\mathbb{R}$ whose distribution derivatives up to order r belong to $L^p_{\rm per}(\mathbb{R})$. The norm is equal to the sum of the L^p norms of these derivatives over one period. Without loss of generality we take the period equal to 2π. Introduce a partition of the interval $[0,2\pi]$ into N equal subintervals of size $\Delta x := 2\pi/N$.

Because of the periodicity, the trapezoidal rule is

$$\int\limits_0^{2\pi} f(x)\,dx \approx T_N(f) := \Delta x \sum_{j=0}^{N-1} f(x_j), \quad x_j := j\Delta x, \quad j = 0,1,\dots,N-1. \tag{1.1}$$

The error is equal to

$$E_N(f) := T_N(f) - \int\limits_0^{2\pi} f(x)\,dx.$$

Write f as the sum of its Fourier series,

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx}, \qquad c_n := \frac{1}{2\pi}\int\limits_0^{2\pi} f(x)e^{-inx}\,dx.$$

Denote by $\mathcal{P}(m)$ the set of all trigonometric polynomials of degree at most m, that is, functions of the form

$$\sum_{n=-m}^{m} a_n e^{inx}, \qquad a_n \in \mathbb{C}.$$

Summing finite geometric series shows that $T_N(e^{inx}) = 0$ for $0 < |n| < N$ and it follows that T_N exactly integrates trigonometric polynomials of degree $N-1$. Therefore, for any $P \in \mathcal{P}(N-1)$,

$$E_N(f) = E_N(f-P) = T_N(f-P) - \int\limits_0^{2\pi} (f(x) - P(x))\,dx.$$

This yields the bound in terms of best trigonometric approximation

$$|E_N(f)| \le 4\pi \inf_{P\in\mathcal{P}(N-1)} \|f-P\|_{L^\infty}. \tag{1.2}$$

Spectral accuracy then follows for infinitely smooth f thanks to the rapid decay of the Fourier coefficients.

Our estimate proceeds differently. For any integer k the functions e^{inx} and $e^{i(n+kN)x}$ agree at the nodes for T_N. Therefore, $T_N(e^{inx}) = T_N(e^{i(n+kN)x})$ and thus, $T_N(e^{inx}) = 0$ for all $n \neq kN$ and $T_N(e^{inx}) = 2\pi$ for all $n = kN$. This can also be checked by summing the corresponding finite geometric series. It follows that

$$E_N(f) = 2\pi \sum_{0\neq k\in\mathbb{Z}} c_{kN} = 2\pi \sum_{0\neq k\in\mathbb{Z}} \Big((ikN)^r c_{kN}\Big)\frac{1}{(ikN)^r}\,. \tag{1.3}$$

This involves only a small fraction of the Fourier coefficients c_n with $|n| \geq N$.

Theorem 1.1 *If $f \in W^{r,1}_{\text{per}}$ and $1 < r \in \mathbb{N}$, then the error of the trapezoidal quadrature rule* (1.1) *satisfies*

$$|E_N(f)| \leq \frac{C_r \left\|f^{(r)}\right\|_{L^1([0,2\pi])}}{N^r}, \qquad f^{(r)} := \frac{d^r f}{dx^r}, \qquad C_r := 2\sum_{k=1}^{\infty}\frac{1}{k^r}. \tag{1.4}$$

Remark 1.2 *The result is interesting only for $r > 2$, since for $1 \leq r \leq 2$ the N^{-r} convergence rate can be established using standard arguments even in the case of nonperiodic f.*

Remark 1.3 *Analogous estimates are true for noninteger r. For $r \geq 2$ they can be obtained by interpolation between integer values.*

Proof Introduce

$$Q_N(x) := \frac{1}{N}\sum_{j=0}^{N-1}\delta(x - x_j), \qquad g_N(x) := \sum_{0\neq k\in\mathbb{Z}}\frac{1}{(ikN)^r}e^{ikNx},$$

where δ is the Dirac delta-function. The nth Fourier coefficient of Q_N is equal to $T_N(e^{-inx})/(2\pi)^2$ so

$$Q_N(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{ikNx}.$$

This together with $f^{(r)}(x) = \sum_{n\in\mathbb{Z}} (in)^r\, c_n\, e^{inx}$ yields

$$\sum_{0\neq k\in\mathbb{Z}} (ikN)^r c_{kN}\; e^{ikNx} = Q_N * f^{(r)}(x),$$

where $*$ denotes convolution of periodic functions. Equation (1.3) then implies

$$E_N(f) = \int_0^{2\pi} Q_N * f^{(r)}(x)\; g_N(-x)\, dx\,. \tag{1.5}$$

Since Q_N is a measure with total variation per period equal to 1, one has

$$\left\| \sum_{0\neq k\in\mathbb{Z}} (ikN)^r c_{kN} e^{ikNx} \right\|_{L^1([0,2\pi])} = \left\| Q_N * f^{(r)} \right\|_{L^1([0,2\pi])} \le \left\| f^{(r)} \right\|_{L^1([0,2\pi])}. \tag{1.6}$$

On the other hand,

$$\|g_N\|_{L^\infty} \le \sum_{0\neq k\in\mathbb{Z}} \frac{1}{|kN|^r} = \frac{2}{N^r} \sum_{k=1}^{\infty} \frac{1}{k^r}, \tag{1.7}$$

and the sum is finite for $r > 1$. Combining (1.5), (1.6) and (1.7) yields (1.4).

Remark 1.4 *The order of accuracy established in Theorem* 1.1 *for the trapezoidal rule is also true for other quadratures based on piecewise polynomial interpolations. This is so since such quadratures applied to periodic functions can be rewritten as a convex combination of trapezoidal rules with shifted nodes (see* [2]*).*

2 A lower bound and comparison with best trigonometric approximation

For integer $r > 1$, $W^{r,p}_{\text{per}}$ consists of C^{r-1} functions so that $f^{(r-1)}$ is absolutely continuous with derivative in L^p_{per}. For $p > 1$, $f^{(r-1)}$ belongs to the Hölder class $C^{r-1,\alpha}$ with $\alpha = 1 - 1/p$. The right-hand side of (1.2) is the error in best polynomial approximation. That error has a precise estimate in terms of the modulus of continuity of $f^{(r-1)}$ (see [1, 3]). The rate of best approximation is different for the different spaces $W^{r,p}_{\text{per}}$ with $r > 1$ fixed and $1 \le p \le \infty$. In contrast, the order of convergence of the trapezoidal rule is essentially independent of p as the following example shows.

Example 2.1 *Define f by the lacunary Fourier series:*

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{(2^n)^r} e^{i2^n x}.$$

Then $\int_0^{2\pi} f(x)\,dx = 0$. In addition, $f \in W^{r-\varepsilon,\infty}_{\text{per}}$ for all $\varepsilon > 0$ and the error in the trapezoidal approximation $T_{2^N}(f)$ is exactly equal to

$$E_{2^N}(f) = 2\pi \sum_{2^n \text{ is a multiple of } 2^N} \frac{1}{(2^n)^r} = 2\pi \sum_{k=0}^{\infty} \frac{1}{(2^{N+k})^r} = \frac{1}{(2^N)^r} \frac{2^{r+1}\pi}{2^r - 1}.$$

As this is $\mathcal{O}((2^N)^{-r})$, the rate of convergence for $W^{r,\infty}_{\text{per}}$ cannot be better, in the sense of a higher power of $1/N$, than that for $W^{r,1}_{\text{per}}$.

3 Conclusion

The trapezoidal rule and other quadrature formulae based on piecewise polynomial interpolations have error $\mathcal{O}((\Delta x)^r)$ for functions in $W^{r,p}_{\text{per}}$. The rate is independent of p and is optimal in the sense that no higher power of Δx is possible. The error is that which is predicted by approximation theory for functions in $W^{r,\infty}_{\text{per}}$ and it is interesting that it remains true for the elements of $W^{r,1}_{\text{per}}$, for which the best approximation by trigonometric polynomials of degree $N-1$ is not as small as $\mathcal{O}((\Delta x)^{r-1+\varepsilon})$. For $W^{r,1}_{\text{per}}$, the rate of convergence is essentially a full order more rapid than that given by best approximation as in (1.2).

Acknowledgment

Most of this work was done in Fall 2005, when A. Kurganov visited the Department of Mathematics of the University of Michigan during the semester following hurricane Katrina. A. Kurganov thanks the chairman Professor A. Bloch and all members of the department for their extremely warm hospitality.

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A Note on the Oseen Kernels

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Summary. The Oseen operators are $\Delta^{-1}\partial_{x_j}\partial_{x_k}e^{t\Delta}$, where Δ is the standard Laplace operator and $t \in \mathbb{R}_+$. We give an explicit expression for the kernels of these Fourier multipliers which involves the incomplete gamma function and the confluent hypergeometric functions of the first kind. This explicit expression provides directly the classical decay estimates with sharp bounds. Although the computations are elementary and the definition of the Oseen kernels goes back to the 1911 paper of this author, we were not able to find the simple explicit expression below in the literature.

2000 AMS Subject Classification: Primary: 35Q30, 33C15

Key words: Oseen kernels, singular integrals, hypergeometric function.

1 Introduction

The (Marcel) Riesz operators $(R_j)_{1\le j\le n}$ are the following Fourier multipliers (we use the notation $\hat{u}$ for the Fourier transform of u: our normalization is given in formula (3.1) of our appendix)

$$(\widehat{R_j u})(\xi) = \xi_j|\xi|^{-1}\hat{u}(\xi), \qquad R_j = D_j/|D| = (-\Delta)^{-1/2}\frac{\partial}{i\partial x_j}. \tag{1.1}$$

The R_j are selfadjoint bounded operators on $L^2(\mathbb{R}^n)$ with norm 1. The Riesz operators are the natural multidimensional generalization of the Hilbert transform, given by the convolution with pv $\frac{i}{\pi x}$ which is the one-dimensional Fourier multiplier by $\operatorname{sign}\xi$. These operators are the paradigmatic singular integrals, introduced by Calderón and Zygmund, and are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and send L^1 into L^1_w. However, they are not continuous on

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_10,

the Schwartz class, because of the singularity at the origin. The Leray–Hopf projector[1] is the following matrix valued Fourier multiplier:

$$\mathbf{P}(\xi) = \mathrm{Id} - \frac{\xi \otimes \xi}{|\xi|^2} = \big(\delta_{jk} - |\xi|^{-2}\xi_j\xi_k\big)_{1\le j,k\le n}, \quad \mathbf{P} = \mathbf{P}(D) = \mathrm{Id} - R \otimes R. \tag{1.2}$$

We can also consider the $n \times n$ matrix of operators given by $\mathbf{Q} = R \otimes R = (R_j R_k)_{1\le j,k\le n}$ sending the vector space of $L^2(\mathbb{R}^n)$ vector fields into itself. The operator $\mathbf{Q}$ is selfadjoint and is a projection since $\sum_l R_l^2 = \mathrm{Id}$ so that $\mathbf{Q}^2 = (\sum_l R_j R_l R_l R_k)_{j,k} = \mathbf{Q}$. As a result the operator

$$\mathbf{P} = \mathrm{Id} - R \otimes R = \mathrm{Id} - |D|^{-2}(D \otimes D) = \mathrm{Id} - \Delta^{-1}(\nabla \otimes \nabla) \tag{1.3}$$

is also an orthogonal projection, the Leray–Hopf projector (a.k.a. the Helmholtz–Weyl projector); the operator $\mathbf{P}$ is in fact the orthogonal projection onto the closed subspace of L^2 vector fields with null divergence. We have for a vector field $u = \sum_j u_j \partial_j$, the identity $\operatorname{grad}\operatorname{div} u = \nabla(\nabla \cdot u)$, and thus

$$\operatorname{grad}\operatorname{div} = \nabla \otimes \nabla = \Delta\ R \otimes R, \quad \text{so that} \tag{1.4}$$

$$\mathbf{Q} = R \otimes R = \Delta^{-1} \operatorname{grad}\operatorname{div}, \quad \operatorname{div} R \otimes R = \operatorname{div}, \tag{1.5}$$

which implies $\operatorname{div} \mathbf{P}u = \operatorname{div} u - \operatorname{div}(R \otimes R)u = 0$, and if $\operatorname{div} u = 0$, we have $\mathbf{Q}u = 0$ and $u = \mathbf{Q}u + \mathbf{P}u = \mathbf{P}u$. This operator plays an important role in fluid mechanics since the Navier–Stokes system ([7], [3], [6]) for incompressible fluids can be written as

$$\begin{cases} \partial_t v + \mathbf{P}((v \cdot \nabla)v) - \nu \triangle v = 0, \\ \mathbf{P}v = v, \\ v_{|t=0} = v_0. \end{cases} \tag{1.6}$$

As already stated for the Riesz operators, $\mathbf{P}$ is not a classical pseudodifferential operator, because of the singularity at the origin: however, it is indeed a Fourier multiplier with the same continuity properties as those of R, and in particular is bounded on L^p for $p \in (1, +\infty)$. In three dimensions the **curl** operator is given by the matrix

$$\mathbf{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \mathbf{curl}^* \tag{1.7}$$

so that $\mathbf{curl}^2 = -\Delta\,\mathrm{Id} + \operatorname{grad}\operatorname{div}$ and (the Biot–Savard law)

$$\mathrm{Id} = (-\Delta)^{-1}\,\mathbf{curl}^2 + \Delta^{-1} \operatorname{grad}\operatorname{div} = (-\Delta)^{-1}\,\mathbf{curl}^2 + \mathrm{Id} - \mathbf{P}, \tag{1.8}$$

which gives

[1] That projector is also called the Helmholtz–Weyl projector by some authors.

$$\mathbf{curl}^2 = -\Delta \mathbf{P}, \tag{1.9}$$

so that $[\mathbf{P}, \mathbf{curl}] = 0$ and

$$\mathbf{P}\,\mathbf{curl} = \mathbf{curl}\,\mathbf{P} = \mathbf{curl}(-\Delta)^{-1}\,\mathbf{curl}^2 = \mathbf{curl}\big(\mathrm{Id} - \Delta^{-1}\,\mathrm{grad}\,\mathrm{div}\big) = \mathbf{curl} \tag{1.10}$$

since $\mathbf{curl}\,\mathrm{grad} = 0$ (note also that the transposition of the latter gives $\mathrm{div}\,\mathbf{curl} = 0$). The solutions of (1.6) are satisfying

$$v(t) = e^{t\nu\Delta}v_0 - \int_0^t e^{(t-s)\nu\Delta}\mathbf{P}\nabla\big(v(s)\otimes v(s)\big)ds.$$

2 The action of the Leray projector on Gaussian functions

We want now to compute the action of $\mathbf{P}$ on Gaussian functions.

Lemma 2.1 *Let $n \geq 1$ be an integer, $1 \leq j, k \leq n$ and $a > 0$. Then, with $u_a(x) = a^{n/2}e^{-\pi a|x|^2}$, we have*

$$\text{for } j \neq k,\ (R_jR_ku_a)(x) = -x_jx_k|x|^{-n-2}\gamma(1+\frac{n}{2}, a\pi|x|^2)\pi^{-n/2}, \tag{2.1}$$

$$(R_j^2u_a)(x) = -x_j^2|x|^{-n-2}\gamma(1+\frac{n}{2}, a\pi|x|^2)\pi^{-n/2} + \frac{1}{2}|x|^{-n}\gamma(\frac{n}{2}, a\pi|x|^2)\pi^{-n/2}, \tag{2.2}$$

where γ is the incomplete gamma function (see below a reminder).

A reminder. We recall the definition of the (lower) incomplete Gamma function (see e.g. [2], [1]),

$$\begin{aligned}\gamma(a,x) &= \int_0^x t^{a-1}e^{-t}dt \\ &= a^{-1}x^ae^{-x}{}_1F_1(1;1+a;x) = a^{-1}x^a{}_1F_1(a;1+a;-x),\end{aligned} \tag{2.3}$$

where ${}_1F_1$ is the confluent hypergeometric function of the first kind. Also for n positive integer, we have

$$\gamma(n,x) = (n-1)!\big(1 - e^{-x}\sum_{0\leq k\leq n-1}\frac{x^k}{k!}\big) = \Gamma(n)\big(1 - e^{-x}\sum_{0\leq k<n}\frac{x^k}{k!}\big).$$

The confluent hypergeometric function has a hypergeometric series given by

$${}_1F_1(a;b;z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots = \sum_{k\geq 0}\frac{(a)_k}{(b)_k}\frac{z^k}{k!},$$

where $(x)_n$ stands for the Pochhammer symbol:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\dots(x+n-1).$$

We note also the following identity:

$$\forall a \in \mathbb{C}\backslash\mathbb{Z}_-^*, \quad {}_1F_1(1; 1+a; z) = \sum_{k\ge 0} \frac{z^k}{(a+1)\dots(a+k)} \tag{2.4}$$

which is an entire function of the variable z for these values of a; as a result, we can write for $\operatorname{Re} a > 0, x \ge 0$,

$$\gamma(a,x) = a^{-1}x^a e^{-x} \sum_{k\ge 0} \frac{x^k}{(a+1)\dots(a+k)}, \tag{2.5}$$

and this implies that

$$\forall a > 0, \forall x \ge 0, \quad a^{-1}x^a e^{-x} \le \gamma(a,x) \le \min\bigl(\Gamma(a), a^{-1}x^a\bigr). \tag{2.6}$$

We have also

$$\gamma(1+a,x) = a\gamma(a,x) - x^a e^{-x}. \tag{2.7}$$

Remark 2.2 *The above lemma can be generalized easily to the case where* A *is a complex-valued symmetric matrix with a positive definite real part, with* $u_A(x) = (\det A)^{1/2} e^{-\pi\langle Ax,x\rangle}$, *with* $(\det A)^{1/2} = e^{\frac12 \operatorname{trace} \operatorname{Log} A}$ *(see the appendix for the choice of the determination of* $\operatorname{Log} A$*).*

Proof (Proof of the lemma) We consider for $t > 0$ the smooth function

$$F_{j,k}(t,x) = \int_{\mathbb{R}^n} e^{2i\pi x\xi} e^{-t4\pi^2|\xi|^2} \xi_j \xi_k |\xi|^{-2} d\xi, \tag{2.8}$$

and we note that

$$\begin{aligned}\frac{\partial F_{jk}}{\partial t}(t,x) &= -4\pi^2 \int e^{2i\pi x\xi} e^{-t4\pi^2|\xi|^2} \xi_j \xi_k d\xi \\ &= -4\pi^2 \frac{1}{(2i\pi)^2} \partial_{x_j}\partial_{x_k} \int e^{2i\pi x\xi} e^{-t4\pi^2|\xi|^2} d\xi = \partial_{x_j}\partial_{x_k}(e^{-\frac{|x|^2}{4t}})(4\pi t)^{-n/2},\end{aligned} \tag{2.9}$$

so that

$$\text{for } j \ne k,\ \frac{\partial F_{jk}}{\partial t}(t,x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} (\frac{x_j x_k}{4t^2}), \tag{2.10}$$

$$\text{for } j = k,\ \frac{\partial F_{jj}}{\partial t}(t,x) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} (\frac{x_j^2}{4t^2} - \frac{1}{2t}). \tag{2.11}$$

Since we have also $F_{j,k}(+\infty, x) = 0$, we obtain for $j \neq k$, $x \neq 0$,

$$\begin{aligned} F_{jk}(\frac{1}{4\pi}, x) &= \int_{+\infty}^{1/4\pi} (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} \frac{x_j x_k}{4t^2} dt \\ &= -x_j x_k \int_0^{\pi|x|^2} s^{n/2} |x|^{-n} e^{-s} 4s^2 |x|^{-4} |x|^2 s^{-2} ds \pi^{-n/2} \\ &= -x_j x_k |x|^{-n-2} \int_0^{\pi|x|^2} s^{n/2} e^{-s} ds \pi^{-n/2}, \end{aligned} \tag{2.12}$$

i.e., for $j \neq k$,

$$\int_{\mathbb{R}^n} e^{2i\pi x\xi} e^{-\pi|\xi|^2} \xi_j \xi_k |\xi|^{-2} d\xi = -x_j x_k |x|^{-n-2} \int_0^{\pi|x|^2} s^{n/2} e^{-s} ds \pi^{-n/2}. \tag{2.13}$$

For $j = k$, we have

$$\begin{aligned} F_{jj}(\frac{1}{4\pi}, x) &= \int_{+\infty}^{1/4\pi} (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} (\frac{x_j^2}{4t^2} - \frac{1}{2t}) dt \\ &= -x_j^2 |x|^{-n-2} \int_0^{\pi|x|^2} s^{n/2} e^{-s} ds \pi^{-n/2} \\ &\quad + \frac{1}{2} |x|^{-n} \int_0^{\pi|x|^2} s^{\frac{n}{2}-1} e^{-s} ds \pi^{-n/2}, \end{aligned} \tag{2.14}$$

so that

$$\begin{aligned} &\int_{\mathbb{R}^n} e^{2i\pi x\xi} e^{-\pi|\xi|^2} \xi_j^2 |\xi|^{-2} d\xi \\ &= -x_j^2 |x|^{-n-2} \int_0^{\pi|x|^2} s^{n/2} e^{-s} ds \pi^{-n/2} + \frac{1}{2} |x|^{-n} \int_0^{\pi|x|^2} s^{\frac{n}{2}-1} e^{-s} ds \pi^{-n/2}. \end{aligned} \tag{2.15}$$

As a consequence, for $t > 0, j \neq k$, we have

$$F_{jk}(t, x) = -\gamma(1 + \frac{n}{2}, \frac{|x|^2}{4t}) \pi^{-n/2} \frac{x_j x_k}{|x|^{2+n}}, \tag{2.16}$$

and

$$F_{jj}(t, x) = -\gamma(1 + \frac{n}{2}, \frac{|x|^2}{4t}) \pi^{-n/2} \frac{x_j^2}{|x|^{2+n}} + \frac{1}{2} \gamma(\frac{n}{2}, \frac{|x|^2}{4t}) \pi^{-n/2} \frac{1}{|x|^n}. \tag{2.17}$$

As a result, we have indeed, with $a > 0$, $j \neq k$,

$$(R_j R_k u_a)(x) = \int_{\mathbb{R}^n} e^{2i\pi x\xi} e^{-\pi a^{-1}|\xi|^2} \xi_j \xi_k |\xi|^{-2} d\xi \tag{2.18}$$

$$= a^{n/2} \int_{\mathbb{R}^n} e^{2i\pi a^{1/2} x\xi} e^{-\pi|\xi|^2} \xi_j \xi_k |\xi|^{-2} d\xi \tag{2.19}$$

$$= a^{n/2} F_{jk}(\frac{1}{4\pi}, a^{1/2}x) \tag{2.20}$$

$$= -x_j x_k |x|^{-n-2} \gamma(1+\frac{n}{2}, a\pi|x|^2)\pi^{-n/2}, \tag{2.21}$$

and for $j = k$,

$$(R_j^2 u_a)(x) = \int_{\mathbb{R}^n} e^{2i\pi x\xi} e^{-\pi a^{-1}|\xi|^2} \xi_j^2 |\xi|^{-2} d\xi \tag{2.22}$$

$$= a^{n/2} \int_{\mathbb{R}^n} e^{2i\pi a^{1/2} x\xi} e^{-\pi|\xi|^2} \xi_j^2 |\xi|^{-2} d\xi \tag{2.23}$$

$$= a^{n/2} F_{jj}(\frac{1}{4\pi}, a^{1/2}x) \tag{2.24}$$

$$= -x_j^2 |x|^{-n-2} \gamma(1+\frac{n}{2}, a\pi|x|^2)\pi^{-n/2} + \frac{1}{2}|x|^{-n}\gamma(\frac{n}{2}, a\pi|x|^2)\pi^{-n/2}. \tag{2.25}$$

□

Theorem 2.3 *Let $n \geq 1$ be an integer and $\Delta = \sum_{1\leq j\leq n} \partial_{x_j}^2$ be the standard Laplace operator on $\mathbb{R}^n$. For $t \geq 0$, we define the Oseen matrix operator*

$$\Omega(t) = \Delta^{-1}(\nabla \otimes \nabla)e^{t\Delta} = (I - \mathbf{P})e^{t\Delta} = \Delta^{-1}(\partial_{x_j} \otimes \partial_{x_k})_{1\leq j,k\leq n} e^{t\Delta}. \tag{2.26}$$

The operator $\Omega(t)$ is the Fourier multiplier by the matrix $\Omega(t,\xi) = |\xi|^{-2}(\xi \otimes \xi)e^{-4\pi t|\xi|^2}$ and is given by the convolution (w.r.t. the variable x) with the matrix $\big(F_{jk}(t,x)\big)_{1\leq j,k\leq n}$ where

$$\text{for } j \neq k,\ F_{jk}(t,x) = -x_j x_k |x|^{-n-2} \gamma(1+\frac{n}{2}, \frac{|x|^2}{4t})\pi^{-n/2}, \tag{2.27}$$

$$F_{jj}(t,x) = -x_j^2 |x|^{-n-2} \gamma(1+\frac{n}{2}, \frac{|x|^2}{4t})\pi^{-n/2} + \frac{1}{2}|x|^{-n}\gamma(\frac{n}{2}, \frac{|x|^2}{4t})\pi^{-n/2}, \tag{2.28}$$

$$F_{jj}(t,x) = \gamma(\frac{n}{2}, \frac{|x|^2}{4t})\pi^{-n/2}\frac{1}{2}|x|^{-n-2}\big(|x|^2 - nx_j^2\big) + x_j^2|x|^{-2}(4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}. \tag{2.29}$$

On $t > 0$, the functions F_{jk} are real analytic functions of the variable $t^{-1/2}x$ multiplied by $t^{-n/2}$. We have also

$$F_{jk}(t,x) = (4\pi t)^{-n/2} F_{jk}\big(\frac{1}{4\pi}, x(4\pi t)^{-1/2}\big), \tag{2.30}$$

and with $|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}$,

$$|F_{jk}(t,x)| \leq |x|^{-n}\frac{n+1}{|\mathbb{S}^{n-1}|}, \qquad |F_{jk}(t,x)| \leq \big(\frac{|x|^2}{2(n+2)t} + \frac{1}{n}\big)(4\pi t)^{-n/2}. \tag{2.31}$$

Moreover we have

$$\text{for } j \neq k,\ \ F_{jk}(t,x) = -\frac{2}{n+2}\frac{x_j x_k}{4t}(4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} {}_1F_1(1; 2+\frac{n}{2}; \frac{|x|^2}{4t}) \tag{2.32}$$

and

$$\begin{aligned} F_{jj}(t,x) = &-\frac{2}{n+2}\frac{x_j^2}{4t}(4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}} {}_1F_1(1; 2+\frac{n}{2}; \frac{|x|^2}{4t}) \\ &+ \frac{1}{n}(4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} {}_1F_1(1; 1+\frac{n}{2}; \frac{|x|^2}{4t}). \end{aligned} \tag{2.33}$$

Proof (The proof is an immediate consequence of (2.16), (2.17), (2.6)*.)*

Remark 2.4 *This theorem provides a direct proof, using special functions, of the estimates established in a more general context in* [4] *as well as those stated on page* 27 *of* [6]*.*

Remark 2.5 *We get easily from the first part of the previous theorem that the kernel of the operator* $I - \mathbf{P}$*, which is the matrix Fourier multiplier* $|\xi|^{-2}(\xi \otimes \xi)$*, is the singular integral given by the (principal-value) convolution with the matrix* $(f_{jk}(x))$ *where*

$$\text{for } j \neq k,\ \ f_{jk}(x) = -x_j x_k |x|^{-n-2}\Gamma(1+\frac{n}{2})\pi^{-n/2} = -x_j x_k |x|^{-n-2}\frac{n}{|\mathbb{S}^{n-1}|}, \tag{2.34}$$

$$f_{jj}(x) = |x|^{-n-2}(|x|^2 - n x_j^2)|\mathbb{S}^{n-1}|^{-1} + n^{-1}\delta_0(x). \tag{2.35}$$

We note also that the functions $g_{jk} = f_{jk} - n^{-1}\delta_{j,k}\delta_0$ *are homogeneous of degree* $-n$ *on* $\mathbb{R}^n \backslash \{0\}$ *with integral* 0 *on* $\mathbb{S}^{n-1}$ *so that the principal value*

$$\langle T_{jk}, \varphi \rangle = \lim_{\varepsilon \to 0_+} \int_{|x| \geq \varepsilon} g_{jk}(x)\varphi(x)dx$$

actually defines a homogeneous distribution T_{jk} *of degree* $-n$ *on* $\mathbb{R}^n$ *([5]).*

3 Appendix

The Fourier transformation

The Fourier transform of a function u in the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ is defined by the formula

$$\hat{u}(\xi) = \int e^{-2i\pi x\xi} u(x)dx, \tag{3.1}$$

and it is an isomorphism of $\mathscr{S}(\mathbb{R}^n)$ so that $u(x) = \int e^{2i\pi x\xi}\hat{u}(\xi)d\xi$. That isomorphism extends to an isomorphism of the temperate distributions $\mathscr{S}'(\mathbb{R}^n)$ via the duality formula $\langle \widehat{T}, \varphi \rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\varphi} \rangle_{\mathscr{S}',\mathscr{S}}$. The Fourier transform is also a unitary transformation of $L^2(\mathbb{R}^n)$.

The logarithm of a nonsingular symmetric matrix

The set $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, so that we can define the principal determination of the logarithm for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by the formula

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\zeta}{\zeta}. \tag{3.2}$$

The function Log is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$ and we have $\operatorname{Log} z = \ln z$ for $z \in \mathbb{R}_+^*$ and by analytic continuation $e^{\operatorname{Log} z} = z$ for $z \in \mathbb{C}\backslash\mathbb{R}_-$. We get also by analytic continuation that $\operatorname{Log} e^z = z$ for $|\operatorname{Im} z| < \pi$.

Let Υ_+ be the set of symmetric nonsingular $n \times n$ matrices with complex entries and nonnegative real part. The set Υ_+ is star-shaped with respect to the Id: for $A \in \Upsilon_+$, the segment $[1, A] = \big((1-t)\operatorname{Id} + tA\big)_{t\in[0,1]}$ is obviously made with symmetric matrices with nonnegative real part which are invertible, since for $0 \le t < 1$, $\operatorname{Re}\big((1-t)\operatorname{Id} + tA\big) \ge (1-t)\operatorname{Id} > 0$ and for $t = 1$, A is assumed to be invertible. We can now define for $A \in \Upsilon_+$

$$\operatorname{Log} A = \int_0^1 (A - I)\big(I + t(A - I)\big)^{-1} dt. \tag{3.3}$$

We note that A commutes with $(I + sA)$ (and thus with $\operatorname{Log} A$), so that, for $\theta > 0$,

$$\begin{aligned}\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = \int_0^1 &\big(I + t(A + \theta I - I)\big)^{-1} dt \\ &- \int_0^1 \big(A + \theta I - I\big)t\big(I + t(A + \theta I - I)\big)^{-2} dt,\end{aligned}$$

and since $\frac{d}{dt}\Big\{\big(I + t(A + \theta I - I)\big)^{-1}\Big\} = -\big(I + t(A + \theta I - I)\big)^{-2}(A + \theta I - I)$, we obtain by integration by parts $\frac{d}{d\theta}\operatorname{Log}(A + \theta I) = (A + \theta I)^{-1}$. As a result, we find that for $\theta > 0, A \in \Upsilon_+$, since all the matrices involved are commuting,

$$\frac{d}{d\theta}\Big((A + \theta I)^{-1} e^{\operatorname{Log}(A+\theta I)}\Big) = 0,$$

so that, using the limit $\theta \to +\infty$, we get that $\forall A \in \Upsilon_+, \forall \theta > 0,\ e^{\operatorname{Log}(A+\theta I)} = (A + \theta I)$, and by continuity

$$\forall A \in \Upsilon_+, \quad e^{\operatorname{Log} A} = A, \quad \text{which implies} \quad \det A = e^{\operatorname{trace}\operatorname{Log} A}. \tag{3.4}$$

Using (3.4), we can define for $A \in \Upsilon_+$, using (3.3),

$$(\det A)^{-1/2} = e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} = |\det A|^{-1/2} e^{-\frac{i}{2}\operatorname{Im}(\operatorname{trace}\operatorname{Log} A)}. \tag{3.5}$$

- When A is a positive definite matrix, $\operatorname{Log} A$ is real-valued and $(\det A)^{-1/2} = |\det A|^{-1/2}$.

- When $A = -iB$ where B is a real nonsingular symmetric matrix, we note that $B = PD^tP$ with $P \in O(n)$ and D diagonal. We see directly on the formulas (3.3),(3.2) that
$$\operatorname{Log} A = \operatorname{Log}(-iB) = P(\operatorname{Log}(-iD))^tP, \quad \operatorname{trace}\operatorname{Log} A = \operatorname{trace}\operatorname{Log}(-iD)$$
and thus, with (μ_j) the (real) eigenvalues of B, we have $\operatorname{Im}(\operatorname{trace}\operatorname{Log} A) = \operatorname{Im}\sum_{1\le j\le n}\operatorname{Log}(-i\mu_j)$, where the last Log is given by (3.2). Finally we get
$$\operatorname{Im}(\operatorname{trace}\operatorname{Log} A) = -\frac{\pi}{2}\sum_{1\le j\le n}\operatorname{sign}\mu_j = -\frac{\pi}{2}\operatorname{sign} B$$
where $\operatorname{sign} B$ is the signature of B. As a result, we have when $A = -iB$, B real symmetric nonsingular matrix,
$$(\det A)^{-1/2} = |\det A|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign}(iA)} = |\det B|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} B}. \tag{3.6}$$

Proposition 3.1 *Let A be a symmetric nonsingular $n\times n$ matrix with complex entries such that $\operatorname{Re} A \ge 0$. We define the Gaussian function v_A on $\mathbb{R}^n$ by $v_A(x) = e^{-\pi\langle Ax,x\rangle}$. The Fourier transform of v_A is*
$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}, \tag{3.7}$$
where $(\det A)^{-1/2}$ is defined according to formula (3.5). In particular, when $A = -iB$ with a symmetric real nonsingular matrix B, we get
$$\operatorname{Fourier}(e^{i\pi\langle Bx,x\rangle})(\xi) = \widehat{v_{-iB}}(\xi) = |\det B|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sign} B} e^{-i\pi\langle B^{-1}\xi,\xi\rangle}.$$

Proof. Let us define Υ_+^* as the set of symmetric $n\times n$ complex matrices with a positive definite real part (naturally these matrices are nonsingular since $Ax = 0$ for $x \in \mathbb{C}^n$ implies $0 = \operatorname{Re}\langle Ax,\bar{x}\rangle = \langle(\operatorname{Re} A)x,\bar{x}\rangle$, so that $\Upsilon_+^* \subset \Upsilon_+$).

Let us assume first that $A \in \Upsilon_+^*$; then the function v_A is in the Schwartz class (and so is its Fourier transform). The set Υ_+^* is an open convex subset of $\mathbb{C}^{n(n+1)/2}$ and the function $\Upsilon_+^* \ni A \mapsto \widehat{v_A}(\xi)$ is holomorphic and given on $\Upsilon_+^* \cap \mathbb{R}^{n(n+1)/2}$ by (3.7). On the other hand, the function $\Upsilon_+^* \ni A \mapsto e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi,\xi\rangle}$ is also holomorphic and coincides with previous one on $\mathbb{R}^{n(n+1)/2}$. By analytic continuation this proves (3.7) for $A \in \Upsilon_+^*$.

If $A \in \Upsilon_+$ and $\varphi \in \mathscr{S}(\mathbb{R}^n)$, we have $\langle\widehat{v_A},\varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int v_A(x)\hat{\varphi}(x)dx$ so that $\Upsilon_+ \ni A \mapsto \langle\widehat{v_A},\varphi\rangle$ is continuous and thus (note that the mapping $A \mapsto A^{-1}$ is a homeomorphism of Υ_+), using the previous result on Υ_+^*,
$$\langle\widehat{v_A},\varphi\rangle = \lim_{\varepsilon\to 0_+}\langle\widehat{v_{A+\varepsilon I}},\varphi\rangle = \lim_{\varepsilon\to 0_+}\int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log}(A+\varepsilon I)} e^{-\pi\langle(A+\varepsilon I)^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi$$
(by continuity of Log on Υ_+ and domin. cv.)
$$= \int e^{-\frac{1}{2}\operatorname{trace}\operatorname{Log} A} e^{-\pi\langle A^{-1}\xi,\xi\rangle}\varphi(\xi)d\xi,$$
which is the sought result.

Some standard examples of Fourier transform

Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \le 0$. With the notation of this section, we have, with δ_0 the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\text{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi} 2\widehat{\delta_0}(\xi) = \widehat{D\,\text{sign}}(\xi) = \xi\widehat{\text{sign}}\xi$$

so that $\xi\big(\widehat{\text{sign}}\xi - \frac{1}{i\pi} pv(1/\xi)\big) = 0$ and $\widehat{\text{sign}}\xi - \frac{1}{i\pi} pv(1/\xi) = c\delta_0$ with $c = 0$ since the lhs is odd. We get

$$\widehat{\text{sign}}(\xi) = \frac{1}{i\pi} pv \frac{1}{\xi}, \qquad \widehat{pv(\frac{1}{\pi x})} = -i \,\text{sign}\, \xi, \qquad \hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi} pv(\frac{1}{\xi}). \quad (3.8)$$

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Instability Behavior and Loss of Regularity

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Summary. Solutions to weakly hyperbolic Cauchy problems contain as one of the most important properties the so-called *loss of regularity*. Recently authors have begun to understand how to show that the loss really appears. In this note we describe several models and explain different ways how to attack the question that a ν loss of regularity really appears.

2000 AMS Subject Classification: Primary: 35L80, 35L15, 35B40

Key words: Weakly hyperbolic equations, oscillations, degeneracies, loss of regularity, Floquet theory, theory of special functions.

1 Introduction

It is well-known that solutions to degenerate hyperbolic Cauchy problems show in many cases the effect of *loss of regularity or loss of derivatives*. In the pioneering paper [5] the authors discussed among other things strictly hyperbolic Cauchy problems with low regularity in coefficients. Let us explain the results by the model case

$$u_{tt} - a(t)u_{xx} = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ a(t) \geq C > 0.$$

1. If $\int_0^{T-\tau} |a(t+\tau) - a(t)|dt \leq A\tau,\ \tau \in [0, T/2]$, then the Cauchy problem is H^s (C^∞) well-posed *without any loss of derivatives.*
2. If $\int_0^{T-\tau} |a(t+\tau)-a(t)|dt \leq A\tau(|\log\tau|+1),\ \tau \in (0, T/2]$, then the Cauchy problem is H^∞ (C^∞) well-posed *with an at most finite loss of derivatives.*

There are other examples which describe a deteriorating behavior of oscillating coefficients (here near $t = 0$).

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_11,

1. [7] The Cauchy problem

$$u_{tt} - \Big(2 + \cos\Big(\log\frac{1}{t}\Big)^{\alpha}\Big)u_{xx} = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

   is C^∞ well-posed if and only if $\alpha \le 2$.
2. [15] The Cauchy problem

$$u_{tt} - e^{-2t^{-\alpha}}\Big(2 + \cos\Big(\frac{1}{t}\Big)\Big)u_{xx} = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

   is C^∞ well-posed if and only if $\alpha \ge 1/2$.

All these cited results have common features.

1. A threshold is explained between finite loss or infinite loss of derivatives.
2. Sufficient conditions explain that at most a finite loss of derivatives appears.
3. Counterexamples show that the infinite loss of derivatives really appears.

At the moment we have only some model cases for which it is shown that a precise "finite" loss of regularity really appears. In [16] the authors studied systematically the finite degenerate case, and explained loss of regularity and difference of regularities of data. By application of the *theory of confluent hypergeometric functions* they arrive at the following conclusion:

Proposition 1.1 *Let us consider the weakly hyperbolic Cauchy problem*

$$u_{tt} - t^{2l}u_{xx} - at^{l-1}u_x = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

under the assumption $u_0 \in H^s$, $u_1 \in H^{s-\frac{1}{l+1}}$, *respectively. Then there exists a unique solution*

$$\begin{cases} u \in C([0,T],H^s) \cap C^1([0,T],H^{s-1-\frac{|a|-l}{2(l+1)}}) & \text{if} \quad |a| \le l, \\ u \in C([0,T],H^{s-\frac{|a|-l}{2(l+1)}}) \cap C^1([0,T],H^{s-1-\frac{|a|-l}{2(l+1)}}) & \text{if} \quad |a| \ge l. \end{cases}$$

Remark 1.2 *This proposition explains that a loss of regularity for the solution itself appears only for* $|a| > l$. *The loss is* $\langle D_x\rangle^{\frac{|a|-l}{2(l+1)}}$ *and the difference of regularities of data is* $\langle D_x\rangle^{\frac{1}{l+1}}$. *Moreover, the statement generalizes a well-known result for the classical wave equation* ($l = a = 0$).

Let us recall a regularity result from [1]. For this reason we introduce the Cauchy problem with *infinite or flat degeneracy*

$$u_{tt} - \Big(\frac{1}{t^2}e^{-\frac{1}{t}}\Big)^2 u_{xx} - a\frac{1}{t^4}e^{-\frac{1}{t}}u_x = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad (1.1)$$

with a real coefficient a. Then the results from [1] yield the following statement:

Proposition 1.3 *Let us suppose* $u_0 \in (\log\langle D_x\rangle)^{-1}H^s$ *and* $u_1 \in H^s$ *for the Cauchy problem* (1.1)*. Then there exists a unique solution*

$$\begin{cases} u \in C([0,T],H^s) \cap C^1([0,T],H^{s-\frac{|a|+1}{2}}) & \text{if} \quad |a|<1, \\ u \in C([0,T],H^{s-\frac{|a|-1}{2}}) \cap C^1([0,T],H^{s-\frac{|a|+1}{2}}) & \text{if} \quad |a| \geq 1. \end{cases}$$

In the following proposition we give a more precise statement.

Theorem 1.4 *Consider the Cauchy problem* (1.1) *under the assumptions* $u_0 \in H^s, u_1 \in \log\langle D_x\rangle H^s$*, then there is a unique solution*

$$\begin{cases} u \in C([0,T],H^s) \cap C^1([0,T],(\log\langle D_x\rangle)H^{s-\frac{|a|+1}{2}}) & \text{if } |a|<1, \\ u \in C([0,T],(\log\langle D_x\rangle)H^{s-\frac{|a|-1}{2}}) \cap C^1([0,T],(\log\langle D_x\rangle)H^{s-\frac{|a|+1}{2}}) & \text{if } |a| \geq 1. \end{cases}$$

Remark 1.5 *Let us compare the statements of Proposition* 1.3 *and Theorem* 1.4.

In Proposition 1.3 *the following properties of solutions to the Cauchy problem* (1.1) *are shown:*

- *The difference of regularity of data is* $\log\langle D_x\rangle$.
- *The threshold for a higher loss of regularity for the solution is* $|a|=1$.
- *For all real* a *there appears at least the loss of regularity* $\log\langle D_x\rangle$.

In Theorem 1.4 *the following properties of solutions to the Cauchy problem* (1.1) *are shown:*

- *The difference of regularity of data is* $\log\langle D_x\rangle$.
- *The threshold for the loss of regularity is* $|a|=1$.
- *For all real* $|a|<1$ *there is no loss of regularity for the solution itself.*

The statements of Proposition 1.1, Theorem 1.4 and the example from [15] can be concluded from the following general result which is proved in [10].

Proposition 1.6 *Let us consider with a real constant* a *the Cauchy problem*

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} - a\frac{\lambda^2(t)}{\Lambda(t)}u_x = 0,\ u(0,x)=u_0(x),\ u_t(0,x)=u_1(x),$$

under the following assumptions to the function $\lambda = \lambda(t) \in C[0,T] \cap C^2(0,T]$ *describing the degeneracy of the coefficient at* $t=0$*:*

$$\begin{cases} \lambda(0)=0,\ \lambda'(t)>0,\ t\in(0,T], \\ d_0\frac{\lambda(t)}{\Lambda(t)} \leq \frac{\lambda'(t)}{\lambda(t)} \leq d_1\frac{\lambda(t)}{\Lambda(t)},\ d_0,d_1>0,\ \Lambda(t)=\int_0^t \lambda(s)ds, \\ |\lambda''(t)| \leq d_2\lambda(t)(\frac{\lambda(t)}{\Lambda(t)})^2. \end{cases}$$

We introduce a positive and monotonously decreasing continuous function $\nu = \nu(t)$, $t \in (0,T]$, which measures the oscillating behavior of the coefficient. Finally, we suppose the following conditions to the function $b = b(t) \in C^2(0,T]$ describing the oscillating behavior of the coefficient at $t = 0$:

$$\begin{cases} c_0 := \inf\limits_{t\in(0,T]} b(t) \le b(t) \le c_1 := \sup\limits_{t\in(0,T]} b(t),\ t\in(0,T],\ c_0, c_1 > 0, \\ |b^{(k)}(t)| \le C_k\big(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\big)^k,\ k = 1, 2. \end{cases}$$

Under these assumptions, if $u_0 \in H^s$, $u_1 \in (\Lambda^{-1}(\frac{N}{\langle D_x\rangle}))^{-1}H^s$, N is a fixed large positive constant, then there is a unique solution u belonging to the following function spaces:

1. *when $0 < \lim_{t\to 0} \frac{\Lambda(t)^{\frac{|a|}{c_0}}}{\lambda(t)} \le +\infty$, then*

$$u \in C\Big([0,T], \exp\Big(C_\alpha \nu\Big(\Big(\frac{\Lambda}{\nu}\Big)^{-1}\Big(\frac{N_2}{\langle D_x\rangle}\Big)\Big)\Big)H^s\Big)$$

$$\bigcap C^1\Big([0,T], \frac{\lambda^{\frac{1}{2}}\big((\frac{\Lambda}{\nu})^{-1}(\frac{N_2}{\langle D_x\rangle})\big)}{\Lambda^{\frac{|a|}{2c_0}}\big((\frac{\Lambda}{\nu})^{-1}(\frac{N_2}{\langle D_x\rangle})\big)} \exp\Big(C_\alpha \nu\Big(\Big(\frac{\Lambda}{\nu}\Big)^{-1}\Big(\frac{N_2}{\langle D_x\rangle}\Big)\Big)\Big)H^{s-1}\Big)$$

with nonnegative constants C_α and N_2,

2. *when $\lim_{t\to 0} \frac{\Lambda(t)^{\frac{|a|}{c_0}}}{\lambda(t)} = 0$, then*

$$u \in C\Big([0,T], \frac{\lambda^{\frac{1}{2}}\big((\frac{\Lambda}{\nu})^{-1}(\frac{N_2}{\langle D_x\rangle})\big)}{\Lambda^{\frac{|a|}{2c_0}}\big((\frac{\Lambda}{\nu})^{-1}(\frac{N_2}{\langle D_x\rangle})\big)} \exp\Big(C_\alpha \nu\Big(\Big(\frac{\Lambda}{\nu}\Big)^{-1}\Big(\frac{N_2}{\langle D_x\rangle}\Big)\Big)\Big)H^s\Big)$$

$$\bigcap C^1\Big([0,T], \frac{\lambda^{\frac{1}{2}}\big((\frac{\Lambda}{\nu})^{-1}(\frac{N_2}{\langle D_x\rangle})\big)}{\Lambda^{\frac{|a|}{2c_0}}\big((\frac{\Lambda}{\nu})^{-1}(\frac{N_2}{\langle D_x\rangle})\big)} \exp\Big(C_\alpha \nu\Big(\Big(\frac{\Lambda}{\nu}\Big)^{-1}\Big(\frac{N_2}{\langle D_x\rangle}\Big)\Big)\Big)H^{s-1}\Big)$$

with nonnegative constants C_α and N_2.

Remark 1.7 *We restrict ourselves to the term of smaller order $a\frac{\lambda^2(t)}{\Lambda(t)}u_x$. In this way we assume sharp Levi conditions which are connected with the term $\lambda^2(t)$ in the principal part describing the degeneracy at $t = 0$. This is reasonable. The paper* [9] *shows that it is very difficult to determine precise Levi conditions which are connected with the oscillating term $b^2(t)$ in the principal part.*

Proposition 1.6 yields sufficient conditions under which no *loss of derivatives appears for solutions to weakly hyperbolic Cauchy problems* ($\nu \equiv$ const). Moreover, it gives some more precise (to the usual ones) explanations.

1. A threshold is explained between no loss and loss.
2. Sufficient conditions explain that at most an arbitrary small loss or finite loss of derivatives appears.

In the present paper we are interested in the following question:

Are we able to show by constructing examples that the arbitrary small loss or finite loss or infinite loss of derivatives really appears?

A first method is to use the *theory of special functions*. This will be done in Section 2 to prove Theorem 1.4, that is, the more precise statement in comparison with the statement from [1] for (1.1). In this way we get in addition that the *small loss or finite loss really appears*.

In Section 3 we will discuss the optimality of our results concerning the infinite loss of regularity by the application of *Floquet theory*. We will prove the infinite loss for a family of coefficients in (1.1) with $a = 0$ having an oscillating behavior arbitrarily close to the critical case. The family is produced by an arbitrary given periodic function $b = b(t)$. We denote as critical case the case which yields H^∞ well-posedness with a finite loss of derivatives.

The question for the optimality of our results in the case of finite loss or arbitrary small loss of regularity is considered in detail in Section 4. Here we choose families of Cauchy problems satisfying uniform assumptions from Proposition 1.6 for the critical or for cases better than the critical one and show that the loss of regularity really appears.

Some concluding remarks in Section 5 complete the paper.

2 Proof of Theorem 1.4

Let us recall the Cauchy problem with infinite or flat degeneracy

$$u_{tt} - \Big(\frac{1}{t^2}e^{-\frac{1}{t}}\Big)^2 u_{xx} - a\frac{1}{t^4}e^{-\frac{1}{t}}u_x = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \quad (2.1)$$

with a real constant a.

Proof. We follow the approach of [1]. After partial Fourier transformation we have

$$\hat{u}_{tt} + \Big(\frac{1}{t^2}e^{-\frac{1}{t}}\Big)^2 \xi^2\hat{u} - ia\frac{1}{t^4}e^{-\frac{1}{t}}\xi\hat{u} = 0.$$

The transformations

$$\tau = \xi\Lambda(t),\ \hat{u}(t,\xi) = tv(\tau)$$

imply

$$v_{\tau\tau} + \frac{1}{\tau}v_\tau + (1 - \frac{ia}{\tau})v = 0.$$

Finally, we apply the transformations

$$\tau = \frac{z}{2i}, \; w(z) = e^{\frac{z}{2}} v(\tau)$$

which give

$$z w_{zz}(z) + (1-z) w_z(z) - \frac{1+a}{2} w(z) = 0.$$

This is *Kummer's equation (logarithmic case)*, that is,

$$z w_{zz}(z) + (\gamma - z) w_z(z) - \alpha w(z) = 0 \text{ with } \gamma = 1 \text{ and } \alpha = \frac{1+a}{2}.$$

Lemma 2.1 *(see* [2]*) The functions*

$$v_1(t,\xi) = t e^{-i\Lambda(t)\xi} \Psi(\alpha, 1; 2i\Lambda(t)\xi), \quad v_2(t,\xi) = t e^{i\Lambda(t)\xi} \Psi(1-\alpha, 1; -2i\Lambda(t)\xi)$$

form a fundamental system of solutions for $t \geq 0$. Here $\alpha = \frac{1+a}{2}$.

Any solution $\hat{u}(t,\xi)$ can be written as follows:

$$\hat{u}(t,\xi) = p_0(t,\xi)\hat{u}(0,\xi) + p_1(t,\xi)\hat{u}_t(0,\xi),$$

where

$$\begin{cases} p_0(t,\xi) = W(0,\xi)^{-1}\big(v_{2,t}(0,\xi)v_1(t,\xi) - v_{1,t}(0,\xi)v_2(t,\xi)\big), \\ p_1(t,\xi) = W(0,\xi)^{-1}\big(-v_2(0,\xi)v_1(t,\xi) + v_1(0,\xi)v_2(t,\xi)\big), \\ W(0,\xi) = v_1(0,\xi)v_{2,t}(0,\xi) - v_{1,t}(0,\xi)v_2(0,\xi), \end{cases}$$

and

$$\begin{cases} v_{1,t}(t,\xi) = e^{-i\Lambda(t)\xi}(1 - t^{-1} i\xi\Lambda(t))\Psi(\alpha,1;2i\Lambda(t)\xi) \\ \qquad\qquad + 2i\xi t^{-1}\Lambda(t) e^{-i\Lambda(t)\xi}\Psi_z(\alpha,1;2i\Lambda(t)\xi), \\ v_{2,t}(t,\xi) = e^{i\Lambda(t)\xi}(1 + t^{-1} i\xi\Lambda(t))\Psi(1-\alpha,1;-2i\Lambda(t)\xi) \\ \qquad\qquad - 2i\xi t^{-1}\Lambda(t) e^{i\Lambda(t)\xi}\Psi_z(1-\alpha,1;-2i\Lambda(t)\xi). \end{cases}$$

Let us introduce some basic properties for the functions $\Psi = \Psi(\alpha,\gamma,z)$ (see [2]).

Taking into account the asymptotic expansion

$$\Psi(\alpha,1;z) = -\frac{1}{\Gamma(\alpha)}\big(\log z + \psi(\alpha) - 2\gamma\big) + o(|z\log z|) \text{ for } z \to 0,$$

where the constant γ is *Euler's constant* (a positive number belonging to the interval $(0,1)$) and where the function $\psi = \psi(z)$ is the logarithmic derivative $d_z \log\Gamma(z)$ of the Gamma function $\Gamma = \Gamma(z)$. Applying the formula

$$\Psi_z^{(n)}(\alpha,1;z) = (-1)^n(\alpha)_n\Psi(\alpha+n,1+n;z)$$

for $n = 1$, where

$$(\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

and

$$\Psi(\alpha,\gamma;z) = z^{1-\gamma}\frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} + O(|\log z|) \text{ when } \operatorname{Re}\gamma \geq 1 \text{ for small z},$$

we obtain

$$\begin{cases} v_{1,t}(0,\xi) = -\frac{1}{\Gamma(\alpha)}\big(\log|2\xi| + i(\operatorname{sign}\xi)\frac{\pi}{2} + \psi(\alpha) - 2\big), \\ v_{2,t}(0,\xi) = -\frac{1}{\Gamma(1-\alpha)}\big(\log|2\xi| - i(\operatorname{sign}\xi)\frac{\pi}{2} + \psi(1-\alpha) - 2\big), \\ v_1(0,\xi) = \frac{1}{\Gamma(\alpha)}, \quad v_2(0,\xi) = \frac{1}{\Gamma(1-\alpha)}. \end{cases}$$

Therefore, we have

$$\begin{aligned} p_0(t,\xi) = {} & \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\psi(\alpha) - \psi(1-\alpha) + i(\operatorname{sign}\xi)\pi} \\ & \times \Big(-\frac{1}{\Gamma(1-\alpha)}\big(\log|2\xi| - i(\operatorname{sign}\xi)\frac{\pi}{2} + \psi(1-\alpha) - 2\big) \\ & \times\; te^{-i\Lambda(t)\xi}\Psi(\alpha,1;2i\Lambda(t)\xi) + \frac{1}{\Gamma(\alpha)}\big(\log|2\xi| + i(\operatorname{sign}\xi)\frac{\pi}{2} + \psi(\alpha) - 2\big) \\ & \times\; te^{i\Lambda(t)\xi}\Psi(1-\alpha,1;-2i\Lambda(t)\xi)\Big); \\ p_1(t,\xi) = {} & \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\psi(\alpha) - \psi(1-\alpha) + i(\operatorname{sign}\xi)\pi}\Big(\frac{1}{\Gamma(\alpha)}te^{i\Lambda(t)\xi}\Psi(1-\alpha,1;-2i\Lambda(t)\xi) \\ & - \frac{1}{\Gamma(1-\alpha)}te^{-i\Lambda(t)\xi}\Psi(\alpha,1;2i\Lambda(t)\xi)\Big). \end{aligned}$$

Now we divide the extended phase space into two zones. First let us derive an estimate in the *pseudo-differential zone* $Z_{pd}(N,M) = \{(t,\xi) \in [0,T] \times \{|\xi| \geq M\} : \Lambda(t)|\xi| \leq N\}$.

Lemma 2.2 *In the pseudo-differential zone we have the following estimate:*

$$|\hat{u}(t,\xi)| \leq C(N,M)\Big(|\hat{u}(0,\xi)| + \frac{1}{\log\langle\xi\rangle}|\hat{u}_t(0,\xi)|\Big).$$

Proof. First, from the definition of the pseudo-differential zone, we have

$$t \leq \frac{1}{\log|\xi| - \log N} \quad \text{implies} \quad t\log|\xi| \leq C(N,M).$$

We apply the following properties of Ψ-functions and of Φ-functions:

$$\Phi(\alpha,\gamma;z)=\exp(z)\Phi(\gamma-\alpha,\gamma;-z),$$

$$\Psi(\alpha,n+1;z)=\frac{(-1)^{n-1}}{n!\Gamma(\alpha-n)}\Big(\Phi(\alpha,n+1;z)\log z + \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(n+1)_r}(\psi(\alpha+r)-\psi(1+r)-\psi(1+n+r))\frac{z^r}{r!}\Big) + \frac{(n-1)!}{\Gamma(\alpha)}\sum_{r=0}^{n-1}\frac{(\alpha-n)_r}{(1-n)_r}\frac{z^{r-n}}{r!}$$

for $n=0,1,2,\ldots,$ and the last sum has to be omitted if $n=0$.

Straightforward calculations bring the following asymptotic behavior of $p_0(t,\xi)$ and $p_1(t,\xi)$ in the pseudo-differential zone:

$$p_0(t,\xi)\sim\frac{\Phi(1-\alpha,1;-2i\Lambda(t)\xi)}{\psi(\alpha)-\psi(1-\alpha)+i(\text{sign}\xi)\pi} \times\ e^{i\Lambda(t)\xi}\Big(t\log(2\Lambda(t)|\xi|)(-i(\text{sign}\xi)\pi+\psi(1-\alpha)-\psi(\alpha)) + it(\text{sign}\xi)\frac{\pi}{2}(2\log|2\xi|+\psi(1-\alpha)+\psi(\alpha)-4)\Big) + e^{i\Lambda(t)\xi}\frac{t(\psi(\alpha)-2\psi(1))(\log|2\xi|-i(\text{sign}\xi)\frac{\pi}{2}+\psi(1-\alpha)-2)}{\psi(\alpha)-\psi(1-\alpha)+i(\text{sign}\xi)\pi} - e^{i\Lambda(t)\xi}\frac{t(\psi(1-\alpha)-2\psi(1))(\log|2\xi|+i(\text{sign}\xi)\frac{\pi}{2}+\psi(\alpha)-2)}{\psi(\alpha)-\psi(1-\alpha)+i(\text{sign}\xi)\pi}.$$

For the representation of p_1 we get

$$p_1(t,\xi)=\frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\psi(\alpha)-\psi(1-\alpha)+i(\text{sign}\xi)\pi}\Big(\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(1-\alpha)}te^{-i\Lambda(t)\xi} \times\ \Big(e^{2i\Lambda(t)\xi}\Phi(1-\alpha,1;-2i\Lambda(t)\xi)\log(2i\Lambda(t)\xi) + \sum_{r=0}^{\infty}\frac{(\alpha)_r}{(1)_r}(\psi(\alpha+r)-2\psi(1))\frac{(2i\Lambda(t)\xi)^r}{r!}\Big) + \frac{1}{\Gamma(\alpha)}\frac{-1}{\Gamma(1-\alpha)}te^{i\Lambda(t)\xi}\Big(\Phi(1-\alpha,1;-2i\ \Lambda(t)\xi)\log(-2i\Lambda(t)\xi)$$

$$+ \sum_{r=0}^{\infty} \frac{(1-\alpha)_r}{(1)_r}(\psi(1-\alpha+r) - 2\psi(1))\frac{(-2i\Lambda(t)\xi)^r}{r!}\Big)\Big)$$

$$\sim\ e^{-i\Lambda(t)\xi}\Big(\frac{it(\mathrm{sign}\xi)\pi\Phi(1-\alpha,1;-2i\Lambda(t)\xi)}{\psi(\alpha)-\psi(1-\alpha)+i(\mathrm{sign}\xi)\pi}+\frac{t\psi(\alpha)-t\psi(1-\alpha)}{\psi(\alpha)-\psi(1-\alpha)+i(\mathrm{sign}\xi)\pi}\Big),$$

where in the representation for p_0 and p_1 the sign $\sim$ means modulo terms behaving like $(\log\langle\xi\rangle)^{-1}$. From the definition of the zone and the special structure of $\Lambda(t)$, here $\frac{1}{t}$ plays a significant role, we know that $t\log(2\Lambda(t)|\xi|)$ remains bounded in a right-sided neighborhood of $\Lambda(t)\xi = 0$. Furthermore, $\Phi(\alpha,\gamma,z)$ is analytic with respect to z. Keeping in mind these observations we deduce immediately that

$$|p_0(t,\xi)| \le C(N,M), \quad |p_1(t,\xi)| \le \frac{C(N,M)}{\log\langle\xi\rangle}.$$

This implies the desired statement.

Now let us consider the *hyperbolic zone* which is defined by

$$Z_{hyp}(N,M) = \{(t,\xi)\in[0,T]\times\{|\xi|\ge M\} : \Lambda(t)|\xi| \ge N\}.$$

Lemma 2.3 *In the hyperbolic zone we have the following estimates:*

$$|\hat{u}(t,\xi)| \le \begin{cases} C(N,M)\Big(|\hat{u}(0,\xi)| + \frac{1}{\log\langle\xi\rangle}|\hat{u}_t(0,\xi)|\Big) & \text{for } |a|<1,\\ C(N,M)\langle\xi\rangle^{\frac{|a|-1}{2}}\Big(\log\langle\xi\rangle|\hat{u}(0,\xi)| + |\hat{u}_t(0,\xi)|\Big) & \text{for } |a|\ge 1.\end{cases}$$

Proof. In this zone we will apply the asymptotic behavior of the Ψ-functions for large $|z|$ with $0 < \arg z < \pi$:

1. $|\Psi(\alpha,1;z)| \le C(\alpha,1)|z|^{\mathrm{Re}(-\alpha)}$,
2. $|\Psi(1-\alpha,1;-z)| \le C(\alpha,1)|z|^{\mathrm{Re}(\alpha-1)}$.

First, we consider the term $t\log|\xi|$. From

$$t \ge t_\xi = \frac{1}{\log|\xi| - \log N}$$

we conclude

$$\log|\xi| - \log N \ge \frac{1}{t}, \quad \log|\xi| \ge \frac{1}{t}\Big(1 + t\ \log N\Big) \ge \frac{1}{t}\Big(\frac{1}{1-t}\Big),$$

where N is large and T is small enough. A small T is sufficient for our considerations. Hence,

$$t\log|\xi| - t^2\log|\xi| \ge 1, \quad t^2\log|\xi| \le t\log|\xi| - 1 \text{ (for all } t \ge t_\xi),$$

$$t\log|\xi| \le \log|\xi| - \frac{1}{t} = \log \Lambda(t)|\xi|,$$

respectively. Now let us use the representations for $p_0(t,\xi)$ and $p_1(t,\xi)$. Then we can estimate as follows:

$$\begin{aligned}
|p_0(t,\xi)| = \Big| & \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\psi(\alpha)-\psi(1-\alpha)+i(\mathrm{sign}\xi)\pi} \\
& \times \Big(-\frac{1}{\Gamma(1-\alpha)}(\log|2\xi| - i(\mathrm{sign}\xi)\frac{\pi}{2} + \psi(1-\alpha) - 2\gamma) \\
& \times\ te^{-i\Lambda(t)\xi}\Psi(\alpha,1;2i\Lambda(t)\xi) \\
& +\ \frac{1}{\Gamma(\alpha)}(\log|2\xi| + i(\mathrm{sign}\xi)\frac{\pi}{2} + \psi(\alpha) - 2\gamma) \\
& \times\ te^{i\Lambda(t)\xi}\Psi(1-\alpha,1;-2i\Lambda(t)\xi)\Big)\Big| \\
\le\ & C(\alpha)t\log|\xi|\Big(|\Psi(1-\alpha,1;-2i\Lambda(t)\xi)| + |\Psi(\alpha,1;2i\Lambda(t)\xi)|\Big) \\
\le\ & C(\alpha)t\log|\xi|(\Lambda(t)|\xi|)^{\frac{|a|-1}{2}}; \\
|p_1(t,\xi)| = \Big| & \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\psi(\alpha)-\psi(1-\alpha)+i(\mathrm{sign}\xi)\pi}\Big(\frac{1}{\Gamma(\alpha)}te^{i\Lambda(t)\xi}\Psi(1-\alpha,1;-2i\Lambda(t)\xi) \\
& -\ \frac{1}{\Gamma(1-\alpha)}te^{-i\Lambda(t)\xi}\Psi(\alpha,1;2i\Lambda(t)\xi)\Big)\Big| \\
\le\ & C(\alpha)t\Big(|\Psi(1-\alpha,1;-2i\Lambda(t)\xi)| + |\Psi(\alpha,1;2i\Lambda(t)\xi)|\Big) \\
\le\ & C(\alpha)t(\Lambda(t)|\xi|)^{\frac{|a|-1}{2}}.
\end{aligned}$$

Taking account of $t\log|\xi| \le \log(\Lambda(t)|\xi|)$ the statement of this lemma follows immediately.

Summarizing the statements from Lemmas 2.2 and 2.3 we have

$$|\hat{u}(t,\xi)| \le \begin{cases} C\Big(|\hat{u}(0,\xi)| + \frac{1}{\log\langle\xi\rangle}|\hat{u}_t(0,\xi)|\Big) & \text{for } |a|<1, \\ C\langle\xi\rangle^{\frac{|a|-1}{2}}\log\langle\xi\rangle\Big(|\hat{u}(0,\xi)| + \frac{1}{\log\langle\xi\rangle}|\hat{u}_t(0,\xi)|\Big) & \text{for } |a|\ge 1. \end{cases}$$

Similarly, using the above-introduced approach we obtain the following estimate for $D_t\hat{u}$:

$$|D_t\hat{u}(t,\xi)| \le C\langle\xi\rangle^{\frac{|a|+1}{2}}\log\langle\xi\rangle\Big(|\hat{u}(0,\xi)| + \frac{1}{\log\langle\xi\rangle}|\hat{u}_t(0,\xi)|\Big) \text{ for } |a|\ge 0.$$

Thus all statements of Theorem 1.4 are proved.

3 Optimality of conditions for infinite loss of regularity

To study the optimality of results concerning the influence of oscillations we restrict ourselves to the weakly hyperbolic Cauchy problem

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

To prove the optimality of conditions for the infinite loss of regularity we treat the weakly hyperbolic Cauchy problem with a special structure of the coefficients

$$\begin{cases} u_{tt} - \lambda^2(t)b^2\Big(\Big(\log \frac{1}{\Lambda(t)}\Big)^2 \log^{[n]} \frac{1}{\Lambda(t)}\Big)u_{xx} = 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \end{cases} \tag{3.1}$$

where $n \geq 1$ and $b = b(s)$ is a positive, 1-periodic, nonconstant function belonging to C^2. If $n = 0$, then the assumptions of Proposition 1.6 are satisfied with $\nu(t) = \log \frac{1}{\Lambda(t)}$, consequently, we have *at most a finite loss of regularity.* If $n \geq 1$, then we arrive with increasing n *arbitrary close* at the critical case $n = 0$, thus we expect *an infinite loss of regularity.* That it is really so we will prove by application of Floquet theory.

Theorem 3.1 *Let us consider under the above assumptions the Cauchy problem* (3.1)*. If $n \geq 1$, then the Cauchy problem is not H^∞ (C^∞) well-posed, that is, there exists an infinite loss of regularity.*

Proof. Philosophy of the proof

Let us suppose that the Cauchy problem (3.1) is C^∞ well-posed. Then due to the cone of dependence property it is sufficient to study H^∞ well-posedness. Taking into account that (3.1) becomes strictly hyperbolic for $t > 0$ with a smooth coefficient we expect even uniform H^∞ well-posedness on $[0,T]$. Then according to [12] the property of uniform H^∞ well-posedness means the existence of two nonnegative constants r and C such that for any solution u, for any $t^{(1)}$, $t^{(2)} \in [0,T]$ and for any $\xi \in \mathbb{R}$, the partial Fourier transform $\hat{u}$ satisfies the following estimate:

$$|\hat{u}(t^{(2)},\xi)| + \Big|\frac{d}{dt}\hat{u}(t^{(2)},\xi)\Big| \leq C(1+|\xi|)^r\Big(|\hat{u}(t^{(1)},\xi)| + \Big|\frac{d}{dt}\hat{u}(t^{(1)},\xi)\Big|\Big).$$

At the end of this proof we will show that there exist solutions $\{\hat{u}_m(t,\xi)\}_m$ to the partial Fourier transformed equation from (3.1), that is, to

$$\hat{u}_{tt} + \lambda^2(t)b^2\Big(\Big(\log \frac{1}{\Lambda(t)}\Big)^2 \log^{[n]} \frac{1}{\Lambda(t)}\Big)\xi^2\hat{u} = 0$$

accompanied by a sequence of frequencies $\{\xi_m\}_m$ and a sequence of time-pairs $\{(t_m^{(1)}, t_m^{(2)})\}_m$ $(0 < t_m^{(1)} < t_m^{(2)} < T)$, with $\lim_{m\to\infty} \xi_m = \infty$, $\lim_{m\to\infty} t_m^{(1)} = \lim_{m\to\infty} t_m^{(2)} = 0$, and

$$|\hat{u}_m(t_m^{(2)},\xi_m)| + \Big|\frac{d}{dt}\hat{u}_m(t_m^{(2)},\xi_m)\Big|$$
$$\geq C_0 \exp(C_1 \log\xi_m (\log^{[n]}\xi_m)^{\frac{1}{2}})\Big(|\hat{u}_m(t_m^{(1)},\xi_m)| + \Big|\frac{d}{dt}\hat{u}_m(t_m^{(1)},\xi_m)\Big|\Big).$$

This gives the contradiction to the H^∞ well-posedness.

We will only prove the statement for $n \geq 2$; some minor modifications in the approach give the result in the case $n = 1$.

Step 1: *Derivation of an auxiliary differential equation*

Setting $s = \log^{[n]} \frac{1}{\Lambda(t)} (\log \frac{1}{\Lambda(t)})^2$ let us define $w(s,\xi) := \tau^{\frac{1}{2}}(s)\hat{u}(t(s),\xi)$, where $\tau(s) := -\frac{ds}{dt}(t(s))$. Then we obtain the auxiliary differential equation

$$w_{ss}(s,\xi) + b^2(s)\lambda(s,\xi)w(s,\xi) = 0, \quad (s,\xi) \in [s(T),\infty) \times \mathbb{R},$$

where

$$\lambda(s,\xi) = \lambda_1(s,\xi) + \lambda_2(s),\ \lambda_1(s,\xi) = \frac{\lambda^2(t(s))|\xi|^2}{\tau^2(s)},$$

$$\lambda_2(s) = \frac{\theta(s)}{b^2(s)\tau^2(s)},\ \theta = \frac{(\tau')^2 - 2\tau''\tau}{4}.$$

Straightforward calculations yield

$$\begin{aligned}
\frac{\tau'(s)^2 - 2\tau''(s)\tau(s)}{\tau^2(s)} &= \frac{1}{\tau^4(s)}\Big(3\Big(\frac{d^2s}{dt^2}\Big)^2 - 2\frac{d^3s}{dt^3}\frac{ds}{dt}\Big)\\
&\sim \Big(2\frac{\lambda(t)}{\Lambda(t)}\log\frac{1}{\Lambda(t)}\log^{[n]}\frac{1}{\Lambda(t)}\Big)^{-4}\Big(12\Big(\frac{\lambda(t)}{\Lambda(t)}\Big)^4\Big(\log^{[n]}\frac{1}{\Lambda(t)}\Big)^2\\
&+ 12\Big(\frac{\lambda'(t)\Lambda(t) - \lambda^2(t)}{\Lambda^2(t)}\log\frac{1}{\Lambda(t)}\log^{[n]}\frac{1}{\Lambda(t)}\Big)^2\\
&- 8\frac{\lambda(t)}{\Lambda(t)}\frac{\lambda''(t)\Lambda(t) - \lambda'(t)\lambda(t)}{\Lambda^2(t)}\Big(\log\frac{1}{\Lambda(t)}\log^{[n]}\frac{1}{\Lambda(t)}\Big)^2\\
&+ 16\Big(\frac{\lambda(t)}{\Lambda(t)}\Big)^2\frac{\lambda'(t)\Lambda(t) - \lambda^2(t)}{\Lambda^2(t)}\Big(\log\frac{1}{\Lambda(t)}\log^{[n]}\frac{1}{\Lambda(t)}\Big)^2\Big).
\end{aligned}$$

From here we know that $\lim_{s\to\infty}\lambda_2(s) = 0$.

For the further calculations we need

$$s = \log^{[n]}\frac{1}{\Lambda(t)}\Big(\log\frac{1}{\Lambda(t)}\Big)^2 \text{ implies } \Big(\log\frac{1}{\Lambda(t)}\Big)^2 \sim \frac{s}{\log^{[n-1]}s^{\frac{1}{2}}}.$$

Step 2: *Asymptotics*

Actually we define a function $s_\xi = s(\xi)$ implicitly by

$$\lambda(s,\xi) \sim \frac{1}{4}\frac{1}{s\log^{[n-1]} s^{\frac{1}{2}} \exp\Big(2\Big(\frac{s}{\log^{[n-1]} s^{\frac{1}{2}}}\Big)^{\frac{1}{2}}\Big)}|\xi|^2 + \lambda_2(s) = \lambda_0,$$

where λ_0 will be chosen later. This definition implies $s_\xi \to \infty$ if $\xi \to \infty$. For this reason the asymptotical behavior of λ_1 and λ_2 around large s_ξ is of interest.

Lemma 3.2 *For* $0 \le \delta \le \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$, *and* s_ξ *large enough we have*

$$|\lambda_1(s_\xi - \delta,\xi) - \lambda_1(s_\xi,\xi)| \le C\lambda_1(s_\xi,\xi)(\log^{[n]}|\xi|)^{-\frac{1}{2}},$$

$$|\lambda_2(s_\xi - \delta) - \lambda_2(s_\xi)| \le C\lambda_2(s_\xi)((\log|\xi|\log^{[n]}|\xi|)^{\frac{1}{2}})^{-1}.$$

Summarizing we have

$$|\lambda(s_\xi - \delta,\xi) - \lambda(s_\xi)| \le C\lambda(s_\xi,\xi)(\log^{[n]}|\xi|)^{-\frac{1}{2}}.$$

Proof. From the representation of λ_0 we get

$$\frac{1}{4}(\log|\xi|)^2 \le \frac{s_\xi}{\log^{[n-1]} s_\xi^{\frac{1}{2}}} \le (\log|\xi|)^2.$$

Consequently, we have

$$C_1(\log|\xi|)^2\log^{[n]}|\xi| \le s_\xi \le C_2(\log|\xi|)^2\log^{[n]}|\xi|.$$

Thus, for large ξ and $0 \le \delta \le \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$ we have

$$\Big|\frac{\delta}{s_\xi}\Big| \le C\frac{1}{\log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}},$$

and, consequently,

$$\begin{aligned}
&|\lambda_1(s_\xi - \delta,\xi) - \lambda_1(s_\xi,\xi)| \\
&\quad = \Big|\frac{|\xi|^2}{(s_\xi - \delta)\log^{[n-1]}(s_\xi - \delta)^{\frac{1}{2}} \exp\Big(2\Big(\frac{s_\xi - \delta}{\log^{[n-1]}(s_\xi - \delta)^{\frac{1}{2}}}\Big)^{\frac{1}{2}}\Big)} - \lambda_1(s_\xi,\xi)\Big| \\
&\quad \sim \Big|\frac{|\xi|^2}{(s_\xi - \delta)\log^{[n-1]} s_\xi^{\frac{1}{2}} \exp\Big(2\Big(\frac{s_\xi - \delta}{\log^{[n-1]}(s_\xi - \delta)^{\frac{1}{2}}}\Big)^{\frac{1}{2}}\Big)} - \lambda_1(s_\xi,\xi)\Big| \\
&\quad \sim \lambda_1(s_\xi,\xi)\Big|\Big(1 - \frac{\delta}{s_\xi}\Big)^{-1} \exp\Big(-2\Big(\frac{s_\xi}{\log^{[n-1]} s_\xi^{\frac{1}{2}}}\Big)^{\frac{1}{2}}\Big(\Big(1 - \frac{\delta}{s_\xi}\Big)^{\frac{1}{2}} - 1\Big)\Big) - 1\Big| \\
&\quad \le C\lambda_1(s_\xi,\xi)(\log^{[n]}|\xi|)^{-\frac{1}{2}}.
\end{aligned}$$

Since $\lambda_2(s) \sim \frac{1}{s\log^{[n-1]} s^{\frac{1}{2}}}$ we may conclude

$$
\begin{aligned}
&|\lambda_2(s_\xi-\delta)-\lambda_2(s_\xi)| \\
&\sim \Big|\frac{1}{(s_\xi-\delta)\log^{[n-1]}(s_\xi-\delta)^{\frac{1}{2}}}-\lambda_2(s_\xi)\Big| \leq \Big|\frac{1}{(s_\xi-\delta)\log^{[n-1]}(s_\xi)^{\frac{1}{2}}}-\lambda_2(s_\xi)\Big| \\
&\sim \lambda_2(s_\xi)\Big|\Big(1-\frac{\delta}{s_\xi}\Big)^{-1}-1\Big| \leq C\lambda_2(s_\xi)\frac{1}{\log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}}.
\end{aligned}
$$

Remark 3.3 *Taking into consideration $\lambda_2(s) \to 0$ for $s \to \infty$ we see from the last estimate that $\lambda_2(s)$ plays a negligible role, so in the following steps it is sufficient to restrict ourselves to $\lambda_1(s,\xi)$. Namely, we treat $\lambda(s,\xi)=\lambda_1(s,\xi)$.*

Step 3: *Application of Floquet's theory*

We are interested in the fundamental solution $X = X(s, s_0)$ as the solution to the Cauchy problem

$$\frac{d}{ds}X = \begin{pmatrix} 0 & -\lambda_0 b^2(s) \\ 1 & 0 \end{pmatrix} X =: A(s)X, \qquad X(s_0,s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is clear that $X(s_0-1,s_0)$ is independent of $s_0 \in \mathbb{N}$.

Lemma 3.4 *(Floquet's theory, see* [11]*) Let $b = b(s) \in C^2$ be a nonconstant and positive function on $\mathbb{R}$ which is* 1*-periodic, then there exists a positive real number $\lambda_0 > 0$ such that λ_0 belongs to an interval of instability for $w_{ss} + \lambda_0 b^2(s)w = 0$, that is, $X(s_0-1,s_0)$ has eigenvalues μ_0 and μ_0^{-1} satisfying $|\mu_0| > 1$.*

Set

$$X(s_\xi-1,s_\xi) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

The lemma implies that the eigenvalues of this matrix are μ_0 and μ_0^{-1}. Hence,

$$x_{11}+x_{22} = \mu_0+\mu_0^{-1}.$$

Thus

$$|x_{11}-\mu_0|+|x_{22}-\mu_0| \geq |\mu_0-\mu_0^{-1}|,$$

from which follows

$$\max\{|x_{11}-\mu_0|,|x_{22}-\mu_0|\} \geq \frac{1}{2}|\mu_0-\mu_0^{-1}|.$$

We assume that

$$|x_{11}-\mu_0| \geq \frac{1}{2}|\mu_0-\mu_0^{-1}|.$$

The other case can be treated similarly. And we also have

$$|x_{22} - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}|.$$

Now we consider the following equation for an integer $m \geq 0$:

$$w_{ss} + \frac{1}{4}\frac{1}{(s_\xi - m + s)\log^{[n-1]}(s_\xi - m + s)^{\frac{1}{2}} \exp\left(2\left(\frac{s_\xi - m + s}{\log^{[n-1]}(s_\xi - m + s)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right)} \times |\xi|^2 b^2(s_\xi + s)w = 0.$$

Let $X_m(s, s_1)$ be the solution of the associated first-order system

$$\frac{d}{ds}X_m(s, s_1) = \begin{pmatrix} 0 & -\frac{1}{4}\frac{1}{(s_\xi - m + s)\log^{[n-1]}(s_\xi - m + s)^{\frac{1}{2}} \exp\left(2\left(\frac{s_\xi - m + s}{\log^{[n-1]}(s_\xi - m + s)^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right)}|\xi|^2 b^2(s_\xi + s) \\ 1 & 0 \end{pmatrix} \times X_m(s, s_1), \quad X_m(s_1, s_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 3.5 *It holds*

$$\max_{s, s_1 \in [-1, 0]} \|X_m(s, s_1)\| \leq \exp(C\lambda_0) \text{ for } 0 \leq m \leq \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}.$$

Proof. Using (A_m denotes the coefficient matrix from the above system)

$$X_m(s, s_1) = I + \sum_{j=1}^{\infty}\int_{s_1}^{s} A_m(r_1, \xi)\int_{s_1}^{r_1} A_m(r_2, \xi)\cdots\int_{s_1}^{r_{j-1}} A_m(r_j, \xi)dr_j \cdots dr_1$$

according to Lemma 3.2 we have

$$\begin{aligned}\max_{s, s_1 \in [-1, 0]} \|X_m(s, s_1)\| &\leq \exp(1 + b_1^2|\lambda(s_\xi - m, \xi) - \lambda(s_\xi, \xi) + \lambda_0|) \\ &\leq \exp(1 + b_1^2(1 + \varepsilon)\lambda_0) \leq \exp(C\lambda_0),\end{aligned}$$

where $b_1 = \max_t\{b(t)\}$ and $0 \leq m \leq \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$.

Lemma 3.6 *It holds*

$$\|X_m(-1, 0) - X(s_\xi - 1, s_\xi)\| \leq C(\log^{[n]}|\xi|)^{-\frac{1}{2}}$$

for $0 \leq m \leq \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$.

Proof. First, note that $X(s_\xi + s, s_\xi) = X(s,0)$, since $s_\xi \in \mathbb{N}$ and $b(s)$ is 1-periodic. Observe that

$$\frac{d}{ds}X_m(s,0) = \begin{pmatrix} 0 & -\lambda(s_\xi,\xi)b^2(s) \\ 1 & 0 \end{pmatrix} X_m(s,0) + \begin{pmatrix} 0 & \Big(\lambda(s_\xi,\xi) - \lambda(s_\xi - m + s,\xi)\Big)b^2(s) \\ 0 & 0 \end{pmatrix} X_m(s,0)$$

with $X_m(0,0) = I$. Thus

$$\tfrac{d}{ds}\Big(X_m(s,0) - X(s,0)\Big) = \begin{pmatrix} 0 & -\lambda(s_\xi,\xi)b^2(s) \\ 1 & 0 \end{pmatrix} \Big(X_m(s,0) - X(s,0)\Big) + \begin{pmatrix} 0 & \Big(\lambda(s_\xi,\xi) - \lambda(s_\xi - m + s,\xi)\Big)b^2(s) \\ 0 & 0 \end{pmatrix} X_m(s,0)$$

with initial data $X_m(0,0) - X(0,0) = 0$. By Lemma 3.2 it follows

$$|\lambda(s_\xi,\xi) - \lambda(s_\xi - m + s,\xi)| \le C\lambda_0(\log^{[n]}|\xi|)^{-\frac{1}{2}}$$

for $0 \le m \le \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$. Hence,

$$\|X_m(s,0) - X(s,0)\| \le \int_0^s C\lambda_0\|X_m(r,0) - X(r,0)\|dr + \int_0^s C\lambda_0(\log^{[n]}|\xi|)^{-\frac{1}{2}}\|X_m(r,0)\|dr.$$

So by Lemma 3.5, Gronwall's inequality and the hypothesis on m we may conclude

$$\|X_m(-1,0) - X(-1,0)\| \le C\lambda_0(\log^{[n]}|\xi|)^{-\frac{1}{2}}.$$

This completes the proof of the lemma.

Lemma 3.7 *It holds*

$$\|X_m(-1,0) - X_{m-1}(-1,0)\| \le C(\log|\xi|\log^{[n]}|\xi|)^{-1}$$

for $0 < m \le \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$.

Proof. First,

$$\frac{d}{ds}\Big(X_m(s,0) - X_{m-1}(s,0)\Big) = \begin{pmatrix} 0 & -\lambda(s_\xi - m + s,\xi)b^2(s) \\ 1 & 0 \end{pmatrix} \Big(X_m(s,0) - X_{m-1}(s,0)\Big) + \begin{pmatrix} 0 & \Big(\lambda(s_\xi - (m-1) + s,\xi) - \lambda(s_\xi - m + s,\xi)\Big)b^2(s) \\ 0 & 0 \end{pmatrix} X_{m-1}(s,0)$$

with initial data $X_m(0,0) - X_{m-1}(0,0) = 0$. Applying the same techniques as in the proof to Lemma 3.2 we have for $0 < m \le \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$ the estimates

$$\begin{aligned}
&|\lambda(s_\xi - m + r, \xi) - \lambda(s_\xi - (m-1) + r, \xi)| \\
&\sim \ \lambda(s_\xi - (m-1) + r, \xi)\Big|\Big(1 - \frac{1}{s_\xi - (m-1) + r}\Big)^{-1} \\
&\quad \times \ \exp\Big(-2\Big(\frac{s_\xi - (m-1) + r}{\log^{[n-1]}(s_\xi - (m-1) + r)^{\frac{1}{2}}}\Big)^{\frac{1}{2}} \\
&\quad \times \ \Big(\Big(1 - \frac{1}{s_\xi - (m-1) + r}\Big)^{\frac{1}{2}} - 1\Big)\Big) - 1\Big| \\
&\le \ \frac{C\lambda_0}{s_\xi - (m-1) + r}\Big(\frac{s_\xi - (m-1) + r}{\log^{[n-1]}(s_\xi - (m-1) + r)^{\frac{1}{2}}}\Big)^{\frac{1}{2}} \\
&\le \ C\lambda_0(\log|\xi|\log^{[n]}|\xi|)^{-1}.
\end{aligned}$$

By a similar argument as that used in the proof to Lemma 3.6 we get

$$\|X_m(-1,0) - X_{m-1}(-1,0)\| \le C\lambda_0(\log|\xi|\log^{[n]}|\xi|)^{-1}$$

for $0 < m \le \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$, as required.

Using

$$\det(I\mu_0 - X(s_\xi - 1, s_\xi)) = 0, \quad \det(I\mu_m - X_m(-1,0)) = 0$$

by Lemma 3.6 we have

$$\|X_m(-1,0) - X(s_\xi - 1, s_\xi)\| \le C(\log^{[n]}|\xi|)^{-\frac{1}{2}}.$$

Therefore the matrix $X_m(-1,0)$ with the property $\det X_m(-1,0) = 1$ has eigenvalues μ_m and μ_m^{-1} which satisfy

$$|\mu_m - \mu_0| \le C(\log^{[n]}|\xi|)^{-\frac{1}{2}} \le \varepsilon$$

for any given positive ε and for sufficiently large s_ξ. Choosing $\varepsilon \le \frac{|\mu_0|-1}{2}$ we have

$$|\mu_m| \ge \frac{1}{2}(|\mu_0| + 1) \ge 1 + \varepsilon.$$

So the eigenvalues μ_m and μ_m^{-1} are uniformly distinct for every m. Let us denote

$$X_m(-1,0) = \begin{pmatrix} x_{11}(m) & x_{12}(m) \\ x_{21}(m) & x_{22}(m) \end{pmatrix}.$$

Obviously, we have

$$|x_{11}(m)-\mu_m| \geq |x_{11}-\mu_0| - (|x_{11}(m)-x_{11}|+|\mu_0-\mu_m|) \geq \frac{1}{4}|\mu_0-\mu_0^{-1}|.$$

Analogously, we have

$$|x_{22}(m)-\mu_m^{-1}| \geq \frac{1}{4}|\mu_0-\mu_0^{-1}|.$$

According to Lemma 3.7 we have

$$|x_{ij}(m)-x_{ij}(m-1)| \leq C(\log|\xi|\log^{[n]}|\xi|)^{-1}.$$

Immediately we conclude

$$|\mu_m-\mu_{m-1}| \leq C(\log|\xi|\log^{[n]}|\xi|)^{-1}.$$

Step 4: *Energy estimate for a model problem*

Lemma 3.8 *Let m_0 be the largest integer satisfying $0 \leq m \leq \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$. Then for the solution $w=w(s,\xi)$ of*

$$w_{ss}(s,\xi)+b^2(s)\lambda(s,\xi)w(s,\xi)=0, \quad (s,\xi)\in[s(1),\infty)\times\mathbb{R},$$

with the initial data

$$\frac{d}{ds}w(s_\xi,\xi)=\frac{x_{12}(0)}{\mu_0-x_{11}(0)}, \quad w(s_\xi,\xi)=1,$$

we have

$$\left|\frac{d}{ds}w(s_\xi-m_0-1,\xi)\right|+|w(s_\xi-m_0-1,\xi)| \geq C\exp(C_1\log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}).$$

Proof. The function $w=w(s_\xi-m_0+s,\xi)$ satisfies

$$\frac{d^2}{ds^2}w+\lambda(s_\xi-m+s,\xi)b(s_\xi+s)w=0$$

with $m=m_0$. Thus

$$\begin{pmatrix} \frac{d}{ds}w(s_\xi-m_0-1,\xi) \\ w(s_\xi-m_0-1,\xi) \end{pmatrix}$$

$$=X_{m_0}(-1,0)X_{m_0-1}(-1,0)\cdots X_0(-1,0)\begin{pmatrix} \frac{d}{ds}w(s_\xi,\xi) \\ w(s_\xi,\xi) \end{pmatrix}.$$

The matrix

$$B_m = \begin{pmatrix} \frac{x_{12}(m)}{\mu_m - x_{11}(m)} & 1 \\ 1 & \frac{x_{21}(m)}{\mu_m^{-1} - x_{22}(m)} \end{pmatrix}$$

is a diagonalizer for $X_m(-1,0)$, that is,

$$X_m(-1,0)B_m = B_m \begin{pmatrix} \mu_m & 0 \\ 0 & \mu_m^{-1} \end{pmatrix}.$$

Since $\det X_m(-1,0) = 1$ and the trace of $X_m(-1,0)$ is $\mu_m + \mu_m^{-1}$ we get

$$\det B_m = \frac{\mu_m - \mu_m^{-1}}{\mu_m^{-1} - x_{22}(m)}.$$

Lemma 3.5 indicates that

$$|\mu_m^{-1} - x_{22}(m)| \le C, \quad |x_{ij}(m)| \le C,$$

and since $|\mu_m| \ge 1 + \varepsilon$, so

$$|\det B_m| \ge C > 0, \quad \|B_m\| \le C, \quad \|B_m^{-1}\| \le C$$

for $0 \le m \le \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}}$. Then from

$$|x_{ij}(m) - x_{ij}(m-1)| \le C(\log|\xi| \log^{[n]}|\xi|)^{-1}$$

we get

$$\|B_{m-1}^{-1}B_m - I\| = \|B_{m-1}^{-1}(B_m - B_{m-1})\| \le C(\log|\xi| \log^{[n]}|\xi|)^{-1}.$$

If we denote $G_m := B_{m-1}^{-1}B_m - I$, then

$$X_{m_0}(-1,0)X_{m_0-1}(-1,0)\cdots X_0(-1,0)$$

$$= B_{m_0} \begin{pmatrix} \mu_{m_0} & 0 \\ 0 & \mu_{m_0}^{-1} \end{pmatrix} (I + G_{m_0}) \cdots (I + G_1) \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix} B_0^{-1}.$$

We shall show that the $(1,1)$ element y_{11} of the matrix

$$\begin{pmatrix} \mu_{m_0} & 0 \\ 0 & \mu_{m_0}^{-1} \end{pmatrix} (I + G_{m_0}) \cdots (I + G_1) \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix}$$

can be estimated to below by $C_0 \exp(C_1 \log|\xi|(\log^{[n]}|\xi|)^{\frac{1}{2}})$. First we recall

$$\|B_{m-1}^{-1}B_m - I\| = \|B_{m-1}^{-1}(B_m - B_{m-1})\| \le C(\log|\xi| \log^{[n]}|\xi|)^{-1}.$$

It holds

$$\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} \prod_{k=1}^{m_0} \mu_k & 0 \\ 0 & \prod_{k=1}^{m_0} \mu_k^{-1} \end{pmatrix} + M_1 + \cdots + M_{m_0},$$

where M_l is the matrix of the sum of terms containing exactly l of the matrices G_k, $k = 1, \ldots, m_0$. We have the estimate

$$\|M_l\| \le \left(\prod_{k=1}^{m_0} |\mu_l| \right) \left(\sum_{1 \le i_1 < \cdots < i_l \le m_0} \prod_{j=1}^{l} \|G_{ij}\| \right).$$

Using

$$\|G_l\| \le C(\log|\xi| \log^{[n]}|\xi|)^{-1}, \; l = 1, \ldots, m_0,$$

gives

$$\|M_l\| \le \left(\prod_{k=1}^{m_0} |\mu_l| \right) \binom{m_0}{l} C(\log|\xi| \log^{[n]}|\xi|)^{-l}.$$

Consequently,

$$|y_{11}| \ge \left(\prod_{k=1}^{m_0} |\mu_k| \right) \left(2 - \left(1 + \frac{C}{\log|\xi| \log^{[n]}|\xi|} \right)^{\log|\xi| \left(\log^{[n]}|\xi| \right)^{1/2}} \right) \ge \frac{1}{2} \prod_{k=1}^{m_0} |\mu_k|.$$

On the other hand, $|y_{mp}| \le c \prod_{k=1}^{m_0} |\mu_k|$, $c > 0$, is sufficiently small, for $(m, p) \neq (1, 1)$. Thus, the statement of the lemma follows directly from

$$\prod_{k=1}^{m_0} |\mu_k| \ge C(1+\varepsilon)^{m_0} \ge C(1+\varepsilon)^{\log|\xi| (\log^{[n]}|\xi|)^{\frac{1}{2}}}.$$

Step 5: *Conclusion*

Now we take a sequence $\{\xi_m\}_m$ of positive frequencies satisfying $\xi_m \to \infty$ as $m \to \infty$. For large ξ_m let $w_m(s, \xi)$ be the solution from Lemma 3.8 with $\xi = \xi_m$ and let us set

$$t_m^{(1)} = t(s_{\xi_m}), \; t_m^{(2)} = t(s_{\xi_m} - m_0(\xi_m) - 1), \; \hat{u}_m(t, \xi_m) = \tau^{-\frac{1}{2}}(s(t)) w_m(s(t), \xi_m).$$

Then

$$\Lambda^{-1}\Big(\frac{c_1}{|\xi_m|} \Big) \le t_m^{(1)} < t_m^{(2)} \le \Lambda^{-1}\Big(\frac{c_2}{|\xi_m|} \Big),$$

thus both sequences $\{t_m^{(1)}\}_m$, $\{t_m^{(2)}\}_m$ are zero sequences. For $0 < t \le T$ we conclude

$$\Big| \frac{d}{ds} w_m(s(t), \xi) \Big| \le \frac{1}{2} \tau_s(s(t)) \tau^{-\frac{1}{2}}(s(t)) |\hat{u}_m(t, \xi)| + \tau^{-\frac{1}{2}}(s(t)) \Big| \frac{d}{dt} \hat{u}_m(t, \xi) \Big|.$$

Hence,

$$\begin{aligned}
&|w_m(s(t),\xi)| + \Big|\frac{d}{ds}w_m(s(t),\xi)\Big| \\
&\quad\le \tau^{\frac12}(s(t))\Big(1+\frac{\tau_s(s(t))}{2\tau(s(t))}\Big)|\hat u_m(t,\xi)| + \tau^{-\frac12}(s(t))\Big|\frac{d}{dt}\hat u_m(t,\xi)\Big| \\
&\quad\le 2\tau^{\frac12}(s(t))|\hat u_m(t,\xi)| + \tau^{-\frac12}(s(t))\Big|\frac{d}{dt}\hat u_m(t,\xi)\Big| \\
&\quad\le C\tau^{\frac12}(s(t))\Big(|\hat u_m(t,\xi)| + \Big|\frac{d}{dt}\hat u_m(t,\xi)\Big|\Big), \\
&|\hat u_m(t,\xi)| + \Big|\frac{d}{dt}\hat u_m(t,\xi)\Big| \le C\tau^{\frac12}(s(t))\Big(|w_m(s(t),\xi)| + \Big|\frac{d}{ds}w_m(s(t),\xi)\Big|\Big).
\end{aligned}$$

Now we will apply the first inequality for $t = t_m^{(2)}$, the second inequality for $t = t_m^{(1)}$ and Lemma 3.8. Summarizing gives

$$\begin{aligned}
&|\hat u_m(t_m^{(2)},\xi_m)| + \Big|\frac{d}{dt}\hat u_m(t_m^{(2)},\xi_m)\Big| \\
&\quad\ge\ C\tau^{-\frac12}(s(t_m^{(2)}))\Big(|w_m(s(t_m^{(2)}),\xi_m)| + \Big|\frac{d}{ds}w_m(s(t_m^{(2)}),\xi_m)\Big|\Big) \\
&\quad\ge C\tau^{-\frac12}(s(t_m^{(2)}))\exp(C_1\log|\xi_m|(\log^{[n]}|\xi_m|)^{\frac12}) \\
&\quad\ge\ C\tau^{-\frac12}(s(t_m^{(2)}))\tau^{-\frac12}(s(t_m^{(1)}))\exp(C_1\log|\xi_m|(\log^{[n]}|\xi_m|)^{\frac12}) \\
&\qquad\times\ \Big(|\hat u_m(t_m^{(1)},\xi_m)| + \Big|\frac{d}{dt}\hat u_m(t_m^{(1)},\xi_m)\Big|\Big) \\
&\quad\ge\ C\exp(C_2\log|\xi_m|(\log^{[n]}|\xi_m|)^{\frac12})\Big(|\hat u_m(t_m^{(1)},\xi_m)| + \Big|\frac{d}{dt}\hat u_m(t_m^{(1)},\xi_m)\Big|\Big),
\end{aligned}$$

where C and C_2 are used as universal positive constants. In fact, for the infinite degenerate case we calculate

$$s(t) = \frac{1}{t^2}\log^{[n-1]}\frac1t,\quad t\sim\sqrt{\frac{\log^{[n-1]}s^{\frac12}}{s}},$$

$$\tau(s) = 2\frac{1}{t^3}\log^{[n-1]}\frac1t\Big|_{t=t(s)} \sim 2s\sqrt{\frac{s}{\log^{[n-1]}s^{\frac12}}},$$

$$s_\xi,\ s_\xi - m_0 - 1 \sim (\log|\xi|)^2\log^{[n]}|\xi|,$$

$$\tau(s_\xi),\ \tau(s_\xi - m_0 - 1) \sim \frac{s_\xi^{\frac32}}{(\log^{[n-1]}s_\xi^{\frac12})^{\frac12}} \sim \frac{(\log|\xi|)^3(\log^{[n]}|\xi|)^{\frac32}}{\Big(\log^{[n-1]}\Big((\log|\xi|)(\log^{[n]}|\xi|)^{\frac12}\Big)\Big)^{\frac12}}.$$

So the conclusion holds. Furthermore, for the finite degenerate case we calculate

$$s(t) = \Big(\log \frac{l+1}{t^{l+1}}\Big)^2 \log^{[n]} \frac{l+1}{t^{l+1}},$$

$$t \sim (l+1)^{\frac{1}{l+1}} \exp\Big(-\frac{1}{l+1}\Big(\frac{s}{\log^{[n-1]} s^{\frac12}}\Big)^{\frac12}\Big),$$

$$\tau(s) = 2\frac{l+1}{t}\log\frac{l+1}{t^{l+1}}\log^{[n]}\frac{l+1}{t^{l+1}}\Big|_{t=t(s)}$$

$$\sim\ 2(l+1)\Big(\frac{1}{l+1}\Big)^{\frac{1}{l+1}} s\,\frac{\exp\Big(\frac{1}{l+1}\Big(\frac{s}{\log^{[n-1]} s^{\frac12}}\Big)^{\frac12}\Big)}{\Big(\frac{s}{\log^{[n-1]} s^{\frac12}}\Big)^{\frac12}},$$

$$s_\xi,\ s_\xi - m_0 - 1 \sim (\log|\xi|)^2 \log^{[n]}|\xi|,$$

$$\tau(s_\xi),\ \tau(s_\xi - m_0 - 1)$$

$$\sim\ \log|\xi|(\log^{[n]}|\xi|)^{\frac12}\Big(\log^{[n-1]}\Big(\log|\xi|(\log^{[n]}|\xi|)^{\frac12}\Big)\Big)^{\frac12}$$

$$\times\ \exp\Big(\frac{1}{l+1}\frac{\log|\xi|(\log^{[n]}|\xi|)^{\frac12}}{(\log^{[n-1]}(\log|\xi|(\log^{[n]}|\xi|)^{\frac12}))^{\frac12}}\Big).$$

Since $\xi_m \to \infty$ as $m \to \infty$ the conclusion still holds. The proof of our theorem is complete.

Remark 3.9 *We are able to construct in* (3.1) *coefficients for which the corresponding Cauchy problem is not C^∞ (H^∞) well-posed. It seems to be a challenge to apply Floquet theory to check optimality of conditions for finite loss or arbitrary small loss. The question for the finite loss will be discussed in the next section.*

Remark 3.10 *If we choose $\lambda(t) = \exp(-t^{-\alpha})$, $\alpha > 0$, then due to Theorem* 3.1 *the Cauchy problems*

$$u_{tt} - e^{-2t^{-\alpha}}\Big(2 + \cos\Big(\frac{1}{t^{2\alpha}}\log^{[n-1]}\frac{1}{t^\alpha}\Big)\Big)u_{xx} = 0,$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ n \ge 1,$$

are not C^∞ (H^∞) well-posed. This improves the result from [15] *which is cited in the Introduction.*

If we choose $\lambda(t) = t^l$, $l > 0$, then due to Theorem 3.1 *the Cauchy problems*

$$u_{tt} - t^{2l}\Big(2 + \sin\Big(\Big(\log\frac{1}{t}\Big)^2\Big(\log^{[n]}\frac{1}{t}\Big)\Big)\Big)u_{xx} = 0,$$

$$u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ n \ge 1,$$

are not C^∞ (H^∞) well-posed.

4 Optimality of conditions for finite loss of regularity

Let us treat the weakly hyperbolic Cauchy problem ($b(t) \equiv 1$ in Proposition 1.6)

$$u_{tt} - \lambda^2(t)u_{xx} - a\frac{\lambda^2(t)}{\Lambda(t)}u_x = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x).$$

Comparing Proposition 1.6 with the statements from Proposition 1.1 and Theorem 1.4 shows that we are able to describe for some examples of functions $\lambda = \lambda(t)$ in *an optimal way* the influence of degeneracies in coefficients at $t = 0$ on the *loss of regularity*. For these examples we understand under which assumptions to a the finite loss really appears.

In this section we are interested in the influence of the oscillating parts, thus we restrict ourselves to (3.1). If we set $n = 0$ in (3.1), then the assumptions of Proposition 1.6 are satisfied with $\nu(t) = \log \frac{1}{\Lambda(t)}$, consequently, we have *at most a finite loss of regularity*. In this section we want to explain that we also have *at least a finite loss of regularity*; this means that the finite loss of regularity really appears.

We follow a strategy that was proposed in [6]. There the model under consideration has been with $k \in \mathbb{N}$:

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(kt)u_{x_i x_j} = 0,\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),$$

under the following assumptions:

1. The coefficients $a_{ij} = a_{ij}(t)$ are nonconstant, 1-periodic and belong to $L^1_{loc}(\mathbb{R}_+)$,
2. $\sum_{i,j=1}^{n} a_{ij}(t)\xi_i\xi_j \geq 0$ (weak hyperbolicity assumption).

The authors proved that for any $\delta > 0$ there exist two analytic initial data u_0 and u_1 such that the sequence $\{u_k(t,\cdot)\}$ is unbounded in $D'(\mathbb{R}^n)$ for any $t > \delta$. This result shows that if a positive integer parameter k (homogenization in t) produces a family of faster and faster oscillating coefficients, the solutions may show a uniformly unstable behavior in a certain sense, although for every fixed k the Cauchy problem is well-posed in the analytic frame with respect to the spatial variables.

Probably inspired by this result the authors used in [3] an instability argument to show for a strictly hyperbolic Cauchy problem with low regular time-dependent coefficients ($\lambda \equiv 1$ in (3.1) or (4.1)) that the finite loss really appears. There the *family of coefficients* $\{b_k\}$ *is constructed by choosing a fixed periodic coefficient* $b = b(t)$ from [5] and by including into this coefficient the idea of homogenization.

In the following we generalize the ideas to construct counterexamples from [3] to the present *weakly hyperbolic situation by including a pure Floquet effect*. A *pure Floquet effect* means that on the one hand we want to construct a *family of coefficients for an arbitrary given 1-periodic, positive, nonconstant and smooth function* $b = b(t)$ and on the other hand the proof bases on the following lemma which we obtain by repeating the approach of Section 2 from [14].

Lemma 4.1 *Let* $b = b(t)$ *be a* 1*-periodic function which is positive, non-constant, smooth, but constant in a small neighborhood of* 0*. Let us consider the Cauchy problem*

$$u_{tt} - b^2(t)u_{xx} = 0,\ u(0,x) = \exp(ix\xi),\ u_t(0,x) = 0,$$

where the positive real ξ^2 *belongs to an interval of instability for the coefficient* $b^2(t)$*. Then there exists a unique solution* $u = u(t,x) = \exp(ix\xi)w(t)$*, where* w *satisfies the asymptotic relation* $|w(M)| \sim |\mu_0|^M$ *for all sufficiently large* $M \in \mathbb{N}$*.*

Let us consider the family of weakly hyperbolic Cauchy problems

$$u_{tt} - \lambda^2(t)b_k^2(t)u_{xx} = 0,\ u(t_k,x) = u_{0,k}(x),\ u_t(t_k,x) = u_{1,k}(x),\ t_k \in [0,T]. \tag{4.1}$$

It is clear that Proposition 1.6 yields the following corollary:

Corollary 4.2 *Let us consider the family of Cauchy problems* (4.1) *under the assumption that the coefficients* $\lambda(t)$ *and* $b_k(t)$ *satisfy the conditions of Proposition* 1.6 *with constants which are independent of* k, t_k, *and with* $\nu(t) = \log\frac{1}{\Lambda(t)}$*. If* $u_{0,k} \in H^s$, $u_{1,k} \in (\Lambda^{-1}(\frac{N}{\langle D_x\rangle}))^{-1}H^s$, N *is a fixed large positive constant, then there are unique solutions* u_k *belonging to*

$$C([0,T],\langle D_x\rangle^{C_\alpha}H^s) \cap C^1([0,T],\langle D_x\rangle^{C_\alpha}H^{s-1}),$$

where the positive constant C_α *is independent of* k *and* t_k*.*

In the following theorem we will prove that we have *at least* a finite loss of regularity, too. We will *prove the statement only for the infinite degenerate case*, for the finite degenerate case see Remark 4.5.

Theorem 4.3 *Under the assumptions of Proposition* 1.6 *and the additional assumption* $\frac{\Lambda(t)}{\lambda(t)} = o(t)$ *there exists to an arbitrary given 1-periodic, positive, nonconstant and smooth function* $b = b(t)$ *a family of coefficients* $\{b_k = b_k(t)\}_k$ *satisfying the conditions of Proposition* 1.6 *with constants which are independent of* k*, there exists a family of data* $\{u_{0,k} = u_{0,k}(x),\ u_{1,k} = u_{1,k}(x)\}_k$ *belonging to* $H^s(\mathbb{R}) \times (\Lambda^{-1}(\frac{N}{|D_x|}))^{-1}H^s(\mathbb{R})$ *and which are prescribed on* $t = t_k^{(1)}$*, and, finally, there exist two zero sequences* $\{t_k^{(1)}\}_k$, $\{t_k^{(2)}\}_k$ *such that the following estimates hold for the sequence of solutions* $\{u_k\}_k$*:*

$$\|u_k(t_k^{(2)},\cdot)\|_{H^{s-p_0}(\mathbb{R})} \geq C_k \|u_k(t_k^{(1)},\cdot)\|_{H^s(\mathbb{R})},$$

where the positive constant p_0 *is independent of* k *and where* $\sup_k C_k = \infty$.

Proof. Step 1: *Auxiliary sequences*

For the approach we need several sequences of parameters:

- **(A1)** sequences $\{t_k\}_k$, $\{\rho_k\}_k$, and $\{\delta_k\}_k$ tending to 0,
- **(A2)** a sequence $\{h_k\}_k$ tending to ∞.

We introduce two other sequences $\{t_k'\}_k$ and $\{t_k''\}_k$ which are defined by $t_k' = t_k + \rho_k$ and $t_k'' = t_k - \rho_k$. Finally, we define three sequences of intervals $\{I_k\}_k$, $\{I_k'\}_k$ and $\{I_k''\}_k$ by

$$I_k = \Big[t_k - \frac{\rho_k}{2}, t_k + \frac{\rho_k}{2}\Big], \quad I_k' = \Big[t_k' - \frac{\rho_k}{2}, t_k' + \frac{\rho_k}{2}\Big], \quad I_k'' = \Big[t_k'' - \frac{\rho_k}{2}, t_k'' + \frac{\rho_k}{2}\Big].$$

To guarantee that I_k, I_k', I_k'' are contained in $(0, T]$ we assume

- **(A3)** $\rho_k = o(t_k)$ for $k \to \infty$.

Step 2: *Construction of a family of coefficients*

Therefore we need an increasing function $\mu \in C^\infty(\mathbb{R})$ which is defined by

$$\mu(x) = \begin{cases} 0, \ x \in (-\infty, -\frac{1}{3}], \\ 1, \ x \in [\frac{1}{3}, +\infty). \end{cases}$$

Now we are at the position to introduce the family of coefficients $\{a_k = a_k(t)\}_k$ which is defined by

$$a_k(t) = \begin{cases} \lambda^2(t), & t \in \ [0,T] \setminus (I_k' \cup I_k \cup I_k''); \\ \delta_k b^2(h_k(t - t_k)), & t \in I_k; \\ \delta_k b(0)^2 (1 - \mu(\frac{t-t_k'}{\rho_k})) + \lambda^2(t)\mu(\frac{t-t_k'}{\rho_k}), & t \in I_k'; \\ \delta_k b(0)^2 \mu(\frac{t-t_k''}{\rho_k}) + \lambda^2(t)(1 - \mu(\frac{t-t_k''}{\rho_k})), & t \in I_k''. \end{cases}$$

Taking into account the definition of b the parameters h_k and ρ_k should satisfy the assumption

- **(A4)** $\dfrac{h_k \rho_k}{2} \in \mathbb{N}$.

To have a connection to the structure of the coefficient from Proposition 1.6 we introduce the family $\{b_k = b_k(t)\}_k$ of *oscillating parts* of $\{a_k\}_k$ by

$$b_k^2(t) = \begin{cases} 1, & t \in [0,T] \setminus (I_k' \cup I_k \cup I_k''); \\ \frac{1}{\lambda^2(t)} \delta_k b^2(h_k(t-t_k)), & t \in I_k; \\ \frac{1}{\lambda^2(t)} \delta_k b(0)^2 (1 - \mu(\frac{t-t_k'}{\rho_k})) + \mu(\frac{t-t_k'}{\rho_k}), & t \in I_k'; \\ \frac{1}{\lambda^2(t)} \delta_k b(0)^2 \mu(\frac{t-t_k''}{\rho_k}) + (1 - \mu(\frac{t-t_k''}{\rho_k})), & t \in I_k''. \end{cases}$$

Step 3: *Concrete choice of parameters*

We choose the sequences of parameters as

$$t_k = \Lambda^{-1}(\exp(-k)), \quad \rho_k = \Big[\Big(\frac{\Lambda(t_k)}{\lambda(t_k)} \Big)^{-1} \Big]^{-1},$$

$$\delta_k = [\lambda(t_k)^{-1}]^{-2}, \quad h_k = 2\Big[\frac{\lambda(t_k)}{\Lambda(t_k)} \Big] \Big[\log \frac{1}{\Lambda(t_k)} \Big],$$

where $[a]$ denotes the integer part of a. It is clear that these sequences satisfy the assumptions (A1), (A2), (A4) and together with the assumption of the theorem the condition (A3). Moreover, we suppose

- (**A5**) $d_0 \le \inf_k \frac{\lambda(t_k)}{\lambda(t_k + \frac{\rho_k}{2})} \le \sup_k \frac{\lambda(t_k)}{\lambda(t_k + \frac{\rho_k}{2})} \le d_1$

with positive constants d_0 and d_1.

Example 4.4 *We give some functions λ satisfying* (A5)*. If*

$$\lambda(t) = \frac{d}{dt} \exp\Big(-\exp^{[n]} \frac{1}{t} \Big) = \frac{1}{t^2} \exp\Big(-\exp^{[n]} \frac{1}{t} \Big) \exp^{[n]} \frac{1}{t} \cdots \exp \frac{1}{t},$$

then $\Lambda(t) = \exp\Big(-\exp^{[n]} \frac{1}{t} \Big)$*, and* $\Lambda^{-1}(s) = \frac{1}{\log^{[n+1]} \frac{1}{s}}$*. According to our choice*

$$t_k = \Lambda^{-1}\Big(\exp(-\exp^{[n]} k) \Big) = \frac{1}{k} \text{ with } k \in \mathbb{N}^+,$$

$$\rho_k \sim \frac{1}{2} \frac{\Lambda(t_k)}{\lambda(t_k)} = \frac{1}{2} \frac{t_k^2}{\exp^{[n]} \frac{1}{t_k} \cdots \exp \frac{1}{t_k}} = \frac{1}{2k^2 \exp^{[n]} k \cdots \exp k}.$$

Consequently, $\lim_{k\to\infty} \frac{\lambda(t_k)}{\lambda(t_k \pm \frac{1}{2}\rho_k)} = 1$*. Note that when* $n = 0$*, it is Aleksandrian's example.*

Step 4: *Properties of* b_k

Now we will check that the coefficients b_k satisfy all the assumptions from Proposition 1.6. The assumption (A5) implies that

$$0 < b_0 \le \inf_{t\in[0,T]} b_k(t) \le \sup_{t\in[0,T]} b_k(t) \le b_1 < \infty,$$

where the constants b_0 and b_1 are independent of k.

Straightforward calculations yield in the intervals I_k, I'_k and I''_k representations for b_k, b'_k and b''_k.

Taking account of the assumptions (A1) to (A5) for λ, in particular from (A5) we conclude $\Lambda(t) \sim \Lambda(t_k)$ on I_k, and by the choice of parameters from step 3 we are able to conclude on $I_k \cup I'_k \cup I''_k$ that

$$|b'_k(t)| \le C\frac{\lambda(t)}{\Lambda(t)}\log\frac{1}{\Lambda(t)}; \quad |b''_k(t)| \le C\Big(\frac{\lambda(t)}{\Lambda(t)}\log\frac{1}{\Lambda(t)}\Big)^2,$$

where the constant C is independent of k.

Step 5: *Concrete choice of data*

Let $\chi = \chi(r) \in [0,1]$ be a cut-off function from $C_0^\infty(\mathbb{R})$, where $\chi \equiv 1$ for $|r| \le 1$ and $\chi \equiv 0$ for $|r| \ge 2$. Then we choose for large k the following data:

$$u_{0,k}(x) = \exp\Big(i\frac{h_k}{\sqrt{\delta_k}}x\xi\Big)\chi\Big(\frac{x}{(\log\frac{1}{\Lambda(t_k)})^2 P_k}\Big), \quad u_{1,k}(x) = 0 \quad \text{for all } x \in \mathbb{R}\,,$$

where

$$P_k = 2\pi\frac{\sqrt{\delta_k}}{h_k\xi} \sim \Lambda(t_k)\Big(\log\frac{1}{\Lambda(t_k)}\Big)^{-1}.$$

Then for $s \ge 0$ the norm $\|u_{0,k}\|_{H^s(\mathbb{R})}$ can be estimated in the following way:

$$\|u_{0,k}\|_{H^s(\mathbb{R})} \le C\Big(\Big(\frac{h_k}{\sqrt{\delta_k}}\Big)^s + \frac{1}{(\Lambda(t_k)\log\frac{1}{\Lambda(t_k)})^s} + 1\Big)\Big(\log\frac{1}{\Lambda(t_k)}\Big)\sqrt{P_k}. \quad (4.2)$$

Step 6: *Auxiliary Cauchy problems on* I_k

Now let us study the family of Cauchy problems

$$u_{tt} - \delta_k b^2(h_k(t-t_k))u_{xx} = 0,\ u(t_k,x) = u_{0,k}(x),\ u_t(t_k,x) = 0,\ t \in I_k.$$

Later we are interested in the unique solution $u_k = u_k(t_k + \frac{\rho_k}{2}, x)$ on the set $\{|x| \le P_k\}$. Let us determine the domain of dependence of the solutions u_k on $t = t_k + \frac{\rho_k}{2}$ over the set $\{|x| \le P_k\}$ with respect to the datum given on $t = t_k$. If x is taken on $t = t_k \pm \frac{\rho_k}{2}$ from $\{|x| \le P_k\}$, then the solution $u(t_k + \frac{\rho_k}{2}, x)$ will be influenced by the datum on the set $\{|x| \le P_k + O(\frac{\rho_k\sqrt{\delta_k}}{2})\}$. Using $\rho_k\sqrt{\delta_k} = O(\Lambda(t_k))$ and $P_k = \Lambda(t_k)(\log\frac{1}{\Lambda(t_k)})^{-1}$ we have to take into consideration the datum on the set $\{|x| \le O(\log(\frac{1}{\Lambda(t_k)})P_k)\}$. Using the structure of $u_{0,k}$ we have $u_{0,k}(x) = \exp\Big(i\frac{h_k}{\sqrt{\delta_k}}x\xi\Big)$ on this set.

We apply the transformation $s = h_k(t - t_k)$, $v(s,x) := u(t,x)$, thus

$$v_{ss} - \frac{\delta_k}{h_k^2} b^2(s) v_{xx} = 0, \quad v(0,x) = u_{0,k}(x), \quad v_s(0,x) = 0,\ s \in \Big[-\frac{h_k\rho_k}{2}, \frac{h_k\rho_k}{2}\Big].$$

Using Lemma 4.1 we have a unique solution in the form $u_k(s,x) = u_{0,k}(x)w(s)$, where $w = w(s)$ satisfies

$$w''(s) + \xi^2 b^2(s) w(s) = 0,\ w(0) = 1,\ w'(0) = 0,\ s \in \Big[-\frac{h_k\rho_k}{2}, \frac{h_k\rho_k}{2}\Big],$$

and where ξ is chosen as in Lemma 4.1. By Lemma 4.1 and after transforming back we arrive at

$$u_k\Big(t_k + \frac{\rho_k}{2}, x\Big) = \exp\Big(i\frac{h_k}{\sqrt{\delta_k}}x\xi\Big) w\Big(\frac{\rho_k h_k}{2}\Big), \quad u_k(t_k,x) = \exp\Big(i\frac{h_k}{\sqrt{\delta_k}}x\xi\Big) w(0),$$

where $\quad \Big|w\Big(\frac{\rho_k h_k}{2}\Big)\Big| \sim |\mu_0|^{\frac{\rho_k h_k}{2}}.$

Step 7: At least a finite loss

Let us determine the norm $\|u_k(t_k + \frac{\rho_k}{2}, \cdot)\|_{H^{s-p_0}(\{|x|\le P_k\})}$. It holds

$$\|u_k(t_k + \frac{\rho_k}{2}, \cdot)\|_{H^{s-p_0}(\{|x|\le P_k\})} \sim \Big(\Big(\frac{h_k}{\sqrt{\delta_k}}\Big)^{s-p_0} + 1\Big)\sqrt{P_k}|\mu_0|^{\frac{\rho_k h_k}{2}}. \tag{4.3}$$

From

$$h_k\rho_k \sim \log\frac{1}{\Lambda(t_k)} \sim \log\frac{h_k}{\sqrt{\delta_k}} \sim \log\langle D_x\rangle,$$

(4.2) and (4.3) we get with a sufficiently small p_0 depending on $\log|\mu_0|$ the estimate

$$\|u_k(t_k^{(2)}, \cdot)\|_{H^{s-p_0}(\mathbb{R})} \ge \|u_k(t_k^{(2)}, \cdot)\|_{H^{s-p_0}(\{|x|\le P_k\})} \ge C_k\|u_k(t_k^{(1)}, \cdot)\|_{H^s(\mathbb{R})},$$

where $\sup_k C_k = \infty$. From here, we know that the *finite loss of regularity really appears.*

Remark 4.5 *The finite degenerate case (without necessity of the condition* $\frac{\Lambda(t)}{\lambda(t)} = o(t)$*) can be discussed in the same way. We choose the sequences* $\{\rho_k\}_k = \{2^{-(k+3)}\}_k$, $\{t_k\}_k = \{2^{-k}\}_k$, $\{\delta_k\}_k = \{\lambda^2(t_k) = 2^{-2k\ell}\}_k$, $\{h_k = 32k2^k\}_k$.

5 Concluding remarks

There exist several open problems which we want to describe in this section.

Remark 5.1 *Let us consider*

$$u_{tt} - \lambda^2(t)b^2(t)u_{xx} - a\frac{\lambda^2(t)}{\Lambda(t)}u_x = 0, \; u(0,x) = u_0(x), \; u_t(0,x) = u_1(x).$$

As far as the authors know the following problems seem to be open:

1. *Let us set* $a = 0$*. Find examples for the coefficient* $\lambda(t)b(t)$ *(with an oscillating part) yielding as precise loss a finite loss of regularity for the corresponding Cauchy problem.*
2. *Prove by using the Floquet effect and instability argument that an arbitrary small loss really appears. In particular it is interesting to observe that for a suitable family* $\{b_k\}_k$ *we have with* $a = 0$ *that an arbitrary small loss appears, but if we choose* $a \neq 0$*, then a finite loss may appear.*
3. *If* $b \equiv 1$*, then one should understand under which assumptions to the constant* a *the finite loss really appears.*

Remark 5.2 *In a forthcoming note we will apply our approach to understand the optimality of assumptions to describe a blow-up of energy at* ∞ *with a certain blow-up rate in wave models, a recent research topic, was initiated in the papers* [4], [8] *and* [13].

Acknowledgment

The first author would like to thank the Department of Mathematics and Computer Science of TU Bergakademie Freiberg for the warm hospitality and assistance during his research stay from October 2006 to September 2007. Both authors are grateful to Professor Del Santo who introduced to them during their stay at the University of Trieste the basic ideas about the method how to construct examples for the infinite loss of regularity from [5].

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Decay Estimates for Variable Coefficient Wave Equations in Exterior Domains

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Summary. In this article we consider variable coefficient, time-dependent wave equations in exterior domains $\mathbb{R} \times (\mathbb{R}^n \setminus \Omega)$, $n \geq 3$. We prove localized energy estimates if Ω is star-shaped, and global in time Strichartz estimates if Ω is strictly convex.

2000 AMS Subject Classification: Primary: 35L20, Secondary: 34L25

Key words: Wave equation, local energy decay, exterior domains

1 Introduction

Our goal, in this article, is to prove analogs of the well-known Strichartz estimates and localized energy estimates for variable coefficient wave equations in exterior domains. We consider long-range perturbations of the flat metric, and we take the obstacle to be star-shaped. The localized energy estimates are obtained under a smallness assumption for the long-range perturbation. Global-in-time Strichartz estimates are then proved assuming the local-in-time Strichartz estimates, which are known to hold for strictly convex obstacles.

For the constant coefficient wave equation $\Box = \partial_t^2 - \Delta$ in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 2$, we have that solutions to the Cauchy problem

$$\Box u = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \tag{1.1}$$

*The work of the first author was supported in part by NSF grant DMS0800678. The work of the second author was supported in part by NSF grants DMS0354539 and DMS0301122.

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_12,

satisfy the Strichartz estimates[1]

$$\||D_x|^{-\rho_1}\nabla u\|_{L^{p_1}L^{q_1}} \lesssim \|\nabla u(0)\|_{L^2} + \||D_x|^{\rho_2}\Box u\|_{L^{p_2'}L^{q_2'}},$$

for Strichartz admissible exponents (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2). Here, exponents (ρ, p, q) are called Strichartz admissible if $2 \le p, q \le \infty$,

$$\rho = \frac{n}{2} - \frac{n}{q} - \frac{1}{p}, \quad \frac{2}{p} \le \frac{n-1}{2}\Big(1 - \frac{2}{q}\Big),$$

and $(\rho, p, q) \neq (1, 2, \infty)$ when $n = 3$.

The Strichartz estimates follow via a TT^* argument and the Hardy–Littlewood–Sobolev inequality from the dispersive estimates,

$$\||D_x|^{-\frac{n+1}{2}(1-\frac{2}{q})}\nabla u(t)\|_{L^q} \lesssim t^{-\frac{n-1}{2}(1-\frac{2}{q})}\|u_1\|_{L^{q'}}, \qquad 2 \le q < \infty$$

for solutions to (1.1) with $u_0 = 0$, $f = 0$. This in turn is obtained by interpolating between an $L^2 \to L^2$ energy estimate and an $L^1 \to L^\infty$ dispersive bound which provides $O(t^{-(n-1)/2})$ type decay. Estimates of this form originated in the work [25], and as stated are the culmination of several subsequent works. The endpoint estimate $(p, q) = \Big(2, \frac{2(n-1)}{n-3}\Big)$ was most recently obtained in [8], and we refer the interested reader to the references therein for a more complete history.

The second estimate which shall be explored is the localized energy estimate, a version of which states

$$\begin{aligned} \sup_j \|\langle x\rangle^{-1/2}\nabla u\|_{L^2(\mathbb{R}\times\{|x|\in[2^{j-1},2^j]\})} \\ \lesssim \|\nabla u(0)\|_{L^2} + \sum_k \|\langle x\rangle^{1/2}\Box u\|_{L^2(\mathbb{R}\times\{|x|\in[2^{k-1},2^k]\})} \end{aligned} \tag{1.2}$$

in the constant coefficient case. These estimates can be proved using a positive commutator argument with a multiplier which is roughly of the form $f(r)\partial_r$ when $n \ge 3$ and are quite akin to the bounds found in, e.g., [16], [24], [9], [20], [7], and [23]. See also [1], [12], [13] for certain estimates for small perturbations of the d'Alembertian.

Variants of these estimates for constant coefficient wave equations are also known in exterior domains. Here, u is replaced by a solution to

$$\Box u = F, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n \backslash \Omega$$

where Ω is a bounded set with smooth boundary. The localized energy estimates have played a key role in proving a number of long time existence results for nonlinear wave equations in exterior domains. See, e.g., [7] and

[1] Here and throughout, we shall use ∇ to denote a space-time gradient unless otherwise specified with subscripts.

[11, 12] for their proof and application. Here, it is convenient to assume that the obstacle Ω is star-shaped, though certain estimates are known (see e.g. [11], [3]) in more general settings. Exterior to star-shaped obstacles, the estimates for small perturbations of $\Box$ continue to hold (see [11]). This, however, only works for $n \geq 3$, and the bound which results is not strong enough in order to prove the Strichartz estimates which we desire. As such, we shall, in the sequel, couple this bound with certain frequency localized versions of the estimate in order to prove the Strichartz estimates. For time-independent perturbations, one may permit more general geometries. See, e.g., [3].

Certain global-in-time Strichartz estimates are also known in exterior domains, but, except for certain very special cases (see [4], [2], which are closely based on [21]), require that the obstacle be strictly convex. Local-in-time estimates were shown in [19] for convex obstacles, and using these estimates, global estimates were constructed in [20] for n odd and [3] and [14] for general n. See also [6].

In the present article, we explore variable coefficient cases of these estimates. Here, $\Box$ is replaced by the second order hyperbolic operator

$$P(t,x,D) = D_i a^{ij}(t,x) D_j + b^i(t,x) D_i + c(t,x),$$

where $D_0 = D_t$ is understood. We assume that (a^{ij}) has signature $(n,1)$ and that $a^{00} < 0$, i.e., that time slices are space-like. We shall then consider the initial value boundary value problem

$$Pu = f, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n \backslash \Omega. \quad (1.3)$$

When $\Omega = \emptyset$ and $b^i \equiv c \equiv 0$, the problem of proving Strichatz estimates is understood locally, and of course, localized energy estimates are trivial locally-in-time. For smooth coefficients, Strichartz estimates were first proved in [15] using Fourier integral operators. Using a wave packet decomposition, Strichartz estimates were obtained in [17] for $C^{1,1}$ coefficients in spatial dimensions $n = 2, 3$. Using instead an approach based on the FBI transform, these estimates were extended to all dimensions in [27, 28, 29]. For rougher coefficients, the Strichartz estimates as stated above are lost (see [18], [22]) and only certain estimates with losses are available [28, 29]. When the boundary is nonempty, far less is known, and we can only refer to the results of [19] for smooth time independent coefficients, $b^i \equiv c \equiv 0$, and Ω strictly geodesically convex. The proof of these estimates is quite involved and uses a Melrose–Taylor parametrix to approximate the reflected solution.

For the boundaryless problem, global-in-time localized energy estimates and Strichartz estimates were recently shown in [13] for small, C^2, long-range perturbations. The former follow from a positive commutator argument with a multiplier which is akin to what we present in the sequel. For the latter, an outgoing parametrix is constructed using a time-dependent FBI transform in a fashion which is reminiscent to that of the preceding work [26] on Schrödinger equations. Upon conjugating the half-wave equation by the FBI transform, one obtains a degenerate parabolic equation due to a nontrivial second-order term in the asymptotic expansion. Here, the bounds from [26], which are based on the maximum principle, may be cited. The errors in this parametrix construction are small in the localized energy spaces, which again are similar to those below, and it is shown that the global Strichartz estimates follow from the localized energy estimates.

The aim of the present article is to combine the approach of [13] with analogs of those from [20], [3], and [14] to show that global-in-time Strichartz estimates in exterior domains follow from the localized energy estimates and local-in-time Strichartz estimates for the boundary value problem. As we shall show the localized energy estimates for small perturbations outside of star-shaped obstacles, the global Strichartz estimates shall then follow for convex obstacles from the estimates of [19].

Let us now more precisely describe our assumptions. We shall look at certain long-range perturbations of Minkowski space. To state this, we set

$$D_0 = \{|x| \le 2\}, \quad D_j = \{2^j \le |x| \le 2^{j+1}\}, \quad j = 1, 2, \dots$$

and

$$A_j = \mathbb{R} \times D_j, \qquad A_{<j} = \mathbb{R} \times \{|x| \le 2^j\}.$$

We shall then assume that

$$\sum_{j\in\mathbb{N}} \sup_{A_j\cap(\mathbb{R}\times\mathbb{R}^n\setminus\Omega)} \langle x\rangle^2|\nabla^2 a(t,x)| + \langle x\rangle|\nabla a(t,x)| + |a(t,x) - I_n| \le \varepsilon \tag{1.4}$$

and, for the lower order terms,

$$\sum_{j\in\mathbb{N}} \sup_{A_j\cap(\mathbb{R}\times\mathbb{R}^n\setminus\Omega)} \langle x\rangle^2|\nabla b(t,x)| + \langle x\rangle|b(t,x)| \le \varepsilon, \tag{1.5}$$

$$\sum_{j\in\mathbb{N}} \sup_{A_j\cap(\mathbb{R}\times\mathbb{R}^n\setminus\Omega)} \langle x\rangle^2|c(t,x)| \le \varepsilon. \tag{1.6}$$

If ε is small enough, then (1.4) precludes the existence of trapped rays, while for arbitrary ε it restricts the trapped rays to finitely many dyadic regions.

We now define the localized energy spaces that we shall use. We begin with an initial choice which is convenient for the local energy estimates but not so much for the Strichartz estimates. Precisely, we define the localized energy space LE_0 as

$$\|\varphi\|_{LE_0} = \sup_{j\ge 0}\Big(2^{-j/2}\|\nabla\varphi\|_{L^2(A_j\cap(\mathbb{R}\times\mathbb{R}^n\setminus\Omega))} + 2^{-3j/2}\|\varphi\|_{L^2(A_j\cap(\mathbb{R}\times\mathbb{R}^n\setminus\Omega))}\Big),$$

while for the forcing term we set

$$\|f\|_{LE_0^*} = \sum_{k\geq 0} 2^{k/2}\|f\|_{L^2(A_k\cap(\mathbb{R}\times\mathbb{R}^n\setminus\Omega))}.$$

The local energy bounds in these spaces shall follow from the arguments in [12].

On the other hand, for the Strichartz estimates, we shall introduce frequency localized spaces as in [13], as well as the earlier work [26]. We use a Littlewood–Paley decomposition in frequency,

$$1 = \sum_{k=-\infty}^{\infty} S_k(D), \quad \text{supp } s_k(\xi) \subset \{2^{k-1} < |\xi| < 2^{k+1}\}$$

and for each $k \in \mathbb{Z}$, we use

$$\|\varphi\|_{X_k} = 2^{-k^-/2}\|\varphi\|_{L^2(A_{<k^-})} + \sup_{j\geq k^-} \||x|^{-1/2}\varphi\|_{L^2(A_j)}$$

to measure functions of frequency 2^k. Here $k^- = \frac{|k|-k}{2}$. We then define the global norm

$$\|\varphi\|_X^2 = \sum_{k=-\infty}^{\infty} \|S_k\varphi\|_{X_k}^2.$$

Then for the local energy norm we use

$$\|\varphi\|_{LE_\infty}^2 = \|\nabla\varphi\|_X^2.$$

For the inhomogeneous term we introduce the dual space $Y = X'$ with norm defined by

$$\|f\|_Y^2 = \sum_{k=-\infty}^{\infty} \|S_k f\|_{X_k'}^2.$$

To relate these spaces to the LE_0 respectively LE_0^* we use Hardy-type inequalities which are summarized in the following proposition:

Proposition 1.1 *We have*

$$\sup_j \||x|^{-1/2}u\|_{L^2(A_j)} \lesssim \|u\|_X \tag{1.7}$$

and

$$\|u\|_Y \lesssim \sum_j \||x|^{1/2}u\|_{L^2(A_j)}. \tag{1.8}$$

In addition,

$$\||x|^{-3/2}\varphi\|_{L^2} \lesssim \|\nabla_x\varphi\|_X, \quad n\geq 4. \tag{1.9}$$

The first bound (1.7) is a variant of a Hardy inequality, see [13, (16), Lemma 1], and also [26]. The second (1.8) is its dual. The bound (1.9), proved in [13, Lemma 1], fails in dimension three.

Now we turn our attention to the obstacle problem. For R fixed so that $\Omega \subset \{|x| < R\}$, we select a smooth cutoff χ with $\chi \equiv 1$ for $|x| < 2R$ and $\operatorname{supp} \chi \subset \{|x| < 4R\}$. We shall use χ to partition the analysis into a portion near the obstacle and a portion away from the obstacle. In particular, we define the localized energy space $LE \subset LE_0$ as

$$\|\varphi\|_{LE}^2 = \|\varphi\|_{LE_0}^2 + \|(1-\chi)\varphi\|_{LE_\infty}^2.$$

For the forcing term, we will respectively construct $LE^* \supset LE_0^*$ by

$$\|f\|_{LE^*}^2 = \|\chi f\|_{LE_0^*}^2 + \|(1-\chi)f\|_Y^2, \qquad n \geq 4.$$

This choice is no longer appropriate in dimension $n = 3$, as otherwise the local L^2 control of the solution is lost. Instead we simply set

$$\|f\|_{LE^*}^2 = \|f\|_{LE_0^*}^2, \qquad n = 3.$$

Using these space, we now define what it means for a solution to satisfy our stronger localized energy estimates.

Definition 1.1 *We say that the operator P satisfies the localized energy estimates if for each initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ and each inhomogeneous term $f \in LE^*$, there exists a unique solution u to* (1.3) *with $u \in LE$ which satisfies the bound*

$$\|u\|_{LE} + \left\|\frac{\partial u}{\partial \nu}\right\|_{L^2(\partial\Omega)} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{LE^*}. \tag{1.10}$$

We prove that the localized energy estimates hold under the assumption that P is a small perturbation of the d'Alembertian:

Theorem 1.1 *Let Ω be a star-shaped domain. Assume that the coefficients a^{ij}, b^i, and c satisfy* (1.4), (1.5), *and* (1.6) *with an ε which is sufficiently small. Then the operator P satisfies the localized energy estimates globally-in-time for $n \geq 3$.*

These results correspond to the $s = 0$ results of [13]. Some more general results are also available by permitting $s \neq 0$, but for simplicity we shall not provide these details.

Once we have the local energy estimates, the next step is to prove the Strichartz estimates. To do so, we shall assume that the corresponding Strichartz estimate holds locally-in-time.

Definition 1.2 *For a given operator P and domain Ω, we say that the local Strichartz estimate holds if*

$$\|\nabla u\|_{|D_x|^{\rho_1} L^{p_1} L^{q_1}([0,1]\times\mathbb{R}^n\setminus\Omega)} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{|D_x|^{-\rho_2} L^{p_2'} L^{q_2'}([0,1]\times\mathbb{R}^n\setminus\Omega)} \tag{1.11}$$

for any solution u to (1.3)*.*

As mentioned previously, (1.11) is only known under some fairly restrictive hypotheses. We show a conditional result which says that the global-in-time Strichartz estimates follow from the local-in-time estimates as well as the localized energy estimates.

Theorem 1.2 *Let Ω be a domain such that P satisfies both the localized energy estimates and the local Strichartz estimate. Let a^{ij}, b^i, c satisfy* (1.4)*,* (1.5)*, and* (1.6)*. Let (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) be two Strichartz pairs. Then the solution u to* (1.3) *satisfies*

$$\|\nabla u\|_{|D_x|^{\rho_1} L^{p_1} L^{q_1}} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{|D_x|^{-\rho_2} L^{p_2'} L^{q_2'}}. \tag{1.12}$$

Notice that this conditional result does not require the ε in (1.4), (1.5), and (1.6) to be small. We do, however, require this for our proof of the localized energy estimates which are assumed in Theorem 1.2.

As an example of an immediate corollary of the localized energy estimates of Theorem 1.1 and the local Strichartz estimates of [19], we have:

Corollary 1.1 *Let $n \geq 3$, and let Ω be a strictly convex domain. Assume that the coefficients a^{ij}, b^i, and c are time-independent in a neighborhood of Ω and satisfy* (1.4)*,* (1.5)*, and* (1.6) *with an ε which is sufficiently small. Let (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) be two Strichartz pairs which satisfy*

$$\frac{1}{p_1} = \Big(\frac{n-1}{2}\Big)\Big(\frac{1}{2} - \frac{1}{q_1}\Big), \quad \frac{1}{p_2'} = \Big(\frac{n-1}{2}\Big)\Big(\frac{1}{2} - \frac{1}{q_2'}\Big).$$

Then the solution u to (1.3) *satisfies*

$$\|\nabla u\|_{|D_x|^{\rho_1} L^{p_1} L^{q_1}} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{|D_x|^{-\rho_2} L^{p_2'} L^{q_2'}}. \tag{1.13}$$

This paper is organized as follows. In the next section, we prove the localized energy estimates for small perturbations of the d'Alembertian exterior to a star-shaped obstacle. In the last section, we prove Theorem 1.2 which says that global-in-time Strichartz estimates follow from the localized energy estimates as well as the local Strichartz estimates.

2 The localized energy estimates

In this section, we shall prove Theorem 1.1.

By combining the inclusions $LE \subset LE_0$, $LE_0^* \subset LE^*$ and the bounds (1.9), (1.5), and (1.6), one can easily prove the following which permits us to treat the lower order terms perturbatively. See also [13, Lemma 3].

Proposition 2.1 *Let b^i, c be as in* (1.5) *and* (1.6)*, respectively. Then,*

$$\|b\nabla u\|_{LE^*} \lesssim \varepsilon \|u\|_{LE}, \tag{2.1}$$

$$\|cu\|_{LE^*} \lesssim \varepsilon \|u\|_{LE}. \tag{2.2}$$

We now look at the proof of the localized energy estimates. Due to Proposition 2.1 we can assume that $b = 0, c = 0$. To prove the theorems, we use positive commutator arguments. We first do the analysis separately in the two regions.

2.1 Analysis near Ω and classical Morawetz-type estimates

Here we sketch the proof from [12] which gives an estimate which is similar to (1.2) for small perturbations of the d'Alembertian. This estimate shall allow us to gain control of the solution near the boundary. It also permits local L^2 control of the solution, not just the gradient in three dimensions. The latter is necessary as the required Hardy inequality which can be utilized in higher dimensions corresponds to a false endpoint estimate in three dimensions.

The main estimate is the following:

Proposition 2.2 *Let Ω be a star-shaped domain. Assume that the coefficients a^{ij}, b^i, and c satisfy* (1.4), (1.5), *and* (1.6)*, respectively, with an ε which is sufficiently small. Suppose that φ satisfies $P\varphi = F$, $\varphi|_{\partial\Omega} = 0$. Then*

$$\|\varphi\|_{LE_0} + \|\nabla\varphi\|_{L^\infty L^2} + \|\partial_\nu \varphi\|_{L^2(\partial\Omega)} \lesssim \|\nabla\varphi(0)\|_2 + \|F\|_{LE_0^*}. \tag{2.3}$$

Proof We provide only a terse proof. The interested reader can refer to [12] for a more detailed proof. For $f = \frac{r}{r+\rho}$, where ρ is a fixed positive constant, we use a multiplier of the form

$$\partial_t\varphi + f(r)\partial_r\varphi + \frac{n-1}{2}\frac{f(r)}{r}\varphi.$$

By multiplying $P\varphi$ and integrating by parts, one obtains

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^n\setminus\Omega} &\frac{1}{2}f'(r)(\partial_r\varphi)^2 + \Big(\frac{f(r)}{r} - \frac{1}{2}f'(r)\Big)|\nabla\!\!\!/\varphi|^2 \\ &+ \frac{1}{2}f'(r)(\partial_t\varphi)^2 - \frac{n-1}{4}\Delta\Big(\frac{f(r)}{r}\Big)\varphi^2\,dxdt \\ &- \frac{1}{2}\int_0^T\int_{\partial\Omega}\frac{f(r)}{r}(\partial_\nu\varphi)^2\langle x,\nu\rangle(a^{ij}\nu_i\nu_j)\,d\sigma dt + (1+O(\varepsilon))\|\nabla\varphi(T)\|_2^2 \\ &\lesssim \|\nabla\varphi(0)\|_2^2 + \int_0^T\int_{\mathbb{R}^n\setminus\Omega}|F|\Big(|\partial_t\varphi| + |f(r)\partial_r\varphi| + \Big|\frac{f(r)}{r}\varphi\Big|\Big)\,dx\,dt \\ &+ \int_0^T\int_{\mathbb{R}^n\setminus\Omega} O\Big(\frac{|a-I|}{r} + |\nabla a|\Big)|\nabla\varphi|\Big(|\nabla\varphi| + \Big|\frac{\varphi}{r}\Big|\Big)\,dx\,dt \\ &\lesssim \|\nabla\varphi(0)\|_2^2 + \|F\|_{LE_0^*(0,T)}\|\varphi\|_{LE_0(0,T)} + \varepsilon\|\varphi\|^2_{LE_0(0,T)}.\end{aligned} \tag{2.4}$$

Here, we have used the Hardy inequality $\||x|^{-1}\varphi\|_2 \lesssim \|\nabla\varphi\|_2$, $n \geq 3$, as well as (1.4).

All terms on the left are nonnegative. By direct computation, the first term controls

$$\rho^{-1}\|\nabla\varphi\|^2_{L^2([0,T]\times\{|x|\approx\rho\})} + \rho^{-3}\|\varphi\|^2_{L^2([0,T]\times\{|x|\approx\rho\})}.$$

Taking a supremum over dyadic ρ provides a bound for the $\|\varphi\|_{LE_0(0,T)}$. In the second term we have $-\langle x,\nu\rangle \gtrsim 1$, which follows from the assumption that Ω is star-shaped, and also $a^{ij}\nu_i\nu_j \gtrsim 1$ which follows from (1.4). By simply taking $\rho = 1$, one can bound the third term on the left of (2.3) by the right side of (2.4). Thus we obtain

$$\begin{aligned}\|\varphi\|_{LE_0(0,T)} &+ \|\nabla\varphi(T)\|_{L^\infty L^2} + \|\partial_\nu\varphi\|_{L^2(\partial\Omega)} \\ &\lesssim \|\nabla\varphi(0)\|_2^2 + \|F\|_{LE_0^*(0,T)}\|\varphi\|_{LE_0(0,T)} + \varepsilon\|\varphi\|^2_{LE_0(0,T)}.\end{aligned}$$

The LE_0 terms on the right can be bootstrapped for ε small which yields (2.3).

2.2 Analysis near ∞ and frequency localized estimates

In this section, we briefly sketch the proof from [13] for some frequency localized versions of the localized energy estimates for the boundaryless equation. The main estimate here, which is from [13], is the following.

Proposition 2.3 *Suppose that a^{ij} are as in Theorem 1.1 and $b = 0$, $c = 0$. Then for each initial data $(u_0, u_1) \in \dot{H}^1 \times L^2$ and each inhomogeneous term $f \in Y \cap L^1L^2$, there exists a unique solution u to the boundaryless equation*

$$Pu = f, \quad u(0) = u_0, \quad \partial_t u(0) = u_1$$

satisfying

$$\|\nabla u\|_{L^\infty L^2 \cap X} \lesssim \|\nabla u(0)\|_{L^2} + \|f\|_{L^1 L^2 + Y}. \tag{2.5}$$

The proof here uses a multiplier of the form

$$D_t + \delta_0 Q + i\delta_1 B.$$

Here the parameters are chosen so that

$$\varepsilon \ll \delta_1 \ll \delta \ll \delta_0 \ll 1.$$

The multiplier Q is given by

$$Q = \sum_k S_k Q_k S_k$$

where Q_k are differential operators of the form

$$Q_k = (D_x x \varphi_k(|x|) + \varphi_k(|x|) x D_x).$$

The φ_k are functions of the form

$$\varphi_k(x) = 2^{-k^-} \psi_k(2^{-k^-} \delta x)$$

where for each k the functions ψ_k have the following properties:

(i) $\psi_k(s) \approx (1+s)^{-1}$ for $s > 0$ and $|\partial^j \psi_k(s)| \lesssim (1+s)^{-j-1}$ for $j \leq 4$,
(ii) $\psi_k(s) + s\psi_k'(s) \approx (1+s)^{-1}\alpha_k(s)$ for $s > 0$,
(iii) $\psi_k(|x|)$ is localized at frequency $\ll 1$.

The α_k are slowly varying functions that are related to the bounds of the individual summands in (1.4). This construction is reminiscent of those in [26], [10], and [13].

For the Lagrangian term B, we fix a function b satisfying

$$b(s) \approx \frac{\alpha(s)}{1+s}, \quad |b'(s)| \ll b(s).$$

Then, we set $B = \sum_k S_k 2^{-k^-} b(2^{-k^-} x) S_k$.

The computations, which are carried out in detail in [13], are akin to those outlined in the previous section.

2.3 Proof of Theorem 1.1

Consider first the three-dimensional case. For $f \in LE^* = LE_0^*$ we can use Proposition 2.2 to obtain

$$\|u\|_{LE_0} + \|\nabla u\|_{L^\infty L^2} + \|\partial_\nu u\|_{L^2(\partial\Omega)} \lesssim \|\nabla u(0)\|_2 + \|f\|_{LE_0^*}.$$

It remains to estimate $\|(1-\chi)u\|_{LE_\infty}$ with χ as in the definition of LE. By (2.5) we have

$$\|(1-\chi)u\|_{LE_\infty} \lesssim \|\nabla(1-\chi)u(0)\|_{L^2} + \|P[(1-\chi)u]\|_Y$$

$$\lesssim \|\nabla u(0)\|_{L^2} + \|P[(1-\chi)u]\|_{LE_0^*}.$$

Finally, to bound the last term we write

$$P[(1-\chi)u] = -[P,\chi]u + (1-\chi)f.$$

The commutator has compact spatial support; therefore

$$\|P[(1-\chi)u]\|_{LE_0^*} \lesssim \|u\|_{LE_0} + \|f\|_{LE_0^*}$$

and the proof is concluded.

Consider now higher dimensions $n \geq 4$. For fixed $f \in LE^*$, we first solve the boundaryless problem

$$Pu_\infty = (1-\chi)f \in Y, \qquad u_\infty(0) = 0,\ \partial_t u_\infty(0) = 0$$

using Proposition 2.3. We consider χ_∞ which is identically 1 in a neighborhood of infinity and vanishes on supp χ. For the function $\chi_\infty u_\infty$ we use the Hardy inequalities in Proposition 1.1 to write

$$\|\chi_\infty u_\infty\|_{LE} \approx \|\nabla(\chi_\infty u_\infty)\|_X \lesssim \|\nabla u_\infty\|_X \lesssim \|(1-\chi_\infty)f\|_Y.$$

The remaining part $\psi = u - \psi_\infty u_\infty$ solves

$$P\psi = \chi_\infty f + [P,\chi_\infty]u_\infty;$$

therefore

$$\|P\psi\|_{LE_0^*} \lesssim \|f\|_{LE^*} + \|u_\infty\|_{LE_0} \lesssim \|f\|_{LE^*} + \|\nabla u_\infty\|_X \lesssim \|f\|_{LE^*}.$$

Then we estimate ψ as in the three dimensional case. The proof is concluded.

3 The Strichartz estimates

In this final section, we prove Theorem 1.2, the global Strichartz estimates. We use fairly standard arguments to accomplish this. In a compact region about the obstacle, we prove the global estimates using the local Strichartz estimates and the localized energy estimates. Near infinity, we use [13]. The two regions can then be glued together using the localized energy estimates.

We shall utilize the following two propositions. The first gives the result when the forcing term is in the dual localized energy space.

Proposition 3.1 *Let (ρ, p, q) be a Strichartz pair. Let Ω be a domain such that P satisfies both the localized energy estimates and the homogeneous local Strichartz estimate with exponents (ρ, p, q). Then for each $\varphi \in LE$ with $P\varphi \in LE^*$, we have*

$$\||D_x|^{-\rho}\nabla\varphi\|^2_{L^pL^q} \lesssim \|\nabla\varphi(0)\|^2_{L^2} + \|\varphi\|^2_{LE} + \|P\varphi\|^2_{LE^*}. \tag{3.1}$$

The second proposition allows us to gain control when the forcing term is in a dual Strichartz space.

Proposition 3.2 *Let (ρ_1, p_1, q_1) and (ρ_2, p_2, q_2) be Strichartz pairs. Let Ω be a domain such that P satisfies both the localized energy estimates and the local Strichartz estimate with exponents (ρ_1, p_1, q_1), (ρ_2, p_2, q_2). Then there is a parametrix K for P with*

$$\|\nabla Kf\|^2_{L^\infty L^2} + \|Kf\|^2_{LE} + \||D_x|^{-\rho_1}\nabla Kf\|^2_{L^{p_1}L^{q_1}} \lesssim \||D_x|^{\rho_2} f\|^2_{L^{p_2'}L^{q_2'}} \tag{3.2}$$

and

$$\|PKf - f\|_{LE^*} \lesssim \||D_x|^{\rho_2} f\|_{L^{p_2'}L^{q_2'}}. \tag{3.3}$$

We briefly delay the proofs and first apply the propositions to prove Theorem 1.2.

Proof (Proof of Theorem 1.2*)* For

$$Pu = f + g, \quad f \in |D_x|^{-\rho_2} L^{p_2'} L^{q_2'},\ g \in LE^*,$$

we write

$$u = Kf + v.$$

The bound for ∇Kf follows immediately from (3.2).

To bound v, we note that

$$Pv = (1 - PK)f + g.$$

Applying (3.1) and the localized energy estimate, we have

$$\||D_x|^{-\rho_1}\nabla v\|_{L^{p_1}L^{q_1}} \lesssim \|\nabla u(0)\|_{L^2} + \|\nabla Kf\|_{L^\infty L^2} + \|(1-PK)f\|_{LE^*} + \|g\|_{LE^*}.$$

The Strichartz estimates (1.12) then follow from (3.2) and (3.3).

Proof (Proof of Proposition 3.1*)* We assume $P\varphi \in Y$, and we write

$$\varphi = \chi\varphi + (1-\chi)\varphi$$

with χ as in the definition of the LE norm. Since, using (1.8), the fundamental theorem of calculus, and (1.7), we have

$$\|[P,\chi]\varphi\|_{LE^*} \lesssim \|\varphi\|_{LE},$$

it suffices to show the estimate for $\varphi_1 = \chi\varphi$, $\varphi_2 = (1-\chi)\varphi$ separately.

To show (3.1) for φ_1, we need only assume that φ_1 and $P\varphi_1$ are compactly supported, and we write

$$\varphi_1 = \sum_{j\in\mathbb{Z}} \beta(t-j)\varphi_1$$

for an appropriately chosen, smooth, compactly supported function β. By commuting P and $\beta(t-j)$, we easily obtain

$$\sum_{j\in\mathbb{N}} \|\beta(t-j)\varphi_1\|^2_{LE} + \|P(\beta(t-j)\varphi_1)\|^2_{L^1L^2} \lesssim \|\varphi_1\|^2_{LE} + \|P\varphi_1\|^2_{LE^*}.$$

Here, as above, we have also used (1.8), the fundamental theorem of calculus, and (1.7). Applying the homogeneous local Strichartz estimate to each piece $\beta(t-j)\varphi_1$ and using Duhamel's formula, the bound (3.1) for φ_1 follows immediately from the square summability above.

On the other hand, φ_2 solves a boundaryless equation, and the estimate (3.1) is just a restatement of [13, Theorem 7] with $s=0$. This follows directly when $n \geq 4$ and easily from (1.8) when $n=3$.

Proof (Proof of Proposition 3.2*)* We split f in a fashion similar to the above:

$$f = \chi f + (1-\chi)f = f_1 + f_2.$$

For f_1, we write

$$f_1 = \sum_j \beta(t-j) f_1$$

where β is supported in $[-1,1]$. Let ψ_j be the solution to

$$P\psi_j = \beta(t-j)f_1.$$

By the local Strichartz estimate, we have

$$\||D_x|^{-\rho_1}\nabla\psi_j\|_{L^{p_1}L^{q_1}(E_j)} + \|\nabla\psi_j\|_{L^\infty L^2(E_j)} \lesssim \|\beta(t-j)|D_x|^{\rho_2} f_1\|_{L^{p_2'}L^{q_2'}}$$

where $E_j = [j-2, j+2] \times (\{|x|<2\} \cap \mathbb{R}^n\backslash\Omega)$. Letting $\tilde{\beta}(t-j,r)$ be a cutoff which is supported in E_j and is identically one on the support of $\beta(t-j)\chi$, set $\varphi_j = \tilde{\beta}(t-j,r)\psi_j$. Then,

$$\||D_x|^{-\rho_1}\nabla\varphi_j\|_{L^{p_1}L^{q_1}} + \|\nabla\varphi_j\|_{L^\infty L^2} \lesssim \|\beta(t-j)|D_x|^{\rho_2} f_1\|_{L^{p_2'}L^{q_2'}}. \tag{3.4}$$

Moreover,

$$P\varphi_j - \beta(t-j)f_1 = [P, \tilde\beta(t-j,r)]\psi_j,$$

and thus,

$$\|P\varphi_j - \beta(t-j)f_1\|_{L^2} \lesssim \|\beta(t-j)|D_x|^{-\rho_2} f_1\|_{L^{p_2'}L^{q_2'}}. \tag{3.5}$$

Setting

$$Kf_1 = \sum_j \varphi_j$$

and summing the bounds (3.4) and (3.5) yields the desired result for f_1.

For f_2, we solve the boundaryless equation

$$P\psi = f_2.$$

For a second cutoff $\tilde\chi$ which is 1 on the support of $1-\chi$ and vanishes for $\{r < R\}$, we set

$$Kf_2 = \tilde\chi\psi.$$

The following lemma, which is in essence from [13, Theorem 6], applied to ψ then easily yields the desired bounds.

Lemma 3.1 *Let $f \in |D_x|^{-\rho_2} L^{p_2'} L^{q_2'}$. Then the forward solution ψ to the boundaryless equation $P\psi = f$ satisfies the bound*

$$\|\nabla\psi\|^2_{L^\infty L^2} + \|\psi\|^2_{LE} + \||D_x|^{-\rho_1}\nabla\psi\|^2_{L^{p_1}L^{q_1}} \lesssim \||D_x|^{\rho_2} f\|^2_{L^{p_2'}L^{q_2'}}. \tag{3.6}$$

It remains to prove the lemma. From [13, Theorem 6], we have that

$$\|\nabla\psi\|^2_X + \||D_x|^{-\rho_1}\nabla\psi\|^2_{L^{p_1}L^{q_1}} \lesssim \||D_x|^{\rho_2} f\|^2_{L^{p_2'}L^{q_2'}}. \tag{3.7}$$

By (1.7) we have

$$\sup_{j\geq 0} 2^{-j/2}\|\nabla\psi\|_{L^2(A_j)} \lesssim \|\nabla\psi\|x.$$

It remains only to show the uniform bound

$$2^{-\frac{3j}{2}}\|\psi\|_{L^2(A_j)} \lesssim \||D_x|^{\rho_2} f\|_{L^{p_2'}L^{q_2'}} \tag{3.8}$$

when $n = 3$. Let $H(t,s)$ be the forward fundamental solution to P. Then

$$\psi(t) = \int_{-\infty}^t H(t,s)f(s)ds.$$

Therefore (3.8) can be rewritten as

$$2^{-\frac{3j}{2}} \left\| \int_{-\infty}^{t} H(t,s) f(s) ds \right\|_{L^2(A_j)} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p_2'} L^{q_2'}}.$$

Since $p_2' < 2$ for Strichartz pairs in $n = 3$, by the Christ-Kiselev lemma [5] (see also [20]) it suffices to show that

$$2^{-\frac{3j}{2}} \left\| \int_{-\infty}^{\infty} H(t,s) f(s) ds \right\|_{L^2(A_j)} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p_2'} L^{q_2'}}. \tag{3.9}$$

The function

$$\psi_1(t) = \int_{-\infty}^{\infty} H(t,s) f(s) ds$$

solves $P\psi_1 = 0$, and from (3.7) we have

$$\|\nabla \psi_1\|_{L^\infty L^2} \lesssim \| |D_x|^{\rho_2} f \|_{L^{p_2'} L^{q_2'}}.$$

On the other hand, from (2.3) with $P\psi_1 = 0$ and $\Omega = \emptyset$, we obtain

$$2^{-\frac{3j}{2}} \|\psi_1\|_{L^2(A_j)} \lesssim \|\nabla \psi_1(0)\|_2^2.$$

Hence (3.9) follows, and the proof is concluded.

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On Gevrey Well-Posedness of the Cauchy Problem for Some Noneffectively Hyperbolic Operators

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Dedicated to Professor Ferruccio Colombini

Summary. We give an example of second-order hyperbolic operator for which the Cauchy problem is not Gevrey 6 well-posed for any lower order term. This phenomenon is caused by the existence of a null bicharacteristic landing on the double characteristic manifold. We also give an example of second-order hyperbolic operator for which the Cauchy problem is Gevrey 4 well-posed for any lower order term but not Gevrey s well-posed for $s > 4$ with some lower order term.

2000 AMS Subject Classification: Primary 35L15; Secondary 35L30

Key words: Noneffectively hyperbolic, null bicharacteristic, Gevrey well-posed, Cauchy problem.

1 Introduction

Let us consider

$$p(x, D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 \tag{1.1}$$

in $\mathbb{R}^3$. The double characteristic manifold is given by

$$\Sigma = \{\xi_0 = 0, x_1 = 0, \xi_1 = 0\}, \quad \xi_2 \neq 0.$$

A null bicharacteristic is an integral curve of the Hamilton system

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_13,

$$\begin{cases} \dot{x} = \dfrac{\partial p}{\partial \xi}(x,\xi), \\ \dot{\xi} = -\dfrac{\partial p}{\partial x}(x,\xi) \end{cases}$$

on which $p(x, \xi)$ vanishes. Let $\rho \in \Sigma$ and consider the linearization of the Hamilton system at ρ

$$\dot{X} = F_p(\rho)X, \quad X = (x, \xi)$$

where

$$F_p(\rho) = \begin{pmatrix} \dfrac{\partial^2 p}{\partial x \partial \xi}(\rho) & \dfrac{\partial^2 p}{\partial \xi \partial \xi}(\rho) \\ -\dfrac{\partial^2 p}{\partial x \partial x}(\rho) & -\dfrac{\partial^2 p}{\partial \xi \partial x}(\rho) \end{pmatrix}.$$

We note that $p(x, D)$ is a model operator of noneffectively hyperbolic operator, that is, $\mathrm{Sp}(F_p(\rho)) \subset i\mathbb{R}$, $\rho \in \Sigma$ such that

$$\mathrm{Ker} F_p^2(\rho) \cap \mathrm{Im} F_p^2(\rho) \neq \{0\}, \quad \rho \in \Sigma \tag{1.2}$$

where $\mathrm{Sp}(F_p)$ denotes the spectrum of F_p. A main feature of this operator is the existence of a null bicharacteristic

$$x_1 = -\frac{x_0^2}{4}, \;\; x_2 = \frac{x_0^5}{80}, \;\; \xi_0 = 0, \;\; \xi_1 = \frac{x_0^3}{8}, \;\; \xi_2 = \text{constant} \neq 0$$

(parametrized by x_0) tangent to Σ when $x_0 \to 0$. Then we have

Theorem 1.1 *The Cauchy problem for $P(x, D) = p(x, D) + \sum_{j=0}^{2} b_j D_j$ is not locally solvable at the origin in Gevrey class of order $s > 6$ for any b_0, b_1, $b_2 \in \mathbb{C}$.*

To prove this result we recall

Proposition 1.2 ([1]) *The Cauchy problem for $P(x, D) = p(x, D) + \sum_{j=0}^{1} b_j D_j$ is not locally solvable in Gevrey class of order $s > 5$ for any b_0, $b_1 \in \mathbb{C}$.*

Thus in order to prove Theorem 1.1 we may assume that $b_2 \neq 0$. Moreover, making a change of coordinates; $x_2 \to -x_2$ if necessary, we may assume that $b_2 \in \mathbb{C} \backslash \mathbb{R}^+$. In Section 2, following [2], [4] we construct an asymptotic solution U_λ to $PU_\lambda = 0$ which contradicts the a priori estimate, derived in Section 4, when $\lambda \to \infty$ and hence finally proves Theorem 1.1.

The same reasoning can be applied to the operator

$$p(x, D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 \tag{1.3}$$

to get

Proposition 1.3 *Assume that* $b_2 \neq 0$. *Then the Cauchy problem for* $p(x, D) + \sum_{j=0}^{2} b_j D_j$ *is not locally solvable at the origin in Gevrey class of order* $s > 4$.

Changing $x_1 D_2$ by D_1 in (1.3) we turn to consider

$$p(x, D) = -D_0^2 + 2D_0 D_1 + x_1^2 D_2^2 \tag{1.4}$$

where the double characteristic manifold is again Σ and $F_p(\rho)$ verifies (1.2) while there is no null bicharacteristic landing on Σ. We study the Cauchy problem for

$$P(x, D) = p(x, D) + S D_2, \quad S \in \mathbb{C}.$$

Then we have

Proposition 1.4 ([3]) *For any* $S \in \mathbb{C}$ *the Cauchy problem is Gevrey* s *well-posed for any* $1 \leq s < 4$.

Note that this result was examined in [3] using the explicit formulas of the fundamental solution of the Cauchy problem for P. Our proof given in Section 6 is based on energy estimates and is available for proving Gevrey 4 well-posedness for noneffectively hyperbolic operators generalizing P. Remark that it was also examined in [3] that if $S \neq 0$, then the Cauchy problem is not Gevrey s well-posed for $s > 4$.

2 Asymptotic solution

Let us consider

$$P = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 + \sum_{j=0}^{2} b_j D_j, \quad b_j \in \mathbb{C}.$$

Make a change of variables;

$$x_0 = \lambda^{-1} y_0, \;\; x_1 = \lambda^{-2} y_1, \;\; x_2 = \lambda^{-4} y_2$$

so that we have

$$P_\lambda = -\lambda^{-2} D_0^2 + 2\lambda^{-1} y_1 D_0 D_2 + D_1^2 + \lambda^{-2} y_1^3 D_2^2 + b_2 D_2 + \lambda^{-2} b_1 D_1 + \lambda^{-3} b_0 D_0.$$

We switch the notation to x and set $b_2 = b$ so that we study

$$P_\lambda = -\lambda^{-2} D_0^2 + 2\lambda^{-1} x_1 D_0 D_2 + D_1^2 + \lambda^{-2} x_1^3 D_2^2 + b D_2 + \lambda^{-2} b_1 D_1 + \lambda^{-3} b_0 D_0.$$

Let us denote

$$E_\lambda = \exp\left(i\lambda^2 x_2 + i\lambda \varphi(x)\right)$$

and compute $\lambda^{-1}E_\lambda^{-1}P_\lambda E_\lambda$ which yields

$$\lambda^{-1}E_\lambda^{-1}P_\lambda E_\lambda = \lambda\big\{2x_1\varphi_{x_0} + \varphi_{x_1}^2 + x_1^3 + b\big\}$$
$$+\big\{2x_1D_0 + 2\varphi_{x_1}D_1 + 2x_1\varphi_{x_0}\varphi_{x_2} + b\varphi_{x_2} + 2x_1^3\varphi_{x_1} - i\varphi_{x_1x_1}\big\}$$
$$+\lambda^{-1}h^{(1)}(x,D) + \lambda^{-2}h^{(2)}(x,D) + \lambda^{-3}h^{(3)}(x,D)$$

where $h^{(i)}(x,D)$ are differential operators of order 2. We first assume that

$$\mathsf{Im} b \neq 0.$$

Take y_1 small so that

$$\mathsf{Im}\frac{b}{2y_1} > 0$$

and work near the point $(x_0, x_1, x_2) = (t, y_1, 0) = x^*$. We solve the equation

$$2x_1\varphi_{x_0} + \varphi_{x_1}^2 + x_1^3 + b = 0 \tag{2.1}$$

imposing the condition

$$\varphi = (x_1 - y_1) + i(x_1 - y_1)^2 + ix_2^2 \quad \text{on} \quad x_0 = t.$$

Noticing $\varphi = (x_1-y_1)+i(x_1-y_1)^2+ix_2^2+\varphi_{x_0}(t,x_1,x_2)(x_0-t)+O((x_0-t)^2)$ we conclude

$$\mathsf{Im}\varphi = (x_1 - y_1)^2 + x_2^2 + \big\{\mathsf{Im}\varphi_{x_0}(t, y_1, 0) + R(x)\big\}(x_0 - t)$$

where $R(x) = O(|x - x^*|)$. Note that

$$\varphi_{x_0}(x^*) = \frac{-1-b}{2y_1} - \frac{y_1^2}{2}$$

and hence $\mathsf{Im}\varphi_{x_0}(x^*) < 0$. Writing $\alpha = \mathsf{Im}\varphi_{x_0}(x^*)$ we have

$$\begin{aligned}
\mathsf{Im}\varphi &= (x_1 - y_1)^2 + x_2^2 + \alpha(x_0 - t) + \frac{1}{2}(\varepsilon^{-1}(x_0 - t)\\
&\quad + \varepsilon R(x))^2 - \frac{\varepsilon^{-2}}{2}(x_0 - t)^2 - \frac{\varepsilon^2}{2}R(x)^2\\
&= (x_1 - y_1)^2 + x_2^2 + (x_0 - t)^2 - \frac{\varepsilon^2}{2}R(x)^2\\
&\quad + \big\{\alpha - (\frac{\varepsilon^{-2}}{2} + 1)(x_0 - t)\big\}(x_0 - t) + \frac{1}{2}(\varepsilon^{-1}(x_0 - t) + \varepsilon R(x))^2\\
&= |x - x^*|^2 - \frac{\varepsilon^2}{2}R(x)^2 + \frac{1}{2}(\varepsilon^{-1}(x_0 - t) + \varepsilon R(x))^2\\
&\quad + \big\{\alpha - (\frac{\varepsilon^{-2}}{2} + 1)(x_0 - t)\big\}(x_0 - t).
\end{aligned}$$

Thus $-\mathsf{Im}\varphi$ attains its strict maximum at x^* in the set $\{x; |x-x^*| < \delta, x_0 \leq t\}$ if $\delta > 0$ is small enough. Let L be a compact set in $\mathbb{R}^3$. For $t \in \mathbb{R}$ let us denote $L_-^t = \{x \in L \mid x_0 \leq t\}$ and $L_t^+ = \{x \in L \mid x_0 \geq t\}$. Then we have

Lemma 2.1 *Let K be a small compact neighborhood of x^*. Then we have*

$$\sup_{x\in K_-^{t+\tau}} \{-\mathsf{Im}\varphi(x)\} \le 2|\alpha|\tau$$

for any small $\tau > 0$. Let $\delta > 0$ be small. Then there exist $c(\delta) > 0$ and $\tau(\delta) > 0$ such that

$$\sup_{x\in K_-^{t+\tau}\cap\{|x-x^*|\ge\delta\}} \{-\mathsf{Im}\varphi(x)\} \le -c(\delta)$$

for any $\tau \le \tau(\delta)$.

Let us denote

$$\lambda^{-1}P_\lambda E_\lambda = E_\lambda Q_\lambda, \quad Q_\lambda = Q_0(x,D) + Q_1(x,\lambda,D)$$

where

$$\begin{cases} Q_0(x,D) = 2x_1D_0 + 2\varphi_{x_1}D_1 + 2x_1\varphi_{x_0}\varphi_{x_2} + b\varphi_{x_2} + 2x_1^3\varphi_{x_1} - i\varphi_{x_1x_1}, \\ Q_1(x,\lambda,D) = \lambda^{-1}h^{(1)}(y,D) + \lambda^{-2}h^{(2)}(x,D) + \lambda^{-3}h^{(3)}(x,D). \end{cases}$$

Let us set $V_\lambda = \sum_{n=0}^N v_\lambda^{(n)}$ and determine $v_\lambda^{(n)}$ by solving the Cauchy problem

$$\begin{cases} Q_0(x,D)v_\lambda^{(n)} = -g_\lambda^{(n)} = -Q_1v_\lambda^{(n-1)}, \\ v_\lambda^{(0)}(t,x_1,x_2) = 1, \\ v_\lambda^{(n)}(t,x_1,x_2) = 0, \quad n \ge 1 \end{cases}$$

where $v_\lambda^{(-1)} = 0$ so that $Q_\lambda V_\lambda = Q_1(x,\lambda,D)v_\lambda^{(N)}$. Hence

$$\lambda^{-1}P_\lambda E_\lambda V_\lambda = E_\lambda Q_1(x,\lambda,D)v_\lambda^{(N)}. \tag{2.2}$$

We turn to the case

$$b \in \mathbb{R}, \quad b < 0.$$

We write $b = -\gamma^2$, $\gamma > 0$. We solve the equation (2.1) under the condition

$$\varphi = -i(x_0 - t) + ix_2^2 \quad \text{on} \;\; x_1 = 0.$$

That is, one solves the equation $\varphi_{x_1} = \sqrt{\gamma^2 - x_1^3 - 2x_1\varphi_{x_0}}$. It is clear that

$$\varphi_{x_1} = \big(\gamma + i\frac{x_1}{\gamma}\big) + O(x_1^2).$$

One can write

$$\varphi = -i(x_0 - t) + ix_2^2 + \big(\gamma + i\frac{x_1}{\gamma}\big)x_1 + R(x)$$

where $R(x) = O(x_1^3)$. Note that

$$\begin{aligned}\mathsf{Im}\varphi &= -(x_0 - t) + x_2^2 + \gamma^{-1}x_1^2 + R(x)\\ &= (x_0 - t)^2 + \gamma^{-1}x_1^2 + x_2^2 + R(x) + \big\{-1 - (x_0 - t)\big\}(x_0 - t)\end{aligned}$$

and hence the same assertion as Lemma 2.1 holds. Noting that φ_{x_1} is different from zero in an open neighborhood of $x^* = (t, 0, 0)$ we can solve the transport equation in the x_1 direction. The rest of the proof is just a repetition.

3 Lemmas

To estimate $E_\lambda V_\lambda$ constructed in the previous section we apply the methods of majorant following [4]. Consider $Qv = \sum_{|\alpha|\le 1} b_\alpha D^\alpha$ where we assume that the coefficient of D_0 is different from zero near $x = x^*$.

Lemma 3.1 *Let $Qv = g$ and let*

$$\Phi(\tau, \eta; v) = \sum_\alpha \frac{\tau^{\alpha_0}\eta^{|\alpha'|}}{\alpha!}|D^\alpha v(x^*)|.$$

Then we have

$$\frac{\partial}{\partial\tau}\Phi(\tau, \eta; v) \ll C(\tau, \eta)\frac{\partial}{\partial\eta}\Phi(\tau, \eta; v) + C(\tau, \eta)\Phi(\tau, \eta; g)$$

with some holomorphic $C(\tau, \eta)$ at $(0, 0)$ with $C(\tau, \eta) \gg 0$ which depends only on Q.

Proof. Note that

$$\frac{\partial}{\partial\tau}\Phi(\tau, \eta; v) = \sum_\beta \frac{\tau^{\beta_0}\eta^{|\beta'|}}{\beta!}|D^\beta(D_0 v)(x^*)| = \Phi(\tau, \eta; D_0 v).$$

On the other hand, from $Qv = g$ one sees $D_0 v = \sum_{j=1}^n c_j D_j v + c_0 v$. Since $\Phi(\tau, \eta; fg) \ll \Phi(\tau, \eta; f)\Phi(\tau, \eta; g)$ and hence

$$\frac{\partial}{\partial\tau}\Phi(\tau, \eta; v) \ll C(\tau, \eta)\big(\sum_{j=1}^n \Phi(\tau, \eta; D_j v) + \Phi(\tau, \eta; g)\big).$$

To conclude the assertion it is enough to note

$$\frac{\partial\Phi}{\partial\eta} \gg \sum_{\alpha_j\ge 1} \frac{|\alpha'|\tau^{\alpha_0}\eta^{|\alpha'|-1}}{\alpha!}|D^{\tilde\alpha}(D_j v)(x^*)|,$$

$$\frac{|\alpha'|\tau^{\alpha_0}\eta^{|\alpha'|-1}}{\alpha!} = \frac{|\alpha'|\tau^{\tilde\alpha_0}\eta^{|\tilde\alpha'|}}{\alpha_j\tilde\alpha!} \ge \frac{\tau^{\tilde\alpha_0}\eta^{|\tilde\alpha'|}}{\tilde\alpha!}.$$

Lemma 3.2 *Assume $Qv = g$ and*

$$\begin{cases} \dfrac{\partial}{\partial \tau}\Phi^*(\tau,\eta) \gg C(\tau,\eta)\dfrac{\partial}{\partial \eta}\Phi^*(\tau,\eta) + C(\tau,\eta)\Phi(\tau,\eta;g), \\ \Phi^*(0,\eta) \gg \Phi(0,\eta;v). \end{cases}$$

Then we have

$$\Phi(\tau,\eta;v) \ll \Phi^*(\tau,\eta).$$

Proof. Let $\tilde{\Phi}$ be a solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial \tau}\tilde{\Phi}(\tau,\eta) = C(\tau,\eta)\dfrac{\partial}{\partial \eta}\tilde{\Phi}(\tau,\eta) + C(\tau,\eta)\Phi(\tau,\eta;g), \\ \tilde{\Phi}(0,\eta) = \Phi^*(0,\eta). \end{cases}$$

Then it is clear that $\Phi(\tau,\eta;v) \ll \tilde{\Phi}(\tau,\eta) \ll \Phi^*(\tau,\eta)$.

Lemma 3.3 *Assume $0 < a \le ka_1$ and $0 < b \le kb_1$ with some $0 < k < 1$. Then we have*

(i) $\Big(1 - \dfrac{\eta}{b} - \dfrac{\tau}{a}\Big)^{-1}\Big(1 - \dfrac{\eta}{b_1} - \dfrac{\tau}{a_1}\Big)^{-1} \ll (1-k)^{-1}\Big(1 - \dfrac{\eta}{b} - \dfrac{\tau}{a}\Big)^{-1}$,

(ii) $\Big(1 - \dfrac{\eta}{b}\Big)^{-1}\Big(1 - \dfrac{\tau}{a}\Big)^{-1} \ll \Big(1 - \dfrac{\eta}{b} - \dfrac{\tau}{a}\Big)^{-1}$.

Proof. The assertion (i) follows from

$$\begin{aligned} &\Big\{\sum\big(\frac{\eta}{b} + \frac{\tau}{a}\big)^n\Big\}\Big\{\sum\big(\frac{\eta}{b_1} + \frac{\tau}{a_1}\big)^n\Big\} \\ &= \sum_{n,m}\big(\frac{\eta}{b} + \frac{\tau}{a}\big)^n\big(\frac{\eta}{b_1} + \frac{\tau}{a_1}\big)^m \ll \sum_{n,m} k^m\big(\frac{\eta}{b} + \frac{\tau}{a}\big)^{n+m} \\ &\ll \sum_m k^m \sum_n \big(\frac{\eta}{b} + \frac{\tau}{a}\big)^n. \end{aligned}$$

Here we recall that if $\varphi(\tau,\eta)$ is holomorphic in a neighborhood of $\{(\tau,\eta) \mid |\eta| \le b, |\tau| \le a\}$, then we have

$$\varphi(\tau,\eta) \ll (1 - \frac{\tau}{a})^{-1}(1 - \frac{\eta}{b})^{-1} \sup_{|\tau|=a,|\eta|=b} |\varphi(\tau,\eta)|$$

from the Cauchy's integral formula. Assume that

$$C(\tau,\eta) \ll (1 - \frac{\tau}{a_1})^{-1}(1 - \frac{\eta}{b_1})^{-1}B \ll \big(1 - \frac{\tau}{a_1} - \frac{\eta}{b_1}\big)^{-1}B.$$

Lemma 3.4 *Assume that $Qv = g$ and*

$$\Phi(0,\eta;v) \ll \omega^{-1}\big(1-\frac{\eta}{b}\big)^{-n},\ \ \Phi(\tau,\eta;g) \ll L\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-n}e^{M\tau\omega}.$$

We also assume that $Ba/b \le (1-k)$ and $B \le (1-k)M$. Then we have

$$\Phi(\tau,\eta;v) \ll L\omega^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-n}e^{M\tau\omega}.$$

Proof. Let us denote $(L \ge 1)$

$$\Phi^* = L\omega^{-1}\left(1-\frac{\tau}{a}-\frac{\eta}{b}\right)^{-n}e^{M\tau\omega}.$$

It is easy to see by Lemma 3.3 that

$$\frac{\partial\Phi^*}{\partial\tau} \gg C(\tau,\eta)\frac{\partial\Phi^*}{\partial\eta}+C(\tau,\eta)\Phi(\tau,\eta;g).$$

Then the assertion follows from Lemma 3.2.

Let us denote

$$\Phi^n_\lambda = \Phi(\tau,\eta;v^{(n)}_\lambda)$$

and hence $\Phi^n_\lambda(0,\eta)=0$ for $n\ge 1$ and $\Phi^0_\lambda(0,\eta)=1$. We assume that

$$\Phi^n_\lambda(\tau,\eta) \ll A^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!\left(1-\frac{\tau}{a}-\frac{\eta}{b}\right)^{-k-1}e^{M\tau\omega}. \tag{3.1}$$

For $n=0$ this holds clearly. Suppose that (3.1) holds for $\le n-1$. Let

$$g = \left(\sum_{j=1}^{3}\lambda^{-j}h^{(j)}(x,D)\right)v^{(n-1)}_\lambda = Q_1(x,\lambda,D)v^{(n-1)}_\lambda$$

and we first show that

$$\Phi(\tau,\eta;g) \ll A^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}.$$

As for terms $c(x)D^\alpha u$ with $|\alpha|\le 2$ we have

$$\begin{aligned}\Phi(\tau,\eta;cD^\alpha u) &\ll C\big(1-\frac{\tau}{a_1}-\frac{\eta}{b_1}\big)^{-1}\Phi(\tau,\eta;D^\alpha u)\\ &\ll C\big(1-\frac{\tau}{a_1}-\frac{\eta}{b_1}\big)^{-1}\Big[\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\tau\partial\eta}+\frac{\partial^2}{\partial\eta^2}\Big]\Phi(\tau,\eta;u).\end{aligned}$$

We now estimate

$$\Big[\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial\tau\partial\eta}+\frac{\partial^2}{\partial\eta^2}\Big]\sum_{k=0}^{2(n-1)}\omega^{n-1-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}$$

which is bounded by

$$\begin{aligned}\sum_{k=0}^{2(n-1)}\big(&M^2\omega^{n+1-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}\\ &+2M\omega^{n-k}(k+1)!a^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-2}\\ &+\omega^{n-1-k}(k+2)!a^{-2}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-3}\\ &+M\omega^{n-k}(k+1)!b^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-2}\\ &+\omega^{n-1-k}(k+2)!a^{-1}b^{-1}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-3}\\ &+\omega^{n-1-k}(k+2)!b^{-2}\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-3}\big)\\ &\ll\omega\big\{M^2+2Ma^{-1}+a^{-2}+Mb^{-1}+a^{-1}b^{-1}+b^{-2}\big\}\\ &\cdot\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}\end{aligned}$$

up to the factor $A^n\lambda^{-n+1}e^{M\tau\omega}$. Taking A so that

$$A\ge\big\{M^2+2Ma^{-1}+a^{-2}+Mb^{-1}+a^{-1}b^{-1}+b^{-2}\big\}$$

we conclude that

$$\Phi(\tau,\eta;g)\ll A^{n+1}\lambda^{-n}\omega\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}.$$

Recalling that $\Phi^n_\lambda(0,\eta)=0\ll\omega^{-1}\big(1-\frac{\eta}{b}\big)^{-1}$, $n\ge 1$ for any ω and applying Lemma 3.4 we see

Lemma 3.5 *We have*

$$\Phi^n_\lambda(\tau,\eta)\ll A^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\tau}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\tau\omega}$$

for any $\omega\ge 1$.

Lemma 3.6 *There are* $h>0$ *and* $\delta>0$ *such that*

$$\sum_\alpha\frac{h^{|\alpha|}}{\alpha!}\sup_{|x-x^*|\le\delta}|D^\alpha v^{(n)}_\lambda(x)|\le B^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!e^{M_1\omega}.$$

Proof. Note that

$$\sum_{\alpha}\frac{\eta^{|\alpha|}}{\alpha!}|D^{\alpha}v_{\lambda}^{(n)}(x^*)|\leq A^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!\big(1-\frac{\eta}{a}-\frac{\eta}{b}\big)^{-k-1}e^{M\eta\omega}$$

and hence for $0<\eta\leq\eta_0$ we have

$$\sum_{\alpha}\frac{\eta^{|\alpha|}}{\alpha!}|D^{\alpha}v_{\lambda}^{(n)}(x^*)|\leq B^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!e^{M\eta_0\omega}.$$

This shows that

$$|v_{\lambda}^{(n)}(x)|\leq\sum_{\alpha}\frac{|D^{\alpha}v_{\lambda}^{(n)}(x^*)|}{\alpha!}|(x-x^*)^{\alpha}|\leq B^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!e^{M_1\omega}$$

for $|x-x^*|\leq\eta_0$. From the Cauchy's inequality it follows that

$$\sup_{|x-x^*|\leq\eta_0/2}|D^{\alpha}v_{\lambda}^{(n)}(x)|\leq(\eta_0/2)^{-|\alpha|}\alpha!B^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!e^{M_1\omega}$$

and hence we have

$$\sum_{\alpha}\frac{h^{|\alpha|}}{\alpha!}\sup_{|x-x^*|\leq\delta}|D^{\alpha}v_{\lambda}^{(n)}(x)|\leq B^{n+1}\lambda^{-n}\sum_{k=0}^{2n}\omega^{n-k}k!e^{M_1\omega}$$

for $2h<\eta_0$ and $2\delta<\eta_0$ with a possibly different B.

Let us define

$$V_{\lambda}(x)=\sum_{n=0}^{N}v_{\lambda}^{(n)}(x)$$

where N and ω are chosen so that

$$\omega=4N,\quad\lambda=\omega Be^{L}$$

where L will be determined later. Then we have for $n\leq N$

$$\sum_{k=0}^{2n}\omega^{n-k}k!e^{M_1\omega}\leq\omega^{n}e^{M_1\omega}\sum_{k=0}^{2n}\Big(\frac{k}{\omega}\Big)^{k}\leq\omega^{n}e^{M_1\omega}\sum_{k=0}^{2n}\Big(\frac{1}{2}\Big)^{k}$$

and hence

$$\begin{aligned}\sum_{\alpha}\frac{h^{|\alpha|}}{\alpha!}\sup_{|x-x^*|\leq\delta}|D^{\alpha}v_{\lambda}^{(n)}(x)|&\leq B^{n+1}\lambda^{-n}\omega^{n}e^{M_1\omega}\\&\leq B^{n+1}(B^{-1}e^{-L})^{n}e^{M_1\omega}=Be^{-Ln+M_1\omega}.\end{aligned}$$

In particular one has

$$\sum_\alpha \frac{h^{|\alpha|}}{\alpha!} \sup_{|x-x^*|\le\delta} |D^\alpha v_\lambda^{(N)}(x)| \le Be^{-LN+4M_1N} = Be^{-e^{-L}(L-4M_1)\lambda/4B}.$$

On the other hand, we see

$$\sum_\alpha \frac{h^{|\alpha|}}{\alpha!} \sup_{|x-x^*|\le\delta} |D^\alpha V_\lambda(x)| \le \sum_{n=0}^{N} B^{n+1}\lambda^{-n}\omega^n e^{M_1\omega}$$
$$= e^{M_1\omega} B \sum_{n=0}^{N} \Big(\frac{B\omega}{\lambda}\Big)^n \le e^{M_1\omega}B = Be^{4M_1N} = Be^{e^{-L}M_1\lambda/B}.$$

4 A priori estimate

In this section assuming that the Cauchy problem for $P(x,D)$ is Gevrey s well-posed we derive a priori estimate following [6], [4]. Let L be a compact set in $\mathbb{R}^3$. Then let us denote by $\gamma_0^{(s)}(L)$ the set of all $f(x) \in C_0^\infty(L)$ such that

$$|\partial_x^\alpha f(x)| \le CA^{|\alpha|}(\alpha!)^s, \quad \alpha \in \mathbb{N}^3 \tag{4.1}$$

with some $C>0$, $A>0$. Let $h>0$ be given. We denote by $\gamma_0^{(s),h}(L)$ the set of all $f(x) \in \gamma_0^{(s)}(L)$ verifying (4.1) with $A = h^{-1}$ and some $C>0$. Note that $\gamma_0^{(s),h}(L)$ is a Banach space equipped with the norm

$$\sup_{x,\alpha} \frac{h^{|\alpha|}|\partial_x^\alpha f(x)|}{(\alpha!)^s}.$$

Consider

$$P_\lambda = P(\lambda^{-\sigma}x, \lambda^\sigma \xi)$$

where $\lambda^{-\sigma}x = (\lambda^{-\sigma_0}x_0, \lambda^{-\sigma_1}x_1, \lambda^{-\sigma_2}x_2)$ and $\sigma_j \ge 0$. Then we have:

Lemma 4.1 *Assume that the Cauchy problem for P is Gevrey s well-posed near the origin. Let W be a compact neighborhood of the origin. Then there are $c>0$, $C>0$ such that*

$$|u|_{C^0(W_-^t)} \le C\exp\big(c(\lambda^{\sigma_0}/\tau)^{1/(s-\kappa)}\big)\exp\big(\lambda^{\bar\sigma/s'}\big)\sum_\alpha \sup_{x_0\le t+\tau} \frac{h^{|\alpha|}|\partial_x^\alpha P_\lambda u|}{(\alpha!)^{(s-s')}}$$

for any $u \in \gamma_0^{(s),h}(W_0^+)$, any $t>0$, $\tau>0$, any $1<s'<s$, any $1<\kappa<s$ where $\bar\sigma = \max_j\{\sigma_j\}$.

Proof. Assume that the Cauchy problem for P is Gevrey s well-posed. Let $h > 0$ and K be a compact neighborhood of the origin. From the standard arguments it follows that there is a neighborhood of the origin D such that for any $f(x) \in \gamma_0^{(s),h}(K_0^+)$ there is a $u \in C^2(D)$ satisfying $Pu = f$ in D and $u = 0$ in $x_0 \le 0$ such that

$$|u|_{C^0(L)} \le C \sum_\alpha \sup \frac{h^{|\alpha|}|\partial_x^\alpha f(x)|}{(\alpha!)^s}$$

where $L \subset D$ is a fixed compact set. We may assume that $K \subset D$ (see for example [7]). Thus we have

$$|u|_{C^0(L)} \le C \sum_\alpha \sup \frac{h^{|\alpha|}|\partial_x^\alpha Pu|}{(\alpha!)^s}, \quad \forall u(x) \in \gamma_0^{(s),h}(K_0^+).$$

Let $\chi(r) \in \gamma^{(\kappa)}(\mathbb{R})$, $\kappa < s$, such that $\chi(r) = 1$ for $r \le 0$, $\chi(r) = 0$ for $r \ge 1$ and set $\chi_1(x_0) = \chi((x_0 - t)/\tau)$ so that

$$\begin{cases} \chi_1(x_0) = 1 & x_0 \le t, \\ \chi_1(x_0) = 0 & x_0 \ge t + \tau. \end{cases}$$

Let $u \in \gamma_0^{(s),h}(K_0^+)$ and consider $\chi_1 Pu$. Let $v \in C^2(D)$ be a solution to $Pv = \chi_1 Pu$ with $v = 0$ for $x_0 \le 0$. Since $Pv = Pu$ for $x_0 \le t$ and hence

$$|u|_{C^0(L_-^t)} = |v|_{C^0(L_-^t)} \le C \sum_\alpha \sup \frac{h^{|\alpha|}|\partial_x^\alpha(\chi_1 Pu)|}{(\alpha!)^s}.$$

Recall that $|\partial_x^\beta \chi_1(x)| \le C^{|\beta|+1}(\beta!)^\kappa \tau^{-|\beta|}$ and hence

$$\begin{aligned}
\sum_\alpha \sup \frac{h^{|\alpha|}|\partial_x^\alpha(\chi_1 Pu)|}{(\alpha!)^s} &\le \sum \sup \frac{\alpha!}{\alpha_1!\alpha_2!} \frac{h^{|\alpha|}|\partial_x^{\alpha_1}\chi_1||\partial_x^{\alpha_2}Pu|}{(\alpha!)^s} \\
&\le \sum \sup \frac{1}{\alpha_1!\alpha_2!} \frac{h^{|\alpha|}|\partial_x^{\alpha_1}\chi_1||\partial_x^{\alpha_2}Pu|}{(\alpha_1!)^{s-1}(\alpha_2!)^{s-1}} \\
&\le \sum_{\alpha_1} \sup \frac{h^{|\alpha_1|}|\partial_x^{\alpha_1}\chi_1|}{(\alpha_1!)^s} \sum_{\alpha_2} \sup_{x_0 \le t+\tau} \frac{h^{|\alpha_2|}|\partial_x^{\alpha_2}Pu|}{(\alpha_2!)^s}.
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{\alpha_1} \sup \frac{h^{|\alpha_1|}|\partial_x^{\alpha_1}\chi_1|}{(\alpha_1!)^s} &\le \sum_{\alpha_1} \frac{C^{|\alpha_1|+1}\tau^{-|\alpha_1|}h^{|\alpha_1|}}{(\alpha_1!)^{s-\kappa}} \\
&\le C \exp\Big(c\big(\frac{1}{\tau}\big)^{1/(s-\kappa)}\Big) \sum_{\alpha_1}(Ch)^{|\alpha_1|} \\
&\le C_h \exp\Big(c\big(\frac{1}{\tau}\big)^{1/(s-\kappa)}\Big)
\end{aligned}$$

we have

$$|u|_{C^0(L^t_-)} \le C_h \exp\Big(c\big(\frac{1}{\tau}\big)^{1/(s-\kappa)}\Big)\sum_{\alpha}\sup_{x_0\le t+\tau}\frac{h^{|\alpha|}|\partial_x^\alpha Pu|}{(\alpha!)^s}. \qquad (4.2)$$

Let $u \in \gamma_0^{(s),h}(W_0^+)$. Then it is clear that $u(\lambda^\sigma x) \in \gamma_0^{(s),h}(K_0^+)$ for large λ. For $v(x) = u(\lambda^\sigma x)$ we apply the inequality (4.2) with $t = \lambda^{-\sigma_0}\hat{t}$, $\tau = \lambda^{-\sigma_0}\hat{\tau}$ to get

$$|v|_{C^0(L^t_-)} \le C_h \exp\Big(c\big(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\big)^{1/(s-\kappa)}\Big)\sum_{\alpha}\sup_{x_0\le t+\tau}\frac{h^{|\alpha|}|\partial_x^\alpha Pv|}{(\alpha!)^s}$$

where $Pv = Pu(\lambda^\sigma x) = (P_\lambda u)(\lambda^\sigma x)$ and hence

$$\partial^\alpha\big[(P_\lambda u)(\lambda^\sigma u)\big] = \lambda^{\langle\sigma,\alpha\rangle}(\partial_x^\alpha P_\lambda u)(\lambda^\sigma x).$$

Thus we have

$$\begin{aligned}
|u|_{C^0(W^{\hat{t}}_-)} &\le C_h e^{c\left(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\right)^{1/(s-\kappa)}}\sum_{\alpha}\sup_{x_0\le\hat{t}+\hat{\tau}}\frac{h^{|\alpha|}\lambda^{\bar{\sigma}|\alpha|}|\partial_x^\alpha(P_\lambda u)(x)|}{(\alpha!)^s}\\
&= C_h e^{c\left(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\right)^{1/(s-\kappa)}}\sum_{\alpha}\sup_{x_0\le\hat{t}+\hat{\tau}}\frac{h^{|\alpha|}\lambda^{\bar{\sigma}|\alpha|}|\partial_x^\alpha(P_\lambda u)(x)|}{(\alpha!)^{s'}(\alpha!)^{s-s'}}\\
&\le C_h e^{c\left(\frac{\lambda^{\sigma_0}}{\hat{\tau}}\right)^{1/(s-\kappa)}} e^{c\lambda^{\bar{\sigma}/s'}}\sum_{\alpha}\sup_{x_0\le\hat{t}+\hat{\tau}}\frac{h^{|\alpha|}|\partial_x^\alpha(P_\lambda u)(x)|}{(\alpha!)^{s-s'}}.
\end{aligned}$$

This proves the assertion.

5 Proof of Theorem 1.1

Take $\chi(x) \in \gamma_0^{(\kappa)}(W_0^+)$ such that $\chi(x) = 1$ in a neighborhood of x^* supported in $\{|x - x^*| \le \delta\}$ and $1 < \kappa < s$. Let us set $U_\lambda = E_\lambda V_\lambda\chi \in \gamma_0^{(s),h}(W_0^+)$ and note $|U_\lambda(x^*)| = 1$. Then we have from (2.2)

$$\begin{aligned}
P_\lambda U_\lambda &= (P_\lambda E_\lambda V_\lambda)\chi + \sum_{|\alpha|\le 1, 1\le|\beta|\le 2} c_{\alpha,\beta}(x,\lambda)\partial_x^\alpha(E_\lambda V_\lambda)\partial_x^\beta\chi\\
&= E_\lambda Q_1 v_\lambda^{(N)}\chi + \sum_{|\alpha|\le 1, 1\le|\beta|\le 2} c_{\alpha,\beta}(x,\lambda)\partial_x^\alpha(E_\lambda V_\lambda)\partial_x^\beta\chi.
\end{aligned}$$

To estimate the right-hand side we note

Lemma 5.1 *Let* $a = \sup_{x\in K}\{-\mathsf{Im}\varphi(x)\}$. *Then we have*

$$\sum_{\alpha}\sup_{K}\frac{h^{|\alpha|}|\partial_x^\alpha E_\lambda|}{(\alpha!)^s} \le C_h \exp(\lambda^{2/s} + a\lambda).$$

Proof. Recall that $E_\lambda = \exp(i\lambda^2 x_2 + i\lambda\varphi(x))$. Since $\varphi(x)$ is real analytic in a neighborhood K of x^*, it is not difficult to check that

$$|\partial_x^\alpha e^{i\lambda\varphi(x)}| \le C^{|\alpha|+1}(\lambda+|\alpha|)^{|\alpha|}e^{-\lambda\mathsf{Im}\varphi(x)}, \quad x \in K$$

and hence we have

$$\sum_\alpha \sup_K \frac{h^{|\alpha|}|\partial_x^\alpha e^{i\lambda\varphi(x)}|}{(\alpha!)^s} \le C_h e^{c\lambda^{1/s}+a\lambda}.$$

This proves the assertion.

Recall that $-\mathsf{Im}\varphi(x) \le -\delta$ if $x \in \text{supp}[\partial_x^\beta\chi] \cap \{x_0 \le t+\tau\}$, $0 < \tau \le \tau_0$. Then it follows that

$$\sum_\gamma \sup_{x_0 \le t+\tau} \frac{h^{|\gamma|}|\partial_x^\gamma(\partial_x^\alpha(E_\lambda V_\lambda)\partial_x^\beta\chi)|}{(\gamma!)^s} \le C_h \exp(c\lambda^{2/s} - \delta\lambda + e^{-L}M_1B^{-1}\lambda).$$

Assuming $s > 2$ and take L large so that $e^{-L}M_1B^{-1} < \delta$ we see that the left-hand side is bounded by $e^{-\delta'\lambda}$. We turn to $E_\lambda Q_1 v_\lambda^{(N)}\chi$. Recalling that $-\mathsf{Im}\varphi(x) \le 2a\tau$ if $x \in \text{supp}[\chi] \cap \{x_0 \le t+\tau\}$ we see

$$\sum_\alpha \sup_{x_0 \le t+\tau} \frac{h^{|\alpha|}|\partial_x^\alpha(E_\lambda Q_1 v_\lambda^{(N)}\chi)|}{(\alpha!)^s}$$
$$\le C_h \exp(c\lambda^{2/s} + 2a\tau\lambda - e^{-L}(L-4M_1)(4B)^{-1}\lambda).$$

Take $L > 4M_1$ and choose $\tau > 0$ so that

$$2a\tau - e^{-L}(L-4M_1)(4B)^{-1} < 0,$$

then the right-hand side is bounded by $e^{-\delta''\lambda}$. Let

$$s > 6.$$

Then we can choose $s' > 4$ such that $s - s' > 2$ and hence

$$\sum_\alpha \sup_{x_0 \le t+\tau} \frac{h^{|\alpha|}|\partial_x^\alpha(P_\lambda U_\lambda)|}{(\alpha!)^{s-s'}} \le C_h e^{-\delta_1\lambda}$$

with some $\delta_1 > 0$. Since $\bar\sigma = 4$ and $\sigma_0 = 1$, taking $1 < \kappa$ small we have $\bar\sigma/s' < 1$ and $\sigma_0/(s-\kappa) < 1$ and hence we conclude that

$$|U_\lambda|_{C^0(W_-^t)} \le C_h e^{-c\lambda+o(\lambda)}$$

as $\lambda \to \infty$. This gives a contradiction because

$$|U_\lambda(x^*)| = 1.$$

6 Proof of Propositions 1.3 and 1.4

We first give a sketch of the proof of Proposition 1.3. Let us make a change of variables:

$$x_0 = y_0,\ \ x_1 = \lambda^{-1}y_1,\ \ x_2 = \lambda^{-2}y_2.$$

Then we see that

$$P_\lambda = -\lambda^{-2}D_0^2 + 2\lambda^{-1}x_1D_0D_2 + D_1^2 + b_2D_2 + \lambda^{-1}b_1D_1 + \lambda^{-2}b_0D_0$$

and hence with $E_\lambda = \exp\left(i\lambda^2x_2 + i\lambda\varphi(x)\right)$ we have

$$\begin{aligned}\lambda^{-1}E_\lambda^{-1}P_\lambda E_\lambda = \lambda\{&2x_1\varphi_{x_0} + \varphi_{x_1}^2 + b_2\}\\ &+ \{2x_1D_0 + 2\varphi_{x_1}D_1 + 2x_1\varphi_{x_0}\varphi_{x_2} + b_2\varphi_{x_2} - i\varphi_{x_1x_1}\}\\ &+ \lambda^{-1}h^{(1)}(x,D) + \lambda^{-2}h^{(2)}(x,D) + \lambda^{-3}h^{(3)}(x,D).\end{aligned}$$

One can construct an asymptotic solution V_λ in exactly the same way as we did proving Theorem 1.1 and set $U_\lambda = E_\lambda V_\lambda\chi$. Recall Lemma 4.1

$$|U_\lambda|_{C^0(W_-^t)} \le Ce^{c\tau^{-1/(s-\kappa)}}e^{\lambda^{\bar\sigma/s'}}\sum_\alpha \sup_{x_0\le t+\tau}\frac{h^{|\alpha|}|\partial_x^\alpha P_\lambda U_\lambda|}{(\alpha!)^{s-s'}}$$

where we now take $\bar\sigma = 2$. Let

$$s > 4.$$

We can choose $s' > 2$ so that $s - s' > 2$ and hence we conclude that

$$|U_\lambda|_{C^0(W_-^t)} \le C_he^{-c\lambda+o(\lambda)}$$

as $\lambda \to \infty$. This is a contradiction.

We next give a sketch of deriving a priori estimates for P in (1.4). After Fourier transform with respect to x_2 it is enough to study

$$P = -D_0^2 + 2D_0D_1 + x_1^2\lambda^2 + S\lambda;$$

here we have set $\xi_2 = \lambda$ for simplicity of notations. We may assume that $\lambda > 0$. Let us set

$$\varphi = \sqrt{x_1^2 + \lambda^{-1}} - x_1, \quad w(x_1,\lambda) = \sqrt{x_1^2 + \lambda^{-1}}.$$

Note that $|\log\varphi| \le \frac{1}{2}\log\lambda$ for large λ when $|x_1|$ is bounded. We consider the operator

$$\tilde P = e^{-\gamma\lambda^{1/4}(\log\varphi + x_0)}Pe^{\gamma\lambda^{1/4}(\log\varphi + x_0)}.$$

Note that

$$e^{-\gamma\lambda^{1/4}\log\varphi}D_1e^{\gamma\lambda^{1/4}\log\varphi}=D_1-i\gamma\lambda^{1/4}\frac{\partial_{x_1}\varphi}{\varphi}=D_1+i\gamma\lambda^{1/4}w^{-1},$$
$$e^{-\gamma\lambda^{1/4}x_0}D_0e^{\gamma\lambda^{1/4}x_0}=D_0-i\gamma\lambda^{1/4}.$$

Then it is easy to see that

$$\begin{aligned}\tilde{P}&=-(D_0-i\gamma\lambda^{1/4})^2+2(D_1+i\gamma\lambda^{1/4}w^{-1})(D_0-i\gamma\lambda^{1/4})+x_1^2\lambda^2+S\lambda\\&=-\Lambda^2+2B\Lambda+Q+(S-1)\lambda\end{aligned}$$

where

$$\Lambda=D_0-i\gamma\lambda^{1/4},\ \ B=D_1+i\gamma\lambda^{1/4}w^{-1},\ \ Q=x_1^2\lambda^2+\lambda.$$

Denoting by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and the norm in $L^2(\mathbb{R}_{x_1})$, respectively, we apply the energy identity (see for example [1])

$$\begin{aligned}\mathsf{Im}(\tilde{P}u,\Lambda u)&=\frac{d}{dx_0}\big(\|\Lambda u\|^2+(\mathsf{Re}Qu,u)\big)+2((\mathsf{Im}B)\Lambda u,\Lambda u)\\&\quad+2\gamma\lambda^{1/4}(\Lambda u,\Lambda u)+2\gamma\lambda^{1/4}((\mathsf{Re}Q)u,u)+\mathsf{Im}((S-1)\lambda u,\Lambda u)\\&\geq\frac{d}{dx_0}\big(\|\Lambda u\|^2+(Qu,u)\big)+2\gamma\lambda^{1/4}(w^{-1}\Lambda u,\Lambda u)+2\gamma\lambda^{1/4}\|\Lambda u\|^2\\&\quad+2\gamma\lambda^{1/4}(Qu,u)+\lambda\mathsf{Im}((S-1)u,\Lambda u).\end{aligned}$$

It is enough to estimate the last term on the right-hand side. Note that

$$\begin{aligned}2\lambda|\mathsf{Im}((S-1)u,\Lambda u)|&\leq 2\lambda|S-1||(u,\Lambda u)|\\&\leq\gamma^{-1}|S-1|^2|(\lambda^{7/4}wu,u)|+\gamma|(\lambda^{1/4}w^{-1}\Lambda u,\Lambda u)|.\end{aligned}$$

On the other hand, noting $\lambda^2w(x_1,\lambda)^2=Q$, we see

$$|(\lambda^{7/4}wu,u)|=\Big|(\frac{\lambda^{1/4}\lambda^2w^2}{\lambda^{1/2}w}u,u)\Big|\leq(\lambda^{1/4}Qu,u)$$

because $\lambda^{1/2}w\geq 1$ and hence

$$\begin{aligned}\mathsf{Im}(\tilde{P}u,\Lambda u)&\geq\frac{d}{dx_0}\big(\|\Lambda u\|^2+(Qu,u)\big)\\&\quad+\gamma\Big(\lambda^{1/4}\|w^{-1/2}\Lambda u\|^2+2\lambda^{1/4}\|\Lambda u\|^2+\lambda^{1/4}(Qu,u)\Big)\end{aligned}$$

for $\gamma\geq|S-1|$. Since $2|\mathsf{Im}(\tilde{P}u,\Lambda u)|\leq\gamma^{-1}\lambda^{-1/4}\|\tilde{P}u\|^2+\gamma\lambda^{1/4}\|\Lambda u\|^2$ we get

$$\begin{aligned}\gamma^{-1}\int_{-\infty}^t\lambda^{-1/4}\|\tilde{P}u\|^2dx_0\geq\|\Lambda u(t)\|^2+(Qu(t),u(t))&\\+\gamma\int_{-\infty}^t\Big\{\lambda^{1/4}\|w^{-1/2}\Lambda u\|^2&\\+\lambda^{1/4}\|\Lambda u\|^2+\lambda^{1/4}(Qu,u)\Big\}dx_0.&\end{aligned}$$

Replacing $e^{\gamma\lambda^{1/4}(\log\varphi+x_0)}u$ by v we have

Proposition 6.1 *We have*

$$\gamma^{-1}\int_{-\infty}^{t}\lambda^{-1/4}\|e^{-\gamma\lambda^{1/4}(\log\varphi+x_0)}Pv\|^2dx_0$$
$$\geq \|e^{-\gamma\lambda^{1/4}(\log\varphi+t)}D_0v(t)\|^2+(e^{-2\gamma\lambda^{1/4}(\log\varphi+t)}Qv(t),v(t))$$
$$+\gamma\int_{-\infty}^{t}\lambda^{1/4}\big\{\|e^{-\gamma\lambda^{1/4}(\log\varphi+x_0)}D_0v\|^2+(e^{-2\gamma\lambda^{1/4}(\log\varphi+x_0)}Qv,v)\big\}dx_0$$

for any smooth $v(x_0,x_1,\lambda)$ *vanishing for large* $|x_1|$.

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Singularities of the Scattering Kernel Related to Trapping Rays

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Dedicated to Ferruccio Colombini on the occasion of his 60th birthday

Summary. An obstacle $K \subset \mathbb{R}^n$, $n \geq 3$, n odd, is called trapping if there exists at least one generalized bicharacteristic $\gamma(t)$ of the wave equation staying in a neighborhood of K for all $t \geq 0$. We examine the singularities of the scattering kernel $s(t, \theta, \omega)$ defined as the Fourier transform of the scattering amplitude $a(\lambda, \theta, \omega)$ related to the Dirichlet problem for the wave equation in $\Omega = \mathbb{R}^n \setminus K$. We prove that if K is trapping and $\gamma(t)$ is nondegenerate, then there exist reflecting (ω_m, θ_m)-rays δ_m, $m \in \mathbb{N}$, with sojourn times $T_m \to +\infty$ as $m \to \infty$, so that $-T_m \in \text{sing supp}\, s(t, \theta_m, \omega_m)$, $\forall m \in \mathbb{N}$. We apply this property to study the behavior of the scattering amplitude in $\mathbb{C}$.

2000 AMS Subject Classification: Primary: 35P25, Secondary: 47A40, 35L05

Key words: Scattering amplitude, reflecting rays, trapping trajectories, sojourn time.

1 Introduction

Let $K \subset \{x \in \mathbb{R}^n,\ |x| \leq \rho\}$, $n \geq 3$, n odd, be a bounded domain with C^∞ boundary ∂K and connected complement $\Omega = \overline{\mathbb{R}^n \setminus K}$. Such K is called an *obstacle* in $\mathbb{R}^n$. In this paper we consider the Dirichlet problem for the wave

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_14,

equation, however in a similar way one can deal with other boundary value problems. Given two directions $(\theta, \omega) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, consider the *outgoing solution* $v_s(x, \lambda)$ of the problem

$$\begin{cases} (\Delta + \lambda^2)\, v_s = 0 \text{ in } \mathring{\Omega}, \\ v_s + e^{-\mathbf{i}\lambda\langle x, \omega\rangle} = 0 \text{ on } \partial K, \end{cases}$$

satisfying the so-called $(\mathbf{i}\lambda)$ - outgoing Sommerfeld radiation condition:

$$v_s(r\theta, \lambda) = \frac{e^{-\mathbf{i}\lambda r}}{r^{(n-1)/2}}\Big(a(\lambda, \theta, \omega) + \mathcal{O}\Big(\frac{1}{r}\Big)\Big), \;\; x = r\theta, \text{ as } |x| = r \longrightarrow \infty\,.$$

The leading term $a(\lambda, \theta, \omega)$ is called the *scattering amplitude* and we have the following representation:

$$a(\lambda, \theta, \omega) = \frac{(\mathbf{i}\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}} \int_{\partial K} \Big(\mathbf{i}\lambda \langle \nu(x), \theta\rangle e^{\mathbf{i}\lambda\langle x, \theta - \omega\rangle} - e^{\mathbf{i}\lambda\langle x, \theta\rangle} \frac{\partial v_s}{\partial \nu}(x, \lambda)\Big) dS_x\,, \tag{1.1}$$

where $\langle \bullet, \bullet \rangle$ denotes the inner product in $\mathbb{R}^n$ and $\nu(x)$ is the unit normal to $x \in \partial K$ pointing into Ω (see [9], [13]).

Throughout this note we assume that $\theta \neq \omega$. The *scattering kernel* $s(t, \theta, \omega)$ is defined as the Fourier transform of the scattering amplitude

$$s(t, \theta, \omega) = \mathcal{F}_{\lambda \to t}\Big(\Big(\frac{\mathbf{i}\lambda}{2\pi}\Big)^{(n-1)/2} \overline{a(\lambda, \theta, \omega)}\Big)\,,$$

where $\Big(\mathcal{F}_{\lambda \to t}\varphi\Big)(t) = (2\pi)^{-1} \int e^{\mathbf{i}t\lambda} \varphi(\lambda) d\lambda$ for functions $\varphi \in \mathcal{S}(\mathbb{R})$. Let $V(t, x; \omega)$ be the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta)\, V = 0 \text{ in } \mathbb{R} \times \mathring{\Omega}, \\ V = 0 \text{ on } \mathbb{R} \times \partial K, \\ V|_{t < -\rho} = \delta(t - \langle x, \omega \rangle). \end{cases}$$

Then we have

$$s(\sigma, \theta, \omega) = (-1)^{(n+1)/2} 2^{-n} \pi^{1-n} \int_{\partial K} \partial_t^{n-2} \partial_\nu V(\langle x, \theta\rangle - \sigma, x; \omega) dS_x\,,$$

where the integral is interpreted in the sense of distributions.

The singularities of $s(t, \theta, \omega)$ with respect to t can be observed since at these times we have some nonnegligible picks of the scattering amplitude. For example, if K is strictly convex, for fixed $\theta \neq \omega$ we have only one singularity at $t = -T_\gamma$ related to the sojourn time of the unique (ω, θ)-reflecting ray γ (see [8]). For general nonconvex obstacles the geometric situation is much more complicated since we have different type of rays incoming with direction ω and outgoing in direction θ for which an asymptotic solution related to the rays is impossible to construct. In many problems, such as those concerning local decay of energy, behavior of the cut-off resolvent of the Laplacian, the existence of resonances, etc., the difference between nontrapping and trapping obstacles is quite significant. In recent years many authors have studied mainly trapping obstacles with some very special geometry and the case of several strictly convex disjoint obstacles has been investigated both from a mathematical and a numerical analysis point of view.

In this work our purpose is the study of the obstacles having at least one (ω, θ)-trapping ray γ which in general could be nonreflecting (see Section 2 for the definition of an (ω, θ)-ray). No assumptions are made on the geometry of the obstacle outside some small neighborhood of γ and no information is required about other possible (ω, θ)-rays. Our aim is to examine if the existence of γ may create an **infinite number** of delta-type singularities $T_m \to \infty$ of $s(-t, \theta_m, \omega_m)$, in contrast to the nontrapping case where $s(t, \theta, \omega)$ is C^∞ smooth for $|t| \geq T_0 > 0$ and all $(\theta, \omega) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. On the other hand, it is important to stress that the scattering amplitude and the scattering kernel are global objects and their behavior depends on all (ω, θ)-rays so any type of cancellation of singularities may occur. The existence of a trapping ray influences the singularities of $s(t, \theta, \omega)$ if we assume that γ is nondegenerate which is a local condition (see Section 3). Thus our result says that from the scattering data related to the singularities of $s(t, \theta, \omega)$ we can "hear" whether K is trapping or not.

The proof of our main result is based on several previous works [13], [14], [15], [16], [19], and our purpose here is to show how the results of these works imply the existence of an infinite number of singularities. The reader may consult [18] for a survey on the results mentioned above.

2 Scattering kernel

We start with the definition of the so-called reflecting (ω, θ)-rays. Given two directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, consider a curve $\gamma \in \Omega$ having the form

$$\gamma = \cup_{i=0}^{m} l_i, \;\; m \geq 1,$$

where $l_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, ..., m-1$, $x_i \in \partial K$, and l_0 (resp. l_m) is the infinite segment starting at x_1 (resp. at x_m) and having direction $-\omega$ (resp. θ). The curve γ is called a *reflecting* (ω, θ)-ray in Ω if for $i = 0, 1, ..., m-1$ the segments l_i and l_{i+1} satisfy the law of reflection at x_{i+1} with respect to ∂K. The points $x_1, ..., x_m$ are called *reflection points* of γ and this ray is called *ordinary reflecting* if γ has no segments tangent to ∂K.

Next, we define two notions related to (ω, θ)-rays. Fix an arbitrary open ball U_0 with radius $a > 0$ containing K and for $\xi \in \mathbb{S}^{n-1}$ introduce the hyperplane Z_ξ orthogonal to ξ, tangent to U_0 and such that ξ is pointing into the interior of the open half space H_ξ with boundary Z_ξ containing U_0. Let $\pi_\xi : \mathbb{R}^n \longrightarrow Z_\xi$ be the orthogonal projection. For a reflecting (ω, θ)-ray γ in Ω with successive reflecting points $x_1, ..., x_m$ the *sojourn time* T_γ of γ is defined by

$$T_\gamma = \|\pi_\omega(x_1) - x_1\| + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| + \|x_m - \pi_{-\theta}(x_m)\| - 2a\,.$$

Obviously, $T_\gamma + 2a$ coincides with the length of the part of γ that lies in $H_\omega \cap H_{-\theta}$. The sojourn time T_γ does not depend on the choice of the ball U_0 and

$$T_\gamma = \langle x_1, \omega\rangle + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| - \langle x_m, \theta\rangle\,.$$

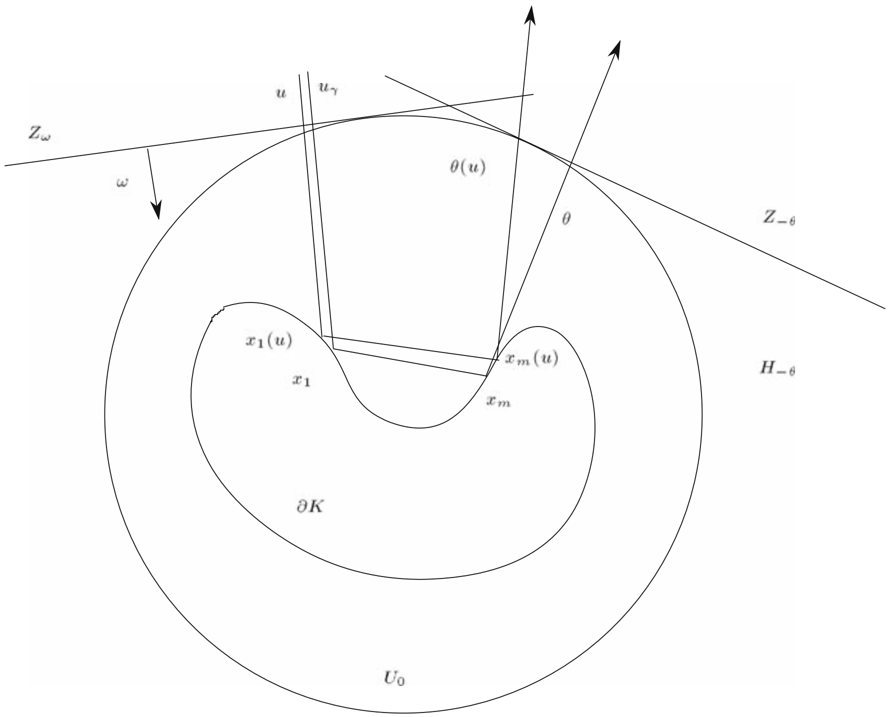

Fig. 1.

Given an ordinary reflecting (ω, θ)-ray γ set $u_\gamma = \pi_\omega(x_1)$. Then there exists a small neighborhood W_γ of u_γ in Z_ω such that for every $u \in W_\gamma$ there is a unique direction $\theta(u) \in \mathbb{S}^{n-1}$ and points $x_1(u), ..., x_m(u)$ which are the successive reflection points of a reflecting $(u, \theta(u))$-ray in Ω with $\pi_\omega(x_1(u)) = u$ (see Figure 1). We obtain a smooth map

$$J_\gamma : W_\gamma \ni u \longrightarrow \theta(u) \in \mathbb{S}^{n-1}$$

and $dJ_\gamma(u_\gamma)$ is called a *differential cross section* related to γ. We say that γ is *nondegenerate* if

$$\det dJ_\gamma(u_\gamma) \neq 0\,.$$

The notion of sojourn time as well as that of differential cross section are well known in the physical literature and the definitions given above are due to Guillemin [5].

For nonconvex obstacles there exist (ω, θ)-rays with some tangent and/or gliding segments. To give a precise definition one has to involve the generalized bicharacteristics of the operator $\Box = \partial_t^2 - \Delta_x$ defined as the trajectories of the generalized Hamilton flow $\mathcal{F}_t$ in Ω generated by the symbol $\sum_{i=1}^n \xi_i^2 - \tau^2$ of $\Box$ (see [11] for a precise definition). In general, $\mathcal{F}_t$ is not smooth and in some cases there may exist two different integral curves issued from the same point in the phase space (see [23] for an example). To avoid this situation in the following we assume that the following generic condition is satisfied.

$(\mathcal{G})$ If for $(x, \xi) \in T^*(\partial K)$ the normal curvature of ∂K vanishes of infinite order in direction ξ, then ∂K is convex at x in direction ξ.

Given $\sigma = (x, \xi) \in T^*(\Omega) \setminus \{0\} = \dot{T}^*(\Omega)$, there exists a unique generalized bicharacteristic $(x(t), \xi(t)) \in \dot{T}^*(\Omega)$ such that $x(0) = x,\ \xi(0) = \xi$ and we define $\mathcal{F}_t(x, \xi) = (x(t), \xi(t))$ for all $t \in \mathbb{R}$ (see [11]). We obtain a flow $\mathcal{F}_t : \dot{T}^*(\Omega) \longrightarrow \dot{T}^*(\Omega)$ which is called the *generalized geodesic flow* on $\dot{T}^*(\Omega)$. It is clear that this flow leaves the *cosphere bundle* $S^*(\Omega)$ invariant. The flow $\mathcal{F}_t$ is discontinuous at points of transversal reflection at $\dot{T}^*_{\partial K}(\Omega)$ and to make it continuous, consider the *quotient space* $\dot{T}^*(\Omega)/\sim$ of $\dot{T}^*(\Omega)$ with respect to the following equivalence relation: $\rho \sim \sigma$ if and only if $\rho = \sigma$ or $\rho, \sigma \in T^*_{\partial K}(\Omega)$ and either $\lim_{t \nearrow 0} \mathcal{F}_t(\rho) = \sigma$ or $\lim_{t \searrow 0} \mathcal{F}_t(\rho) = \sigma$. Let Σ_b be the image of $S^*(\Omega)$ in $\dot{T}^*(\Omega)/\sim$. The set Σ_b is called the *compressed characteristic set.* Melrose and Sjöstrand ([11]) proved that the natural projection of $\mathcal{F}_t$ on $\dot{T}^*(\Omega)/\sim$ is continuous.

Now a curve $\gamma = \{x(t) \in \Omega :\ t \in \mathbb{R}\}$ is called an (ω, θ)*-ray* if there exist real numbers $t_1 < t_2$ so that

$$\hat{\gamma}(t) = (x(t), \xi(t)) \in S^*(\Omega)$$

is a *generalized bicharacteristic* of $\Box$ and

$$\xi(t) = \omega \text{ for } t \leq t_1,\ \ \xi(t) = \theta \text{ for } t \geq t_2,$$

provided that the time t increases when we move along $\hat{\gamma}$. Denote by $\mathcal{L}_{(\omega,\theta)}(\Omega)$ the *set of all* (ω,θ)*-rays* in Ω. The *sojourn time* T_δ of $\delta \in \mathcal{L}_{(\omega,\theta)}(\Omega)$ is defined as the length of the part of δ lying in $H_\omega \cap H_{-\theta}$.

It was proved in [12], [3] (cf. also Chapter 8 in [14] and [10]) that for $\omega \neq \theta$ we have

$$\text{sing supp}_t\ s(t,\theta,\omega) \subset \{-T_\gamma : \gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega)\}. \tag{2.1}$$

This relation was established for convex obstacles by Majda [9] and for some Riemann surfaces by Guillemin [5]. The proof in [12], [3] deals with general obstacles and is based on the results in [11] concerning propagation of singularities.

In analogy with the well-known Poisson relation for the Laplacian on Riemannian manifolds, (2.1) is called *the Poisson relation for the scattering kernel*, while the set of all T_γ, where $\gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega)$, $(\omega,\theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, is called the *scattering length spectrum* of K.

To examine the behavior of $s(t,\theta,\omega)$ near singularities, assume that γ is a fixed *nondegenerate ordinary reflecting* (ω,θ)-ray such that

$$T_\gamma \neq T_\delta \ \text{ for every } \ \delta \in \mathcal{L}_{(\omega,\theta)}(\Omega) \setminus \{\gamma\}. \tag{2.2}$$

By using the continuity of the generalized Hamiltonian flow, it is easy to show that

$$(-T_\gamma - \varepsilon, -T_\gamma + \varepsilon) \cap \text{sing supp}_t\ s(t,\theta,\omega) = \{-T_\gamma\} \tag{2.3}$$

for $\varepsilon > 0$ sufficiently small. For strictly convex obstacles and $\omega \neq \theta$ every (ω,θ)-ray is nondegenerate and (2.3) is obviously satisfied. For general nonconvex obstacles one needs to establish some global properties of (ω,θ)-rays and choose (ω,θ) so that (2.3) holds. The singularity of $s(t,\theta,\omega)$ at $t = -T_\gamma$ can be investigated by using a global construction of an asymptotic solution as a Fourier integral operator (see [6], [12] and Chapter 9 in [14]), and we have the following:

Theorem 2.1 ([12]) *Let γ be a nondegenerate ordinary reflecting (ω,θ)-ray and let $\omega \neq \theta$. Then under the assumption* (2.3) *we have*

$$-T_\gamma \in \text{sing supp}_t\ s(t,\theta,\omega) \tag{2.4}$$

and for t close to $-T_\gamma$ the scattering kernel has the form

$$s(t,\theta,\omega) = \Big(\frac{1}{2\pi \mathbf{i}}\Big)^{(n-1)/2} (-1)^{m_\gamma - 1} \exp\Big(\mathbf{i}\frac{\pi}{2}\beta_\gamma\Big) \tag{2.5}$$

$$\times \Big|\frac{\det dJ_\gamma(u_\gamma)\langle \nu(q_1),\omega\rangle}{\langle \nu(q_m),\theta\rangle}\Big|^{-1/2} \delta^{(n-1)/2}(t + T_\gamma) + \text{lower order singularities.}$$

Here m_γ is the number of reflections of γ, q_1 (resp. q_m) is the first (resp. the last) reflection point of γ and $\beta_\gamma \in \mathbb{Z}$.

For strictly convex obstacles we have

$$m_{\gamma_+} = 1,\ \beta_{\gamma_+} = -\frac{n-1}{2},\ q_1 = q_m,$$

$\theta - \omega$ is parallel to $\nu(q_1)$ and

$$|\det dJ_{\gamma_+}(u_{\gamma_+})| = 4|\theta - \omega|^{n-3}\mathcal{K}(x_+),$$

where γ_+ is the unique (ω,θ)-reflecting ray at x_+, u_{γ_+} is the corresponding point on Z_ω and $\mathcal{K}(x_+)$ is the Gauss curvature at x_+. Thus we obtain the result of Majda [8] (see also [9]) describing the leading singularity at $-T_{\gamma_+}$.

To obtain an equality in the Poisson relation (2.1), one needs to know that every (ω,θ)-ray produces a singularity. To achieve this, a natural way to proceed would be to ensure that the properties (2.2), (2.3) hold. It is clear that these properties depend on the global behavior of the (ω,θ)-rays in the exterior of the obstacle, and in this regard the existence of (ω,θ)-rays with tangent or gliding segments leads to considerable difficulties. Moreover, different ordinary reflecting rays could produce singularities which mutually cancel. By using the properties of (ω,θ)-rays established in [15], [16], as well as the fact that for almost all directions (ω,θ), the (ω,θ)-rays are ordinary reflecting (see [19]), the following was derived in [19]:

Theorem 2.2 ([19]) *There exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ such that for each $(\omega,\theta)\in\mathcal{R}$ the only (ω,θ)-rays in Ω are ordinary reflecting (ω,θ)-rays and*

$$\text{sing supp}_t\, s(t,\theta,\omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}\ .$$

This result is the basis for several interesting inverse scattering results (see [20], [21]).

3 Trapping obstacles

Given a generalized bicharacteristic γ in $S^*(\Omega)$, its projection $\tilde{\gamma} = \sim(\gamma)$ in Σ_b is called a *compressed generalized bicharacteristic*. Let U_0 be an open ball containing K and let C be its boundary sphere. For an arbitrary point $z = (x,\xi)\in\Sigma_b$, consider the compressed generalized bicharacteristic

$$\gamma_z(t) = (x(t),\xi(t)) \in \Sigma_b$$

parametrized by the time t and passing through z for $t = 0$. Denote by $T(z)\in\mathbb{R}^+\cup\infty$ the maximal $T > 0$ such that $x(t)\in U_0$ for $0\le t\le T(z)$. The so-called *trapping set* is defined by

$$\Sigma_\infty = \{(x,\xi) \in \Sigma_b : \, x \in C, \, T(z) = \infty\} \, .$$

It follows from the continuity of the compressed generalized Hamiltonian flow that the trapping set Σ_∞ is closed in Σ_b. For simplicity, in the following the compressed generalized bicharacteristics will be called simply generalized ones. The obstacle K is called *trapping* if $\Sigma_\infty \neq \emptyset$, i.e., when there exists at least one point $(\hat{x}, \hat{\xi}) \in C \times \mathbb{S}^{n-1}$ such that the (generalized) trajectory issued from $(\hat{x}, \hat{\xi})$ stays in U_0 for all $t \geq 0$. This provides some information about the behavior of the rays issued from the points (y, η) sufficiently close to $(\hat{x}, \hat{\xi})$; however, in general it does not yield any information about the geometry of (ω, θ)-rays.

Now for every trapping obstacle we have the following:

Theorem 3.1 ([15], [18]) *Let the obstacle K be trapping and satisfy the condition $(\mathcal{G})$. Then there exists a sequence of ordinary reflecting (ω_m, θ_m)-rays γ_m with sojourn times $T_{\gamma_m} \longrightarrow \infty$.*

To prove this we use the following:

Proposition 3.2 ([7], [19]) *The set of points $(x, \xi) \in S^*_C(\Omega) = \{(x,\xi) \in T^*(\Omega) : x \in C, \, |\xi| = 1\}$ such that the trajectory $\{\mathcal{F}_t(x,\xi) : t \geq 0\}$ issued from (x, ξ) is bounded has Lebesgue measure zero in $S^*_C(\Omega)$.*

Proof Assume K is trapping and satisfies the condition $(\mathcal{G})$. We will establish the existence of (ω, θ_m)-rays with sojourn times $T_m \to \infty$ for some $\omega \in \mathbb{S}^{n-1}$ suitably fixed. It is easy to see that $\Sigma_b \setminus \Sigma_\infty \neq \emptyset$. Since K is trapping, we have $\Sigma_\infty \neq \emptyset$, so the boundary $\partial \Sigma_\infty$ of Σ_∞ in Σ_b is not empty. Fix an arbitrary $\hat{z} \in \partial \Sigma_\infty$ and take an arbitrary sequence $z_m = (0, x_m, 1, \xi_m) \in \Sigma_b$, so that $z_m \notin \Sigma_\infty$ for every $m \in \mathbb{N}$ and $z_m \longrightarrow \hat{z}$. Consider the compressed generalized bicharacteristics $\delta_m = (t, x_m(t), 1, \xi_m(t))$ passing through z_m for $t = 0$ with sojourn times $T_{z_m} < \infty$. If the sequence $\{T_{z_m}\}$ is bounded, one gets a contradiction with the fact that $\hat{z} \in \Sigma_\infty$. Thus, $\{T_{z_m}\}$ is unbounded, and replacing the sequence $\{z_m\}$ by an appropriate subsequence we may assume that $T_{z_m} \longrightarrow +\infty$. Setting

$$y_m = x_m(T(z_m)) \in C, \, \omega_m = \xi_m(T(z_m)) \in \mathbb{S}^{n-1}$$

and passing again to a subsequence if necessary, we may assume that $y_m \to z_0 \in C$, $\omega_m \to \omega_0 \in \mathbb{S}^{n-1}$. Then for the generalized bicharacteristic $\delta_\mu(t) = (t, x(t), 1, \xi(t))$ issued from $\mu = (0, z_0, 1, \omega_0)$ we have $T(\delta_\mu) = \infty$. Next, consider the hyperplane Z_{ω_0} passing through z and orthogonal to ω_0 and the set of points Z_∞ such that the generalized bicharacteristic γ_u issued from $u \in Z_\infty$ with direction ω_0 satisfies the condition $T(\gamma_u) = \infty$. The set $Z_\infty \cap Z_{\omega_0}$ is closed in Z_{ω_0} and $Z_{\omega_0} \setminus Z_\infty \neq \emptyset$. Repeating the above argument, we obtain rays γ_m with sojourn times $T_{\gamma_m} \longrightarrow +\infty$. Using Proposition 3.2, we may assume that each ray γ_m is unbounded in both directions, i.e., γ_m

is an (ω_0, θ_m)-ray for some $\theta_m \in \mathbb{S}^{n-1}$. Moreover, according to results in [11] and [15], these rays can be approximated by ordinary reflecting ones, so we may assume that each γ_m is an ordinary reflecting (ω_m, θ_m)-ray for some $\omega_m, \theta_m \in \mathbb{S}^{n-1}$. This completes the proof.

To show that the rays γ_m constructed in Theorem 3.1 produce singularities, we need to check the condition (2.3). In general the ordinary reflecting ray γ_m could be degenerate and we have to replace γ_m by another ordinary reflecting nondegenerate $(\theta'_m,\ \omega'_m)$-ray γ'_m with sojourn time T'_m sufficiently close to T_{γ_m}. Our argument concerns the rays issued from a small neighborhood $W \subset C \times \mathbb{S}^{n-1}$ of the point $(z_0, \omega_0) \in C \times \mathbb{S}^{n-1}$ introduced in the proof of Theorem 3.1.

Let $\mathcal{O}(W)$ be the set of all pairs of directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ such that there exists an ordinary reflecting (ω, θ)-ray issued from $(x, \omega) \in W$ with *outgoing* direction $\theta \in \mathbb{S}^{n-1}$. To obtain convenient approximations with (ω, θ)-rays issued from W, it is desirable to know that $\mathcal{O}(W)$ has a positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ for all sufficiently small neighborhoods $W \subset C \times \mathbb{S}^{n-1}$ of (z_0, ω_0). Roughly speaking this means that the trapping generalized bicharacteristic $\delta_\mu(t)$ introduced above is nondegenerate in some sense. More precisely, we introduce the following:

Definition 3.3 *The generalized bicharacteristic γ issued from $(y, \eta) \in C \times \mathbb{S}^{n-1}$ is called weakly nondegenerate if for every neighborhood $W \subset C \times \mathbb{S}^{n-1}$ of (y, η) the set $\mathcal{O}(W)$ has a positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$.*

The above definition generalizes that of a nondegenerate ordinary reflecting ray γ given in Section 2. Indeed, let γ be an ordinary reflecting nondegenerate (ω_0, θ_0)-ray issued from $(x_0, \omega_0) \in C \times \mathbb{S}^{n-1}$. Let $Z = Z_{\omega_0}$ and consider the C^∞ map

$$D = X \times \Gamma \ni (x, \omega) \longrightarrow f(x, \omega) \in \mathbb{S}^{n-1},$$

where $X \subset Z$ is a small neighborhood of x_0, $\Gamma \subset \mathbb{S}^{n-1}$ is a small neighborhood of ω_0, and $f(x, \omega)$ is the outgoing direction of the ray issued from x in direction ω. We have $\det f'_x(x_0, \omega_0) \neq 0$ and we may assume that D is chosen small enough so that $\det f'_x(x, \omega) \neq 0$ for $(x, \omega) \in \bar{D}$. Set

$$\max_{(x,\omega)\in \bar{D}} \|(f'_x(x, \omega))^{-1}\| = \frac{1}{\alpha}.$$

Then for small $\varepsilon > 0$ we have $\|f'_x(x, \omega) - f'_x(x_0, \omega_0)\| \leq \frac{\alpha}{4}$, provided $\|x - x_0\| < \varepsilon$, $\|\omega - \omega_0\| < \varepsilon$ and

$$X_\varepsilon = \{x \in Z : \ \|x - x_0\| < \varepsilon\} \subset X, \ \ \Gamma_\varepsilon = \{\omega \in \mathbb{S}^{n-1} : \ \|\omega - \omega_0\| < \varepsilon\} \subset \Gamma.$$

Next consider the set

$$\varXi_\varepsilon = \left\{\theta \in \mathbb{S}^{n-1} :\ \|\theta - \theta_0\| < \frac{\varepsilon\alpha}{4}\right\}.$$

Then taking $\varepsilon' \in (0, \varepsilon)$ so that $\|f(x_0, \omega) - \theta_0\| < \frac{\varepsilon\alpha}{4}$ for $\omega \in \varGamma_{\varepsilon'}$, and applying the inverse mapping theorem (see Section 5 in [15]), we conclude that for every fixed $\omega \in \varGamma_{\varepsilon'}$ and every fixed $\theta \in \varXi_{\varepsilon'}$ we can find $x_{(\omega,\theta)} \in X_\varepsilon$ with $f(x_{(\omega,\theta)}, \omega) = \theta$. Consequently, the corresponding set of directions $\varGamma_{\varepsilon'} \times \varXi_{\varepsilon'} \subset \mathcal{O}(W)$ has positive measure in $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. This argument works for every neighborhood of (x_0, ω_0), so γ is weakly nondegenerate according to Definition 3.3.

Remark 3.4 *In general a weakly nondegenerate ordinary reflecting ray does not need to be nondegenerate. To see this, first notice that the set of those* $(y, \eta) \in C \times \mathbb{S}^{n-1}$ *that generate weakly nondegenerate bicharacteristics is closed in* $C \times \mathbb{S}^{n-1}$. *Now consider the special case when* K *is convex with vanishing Gauss curvature at some point* $x_0 \in \partial K$ *and strictly positive Gauss curvature at any other point of* ∂K. *Consider a reflecting ray* γ *in* $\mathbb{R}^n$ *with a single reflection point at* x_0. *Then, as is well-known,* γ *is degenerate, that is, the differential cross section vanishes. However, arbitrarily close to* γ *we can choose an ordinary reflecting ray* δ_m *with a single reflection point* $x_m \neq x_0$. *Then* δ_m *is nondegenerate and hence it is weakly nondegenerate. Thus,* γ *can be approximated arbitrarily well with weakly nondegenerate rays, and therefore* γ *itself is weakly nondegenerate.*

Now we have a stronger version of Theorem 3.1.

Theorem 3.5 *Let the obstacle* K *have at least one trapping weakly nondegenerate bicharacteristic* δ *issued from* $(y, \eta) \in C \times \mathbb{S}^{n-1}$ *and let* K *satisfy* $(\mathcal{G})$. *Then there exists a sequence of ordinary reflecting nondegenerate* (ω_m, θ_m)*-rays* γ_m *with sojourn times* $T_{\gamma_m} \longrightarrow \infty$.

Proof Let $W_m \subset C \times \mathbb{S}^{n-1}$ be a neighborhood of (y, η) such that for every $z \in W_m$ the generalized bicharacteristic γ_z issued from z satisfies the condition $T(\gamma_z) > m$. The continuity of the compressed generalized flow guarantees the existence of W_m for all $m \in \mathbb{N}$. Moreover, we have $W_{m+1} \subset W_m$. Consider the *open subset* F_m of $C \times \mathbb{S}^{n-1} \times C \times \mathbb{S}^{n-1}$ consisting of those (x, ω, z, θ) such that $(x, \omega) \in W_m$ and there exists an ordinary reflecting (ω, θ)-ray issued from $(x, \omega) \in W_m$ and passing through z with direction θ.

The projection $F_m \ni (x, \omega, z, \theta) \longrightarrow (\omega, \theta)$ is smooth and Sard's theorem implies the existence of a set $\mathcal{D}_m \subset \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ with measure zero so that if $(\omega, \theta) \notin \mathcal{D}_m$ the corresponding (ω, θ)-ray issued from $(x, \omega) \in W_m$ is non-degenerate. Then the set $\mathcal{O}(W_m) \setminus \mathcal{D}_m$ has a positive measure and taking $(\omega_m, \theta_m) \in \mathcal{O}(W_m) \setminus \mathcal{D}_m$ we obtain an ordinary reflecting nondegenerate (ω_m, θ_m)-ray δ_m with sojourn time T_m issued from $z_m \in W_m$. Next we choose

$$q(m) > \max\{m+1, T_m\},\ q(m) \in \mathbb{N}$$

and repeat the same argument for $W_{q(m)}$ and $F_{q(m)}$. This completes the proof.

Remark 3.6 *In general, a generalized trapping ray δ can be weakly degenerate if its reflection points lie on flat regions of the boundary. In the case when K is a finite disjoint union of several convex domains sufficient conditions for a trapping ray to be weakly nondegenerate are given in* [16]. *On the other hand, we expect that the sojourn time T_γ of an ordinary reflecting ray γ may produce a singularity of the scattering kernel if the condition (2.3) is replaced by some weaker one. For this purpose one needs a generalization of Theorem* 2.1 *based on the asymptotics of oscillatory integrals with degenerate critical points.*

Now assume that γ is an ordinary reflecting nondegenerate (ω, θ)-ray with sojourn time T_γ issued from $(x, \xi) \in C \times \mathbb{S}^{n-1}$. For such a ray the condition (2.3) is not necessarily fulfilled. Since γ is nondegenerate, there are no (ω, θ)-rays δ with sojourn time T_γ issued from points in a small neighborhood of (x, ξ). This is not sufficient for (2.3) and we must take into account all (ω, θ)-rays. The result in [19] says that for almost all directions $(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ all (ω, θ)-rays are reflecting ones and the result in [15] implies the property (2.2) for the sojourn times of ordinary reflecting rays (ω, θ)-ray, provided that (ω, θ) is outside some set of measure zero. Thus we can approximate (ω, θ) by directions (ω', θ') for which the above two properties hold. Next, the fact that γ is nondegenerate combined with the inverse mapping theorem make it possible to find an ordinary reflecting nondegenerate (ω', θ')-ray γ' with sojourn time T'_γ sufficiently close to T_γ so that (2.2) and (2.3) hold for γ'. We refer to Section 5 in [15] for details concerning the application of the inverse mapping theorem. Finally, we obtain the following:

Theorem 3.7 *Under the assumptions of Theorem* 3.5 *there exists a sequence $(\omega_m, \theta_m) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and ordinary reflecting nondegenerate (ω_m, θ_m)-rays γ_m with sojourn times $T_m \longrightarrow \infty$ so that*

$$-T_m \in \text{sing supp}\, s(t, \omega_m, \theta_m),\ \ \forall m \in \mathbb{N}. \tag{3.1}$$

The relation (3.1) was called property (S) in [15] and it was conjectured that every trapping obstacle has the property (S). The above result says that this is true if the generalized Hamiltonian flow is continuous and if there is at least one weakly nondegenerate trapping ray δ. The assumption that δ is weakly nondegenerate has been omitted in Theorem 8 in [18].

4 Trapping rays and estimates of the scattering amplitude

The scattering resonances are related to the behavior of the modified resolvent of the Laplacian. For $\Im\lambda < 0$ consider the outgoing resolvent $R(\lambda) = (-\Delta - \lambda^2)^{-1}$ of the Laplacian in Ω with Dirichlet boundary conditions on ∂K. The outgoing condition means that for $f \in C_0^\infty(\Omega)$ there exists $g(x) \in C_0^\infty(\mathbb{R}^n)$ so that we have

$$R(\lambda)f(x) = R_0(\lambda)g(x), \ |x| \to \infty,$$

where

$$R_0(\lambda) = (-\Delta - \lambda^2)^{-1} : L^2_{\rm comp}(\mathbb{R}^n) \longrightarrow H^2_{\rm loc}(\mathbb{R}^n)$$

is the outgoing resolvent of the free Laplacian in $\mathbb{R}^n$. The operator

$$R(\lambda) : L^2_{\rm comp}(\Omega) \ni f \longrightarrow R(\lambda)f \in H^2_{\rm loc}(\Omega)$$

has a meromorphic continuation in $\mathbb{C}$ with poles λ_j, $\Im\lambda_j > 0$, called *resonances* ([7]). Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\chi(x) = 1$ on a neighborhood of K. It is easy to see that the *modified resolvent*

$$R_\chi(\lambda) = \chi R(\lambda)\chi$$

has a meromorphic continuation in $\mathbb{C}$ and the poles of $R_\chi(\lambda)$ are independent of the choice of χ. These poles coincide with their multiplicities with those of the resonances. On the other hand, the scattering amplitude $a(\lambda, \theta, \omega)$ also admits a meromorphic continuation in $\mathbb{C}$ and the poles of this continuation and their multiplicities are the same as those of the resonances (see [7]). From the general results on propagation of singularities given in [11], it follows that if K is nontrapping, there exist $\varepsilon > 0$ and $d > 0$ so that $R_\chi(\lambda)$ has no poles in the domain

$$U_{\varepsilon,d} = \{\lambda \in \mathbb{C} : \ 0 \le \Im\lambda \le \varepsilon \log(1 + |\lambda|) - d\}.$$

Moreover, for nontrapping obstacles we have the estimate (see [24])

$$\|R_\chi(\lambda)\|_{L^2(\Omega) \longrightarrow L^2(\Omega)} \le \frac{C}{|\lambda|} e^{C|\Im\lambda|}, \ \forall \lambda \in U_{\varepsilon,d}.$$

We conjecture that the existence of singularities $t_m \longrightarrow -\infty$ of the scattering kernel $s(t, \theta_m, \omega_m)$ implies that for every $\varepsilon > 0$ and $d > 0$ we have resonances in $U_{\varepsilon,d}$.

Here we prove a weaker result assuming an estimate of the scattering amplitude.

Theorem 4.1 *Suppose that there exist* $m \in \mathbb{N}$, $\alpha \ge 0$, $\varepsilon > 0$, $d > 0$ *and* $C > 0$ *so that* $a(\lambda, \theta, \omega)$ *is analytic in* $U_{\varepsilon,d}$ *and*

$$|a(\lambda, \theta, \omega)| \le C(1 + |\lambda|)^m e^{\alpha|\Im\lambda|}, \ \forall (\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \ \forall \lambda \in U_{\varepsilon,d}. \quad (4.1)$$

Then if K *satisfies* $(\mathcal{G})$, *there are no trapping weakly nondegenerate rays in* Ω.

The proof of this result follows directly from the statement in Theorem 2.3 in [15]. In fact, if there exists a weakly nondegenerate trapping ray, we can apply Theorem 3.7, and for the sequence of sojourn times $\{-T_m\}$, $T_m \to \infty$, related to a weakly nondegenerate ray δ, an application of Theorem 2.1 yields a sequence of delta type isolated singularities of the scattering kernel. The existence of these singularities combined with the estimate (4.1) leads to a contradiction since we may apply the following:

Lemma 4.2 ([15]) *Let $u \in \mathcal{S}'(\mathbb{R})$ be a distribution. Assume that the Fourier transform $\hat{u}(\xi), \xi \in \mathbb{R}$, admits an analytic continuation in*

$$W_{\varepsilon,d} = \{\xi \in \mathbb{C} :\ d - \varepsilon \log(1 + |\xi|) \leq \Im\xi \leq 0\},\ \varepsilon > 0,\ d > 0$$

such that for all $\xi \in W_{\varepsilon,d}$ we have

$$|\hat{u}(\xi)| \leq C(1 + |\xi|)^N e^{\gamma|\Im\xi|},\ \gamma \geq 0.$$

Then for each $q \in \mathbb{N}$ there exists $t_q < \tau$ and $v_q \in C^q(\mathbb{R})$ such that $u = v_q$ for $t \leq t_q$.

Here the Fourier transform $\hat{u}(\xi) = \int e^{-\mathbf{i}t\xi} u(t)dt$ for $u \in C_0^\infty(\mathbb{R})$ and for $\lambda \in \mathbb{R}$ we have

$$\hat{s}(\lambda, \theta, \omega) = \Big(\frac{\mathbf{i}\lambda}{2\pi}\Big)^{(n-1)/2} \overline{a(\lambda, \theta, \omega)} = \Big(\frac{\mathbf{i}\lambda}{2\pi}\Big)^{(n-1)/2} a(-\lambda, \theta, \omega).$$

Thus $\hat{s}(\lambda, \theta, \omega)$ admits an analytic continuation in $W_{\varepsilon,d}$ and the estimate (4.1) implies an estimate for $\hat{s}(\lambda, \theta, \omega)$ in $W_{\varepsilon,d}$.

It is easy to see that the analyticity of $R_\chi(\lambda)$ in $U_{\varepsilon,d}$ and the estimate

$$\|R_\chi(\lambda)\|_{L^2(\Omega) \longrightarrow L^2(\Omega)} \leq C'(1 + |\lambda|)^{m'} e^{\alpha'|\Im\lambda|},\ \ \forall\lambda \in U_{\varepsilon,d} \tag{4.2}$$

with $m' \in \mathbb{N}$, $\alpha' \geq 0$, imply (4.1) with suitable m and α. This follows from the representation of the scattering amplitude involving the cut-off resolvent $R_\psi(\lambda)$ (see [15], [17]) with $\psi \in C_0^\infty(\mathbb{R}^n)$ having support in $\{x \in \mathbb{R}^n :\ 0 < a' \leq |x| \leq b'\}$. Moreover, we can take $a' < b'$ arbitrary large. More precisely, let $\varphi_a \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function such that $\varphi_a(x) = 1$ for $|x| \leq \rho$. Set

$$F_a(\lambda, \omega) = [\varDelta\varphi_a + 2\mathbf{i}\lambda\langle\nabla\varphi_a, \omega\rangle]e^{\mathbf{i}\lambda\langle x,\omega\rangle}.$$

Let $\varphi_b(x) \in C_0^\infty(\mathbb{R}^n)$ be such that $\varphi_b(x) = 1$ on a neighborhood of K and $\varphi_a(x) = 1$ on supp φ_b. The scattering amplitude $a(\lambda, \theta, \omega)$ has the representation

$$a(\lambda, \theta, \omega) = c_n \lambda^{(n-3)/2} \int_\Omega e^{-\mathbf{i}\lambda\langle x,\theta\rangle} \Big[(\varDelta\varphi_b)R(\lambda)F_a(\lambda, \omega)$$
$$+2\langle\nabla_x\varphi_b, \nabla_x(R(\lambda)F_a(\lambda, \omega))\rangle\Big] dx$$

with a constant c_n depending on n and this representation is independent of the choice of φ_a and φ_b. In particular, if the estimate (4.2) holds, then the obstacle K has no trapping weakly nondegenerate rays.

Consider the cut-off resolvent $R_\psi(\lambda)$ with supp $\psi \subset \{x \in \mathbb{R}^n : \ 0 < a' < |x| < b'\}$. For $\lambda \in \mathbb{R}$ and sufficiently large a' and b', Burq [2] (see also Cardoso and Vodev [4]) established the estimate

$$\|R_\psi(\lambda)\|_{L^2(\Omega) \longrightarrow L^2(\Omega)} \leq \frac{C_2}{1+|\lambda|}, \ \lambda \in \mathbb{R} \tag{4.3}$$

without any geometrical restriction of K. On the other hand, if we have resonances converging sufficiently fast to the real axis, the norm

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)}$$

with $\chi = 1$ on K increases like $\mathcal{O}(e^{\mathbb{C}|\lambda|})$ for $\lambda \in \mathbb{R}$, $|\lambda| \to \infty$. Thus the existence of trapping rays influences the estimates of $R_\chi(\lambda)$ with $\chi(x)$ equal to 1 on a neighborhood of the obstacle and the behaviors of the scattering amplitude $a(\lambda, \theta, \omega)$ and the cut-off resolvent $R_\chi(\lambda)$ for $\lambda \in \mathbb{R}$ are rather different if we have trapping rays.

It is interesting to examine the link between the estimates for $a(\lambda, \theta, \omega)$ and the cut-off resolvent $R_\chi(\lambda)$ for $\lambda \in U_{\varepsilon,d}$. In this direction we have the following:

Theorem 4.3 *Under the assumptions of Theorem* 4.1 *for* $a(\lambda, \theta, \omega)$ *the cut-off resolvent* $R_\chi(\lambda)$ *with arbitrary* $\chi \in C_0^\infty(\mathbb{R}^n)$ *satisfies the estimate (4.2) in* $U_{\varepsilon,d}$ *with suitable* $C' > 0$, $m' \in \mathbb{N}$ *and* $\alpha' \geq 0$.

Proof The poles of $a(\lambda, \theta, \omega)$ in $\{z \in \mathbb{C} : \ \Im\lambda > 0\}$ coincide with the poles of the scattering operator

$$S(\lambda) = I + K(\lambda) : L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1}),$$

where $K(\lambda)$ has kernel $a(\lambda, \theta, \omega)$. Thus the estimate (4.1) of $a(\lambda, \theta, \omega)$ leads to an estimate of the same type for the norm of the scattering operator $S(\lambda)$ for $\lambda \in U_{\varepsilon,d}$. Notice that $S^{-1}(\lambda) = S^*(\bar{\lambda})$ for every $\lambda \in \mathbb{C}$ for which the operator $S(\lambda)$ is invertible. Moreover, the resonances λ_j are symmetric with respect to the imaginary axe $\mathbf{i}\mathbb{R}$.

Consider the energy space $H = H_D(\Omega) \oplus L^2(\Omega)$, the unitary group $U(t) = e^{\mathbf{i}tG}$ in H related to the Dirichlet problem for the wave equation in Ω and the semigroup $Z^b(t) = P_+^b U(t) P_-^b$, $t \geq 0$, introduced by Lax and Phillips ([7]). Here $P_\pm^b$ are the orthogonal projections on the orthogonal complements of the Lax–Phillips spaces $D_\pm^b$, $b > \rho$ (see [7] for the notation). Let B^b be the generator of $Z^b(t)$. The eigenvalues z_j of B^b are independent of b, the poles of the scattering operator $S(\lambda)$ are $\{-\mathbf{i}z_j \in \mathbb{C},\ z_j \in \text{ spec } B^b\}$ and the multiplicities of z_j and $-\mathbf{i}z_j$ coincide. Given a fixed function $\chi \in C_0^\infty(\mathbb{R}^n)$, equal to 1 on K, we can choose $b > 0$ so that $P_\pm^b \chi = \chi P_\pm^b = \chi$. We fix $b > 0$

with this property and will write below B, $P_\pm$ instead of B^b, $P^b_\pm$. Changing the outgoing representation of H, we may introduce another scattering operator $S_1(\lambda)$ (see Chapter III in [7]) which is an operator-valued inner function in $\{\lambda \in \mathbb{C} :\ \Im\lambda \le 0\}$ and

$$\|S_1(\lambda)\|_{L^2(\mathbb{S}^{n-1})\to L^2(\mathbb{S}^{n-1})} \le 1,\ \Im\lambda \le 0. \tag{4.4}$$

The estimate (4.4) is not true for the scattering operator $S(\lambda) = I + K(\lambda)$ related to the scattering amplitude. On the other hand, the link between the outgoing representations of H introduced in Chapters III and V in [7] implies the equality

$$S_1(\lambda) = e^{-\mathbf{i}\beta\lambda} S(\lambda),\ \beta > 0. \tag{4.5}$$

The following estimate established in Theorem 3.2 in [7] plays a crucial role:

$$\|(\mathbf{i}\lambda - B)^{-1}\|_{H\to H} \le \frac{3}{2|\Im\lambda|}\|S_1^{-1}(\bar{\lambda})\|_{L^2(\mathbb{S}^{n-1})\to L^2(\mathbb{S}^{n-1})},\ \forall\lambda \in U_{\varepsilon,d}\setminus\mathbb{R}.$$

Since $S^{-1}(\lambda) = S^*(\bar{\lambda})$ for all $\lambda \in \mathbb{C}$ for which $S(\lambda)$ is invertible, the estimates (4.1) and (4.5) imply

$$\begin{aligned}\|(\mathbf{i}\lambda - B)^{-1}\|_{H\to H} &\le \frac{3e^{\beta|\Im\lambda|}}{2|\Im\lambda|}\|S(\lambda)\|_{L^2(\mathbb{S}^{n-1})\to L^2(\mathbb{S}^{n-1})} \\ &\le C_1(1+|\lambda|)^{m'}\frac{e^{\alpha'|\Im\lambda|}}{|\Im\lambda|},\ \forall\lambda \in U_{\varepsilon,d}\setminus\mathbb{R}.\end{aligned}$$

For $\operatorname{Re}\lambda > 0$ we have

$$\chi(\lambda - B)^{-1}\chi = \int_0^\infty e^{-\lambda t}\chi P_+ U(t) P_-\chi dt = -\mathbf{i}\chi(-\mathbf{i}\lambda - G)^{-1}\chi$$

and by an analytic continuation we obtain this equality for $\lambda \in \mathbf{i}U_{\varepsilon,d}$. By using the relation between $R_\chi(\lambda)$ and $\chi(\lambda - G)^{-1}\chi$, we deduce the estimate

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \le C_2(1+|\lambda|)^{m'}\frac{e^{\alpha'|\Im\lambda|}}{|\Im\lambda|}$$

for $\Im\lambda = \varepsilon\log(1+|\lambda|) - d$, $|\operatorname{Re}\lambda| \ge c_0$. On the other hand, $\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)}$ is bounded for $\Im\lambda = -c_1 < 0$ and have the estimate (see for example [22])

$$\|R_\chi(\lambda)\|_{L^2(\Omega)\to L^2(\Omega)} \le Ce^{C|\lambda|^n},\ \Im\lambda \le \varepsilon\log(1+|\lambda) - d.$$

Then an application of the Phragmen–Lindelöf theorem yields the result.

It is an interesting open problem to show that the analyticity of $a(\lambda,\theta,\omega)$ in $U_{\varepsilon,d}$ implies the estimate (4.1) with suitable m, α and C without any information for the *geometry* of the obstacle. The same problem arises for the strip $V_\delta = \{\lambda \in \mathbb{C} :\ 0 \le \Im\lambda \le \delta\}$ and we have the following:

Conjecture. *Assume that the scattering amplitude $a(\lambda, \theta, \omega)$ is analytic in V_δ. Then there exist constants $C_1 > 0$, $C \geq 0$ such that*

$$|a(\lambda, \theta, \omega)| \leq C_1 e^{C|\lambda|^2}, \ \forall(\omega, \theta) \in \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}, \ \forall \lambda \in V_\delta.$$

For $n = 3$ this conjecture is true since we may obtain an exponential estimate $\mathcal{O}(e^{C|\lambda|^2})$ for the cut-off resolvent $R_\chi(\lambda)$, $\lambda \in V_\delta$ (see for more details [1]).

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Analytic Hypoellipticity for a Sum of Squares of Vector Fields in $\mathbb{R}^3$ Whose Poisson Stratification Consists of a Single Symplectic Stratum of Codimension Four

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Summary. We prove analytic hypoellipticity for a sum of squares of vector fields in $\mathbb{R}^3$ all of whose Poisson strata are equal and symplectic of codimension four, extending in a model setting the recent general result of Cordaro and Hanges in codimension two [2]. The easy model we study first and then its easy generalizations possess a divisibility property reminiscent of earlier work of the author and Derridj in [3] and Grigis–Sjöstrand in [4].

2000 AMS Subject Classification: 35H10, 35H12.

Key words: Analytic, Sum of squares, Poisson stratification.

1 Introduction and statement of theorems

In $\mathbb{R}^3$, we consider sums of squares of four vector fields whose characteristic variety and all subsequent Poisson–Treves strata are symplectic of codimension four (in fact all the same):

$$P = D_x^2 + (D_y + x^2 D_t)^2 + (x^3 D_t)^2 + (y^3 D_t)^2 = \sum_1^4 X_j^2.$$

Hörmander's condition being satisfied at rank 3, P is subelliptic with $\varepsilon = 1/4$ and hence Gevrey hypoelliptic in G^s for all $s \geq 4$, but we will show that in fact P is analytic hypoelliptic.

The conjecture of Treves [11] states that the operator should be analytic hypoelliptic if (and only if) the characteristic variety and all the iterated

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_15,

Poisson strata are symplectic. Here we conclude analytic hypoellipticity for a restricted class, extending recent (unpublished) work by Cordaro and Hanges when the codimension was equal to two.

This model can be generalized as long as a divisibility property reminiscent of earlier work of the author in [3] and Grigis–Sjöstrand in [4] is maintained.

Theorem 1 *Let*

$$P = D_x^2 + \left(D_y + x^2 D_t\right)^2 + \left(x^3 D_t\right)^2 + \left(y^3 D_t\right)^2 = \sum_1^4 X_j^2.$$

Then P is analytic hypoelliptic.

Theorem 2 *Let $\ell, p > 0$ be arbitrary. And let*

$$P_g = D_x^2 + (D_y + g(x)D_t)^2 + (x^\ell D_t)^2 + (y^p D_t)^2 = \sum_1^4 X_{j,g}^2.$$

Then P_g is analytic hypoelliptic whenever there exists $k_0 > 0$ such that the real analytic function g satisfies $g^{(k)}(0) = 0, k < k_0$ but $g^{(k_0)}(0) \neq 0$.

Remark 1. The last two vector fields ensure that the characteristic variety and all the deeper Poisson–Treves strata, all equal to $\{x = y = \xi = \eta = 0\}$, are symplectic.

Remark 2. In the analogue of the strictly pseudoconvex case, here the model would have just x in place of x^2, we know that the condition of divisibility would always be satisfied.

Remark 3. The presence of the last two vector fields, in addition to ensuring that the layers are symplectic, means that for $(x, y) \neq (0, 0)$, the operator is elliptic, microlocally near $(x, y, t; \xi, \eta, \tau) = (0, 0, 0; 0, 0, 1)$ and hence the result is known. Thus only $(x, y) = (0, 0)$ is in doubt, which means that any localization $\varphi(x, y, t) = \varphi_1(x, y)\varphi_2(t)$ may be done only in the variable t, since any derivative falling on φ_1 will be nonzero only for (x, y) away from $(0, 0)$.

Remark 4. Each of the four vector fields may be multiplied by nonzero real analytic functions with no change in the statement of the theorems.

2 The proof in the case of Theorem 1

By subellipticity, we know that distribution solutions are smooth. To prove the theorem, then, it will suffice to show that, locally uniformly near the origin,

$$|D^p u| \le C^{p+1} p!$$

since for $x^2 + y^2 \neq 0$ our operator is elliptic, any localization may be taken in the variable t alone. And since the only points in the cotangent space which *might* be in the analytic wave front set of the solution are those where the operator is characteristic, namely, $(0,0,0;0,0,\tau)$ with $\tau \neq 0$, it suffices to estimate high derivatives in the t variable, and it will suffice to do so (locally) in L^2 norm. The *a priori* estimate for P is merely the maximal estimate, with the subelliptic portion used only as a starting point to control the initial norm:

$$\sum_{j=1}^{4} \|X_j v\|_0^2 + \|v\|_{\frac{1}{4}}^2 \lesssim |(Pv,v)| + \|v\|_0^2, \qquad v \in C_0^\infty.$$

Now merely introducing $v = \varphi(t) D_t^p u$ into the estimate will lead to derivatives on φ which cannot be immediately controlled:

$$[X_2, \varphi D_t^p]\, u \rightsquigarrow x^2 \varphi' D_t^p u$$

since while we may have started with the maximally controlled vector field X_2 we now lack such a vector field on the right with which to use the estimate again to maximal advantage.

Thus we introduce a variant of the effective localization of high powers of D_t which we have used before. And to avoid introducing too much notation at once, we write out, analogous to the naïve first localization φD_t^p above, a third-order version and take its bracket with X_2. Noting that $[X_2, \frac{x}{2}X_1] = -x^2 D_t$, we have, with

$$\left(D_t^1\right)'_\varphi \equiv \varphi(t) D_t + \varphi' \left(\frac{x}{2} X_1\right),$$

$$\left[X_2, \left(D_t^1\right)_\varphi D_t^{p-1}\right]$$

$$= \left(x^2 \varphi' D_t - x^2 \varphi' D_t + x^2 \varphi'' \left(\frac{x}{2} X_1\right)\right) D_t^{p-1} = x^2 \varphi'' \left(\frac{x}{2} X_1\right) D_t^{p-1}$$

and, with

$$\left(D_t^2\right)'_\varphi \equiv \varphi(t) D_t^2 + \varphi' \left(\frac{x}{2} X_1\right) D_t + \varphi'' \left(\frac{x}{2} X_1\right)^2 /2!,$$

$$\begin{aligned}
&\left[X_2, \left(D_t^2\right)_\varphi D_t^{p-2}\right] \\
&= \left\{x^2 \varphi' D_t^2 - x^2 \varphi' D_t^2\right\} D_t^{p-2} + \left\{x^2 \varphi'' \left(\frac{x}{2} X_1\right) D_t \right. \\
&\quad \left. + \left[X_2, \varphi'' \left(\frac{x}{2} X_1\right)^2 /2!\right]\right\} D_t^{p-2} \\
&= \left\{x^2 \varphi'' \left(\frac{x}{2} X_1\right) D_t - x^2 \varphi'' \left(\frac{x}{2} X_1\right) D_t\right\} D_t^{p-2}
\end{aligned}$$

$$+\left\{\varphi''\left[\frac{x}{2}X_1,\left[X_2,\frac{x}{2}X_1\right]\right]/2!+x^2\varphi'''\left(\frac{x}{2}X_1\right)^2/2!\right\}D_t^{p-2}$$
$$=\left\{\varphi''\left[\frac{x}{2}X_1,-x^2D_t\right]/2!+x^2\varphi'''\left(\frac{x}{2}X_1\right)^2/2!\right\}D_t^{p-2}$$
$$=-\varphi''x^2D_t^{p-1}/2!+x^2\varphi'''\left(\frac{x}{2}X_1\right)^2D_t^{p-2}/2!.$$

In other words, even though we expect the last term (a term where we have managed to eliminate all D_t at least to the level where we have corrected φD_t^p, and which can be corrected all the way until $p=0$, in similar fashion), there is the additional term $-\varphi''x^2D_t/2!$ which will require a new cascade of correcting terms. It arose from a double commutator $[\frac{x}{2}X_1,[X_2,\frac{x}{2}X_1]]$ which never occurred in the nondegenerate, or strictly pseudo-convex, case, where the first brackets one encountered had constant coefficients.

That being said, we propose to use *not* just

$$\left(D_t^2\right)'_\varphi=\varphi(t)D_t^2+\varphi'\frac{x}{2}X_1D_t+\varphi''\left(\frac{x}{2}X_1\right)^2/2!$$

as the second-order modification of $\varphi(t)D_t^2$ but rather

$$(T^2)_\varphi\equiv\varphi(t)D_t^2+\varphi'\frac{(\frac{x}{2}X_1)^1}{1!}D_t+\varphi''\left\{\frac{(\frac{x}{2}X_1)^2}{2!}-\frac{1}{2}\frac{(\frac{x}{2}X_1)^1}{1!}\right\}$$
$$=\sum_0^2\varphi^{(j)}A^j_{j'}\frac{(\frac{x}{2}X_1)^{j'}}{j'!}$$

(with $A^0_0=1,A^1_1=1,A^1_0=0,A^2_2=1,A^2_1=-\frac{1}{2},A^2_0=0$) so that

$$[X_2,(T^2)_\varphi]\text{ is free of }D_t,$$

and we are far more confident that we can handle terms entirely free of D_t. We note in passing that the brackets of $(T^2)_\varphi$ with X_3 or X_4 will contain factors of x^3 or y^3 which are extremely powerful as they can ‘convert’ D_t into a good vector field (X_3 or X_4 again), while brackets with X_1 are good as well—we will see that bracketing with X_1 leaves a free X_1 on the left modulo manageable terms, and this X_1 will allow us to use the maximal estimate effectively.

The general form of the modification of φD_t^p requires then a sum of p terms, each with some $\varphi^{(j)}$ followed by suitable polynomials in

$$M=\frac{x}{2}X_1$$

of degree j, which we write (for $j\geq 1, N_0$ being equal to 1) as

$$N_j=\sum_{j'=0}^{j}A^j_{j'}\frac{M^{j'}}{j'!}$$

with the coefficients $A^j_{j'}$ carefully chosen, all $A^j_j = 1$. The relation $[M, X_2] = x^2 D_t$ might more suggestively be written using the notation $X_2 = X_2' + X_2''$ with $X_2' = D_y$ and $X_2'' = x^2 D_t$. Then

$$[M, X_2] = [M, X_2''] = X_2'' = x^2 D_t$$

and so

$$[M, [M, \ldots [M, X_2]] \ldots] = [M, [M, \ldots [M, X_2'']] \ldots] = X_2'' = x^2 D_t$$

and hence

$$[M^b, X_2] = [M^b, X_2''] = \sum_{b'=1}^{b} \binom{b}{b'} ad^{b'}_M (X_2'') M^{b-b'} = x^2 \sum_{b'=1}^{b} \binom{b}{b'} M^{b-b'} D_t.$$

Thus with factorials,

$$\left[\frac{M^b}{b!}, X_2\right] = x^2 \sum_{b'=1}^{b} \frac{1}{b'!} \frac{M^{b-b'}}{(b-b')!} D_t$$

or together with the coefficients $A^j_{j'}$,

$$[N_j, X_2] = x^2 \sum_{j'=1}^{j} A^j_{j'} \sum_{j''=1}^{j'} \frac{1}{j''!} \frac{M^{j'-j''}}{(j'-j'')!} D_t.$$

We proceed to define, quite generally,

$$(T^p)_\varphi \equiv \sum_{j=0}^{p} \varphi^{(j)} N_j D_t^{p-j}$$

with $\varphi(t) = \varphi_{p_0}(t) \equiv 1$ on one neighborhood of $t = 0$ and supported in a larger one where Pu is known to be analytic for $x^2 + y^2$ small and satisfying, for N given, the Ehrenpreis-type bounds on derivatives:

$$|D^k \varphi_{p_0}| \le C^{k+1} p_0^k, \qquad k \le p_0,$$

with the constant C independent of p_0. To construct such φ_{p_0} one merely convolves p_0 copies of a standard bump function whose support has width proportional to $1/p_0$ with the characteristic function of a small interval about 0 in t.

What will be crucial in the bracket $[X_2, (T^p)_\varphi]$ is that it contain *no* residual D_t, since any residual D_t is coupled with an unacceptable number of derivatives on the localizing function.

We have, since $[X_2, N_0] = [X_2, 1] = 0$,

$$
\begin{aligned}
[X_2, (T^p)_\varphi] &= \left[X_2, \sum_{j=0}^{p} \varphi^{(j)} N_j D_t^{p-j} \right] \\
&= \sum_{j=0}^{p} \left[X_2, \varphi^{(j)} \right] N_j D_t^{p-j} + \sum_{j=1}^{p} \varphi^{(j)} [X_2, N_j] D_t^{p-j} \\
&= \sum_{j=0}^{p} x^2 \varphi^{(j+1)} N_j D_t^{p-j} \\
&\quad - \sum_{j=1}^{p} x^2 \varphi^{(j)} \sum_{j'=1}^{j} A_{j'}^{j} \sum_{j''=1}^{j'} \frac{1}{j''!} \frac{M^{j'-j''}}{(j'-j'')!} D_t^{p-(j-1)} \\
&= \sum_{j=1}^{p} x^2 \varphi^{(j)} N_{j-1} D_t^{p-(j-1)} \\
&\quad - \sum_{j=1}^{p} x^2 \varphi^{(j)} \sum_{j'=1}^{j} A_{j'}^{j} \sum_{j''=1}^{j'} \frac{1}{j''!} \frac{M^{j'-j''}}{(j'-j'')!} D_t^{p-(j-1)}
\end{aligned}
$$

by the above expression for $[N_j, X_2]$.

To make this expression exactly zero (except for terms which contain no D_t) we must match derivatives on φ, powers of D_t and powers of M in the two terms. This may be seen clearly with a shift of index (after factoring out x^2):

$$
N_{j-1} = \sum_{j'=0}^{j-1} A_{j'}^{j-1} \frac{M^{j'}}{j'!} = \sum_{j'=1}^{j} A_{j'}^{j} \sum_{j''=1}^{j'} \frac{1}{j''!} \frac{M^{j'-j''}}{(j'-j'')!}
$$

or, equating powers of M, after writing $\tilde{j} = j' - j''$ on the right and $\tilde{j} = j'$ on the left,

$$
\sum_{\tilde{j}=0}^{j-1} A_{\tilde{j}}^{j-1} \frac{M^{\tilde{j}}}{\tilde{j}!} = \sum_{\tilde{j}=0}^{j-1} \sum_{j''=1}^{j-\tilde{j}} A_{\tilde{j}+j''}^{j} \frac{1}{j''!} \frac{M^{\tilde{j}}}{\tilde{j}!},
$$

which forces, for all $\tilde{j} \leq j-1$,

$$
A_{\tilde{j}}^{j-1} = \sum_{j''=1}^{j-\tilde{j}} A_{\tilde{j}+j''}^{j} \frac{1}{j''!}
$$

together with the conditions $A_r^r = 1$ for all r and some freedom, for example assigning the values of $A_0^j, j > 0$, or the values of $A_k^p, k > 0$. In practical terms, the former amounts to adding localizations of lower powers of D_t to that of the highest power.

However, as shown in [3] and Hirzebruch's book [5] Lemma 1.7.1, the unique solution with $A_q^q = 1$ for all q and $A_0^q = (-1)^q$ for all q is given by

$$A_s^r = \left(\left(\frac{t}{e^t-1}\right)^{r+1}\right)^{(r-s)}(0)/(r-s)!$$
$$= \text{ the coefficient of } t^{r-s} \text{ in } \left(\frac{t}{e^t-1}\right)^{r+1},$$

and these coefficients satisfy the desired estimates: there exists a constant C such that for $s \le r$,

$$|A_s^r| \le C^r.$$

In addition, one may 'raise and lower' indices nicely:

Proposition 2.1 *For any p, q, and c, we may shift lower indices:*

$$A_q^p = \sum_{j=0}^{p-q} S_j^{c+j} A_{q-c}^{p-(c+j)}.$$

Lemma 2.1 *There exists a constant C such that for $s \le r$ we have*

$$|S_s^r| \le C^r.$$

An easy calculation shows that

$$[X_1, N_j] = -\sum_{j'=0}^{j} A_{j'}^j \sum_{j''=1}^{j'} \frac{ad_M^{j''}(X_1)}{j''!}\frac{M^{j'-j''}}{(j'-j'')!}$$
$$= -X_1 \circ \sum_{j'=0}^{j} A_{j'}^j \sum_{j''=1}^{j'} \frac{\left(-\frac{1}{2}\right)^{j''}}{j''!}\frac{M^{j'-j''}}{(j'-j'')!}$$
$$= \sum_{1\le j''+\ell\le j} \frac{\left(-\frac{1}{2}\right)^{j''}}{j''!} S_\ell^{j''+\ell}\left\{-X_1 \circ N_{j-(j''+\ell)}\right\}.$$

Similarly when we take the bracket with $X_3 = x^3 D_t$ we obtain

$$[X_3, N_j] = -X_3 \circ \sum_{1\le j''\le j'\le j} A_{j'}^j \frac{\left(\frac{3}{2}\right)^{j''}}{j''!}\frac{M^{j'-j''}}{(j'-j'')!}$$
$$= \sum_{1\le j''+\ell\le j} \frac{\left(\frac{3}{2}\right)^{j''}}{j''!} S_\ell^{j''+\ell}\{-X_3 \circ N_{j-(j''+\ell)}\}.$$

Lastly, the bracket with X_4 is zero.

Thus in all,

$$[X_1, (T^p)_\varphi] = X_1 \circ \sum_{1 \le p'' \le p' \le p} R^{p'}_{p''} (T^{p-p'})_{\varphi_{p''}},$$
$$[X_2, (T^p)_\varphi] = \underline{C^p}\, \varphi^{(p+1)} X^p / p!,$$
$$[X_3, (T^p)_\varphi] = X_3 \circ \sum_{1 \le p'' \le p' \le p} \tilde{R}^{p'}_{p''} (T^{p-p'})_{\varphi_{p''}}$$

and

$$[X_4, (T^p)_\varphi] = 0,$$

where $\varphi_{p''}$ denotes a derivative of φ of order at most p'' and

$$|R^{p'}_{p''}| + |\tilde{R}^{p'}_{p''}| \le C^{p'}.$$

Thus we insert $v = (T^p)_\varphi u$ in the *a priori* estimate

$$\sum_{j=1}^{4} \|X_j (T^p)_\varphi u\|_0 + \|(T^p)_\varphi u\|_{\frac{1}{4}} \lesssim |(P(T^p)_\varphi u, (T^p)_\varphi u)| + \|(T^p)_\varphi u\|_0^2$$

and, with $Pu = \alpha$ analytic, write

$$(P(T^p)_\varphi u, (T^p)_\varphi u) = ((T^p)_\varphi \alpha, (T^p)_\varphi u) + E,$$

where the error E is expressed in terms of the above commutators in ways which by now are more or less obvious. In view of the beautiful bracket relations, we may iterate the use of the maximal estimate until we have pure X derivatives, then continue to use it until once again we have pure D_t derivatives. Removing that localizing function from the norm and introducing a new one as in [7], [8], as many as $\log_2 p_0$ times we arrive at an overall bound of

$$C^{p_0+1} p_0^{p_0} \text{ or } C^{p_0+1} p_0!$$

for derivatives of the solution of order p_0, locally, which proves analyticity.

3 The proof in the case of Theorem 2

For Theorem 2, the bracket of X_2 with φD_t is

$$[X_2, \varphi D_t] = g(x) D_t$$

so that if we define

$$M = \frac{g}{g'} \frac{\partial}{\partial x},$$

we will have

$$[M, X_2] = g D_t, \quad [M, [M, X_2]] = g D_t, \quad \text{etc.}$$

just as before, except that $\frac{g}{g'}$ takes the place of $\frac{x}{2} = \frac{x^2}{2x}$.

This time brackets with X_1 behave as follows:

$$[X_1, N_j] = \sum_{j'=0}^{j} A^j_{j'} \sum_{j''=1}^{j'} \frac{ad^{j''}_M(X_1)}{j''!} \frac{M^{j'-j''}}{(j'-j'')!}$$
$$= \sum_{j'=0}^{j} \sum_{j''=1}^{j'} \frac{M^{j''}\left(\frac{g}{g'}\right)}{j''!} \circ X_1 \circ A^j_{j'} \frac{M^{j'-j''}}{(j'-j'')!}$$
$$= \sum_{1 \le j''+\ell \le j} \frac{M^{j''}\left(\frac{g}{g'}\right)}{j''!} S^{j''+\ell}_{\ell} \{X_1 \circ N_{j-(j''+\ell)}\},$$

brackets with X_3 similarly as above, and brackets with X_4 are again zero.

With coefficients, the analysis is not fundamentally changed, and details have been worked out in the related case of pseudo-convex domains in $\mathbb{C}^n$ or pseudo-convex C-R manifolds in [3].

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Multidimensional Soliton Integrodifferential Systems

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Summary. The article presents preliminary results on applications of the theory of noncommutative KdV equation $u_t = \partial^3 u - 3\,(u\partial u + (\partial u)\,u)$ (recently developed in [Treves, 2007]) to algebras of matrices, first of finite rank and then of infinite rank. The resulting differential equations in these algebras can only make sense in a noncommutative setup, as the basic "space derivation" is commutation with another (fixed) matrix. The infinite rank situation is reinterpreted, via Hermite expansion, in the algebra of bounded linear operators on Schwartz space $\mathcal{S}\,(\mathbb{R}^n)$. Special choices of the "space derivation" as commutation with partial differential operators can be identified to evolution equations whose linear part is partial differential (in $\mathbb{R}^{2n+1}$) and the nonlinear part is integrodifferential: a partial differential operator (in $\mathbb{R}^{2n}$) acting on the square of the unknown u in the sense of Volterra composition. The choice of the harmonic oscillator $D_x^2 + x^2$ (when $n = 1$) is particularly amenable to Hermite expansion approach. Existence and uniqueness of global solutions in the Cauchy problem can be proved for special initial data (in $\mathbb{R}^{2+1}$)

2000 AMS Subject Classification: Primary: 37J35, Secondary: 35Q53

Key words: Noncommutative differential algebras, Bäcklund transformations, Korteweg–de Vries hierarchy.

1 Basic facts of noncommutative KdV theory

We are going to use some of the main properties of the following noncommutative version of the classical KdV equation:

$$u_t = \partial^3 u - 3\,(u\partial u + (\partial u)\,u)\,. \tag{1.1}$$

*This paper is in final form and no version of it will be submitted for publication elsewhere.

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_16,

The unknown u in (1.1) is a smooth function of time t valued in some noncommutative (but associative) differential algebra $(\mathbb{A}, \partial)$. In standard (i.e., commutative) KdV theory $\mathbb{A}$ is usually an algebra of smooth functions of the real variable $x \in \mathbb{R}$, either periodic or rapidly decaying at infinity, or a suitable algebra of Laurent series in a "complex variable" x; and $\partial = \frac{\partial}{\partial x}$. In the noncommutative theory a much more varied choice of $(\mathbb{A}, \partial)$ is possible. In the present article our focus shall be mainly on algebras of $n \times n$ matrices (with complex entries and, possibly, with $n = +\infty$) and on derivations $\partial = \partial_X$ of the type $Y \longrightarrow [X, Y]$, $X, Y \in \mathbb{A}$, an object that makes no sense in the commutative setup.

1.1 Noncommutative setup

In order to state the fundamental properties of Eq. (1.1) we introduce the universal monogenic differential algebra $\mathfrak{P}$: this is the algebra of polynomials in the noncommuting indeterminates $\xi_0, \xi_1, ..., \xi_\nu, ...$, equipped with the *chain rule derivation* $\mathfrak{d}$: $\mathfrak{d}\xi_i = \xi_{i+1}$ (for details on this and on what follows see [Treves, 2007]). We denote by $\mathfrak{P}_\circ$ the maximal ideal in $\mathfrak{P}$, consisting of the polynomials that "vanish at the origin", i.e., without a constant term. The restriction of $\mathfrak{d}$ to $\mathfrak{P}_\circ$ is injective. Note that $\mathfrak{d}\mathfrak{P} \subset \mathfrak{P}_\circ$. We denote by $[\mathfrak{P}, \mathfrak{P}]$ the linear span over $\mathbb{C}$ of the commutators $[P, Q]$; we have $[\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{P}_\circ$. Given any derivation D in the algebra $\mathfrak{P}$ the Leibniz formula implies $D\,[\mathfrak{P}, \mathfrak{P}] \subset [\mathfrak{P}, \mathfrak{P}]$. In certain circumstances it is convenient to reason in the "completion" $\widehat{\mathfrak{P}}$ of $\mathfrak{P}$, the algebra of formal power series in the indeterminates $\xi_0, \xi_1, ..., \xi_\nu, ...$. The extension of $\mathfrak{d}$ to $\widehat{\mathfrak{P}}$ is self-evident.

In the applications to a specific differential algebra $(\mathbb{A}, \partial)$ one selects an element $a \in \mathbb{A}$ and replaces ξ_j by $\partial^j a$, $j \in \mathbb{Z}_+$. If $P = P\left(\xi_0, \xi_1, ..., \xi_\nu\right) \in \mathfrak{P}$, we write $P\,[a] = P\left(a, \partial a, ..., \partial^\nu a\right)$. The term *chain rule derivation* is due to the self-evident formula

$$\partial\left(P\,[a]\right) = \left(\mathfrak{d}P\right)[a]\,. \tag{1.2}$$

We must introduce two types of derivatives special to the noncommutative setup: *replacement derivatives* and *twisted derivatives.*

Definition 1.1 *Let $k \in \mathbb{Z}_+$ and $F \in \mathfrak{P}$. We denote by $\left(F\frac{\partial}{\partial \xi_j}\right)^{\checkmark}$ the derivation D of $\mathfrak{P}$ such that $D\xi_j = F$ and $D\xi_k = 0$ if $k \neq j$.*

With this notation we can write

$$\mathfrak{d} = \sum_{j=0}^{\infty}\left(\xi_{j+1}\frac{\partial}{\partial \xi_j}\right)^{\checkmark}. \tag{1.3}$$

Definition 1.2 *We shall denote by $\frac{\partial^{tw}}{\partial \xi_k}$ the linear endomorphism of $\mathfrak{P}$ defined as follows: If $I = (i_1, ..., i_\nu)$ is any nonvoid multi-index and $\xi_I = \xi_{i_1} \cdots \xi_{i_\nu}$, then*

$$\frac{\partial^{tw}}{\partial \xi_k} \xi_I = \sum_{\alpha=1}^{\nu} \varepsilon\left(i_\alpha, k\right) \xi_{i_{\alpha+1}} \cdots \xi_{i_\nu} \xi_{i_1} \cdots \xi_{i_{\alpha-1}}$$

with $\varepsilon\left(i_\alpha, k\right) = 1$ *if* $i_\alpha = k$ *and* $\varepsilon\left(i_\alpha, k\right) = 0$ *if* $i_\alpha \neq k$; $\frac{\partial^{tw}}{\partial \xi_k} 1 = 0$. *We shall refer to* $\frac{\partial^{tw}}{\partial \xi_k}$ *as the* ***twisted partial derivative*** *with respect to* ξ_k.

The linear endomorphism $\frac{\partial^{\mathrm{tw}}}{\partial \xi_k}$ is *not* a derivation of $\mathfrak{P}$: it does not satisfy the Leibniz formula. We have

$$\forall P, Q \in \mathfrak{P},\ \forall k \in \mathbb{Z}_+,\ \frac{\partial^{\mathrm{tw}}}{\partial \xi_k} [P, Q] = 0. \tag{1.4}$$

The next definition is crucial to the "Hamiltonian" theory of Eq. (1.1):

Definition 1.3 *By the* ***twisted variational derivative*** *we shall mean the linear operator in* $\mathfrak{P}$,

$$P \longrightarrow \frac{\delta^{tw} P}{\delta \xi} = \sum_{j=0}^{\infty} (-1)^j\, \mathfrak{d}^j \left(\frac{\partial^{tw} P}{\partial \xi_j} \right).$$

The twisted variational derivative $\frac{\delta^{\mathrm{tw}}}{\delta \xi}$ plays in the noncommutative setup the role played in the commutative setup by the variational derivative $\frac{\delta}{\delta \xi}$ of Gelfand–Dickey ([G-D, 1975], [G-D, 1976], [G-D, 1977]). The next statement is easy to prove.

Proposition 1.1 *The null-space of* $\frac{\delta^{tw}}{\delta \xi}$ *in* $\mathfrak{P}_\circ$ *is exactly equal to the vector sum* $[\mathfrak{P}, \mathfrak{P}] + \mathfrak{d}\mathfrak{P}$.

A polynomial $R \in \mathfrak{P}$ is said to be *Hamiltonian* if there is $Q \in \mathfrak{P}$ such that $R = \mathfrak{d} \frac{\delta^{\mathrm{tw}} Q}{\delta \xi}$; often the polynomial $\frac{\delta^{\mathrm{tw}} Q}{\delta \xi}$ itself is called a *hamiltonian.*

Let $P \in \mathfrak{P}$. In the commutative setup a polynomial Q is said to be conserved for the evolution equation $u_t = P[u]$ (or for P) if $P \frac{\delta Q}{\delta \xi} \in \mathfrak{d}\mathfrak{P}$. This concept is much too restrictive; it must be replaced by that of trace-conserved polynomials.

Definition 1.4 *A polynomial* $Q \in \mathfrak{P}$ *is said to be* ***trace-conserved*** *for* $P \in \mathfrak{P}$ *if* $P \frac{\delta^{tw} Q}{\delta \xi} \in [\mathfrak{P}, \mathfrak{P}] + \mathfrak{d}\mathfrak{P}$.

It follows directly from Proposition 1.1 that every polynomial belonging to $[\mathfrak{P}, \mathfrak{P}] + \mathfrak{d}\mathfrak{P}$ is trace-conserved for any polynomial $P \in \mathfrak{P}$. In view of this fact it is convenient to mod out the linear subspace $[\mathfrak{P}, \mathfrak{P}] + \mathfrak{d}\mathfrak{P}$. Actually the quotient map $\mathfrak{P} \longrightarrow \mathfrak{P} / \left([\mathfrak{P}, \mathfrak{P}] + \mathfrak{d}\mathfrak{P}\right)$ can be factored as the composition of two (commuting) quotient maps: the projection $\mathfrak{P} \longrightarrow \mathfrak{P}/\mathfrak{d}\mathfrak{P}$, customarily denoted by $P \longrightarrow \int P$; and the quotient maps $\mathrm{Tr} : \mathfrak{P} \longrightarrow \mathfrak{P}/[\mathfrak{P},\mathfrak{P}]$. To say that a polynomial Q is trace-conserved for $P \in \mathfrak{P}$ is the same as saying that $\int \mathrm{Tr}\, Q = 0$.

If $(\mathbb{A},\partial)$ is a differential algebra generated by a single element a and if we replace ξ_j by $\partial^j a$ for every $j \in \mathbb{Z}_+$ the map Tr corresponds to the quotient map $\mathrm{Tr}_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}/[\mathbb{A},\mathbb{A}]$. If $\mathbb{A}$ is an algebra of $N \times N$ matrices, the kernel of $\mathrm{Tr}_{\mathbb{A}}$ consists of matrices whose trace vanishes. But there might be matrices $a \in \mathbb{A}$ whose trace (in the ordinary sense: $\mathrm{tr}\, a$) is zero and yet $\mathrm{Tr}_{\mathbb{A}}\, a \neq 0$. Indeed, $\mathrm{tr}\, a = 0 \Longleftrightarrow a \in [\mathbf{M}_N(\mathbb{C}), \mathbf{M}_N(\mathbb{C})]$, which might not be equivalent to $a \in [\mathbb{A},\mathbb{A}]$.

Let now $u(t)$ be a $\mathcal{C}^\infty$ function of t valued in $\mathbb{A}$ which is a solution of the evolution equation $u_t = P[u]$, $P \in \mathfrak{P}$. We point out that $\left(\partial^k u\right)_t = \left(\mathfrak{d}^k P\right)[u]$ for all $k \in \mathbb{Z}_+$. Given any polynomial $Q \in \mathfrak{P}$ the chain rule shows directly that

$$\partial_t Q[u(t)] = \sum_{k=0}^{\infty} \left(\left(\mathfrak{d}^k P \frac{\partial}{\partial \xi_k} \right)^{\surd} Q \right) [u(t)] \tag{1.5}$$

(cf. Definition 1.1). It is easily seen that

$$\sum_{k=0}^{\infty} \left(\mathfrak{d}^k P \frac{\partial}{\partial \xi_k} \right)^{\surd} Q - P \frac{\delta^{\mathrm{tw}} Q}{\delta \xi} \in [\mathfrak{P},\mathfrak{P}] + \mathfrak{d}\mathfrak{P}.$$

If Q is trace-conserved for P, we conclude that

$$\int \mathrm{Tr} \sum_{k=0}^{\infty} \left(\mathfrak{d}^k P \frac{\partial}{\partial \xi_k} \right)^{\surd} Q = 0. \tag{1.6}$$

In view of (1.5) we rewrite this as $\frac{d}{dt} \int \mathrm{Tr}_{\mathbb{A}}\, Q[u(t)] = 0$, which we interpret as saying that $\int \mathrm{Tr}_{\mathbb{A}}\, Q[u(t)]$ is a *constant of motion*. In the present context this is an abstract object, since $\int \mathrm{Tr}_{\mathbb{A}}\, Q[u(t)]$ is an element of $\mathbb{A}/([\mathbb{A},\mathbb{A}] + \partial\mathbb{A})$. But in applications it can be given a concrete meaning, as the following two examples show.

Example 1. Let $\mathbb{A}$ be a differential subalgebra of the algebra of $n \times n$ matrices $\mathbf{M}_n(\mathbb{B})$ $(n \geq 2)$ with entries in a *commutative* differential algebra $(\mathbb{B},\partial)$ (with scalar field $\mathbb{C}$). If necessary we adjoin to $\mathbb{A}$ the $n \times n$ identity matrix I_n as the unit element. The derivation in $\mathbb{A}$ is ∂ acting entrywise (with $\partial I_n = 0$). We avail ourselves of the fact that the trace of any matrix $a \in \mathbf{M}_n(\mathbb{B})$, $\mathrm{tr}\, a \in \mathbb{B}$, vanishes if $a \in [\mathbb{A},\mathbb{A}]$. Let $Q \in \mathfrak{P}$ be a trace-conserved polynomial of $P \in \mathfrak{P}$; (1.6) entails that $\frac{d}{dt} \mathrm{tr}\, Q[u(t)] = \partial \Phi[u(t)]$ for some "flux" $\Phi \in \mathfrak{P}$ and all solutions $u \in \mathcal{C}^\infty(\mathbb{R};\mathbb{A})$ of the evolution equation $u_t = P[u]$.

More concretely, take $\mathbb{B}$ to be either $\mathcal{C}^\infty\left(\mathbb{S}^1\right)$ or the Schwartz space $\mathcal{S}(\mathbb{R})$, and $\partial = \frac{d}{dx}$. In both these choices, for a function φ to belong to $\partial\mathbb{B}$ it is necessary and sufficient that $\int_{\mathbb{X}} \varphi(x)\, dx = 0$ ($\mathbb{X} = \mathbb{S}^1$ or $\mathbb{R}$). We can state that if $u(t,x)$ is a $\mathcal{C}^\infty$ function of $t \in \mathbb{R}$ valued in $\mathbf{M}_n(\mathbb{B})$ satisfying the evolution equation $u_t = P[u]$ and if $Q \in \mathfrak{P}$ is a trace-conserved polynomial of P, then the complex function $t \longrightarrow \int_{\mathbb{X}} \operatorname{tr} Q[u(t,x)]\, dx$ is constant.

Example 2. Select a matrix $X \in \mathbf{M}_n(\mathbb{C})$ and let $\mathbb{A}$ be a subalgebra of $\mathbf{M}_n(\mathbb{C})$ $(n \geq 2)$ stable under the map $\operatorname{Ad} X$. As derivation ∂ we take $\partial_X Y = \operatorname{Ad} X(Y) = [X,Y]$. If $Q \in \mathfrak{P}$ is a trace-conserved polynomial of P and $u(t) \in \mathcal{C}^\infty(\mathbb{R};\mathbb{A})$ is a solution of the evolution equation $u_t = P[u]$, then the $n \times n$ matrix $\partial_t Q[u(t)]$ belongs to $[\mathbb{A},\mathbb{A}] + \partial_X \mathbb{A} \subset [\mathbf{M}_n(\mathbb{C}), \mathbf{M}_n(\mathbb{C})]$ and therefore its (ordinary) trace vanishes. We conclude that the complex function $t \longrightarrow \operatorname{tr} Q[u(t)]$ is constant.

We also note that if $P = \mathfrak{d}\frac{\delta^{\mathrm{tw}} Q}{\delta \xi}$, then Q is trace-conserved for P. Indeed,

$$P\frac{\delta^{\mathrm{tw}} Q}{\delta \xi} - \mathfrak{d}\left(\frac{1}{2}\left(\frac{\delta^{\mathrm{tw}} Q}{\delta \xi}\right)^2\right) \in [\mathfrak{P},\mathfrak{P}].$$

1.2 Fundamental properties of the noncommutative KdV equation

To find the trace-conserved polynomials of the KdV equation we follow the approach of Gelfand–Dickey ([G-D, 1977]; see also [Dickey, 2003]). We introduce the symbolic calculus of classical pseudodifferential operators in a single varible, but with coefficients in a differential algebra $(\mathbb{A},\partial)$. The symbols are Laurent series

$$\sigma(z) = \sum_{n=-\infty}^{N} c_n z^n \tag{1.7}$$

with coefficients $c_n \in \mathbb{A}$ and $N \in \mathbb{Z}$; N may vary with σ; assuming that $c_N \neq 0$ we refer to N as the *order* of the series (1.7). There are *two* commuting derivations in $\mathbb{A}[z] \oplus \mathbb{A}\left[\left[z^{-1}\right]\right]$, $\frac{d}{dz}$ and ∂, with ∂ acting coefficientwise. Below we use the standard decomposition of symbols (1.7):

$$\sigma_+ = \sum_{n=0}^{N} c_n z^n,\ \sigma_- = \sum_{n=1}^{\infty} c_{-n} z^{-n}. \tag{1.8}$$

We shall refer to the leading coefficient in $\sigma_{\text{-}}(z)$, c_{-1}, as the *residue* of σ and denote it by $\operatorname{Res}\sigma$. The symbol composition is the standard one:

$$\sigma_1 \# \sigma_2 = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{d^n \sigma_1}{dz^n}\right) \partial^n \sigma_2. \tag{1.9}$$

The right-hand side in (1.9) is supposed to be rewritten in the form (1.7): if $\sigma_1 = a z^m$ $(m \in \mathbb{Z})$, then

$$\frac{1}{n!}\left(\frac{d^n\sigma_1}{dz^n}\right)\partial^n\sigma_2 = \frac{m\,(m-1)\cdots(m-n+1)}{n!}a\,(\partial^n\sigma_2)\,z^{m-n}.$$

We write

$$[\sigma_1,\sigma_2]_\# = \sigma_1\#\sigma_2 - \sigma_2\#\sigma_1. \tag{1.10}$$

The composition law (1.9) is associative but not commutative (even if $\mathbb{A}$ is commutative). We denote by $\mathbf{Symb}\,(\mathbb{A})$ the set $\mathbb{A}\,[z]\oplus\mathbb{A}\left[\left[z^{-1}\right]\right]$ equipped with ordinary addition and with the multiplication (1.9). The elements of $\mathbf{Symb}\,(\mathbb{A})$ will be referred to as *symbols* valued in $\mathbb{A}$.

We shall always assume that $\mathbb{A}$ has a unit element, 1; we denote by $\mathbf{1}$ the symbol "identically equal" to 1. For every $\sigma\in\mathbf{Symb}\,(\mathbb{A})$,

$$\sigma\#\mathbf{1} = \mathbf{1}\#\sigma = \sigma. \tag{1.11}$$

Thus $\mathbf{Symb}\,(\mathbb{A})$ is a noncommutative ring with a unit element.

In connection with residues the following proposition expresses an important property of commutators:

Proposition 1.2 *If* $\sigma_1,\sigma_2\in\mathbf{Symb}\,(\mathbb{A})$, *then* $\mathrm{Res}\,[\sigma_1,\sigma_2]_\#\in[\mathbb{A},\mathbb{A}]+\partial\mathbb{A}$.

This said, we focus on $\mathbf{Symb}\,(\mathfrak{P})$, more specifically on the symbol $z^2-\xi_0$ of the *Sturm–Liouville* differential operator $\mathfrak{d}^2-\xi_0$. It is readily seen that the half-integral powers of $z^2-\xi_0$ are well-defined symbols belonging to $\mathbf{Symb}\,(\mathfrak{P})$. We use the following notation, for each $m\in\mathbb{Z}_+$,

$$S_m\,(\xi) = -2^{2m+1}\,\mathrm{Res}\left(z^2-\xi_0\right)^{\#\left(m+\frac{1}{2}\right)},\; R_m\,(\xi) = \mathfrak{d}S_m\,(\xi)\,. \tag{1.12}$$

To give an idea of the expressions of the polynomials S_m here is a short list of those of small degree:

$$\begin{aligned}
S_0 &= \xi_0 \qquad\qquad (1.13)\\
S_1 &= \xi_2 - 3\xi_0^2;\\
S_2 &= \xi_4 - 5\,(\xi_0\xi_2+\xi_2\xi_0) - 5\xi_1^2 + 10\xi_0^3;\\
S_3 &= \xi_6 - 7\,(\xi_4\xi_0+\xi_0\xi_4) - 14\,(\xi_3\xi_1+\xi_1\xi_3)\\
&\quad - 21\xi_2^2 + 21\xi_2\xi_0^2 + 28\xi_0\xi_2\xi_0 + 21\xi_0^2\xi_2\\
&\quad + 28\xi_1^2\xi_0 + 28\xi_0\xi_1^2 + 14\xi_1\xi_0\xi_1 - 35\xi_0^4.
\end{aligned}$$

We have $R_0=\xi_1$; $R_1=\xi_3-3\,(\xi_0\xi_1+\xi_1\xi_0)$ is the (noncommutative) KdV polynomial. The sequence $\{R_m\}_{m=0,1,2,\ldots}$ makes up the *KdV hierarchy.*

Theorem 1 *For every pair of integers* m,n, S_m *is a trace-conserved polynomial of* R_n.

The next statement embodies the "Hamiltonian nature" of the KdV hierarchy.

Theorem 2 *For each $m \in \mathbb{Z}_+$,*

$$S_m = -\frac{1}{2\,(2m+3)}\frac{\delta^{tw} S_{m+1}}{\delta\xi}. \tag{1.14}$$

In particular, the polynomial S_m is a hamiltonian. It also follows that

$$R_{m-1} = (-1)^m \,\mathfrak{d}\frac{\delta^{\mathrm{tw}} Q_m}{\delta\xi} \tag{1.15}$$

with $Q_m \in \mathfrak{P}$ any polynomial congruent to $\frac{(-1)^m}{2(2m+1)} S_m \mod ([\mathfrak{P},\mathfrak{P}] + \mathfrak{d}\mathfrak{P})$. It is convenient to have compact expressions of such polynomials:

$$Q_1 = \frac{1}{2}\xi_0^2, \tag{1.16}$$

$$Q_2 = \frac{1}{2}\xi_1^2 + \xi_0^3,$$

$$Q_3 = \frac{1}{2}\xi_2^2 + 5\xi_0\xi_1^2 + \frac{5}{2}\xi_0^4,$$

$$Q_4 = \frac{1}{2}\xi_3^2 + 7\xi_0\xi_2^2 + \frac{35}{3}\xi_0^2\xi_1^2 + \frac{70}{3}\left(\xi_0\xi_1\right)^2 + 7\xi_0^5.$$

Furthermore there is an ascending recurrence relation between the S_m mimicking the *Lenard relation* in the commutative setup (see [Lax, 1977], [Lax, 1978]). Define the operator

$$\mathcal{L}P = \mathfrak{d}^3 P - 2\left((\mathfrak{d}P)\,\xi_0 + \xi_0\mathfrak{d}P\right) - \left(P\xi_1 + \xi_1 P\right) + \left[\xi_0, \mathfrak{d}^{-1}\left[\xi_0, P\right]\right] \tag{1.17}$$

acting on polynomials $P \in \mathfrak{P}_\circ$ such that $[\xi_0, P] \in \mathfrak{d}\mathfrak{P}$. For each $m \in \mathbb{Z}_+$ it can be shown that $[\xi_0, S_m] \in \mathfrak{d}\mathfrak{P}$ and that

$$R_{m+1} = \mathfrak{d}S_{m+1} = \mathcal{L}S_m. \tag{1.18}$$

It can also be shown that the Sturm–Liouville operator $\mathfrak{d}^2 - \xi_0$ and the differential operator $P_m(\xi, \mathfrak{d})$ with symbol $P_m(\xi, z) = -4^m \Lambda_+^{\#\left(m+\frac{1}{2}\right)}(\xi, z)$ form a *Lax pair* defining the noncommutative mth KdV polynomial R_m ([Lax, 1977]):

$$R_m(\xi) = -\left[P_m(\xi, \mathfrak{d}), \mathfrak{d}^2 - \xi_0\right]. \tag{1.19}$$

For proofs of the statements in this section see [Treves, 2007].

1.3 Traveling wave solutions in the abstract noncommutative setup

It is convenient to take as framework the *completion* $\widehat{\mathfrak{P}}$ of the ring $\mathfrak{P}$; $\widehat{\mathfrak{P}}$ is the ring of formal power series

$$f(\xi) = \sum_I c_I \xi_I, \tag{1.20}$$

where now the sum is allowed to range over infinite sets of multi-indices. The completions $\widehat{\mathfrak{P}}$, $\widehat{\mathfrak{P}}_\circ$ are Fréchet spaces for the topology of convergence of the individual coefficients; $\widehat{\mathfrak{P}}_\circ$ is the maximal ideal of $\widehat{\mathfrak{P}}$ (still defined by the vanishing of $c_\emptyset$).

In this setup the concept of *translation* makes sense:

Definition 1.5 *The* ***translate*** *of a series $f \in \widehat{\mathfrak{P}}$ by a number $\tau \in \mathbb{C}$ will be the series* $\exp(-\tau\mathfrak{d}) f$.

If $(\mathbb{A}, \partial)$ is a differential algebra and $a \in \mathbb{A}$, the operation $\exp(-\tau\partial)\, a$ does not make sense, unless special properties are hypothesized, about the algebra $\mathbb{A}$ or the element a. If h is an entire holomorphic function on $\mathbb{C}$ and $\partial = \frac{\partial}{\partial z}$, then $\exp(-\tau\partial)\, h(z) = h(z-\tau)$. Likewise, if h is a real-analytic function on $\mathbb{R}$ and if $\tau \in \mathbb{R}$, then $\exp(-\tau\partial)\, h(x) = h(z-x)$.

Let $u_\circ \in \mathbb{A}$ have the property that $\exp(-\tau\partial)\, u_\circ$ is a well-defined element of $\mathbb{A}$ for all real numbers $\tau \in (-T, T)$ (with $0 < T \leq +\infty$); we can ask whether the $\mathbb{A}$-valued function $u(t) = \exp(-vt\partial)\, u_\circ$ ($v > 0$, $0 < t < T/v$) is a solution of the KdV equation (1.1), i.e.,

$$u'(t) = \partial\left(\partial^2 u(t) - 3u^2(t)\right). \tag{1.21}$$

This means that

$$\partial\left(\partial^2 u_\circ - 3u_\circ^2 + vu_\circ\right) = 0. \tag{1.22}$$

Equation (1.22) is satisfied if the following *eigenvalue equation* is satisfied:

$$\partial^2 u_\circ - 3u_\circ^2 = -vu_\circ. \tag{1.23}$$

It is not unreasonable to refer to $\exp(-vt\partial)\, u_\circ$ as a **traveling wave solution**.

2 Finite-dimensional systems

2.1 Matrix systems

In this subsection we take a closer look at the KdV equation in the ring $\mathbf{M}_n(\mathbb{C})$ of $n \times n$ matrices with complex entries ($2 \leq n \in \mathbb{Z}$). We suppose given a matrix $X \in \mathbf{M}_n(\mathbb{C})$ and write $\partial_X Y = XY - YX$, $Y \in \mathbf{M}_n(\mathbb{C})$; the corresponding KdV equation reads

$$\frac{dU}{dt} = \partial_X \left(\partial_X^2 U - 3U^2\right), \tag{2.1}$$

where $U = U(t) = (u_{jk}(t))_{1 \le j,k \le n}$ is a smooth function of t valued in $\mathbf{M}_n(\mathbb{C})$. Equation (2.1) is a system of n^2 ODEs in the n^2 unknowns $u_{jk}(t)$. The fundamental theorem of ODE theory states that solutions $U(t)$ exist, defined and analytic in some interval $|t| < T$ with $T > 0$ depending on X and on the initial value $U(0)$. Since the right-hand side in (2.1) is a commutator of matrices the (ordinary) trace of both sides in (2.1) vanishes identically and $\operatorname{tr} U$ is a constant of motion.

For the sake of simplicity we are going to suppose X diagonalizable; let $\Gamma \in \mathbf{GL}(n,\mathbb{C})$ be such that $\Lambda = \Gamma X \Gamma^{-1}$ is diagonal, and let λ_i, $i = 1, ..., n$, denote the diagonal entries of Λ. We have

$$\Gamma(\partial_X U)\Gamma^{-1} = \partial_\Lambda \left(\Gamma U \Gamma^{-1}\right)$$

and therefore, if we define $V = \Gamma U \Gamma^{-1} = (v_{jk})_{1 \le j,k \le n}$, (2.1) is transformed into

$$\frac{dV}{dt} = \partial_\Lambda \left(\partial_\Lambda^2 V - 3V^2\right). \tag{2.2}$$

We have $\partial_\Lambda V = ((\lambda_j - \lambda_k) v_{j,k})_{1 \le j,k \le n}$ and

$$\frac{dv_{j,k}}{dt} = (\lambda_j - \lambda_k)\left((\lambda_j - \lambda_k)^2 v_{j,k} - 3\sum_{\ell=1}^{n} v_{j,\ell} v_{\ell,k}\right); \tag{2.3}$$

(2.3) implies $\frac{dv_{i,i}}{dt} = 0$ for every $i = 1, ..., n$:

Proposition 2.1 *Each diagonal entry $v_{i,i}$ ($i = 1, ..., n$) is time-independent.*

We can rewrite (2.3) as follows:

$$\frac{dv_{j,k}}{dt} = (\lambda_j - \lambda_k)\left(\left((\lambda_j - \lambda_k)^2 - 3(v_{j,j} + v_{k,k})\right) v_{j,k} - 3\sum_{\ell=1}^{n} v_{j,\ell} v_{\ell,k}\right). \tag{2.4}$$

2.2 Constants of motion and absence of isospectrality

Throughout this subsection $U(t)$ will be an arbitrary solution of (2.1) defined and analytic in some interval $|t| < T$ (with $T > 0$ depending on U); $V(t) = \Gamma U(t)\Gamma^{-1}$ is a solution of (2.2). By a *constant of motion* we mean a functional $\mathbf{M}_n(\mathbb{C}) \ni A \longrightarrow \Phi[A]$ (valued in some set) having the following property:

(CofM) *If* $U \in \mathcal{C}^{\omega}([-T,T];\mathbf{M}_n(\mathbb{C}))$ *is any solution of* (2.1), *then* $\Phi[U(t)] = \Phi[U(0)]$ *for all* $t \in [-T,T]$.

Often, but not always, $\Phi[U(t)]$ will be a polynomial function of U and its "derivatives" $\partial_X^k U$ $(k = 1, 2, ...)$. But it could also be an eigenvalue or even the entire spectrum of $U(t)$. We shall apply the same terminology to a solution $V \in \mathcal{C}^{\omega}([-T,T];\mathbf{M}_n(\mathbb{C}))$ of (2.2), with ∂_X replaced by ∂_Λ.

At this stage we know that the diagonal entries $v_{i,i}$, $i = 1, ..., n$, are constants of motion.

If $Q(\xi_0, \xi_1, ..., \xi_\nu) \in \mathfrak{P}$ and if we write $Q[U] = Q(U, \partial_X U, ..., \partial_X^\nu U)$ and $Q[V] = Q(V, \partial_\Lambda V, ..., \partial_\Lambda^\nu V)$, then $Q[V] = \Gamma Q[U]\Gamma^{-1}$ and $\operatorname{tr} Q[V] = \operatorname{tr} Q[U]$. These quantities, $\operatorname{tr} Q[V] = \operatorname{tr} Q[U]$, will be constants of motion whenever Q is a trace-conserved polynomial for the noncommutative KdV equation. We know that (1.1) has infinitely many trace-conserved polynomials (Theorem 1). The question arises as to how many of the resulting constants of motion are truly independent when acting on solutions of (2.1) or, equivalently, on solutions of (2.2).

Proposition 2.2 *If* $n = 2$, *the spectrum of a solution* $U(t)$ *of* (2.1) *is a constant of motion.*

Proof. We know that ξ_0 and $\frac{1}{2}\xi_0^2$ are trace-conserved polynomials of (1.1) [cf. (1.16)], implying that $\operatorname{tr} U$ and $\frac{1}{2}\operatorname{tr} U^2$ are constants of motion, whence the statement since the eigenvalues of U are the roots of the equation

$$z^2 - (\operatorname{tr} U)\, z + \frac{1}{2}\left((\operatorname{tr} U)^2 - \operatorname{tr} U^2\right) = 0.$$

Proposition 2.3 *If* $n \geq 3$, *the spectrum of a solution* $U(t)$ *of* (2.1) *is not a constant of motion.*

Proof. We focus on (2.2) or, equivalently, on the system of equations (2.4). It suffices to prove the claim for $n = 3$ since $\mathbf{M}_3(\mathbb{C})$ is naturally embedded in $\mathbf{M}_{3+k}(\mathbb{C})$ $(k = 1, 2, ...)$; it is evident by uniqueness in the initial value problem or directly by inspection of (2.4) that $v_{j,k}(0) = 0$ for $\max(j,k) > 3$ implies $v_{j,k}(t) = 0$ for those same j, k and all t.

We begin by showing that the products $v_{i,j}v_{j,i}$ $(1 \leq i < j \leq 3)$ are not constants of motion. It suffices to consider the case $i = 1$, $j = 2$; we have

$$v_{2,1}v_{1,2}' = (\lambda_1 - \lambda_2)^3 v_{2,1}v_{1,2} - 3(\lambda_1 - \lambda_2)\left((v_{1,1} + v_{2,2})\, v_{2,1}v_{1,2} + v_{1,3}v_{3,2}v_{2,1}\right)$$

$$v_{1,2}v_{2,1}' = -(\lambda_1 - \lambda_2)^3 v_{1,2}v_{2,1} + 3(\lambda_1 - \lambda_2)\left((v_{1,1} + v_{2,2})\, v_{1,2}v_{2,1} + v_{1,2}v_{2,3}v_{3,1}\right)$$

whence

$$(v_{1,2}v_{2,1})' = 3(\lambda_1 - \lambda_2)(v_{1,2}v_{2,3}v_{3,1} - v_{1,3}v_{3,2}v_{2,1}). \tag{2.5}$$

We can select $V(0)$ in such a way that $v_{1,2}v_{2,3}v_{3,1} - v_{1,3}v_{3,2}v_{2,1} \neq 0$ at $t = 0$. From the list (1.16) we know that $\operatorname{tr}\left(\frac{1}{2}[\Lambda, V]^2 + V^3\right)$ is a constant of motion. If the spectrum of V were a constant of motion, the same would be true of $\operatorname{tr} V^3$ and therefore of

$$-\operatorname{tr}\frac{1}{2}[\Lambda, V]^2 = (\lambda_1 - \lambda_2)^2 v_{1,2}v_{2,1} + (\lambda_2 - \lambda_3)^2 v_{2,3}v_{3,2} + (\lambda_3 - \lambda_1)^2 v_{1,3}v_{3,1}.$$

But (2.5) entails

$$-\frac{1}{2}\frac{d}{dt}\operatorname{tr}[\Lambda, V]^2 = 9(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)(v_{1,2}v_{2,3}v_{3,1} - v_{1,3}v_{3,2}v_{2,1}).$$

Remark 1. Proposition 2.3 shows that the conjectured Theorem 4.1 in the paper [Treves, 2007] is false.

We note that transposition transforms (2.2) into

$$\frac{dV^\top}{dt} = -\partial_\Lambda\left(\partial_\Lambda^2 V^\top - 3V^{\top 2}\right).$$

We see that $V^\top(-t)$ is also a solution of (2.2). Suppose the initial value $V(0)$ is symmetric; uniqueness in the initial value problem implies $V(t) = V^\top(-t)$ for all $t \in (-T, T)$.

3 Noncommmutative KdV hierarchy based on Schwartz space

3.1 Schwartz space and its bounded linear operators

We denote by $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, etc., the variable point and coordinates in $\mathbb{R}^n$; the variable point in $\mathbb{R}^{2n}$ will often be denoted by (x, y), or by (ξ, η) on the Fourier transform side. We use standard multi-index notation: $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$; $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$; $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, etc. We write $D_x = (D_{x_1}, ..., D_{x_n})$ with $D_{x_j} = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}$ and $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$. Unless specified otherwise, functions are complex-valued.

The Schwartz space of complex-valued $\mathcal{C}^\infty$ functions in $\mathbb{R}^n$ rapidly decaying at infinity, as well as all their derivatives, is denoted by $\mathcal{S}(\mathbb{R}^n)$; its dual is the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions in $\mathbb{R}^n$. According to the Schwartz kernels theorem the vector space of bounded linear operators $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ can be identified to the tensor product completion $\mathcal{S}(\mathbb{R}_x^n)\widehat{\otimes}\mathcal{S}'(\mathbb{R}_y^n)$ (cf. [Schwartz, 1966], p. 243, also [Treves, 1967 & 2006], Ch. 43): a (unique) bounded linear operator $\mathbf{Op}K : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ corresponds to the kernel distribution $K(x,y) \in \mathcal{S}(\mathbb{R}^n_x) \widehat{\otimes} \mathcal{S}'(\mathbb{R}^n_y)$ through the *Volterra* pairing

$$\mathcal{S}\left(\mathbb{R}^n_y\right) \ni \varphi(y) \longrightarrow \int K(x,y)\,\varphi(y)\,dy, \tag{3.1}$$

where the integral sign stands for the duality bracket between Schwartz functions and tempered distributions. The kernel representing the *compose* of two operators $\mathbf{Op}K_i : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ $(i = 1, 2)$ is the *Volterra product* of the kernels K_i:

$$(K_1 \circ K_2)(x,y) = \int K(x,z)\,K(z,y)\,dz. \tag{3.2}$$

3.2 Differential subalgebras and KdV equation

Let $\mathcal{O}_M(\mathbb{R}^n_x)$ denote the algebra of complex-valued $\mathcal{C}^\infty$ functions χ in $\mathbb{R}^n_x$ whose partial derivatives of all orders grow temperedly at infinity; multiplication $f \longrightarrow \chi f$ is a bounded linear operator on $\mathcal{S}(\mathbb{R}^n_x)$ ([Schwartz, 1966], p. 243 et sq.). Below we deal with a differential operator

$$L(x, D_x) = \sum_{\alpha \in \mathbb{Z}^n_+} c_\alpha(x)\, D^\alpha_x \tag{3.3}$$

whose coefficients c_α belong to $\mathcal{O}_M(\mathbb{R}^n)$; there is an integer m such that $|\alpha| > m \Longrightarrow c_\alpha(x) \equiv 0$ (the smallest such integer m being the *order* of L). The fact that $\mathcal{O}_M(\mathbb{R}^n_x)$ is an algebra for ordinary multiplication implies immediately

Proposition 3.1 *If the coefficients of $L(x, D_x)$ belong to $\mathcal{O}_M(\mathbb{R}^n_x)$, then $K(x,y) \longrightarrow L(x, D_x)K(x,y)$ and $K(x,y) \longrightarrow L(y, D_y)K(x,y)$ are linear maps of $\mathcal{S}(\mathbb{R}^n_x) \widehat{\otimes} \mathcal{S}'(\mathbb{R}^n_y)$ into itself.*

We shall denote by $L^\top(x, D_x)$ the transpose of $L(x, D_x)$; often we write L for $L(x, D_x)$. We also write $(\mathrm{Ad}\,L)\,\mathbf{Op}K = [L, \mathbf{Op}K]$ if $K \in \mathcal{S}(\mathbb{R}^n_x) \widehat{\otimes} \mathcal{S}'(\mathbb{R}^n_y)$; and

$$\partial_L K(x,y) = L(x, D_x)\,K(x,y) - L^\top(y, D_y)\,K(x,y). \tag{3.4}$$

We have

$$(\mathrm{Ad}\,L)\,\mathbf{Op}K = \mathbf{Op}\partial_L K. \tag{3.5}$$

Proposition 3.2 *If the coefficients of $L(x, D_x)$ belong to $\mathcal{O}_M(\mathbb{R}^n)$, ∂_L is a derivation of the Volterra algebra $\mathcal{S}(\mathbb{R}^n_x) \widehat{\otimes} \mathcal{S}'(\mathbb{R}^n_y)$.*

Evident.

The definition of the *trace* of a “finite” tensor $K(x,y) = \sum_{j=1}^{N} \varphi_j(x) u_j(y) \in \mathcal{S}(\mathbb{R}_x^n) \otimes \mathcal{S}'(\mathbb{R}_y^n)$ is obvious:

$$\operatorname{tr} K = \sum_{j=1}^{N} \int \varphi_j(x) u_j(x)\, dx \tag{3.6}$$

where the integral stands for the duality bracket between $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

Proposition 3.3 *Given any $K \in \mathcal{S}(\mathbb{R}_x^n) \widehat{\otimes} \mathcal{S}'(\mathbb{R}_y^n)$ the derivative $\partial_L K$ belongs to the closure of the vector subspace of $\mathcal{S}(\mathbb{R}_x^n) \otimes \mathcal{S}'(\mathbb{R}_y^n)$ consisting of the 2-tensors whose trace vanishes.*

Proof. If $K_\nu(x,y) = \sum_{j=1}^{N_\nu} \varphi_{\nu,j}(x) u_{\nu,j}(y) \in \mathcal{S}(\mathbb{R}_x^n) \otimes \mathcal{S}'(\mathbb{R}_y^n)$ converges to $K(x,y)$, then $\partial_L K_\nu$ converge to $\partial_L K$. By (3.4) and (3.6) we get

$$\operatorname{tr} \partial_L K_\nu(x,y) = \sum_{j=1}^{N_\nu} \int \left(L\varphi_{\nu,j}(x) u_{\nu,j}(x) - \varphi_{\nu,j}(x) L^\top u_{\nu,j}(x) \right) dx = 0.$$

As in (3.4) $L = L(x,D)$ is a differential operator with coefficients in $\mathcal{O}_M(\mathbb{R}^n)$. Let $\mathcal{A}$ be a subalgebra of $\mathcal{S}(\mathbb{R}_x^n) \otimes \mathcal{S}'(\mathbb{R}_y^n)$ stable under the derivation ∂_L. The KdV equation, and the KdV hierarchy, make sense in the differential algebra $(\mathcal{A}, \partial_L)$:

$$\partial_t K = \partial_L^3 K - 3\partial_L (K \circ K). \tag{3.7}$$

Some choices of $\mathcal{A}$ may not contain the Dirac distribution $\delta(x-y)$ (the identity for Volterra composition); in those cases we shall replace $\mathcal{A}$ by the direct sum $\mathcal{A} \oplus \mathbb{C}\delta(x-y)$ and $K(t,x,y) \in \mathcal{A}$ by $\lambda(t)\delta(x-y) + K(t,x,y)$, in which case (3.7) reads

$$\lambda'(t)\delta(x-y) + \partial_t K = \partial_L^3 K - 3\partial_L (K \circ K + 2\lambda(t) K). \tag{3.8}$$

This demands that λ' vanish identically, i.e., λ is a complex constant and (3.8) reduces to

$$\partial_t K = \partial_L^3 K - 6\lambda \partial_L K - 3\partial_L (K \circ K). \tag{3.9}$$

In subalgebras $\mathcal{A}$ stable under the operator $\exp(-6\lambda\partial_L)$ we can replace K by $\widetilde{K} = \exp(6\lambda t \partial_L) K$. As a consequence of the Leibniz formula $\exp(6\lambda t \partial_L)(K \circ K) = \widetilde{K} \circ \widetilde{K}$ the original KdV equation (3.7) will be satisfied by $\widetilde{K}$. In the sequel we shall focus on (3.9).

Example 3. Suppose we take $K(t,x,y) = E(t,x)\delta(x-y)$: $\mathbf{Op}K$ is multiplication by $E \in \mathcal{C}^\infty(-T,T;\mathcal{O}_M(\mathbb{R}^n))$; $\mathbf{Op}(K \circ K)$ is multiplication by E^2 and

$\mathbf{Op}\partial_L K$ is the differential operator $\varphi \longrightarrow L(E\varphi) - EL\varphi$. Suppose L is a vector field: $L = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}$. In this case

$$\partial_L K(x,y) = (LE)(t,x)\,\delta(x-y)$$

and (3.9) is

$$\partial_t E = L^3 E - 6\lambda LE - 3L\left(E^2\right),$$

i.e., the "scalar pseudo-" KdV equation along the integral curves of the vector field L.

4 The constant coefficients case

4.1 Existence of solutions

If $L(x, D_x) = L(D_x)$ has constant coefficients, then $L^\top(y, D_y) = L(-D_y)$ and (3.9) reads

$$\begin{aligned}\partial_t f &= (L(D_x) - L(-D_y))^3 f(x,y) \\ &\quad - 6(L(D_x) - L(-D_y))\left(\lambda f(x,y) + \frac{1}{2}\int_{\mathbb{R}^n} f(x,z,t) f(z,y,t)\,dz\right).\end{aligned} \tag{4.1}$$

It is natural to carry out Fourier transformations with respect to (x, y); we have

$$\widehat{f}(\xi,\eta,t) = \int_{\mathbb{R}^{2n}} e^{-ix\xi - iy\eta} f(x,y)\,dxdy,$$

$$f(x,y,t) = \int_{\mathbb{R}^{2n}} e^{ix\xi + iy\eta} \widehat{f}(\xi,\eta,t)\,\frac{d\xi d\eta}{(2\pi)^{2n}},$$

and

$$\int_{\mathbb{R}^{2n}} f(x,z,t) f(z,y,t)\,dz = \int_{\mathbb{R}^n} \widehat{f}(\xi,\zeta,t)\,\widehat{f}(-\zeta,\eta,t)\,\frac{d\zeta}{(2\pi)^n}.$$

The Fourier transform of (4.1) is

$$\begin{aligned}\partial_t \widehat{f}(\xi,\eta,t) &= \Big((L(\xi) - L(-\eta))^3 - 3\lambda(L(\xi) - L(-\eta))\Big)\widehat{f}(\xi,\eta,t) \\ &\quad - 3(L(\xi) - L(-\eta))\int_{\mathbb{R}^n} \widehat{f}(\xi,\zeta,t)\,\widehat{f}(-\zeta,\eta,t)\,\frac{d\zeta}{(2\pi)^n}.\end{aligned} \tag{4.2}$$

Theorem 3 *To each real number $\rho > 0$ there is $T > 0$ such that the following holds. Given an arbitrary function $\widehat{f}_\circ \in \mathcal{C}\left(\mathbb{R}^{2n}\right)$ whose support is contained in the biball $\left\{(\xi,\eta) \in \mathbb{R}^{2n};\ |\xi| \le \rho,\ |\eta| \le \rho\right\}$ there is a unique real-analytic function $\widehat{f}(\xi,\eta,t)$ of t, $|t| < T$, valued in $\mathcal{C}\left(\mathbb{R}^{2n}\right)$, satisfying (4.2) in $(-T,T)\times\mathbb{R}^{2n}$ and such that $\widehat{f}\Big|_{t=0} = \widehat{f}_\circ$ in $\mathbb{R}^{2n}$. Then $\operatorname{supp}\widehat{f}(\cdot,\cdot,t) \subset \operatorname{supp}\widehat{f}_\circ$ for all $t \in (-T,T)$.*

Proof. The property of the support and consequently the uniqueness of the solution $\widehat{f}(\xi,\eta,t)$ of (4.2) are evident since $\widehat{f}(\xi,\eta,0)=0 \Longrightarrow \partial_t^k \widehat{f}(\xi,\eta,0)=0$ for all $k\in\mathbb{Z}_+^n$. We prove the existence of the solution $\widehat{f}(\xi,\eta,t)$. We use the constants

$$C(\rho)=3\max_{|\xi|\le\rho,|\eta|\le\rho}|L(\xi)-L(-\eta)|,$$

$$M(\rho)=\max_{|\xi|\le\rho,|\eta|\le\rho}\left|(L(\xi)-L(-\eta))^3-3\lambda(L(\xi)-L(-\eta))\right|$$

and

$$\Phi_k=\frac{1}{k!}\max_{(\xi,\eta)\in\mathbb{R}^{2n}}\left|\partial_t^k\widehat{f}(\xi,\eta,0)\right|.$$

We derive from (4.2):

$$(k+1)\Phi_{k+1}\le M(\rho)\Phi_k+C(\rho)\left(\frac{\rho}{2\pi}\right)^{2n}\sum_{\ell=0}^{k}\Phi_{k-\ell}\Phi_\ell. \tag{4.3}$$

Suppose we have proved there are positive constants A, B such that $\Phi_\ell\le AB^\ell$ for all $\ell=0,...,k$ and all $(\xi,\eta)\in\mathbb{R}^{2n}$. We derive from (4.3):

$$\Phi_{k+1}\le\left(\frac{1}{k+1}M(\rho)+C(\rho)\left(\frac{\rho}{2\pi}\right)^{2n}A\right)AB^k.$$

Requiring

$$M(\rho)+C(\rho)\left(\frac{\rho}{2\pi}\right)^{2n}A\le B$$

ensures that $\Phi_{k+1}\le AB^{k+1}$ and, as a consequence, that the solution $\widehat{f}(\xi,\eta,t)$ exists and extends as a holomorphic function of $t\in\mathbb{C}$ in the disk $|t|<T=B^{-1}$.

If $\widehat{f}(\xi,\eta,t)$ has the properties in Theorem 3, its inverse Fourier transform extends as an entire analytic function of exponential type $f(z,w,t)$ of $(z,w)\in\mathbb{C}^{2n}$, more precisely an entire function such that

$$|f(z,w,t)|\le C_\circ\exp(C_1\rho(|\operatorname{Im}z|+|\operatorname{Im}w|)) \tag{4.4}$$

for all $(z,w)\in\mathbb{C}^{2n}$ and all t, $|t|<T$.

4.2 Traveling wave solutions

When $L=L(D_x)$ has constant coefficients, (4.1) might have *traveling wave* solutions of the form $F(x-tv,y-tw)$ with $F(x,y)\in\mathcal{S}(\mathbb{R}_x^n)\widehat{\otimes}\ \mathcal{S}'\left(\mathbb{R}_y^n\right)$, exhibiting something like a *wave front.* The kernel distribution F must satisfy the equation

$$(L(D_x) - L(-D_y))^3 F(x,y) - 3\lambda (L(D_x) - L(-D_y)) F(x,y) \tag{4.5}$$

$$= 3(L(D_x) - L(-D_y)) \int F(x,z) F(z,y)\, dz + (v \cdot \partial_x F + w \cdot \partial_y F)(x,y).$$

A very simple example obtains when $w = v$ and $L(D_x) = v \cdot \partial_x$. Let us use the notation

$$\vartheta_v = v \cdot (\partial_x + \partial_y) = L(D_x) - L(-D_y).$$

In this case (4.5) reduces to

$$\vartheta_v^3 F(x,y) - (3\lambda + 1)\vartheta_v F(x,y) = 3\vartheta_v \int F(x,z) F(z,y)\, dz.$$

A solution is given by $F(x,y) = \frac{1}{3} E(v \cdot x)\, \delta(x-y)$ provided $E \in \mathcal{C}^\omega(\mathbb{R})$ satisfies

$$E'''(x) - (3\lambda + \nu) E'(x) - 2E(x) E'(x) = 0. \tag{4.6}$$

Equation (4.6) is essentially the equation that determines the (single) soliton solutions of the standard KdV equation. First an integration yields

$$E''(x) - (3\lambda + \nu) E(x) - E^2(x) = C_1; \tag{4.7}$$

then multiplying by $2E'(x)$ and one more integration yields

$$E'^2(x) = \frac{2}{3} E^3(x) + (3\lambda + \nu) E^2(x) + 2C_1 E(x) + C_2. \tag{4.8}$$

The elliptic functions solutions of (4.8) are the soliton solutions, standard in the one-dimensional setup. Here we get the traveling wave solutions

$$F(x,y) = \frac{1}{3} E\left(v \cdot x - t|v|^2\right) \delta(x-y); \tag{4.9}$$

the waves travel along the main diagonal $x = y$ in the direction v. Essentially this is a one-dimension phenomenon. Off the diagonal the solutions are identically equal to zero. Note that

$$(\mathbf{Op} F)\varphi(x) = \frac{1}{3} E\left(v \cdot x - t|v|^2\right) \varphi(x).$$

Remark 2. Actually, the same result holds for any first-order linear differential operators of the type $L(x, D_x) = v \cdot \partial_x + a(x)$ with $a \in \mathcal{O}_M(\mathbb{R}^n)$. Indeed, if F is given by (4.9),

$$(L(x, D_x) - L(y, -D_y)) F = v \cdot (\partial_x + \partial_y) F.$$

Remark 3. What we are really doing in the present subsection is to reason within the *commutative* subalgebra of $\mathcal{S}(\mathbb{R}_x^n)\,\widehat{\otimes}\,\mathcal{S}'(\mathbb{R}_y^n)$ made up of the kernel distributions $f(x)\,\delta(x-y)$ with $f \in \mathcal{O}_M(\mathbb{R}^n)$; the corresponding subalgebra of bounded linear operators on $\mathcal{S}(\mathbb{R}_x^n)$ is made up of the multiplication operators $\varphi \longrightarrow f\varphi$. But instead of the standard KdV equation in $\mathbb{R}$ we are looking at the KdV equation for the vector field $v \cdot (\partial_x + \partial_y)$ in $\mathbb{R}^{2n}$. In this case the soliton solutions are supported by the diagonal of $\mathbb{R}_x^n \times \mathbb{R}_y^n$.

5 KdV equation based on the harmonic oscillator

In the remainder of this article we focus on the case $n = 1$.

5.1 KdV equation with $L = D_x^2 + x^2$

We return to (3.9) where we take $L = D_x^2 + x^2$:

$$\begin{aligned}\partial_t K = \left(D_x^2 - D_y^2 + x^2 - y^2\right)^3 K \\ - 6\lambda\left(D_x^2 - D_y^2 + x^2 - y^2\right) K - 3\left(D_x^2 - D_y^2 + x^2 - y^2\right)(K \circ K)\,.\end{aligned} \tag{5.1}$$

We shall reason within a subalgebra $\mathcal{A}$ of the Volterra algebra $\mathcal{S}(\mathbb{R}_x^n)\,\widehat{\otimes}\,\mathcal{S}'(\mathbb{R}_y)$ defined by properties of the Hermite function expansions of its elements (cf. Appendix). We write

$$K(t,x,y) = \sum_{\alpha=0}^{\infty}\sum_{\beta=\alpha}^{\infty} c_{\alpha,\beta}(t)\,\mathcal{H}_\alpha(x)\,\mathcal{H}_\beta(y)\,, \tag{5.2}$$

with $c_{\alpha,\beta}(t) \in \mathcal{C}^\infty(-T,T)$. We have

$$(K \circ K)(t,x,y) = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}\left(\sum_{\gamma=0}^{\infty} c_{\alpha,\gamma}(t)\,c_{\gamma,\beta}(t)\right)\mathcal{H}_\alpha(x)\,\mathcal{H}_\beta(y)\,,$$

and according to (6.4),

$$\partial_L^k K(t,x,y) = 2^k \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty}(\alpha-\beta)^k\, c_{\alpha,\beta}(t)\,\mathcal{H}_\alpha(x)\,\mathcal{H}_\beta(y)\,,\; k \in \mathbb{Z}_+. \tag{5.3}$$

Thus (3.9) is equivalent to an infinite system of ODEs in which the unknowns are the coefficients $c_{\alpha,\beta}(t)$:

$$\begin{aligned}c'_{\alpha,\beta}(t) = 4(\alpha-\beta)\left(2(\alpha-\beta)^2 - 3\lambda\right) c_{\alpha,\beta}(t) \\ -6(\alpha-\beta)\sum_{\gamma\in\mathbb{Z}_+} c_{\alpha,\gamma}(t)\,c_{\gamma,\beta}(t)\,,\end{aligned} \tag{5.4}$$

$(\alpha,\beta) \in \mathbb{Z}_+^2$. We can state:

Proposition 5.1 *Let $\mathcal{A}$ be a subalgebra of $S(\mathbb{R}_x)\,\widehat{\otimes}\,S'(\mathbb{R}_y)$ stable under ∂_L. If $K(t,x,y)\in\mathcal{C}^\infty(\mathcal{I};\mathcal{A})$ is a solution of the KdV equation* (3.9)*, then every diagonal coefficient $c_{\alpha,\alpha}$ of K is constant.*

5.2 Global Cauchy problem in an algebra of upper-triangular matrices

In this subsection we write $\mathfrak{s}$ rather than $\mathfrak{s}_1$ and $\mathfrak{s}'$ rather than $\mathfrak{s}_1'$ (see Appendix). We view the double sequences $(c_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}_+}\in\mathfrak{s}\widehat{\otimes}\mathfrak{s}'$ as $(\infty\times\infty)$ matrices.We shall say that a matrix $(c_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}_+}\in\mathfrak{s}\widehat{\otimes}\mathfrak{s}'$ is **upper-triangular** if $0\le\beta<\alpha\Longrightarrow c_{\alpha,\beta}=0$.

Definition 5.1 *Given a number $\rho>1$ we denote by $\mathbb{A}_\rho^+$ the vector space of upper-triangular matrices $c=(c_{\alpha,\alpha+k})_{\alpha,k\in\mathbb{Z}_+}$ such that*

$$N_\rho(c)=\sum_{\alpha=0}^{\infty}\sum_{k=0}^{\infty}\rho^k\left|c_{\alpha,\alpha+k}\right|<+\infty. \tag{5.5}$$

We denote by $\mathbb{A}^+$ the union of the vector spaces $\mathbb{A}_\rho^+$, $\rho>1$.

If $1<\rho'<\rho$, then $\mathbb{A}_\rho^+\subset\mathbb{A}_{\rho'}^+\subset L^1\left(\mathbb{Z}^2\right)\subset\mathfrak{s}\widehat{\otimes}\mathfrak{s}'$. If $(c_{\alpha,\alpha+k})_{\alpha,k\in\mathbb{Z}_+}\in\mathbb{A}^+$, then for each $\alpha\in\mathbb{Z}_+$ the (row) sequence $\{c_{\alpha,\alpha+k}\}_{k=0,1,\ldots}$ decays exponentially. For each $\rho>1$, $\mathbb{A}_\rho^+$ is a Banach space for the norm $N_\rho(\cdot)$.

Proposition 5.2 *The set $\mathbb{A}_\rho^+$ is a subalgebra of $\mathfrak{s}\widehat{\otimes}\mathfrak{s}'$; it is a Banach algebra for the norm $N_\rho(\cdot)$.*

Proof. The set of upper-triangular matrices forms a subalgebra of $\mathfrak{s}\widehat{\otimes}\mathfrak{s}'$. Let $c^{(i)}=\left(c^{(i)}_{\alpha,\alpha+k}\right)_{\alpha,k\in\mathbb{Z}_+}$, $i=1,2$, belong to $\mathbb{A}_\rho^+$. We have

$$\begin{aligned}
N_\rho\left(c^{(1)}c^{(2)}\right)&=\sum_{\alpha=0}^{\infty}\sum_{k=0}^{\infty}\rho^k\left|\sum_{\ell=0}^{k}c^{(1)}_{\alpha,\alpha+\ell}c^{(2)}_{\alpha+\ell,\alpha+k}\right|\\
&\le\sum_{\alpha=0}^{\infty}\sum_{\ell=0}^{\infty}\rho^\ell\left|c^{(1)}_{\alpha,\alpha+\ell}\right|\sum_{k=\ell}^{\infty}\rho^{k-\ell}\left|c^{(2)}_{\alpha+\ell,\alpha+k}\right|\\
&\le N_\rho\left(c^{(2)}\right)\sum_{\alpha=0}^{\infty}\sum_{\ell=0}^{\infty}\rho^\ell\left|c^{(1)}_{\alpha,\alpha+\ell}\right|=N_\rho\left(c^{(1)}\right)N_\rho\left(c^{(2)}\right).
\end{aligned}$$

We call $\mathcal{A}_\rho^+$ the subalgebra of $\mathcal{S}(\mathbb{R}_x)\,\widehat{\otimes}\,\mathcal{S}'(\mathbb{R}_y)$ consisting of the kernel distributions $K(x,y)$ whose Hermite coefficients are the entries of a matrix $c\in\mathbb{A}_\rho^+$; we define $N_\rho(K)=N_\rho(c)$. We define $\mathcal{A}^+=\bigcup_{\rho>1}\mathcal{A}_\rho^+$. It is evident that

$\mathcal{A}^+$ is a subalgebra of trace-class of $\mathcal{S}(\mathbb{R}_x) \widehat{\otimes} \mathcal{S}'(\mathbb{R}_y)$ (cf. end of Appendix). If both $K_1, K_2 \in \mathcal{A}^+$, then

$$\operatorname{Tr}(K_1 K_2) = (\operatorname{Tr} K_1) \operatorname{Tr} K_2 \tag{5.6}$$

(see Definition 6.1).

Theorem 4 *Let the matrix* $U^\circ = \left(u^\circ_{\alpha,\alpha+k}\right)_{\alpha,k\in\mathbb{Z}_+} \in \mathbb{A}^+_{\rho_\circ}$. *If* $\operatorname{Re}\left(\lambda + u^\circ_{\alpha,\alpha}\right) \leq -R$ *for some* $R > 0$ *and every* $\alpha \in \mathbb{Z}_+$ *and if* $N_{\rho_\circ}(U^\circ) \leq R+4$, *then there is a unique solution* $U(t) = (u_{\alpha,\alpha+k}(t))_{\alpha,k\in\mathbb{Z}_+} \in \mathcal{C}^\infty([0,+\infty); \mathbb{A}^+)$ *of the system of equations* (5.4) *such that* $U(0) = U^\circ$; $U(t)$ *extends as a holomorphic function of* t *valued in* $\mathbb{A}^+$ *in a complex neighborhood of* $(0,+\infty)$.

Proof. We let t vary in a sector of the complex plane,

$$\Sigma_\varepsilon = \{t \in \mathbb{C};\ \ |\operatorname{Im} t| \leq \varepsilon \operatorname{Re} t\}$$

with $\varepsilon > 0$ suitably chosen.

I. **Existence of solution** We have $u_{\alpha,\alpha}(t) = u^\circ_{\alpha,\alpha}$ for all t. According to (5.4) the coefficient $u_{\alpha,\alpha+k}(t)$ $(k \geq 1)$ must be a solution of the equation

$$u'_{\alpha,\alpha+k}(t) + 2kM_{\alpha,k}u_{\alpha,\alpha+k}(t) = 6k \sum_{\alpha<\gamma<\alpha+k} u_{\alpha,\gamma}(t)\, u_{\gamma,\alpha+k}(t)\,, \tag{5.7}$$

where

$$M_{\alpha,k} = 4k^2 - 6\lambda - 3\left(u^\circ_{\alpha,\alpha} + u^\circ_{\alpha+k,\alpha+k}\right).$$

We note right away that our hypothesis on the diagonal entries of U° entails

$$\operatorname{Re} M_{\alpha,k} \geq 4k^2 + 6R; \tag{5.8}$$

we also have

$$|\operatorname{Im} M_{\alpha,k}| \leq \kappa = 3\,|\operatorname{Im}(2\lambda - \operatorname{Tr} U^\circ)|\,. \tag{5.9}$$

To determine the functions $u_{\alpha,\alpha+k}(t)$ we reason by induction on k. If $k = 1$, (5.7) reads

$$u'_{\alpha,\alpha+1}(t) + 2M_{\alpha,\alpha+1}u_{\alpha,\alpha+1}(t) = 0$$

whose solution is

$$u_{\alpha,\alpha+1}(t) = u^\circ_{\alpha,\alpha+1} \exp(-2M_{\alpha,\alpha+1}t)\,.$$

In this case (5.8) and (5.9) entail

$$|u_{\alpha,\alpha+1}(t)| \leq \left|u^\circ_{\alpha,\alpha+1}\right| \exp(-4 \operatorname{Re} t + 2\kappa\,|\operatorname{Im} t|)$$

whence

$$\left|u_{\alpha,\alpha+1}(t)\right| \leq \left|u^{\circ}_{\alpha,\alpha+1}\right| \tag{5.10}$$

provided $t \in \Sigma_{2/\kappa}$.

In the remainder of the proof we suppose $k \geq 2$. We require that the positive number $\varepsilon < 2\kappa^{-1}$ be so small that

$$\forall t \in \Sigma_{\varepsilon},\ \operatorname{Re}\left(M_{\alpha,k}t\right) \geq 3\left(k^2+R\right)|t|. \tag{5.11}$$

We shall reason by induction, starting from the hypothesis that all $u_{\alpha,\alpha+j}(t)$ have been determined for $1 \leq j \leq k-1$, and that the following inequality holds for all $t \in \Sigma_{\varepsilon}$:

$$\sum_{\alpha=0}^{\infty}\sum_{\ell=1}^{j}\rho_{\circ}^{\ell}\left|u_{\alpha,\alpha+j}(t)\right| \leq \sum_{\alpha=0}^{\infty}\sum_{\ell=1}^{j}\rho_{\circ}^{\ell}\left|u^{\circ}_{\alpha,\alpha+\ell}\right|. \tag{5.12}$$

We derive from (5.7):

$$\begin{aligned} &u_{\alpha,\alpha+k}(t) \\ &\quad = u^{\circ}_{\alpha,\alpha+k}e^{-2kM_{\alpha,k}t} + 6k\sum_{j=1}^{k-1}\int_0^t e^{-2kM_{\alpha,k}(t-s)}u_{\alpha,\alpha+j}(s)\,u_{\alpha+j,\alpha+k}(s)\,ds, \end{aligned}$$

with the integration carried out on the straight-line segment joining the origin to $t \in \Sigma_{\varepsilon}$. We get

$$\begin{aligned} \left|u_{\alpha,\alpha+k}(t)\right| &\leq \left|u^{\circ}_{\alpha,\alpha+k}\right| \\ &\quad + \frac{3|t|}{\operatorname{Re}\left(M_{\alpha,k}t\right)}\sup_{0<\theta<1}\sum_{j=1}^{k-1}\left|u_{\alpha,\alpha+j}(\theta t)\right|\left|u_{\alpha+j,\alpha+k}(\theta t)\right| \\ &\leq \left|u^{\circ}_{\alpha,\alpha+k}\right| + \left(k^2+R\right)^{-1}\sup_{0<\theta<1}\sum_{j=1}^{k-1}\left|u_{\alpha,\alpha+j}(\theta t)\right|\left|u_{\alpha+j,\alpha+k}(\theta t)\right| \end{aligned} \tag{5.13}$$

by applying (5.11). We derive from (5.13):

$$\begin{aligned} \rho_{\circ}^{k}\sum_{\alpha=0}^{\infty}\left|u_{\alpha,\alpha+k}(t)\right| &\leq \rho_{\circ}^{k}\sum_{\alpha=0}^{\infty}\left|u^{\circ}_{\alpha,\alpha+k}\right| \\ &\quad + \left(k^2+R\right)^{-1}\sup_{0<\theta<1}\sum_{j=1}^{k-1}\sum_{\alpha=0}^{\infty}\quad \rho_{\circ}^{j}\left|u_{\alpha,\alpha+j}(\theta t)\right|\rho_{\circ1}^{k-j}\left|u_{\alpha+j,\alpha+k}(\theta t)\right| \\ &\leq \rho_{\circ}^{k}\sum_{\alpha=0}^{\infty}\left|u^{\circ}_{\alpha,\alpha+k}\right| \\ &\quad + \left(k^2+R\right)^{-1}\left(\sum_{\alpha=0}^{\infty}\sum_{j=1}^{k-1}\rho_{\circ}^{j}\left|u^{\circ}_{\alpha,\alpha+j}\right|\right)^2 \end{aligned}$$

$$\leq \rho_\circ^k \sum_{\alpha=0}^{\infty} \left| u^\circ_{\alpha,\alpha+k} \right| + \left(k^2+R\right)^{-1} N_{\rho_\circ}\left(U^\circ\right) \sum_{\alpha=0}^{\infty} \sum_{j=1}^{k-1} \rho_\circ^j \left| u^\circ_{\alpha,\alpha+j} \right| .$$

Since $N_{\rho_\circ}\left(U^\circ\right) \leq \left(k^2+R\right)^{-1}$ we conclude that (5.12) holds also for $j=k$, implying, for all $k \in \mathbb{Z}_+$,

$$\rho_\circ^k \sum_{\alpha=0}^{\infty} \left| u_{\alpha,\alpha+k}\left(t\right) \right| \leq N_{\rho_\circ}\left(U^\circ\right),$$

and therefore, if $1 < \rho < \rho_\circ$,

$$\sum_{k=0}^{\infty} \rho^k \sum_{\alpha=0}^{\infty} \left| u_{\alpha,\alpha+k}\left(t\right) \right| \leq \sum_{k=0}^{\infty} \left(\rho/\rho_\circ\right)^k \rho_\circ^k \sum_{\alpha=0}^{\infty} \left| u_{\alpha,\alpha+k}\left(t\right) \right| \leq \frac{\rho_\circ}{\rho_\circ - \rho} N_{\rho_\circ}\left(U^\circ\right).$$

This proves that $U\left(t\right)$ is a continuous function of $t \in \Sigma_\varepsilon$ valued in $\mathbb{A}_\rho^+$, holomorphic in the interior of Σ_ε. It is directly seen that $U\left(t\right) \in \mathcal{C}\left([0,+\infty);\mathbb{A}_\rho^+\right)$.

Regarding the derivatives of $U\left(t\right)$ we deduce from (5.7):

$$\partial_t^{p+1} u_{\alpha,\alpha+k} + 2kM_{\alpha,k} \partial_t^p u_{\alpha,\alpha+k} = 6k \sum_{\alpha<\gamma<\alpha+k} \sum_{q=0}^{p} \binom{p}{q} \left(\partial_t^{p-q} u_{\alpha,\gamma}\right) \partial_t^q u_{\gamma,\alpha+k}.$$

Induction with respect to $p \in \mathbb{Z}_+$ shows readily that $U\left(t\right) \in \mathcal{C}^\infty\left([0,+\infty);\mathbb{A}^+\right)$.

II. Uniqueness of solution By subtraction we are reduced to proving that if $U\left(t\right) \in \mathcal{C}^\infty\left([0,+\infty);\mathbb{A}^+\right)$ is a solution of the system of equations (5.7) such that $U\left(0\right) = 0$, then $U\left(t\right) = 0$ for all $t \geq 0$. This is equivalent to showing that if (5.7) holds and if $u^\circ_{\alpha,\alpha+k} = 0$ for all $(\alpha,k) \in \mathbb{Z}_+^2$, then $u_{\alpha,\alpha+k}\left(t\right) = 0$ for all $t \geq 0$ and all $(\alpha,k) \in \mathbb{Z}_+^2$. This follows directly from (5.12).

Remark 4. If $U\left(0\right)$ is an $N \times N$ matrix ($N < +\infty$), the solution $U\left(t\right)$ is also an $N \times N$ matrix. This follows directly from the uniqueness of the solution.

5.3 Traveling wave solutions

In this subsection we look at traveling wave solutions in the sense of Subsection 1.3, i.e., of the form $\exp\left(-vt\partial_L\right) u_\circ$, here with $u_\circ \in \mathbb{A}^+$. Our KdV equation is (5.1), i.e.,

$$\partial_t K = \partial_L \left(\partial_L^2 K - 6\lambda K - 3K \circ K\right).$$

We go directly to (1.23) whose version here is the eigenvalue equation

$$\left(D_x^2 - D_y^2 + x^2 - y^2\right)^2 u_\circ(x,y) - 3\int_{\mathbb{R}} u_\circ(x,z)\, u_\circ(z,y)\, dz = (6\lambda - v)\, u_\circ(x,y). \tag{5.14}$$

We shall look at possible solutions in the algebra $\mathcal{A}^+$, i.e., of the kind

$$u_\circ(x,y) = \sum_{\alpha=0}^{\infty}\sum_{k=0}^{\infty} c_{\alpha,\alpha+k}\mathcal{H}_\alpha(x)\,\mathcal{H}_{\alpha+k}(y), \tag{5.15}$$

with $\{c_{\alpha,\alpha+k}\}_{\alpha,k\in\mathbb{Z}_+} \in \mathbb{A}^+$. According to (6.4), (5.14) translates into

$$4k^2 c_{\alpha,\alpha+k} - 3\sum_{\gamma=\alpha}^{\alpha+k} c_{\alpha,\gamma}c_{\gamma,\alpha+k} = (6\lambda - v)\, c_{\alpha,\alpha+k}. \tag{5.16}$$

Proposition 5.3 *Let* $c = \{c_{\alpha,\alpha+k}\}_{\alpha,k\in\mathbb{Z}_+} \in \mathbb{A}^+$ *be the solution of the system of equations* (5.16)*. For each* $\alpha \in \mathbb{Z}_+$ *the following properties hold:*

1. *Either* $c_{\alpha,a} = 0$ *or* $c_{\alpha,\alpha} = \frac{1}{3}v - 2\lambda$.
2. *If* $\lambda \geq \frac{1}{6}v$ *and if* $c_{\alpha,\alpha} \neq 0$*, then* $c_{\alpha,\alpha+k} = 0$ *for all* $k \geq 1$.

Proof. Putting $k = 0$ in (5.16) yields right away

$$\left(c_{\alpha,\alpha} + 2\lambda - \frac{1}{3}v\right) c_{\alpha,\alpha} = 0,\ \alpha \in \mathbb{Z}_+. \tag{5.17}$$

which proves Property #1. For $k \geq 1$ we rewrite (5.16) as

$$\left(4k^2 - 3c_{\alpha,\alpha} - 3c_{\alpha+k,\alpha+k} - 6\lambda + v\right) c_{\alpha,\alpha+k} = 3\sum_{\gamma=\alpha+1}^{\alpha+k-1} c_{\alpha,\gamma}c_{\gamma,\alpha+k}. \tag{5.18}$$

If $c_{\alpha,\alpha} \neq 0$, then $c_{\alpha,\alpha} = \frac{1}{3}v - 2\lambda \leq 0$. Equation (5.18) becomes

$$\left(4k^2 - 3c_{\alpha+k,\alpha+k}\right) c_{\alpha,\alpha+k} = 3\sum_{\gamma=\alpha+1}^{\alpha+k-1} c_{\alpha,\gamma}c_{\gamma,\alpha+k}. \tag{5.19}$$

Since $c_{\alpha+k,\alpha+k} \leq 0$ the vanishing of the right-hand side in (5.19) implies $c_{\alpha,\alpha+k} = 0$. If $k = 1$, the right-hand side is always equal to zero. If $k \geq 2$, induction entails $c_{\alpha,\gamma} = 0$ for all γ, $\alpha < \gamma < \alpha + k$.

Putting $k = 1$ into (5.18) yields

$$(4 - 3c_{\alpha,\alpha} - 3c_{\alpha+1,\alpha+1} - 6\lambda + v)\, c_{\alpha,\alpha+1} = 0. \tag{5.20}$$

We look next at the case in which $c_{\alpha,\alpha} = 0$ for all $\alpha \in \mathbb{Z}_+$.

Proposition 5.4 *Let the sequence* $\{c_{\alpha,\alpha+1}\}_{\alpha=0,1,...} \in \mathfrak{s}'$ *be arbitrary. If* $\lambda = \frac{2}{3} + \frac{1}{6}v$*, there is a distribution*

$$u_\circ(x,y) = \sum_{\alpha=0}^{\infty} \sum_{k=1}^{\infty} c_{\alpha,\alpha+k} \mathcal{H}_\alpha(x) \mathcal{H}_{\alpha+k}(y) \tag{5.21}$$

which is a solution of (5.14).

Proof. Equation (5.20) is satisfied since $4 - 3c_{\alpha,\alpha} - 3c_{\alpha+1,\alpha+1} - 6\lambda + v = 0$. If $k > 1$, we solve (5.19) using induction and taking

$$c_{\alpha,\alpha+k} = \frac{3}{4k^2} \sum_{\ell=1}^{k-1} c_{\alpha,\alpha+\ell} c_{\alpha+\ell,\alpha+k}. \tag{5.22}$$

Remark 5. Suppose $\lambda = \frac{2}{3} + \frac{1}{6}v$. If we combine Propositions 5.3 and 5.4 we see that to each partition $\mathbb{Z}_+ = A \cup B$ ($A \cap B = \varnothing$) there correspond solutions of (5.14) of the following kind: given an arbitrary sequence $\{c_{\beta,\beta+1}\}_{\beta\in B} \in \mathfrak{s}'$, then

$$u_\circ(x,y) = \frac{4}{3} \sum_{\alpha\in A} \mathcal{H}_\alpha(x) \mathcal{H}_\alpha(y) + \sum_{\beta\in B} \sum_{k=1}^{\infty} c_{\beta,\beta+k} \mathcal{H}_\beta(x) \mathcal{H}_{\beta+k}(y)$$

is a solution of (5.14).

Proposition 5.5 *Let the sequence* $\{c_{\alpha,\alpha+1}\}_{\alpha=0,1,...}$ *be such that*

$$\sum_{\alpha=0}^{\infty} |c_{\alpha,\alpha+1}| \leq \rho_\circ^{-1} \tag{5.23}$$

for some number $\rho_\circ > 1$*. Then, whatever the positive number* $\rho < \rho_\circ$*, the solution* $u_\circ$ *of* (5.14) *given by* (5.21) *belongs to* $\mathcal{A}_\rho^+$*.*

Proof. We derive from (5.22):

$$\rho_\circ^k |c_{\alpha,\alpha+k}| \leq \frac{3}{4k^2} \sum_{\ell=1}^{k-1} \rho_\circ^\ell |c_{\alpha,\alpha+\ell}| \rho_\circ^{k-\ell} |c_{\alpha+\ell,\alpha+k}|. \tag{5.24}$$

We claim that, for every $k = 1, 2, ...$,

$$\rho_\circ^k \sum_{\alpha=0}^{\infty} |c_{\alpha,\alpha+k}| \leq 1; \tag{5.25}$$

(5.25) is the same as (5.23) when $k = 1$. Induction on k yields

$$\rho_\circ^k \sum_{\alpha=0}^{\infty} |c_{\alpha,\alpha+k}| \leq \frac{3}{4k^2} \sum_{\ell=1}^{k-1} \sum_{\alpha=0}^{\infty} \rho_\circ^\ell |c_{\alpha,\alpha+\ell}| \rho^{\alpha+k-\ell} |c_{\alpha+\ell,\alpha+k}|$$

$$\leq \frac{3}{4k^2} \left(\sum_{\ell=1}^{k-1} \sum_{\alpha=0}^{\infty} \rho_\circ^\ell |c_{\alpha,\alpha+\ell}| \right)^2 \leq \frac{3}{4}$$

whence (5.25). From (5.25) we conclude, for any ρ, $0 < \rho < \rho_\circ$,

$$\sum_{\alpha=0}^{\infty} \sum_{k=1}^{\infty} \rho^k |c_{\alpha,\alpha+k}| \leq \frac{3}{4} \frac{\rho_\circ}{\rho_\circ - \rho}.$$

As before $L = D_x^2 + x^2$. Let $u_\circ$ be the solution of (5.14) given by (5.21) and let $\zeta = \xi + i\eta \in \mathbb{C}$ be a complex number. We have [cf. (5.3)]

$$\exp(-\zeta \partial_L) u_\circ = \sum_{\alpha=0}^{\infty} \sum_{k=1}^{\infty} c_{\alpha,\alpha+k} \exp(-2k\zeta) \mathcal{H}_\alpha(x) \mathcal{H}_{\alpha+k}(y).$$

If (5.25) holds, then

$$\sum_{\alpha=0}^{\infty} \sum_{k=1}^{\infty} |c_{\alpha,\alpha+k}| \exp(-2k\xi) < +\infty$$

provided $\xi \geq -\frac{1}{2k} \log \rho$; and as a consequence, $\exp(-\zeta \partial_L) u_\circ$ is a holomorphic function of ζ in the open half-plane $\mathrm{Re}\, \zeta \geq -\frac{1}{2k} \log \rho$, valued in the space $L^2(\mathbb{R}^2)$.

6 Appendix: Hermite functions expansion

The definition of the Hermite functions used in this article is

$$\mathcal{H}_m(x) = \frac{1}{\pi^{\frac{1}{4}} \sqrt{2^m m!}} e^{\frac{1}{2}x^2} \frac{d^m}{dx^m} \left(e^{-x^2} \right) = \frac{1}{\pi^{\frac{1}{4}} \sqrt{2^m m!}} \left(\frac{d}{dx} - x \right)^m \left(e^{-\frac{1}{2}x^2} \right). \tag{6.1}$$

We recall that

$$\frac{d\mathcal{H}_m}{dx} - x\mathcal{H}_m = \sqrt{2(m+1)} \mathcal{H}_{m+1}, \tag{6.2}$$

$$\frac{d\mathcal{H}_m}{dx} + x\mathcal{H}_m = -\sqrt{2m} \mathcal{H}_{m-1}, \tag{6.3}$$

$$D_x^2 \mathcal{H}_m + x^2 \mathcal{H}_m = 2(m+1) \mathcal{H}_m, \tag{6.4}$$

and that

$$\int_{-\infty}^{+\infty} \mathcal{H}_p(x) \mathcal{H}_q(x)\, dx = \begin{cases} 0 \text{ if } p \neq q, \\ 1 \text{ if } p = q. \end{cases} \tag{6.5}$$

Let $\mathfrak{s}_n$ denote the vector space of complex sequences $(c_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ rapidly decaying as $|\alpha| \longrightarrow +\infty$: $\forall k \in \mathbb{Z}_+^n$, $\sum_{\alpha\in\mathbb{Z}_+^n} (1+|\alpha|)^k |c_\alpha| < +\infty$. We assume that $\mathfrak{s}_n$ is equipped with the topology defined by the seminorms $(c_\alpha)_{\alpha\in\mathbb{Z}_+^n} \longrightarrow \sum_{\alpha\in\mathbb{Z}_+^n}^k (1+|\alpha|)^k |c_\alpha|$. The dual of $\mathfrak{s}_n$ is the vector space $\mathfrak{s}'_n$ of complex sequences $(c_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ that are tempered: there is $k \in \mathbb{Z}_+^n$ such that $\sum_{\alpha\in\mathbb{Z}_+^n} (1+|\alpha|)^{-k} |c_\alpha| < +\infty$; $\mathfrak{s}'_n$ will be equipped with its strong dual topology. We recall the following result (see [Schwartz, 1966], p. 262):

Theorem 5 *The Hermite functions expansion*

$$(c_\alpha)_{\alpha\in\mathbb{Z}_+^n} \longrightarrow \sum_{\alpha\in\mathbb{Z}_+^n} c_\alpha \mathcal{H}_{\alpha_1}(x_1)\cdots\mathcal{H}_{\alpha_n}(x_n)$$

defines a topological vector space isomorphism of $\mathfrak{s}_n$ (resp. $\mathfrak{s}'_n$) onto the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ [resp. $\mathcal{S}'(\mathbb{R}^n)$].

There is a similar representation for the kernel distributions $K(x,y) \in \mathcal{S}(\mathbb{R}_x^n) \widehat{\otimes}\ \mathcal{S}'(\mathbb{R}_y^n)$. We introduce the completed tensor product $\mathfrak{s}_n\widehat{\otimes}\mathfrak{s}'_n$, i.e., the set of complex multisequences $(c_{\alpha,k})_{\alpha,k\in\mathbb{Z}_+^n}$ that satisfy the following condition:

- *To every $k \in \mathbb{Z}_+$ there is $\ell \in \mathbb{Z}_+$ such that*

$$\sum_{\alpha,k\in\mathbb{Z}_+^n} (1+|\alpha|)^k (1+|k|)^{-\ell} |c_{\alpha,k}| < +\infty.$$

The set $\mathfrak{s}_n\widehat{\otimes}\mathfrak{s}'_n$ is an algebra for the natural matrix multiplication of its elements: if $\Gamma^{(i)} = \left(c_{\alpha,k}^{(i)}\right)_{\alpha,k\in\mathbb{Z}_+}$, $i=1,2$, then

$$\Gamma^{(1)}\Gamma^{(2)} = \left(\sum_{\gamma\in\mathbb{Z}_+^n} c_{\alpha,\gamma}^{(1)} c_{\gamma,k}^{(2)}\right)_{\alpha,k\in\mathbb{Z}_+^n}.$$

Theorem 6 *The map which assigns to the complex multisequence*

$$(c_{\alpha,k})_{\alpha,k\in\mathbb{Z}_+^n} \in \mathfrak{s}_n\widehat{\otimes}\mathfrak{s}'_n$$

the Hermite series

$$K(x,y) = \sum_{\alpha,k\in\mathbb{Z}_+^n} c_{\alpha,k}\mathcal{H}_{\alpha_1}(x_1)\cdots\mathcal{H}_{\alpha_n}(x_n)\mathcal{H}_{k_1}(y_1)\cdots\mathcal{H}_{k_n}(y_n) \qquad (6.6)$$

is an algebra isomorphism of $\mathfrak{s}_n\widehat{\otimes}\ \mathfrak{s}'_n$ onto $\mathcal{S}(\mathbb{R}_x^n)\widehat{\otimes}\ \mathcal{S}'(\mathbb{R}_y^n)$.

The proof of Theorem 6 is an easy application of Theorem 5 and of the definition of the tensor product completion $\mathcal{S}(\mathbb{R}^n_x)\,\widehat{\otimes}\,\mathcal{S}'\left(\mathbb{R}^n_y\right)$.

The identity operator is associated to the Dirac (kernel) distribution

$$\delta(x-y)=\sum_{\alpha\in\mathbb{Z}^n_+}\mathcal{H}_{\alpha_1}(x_1)\cdots\mathcal{H}_{\alpha_n}(x_n)\,\mathcal{H}_{\alpha_1}(y_1)\cdots\mathcal{H}_{\alpha_n}(y_n)\,. \tag{6.7}$$

It is clear that the coefficients of this series satisfy (•): in (6.7) $c_{\alpha,k}$ is the Kronecker index and we can take $\ell=k+n+1$ in (•).

Let

$$K^{(i)}(x,y)=\sum_{\alpha,k\in\mathbb{Z}^n_+}c^{(i)}_{\alpha,k}\mathcal{H}_{\alpha_1}(x_1)\cdots\mathcal{H}_{\alpha_n}(x_n)\,\mathcal{H}_{k_1}(y_1)\cdots\mathcal{H}_{k_n}(y_n)\,,\; i=1,2, \tag{6.8}$$

be two elements of $\mathcal{S}(\mathbb{R}^n_x)\,\widehat{\otimes}\,\mathcal{S}'\left(\mathbb{R}^n_y\right)$. The Hermite coefficients of the Volterra product $\left(K^{(1)}\circ K^{(2)}\right)(z,y)=\int K^{(1)}(x,z)\,K^{(2)}(z,y)\,dz$ are the complex numbers

$$C_{\alpha,k}=\sum_{\gamma\in\mathbb{Z}^n_+}c^{(1)}_{\alpha,\gamma}c^{(2)}_{\gamma,k},\ \alpha,k\in\mathbb{Z}^n_+. \tag{6.9}$$

Definition 6.1 *We say that a multisequence* $\left(c_{\alpha,k}\right)_{\alpha,k\in\mathbb{Z}^n_+}\in\mathfrak{s}_n\widehat{\otimes}\mathfrak{s}'_n$ *is* ***of trace-class*** *if* $\sum_{\alpha\in\mathbb{Z}^n_+}|c_{\alpha,\alpha}|<+\infty$*. If this is so, we say also that the kernel distribution* $K(x,y)\in\mathcal{S}(\mathbb{R}^n_x)\,\widehat{\otimes}\,\mathcal{S}'\left(\mathbb{R}^n_y\right)$ *given by* (6.6) *and* $\mathbf{Op}K$ *are of trace-class. The complex number* $\sum_{\alpha\in\mathbb{Z}^n_+}c_{\alpha,\alpha}$ *is called the* ***trace*** *of* K *or of* $\mathbf{Op}K$ *and denoted by* $\operatorname{Tr}K$ *or* $\operatorname{Tr}\mathbf{Op}K$*.*

The set of kernel distributions of trace-class is a linear subspace, but not a subalgebra, of the Volterra algebra $\mathcal{S}(\mathbb{R}^n_x)\,\widehat{\otimes}\,\mathcal{S}'\left(\mathbb{R}^n_y\right)$. The Dirac distribution $\delta(x-y)$ and the identity operator are not of trace-class.

Remark 6. If the kernel distribution $K\in\mathcal{S}(\mathbb{R}^n_x)\,\widehat{\otimes}\,\mathcal{S}'\left(\mathbb{R}^n_y\right)$ in (6.6) is of trace-class, the series $\sum_{\alpha\in\mathbb{Z}^n_+}c_{\alpha,\alpha}\mathcal{H}_{\alpha_1}(x_1)\cdots\mathcal{H}_{\alpha_n}(x_n)\,\mathcal{H}_{\alpha_1}(y_1)\cdots\mathcal{H}_{\alpha_n}(y_n)$ converges in $L^2\left(\mathbb{R}^n_x\times\mathbb{R}^n_y\right)$. But this does not mean that the restriction of $K(x,y)$ to the diagonal $\operatorname{diag}\left(\mathbb{R}^n_x\times\mathbb{R}^n_y\right)$ is well-defined and that

$$\operatorname{Tr}K=\sum_{\alpha\in\mathbb{Z}^n_+}c_{\alpha,\alpha}=\int_{\mathbb{R}^n}K(x,x)\,dx. \tag{6.10}$$

Proposition 6.1 *Every finite tensor*

$$K(x,y)=\sum_{j=1}^{N}\varphi_j(x)\,u_j(y)\in\mathcal{S}(\mathbb{R}^n_x)\otimes\mathcal{S}'\left(\mathbb{R}^n_y\right)$$

is of trace-class.

Proof. It suffices to prove the claim for a single product $\varphi(x)\,u(y)$, in which case it is an immediate consequence of Theorem 5.

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Selected Lectures in Microlocal Analysis

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Summary. This paper surveys classical results on microlocal analysis. It also includes more recent theorems on propagation at a non-microcharacteristic boundary: these are a boundary microlocal version of Holmgren's uniqueness Theorem.

2000 AMS Subject Classification: 58J32, 58J40

Key words: Microlocal analysis, boundary values.

1 Decomposition in plane waves: analytic wave front set

Our discussion follows the guidelines of Hörmander [6] and Sato–Kashiwara–Kawai [8]. Let x be the coordinates in $\mathbf{R}^n$ and (x, ξ) the canonically associated symplectic coordinates in $T^*\mathbf{R}^n$. Then

$$\omega := \xi \cdot dx, \quad d\omega := d\xi \wedge dx,$$

are the canonical 1- and 2-forms, respectively. We refer to x as the *position* and to ξ as the *frequency*. The spectral decomposition—in the space of frequencies—of the singularity of a function $u = u(x)$, is obtained from the representation of u in plane waves. For an integrable function u, we denote by $\hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-ix\cdot\xi} u(x)\, dx$ its Fourier transform; if $\hat{u}$ is also integrable, we have (inversion formula)

$$u(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi. \tag{1.1}$$

Formula (1.1) provides a "decomposition" of $u(x)$ in the functions $e^{ix\cdot\xi}$ (with "coefficients" $\hat{u}(\xi)$). Note that these exponential functions do not really depend on x but rather on $x \cdot \xi$, that is, they are constant on the planes

A. Bove et al. (eds.), *Advances in Phase Space Analysis of Partial Differential Equations,* DOI 10.1007/978-0-8176-4861-9_17,

$x \cdot \xi$ = cost. They are so-called *plane waves.* We define a set WFA(u) in $\dot{T}^*\mathbf{R}^n := T^*\mathbf{R}^n \setminus (\mathbf{R}^n \times \{0\})$, closed and conic, that is, invariant under multiplication in the fibers by $t \in \dot{\mathbf{R}}^+$.

Definition 1.1 (Hörmander: analytic wave front set) *Let $(x_o, \xi_o) \in \dot{T}^*\mathbf{R}^n$; we say that (x_o, ξ_o) does not belong to the analytic wave front set WFA(u) of u when, for some ε and M and for any ν,*

$$|\hat{u}^\nu(\xi)| \leq c^{\nu+1}\nu!|\xi|^{-\nu} \text{ for any } |\xi| \geq M,\ \Big|\frac{\xi}{|\xi|} - \frac{\xi_o}{|\xi_o|}\Big| < \varepsilon \text{ where } u^\nu = \chi^\nu u$$
$$\text{for a sequence } \chi^\nu \in C_c^\infty \text{ with } \chi^\nu \equiv 1 \text{ in a neighborhood of } x_o. \quad (1.2)$$

In this definition, the regularity with respect to ξ_o is related to the rapid decay at ∞ of the Fourier transform in a conical neighborhood of ξ_o; in particular, the low frequencies are disregarded. For a multiindex $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \sum_j \alpha_j$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdot ... \cdot \partial_{x_n}^{\alpha_n}$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdot ... \cdot \xi_n^{\alpha_n}$. By the identity $\widehat{\partial_x^\alpha u} = (-i)^{|\alpha|}\xi^\alpha \hat{u},\ \alpha \in \mathbb{N}^n$, one readily sees that

$$\pi\, \mathrm{WFA}(u) = \text{sing supp}(u)$$

where $\pi : \dot{T}^*\mathbf{R}^n \to \mathbf{R}^n$ is the canonical projection. For a cone $K \subset \dot{\mathbf{R}}^n$, which is closed and proper, that is, contained in an open half-space, we define the *polar cone* by

$$K^o = \{y :\ y \cdot \xi \geq 0 \text{ for any } \xi \in K\}$$

and set $\Gamma = \mathrm{int}(K^o)$; this is an open convex nonempty cone. Let us consider a decomposition:

$$\dot{\mathbf{R}}^n = \underset{j=1,\dots,N}{\cup} K_j,$$

where the K_j's are a family of closed proper cones with disjoint interiors. This yields a corresponding decomposition in the integration (1.1):

$$\begin{aligned} u(x) &= (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\cdot\xi}\hat{u}(\xi)\, d\xi \\ &= \sum_{j=1}^N (2\pi)^{-n} \int_{K_j} e^{ix\cdot\xi}\hat{u}(\xi) =: \sum_{j=1}^N u_j(x). \end{aligned} \quad (1.3)$$

We then have

- In each u_j we can replace x by $z = x + iy$ provided that $y \in \Gamma_j := \mathrm{int}(K_j^o)$ since there is then exponential decay in the term under integration:

  $$|e^{i(x+iy)\cdot\xi}| = e^{-y\cdot\xi} \text{ with } y \cdot \xi > 0;$$

  in particular, $u_j \in \mathrm{hol}(\mathbf{R}^n + i\Gamma_j)$.

- There are many possible decompositions $u = \sum_j u_j$. Let us point out our attention to a complex neighborhood B of x_o: a term which is holomorphic in $(\mathbf{R}^n+i(\Gamma_i+\Gamma_j))\cap B$ can be given either of the indices i or j. In particular, this is the case of a term which is holomorphic in a neighborhood of x_o in $\mathbf{C}^n$ which can be given any index j. One can consult [2] for a greater account on this subject.
- Let U be a real neighborhood of x_o. If u is represented by a single holomorphic function in the wedge $U + i\Gamma_\varepsilon$ truncated by the condition $|y| < \varepsilon$, then
$$\mathrm{WFA}(u) \subset U \times K \quad \text{where } K = \dot{\Gamma}^o. \tag{1.4}$$
And conversely, if $\mathrm{WFA}(u) \subset U \times K$, then u extends holomorphically to $z \in U + i\Gamma'_\varepsilon$ for any $\Gamma' \subset\subset \Gamma = \mathrm{int}(K^o)$ and for a suitable ε. Clearly, K is only an estimate for the singularity of u in the same way as Γ is for its holomorphic extension.
- Fix a frequency ξ_o, assume $\xi_o \notin \mathrm{WFA}(u)_{x_o}$ and take a neighborhood K_{j_o} of ξ_o such that $K_{j_o} \cap \mathrm{WFA}(u)_{x_o} = \emptyset$. Define $u_{j_o} := \int_{K_{j_o}} \cdot$; from
$$\mathrm{WFA}(u_{j_o})_{x_o} \subset \mathrm{WFA}(u)_{x_o} \cap K_{j_o} = \emptyset,$$
we get that u_{j_o} extends holomorphically at x_o. On the other hand, the other terms $u_j = (2\pi)^{-n} \int_{K_j} \cdot$ for $j \neq j_o$ extend holomorphically to Γ_j with $\Gamma_j^o \not\ni \xi_o$ and satisfy $(x_o, \xi_o) \notin \mathrm{WFA}(u_j)$ as well.

In conclusion, we have proved the following characterization:

$$(x_o, \xi_o) \notin \mathrm{WFA}(u) \text{ if and only if } u|_U = \sum_{j=1}^N u_j|_U$$
$$\text{with } u_j \in \mathrm{hol}(U + i(\Gamma_j)_\varepsilon) \text{ for } \Gamma_j \text{ satisfying } \Gamma_j^o \not\ni \xi_o.$$

We say that a term u_{j_o} in the decomposition of u is *missing* or can be *absorbed* by the others when there is u_j for $j \neq j_o$ which extends holomorphically to $U + i\Gamma_\varepsilon$ where Γ is the convex hull of $\Gamma_{j_o} \cup \Gamma_j$. In this terminology:

$$(x_o, \xi_o) \notin \mathrm{WFA}(u) \text{ iff } u_{j_o} \text{ is missing for a neighborhood } K_{j_o} \text{ of } \xi_o.$$

Remark 1.1 One could define the C^∞ WFA by replacing in (1.2) $c^{\nu+1}\nu!$ by a more general c_ν and by taking a single cut-off function χ instead of a sequence χ_ν; all the above discussion would hold unchanged (except from the second half part of the third itemized sentence).

2 Operators/symbols: symplectic transformations

We deal with linear differential operators $P = P(x, D)$ of the type

$$\sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha$$

with real analytic coefficients $a_\alpha(x)$ defined in $\mathbf{R}^n$ or in an open subset of $\mathbf{R}^n$. We associate to $P(x, D)$ its symbol $\sigma(P) = P(x, \xi)$, its principal symbol $\sigma_p(P) = P_m(x, \xi)$ which is the homogeneous term of order m of $\sigma(P)$ and finally its characteristic variety $\mathrm{char}(P)$ that we also denote by V and which is defined by $V = \{(x, \xi) \in \dot{T}^*\mathbf{R}^n : P_m(x, \xi) = 0\}$. The set of complex zeros of P_m, that is, $V^{\mathbf{C}} = \{(z, \zeta) \in \dot{T}^*\mathbf{C}^n : P_m(z, \zeta) = 0\}$, is in general larger than the complexification of V. (We do not specify that x is taken in an open set of $\mathbf{R}^n$ and z in a complex neighborhood to which the coefficients of P have holomorphic extension.) An elementary example occurs for the partially elliptic operator $P = \sum_{j\le n-1} \partial_{x_j}^2$. This geometric discussion, sheds light into deeper properties of the operator. Suppose that there is no *microlocal* coincidence of $V^{\mathbf{C}}$ with the complexification of V in a neighborhood of a non-zero characteristic point (as for the above operator). Then this circumstance was pointed out by Hörmander in [5] as responsible of the nonexistence of global real analytic solutions in $\mathbf{R}^n$ of the equation $Pu = f$ with constant coefficients.

The microlocal analysis describes the properties of the differential operators by means of the microdifferential operators whose symbols are no more polynomials in ξ but more general analytic functions. For instance, the operator $P^{-1}(x, D)$ associated to the symbol $P^{-1}(x, \xi)$ at points where $P_m(x, \xi) \ne 0$. But these are the tool of the analysis, not the object. The program is, in the beginning, to find suitable transformations in a neighborhood of a point $(x_o, \xi_o) \in \dot{T}^*\mathbf{R}^n$ so that, in some significant cases, the symbol is reduced into a *canonical* form. The aim is to interchange:

$$\sigma(P) \to \begin{cases} 1 & P \text{ noncharacteristic, i.e., } P_m(x_o, \xi_o) \ne 0, \\ \xi_1 & P_m \text{ real simply-characteristic, i.e., } P_m \text{ real,} \\ & P_m(x_o, \xi_o) = 0,\ \partial_\xi P_m(x_o, \xi_o) \ne 0, \\ \sum_{j=1}^m \xi_j^2 & P \text{ transversally elliptic.} \end{cases} \tag{2.1}$$

Once the local geometric transformation in $T^*\mathbf{R}^n$ is found, one is called to *quantize* it, that is, to find its differential counterpart acting on differential operators:

$$P(x, D) \to \begin{cases} \mathrm{id}, \\ \partial_{x_1}, \\ \sum_{j=1}^m \partial_{x_j}^2. \end{cases} \tag{2.2}$$

What is the kind of the suitable transformations? They are required to preserve *moments* in the sense of quantum mechanics, that is, commutators of the operators. But the commutator $[\cdot,\cdot]$ is related to the Poisson bracket $\{\cdot,\cdot\}$ by the following relation. Let $\mathrm{order}([P,Q]) = \mathrm{order}P + \mathrm{order}(Q) - 1$; then

$$\begin{aligned}\sigma_p([P,Q]) &= \{\sigma_p(P), \sigma_p(Q)\}\\ &= d\xi \wedge dx(H_{\sigma_p(P)}, H_{\sigma_p(Q)}),\end{aligned}$$

where $d\xi \wedge dx$ is the canonical 2-form and $H_{\sigma_p(P)}$, $H_{\sigma_p(Q)}$ are the Hamiltonian vector fields. Therefore, the admissible transformations are those which are *symplectic*, that is, which preserve $d\xi \wedge dx$. And they also need to respect the homogeneity, that is, the 1-form ξdx itself. Naturally, so far, the normalization (2.1) only carries a geometric meaning. The further task consists in finding the associated *differential* normalization satisfying (2.2).

3 The theorem of elliptic regularity

We perform here the microlocal reduction $P(x,\xi) \to 1$ and $P(x,D) \to \mathrm{id}$ in a neighborhood of the points (x_o,ξ_o) such that $P_m(x_o,\xi_o) \neq 0$. By the microlocal coincidence $P(x,D) = \mathrm{id}$ at (x_o,ξ_o) we mean the property $(x_o,\xi_o) \notin \mathrm{WFA}(P(x,D)u - u)$ for any u. In general, by the expression "microlocal vanishing at (x_o,ξ_o)" we mean the property "$(x_o,\xi_o) \notin \mathrm{WFA}$".

Theorem 3.1 (Sato: elliptic regularity) *We have*

$$WFA(u) \subset WFA(P(x,D)u) \cup \mathrm{char}(P). \tag{3.1}$$

In particular, if P is elliptic, that is, char$(P) = \emptyset$, and if $P(x,D)u \in C^\omega$, then also $u \in C^\omega$.

Proof. In this proof we follow Bony–Schapira [2]. Let $P_m(x_o,\xi_o) \neq 0$ and $(x_o,\xi_o) \notin \mathrm{WFA}(Pu)$; we wish to prove that $(x_o,\xi_o) \notin \mathrm{WFA}(u)$. Let us choose a neighborhood K_{j_o} of ξ_o and supplement it to a full covering $\dot{\mathbf{R}}^n = K_{j_o} \cup \left(\bigcup_{j\neq j_o} K_j\right)$. We have $Pu = \sum_{j\neq j_o} f_j$, that is, f_{j_o} is "missing" according to the terminology introduced before Remark 1.1. The question is whether

$$u = \sum_{j\neq j_o} u_j,$$

that is, whether u_{j_o} is also "missing". Write

$$\begin{aligned}Pu_{j_o} &= \sum_{j\neq j_o} (-Pu_j + f_j)\\ &= \sum_{j\neq j_o} \tilde{f}_j \text{ for } \tilde{f}_j \in \mathrm{hol}(U + i(\Gamma_j + \Gamma_{j_o})_\varepsilon).\end{aligned}$$

Recall the notation $V = \mathrm{char}(P)$. For suitably small K_{j_o}, we have $V_x \cap K_{j_o} = \emptyset$ for any x close to x_o which yields

$$(\Gamma_j + \Gamma_{j_o})^o \cap V_x = \emptyset. \tag{3.2}$$

Because of (3.2), we can solve by the method of the *noncharacteristic deformation* of [2] the equation

$$P\tilde{u}_j = \tilde{f}_j \text{ in } U + i(\Gamma_j + \Gamma_{j_o})_\varepsilon.$$

In particular, $(x_o, \xi_o) \notin \mathrm{WFA}(\tilde{u}_j)$. Again by (3.2), we have (cf. [2])

$$P\left(u_{j_o} - \sum_{j \neq j_o} \tilde{u}_j\right) = 0 \text{ implies } \left(u_{j_o} - \sum_{j \neq j_o} \tilde{u}_j\right) \in C^\omega_{x_o}.$$

Let us denote by v the C^ω-function on the right side of the above implication. What we have got is

$$u_{j_o} = \sum_{j \neq j_o} \tilde{u}_j + v \text{ with } (x_o, \xi_o) \notin \mathrm{WFA}(\tilde{u}_j) \text{ and } v \in C^\omega,$$

which yields the conclusion of the theorem.

Corollary 3.1 (Holmgren's uniqueness theorem) *Suppose*

$$\begin{cases} Pu = 0, \\ \partial^j_{x_1} u|_{x_1=0} = 0 \; for \; any \; j = 0, ..., m-1, \\ P_m(x_o, \theta) \neq 0 \; for \; \theta = (1, ..., 0). \end{cases}$$

Then, $u \equiv 0$ *in a neighborhood of* $x_o = 0$.

We give the sketch of the proof which is based on the involutivity of WFA. Denote by $\mathrm{ext}(u)$ the extension by 0 of u to $x_1 \leq 0$; we have $P(\mathrm{ext}(u)) = 0$. Application of Theorem 3.1 yields $(x_o, \theta) \notin \mathrm{WFA}(u)$. But if the support of a distribution is contained in $x_1 \leq 0$, then $(0, \theta)$ belongs to its WFA (Theorem 8.5.6 of [6]), unless the distribution is 0.

When the plane $x_1 = 0$ has its conormal θ which satisfies $P_m(x_o, \theta) \neq 0$, then it is said to be *noncharacteristic*. There is a microlocal version of Holmgren's uniqueness theorem. For this, a geometric preliminary is required.

Definition 3.1 *Let* $W^{\mathbb{C}}$ *be a regular involutive manifold passing though* (x_o, ξ_o) *and contained in the plane defined by* $\zeta_1 = 0$. *We say that the plane* $x_1 = 0$ *is non-microcharacteristic for* P *at* $(x_o, \xi_o) \in V \cap W$ *along* W, *when* ∂_{ζ_1} *does not belong to* $C_{(x_o, \xi_o)}(V^{\mathbb{C}}, W^{\mathbb{C}})$, *the Whitney normal cone at* (x_o, ξ_o).

In this situation, the following theorem holds:

Theorem 3.2 (Bony: microlocal Holmgren) *Let $x_1 = 0$ be non-micro-characteristic at (x_o, ξ_o) and u be a solution of $Pu = 0$. Then, microlocally at (x_o, ξ_o):*

$$u|_{x_1>0} = 0 \textit{ implies } u = 0 \ , \tag{3.3}$$

that is, $(x_o, \xi_o) \notin \overline{WFA(u)|_{x_1>0}}$ implies $(x_o, \xi_o) \notin WFA(u)$.

(See [1] for the proof.) In other words, the microlocal 0 propagates through a non-microcharacteristic plane. The statement can be extended to microlocal solutions u: these satisfy $(x_o, \xi_o) \notin \mathrm{WFA}(Pu)$ instead of $Pu = 0$.

4 Propagation in the interior: real simply-characteristic operators

We consider here operators whose principal part P_m is real and satisfies $P_m(x_o, \xi_o) = 0$ but $\partial_\xi(x_o, \xi_o) \neq 0$. We want to motivate the microlocal reduction $P(x, \xi) \to \xi_1$ and $P(x, D) \to \partial_{x_1}$. We have the analytic factorization

$$P_m(x, \xi) = h(x, \xi) \cdot (\partial_\xi P_m(x_o, \xi_o)\delta_\xi + \partial_x P_m(x_o, \xi_o)\delta_x) \text{ for } h(x_o, \xi_o) \neq 0, \tag{4.1}$$

where $\delta_\xi = \xi - \xi_o, \ \delta_x = x - x_o$. Now, h is invertible, as a symbol and also, according to the conclusions of Section 3, as a differential operator. It is also clear that there is a change of symplectic coordinates $\chi : (x, \xi) \mapsto (\tilde{x}, \tilde{\xi})$ from a neighborhood of (x_o, ξ_o) to a neighborhood of $(0; (1, 0...))$ in $T^*\mathbf{R}^n$ which interchanges the linear expression in (4.1) into $\tilde{\xi}_1$. We wish to explain in what sense this yields a reduction $P(x, D) \to \partial_{\tilde{x}_1}$. For this we define the tangent vectors

$$\begin{cases} v := (\partial_x P_m, \partial_\xi P_m) \in T_{(x_o,\xi_o)} T^*\mathbf{R}^n, \\ w := (\partial_\xi P_m, -\partial_x P_m) \in T_{(x_o,\xi_o)} T^*\mathbf{R}^n, \end{cases}$$

whose second is the Hamiltonian vector field H_{P_m}. Assuming that

$$|dP_m(x_o, \xi_o)|^2 = 1,$$

they are related by $d\omega(v, w) = 1$. If we perform the transformation χ and remember that $P_m \to \xi_1$, we get the following description for the transformed vectors $\tilde{v} = \chi' v, \ \tilde{w} = \chi' w$:

$$\begin{cases} \tilde{v} = (\partial_{\tilde{x}} \tilde{\xi}_1 \partial_{\tilde{\xi}} \tilde{\xi}_1) = (0, (1, ..., 0)), \\ \tilde{w} = H_{\tilde{\xi}_1} = ((1, ..., 0), 0). \end{cases}$$

These are still linked by the relation $d\omega(\tilde{v}, \tilde{w}) = \langle d\tilde{\xi}_1, H_{\tilde{\xi}_1} \rangle = 1$. Now, the solutions of $\partial_{\tilde{x}_1}$ are independent of $\tilde{x}_1$, that is, their zeros propagate along the straight lines parallel to $\tilde{w} = H_{\tilde{\xi}_1}$. But if we want to put into a symplectically invariant fashion the statement:

"the support of the solutions of $\partial_{\tilde{x}_1}$ is invariant under the direction $\tilde{w}$"

it turns into:

"the WFA of the solutions of P is invariant for $w = H_P$"

Naturally, this is just a geometric discussion. But this happens to have a "differential" counterpart which is contained in the following theorem for whose proof we refer to [4].

Theorem 4.1 (Hörmander: real simply-characteristic propagation) *Let P_m be real with simple characteristics and let u be a solution of $Pu = 0$. Then the analytic wave front set WFA(u) is invariant under the flow of the Hamiltonian vector field H_{P_m}.*

The theorem applies in particular to the wave operator $\Box = \partial^2_{x_n} - \sum_{j=1}^{n-1} \partial^2_{x_j}$. If ξ_o is a characteristic, that is, a nonnull zero of $\Box(\xi) = 0$, we define the corresponding bicharacteristic line (for a point x_o) as the line parallel to the vector $\partial_\xi \Box(\xi_o)$ in the x-space and whose ξ-value is constantly ξ_o; we denote it by b_{ξ_o}. Therefore, this is the integral curve of $H_\Box$ issued from (x_o, ξ_o). The theorem of propagation says that the WFA of a wave is a union of bicharacteristics. That is, the singularity in the frequency ξ_o propagates along the line b_{ξ_o}. Hence, in the free space, the singularity of a wave propagates along rays. What happens if the wave is no more free?

5 Propagation at the boundary: reflection and diffraction of the light

Suppose that a light ray hits an obstacle. Along which way does the propagation of the analytic singularity take place? We discuss either case of a transversal and a tangential ray. We fix our notations: x_n is chosen as the time coordinate so that our operator is $\Box = \partial^2_{x_n} - \sum_{j \le n-1} \partial^2_{x_j}$, and the obstacle or boundary is given by $x_1 = 0$. The boundary in the $(x_1, ..., x_{n-1})$-space is necessarily *noncharacteristic*, that is, its conormal θ, in our case $\theta = (1, 0, ...)$, does not annihilate P_m (in which case the high frequencies remain high even after restriction to $\xi_1 = 0$. Our wave stays in $x_1 > 0$. Now the situation has become richer: together with the bicharacteristics of the operator, which move in the space of the time x_n and the position $(x_1, ..., x_{n-1})$, there are now also those of the boundary which move in the space of the frequencies ξ. Symplectically, they do not differ. Let $\rho : T^*\mathbf{R}^n|_{x_1=0} \to T^*\mathbf{R}^{n-1}$ be the projection $\xi \mapsto (\xi_2, ..., \xi_n)$ with poles $\pm(1, 0, ...)$; we write ξ' for $(\xi_2, ..., \xi_n)$. The bicharacteristics of the plane $x_1 = 0$, that is the integral curves of H_{x_1}, are the fibers of ρ. We denote by $\xi_o^\pm$ the two characteristics with the same projection ξ'_o and by $\beta_{\xi_o^\pm}$ the components of $b_{\xi_o^\pm}$ which enter in $x_1 > 0$. We denote by $b_{x'_o}$ the bicharacteristic of the boundary which connects $b_{\xi_o^+}$ with $b_{\xi_o^-}$ above the point x'_o. The spectral frequency ξ_o^+ propagates in the free space along $\beta_{\xi_o^+}$. When the wave encounters the obstacle $x_1 = 0$ in x'_o it turns from ξ_o^+ along $b_{x'_o}$ until it reaches ξ_o^-. From this point it leaves the boundary and propagates in the free space into $x_1 > 0$ along $\beta_{\xi_o^-}$. We wish to show that the WFA of a wave is a union of $\beta_{\xi_o^\pm}$. The two bicharacteristics $\beta_{\xi_o^\pm}$ form the same angle with the plane $x_1 = 0$, that is, they reflect above that plane. In particular, this holds for the light rays, their projections over $x_n = \text{const}$.

Remark 5.1 The velocity of propagation of the light along the bicharacterstics $\beta^\pm_{\xi_o}$, that is the ratio $\frac{\sqrt{\sum_{j=1,\dots,n-1}\dot{x}_j^2}}{|\dot{x}_n|}$, is finite. Instead, along the bicharacteristics of the boundary, that is, the integral curves of H_{x_1}, x is constant and in particular the time x_n is. Thus the reflection, that is, the "switch" from ξ^+ to ξ^-, takes no time.

For a wave u in $x_1 > 0$, we define $\gamma(u) = (u|_{x_1=0}, \partial_{x_1}u|_{x_1=0})$ and call $\gamma(u)$ the *traces* of u on $x_1 = 0$. We have the following result for whose proof we refer to [12], [9] (cf. also [11]).

Theorem 5.1 (Schapira, Sjöstrand: reflection of singularities) *The wave front set WFA($\gamma(u)$) can be split into two closed sets $\mathcal{S}^+ \cup \mathcal{S}^- \subset \dot{T}^*\mathbf{R}^{n-1}$ such that*

$$WFA(u) = \bigcup_{\xi' \in \mathcal{S}^\pm} \beta_{\xi_\pm}, \tag{5.1}$$

where $\xi_\pm$ are the two characteristics such that $\rho(\xi_\pm) = \xi'$. In particular, for any $(x'_o, \xi'_o) \in \dot{T}^*\mathbf{R}^{n-1}$, if each of the two bicharacteristics $\beta_{\xi_o^\pm}$ contains $x^\pm$ such that $(x^\pm, \xi_o^\pm) \notin WFA(u)$, it follows that $(x'_o, \xi'_o) \notin WFA(\gamma(u))$.

A similar statement holds for the solutions in $x_1 > 0$ of any equation $Pu = 0$ with P_m real with simple characteristics provided that the bicharacteristics are transversal to the boundary.

Remark 5.2 (diffractive rays) What happens when b_{ξ_o} is no more transversal to the boundary $x_1 = 0$, that is, when $(\xi_o)_1 = 0$? In order to make the geometric setting not too hard, we suppose that the ray is "glancing", in the sense that it is tangent to the boundary at order 2. In this case, there is no more reflection but still the propagation of the singularity takes place along the bicharacteristic; if u is regular in two points of b_{ξ_o} from the two sides of its contact point x_o with the boundary, then the traces of u are regular in the frequency ξ_o at x_o. The result is due to Kataoka [7] and Sjöstrand [12] (cf. also Schapira [10] for another proof).

6 Propagation at the boundary: transversal ellipticity and non-microcharacteristicity

We start by rephrasing the conclusions of the theorem of reflection, and precisely, by restating its last statement as a *boundary microlocal Holmgren theorem.* To see this, we suppose from now on that $P_m((x_o, \xi_o) + \tau\theta)$ has all its zeros in τ which are real. (We could indeed reason in full generality if we would be ready to pay the price of introducing the microsupport at the boundary $\mathrm{WFA}_{\{x_1>0\}}$.) If u is a solution of $Pu = 0$ in $x_1 > 0$, denote by $\mathrm{ext}(u)$ its extension by 0. Note that, for $\xi'_o \in \dot{\mathbf{R}}^{n-1}$, "$\mathrm{WFA}(\gamma(u)) \cap \{(x_o, \xi'_o)\} = \emptyset$" is equivalent to "$\mathrm{WFA}(\mathrm{ext}(u)) \cap \rho^{-1}(x'_o, \xi'_o) = \emptyset$". The last conclusion of Theorem 5.1 is equivalent to the following statement

Theorem 6.1 (boundary microlocal Holmgren's I) *Let P_m be real simply-characteristic in $\rho^{-1}(x'_o, \xi'_o)$ and suppose that its bicharacteristics are transversal to $x_1 = 0$. For the solutions of $P(x, D)u = 0$ in $x_1 > 0$ we have, in a neighborhood of $\rho^{-1}(x'_o, \xi'_o)$, the microlocal implication*

$$u|_{x_1>0} = 0 \textit{ implies } \mathit{ext}(u) = 0. \tag{6.1}$$

There are two relevant classes of operators to which the same conclusion applies. First are the *transversally elliptic* operators, that is, those whose principal symbol can be reduced by a symplectic transformation to the microlocal model $\sum_{j=1,...,m} \xi_j^2$. Then the characteristic variety can be reduced to $V = \{\xi_j = 0,\ j = 1, ..., m\}$ in a neighborhood of $\xi_o = (0, \xi_{o\,m+1}, ...)$. According to Bony–Schapira [3], the propagation in the absence of boundary is ruled by the bicharacteristic foliation, the collection of the m-dimensional integral leaves of the symplectic orthogonal bundle $TV^{\perp\sigma}$; in the canonical model these are the planes $\Sigma = \{(x, \xi) : (x_{m+1}, ..., x_n) = \mathrm{const},\ \xi = \xi_o\}$. In the presence of the boundary condition $x_1 = 0$, we have:

Theorem 6.2 (boundary microlocal Holmgren's II) *Let P be transversally elliptic in $\rho^{-1}(x_o, \xi'_o) \cap V$ and suppose that the bicharacteristic (Hamiltonian) leaves are transversal to $x_1 = 0$. Then, in a neighborhood of $\rho^{-1}(x_o, \xi'_o)$, the solutions u of $Pu = 0$ over $x_1 > 0$ satisfy microlocally the implication* (6.1).

We refer to Schapira–Zampieri [11] and Uchida–Zampieri [13] for the proof. The last case is the merely non-microcharacteristic one, treated by Zampieri in [14]. This contains the preceding: in fact in the special case of transversally elliptic operators, the non-microcharacteristic boundaries are characterized as those which are transversal to the bicharacteristic foliations.

Theorem 6.3 (boundary microlocal Holmgren's III) *Let $x_1 = 0$ be non-microcharacteristic for P in any point of $V \cap \rho^{-1}(x_o, \xi'_o)$. Then, in a neighborhood of $\rho^{-1}(x_o, \xi'_o)$, we have microlocally the implication* (6.1).

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