Boris Kruglikov
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ABEL PRISEN

ABEL SYMPOSIA 5

Differential Equations – Geometry, Symmetries and Integrability

The Abel Symposium 2008

Springer

ABEL SYMPOSIA

Edited by the Norwegian Mathematical Society

Participants to the Abel Symposium 2008. Photo credits: B. Kruglikov

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Differential Equations: Geometry, Symmetries and Integrability

The Abel Symposium 2008

Proceedings of the Fifth Abel Symposium,
Tromsø, Norway, June 17-22, 2008

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Springer

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Preface to the Series

The Niels Henrik Abel Memorial Fund was established by the Norwegian government on January 1. 2002. The main objective is to honor the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics. The prize shall contribute towards raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective the board of the Abel fund has decided to finance one or two Abel Symposia each year. The topic may be selected broadly in the area of pure and applied mathematics. The Symposia should be at the highest international level, and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these Symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The board of the Niels Henrik Abel Memorial Fund is confident that the series will be a valuable contribution to the mathematical literature.

Ragnar Winther
Chairman of the board of the Niels Henrik Abel Memorial Fund

Preface

The Abel Symposium 2008 focused on the modern theory of differential equations and their applications in geometry, mechanics, and mathematical physics. Following the tradition of Monge, Abel and Lie, the scientific program emphasized the role of algebro-geometric methods, which nowadays permeate all mathematical models in natural and engineering sciences. The ideas of invariance and symmetry are of fundamental importance in the geometric approach to differential equations, with a serious impact coming from the area of integrable systems and field theories. More specifically, the following topics were central to the Symposium:

- Integrability methods for linear and non-linear differential equations
- Geometric analysis of the solutions and their moduli spaces
- Symmetries, conservation laws, recursion operators and generalizations
- Lie equations and geometric structures
- Pseudogroup actions and the algebra of differential invariants
- Topics from Hamiltonian and celestial mechanics
- Differential equations within field theories and quantum physics.

The Symposium was hosted by the University of Tromsø, the world's northernmost university, from June 17–22, 2008, and was organized by:

- Ian Anderson, Utah State University
- Alain Chenciner, Université Paris VII and IMCCE
- Boris Kruglikov, University of Tromsø
- Demeter Krupka, Palacky University
- Valentin Lychagin, University of Tromsø
- Eldar Straume, NTNU-Trondheim

Leading researchers from 13 different countries, working in these areas were invited to participate. A total of 43 participants were invited, including five Ph.D. students and post-doctoral researchers from Norway. The daily program consisted of lectures and discussions, held in an open and encouraging atmosphere. The following 28 plenary lectures were given:

1. Alain Albouy: *Projective dynamics of a classical particle or a multiparticle system*,
2. Ian Anderson: *Symmetry reduction and Darboux integrability*,
3. Ugo Bruzzo: *Framed bundles on Hirzebruch surfaces and equivariant cohomology of moduli spaces*,
4. Andreas Cap: *BGG sequences and geometric overdetermined systems*,
5. Alain Chenciner: *Action minimization and global continuation of Lyapunov families stemming from relative equilibria*,
6. Philippe Delanoe: *Transport equations and geometry: a survey*,
7. Boris Dubrovin: *On deformations of integrable hierarchies*,
8. Vladislav Goldberg (together with V. Lychagin): *Abelian equations and differential invariants of planar 4-webs*,
9. Michael Hazewinkel: *Niceness theorems*,
10. Wu-Yi Hsiang: *On the critical optimality for sphere packings of mixtures of two sizes*,
11. Nail Ibragimov: *Symmetries and conservation laws: a general theorem for arbitrary differential equations*,
12. Niky Kamran: *Focal systems for Pfaffian systems with characteristics*,
13. Boris Kruglikov (together with V. Lychagin): *Algebraic aspects of the compatibility of PDEs*,
14. Demeter Krupka: *Differential invariants in gauge theory*,
15. Boris Kupershmidt: *Phase spaces in algebra*,
16. Bernard Malgrange: *Lie pseudogroups and differential Galois theory*,
17. Andrei Marshakov: *Gauge/string duality, integrable equations and Abelian differentials*,
18. John Mather: *Tonelli minimizers and relative Tonelli minimizers*,
19. Vladimir Matveev: *Projective vector fields: Solution of Lie and Schouten problems*,
20. Sergei Merkulov: *Wheeled props, deformation theory and quantization*,
21. Richard Moeckel: *Topics on the three-body problem*,
22. Juan Morales-Ruiz (together with D. Blazquez): *Local and global properties of Lie-Vessiot systems*,
23. Jesu Rodriguez: *Characteristics of PDE in the framework of Lie-Weil jet spaces*,
24. Vladimir Roubtsov: *Partition function of SOS elliptic models with domain wall boundary conditions and projection method*,
25. Per Tomter: *Isometric immersions of homogeneous hyperspheres into complex hyperbolic space*,
26. Alexander Verbovetsky (together with P. Kersten, J. Krasilshchik and R. Vitolo): *Hamiltonian structures for general PDEs*,
27. Raffaele Vitolo (together with M. Modugno and C. Tejero Prieto): *Geometric aspects of the quantization of a rigid body*,
28. Keizo Yamaguchi: *Contact geometry of second order*.

In addition to the above speakers (and organizers), the following invited guests also attended the symposium:

1. David Blazquez, Barcelona
2. Tor Flå, Tromsø
3. Hilja Lisa Huru, Oslo
4. Marte Høyem, Tromsø
5. Per Jakobsen, Tromsø
6. Cathrine Jensen, Tromsø
7. Joseph Krasilshchik, Moscow
8. Alexey Kushner, Astrakhan
9. Einar Mjølhus, Tromsø
10. Martin Rypdal, Tromsø
11. Mahdi Khajeh Salehani, Trondheim
12. Alexey Samokhin, Moscow
13. Lars Sydnes, Trondheim

We are grateful to all the participants for their valuable contributions and for making the symposium a successful event. We would also like to express our gratitude to the Abel Foundation and the Norwegian Mathematical Society for giving us the opportunity to arrange the Abel Symposium 2008.

Moreover, we would like to thank the University of Tromsø for providing us with additional support and practical assistance related to the preparation and organization of the symposium.

Tromsø and Trondheim
January 28, 2009

Boris Kruglikov
Valentin Lychagin
Eldar Straume

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Some Canonical Structures of Cartan Planes in Jet Spaces and Applications

Ricardo J. Alonso, Sonia Jiménez, and Jesús Rodríguez

Abstract The tangent space to a jet manifold at a point has a module structure; this fact allows us to endow the Cartan subspace with a canonical bracket, a point-wise definition of its curvature. This bracket is related with the Spencer differential and an algebraic proof of the criterion on formal integrability given by Kruglikov and Lychagin is outlined.

On the other hand, we define the characteristic vectors of a system of partial differential equations in the framework of differential correspondences and show how the canonical bracket defined above can be used to compute them.

The idea of considering points of a manifold other than its ordinary zero-dimensional ones is very old, and it can be found in the work of Plücker, Grassmann, Weil, Lie, Grothendieck, etc. This is one of the key ideas in Algebraic Geometry.

In his papers about differential equations, Sophus Lie used a wide definition of point of a manifold: the points used by Lie are the ordinary ones, together with all the "infinitesimal submanifolds" of a convenient dimension, up to a certain order. Thus, the jets are present in his work, though in the language of his time (see [16, Sect. 130, p. 541], for instance). Lie did not assume any fibred structure on the manifold considered, so his jets are more general than the jets of sections of a fibre bundle.

For a better understanding of Lie's ideas it is convenient to think of jets of a smooth manifold as ideals of its ring of smooth functions. This point of view, which was introduced in [25] (see also [20]), is a natural continuation of Weil's theory of near points [27] and it allows describing the process of prolongation, the affine structures, the contact system, etc., in terms of the ring of smooth functions of the original manifold, making the fibration unnecessary and simplifying essentially the calculus in local coordinates. Several applications were done, showing the improvement given by this approach with respect to the usual one: Lie equations and pseudogroups [21, 22], formal integrability [23] or differential invariants [24]. Furthermore, following the scheme outlined in [17] and within the

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framework of the above mentioned theory of jets, in [9, 10] a new theory of differential correspondences is developed, with applications to systems of partial differential equations, that clarifies and completes some of the partial results announced by Lie in [17]. A point of view very close to this one allowed the first author to give the first intrinsic formulation and proof of Drach's theorem in [2].

In this paper we use the characterization of the tangent space to a jet bundle given in [20] and the definition of the contact system of [4] to describe in our language the Cartan distribution and define a canonical bracket, which is a point-wise definition of its curvature; this way we find the metasymplectic structure (see [13, 14]). This allows defining in this framework the Spencer sequence. The isotropic horizontal subspaces of the Cartan space are characterized as given by upper order jets; the proof is a Frobenius-like theorem for polynomials. As a consequence of this theorem it is easy to give an algebraic proof of the criterion on formal integrability obtained in [14] by Kruglikov and Lychagin in terms of the vanishing of Weyl's tensor.

Next we define the characteristics of a system of partial differential equations in the language of differential correspondences established in [9, 10], following [3]. Finally, we show how the so-called singular vectors in [26], which include the characteristic ones, can be computed by using the canonical bracket defined above.

Notations and Conventions. In the entirety of this paper M will be an n-dimensional smooth manifold and the word 'submanifold' will mean 'locally closed submanifold.' When S is a locally closed submanifold of M, I_S will be the ideal of $C^\infty(M)$ consisting of the functions vanishing on S. In case of S being only locally closed, we should replace M by the open set U into which S is a closed submanifold. Nevertheless, for simplicity in the exposition, that will be implicitly understood.

1 Jets of Submanifolds

Our work is based in the presentation of jet spaces given in [20], which is a natural continuation of the theory of Weil's near points [27]. A near point of a smooth manifold M is a homomorphism of $\mathbb{R}$-algebras from $C^\infty(M)$ into a Weil algebra (that is, a finite-dimensional, commutative and rational $\mathbb{R}$-algebra). According to our definition, a jet of M is the kernel of a near point of M, and consequently it is an ideal of $C^\infty(M)$. The manifolds of jets defined in this way include Grassmann manifolds or jets of submanifolds as particular cases. A detailed exposition of this theory can be found in [1,5,20]. Here we will use jets of submanifolds only; next we will give the basic definitions and summarize, without proofs, some results which will be used later on.

Let M be an n-dimensional smooth manifold and $\mathcal{A} = C^\infty(M)$; if S is an m-dimensional submanifold of M defined by an ideal I_S of $C^\infty(M)$, for each point $p \in S$ the (m,k)-jet of S at p is the class of all m-dimensional submanifolds of M which have a contact of order k with S at p. We can associate with this jet the ideal $\mathfrak{p}^k_m = I_S + \mathfrak{m}_p^{k+1}$, where $\mathfrak{m}_p$ is the maximal ideal of those functions in $C^\infty(M)$ which vanish at p. This gives a bijection between the set of (m,k)-jets of M and the set of ideals $\mathfrak{p}^k_m$ of $C^\infty(M)$ such that the factor ring $\mathcal{A}\big/\mathfrak{p}^k_m$ is isomorphic to the Weil algebra

$$\mathbb{R}^k_m = \mathbb{R}[t_1,\dots,t_m]\Big/(t_1,\dots,t_m)^{k+1} .$$

When $m = \dim M$, each (m,k)-jet of M has the form $\mathfrak{m}_p^{k+1}$, where $p \in M$.

We will denote by $J^k_m M$ the set of all (m,k)-jets of M. There is a canonical projection $\pi_k\colon J^k_m M \longrightarrow M$ which assigns to each jet $\mathfrak{p}^k_m$ the unique maximal ideal $\mathfrak{p}^0_m = \mathfrak{m}_p$ of $C^\infty(M)$ containing $\mathfrak{p}^k_m$; the point $p \in M$ corresponding to this ideal is called the source of $\mathfrak{p}^k_m$.

If $r \le k$, we have a canonical projection

$$\begin{aligned} \pi^r_k : J^k_m M &\longrightarrow J^r_m M \\ \mathfrak{p}^k_m &\longmapsto \mathfrak{p}^r_m = \mathfrak{p}^k_m + \mathfrak{m}_p^{r+1} \end{aligned}$$

Remark 1. When $k = 1$ we can give the following geometric description of jets: there is a bijection between the $(m,1)$-jets of M with source p and the m-dimensional subspaces of T_pM. In fact, we can associate with each first order jet $\mathfrak{p}^1_m \in J^1_m M$ the vector space

$$L_{\mathfrak{p}^1_m} = \{D_p \in T_pM : D_p(f) = 0\ \forall f \in \mathfrak{p}^1_m\}$$

and conversely, the subspace $L \subseteq T_pM$ defines the jet

$$\mathfrak{p}^1_m = \{f \in \mathfrak{m}_p : D_p(f) = 0\ \forall D_p \in L\}.$$

Therefore $J^1_m M$ is the Grassmann manifold of m-planes tangent to M.

In [20] $J^k_m M$ is endowed with a smooth structure as a quotient of the space of regular (m,k)-velocities of M by the action of the differential group $\mathrm{Aut}(\mathbb{R}^k_m)$; such a smooth structure can be described as follows:

Let U be an open subset of M coordinated by $x_1,\dots,x_n$. Let us choose m of them, for example $x_1,\dots,x_m$, and denote by $\underline{J}^k_m U$ the set of jets $\mathfrak{p}^k_m \in J^k_m U$ such that

$$C^\infty(U)\Big/\mathfrak{p}^k_m \approx \mathbb{R}[x_1,\dots,x_m]\Big/\mathfrak{p}^k_m \cap \mathbb{R}[x_1,\dots,x_m] ;$$

this ring is isomorphic to the algebra of polynomials

$$\mathbb{R}^k_m = \mathbb{R}[x_1,\dots,x_m]\Big/(x_1 - x_1(p),\dots,x_m - x_m(p))^{k+1} ,$$

where $p = \pi_k(\mathfrak{p}^k_m)$, hence for each function $f \in C^\infty(U)$ there is a unique polynomial $P_f \in \mathbb{R}^k_m$ such that $f - P_f \in \mathfrak{p}^k_m$.

For $j = 1,\dots,n-m$ let us denote $y_j = x_{m+j}$; then

$$P_{y_j} = \sum_{|\alpha|\le k} \frac{1}{\alpha!} y_{j\alpha}(\mathfrak{p}^k_m)(x - x(p)^\alpha$$

for suitable numbers $y_{j\alpha}(\mathfrak{p}^k_m)$. Then

$$\mathfrak{p}^k_m = (y_1 - P_{y_1}, \dots, y_{n-m} - P_{y_{n-m}}) + \mathfrak{m}_p^{k+1}$$

and consequently the functions x_i, $y_{j,\alpha}$ are a system of local coordinates in $\underline{J}^k_m U$. This way we obtain an atlas on $J^k_m U$.

Remark 2. $\underline{J}^k_m U$ is an open subset of $J^k_m M$ isomorphic to the space of k-jets of sections of the projection $(x_i, y_j) \longmapsto x_i$. This is the reason why we use the notations x_i, y_j, thus establishing a distinction between the "base coordinates" and the "fibre coordinates". Nevertheless, such a distinction is only formal, because m arbitrary coordinates can be chosen as independent variables and the remainder as functions of them.

This way we see that our theory recovers the original point of view of Lie about jets (see [16], for instance); he used to consider a system of local coordinates $x_1, \dots, x_n$ in an open subset of M, and to think of m of them as independent and the remainder as dependent variables; but in a dynamical way, without fixing them.

A fundamental advantage of this theory of jets is the fact that the prolongations of submanifolds or ideals, the tangent structures and other constructions related to the jet spaces of a manifold M can be described in terms of the ring $\mathcal{A} = C^\infty(M)$; we do not need to change the ring of functions, but only the algebra where mappings or derivations are valued. An important example of this approach is the following characterization of the tangent space $T_{\mathfrak{p}^k_m} J^k_m M$ given in [20] (see [1] for the generalization to A-jets).

Theorem 1. *Let M be a smooth manifold; for each jet $\mathfrak{p}^k_m \in J^k_m M$ the tangent space $T_{\mathfrak{p}^k_m} J^k_m M$ is canonically isomorphic to the vector space $\mathcal{D}_{\mathfrak{p}^k_m}/\mathcal{D}'_{\mathfrak{p}^k_m}$, where $\mathcal{D}_{\mathfrak{p}^k_m} = \mathrm{Der}_{\mathbb{R}}\left(\mathcal{A}, \mathcal{A}/\mathfrak{p}^k_m\right)$ and $\mathcal{D}'_{\mathfrak{p}^k_m} = \{D \in \mathcal{D}_{\mathfrak{p}^k_m} : Df = 0\,\forall f \in \mathfrak{p}^k_m\}$.*

In the local coordinates described above, this identification gives:

$$\begin{aligned} T_{\mathfrak{p}^k_m} J^k_m M &\xrightarrow{\sim} \mathcal{D}_{\mathfrak{p}^k_m}/\mathcal{D}'_{\mathfrak{p}^k_m} \\ \left(\frac{\partial}{\partial x_i}\right)_{\mathfrak{p}^k_m} &\longmapsto \left[\frac{\partial}{\partial x_i}\right] \\ \left(\frac{\partial}{\partial y_{j\alpha}}\right)_{\mathfrak{p}^k_m} &\longmapsto \left[\frac{(x - x(p))^\alpha}{\alpha!}\frac{\partial}{\partial y_j}\right] \end{aligned}$$

where we write $[D]$ for the class of the derivation $D \in \mathcal{D}_{\mathfrak{p}^k_m}$ modulo $\mathcal{D}'_{\mathfrak{p}^k_m}$ (see [20] for the computation).

Remark 3. As a consequence of Theorem 1 there is a canonical structure of $\mathcal{A}/\mathfrak{p}^k_m$-module in $T_{\mathfrak{p}^k_m} J^k_m M$.

Corollary 1. *Each $f \in \mathfrak{p}_m^k$ defines an $\mathcal{A}/\mathfrak{p}_m^k$-linear map*

$$\begin{aligned} \mathfrak{d}_{\mathfrak{p}_m^k} f : T_{\mathfrak{p}_m^k} J_m^k M &\longrightarrow \mathcal{A}/\mathfrak{p}_m^k \\ D_{\mathfrak{p}_m^k} = [D] &\longmapsto Df \end{aligned}$$

where D is any derivation in $\mathcal{D}_{\mathfrak{p}_m^k}$ in the class $[D]$.

The local expression of $\mathfrak{d}_{\mathfrak{p}_m^k} f$ is

$$\mathfrak{d}_{\mathfrak{p}_m^k} f = \sum_i \left[\frac{\partial f}{\partial x_i}\right]_{\mathfrak{p}_m^k} d_{\mathfrak{p}_m^k} x_i + \sum_{j,\alpha} \left[\frac{(x - x(p))^\alpha}{\alpha!} \frac{\partial f}{\partial y_j}\right]_{\mathfrak{p}_m^k} d_{\mathfrak{p}_m^k} y_{j,\alpha}. \tag{1}$$

Corollary 2. *For each jet $\mathfrak{p}_m^k \in J_m^k M$ we have*

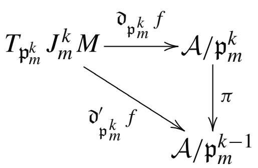
$$T^*_{\mathfrak{p}_m^k} J_m^k M = \{\textit{Real components of}\, \mathfrak{d}_{\mathfrak{p}_m^k} f : f \in \mathfrak{p}_m^k\}$$

Proof. Given a non-vanishing vector $D_{\mathfrak{p}_m^k} \in T_{\mathfrak{p}_m^k} J_m^k M$, there exists at least a function $f \in \mathfrak{p}_m^k$ such that $\mathfrak{d}_{\mathfrak{p}_m^k} f(D_{\mathfrak{p}_m^k}) \neq 0$, hence there is a real component of $\mathfrak{d}_{\mathfrak{p}_m^k} f$ which does not vanish at $\mathfrak{p}_m^k$. □

2 The Contact System

In this section we define the contact system in $J_m^k M$ following [4].

Let $\mathfrak{p}_m^k \in J_m^k M$; since $\mathfrak{p}_m^{k-1} = \mathfrak{p}_m^k + \mathfrak{m}_p^k$, we have a commutative diagram



Definition 1. The distribution $\mathcal{C}$ of tangent vector fields on $J_m^k M$ whose value at each jet $\mathfrak{p}_m^k$ is

$$\mathcal{C}_{\mathfrak{p}_m^k} = \bigcap_{f \in \mathfrak{p}_m^k} \operatorname{Ker}(\mathfrak{d}'_{\mathfrak{p}_m^k} f) \subset T_{\mathfrak{p}_m^k} J_m^k M$$

is the *Cartan distribution* on $J_m^k M$.

The Pfaff system Ω associated with $\mathcal{C}$ is the *contact system* on $J_m^k M$.

Remark 4. Each derivation $D : \mathcal{A} \longrightarrow \mathcal{A}$ defines a tangent vector at $\mathfrak{p}_m^k$; this vector belongs to the Cartan distribution if and only if $D(\mathfrak{p}_m^k) \subseteq \mathfrak{p}_m^{k-1}$.

The reason why we give this definition is that the Pfaff system defined above vanishes over the tangent space to the k-jet prolongation of any m-dimensional submanifold of M, as we will see now.

Let S be an m-dimensional submanifold of M; then $J^k_m S$ is diffeomorphic to a submanifold of $J^k_m M$, called the prolongation of S to $J^k_m M$, whose points are the jets of the form $I_S + \mathfrak{m}_p^{k+1}$, where $p \in S$. From the definitions it follows that

$$J^k_m S = \{\mathfrak{p}^k_m \in J^k_m M : \mathfrak{p}^k_m \supseteq I_S\};$$

if we use the language of Algebraic Geometry, $J^k_m S$ is the set of zeros of the ideal I_S in $J^k_m M$.

The following result is an immediate consequence of the definitions and Theorem 1:

Proposition 1. *If S is an m-dimensional submanifold of M, for each $\mathfrak{p}^k_m \in J^k_m S$ the tangent space $T_{\mathfrak{p}^k_m} J^k_m S$ is isomorphic to the set of classes of derivations in $\mathcal{D}_{\mathfrak{p}^k_m}/\mathcal{D}'_{\mathfrak{p}^k_m}$ killing I_S.*

Let us choose local coordinates $\{x_1, \dots, x_m, y_1, \dots, y_{n-m}\}$ in a neighbourhood of p such that $\mathfrak{p}^k_m = (y_1, \dots, y_{n-m}) + \mathfrak{m}_p^{k+1}$; each m-dimensional submanifold S of M such that $I_S \subseteq \mathfrak{p}^k_m$ is defined locally by equations $y_j = P_j(x_1, \dots, x_m)$, $1 \le j \le n-m$, where $P_j(x) \in \mathfrak{m}_p^{k+1}$. Consequently, the set of values $D_{\mathfrak{p}^k_m}(f)$, where $f \in \mathfrak{p}^k_m$ and $D_{\mathfrak{p}^k_m}$ runs through the tangent spaces to the k-jet prolongations of m-dimensional submanifolds of M, equals $\mathfrak{m}^k_p/\mathfrak{p}^k_m$. Accordingly, for each $f \in \mathfrak{p}^k_m$ the differentials $\mathfrak{d}'_{\mathfrak{p}^k_m} f$ annihilate each tangent vector to the k-jet prolongation of any m-dimensional submanifold of M.

Expression of Ω in Local Coordinates. If $f \in \mathcal{A}$, we denote by $f(\mathfrak{p}^k_m)$ the class of f in $\mathcal{A}/\mathfrak{p}^k_m$. Let us choose local coordinates $x_1, \dots, x_m, y_1, \dots, y_{n-m}$ on $U \subset M$; each jet $\mathfrak{p}^k_m$ whose source p belongs to U is the ideal generated by $\mathfrak{m}_p^{k+1}$ and the $n-m$ functions

$$f_j = y_j - \sum_{|\alpha| \le k} \frac{1}{\alpha!} y_{j,\alpha}(\mathfrak{p}^k_m)(x - x(p))^\alpha;$$

the local expression of $\mathfrak{d}_{\mathfrak{p}^k_m} f$ is

$$\mathfrak{d}_{\mathfrak{p}^k_m} f = \sum_i \frac{\partial f}{\partial x_i}(\mathfrak{p}^k_m) d_{\mathfrak{p}^k_m} x_i + \sum_{j,\alpha} \frac{1}{\alpha!}(x - x(p))^\alpha \frac{\partial f}{\partial y_j}(\mathfrak{p}^k_m) d_{\mathfrak{p}^k_m} y_{j,\alpha}.$$

Consequently,

$$\mathfrak{d}'_{\mathfrak{p}^k_m} f_j = \sum_{|\alpha| \le k-1} \frac{1}{\alpha!}(x - x(p))^\alpha (d_{\mathfrak{p}^k_m} y_{j,\alpha} - \sum_i y_{j,\alpha+1_i}(\mathfrak{p}^k_m) d_{\mathfrak{p}^k_m} x_i)$$

Since $\eth'_{\mathfrak{p}^k_m}(\mathfrak{m}_p^{k+1}) = 0$, the contact system Ω is generated locally by the 1-forms

$$\omega_{j,\alpha} = dy_{j,\alpha} - \sum_i y_{j,\alpha+1_i} dx_i, \tag{2}$$

where 1_i is the m-index with 1 in the ith component and 0 in the remaining ones. This agrees with the usual expression for the contact system.

According to the computations above, the Cartan distribution is spanned locally by the vector fields

$$\partial_i^{(k)} = \frac{\partial}{\partial x_i} + \sum_{\substack{|\alpha| \le k-1 \\ 1 \le j \le n-m}} y_{j,\alpha+1_i} \frac{\partial}{\partial y_{j,\alpha}} \qquad (1 \le i \le m)$$

$$\frac{\partial}{\partial y_{j,\beta}} \qquad (1 \le j \le n-m,\ |\beta| = k).$$

3 The Canonical Bracket

In this section we give a point-wise definition of the metasymplectic structure in $J^k_m M$, that is to say, the curvature of the Cartan distribution, considered by Kruglikov and Lychagin in [14] (see also [13, 19]).

Let $\mathfrak{p}^k_m \in J^k_m M$; each tangent vector $D_{\mathfrak{p}^k_m} \in \mathcal{C}_{\mathfrak{p}^k_m}$ is represented by a derivation $D : \mathcal{A} \longrightarrow \mathcal{A}/\mathfrak{p}^k_m$ such that $D(\mathfrak{p}^k_m) \subseteq \mathfrak{p}^{k-1}_m/\mathfrak{p}^k_m$. Consequently D induces a derivation

$$\overline{D} : \mathcal{A}/\mathfrak{p}^k_m \longrightarrow \mathcal{A}/\mathfrak{p}^{k-1}_m.$$

If $D_1, D_2 \in \mathcal{C}_{\mathfrak{p}^k_m}$ we can define

$$[D_1, D_2] = \overline{D}_1 \circ D_2 - \overline{D}_2 \circ D_1$$

according to the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{D_1} & \mathcal{A}/\mathfrak{p}^k_m \\ {\scriptstyle D_2}\downarrow & & \downarrow{\scriptstyle \overline{D}_2} \\ \mathcal{A}/\mathfrak{p}^k_m & \xrightarrow{\overline{D}_1} & \mathcal{A}/\mathfrak{p}^{k-1}_m \end{array}$$

$[D_1, D_2] : \mathcal{A} \longrightarrow \mathcal{A}/\mathfrak{p}^{k-1}_m$ is a derivation, and its class is a vector belonging to $T_{\mathfrak{p}^{k-1}_m} J^{k-1}_m M$.

Lemma 1. *Let $D : \mathcal{A} \longrightarrow \mathcal{A}$ be a derivation; we have:*

1. *If $D(\mathfrak{p}^k_m) \subseteq \mathfrak{p}^k_m$, then $D(\mathfrak{m}_p) \subseteq \mathfrak{m}_p$; therefore, $D(\mathfrak{p}^r_m) \subseteq \mathfrak{p}^r_m$ for each $r \le k$.*
2. *If $D(\mathfrak{p}^k_m) \subseteq \mathfrak{p}^{k-1}_m$, then $D(\mathfrak{p}^r_m) \subseteq \mathfrak{p}^{r-1}_m$ for each $r \le k$.*

Proof. Let $\{x_i, y_j\}$ be a system of local coordinates around p such that $\mathfrak{p}^k_m = (y_1, \dots, y_{n-m}) + \mathfrak{m}_p^{k+1}$.

In the hypothesis of the first assertion, $D(\mathfrak{m}_p^{k+1}) \subset \mathfrak{p}^k_m$; therefore, for $i = 1, \dots, m$ we have

$$D(x_i^{k+1}) = (k+1)D(x_i)x_i^k \in \mathfrak{m}_p^{k+1},$$

and consequently $D(x_i) \in \mathfrak{m}_p$. On the other hand, $D(y_j) \in \mathfrak{p}^k_m \subset \mathfrak{m}_p$, because $y_j \in \mathfrak{p}^k_m$. Therefore $D(\mathfrak{m}_p) \subseteq \mathfrak{m}_p$.

Since $\mathfrak{p}^r_m = \mathfrak{p}^k_m + \mathfrak{m}_p^{r+1}$, we have:

$$D(\mathfrak{p}^r_m) = D(\mathfrak{p}^k_m) + D(\mathfrak{m}_p^{r+1}) \subseteq \mathfrak{p}^k_m + \mathfrak{m}_p^{r+1} = \mathfrak{p}^r_m.$$

The second assertion is deduced with a similar computation. □

Each derivation from $\mathcal{A}$ into $\mathcal{A}/\mathfrak{p}^k_m$ is the composition of a derivation from $\mathcal{A}$ into itself with the canonical mapping $\mathcal{A} \longrightarrow \mathcal{A}/\mathfrak{p}^k_m$; therefore, we can represent each Cartan vector in $\mathcal{C}_{\mathfrak{p}^k_m}$ by a derivation from $\mathcal{A}$ into $\mathcal{A}/\mathfrak{p}^k_m$ or by a derivation from $\mathcal{A}$ into $\mathcal{A}$ which applies $\mathfrak{p}^k_m$ into $\mathfrak{p}^{k-1}_m$; we will use the same notation for both representatives. By Lemma 1, if $D_1, D_2 : \mathcal{A} \longrightarrow \mathcal{A}$ are derivations representing vectors of $\mathcal{C}_{\mathfrak{p}^k_m}$, then $[D_1, D_2](\mathfrak{p}^k_m) \subseteq \mathfrak{p}^{k-2}_m$, hence

$$[D_1, D_2](\mathfrak{p}^{k-1}_m) = [D_1, D_2](\mathfrak{p}^k_m + \mathfrak{m}^k_p) \subseteq \mathfrak{p}^{k-2}_m + \mathfrak{m}^{k-1}_p = \mathfrak{p}^{k-2}_m,$$

and $[D_1, D_2]$ defines a vector in $\mathcal{C}_{\mathfrak{p}^{k-1}_m}$.

The class of $[D_1, D_2]$ in $T_{\mathfrak{p}^{k-1}_m} J^{k-1}_m M$ depends on D_1, D_2: if we choose other derivations D_1', D_2' representing the same vectors than D_1, D_2, then

$$[D_1, D_2] - [D_1', D_2'] \in L_{\mathfrak{p}^k_m} \subset \mathcal{C}_{\mathfrak{p}^{k-1}_m}.$$

Since $\mathcal{C}_{\mathfrak{p}^{k-1}_m}/L_{\mathfrak{p}^k_m} \approx Q_{\mathfrak{p}^{k-1}_m} J^{k-1}_m M$, we have

Proposition 2 (Canonical bracket). *Let $\mathfrak{p}^k_m \in J^k_m M$. The commutator of derivations defines a skew-symmetric bracket*

$$\Omega_{2,\mathfrak{p}^k_m} : \mathcal{C}_{\mathfrak{p}^k_m} \otimes_{\mathbb{R}} \mathcal{C}_{\mathfrak{p}^k_m} \longrightarrow Q_{\mathfrak{p}^{k-1}_m} J^{k-1}_m M$$

called metasymplectic structure *or, when it is understood as an element of $\bigwedge^2 \mathcal{C}_{\mathfrak{p}^k_m} \otimes_{\mathbb{R}} Q_{\mathfrak{p}^{k-1}_m} J^{k-1}_m M$,* curvature of the Cartan distribution *at $\mathfrak{p}^k_m$.*

Definition 2. An m-dimensional subspace $H \subseteq T_{\mathfrak{p}^k_m} J^k_m M$ is *horizontal* if the dimension of its projection onto $T_p M$ equals m. A subspace $H \subseteq \mathcal{C}_{\mathfrak{p}^k_m}$ is *isotropic* or *involutive* if $\Omega_2|_H = 0$.

Theorem 2. *Let H be an m-dimensional subspace of $T_{\mathfrak{p}^k_m} J^k_m M$. Then:*

$H = L_{\mathfrak{p}^{k+1}_m}$ for some $\mathfrak{p}^{k+1}_m \in J^{k+1}_m M \iff H$ is horizontal and isotropic. In this case, $\mathfrak{p}^{k+1}_m = \{f \in \mathfrak{p}^k_m : Df = 0 \; \forall \; D \in H\}$.

Proof. The necessity of the condition is obvious: $L_{\mathfrak{p}^k_m}$ is horizontal and isotropic.

Conversely, let $x_1, \dots, x_m, y_1, \dots, y_{n-m}$ be local coordinates around p in M such that $\mathfrak{p}^k_m = (y_1, \dots, y_{n-m}) + \mathfrak{m}_p^{k+1}$. Since $H \subseteq \mathcal{C}_{\mathfrak{p}^k_m}$ is horizontal, it is generated by vectors corresponding to derivations

$$D_i = \frac{\partial}{\partial x_i} + \sum_j P_{i,j}(x) \frac{\partial}{\partial y_j} \qquad (1 \le i \le m)$$

P_{ij} homogeneous polynomials, deg $P_{ij} = k$.

The isotropy condition gives

$$0 = \Omega_2(D_i, D_r) = \sum_j \left(\frac{\partial P_{rj}}{\partial x_i} - \frac{\partial P_{ij}}{\partial x_r} \right) \frac{\partial}{\partial y_j}.$$

Hence there exist homogeneous polynomials $P_j(x)$, deg $P_j = k+1$, such that $P_{ij} = \frac{\partial P_j}{\partial x_i}$.

If we replace the coordinates x_i, y_j by $x_i, u_j = y_j - P_j(x)$, we have: $D_i = \frac{\partial}{\partial x_i}$, $\mathfrak{p}^k_m = (u_1, \dots, u_{n-m}) + \mathfrak{m}_p^{k+1}$ and $H = L_{\mathfrak{p}^{k+1}_m}$, where

$$\mathfrak{p}^{k+1}_m = (u_1, \dots, u_{n-m}) + \mathfrak{m}_p^{k+2}. \ \square$$

4 The Spencer Sequence and Formal Integrability

Each jet $\mathfrak{p}^{k+1}_m \in J^{k+1}_m M$ defines a horizontal subspace $L_{\mathfrak{p}^{k+1}_m}$ of $T_{\mathfrak{p}^k_m} J^k_m M$ as follows:

$$L_{\mathfrak{p}^{k+1}_m} = \{[D] : D(\mathfrak{p}^{k+1}_m) = 0\}.$$

Let us denote by $Q_{\mathfrak{p}^k_m} J^k_m M$ the vertical subspace of $T_{\mathfrak{p}^k_m} J^k_m M$ for the projection π^{k-1}_k; then

$$Q_{\mathfrak{p}^k_m} J^k_m M = \{[D] : D(\mathfrak{p}^{k-1}_m) \subseteq \mathfrak{p}^{k-1}_m\}.$$

$\mathfrak{p}^{k+1}_m$ induces a splitting

$$\mathcal{C}_{\mathfrak{p}^k_m} = L_{\mathfrak{p}^{k+1}_m} \oplus Q_{\mathfrak{p}^k_m} J^k_m M$$

The following proposition describes the behaviour of the canonical bracket with respect to these subspaces:

Proposition 3. *Let Ω_2 be the value at $\mathfrak{p}^k_m$ of the canonical bracket. We have:*

1. $\Omega_2|_{L_{\mathfrak{p}^{k+1}_m}} = 0$.
2. $\Omega_2|_{Q_{\mathfrak{p}^k_m} J^k_m M} = 0$.
3. *If $X \in L_{\mathfrak{p}^{k+1}_m}$ and $D \in Q_{\mathfrak{p}^k_m} J^k_m M$, then $\Omega_2(X, D)$ depends only on the projection $\overline{X} \in L_{\mathfrak{p}^1_m}$.*

Proof. The first assertion is obvious. To prove the second one, let us consider two vectors $D_1, D_2 \in Q_{\mathfrak{p}_m^k} J_m^k M$; they are the classes of two derivations $D_1, D_2 : \mathcal{A} \longrightarrow \mathcal{A}$ such that $D_i(\mathcal{A}) \subseteq \mathfrak{p}_m^{k-1}, i = 1, 2$. Then $\overline{D}_1 = \overline{D}_2 = 0$, hence $[D_1, D_2] = 0$.

Finally, let $X \in L_{\mathfrak{p}_m^k}$ and $D \in Q_{\mathfrak{p}_m^k} J_m^k M$. If the projection $\overline{X}$ of X over $L_{p_m^1}$ vanishes, then $X(\mathfrak{m}_p) \subseteq \mathfrak{m}_p$, and $[X, D](\mathfrak{p}_m^{k-1}) \subseteq \mathfrak{p}_m^{k-1}$ (as a derivation from $\mathcal{A}$ into itself), like in Lemma 1; that is, $\Omega_2(X, D) = 0$. □

According to Proposition 3, Ω_2 is completely determined by the values $\Omega_2(X, D)$, where $X \in L_{\mathfrak{p}_m^{k+1}}$ and $D \in Q_{\mathfrak{p}_m^k} J_m^k M$, and these values depend only on the projection $\overline{X}$ of X in $L_{\mathfrak{p}_m^1}$. Therefore we can define the map

$$\begin{aligned} \delta : Q_{\mathfrak{p}_m^k} J_m^k M &\longrightarrow L_{\mathfrak{p}_m^1}^* \otimes Q_{\mathfrak{p}_m^{k-1}} J_m^{k-1} M \\ D &\longmapsto \delta(D) \end{aligned}$$

where, if $\overline{X} \in L_{\mathfrak{p}_m^1}$, $\delta(D)(\overline{X}) = \Omega_2(X, D)$, being $X \in \mathcal{C}_{\mathfrak{p}_m^k}$ any vector over $\overline{X}$.

For $r \geq 0$, δ can be extended to a mapping

$$\delta : \Lambda^r L_{\mathfrak{p}_m^1}^* \otimes Q_{\mathfrak{p}_m^k} J_m^k M \longrightarrow \Lambda^{r+1} L_{\mathfrak{p}_m^1}^* \otimes Q_{\mathfrak{p}_m^{k-1}} J_m^{k-1} M$$

in the standard way: if $\omega \in \Lambda^r L_{\mathfrak{p}_m^1}^* \otimes Q_{\mathfrak{p}_m^k} J_m^k M$ and $\overline{X}_1, \ldots, \overline{X}_{r+1} \in L_{\mathfrak{p}_m^1}$, then

$$\delta(\omega)(\overline{X}_1, \ldots, \overline{X}_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} \Omega_2(X_i, \omega(\overline{X}_1, \ldots, \overset{\vee}{\overline{X}_i}, \ldots, \overline{X}_{r+1})),$$

where for $i = 1, \ldots, r+1$, X_i is a vector belonging to $\mathcal{C}_{\mathfrak{p}_m^k}$ whose projection into $L_{\mathfrak{p}_m^1}$ is $\overline{X}_i$.

Remark 5. Given one of the mappings δ and Ω_2, the other one is determined.

Remark 6. If p is the source of $\mathfrak{p}_m^k$ and we denote $V_p M = T_p M / L_{\mathfrak{p}_m^1}$, then

$$Q_{\mathfrak{p}_m^k} J_m^k M \simeq S^k L_{\mathfrak{p}_m^1}^* \otimes V_p M;$$

thus, the elements of $Q_{\mathfrak{p}_m^k} J_m^k M$ are vectors with coefficients polynomials, and $\Omega_2(X, D)$ is obtained by derivation of the coefficients of D with respect to $\overline{X}$ (see Kuranishi's fundamental identification in [15, 20]).

Remark 7. When we consider jets of sections of a fibre bundle, $L_{\mathfrak{p}_m^1}$ is the tangent space to the base manifold and δ is the Spencer differential [7, 23].

Now we could define the Weyl tensor at a jet and characterize the formal integrability of a system of partial differential equations at a point following Kruglikov and Lychagin [14].

5 Differential Correspondences

Since each jet in M is an ideal of $C^\infty(M)$, the relation of inclusion between ideals gives canonical correspondences between different jet spaces. Next we define the correspondences for first order jets following [9, 10], though most of the constructions and results can be generalized for higher orders.

Definition 3. Given two integers r, m such that $0 \le r \le m \le n$, the *Lie correspondence* $\bigwedge_{m,r} = \bigwedge_{m,r}(M)$ is the subset of the fibred product $J^1_m M \times_M J^1_r M$ consisting of the pairs of jets $(\mathfrak{p}^1_m, \mathfrak{q}^1_r)$ such that $\mathfrak{p}^1_m \subseteq \mathfrak{q}^1_r$ (inclusion as ideals of $C^\infty(M)$).

A geometric interpretation of these correspondences results from thinking of each first order jet over $p \in M$ as a linear subspace of T_pM: the inclusion between the ideals $\mathfrak{p}^1_m \subseteq \mathfrak{q}^1_r$ becomes the inclusion $L_{\mathfrak{p}^1_m} \supseteq L_{\mathfrak{q}^1_r}$ between linear subspaces of T_pM.

The Lie correspondence $\bigwedge_{m,r}$ is endowed with two natural projections, π_1 and π_2, such that the diagram

$$\begin{array}{ccccc} & & \bigwedge_{m,r} & & \\ & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\ J^1_m M & & & & J^1_r M \\ & \searrow & & \swarrow & \\ & & M & & \end{array}$$

is commutative.

Remark 8. At this point it is essential to stop thinking of jets as 'jets of cross-sections of a fibred manifold', because when M is fibred over a manifold, all jets have the same dimension and the above correspondences cannot be established.

In [4] it is pointed out that the value of the contact system at a jet is essentially the jet itself; this idea was used in [9, 10] to characterize the Lie correspondence by means of inclusions between contact systems and to obtain its local equations.

From the definitions it follows that a couple $(\mathfrak{p}^1_m, \mathfrak{q}^1_r) \in \bigwedge_{m,r}$ if and only if the value at $\mathfrak{q}^1_r$ of the contact system on $J^1_r M$ is contained in the value at $\mathfrak{p}^1_m$ of the contact system on $J^1_m M$ (both lifted to the Lie correspondence). Then, if we take local coordinates

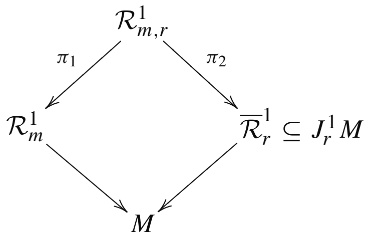
$$\begin{array}{ll} \overbrace{x_1, \dots, x_r}^{i}, \overbrace{x_{r+1}, \dots, x_m}^{h}, \overbrace{y_1, \dots, y_{n-m}}^{j} & \text{in } M, \\ x_i, x_h, y_j, y_{j,i}, y_{j,h} \text{ in } J^1_m M & \\ x_i, x_h, y_j, x_{h,i}, \overline{y}_{j,i} \text{ in } J^1_r M & \end{array}$$

(where the $\overline{y}_{j,i}$ have the same meaning than the $y_{j,i}$ above), then the equations of $\bigwedge_{m,r}$ are

$$\overline{y}_{j,i} - y_{j,i} - \sum_{h=r+1}^{m} y_{j,h} x_{h,i} = 0 \qquad (1 \leq j \leq n-m, 1 \leq i \leq r) \tag{3}$$

For each PDE system $\mathcal{R}^1_m \subseteq J^1_m M$, $\mathcal{R}^1_{m,r}$ will denote the intersection of $\bigwedge_{m,r}$ with $\mathcal{R}^1_m \times_M J^1_r(M)$. The local equations of the correspondence $\mathcal{R}^1_{m,r}$ as a submanifold of $J^k_m M \times_M J^1_r(M)$ are obtained by adding to Equations (3) the local equations of $\mathcal{R}^1_m$.

We have the commutative diagram



where $\overline{\mathcal{R}}^1_r = \pi_2(\mathcal{R}^1_{m,r})$; $\overline{\mathcal{R}}^1_r$ is a SPDE with r independent variables and $n-r$ unknown functions.

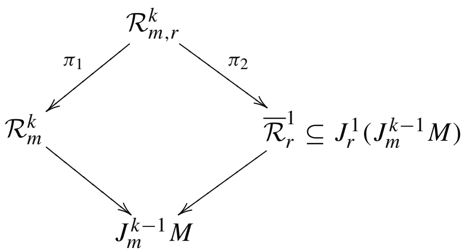
If $\mathcal{R}^k_m$ is a SPDE of order k, it can be considered as a SPDE of first order via the natural immersion

$$J^k_m M \hookrightarrow J^1_m(J^{k-1}_m M)$$

Hence the theory of correspondences can be applied to those systems (here the base-manifold is $J^{k-1}_m M$ instead of M). If we write

$$\mathcal{R}^k_{m,r} = \bigwedge\nolimits_{m,r}(J^{k-1}_m M) \bigcap \left[\mathcal{R}^k_m \times_{J^{k-1}_m M} J^1_r(J^{k-1}_m M)\right],$$

we have the following commutative diagram:



Remark 9. In [10] this kind of correspondences have been applied to show how the integration of a class of involutive PDE systems can be reduced to the integration of first order systems with a single unknown function.

Finally we describe the tangent space to $\bigwedge_{m,r}$ at a couple $(\mathfrak{p}_m^1, \mathfrak{q}_r^1)$; to this end we use the isomorphism between the tangent space $T_{\mathfrak{p}_m^1} J_m^1 M$ and the vector space $\mathcal{D}_{\mathfrak{p}_m^k} / \mathcal{D}'_{\mathfrak{p}_m^k}$ given by Theorem 1.

Let $D \in \mathrm{Der}_{\mathbb{R}}\left(C^\infty(M), C^\infty(M) \big/ \mathfrak{p}_m^1\right)$ and let us denote by $\widetilde{D}$ the derivation from $C^\infty(M)$ into $C^\infty(M) \big/ \mathfrak{q}_r^1$ obtained as the composition of D with the natural projection $\pi : C^\infty(M) \big/ \mathfrak{p}_m^1 \longrightarrow C^\infty(M) \big/ \mathfrak{q}_r^1$; we have the commutative diagram

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{D} & C^\infty(M) \big/ \mathfrak{p}_m^1 \\ & \underset{\widetilde{D}}{\searrow} & \downarrow \pi \\ & & C^\infty(M) \big/ \mathfrak{q}_r^1 \end{array}$$

(Note that the class of $\widetilde{D}$ in $T_{\mathfrak{q}_r^1} J_r^1 M$ depends on the derivation D). In [3] we proved the following results:

Theorem 3. *The tangent space $T_{(\mathfrak{p}_m^1, \mathfrak{q}_r^1)} \bigwedge_{m,r}$ is canonically isomorphic of the vector space of all (fibred) sums $[D] + [\widetilde{D}]$, where $[D]$ is the class of a derivation D from $C^\infty(M)$ into $C^\infty(M) \big/ \mathfrak{p}_m^1$ and $\widetilde{D} = \pi \circ D$.*

Corollary 3. *The subspace of $T_{(\mathfrak{p}_m^1, \mathfrak{q}_r^1)} \bigwedge_{m,r}$ vertical for the projection on the second factor, $\pi_2 \colon \bigwedge_{m,r} \longrightarrow J_m^1 M$, is the set of classes of derivations from $C^\infty(M)$ into $C^\infty(M) \big/ \mathfrak{p}_m^1$ which value in $\mathfrak{q}_r^1 \big/ \mathfrak{p}_m^1$.*

6 Characteristics of Systems of PDE

We finish this paper by showing how this approach can be used to compute the real characteristics of some systems of partial differential equations.

A first approach to characteristics can be achieved through the characteristic 1-forms, which in [23] are defined for systems of partial differential equations which are defined as subbundles of a bundle of jets of sections of a fibre bundle. Furthermore, a relationship between the existence of non-characteristic 1-forms and the possibility of writing locally an undetermined system of PDE in the Cauchy-Kowalewski normal form was established. Since the manifold $J_m^k M$ is locally the set of jets of sections of a fibre bundle and the result mentioned above is local, it remains being valid in general.

Let $\mathcal{R}_m^k$ be a PDE system. Its symbol at $\mathfrak{p}_m^k \in \mathcal{R}_m^k$, denoted by $G_{\mathfrak{p}_m^k}$, is a subspace of $Q_{\mathfrak{p}_m^k} J_m^k M$. Let us write $V_p M = T_p M \Big/ L_{\mathfrak{p}_m^1}$. If $\omega \in L^*_{\mathfrak{p}_m^1}$, then $\omega^k \cdot V_p M$ is identified with a subspace of $Q_{\mathfrak{p}_m^k} J_m^k M$.

Definition 4. ω is non-characteristic for $\mathcal{R}^k_m$ at $\mathfrak{p}^k_m$ if $\omega^k \cdot V_pM$ and $G_{\mathfrak{p}^k_m}$ intersect transversally in $Q_{\mathfrak{p}^k_m} J^k_m M$, and characteristic otherwise.

Remark 10. The condition of transversality in the former definition means that $\omega^k \cdot V_pM + G_{\mathfrak{p}^k_m} = Q_{\mathfrak{p}^k_m} J^k_m M$, so our definition is slightly different from the one given in [12]. This assumption supposes a restriction in the number of equations which define the system of PDE considered; it is adequate for underdetermined systems. Characteristic 1-forms give an unifying approach for the classical definitions of characteristics of underdetermined systems of PDE's, such as Cauchy, Monge or Friedrichs characteristics, and it is motivated by its relationship with the Cauchy-Kowalewski normal form, as we will explain now.

In the notations above, let us suppose that the ideal of $\mathcal{R}^k_m$ is generated by r functions $F_1, \ldots, F_r$ functionally independent in a neighbourhood of $\mathfrak{p}^k_m$ coordinated by functions $x_i, y_j, y_{j\alpha}$. If the canonical projection $\mathcal{R}^k_m \longrightarrow J^{k-1}_m M$ is a submersion at $\mathfrak{p}^k_m$ and $d_p x_m$ is non-characteristic, then the rank of the matrix

$$\left(\frac{\partial(F_1, \ldots, F_r)}{\partial(y_{1,k1_m}, \ldots, y_{n-m,k1_m})} \right)_{\mathfrak{p}^k_m}$$

equals r (where $k1_i$ is the m-index with k in the ith component and 0 in the remaining ones); this fact implies that $\mathcal{R}^k_m$ is underdetermined ($r \leq n-m$). By applying the implicit function theorem it turns out that the system $\mathcal{R}^k_m$ is defined, in a neighbourhood of $\mathfrak{p}^k_m$, by equations

$$y_{j_s,k1_m} = \varphi_{j_s}(x_i, y_{j,\alpha}) \qquad (1 \leq s \leq r) \tag{4}$$

If $\mathcal{R}^k_m$ is determined ($r = n-m$), then the functions φ_{j_s} do not depend on any variable $y_{j,k1_m}$, ($1 \leq j \leq n-m$), hence (4) is the Cauchy-Kowalewski normal form for $\mathcal{R}^k_m$ at $\mathfrak{p}^k_m$. If it is underdetermined ($r \leq n-m$), then Equations (4) can be written in the form

$$y_{s,k1_m} = \varphi_s(x_i, y_{j,\alpha}) \qquad (1 \leq s \leq r) \tag{5}$$

and the functions φ_s may depend on the variables $y_{j,k1_m}$ ($r+1 \leq j \leq n-m$), so that Equations (5) are the Cauchy-Kowalewski normal form for $\mathcal{R}^k_m$, considered as a system of partial differential equations with $y_1, \ldots, y_r$ as unknown functions and $y_{r+1}, \ldots, y_{n-m}$ as parameters. Its general solution depends on arbitrary functions.

Conversely, if the system of equations defining $\mathcal{R}^k_m$ in a neighbourhood of $\mathfrak{p}^k_m$ can be written in the Cauchy-Kowalewski normal form with respect to the variable x_m, then $\omega = d_p x_m$ is non-characteristic for $\mathcal{R}^k_m$ at $\mathfrak{p}^k_m$.

These considerations prove the following theorem (see [23] for details):

Theorem 4. *Let $\mathcal{R}^k_m$ be a system of PDE's such that the canonical projection $\mathcal{R}^k_m \longrightarrow J^{k-1}_m M$ is a submersion at $\mathfrak{p}^k_m$. Then there is a non-characteristic 1-form* *for* $\mathcal{R}^k_m$ *at* $\mathfrak{p}^k_m$ *if and only if* $\mathcal{R}^k_m$ *is underdetermined and it can be written in the Cauchy-Kowalewski normal form in a neighbourhood of* $\mathfrak{p}^k_m$.

Another approach to characteristics can be given in the framework of differential correspondences (see [3]). According to Lie [17], a submanifold of $\mathcal{R}^k_m$ is characteristic if it is contained in several k-jet prolongations of classical solutions of $\mathcal{R}^k_m$, which are tangent along the characteristic. This condition can be translated into our language as follows: The first prolongation $\mathcal{R}^{k+1}_m$ of $\mathcal{R}^k_m$ is a system of PDE whose points can be thought of as the tangent spaces to the k-jet prolongations of the classical solutions of $\mathcal{R}^k_m$; for $r \le m$, an r-dimensional subspace of $T_{\mathfrak{p}^k_m}\mathcal{R}^k_m$ is a jet belonging to $J^1_r(\mathcal{R}^k_m)$, and it is tangent to several solutions of $\mathcal{R}^k_m$ when it contains several jets in $\mathcal{R}^{k+1}_m$ in the fibre of $\mathfrak{p}^k_m$.

Let us consider the intersection $\widehat{\mathcal{R}}_{m,r}$ of the differential correspondence $\bigwedge_{m,r}(J^{k+1}_m M)$ with $\mathcal{R}^{k+1}_m \times_{\mathcal{R}^k_m} J^1_r(\mathcal{R}^k_m)$; we have the commutative diagram

$$\begin{array}{ccccc} & & \widehat{\mathcal{R}}_{m,r} & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ J^1_m(J^k_m M) \supseteq \mathcal{R}^{k+1}_m & & & & J^1_r(\mathcal{R}^k_m) \subseteq J^1_r(J^k_m M) \\ & \searrow & & \swarrow & \\ & & \mathcal{R}^k_m & & \end{array}$$

Definition 5. Let $(\mathfrak{p}^{k+1}_m, \mathfrak{q}^1_r) \in \widehat{\mathcal{R}}_{m,r}$; we say that $\mathfrak{q}^1_r$ is characteristic for $\mathcal{R}^k_m$ at $\mathfrak{p}^k_m$ if the tangent space to $\widehat{\mathcal{R}}_{m,r}$ at $(\mathfrak{p}^{k+1}_m, \mathfrak{q}^1_r) \in \widehat{\mathcal{R}}_{m,r}$ vertical for π_2 is different from 0.

Remark 11. If $\mathfrak{p}^{k+1}_m \in \mathcal{R}^{k+1}_m$, each characteristic jet $\mathfrak{q}^1_{m-1} \in J^1_{m-1}(\mathcal{R}^k_m)$ at $\mathfrak{p}^{k+1}_m$ can be thought of as an $(m-1)$-dimensional subspace $L_{\mathfrak{q}^1_{m-1}}$ of $T_{\mathfrak{p}^k_m}\mathcal{R}^k_m$ contained in a family of m-dimensional subspaces defined by points of $\mathcal{R}^{k+1}_m$. If $\mathfrak{p}^1_m$ is the projection of $\mathfrak{p}^k_m$ into $J^1_m M$ and $\omega \in L^*_{\mathfrak{p}^1_m}$ generates the subspace incident with the projection of $L_{\mathfrak{q}^1_{m-1}}$ into $L_{\mathfrak{p}^1_m}$, then ω is a characteristic 1-form and the tangent space to $\widehat{\mathcal{R}}_{m,r}$ at $(\mathfrak{p}^{k+1}_m, \mathfrak{q}^1_r) \in \widehat{\mathcal{R}}_{m,r}$ vertical for π_2 is isomorphic to $\omega^{k+1} \cdot V_p M$.

When $m = 2$, the characteristic jets $\mathfrak{q}^1_1$ give the characteristic directions (see [3] for examples).

Finally we will establish a relationship between the singular vectors (see [26]) and the canonical bracket defined in Sect. 3; when $m = 2$, singular vectors are also characteristic. Another approach to singular vectors, based in the investigation of singular integral grassmannians, can be found in [18].

Let us assume that the projection $\mathcal{R}^{k+1}_m \longrightarrow \mathcal{R}^k_m$ is onto, let $\mathfrak{p}^k_m \in \mathcal{R}^k_m$ and $\mathfrak{p}^{k+1}_m \in \mathcal{R}^{k+1}_m$ whose projection into $\mathcal{R}^k_m$ is $\mathfrak{p}^k_m$. Since the spaces $L_{\mathfrak{p}^{k+1}_m}$ are isotropic for the canonical bracket, the characteristic vectors belong to more than one m-dimensional horizontal isotropic subspace of $\mathcal{C}_{\mathfrak{p}^k_m}$; this means that it must commute with some element of $G_{\mathfrak{p}^k_m}$.

Each horizontal vector $\overline{X} \in L_{\mathfrak{p}^1_m}$ defines a linear mapping

$$\begin{aligned} i_X\Omega_2 : Q_{\mathfrak{p}^k_m} J^k_m M &\longrightarrow Q_{\mathfrak{p}^{k-1}_m} J^{k-1}_m M \\ D &\longmapsto \Omega_2(X, D) \end{aligned}$$

where X is any vector in $\mathcal{C}_{\mathfrak{p}^k_m}$ over $\overline{X}$; we denote $\varphi_{\overline{X}} : G_{\mathfrak{p}^k_m} \longrightarrow Q_{\mathfrak{p}^{k-1}_m} J^{k-1}_m M$ its restriction to the symbol $G_{\mathfrak{p}^k_m}$ of $\mathcal{R}^k_m$ at $\mathfrak{p}^k_m$.

Definition 6. X is singular if the rank of the mapping above is not maximal, that is to say, if there exists $\overline{Y} \in L_{\mathfrak{p}^1_m}$ such that $\mathrm{rank}(\varphi_{\overline{X}}) < \mathrm{rank}(\varphi_{\overline{Y}})$

Remark 12. The singular vectors are the values of the singular vector fields defined in [26]. Characteristic vectors are singular, and if $m = 2$ all singular vectors are characteristic also. If $\varphi_{\overline{X}} \equiv 0$, then X belongs to each involutive subspace of $\mathcal{C}_{\mathfrak{p}^k_m} \cap T_{\mathfrak{p}^k_m}\mathcal{R}^k_m$, and it is Cartan characteristic.

Let us we finish with some examples; for simplicity we omit the subscripts representing the points where the vectors are valued.

Example 1. Let us define in this language the Monge characteristics of a second order equation $\mathcal{R}^2_2$ given by $F(x, y, z, p, q, r, s, t) = 0$. If $\mathfrak{p}^2_2 \in \mathcal{R}^2_2$, then $\mathcal{C}_{\mathfrak{p}^2_2}$ is generated by the vectors (valued at $\mathfrak{p}^2_2$):

$$\begin{aligned} \partial^{(2)}_x &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + r\frac{\partial}{\partial q} + s\frac{\partial}{\partial q} \\ \partial^{(2)}_y &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + s\frac{\partial}{\partial p} + t\frac{\partial}{\partial q} \\ Q_{\mathfrak{p}^2_2} J^2_2(\mathbb{R}^3) &= \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right\rangle \end{aligned}$$

The vector $\lambda\frac{\partial}{\partial x} + \mu\frac{\partial}{\partial y}$ (valued at p) gives the linear mapping from $Q_{\mathfrak{p}^2_2} J^2_2(\mathbb{R}^3)$ into $Q_{\mathfrak{p}^1_2} J^1_2(\mathbb{R}^3) = \left\langle \frac{\partial}{\partial p}, \frac{\partial}{\partial q} \right\rangle$ whose associated matrix is

$$\begin{pmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \end{pmatrix}$$

If a vector belonging to $G_{\mathfrak{p}^2_2}$ is in the kernel of this mapping, it must verify a homogeneous system of linear equations whose associated matrix is

$$\begin{pmatrix} \lambda & \mu & 0 \\ 0 & \lambda & \mu \\ F_r & F_s & F_t \end{pmatrix}$$

This system has nontrivial solutions if and only if the determinant of the matrix above vanish, and we have the equation

$$\lambda^2 F_t - \lambda\mu F_s + \mu^2 F_t = 0,$$

which is the known equation of the characteristics for a second order equation.

Example 2. Let us consider now an equation $\mathcal{R}^2_m$ of second order with one unknown function z and m independent variables $x_1, \dots, x_m$,

$$F(x_i, z, z_\alpha) = 0,$$

where α runs through the m-indexes such that $|\alpha| \le 2$.

If $\mathfrak{p}^2_m \in \mathcal{R}^2_m$, then $\mathcal{C}_{\mathfrak{p}^2_m}$ is generated by the vectors (valued at $\mathfrak{p}^2_m$):

$$\partial_i^{(2)} = \frac{\partial}{\partial x_i} + \sum_{|\beta|=1} z_{\beta+1_i} \frac{\partial}{\partial z_\beta} \qquad (1 \le i \le m)$$

$$Q_{\mathfrak{p}^2_m} J^2_m(\mathbb{R}^{m+1}) = \left\langle \frac{\partial}{\partial z_\alpha} \right\rangle_{|\alpha|=2}$$

If $|\alpha| = 2$, then

$$\Omega_2\left(\partial_i^{(2)}, \frac{\partial}{\partial z_\alpha}\right) = \frac{\partial}{\partial z_{\alpha-1_i}};$$

therefore, the linear mapping from $Q_{\mathfrak{p}^2_m} J^2_m(\mathbb{R}^{m+1})$ into $Q_{\mathfrak{p}^1_m} J^1_m(\mathbb{R}^{m+1})$ induced by the vector $\overline{X} = \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_i}$ is

$$\sum_{\substack{|\alpha|=2\\ 1\le i\le m}} \lambda_i dz_\alpha \otimes \frac{\partial}{\partial z_{\alpha-1_i}}.$$

If $D = \sum_{|\alpha|=2} A_\alpha \frac{\partial}{\partial z_\alpha} \in G_{\mathfrak{p}^2_m}$ belongs to the kernel of $\varphi_{\overline{X}}$, the coefficients A_α must verify a homogeneous system of linear equations

$$\sum_{i=1}^{m} \lambda_i A_{1_i+1_j} = 0 \qquad (1 \le j \le m)$$

$$\sum_{|\alpha|=2} F_{z_\alpha} A_\alpha = 0$$

and $\overline{X}$ is singular when these equations are linearly dependent. If we compute this condition we obtain that $\overline{X}$ is singular if and only if there exist constants $\mu_1, \dots, \mu_m$ such that

$$F_{z_\alpha} = \sum_{1_i+1_j=\alpha} \lambda_i \mu_j \qquad |\alpha| = 2$$

Such a condition can be written in this form: $\mathcal{R}^2_m$ has singular vectors at $\mathfrak{p}^2_m$ if and only if the quadratic form $\sum_{|\alpha|=2} F_{z_\alpha} t^\alpha$ can be decomposed as a product of two linear factors

$$\sum_{|\alpha|=2} F_{z_\alpha} t^\alpha = \left(\sum_i \lambda_i t_i\right)\left(\sum_j \mu_j t_j\right),$$

which rarely happens if $m > 2$. This example was taken from [26], where the computations are based in the Lie brackets of the vectors belonging to the Cartan distribution.

Example 3. Finally we compute Friedrichs characteristics for a system $\mathcal{R}^1_2$ of two equations

$$F(x_1, x_2, u, v, u_1, u_2, v_1, v_2) = 0$$
$$G(x_1, x_2, u, v, u_1, u_2, v_1, v_2) = 0$$

where x_1, x_2 are the independent variables, u, v the unknown functions and u_1, u_2, v_1, v_2 the first derivatives of u and v with respect to x_1, x_2.

Given $\mathfrak{p}^1_2 \in \mathcal{R}^1_2$, $\mathcal{C}_{\mathfrak{p}^1_2}$ is generated by the vectors (valued at $\mathfrak{p}^1_2$)

$$\partial^{(1)}_{x_1} = \frac{\partial}{\partial x_1} + u_1 \frac{\partial}{\partial u} + v_1 \frac{\partial}{\partial v}$$
$$\partial^{(1)}_{x_2} = \frac{\partial}{\partial x_2} + u_2 \frac{\partial}{\partial u} + v_2 \frac{\partial}{\partial v}$$
$$Q_{\mathfrak{p}^1_2} J^1_2(\mathbb{R}^4) = \left\langle \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2} \right\rangle$$

The vector $\lambda \frac{\partial}{\partial x_1} + \mu \frac{\partial}{\partial x_2}$ (valued at p) gives the linear mapping from $Q_{\mathfrak{p}^1_2} J^1_2(\mathbb{R}^4)$ into $T_p\mathbb{R}^4/L_{\mathfrak{p}^1_2} = \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle$ whose associated matrix is

$$\begin{pmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \lambda & \mu \end{pmatrix}$$

A vector $D_{\mathfrak{p}^2_2} \in G_{\mathfrak{p}^1_2}$ is in the kernel of this linear mapping $\iff$ its coefficients fulfil a homogeneous system of linear equations whose associated matrix is

$$\begin{pmatrix} \lambda & \mu & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ F_{u_1} & F_{u_2} & F_{v_1} & F_{v_2} \\ G_{u_1} & G_{u_2} & G_{v_1} & G_{v_2} \end{pmatrix}$$

This system has nontrivial solutions $\iff$ the determinant of the former matrix vanish, which gives Friedrichs's condition:

$$\lambda^2 \begin{vmatrix} F_{u_2} & F_{v_2} \\ G_{u_2} & G_{v_2} \end{vmatrix} - \lambda\mu \left(\begin{vmatrix} F_{u_2} & F_{v_1} \\ G_{u_2} & G_{v_1} \end{vmatrix} + \begin{vmatrix} F_{u_1} & F_{v_2} \\ G_{u_1} & G_{v_2} \end{vmatrix} \right) + \mu^2 \begin{vmatrix} F_{u_1} & F_{v_1} \\ G_{u_1} & G_{v_1} \end{vmatrix} = 0$$

This example was taken from [6], where it is studied as an application of focal subsystems of a Pfaff system.

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Transformations of Darboux Integrable Systems

Ian M. Anderson and Mark E. Fels

Abstract This article reviews some recent theoretical results about the structure of Darboux integrable differential systems and their relationship with symmetry reduction of exterior differential systems. The symmetry reduction representation of Darboux integrable equations is then used to derive some new and unusual transformations.

1 Introduction

Broadly speaking, a system of partial differential equations $\Delta_1 = 0$ is said to be ***Darboux integrable*** if there exists an auxiliary system of compatible PDE $\Delta_2 = 0$ [20] such that

1. The combined system $\{\, \Delta_1 = 0,\ \Delta_2 = 0\,\}$ is a system of total partial differential equations, that is, one which can be integrated by ODE methods, and
2. The auxiliary system $\Delta_2 = 0$ is parameterized by a number of arbitrary functions, sufficient in number so as to insure that every (local) solution to $\Delta_1 = 0$ can be realized as a solution to $\{\, \Delta_1 = 0,\ \Delta_2 = 0\,\}$.

Partial differential equations which admit closed-form general solutions can be shown to be Darboux integrable ([16], p. 225) but the general definition of Darboux integrability extends well beyond this special case.

The auxiliary equations $\Delta_2 = 0$ are classically referred to as ***intermediate integrals*** for the given system $\Delta_1 = 0$ and, apart from E. Vessiot's remarkable papers [26], [27], the analysis of the method of Darboux has focused exclusively on the existence of these integrals. Vessiot observed that inherent in the integration of Darboux integrable equations are certain ODE systems known as equations of Lie type and this led, for the first time, to a group theoretical formulation of the method

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of Darboux. In a recent article [2], the authors (with P. J. Vassiliou) re-interpreted Vessiot's approach within the more general setting of symmetry reduction of differential systems and used this general setting to develop a new, algorithmic approach for the explicit integration of Darboux integrable systems. We also introduced the concept of a ***non-linear superposition formula*** for differential systems and showed how Darboux integrable systems always admit such a formula. This concept of a non-linear superposition formula provides a new way of looking at Darboux integrable systems which has proven to be quiet useful.

From the outset, it should be emphasized that the group-theoretic tools which arise in this new approach to Darboux integrability are rather different from the more familiar methods due to Sophus Lie – the relevant Lie groups arise, not as symmetry groups for the given equations, but rather from certain canonical normalizations of the structure equations for the exterior differential systems associated to $\Delta_1 = 0$. We cite the theory of equations of Lie type [21](Chap. 3), [24] (Chap. 10) and papers by Cartan [8] and Vessiot [25] as other instances where Lie groups, which do not arise as symmetry groups, have lead to new integration methods for various classes of differential equations.

The best-known example of a Darboux integrable equation is the Liouville equation

$$u_{xy} = e^{u}. \tag{1}$$

The intermediate integrals for this equation are

$$u_{xx} = \frac{1}{2}u_x^2 + f(x) \quad \text{and} \quad u_{yy} = \frac{1}{2}u_y^2 + g(y). \tag{2}$$

It is a simple but nevertheless instructive exercise, to check that the compatibility conditions for (1) and (2) are satisfied. The method by which one derives (2) from (1) is well-established (see, for example [7], [3], [4], [23]). Vessiot's critical observation is that these intermediate integrals may be viewed as a Ricatti equation and these are equations of Lie type for the standard fractional linear action of the Lie group $SL(2)$ on the line.

The method of Darboux is one of the cornerstones of the classical geometric theory of differential equations developed in the nineteenth century by Monge, Ampere, Laplace, Goursat and Darboux. For the following reasons, we believe it remains an important topic:

1. Darboux integrable systems always have infinitely many generalized symmetries and conservation laws ([4], [23]) and consequently are always present in any classification of PDE with these properties.
2. One of the principle goals of the geometric theory of PDE is to relate properties of geometric invariants of PDE to solution techniques and properties of the solutions. Surely Darboux integrable equations afford the simplest situation where such relationships can be developed.
3. The mathematical physics literature contains many ad hoc methods for finding closed-form general solutions to reductions of various fundamental field theories. The method of Darboux, as generalized in [2], provides a completely systematic and algorithmic approach to the derivation of these solutions.

4. The integration of PDE by computer algebra systems is a very active area of current research and classical geometric methods, such as the method of Darboux, provide a powerful complement to differential elimination methods.

The goal of the present article is to summarize the key results of [2] and, as an application, establish some new and rather surprising transformations between various Darboux integrable differential systems.

2 Symmetry Reduction of Exterior Differential Systems and the Method of Darboux

We begin this section with the general definition of symmetric reduction of an EDS ([1]). We illustrate this definition within the familiar context of ODE reduction and make a few comments regarding the general theory. A (non-technical) summary of the application of symmetry reduction to the study of Darboux integrablity, as developed in [2], is presented.

Definition 2.1 Let G be a Lie group acting regularly on a manifold M and let $\mathbf{q}\colon M \to M/G$ be the projection map to the quotient space M/G of M by the orbits of G. Let $\mathcal{I}$ be an EDS on M and suppose that G is a symmetry group of $\mathcal{I}$. Then the G-reduction of $\mathcal{I}$ is the EDS $\mathcal{I}/G$ on M/G defined by

$$\mathcal{I}/G = \{\, \omega \in \Omega^*(M/G) \mid \mathbf{q}^*(\omega) \in \mathcal{I}\,\}. \tag{3}$$

To calculate the reduced differential system $\mathcal{I}/G$, one first calculates the forms $\mathcal{I}_{\mathbf{sb}} \subset \mathcal{I}$ which are semi-basic for the action of G on M. Specifically, if Γ denotes the Lie algebra of infinitesimal generators for the action of G on M, then

$$\mathcal{I}_{\mathbf{sb}} = \{\, \omega \in \mathcal{I} \mid X \lrcorner\, \omega = 0 \text{ for all } X \in \Gamma\,\}. \tag{4}$$

One can interpret $\mathcal{I}_{\mathbf{sb}}$ as the largest differential sub-system of $\mathcal{I}$ for which each $X \in \Gamma$ is a Cauchy characteristic. At this point, a standard result on reduction by Cauchy characteristics (see, for example [5], p. 31) asserts that one can construct a basis for $\mathcal{I}_{\mathbf{sb}}$ in terms of the G-invariant functions on M and their differentials. This basis naturally projects under $\mathbf{q}$ to give a basis for $\mathcal{I}/G$.

As a simple example, let us consider the problem of integrating the 4th order ODE ([19],(7.16))

$$3y''y^{(iv)} - 5(y''')^2 = 0. \tag{5}$$

In terms of standard jet coordinates $\{\, x,\ y,\ y_1,\ y_2,\ y_3\,\}$, the differential system for this ODE is the rank 4 Pfaffian system

$$\mathcal{I} = \{\, \theta^1 = dy - y_1 dx,\ \theta^2 = dy_1 - y_2 dx,\ \theta^3 = dy_2 - y_3 dx,\ \theta^4 = dy_3 - \frac{5y_3^2}{3y_2} dx\} \tag{6}$$

and we take as our symmetry group (with group parameters $(a, b, c) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^*$) the transformation group

$$x = a + cx,\ y = b + cy,\ y_1 = y_1,\ y_2 = \frac{y_2}{c}, y_3 = \frac{y_3}{c^2}. \tag{7}$$

The infinitesimal generators for this action are

$$\Gamma = \{\ \partial_x,\ \partial_y,\ x\partial_x + y\partial_y - y_2\partial_{y_2} - 2y_3\partial_{y_3}\ \}. \tag{8}$$

The orbits of G are two dimensional so that the quotient space M/G has dimension two with coordinates (say) $\{u,\ v\}$. The invariants for action of G on M are y_1 and y_3/y_2^2 so that we may write the projection $\mathbf{q}$ as

$$u = y_1, \quad v = \frac{y_3}{y_2^2}. \tag{9}$$

By solving the linear system

$$X \lrcorner\ (a_1\theta^1 + a_2\theta^2 + a_3\theta^3 + a_4\theta^4) = 0 \quad \text{for all } X \in \Gamma \tag{10}$$

we determine that

$$\mathcal{I}_{\mathbf{sb}} = \{\ \theta^4 - \frac{2y_3}{y_2}\theta^3 + \frac{y_3^2}{3y_2^2}\theta^2\ \} = \{\ dy_3 - \frac{2y_3}{y_2}dy_2 + \frac{y_3^2}{3y_2^2}dy_1\ \}. \tag{11}$$

We substitute $y_3 = vy_2^2$ and $y_1 = u$ into this result to find that the reduced EDS is

$$\mathcal{I}/G = \{\ v^2du + 3dv\ \} \tag{12}$$

and the reduced ODE is

$$\frac{dv}{du} = -\frac{1}{3}v^2. \tag{13}$$

This is precisely the result one would obtain using a step-by-step reduction of (5) following the well-known algorithm as presented in [22], pp. 130–161. In [1] we show how the general solution to (5) can be obtained from the solution to (13) and the group action (7).

Thus, for ODE, the reduction procedure given by Definition 2.1 coincides, more or less, with the usual reduction by differential invariants although it does lead to a new approach for lifting solutions of the reduced system to the original system and for dealing with non-regular group actions and singular orbits [13]. What is of real importance for us here is that Definition 2.1 provides us with a simple, unambiguous approach to the symmetry reduction of partial differential equations within an EDS setting. We should emphasize, however, that if $\mathcal{I}$ is the canonical Pfaffian system for some system of PDE (obtained by the restriction of the contact ideal on the appropriate jet space), then the reduction $\mathcal{I}/G$ may not be the canonical Pfaffian system for a PDE with the same order and same number of dependent and independent variables – even when the Cartan character and Cartan integer remain the same. In this sense, the ODE reduction presented in the foregoing example, where we effortlessly passed from the reduced EDS (12) to the ODE (13) is not reflective of the general situation.

This last remark naturally leads to the ***PDE recognition problem*** for differential systems of the kind discussed by Vessiot (see also Stomark [24], page 274) and more recently by Yamaguchi [28]. The construction (3) also raises a wide range of interesting (and often very challenging) problems regarding the behavior of the various geometric properties for differential systems under reduction. As a simple illustration, we cite Theorem 5.1 in [1], where conditions under which the reduction of a Pfaffian system is Pfaffian are established.

We now frame the theory of Darboux integrablity within the context of symmetry reduction of differential systems. Roughly speaking, a differential system $\mathcal{I}$ is Darboux integrable if (1) it is algebraically generated by 1-forms and 2-forms; (2) if the structure equations for the 1-forms decompose, at the symbol level, into a certain block diagonal form; and (3) if the singular Pfaffian systems determined by this decomposition admit a sufficient number of intermediate integrals. See [2] for the precise definition of Darboux integrablity. This definition generalizes the classical definition of Darboux integrablity for scalar PDE in the plane. We remark that, as with the classical definition, it frequently happens that a differential system is not Darboux integrable but that its prolongation to some order is.

The main results of [2] can be summarized as follows.

Result 1. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be Pfaffian systems on manifolds M_1 and M_2, respectively. Then the differential system $\mathcal{W}_1 + \mathcal{W}_2$ on $M_1 \times M_2$ is Darboux integrable (Theorem 3.1).

Result 2. Let G be a Lie group acting freely on M_1 and M_2 and as symmetries of $\mathcal{W}_1$ and $\mathcal{W}_2$ Asssume that G is transverse to $\mathcal{W}_1$ and $\mathcal{W}_2$. Then the quotient differential system $(\mathcal{W}_1 + \mathcal{W}_2)/G$ is Darboux integrable (Corollary 3.4).

Result 3. Let $\mathcal{I}$ on M be a Darboux integrable differential system with associated singular Pfaffian systems $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$. Then there is a (local) Lie group G and free right and left actions $\hat{\mu}\colon G \times M \to M$ and $\check{\mu}\colon G \times M \to M$ such that $\hat{\mu}$ preserves $\hat{\mathcal{V}}$, $\check{\mu}$ preserves $\check{\mathcal{V}}$, and $\hat{\mu}$ commutes with $\check{\mu}$. (For the complete list of properties which characterize these actions, see Sect. 5.3 of [2].) The Lie group G is called the ***Vessiot group*** for the Darboux integrable differential system $\mathcal{I}$.

Result 4. Let $\mathcal{I}$ on M be a Darboux integrable differential system with associated singular Pfaffian systems $\hat{\mathcal{V}}$ and $\check{\mathcal{V}}$ and Vessiot group G. Let $\mathcal{W}_1$ be the restriction of $\hat{\mathcal{V}}$ to a fixed, maximal integral manifold of $\hat{\mathcal{V}}^\infty$ and let $\mathcal{W}_2$ be the restriction of $\check{\mathcal{V}}$ to a fixed, maximal integral manifold of $\check{\mathcal{V}}^\infty$. Then

$$\mathcal{I} \cong (\mathcal{W}_1 + \mathcal{W}_2)/G. \tag{14}$$

We call the quotient differential systems $(\mathcal{W}_1 + \mathcal{W}_2)/G$ the ***quotient representation*** of the Darboux integrable differential system $\mathcal{I}$. This is Theorem 5.1 in [2].

Result 5. Every Darboux integrable differential system is uniquely characterized by its restricted singular Pfaffian systems $\mathcal{W}_1$ and $\mathcal{W}_2$, the Lie group G, and the actions $\hat{\mu}$ and $\check{\mu}$.

Result 6. The integral manifolds of a Darboux integrable differential system $\mathcal{I}$ can be constructed from the integral manifolds of its restricted singular Pfaffian systems $\mathcal{W}_1$ and $\mathcal{W}_2$ (Corollary 5.12).

Result 2 is extremely important to the entire subject of Darboux integrablity. While, in the past, it has been quite difficult to construct new examples of Darboux integrable systems it is now possible, using Result 2, to create entire new classes of Darboux integrable EDS.

Result 4 shows that every Darboux integrable system can be realized as a reduction of the trivial type of Darboux integrable system consider in Result 1. Regrettably, the present proof of this result is rather difficult. This is primarily because there are many groups actions which satisfy the conclusions of Result 3 but only a very careful and lengthy analysis of the structure equations for the singular Pfaffian systems leads to the correct choice of group actions required for the validity of (14).

Result 5 emphasizes the fact that Darboux integrablity, instead of being studied from the viewpoint of compatibility theory, can now be studied entirely within the setting of group actions and symmetry groups of differential systems. For example, (differential) invariants for the action of the Lie group G on the manifold M_1 and M_2 project under $\mathbf{q}$ to give the intermediate integrals for the EDS $\mathcal{I}$.

Result 6 shows explicitly that the explicit integration of a Darboux integrable $\mathcal{I}$ depends upon the explicit integration of the its restricted singular Pfaffian systems $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$. In particular, one is assured of algebraic, closed-form general solutions for $\mathcal{I}$ whenever $\hat{\mathcal{W}}$ and $\check{\mathcal{W}}$ can be identified with the canonical contact structures on jet spaces. Result 6 is also the key to the symbolic implementation of the method of Darboux.

3 Transformations of Darboux Integrable Systems

The symmetry reduction approach to the method of Darboux described in the previous section allows us to develop a new, coherent transformation theory for Darboux integrable differential systems. The following three Principles summarize the key results obtained thus far. Module various technical transversality conditions these principles are indeed theorems. Precise statements and proofs of these theorems will appear elsewhere.

Principle A (Prolongation). **[i]** *Let $\mathcal{I}$ be a differential system with independence condition $\mathcal{J}$. Let G be a symmetry group of $(\mathcal{I}, \mathcal{J})$ and suppose that $\mathcal{I}/G$ is a (regular) differential system. Then the symmetry group G lifts to a symmetry group of any prolongation (or partial prolongation [7]) $\mathcal{I}^{[\mathfrak{p}]}$ of $\mathcal{I}$ and*

$$(\mathcal{I}/G)^{[\mathfrak{p}]} = (\mathcal{I}^{[\mathfrak{p}]})/G.$$

[ii] *Let* $\mathcal{I}$ *be a differential system with independence condition* $\mathcal{J}$. *If* $\mathcal{I}$ *is Darboux integrable, then every prolongation of* $\mathcal{I}$ *is Darboux integrable.*

[iii] *Let* $\mathcal{I}$ *be a differential system with independence condition* $\mathcal{J}$. *Let G be a symmetry group of* $(\mathcal{I}, \mathcal{J})$ *and suppose that* $\mathcal{I}/G$ *is a (regular)differential system. If* $\mathcal{I}$ *is Darboux integrable, then some prolongation of* $\mathcal{I}/G$ *is Darboux integrable.*

Principle B (Mappings). **[i]** *Let* $\mathcal{I}$ *be a differential system with symmetry group G. Let H be a normal subgroup of G and suppose that* $\mathcal{I}/G$ *and* $\mathcal{I}/H$ *are regular differential systems. Then the quotient group G/H is a symmetry group of* $\mathcal{I}/H$ *and*

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\ \mathbf{q}_H\ } & \mathcal{I}/H \\ & \searrow^{\mathbf{q}_G} & \downarrow \mathbf{q}_{G/H} \\ & & \mathcal{I}/G \end{array}$$

is a commutative diagram of differential systems.

[ii] *Let* $\mathcal{I}$ *and* $\mathcal{J}$ *be two differential systems with a common symmetry group G. Suppose that* $\mathcal{I}/G$ *and* $\mathcal{J}/G$ *are regular differential systems and that* $\mathcal{I}$ *and* $\mathcal{J}$ *are equivalent by a G equivariant diffeomorphism. Then there is an induced equivalence*

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\ \cong\ } & \mathcal{J} \\ \downarrow \mathbf{q}_G & & \downarrow \mathbf{q}_G \\ \mathcal{I}/G & \xrightarrow{\ \cong\ } & \mathcal{J}/G \end{array} .$$

Principle C (Extensions). **[i]** *If* $\pi : \widetilde{\mathcal{I}} \to \mathcal{I}$ *is an integrable extension (see [6]) of a Darboux integrable differential systems* $\mathcal{I}$, *then* $\widetilde{\mathcal{I}}$ *is Darboux integrable.*

[ii] *If* $\pi : \widetilde{\mathcal{I}} \to \mathcal{I}$ *is an integrable extension of a Darboux integrable differential systems* $\mathcal{I}$, *then there are integrable extensions* $\pi_1 : \widetilde{\mathcal{W}}_1 \to \mathcal{W}_1$ *and* $\pi_2 : \widetilde{\mathcal{W}}_2 \to \mathcal{W}_2$ *and symmetry groups G and* $\widetilde{G}$ *of* $\mathcal{W}_i$ *and* $\widetilde{\mathcal{W}}_i$, $i = 1, 2$ *such that the diagram of differential systems*

$$\begin{array}{ccc} \widetilde{\mathcal{W}}_1 + \widetilde{\mathcal{W}}_2 & \xrightarrow{\ \mathbf{q}_{\widetilde{G}}\ } & \widetilde{\mathcal{I}} \\ \pi_1 \times \pi_2 \downarrow & & \downarrow \pi \\ \mathcal{W}_1 + \mathcal{W}_2 & \xrightarrow{\ \mathbf{q}_G\ } & \mathcal{I} \end{array}$$

commutes.

[iii] *There is a normal subgroup H of* $\widetilde{G}$ *such that* $G = \widetilde{G}/H$. *The projection maps* π, π_1 *and* π_2 *are all* $\widetilde{G}$ *equivariant maps.*

4 Linear Equations

In this first example we consider Darboux integrable linear partial differential equations

$$u_{xy} + a(x, y)\, u_x + b(x, y)\, u_y + c(x, y)\, u = 0. \tag{15}$$

We describe the quotient representation for these equations and we show how Principle B leads naturally to the constructions of the classical Laplace transformations between linear Darboux integrable equations.

Let $J^m(\mathbf{R}, \mathbf{R}) \times J^n(\mathbf{R}, \mathbf{R})$ be the product of two jet spaces with standard jet coordinates

$$(x,\ \phi,\ \phi_1,\ \ldots,\ \phi_m,\ y,\ \psi,\ \psi_1\ ,\ldots,\ \psi_n)$$

and let $\mathcal{C}^m + \mathcal{C}^n$ be the sum of the canonical contact systems

$$\begin{aligned}\mathcal{C}^m &= \{\theta_0 = d\phi - \phi_1\, dx,\ \theta_1 = d\phi_1 - \phi_2\, dx,\ \ldots, \theta_{m-1} = d\phi_{m-1} - \phi_m\, dx\}\\ \mathcal{C}^n &= \{\vartheta_0 = d\psi - \psi_1\, dy,\ \vartheta_1 = d\psi_1 - \psi_2\, dy,\ \ldots, \vartheta_{n-1} = d\psi_{n-1} - \psi_n\, dy\}.\end{aligned} \tag{16}$$

Result 1 states that this sum is trivially a Darboux integrable differential system. Principle A states that quotients of $\mathcal{C}^m + \mathcal{C}^n$ will be Darboux integrable. Here we establish precisely which group actions will lead to the Pfaffian systems for (15).

Let G_p be the p dimensional Abelian group acting on $J^m \times J^n$ with infinitesimal generators defined by the prolongation of the vector fields.

$$Z^i = f^i(x)\frac{\partial}{\partial \phi} + f^i(y)\frac{\partial}{\partial \psi}, \quad i = 1, \ldots, p. \tag{17}$$

The functions f^i are smooth and subject only to the condition stated below in Theorem 4.1. By definition, G_p is a symmetry group for the canonical contact system $\mathcal{C}^m + \mathcal{C}^n$.

Theorem 4.1 **[i]** *Let $p = m + n - 3$ and assume that the action of G_p on $J^{n-2} \times J^{m-2}$ is free. Then the quotient differential system*

$$\mathcal{I} = (\mathcal{C}^m + \mathcal{C}^n)/G_p \tag{18}$$

is the standard rank 3 Pfaffian system, defined on a seven manifold, for a linear PDE (15).

[ii] *The Pfaffian system for any Darboux integrable linear PDE* (15) *is a quotient differential system of the type* (18).

Proof. The prolonged infinitesimal actions for G_p are

$$\mathrm{pr} Z^i = \sum_{k=0}^{m} [\frac{d^k f^i}{d\,x^k}(x)]\frac{\partial}{\partial \phi_k} + \sum_{l=0}^{n} [\frac{d^l\, f^i}{d\, y^l}(y)]\frac{\partial}{\partial \psi_l},$$

so that a form

$$\omega = \sum_{k=0}^{n-1} A^k \theta_k + \sum_{l=0}^{n-1} B^l \vartheta_l$$

is semi-basic if and only if

$$\sum_{k=0}^{m-1} [\frac{d^k f^i}{d\,x^k}(x)] A^k \ + \sum_{l=0}^{n-1} [\frac{d^l\, f^i}{d\,y^l}(y)] C^l = 0.$$

With the functions f^i chosen so that action of G_p on $J^{n-2} \times J^{m-2}$ is free, the rank of this linear system of $m+n-3$ equations for $m+n$ unknowns A^k and B^l is maximal. This implies that G_p is transverse to the derived system $\mathcal{C}^{m-1} + \mathcal{C}^{n-1}$ and therefore the quotient EDS is a rank 3 Pfaffian system.

The functions x, y are obviously invariants for this action and there is precisely 1 additional differential invariant U on $J^{(m-2)} \times J^{(n-2)}$. As the solution to the equations $\mathrm{pr}Z^i(U) = 0$, the function U is linear in the variables ϕ_m and ψ_n, this is,

$$U = \sum_{k=0}^{m-2} C^k(x, y)\phi_k + \sum_{l=0}^{n-2} D^l(x, y)\psi_l. \tag{19}$$

We next note the six functions x, y, U, $D_x U$ $D_y U$ and $D_{xy} U$ on $J^{m-1} \times J^{n-1}$ are all G_p invariant. But there can only be five independent G_p invariant functions on $J^{n-1} \times J^{m-1}$ so that these six functions are necessary functionally dependent. In fact, the linearity of these invariants in the variables ϕ_k and ψ_l forces this dependence to be of the form

$$D_{xy}U + a(x, y)D_x U + b(x, y)D_y U + c(x, y)U = 0 \tag{20}$$

for some choice of functions a, b, c. These coefficients determine the PDE (15) for our quotient EDS.

The quotient of $J^m \times J^n$ by G_p is a seven dimensional manifold with coordinates $(x, y, u, u_x, u_u, u_{xx}, u_{yy})$, where the quotient map is defined by

$$x = x,\ \ y = y,\ \ u = U,\ \ u_x = D_x U,\ \ u_y = D_y U,$$
$$u_{xx} = D_{xx}U, \quad u_{yy} = D_{yy}U.$$

On account of (19) and (20), the forms

$$\mathcal{I} = \{\, du - u_x\, dx - u_y\, dy,\ du_x - u_{xx}\, du_x - u_{xy} dy,\ du_y - u_{xy}\, dx - u_{yy}\, dy \,\}, \tag{21}$$

where u_{xy} is given by (15), pullback under $\mathbf{q}$ into $\mathcal{C}^m + \mathcal{C}^n$ and therefore determine the quotient Pfaffian differential system. The system $\mathcal{I}$ is therefore the canonical differential system for PDE of the form (15).

We remark that the intermediate integrals for (15) are given by the projection of the differential invariants for the action of G_p on the individual jet spaces $J^{m+n-1}(x,\phi)$ and $J^{m+n-1}(y,\psi)$. □

This symmetry approach allows one to generate many new families of linear, Darboux integrable, scalar PDE. But, in order to arrive at some simple concrete examples for which we can give complete formulas, we consider the elementary action (17) determined by the functions $f^i(z) = z^i$ for $i = 0\dots 4$ and acting on the jet spaces

$$J^4(\mathbf{R},\mathbf{R})\times J^4(\mathbf{R},\mathbf{R}),\quad J^3(\mathbf{R},\mathbf{R})\times J^5(\mathbf{R},\mathbf{R})\quad \text{and}\quad J^2(\mathbf{R},\mathbf{R})\times J^6(\mathbf{R},\mathbf{R}). \tag{22}$$

In each case the canonical contacts systems on these jet spaces define rank 6 Pfaffian systems on 12 dimensional manifolds. The quotient differential systems

$$\mathbf{q}_1 : \mathcal{C}^4\times\mathcal{C}^4 \to \mathcal{I},\quad \mathbf{q}_2 : \mathcal{C}^3\times\mathcal{C}^5 \to \mathcal{J},\quad \mathbf{q}_3 : \mathcal{C}^2\times\mathcal{C}^6 \to \mathcal{K} \tag{23}$$

are found to be the canonical 3 rank Pfaffian systems for the equations

$$\mathcal{I} : u_{xy} + \frac{6u}{\zeta^2} = 0,\quad \mathcal{J} : v_{xy} - \frac{2v_x}{\zeta} + \frac{6v}{\zeta^2} = 0,\quad \mathcal{K} : w_{xy} - \frac{4w_x}{\zeta} + \frac{4w}{\zeta^2} = 0. \tag{24}$$

Here $\zeta = x - y$. The projection map $\mathbf{q}_1$ is

$$\begin{aligned}
u = U \quad &= \frac{12\phi}{\zeta^2} - \frac{6\phi_1}{\zeta} + \phi_2 - \frac{12\psi}{\zeta^2} - \frac{6\psi_1}{\zeta} - \phi_2,\\
u_x = D_x U \quad &= -\frac{24}{\zeta^3}\phi + \frac{18}{\zeta^2}\phi_1 - \frac{6}{\zeta}\phi_2 + \phi_3 + \frac{24}{\zeta^3}\psi + \frac{6}{\zeta^2}\psi_1,\\
u_y = D_y U \quad &= \frac{24}{\zeta^3}\phi - \frac{6}{\zeta^2}\phi_1 - \frac{24}{\zeta^3}\psi - \frac{18}{\zeta^2}\psi_1 - \frac{6}{\zeta}\psi_2 - \psi_3,\\
u_{xx} = D_{xx} U &= \frac{72}{\zeta^4}\phi - \frac{60}{\zeta^3}\phi_1 + \frac{24}{\zeta^2}\phi_2 - \frac{6}{\zeta}\phi_3 + \phi_4 - \frac{72}{\zeta^4}\phi - \frac{12}{\zeta^3}\phi_1,\\
u_{yy} = D_{yy} U &= \frac{72}{\zeta^4}\phi - \frac{12}{\zeta^3}\phi_1 - \frac{72}{\zeta^4}\psi - \frac{60}{\zeta^3}\psi_1 - \frac{24}{\zeta^2}\psi_2 - \frac{6}{\zeta}\psi_3 - \psi_4,
\end{aligned}$$

with partial prolongations $\mathbf{q}_1^{[0,1]} :\ \mathcal{C}^4\times\mathcal{C}^5 \to \mathcal{I}^{[0,1]}$ and $\mathbf{q}_1^{[0,2]} :\ \mathcal{C}^4\times\mathcal{C}^6 \to \mathcal{I}^{[0,2]}$ defined by

$$\begin{aligned}
u_{yyy} &= \frac{288}{\zeta^5}\phi - \frac{36}{\zeta^4}\phi_1 - \frac{288}{\zeta^5}\psi - \frac{252}{\zeta^4}\psi_1 - \frac{108}{\zeta^3}\psi_2 - \frac{30}{\zeta^2}\psi_3 - \frac{6}{\zeta}\psi_4 - \psi_5, \text{ and}\\
u_{yyyy} &= \frac{1440}{\zeta^6}\phi - \frac{144}{\zeta^5}\phi_1 - \frac{1440}{\zeta^6}\psi - \frac{1296}{\zeta^5}\psi - \frac{576}{\zeta^4}\psi_2 - \frac{168}{\zeta^3}\psi_3\\
&\quad - \frac{36}{\zeta^2}\psi_4 - \frac{6}{\zeta}\psi_5 - \psi_6.
\end{aligned}$$

The projection map $\mathbf{q}_2$ is

$$\begin{aligned}
v = V \quad &= +\frac{24}{\zeta^3}\phi - \frac{6}{\zeta^2}\phi_1 - \frac{24}{\zeta^3}\psi - \frac{18}{\zeta^2}\psi_1 - \frac{6}{\zeta}\psi_2 - \psi_3,\\
v_x = D_x V \quad &= -\frac{72}{\zeta^4}\phi + \frac{36}{\zeta^3}\phi_1 - \frac{6}{\zeta^2}\phi_2 + \frac{72}{\zeta^4}\psi + \frac{36}{\zeta^3}\psi_1 + \frac{6}{\zeta^2}\psi_2,\\
v_y = D_y V \quad &= +\frac{72}{\zeta^4}\phi - \frac{12}{\zeta^3}\phi_1 - \frac{72}{\zeta^4}\psi - \frac{60}{\zeta^3}\psi_1 - \frac{24}{\zeta^2}\psi_2 - \frac{6}{\zeta}\psi_3 - \psi_4,\\
v_{xx} = D_{xx} V &= +\frac{288}{\zeta^5}\phi - \frac{180}{\zeta^4}\phi_1 + \frac{48}{\zeta^3}\phi_2 - \frac{6}{\zeta^2}\phi_3 - \frac{288}{\zeta^5}\psi - \frac{108}{\zeta^4}\psi_1 - \frac{12}{\zeta^3}\psi_2,\\
v_{yy} = D_{yy} V &= +\frac{288}{\zeta^5}\phi - \frac{36}{\zeta^4}\phi_1 - \frac{288}{\zeta^5}\psi - \frac{252}{\zeta^4}\psi_1 - \frac{108}{\zeta^3}\psi_2 - \frac{30}{\zeta^2}\psi_3\\
&\quad - \frac{6}{\zeta}\psi_4 - \psi_5
\end{aligned}$$

with partial prolongations $\mathbf{q}_2^{[1,0]}\colon\ \mathcal{C}^4 \times \mathcal{C}^5 \to \mathcal{J}^{[1,0]}$ and $\mathbf{q}_2^{[0,1]}\colon\ \mathcal{C}^3 \times \mathcal{C}^6 \to \mathcal{J}^{[0,1]}$ defined by

$$\begin{aligned}
v_{xxx} &= -\frac{1440}{\zeta^6}\phi + \frac{1008}{\zeta^5}\phi_1 - \frac{324}{\zeta^4}\phi_2 + \frac{60}{\zeta^3}\phi_3 - \frac{6}{\zeta^2}\phi_4 + \frac{1440}{\zeta^6}\psi\\
&\quad + \frac{432}{\zeta^5}\psi_1 + \frac{36}{\zeta^4}\psi_2,\\
v_{yyy} &= +\frac{1440}{\zeta^6}\phi - \frac{144}{\zeta^5}\phi_1 - \frac{1440}{\zeta^6}\psi - \frac{1296}{\zeta^5}\psi_1 - \frac{576}{\zeta^4}\psi_2\\
&\quad - \frac{168}{\zeta^3}\psi_3 - \frac{36}{\zeta^2}\psi_4 - \frac{6}{\zeta}\psi_5 - \phi_6.
\end{aligned}$$

The projection map $\mathbf{q}_3$ is

$$\begin{aligned}
w = W \quad &= \frac{24}{\zeta^4}\phi - \frac{24}{\zeta^4}\psi - \frac{24}{\zeta^3}\psi_1 - \frac{12}{\zeta^2}\psi_2 - \frac{4}{\zeta}\psi_3 - \psi_4,\\
w_x = D_x W \quad &= -\frac{96}{\zeta^5}\phi + \frac{24}{\zeta^4}\phi_1 + \frac{96}{\zeta^5}\psi + \frac{72}{\zeta^4}\psi_1 + \frac{24}{\zeta^3}\psi_2 + \frac{4}{\zeta^2}\psi_3,\\
w_y = D_y W \quad &= \frac{96}{\zeta^5}\phi - \frac{96}{\zeta^5}\psi - \frac{96}{\zeta^4}\psi_1 - \frac{48}{\zeta^3}\psi_2 - \frac{16}{\zeta^2}\psi_3 - \frac{4}{\zeta}\psi_4 - \psi_5,\\
w_{xx} = D_{xx} W &= \frac{480}{\zeta^6}\phi - \frac{192}{\zeta^5}\phi_1 + \frac{24}{\zeta^4}\phi_2 - \frac{480}{\zeta^6}\psi - \frac{288}{\zeta^5}\psi_1 - \frac{72}{\zeta^4}\psi_2 - \frac{8}{\zeta^3}\psi_3,\\
w_{yy} = D_{yy} W &= \frac{480}{\zeta^6}\phi - \frac{480}{\zeta^6}\psi - \frac{480}{\zeta^5}\psi_1 - \frac{240}{\zeta^4}\psi_2 - \frac{80}{\zeta^3}\psi_3 - \frac{20}{\zeta^2}\psi_4\\
&\quad - \frac{4}{\zeta}\psi_5 - \psi_6,
\end{aligned}$$

with partial prolongations $\mathbf{q}_3^{[1,0]}\colon\ \mathcal{C}^3 \times \mathcal{C}^6 \to \mathcal{K}^{[1,0]}$ and $\mathbf{q}_3^{[2,0]}\colon\ \mathcal{C}^5 \times \mathcal{C}^6 \to \mathcal{K}^{[2,0]}$ defined by

$$w_{xxx} = -\frac{2880}{\zeta^7}\phi + \frac{1440}{\zeta^6}\phi_1 - \frac{288}{\zeta^5}\phi_2 + \frac{24}{\zeta^4}\phi_3 + \frac{2880}{\zeta^7}\psi + \frac{1440}{\zeta^6}\psi_1 + \frac{288}{\zeta^5}\psi_2 + \frac{24}{\zeta^4}\psi_3,\ w_{xxxx} = \frac{20160}{\zeta^8}\phi - \frac{11520}{\zeta^7}\phi_1 + \frac{2880}{\zeta^6}\phi_2 - \frac{384}{\zeta^5}\phi_3 + \frac{24}{\zeta^4}\phi_4 - \frac{20160}{\zeta^8}\psi - \frac{8640}{\zeta^7}\psi_1 - \frac{1440}{\zeta^6}\psi_2 - \frac{96}{\zeta^5}\psi_3.$$

Because the quotients (23) are all with respect to (the prolongations of) the same group action, we can use Principle B to calculate the internal equivalences

$$\begin{array}{ccc} \mathcal{C}^4 \times \mathcal{C}^5 & \xleftrightarrow{\ \mathrm{id}\ } & \mathcal{C}^4 \times \mathcal{C}^5 \\ \Sigma_1^{[0,1]} \uparrow\ \downarrow \mathbf{q}_1^{[0,1]} & & \mathbf{q}_2^{[1,0]} \downarrow\ \uparrow \Sigma_2^{[1,0]} \\ \mathcal{I}^{[0,1]} & \underset{\Psi_1}{\overset{\Phi_1}{\rightleftarrows}} & \mathcal{J}^{[1,0]} \end{array} \qquad \begin{array}{ccc} \mathcal{C}^3 \times \mathcal{C}^6 & \xleftrightarrow{\ \mathrm{id}\ } & \mathcal{C}^3 \times \mathcal{C}^6 \\ \Sigma_2^{[0,1]} \uparrow\ \downarrow \mathbf{q}_2^{[0,1]} & & \mathbf{q}_3^{[1,0]} \downarrow\ \uparrow \Sigma_3^{[1,0]} \\ \mathcal{J}^{[0,1]} & \underset{\Psi_2}{\overset{\Phi_2}{\rightleftarrows}} & \mathcal{K}^{[1,0]} \end{array}$$

$$\text{and}\qquad \begin{array}{ccc} \mathcal{C}^4 \times \mathcal{C}^6 & \xleftrightarrow{\ \mathrm{id}\ } & \mathcal{C}^4 \times \mathcal{C}^6 \\ \Sigma_1^{[0,2]} \uparrow\ \downarrow \mathbf{q}_1^{[0,2]} & & \mathbf{q}_3^{[2,0]} \downarrow\ \uparrow \Sigma_3^{[2,0]} \\ \mathcal{I}^{[0,2]} & \underset{\Psi_3}{\overset{\Phi_3}{\rightleftarrows}} & \mathcal{K}^{[2,0]} \end{array}$$

We choose cross-sections $\phi = \phi_1 = \phi_2 = \phi_3 = \phi_4 = 0$ and, for $\mathbf{q}_1$

$$\psi = -\frac{\zeta^4}{24}u_{xx} - \frac{\zeta^3}{12}u_x,\ \psi_1 = \frac{\zeta^3}{6}u_{xx} + \frac{\zeta^2}{2}u_x,\ \psi_2 = -\frac{\zeta^2}{2}u_{xx} - 2\zeta u_x - u,$$
$$\psi_3 = \zeta u_{xx} + 5u_x - u_y + \frac{6u}{\zeta}, \quad \psi_4 = -u_{xx} - u_{yy} + \frac{6}{\zeta}(u_y - u_x) - \frac{12u}{\zeta^2},$$
$$\psi_5 = -u_{yyy} + \frac{6}{\zeta}u_{yy} - \frac{6}{\zeta^2}u_y, \quad \psi_6 = -u_{yyyy} + \frac{6}{\zeta}u_{yyy} - \frac{12}{\zeta^3}u_y;$$

for $\mathbf{q}_2$

$$\psi = +\frac{\zeta^4}{24}v_x + \frac{\zeta^5}{24}v_{xx} + \frac{\zeta^6}{144}v_{xxx}, \quad \psi_1 = -\frac{2\zeta^3}{9}v_x - \frac{7\zeta^4}{36}v_{xx} - \frac{\zeta^5}{36}v_{xxx},$$
$$\psi_2 = +\zeta^2 v_x + \frac{2\zeta^3}{3}v_{xx} + \frac{\zeta^4}{12}v_{xxx}, \quad \psi_3 = -v - 3\zeta v_x - \frac{3\zeta^2}{2}v_{xx} - \frac{\zeta^3}{6}v_{xxx},$$

$$\psi_4 = +\frac{6}{\zeta}v + \frac{13}{3}v_x - v_y + \frac{5\zeta}{3}v_{xx} + \frac{\zeta^2}{6}v_{xxx},$$
$$\psi_5 = -\frac{6}{\zeta^2}v + \frac{6}{\zeta}v_y - v_{yy}, \quad \psi_6 = -\frac{12}{\zeta^3}v + \frac{6}{\zeta}v_{yy} - v_{yyy},$$

and for $\mathbf{q}_3$

$$\psi = -\frac{\zeta^5}{24}w_x - \frac{\zeta^6}{16}w_{xx} - \frac{\zeta^7}{48}w_{xxx} - \frac{\zeta^8}{576}w_{xxxx}, \quad \psi_5 = \frac{4}{\zeta}w - w_y,$$
$$\psi_1 = \frac{5\zeta^4}{24}w_x + \frac{7\zeta^5}{24}w_{xx} + \frac{13\zeta^6}{144}w_{xxx} + \frac{\zeta^7}{144}w_{xxxx}, \quad \psi_6 = \frac{4}{\zeta^2}w + \frac{4}{\zeta}w_y - w_{yy},$$
$$\psi_2 = -\frac{5\zeta^3}{6}w_x - \frac{25\zeta^4}{24}w_{xx} - \frac{7\zeta^5}{24}w_{xxx} - \frac{\zeta^6}{48}w_{xxxx},$$
$$\psi_3 = \frac{5\zeta^2}{2}w_x + \frac{5\zeta^3}{2}w_{xx} + \frac{5\zeta^4}{8}w_{xxx} + \frac{\zeta^5}{24}w_{xxxx},$$
$$\psi_4 = -w - 4\zeta w_x - 3\zeta^2 w_{xx} - \frac{2\zeta^3}{3}w_{xxx} - \frac{\zeta^4}{24}w_{xxxx}.$$

The internal equivalences Φ_1 Φ_2, Φ_3 and their inverses Ψ_1, Ψ_2, Ψ_3 can now be easily computed as the prolongations of the formulas

$$\Phi_1 = \mathbf{q}_2^{[1,0]} \circ \Sigma_1^{[0,1]} : \ v = u_y, \qquad \Psi_1 = \mathbf{q}_1^{[0,1]} \circ \Sigma_2^{[1,0]} : \ u = -\frac{\zeta^2}{6}v_x,$$
$$\Phi_2 = \mathbf{q}_3^{[1,0]} \circ \Sigma_2^{[0,1]} : \ w = v_y - \frac{2}{\zeta}v, \quad \Psi_2 = \mathbf{q}_2^{[0,1]} \circ \Sigma_3^{[1,0]} : \ v = -\frac{\zeta^2}{4}w_x,$$
$$\Phi_3 = \mathbf{q}_3^{[2,0]} \circ \Sigma_1^{[0,2]} : w = u_{yy} - \frac{2}{\zeta}u_y, \Psi_3 = \mathbf{q}_1^{[0,2]} \circ \Sigma_3^{[2,0]} : u = \frac{\zeta^4}{24}w_{xx} + \frac{\zeta^3}{12}w_x.$$

The maps Φ_1 and Φ_2 are precisely the classical Laplace transformations ([11], Vol 2, p. 23–53, [14], Vol 6, p. 39–104) for the equations defined by $\mathcal{I}$ and $\mathcal{J}$ so that, as promised, we have re-constructed these transformations from the symmetry reduction viewpoint. We emphasize that had these Laplace transformations been unknown to us, we would have discovered them by applying the algorithms of [2] to recognize $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ as the quotients (23) by the same group action (17) (with $f^i(z) = z^i$).

The differential invariants for the action (17) on the individual jet spaces are simply ϕ_5 and ψ_5 and these project under $\mathbf{q}_1^{[1,1]}$, $\mathbf{q}_2^{[2,0]}$ $\mathbf{q}_3^{[3,0]}$ to give the following intermediate integrals

$$I(\mathcal{I}) = u_{xxx} + \frac{6}{\zeta}u_{xx} + \frac{6}{\zeta^2}u_x, \quad J(\mathcal{I}) = -u_{yyy} + \frac{6}{\zeta}u_{yy} - \frac{6}{\zeta^2}u_y,$$

$$I(\mathcal{J}) = \frac{\zeta^2}{6} v_{xxxx} + 2\zeta v_{xxx} + 6v_{xx} + \frac{4}{\zeta} v_x, \quad J(\mathcal{J}) = -\frac{6}{\zeta^2} v + \frac{6}{\zeta} v_y - v_{yy},$$
$$I(\mathcal{K}) = \frac{\zeta^4}{24} w_{xxxxx} + \frac{5\zeta^3}{6} w_{xxxx} + 5\zeta^2 w_{xxx} + 10\zeta w_{xx} + 5w_x,$$
$$J(\mathcal{K}) = -w_y + \frac{4}{\zeta} w.$$

From the orders of these intermediate integrals we can deduce that the Laplace invariants for our three equations vanish at orders

$$h_2(\mathcal{I}) = h_3(\mathcal{J}) = h_4(\mathcal{K}) = 0 \quad \text{and} \quad k_2(\mathcal{I}) = k_1(\mathcal{J}) = k_0(\mathcal{J}) = 0.$$

The inferences of this last computation hold generally [14].

Theorem 4.2 (The Canonical Form for Darboux Integrable Linear PDE) *Let $\mathcal{I}$ be the Pfaffian system for a linear Darboux integrable equation* (15). *Then there is another linear Darboux integrable equation* (15) *with associated Pfaffian system $\mathcal{J}$ such that Laplace invariant $k_0(\mathcal{J}) = 0$ and the appropriate partial prolongations of $\mathcal{I}$ and $\mathcal{J}$ are internally equivalent.*

5 Internal Equivalences of Some Non-linear PDE

We have systematically calculated the quotient representations for all the examples in Goursat [16] and, in so doing, we have uncovered a number of new internal equivalences.

Example 5.1 *The canonical Pfaffian systems (on seven manifolds) for the two equations ([16] p. 124 and p. 134)*

$$\mathcal{I}:\ u_{xy} = e^u \quad \textit{and} \quad \mathcal{J}:\ v_{xy} = v v_x \tag{25}$$

are quotients of the contact systems $\mathcal{C}^3 \times \mathcal{C}^3$ and $\mathcal{C}^2 \times \mathcal{C}^4$ by the diagonal action of $SL(2)$, acting by fractional linear transformations on the dependent variables ϕ and ψ. The two projection maps

$$\mathbf{q}_1 : \mathcal{C}^3 \times \mathcal{C}^3 \to \mathcal{I} \quad \textit{and} \quad \mathbf{q}_2 : \mathcal{C}^2 \times \mathcal{C}^4 \to \mathcal{J}$$

are given by

$$u = \ln \frac{2\phi_1\psi_1}{(\phi+\psi)^2} \quad \textit{and} \quad v = \frac{\psi_2}{\psi_1} - \frac{2\psi_1}{\phi+\psi}, \tag{26}$$

and the prolongations of these equations to order 2. Then, just as in the previous section, we find that there is an internal equivalence

$$\Phi : \mathcal{I}^{[0,1]} \to \mathcal{J}^{[1,0]} \quad \textit{with inverse} \quad \Psi : \mathcal{J}^{[1,0]} \to \mathcal{I}^{[0,1]}$$

which is given by

$$v = u_y \quad \textit{and} \quad u = \log(v_x). \tag{27}$$

The differential invariants for the action of $SL(2)$ *on* $J^3 \times J^3$ *are*

$$x, \quad \frac{2\phi_1\phi_3 - 3\phi_2^2}{\phi_1^2}, \quad y, \quad \frac{2\psi_1\psi_3 - 3\psi_2^2}{\psi_1^2} \tag{28}$$

and these project under $\mathbf{q}_1$ *to the intermediate integrals*

$$I_1 = x, \quad I_2 = u_{xx} - \frac{1}{2}u_x^2, \quad J_1 = y, \quad J_2 = u_{yy} - \frac{1}{2}u_y^2$$

for $\mathcal{I}$*. These, in turn , are transformed by* (27) *to the intermediate integrals*

$$\tilde{I}_1 = x, \quad \tilde{I}_2 = \frac{v_{xxx}}{v_x} - \frac{3}{2}v_{xx}^2, \quad \tilde{J}_1 = y, \quad \tilde{J}_2 = v_y^2 - \frac{1}{2}v^2,$$

for $\mathcal{J}$*.*

Example 5.2 *A more complex example is given by the two systems ([16], p. 186 and p. 231)*

$$\mathcal{I} : z_{xy} = \frac{2}{x+y}\sqrt{z_x z_y} \quad \textit{and} \quad \mathcal{J} : w_{uv} + w^2 w_{vv} + 2ww_v^2 = 0. \tag{29}$$

Remarkably, the quotient representations for these two differential systems are

$$\mathbf{q}_1 : \mathcal{H}^{2,[1]} \times \mathcal{H}^{2,[1]} \to \mathcal{I} \quad \textit{and} \quad \mathbf{q}_2 : \mathcal{H}^{2,[2]} \times \mathcal{H}^2 \to \mathcal{J}, \tag{30}$$

where $\mathcal{H}^2$ *is the rank 2 Pfaffian system defined on a 5 manifold with coordinates* $\{t, \sigma, \phi, \phi_1, \phi_2\}$*, by* $\mathcal{H}^2 = \{d\phi - \phi_1\, dt, d\sigma - \phi_1^2\, dt\}$*, and where the prolongations are given by*

$$\mathcal{H}^{2,[1]} = \mathcal{H}^2 \cup \{\, d\phi_1 - \phi_2\, dt\,\} \quad \textit{and} \quad \mathcal{H}^{2,[2]} = \mathcal{H}^{2,[1]} \cup \{\, d\phi_2 - \phi_3\, dt\,\}. \tag{31}$$

(We write the second copy of $\mathcal{H}^2$ *in terms of the coordinates* $\{s, \tau, \psi, \phi_1, \psi_2\}$ *as* $\{d\psi - \psi_1 ds,\ d\tau - \psi_1^2 dt\}$*.) The symmetry group for the reductions* (30) *is the three dimensional Heisenberg group, acting with infinitesimal generators*

$$\{\partial_\phi - \partial_\psi,\ \partial_\sigma - \partial_\tau,\ t\partial_\phi + \partial_{\phi_1} + 2\phi\partial_\sigma + s\partial_\psi + \partial_{\psi_1} + 2\psi\partial_\tau\}. \tag{32}$$

We are in precisely the situation of Theorem A and therefore $\mathcal{I}^{[1,0]}$ *and* $\mathcal{J}^{[0,1]}$ *are internally equivalent. With* $\zeta = s + t$ *the (prolonged) projection* $\mathbf{q}_1^{[1,0]}$ *is given by*

$$x = t, \quad y = s, \quad z = \sigma + \tau - \frac{(\phi + \psi)^2}{\zeta}, \quad z_x = P^2, \quad z_y = Q^2,$$
$$z_{xx} = 2PP_t, \quad z_{yy} = 2QQ_s, \quad z_{xxx} = 2P_t^2 + 2PP_{tt}, \quad \textit{where}$$
$$P = \frac{\phi + \psi - \zeta\phi_1}{\zeta}, \quad Q = \frac{\phi + \psi - \zeta\psi_1}{\zeta}.$$

The projection $\mathbf{q}_2^{[0,1]}$ *is*

$$u = t, \quad v = \sigma + \tau + \zeta\psi_1^2 - 2(\phi + \psi)\psi_1, \quad w = \psi_1 - \phi_1,$$
$$w_u = -\phi_2 - w^2 w_v, \quad w_v = -\frac{1}{2(\phi + \psi - \zeta\psi)}, \quad w_{vv} = -2\zeta w_v^3,$$
$$w_{uu} = -2\zeta w^4 z_v^3 + 4w^4 w_v^2 + 2\phi_2 w w_v - \phi_3, \quad w_{vvv} = 12\zeta^2 w_v^5 - \frac{2w_v^4}{\psi_2}.$$

Again, we proceed as in Sect. 4 to arrive at the equivalence $\Phi : \mathcal{I}^{[1,0]} \to \mathcal{J}^{[0,1]}$, *given by*

$$\Phi : u = x, \quad v = (x + y)z_y + z, \quad w = \sqrt{z_x} + \sqrt{z_y}$$

with inverse

$$\Psi : x = u, \quad y = -u - \frac{w_{vv}}{2w_v^3}, \quad z = v + \frac{w_v}{2w_{vv}}.$$

Under the mapping Ψ *the intermediate integrals*

$$I_1 = x, \quad I_2 = \frac{z_{xx}}{2\sqrt{z_x}} + \frac{\sqrt{z_x}}{x + y}, \quad J_1 = y, \quad J_2 = \frac{z_{yy}}{2\sqrt{z_y}} + \frac{\sqrt{z_y}}{x + y}$$

for $\mathcal{I}$ *are transformed to the intermediate integrals*

$$\tilde{I}_1 = u, \quad \tilde{I}_2 = w^2 w_v + w_u, \quad \tilde{J}_1 = -\frac{w_{vv}}{2w_v^3} - u, \quad \tilde{J}_2 = \frac{2w_v^5}{w_{vvv} w_v - 3w_{vv}^2}$$

for $\mathcal{J}$. *Note that the invariants* $\tilde{J}_1$ *and* $\tilde{J}_2$ *are of lower order than that suggested by Goursat.*

6 Moutard Equations

Moutard equations are non-linear scalar PDE of the form

$$v_{xy} + \frac{\partial}{\partial x}(A_0 e^v) - \frac{\partial}{\partial y}(B_0 e^{-v}) + C_0 = 0, \tag{33}$$

where the coefficients A_0, B_0 and C_0 are functions of the independent variables x, y. Assuming that $B_0 > 0$, the change of dependent variables $v \to v + \log(B_0)$ transforms (33) to the standard form

$$v_{xy} + \frac{\partial}{\partial x}(Ae^{v}) - \frac{\partial}{\partial y}(e^{-v}) + C = 0, \tag{34}$$

where A and C are functions of x, y. Let $\mathcal{M}$ be the usual rank 3 Pfaffian system for (34) on the seven manifold N with coordinates $(x, y, v, v_x, v_y, v_{xx}, v_{yy})$. Goursat [16] (p. 249) establishes a close relationship between Darboux integrable Moutard equations and Darboux integrable linear equations. From our perspective of symmetry reduction, this relationship is given by

Theorem 6.1 **[i]** *Let $\mathcal{L}^{[1,0]}$ be the rank 4 Pfaffian system for the partially prolonged linear PDE (15), and let S be the 1 dimensional scaling symmetry group of $\mathcal{L}^{[1,0]}$ with infinitesimal generator*

$$W = u\partial_u + u_x\partial_{u_x} + u_y\partial_{u_y} + u_{xx}\partial_{u_{xx}} + u_{yy}\partial_{u_{yy}} + u_{xxx}\partial_{u_{xxx}}. \tag{35}$$

Then the quotient system $\mathcal{L}^{[1,0]}/S$ is the standard Pfaffian system for a Moutard equation (34).

[ii] *Every Moutard system $\mathcal{M}$ is the quotient $\mathbf{q}_S\colon \mathcal{L}^{[1,0]} \to \mathcal{M}$ of a Pfaffian system for a linear equation. The projection map $\mathbf{q}_S$ defines $\mathcal{L}^{[1,0]}$ as a rank 1 integrable extension of $\mathcal{M}$.*

[iii] *The Moutard system $\mathcal{M}$ is Darboux integrable (at some order of prolongation) if and only if $\mathcal{L}^{[1,0]}$ is Darboux integrable.*

Proof. The differential system $\mathcal{L}^{[1,0]}$ is the rank 4 Pfaffian system with generators

$$\begin{aligned} \theta &= du - u_x\,dx - u_y\,dy, & \theta_x &= du_x - u_{xx}\,dx - u_{yy}\,dy,\\ \theta_y &= du - u_{xy}\,dx - u_{yy}\,dy, & \theta_{xx} &= du_{xx} - u_{xxx}\,dx - u_{xxy}\,dy, \end{aligned} \tag{36}$$

where u_{xy} and u_{xxy} are given by the PDE (15) and its x derivative.

A basis for the semi-basic forms $\mathcal{L}^{[1,0]}_{\mathbf{sb}}$, with respect to the symmetry group S, is easily determined to be

$$\vartheta_1 = \theta_x - \frac{u_x}{u}\theta, \quad \vartheta_2 = \theta_y - \frac{u_y}{u}\theta, \quad \vartheta_3 = \theta_{xx} - \frac{u_{xx}}{u}\theta. \tag{37}$$

The quotient map, for the scaling group S, from the eight manifold for $\mathcal{L}^{[1,0]}$ to the seven manifold N is defined by the prolongation of

$$x = x, \quad y = y, \quad v = \frac{u_x}{u}. \tag{38}$$

It is a simple matter to re-write the semi-basic forms (37) in terms of v and its derivatives to deduce that the quotient differential system on N is that of a PDE of the form

$$v_{xy} + \rho\, v_x v_y + \tilde{A}\, v_x + \tilde{B}\, v_y + \tilde{C} = 0, \quad \text{where} \quad \rho = -\frac{1}{(b+v)} \tag{39}$$

and where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are certain functions of x, y, and v. This is not in the form of a Moutard equation but the point transformation

$$\tilde{v} = -\log(b+v)$$

will eliminate the quadratic term $\rho\, v_x v_y$ in (39) and lead to the Moutard equation (34), with

$$A = \frac{\partial b}{\partial y} + ab - c, \quad \text{and} \quad C = \frac{\partial b}{\partial y} - \frac{\partial a}{\partial x}. \tag{40}$$

In other words, instead of using the obvious projection map (38), it is better to use

$$x = x \quad y = y, \quad v = -\log(\frac{u_x}{u} + b). \tag{41}$$

To prove the first part of **[ii]**, we simply check that that for given functions $A(x, y)$ and $B(x, y)$, it is always possible find $a(x, y)$, $b(x, y)$, $c(x, y)$ satisfying equations (40). To prove the second part of **[ii]**, we need only observe that the form θ is a complement to the semi-basic forms (37), relative to (36), and that

$$d\,\theta \equiv 0 \mod \{\theta, \vartheta_1, \vartheta_2, \vartheta_3\}. \tag{42}$$

Part **[iii]** then follows directly from Principles A and C. □

The combination of Theorems 4.1 and 6.1 yield the following corollary.

Corollary 6.2 *For any Darboux integrable Moutard system $\mathcal{M}$, there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{C}^{m+1} \times \mathcal{C}^{n} & \xrightarrow{\mathbf{q}_{G_p}} & \mathcal{L}^{[1,0]} \\ & \searrow^{\mathbf{q}_{K_{p+1}}} & \downarrow \mathbf{q}_S \\ & & \mathcal{M} \end{array} \tag{43}$$

Here $\mathcal{C}^{m+1}$ and $\mathcal{C}^{n}$ are the contact systems (16), K_{p+1} is the $p+1$ dimensional symmetry group with generators (17) and $W = \phi\partial_\phi + \psi\partial_\psi$ (prolonged). The group G_p is defined as in the statement of Theorem 4.1.

Example 6.1 *The quotient of the linear equation $\mathcal{J}$ (the second Pfaffian system in (24)) by the scaling action $v\partial_v$ is the Moutard equation*

$$V_{xy} - 6D_x(\frac{e^V}{\zeta^2}) - D_y(e^{-V}) - \frac{2}{\zeta^2} = 0. \tag{44}$$

The composition of the projections $\mathbf{q}_2$ *and* $\mathbf{q}_S$ *gives the projection map* $\mathbf{q} : \mathcal{C}^4 \times \mathcal{C}^5 \to \mathcal{M}$ *as*

$$V = \log\Big(\frac{\zeta(24\phi - 6\zeta\phi_1 - 24\psi - 18\zeta\psi_1 - 6\zeta^2\psi_2 - \zeta^3\psi_3)}{6(-12\phi + 6\zeta\phi_1 - \zeta^2\phi_2 + 12\psi + 6\zeta\psi_1 + \zeta^2\psi_2)}\Big). \tag{45}$$

7 First Order Linear Systems

In this example we consider the simple class of first order linear PDE

$$u_y = \alpha_0\, u + \beta_0\, v, \qquad v_x = \gamma_0\, u + \delta_0\, v, \tag{46}$$

where the coefficients α_0, β_0, γ_0, δ_0 are functions of the independent variables x, y. A simple scaling of the dependent variables u and v transforms this system to the form

$$u_y = \alpha\, v, \qquad v_x = \beta\, u. \tag{47}$$

We associate to each such system a rank 2 Pfaffian system $\mathcal{S}$ on a six manifold. The quotient representation for Darboux systems of the type (47) can be obtained directly using the arguments of Theorem 4.1 or indirectly using the integrable extensions approach of the previous section.

Let $\mathcal{C}^m$ and $\mathcal{C}^n$ be the contact systems (16) and let G_p be the p dimensional Abelian group acting on $J^m \times J^n$ with infinitesimal generators (17).

Theorem 7.1 *The quotient differential system* $\mathcal{I} = (\mathcal{C}^m + \mathcal{C}^n)/G_p$, *where* $p = m + n - 2$ *is the standard rank 2 Pfaffian system, defined on a six manifold, for a linear PDE system* (47)*.*

Proof. The detailed argument follows the same lines as given for the proof of Theorem 4.1. Here we simply note that on $J^{n-1} \times J^{m-2}$ there are three differential invariants x, y and

$$U = U_0(x, y, \phi_0, \phi_1, \ldots, \phi_{n-2}, \psi_0, \psi_1, \ldots, \psi_{m-2}) + \phi_{n-1}$$

while on $J^{n-2} \times J^{m-1}$ there are three differential invariants x, y and

$$V = V_0(x, y, \phi_0, \phi_1, \ldots, \phi_{n-2}, \psi_0, \psi_1, \ldots, \psi_{m-2}) + \psi_{m-1}.$$

The functions U_0 and V_0 are linear in the jet coordinates ψ_k and ψ_l. These invariants are necessarily related by identities of the form

$$D_y U = \alpha\, V \quad \text{and} \quad D_x V = \beta\, U$$

which determine the coefficients for the quotient system (47). □

Theorem 7.2 *For any Pfaffian system $\mathcal{S}$ defined by (47) there is an integrable extension to a system $\mathcal{L}$ defined by (15). For any Pfaffian system $\mathcal{L}$ defined by (15) there is an integrable extension to the prolonged system $\mathcal{S}^{[1]}$ defined by (47).*

Proof. We first remark that by change of a dependent variable $u \to \mu(x, y)\, u$ one can always transform (15) to an equivalent linear equation with either $b = 0$ or $c = 0$.

If u and v solve (47), then

$$z = e^{\lambda} u - v, \quad \lambda = \lambda(x, y) \tag{48}$$

satisfies a 2nd order linear PDE

$$z_{xy} + a z_x + b z_y + c z = 0 \tag{49}$$

precisely when λ satisfies the Moutard equation

$$\lambda_{xy} + D_x(\alpha e^{\lambda}) - D_y(\beta e^{-\lambda}) = 0. \tag{50}$$

The coefficients of (49) of are given by

$$a = -\lambda_y, \quad b = 0, \quad c = e^{\lambda}\alpha_x + e^{\lambda}\lambda_x \alpha - \alpha\beta. \tag{51}$$

Note that in the special case where $\alpha_y = \beta_x$, then $\lambda = 0$ is a solution to (50).

Conversely, functions $\mu(x, y)$ and $\nu(x, y)$ can be chosen so that for any solution u to the linear equation (49), with $c = 0$, the functions

$$u = \mu(x, y) z_x \quad \text{and} \quad v = \nu(x, y) z_y \tag{52}$$

satisfy a system of the form (47).

The required integrable extensions $\pi_1 \colon \mathcal{S}^{[1]} \to \mathcal{L}$ and $\pi_2 \colon \mathcal{L} \to \mathcal{S}$ are determined by (48) (and its derivatives) and (52). □

Theorem 7.3 *For the Pfaffian system $\mathcal{L}$ associated to a Darboux integrable 2nd order linear PDE (15) or for the Pfaffian system $\mathcal{S}$ associated to a Darboux integrable 1st order linear system (47), there are commutative diagrams*

$$\begin{array}{ccc} \mathcal{C}^m \times \mathcal{C}^n & \xrightarrow{\mathbf{q}_{\tilde{G}_{p-1}}} & \mathcal{L} \\ & \searrow^{\mathbf{q}_{G_p}} & \downarrow \mathbf{q}_H \\ & & \mathcal{S} \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{C}^{m+1} \times \mathcal{C}^{n+1} & \xrightarrow{\mathbf{q}_{\tilde{G}_{p-1}}} & \mathcal{S}^{[1]} \\ & \searrow^{\mathbf{q}_{G_p}} & \downarrow \mathbf{q}_H \\ & & \mathcal{L} \end{array}. \tag{53}$$

The infinitesimal generators for G_p are given by (17), $\tilde{G}_{p-1}$ is a subgroup of G_p and $G_p = \tilde{G}_{p-1} \oplus H$. In the first diagram G_p is of dimension $p = n + m - 2$ while in the second diagram G_p has dimension $p = n + m - 1$.

Example 7.1 *As a simple example, let G_i be the four dimensional Abelian subgroup of the 5 dimensional group* (17) *(with $f^i(z) = z^i$, $i = 0 \dots 4$) obtained by removing the vector Z^i. The quotients $S_i^{[1]}$ of $\mathcal{C}^4 \times \mathcal{C}^4$ by G_i give the first prolongations of the systems*

$$u_y = \alpha_i\, v, \quad v_x = \beta_i\, u,$$

with

$$\alpha_0 = \frac{2y}{x(x-y)}, \quad \beta_0 = -\frac{2x}{y(x-y)}, \quad \alpha_3 = \frac{x+3y}{x^2-y^2}, \quad \beta_3 = -\frac{3x+y}{x^2-y^2},$$
$$\alpha_1 = \frac{y(3x+y)}{x(x^2-y^2)}, \quad \beta_1 = -\frac{x(x+3y)}{y(x^2-y^2)}, \quad \alpha_4 = \frac{2}{x-y}, \quad \beta_4 = -\frac{2}{x-y},$$
$$\alpha_2 = \frac{6y(x+y)}{(x-y)(x^2+4xy+y^2)}, \quad \beta_2 = -\frac{6x(x+y)}{(x-y)(x^2+4xy+y^2)}.$$

These coefficients all satisfy $\alpha_y = \beta_x$ and therefore all the Pfaffian system $S_i^{[1]}$ quotient to same(!) Pfaffian systems $\mathcal{I}$, defined by (24) *(with u replaced by z) via the projection map $z = u - v$.*

8 Goursat's Equation

Goursat ([15], [18]) showed that the non-linear equation

$$u_{xy} = 2A(x, y)\sqrt{u_x u_y} \tag{54}$$

can be linearized to the first order system

$$P_x = AQ, \quad Q_y = AP \tag{55}$$

by setting

$$u_x = P^2 \qquad u_y = Q^2, \tag{56}$$

or, alternatively, to the second order linear equation

$$v_{xy} - \frac{A_x}{2A} v_y - Av = 0 \tag{57}$$

by setting $v_x = u^2$.

In this example, we shall use Principle C to determine the general form of the quotient representation for Darboux integrable systems of the type (54) and we shall study, in some detail, the special case $A = \dfrac{n}{x+y}$.

Let $\mathcal{G}_A$ be the canonical rank 3 Pfaffian system (on a seven dimensional manifold) for (54) and let $\mathcal{S}_A$ the canonical rank 2 Pfaffian system (on a six dimensional manifold) for (55). Then (56) defines a map $\pi : \mathcal{G}_A \to \mathcal{S}_A$ for which $\mathcal{G}_A$ is an integrable extension of $\mathcal{S}_A$. Principles A and C show that (54) is Darboux integrable at some prolonged order whenever (55) is Darboux integrable (prolonged to the same order).

We know from Sect. 7 that the linear system (55) is Darboux integrable if it is the quotient of jet spaces $\mathcal{C}^m \times \mathcal{C}^n$ by an Abelian Lie group G of dimension $m + n - 2$, acting freely. We then infer, again by Principle C, that (54) will be the quotient of Pfaffian systems $\mathcal{D}^{m+1} \times \mathcal{E}^{n+1}$ by a Lie group $\tilde{G}$, where $\mathcal{D}^{m+1}$ is a 1 dimensional integrable extension of $\mathcal{C}^m$ and $\mathcal{E}^{n+1}$ is a 1 dimensional integrable extension of $\mathcal{C}^n$. Moreover, there are projection maps $\pi_1 : \mathcal{D}^{m+1} \to \mathcal{C}^m$ and $\pi_2 : \mathcal{E}^{n+1} \to \mathcal{C}^n$ such that the diagram

$$\begin{array}{ccc} \mathcal{D}^{m+1} + \mathcal{E}^{n+1} & \xrightarrow{\mathbf{q}_{\tilde{G}}} & \mathcal{G}_A \\ \pi_1 \times \pi_2 \Big\downarrow & & \Big\downarrow \pi \\ \mathcal{C}^m + \mathcal{C}^n & \xrightarrow{\mathbf{q}_G} & \mathcal{S}_A \end{array} \tag{58}$$

commutes.

Principle C also implies that $\tilde{G}$ is a 1-step solvable Lie group of dimension $n + m - 1$.

If we denote the fiber coordinate for the projection map π_1 by σ, then generators for the action of $\tilde{G}$ on $\mathcal{D}^{m+1}$ may be taken to be of the form

$$W = \frac{\partial}{\partial \sigma} \quad \text{and} \quad \tilde{X}_i = f_i(x)\frac{\partial}{\partial \phi} + f_i'(x)\frac{\partial}{\partial \phi} + \cdots + f_i^{(m)}(x)\frac{\partial}{\partial \phi_m} + \xi_i \frac{\partial}{\partial \sigma}, \tag{59}$$

where $\xi_i = \xi_i(x, \phi, \phi_1, \ldots, \phi_{m-1}, \sigma)$. The integrable extension $\mathcal{D}^{m+1}$ can be written as

$$\mathcal{D}^{m+1} = \mathcal{C}^m \cup \{d\sigma - H(x, \phi, \phi_1, \ldots, \phi_{m-1})\, dx\}.$$

Theorem 8.1 *The Goursat equation*

$$u_{xy} = 2A(x, y)\sqrt{u_x u_y}$$

is Darboux integrable if and only if it is the quotient of a pair of Monge equations

$$\frac{d\sigma}{dx} = H(x, \phi, \frac{d\phi}{dx}, \frac{d^2\phi}{dx^2} \ldots)$$

by a product action of an Abelian or 1-step solvable group with infinitesimal generators (59).

Just as with the case of linear equations (Sect. 5), the functions $f_i(x)$ can be prescribed arbitrarily. The functions H and ξ_i can be determined directly from the symmetry condition

$$Z_i(H) = D_x(\xi_i) \quad \text{which implies that} \quad E(Z_i(H)) = Z_i(E(H)) = 0, \tag{60}$$

where E is the Euler–Lagrange operator for the variable ϕ. This over-determined system of equations can be used to first determine H and then the coefficients ξ_i (independent of the method contained in the proof of Principle C).

The special case $A = \dfrac{2n}{x+y}$, corresponding to the choice of functions $f_i = x^i$, is easily solved.

Theorem 8.2 *The standard differential system $\mathcal{G}_n$ for the Goursat system $u_{xy} = \dfrac{2n}{x+y}\sqrt{u_x u_y}$ is the quotient differential system*

$$\mathcal{G}_n = \left(\mathcal{H}^{n+1,[1]} + \mathcal{H}^{n+1,[1]}\right)/G_{2n+1}, \tag{61}$$

where $\mathcal{H}^{n+1}$ is the rank $n+1$ Pfaffian system for the generalized Hilbert-Cartan equation

$$\frac{d\sigma}{dx} = \left[\frac{d^n\phi}{dx^n}\right]^2, \tag{62}$$

$\mathcal{H}^{n+1,[1]}$ is the first prolongation of $\mathcal{H}^{n+1}$, and G_{2n+1} is the $2n+1$dimensional, 1 step nilpotent Lie group with infinitesimal generators (59) for $f_i = x^i$, $i = 0 \ldots 2n-1$.

The Monge system (62) enjoys a number of remarkable properties which we describe in the Sect. 9.

As in Sect. 4, the explicit formulas for the quotient maps $\mathbf{q}_n \colon \mathcal{H}^{n+1,[1]} + \mathcal{H}^{n+1,[1]} \to \mathcal{G}_n$ are determined by the prolongation of the lowest order joint differential invariants for the diagonal actions of the group G_{2n+1}. Let $\gamma = x + y$, $\Phi_k = \gamma^k \phi_k$, and $\Psi_k = \gamma^k \psi_k$. Then, for $n = 1$, the infinitesimal generators are $\{\, \partial_\sigma, \partial_\phi, t\partial_\phi + \partial_{\phi_1} + 2\phi\partial_\sigma \,\}$ and the joint invariant is

$$u_1 = \sigma_1 + \tau_1 + \frac{1}{\gamma}\begin{bmatrix}\Phi_0\\ \Psi_0\end{bmatrix}^t \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}\Phi_0\\ \Psi_0\end{bmatrix}; \tag{63}$$

for $n = 2$ the infinitesimal generators are vector fields X_i^+, $i = 2, \ldots, 6$ (see (74)) and the joint invariant is

$$u_2 = \sigma_2 + \tau_2 + \frac{1}{\gamma^3}\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Psi_0\\ \Psi_1\end{bmatrix}^t \begin{bmatrix}-12 & 6 & -12 & 6\\ 6 & -4 & 6 & -2\\ -12 & 6 & -12 & 6\\ 6 & -2 & 6 & -4\end{bmatrix}\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Psi_0\\ \Psi_1\end{bmatrix}; \tag{64}$$

for $n = 3$ the generators are $\{W, X_0, X_1, X_2, X_3, X_4, X_5\}$ (see (76)) and the joint invariant is

$$u_3 = \sigma_3 + \tau_3 + \frac{1}{\gamma^5}\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\\ \Psi_0\\ \Psi_1\\ \Psi_2\end{bmatrix}^t \begin{bmatrix} -720 & 360 & -60 & -720 & 360 & -60\\ 360 & -192 & 36 & 360 & -168 & 24\\ -60 & 36 & -9 & -60 & 24 & -3\\ -720 & 360 & -60 & -720 & 360 & -60\\ 360 & -168 & 24 & 360 & -192 & 36\\ -60 & 24 & -3 & -60 & 36 & -9 \end{bmatrix}\begin{bmatrix}\Phi_0\\ \Phi_1\\ \Phi_2\\ \Psi_0\\ \Psi_1\\ \Psi_2\end{bmatrix}. \tag{65}$$

We can use Theorem 8.2 to establish, in a rather novel fashion a connection between the systems $\mathcal{G}_{n+1}$ and $\mathcal{G}_n$. We start with the simple observation that the transformation $\phi = \Phi'$ defines the equation $\sigma' = [\phi^{(n)}]^2$ as the quotient of $\sigma' = [\Phi^{(n+1)}]^2$ by the 1 dimensional group L with generator ∂_{Φ_0}. This, together with Theorem 8.2, then yields

$$\begin{array}{ccc} & & \mathcal{H}^{n+1,[1]} + \mathcal{H}^{n+1,[1]} \\ & \mathbf{q}_L \times \mathbf{q}_L \nearrow & \Big\downarrow \mathbf{q}_{G_{2n+1}} \\ \mathcal{H}^{n+2,[1]} + \mathcal{H}^{n+2,[1]} & \xrightarrow[\pi_{n+3}]{} & \mathcal{G}_n \end{array} \quad . \tag{66}$$

From this diagram and (61) we arrive at

$$\begin{array}{ccc} & \mathcal{H}^{n+2,[1]} + \mathcal{H}^{n+2,[1]} & \\ \mathbf{q}_{G_{2n+3}} \swarrow & & \searrow \pi_{n+3} \\ \mathcal{G}_{n+1} & & \mathcal{G}_n \end{array} \quad . \tag{67}$$

Now, because G_{2n+3} is a solvable group, every solution or integral manifold for $\mathcal{G}_{n+1}$ determines, by quadratures, an integral manifold to $\mathcal{H}^{n+2,[1]} \times \mathcal{H}^{n+2,[1]}$ (see [1], Theorem 6.2) which then projects under π_{n+3} to an integral manifold for $\mathcal{G}_n$. Because the projection map π_{n+3} is invariant with respect to the flows of all the generators of G_{2n+3} except the last one, the explicit formulas for this construction turn out to be remarkably simple.

Theorem 8.3 *If* $U(x, y)$ *solves* $U_{xy} = \dfrac{2(n+1)}{(x+y)}\sqrt{U_x U_y}$ *and* $\lambda(x, y)$ *solves*

$$\lambda_x = \frac{(2n+1)\sqrt{U_x}}{(x+y)^{n+1}}, \quad \lambda_y = -\frac{(2n+1)\sqrt{U_y}}{(x+y)^{n+1}} \tag{68}$$

then

$$V(x, y) = \frac{(x + y)^{2n}}{2n + 1}\lambda^2 + U(x, y) \tag{69}$$

solves the equation $V_{xy} = \dfrac{2n}{(x + y)}\sqrt{V_x V_y}$.

This final theorem suggests the very intriguing possibility of adapting these Lie group theoretic methods towards the construction of Bäcklund transformations between various Darboux integrable systems ([10], [29]).

9 The Monge Equations $\sigma' = [\phi^{(n)}]^2$

Here we review some of the basic properties for the rank $n+1$ Pfafian systems $\mathcal{H}^{n+1}$ defined by the Monge equation $\sigma' = [\phi^{(n)}]^2$. For $n = 1$, the Pfaffian system

$$\mathcal{H}^2 = \{\, d\phi - \phi_1 dx,\ d\sigma - \phi_1^2 dx\,\} \tag{70}$$

has derived flag dimensions [2, 1, 0] and is therefore contact equivalent, by Engel's theorem (see, for example, [5], p. 50), to

$$\mathcal{C}^2(\mathbf{R}, \mathbf{R}) = \{\, d\Phi - \Phi_1 dX,\ d\Phi_1 - \Phi_2 dX\,\}. \tag{71}$$

An explicit equivalence is given by

$$x = \Phi_2, \quad \phi = X\Phi_2 - \Phi_1, \quad \phi_1 = X, \quad \sigma = X^2\Phi_2 - 2X\Phi_1 + 2\Phi. \tag{72}$$

For $n = 2$, the Pfaffian system

$$\mathcal{H}^3 = \{\, d\phi - \phi_1 dx,\ d\phi_1 - \phi_2 dx,\ d\sigma - \phi_2^2 dx\,\} \tag{73}$$

has derived flag dimensions [3, 2, 0] (the generic flag dimensions) and, amongst all generic rank 3 Pfaffian systems in five variables, has the symmetry algebra of largest dimension, namely, the real split form of the exceptional Lie algebra $\mathfrak{g}_2$. The generators for this symmetry algebra are:

$$H_1 = 2x\partial_x + 3\phi\partial_\phi + \phi_1\partial_{\phi_1} - \phi_2\partial_{\phi_2}, \quad H_2 = -(\phi\partial_\phi + \phi_1\partial_{\phi_1} + \phi_2\partial_{\phi_2} + 2\sigma\partial_\sigma),$$
$$X_1^+ = \frac{1}{2}x^2\partial_x + \frac{3}{2}\phi x\partial_\phi + (\frac{3}{2}\phi + \frac{1}{2}x\phi_1)\partial_{\phi_1} + (2\phi_1 - \frac{1}{2}\phi_2 x)\partial_{\phi_2} + 2\phi_1^2\partial_\sigma,$$
$$X_2^+ = \partial_\phi, \quad X_3^+ = x\partial_\phi + \partial_{\phi_1}, \quad X_4^+ = \frac{1}{2}x^2\partial_\phi + x\partial_{\phi_1} + \partial_{\phi_2} + 2\phi_1\partial_\sigma,$$
$$X_5^+ = \frac{1}{6}x^3\partial_\phi + \frac{1}{2}x^2\partial_{\phi_1} + x\partial_{\phi_2} + (2x\phi_1 - 2\phi)\partial_\sigma, \quad X_6^+ = \partial_\sigma, \quad X_1^- = \partial_x,$$

$$
\begin{aligned}
X_2^- &= (\frac{4}{3}\phi_1 x^2 - 2\phi x - \frac{1}{3}\phi_2 x^3)\partial_x + (\frac{1}{6}x^3\sigma + \frac{2}{3}\phi_1^2 x^2 - 2\phi^2 - \frac{1}{3}x^3\phi_2\phi_1)\partial_\phi, \\
&+(\frac{1}{2}x^2\sigma + \frac{2}{3}\phi_1^2 x - 2\phi\phi_1 - \frac{1}{6}\phi_2^2 x^3)\partial_{\phi_1} + (\sigma x - \frac{4}{3}\phi_1^2 + \frac{2}{3}x\phi_1\phi_2 - \frac{1}{3}\phi_2^2 x^2)\partial_{\phi_2} \\
&+(2\sigma x\phi_1 - 2\sigma\phi - \frac{1}{9}x^3\phi_2^3 - \frac{8}{9}\phi_1^3)\partial_\sigma, \\
X_3^- &= (\frac{8}{3}x\phi_1 - 2\phi - \phi_2 x^2)\partial_x + (\frac{1}{2}x^2\sigma + \frac{4}{3}\phi_1^2 x - x^2\phi_2\phi_1)\partial_\phi \\
&+(\sigma x + \frac{2}{3}\phi_1^2 - \frac{1}{2}\phi_2^2 x^2)\partial_{\phi_1} + (\sigma + \frac{2}{3}\phi_2\phi_1 - \frac{2}{3}\phi_2^2 x)\partial_{\phi_2} + (2\phi_1\sigma - \frac{1}{3}\phi_2^3 x^2)\partial_\sigma, \\
X_4^- &= (\frac{8}{3}\phi_1 - 2\phi_2 x)\partial_x + (\sigma x + \frac{4}{3}\phi_1^2 - 2x\phi_1\phi_2)\partial_\phi + (\sigma - \phi_2^2 x)\partial_{\phi_1} \\
&-\frac{2}{3}\phi_2^2\partial_{\phi_2} - \frac{2}{3}x\phi_2^3\partial_\sigma, \\
X_5^- &= -2\phi_2\partial_x + (\sigma - 2\phi_2\phi_1)\partial_\phi - \phi_2^2\partial_{\phi_1} - \frac{2}{3}\phi_2^3\partial_\sigma, \\
X_6^- &= (\frac{2}{3}\phi_1^2 - \phi\phi_2)\partial_x + (\frac{1}{2}\sigma\phi + \frac{4}{9}\phi_1^3 - \phi_2\phi\phi_1)\partial_\phi + (\frac{1}{2}\phi_1\sigma - \frac{1}{2}\phi\phi_2^2)\partial_{\phi_1} \\
&+(-\frac{1}{3}\phi_1\phi_2^2 + \frac{1}{2}\sigma\phi_2)\partial_{\phi_2} + (\frac{1}{2}\sigma^2 - \frac{1}{3}\phi\phi_2^3)\partial_\sigma.
\end{aligned}
\tag{74}
$$

The vector fields X_i^+ have positive weight, the vectors H_1, H_2 define a Cartan subalgebra for $\mathfrak{g}_2$, and the vectors X_i^- have negative weight.

For $n = 3$, the Pfaffian system

$$
\mathcal{H}^4 = \{\, d\phi - \phi_1\, dx,\ d\phi_1 - \phi_2\, dx,\ d\phi_2 - \phi_2\, dx,\ d\sigma - \phi_3^2\, dx \,\} \tag{75}
$$

has derived flag dimensions [4, 3, 1, 0] and, amongst all rank 4 Pfaffian systems in six variables with such derived flag dimensions, has the symmetry algebra of largest dimension. In this case the symmetry algebra has Levi decomposition $\mathfrak{sl}(2) \ltimes \mathfrak{r}$, where the radical $\mathfrak{r}$ is an eight dimensional solvable algebra. The explicit formulas for this algebra are:

$$
\begin{aligned}
&W = \partial_\sigma, \quad X_0 = \partial_\phi, \quad X_1 = x\,\partial_\phi + \partial_{\phi_1}, \quad X_2 = \frac{1}{2}x^2\,\partial_\phi + x\,\partial_{\phi_1} + \partial_{\phi_2}, \\
&X_3 = \frac{1}{6}x^3\,\partial_\phi + \frac{1}{2}x^2\,\partial_{\phi_1} + x\,\partial_{\phi_2} + \partial_{\phi_3} + 2\phi_2\,\partial_\sigma, \\
&X_4 = \frac{1}{24}x^4\,\partial_\phi + \frac{1}{6}x^3\,\partial_{\phi_1} + \frac{1}{2}x^2\,\partial_{\phi_2} + x\,\partial_{\phi_3} + (-2\phi_1 + 2x\phi_2)\,\partial_\sigma, \\
&X_5 = \frac{1}{120}x^5\partial_\phi + \frac{1}{24}x^4\partial_{\phi_1} + \frac{1}{6}x^3\,\partial_{\phi_2} + \frac{1}{2}x^2\,\partial_{\phi_3} + (2\phi - 2x\phi_1 + x^2\phi_2)\,\partial_\sigma \\
&R = \phi\,\partial_\phi + \phi_1\,\partial_{\phi_1} + \phi_2\,\partial_{\phi_2} + \phi_3\,\partial_{\phi_3} + 2\sigma\,\partial_\sigma, \quad S_0 = \partial_x, \\
&S_1 = 2x\partial_x + 5\phi\,\partial_\phi + 3\phi_1\,\partial_{\phi_1} + \phi_2\partial_{\phi_2} - \phi_3\,\partial_{\phi_3}, \quad S_2 = x^2\,\partial_x + 5x\phi\,\partial_\phi \\
&+(5\phi + 3x\phi_1)\,\partial_{\phi_1} + (8\phi_1 + x\phi_2)\partial_{\phi_2} + (9\phi_2 - x\phi_3)\,\partial_{\phi_3} + 9\phi_2^2\,\partial_\sigma.
\end{aligned}
\tag{76}
$$

The vector fields S_0, S_1, S_2 define the semi-simple part, the nilradical is the seven dimensional 1 step nilpotent subalgebra given by $\{W, X_0, X_1, \ldots, X_5\}$.

For $n \geq 3$ this pattern persists. The derived flag for $\mathcal{H}^{n+1}$ is $[n, n-1, n-3, n-4, n-5, \ldots]$ and the symmetry algebra is $\mathfrak{sl}(2) \ltimes \mathfrak{r}$, where the radical has dimension $2n+2$. In all cases the nilradical is a 1 step nilpotent algebra of dimension $2n+1$.

The Pfaffian systems $\mathcal{H}^{n+1}$ are also the canonical (flat) models in the Tanaka theory associated to the unique $2n+1$ graded nilpotent Lie algebras with grading $[2, 1, 2, 1, 1, \ldots]$. See Doubrov and Zelenko [12], Theorem 3.

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Differential Geometric Heuristics for Riemannian Optimal Mass Transportation

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Abstract We give an account on Otto's geometrical heuristics for realizing, on a compact Riemannian manifold M, the L^2 Wasserstein distance restricted to smooth positive probability measures, as a Riemannian distance. The Hilbertian metric discovered by Otto is obtained as the base metric of a Riemannian submersion with total space, the group of diffeomorphisms of M equipped with the Arnol'd metric, and projection, the push-forward of a reference probability measure. The expression of the horizontal constant speed geodesics (time dependent optimal mass transportation maps) is derived using the Riemannian geometry of M as a guide.

1 Optimal Mass Transportation Diffeomorphims

Let M be a compact connected n-dimensional manifold (all objects are C^∞ unless otherwise specified; so, a measure admits a smooth density in each chart). We may view measures as n-forms of odd type [dRh55], hence freely consider the pull-back measure $\phi^*\nu$ of a measure ν by a map $\phi : M \to M$. Pulling-back does not preserve the total mass: $\int_M d(\phi^*\nu) \not\equiv \int_M d\nu$, unless ϕ is a diffeomorphism. This is in contrast with the push-forward (also called *transport*) of a measure μ by a map $\phi : M \to M$, denoted by $\phi_\# \mu$, which may be defined (via the Riesz representation theorem [Rie09]) by:

$$\int_M u \, d\nu := \int_M (u \circ \phi) \, d\mu, \text{ with } \nu = \phi_\# \mu, \tag{1}$$

where u stands for an arbitrary continuous real function on M. Here, the measure $\phi_\# \mu$ is not necessarily smooth (even though μ and ϕ are), but it certainly is if ϕ is a diffeomorphism (if so, exercise: check that $\phi_\# \mu = (\phi^{-1})^* \mu$). In any case, the

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total mass is preserved (letting $u = 1$ in (1)). In the sequel, we normalize the total mass equal to 1, all maps from M to itself are diffeomorphisms and we restrict the transport to (smooth) positive probability measures, the set of which we denote by Prob. The latter is a convex domain in an *affine* space modelled on the Fréchet space Mes_0 of measures with zero average on M. In particular, we will freely use the fact that the tangent bundle $T\,\mathrm{Prob}$ is trivial, equal to $\mathrm{Prob} \times \mathrm{Mes}_0$. As readily checked, the transport yields a *right action* on Prob of the group of diffeomorphisms of M. From now on, we endow the manifold M with a Riemannian metric g:

Question Q: using the metric g and the above right action, how can one find good notions of distance and shortest path in Prob?

Given measures $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$, optimal transport theory provides an answer which we now describe. First of all, a criterion of optimality is required. Following Brenier and McCann [Bre91, McC01], it is defined by choosing the *cost-function* given by: $\forall (p, q) \in M \times M$, $c(p,q) := \frac{1}{2} d_g^2(p,q)$, where d_g stands for the geodesic distance in M, and by looking for a minimizer of the total transport cost functional:

$$C_\mu(\phi) := \int_M \frac{1}{2} d_g^2(m, \phi(m))\, d\mu\ ,$$

among all Borel maps $\phi : M \to M$ satisfying $\phi_\# \mu = \nu$. Such a minimization problem is called a Monge's problem, after Gaspard Monge who was the first to consider such a problem, in the Euclidean space for the total work functional $\int_{\mathbb{R}^n} |x - \phi(x)| d\mu$ [Mon81].

An essential tool for solving a Monge's problem is the notion of c-convexity. Dropping temporarily smoothness, a real function f on M is called c-convex on M if it can be written $f = h^c$ for some real function h, where:

$$\forall m \in M, \quad h^c(m) := \sup_{p \in M} [-h(p) - c(m, p)],$$

and $c = \frac{1}{2} d_g^2$. If so, f is Lipschitz, thus differentiable outside a subset $S \subset M$ of zero Riemannian volume measure (Rademacher's theorem); moreover, the gradient $\nabla f : M \backslash S \to TM$ is Borel measurable [McC01]. The map $h \mapsto h^c$ is often called the c-transform on M [CMS01] (thought of as a kind of Legendre transform) and a c-convex function $f = h^c$ satisfies the involution identity: $f = (f^c)^c$ [R-R98].
Setting $\exp : TM \to M$ for the Riemannian exponential map, we can now state the main result of the landmark paper [McC01]:

Theorem 1 (McCann). *Given* $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$*, there exists a c-convex function* $f : M \to \mathbb{R}$*, unique up to addition of a constant, which satisfies the equation:*

$$\exp(\nabla f)_\# \mu = \nu\ . \tag{2}$$

Moreover, the Borel map $\exp(\nabla f) : M \to M$ *is the unique minimizer for our Monge's problem (modulo discrepancies on a subset of zero* μ*-measure).*

The quantity

$$\mathrm{W}_2(\mu, \nu) := \sqrt{C_\mu(\exp(\nabla f))} \equiv \sqrt{\int_M \frac{1}{2}|\nabla f|^2 \, d\mu}$$

with μ, ν, f as in Theorem 1 defines a *distance* in Prob [Vil08, Chap. 6] (see also [Vil03] and Theorem 3 below); let Prob_2 denote the completion of Prob for this distance. The complete metric space $(\mathrm{Prob}_2, \mathrm{W}_2)$ is called the L^2 Wasserstein space associated to (M, g), and W_2, the Wasserstein distance. The following star-shapedness property holds:

Lemma 1 ([CMS01] Lemma 5.1). *For each* $t \in [0, 1]$*, the function* tf *is c-convex on* M *if* f *is so.*

With Lemma 1 at hand, we infer from Theorem 1 that the path given by:

$$t \in [0, 1] \to \mu_t := \exp(t\nabla f)_\# \mu \in \mathrm{Prob}_2 \tag{3}$$

is W_2-*minimizing* from $\mu_0 = \mu$ to $\mu_1 = \nu$. We thus have got an answer to Question Q, except for the smoothness of the measures μ_t for $t \in (0, 1)$.

Indeed, the smoothness of the data (M, μ, ν, g) does not always imply that of the optimal transport map given by Theorem 1. Recently, this question has been intensively investigated (see [Vil08, Chap. 12] and references therein). However, anytime the given measures μ and ν are close enough[1] in Prob, the c-convex solution of (2) must be smooth [Del04, Theorem 1]. By combining Theorem 1 with Theorem 5 of Appendix 1 below, we can state a result in the smooth category, namely:

Theorem 2. *Given* $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$*, assume the existence of a smooth solution* f *of the partial differential equation:*

$$\exp(\nabla f)^* \nu = \mu \, . \tag{4}$$

The function f *must be c-convex on* M *and satisfy (2). Moreover, the path (3) ranges in* Prob *and, for each* $t \in [0, 1]$*, the map* $\exp(t\nabla f)$ *is a diffeomorphism.*

At this stage, the reader may not realize how natural, from the Riemannian geometric viewpoint, are the answers to Question Q given by the two preceding theorems. The goal of this paper is to convince ourselves that they are, indeed, completely natural. To do so, we give below an exhaustive account on the beautiful heuristics discovered by Félix Otto [Ott01] (see also [Lot08, K-L08]). We will

[1] in Fréchet topology, of course (cf. supra)

proceed stepwise, in a pretty self-contained way,[2] working mostly in the group of diffeomorphisms of M rather than in Prob, with elementary tools from (finite-dimensional) Riemannian geometry and Poisson's type equations. Hopefully, it will serve as a complement to John Lott's recent paper [Lot08] written in the spirit of infinite-dimensional calculations performed straight in Prob. It will also prepare the reader for further studies e.g. in the sub-Riemannian setting [K-L08].

Finally, as regards the geometry of equation (4), we would like to mention that (4) admits a (non-homogeneous) Monge–Ampère structure in Lychagin's sense [Lyc79] hence Lie solutions [Del08] which would deserve a deeper study.

2 Geometry of the Group of Diffeomorphisms, After Arnol'd

Henceforth, we set Diff for the group of diffeomorphisms of the manifold M and, fixing $\lambda \in \mathrm{Prob}$, we single out the subgroup Diff_λ of diffeomorphisms which preserve the reference measure λ (pushing it to itself).

2.1 Rearrangement Classes

Let us consider the map $\mathcal{P}_\lambda : \mathrm{Diff} \to \mathrm{Prob}$ defined by

$$\forall \phi \in \mathrm{Diff}, \quad \mathcal{P}_\lambda(\phi) := \phi_\# \lambda \ .$$

It yields a partition of Diff into countersets:

$$\mathrm{c}_\mu := \{\phi \in \mathrm{Diff},\ \phi_\# \lambda = \mu\}, \quad \mu \in \mathcal{P}_\lambda(\mathrm{Diff}),$$

including the one which contains $\mathbb{I}$ (the identity of M), namely $\mathrm{c}_\lambda \equiv \mathrm{Diff}_\lambda$. Moreover, two diffeomorphisms ϕ and ψ lie in the same counterset c_μ if and only if:

$$\exists \xi \in \mathrm{Diff}_\lambda, \quad \phi = \psi \circ \xi \ . \tag{5}$$

In other words, letting Diff_λ act on Diff by right composition and considering (5) as an equivalence relation, we have for the quotient space:

$$\mathrm{Diff} \,/\, \mathrm{Diff}_\lambda = \{\mathrm{c}_\mu,\ \mu \in \mathcal{P}_\lambda(\mathrm{Diff})\};$$

each counterset c_μ may thus be viewed as a coset, called by Brenier [Bre91] a *rearrangement class*.

---

[2] except for the last part of Appendix 1.

2.2 Tangent Bundle

For $t \in \mathbb{R}$ close to 0, let $t \mapsto \psi_t \in \mathrm{Diff}$ be a path satisfying $\psi_0 = \mathbb{I}$. On the one hand $\frac{d\psi_t}{dt}|_{t=0}$ lies in the tangent space $T_{\mathbb{I}}\,\mathrm{Diff}$, on the other hand we have:

$$\forall m \in M, \quad \frac{d\psi_t(m)}{dt}|_{t=0} \in T_m M \ .$$

So $T_{\mathbb{I}}\,\mathrm{Diff}$ coincides with the vector fields on M, a Fréchet space henceforth denoted by Vec.

Fixing an arbitrary $\phi \in \mathrm{Diff}$, let $t \mapsto \phi_t$ be a path satisfying $\phi_0 = \phi$. How can we view the tangent vector $\frac{d\phi_t}{dt}|_{t=0} \in T_\phi\,\mathrm{Diff}$? Sticking to the right composition, we may write $\phi_t = \psi_t \circ \phi$ with ψ_t as above, getting:

$$\frac{d\phi_t}{dt}|_{t=0} = \frac{d\psi_t}{dt}|_{t=0} \circ \phi \ .$$

We conclude:

$$T_\phi\,\mathrm{Diff} = \{V \circ \phi, \ V \in \mathrm{Vec}\} \ . \tag{6}$$

2.3 The Arnol'd Metric

Following Arnol'd [Arn66], let us define on the tangent bundle $T\,\mathrm{Diff}$ the following field of Hilbertian scalar products:

$$\forall \phi \in \mathrm{Diff}, \ \forall (V, W) \in \mathrm{Vec}^2, \quad \langle V \circ \phi, W \circ \phi \rangle_\phi := \int_M \frac{1}{2} g(V \circ \phi, W \circ \phi)\, d\lambda \ .$$

Observing that

$$\phi \in \mathrm{c}_\mu \Longrightarrow \langle V \circ \phi, W \circ \phi \rangle_\phi = \int_M \frac{1}{2} g(V, W)\, d\mu \ ,$$

we infer that the Arnol'd metric is right-invariant along each rearrangement class $\mathrm{c}_\mu \in \mathrm{Diff}/\mathrm{Diff}_\lambda$ (originally, Arnol'd restricted it to Diff_λ with the idea that the resulting geodesics would describe the motion of an incompressible fluid in the manifold M, see [Arn66, E-M70]).

Using the Arnol'd metric, we can define the length of paths in Diff, hence a *distance* on Diff; let us denote it by d_A. Given $(\phi, \psi) \in \mathrm{Diff}^2$, we thus have:

$$\mathrm{d}_A(\phi, \psi) = \inf \int_0^1 \sqrt{\left\langle \frac{d\phi_t}{dt}, \frac{d\phi_t}{dt} \right\rangle_{\phi_t}}\, dt \ ,$$

where the infimum runs over all paths $t \in [0, 1] \to \phi_t \in \mathrm{Diff}$ such that $\phi_0 = \phi$ and $\phi_1 = \psi$.

3 The Riemannian Submersion $\mathcal{P}_\lambda$: Diff → Prob, after Moser, Ebin–Marsden and Otto

With the view of improving the way of solving some nonlinear heat equations, Félix Otto (working in $\mathbb{R}^n$) [Ott01] advocated the use of a new gradient flow on Prob which he had the idea to construct with a metric inherited from the Arnol'd one via the projection $\mathcal{P}_\lambda$. In the present section, we implement the latter idea stepwise. The reader will find in [K-L08] a parallel theory outlined for the sub-Riemannian case.

We require notations: Funct will denote the Fréchet space of smooth real-valued functions on M, and for each $\mu \in$ Prob, Funct_0^μ will denote the subspace of functions $f \in$ Funct such that $f\mu \in \text{Mes}_0$. Auxiliary material for this section may be found in Appendix 2.

3.1 The Submersion

The first step is Moser's famous result on volume forms [Mos65].

Proposition 1 (Moser). *The map* $\mathcal{P}_\lambda$: Diff → Prob *is onto.*

Proof. Following [Mos65, E-M70], let us construct a right-inverse for the map $\mathcal{P}_\lambda$. Given an arbitrary $\mu \in$ Prob, consider the linear interpolation path $t \in [0,1] \to \mu_t := t\lambda + (1-t)\mu \in$ Prob. By Corollary 5 of Appendix 2, for each $t \in [0,1]$, there exists a unique $f_t \in \text{Funct}_0^{\mu_t}$ solving the equation:

$$\text{div}_{\mu_t}(\nabla f_t)\, \mu_t = -\frac{d\mu_t}{dt}\,. \tag{7}$$

The map $t \mapsto f_t$ is smooth and, from (7), the flow $t \in [0,1] \to \phi_t \in$ Diff of the time-dependent vector field ∇f_t on M satisfies:

$$\frac{d}{dt}\left(\phi_{t\#}\mu_t\right) = 0, \quad \phi_0 = \mathbb{I}\,.$$

We thus have $\phi_{t\#}\mu_t = \mu_0$ hence, in particular: $\mathcal{P}_\lambda(\phi_1) = \mu$ □

Let us denote by $\mathcal{M}_\lambda(\mu)$ the diffeomorphism ϕ_1 just constructed. The map $\mathcal{M}_\lambda$: Prob → Diff is a right-inverse for $\mathcal{P}_\lambda$ such that $\mathcal{M}_\lambda(\lambda) = \mathbb{I}$.

Corollary 1 (Ebin–Marsden). *The map* $\mathcal{P}_\lambda$: Diff → Prob *is a submersion.*

Proof. As observed in [E-M70], the map $\mathcal{M}_\lambda$ yields a *factorization* of Diff; specifically, setting for each $\phi \in$ Diff,

$$\text{D}_\lambda(\phi) := [\mathcal{M}_\lambda\,(\mathcal{P}_\lambda(\phi))]^{-1} \circ \phi\,,$$

and recalling (5), we get a map $\mathrm{D}_\lambda : \mathrm{Diff} \to \mathrm{Diff}_\lambda$ such that the following factorization identically holds in Diff:

$$\phi \equiv \mathcal{M}_\lambda\,(\mathcal{P}_\lambda(\phi)) \circ \mathrm{D}_\lambda(\phi)\ .$$

In other words, as do Ebin and Marsden, we may declare that the map:

$$\phi \in \mathrm{Diff} \to (\mathcal{P}_\lambda(\phi), \mathrm{D}_\lambda(\phi)) \in \mathrm{Prob} \times \mathrm{Diff}_\lambda$$

is a diffeomorphism (global and onto). The latter makes the map $\mathcal{P}_\lambda$ read merely like a projection; so, indeed, it is a submersion. □

Proposition 2. *The tangent map to $\mathcal{P}_\lambda$ is onto with direct kernel.*

Proof. A straightforward calculation, using (6) and Definition 1 of Appendix 2, yields for the tangent map to $\mathcal{P}_\lambda$ the following important expression:

$$\forall \mu \in \mathrm{Prob},\ \forall \phi \in \mathrm{c}_\mu,\ \forall V \in \mathrm{Vec}, \quad T_\phi \mathcal{P}_\lambda(V \circ \phi) = \mathrm{div}_\mu(V)\mu\ . \tag{8}$$

Combining it with Corollaries 5 and 6 of Appendix 2 yields the proposition (for the notion of *direct* factor, see e.g. [Lan62]). □

3.2 Helmholtz Splitting

From Proposition 1 and Corollary 1, for each $\mu \in \mathrm{Prob}$, we have $\mathcal{P}_\lambda^{-1}(\mu) = \mathrm{c}_\mu$ and this fiber is a *submanifold* of Diff diffeomorphic to Diff_λ. Given $\phi \in \mathrm{c}_\mu$, let us identify the (so-called *vertical*) subspace $T_\phi \mathrm{c}_\mu$ of $T_\phi\,\mathrm{Diff}$. Pick a path $t \mapsto \phi_t \in \mathrm{c}_\mu$ with $\phi_0 = \phi$ and differentiate with respect to t at $t = 0$ the identity: $\mathcal{P}_\lambda(\phi_t) = \mu$. Recalling (8), we get the equation $\mathrm{div}_\mu(V) = 0$ satisfied by the vector field V such that $V \circ \phi = \frac{d\phi_t}{dt}|_{t=0}$. In other words, we have:

$$T_\phi \mathrm{c}_\mu = \left\{ V \circ \phi,\ V \in \mathrm{Ker\,div}_\mu \right\}. \tag{9}$$

Regarding the orthogonal complement of $T_\phi \mathrm{c}_\mu$ in $T_\phi\,\mathrm{Diff}$ for the Arnol'd metric, the so-called *horizontal* subspace at ϕ, we can write from the definition of the Arnol'd metric and (9):

$$\forall W \in \mathrm{Vec},\ (W \circ \phi) \in T_\phi \mathrm{c}_\mu^\perp \iff \forall V \in \mathrm{Ker\,div}_\mu,\ \int_M g(V, W)d\mu = 0\ .$$

By Corollary 6 of Appendix 2 (Helmholtz decomposition), we conclude:

$$\forall W \in \mathrm{Vec},\ (W \circ \phi) \in T_\phi \mathrm{c}_\mu^\perp \iff \exists f \in \mathrm{Funct}_0^\mu,\ W = \nabla f\ .$$

Setting $\mathcal{V}_\phi$ and $\mathcal{H}_\phi$ respectively for the vertical and horizontal tangent subspaces to Diff at ϕ, we may summarize the situation as follows:

Proposition 3. *At each $\phi \in \mathrm{c}_\mu$, the following splitting holds:*

$$T_\phi \,\mathrm{Diff} = \mathcal{V}_\phi \oplus \mathcal{H}_\phi \ ,$$

with the vertical subspace $\mathcal{V}_\phi = T_\phi \mathrm{c}_\mu$ given by (9) and the horizontal subspace, by:

$$\mathcal{H}_\phi = \{\nabla f \circ \phi, \ f \in \mathrm{Funct}_0^\mu\} \ .$$

Moreover, the factors of the splitting are orthogonal for the Arnol'd metric and they vary smoothly with the diffeomorphism ϕ.

3.3 Horizontal Lift

A path $t \mapsto \phi_t \in$ Diff is called *horizontal* if: $\forall t, \ \frac{d\phi_t}{dt} \in \mathcal{H}_{\phi_t}$. It is the *horizontal lift* of a path $t \mapsto \mu_t \in$ Prob if it is horizontal satisfying $\mu_t = \mathcal{P}_\lambda(\phi_t)$.

Proposition 4. *Each path $t \mapsto \mu_t \in$ Prob admits a unique horizontal lift passing, at some time $t = t_0$, through a given diffeomorphism of $\mathrm{c}_{\mu_{t0}}$.*

Proof. In order to prove the uniqueness, let $t \mapsto \phi_t \in$ Diff and $t \mapsto \psi_t \in$ Diff be two horizontal lifts of the same path $t \mapsto \mu_t \in$ Prob with $\phi_{t_0} = \psi_{t_0}$. Set $\dot{\phi}_t = \nabla f_t \circ \phi_t$ (resp. $\dot{\psi}_t = \nabla h_t \circ \psi_t$), with f_t (resp. h_t) in $\mathrm{Funct}_0^{\mu_t}$, and differentiate with respect to t the equation $\mathcal{P}_\lambda(\phi_t) = \mathcal{P}_\lambda(\psi_t)$. Recalling (8), we get:

$$\mathrm{div}_{\mu_t}(\nabla(f_t - h_t)) = 0 \ ,$$

hence $f_t = h_t$ by Theorem 6 (Appendix 2). In particular, the time-dependent vector fields ∇f_t and ∇h_t have the same flow θ_t, so indeed:

$$\phi_t = \theta_{t-t_0} \circ \phi_{t_0} \equiv \theta_{t-t_0} \circ \psi_{t_0} = \psi_t \ .$$

As for the existence, given a path $t \mapsto \mu_t \in$ Prob defined near $t = t_0$ and a diffeomorphism $\psi_0 \in \mathrm{c}_{\mu_{t_0}}$, Corollary 5 of Appendix 2 provides for each t a solution f_t of the equation:

$$\Delta_{\mu_t} f_t \ \mu_t = \frac{d\mu_t}{dt} \ . \qquad (10)$$

From (8), the flow ϕ_t of the time-dependent vector field ∇f_t is such that the path $t \mapsto \psi_t = \phi_{t-t_0} \circ \psi_0 \in$ Diff is a horizontal lift of $t \mapsto \mu_t \in$ Prob passing through ψ_0 at $t = t_0$, as required. □

We will sometimes call (10) the *horizontal lift* equation.

3.4 The Otto Metric

Following Otto [Ott01] (see also [Lot08]), for each $\mu \in \text{Prob}$, we equip the tangent space $T_\mu \text{Prob}$ with the Hilbertian scalar product such that, for each $\phi \in \mathrm{c}_\mu$, the restriction of the tangent map $T_\phi \mathcal{P}_\lambda$ to the horizontal subspace $\mathcal{H}_\phi$ is an *isometry*. Recalling (8), we see that it must be defined by[3]:

$$\forall (\nu, \nu') \in T_\mu \text{Prob} \times T_\mu \text{Prob}, \quad \langle \nu, \nu' \rangle_\mu := \int_M g(\nabla f, \nabla f') d\mu \,, \tag{11}$$

with f given by:

$$\text{div}_\mu(\nabla f)\, \mu = \nu$$

(recalling Corollary 5 of Appendix 2) and similarly for f' with ν'. By construction, when Prob (resp. Diff) is endowed with the Otto (resp. Arnol'd) metric, the map $\mathcal{P}_\lambda$ becomes a *Riemannian submersion* (see e.g. [C-E75, pp. 65–68], [FIP04] and references therein).

3.5 The L^2 Wasserstein Distance

Given $(\mu, \nu) \in \text{Prob} \times \text{Prob}$, recall that the L^2 Wasserstein distance $\mathrm{W}_2(\mu, \nu)$ is given by: $\mathrm{W}_2(\mu, \nu) = \inf \sqrt{C_\mu(\phi)}$ where the total cost functional C_μ was defined in Sect. 1 and the infimum is taken over all measurable maps $\phi : M \to M$ such that $\phi_\# \mu = \nu$.

Using the Otto metric, we can define in Prob the notion of arclength, hence an alternative distance (by the usual length infimum procedure) which we denote by d_O. The fundamental result of [Ott01, Lot08] is the following[4]:

Theorem 3 (Otto–Lott). *The Otto distance* d_O *coincides on* Prob *with the* L^2 *Wasserstein distance* W_2.

The inequality $\mathrm{W}_2 \leq \mathrm{d}_O$ is fairly straightforward to prove. For completeness, let us prove it here.

Pick a constant speed path $t \in [0, 1] \to \mu_t \in \text{Prob}$ with $\mu_0 = \mu$, $\mu_1 = \nu$, and let $t \in [0, 1] \to \phi_t \in \text{Diff}$ be its horizontal lift, given by Proposition 4. From the definition of W_2, we have:

$$\mathrm{W}_2^2(\mu, \nu) \leq \int_M \frac{1}{2} d_g^2 \left[m, (\phi_1 \circ \phi_0^{-1})(m) \right] \, d\mu = \int_M \frac{1}{2} d_g^2(\phi_0(m), \phi_1(m)) \, d\lambda \,.$$

[3] using (23), we also have: $\langle \nu, \nu' \rangle_\mu = \int_M f d\nu' = \int_M f' d\nu$

[4] which implies that W_2 is, indeed, a distance.

Moreover, recalling the definition of the Arnol'd metric, we may write:

$$\int_M \frac{1}{2} d_g^2(\phi_0(m), \phi_1(m))\, d\lambda \le \int_M \frac{1}{2} \left(\int_0^1 |\dot{\phi}_t(m)| dt \right)^2 d\lambda \le \int_0^1 \langle \dot{\phi}_t, \dot{\phi}_t \rangle_{\phi_t} dt \ ,$$

where the latter inequality is derived by applying Schwarz inequality followed by Fubini theorem. From the definition of the Otto metric and since the path $t \in [0,1] \mapsto \mu_t \in \text{Prob}$ has constant speed (we set L for its length), combining the above inequalities yields: $W_2(\mu, \nu) \le L$. Taking the infimum of the right-hand side over all (constant speed) paths in Prob going from μ to ν, we get the desired result. □

The reversed inequality is more tricky; it will be proved below (Corollary 3) in a different way than in [Lot08].

4 Geodesics

In this section, we will investigate the properties of the *horizontal geodesics* in the group Diff as total space of the Riemannian submersion precedingly defined.

4.1 A Sufficient Condition for Geodesicity in Diff

What is a reasonable notion of shortest path $t \in [0,1] \to \phi_t \in \text{Diff}$ between two given diffeomorphisms ϕ_0 and ϕ_1? A naive guess prompts us, for each $m \in M$, to interpolate between the image points $\phi_0(m)$ and $\phi_1(m)$ by means of a constant speed minimizing geodesic in M (unique provided its end points are located close enough). It motivates the following condition:

Condition G: *for each $m \in M$, the path $t \in [0,1] \to \phi_t(m) \in M$ is minimizing with constant speed (MCS, for short).*

The next result is classical [E-M70]:

Proposition 5. *Let $t \in [0,1] \to \phi_t \in$ Diff be a path from ϕ_0 to ϕ_1. If it satisfies Condition G, it must be MCS in* Diff *for the Arnol'd metric.*

Proof. The constant speed (CS) property is trivial; let us focus on the minimizing one. For each CS path $t \in [0,1] \to \psi_t \in \text{Diff}$ with $\psi_0 = \phi_0, \psi_1 = \phi_1$, Fubiny theorem yields for its length L_ψ the equality:

$$L_\psi^2 = \int_0^1 \langle \dot{\psi}_t, \dot{\psi}_t \rangle_{\psi_t} dt = \int_M \frac{1}{2} \left(\int_0^1 |\dot{\psi}_t(m)|^2 dt \right) d\lambda \ .$$

Schwarz inequality implies:

$$L_{\psi}^2 \geq \int_M \frac{1}{2} \left(\int_0^1 |\dot{\psi}_t(m)| dt \right)^2 d\lambda \geq \int_M \frac{1}{2} d_g^2(\phi_0(m), \phi_1(m))\, d\lambda$$

and, taking the infimum of the left-hand side on such paths ψ_t, we conclude:

$$\mathrm{d}_A^2(\phi_0, \phi_1) \geq \int_M \frac{1}{2} d_g^2(\phi_0(m), \phi_1(m))\, d\lambda \ .$$

But the squared length L_{ϕ}^2 of the path $t \in [0, 1] \to \phi_t \in \mathrm{Diff}$ (for the Arnol'd metric) is equal to the latter right-hand side, due to Condition G. In other words, we have $L_{\phi} \leq \mathrm{d}_A(\phi_0, \phi_1)$ therefore, indeed, the aforementioned path is minimizing.

□

We defer to Sect. 4.3 (Proposition 9) a proof, in the same spirit (avoiding to compute the Levi-Civita connection of the Arnol'd metric as in [E-M70, Theorem 9.1]), of a partial converse to Proposition 5.

4.2 Short Horizontal Segments

Throughout this section, we fix an arbitrary couple $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$ of distinct but *suitably close* probability measures, and a diffeomorphism $\phi \in \mathrm{c}_{\mu}$. We look for a horizontal path $t \in [0, 1] \to \phi_t \in \mathrm{Diff}$ starting from ϕ, such that its projection $\mu_t := \mathcal{P}_{\lambda}(\phi_t)$ satisfies $\mu_1 = \nu$ and realizes the distance $\mathrm{d}_O(\mu, \nu)$.

Choice of a Candidate Path

Since the path $t \in [0, 1] \to \phi_t \in \mathrm{Diff}$ should be minimizing, we assume that it satisfies Condition G. If so, we must have:

$$\exists V \in \mathrm{Vec}, \quad \phi_t = \exp(tV) \circ \phi \ .$$

Moreover, since the path is horizontal and $\dot{\phi}_0 = V \circ \phi$, we infer:

$$\exists f \in \mathrm{Funct}_0^{\mu}, \quad V = \nabla f \ ,$$

with f unique. Several questions arise, namely: does there exists an actual candidate path:

$$t \in [0, 1] \to \phi_t = \exp(t \nabla f) \circ \phi \in \mathrm{Diff} \tag{12}$$

satisfying $\mathcal{P}_{\lambda}(\phi_1) = \nu$? Is that path horizontal? Does it realize the Arnol'd distance between its end points?

Let us focus for the moment on the first question and consider, near $u = 0$, the local map E_μ defined by:

$$u \in \mathrm{Funct}_0^\mu \to E_\mu(u) := \exp(\nabla u)_\# \mu \in \mathrm{Prob}\,.$$

Proposition 6. *The map E_μ is a diffeomorphism of a neighborhood of 0 in* Funct_0^μ *to a neighborhood of μ in* Prob.

Proof. The linearization at 0 of the map E_μ is readily found equal to:

$$\forall v \in \mathrm{Funct}_0^\mu, \quad dE_\mu(0)(v) = \Delta_\mu v\ \mu \in \mathrm{Mes}_0\ .$$

By Theorem 6 of Appendix 2, the map $dE_\mu(0) : \mathrm{Funct}_0^\mu \to \mathrm{Mes}_0$ is an elliptic isomorphism. Recalling that Prob is a domain in an affine space modelled on Mes_0, the proposition follows from the *elliptic* inverse function theorem [Del90]. □

Using the local diffeomorphism E_μ, for each $\nu \in$ Prob close enough to μ, we let $f := E_\mu^{-1}(\nu)$ in the path (12) and verify that, indeed, it satisfies:

$$\mathcal{P}_\lambda(\phi_1) = (\exp(\nabla f) \circ \phi)_\# \lambda = \exp(\nabla f)_\# \mu = E_\mu(f) = \nu$$

as required, with $\exp(t\nabla f) \in$ Diff for each $t \in [0, 1]$. The first question (local existence) is thus settled.

Remark 1. Since the function $f = E_\mu^{-1}(\nu) \in \mathrm{Funct}_0^\mu$ is small, it has the following property:

Property NC: *For each $m \in M$ and $t \in [0, 1]$, the points m and $\exp_m(t\nabla_m f)$ are not cut points of each other.*

The latter readily implies the existence, for each $t \in [0, 1]$, of a *unique* vector field $Z_t(f) \in$ Vec such that:

$$\exp(Z_t(f)) = [\exp(t\nabla f)]^{-1} \quad \text{in Diff}\,. \tag{13}$$

Horizontality

Let us turn to the second question.

Proposition 7. *The path (12) is horizontal.*

Proof. We require two preliminary steps.
Step 1: from a Riemannian lemma (see Appendix 3), we have:

$$\forall m \in M, \forall t \in [0, 1], \forall v \in T_pM \text{ with } p = \exp_m(t\nabla_m f),$$

$$g_p\left[d\exp_m(t\nabla_m f)(\nabla_m f), v\right] = g_m\left[\nabla_m f, d\exp_p(Z_t(f)_p)(v)\right] ,$$

where the vector field $Z_t(f)$ is the one defined in the previous remark.

Step2: we have,

$$\forall t \in [0,1], \forall \xi \in \operatorname{Ker}\operatorname{div}_{\mu_t} \text{ (with } \mu_t = \mathcal{P}_\lambda(\phi_t)),$$

$$d\exp_{\exp(t\nabla f)}(Z_t(f))(\xi \circ \exp(t\nabla f)) \in \operatorname{Ker}\operatorname{div}_\mu .$$

Indeed, fix $t \in [0,1]$ and $\xi \in \operatorname{Ker}\operatorname{div}_{\mu_t}$, set for short $Z_t(f) = Z_t$ and consider the vector field:

$$\zeta = d\exp_{\exp(t\nabla f)}(Z_t)(\xi \circ \exp(t\nabla f)) \in \mathrm{Vec}\,.$$

Let Ψ_τ be the flow of ξ and Θ_τ the composed map given by

$$\Theta_\tau := \exp(Z_t) \circ \Psi_\tau \circ \exp(t\nabla f)\,.$$

The one-parameter map Θ_τ satisfies, on the one hand $\frac{d\Theta_\tau}{d\tau}|_{\tau=0} = \zeta$, on the other hand:

$$(\Theta_\tau)_\# \mu = \exp(Z_t)_\# \mu_t = \mu$$

since Ψ_τ preserves the measure μ_t. Therefore, indeed, we have:

$$\operatorname{div}_\mu(\zeta)\,\mu = \frac{d}{d\tau}(\Theta_\tau)_\# \mu|_{\tau=0} = 0 \quad \square$$

We are in position to prove Proposition 7. Fix $t \in [0,1]$ and set $\dot{\phi}_t = V_t \circ \phi_t$. From Helmholtz decomposition (Corollary 6 of Appendix 2), it suffices to pick an arbitrary $\xi \in \operatorname{Ker}\operatorname{div}_{\mu_t}$ and check that the integral $\int_M g(V_t,\xi)\,d\mu_t$ vanishes. We compute:

$$\int_M g(V_t,\xi)\,d\mu_t = \int_M g(\dot{\phi}_t, \xi\circ\phi_t)\,d\lambda = \int_M g(d\exp(t\nabla f)(\nabla f), \xi\circ\exp(t\nabla f))\,d\mu,$$

hence, by Step 1: $\int_M g(V_t,\xi)\,d\mu_t = \int_M g(\nabla f,\zeta)\,d\mu$, with ζ defined (from ξ and t) as in the proof of Step 2. Now Step 2 implies the desired vanishing. $\square$

From Proposition 7 combined with Proposition 3, we immediately get:

Corollary 2. *For $\mu_t = \mathcal{P}_\lambda(\phi_t)$ with ϕ_t given by (12) and for each $t \in [0,1]$, there exists a unique $f_t \in \mathrm{Funct}_0^{\mu_t}$, depending smoothly on t, such that:*

$$\frac{d}{dt} exp(t\nabla f) = \nabla f_t \circ exp(t\nabla f)\,. \tag{14}$$

The smoothness of $t \mapsto f_t$ follows from the, linear elliptic, horizontal lift equation (10) satisfied by f_t. For later use, we observe that $f_0 \equiv f$.

Minimization Property and Equality $\mathbf{W_2 = d_{\mathcal{O}}}$

Finally, does our candidate path (12) realize the Arnol'd distance between its end points ? Having no direct grasp on the question, let us just go ahead with what we can prove and try to find the answer on the way – a typical scientific attitude.[5] Doing so, we will record the following result, established differently in [Lot08, Proposition 4.24].

Proposition 8. *Let $t \in [0,1] \to \phi_t \in \mathrm{Diff}$ be given by (12). Up to addition of a function of t only, the path $t \in [0,1] \to f_t \in \mathrm{Funct}_0^{\mu_t}$ associated to it in Corollary 2 satisfies the equation:*

$$\frac{\partial f_t}{\partial t} + \frac{1}{2}|\nabla f_t|^2 = 0\,. \tag{15}$$

Moreover, the function f must be c-convex on M.

Proof. Since the path (12) satisfies Condition G, we have:

$$\forall m \in M, \quad \frac{\nabla}{dt}\left(\frac{d\phi_t}{dt}(m)\right) = 0$$

which shows that its (Eulerian) velocity field $V_t = \nabla f_t$ satisfies the so-called [K-M07] inviscid Burgers equation:

$$\frac{\partial V_t}{\partial t} + \nabla_{V_t} V_t = 0\,. \tag{16}$$

The latter yields for f_t the equation:

$$\nabla\left(\frac{\partial f_t}{\partial t} + \frac{1}{2}|\nabla f_t|^2\right) = 0$$

or else, (15) as claimed.

Regarding the c-convexity on M of the function f, let us stress that it is not new; it holds because $\exp(\nabla f) \in \mathrm{Diff}$ [Del04, Proposition 2]. Alternatively, though, we will derive it now from (15), thus bringing to light how it originates from Condition G.

As indicated in [Lot08, Remark 4.27] (see also [Vil08]), the solution of (15) in $\mathrm{Funct}_0^{\mu_t}$ is equal to $f_t = \widetilde{f}_t - \int_M \widetilde{f}_t \, d\mu_t$ with $\widetilde{f}_t$ given by the Hopf–Lax–Oleinik formula:

[5] *street-lamp paradigm*, as René Thom used to call it

$$\forall m \in M, \quad \widetilde{f}_t(m) = \inf_{p \in M} \left[f(p) + \frac{1}{2t}\, d_g^2(p,m) \right] .$$

For $t = 1$, we infer:

$$\forall m \in M, \quad -f_1(m) = \sup_{p \in M} \left[\int_M \widetilde{f}_1 \, d\nu - f(p) - \frac{1}{2}\, d_g^2(p,m) \right] ,$$

so the function $-f_1$ is c-convex on M. It is convenient to set $f^c := -f_1$ with a slight abuse of notation due to the Funct_0^ν normalization. The function f^c is easily seen to satisfy:

$$\nabla f^c \circ \exp(\nabla f) = -\frac{d}{dt} \exp(t \nabla f)|_{t=1}$$

hence also:

$$\exp(\nabla f^c) \circ \exp(\nabla f) = \mathbb{I} , \tag{17}$$

or else: $\nabla f^c = Z_1(f)$, with the auxiliary notation introduced in Remark 1. So we may repeat the arguments of this section with the reversed path:

$$t \in [0,1] \to \psi_t := \exp(t \nabla f^c) \circ \phi_1$$

instead of the original one (12), thus switching (μ, f) and (ν, f^c). Doing so, we set $\nu_t := \mathcal{P}_\lambda(\psi_t) \equiv \mu_{1-t}$, let $f_t^c \in \mathrm{Funct}_0^{\nu_t}$ be given by Corollary 2 and reach the conclusion that the function $-f_1^c \equiv f$ is, indeed, c-convex on M. □

Equation (15), with f_t solving the horizontal lift equation (10), may be viewed in Prob equipped with the Otto metric as the *geodesic equation* bearing on the path $t \mapsto \mu_t$ [0-V00, Ott01, Lot08].

To further specify how the functions f and f^c are related, let us establish a key result [McC01, Lemma 7] by means of an elementary (smooth) proof.

Lemma 2. *Set $F_f : M \times M \to \mathbb{R}$ for the auxiliary function given by:*

$$F_f(q_1, q_2) := f(q_1) + f^c(q_2) + \frac{1}{2} d_g^2(q_1, q_2) ,$$

and Σ_f, for the submanifold of $M \times M$ defined by:

$$\Sigma_f := \left\{ (q_1, q_2) \in M^2, \ q_2 = \exp_{q_1}(\nabla_{q_1} f) \right\} .$$

The function F_f is constant on Σ_f where it assumes a global minimum equal to $\int_M \widetilde{f}_1 \, d\nu$.

Proof. From the above expression of $-f_1 = f^c$, we know that $F_f \geq \int_M \widetilde{f}_1 \, d\nu$. By a classical property of the function $\frac{1}{2} d_g^2$ (recalled in Appendix 3), the function

F_f satisfies $dF_f = 0$ at each point of Σ_f. In particular, it must be *constant* on the submanifold Σ_f since the latter is connected. To evaluate that constant, recalling:

$$f^c(q_2) = \sup_{q_1 \in M} \left[\int_M \widetilde{f}_1 \, d\nu - f(q_1) - \frac{1}{2} d_g^2(q_1, q_2) \right] ,$$

we pick for q_2 a point $m \in M$ where f assumes its global *minimum* (such a point exists because M is compact). With $q_2 = m$, we observe that the (continuous) real function: $q_1 \in M \to -f(q_1) - \frac{1}{2} d_g^2(q_1, m)$ must assume a global *maximum* at $q_1 = m$. So $f^c(m) = \int_M \widetilde{f}_1 \, d\nu - f(m)$ and $F_f = \int_M \widetilde{f}_1 \, d\nu$ on Σ_f as claimed. □

With Lemma 2 at hand, we can cope with the minimization question:

Corollary 3. *The path* $t \in [0, 1] \to \phi_t \in \mathrm{Diff}$ *given by (12) realizes the Arnol'd distance between its end points. Moreover,* $\mathrm{W}_2 = \mathrm{d}_O$.

Proof. Let $\psi : M \to M$ be a Borel map satisfying $\psi_\# \mu = \nu$. By Lemma 2, it satisfies $\int_M F_f(m, \psi(m)) \, d\mu \geq \int_M \widetilde{f}_1 \, d\nu$; since $f \in \mathrm{Funct}_0^\mu$ and $f^c \in \mathrm{Funct}_0^\nu$, we may readily rewrite this inequality as:

$$\int_M \frac{1}{2} d_g^2(m, \psi(m)) \, d\mu \geq \int_M \widetilde{f}_1 \, d\nu ,$$

with equality holding if $\psi = \exp(\nabla f)$. From the latter, we infer:

$$\int_M \frac{1}{2} d_g^2(m, \psi(m)) \, d\mu \geq \int_M \frac{1}{2} d_g^2(m, \exp_m(\nabla_m f)) \, d\mu .$$

Taking the infimum of the left-hand side over all measurable maps ψ such that $\psi_\# \mu = \nu$, we obtain:

$$\mathrm{W}_2(\mu, \nu) = \sqrt{\int_M \frac{1}{2} |\nabla f|^2 \, d\mu} .$$

The latter right-hand side is nothing but the Arnol'd length of the path (12) which, by Proposition 7, is horizontal; it thus coincides with the Otto length of the path $t \in [0, 1] \to \mu_t = \mathcal{P}_\lambda(\phi_t) \in \mathrm{Prob}$. Recalling the inequality $\mathrm{W}_2 \leq \mathrm{d}_O$ proved above (after Theorem 3), we conclude that the path (12) is, indeed, minimizing for the Arnol'd distance and that we have:

$$\mathrm{d}_A(\phi, \phi_1) = \mathrm{d}_O(\mu, \nu) = \mathrm{W}_2(\mu, \nu) .$$

Now, the present proof shows that the equality $\mathrm{W}_2 = \mathrm{d}_O$ holds near the diagonal of $\mathrm{Prob} \times \mathrm{Prob}$. As in any length space, this result suffices to conclude that it holds everywhere. □

4.3 Horizontal Segments in the Large

Dropping the closeness assumption on the assigned probability measures $(\mu, \nu) \in$ Prob $\times$ Prob, should we still consider the path (12) as a good candidate to solve the problem posed at the beginning of Sect. 4.2? Deferring till Sect. 5 a tentative answer to the question, let us record global arguments in favour of such a path.

4.3.1 A Reinhart Lemma

It is a standard fact from Riemannian foliations theory, which goes back to [Rei59], that the horizontal distribution $\phi \in \text{Diff} \to \mathcal{H}_\phi$ is totally geodesic. Specifically, we have:

Lemma 3. *Let* $t \in I \to \phi_t \in \text{Diff}$ *be a geodesic (for the Arnol'd metric) defined on some interval* $I \subset \mathbb{R}$*. If, for some value of the parameter* t*, the velocity* $\dot{\phi}_t = \frac{d\phi_t}{dt}$ *is horizontal, it remains so for all* $t \in I$*.*

Proof. Let us argue by *connectedness* on the closed non-empty subset:

$$\mathcal{T} := \left\{ t \in I,\ \dot{\phi}_t \in \mathcal{H}_{\phi_t} \right\} .$$

We only have to prove that $\mathcal{T}$ is relatively open in the interval I. To do so, fix $T \in \mathcal{T}$ and set:

$$\dot{\phi_T} = \nabla f_T \ \circ \ \phi_T \ .$$

By Proposition 7 combined with Corollary 3, for $\epsilon > 0$ small enough, the path

$$t \in (T - \epsilon, T + \epsilon) \cap I \to \psi_t := \exp\left((t - T)\nabla f_T\right) \circ \phi_T \in \text{Diff}$$

is a horizontal geodesic. Since its position ψ_T and velocity $\dot{\psi_T}$ at time $t = T$ coincide with those of our original path $t \mapsto \phi_t$, both paths must coincide hence $(T - \epsilon, T + \epsilon) \cap I \subset \mathcal{T}$ as desired. □

4.3.2 Necessity of Condition G

Let us provide a metric proof of the following partial converse to Proposition 5 (the full converse is proved differently in [E-M70]).

Proposition 9. *Any horizontal minimizing constant speed (HMCS, for short) geodesic for the Arnol'd metric must satisfy Condition G.*

Proof. Let the path $t \in [0, 1] \to \phi_t \in \text{Diff}$ be HMCS. The constant speed assumption means that the total kinetic energy of the motion ϕ_t on the manifold M at time t, namely the quantity:

$$E(t) := \int_M \frac{1}{2} |\dot{\phi}_t|^2_{\phi_t} d\mu \ ,$$

is independent of $t \in [0, 1]$. Its constancy implies that the squared Arnol'd distance:

$$\mathrm{d}_A^2(\phi_0, \phi_1) = \left(\int_0^1 \sqrt{\langle \dot{\phi}_t, \dot{\phi}_t \rangle_{\phi_t}} \, dt \right)^2$$

is equal to the total energy of the geodesic $t \in [0, 1] \to \phi_t \in \mathrm{Diff}$, namely to $E = \int_0^1 E(t) dt$. Fubiny theorem thus provides:

$$\mathrm{d}_A^2(\phi_0, \phi_1) = \int_M \frac{1}{2} \left(\int_0^1 |\dot{\phi}_t(m)|^2_{\phi_t(m)} dt \right) d\mu$$

hence, by Schwarz inequality:

$$\mathrm{d}_A^2(\phi_0, \phi_1) \geq \int_M \frac{1}{2} \left(\int_0^1 |\dot{\phi}_t(m)|_{\phi_t(m)} dt \right)^2 d\mu \ . \tag{18}$$

Since $\int_0^1 |\dot{\phi}_t(m)|_{\phi_t(m)} dt$ is the length of the path $t \in [0, 1] \to \phi_t(m) \in M$, we have identically:

$$\int_0^1 |\dot{\phi}_t(m)|_{\phi_t(m)} dt \geq d_g(\phi_0(m), \phi_1(m)) \ . \tag{19}$$

Combining the two inequalities yields:

$$\mathrm{d}_A^2(\phi_0, \phi_1) \geq \int_M \frac{1}{2} d_g^2(\phi_0(m), \phi_1(m)) \, d\mu \ ,$$

or else, setting $\mu_t := \mathcal{P}_\lambda(\phi_t)$ and $\psi := \phi_1 \circ \phi_0^{-1}$,

$$\mathrm{d}_A^2(\phi_0, \phi_1) \geq \int_M \frac{1}{2} d_g^2(m, \psi(m)) \, d\mu_0 \ .$$

Noting that $\psi_\# \mu_0 = \mu_1$ and recalling the second part of Corollary 3, we conclude:

$$\mathrm{d}_A^2(\phi_0, \phi_1) \geq \mathrm{d}_O^2(\mu_0, \mu_1) \ . \tag{20}$$

But the path $t \in [0, 1] \to \phi_t \in \mathrm{Diff}$ being minimizing horizontal, the projection $\mathcal{P}_\lambda$ restricted along it is an isometry, hence equality must hold in (20). It implies that it must also hold in (18) and (19). For each $m \in M$, equality in (19) forces the path $t \in [0, 1] \mapsto \phi_t(m) \in M$ to be minimizing, while equality in (18) forces it to have constant speed. □

5 Conclusion: Heuristical Statement

Back to the global question stated at the beginning of Sect. 4.3, we are now in position to prove the following heuristical result:

Theorem 4. *Given* $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$*, the following properties are equivalent:*

(i) *there exists a constant speed geodesic* $t \in [0,1] \to \mu_t \in \mathrm{Prob}$ *with end points* $\mu_0 = \mu$, $\mu_1 = \nu$, *which is minimizing for the Otto metric;*
(ii) *there exists a unique function* $f \in \mathrm{Funct}_0^\mu$ *c-convex on M solving the transport equation (2);*
(iii) *there exists a unique function* $f \in \mathrm{Funct}_0^\mu$ *solving the Monge–Ampère equation (4);*
(iv) *there exists a unique function* $f \in \mathrm{Funct}_0^\mu$ *c-convex on M such that the path* $t \in [0,1] \to \mu_t := \exp(t\nabla f)_{\#}\mu \in \mathrm{Prob}$ *is MCS with* $\exp(t\nabla f) \in \mathrm{Diff}$ *and* $\mu_1 = \nu$.

Proof. The implication $(iv) \Rightarrow (ii)$ is trivial, while $(ii) \Rightarrow (iii)$ follows from Theorem 5 of Appendix 1. The implication $(iii) \Rightarrow (i)$ holds by Theorem 2 combined with the remark which follows Lemma 1 (see (3)). We are thus left with proving that $(i) \Rightarrow (iv)$, which we now do.

Assume (i) and let $t \in [0,1] \to \phi_t \in \mathrm{Diff}$ be a horizontal lift of the path μ_t. By Proposition 9, it must satisfy Condition G hence, being horizontal at $t = 0$, it can be expressed as:

$$\phi_t = \exp(t\nabla f) \circ \phi_0$$

for a unique $f \in \mathrm{Funct}_0^\mu$. The horizontality of this expression for all time is now guaranted by Lemma 3. By writing $\exp(t\nabla f) = \phi_t \circ \phi_0^{-1}$, we see that $\exp(t\nabla f) \in \mathrm{Diff}$ while, by the final statement of Proposition 8, we know that the function f is c-convex on M. □

Appendix 1: Jacobian Equation and Related Properties of Smooth Transport Maps

Given $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$ and a smooth map $\phi : M \to M$ pushing μ to ν, if ϕ is a diffeomorphism, it must satisfy the (pointwise) equation:

$$\phi^* \nu = \mu \,. \tag{21}$$

Indeed, making the change of variable $p \mapsto m = \phi(p)$ in the left-hand integral of (1), we get:

$$\int_M (u \circ \phi)\, d(\phi^* \nu) = \int_M (u \circ \phi) d\mu$$

which yields (21) since the function u is arbitrary. One often calls (21) the *Jacobian equation* of the (μ to ν) transport. Here, we wish to weaken the assumption on ϕ:

Proposition 10. *Assume only that the smooth map $\phi : M \to M$ pushing μ to ν is one-to-one. If so, it must still satisfy the Jacobian equation (21).*

Proof. Let E be the set of points of M at which the Jacobian equation is satisfied. If $m \in E$, (21) implies that m is not critical for the map ϕ since μ and ν nowhere vanish. The inverse function theorem implies the existence of a small enough ball B around m such that ϕ induces a *diffeomorphism* from B to its image Ω. Since ϕ is one-to-one on M pushing μ to ν, we have: $\mu(B) = \nu(\Omega)$. Restricting ϕ and μ to B, ν to Ω, we can argue as above and conclude that B lies in E. So the set E is both closed (since ϕ is smooth) and *open* in the manifold M. By connectedness, the proof is reduced to showing that E is non-empty.

We prove the latter by contradiction. If $E = \emptyset$, Sard theorem [Mil65] implies that the image set $\phi(M)$ has zero measure. Besides, it is closed, since M is compact. So we may pick a function u supported inside its *complement* (a dense open subset of M) with $\int_M u \, d\nu \neq 0$. With this choice of u in (1), we reach a contradiction. □

Although the result just proved is certainly well-known, we did not find any simple proof of it in the literature (see e.g. [Vil08, Chapter 11] and references therein). It implies at once the following corollary:

Corollary 4. *Given $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$ and a smooth map $\phi : M \to M$, the following properties are equivalent:*

(i) ϕ *is one-to-one and satisfies* $\phi_{\#}\mu = \nu$;
(ii) ϕ *satisfies the Jacobian equation* $\phi^*\nu = \mu$.

In either case, the map ϕ must be a diffeomorphism.

Indeed, we know that $(i) \Rightarrow (ii)$ by Proposition 10, while (ii) implies that ϕ is a diffeomorphism [Del08, Lemma 4] and (i) follows by the above change of variable argument. □

Regarding maps of the form $\phi = \exp(\nabla f)$, we obtain:

Theorem 5. *Given $(\mu, \nu) \in \mathrm{Prob} \times \mathrm{Prob}$ and a smooth real function f on M, the following properties are equivalent:*

(i) *the function f is c-convex on M and satisfies* $\exp(\nabla f)_{\#}\mu = \nu$;
(ii) *the map* $\exp(\nabla f)$ *is one-to-one and satisfies* $\exp(\nabla f)_{\#}\mu = \nu$;
(iii) *the function f satisfies the Monge–Ampère equation* $\exp(\nabla f)^*\nu = \mu$.

If so, for each $t \in [0, 1]$, the map $\exp(t\nabla f)$ is a diffeomorphism and the function tf is c-convex on M.

Proof. The equivalence between (ii) and (iii) holds by Corollary 4. One can readily infer (iii) from (i) relying on [CMS01, Theorem 4.2]. Assuming (iii), recalling that $\exp(\nabla f)$ must be a diffeomorphism (cf. supra), the c-convexity of f is established in [Del04, Proposition 2]. The final statement of the theorem holds, assuming (iii), by [Del08, Theorem 2, Proposition 5 and Remark 6] (alternatively, the c-convexity of tf follows also from (i) and Lemma 1). □

Remark 2. The reader may get confused by the openings of [Del04,Del08] because, in both papers, we viewed smooth measures like n-forms and their push-forward by a diffeomorphism ϕ, like the (pointwise!) pull-back by ϕ^{-1} (which is, of course, *stronger* than the measure transport definition). So, for instance in [Del08, p. 327–328], he should assume the c-convexity of the solution of the optimal transport equation (whereas, in a pointwise acceptance of that equation, it is a priori guaranted).

Appendix 2: The Helmholtz Decomposition of Vector Fields

Given a smooth positive measure μ on the compact manifold M (with μ taken in Prob to comply with the normalization of this paper), we can associate to it a differential operator, called the *divergence* (with respect to μ), as follows:

Definition 1. The divergence operator $\operatorname{div}_\mu : \mathrm{Vec} \to \mathrm{Funct}_0^\mu$ is defined, for each vector field V on M with flow ϕ_t, by the formula:

$$\operatorname{div}_\mu(V)\,\mu = \frac{d}{dt}\left(\phi_{t\#}\mu\right)_{t=0} \in \mathrm{Mes}_0\,.$$

Let us record the main properties of the divergence operator.

Proposition 11. *Given* $V \in \mathrm{Vec}$*, the measure* $\operatorname{div}(V)_\mu\,\mu$ *satisfies:*

$$\int_M f\,\operatorname{div}_\mu(V)\,d\mu \equiv \int_M df(V)\,d\mu\,, \tag{22}$$

for each $f \in \mathrm{Funct}$.

Proof. Fix $V \in \mathrm{Vec}$ and set ϕ_t for the flow of V. For each $f \in \mathrm{Funct}$ and each real t, Definition 1 yields:

$$\int_M f\,d(\phi_{t\#}\mu) = \int_M (f \circ \phi_t)\,d\mu\,,$$

and (22) is obtained by differentiating both sides with respect to t at $t = 0$. □

From Proposition 11, using the Riemannian metric g, we can rewrite the identity (22) as:

$$\forall (V,f) \in \mathrm{Vec} \times \mathrm{Funct}_0^\mu,\quad \int_M f\,\operatorname{div}_\mu(V)\,d\mu = \int_M g(\nabla f, V)\,d\mu\,, \tag{23}$$

which shows that, with $\nabla : \text{Funct} \to \text{Vec}$ restricted to Funct_0^μ, the divergence and gradient operators are formally *adjoint* of each other with respect to the following L^2 scalar products:

$$\langle f, f' \rangle := \int_M f f' d\mu \,, \quad \langle V, V' \rangle := \int_M g(V, V') d\mu \,,$$

defined in Funct_0^μ and Vec respectively. We will use the following key result:

Theorem 6. *The second order differential operator* $\Delta_\mu : \text{Funct}_0^\mu \to \text{Funct}_0^\mu$ *defined by:*

$$\forall f \in \text{Funct}_0^\mu, \quad \Delta_\mu f := \text{div}_\mu(\nabla f) \,,$$

is self-adjoint and elliptic. Moreover, it is an automorphism.

Let us call the operator Δ_μ the μ-Laplacian (when μ is the Lebesgue measure of the metric g, it coincides with the Laplacian of g). To see that the μ-Laplacian is one-to-one on Funct_0^μ, pick a function f in its kernel and infer from (23) with $V = \nabla f$ that $\nabla f = 0$ hence $f = 0$. The proof that Δ_μ is onto (thus an automorphism, by the open mapping theorem) relies on its self-adjointness (which holds by construction) combined with standard elliptic regularity theory and the Fredholm alternative. We skip the argument since it is lengthy but classical (see e.g. [G-T83] [Bes87, Appendix]).

Corollary 5. *Assume* $\mu \in \text{Prob}$ *and let* $\tilde{\mu} \in \text{Mes}_0$. *There exists a unique* $f \in \text{Funct}_0^\mu$ *solving the equation:*

$$\Delta_\mu f \; \mu = \tilde{\mu} \,.$$

Corollary 6 (Helmholtz decomposition). *The following splitting holds, with* L^2 *orthogonality (relative to* g *and* μ*) of its factors:*

$$\text{Vec} = \text{Im}\, \nabla \; \oplus \; \text{Ker}\, \text{div}_\mu \,.$$

The first corollary follows at once from Theorem 6. To prove Corollary 6, pick $V \in \text{Vec}$ and use Theorem 6 to solve uniquely for $f \in \text{Funct}_0^\mu$ the equation:

$$\Delta_\mu f = \text{div}_\mu(V) \,.$$

The latter implies $(V - \nabla f) \in \text{Ker}\, \text{div}_\mu$ so the splitting, indeed, holds. The orthogonality of its factors now follows from (23). □

Note that the weak form of the preceding equation, namely:

$$\int_M g(\nabla f, \nabla f') \, d\mu = \int_M g(V, \nabla f') \, d\mu \,,$$

can be solved just by the Riesz representation theorem applied in the completion of Funct_0^μ for the Hilbert scalar product defined by the left-hand side, since the linear form defined by the right-hand side is continuous (by Schwarz inequality).

Appendix 3: Complement to Gauss Lemma

In this appendix, we provide a result[6] of local Riemannian geometry, namely an adjointness property of the exponential map, which may be viewed as a complement to Gauss Lemma (see e.g. [C-E75, p. 6] [doC92, pp. 69–70]).

Lemma 4. *Let (M, g) be a Riemannian manifold and $(m, p) \in M \times M$, a couple of points, not cut points of each other, which may be joined by a minimizing geodesic. Set $v \in T_m M$ and $v^c \in T_p M$ for the (unique) tangent vectors of smallest length such that:*

$$p = \exp_m(v) \text{ and } m = \exp_p(v^c).$$

For each couple of tangent vectors $(w, z) \in T_m M \times T_p M$, we have:

$$g_p\left(d\exp_m(v)(w), z\right) = g_m\left(w, d\exp_p(v^c)(z)\right).$$

In other words, the linear isomorphisms:

$$d\exp_m(v) : (T_m M, g_m) \to (T_p M, g_p),$$

$$d\exp_p(v^c) : (T_p M, g_p) \to (T_m M, g_m)$$

are adjoint *of each other.*

Proof. Setting $c = \frac{1}{2} d_g^2$, for short, the function c is smooth in a neighborhood $\mathcal{N}$ of (m, p) in $M \times M$; we denote by (q_1, q_2) the generic point of $\mathcal{N}$. As well known, for each $q \in M$ and $u \in T_q M$ of smallest length such that $(q, \exp_q(u)) \in \mathcal{N}$, we have (see e.g. [Jos95, p. 256]):

$$-\left(d_{q_1} c\right)(q, \exp_q(u))(.) \equiv g_q(u, .).$$

Let us differentiate this identity with respect to $u \in T_q M$ and read the result at $(q, u) = (m, v)$. We get:

$$\forall w \in T_m M, \quad -\left(d^2_{q_1, q_2} c\right)(q, p)\left[., d\exp_m(v)(w)\right] = g_m(w, .). \tag{24}$$

Similarly, for each $\tilde{q} \in M$ and $\tilde{u} \in T_{\tilde{q}} M$ of smallest length such that $(\exp_{\tilde{q}}(\tilde{u}), \tilde{q}) \in \mathcal{N}$, we have:

$$-\left(d_{q_2} c\right)(\exp_{\tilde{q}}(\tilde{u}), \tilde{q})(.) \equiv g_{\tilde{q}}(\tilde{u}, .),$$

which yields at $(\tilde{q}, \tilde{u}) = (p, v^c)$:

$$\forall z \in T_p M, \quad -\left(d^2_{q_1, q_2} c\right)(q, p)\left[d\exp_p(v^c)(z), .\right] = g_p(z, .). \tag{25}$$

[6] which we could not find in the literature.

Now, applying (24) to the vector $d\exp_p(v^c)(z) \in T_m M$ and (25), to the vector $d\exp_m(v)(w) \in T_p M$, produces the *same result*, due to the symmetry of the quadratic form $d^2_{q_1,q_2}c$ (Schwarz theorem). Identifying the resulting right-hand sides, we obtain the lemma □

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On Rank Problems for Planar Webs and Projective Structures

Vladislav V. Goldberg and Valentin V. Lychagin

Abstract We present some old and recent results on rank problems and linearizability of geodesic planar webs.

1 Introduction

In this paper we continue our studies of geodesic planar webs [13].

We give a modification of the Abel elimination method. This method allows one to find all abelian relations admitted by a planar web and therefore to determine the rank of the web. It requires to solve step-by-step a series of ordinary differential equations. In [25] (see also [24]) the same modification is given by a little bit different approach.

On the other hand, we present the method of finding the web rank by means of differential invariants of the web, i.e., the determination of the web rank without solving the differential equations. Pantazi [22] found some necessary and sufficient conditions for a planar web to be of maximum rank. The paper [22] was followed by the papers [23] and [20]. Pirio in [24] presented a more detailed exposition of results of Pantazi in [22] and [23] and Mihăileanu in [20]. The characterization of webs of maximum rank in [22] and [20] is not given in terms of the web invariants.

We give also an alternative construction (the previous one was given in [13]) of the unique projective structure associated with a planar 4-web. Note that Theorem 7 was first proved in [6] (see Sect. 29, p. 246) and that the result in [6] was recently generalized in [26] for any dimension. Our method exploits differential forms and gives an explicit formula for the projective connection. Remark that this method, as well as one in [13], can be used in any dimension. Presence of the projective structure allows us to connect a differential invariant (which we call the Liouville tensor) with any planar 4-web. This tensor gives a criterion for linearizability of geodesic planar webs (cf. [4]).

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2 Planar Webs

All constructions in the paper are local, and we do not specify domains in which they are valid. Functions, differential forms, etc. are real and of class C^∞.

A planar d-web is given by d one-dimensional foliations in the plane which are in general position, i.e., the directions corresponding to different foliations are distinct. The local diffeomorphisms of the plane act in the natural way on d-webs, and they say that two d-webs are (locally) *equivalent* if there exists a local diffeomorphism which sends one d-web to another.

Because all 2-webs are locally equivalent, we begin with 3-webs.

A 3-web can be defined either by three differential 1-forms, say, ω_1, ω_2, and ω_3, where

$$\omega_1 \wedge \omega_2 \neq 0, \quad \omega_2 \wedge \omega_3 \neq 0, \quad \omega_1 \wedge \omega_3 \neq 0,$$

or by the first integrals of the foliations, say, f_1, f_2, and f_3, where

$$df_1 \wedge df_2 \neq 0, \quad df_2 \wedge df_3 \neq 0, \quad df_1 \wedge df_3 \neq 0.$$

The above functions f_1, f_2, and f_3 are called *web functions*.

Remark that the web functions are defined up to *gauge transformations*

$$f_i \mapsto \Phi_i(f_i),$$

where $i = 1, 2, 3$, and $\Phi_i : \mathbb{R} \to \mathbb{R}$ are local diffeomorphisms of the line.

The implicit function theorem states that there is a relation

$$W(f_1, f_2, f_3) = 0$$

for these functions.

The above relation is called (see, for example, [5]) the *web equation*.

Any pair of functions in this equation is locally indistinguishable and can be viewed as local coordinates on the plane.

Keeping in mind this observation, we consider a space $\mathbb{R}^3$ with coordinates u_1, u_2, and u_3 and two-dimensional surface

$$\Sigma \subset \mathbb{R}^3$$

given by the equation

$$W(u_1, u_2, u_3) = 0.$$

We say that $\Sigma \subset \mathbb{R}^3$ is a *web surface* if any two functions u_i and u_j are local coordinates on Σ.

In the case of d-webs one can choose d local first integrals of the corresponding foliations, say, $f_1, f_2, f_3 \ldots, f_d$, which are also called *web functions*. They define a map

$$\sigma : \mathbb{R}^2 \to \mathbb{R}^d,$$

where

$$\sigma : (x, y) \in \mathbb{R}^2 \mapsto (u_1 = f_1(x, y), \ldots, u_d = f_d(x, y)) \in \mathbb{R}^d.$$

The image Σ of this map is a two-dimensional surface in $\mathbb{R}^d$. Remark that any pair of functions u_i, u_j are local coordinates on Σ.

From this point of view, the local theory of planar d-webs is just a geometry of web surfaces in $\mathbb{R}^d$ considered with respect to the gauge transformations.

3 Basic Constructions

Let us begin with 3-webs, and let differential 1-forms ω_1, ω_2, and ω_3 define such a web. These forms are determined up to multipliers $\omega_i \leftrightarrow \lambda_i \omega_i$, where λ_i are smooth nonvanishing functions. Hence, these forms can be normalized in such a way that

$$\omega_1 + \omega_2 + \omega_3 = 0 \tag{1}$$

with only possible scaling $\omega_i \leftrightarrow \lambda \omega_i$.

One can prove that in this case there is a unique differential 1-form γ such that the so-called *structure equations*

$$d\omega_i = \omega_i \wedge \gamma \tag{2}$$

hold for all $i = 1, 2, 3$ (see [4]).

The form γ determines the *Chern connection* Γ in the cotangent bundle T^*M with the following covariant differential:

$$d_\Gamma : \omega_i \longmapsto -\omega_i \otimes \gamma.$$

The curvature of this connection is equal to

$$R_\Gamma : \omega_i \longmapsto -\omega_i \otimes d\gamma.$$

If we write

$$d\gamma = K\omega_1 \wedge \omega_2,$$

then the function K is called the *curvature function* of the 3-web.

Note that the curvature form $d\gamma$ is an invariant of the 3-web while the curvature function K is a relative invariant of the web of weight two.

Let $\langle \partial_1, \partial_2 \rangle$ be the basis dual to $\langle \omega_1, \omega_2 \rangle$. We put $\partial_3 = \partial_2 - \partial_1$. Then leaves of the 3-web are trajectories of the vector fields ∂_2, ∂_1, and ∂_3.

The form γ can be decomposed as follows:

$$\gamma = g_1\omega_1 + g_2\omega_2,$$

where g_1 and g_2 are smooth functions.

Moreover, in this case one has (see [11])

$$[\partial_1, \partial_2] = -g_2\partial_1 + g_1\partial_2 \tag{3}$$

and

$$K = \partial_1 (g_2) - \partial_2 (g_1). \tag{4}$$

Remark also that the covariant derivatives with respect to the Chern connection have the form

$$\nabla_X(\omega_i) = -\gamma(X)\omega_i$$

and

$$\nabla_X(\partial_i) = -\gamma(X)\partial_i.$$

It shows that the leaves of all three foliations are geodesic with respect to the Chern connection.

Let $d_\nabla : \Omega^1(M) \to \Omega^1(M) \otimes \Omega^1(M)$ be the covariant differential with respect to the Chern connection.

The induced connection in the tangent bundle gives the differential

$$d_\nabla^* : \mathcal{D}(M) \to \mathcal{D}(M) \otimes \Omega^1(M),$$

where

$$d_\nabla : \partial_i \to \partial_i \otimes \gamma.$$

In a similar way the Chern connection induces the covariant differential in the tensor bundles.

Let us denote by $\Theta^{p,q}(M) = (\mathcal{D}(M))^{\otimes p} \otimes \left(\Omega^1(M)\right)^{\otimes q}$ the module of tensors of type (p, q).

Then the covariant differential

$$d_\nabla : \Theta^{p,q}(M) \to \Theta^{p+1,q}(M)$$

acts as follows:

$$d_\nabla : u\partial_{j_1} \otimes \cdots \otimes \partial_{j_p} \otimes \omega_{i_1} \otimes \cdots \otimes \omega_{i_q} \longmapsto \partial_{j_1} \otimes \cdots \otimes \partial_{j_p} \otimes \omega_{i_1} \otimes \cdots \otimes \omega_{i_q} \otimes (du + (p-q)\gamma u)$$

where $u \in C^\infty(M)$.

We say that u is of weight $k = q - p$ and call the form

$$\delta^{(k)}(u) \stackrel{\text{def}}{=} du - ku\gamma \tag{5}$$

the *covariant differential* of u.

Decomposing the form $\delta^{(k)}(u)$ in the basis $\{\omega_1, \omega_2\}$, we obtain

$$\delta^{(k)}(u) = \delta_1^{(k)}(u)\ \omega_1 + \delta_2^{(k)}(u)\ \omega_2,$$

where

$$\delta_i^{(k)}(u) = \partial_i(u) - (k)\, g_i u$$

are the covariant derivatives of u with respect to the Chern connection, $i = 1, 2$.

Note that $\delta_1^{(k)}(u)$ and $\delta_2^{(k)}(u)$ are of weight $k+1$.

One can check that the covariant derivatives satisfy the classical Leibnitz rule

$$\delta_i^{(k+l)}(uv) = \delta_i^{(k)}(u)\ v + u\, \delta_i^{(l)}(v)$$

if u is of weight k and v is of weight l, and the commutation relation

$$\delta_2^{(s+1)} \circ \delta_1^{(s)} - \delta_1^{(s+1)} \circ \delta_2^{(s)} = sK \tag{6}$$

Note that the curvature K is of weight two.

In what follows, we shall omit the superscript indicating the weight in the cases when the weight is known.

For the general d-web defined by differential 1-forms $\omega_1, \omega_2,$ and $\omega_3, \ldots, \omega_d$ we normalize $\omega_1, \omega_2,$ and ω_3 as above and choose ω_i for $i \geq 4$ in such a way that the normalizations

$$a_i\omega_1 + \omega_2 + \omega_{i+2} = 0 \tag{7}$$

hold for $i = 1, \ldots, d-2$, with $a_1 = 1$.

Note that $a_i \neq 0, 1$ for $i \geq 2$.

Moreover, for any fixed i, the value $a_i(x)$, of the function a_i at the point x is the cross-ratio of the four straight lines in the cotangent space T_x^* generated by the covectors $\omega_{1,x}, \omega_{2,x}, \omega_{3,x}$, and $\omega_{i+2,x}$, and therefore it is a web invariant. The functions a_i are called the *basic invariants* (cf. [10] or [9], pp. 302–303) of the web.

Because of locality of our consideration, one can choose a function f in such a way that $\omega_3 = df$ and find coordinates x, y such that $\omega_1 \wedge dx = 0$ and $\omega_2 \wedge dy = 0$.

Let also $\omega_{i+3} \wedge dg_i = 0$, for some functions $g_i(x, y), i = 1, \ldots, d-3$.

Then $\omega_1 = -f_x dx$ and $\omega_2 = -f_y dy$.

The dual basis $\{\partial_1, \partial_2\}$ has the form

$$\partial_1 = -f_x^{-1}\partial_x, \quad \partial_2 = -f_y^{-1}\partial_y,$$

and the connection form is

$$\gamma = -H\omega_3,$$

where

$$H = \frac{f_{xy}}{f_x\ f_y}$$

The curvature function has the following explicit expression:

$$K = -\frac{1}{f_x f_y}\left(\log \frac{f_x}{f_y}\right)_{xy}, \tag{8}$$

and the basic invariants have the form

$$a_i = \frac{f_y g_{i,x}}{f_x g_{i,y}}$$

for $i = 1, \ldots, d-3$.

Note that if a three-web W_3 is given by a web equation $W(u_1, u_2, u_3) = 0$, then the curvature K is expressed as follows (see [5], Sect. 9):

$$K = A_{12} + A_{23} + A_{31}, \tag{9}$$

where

$$A_{rs} = \frac{1}{W_r W_s} \frac{\partial^2}{\partial u_r \partial u_s} \log \frac{W_r}{W_s},$$

and subscripts r and s mean the partial derivatives of the function with respect to the variables u_r and u_s, where $r, s = 1, 2, 3$.

Recall that a planar d-web is said to be (locally) *parallelizable* if it is (locally) equivalent to a d-web of parallel straight lines in the affine plane.

It is known (see, for example, [5], Sect. 8) that *a planar* 3-*web is locally parallelizable if and only if* $K = 0$.

For planar d-webs, $d \geq 4$, the following statement holds (cf. [10] or [9], Sect. 7.2.1 for $d = 4$): *a planar d-web $\langle \omega_1, \omega_2, \omega_3, \omega_4, \ldots, \omega_d \rangle$ is locally parallelizable if and only if its* 3-*subweb $\langle \omega_1, \omega_2, \omega_3 \rangle$ is locally parallelizable* (*i.e.,* $K = 0$)*, and all basic invariants a_i are constants.*

4 Rank

We begin with an interpretation of the classical Abel addition theorem (see [2]) in terms of planar webs (cf. [5]).

Let us consider linear webs on the affine plane, i.e., such planar webs leaves of which are straight lines. There is an elegant method to construct such webs. Take a straight line $rx + sy = 1$ on the affine plane and assume that the coefficients $(r, s) \in \mathbb{R}^2$ satisfy an algebraic equation $P_d(r, s) = 0$ of degree d.

Given (x, y), then the system

$$\begin{cases} rx + sy = 1, \\ P_d(r, s) = 0 \end{cases}$$

has at most d roots.

Assume that in a domain on the plane (x, y) the above system has exactly d roots. Then in this domain we have a linear d-web.

Take now a cubic polynomial

$$P_3(s,t) = s^2 - 4r^3 - g_2 r - g_3,$$

where g_2 and g_3 are constants.

Then the system

$$\begin{cases} rx + sy = 1, \\ s^2 - 4r^3 - g_2 r - g_3 = 0 \end{cases}$$

in the domain

$$x^4 - 24xy^2 - 12g_2 y^4 > 0, \ \ y \neq 0,$$

has three distinct real roots and consequently there are three pairwise distinct straight lines $(r_i(x,y), s_i(x,y))$, passing through the point (x,y). In other words, we have a linear 3-web.

Assume that

$$g_2^3 - 27g_3^2 \neq 0.$$

Then the solutions of the equation $s^2 - 4r^3 - g_2 r - g_3 = 0$ can be parameterized by the Weierstrass elliptic function with the invariants g_2 and g_3:

$$r = \wp(t), s = \wp'(t).$$

Hence, the roots $(r_i(x,y), s_i(x,y))$ correspond to three solutions $(t_i(x,y))$ of the equation

$$\wp(t)x + \wp'(t)y - 1 = 0.$$

Let us put

$$f(t) = \wp(t)x + \wp'(t)y - 1$$

and compute the integral

$$\int t \frac{f'(t)}{f(t)} dt$$

along the boundary of the period parallelogram of the Weierstrass function. We get

$$t_1(x,y) + t_2(x,y) + t_3(x,y) = \text{const}.$$

This is the *abelian relation*.

This relation can be understood geometrically if we note that, by the construction, the functions $t_1\,(x,y)$, $t_2\,(x,y)$, and $t_3\,(x,y)$ are first integrals of the corresponding 3-web.

In more general case, let us consider an arbitrary planar d-web defined by d web functions

$$f_1, \ldots, f_d.$$

Then by an *abelian relation* we mean a relation

$$F_1(f_1) + \cdots + F_d(f_d) = \text{const}.$$

given by d functions $(F_1, \ldots, F_d)$ of one variable.

We say that two abelian relations $(F_1, \ldots, F_d)$ and $(G_1, \ldots, G_d)$ are *equivalent* if

$$F_i = G_i + \underset{i}{\text{const}},$$

for all $i = 1, \ldots, d$.

The set of equivalence classes of abelian relations admits the natural vector space structure with respect to addition

$$(F_1, \ldots, F_d) + (G_1, \ldots, G_d) = (F_1 + G_1, \ldots, F_d + G_d)$$

and multiplication by numbers

$$\alpha(F_1, \ldots, F_d) = (\alpha F_1, \ldots, \alpha F_d).$$

The dimension of this vector space is called the *rank* of the web.

In the case when d-web is defined by differential 1-forms

$$\omega_1, \ldots, \omega_d,$$

the differentiation of the abelian relation leads us to the *abelian equation*

$$\lambda_1\omega_1 + \cdots + \lambda_d\omega_d = 0,$$

for functions $\lambda_1, \ldots, \lambda_d$ under the condition that all differential 1-forms $\lambda_i\omega_i$ are closed:

$$d(\lambda_i\omega_i) = 0.$$

The abelian equation is a system of the first-order linear PDEs for the functions $(\lambda_1, \ldots, \lambda_d)$, and the rank of the web is the dimension of the solution space.

Example 1. The following example illustrates the above constructions for 3-webs.

Consider the 3-web W_3 given by web functions

$$x, y, f(x, y).$$

Then

$$\omega_1 = -f_x dx, \quad \omega_2 = -f_y dy, \quad \omega_3 = df,$$

and the condition

$$\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 = 0$$

implies

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda.$$

The abelian relations take now the form

$$\begin{cases} (\lambda f_x)_y &= 0, \\ (\lambda f_y)_x &= 0, \\ \lambda_x f_y - \lambda_y f_x &= 0, \end{cases}$$

or

$$\begin{cases} (\ln \lambda)_x = -(\ln f_x)_y, \\ (\ln \lambda)_y = -(\ln f_y)_x. \end{cases}$$

The compatibility condition for this system has the form

$$(\ln f_x)_{xy} = (\ln f_y)_{xy}$$

or

$$K = 0.$$

So, we can conclude this consideration by the following statement: *rank of a* 3-*web does not exceed one, and the rank equals to one if and only if the* 3-*web is parallelizable*.

5 Abel's Method

In this section we discuss the rank problem in the classical setting. A method of finding the rank, or in other terms, a method of solving abelian relations was proposed by Abel himself (see [1]). This method is just a consistent elimination of the functions from the abelian relation by using only differentiation.

Let us consider a planar d-web defined by web functions $f_1, \ldots, f_d$ and the corresponding abelian relation

$$F_1(f_1) + \cdots + F_d(f_d) = \text{const}. \tag{10}$$

Modifying Abel's method and adjusting it to (10), we can explain it as follows:

(a) Taking the differential of (10), we get

$$F_1' \, df_1 + F_2' \, df_2 + \cdots + F_d' \, df_d = 0. \tag{11}$$

(b) Taking the wedge product of (11) with df_1, we eliminate F_1 and get the following equation:

$$F_2' + J_{21}^{31} F_3' + \cdots + J_{21}^{d1} F_d' = 0, \tag{12}$$

where

$$J^{ij}_{kl} = \frac{\partial(f_i, f_j)}{\partial(f_k, f_l)}$$

is the Jacobian of the functions f_i and f_j with respect to functions f_k and f_l.

(c) Taking the wedge product of the differential of (12) with df_2, we eliminate F_2 and get the following equation:

$$J^{31}_{21} F''_3 + a_3 F'_3 + \cdots = 0, \tag{13}$$

where a_3 is a certain function.

(d) Divide (13) by the first coefficient and take the differential of the obtained equation; if F''_3 appears in differentiation, take its value from (13). This gives the following equation:

$$F'''_3 df_3 + b_3 F'_3(f_3) + \cdots = 0, \tag{14}$$

where b_3 is a certain function.

(e) Taking the exterior product of (14) with df_3, we eliminate F'''_3 and get the following equation:

$$c_3 F'_3(f_3) + \cdots = 0, \tag{15}$$

where c_3 is a certain function.

(f) Dividing (15) by c_3 and taking the wedge product of the differential of the obtained equation and df_3, we eliminate the function F_3.

(g) Use the procedure outlined above to eliminate the functions $F_4, \ldots, F_{d-1}$. Finally, we obtain a linear differential equation with respect to the function $F_d(f_d)$. This equation can be viewed as family of homogeneous ordinary linear differential equations.

(h) Substitute the solution $F_d(f_d)$ into (10) and apply the outlined procedure to find another function, say, $F_{d-1}(f_{d-1})$.

On Abel's elimination method as well as on less general method of monodromy see [25].

Below we give a few examples of application of the Abel method.

5.1 3-Webs

Here we apply the Abel elimination method for 3-webs to show once more that a planar 3-web admits a nontrivial abelian relation if and only if the 3-web is parallelizable.

Suppose that a 3-web is given by the web functions $f(x, y)$, x, and y and let

$$F(f) + G(x) + H(y) = 0. \tag{16}$$

be an abelian relation.

Take the differential of (16):

$$F'(f)\,df + G'(x)\,dx + H'(y)\,dy = 0, \tag{17}$$

and the wedge product of (17) with df:

$$-f_y\,G'(x) + f_x\,H'(y) = 0. \tag{18}$$

Then

$$G'(x) - \frac{f_x}{f_y}\,H'(y) = 0. \tag{19}$$

Taking the wedge product of the differential of (19) with dx, we get

$$H''(y) + \left(\log \frac{f_x}{f_y}\right)_y H'(y) = 0. \tag{20}$$

In order (20) has a nontrivial solution, it is necessary and sufficient that the function $\left(\log \frac{f_x}{f_y}\right)_y$ does not depend on x, i.e.,

$$\left(\log \frac{f_x}{f_y}\right)_{xy} = 0. \tag{21}$$

This means that $K = 0$, i.e., the 3-web is parallelizable.

5.2 4-Webs of Rank Three

Assume that a 4-web is given by the following web functions:

$$f = x + y,\ \ g = xy,\ \ x,\ \ y.$$

We will apply the Abel elimination method to find all abelian relations admitted by this web.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{22}$$

be an abelian relation.

Taking the differential of (22):

$$F'(f)\,df + G'(g)\,dg + H'(x)dx + K'(y)\,dy = 0, \tag{23}$$

and the wedge product of (23) with dy, we eliminate $K'(y)$:

$$F'(f) + y\,G'(g) + H'(x) = 0. \tag{24}$$

Once more, taking the differential of (24):

$$F''(f)\,df + y\,G''(g)\,dg + G'(g)\,dy + H''(x)\,dx = 0, \tag{25}$$

and the wedge product of (25) with dg, we eliminate $G''(g)$:

$$(x - y)\,F''(f) - y\,G'(g) + x/,\,H''(x) = 0. \tag{26}$$

Using (24), we eliminate $G'(g)$ in (26):

$$(x - y)\,F''(f) + F'(f) + H'(x) + x H''(x) = 0. \tag{27}$$

Dividing (27) by x, taking the differential of the result and taking the wedge product of the differential with dx, we eliminate $H(x)$ and arrive at the equation

$$F'''(f) = 0. \tag{28}$$

Up to an arbitrary constant, the solution of (28) is

$$F(f) = a\,f^2 + b\,f, \tag{29}$$

where a and b are arbitrary constants. By (29), (27) gives

$$x\,H''(x) + H'(x) + 2ax = 0. \tag{30}$$

Up to an arbitrary constant, the solution of (30) is

$$H(x) = -a\,x^2 - b\,x + k\log x, \tag{31}$$

where k is an arbitrary constant.

By (29) and (31), (26) gives

$$G'(g) = -a - \frac{k}{g}. \tag{32}$$

Up to an arbitrary constant, the solution of (32) is

$$G(g) = -2a\,g - k\log g, \tag{33}$$

Now by (29), (31) and (33), we find from (22) that

$$K(y) = -a\,y^2 - b\,y + k\,\log y. \tag{34}$$

Thus, the rank is equal to three, and we have the following three independent abelian relations:

$(a = 0, b = 0, k = -1)$

$$x + y - f = 0;$$

$(a = -1, b = 0, k = 0)$

$$x^2 + y^2 + (-f^2) + (2g) = 0;$$

$(a = 0, b = -1, k = 0)$

$$\log x + \log y + (-\log g) = 0.$$

The fact that the rank of this web is three was also proved in [12] using differential invariants of webs.

5.3 4-Webs of Rank Two

Consider a 4-web given by the following web functions:

$$f = x^2 + y^2,\ \ g = x + y,\ \ x,\ \ y.$$

We will apply the Abel elimination method and find all abelian relations admitted by this web.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{35}$$

be an abelian relation.

Taking the differential:

$$F'(f)\,df + G'(g)\,dg + H'(x)\,dx + K'(y)\,dy = 0, \tag{36}$$

and the wedge product of (36) with dy, we eliminate $K(y)$:

$$2x\ F'(f) + G'(g) + H'(x) = 0. \tag{37}$$

Once more, taking the differential of (37):

$$2x\ F''(f)\,df\ + 2F'(f)\,dx + G''(g)\,dg + H''(x)\,dx = 0, \tag{38}$$

and the wedge product of (38) with dx, we eliminate $H(x)$:

$$2(g^2 - f)\ F''(f) + G''(g) = 0. \tag{39}$$

Taking the wedge product of the differential of (39) with dg, we eliminate $G(g)$:

$$2(g^2 - f)\, F'''(f) - 2F''(f) = 0. \tag{40}$$

Equation (40) is equivalent to the system

$$\begin{cases} -2f\, F'''(f) - 2F''(f) = 0, \\ F'''(f) = 0. \end{cases} \tag{41}$$

Therefore, up to an additive constant,

$$F(f) = kf,$$

where k is a constant.

Now it follows from (39) that, up to an additive constant,

$$G(g) = bg,$$

where b is a constant.

Equation (37) implies that, up to an additive constant,

$$H(x) = -kx^2 - bx,$$

and (35) gives that

$$K(y) = -ky^2 - by.$$

Thus, the rank of the web is equal to two, and we have the following two basic abelian relations:

$(k = 0, b = 1)$

$$(x + y) + (-x) + (-y) = 0,$$

$(k = 1, b = 0)$

$$(x^2 + y^2) + (-x^2) + (-y^2) = 0.$$

In [12] using differential invariants of webs, it was shown that this 4-web is of rank two.

5.4 4-Webs of Rank One

Assume that a 4-web is given by the following web functions:

$$f = \frac{(x - y)^2}{x},\ \ g = \frac{(x - y)^2}{y},\ \ x,\ \ y.$$

We will apply the Abel elimination method to find all abelian relations admitted by this web.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{42}$$

be an abelian relation.

Taking the differential of (42):

$$F'(f)\,df + G'(g)\,dg + H'(x)dx + K'(y)\,dy = 0, \tag{43}$$

and the wedge product of (43) with df, we eliminate $F(f)$:

$$G'(g) - \frac{2xy^2}{(x-y)^3}H'(x) - \frac{(x+y)y^2}{(x-y)^3}K'(y) = 0. \tag{44}$$

Once more, taking the differential of (44):

$$\begin{aligned} &G''(g)\,dg - \frac{2xy^2}{(x-y)^3}H''(x) - \frac{2y(2x+y)(-ydx+xdy)}{(x-y)^3}H'(x) \\ &-\frac{(x+y)y^2}{(x-y)^3}K''(y) - \frac{2y(x+2y)(-ydx+xdy)}{(x-y)^3}K'(y) = 0, \end{aligned} \tag{45}$$

and the wedge product of (45) with dg, we eliminate $G(g)$:

$$H''(x) + \frac{2x+y}{x(x+y)}\,H'(x) + \frac{y}{x}\,K''(y) + \frac{x+2y}{x(x+y)}\,K'(y) = 0. \tag{46}$$

Taking the wedge product of the differential of (46) with dx, we eliminate $H''(x)$:

$$H'(x) - \frac{(x+y)^2y}{x}K'''(y) - \frac{(2x+3y)(x+y)}{x}\,K''(y) - K'(y) = 0. \tag{47}$$

Taking the wedge product of the differential of (47) with dx, we eliminate $H'(x)$:

$$(x+y)yK^{iv}(y) + 3(x+2y)\,K'''(y) + 6K''(y) = 0. \tag{48}$$

Equation (48) is equivalent to the system

$$\begin{cases} y\,K^{iv}(y) + 3xK'''(x) = 0, \\ y^2\,K^{iv}(y) + 6y\,K'''(y) + 6K''(y) = 0. \end{cases} \tag{49}$$

It follows from (49) that

$$y\,K'''(y) + 2K''(y) = 0. \tag{50}$$

Up to an additive constant, the solution of (50) is

$$K(y) = -k \log y + by, \tag{51}$$

where k and b are arbitrary constants.

It follows from (47) and (51) that

$$H'(x) = \frac{k}{x} + b. \tag{52}$$

Up to an additive constant, the solution of (52) is

$$H(x) = k \log x + bx. \tag{53}$$

It follows from (44), (51) and (53) that

$$G'(g) = -\frac{k}{g} + \frac{by(3x+y)}{g(x-y)}. \tag{54}$$

Equation (54) implies that

$$b = 0 \tag{55}$$

and that, up to an additive constant,

$$G(g) = -k \log g, \quad H(x) = k \log x, \quad K(y) = -k \log y. \tag{56}$$

Finally, (42) and (56) give

$$F(f) = k \log f, \tag{57}$$

and the 4-web admits only one independent abelian relation

$$\log f - \log g + \log x - \log y = 0. \tag{58}$$

Using differential invariants of webs introduced in [12], one can show that this 4-web is of rank one.

5.5 4-Webs of Rank Zero

Assume that a 3-web is given by the following web functions:

$$f = (x+y)e^x, \;\; g = xy, \;\; x, \;\; y.$$

We will apply the Abel elimination method to show that this web admits no abelian relations.

Let

$$F(f) + G(g) + H(x) + K(y) = 0 \tag{59}$$

be an abelian relation.

Taking the differential of (59):

$$F'(f)\,df + G'(g)\,dg + H'(x)\,dx + K'(y)\,dy = 0, \tag{60}$$

and the wedge product of (60) with df, we eliminate $F(f)$:

$$[(x+y)x + x - y]\,G'(g) - H'(x) + (1+x+y)\,K'(y) = 0. \tag{61}$$

Once more, taking the differential of (61):

$$\begin{aligned}&[(x+y)x + x - y]\,G''(g)\,dg - H''(x)\,dx + (1+x+y)\,K''(y)\,dy\\ &+ [(2x+y+1)\,dx + (x-1)\,dy]\,G'(g) + (dx+dy)\,K'(y) = 0,\end{aligned} \tag{62}$$

and the wedge product of (62) with dx, we eliminate $H(x)$:

$$x\,[(x+y)x + x - y]\,G''(g) + (x-1)\,G'(g) + K''(y) + K'(y) = 0. \tag{63}$$

Taking the differential of (63):

$$\begin{aligned}&x\,[(x+y)x + x - y]\,G'''(g)\,dg + (x-1)\,G''(g)\,dg\\ &+K'''(y)\,dy + K''(y)\,dy + G''(g)\,dx = 0,\end{aligned} \tag{64}$$

and the wedge product of (64) with dy, we eliminate $K(y)$:

$$g\,(x^2 + g + x - \frac{g}{x}]\,G'''(g) + (3g - \frac{2g}{x} + 3x^2 + 2x + 1)\,G''(g) = 0. \tag{65}$$

Equation (65) is equivalent to the system

$$\begin{cases} g^2 G'''(g) + (3g+1)\,G''(g) = 0,\\ G''(g) = 0,\\ G'''(g) = 0,\end{cases}$$

i.e., to the equation $G''(g) = 0$. Up to an arbitrary constant, the solution of the latter equation is $G = ag$, where a is an arbitrary constant.

If $G = ag$, then (63) becomes

$$K''(y) + K'(y) + a\,(x-1) = 0.$$

It follows that $a = 0$ and $G(g) = 0$. The equation for $K(y)$ becomes $K''(y) + K'(y) = 0$. Up to an arbitrary constant, its solution is $K(y) = -be^{-y}$, where b is an arbitrary constant.

Now (61) becomes

$$H'(x) = b(1+y)e^{-y} + b\,x e^{-y}.$$

It follows that $b = 0$ and $H'(x) = 0$. Hence, up to an arbitrary constant, $H(x) = 0$ and $K(y) = 0$.

Finally (59) implies that $F(f) = 0$.

Thus, the web under consideration admits no abelian relations.

6 Abelian Differential Equations

In this section we discuss properties of abelian equations.

Recall that the abelian equation for a planar d-web given by differential 1-forms

$$\omega_1, \dots, \omega_d$$

is a first-order PDE system for functions $\lambda_1, \dots, \lambda_d$ of the form

$$\lambda_1\omega_1 + \cdots + \lambda_d\omega_d = 0,$$
$$d(\lambda_1\omega_1) = \cdots = d(\lambda_d\omega_d) = 0.$$

Let us write down the abelian equation in more explicit form. To this end, we choose a 3-subweb, say, the 3-web given by

$$\omega_1, \omega_2, \omega_3,$$

and normalize the d-web as it was done earlier:

$$a_1\omega_1 + \omega_2 + \omega_3 = 0, \quad a_2\omega_1 + \omega_2 + \omega_4 = 0, \dots, \quad a_{d-2}\omega_1 + \omega_2 + \omega_d = 0,$$

with $a_1 = 1$ and $d\omega_3 = 0$.

It is easy to see that, if $i \leq 3$, then, due to the structure equations, we get

$$d(\lambda\omega_i) = d\lambda \wedge \omega_i + \lambda d\omega_i = (d\lambda - \lambda\gamma) \wedge \omega_i$$

or

$$d(\lambda\omega_i) = \delta(\lambda) \wedge \omega_i,$$

if we consider λ as a function of weight one.

Assuming that all λ_i are functions of weight one and the functions a_i are of weight 0, we get

$$d(\lambda_1\omega_1) = -\delta_2(\lambda_1)\,\omega_1 \wedge \omega_2,$$
$$d(\lambda_2\omega_2) = \delta_1(\lambda_1)\,\omega_1 \wedge \omega_2,$$
$$d(\lambda_3\omega_3) = (\delta_2(\lambda_3) - \delta_1(\lambda_3))\,\omega_1 \wedge \omega_2,$$
$$d(\lambda_i\omega_i) = (\delta_2(a_{i-2}\lambda_i) - \delta_1(\lambda_i))\,\omega_1 \wedge \omega_2$$

for $i = 4, \ldots, d$.

The normalization condition $\sum_1^d \lambda_i \omega_i = 0$ implies that

$$\lambda_1 = a_1 u_1 + \cdots + a_{d-2} u_{d-2},$$
$$\lambda_2 = u_1 + \cdots + u_{d-2},$$

where

$$u_1 = \lambda_3, \ldots, u_{d-2} = \lambda_d.$$

Therefore the abelian equation can be written in the explicit form as the following PDE system:

$$\Delta_1 (u_1) = \cdots = \Delta_{d-2} (u_{d-2}) = 0,$$
$$\delta_1 (u_1) + \cdots + \delta_1 (u_{d-2}) = 0,$$

where $\Delta_i = \delta_1 - \delta_2 \circ a_i$.

Let

$$\pi : \mathbb{R}^{d-2} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

be the trivial vector bundle, where $\pi : (u_1, \ldots, u_{d-2}, x, y) \mapsto (x, y)$.

Denote by $\mathfrak{A}_1 \subset \mathbf{J}^1 (\pi)$ the subbundle of the 1-jet bundle corresponding to the abelian equation, and by $\mathfrak{A}_k \subset \mathbf{J}^k (\pi)$ the $(k-1)$-prolongation of $\mathfrak{A}_1$.

Let

$$\pi_{k,k-1} : \mathfrak{A}_k \longrightarrow \mathfrak{A}_{k-1}$$

be the restrictions of the natural jet projections

$$\pi_{k,k-1} : \mathbf{J}^k (\pi) \longrightarrow \mathbf{J}^{k-1} (\pi).$$

Then, if $k \le d-2$, one can easily check that $\mathfrak{A}_k$ are vector bundles, the maps $\pi_{k,k-1}$ are projections and

$$\dim \operatorname{Ker} \pi_{k,k-1} = d - k - 2.$$

In other words, we have the following tower of vector bundles:

$$\mathbb{R}^2 \overset{\pi}{\longleftarrow} \mathbb{R}^{d+2} \overset{\pi_{1,0}}{\longleftarrow} \mathfrak{A}_1 \overset{\pi_{2,1}}{\longleftarrow} \mathfrak{A}_2 \overset{\pi_{3,1}}{\longleftarrow} \cdots \overset{\pi_{d-3,d-4}}{\longleftarrow} \mathfrak{A}_{d-3} \overset{\pi_{d-2,d-3}}{\longleftarrow} \mathfrak{A}_{d-2}.$$

The last projection

$$\pi_{k,k-1} : \mathfrak{A}_{d-2} \longrightarrow \mathfrak{A}_{d-3}$$

is an isomorphism, and geometrically it can be viewed as a linear Cartan connection (see [19]) in the vector bundle

$$\pi_{d-3} : \mathfrak{A}_{d-3} \to \mathbb{R}^2.$$

This proves that *the abelian equation is formally integrable if and only if this linear connection is flat*.

It is easy to see that the dimension of this bundle is equal to $(d-2)(d-1)/2$.

The dimension of the solution space is the rank of the corresponding d-web. The above computation shows that the rank of a d-web is finite-dimensional and does not exceed

$$\frac{(d-1)(d-2)}{2}.$$

This result was first established by Bol [7] (see also [5]).

The compatibility conditions for the abelian equation can be found (see [12]) using multi-brackets (see [15]).

These conditions have the form

$$\varkappa = \Box_1 u_1 + \cdots + \Box_{d-2} u_{d-2} = 0,$$

where

$$\Box_i = \Delta_1 \cdots \Delta_{d-2} \cdot \delta_1 - \Delta_1 \cdots \Delta_{i-1} \cdot \delta_1 \cdot \Delta_{i+1} \cdots \Delta_{d-2} \cdot \Delta_i$$

are linear differential operators of order not exceeding $d-2$.

Summarizing, we get the following

Theorem 1. *A d-web is of maximum rank if and only if $\varkappa = 0$ on $\mathfrak{A}_{d-2}$.*

Remark that $\varkappa$ can be viewed as a linear function on the vector bundle $\mathfrak{A}_{d-2}$, and therefore the above theorem imposes $(d-1)\,(d-2)\,/2$ conditions on the d-web (or on $d-2$ web functions) in order the web has the maximum rank. A calculation of these conditions is pure algebraic, and we shall illustrate this calculation below for planar 3-, 4- and 5-webs. Note also that expressions for $\varkappa$ in the case of general d-webs are extremely cumbersome while for concrete d-webs it is not the case.

7 Rank of 4-Webs

7.1 The Obstruction

In order to simplify notations, we put $a_2 = a$ in the normalization for 4-webs :

$$\omega_1 + \omega_2 + \omega_3 = 0,$$
$$a\omega_1 + \omega_2 + \omega_4 = 0,$$

and reserve the subscripts for the covariant derivatives of a. Thus, $a_2 = \delta_2\,(a)$, etc.

For abelian equations we shall use the functions u and v, where $u = u_1$ and $v = u_2$.

Then the abelian equations have the form

$$(u + av)\,\omega_1 + (u + v)\,\omega_2 + u\omega_3 + v\omega_4 = 0,$$

where

$$\lambda_1 = u + av, \lambda_2 = u + v, \lambda_3 = u, \lambda_4 = v,$$

and the functions u and v satisfy the equations

$$\begin{aligned} \delta_1(u) - \delta_2(u) &= 0, \\ \delta_1(v) - \delta_2(av) &= 0, \\ \delta_1(u) + \delta_1(v) &= 0. \end{aligned}$$

In the case of 4-webs, the tower of prolongations has the form

$$\mathbb{R}^2 \xleftarrow{\pi} \mathbb{R}^4 \xleftarrow{\pi_{1,0}} \mathfrak{A}_1 \xleftarrow{\pi_{2,1}} \mathfrak{A}_2,$$

where the isomorphism $\pi_{2,1} : \mathfrak{A}_2 \to \mathfrak{A}_1$ defines a linear Cartan connection on the three-dimensional vector bundle

$$\pi_1 : \mathfrak{A}_1 \to \mathbb{R}^2.$$

In what follows, we use coordinates in the jet spaces adjusted to the Chern connection and weight. Thus, for example, $u_{k,l}$ stands for the operator $\delta_1^k \delta_2^l$.

In these coordinates, the abelian equation takes the following form:

$$\begin{aligned} u_1 - u_2 &= 0, \\ v_1 - av_2 - a_2 v &= 0, \\ u_1 + v_1 &= 0, \end{aligned}$$

and the obstruction

$$\varkappa = (\Delta_1\Delta_2\delta_1 - \delta_1\Delta_1\Delta_2)u + (\Delta_1\Delta_2\delta_1 - \Delta_1\delta_1\Delta_2)v$$

equals

$$\varkappa = c_0 v_2 + c_1 v + c_2 u,$$

where $c_0, c_1,$ and c_2 are certain functions of the curvature function K, the basic invariant a and their covariant derivatives K_i and a_i, a_{ij} (see formula (1) in [12]).

The coefficient c_0 in the expression of $\varkappa$ has an intrinsic geometric meaning.

Namely, by the *curvature function* of a 4-web we mean the arithmetic mean of the curvatures of its 3-subwebs [1, 2, 3], [1, 2, 4], [1, 3, 4] and [2, 3, 4].

Then (see [12]) the coefficient c_0 equals the curvature function of the 4-web.

7.2 4-*Webs of Maximum Rank*

A planar 4-web has the maximum rank three if and only if the obstruction $\varkappa$ identically equals zero, i.e., if and only if $c_0 = c_1 = c_2 = 0$. Computing these coefficients leads us to the following result (see [12]).

Theorem 2. *A planar* 4-*web is of maximum rank if and only if the following relations hold:*

$$
\begin{aligned}
c_0 &= K + \frac{a_{11} - aa_{22} - 2\,(1-a)\,a_{12}}{4a(1-a)} + \frac{(-1+2a)\,a_1^2 - a^2 a_2^2 + 2\,(1-a)^2\,a_1 a_2}{4\,(1-a)^2\,a^2},\\
c_1 &= \frac{K_2 - K_1}{4(1-a)} + \frac{(a-4)\,a_1 + \left(11 - 20a + 12a^2\right) a_2}{12\,(1-a)^2\,a} K + \frac{a_{112} - a_{122}}{4a(1-a)}\\
&\quad + \frac{a_1 - aa_2}{4a^2(1-a)} a_{22} + \frac{(2a-1)\,(a_1 - aa_2)}{4\,(1-a)^2\,a^2} a_{12} - \frac{a_2^2\,((1-2a)\,a_1 + aa_2)}{4\,(1-a)^2\,a^2},\\
c_2 &= \frac{aK_2 - K_1}{4a(1-a)} + \frac{(1-2a)\,a_1 - (a-2)\,aa_2}{4\,(1-a)^2\,a^2} K.
\end{aligned}
$$

Vanishing of the coefficients c_1 and c_2 for 4-webs with $c_0 = 0$ is equivalent to linearizability of the web (see [4]). Therefore the above theorem can be formulated in pure geometric terms:

Theorem 3. *A* 4-*web is of maximum rank three if and only if it is linearizable and its curvature vanishes.*

Theorem 3 leads to interesting results in web geometry:

1. A linearizable planar 4-web is of maximum rank if and only if its curvature vanishes.
2. A planar 4-web of maximum rank is linearizable (algebraizable) (Poincaré).
3. If a planar 4-web with a constant basic invariant a has maximum rank, then it is parallelizable.
4. Parallelizable planar 4-webs have maximum rank.
5. The Mayrhofer 4-webs are of maximum rank.

Recall that a 4-web is called the *Mayrhofer* web if all 3-subwebs of this web are parallelizable.

7.3 4-*Webs of Maximum Rank and Surfaces of Double Translation*

A surface $S \subset \mathbb{R}^3$ is a *surface of translation* in if it admits a vector parametric representation $r = R(u, v)$, where $R(u, v)$ is a solution of the wave equation

$$R_{uv} = 0.$$

Then

$$r = f(u) + g(v), \tag{66}$$

or, in components of vectors,

$$\begin{cases} x = f^1(u) + g^1(v), \\ y = f^2(u) + g^2(v), \\ z = f^3(u) + g^3(v). \end{cases}$$

A surface S is a *surface of double translation* if in addition to representation (66) it also admits a representation

$$r = h(s) + k(t), \tag{67}$$

such that the coordinate functions $u, v, s,$ and t on the surface are pairwise independent.

In other words, they define a 4-web on the surface S. If S is a surface of double translation, then it follows from (66) and (67) that

$$f^i(u) + g^i(v) - h^i(s) - k^i(t) = 0 \tag{68}$$

for $i = 1, 2, 3$. These relations can be viewed as abelian relations for the 4-web mentioned above.

If the surface S does not belong to a plane, then (68) gives three independent abelian relations for the web. Therefore, this web has the maximum rank, and as we have seen earlier, it is linearizable (algebraizable). This result was first proved by Sophus Lie in the form.

Theorem 4. ([16]) *If S is a surface of double translation not belonging to a plane, then the curves $f'(u), g'(v), h'(s)$ and $k'(t)$ belong to an algebraic curve of degree four.*

More on the subject, its further developments and references one can find in [8] and [3].

7.4 4-Webs of Rank Two

As we have seen, a 4-web admits an abelian equation (has a positive rank) if and only if the equation

$$c_0 v_2 + c_1 v + c_2 u = 0 \tag{69}$$

has a nonzero solution.

Suppose that $c_0 = 0$. Then if two other coefficients $c_1 = c_2 = 0$, then a 4-web is of maximum rank three. If $c_0 = 0$ but one of the coefficients c_1 or c_2 is not 0, then

$c_1 v + c_2 u = 0$ and then, say u, satisfies a first-order PDE system of two equations. Therefore, the 4-web admits not more than one abelian equation (i.e., it is of rank one or zero).

Assume that $c_0 \neq 0$. Then we can find all first derivatives u_i and v_j from the abelian equation and (69):

$$\begin{aligned} u_1 &= \frac{ac_1 - a_2 c_0}{c_0} v + \frac{ac_2}{c_0} u, \\ u_2 &= \frac{ac_1 - a_2 c_0}{c_0} v + \frac{ac_2}{c_0} u, \\ v_1 &= \frac{a_2 c_0 - ac_1}{c_0} v - \frac{ac_2}{c_0} u, \\ v_2 &= -\frac{c_1}{c_0} v - \frac{c_2}{c_0} u. \end{aligned}$$

Therefore, a 4-web has rank two if and only if the above system is compatible.

Theorem 5. *A planar* 4*-web is of rank two if and only if* $c_0 \neq 0$, *and*

$$G_{ij} = 0, \;\; i, j = 1, 2, \tag{70}$$

where

$$\begin{aligned} G_{11} &= ac_0(c_{2,2} - c_{2,1}) + ac_2(c_{0,1} - c_{0,2}) - a\,(1-a)\,c_1 c_2 \\ &\quad + (2a_2 - a_1 - aa_2)\,c_0 c_2 - K c_0^2, \\ G_{12} &= ac_0(c_{1,2} - c_{1,1}) + ac_1(c_{0,1} - c_{0,2}) - a\,(1-a)\,c_1^2 \\ &\quad + (2a_2 - a_1 - 2aa_2)\,c_0 c_1 + \left(a_2^2 + a_{12} - a_{22}\right) c_0^2, \\ G_{21} &= c_0(c_{2,1} - ac_{2,2}) + c_2(ac_{0,2} - c_{0,1}) - 2a_2 c_0 c_2 + a\,(1-a)\,c_2^2, \\ G_{22} &= c_0(c_{1,1} - ac_{1,2}) + c_1(ac_{0,2} - c_{0,1}) + a\,(1-a)\,c_1 c_2 - a_2 c_0 c_1 \\ &\quad - a_2(1-a) c_0 c_2 + (a_{22} - K)\,c_0^2. \end{aligned}$$

Example 2. Consider the planar 4-web with the following web functions

$$x, \;\; y, \;\; \frac{x}{y}, \;\; xy(x+y).$$

The linearizability conditions (see [4]) for this web are not satisfied, and therefore, this 4-web is not linearizable, but in this case $G_{11} = G_{12} = G_{21} = G_{22} = 0$. Hence, the 4-web is of rank two.

This example gives us the following important property:
General 4*-webs of rank two are not linearizable.*

7.5 4-Webs of Rank One

As we have seen earlier, a 4-web can be of rank one if $c_0 = 0$ but one of the coefficients c_1 and c_2 of (69) is not 0 or if $c_0 \neq 0$. The following theorem outlines the four cases when a 4-web can be of rank one.

Theorem 6. *A planar* 4*-web is of rank one if and only if one of the following conditions holds:*

1. $c_0 = 0,\ J_1 = J_2 = 0,$ *where*

$$
\begin{aligned}
J_1 &= a_2 c_1 c_2 (c_1 - c_2) + a c_2^2 (c_{1,2} - c_{1,1}) \\
&\quad + c_1 c_2 (c_{1,1} + a(c_{2,1} - c_{1,2} - c_{2,2})) + c_1^2 (a c_{2,2} - c_{2,1}), \\
J_2 &= c_1^2 (c_1 - c_2)^2 K + \left(c_{1,11} - c_{1,12}\right) c_1 c_2 (c_2 - c_1) \\
&\quad + c_1^2 (c_1 - c_2) \left(c_{2,11} - c_{2,12}\right) - c_2 (2c_1 - c_2) c_{1,1} (c_{1,2} - c_{1,1}) \\
&\quad + c_1^2 c_{2,1} (c_{1,2} - c_{2,2} + c_{2,1}) + c_1^2 c_{1,1} (c_{2,2} - 2c_{2,1})
\end{aligned}
$$

   and $c_1 \neq c_2,\ c_1 \neq 0.$
2. $c_0 = 0, c_1 = c_2 \neq 0,$ *and* $J_3 = 0,$ *where*

$$J_3 = (a_{22} - a_{12})(1 - a) + a_2(a_2 - a_1) - (1 - a)^2 K.$$

3. $c_0 = 0, c_1 = 0, c_2 \neq 0,$ *and* $J_4 = 0,$ *where*

$$J_4 = a_{12} a - a_1 a_2 - K a^2.$$

4. $c_0 \neq 0,$ *and* $J_{10} = J_{11} = J_{12} = 0,$ *where*

$$J_{10} = G_{11} G_{22} - G_{21} G_{12},$$

$$
\begin{aligned}
J_{11} &= c_0 (G_{21,1} G_{22} - G_{22,1} G_{21}) + (a_2 c_0 - a c_1) G_{21}^2 \\
&\quad + (a c_2 - a_2 c_0 + a c_1) G_{21} G_{22} - a c_2 G_{22}^2,
\end{aligned}
$$

$$
\begin{aligned}
J_{12} &= c_0 (G_{21,2} G_{22} - G_{22,2} G_{21}) + (a_2 c_0 - a c_1) G_{21}^2 \\
&\quad + a (c_2 - c_1) G_{21} G_{22} - c_2 G_{22}^2.
\end{aligned}
$$

Proof. See [12].

Example 3. Consider the planar 4-web with the following web functions

$$x,\ y,\ \frac{x y^2}{(x - y)^2},\ \frac{x^2 y}{(x - y)^2}.$$

For this web, we have $c_0 = 0$ and $J_1 = J_2 = 0$. Thus, we have the web of type 1 as indicated in Theorem 6, and this 4-web is of rank one.

In this example the 4-web is not linearizable. Therefore,
General 4-*webs of rank one are not linearizable.*

8 Planar 5-Webs of Maximum Rank

Let us consider a planar 5-web in the standard normalization

$$\omega_1 + \omega_2 + \omega_3 = 0, \quad a\omega_1 + \omega_2 + \ \omega_4 = 0, \quad b\omega_1 + \omega_2 + \ \omega_5 = 0,$$

where a and b are the basic invariants of the web.

The abelian equation for such a web has the form

$$(w + au + bv)\omega_1 + (w + u + v)\omega_2 + w\omega_3 + u\omega_4 + v\omega_5 = 0,$$

where we have

$$\lambda_1 = w + au + bv, \quad \lambda_2 = w + u + v,$$

and

$$\lambda_3 = w, \quad \lambda_4 = u, \quad \lambda_5 = v.$$

The functions w, u, and v satisfy the abelian equation

$$\begin{aligned} &\delta_1 (w) - \delta_2 (w) = 0, \quad \delta_1 (u) - \delta_2 (au) = 0, \\ &\delta_1 (v) - \delta_2 (bv) = 0, \quad \delta_1 (w) + \delta_1 (u) + \delta_1 (v) = 0, \end{aligned}$$

and their compatibility condition takes the form

$$\begin{aligned} \varkappa = &(\Delta_1\Delta_2\Delta_3\delta_1 - \delta_1\Delta_2\Delta_3\Delta_1) (w) + (\Delta_1\Delta_2\Delta_3\delta_1 - \Delta_1\delta_1\Delta_3\Delta_2) (u) \\ &+ (\Delta_1\Delta_2\Delta_3\delta_1 - \Delta_1\Delta_2\delta_1\Delta_3) (v) = 0. \end{aligned}$$

In the canonical coordinates in the jet bundles, the abelian equation has the form

$$\begin{aligned} &u_1 + v_1 + w_1 = 0, \qquad v_1 - bv_2 - b_2 v = 0, \\ &u_1 - au_2 - a_2 u = 0, \quad w_1 - w_2 = 0, \end{aligned}$$

and the obstruction $\varkappa$ equals

$$c_0 w_{22} + c_1 w_2 + c_2 v_2 + c_3 w + c_4 u + c_5 v = 0,$$

where the explicit form of expressions for the coefficients c_9, c_1, c_2, c_3, c_4 and c_5 can be found in [12].

This gives the following result [12]:

A planar 5-web is of maximum rank if and only if the invariants c_0, c_1, c_2, c_3, c_4 *and* c_5 *vanish.*

Similar to the case of 4-webs, the coefficient c_0 in the expression of $\varkappa$ for 5-webs has an intrinsic geometric meaning.

Namely, we call the *curvature function* of a 5-web the arithmetic mean of the curvature functions of its ten 3-subwebs.

The straightforward calculation shows that the curvature function equals to c_0.

In other words [12], the curvature of a planar 5-web of maximum rank equals zero.

For the case of planar 5-webs with constant basic invariants a and b the invariants c_i, for $i = 0, 1, 2, 3, 4, 5,$ vanish (and the web is of maximum rank) if and only if this web is parallelizable [12].

Example 4. We consider the Bol 5-web with the web functions

$$x, y, \frac{x}{y}, \frac{1-y}{1-x}, \frac{x-xy}{y-xy}.$$

For this web we have

$$K = 0,\ a = \frac{xy-x}{xy-y},\ b = \frac{y-1}{x-1},$$

and $c_i = 0$, for $i = 0, 1, \ldots, 5$.

Thus, the 3-web is of maximum rank.

Using the linearizability conditions for planar 5-webs [4], we see that the Bol 5-web is not linearizable.

The above example leads us to the following important observation:

General planar 5-webs of maximum rank are not linearizable.

9 Projective Structures and Planar 4-Webs

In this section we give more direct construction of the projective structure associated with 4-webs (see [13]).

Remind that an affine connection ∇ on the plane determines a covariant differential

$$d_\nabla : \Omega^1\left(\mathbb{R}^2\right) \to \Omega^1\left(\mathbb{R}^2\right) \otimes \Omega^1\left(\mathbb{R}^2\right).$$

This differential splits into the sum

$$d_\nabla = d_\nabla^a \oplus d_\nabla^s,$$

where

$$d^a_\nabla : \Omega^1\left(\mathbb{R}^2\right) \to \Omega^2\left(\mathbb{R}^2\right)$$

is the skew-symmetric part, and

$$d^s_\nabla : \Omega^1\left(\mathbb{R}^2\right) \to S^2\left(\Omega^1\right)\left(\mathbb{R}^2\right)$$

is the symmetric one.

The connection is torsion-free if and only the skew-symmetric part coincides with the de Rham differential:

$$d^a_\nabla = d.$$

A foliation given by a differential 1-form ω is geodesic (i.e., all leaves of the foliation are geodesics) with respect to connection ∇ if and only if (see [13]):

$$d^s_\nabla(\omega) = \theta \cdot \omega$$

for some differential 1-form θ.

Remark that it follows from the above formula that two affine connections, say, ∇ and ∇', are projectively equivalent (i.e., have the same geodesics) if and only if there exists a differential 1-form ρ such that

$$d^s_\nabla(\omega) - d^s_{\nabla'}(\omega) = \rho \cdot \omega$$

for all differential 1-forms ω.

Assume that a 4-web is given by differential 1-forms $\omega_i, i = 1, 2, 3, 4,$ which are normalized

$$\omega_1 + \omega_2 + \omega_3 = 0,$$
$$a\omega_1 + \omega_2 + \omega_4 = 0,$$

and

$$d\omega_3 = 0.$$

Let ∇ be a torsion-free connection for which all foliations $\omega_i = 0,\ i = 1, 2, 3, 4,$ are geodesics.

We call such 4-webs *geodesic*.

Then

$$d^s_\nabla(\omega_i) = \theta_i \cdot \omega_i$$

for all $i = 1, 2, 3, 4$ and some differential 1-forms θ_i.

Differentiating the normalization conditions, we get

$$\theta_1 \cdot \omega_1 + \theta_2 \cdot \omega_2 + \theta_3 \cdot \omega_3 = 0,$$
$$da \cdot \omega_1 + a\theta_1 \cdot \omega_1 + \theta_2 \cdot \omega_2 + \theta_4 \cdot \omega_4 = 0.$$

If

$$\theta_i = A_i \omega_1 + B_i \omega_2$$

for all $i = 1, 2, 3, 4$, then the above system is just a system of linear equations for coefficients A_i and B_i.

Solving this system, we find that

$$A_2 = A_1 + z, \quad B_2 = B_1 + z,$$
$$A_3 = A_1, \qquad B_3 = B_1 + z,$$
$$A_4 = A_1 + \frac{a_1}{a}, \ B_4 = B_1 + z,$$

where

$$z = \frac{a_1 - a a_2}{a(1-a)}.$$

In other words, the affine connection is completely determined by the differential 1-form θ_1, and

$$\theta_2 = \theta_1 - z\omega_3,$$
$$\theta_3 = \theta_1 + z\omega_2,$$
$$\theta_4 = \theta_1 + \frac{a_1}{a}\omega_1 + z\omega_2.$$

This shows that all such affine connections are projectively equivalent. Taking the representative with

$$\theta_1 = \frac{z}{2}\omega_3,$$

we get the following result (this result was first obtained in [6], Sect. 29, p. 246):

Theorem 7. *There is a unique projective structure associated with a planar* 4*-web in such a way that the* 4*-web is geodesic with respect to the structure.*

The projective structure is an equivalence class of the torsion-free affine connection ∇ *with the following symmetric differential:*

$$d^s_\nabla(\omega_1) = \frac{z}{2}\,\omega_3 \cdot \omega_1,$$
$$d^s_\nabla(\omega_2) = -\frac{z}{2}\,\omega_3 \cdot \omega_2.$$

We say that a planar d-web is *geodesic* with respect to an affine connection if all leaves of all foliations are geodesic.

The above theorem gives a criterion for a d-web to be geodesic. For simplicity we take the case of 5-webs. Let a 5-web be given by differential 1-forms ω_i, $i = 1, 2, 3, 4, 5$, which are normalized as follows:

$$\omega_1 + \omega_2 + \omega_3 = 0,$$
$$a\omega_1 + \omega_2 + \omega_4 = 0,$$
$$b\omega_1 + \omega_2 + \omega_5 = 0,$$

and

$$d\omega_3 = 0.$$

This web is geodesic if and only if the fifth foliation $\omega_5 = 0$ is geodesic with respect to the canonical projective structure determined by the 4-web ($\omega_i, i = 1, 2, 3, 4$).

We have

$$d^s_\nabla(\omega_5) = \theta_5 \cdot \omega_5 + (zb(1-b) - (b_1 - bb_2))\omega_1^2.$$

Therefore, in order to have a geodesic 5-web, the last term should vanish.

Theorem 8. *A* 5*-web is geodesic if and only if the basic invariants a and b satisfy the following condition:*

$$\frac{a_1 - aa_2}{a(1-a)} = \frac{b_1 - bb_2}{b(1-b)}. \tag{71}$$

The linearizability problem (see [4]) for planar webs can be reformulated now as follows: *a planar d-web is linearizable if and only if the web is geodesic and the canonical projective structure of one of its* 4*-subwebs is flat.*

The flatness of a projective structure can be checked by the Liouville tensor (see [18], [17], [14]). This tensor can be constructed as follows (see, for example, [21]). Let ∇ be a representative of the canonical projective structure, and Ric be the Ricci tensor of the connection ∇. Define a new tensor $\mathfrak{P}$ as

$$\mathfrak{P}(X, Y) = \frac{2}{3} Ric(X, Y) + \frac{1}{3} Ric(Y, X)$$

for all vector fields X, Y.

The Liouville tensor $\mathfrak{L}$ is defined as follows:

$$\mathfrak{L}(X, Y, Z) = \nabla_X(\mathfrak{P})(Y, Z) - \nabla_Y(\mathfrak{P})(X, Z)$$

for all vector fields X, Y, Z.

The tensor is skew-symmetric in X and Y, and therefore it belongs to

$$\mathfrak{L} \in \Omega^1(\mathbb{R}^2) \otimes \Omega^2(\mathbb{R}^2).$$

It is known (see [18], [21], [17], [14]) that *the Liouville tensor depends on the projective structure defined by ∇ and vanishes if and only if the projective structure is flat.*

For the case of the projective structure associated with a planar 4-web we shall call this tensor the *Liouville tensor* of the 4-web

Consider three invariants:

$$w = \frac{f_y}{f_x}, \quad \alpha = \frac{aa_y - wa_x}{wa(1-a)}, \quad k = (\log w)_{xy}. \tag{72}$$

Then the Liouville tensor has the form

$$\mathfrak{L} = \left(L_1 \omega_1 + \frac{L_2}{w} \omega_2 \right) \otimes \omega_1 \wedge \omega_2,$$

where L_1 and L_2 are relative differential invariants of order three. The explicit formulas for these invariants are

$$\begin{aligned} 3L_1 &= w(-(kw)_x + \alpha_{xx} + \alpha\alpha_x) + (\alpha w_{xx} + (\alpha^2 + 3\alpha_x)w_x - 2\alpha_{xy} - 2\alpha\alpha_y) \\ &\quad + w^{-1}(-\alpha w_{xy} - 2\alpha_y w_x + \alpha w_x^2) + w^{-2}\alpha w_x w_y, \\ 3L_2 &= w^2(-(kw^{-1})_y + 2\alpha\alpha_x) + w(2\alpha^2 w_x - 2\alpha_{xy} - \alpha\alpha_y) \\ &\quad + (-\alpha w_{xy} - 2\alpha_y w_x + \alpha_{yy}) + w^{-1}(\alpha w_x w_y - \alpha_y w_y). \end{aligned} \tag{73}$$

Summarizing, we get the following result.

Theorem 9. *A planar d-web is linearizable if and only if the web is geodesic and the Liouville tensor of one of its* 4*-subwebs vanishes.*

Corollary 1. *If the basic invariants of all* 4*-subwebs of a d-web are constants, then the d-web is linearizable if and only if it is parallelizable.*

Proof. First of all, the web is geodesic because of conditions (71).

Moreover, for a 4-subweb, condition $a = \text{const}.$ implies $\alpha = 0$, and by Theorem 9 and (73), the 4-web is linearizable if and only if

$$(kw)_x = 0,$$

and

$$(kw^{-1})_y = 0.$$

Then $w = A(x)B(y)$ and by (8), $K = 0$. Therefore, due to Sect. 3, the 4-web is parallelizable.

The d-web is parallelizable too, because it geodesic and has constant basic invariants.

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Niceness Theorems*

Michiel Hazewinkel

Abstract Many constructions in mathematics give unreasonably nice results. In particular much compatible structure tends to imply that these structures are very regular. Also many counterexamples have nice underlying structures. This paper is a first attempt to analyze these phenomena.

1 Introduction and Statement of the Problems

In this lecture I aim to raise a new kind of question. It appears that many important mathematical objects (including counterexamples) are unreasonably nice, beautiful and elegant. They tend to have (many) more (nice) properties and extra bits of structure than one would a priori expect.

The question is why this happens and whether this can be understood.[1]

These ruminations started with the observation that it is difficult for, say, an arbitrary algebra to carry additional compatible structure. To do so it must be nice,

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* The first time I lectured on this at the WCAA conference, Wilfrid Laurier University in Waterloo, Canada, May 2008, the chairman summed up my lecture as follows: "If it is true it is beautiful, if it is beautiful it is probably true". I also lectured on the same subject at the Abel symposium meeting in Tromsø, Norway in June 2008.

The present screed expands on those first lectures a great deal. Yet, in spite of its length it is just a beginning: a first scratching at the edges of a great and fascinating problem that deserves devoted attention.

[1] There is of course the "anthropomorphic principle" answer, much like the question of the existence of (intelligent) life in this universe. It goes something like this. If these objects wern't nice and regular we would not be able to understand and describe them; we can see/understand only the elegant and beautiful ones. I do not consider this answer good enough though there is something in it. So the search is also on for ugly brutes of mathematical objects.

Also this anthropomorphic argument raises the subsidiary question of why we can only understand/describe beautiful/regular things. There are aspects of (Kolmogorov) complexity and information theory involved here.

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i.e., as an algebra be regular (not in the technical sense of this word), homogeneous, everywhere the same, ... It is for instance very difficult to construct an object that has addition, multiplication and exponentiation, all compatible in the expected ways.

The present scribblings are just a first attempt to identify and describe the phenomenon. Basically this is a prepreprint and it touches just the fringes of the subject. There is much more to be said and there are many more examples than remarked upon here.

This lecture is about lots of examples of this phenomenon such as Daniel Kan's observation that a group carries a comonoid structure in the category of groups if and only if it is a free group, the Milnor-Moore and Leray theorems in the theory of Hopf algebras, Grassmann manifolds and classifying spaces, and especially the star example: the ring of commutative polynomials over the integers in countably infinite indeterminates. This last one occurs all over the place in mathematics and has more compatible structures that can be believed. For instance it occurs as the algebra of symmetric functions in infinitely many variables, as the cohomology and homology of the classifying space **BU**, as the sum of the representation rings of the symmetric groups, as the free lambda-ring on one variable, as the representing ring of the Witt vectors, as the ring of rational representation of $\mathcal{GL}_\infty$, as the underlying ring of the universal formal group, ...

To start with, here is a preliminary list of the kind of phenomena I have in mind.

1. Objects with a great deal of compatible structure tend to have a nice regular underlying structure and/or additional nice properties: "Extra structure simplifies the underlying object". As indicated above this sort of thing was the starting point.
2. *Universal objects.* That is mathematical objects which satisfy a universality property. They tend to have:
   (a) a nice regular underlying structure
   (b) additional universal properties (sometimes seemingly completely unrelated to the defining universal property)
3. Nice objects tend to be large and inversely large objects of one kind or another tend to have additional nice properties. For instance, large projective modules are free (Hyman Bass, [16 Bass]).
4. Extremal objects tend to be nice and regular. (The symmetry of a problem tends to survive in its extremal solutions is one of the aspects of this phenomenon; even when (if properly looked at) there is bifurcation (symmetry breaking) going on.)
5. Uniqueness theorems and rigidity theorems often yield nice objects (and inversely). They tend to be unreasonably well behaved. I.e. if one asks for an object with such and such properties and the answer is unique the object involved tends to be very regular. This is not unrelated to 4.

Concrete examples of all these kinds of phenomena will be given below (Sect. 2) as well as a (pitiful) few first explanatory general theorems (Sect. 3).

The "niceness phenomenon" is not limited to theorems saying that, e.g. in suitable circumstances an object is free; it also extends to counter examples: many of them are very regular in their construction. This can, for instance, take the form of a simple construction repeated indefinitely. Some examples are in Sect. 2.6.

All in all I detect in present day mathematics a strong tendency towards the study of things that in some sense have low Kolmogorov complexity.

2 Examples

2.1 Lots of Compatible Structure Examples

2.1.1 Groups in the Category of Groups

To start with here is an observation of Daniel Kan, {Kan, 1958 #1}, which has moreover the distinction of being one of the first results of this kind and of admitting a nice (sic!) pictorial illustration.

First, here is the abstract setting. Let $\mathfrak{C}$ be a category with a terminal objet and products. For example the category ***Group*** of groups where the product is the direct product and the terminal object is the one element group.

A group object in such a category $\mathfrak{C}$ is an object $G \in \mathfrak{C}$ equipped with a morphism $m : G \times G \longrightarrow G$ (multiplication), a morphism $e : T \longrightarrow G$ (unit element) where T is the terminal object of the category $\mathfrak{C}$, and a morphism $\iota : G \longrightarrow G$ (inverse) such that the categorical versions of the standard group axioms hold. This means that the following diagrams are supposed to be commutative.

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \mathrm{id}} & G \times G \\ \downarrow{\scriptstyle \mathrm{id}\times m} & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G \end{array} \qquad \text{(associativity)} \tag{1}$$

$$\begin{array}{ccc} G \times T & \xrightarrow{\mathrm{id}\times e} & G \times G \\ \uparrow{\scriptstyle \cong} & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{=} & G \end{array} \qquad \begin{array}{ccc} T \times G & \xrightarrow{e \times \mathrm{id}} & G \times G \\ \uparrow{\scriptstyle \cong} & & \downarrow{\scriptstyle m} \\ G & \xrightarrow{=} & G \end{array} \qquad \text{(unit)} \tag{2}$$

$$\begin{array}{ccc} G & \xrightarrow{(\mathrm{id},\iota)} & G \times G \\ \downarrow & & \downarrow{\scriptstyle m} \\ T & \xrightarrow{e} & G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{(\iota,\mathrm{id})} & G \times G \\ \downarrow & & \downarrow{\scriptstyle m} \\ T & \xrightarrow{e} & G \end{array} \qquad \text{(inverse)} \tag{3}$$

where the vertical arrow on the left hand side of the two diagrams (3) is the unique morphism in the category $\mathfrak{C}$ to the terminal object and the vertical isomorphisms on the left of (2) are the canonical isomorphisms of an object with the product of that object with the terminal object.

In the case of the category of groups this means that a group object is a group (with composition law denoted + (though it is not clear yet that it is commutative) with a second composition law, denoted $*$ that is distributive over the first composition law in the sense that the following identity holds

$$(a+b)*(a'+b') = (a*a') + (b*b') \tag{4}$$

This comes from the requirement that $*$ must be a morphism in the category **Group**. Let 0 be the unit element for the composition law + and 1 the unit element for the composition law $*$. Putting $b = a' = 0$ in (4) gives

$$a*b' = (a*0) + (0*b') \tag{5}$$

On the other hand putting in $a' = b' = 1$ in (4) gives

$$a+b = (a+b)*(1+1)$$

and multiplying this with the inverse of $a+b$ for the star composition gives $1 = 1+1$ and hence $1 = 0$. Put this in (5) to find that $a*b' = a+b'$ showing that the compositions are the same and then (4) immediately gives that both are Abelian.

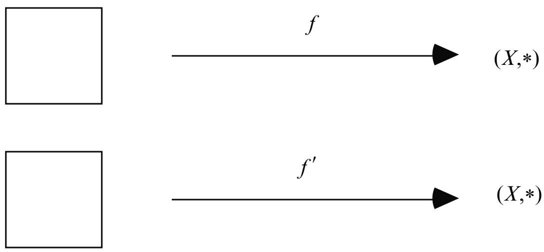
Thus a group object in the category of groups is Abelian and the second composition law is the same as the first.

Actually this can be proved more generally for monoid objects in the category of groups, [60 Kan].

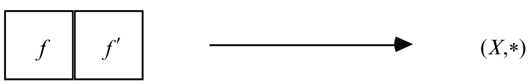
There is a nice illustration of this in homotopy theory (and that is where the idea came from). This goes as follows. The second homotopy group, $\pi_2(X,*)$, of a based space $(X,*)$ is, as a set, the set of all homotopy classes of maps from the disk into X that take the boundary circle into the base point $*$ of X.

For illustrational (and conceptual) purposes it is easier to think of homotopy classes of maps from the unit filled square to X that take the boundary to the base point. Homotopically, of course, this makes no difference.

Now let

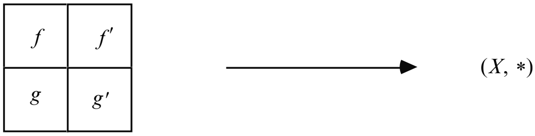


be two such maps. They can be glued together horizontally to give a map of the same kind (up to homotopy):



and this induces a composition on $\pi_2(X, *)$ turning it into a group. Of course the two maps can also be glued together vertically, inducing another, a priori different, group structure.

Now take four such maps f, f', g, g'. Then first gluing f, f' and g, g' together horizontally and then gluing the two results together vertically gives a map that can be depicted



Obviously the same result is obtained by first gluing f and g together vertically, gluing f' and g' together vertically, and gluing the results together horizontally. This establishes the relation (4) in the present case and shows that $\pi_2(X, *)$ is Abelian.[2]

2.1.2 Comonoids in the Category of Groups

Dually there is the notion of a cogroup object in a category.

For this let $\mathfrak{C}$ be a category with direct sums and an initial object. Again the category of groups is an example with the one element group as initial object. The categorical direct sum in *Group* is what in group theory is called the free product.

A cogroup object in such a category $\mathfrak{C}$ is an object $C \in \mathfrak{C}$ together with a comutiplication $\mu : C \longrightarrow C \Sigma C$, a coinverse $\iota : C \longrightarrow C$, and a counit morphism $\varepsilon : C \longrightarrow I$. Here $I \in \mathfrak{C}$ is the initial object and Σ stands for the direct sum in $\mathfrak{C}$. These bits of structure are supposed to satisfy the dual axioms to those for a group object depicted by diagrams (1)–(3) that is the diagrams obtained by reversing all arrows (and replacing m by μ and e by ε) must be commutative. For a comonoid object leave out the coinverse and (the dual of) diagram (3).

It is now a theorem, [60 Kan], that the underlying group of a comonoid object in the category of groups is free as a group.

This has much to do with the fact that the categorical direct sum in *Group* is given by the free product construction.

2.1.3 Hopf's Theorem on the Cohomology of H-Spaces

An H-space is a based topological space $(X, *)$ together with a continuous map $m : X \times X \longrightarrow X$ such that $x \mapsto m(x, *)$ and $x \mapsto m(*, x)$ are homotopic to the

[2] In the present case of homotopy groups it can of course also easily be shown directly that vertical gluing and horizontal gluing give the same result and this is how things are done traditionally in text books; see e.g. {Hu, 1959 #6}.

identity.[3] The result of Heinz Hopf, [56 Hopf], see also [34 Félix et al.], alluded to is now as follows.

Let k be a field of characteristic zero and X a path connected H-space such that $H_*(X; k)$ is of finite type then $H^*(X; k)$ is a free graded-commutative graded algebra.

Here 'finite type' means that each $H_i(X; k)$ is finite dimensional and the cohomology algebra is graded-commutative (= commutative in the graded sense), i.e. $xy = (-1)^{\text{degree}(x)\text{degree}(y)}yx$. Thus the seemingly weak extra bit of structure 'H-space' has a profound influence on the (cohomological) structure of a space.

2.1.4 Intermezzo

Hopf algebras. Let R be a unital commutative ring. A graded module over R is simply a collection of modules over R indexed by the nonnegative integers.[4] Or, equivalently, it is a direct sum

$$M = \bigoplus_{i \in \mathbf{N} \cup \{0\}} M_i \tag{6}$$

An element $x \in M_i$ is said to be homogeneous of degree i. A graded module (6) is said to be of finite type if each of the M_i is of finite rank over the base ring R.

The tensor product of two graded modules M, N is graded by assigning degree $i + j$ to the elements from $M_i \otimes N_j$.

A graded algebra over R is a graded module (6) equipped with a graded associative multiplication and a unit element

$$m : M \otimes M \longrightarrow M, m(M_i \otimes M_j) \subset M_{i+j}; 1 \in M_0 \tag{7}$$

There are two notions of commutativity for graded algebras: (ordinary) commutativity, which means $xy = yx$, and graded-commutativity, which means $xy = (-1)^{\deg(x)\deg(y)}yx$. Both occur frequently in the literature and both will occur in the present paper.[5]

Correspondingly there are two versions for the multiplication in the tensor product of (the underlying graded modules) of graded rings, viz.

$$\begin{aligned} (x \otimes y)(x' \otimes y') &= xx' \otimes yy' \\ (x \otimes y)(x' \otimes y') &= (-1)^{\deg(y)\deg(x')}xx' \otimes yy' \end{aligned} \tag{8}$$

[3] Often in the literature for an H-space it is also required that the 'multiplication' m is associative up to homotopy. For the present result that is not required.

[4] These will be the only kind of gradings occurring

[5] If all the odd degree summands of the graded ring are zero the two notions agree. This can be used to unify things.

where in the second equation the elements x, x', y, y' are supposed to be homogeneous. The sign factor in the second equation of (8) is needed to ensure that the tensor product of two graded-commutative graded algebras is graded-commutative (as of course one wants it to be).

Dually a graded coalgebra over R is a graded module equipped with a coassociative comultiplication and a counit

$$\mu : M \longrightarrow M \otimes M, \mu(M_n) \subset \bigoplus_{i+j=n} M_i \otimes M_j; \varepsilon : M \longrightarrow R, \varepsilon(M_i) = 0 \text{ for } i > 0$$

Just as in the algebra case there are two notions of cocommutativity and two ways to define a coalgebra structure on the tensor product of two graded coalgebras. These two are as follows. Let C and D be two graded coalgebras with comultiplications μ_C, μ_D. Write

$$\mu_C(x) = \sum x'_i \otimes x''_i, \qquad \mu_D(y) = \sum y'_j \otimes y''_j$$

as sums of tensor products of homogeneous elements. Then the two graded coalgebra structures alluded to are

$$\begin{aligned} x \otimes y &\mapsto \sum x'_i \otimes y'_j \otimes x''_i \otimes y''_j \\ x \otimes y &\mapsto \sum (-1)^{\deg(y'_j)\deg(x'_i)} x'_i \otimes y'_j \otimes x''_i \otimes y''_j \end{aligned} \tag{9}$$

Next, a graded bialgebra B is a comonoid object in the category of graded algebras or, equivalently, a monoid object in the category of graded coalgebras. Here again there are two versions depending on what algebra and coalgebra structures are taken on $B \otimes B$. First there is an 'ordinary' bialgebra which happens to carry a grading. In this case the algebra and coalgebra structures are given by the first formulas of (8) and (9) Second there is the 'grade-twist' version in which the algebra and coalgebra structures on the tensor product are given be the second formulas from (8) and (9). Here 'ordinary twist' and 'grade twist', respectively, refer to the morphisms

$$x \otimes y \mapsto y \otimes x, x \otimes y \mapsto (-1)^{\deg(x)\deg(y)} y \otimes x$$

which make their appearance when the conditions are written out explicitly in terms of diagrams or elements.

Finally, a graded Hopf algebra is a graded bialgebra that in addition carries a so-called antipode. That is a morphism ι of graded modules of degree 0 (so that $\iota(H_i) \subset H_i$) that satisfies

$$m(\mathrm{id} \otimes \iota)\mu = e\varepsilon \text{ and } m(\iota \otimes \mathrm{id})\mu = e\varepsilon$$

A graded Hopf algebra over R is connected if the grade zero part H_0 is equal to R so that e and ε induce isomorphism of R with H_0.

An element x in a graded Hopf algebra (or bialgebra) is called primitive if it satisfies

$$\mu(x) = 1 \otimes x \div x \otimes 1 \tag{10}$$

These form a graded submodule $P(H)$ of the Hopf algebra H.

In the case of an 'ordinary twist' Hopf algebra the commutator product

$$[x, y] = xy - yx \tag{11}$$

turns $P(H)$ into a Lie algebra (that happens to carry a grading such that the Lie bracket is of degree 0.

In the case of a 'graded twist' Hopf algebra take

$$[x, y] = xy - (-1)^{\deg(x)\deg(y)} yx \tag{12}$$

to obtain a graded Lie algebra. That is a module equipped with a bilinear product [,] that satisfies graded anticommutativity and the graded Jacobi identity:

$$[x, y] = (-1)^{\text{degree}(x)\text{degree}(y)}[y, x]$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{\text{degree}(x)\text{degree}(y)}[y, [x, z]]$$

2.1.5 Milnor–Moore Theorem (Topological Incarnation)

Let PX be the Moore path space of a path connected based topological space $(X, *)$. That is the space of paths starting from $*$ with specified length (which is what the adjective 'Moore' means in this context). Assigning to a path its endpoint defines a continuous map $PX \longrightarrow X$, which is a fibration with ΩX, the space of Moore loops, as its fibre (over $*$). As PX ia contractible the long exact homotopy sequence attached to this fibration gives isomorphisms $\pi_n(X) \longrightarrow \pi_{n-1}(\Omega X)$. This can be used to transfer the Whitehead products $\pi_m(X) \times \pi_n(X) \longrightarrow \pi_{m+n=1}(X)$ to a Lie product (of degree zero) $(\pi_*(\Omega X) \otimes k) \times (\pi_*(\Omega X) \otimes k) \xrightarrow{m_{\Omega X}} \pi_*(\Omega X) \otimes k$, defining a graded Lie algebra L_X.

Composition of loops turns ΩX into a topological monoid and, up to homotopy there is an inverse as well. Using the Alexander – Whitney and Eilenberg – Zilber chain complex equivalences, see {Félix, 2001 #21}, p. 53ff, and the fact that taking homology of chain complexes commutes with tensor products, ibid. p. 48, the composition $\Omega X \times \Omega X \longrightarrow \Omega X$ and diagonal $\Delta : \Omega X \longrightarrow \Omega X \times \Omega X$ induce an algebra and coalgebra structure on $H_*(\Omega X)$. Moreover, essentially because a loop in a product $X \times Y$ is a pair of loops and composition of loops seen this way goes component-wise, the comultiplication morphism $H_*(\Omega X) \longrightarrow H_*(\Omega X) \otimes H_*(\Omega X)$ is an algebra morphism,[6] ibid. p. 225.

[6] This the origin of the unfortunate but frequently used notation 'Δ' for the comultiplication in a Hopf algebra.

All in all this turns $H_*(\Omega X)$ into a graded connected Hopf algebra (of the 'graded twist' kind).

Now let the coefficients ring used when taking cohomology be a field of characteristic zero.

2.1.6 Theorem ([79 Milnor et al.]

See also [34 Félix et al.], p. 293). Let X be a simply connected path connected topological space. Then the Hurewicz homomorphism for ΩX is an isomorphism of graded Lie algebras of L_X onto the graded Lie algebra of primitives of $H_*(\Omega X; k)$ and this isomorphism extends to an isomorphism of graded Hopf algebras of the universal enveloping algebra UL_X with $H_*(\Omega X; k)$.[7]

There is also a purely algebraic theorem that goes by the name 'Milnor-Moore theorem'. That one involves the notion of the universal enveloping algebra of a Lie algebra and will be discussed in Sect. 2.3

To conclude this Sect. 2.1 let me briefly mention two more simple results that, I feel, qualify as 'niceness theorems'. Both say that the presence of a Hopf algebra (bialgebra) structure has implications for the underlying algebra.

2.1.7 Cartier's Theorem on Nilpotents in Group Schemes

Let H be a finite dimensional Hopf algebra over a field of characteristic zero. Then the underlying algebra has no nilpotents. Actually a much stronger statement holds, see [28 Dolgachev]. The usual statement is: A group scheme of finite type over a field of characteristic zero is smooth. See loc. cit. and [102 Voskresenskii], p. 7.

In characteristic $p > 0$, Cartier's theorem does not hold. On $k[X]/(X^p)$ where k is a field of characteristic $p > 0$, there are the two comultiplications

$$X \mapsto 1 \otimes X + X \otimes 1, \quad X \mapsto 1 \otimes X + X \otimes 1 + X \otimes X$$

and both define a bialgebra, and in fact Hopf algebra structure on $k[X]/(X^p)$. These two Hopf algebras (finite group schemes) are traditionally denoted α_p and μ_p.

Let k be a field and n an integer ≥ 2. Then there is no bialgebra structure on the algebra $M^{n \times n}(k)$ of $n \times n$ matrices over k. See [26 Dascalescu et al.], p. 173.

It is a completely unknown which products of matrix algebras do carry (admit) a bialgebra structure.

Much of mathematics concerns statements as to what consequences follow from what assumptions. So it can be argued that there is nothing particularly special about the results described above. However, I feels that there is something special, something particularly elegant, about the results described. Part of the general problem is to understand why and in what sense.

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[7] Universal enveloping algebras are the topic of Sect. 2.2.1.

Several of the theorems above are 'freeness theorems'. they say that in the presence of suitable extra structure an object is free. Here follow five more. For the first three the 'extra structure' is that the object in question is imbedded in a free object. In some categories that means nothing; in others it is a strong bit of extra structure. Just what categorical properties rule this behaviour is completely unknown.

2.1.8 Nielsen–Schreier Theorem

A subgroup of a free group is free, [92 Schreier; 84 Nielsen]; [99 Suzuki], p. 181.

2.1.9 Shirshov–Witt Theorem

Lic subalgebras of a free Lie algebra are free, [97 Shirshov; 105 Witt]. There is also, up to a point, a braided version, [61 Kharchenko].

2.1.10 Bergman Centralizer Theorem

The centralizer of a non–scalar element in a free power series ring $k\langle\langle X\rangle\rangle$ is of the form $k[[c]]$, [18 Bergman]; [25 Cohn], p. 244. Here c is a single element!

2.1.11 *H*-Spaces

The fundamental group of a cogroup object in the homotopy category of 'nice' based topological spaces is free. See [19 Berstein]. These objects are sometimes called H'-spaces (as a kind of dual or opposite object to H-spaces).

2.1.12 Bott–Samelson Theorem

The homology algebra $H_*(\Omega\Sigma X; k)$ is a free algebra generated by $H_*(X; k)$, [22 Bott et al.; 19 Berstein].

Here Σ is the suspension functor and Ω is the loop space functor on based topological spaces. These are adjoint and there results a topological morphism $X \longrightarrow \Omega\Sigma X$. The multiplication comes from the fact that loops at the base point can be composed making a loop space an H-space.

2.2 Universal Object Examples

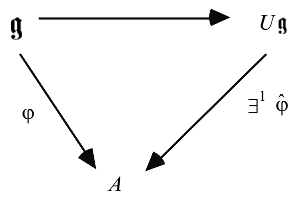
Here the theme is that objects that are defined in terms of some universal property have a tendency to pick up extra bits a structure.

2.2.1 The Universal Enveloping Algebra of a Lie Algebra

Let A be a unital associative algebra over a unital commutative base algebra R. Associated to A there is a Lie algebra structure on A defined by the commutator difference

$$[x, y]_A = xy - yx \tag{13}$$

Let $\mathfrak{g}$ be a Lie algebra. A Lie morphism from $\mathfrak{g}$ to a unital associative algebra A is a module morphism $\varphi : \mathfrak{g} \longrightarrow A$ such that $\varphi([x, y]_{\mathfrak{g}}) = [\varphi x, \varphi y]_A$. The universal enveloping algebra on $\mathfrak{g}$ is a unital associative algebra $U\mathfrak{g}$ together with a Lie morphism $i : \mathfrak{g} \longrightarrow U\mathfrak{g}$ such that for each Lie morphism $\varphi : \mathfrak{g} \longrightarrow A$ there is a unique morphism of associative algebras $\tilde{\varphi} : U\mathfrak{g} \longrightarrow A$ such that $\tilde{\varphi} \circ i = \varphi$. Pictorially (in diagram form) this can be rendered as follows



The associative unital algebra $U\mathfrak{g}$ is a very nice one. For instance there is the Poincaré–Birkhoff–Witt theorem that specifies (under suitable circumstances) a monomial basis for it. This results basically from the construction of $U\mathfrak{g}$. (And one wonders whether this PBW theorem can be deduced directly from the characterizing universality property.)

What is of interest in the present setting is that the universality property immediately implies that $U\mathfrak{g}$ has more structure; in fact that it is a Hopf algebra. This arises as follows. Consider the associative algebra $U\mathfrak{g} \otimes U\mathfrak{g}$ and the morphism $x \mapsto 1 \otimes x + x \otimes 1$ from $\mathfrak{g}$ into it. It is immediate that this is a Lie morphism and hence there is a corresponding (unique) morphism of associative algebras $U\mathfrak{g} \longrightarrow U\mathfrak{g} \otimes U\mathfrak{g}$. It is immediate that this turns $U\mathfrak{g}$ into a Hopf algebra.

There is a completely analogous picture for graded Lie algebras.

Of course the universal problem described here is an instance of an adjoint functor situation. Let **Lie** be the category of Lie algebras (over R) and **Alg** the category of unital associative algebras (over R). Then associating to an associative algebra A its commutator difference product is a (forgetful) functor $V : \mathbf{Alg} \longrightarrow \mathbf{Lie}$ and $\mathfrak{g} \longrightarrow U\mathfrak{g}$ is a functor the other way that is left adjoint to it:

$$\mathbf{Lie}(\mathfrak{g}, V(A)) \cong \mathbf{Alg}(U\mathfrak{g}, A) \tag{14}$$

In the case of a forgetful functor a left adjoint to it yields what are often called free objects (as in this case). Thus $U\mathfrak{g}$ is the free associative algebra on the Lie algebra $\mathfrak{g}$.

A right adjoint functor to a forgetful functor gives cofree objects. An example of a cofree construction will occur below.

The very important notion of adjointness is due to Daniel Kan, [59 Kan] and as Saunders Mac Lane says in the preface of [74 Mac Lane] "Adjoint functors arise everywhere".

If (F, G) is an adjoint functor pair, i.e. $\mathfrak{C}(FX, Y) \cong \mathfrak{D}(X, GY)$ functorially (loosely formulated), one expects niceness properties for both the FX's and the GY's. And indeed many niceness results fall into this scope with the proviso that often these objects pick up extra properties which are not implicit in the adjoint situation.

2.2.2 The Group Algebra of a Group

Much the same picture holds for the group algebra of a group. Except much easier. Here the 'forgetful functor' assigns to an algebra A its group A^* of invertible elements. Recall that the group algebra kG of a group is the free module over k with basis G and the multiplication determined on this basis by the group multiplication. The adjointness equation now is:

$$\mathbf{Group}(G, A^*) \cong \mathbf{Alg}_k(kG, A) \tag{15}$$

There is again a free Hopf algebra structure. For this, to put things formally on the same footing as in the case of the universal enveloping algebra, consider the morphism

$$G \longrightarrow (kG \otimes kG)^*, g \mapsto g \otimes g$$

which by the adjointness equation (15), gives rise to a morphism of algebras $kG \longrightarrow kG \otimes kG$ turning kG into a bialgebra (and a Hopf algebra using the group inverse). Of course in this case things are so simple that it is not worthwhile to go through this yoga.

2.2.3 Free Algebras

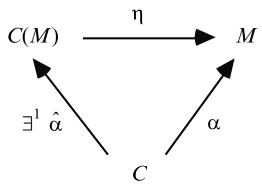
Everyone knows how to construct the free algebra over a module (or a set). The tensor algebra does the job and that is a very nice structure. Less known is that this also works in the setting **CoAlg – HopfAlg**, where **CoAlg** and **HopfAlg** are suitable categories of coalgebras and Hopf algebras over a suitable base ring. See [81 Moore] and [54 Hazewinkel]. This gives the free Hopf algebra on a coalgebra.

2.2.4 Cofree Coalgebras

Given a module M, the cofree coalgebra over[8] M would be a coalgebra $C(M)$ together with a module morphism $C(M) \xrightarrow{\eta} M$ such that for each coalgebra C

[8] It pays to be terminologically careful in this context. I prefer to speak of the free algebra *on* a module and the cofree coalgebra *over* a module.

together with a morphism of modules $C \xrightarrow{\alpha} M$ there is a unique morphism of coalgebras $\hat{\alpha} : C \longrightarrow C(M)$ such that $\eta\hat{\alpha} = \alpha$.



Whether the cofree coalgebra over a module always exist is not quite settled, [46 Hazewinkel]; they certainly exist in many cases. In the connected graded context they always exist and are given by the tensor coalgebra, again a very nice structure.

And in this connected graded context there is the **Alg–HopfAlg** version of the cofree Hopf algebra over an algebra, [54 Hazewinkel; 81 Moore].

2.2.5 The Classifying Spaces $\mathbf{BU}_n$

A completely different kind of universal object is formed by the complex Grassmannians and their inductive limits the classifying spaces $\mathbf{BU}_n$.

Consider the complex vector space $\mathbf{C}^{n+r}$ and define the complex Grassmannian

$$\mathbf{Gr}_n(\mathbf{C}^{n+r}) = \{V : V \text{ is an } n\text{-dimensional subspace of } \mathbf{C}^{n+r}\} \tag{16}$$

This set has a natural structure of a smooth manifold (in fact a complex analytic manifold). Letting r go to infinity (which technically means taking an inductive limit) gives the classifying space

$$\mathbf{BU}_n = \varinjlim \mathbf{Gr}_n(\mathbf{C}^{n+r}) = \mathbf{Gr}_n(\mathbf{C}^{\infty}) \tag{17}$$

It is also perfectly possible to define and work directly with the most right hand side of (17). There is a canonical complex vector bundle over $\mathbf{BU}_n$ which is colloquially defined by saying the fibre over $x \in \mathbf{BU}_n$ "is x". More precisely this canonical vector bundle γ_n is

$$\gamma_n = \{(x, v) : x \in \mathbf{BU}_n, v \in x\} \text{ with projection } (x, v) \mapsto x, \gamma_n \longrightarrow \mathbf{BU}_n \tag{18}$$

There is now the following universality/clasifying property. For every paracompact space X with an n-dimensional complex vector bundle ξ over it there is a map $f_\xi : X \longrightarrow \mathbf{BU}_n$ such that ξ is isomorphic (as a vector bundle) to the pullback $f_\xi^*(\gamma_n)$. Moreover f_ξ is unique up to homotopy.

The remarkable thing here is that the classifying spaces $\mathbf{BU}_n$ are so elegant and simple (as are the universal bundles over them). There are more nice properties. Jumping the gun a little – these spaces will return later – the cohomology of these spaces is particularly nice

$$H^*(\mathbf{BU}_n;\mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots, c_n], \deg(c_r) = 2r \tag{19}$$

All this can be found in [58 Husemoller; 80 Milnor et al.] (and many other books).

2.3 Niceness Theorems for Hopf Algebras

The structure of a Hopf algebra is a heavy one. Indeed at one time they were thought to be so rare that each and every one deserves the most careful study, {Kaplansky, #62}. This is not anymore the case. Hopf algebras abound. Still the structure is not strong enough to produce good niceness theorems. However if one adds conditions like graded and connected some strong structure theorems emerge. These are, e.g. the Leray and Milnor–Moore theorems which will both be described immediately below. In addition there is the Zelevinsky theorem, a structure theorem due to Grünenfeld, {Grünenfeld, #63} and much more, see, e.g. {Masuoka, 2007 #64}. However, whether the various available classification theorems for Hopf algebras qualify as niceness theorems is debatable. I think mostly not.

2.3.1 The Leray Theorem on Commutative Hopf Algebras

Let H be a commutative graded connected Hopf algebra of finite type over a field of characteristic zero. Then the underlying algebra is commutative free. There is also a graded commutative version. The original theorem appears in [68 Leray]. For an up-to-date short account see [87 Patras]. There are all kinds of generalizations, e.g. to an operadic setting, see [86 Patras; 72 Livernet; 35 Fressé]

2.3.2 The Milnor–Moore Theorem on Cocommutative Hopf Algebras

Let H be a cocommutative graded connected Hopf algebra of finite type over a field of characteristic zero. Then the underlying algebra is the universal enveloping algebra of the Lie algebra of primitives $P(H)$ of H, [79 Milnor et al.].

This is the algebraic incarnation referred to in Sect. 2.1.5.

The Milnor–Moore theorem is a dual of the Leray theorem. To realize this recall from Sect. 2.2 above that $U\mathfrak{g}$ is the free object in **Ass** on the object $\mathfrak{g} \in$ **Lie**.

2.4 Large vs. Nice

There is a tendency for (really) nice objects to be big (or very small). A prime example is

$$\mathbf{Symm} = \mathbf{Z}[h_1, h_2, h_3, \cdots] \tag{20}$$

the ring of polynomials over the integers in countably infinite many commuting variables over the integers. This object will be discussed in some detail further on.

Inversely big objects have a better change of being nice.

In this Sect. 2.4.1 give some examples of this phenomenon.

2.4.1 Big Projective Models are Free

This result is due to Hyman Bass, [16 Bass]. For a precise statement see loc. cit. (corollary 3.2). The key ingredient is the following elegant observation.[9]

If $P \oplus Q \cong F$ with F a non–finitely generated free module, then $P \oplus F \cong F$.

The proof is simplicity itself and clearly shows the power and usefulness of infinity.

$$\begin{aligned} F &\cong F \oplus F \oplus \cdots \cong P \oplus Q \oplus P \oplus Q \oplus \cdots \\ &\cong P \oplus F \oplus F \oplus \cdots \cong P \oplus F \end{aligned}$$

2.4.2 General Linear Groups in Various Dimensions

Let k be the field of real numbers, complex numbers or even the quaternions. The general linear groups $\mathbf{GL}_n(k)$ for finite natural numbers are homotopically and cohomologically far from trivial.

Things change drastically in infinite dimension.

2.4.3 Kuiper's Theorem, [65 Kuiper]

Let H be real or complex or quaternionic Hilbert space. Then the general linear group $\mathbf{GL}(H)$ is contractible.

There is also an important equivariant extension due to Graeme Segal, [96 Segal].

Much related is Bessaga's theorem, [20 Bessaga; 21 Bessaga et al.], to the effect that every infinite dimensional Hilbert space is diffeomorphic with its unit sphere.

Kuiper's famous theorem is the key to the classification of Hilbert manifolds, [23 Burghelea et al.; 31 Eells et al.; 32 Eells et al.; 82 Moulis; 83 Moulis].

Here is a table on differential topology in various dimensions as things seem to be constituted at present.

---

[9] Hyman Bass calls it "an elegant little swindle".

1	2	3	4	5	6	⋯	< infinity	infinity
Real easy	Easy	difficult	difficult; boapw	good techniques	good techniques		good techniques	real nice

Here 'good techniques' refers mainly to Smale's handiebody theory. The acronym 'boapw' means 'best of all possible worlds' and refers to the fact that all $\mathbf{R}^n$ for $n \neq 4$ have a unique differentiable structure, but $\mathbf{R}^4$ has over countably infinite different differentiable structures.[10]

2.5 *Extremal Objects and Niceness*

In the world of optimization theory and variational calculus and analysis it is relatively well known that extremal objects tend to be nice (have lots of symmetry), even when bifurcation occurs.

There are also various notions of minimality in algebra and topology and these also tend to be 'nice'. For instance the Sullivan minimal models for rational homotopy, see [34 Félix et al.], are definitely nice.

In the world of operads and PROP's, etc. there are by way of example the following theorems, see [78 Merkulov].

– The minimal resolution of $\mathcal{Ass}$ is a differential graded *free* operad.
– The minimal resolution of $\mathcal{LieB}$ is a *free* differential graded PROP.

Sullivan minimal models and operads, PROP's, etc are highly technical notions and giving details would take me far beyond the scope and intentions of this paper.

I have no doubt that there are more niceness results for minimal resolutions.[11]

2.6 *Uniqueness and Rigidity and Niceness*

For instance **Symm**, see (20) above and below, is unique and rigid as a coring object in the category of unital commutative rings and **MPR**, the Reutenauer-Malvenuto-Poirier Hopf algebra of permutations is rigid and likely unique, see [51 Hazewinkel; 53 Hazewinkel]. And indeed they are very nice objects.

[10] This is a fact that tends to make 'multiple world' enthusiasts happy.

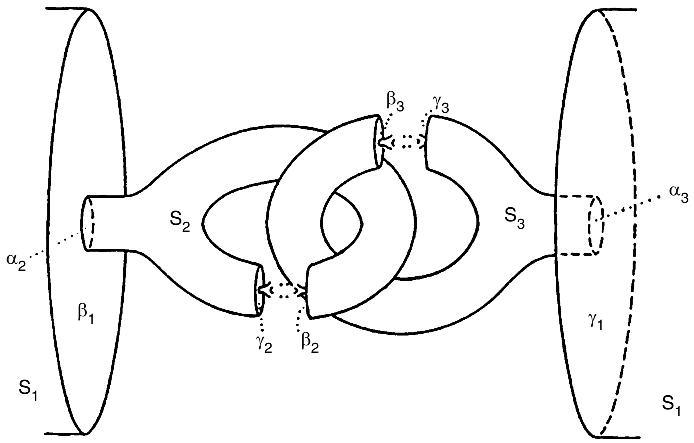
[11] There are at least three meanings for the word 'resolution' and the phrase 'minimal resolution' in mathematics: resolution of singularities, resolution of a module in homological algebra, resolution in (automatic) theorem proving. Outside mathematics there are many more additional meanings.

2.7 Counterexamples and Paradoxical Objects

Not only objects and constructions can exhibit the 'niceness phenomenon' but also counterexamples. This subsection contains a few examples of that.

2.7.1 The Alexander Horned Sphere

First the construction as illustrated by the picture below. Take a hollow cylinder closed at both ends and bend around so that the two ends face each other. Now from each end extrude a horn and interlock them as shown; there result two locations of disks facing each other. Repeat ad infinitum.



The Alexander horned sphere together with its interior is (homeomorphic to) a topological 3-ball. The exterior is not simply connected. This shows that the analogue of the Jordan-Schönflies theorem from dimension 2 does hold in dimension 3.

For some more information on the Alexander horned sphere and its uses see [1; 2].

Somewhat surprisingly (to me in any case), the filled Alexander horned sphere can be used for a monohedral tiling of $\mathbf{R}^3$, [101 Tang].

2.7.2 The Approximation Property

A Banach space is said to have the approximation property if every compact operator is a limit of finite rank operators.

Equivalently a Banach space X has the approximation property if for every compact subset $K \subset X$ and every $\varepsilon > 0$ there is an operator $T : X \longrightarrow X$ of finite rank such that $\|Tx - x\| < \varepsilon$ for all $x \in K$.

Every Banach space with a (Schauder) basis has the approximation property. This includes Hilbert spaces and the l^p spaces.

However, not every Banach space has the approximation property. In 1973 Per Enflo, [33 Enflo], constructed a counterexample.

I do not think this counterexample qualifies as a nice one. However the very nice Banach space of bounded operators on l^2 is also a counterexample, [100 Szankowski].[12]

2.7.3 The Banach–Tarski Paradox

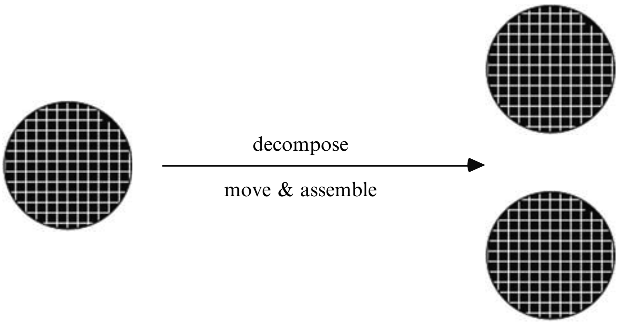
In 1924 Stefan Banach and Alfred Tarski proved the following bizarre seeming statement, [15 Banach et al.].

For two bounded subsets A, B of a Euclidean space of dimension at least three with nonempty interior there exist finite decompositions into disjoint subsets

$$A = A_1 \cup \cdots \cup A_k \quad B = B_1 \cup \cdots \cup B_k$$

such that A_i is congruent to B_i for all $i = \mathrm{I}, \cdots, k$. I.e. A_i becomes B_i under a Euclidean motion.

This is now known as the strong form of the Banach-Tarski paradox. It does not hold in dimensions 1 and 2. A consequence is



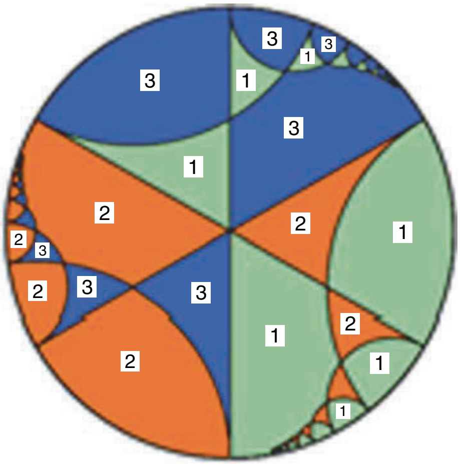
A solid ball can be decomposed into a finite number of point sets that can be reassembled to form two balls identical to the original.

Here 'move' means a Euclidean space move: a combination of translations, rotations and reflections. For some more information on the Banach-Tarski paradox see [103 Wagon; 3; 4].

Thus 'move' is simple enough. The decomposition, however, is complicated. For one thing at least some of the components must be nonmeasurable. Also things are in three dimensions and Cantor-like sets in three dimensions are difficult to visualize.

[12] In article [3] 'Szankowski' is rendered 'Shankovskii' which makes it to find the paper.

Fortunately Stan Wagon found a two dimensional analogue in hyperbolic space and the picture is remarkably beautiful



2.7.4 Julia and Fatou Sets

Here is a question not untypical of those that were asked in general (point-set) topology almost a century ago when people started to realize just how strange topological spaces could be.

Is it possible to divide the square into three regions so that the boundary between two of them is also the boundary between the two other pairs of regions.

The first answer was given by L E J Brouwer in the form of a simple construction repeated ad infinitum. However, the resulting picture is absolutely not beautiful. Nowadays there are the basins of attraction of discrete dynamical systems such as $x \mapsto x^3 - 1$ which has three basins of attraction (Fatou sets), one for each of the roots of $x^3 - 1$ and each pair has the same boundary (Julia set), see [8].

This is part of the world of fractals and (deterministic) chaos, [95 Schuster], and many of the pictures are extraordinarily beautiful, [88 Peitgen et al.][13].

2.7.5 Sorgenfrey Line

As a set the Sorgenfrey is the set of real numbers. It is given a topology by taking as a basis the halfopen intervals $[a, b), a < b$. This topology is finer than the usual one. For instance the sequence $\{n^{-1}\}_{n \in \mathbf{N}}$ converges to zero but $\{-n^{-1}\}_{n \in \mathbf{N}}$ does not. The Sorgenfrey line serves as a counterexample to several topological

[13] Beautiful and arresting enough that the Sparkasse in Bremen organized an exhibition of them in 1984.

properties, [98 Steen et al.]. The point here (as far as this paper is concerned) is not that such counterexamples exist but that there is such a nice regular one. There is also a Sorgenfrey plane, loc. cit.

For some more information see also [73 Lukes; 10].

2.7.6 Exotic Spheres

A further example that fits in this section is that of exotic spheres (Milnor spheres). This deals with existence of differentiable structures on topological spheres, especially the seven dimensional ones, that differ from the standard one. They were the first examples of this phenomenon of distinct differentiable structures on the same topological manifold. This topic is rather more technical, and so I content myself with giving two references to internet accessible documents, [7; 91 Rudyak].

2.8 An Excursion into Formal Group Theory

A one dimensional formal group law over a commutative unital ring A is a power series $F(X, Y)$ in two variables with coefficients in A such that

$$F(X, 0) = X, F(0, Y) = Y, F(X, F(Y, Z)) = F(F(X, Y), Z) \tag{21}$$

Two examples are the multiplicative formal group law and the additive formal group law

$$\hat{\mathbf{G}}_m(X, Y) = X + Y + XY, \quad \hat{\mathbf{G}}_a(X, Y) = X + Y \tag{22}$$

Both examples are nontypical in that they are polynomial; polynomial formal group laws are very rare.

More generally for any n, including $n = \infty$, an n–dimensional formal group over A is an n–tuple of power series in two groups of n indeterminates $F(X; Y)$ such that

$$F(X; 0) = X, F(0; Y) = Y, F(X; F(Y; Z)) = F(F(X; Y); Z) \tag{23}$$

However, certainly from the point of view of applications, one dimensional formal groups are by far the most important, especially one dimensional formal groups over the integers, rings of integers of algebraic number fields, and over polynomial rings over the integers.

The only other that currently seems important is the infinite dimensional formal group $\hat{W}$ of the Witt vectors which is defined by the same polynomials that define the addition of Witt vectors; see the next Sect. 2.9.

A standard reference for formal groups is [45 Hazewinkel].

2.8.1 Lazard Commutativity Theorem

Let A be a ring that has no elements that are simultaneously torsion and nilpotent. Then every one dimensional formal group over A is commutative; i.e. satisfies $F(X, Y) = F(Y, X)$.

2.8.2 Universal Formal Groups

Given a formal group $F(X, Y)$ over A and a morphism of rings $\alpha : A \longrightarrow B$ one obtains a formal group $\alpha_* F(X, Y)$ over B by applying α to the coefficients of $F(X, Y)$.

A one dimensional commutative formal group $F_L(X, Y)$ over a ring L is called universal[14] if for every one dimensional formal group $F(X, Y)$ over a ring A there is a unique morphism of rings $\alpha^F : L \longrightarrow A$ such that $\alpha^F_* F_L(X, Y) = F(X, Y)$.

That such a thing exists and is unique is a triviality. What is very remarkable is the theorem of Lazard, [66 Lazard], that L is the ring of polynomials in an infinity of indeterminates over the integers. The standard proof is a bitch and highly computational.

2.8.3 Morphisms

A morphism of formal groups from an m–dimensional formal group $F(X; Y)$ to an n–dimensional formal group $G(X; Y)$ is an n–tuple of power series in m indeterminates $\varphi(X)$ such that

$$\varphi(0) = 0, G(\varphi(X); \varphi(Y)) = \varphi(F(X; Y))$$

If $\varphi(X) \equiv X$ mod (degree 2) the morphism is said to be strict.

2.8.4 Logarithms

Let A be a ring of characteristic zero so that the canonical ring morphism $A \longrightarrow A \otimes_{\mathbf{Z}} \mathbf{Q} = A_{\mathbf{Q}}$ is injective; let $F(X, Y)$ be a one dimensional formal group over A. Then over $A_{\mathbf{Q}}$ there exists a power series $f(X) = X + a_2 X^2 + \cdots$ such that

$$F(X, Y) = f^{-1}(f(X) + f(Y)) \tag{24}$$

Here f^{-1} is the compositional inverse of f, i.e. $f^{-1}(f(X)) = X$. This f is called the logarithm of F. In the case of the multiplicative formal group, see (22), the

[14] This is a rather different ‘universal’ than e.g. in ‘universal enveloping algebra’. The ‘L’ in these sentences stands for Lazard.

logarithm is

$$\log(1+X) = X - 2^{-1}X^2 + 3^{-1}X^3 - 4^{-1}X^4 + \cdots$$

Indeed, $\log(1+X+Y+XY) = \log(1+X) + \log(1+Y)$. The terminology derives from this example. The logarithm of a formal group is a strict isomorphism of the formal group to the additive formal group; but over $A_{\mathbf{Q}}$.

It is at the level of logarithms that the recursive structure of formal groups appears; a recursive structure that was totally unexpected.

There are also logarithms for more dimensional commutative formal groups.

2.8.5 *p*–Typical Formal Groups

A one dimensional formal group over a characteristic 0 ring is p–typical if its logarithm is of the form

$$f(X) = X + b_1X^p + b_2X^{p^2} + \cdots$$

There is a better definition, see [45 Hazewinkel], which works always and also in the more dimensional case. But this one will do for the purposes of the present paper.

Over a $\mathbf{Z}(p)$–algebra every formal group is strictly isomorphic to a p-typical one, [24 Cartier]. If the ring over which the formal group is defined is of characteristic zero the isomorphism is easily described: take the logarithm and change all coefficients of non-p-powers of X to zero.

2.8.6 The Universal *p*–Typical Formal Group, [47 Hazewinkel]

Take a prime number p and consider the following ring with endomorphism

$$\mathbf{Z}[V] = \mathbf{Z}[V_1, V_2, V_3, \cdots], \psi(V_n) = V_n^p \tag{25}$$

Define

$$a_n(V) = \sum_{i_1+\cdots+i_r=n} p^{-r} V_{i_1} V_{i_2}^{p^{i_1}} V_{i_3}^{p^{i_1+i_2}} \cdots V_{i_r}^{p^{i_1+\cdots+i_{r-1}}} \tag{26}$$

Thus the first few of these polynomials are

$$a_1(V) = p^{-1}V_1,\ a_2(V) = p^{-2}V_1V_1^p + p^{-1}V_2,$$
$$a_3(V) = p^{-3}V_1\mathrm{V}_1^{\mathrm{p}}V_1^{p^2} + p^{-2}V_1V_2^p + p^{-2}V_2V_1^{p^2} + p^{-1}V_3$$

This sequence of polynomials has both a left and a right recursive structure.

The left recursive structure is

$$a_n(V) = \sum_{i=1}^{n} p^{-1} V_i \psi^i (a_{n-i}(V)) \quad \text{(where } a_0(V) = 1\text{)}$$

and the right recursive structure is

$$pa_n(V) = a_{n-1}(V)V_1^{p^{n-1}} + a_{n-2}(V)V_2^{p^{n-2}} + \cdots + a_1(V)V_{n-1}^{p} + V_n$$

Now consider

$$\begin{aligned} f_V(X) &= X + a_1(V)X^p + a_2(V)X^{p^2} + a_3(V)X^{p^3} + \cdots \\ F_V(X, Y) &= f_V^{-1}(f_V(X) + f_V(Y)) \end{aligned} \tag{27}$$

The left recursive structure is used to prove that $F_V(X, Y)$ is integral, i.e. has its coefficients in $\mathbf{Z}[V]$ and hence is a formal group over $\mathbf{Z}[V]$ and, subsequently, to prove that it is the universal p–typical formal group which means that every p–typical formal group can be obtained from it by a suitable ring morphism from $\mathbf{Z}[V]$.

The right recursive structure then leads to important applications to, e.g. complex cobordism theory in algebraic topology and Dirichlet series in number theory.

The important thing here is not that a universal p–typical formal group exists but that it has these very simple and elegant recursive structures.

The universal p–typical formal groups can be simply fitted together to give a construction of the universal formal group.

2.8.7 Formal Groups from Cohomology

Let h^* be a multiplicative extraordinary cohomology theory with first Chern classes. What all these words really mean is not so important at the present stage. Suffice that many of the better known cohomology theories are like this. The point is that under these circumstances there is a universal formula for the first Chern class of a tensor product of line bundles in terms of the first Chern classes of the factors.

$$c_1(\xi \otimes \eta) = \sum_{ij} a_{ij} c_1(\xi)^i c_1(\eta)^j$$

defining a formal group over $h^*(pt)$.

$$F_{h^*}(X, Y) = \sum a_{ij} X^i Y^j, \; a_{ij} \in h^*(pt)$$

Here are some examples.

$h^* = H^*$, ordinary cohomology, $F_H = \hat{\mathbf{G}}_a$, the additive formal group

$h^* = K^*$, complex K–theory, $F_K(X, Y) = X + Y + uXY$, where u is the Bott periodicity element; a version of the multiplicative formal group.

$h^* = \mathbf{MU}^*$, complex cobordism. In this case the formal group has logarithm $f_{\mathbf{MU}}(X) = \sum_{n=0}^{\infty} \frac{[\mathbf{CP}^n]}{n+1} X^{n+1}$. Here $\mathbf{CP}^n$ is n–dimensional complex projective space and $[\mathbf{CP}^n]$ is its complex cobordism class in $\mathbf{MU}^*(pt)$. This profound result is due to A S Mishchenko, see appendix 1 of [85 Novikov].

$h^* = \mathbf{BP}^*$, Brown–Petersen cohomology, the 'prime p part' of complex cobordism. Its formal group is the p–typification of the one of complex cobordism, so that its logarithm is $f_{\mathbf{BP}}(X) = \sum_{r=0}^{\infty} \frac{[\mathbf{CP}^{p^r-1}]}{p^r} X^{p^r}$.

For more details see [45 Hazewinkel] and the references given there and especially [90 Ravenel].

There is more. The formal group of complex cobordism is the universal one, [89 Quillen].

The remarkable, elegant and nice aspect here is that in terms of cobordism the universal formal group is so simple and regular.

It follows from the Quillen theorem that $F_{\mathbf{BP}}(X, Y)$ with logarithm $f_{\mathbf{BP}}(X)$ is the universal p-typical formal group law. But there is also an explicit construction of the universal p-typical formal group law, (27). This has all kinds of consequences for complex cobordism and Brown-Petersen cohomology, see [49 Hazewinkel; 45 Hazewinkel; 90 Ravenel].

Quillen's theorem also goes a fair way towards establishing that complex cobordism is the most general cohomology theory.

2.9 *The Amazing Witt Vectors and Their Gracious Applications*[15]

Let **CRing** be the category of unital commutative associative rings. The big Wiit vectors constitute a functor $W : \mathbf{CRing} \longrightarrow \mathbf{CRing}$ which has an amazing number of universality properties. For a fair amount of information on this functor see [54 Hazewinkel] and the references quoted there.

2.9.1 Definition of the Functor of the Big Witt Vectors

As a set $W(A) = \Lambda(A)$ is the set of all power series with coefficients in A with constant term 1.

$$W(A) = \Lambda(A) = \{1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots : a_i \in A\} \qquad (28)$$

[15] Free after Terry Pratchett, The amazing Maurice and his educated rodents, Corgi, 2002 (and a multitude of other sources).

Multiplication of such power series defines an Abelian group structure on $W(A)$ with as neutral element the power series 1. This is the underlying group of the to be defined ring structure on $W(A)$. The multiplication on $W(A)$ is uniquely determined by the requirement that the very special power series $(1-xt)^{-1}$ multiply as

$$(1-xt)^{-1} * (1-yt)^{-1} = (1-xyt)^{-1} \tag{29}$$

and the demands of distributivity (of multiplication over addition) and functoriality. Just how this works will be indicated immediately below.

The functoriality of $W(-)$ is component-wise, i.e. it is given by

$$W(f)(1+a_1t+a_2t^2+a_3t^3+\cdots) = 1+f(a_1)t+f(a_2)t^2+f(a_3)t^3+\cdots \tag{30}$$

The functor W is obviously representable by the ring $\mathbf{Symm} = \mathbf{Z}[h_1, h_2, h_3, \cdots]$ of polynomials in a countable infinity of indeterminates over the integers. The functorial correspondence is:

$$1+a_1t+a_2t^2+a_3t^3+\cdots \leftrightarrow f: \mathbf{Symm} \to A,\ f(h_n)=a_n \tag{31}$$

It is convenient to view the h_n as the complete symmetric functions in another countably infinite set of indeterminates $\xi_1, \xi_2, \xi_3, \cdots$ which can be encoded as

$$1+h_1t+h_2t^2+h_3t^3+\cdots = \prod_i \frac{1}{(1-\xi_i t)} \tag{32}$$

Now let $h'_1, h'_2, h'_3, \cdots$ be a second set of commuting indeterminates viewed as the complete symmetric functions in $\eta_1, \eta_2, \eta_3, \cdots$. that commute with the ξ. Then distributivity requires that

$$(1+h_1t+h_2t^2+h_3t^3+\cdots)*(1+h'_1t+h'_2t^2+h'_3t^3+\cdots) = \prod_{i,j} \frac{1}{(1-\xi_i\eta_j t)} \tag{33}$$

This makes sense because the right hand side of (33) is symmetric in the ξ and in the η and so, by the fundamental symmetric functions theorem there are unique polynomials

$$\Pi_1(h_1; h'_1),\ \Pi_2(h_1, h_2; h'_1, h'_2),\ \Pi_3(h_1, h_2, h_3; h'_1, h'_2, h'_3), \cdots \tag{34}$$

such that

$$\begin{aligned}&(1+h_1t+h_2t^2+h_3t^3+\cdots)*(1+h'_1t+h'_2t^2+h'_3t^3+\cdots)\\ &= 1+\Pi_1(h_1; h'_1)t+\Pi_2(h_1, h_2; h'_1, h'_2)t^2+\Pi_3(h_1, h_2, h_3; h'_1, h'_2, h'_3)t^3+\cdots\end{aligned} \tag{35}$$

(That the multiplication polynomials Π_n depend only on the first n h_i and h'_i is easily seen by degree considerations.)

By functoriality these polynomials determine the multiplication on each $W(A)$ in the sense that for $a(t) = 1 + a_1t + a_2t^2 + a_3t^3 + \cdots$ and $b(t) = 1 + b_1t + b_2t^2 + b_3t^3 + \cdots$ in $W(A)$ their product is

$$a(t)*b(t) = 1 + \Pi_1(a_1; b_1)t + \Pi_2(a_1, a_2; b_1, b_2)t^2 + \Pi_3(a_1, a_2, a_3; b_1, b_2, b_3)t^3 + \cdots$$

Of course the sum in $W(A)$ is also defined by universal polynomials. These are

$$\Sigma_n(h_1, \cdots, h_n; h'_1, \cdots, h'_n) = \sum_{i+j=n} h_i h'_j \text{ where } \quad h_0 = h'_0 = 1 \tag{36}$$

Another way of expressing most of this is to say that

$$\begin{aligned} h_n &\mapsto \Sigma_n(h_1 \otimes 1, \cdots, h_n \otimes 1; 1 \otimes h_1, \cdots, 1 \otimes h_n) \\ h_n &\mapsto \Pi_n(h_1 \otimes 1, \cdots, h_n \otimes 1; 1 \otimes h_1, \cdots, 1 \otimes h_n) \end{aligned} \tag{37}$$

define on **Symm** (most of) the structure of a coring object in the category **CRing**, which hence, via (31) defines a functorial ring structure on the $W(A)$.

2.9.2 Lambda Rings and Sigma Rings

A pre-sigma-ring (pre-σ-ring) is a unital commutative ring A that comes with extra nonlinear operators that behave (in a very real sense) like symmetric powers. That is, there are operators

$$\sigma_i : A \longrightarrow A, i = 1, 2, \cdots ; \sigma_1 = \text{id} \tag{38}$$

such that

$$\sigma_n(x + y) = \sigma_n(x) + \sum_{i=1}^{n-1} \sigma_i(x)\sigma_{n-i}(y) + \sigma_n(y) \tag{39}$$

It is often useful to have the notation σ_0 for the operator that takes the constant value 1. This notion is equivalent to the better known one of a pre-lambda-ring (pre-λ-ring) but works out just a bit better notationally. The two sets of operations are related by the Wronski-like relations

$$\sum_{i=0}^{n} (-1)^i \sigma_i(x)\lambda_{n-i}(x) = 0$$

The lambda operations behave like exterior powers.

Let $\varphi : A \longrightarrow B$ be a morphism in **CRing** and let both A and B carry pre-sigma-ring structures. Then the morphism is said to be a morphism of pre-sigma-rings if it commutates with the sigma operations, i.e. $\varphi\left(\sigma_n^A(x) = \sigma_n^B(\varphi(x))\right.$. A pre-sigma-ring is a sigma ring if the operations satisfy certain universal formulas when iterated and when applied to a product. This is conveniently formulated as follows.

Consider the ring of big Witt vectors $W(A)$ and write an element of it (formally) as

$$a(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots = \Pi_i \frac{1}{(1 - \xi_i t)}$$

Then

$$\sigma_n(a(t)) = \prod_{i_1 \leq i_2 \leq \cdots \leq i_n} (1 - \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n} t)^{-1} \tag{40}$$

(when written out in terms of the a_i which can be done by the usual symmetric function yoga). This defines a pre-sigma-ring structure on $W(A)$

A pre-sigma-ring A is a sigma-ring if

$$\sigma_1 : A \longrightarrow W(A), x \mapsto 1 + \sigma_1(x)t + \sigma_2(x)t^2 + \sigma_3(x)t^3 + \cdots$$

is a morphism of pre-sigma rings. It is a theorem that $W(A)$ is in fact a sigma-ring. This involves the study of the morphism

$$\sigma^{W(A)} : W(A) \longrightarrow W(W(A)) \tag{41}$$

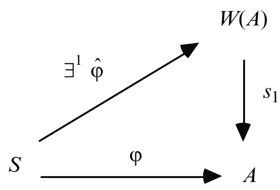
which I like to call the Artin-Hasse exponential.[16]

A ring morphism between sigma-rings is a sigma-ring morphism if it is a morphism of pre-sigma-rings. Let **SigmaRing** be the category of sigma-rings.

Let

$$s_1 : W(A) \longrightarrow A, a(t) \mapsto a_1 \tag{42}$$

be the morphism of rings that assigns to a 1-power-series its first coefficient. The Witt vectors now have the following universality property. Let S be a sigma-ring, A a ring and $\varphi : S \longrightarrow A$ a morphism of rings, then there is a unique morphism of sigma-rings $\hat{\varphi} : S \longrightarrow W(A)$ such that the following diagram commutes



So $W(A) \xrightarrow{s_1} A$ is the cofree sigma-ring over the ring A. Or in other words the functor $W(-)$: **Cring** $\longrightarrow$ **SigmaRing** is right adjoint to the functor the other way that forgets about the sigma structure.

[16] A distant relative of this morphism, viz $W_{p^\infty}(k) \longrightarrow W(W_{p^\infty}(k))$ plays an important role in class field theory. Here k is a finite field and W_{p^∞} is the quotient of the p-adic Witt vectors of the big Witt vectors.

2.9.3 The Comonad Structure on the Big Witt Vectors

A comonad (also called cotriple) (T, μ, ε) in a category $\mathfrak{C}$ is an endo functor T of $\mathfrak{C}$ together with a morphism of functors $\mu : T \longrightarrow TT$ and a morphism of functors $\varepsilon : T \longrightarrow \text{id}$ such that

$$(T\mu)\mu = (\mu T)\mu, \quad (\varepsilon T)\mu = \text{id} = (T\varepsilon)\mu \tag{43}$$

And a coalgebra for the comonad (T, μ, ε) is an object in the category $\mathfrak{C}$ together with a morphism $\sigma : C \longrightarrow TC$ such that

$$\varepsilon_C \sigma = \text{id}, \ (T(\sigma)\sigma = (\mu_{TC})\sigma \tag{44}$$

It is now a theorem that the Artin–Hasse exponential (41), which is functorial, together with the functorial morphism (42) form a cotriple and that the coalgebras for this cotriple are precisely the sigma-rings.

2.9.4 The Sigma and Lambda Ring Structures on Symm

Consider

$$\mathbf{Symm} = \mathbf{Z}[h_1, h_2, h_3, \cdots] \subset \mathbf{Z}[\xi_1, \xi_2, \xi_3, \cdots] \tag{45}$$

as before. There is a unique sigma-ring structure on $\mathbf{Z}[\xi]$ determined by

$$\sigma_n(\xi_i) = \xi_i^n \tag{46}$$

(The corresponding lambda operations are $\lambda_1(\xi_i) = \xi_i, \quad \lambda_n(\xi_i) = 0$ for $n \geq 2$ so that the ξ_i are like line bundles and this is a good way of thinking about them.) The subring **Symm** is stable under these operations and so there is an induced sigma-ring structure on **Symm**.

It is now a theorem that **Symm** with this particular sigma-ring structure is the free sigma-ring one generator. More precisely:

For every sigma ring S and element $x \in S$ there is a unique morphism of sigma-rings **Symm** $\longrightarrow S$ that takes h_1 into x.

The universality properties described in Sects. 2.9.2–2.9.4 are far from unrelated; see Sect. 3.3 below.

A totally different universality property of the Witt vectors is the following one,

2.9.5 Cartier's First Theorem

The (infinite dimensional) formal group of the Witt vectors 'is' the sequence of addition polynomials $\Sigma_1, \Sigma_2, \cdots$ in $X_1, X_2, \cdots; Y_1, Y_2, \cdots$. This formal group is denoted $\hat{W}$. A fourth universality property off the Witt vectors holds in this setting.

Given two formal groups F and G of dimensions m and n, respectively, a morphism of formal groups $\alpha : F \longrightarrow G$ is an n-tuple of power series with zero constant terms $\alpha_1, \cdots \alpha_n$ in m variables such that

$$G(\alpha_1(X), \cdots, \alpha_n(X); \alpha_1(Y), \cdots, \alpha_n(Y)) = (\alpha_1(F(X,Y)), \cdots, \alpha_n(F(X,Y))) \tag{47}$$

A curve in an n-dimensional formal group F is simply an n-tuple of power series in one variable, say, t. In $\hat{W}$ consider the particular curve $\gamma_0(t) = (t, 0, 0, \cdots)$. Then Cartier's first theorem says that for every formal group F and curve $\gamma(t)$ in it there is a unique morphism of formal groups $\hat{W} \longrightarrow F$ that takes $\gamma_0(t)$ into $\gamma(t)$.

2.10 The Star Example: Symm

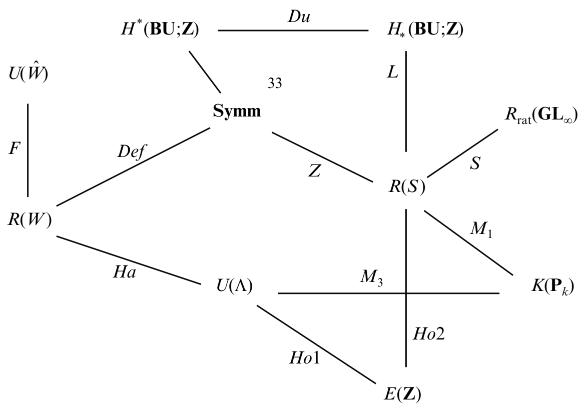
Here is a list of most of the objects with which this subsection will be concerned. Those which have not already been defined above will be described in Sect. 3.3.

— **Symm** $= \mathbf{Z}[h_1, h_2, \cdots] = \mathbf{Z}[c_1, c_2, \cdots] \subset \mathbf{Z}[\xi_1, \xi_2, \cdots]$, the ring of symmetric functions in an infinity of indeterminates. Here h_n is the n-th complete symmetric function in the ξ's and the c_n stand for the elementary symmetric functions. I am writing c_n rather than e_n because in the present context the c_n will correspond to Chern classes.
— $U(\Lambda)$, the universal lambda ring on one generator
— $R(W)$, The representing ring of the functor of the big Witt vectors; see Sect. 2.9 above.
— $R(S) = \bigoplus_{n=0}^{\infty} R(S_n)$, the direct sum of the rings of (the Grothendieck groups) of complex representations of the symmetric groups with the socalled exterior product; if ρ is a representation of S_r and σ is a representation of the symmetric group on s letters S_s then $\rho\sigma = \mathrm{Ind}_{S_r \times S_s}^{S_{r+s}}(\rho \otimes \sigma)$. By decree '$R(S_0)$' is equal to $\mathbf{Z}$. There is also a comultiplication: if σ is a representation of S_n, $\mu(\sigma) = \sum_{r+s=n} \mathrm{Res}_{S_r \times S_s}^{S_n}(\sigma)$. Together with obvious unit and counit morphisms this defines a Hopf algebra. (The antipode comes for free because of the graded connected situation.)
— $R_{\mathrm{rat}}(\mathrm{GL}_\infty)$, the (Grothendieck) ring of rational representations of the infinite linear group.
— $E(\mathbf{Z})$, the value of the exponential functor from [55 Hoffman] on the ring of integers.
— $U(\hat{W})$, the covariant bialgebra of the formal group of the Witt vectors.
— $H^*(\mathbf{BU}; \mathbf{Z})$, the cohomology of the classifying space of complex vector bundles, **BU**.
— H_* (**BU**;**Z**), the homology of the classifying space **BU**.

These are all isomorphic and that implies that **Symm** is very rich in structure indeed. Nor is that all. For instance each of the components $R(S_n)$ of $R(S)$ is a lambda ring in its own right (inner plethysm). Further the functor of the big Witt vectors is lambda ring valued. However, this paper is not about **Symm** and its extraordinarily rich structure,[17] but about niceness results. That includes 'nice proofs'. That is proofs of isomorphism between all these objects that derive from their universality, (co)freeness, … properties and rely minimally on calculations. To what extent there are currently such proofs will be discussed below in Sect. 3.3.

Two more objects that fit in this picture are the rational Witt vector functor in its role in the K-theory of endomorphisms, [12 Almkvist], and the Grothendieck group $K(\mathbf{P}_A)$ of polynomial functors $\mathbf{Mod}_A \longrightarrow \mathbf{Mod}_k$, where A is an algebra over a field k and ${}_A\mathbf{Mod}$ is the category of right A–modules, [75 Macdonald]. If $A = k$ this object is again isomorphic to the nine objects listed above.

The various isomorphisms and relations concerning whom I think I have something to say are depicted in the diagram below.



The bottom object here, viz $E(\mathbf{Z})$ has not yet been described in any way. It is again defined by an adjoint functor situation and, again, it is one which picks up extra structure. It will be described and discussed briefly in Sect. 3.3.

Also it seems from the diagram that the Hopf algebra $R(S) = \bigoplus_{n=0}^{\infty} R(S_n)$ is the central object rather then **Symm**.

2.11 Product Formulas

The simplest (arithmetic) product formula concerns the real and p-adic absolute values of a rational number

$$|a|_\infty = \prod_p |a|_p^{-1}$$

[17] I plan a future paper on that; meanwhile see [Hazewinkel, 2008 #65].

where the product on the right is over all prime numbers p. There are more formulas of this type. This leads to a view of things that is expressed as follows by Yuri Manin in [77 Manin], Reflections on arithmetical physics, pp 149ff.

"Now we can see the following pattern:

- (At least some) essential notions of real and complex calculus have their adèlic counterparts
- Adèlic objects have a strong tendency to be simpler than their Archimedean components, e.g. the adèlic fundamental domains of arithmetical discrete subgroups of semisimple groups usually have volume 1 (the Siegel-Tamagawwa-Weil philosophy ...)
- Due to this fact and to product formulas like (2) or (3) embodying the idea of democracy for all topologies, information on the real component of an adèlic object can be read off either from the real component or the product of the p-adic components for all p's.

With some strain one can generalize and state the following principle which is the main conjecture of this talk.

On the fundamental level our world is neither real, nor p-adic; it is adèlic. For some reasons reflecting the physical nature of our kind of living matter (e.g., the fact that we are built of massive particles), we tend to project the adèlic picture onto its real side. We can equally well spiritually project it upon its non-Archimedean side and calculate most important things arithmetically."

There are applications of this idea to the Polyakov measure (Polyakov partition function), loc. cit., string theory, [36 Freund et al.], Yang-Mills theory, [13 Asok et al.], and much more, see, for a start, (the bibliography of) [63 Khrennikov]. Add to this that the p-adic versions are often easier to handle and one finds some good justification for the discipline of p-adic physics.

3 Some First Results and Theorems

3.1 Freeness Theorems

The only general freeness theorem that I know about is the one from [35 Fressé]. This one says that cogroups (cogroup objects) in the category of algebras over an operad are free. This covers for instance one of the Kan results, the Leray theorem, the Milnor–Moore theorem and probably several more. At this stage it is unclear how far it goes.

I don't think it can be made to take care of the subobject freeness theorems; but there probably is a general theorem, yet to be formulated and proved, that can take care of those.

3.2 On the Lazard Universal Formal Group Theorem

The Lazard universal formal group theorem says that there exists a universal (one dimensional) commutative formal group (trivial) and that the underlying ring is free commutative polynomial in an infinity of indeterminates (surprising and far from trivial). The standard proof is long, laborious, and computational, even when simplified and streamlined as in [38 Fröhlich], see also [90 Ravenel].

Having a candidate universal formal group available, as in [45 Hazewinkel; 48 Hazewinkel] helps a great deal, see [45 Hazewinkel], pp 27–30. But the proof is still mainly computational; also the construction of the candidate universal formal group involves choices of coefficients, which mars things. One dreams of a proof which mainly relies on universality properties.

In this connection there is a rather different proof due to Cristian Lenart, [67 Lenart], which seems to have promising aspects. One ingredient, which I consider promising, is the following. Consider the power series

$$f_b(X) = X + b_2X^2 + b_3X^3 + \cdots$$

over $\mathbf{Z}[b]$. Here the b's are indeterminates. Now form

$$F_b(X, Y) = f_b\left(f_b^{-1}(X) + f_b^{-1}(Y)\right)$$

This is of course a formal group over $\mathbf{Z}[b]$. It is proved[18] in loc. cit. that the coefficients of $F_b(X, Y)$ generate a free polynomial subring, L, of $\mathbf{Z}[b]$ and that regarded as a formal group over the subring L $F_b(X, Y)$ is universal. Of course L is truly smaller than $\mathbf{Z}[b]$. To start with $2b_2 \in L$, but $b_2 \notin L$.

This next bit is pure speculation. The first Cartier theorem on formal groups says that the formal group of Witt vectors, $\hat{W}$, represents the functor 'curves'. This is a rather different universality property for formal groups. The covariant bialgebra of $\hat{W}$ is **Symm**. One wonders whether this can be used to prove the Lazard theorem.

3.3 Objects and Isomorphisms in Connection with Symm

This whole section is concerned with the objects and isomorphisms in the diagram at the end of Sect. 2.10.

[18] The result is nice; I consider the proof highly unsatisfactory.

3.3.1 The Isomorphism 'Ha' Between R(W), the Representing Ring of the Functor of the Big Witt Vectors and $U(\Lambda)$, the Free Lambda Ring on One Generator

Here is a synopsis of the relevant bits of structure. The ring $R(W)$ represents a (covariant) functor that carries a comonad structure, and the coalgebras for this comonad are precisely the lambda rings. That is all that is needed.

Let $\mathfrak{C}$ be a category and let (T, μ, ε) be a comonad in $\mathfrak{C}$. Now let $(Z, z \in T(Z))$ represent the functor T. That is, there is a functorial bijection $\mathfrak{C}(Z, A) \longrightarrow T(A)$, $f \mapsto T(f)(z)$. The monad structure gives in particular a morphism $\sigma : Z \longrightarrow TZ$, viz the image of id_Z under $\mu_z : T(Z) = \mathfrak{C}(Z, Z) \longrightarrow T(T(Z)) = \mathfrak{C}(Z, T(Z))$. This defines a 'coalgebra for T' structure on $\dot{Z}$. Now let (A, σ) be a coalgebra for the comonad T and let a be an element of A. Consider the element $\sigma(a) \in T(A) = \mathfrak{C}(Z, A)$. This gives a unique morphism of T-coalgebras that takes z into a. There are of course a number of things to verify both at this categorical level and to check that these categorical considerations fit with the explicit constructions carried out in the previous subsections. This is straightforward.

Thus the isomorphism 'Ha' is a special case of a quite general theorem and the proof uses no special properties but only universal and other categorical notions. This is the kind of proof I would like to have for all the isomorphisms in the diagram.

3.3.2 The Isomorphism 'Z' Between R(S) and Symm

This is handled by the Zelevinsky theorem, [107 Zelevinsky] and [52 Hazewinkel], Chap. 3. The Zelevinsky theorem deals with PSH algebras (over the integers). The acronym 'PSH' stands for 'Positive–Selfadjoint–Hopf. Actually it is about (nontrivial) graded connected Hopf algebras with a distinguished (preferred) homogenous basis. The Hopf algebra is also supposed to be of finite type so that each homogenous component is a free Abelian group of finite rank.

An inner product is defined by declaring this basis to be orthonormal. The positive elements of the Hopf algebra are the nonnegative (integer coefficient) linear combinations of the distinguished basis elements. Let m and μ denote the multiplication and comultiplication, respectively.

Selfadjoint (selfdual) now means

$$\langle m(x \otimes y), z\rangle = \langle x \otimes y, \mu(z)\rangle$$

and positivity means that if the elements of the distinguished basis are denoted by ω_i, etc., and

$$m(\omega_i \otimes \omega_j) = \sum_r a_{ij}^r \omega_r, \quad \mu(\omega_i) = \sum_{r,s} b_i^{r,s} \omega_r \otimes \omega_s$$

then $a_{ij}^r \geq 0$ and $b_i^{r,s} \geq 0$.

Suppose now that there is precisely one among the distinguished basis elements that is primitive,[19] then (the main part of) the Zelevinsky theorem says that the Hopf algebra in question is isomorphic (as a Hopf algebra) to **Symm**, possibly degree shifted.

An example of a PSH algebra is $R(S)$:

- The distinguished basis is formed by the irreducible representations of the various S_n
- The positive elements are the real (as opposed to virtual) representations, and so multiplication and comultiplication are positive.
- The selfadjointness comes from Frobenius reciprocity
- The Hopf property is handled by (a consequence of) the Mackey double coset theorem.

Using the isomorphism all structure can be transferred making **Symm** also a PSH algebra. An odd thing is that this is not proved directly. The distinguished basis turns out to be formed by the Schur functions. The problem is positivity. There seems to be no direct proof in the literature that the product of two Schur functions is a nonnegative linear combination of Schur functions.

I used to think that this theorem did not count in the context of the diagram because it uses such seemingly non-algebraic things as positivity and distinguished basis. However in the setting of $R(S)$ these are, see above, entirely natural.

There is one more thing I would like to say in this context. The fourth and final step of the proof of the Zelevinsky theorem (in the presentation of [52 Hazewinkel]) essential use is made of something called the Bernstein morphism. This is a morphism

$$H \longrightarrow H \otimes \mathbf{Symm}$$

defined for any commutative associative graded connected Hopf algebra H. If one takes $H = \mathbf{Symm}$ this is precisely the morphism that defines the multiplication on the big Witt vectors. This is a "coincidence" that cries out for further investigation. For a completely different way of establishing that **Symm** and $R(S)$ are isomorphic see [14 Atiyah]. For still another and very elegant proof of this result see [70 Liulevicius; 71 Liulevicius]. It seems that the theorem actually goes back to Frobenius, [37 Frobenius].

3.3.3 The Isomorphism 'S' from R(S) to $R_{rat}(\mathrm{GL}_\infty)$

This is Schur–Weyl duality which has its origins in Schur's thesis of 1901, [94 Schur]. The subject of Schur-Weyl duality has by now evolved into what is practically a small specialism of its own. A search in the MathSci database gave 72 hits. There are quantum and super versions and there are interrelations with such diverse fields as quantum and statistical mechanics, tilting theory, combinatorics, random

[19] There is always at least one because of graded connectedness (and nontriviality).

walks on unitary groups, ... A selection of references is [9; 17 Benkart et al.; 27 Dipper et al.; 29 Doty; 30 Duchesne; 39 Fulton et al.; 40 Goodman; 41 Goodman et al.; 42 Green; 57 Howe; 64 Klink et al.; 69 Lévy; 94 Schur; 93 Schur; 104 Weyl; 106 Zelditch], [14 Atiyah; 75 Macdonald].

Here is what is probably the simplest incarnation of Schur–Weyl duality. Let V be a finite dimensional vector space over a field of characteristic 0. Form the n-th tensor product

$$T^n(V) = V \otimes \cdots \otimes V$$

The symmetric group S_n acts on this by permuting the factors, which gives a finite dimensional representation of S_n that can be decomposed into its isotypic components

$$T^n(V) = \bigoplus_{\pi} \mathrm{Hom}_{kS_n}(E_\pi, T^n(V) \otimes E_\pi = \bigoplus_{\pi} F_\pi(V) \otimes E_\pi \tag{48}$$

functorially in V. Here the E_π are the distinct irreducible kS_n modules. If now $A : V \longrightarrow V$ is a linear transformation $F_\pi(A) : F_\pi(V) \longrightarrow F_\pi(V)$ is an 'invariant matrix' in the sense of Schur, [94 Schur]. This is taken from [75 Macdonald].

Taking invertible A one obtains a representation $F_\pi(V)$ of $\mathbf{GL}(V)$. This can also be seen as coming from the action of $\mathbf{GL}(V)$ on $T^n(V)$ defined by $g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$, noting that this action commutes with the S_n action on $T^n(V)$ and using the double commutant theorem.

The middle term in (48) makes it clear that this is some kind of duality. What I would really like is to have is a pairing $R(S) \times R_{\mathrm{rat}}(\mathbf{GL}) \longrightarrow \mathbf{Z}$ defined directly, which then gives this duality. At the 'finite level' described above this can probably be done by looking at the trace form $\langle X, Y \rangle = \mathrm{Trace(XY)}$ on $\mathrm{End}(T^n(V))$, [41 Goodman et al.], Sect. 9.1; [106 Zelditch], page 19. But not it seems without bringing in a lot of representation theory.

3.3.4 On a Possible Isomorphism 'L' Between R(S) and H_*(BU; Z)

This is mostly speculative. First both rings (as Abelian groups) have a natural basis indexed by partitions. Second there is a bit of positive evidence in [44 Hazewinkel et al.], where in Sect.11 a (nontrivial, i.e. with jumps) family of representations is constructed of S_{n+m} that is parametrized by the Grassmann manifold $\mathbf{Gr}_n(\mathbf{C}^{n+m})$.

3.3.5 On the Isomorphism 'Du' Between H^*(BU; Z) and H_*(BU; Z)

This is a matter of homology – cohomology duality for oriented manifolds. Plus autoduality of the Hopf algebras involved. (Both carry natural Hopf algebra structures.)

3.3.6 On the Isomorphism 'SP' Between $H^*(\mathbf{BU}; \mathbf{Z})$ and Symm

First one defines Chern classes, for instance as in [80 Milnor et al.], Chap. 14; see [5] for a totally different method; the definition of the first Chern class that is in [62 Kharshiladze] is one I particularly like.

The i-th Chern class associates to a complex vector bundle V over a suitable space X an element $c_i(V)$ of the cohomology group $H^{2i}(X; \mathbf{Z})$. One of the more important properties of the Chern classes is 'functoriality'. Let $f : Y \longrightarrow X$ be continuous and let f^*V be the vector bundle pullback of V. Then

$$c_i(f^*V) = f^*(c_i(V))$$

(The notation is a bit unfortunate in that there are two different f^* in the formula; but is traditional). A second important property is the 'Whitney sum formula'. Let

$$c(V) = 1 + c_1(V) + c_2(V) + \cdots$$

be the socalled total Chern class (also sometimes called complete Chern class). Let W be a second complex vector bundle over X. Then

$$c(V \oplus W) = c(V)c(W)$$

where on the right hand side the cohomology cup product is used. And in fact together with $c_0(V) = 1$ and a normalization condition that specifies the total Chern class of the canonical (tautological) line bundle over the complex projective spaces $\mathbf{Gr}_1(\mathbf{C}^n)$ these two properties completely determine the Chern classes. See also [43 Hatcher], theorem 3.2 on page 78.

Next one calculates the cohomology of the classifying spaces $\mathbf{BU}_n$ to be

$$H^*(\mathbf{BU}_n; \mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots, c_n]$$

where the c_i are the Chern classes of the canonical vector bundle γ_n over $\mathbf{BU}_n$. For instance with induction starting with the very simple case $\mathbf{BU}_1 = \mathbf{CP}^\infty$ which has a CW complex cell decomposition with precisely one cell in every even dimension. This is the way it is done in [80 Milnor et al.]. One can also use spectral sequences. It follows that

$$H^*(\mathbf{BU}; \mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots, c_n, \cdots]$$

which is isomorphic, at least as rings, to **Symm**. This is precisely the kind of calculatory proof that I do not like.

However, there is the following aspect. It is often a good idea to view **Symm** as the symmetric functions in an infinity of indeterminates

$$\mathbf{Symm} \subset \mathbf{Z}[\xi_1, \xi_2, \xi_3, \cdots]$$

Now on the topological side consider the canonical line bundle $\gamma_1 \longrightarrow \mathbf{BU}_1$ and take the n-fold product $\gamma_1 \times \cdots \times \gamma_1$. This is an n-dimensional bundle over the n-fold product $\mathbf{BU}_1 \times \cdots \times \mathbf{BU}_1$. The cohomology of this space is

$$\mathbf{Z}[\eta_1, \cdots, \eta_n]$$

where η_i is the first Chern class of the i-th γ_1. Also by the Whitney sum formula

$$c(\gamma_1 \times \cdots \times \gamma_1) = (1 + \eta_1) \cdots (1 + \eta_n)$$

Now by the classifying space property of $\mathbf{BU}_n$ there is a homotopy class of maps $f : \mathbf{BU}_1 \times \cdots \times \mathbf{BU}_1 \longrightarrow \mathbf{BU}_n$ such that the pullback of γ_n by f is $\gamma_1 \times \cdots \times \gamma_1$. Using functoriality it follows that f^* takes $c_i \in H^*(\mathbf{BU}_n; \mathbf{Z})$ to the i-th elementary symmetric function in the η's and that $H^*(\mathbf{BU}_n; \mathbf{Z})$ manifests itself as the ring of symmetric functions in $\mathbf{Z}[\eta_1, \cdots, \eta_n]$. This is taken from page 189 of [80 Milnor et al.]; see also [6] for a slightly different formulation of the same idea.

Add to this that the Chern classes of the γ_n (the universal Chern classes) can be described explicitly in terms of Schubert cycles, [5], and, possibly, this can be worked up to a much less calculatory proof of the isomorphism 'SP'.

3.3.7 The Isomorphism 'F' Between R(W) and $U(\hat{W})$

Consider an n-dimensional formal group F over a (unital commutative associative) ring A. Here n can be infinity. It is given by n power series in $2n$ indeterminates grouped in two groups of n indeterminates with coefficients in A. Let $R(F)$ be the ring of power series over A in n indeterminates. Then the n power series of the formal group F define a bialgebra like structure

$$R(F) \longrightarrow R(F)\hat{\otimes} R(F)$$

This object is called the contravariant bialgebra of the formal group. (It is really needed (in general) to take the completed tensor product; even for $n = 1$ one has $A[[X]] \otimes A[[Y]] \underset{\neq}{\subset} A[[X, Y]]$.)

$R(F)$ is given the usual power series topology. Now form

$$U(F) = \mathbf{Mod}_{A,\mathrm{cont}}(R(F), A) \tag{49}$$

This is the covariant bialgebra (in fact Hopf algebra) of the formal group F. Inversely one can obtain $R(F)$ from $U(F)$; just how will not be needed here.

In the case of the formal group $\hat{W}$ of the Witt vectors (over the integers) the power series defining it are in fact polynomials. And thus the restriction to

$$R(W) = \mathbf{Z}[X_1, X_2, \cdots] \subset \mathbf{Z}[[X_1, X_2, \cdots]] = R(\hat{W})$$

of $R(\hat{W}) \longrightarrow R(\hat{W})\hat{\otimes}R(\hat{W})$ lands in $R(W) \otimes R(W)$. As the polynomials are dense in the power series, in this polynomial case, formula (49) is equivalent to

$$U(\hat{W}) = \mathbf{Mod}_{\mathbf{Z}}(R(W), \mathbf{Z})$$

and thus the isomorphism 'F' is a consequence of the autoduality of $R(W) =$ **Symm**.

3.3.8 On the Isomorphisms 'M1' and 'M3' Between R(S), $K(\mathbf{P}_k)$ and $U(\Lambda)$

One sees from formula (48) in Sect. 3.3.3 that each irreducible representation of S_n defines a functor of $\mathbf{V}_k$ to itself that is polynomial. Here V_k is the category of finite dimensional vector spaces over the field k, and polynomial means that for each pair of vector spaces U,V the mapping $F : \mathrm{Hom}(U, V) \longrightarrow \mathrm{Hom}(F(U), F(V))$ is polynomial. Let now $\mathbf{P}_k$ be the category of polynomial functors $\mathbf{V}_k \longrightarrow \mathbf{V}_k$ of bounded degree and $K(\mathbf{P}_k)$ its Grothendieck group. Then the remarks just made practically establish the isomorphism 'M1'.

Next, $K(\mathbf{P}_k)$ carries a λ-ring structure induced by composition with the exterior powers $\Lambda^i : \mathbf{V}_k \longrightarrow \mathbf{V}_k$. It turns out that it thus becomes the free λ-ring on one generator, [14 Atiyah; 76 Macdonald]. This is 'M3'.

It still needs to be sorted out whether the composition of 'M1' and 'M3' equals the composition of 'Z' and 'Ha'.

The main aim of [76 Macdonald] is to generalize this in various ways. Let A be a k algebra, $\mathbf{V}_A$ the category of finitely generated projective left A modules, $\mathbf{P}_A$ the category of polynomial functors $\mathbf{V}_A \longrightarrow \mathbf{V}_k$ of bounded degree and $K(\mathbf{P}_A)$ its Grothendieck group. Then $K(\mathbf{P}_A)$ is the free λ-ring generated by the classes of the functors $P \mapsto E \otimes_A P$ where E runs through a complete set of non-isomorphic finite dimensional simple right A–modules.

When applied to the group ring of a finite group there is also the result that $\bigoplus_{n\geq 0} R(G \sim S_n)$ is the free λ-ring on the irreducible representation of G. (Here $G \sim S_n$ is the wreath product of G and S_n.

Thus 'M1' and 'M3' are just the simplest cases of much more general results, which makes them nicer in my view.

3.3.9 On the Object E(Z) and the Isomorphisms 'Ho1' and 'Ho2'

Peter Hoffman noted that there is a nice functor E, denoted 'exp' in [55 Hoffman] that makes some of what went before more elegant.

Let **Ab** be the category of Abelian groups and **GrRing** that of (unital ungraded–commutative) graded rings. An object R of **GrRing** is a direct sum of Abelian groups R_i together with multiplications $R_i \otimes R_j \longrightarrow R_{i+j}$ making $\oplus_i R_i$ a unital commutative ring. As in the case of the big Witt vectors one considers the "1-units". To be precise consider the functor

$$^\wedge : \mathbf{GrRing} \longrightarrow \mathbf{Ab} \text{ defined by } \hat{R} = 1 + \prod_{i=1}^{\infty} R_i \tag{50}$$

where the Abelian group structure is given by multiplication.

Note that the functor of the big Witt vectors is given by $S \mapsto S[[t]] \mapsto S[[t]]^\wedge$. What this means is completely unexplored.

The functor (50) has a left adjoint $\mathbf{Ab} \longrightarrow \mathbf{GrRing}$, here denoted E, so that there is the functorial equality

$$\mathbf{GrRing}(E(A), R) = \mathbf{Ab}(A, \hat{R})$$

As a left adjoint $E(A)$ should be thought of as some kind of free object and, as is so often the case with functors that are part of an adjunction it picks up all kinds of extra structure. In this case it is first of all a Hopf algebra (as happened with the universal enveloping algebra). This comes from the observation that

$$E(A \oplus B) = E(A) \otimes E(B)^{20}$$

$E(A)$ carries a natural λ-ring structure. (Though I find the construction very difficult and, frankly, definitely on the ugly side.) However it is worth exploring further as it goes through the notion of what the author calls an ω-ring, a notion equivalent to that of a λ-ring but whose axioms only involve linear maps. This gives one a shot at solving a rather vexing matter. **Symm** is a λ-ring; it is also selfdual. So, morally speaking, there should be something like a 'dual λ-ring structure' on it.

Returning to the paper [55 Hoffman], the main theorems appear to be

$$\bigoplus_{n\geq 0} R(G \sim S_n) \cong E(R(G))$$

$E(A)$ is the free λ-ring generated by A.

which are very nice results showing that the functor E merits further attention.

3.3.10 The K-Theory of Endomorphisms

Let A be a unital commutative ring. Consider the category End(A) of pairs (P, f) where P is a finitely generated A-module and f an endomorphism of P. A morphism $\varphi : (P, f) \longrightarrow (Q, g)$ in End(A) is a morphism φ of A-modules that commutes with the given endomorphisms, i.e. $g\varphi = \varphi f$. There is an obvious notion

[20] This formula also illustrates that 'exponential' or 'exp' is a most apt appellation.

of exact sequence in End(A) and so one can form the Grothendieck group and ring[21] $K(\mathrm{End}(A))$, the study of which was initiated by Gert Almkvist, [11 Almkvist; 12 Almkvist].

Given $(P, f) \in \mathrm{End}(A)$ let Q be a finitely generated module such that $P \oplus Q$ is free and consider the endomorphism $f \oplus 0$ of this module and its characteristic polynomial $\det(1 + t(f \oplus 0))$. This is a polynomial in t that does not depend on Q. This induces a homomorphism $K(\mathrm{End}(A)) \longrightarrow W(A)$, where $W(-)$ is the functor of the big Witt vectors, that is obviously zero on $K(A)$. (The projective modules over A are imbedded in End(A) as pairs (A,0)). Thus there results a morphism (of rings in fact)

$$c : K(\mathrm{End}(A)/K(A) = W_0(A) \longrightarrow W(A)$$

functorial in A. Almkvist now proves:

The morphism c is injective for all A and the image of c (for a given A) consists of all power series $1 + a_1 t + a_2 t^2 + \cdots$ that can be written in the form

$$1 + a_1 t + a_2 t^2 + \cdots = \frac{1 + b_1 t + b_2 t^2 + \cdots + b_r t^r}{1 + d_1 t + d_2 t^2 + \cdots + d_n t^n} \quad \text{with } b_i, d_j \in A$$

For obvious reasons I call these rational Witt vectors.

A first question is now whether this functor $W_0(-)$ is representable. It is, [50 Hazewinkel]. This requires some preparation. Consider the ring

$$\mathbf{Z}[X] = \mathbf{Z}[X_1, X_2, X_3, \cdots]$$

of polynomials in a countable infinity of commuting indeterminates. Form the Hankel matrix

$$\begin{pmatrix} 1 & X_1 & X_2 & X_3 & \cdots \\ X_1 & X_2 & X_3 & X_4 & \cdots \\ X_2 & X_3 & X_4 & X_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Now let J_n be the ideal in $\mathbf{Z}[X]$ generated by all $(n + 1) \times (n + 1)$ minors of this Hankel matrix. These ideals define a topology on $\mathbf{Z}[X]$ which for the present purposes **1** will call the J-topology. The representability result is now as follows.

For each rational Witt vector $a(t) = 1 + a_1 t + a_2 t^2 + \cdots \in W_0(A)$ let $\varphi_{a(t)} : \mathbf{Z}[X] \longrightarrow A$ be the ring morphism defined by $X_i \mapsto a_i$. Then $a(t) \mapsto \varphi_{a(t)}$ is a functorial and injective morphism from $W_0(A)$ to ring morphisms $\mathbf{Z}[X] \longrightarrow A$ that are continuous with respect to the J-topology on $\mathbf{Z}[X]$ and the discrete topology on A. If A is Fatou, so in particular if A is integral and Noetherian, the correspondence is bijective.

[21] The multiplication is induced by the tensor product.

Here Fatou is a technical condition that is of no particular importance for this paper. Suffice it to say that a Noetherian integral domain is Fatou. Incidentally the quotient rings $\mathbf{Z}[X]/J_n$ are integral domains, but they are not Noetherian and not Fatou.

For a host of other results, including a determination of the operations in the K-theory of endomorphisms, see [11 Almkvist; 12 Almkvist; 50 Hazewinkel].

3.3.11 Leftovers

- **Symm** is an object with an enormous amount of compatible structure: Hopf algebra, inner product, selfdual (as a Hopf algebra), PSH, coring object in the category of rings, ring object in the category of corings (up to a little bit of unit trouble), Frobenius and Verschiebung endomorphisms, free algebra on the cofree coalgebra over **Z** (and the dual of this: cofree coalgebra over the free algebra on one element), several levels of lambda ring structure, ...

  The question arises which ones of these have natural interpretations in the other nine incarnations occurring in the diagram (and whether the isomorphisms indicated are the right ones for preserving these structures).
- **Symm** represents the functor of the big Witt vectors $W(A) = \{1 + a_1 t + a_2 t^2 + \cdots : a_i \in A\}$.

  Now **Hopf**(**Symm**, **Symm**) $= W(\mathbf{Z})$, [70 Liulevicius]. This comes about because on the one hand **Symm** is the free algebra on the cofree coalgebra over **Z**, and on the other the cofree coalgebra over the free algebra over **Z**.
  This is a curiosity that certainly merits some thought and one wonders whether something similar occurs elsewhere.

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The Polynomial Algebra and Quantizations of Electromagnetic Fields

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Abstract We consider quantizations and braidings of modules with grading by an abelian group. In particular, we investigate modules with grading by $\mathbb{Z}^n$ and the algebra of polynomials in n variables. We find quantizations of this algebra and quantization of its differential structures. Exploiting the fact that the electromagnetic field tensor can be described by a curvature, the quantizations of the curvature of the polynomial algebra (for $n = 4$) give quantizations of the electromagnetic field tensor and Maxwell's equations.

The algebra of polynomials in n variables, $\mathbb{C}[x_1, \ldots, x_n]$, have a natural grading by $\mathbb{Z}^n$. The framework for quantizations and braidings of modules with grading by an abelian group as found in the papers [5], [6] and [7], are here applied to investigate modules with grading by $\mathbb{Z}^n$ and to find quantizations of the algebra of polynomials. In [6], the results concerning quantizations and braidings for grading by a finite abelian group can be found, and they carry over to the infinite case without change, see [7].

We are particularly concerned with quantizations of differential structures, see [9] and [8]. More precisely, with the same framework for finding quantizations of the algebra $\mathbb{C}[x_1, \ldots, x_n]$, we find quantizations of its derivations, connections and curvatures by the $\mathbb{Z}^n$-grading.

The exposition will be kept at a simple level with examples that nicely illustrate the subject as described in the papers mentioned.

We will end by applying the results obtained to find quantizations of the electromagnetic 2-form which in turn give quantizations of Maxwell's equations. By exploiting the fact that the electromagnetic field tensor can be described in terms of connections and curvatures for $\mathbb{C}[t, x, y, z]$, the quantizations of the curvature of the polynomial algebra (for $n = 4$) give quantizations of the electromagnetic 2-form. With the quantizations already given for curvatures the desired quantizations are easily obtained.

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1 Quantizations and Braidings of $\mathbb{Z}^n$-Graded Modules

We shall repeat some results from [6] and [7] needed for describing quantizations and braidings of $\mathbb{Z}^n$-graded modules.

A braiding σ in a monoidal category C is a natural isomorphism

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X,$$

for objects X and Y, that satisfies the Mac Lane coherence conditions, see [11]. A quantization is a natural isomorphism

$$q_{X,Y} : X \otimes Y \to X \otimes Y,$$

for objects X, Y in C that satisfies the Lychagin coherence condition, see [10]. Quantizations act on the set of braidings as follows

$$\left(\sigma_q\right)_{X,Y} = q_{Y,X}^{-1} \circ \sigma_{X,Y} \circ q_{X,Y}, \tag{1}$$

and σ_q is a braiding too. We will here consider quantizations of categories equipped with trivial braidings, that is, with classical commutative structure. After quantization, however, the braiding is $q_{Y,X}^{-1} \circ \tau_{X,Y} \circ q_{X,Y}$, where τ is the twist.

Let G be an abelian group such that the graded tensor product can be defined, and consider the monoidal category of G-graded modules. From [4] we have that any symmetry σ of the monoidal category of G-graded modules over $\mathbb{C}$ depends only on the grading and is represented by a 2-cochain $\sigma : G \times G \to \mathbb{C}^*$, [2], that satisfies the bihomomorphism and symmetry conditions, where

$$\sigma_{X,Y} : x \otimes y \longmapsto \sigma\left(|x|, |y|\right) y \otimes x,$$

for homogeneous $x \in X$, $y \in Y$, with grading $|x|$ and $|y|$ respectively. Any quantization q of the monoidal category of G-graded modules is a representative of the 2^{nd} cohomology of the abelian group G with coefficients in $\mathbb{C}^*$, $q : G \times G \to \mathbb{C}^*$, where

$$q_{X,Y} : x \otimes y \longmapsto q\left(|x|, |y|\right) x \otimes y,$$

for homogeneous $x \in X$, $y \in Y$.

2 The Algebra of Polynomials $\mathbb{C}[x_1, \ldots, x_n]$

The main object for which we shall explore quantizations, is the algebra of polynomials in n variables, $A = \mathbb{C}[x_1, \ldots, x_n]$. We consider the quantizations of $\mathbb{C}[x_1, \ldots, x_n]$ as a $\mathbb{Z}^n$-graded algebra where $x_1^{k_1} \cdots x_n^{k_n}$ has grade $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. Note that the braiding is trivial, i.e. A is commutative in the classical sense.

Quantizations of $\mathbb{Z}^n$-graded modules are 2-cocycles

$$q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{C}^*,$$
$$q(k, l) = \exp(i \langle Pk, l \rangle), \tag{2}$$

for $k, l \in \mathbb{Z}^n$, where P is a $n \times n$ skew-symmetric matrix with complex entries.

The quantization A_q of $A = \mathbb{C}[x_1, \ldots, x_n]$ is a "deformation" of its multiplication where

$$f *_q g = \exp(i \langle Pk, l \rangle) fg, \tag{3}$$

$f \in (A_q)_k, g \in (A_q)_l$. Particularly, we get the following multiplication for the variables $x_1, \ldots, x_n$,

$$x_i *_q x_j = \exp(ip_{ji}) x_i x_j,$$
$$x_i *_q x_i = x_i^2,$$
$$x^k *_q x^k = \left(x^k\right)^2.$$

Since σ is trivial to start with, the quantized algebra A_q is equipped with the quantized braided commutativity in (1) with $\sigma = \tau$, and

$$x_i *_q x_j - x_j *_q x_i = \exp(ip_{ji}) x_i x_j - \exp(-ip_{ji}) x_j x_i,$$
$$x^k *_q x^l - x^l *_q x^k = \exp(i \langle Pk, l \rangle) x^k x^l - \exp(-i \langle Pl, k \rangle) x^l x^k$$

where $x^k = x_1^{k_1} \cdots x_n^{k_n}$, $x^l = x_1^{l_1} \cdots x_n^{l_n}$. We see that A_q is the quantum space.

Example 1. Consider the algebra of polynomials in two variables, $A = \mathbb{C}[x, y]$, where the homogeneous elements of A of grading $(m, n) \in \mathbb{Z}^2$ are of the form $x^m y^n$. As a $\mathbb{Z}^2$-graded algebra $A = \mathbb{C}[x, y]$ has quantizations represented by 2-cocycles $q : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*$,

$$q(z, z') = \exp(i \langle Pz, z' \rangle), \tag{4}$$

for $z, z' \in \mathbb{Z}^2$, where

$$P = \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix},$$

$h \in \mathbb{C}$, see [10]. A_q is the quantum plane where for the variables x and y the multiplication is

$$x *_q y = \exp(-ih) xy, \; y *_q x = \exp(ih) yx,$$
$$x *_q x = x^2, \; y *_q y = y^2,$$

and A_q is equipped with the following commutativity

$$x *_q y - y *_q x = \exp(-ih) xy - \exp(ih) yx = -2i \sin(h).$$

Example 2. There are applications to harmonic oscillators, see [10]. Consider the one-dimensional oscillator with Hamiltonian

$$H = p^2 + q^2.$$

The system has a non-trivial group of symmetries and the corresponding algebra of functions is graded by $\mathbb{Z}^2$, i.e. $A = \mathbb{C}[p, q]$. The quantization of A, A_q, is represented by (4) and is as described above.

2.1 Quantizations of Derivations of $\mathbb{C}[x_1, \ldots, x_n]$

Derivations of $\mathbb{C}[x_1, \ldots, x_n]$ of degree $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ are of the form

$$W = w_1(x)\,\partial_1 + \cdots + w_n(x)\,\partial_n,$$

where

$$w_i(x) = w_i(x_1, \ldots, x_n) \in A_{(k_1, \ldots, k_i+1, \ldots, k_n)},$$

and ∂_i has degree $-1_i = (0, \ldots, -1, \ldots, 0)$ with -1 in the i^{th} place.

In general for any monoidal category, the quantization W_q of a derivation is a quantization of the action of the derivation, and which satisfies the quantized braided Leibniz rule, see [5]. For $\mathbb{C}[x_1, \ldots, x_n]$ the quantization W_q in terms of the original $W \in Der(A)$ is

$$W_q(f) = \exp(i\langle Pk, l\rangle)\, W(f),$$

for $W \in Der_k(A)$, $f \in A_l$, and W_q satisfies the following quantized Leibniz rule

$$W_q(f *_q g) = W_q(f) *_q g + \exp(2i\langle Pk, l\rangle)\, f *_q W_q(g) \tag{5}$$
$$= \exp(i(\langle Pk, l\rangle + \langle Pk, m\rangle + \langle Pl, m\rangle))\,(W(f)\,g + fW(g)) \tag{6}$$

where $W \in Der_k(A)$, $f \in A_l$, $g \in A_m$.

Consider $x^k = x_1^{k_1} \cdots x_n^{k_n}$ and $x^l = x_1^{l_1} \cdots x_n^{l_n}$. Let $W = x^k \partial_i$. The grade of W is $k - 1_i = (k_1, \ldots, k_i - 1, \ldots, k_n)$, hence

$$\begin{aligned} W_q(x_l) &= \exp(i\langle P(k - 1_i), l\rangle)\, W\left(x^l\right) \\ &= \exp(i\langle P(k - 1_i), l\rangle)\, l_i x_1^{l_1+k_1} \cdots x_i^{l_i+k_i-1} \cdots x_n^{l_n+k_n}. \end{aligned}$$

In particular,

$$\begin{aligned} (\partial_i)_q\left(x^l\right) &= \exp(i\langle P(-1_i), l\rangle)\,\partial_i\left(x^l\right) \\ &= \exp\left(-\left(p_{1i}l_1 + \cdots + p_{(i-1)i}l_{i-1} + p_{(i+1)i}l_{i+1} + \cdots + p_{ni}l_n\right)\right) \\ &\quad \times l_i x_1^{l_1} \cdots x_i^{l_i-1} \cdots x_n^{l_n}. \end{aligned}$$

The quantization of composition of derivations is as for multiplications, that is, $W_q *_q Z_q = \exp(i\langle Pk, l\rangle)\, W_q *_q Z_q$. The bracket, or commutator, is also quantized,

$$\begin{aligned}
\left[W_q, Z_q\right]_q^{\sigma_q}(f) &= W_q *_q Z_q - \exp(2i\langle Pk, l\rangle) Z_q *_q W_q (f) \\
&= \big(\exp(i\langle Pk, l\rangle) W_q Z_q - \exp(2i\langle Pk, l\rangle)\exp(i\langle Pl, k\rangle) Z_q W_q\big)(f) \\
&= \exp(i\langle Pk, l\rangle)\big(W_q Z_q - Z_q W_q\big)(f) \\
&= \exp(i\langle Pk, l\rangle)\exp(i(\langle Pk, l\rangle + \langle Pk, m\rangle + \langle Pl, m\rangle)) WZ(f) \\
&\quad - \exp(i\langle Pk, l\rangle)\exp(i(\langle Pl, k\rangle + \langle Pl, m\rangle + \langle Pk, m\rangle)) ZW(f) \\
&= \exp(i(\langle Pl, m\rangle + \langle Pk, m\rangle))(\exp(2i\langle Pk, l\rangle) WZ - ZW)(f)
\end{aligned}$$

for $W \in Der_k(A)$, $Z \in Der_l(A)$, $f \in A_m$. For example is the quantized bracket applied to $(\partial_i)_q$ and $\left(\partial_j\right)_q$ as follows

$$\begin{aligned}
&\left[(\partial_i)_q, \left(\partial_j\right)_q\right]_q^{\sigma_q}(f) \\
&= \exp(i(\langle(-p_{1i}, \ldots, -p_{(i-1)i}, 0, -p_{(i+1)i}, \ldots, -p_{ni}), m\rangle \\
&\quad + \langle(-p_{1j}, \ldots, -p_{(j-1)j}, 0, -p_{(j+1)j}, \ldots, -p_{nj}), m\rangle)) \\
&\quad \times \left(\exp\left(-2ip_{ij}\right)\partial_i\partial_j(f) - \partial_j\partial_i(f)\right)
\end{aligned}$$

Example 3. Let $n = 2$ and $P = \begin{bmatrix} 0 & h \\ -h & 0 \end{bmatrix}$. Derivations of $A = \mathbb{C}[x, y]$ are of the form

$$W = a(x, y)\partial_x + b(x, y)\partial_y$$

where ∂_x and ∂_y has degree $(-1, 0)$ and $(0, -1)$ respectively. Consider the derivation $W = x_2\partial_1$. Then

$$(W)_q(x_1) = \exp(ih) x_2$$

The quantization of the commutator applied to $(\partial_1)_q$ and $(\partial_2)_q$ is

$$\begin{aligned}
\left[(\partial_1)_q, (\partial_2)_q\right]_q^{\sigma_q}(f) &= \exp(ih(m_2 - m_1))(\exp(-2ih)\partial_1\partial_2 - \partial_2\partial_1)(f), \\
\left[(\partial_2)_q, (\partial_1)_q\right]_q^{\sigma_q}(f) &= \exp(ih(m_2 - m_1))(\exp(2ih)\partial_2\partial_1 - \partial_1\partial_2)(f) \\
&= -\exp(4ih)\left[(\partial_1)_q, (\partial_2)_q\right]_q^{\sigma_q}(f),
\end{aligned}$$

for $f \in A_m, m = (m_1, m_2)$.

2.2 *Quantizations of Connections and Curvatures of $\mathbb{C}[x_1, \ldots, x_n]$*

Consider $A = E$ as an A-module. Any derivation $\left(\bar{W}, W\right) \in Der^A(E)$ of degree k is of the form

$$\bar{W} = w_1(x)\partial_1 + \cdots + w_n(x)\partial_n + \lambda_W(x),$$

where

$$\lambda_W(x) = \lambda_W(x_1, \ldots, x_n) \in A_k.$$

A connection

$$\nabla : Der(A) \to Der^A(E), W \mapsto \bar{W},$$

in the A-module A is given by a differential 1-form

$$\lambda = \lambda_1(x)\, dx_1 + \cdots + \lambda_n(x)\, dx_n,$$

$\lambda_i(x) \in A_{(0,\ldots,-1,\ldots,0)}$, where -1 is in the i^{th} place. The pair $\left(\bar{W}, W\right) \in Der^A(E)$ of degree $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ is then connected by λ as follows

$$\nabla_\lambda : W = \sum_{i=1}^{n} w_i(x)\, \partial_{x_i} \longmapsto \bar{W} = W + \lambda_W = \sum_{i=1}^{n} w_i(x)\left(\partial_{x_i} + \lambda_i(x)\right).$$

The curvature of ∇_λ applied to two derivations W and Z in $Der(A)$ is

$$\begin{aligned} K_{\nabla_\lambda}(W, Z) &= W(\lambda_Z) - Z(\lambda_W) - \lambda_{[W,Z]} \\ &= d\lambda(W, Z). \end{aligned} \tag{7}$$

In general for any monoidal category C, a quantization of a connection ∇ is $\nabla_q : Der^{\sigma_q}\left(A_q\right) \to Der^{(\sigma_q, A_q)}\left(E_q\right)$, defined by

$$\nabla_q \overset{def}{=} Q_q \circ \nabla \circ Q_q^{-1},$$

and ∇_q is a σ_q-connection of E_q. The σ_q-curvature of ∇_q is $K_{\nabla_q} : Der^{\sigma_q}\left(A_q\right) \otimes Der^{\sigma_q}\left(A_q\right) \to End_{A_q}\left(E_q\right)$, defined

$$K_{\nabla_q}\left(W_q, Z_q\right) = \left[\nabla_q\left(W_q\right), \nabla_q\left(Z_q\right)\right]_q^{\sigma_q} - \nabla_q\left(\left[W_q, Z_q\right]_q^{\sigma_q}\right). \tag{8}$$

Hence, for $A = \mathbb{C}[x_1, \ldots, x_n]$ a quantization of the connection ∇_λ given by $\lambda = \lambda_1 dx_1 + \cdots + \lambda_n dx_n$ is

$$(\nabla_\lambda)_q\left(W_q\right)(f) = \exp\left(i\,\langle Pk, m\rangle\right)\nabla_\lambda(W)(f), \tag{9}$$

for $W \in Der_k(A)$, $f \in A_m$. The quantization of the curvature for A as a $\mathbb{Z}^n$-graded algebra is

$$\begin{aligned} K_{(\nabla_\lambda)_q}\left(W_q, Z_q\right) &= \left[W_q + \lambda_{Wq}, Z_q + \lambda_{Zq}\right]_q^{\sigma_q} - \left[W_q, Z_q\right]_q^{\sigma_q} - \lambda_{\left[W_q, Z_q\right]_q^{\sigma_q}} \\ &= \left(W_q + \lambda_{Wq}\right) *_q \left(Z_q + \lambda_{Zq}\right) \\ &\quad - \exp\left(2i\,\langle Pk, l\rangle\right)\left(Z_q + \lambda_{Zq}\right) *_q \left(W_q + \lambda_{Wq}\right) \\ &\quad - \left[W_q, Z_q\right]_q^{\sigma_q} - \lambda_{W_q *_q Z_q - \exp(2i\langle Pk, l\rangle) Z_q *_q W_q} \end{aligned}$$

$$
\begin{aligned}
&= \exp\left(i\left\langle Pk, l\right\rangle\right)\left(W_q + \lambda_{Wq}\right) *_q \left(Z_q + \lambda_{Zq}\right) \\
&\quad - \exp\left(2i\left\langle Pk, l\right\rangle\right)\exp\left(i\left\langle Pl, k\right\rangle\right)\left(Z_q + \lambda_{Zq}\right)\left(W_q + \lambda_{Wq}\right) \\
&\quad - \left[W_q, Z_q\right]_q^{\sigma_q} - \lambda_{\exp(i\langle Pk,l\rangle) W_q Z_q - \exp(2i\langle Pk,l\rangle)\exp(i\langle Pl,k\rangle) Z_q W_q} \\
&= \exp\left(i\left\langle Pk, l\right\rangle\right)\left(W_q\left(\lambda_{Zq}\right) - Z_q\left(\lambda_{Wq}\right) - \lambda_{[W_q, Z_q]}\right),
\end{aligned}
$$

hence,

$$
K_{(\nabla_\lambda)_q}\left(W_q, Z_q\right) = \exp\left(i\left\langle Pk, l\right\rangle\right) K_{\nabla_\lambda}\left(W_q, Z_q\right) \tag{10}
$$

for $Z \in Der_l(A)$.

3 Electromagnetic Fields

Consider $n = 4$ and $A = \mathbb{C}[t, x, y, z]$. We shall explore the fact that the derivations of A is connected to the vector potential of a electromagnetic field. Consider the electromagnetic field

$$
\begin{aligned}
E &= -\partial_t V - \nabla \Phi, \\
B &= \nabla \times V,
\end{aligned}
$$

where V is the vector potential and Φ is the scalar potential, see [3]. The components of E and B are $E = \left\{E_x, E_y, E_z\right\}$ and $B = \left\{B_x, B_y, B_z\right\}$. Let λ be the differential 1-form

$$
\lambda = \Phi d_t - V_x d_x - V_y d_y - V_z d_z.
$$

The electromagnetic 2-form is then

$$
F = d\lambda = (E_x dy + e_y dy + e_z dz)dt + B_x dy dz - B_y dx dz + B_z dx dy.
$$

Maxwell's equations take the form

$$
dF = 0, d * F = 0, \tag{11}
$$

where $d* : \Omega_p \to \Omega_{p-1}$ is the adjoint operator determined by $d : \Omega_p \to \Omega_{p+1}$, see [1].

Since $d\lambda$ applied to two derivations W and Z of A is $d\lambda(W, Z) = W(\lambda_Z) - Z(\lambda_W) - \lambda_{[W,Z]}$, by (7),

$$
F(W, Z) = K_{\nabla_\lambda}(W, Z), \tag{12}
$$

where the connection ∇_λ is defined by the 1-form λ. With the quantization of the curvature (8) we can define the quantization of the electromagnetic form, F_q, by

$$
F_q\left(W_q, Z_q\right) = K_{(\nabla_\lambda)_q}\left(W_q, Z_q\right). \tag{13}
$$

Hence, by (10),

$$F_q\left(W_q, Z_q\right) = \exp\left(i\left\langle Pk, l\right\rangle\right) F\left(W_q, Z_q\right) \tag{14}$$

where $W_q \in Der_k^{\sigma_q}\left(A_q\right)$, $Z_q \in Der_l^{\sigma_q}\left(A_q\right)$. This gives the following quantizations of Maxwell's equations

$$dF_q = 0, \tag{15}$$

$$d * F_q = 0. \tag{16}$$

The operators d and $d*$ also need to be quantized, but in the monoidal category of $\mathbb{Z}^n$-graded modules these operators are graded by zero, hence the quantization is trivial. (However, if we consider d and $d*$ as derivations in the alternating algebra which is graded by $\mathbb{Z}$, these have respectively the grading 1 and -1. We then can obtain non-trivial quantizations, but we will not explore this further here.)

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A Bridge Between Lie Symmetries and Galois Groups

Nail H. Ibragimov

Abstract A bridge between Lie symmetry groups for differential equations and Galois groups for algebraic equations is suggested. It is based on calculation of Lie symmetries for algebraic equations and their restriction of the roots of the equations under consideration. The approach is illustrated by several examples. An alternative representation of Lie symmetries, called the Galois representation, is provided for differential equations.

1 Introduction

It was mentioned in several lectures during the *Abel Symposium 2008* that conceptual similarities and dissimilarities between Lie symmetries and Galois groups are not clarified sufficiently in the literature. I was naturally interested in this matter and found a certain satisfactory, at least for myself, answer to this question some 20 years ago. Namely, I constructed the Galois groups for several simple algebraic equations by first calculating their Lie symmetries and then restricting the symmetry group to the roots of the equation in question. I briefly described this approach in the brochure [1, pp. 429–443], (see also [2, pp. 251–254]), but did not publish in a paper available to a wide audience of mathematicians. Therefore it is presented here. In addition to the calculations presented in [1], it is shown in this paper that the procedure can be applied to differential equations as well. As a result we obtain another representation (let us call it the Galois representation) of Lie symmetries of differential equations.

The approach will be illustrated by applying it to the following equations:

$$x^2 + 1 = 0, \tag{1}$$

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$$x^4 - x^2 + 1 = 0, \tag{2}$$

$$x^4 + x^3 + x^2 + x + 1 = 0. \tag{3}$$

It is known from the literature on Galois groups that the Galois group $\mathcal{G}$ of (1) is

$$\mathcal{G} = \{1, (x_1, x_2)\}, \tag{4}$$

where x_1, x_2 are the roots of (1), (x_1, x_2) is the permutation of the roots, and 1 is the identical permutation (unit of the group). It is also known that the Galois groups of (2) and (3) are

$$\mathcal{G} = \{1, (x_1, x_2)(x_3, x_4), (x_1, x_3)(x_2, x_4), (x_1, x_4)(x_2, x_3)\} \tag{5}$$

and

$$\mathcal{G} = \{1, (x_1, x_2, x_4, x_3), (x_1, x_3, x_4, x_2), (x_1, x_4)(x_2, x_3)\}, \tag{6}$$

respectively. In (5), x_1, x_2, x_3, x_4 denote the roots of (2), and in (6) the roots of (3).

2 Lie Symmetries of Differential Equations

Recall that Lie symmetries of differential equations, e.g., of second-order ordinary differential equations

$$y'' + f(x, y, y') = 0, \tag{7}$$

are *invertible* transformations

$$\bar{x} = \phi(x, y), \quad \bar{y} = \psi(x, y) \tag{8}$$

of the independent and dependent variables mapping (7) to an equation of the same form. Specifically, the following equation holds:

$$\big[\bar{y}'' + f(\bar{x}, \bar{y}, \bar{y}')\big]_{y''=-f} = 0, \tag{9}$$

or, equivalently,

$$\bar{y}'' + f(\bar{x}, \bar{y}, \bar{y}') = \mu(x, y, \ldots)\big[y'' + f(x, y, y')\big], \tag{10}$$

where $\bar{y}'$ and $\bar{y}''$ are the first and second derivatives of $\bar{y}$ with respect to $\bar{x}$, and μ is a certain coefficient (in general, variable) depending on the transformation (8). In Lie's theory, the symmetries (8) depend on continuous parameters, and the set of all such symmetries for a given differential equation forms a *local group* known as a group admitted by the differential equation in question.

For example, the equation

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}\,\cdot \tag{11}$$

admits the two-parameter local group generated by the operators

$$X_1 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}\,, \quad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,\cdot \tag{12}$$

Solving the Lie equations for the generators X_1 and X_2, we obtain the admitted one-parameter groups of transformations

$$T_a: \quad \bar{x} = \frac{x}{1-ax}\,, \quad \bar{y} = \frac{y}{1-ax} \tag{13}$$

and

$$T_b: \quad \bar{x} = x\,\mathrm{e}^{2b}, \quad \bar{y} = y\,\mathrm{e}^{b}, \tag{14}$$

respectively, where a and b are arbitrary parameters. The reckoning shows that the condition (10) for the Lie symmetries (13) and (14) of (11) is satisfied in the following forms:

$$\bar{y}\,'' - \frac{\bar{y}\,'}{\bar{y}^{\,2}} + \frac{1}{\bar{x}\bar{y}} = (1-ax)\left[y'' - \frac{y'}{y^2} + \frac{1}{xy}\right],$$

and

$$\bar{y}\,'' - \frac{\bar{y}\,'}{\bar{y}^{\,2}} + \frac{1}{\bar{x}\bar{y}} = \mathrm{e}^{-3b}\left[y'' - \frac{y'}{y^2} + \frac{1}{xy}\right],$$

respectively. Besides, (11) has the discrete symmetry provided by the reflection with respect to the x axis:

$$T_*: \quad \bar{x} = x, \quad \bar{y} = -y. \tag{15}$$

For this transformation the condition (10) is written

$$\bar{y}\,'' - \frac{\bar{y}\,'}{\bar{y}^{\,2}} + \frac{1}{\bar{x}\bar{y}} = -\left[y'' - \frac{y'}{y^2} + \frac{1}{xy}\right].$$

The composition of the transformations (13), (14) and (15) provides the general group G of the Lie symmetries for (11).

3 Symmetries of Algebraic Equations

Let us consider algebraic equations of the nth degree:

$$P_n(x) \equiv C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_{n-1}x + C_n = 0. \tag{16}$$

The symmetries of algebraic equations are defined as in the case of differential equations. Namely, let a transformation

$$\bar{x} = f(x) \tag{17}$$

convert (16) of the nth degree into an algebraic equation

$$\overline{P}_n(\bar{x}) \equiv \bar{C}_0\bar{x}^n + \bar{C}_1\bar{x}^{n-1} + \bar{C}_2\bar{x}^{n-2} + \cdots + \bar{C}_{n-1}\bar{x} + \bar{C}_n = 0. \tag{18}$$

In general, the coefficients $\bar{C}_i$ in (18) will not coincide with the coefficients C_i in the original equation (16). If they coincide, we will say that the transformation (17) is a symmetry (a Lie symmetry) of (16). We can define the symmetry of algebraic equations as follows.

Definition 1. The transformation (17) is called a symmetry of (16) if (18) coincides with the equation $P_n(\bar{x}) = 0$ whenever x solves (16).

The *determining equation* for symmetries of algebraic equations can also be written as for differential equations, either in the form (9):

$$P_n(\bar{x})\big|_{P_n(x)=0} = 0, \tag{19}$$

or in the form (10):

$$P_n(\bar{x}) = \mu(x)P_n(x), \tag{20}$$

where

$$P_n(\bar{x}) = C_0\bar{x}^n + C_1\bar{x}^{n-1} + C_2\bar{x}^{n-2} + \cdots + C_{n-1}\bar{x} + C_n.$$

The requirement that the transformation (17) maps any algebraic equation into an algebraic equation is rather restrictive. In particular, if one considers only uniquely invertible transformations, then one can show that the general form of invertible transformations (17) converting *every* equation (16) into an algebraic equation (18) is provided by the linear fractional transformations

$$\bar{x} = \frac{ax + \varepsilon}{b + \delta x} \tag{21}$$

with complex coefficients satisfying the invertibility condition

$$ab - \varepsilon\delta \neq 0. \tag{22}$$

If we do not require existence of the uniquely determined inverse transformation, we can use transformations (17) given by the rational fractions

$$\bar{x} = \frac{A_0x^r + A_1x^{r-1} + \cdots + A_r}{B_0x^s + B_1x^{s-1} + \cdots + B_s}\,. \tag{23}$$

The transformation (23) maps algebraic equations into algebraic equations (in general, not of the same degree). It was considered by E.W. Tschirnhausen in 1683 and is known as *Tschirnhausen's transformation* .

In what follows, we will use the following simple result concerning (16) that are symmetric in their coefficients.

Lemma 1. *The equations (16) whose coefficients satisfy the conditions*

$$C_n = C_0, \quad C_{n-1} = C_1, \quad \dots \tag{24}$$

have the symmetry

$$\bar{x} = \frac{1}{x} \cdot \tag{25}$$

Proof. Indeed,

$$\begin{aligned} P_n(\bar{x}) &= C_0 \frac{1}{x^n} + C_1 \frac{1}{x^{n-1}} + \cdots + C_{n-1} \frac{1}{x} + C_n \\ &= x^{-n} \Big[C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n \Big]. \end{aligned}$$

Therefore the conditions (24) yield

$$P_n(\bar{x}) = x^{-n}\, P_n(\bar{x}).$$

Hence, the determining equation (20) is satisfied with $\mu(x) = x^{-n}$. □

3.1 First Example

Let us find the symmetries of the form (21) for the quadratic equation (1),

$$x^2 + 1 = 0.$$

The determining equation (19) is written

$$(\bar{x}^2 + 1)\big|_{x^2=-1} = 0. \tag{26}$$

Substituting (21) in $\bar{x}^2 + 1$, we have:

$$\bar{x}^2 + 1 = (\delta x + b)^{-2} \big[(a^2 + \delta^2) x^2 + 2(a\varepsilon + b\delta) x + b^2 + \varepsilon^2 \big].$$

Therefore the determining equation (26) becomes

$$2(a\varepsilon + b\delta)x + b^2 + \varepsilon^2 - a^2 - \delta^2 = 0$$

and yields the following system of two equations:

$$a\varepsilon + b\delta = 0, \quad b^2 + \varepsilon^2 - a^2 - \delta^2 = 0. \tag{27}$$

In the case $\delta = 0$ we obtain, using the first equation (27) and the condition (22), that $\varepsilon = 0$. Then the second equation (27) becomes $b = \pm a$. Hence, we have obtained the following two symmetries of the form (21) with $\delta = 0$:

$$\bar{x} = x \quad \text{and} \quad \bar{x} = -x. \tag{28}$$

If $\delta \neq 0$, (27) are written:

$$b = -\frac{a\varepsilon}{\delta}, \quad (a^2 + \delta^2)(\varepsilon^2 - \delta^2) = 0.$$

These equations together with the condition (22) yield $\varepsilon^2 - \delta^2 = 0$. Hence,

$$\delta = \varepsilon, \ \ b = -a, \quad \text{or} \quad \delta = -\varepsilon, \ \ b = a,$$

where $\varepsilon \neq 0$. Thus, we have two types of transformations:

$$\bar{x} = \frac{ax + \varepsilon}{a - \varepsilon x} \quad \text{and} \quad \bar{x} = \frac{ax + \varepsilon}{\varepsilon x - a} \cdot \tag{29}$$

If $a = 0$, these transformations reduce to (see also Lemma 1)

$$\bar{x} = \frac{1}{x} \quad \text{and} \quad \bar{x} = -\frac{1}{x} \cdot \tag{30}$$

If $a \neq 0$, the first transformation (29) provides a one-parameter local group:

$$T_\alpha : \ \ \bar{x} = \frac{x + \alpha}{1 - \alpha x}, \tag{31}$$

while the second transformation (29) can be written in the form

$$S_\beta : \ \ \bar{x} = \frac{x + \beta}{\beta x - 1}. \tag{32}$$

Note that (28) are obtained from (31)–(32) by letting $\alpha = \beta = 0$, whereas (30) can be obtained from (31)–(32) by letting $\alpha = \beta = \infty$. The transformations (31)–(32) form a group. Indeed, the composition $S_\beta \circ T_\alpha$ acts as follows:

$$S_\beta\big(T_\alpha(x)\big) = \frac{\frac{x+\alpha}{1-\alpha x} + \beta}{\beta\frac{x+\alpha}{1-\alpha x} - 1} = \frac{(1 - \alpha\beta)x + \alpha + \beta}{(\alpha + \beta)x + \alpha\beta - 1} = \frac{x + \frac{\alpha+\beta}{1-\alpha\beta}}{\frac{\alpha+\beta}{1-\alpha\beta}x - 1} \cdot$$

Hence,

$$S_\beta \circ T_\alpha = S_\gamma, \quad \gamma = \frac{\alpha + \beta}{1 - \alpha\beta} \cdot$$

The similar calculations show that

$$T_\alpha \circ S_\beta = S_\delta, \quad \delta = \frac{\beta - \alpha}{1 + \alpha\beta} \cdot$$

Other group properties are obviously satisfied. Thus, we have obtained the following result.

Theorem 1. *The group of symmetries of the form (21) for (1) is generated by the transformations (31)–(32), where the parameters α and β range over the extended complex plane.*

3.2 Second Example

Let us find the symmetries of the form (21) for (2),

$$x^4 - x^2 + 1 = 0.$$

The determining equation (19) is written

$$(\bar{x}^4 - \bar{x}^2 + 1)\big|_{x^4 = x^2 - 1} = 0.$$

Inspecting this equation as in the first example, we obtain the following four symmetries of the form (21):

$$I: \ \bar{x} = x; \quad S: \ \bar{x} = -x; \quad R: \ \bar{x} = \frac{1}{x}; \quad T: \ \bar{x} = -\frac{1}{x} \cdot \tag{33}$$

Let us verify that, e.g., the transformation R is a symmetry. We have:

$$\bar{x}^4 - \bar{x}^2 + 1 = \frac{1}{x^4} - \frac{1}{x^2} + 1 = \frac{1}{x^4}\Big(x^4 - x^2 + 1\Big).$$

It follows that $\bar{x}^4 - \bar{x}^2 + 1 = 0$ whenever $x^4 - x^2 + 1 = 0$.

The transformations (33) form a group. Indeed:

$$S \circ R = R \circ S = T, \quad S \circ T = T \circ S = R, \quad T \circ R = R \circ T = S$$

and $S^{-1} = S, \ R^{-1} = R, \ T^{-1} = T$.

3.3 Third Example

Consider now (3),

$$P_4(x) \equiv x^4 + x^3 + x^2 + x + 1 = 0. \tag{34}$$

This equation satisfies the conditions (24) and therefore has the symmetry (21), $\bar{x} = x^{-1}$. The reckoning shows that the transformation (21) together with the identical transformation $\bar{x} = x$ are the only symmetries of the form (21). In order to find more symmetries, we can search them among Tschirnhausen's transformations. However, we observe that the following statement is valid.

Theorem 2. *The transformation*

$$\bar{x} = x^n \tag{35}$$

with any integer n indivisible by 5 *is a symmetry for (34).*

Proof. We have to substitute $\bar{x} = x^n$ in the determining equation (19):

$$P_4(\bar{x})\big|_{P_4(x)=0} \equiv (\bar{x}^4 + \bar{x}^3 + \bar{x}^2 + \bar{x} + 1)\big|_{x^4=-(x^3+x^2+x+1)} = 0. \tag{36}$$

Here we should take into account not only the equation

$$x^4 = -(x^3 + x^2 + x + 1) \tag{37}$$

but also the equations obtained from (37) by repeated multiplication by x :

$$x^5 = x \cdot x^4 = -(x^4 + x^3 + x^2 + x) = 1, \quad x^6 = x \cdot x^5 = x, \ldots.$$

Thus, we extend (37) as follows:

$$\begin{array}{lllll} x^5 = 1, & x^6 = x, & x^7 = x^2, & x^8 = x^3, & x^9 = x^4, \\ x^{10} = 1, & x^{11} = x, & x^{12} = x^2, & x^{13} = x^3, & x^{14} = x^4, \\ x^{15} = 1, & \ldots\ldots. & \ldots\ldots. & \ldots\ldots. & \ldots\ldots. \end{array} \tag{38}$$

Let us verify that the transformation

$$\bar{x} = x^2$$

satisfies the determining equation (19). Using (37)–(38), we have:

$$\begin{aligned} P_4(\bar{x})\big|_{P_4(x)=0} &= (x^8 + x^6 + x^4 + x^2 + 1)\big|_{P_4(x)=0} \\ &= x^3 + x + x^4 + x^2 + 1 = 0. \end{aligned}$$

Likewise, we verify that the transformation

$$\bar{x} = x^3$$

also satisfies the determining equation (19). In this case we have:

$$P_4(\bar{x})\big|_{P_4(x)=0} = (x^{12} + x^9 + x^6 + x^3 + 1)\big|_{P_4(x)=0}$$
$$= x^2 + x^4 + x + x^3 + 1 = 0.$$

The transformation

$$\bar{x} = x^4$$

is a symmetry because it is the repeated action of the symmetry transformation $\bar{x} = x^2$.

The transformation

$$\bar{x} = x^5$$

is not a symmetry because for this transformation (38) yield

$$P_4(\bar{x})\big|_{P_4(x)=0} = x^{20} + x^{15} + x^{10} + x^5 + 1 = 5.$$

The composition of the above symmetries shows that the transformations (35) with all positive $n \neq 5m$ $(m = 0, 1, 2, \ldots)$ are symmetries for (34). Finally, the proof for the transformation (35) with the negative values of n is obtained by taking the composition of the transformations (35) with the positive n and the symmetry (22).

Thus, we have proved that the transformations (35) are symmetries for (34), provided that $n \neq 5m,\ (m = 0, \pm 1, \pm 2, \ldots)$. □

4 Derivation of the Galois Groups from Symmetries

4.1 First Example

Let us construct the Galois group for (1),

$$x^2 + 1 = 0,$$

using its symmetry group G consisting of the transformations (31)–(32). Since (1) is invariant under the group G, the roots

$$x_1 = i, \quad x_2 = -i$$

of (1) are merely permuted among themselves (or individually unaltered) by the transformations (31)–(32). Consequently, the restriction of the group G on the set $\{x_1, x_2\}$ is well defined. It is manifest that this restriction is a group. It will be called the *induced symmetry group*, or briefly, the *induced group* and denoted by $\mathcal{G}$. The induced group comprises permutations of the roots x_1, x_2. Let us find these permutations.

The action of the transformations (31) and (32) on the roots are as follows:

$$T_\alpha(x_1) = \frac{i+\alpha}{1-\alpha i} = i = x_1, \quad T_\alpha(x_2) = \frac{-i+\alpha}{1+\alpha i} = -i = x_2,$$

$$S_\beta(x_1) = \frac{i+\beta}{\beta i - 1} = -i = x_2, \quad S_\beta(x_2) = \frac{-i+\beta}{-\beta i - 1} = i = x_1.$$

We see that the restriction of T_α on the roots is the identical transformation (the unit of the group $\mathcal{G}$) which is denoted by 1. We also see that S_β permutes the roots. This permutation is denoted by (x_1, x_2). Thus, the induced group $\mathcal{G}$ comprises two elements:

$$\mathcal{G} = \{1,\ (x_1, x_2)\}. \tag{39}$$

Comparison with (4) shows that the *Galois group* of (1) coincides with the induced group (39).

4.2 Second Example

Let us construct the Galois group for (2),

$$x^4 - x^2 + 1 = 0,$$

using its symmetry group G consisting of the transformations I, S, R, T given in (33). The roots of (2) are

$$x_1 = \sqrt{\frac{1}{2}(1+i\sqrt{3})}, \quad x_2 = -x_1, \quad x_3 = \sqrt{\frac{1}{2}(1-i\sqrt{3})}, \quad x_4 = -x_3.$$

Denoting by $\widetilde{I}, \widetilde{S}, \widetilde{R}, \widetilde{T}$ the restriction of I, S, R, T on the roots, we obtain $\widetilde{I} = 1$ (the unit) and the following permutations:

$$\widetilde{S} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \end{pmatrix}, \quad \widetilde{R} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_4 & x_1 & x_2 \end{pmatrix}, \quad \widetilde{T} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}.$$

They are also denoted by

$$(x_1, x_2)(x_3, x_4), \quad (x_1, x_3)(x_2, x_4), \quad (x_1, x_4)(x_2, x_3).$$

Thus the induced group $\mathcal{G}$ in this case comprises four elements:

$$\mathcal{G} = \{1,\ (x_1, x_2)(x_3, x_4),\ (x_1, x_3)(x_2, x_4),\ (x_1, x_4)(x_2, x_3)\}. \tag{40}$$

Comparison with (5) shows that the *Galois group* of (2) coincides with the induced group (40).

4.3 Third Example

Consider (3),

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

We know from Theorem 2 that (3) has the infinite group G of symmetries given by the transformations (35). Let us single out from this infinite group the transformations

$$\bar{x} = x, \quad \bar{x} = x^2, \quad \bar{x} = x^3, \quad \bar{x} = \frac{1}{x}, \tag{41}$$

i.e., the transformations (35) with $n = 1, 2, 3, -1$. The roots of (3) are

$$x_1 = \epsilon, \quad x_2 = \epsilon^2, \quad x_3 = \epsilon^3, \quad x_4 = \epsilon^4, \tag{42}$$

where

$$\epsilon = \mathrm{e}^{2\pi i/5}.$$

The reckoning shows that the restriction of the transformations (41) to the roots (42) yields four permutations, namely the unit 1 and the permutations

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_4 & x_1 & x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_3 & x_1 & x_4 & x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}$$

denoted by

$$(x_1, x_2, x_4, x_3), \quad (x_1, x_3, x_4, x_2), \quad (x_1, x_4)(x_2, x_3).$$

One can verify that the restriction of the arbitrary transformations (35) to the roots (42) does not give new permutations. Hence, in this case also the induced symmetry group comprises four elements:

$$\mathcal{G} = \{1,\ (x_1, x_2, x_4, x_3),\ (x_1, x_3, x_4, x_2),\ (x_1, x_4)(x_2, x_3)\}. \tag{43}$$

Comparison with (6) shows that the *Galois group* of (3) coincides with the induced group (43).

4.4 A New Definition of Galois Groups

I summarize the above observations in the following definition of Galois groups for algebraic equations.

Definition 2. Let G be the group of symmetries (17) of an algebraic equation (16). The restriction of the group G to the roots of (16) is called the Galois group of (16).

This definition provides a parallel between the role of symmetries in solvability of differential equations by quadrature and solvability of algebraic equation in radicals. Namely, we know due to S. Lie that if an nth-order ordinary differential equations admits a solvable n-parameter symmetry group, then the equation in question can be integrated by quadratures. On the other hand, it is known from the theory of Galois groups that an algebraic equation of degree n is solvable by radicals if its Galois group is an nth-order solvable group. Now we can formulate this statement in terms of symmetry groups:

An algebraic equation of degree n is solvable by radicals if it has a symmetry group G such that the induced group G *is an* nth-*order solvable group.*

The above discussion manifests the known fact that Lie's motivation in constructing his theory was search for a proper extension of the Abel and Galois theory to differential equations.

5 The Galois Representation of Lie Symmetries of Differential Equations

Let us apply to Lie symmetries of differential equations the above idea of restriction of the symmetry group to solutions of the equation under consideration. This will give an alternative representation of Lie symmetries. Namely, since any Lie symmetry of a differential equation maps every solution of this equation into a solution of the same equation, we can introduce the following definition.

Definition 3. Let G be group of Lie symmetries of a differential equation. The restriction of the group G to the solution of the equation in question is called the Galois representation of the symmetry group G.

I will illustrate this approach by the second-order nonlinear ordinary differential equation (11):

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}\cdot$$

The symmetries (12) of (11) generate a solvable two-parameter group. Accordingly, (11) can be integrated by quadratures. The integration of this equation using its symmetries (12) is described in [2], Sect. 12.2.4, where the general solution to (11) is given by the following four equations:

$$\begin{aligned}
&\text{(i)} \quad y = Kx,\\
&\text{(ii)} \quad y = \sqrt{2x + Cx^2},\\
&\text{(iii)} \; y = -\sqrt{2x + Cx^2},\\
&\text{(iv)} \; Ay + Bx + x\ln\left|A\frac{y}{x} - 1\right| + A^2 = 0,
\end{aligned} \tag{44}$$

where A, B, C, K are arbitrary constants, $A \neq 0$.

Let us find the action of the symmetry transformations (13), (14) and (15) on the solutions (44).

I will provide the detailed calculations for the transformation T_a given by (13). The inverse transformation T_a^{-1} is written

$$T_a^{-1}: \quad x = \frac{\bar{x}}{1 + a\bar{x}}\,, \quad y = \frac{\bar{y}}{1 + a\bar{x}}\,.$$

Substituting these expressions in (44)(i) we have:

$$\frac{\bar{y}}{1 + a\bar{x}} = K\,\frac{\bar{x}}{1 + a\bar{x}}\,, \quad \text{whence} \quad \bar{y} = K\bar{x}.$$

Thus, T_a does not change the solution (i). Now we act by T_a on (ii) and get:

$$\frac{\bar{y}}{1 + a\bar{x}} = \sqrt{2\frac{\bar{x}}{1 + a\bar{x}} + C\frac{\bar{x}^2}{(1 + a\bar{x})^2}} = \frac{1}{1 + a\bar{x}}\sqrt{2\bar{x} + (C + 2a)\bar{x}^2},$$

whence $\bar{y} = \sqrt{2\bar{x} + (C + 2a)\bar{x}^2}$. Thus, T_a maps the solution (ii) to the solution (ii) with C replaced by $C + 2a$. The same result will be for the solution (iii). Furthermore, the similar calculations with the solution (iv) shows that it becomes $A\bar{y} + (B + aA^2)\bar{x} + \bar{x}\ln\left|A\frac{y}{x} - 1\right| + A^2 = 0$. Hence, T_a does not change A in the solution (iv) and replaces B by $B + aA^2$. We can represent the action of the Lie symmetry (13) on the solutions (44) as follows:

$$T_a: \quad \overline{K} = K, \quad \overline{C} = C + 2a, \quad \bar{A} = A, \quad \overline{B} = B + aA^2. \tag{45}$$

Proceeding likewise with the symmetry (14) written in the form

$$x = \bar{x}\,\mathrm{e}^{-2b}, \quad y = \bar{y}\,\mathrm{e}^{-b},$$

we obtain the following action of T_b on the solutions (44):

$$T_b: \quad \overline{K} = K\,\mathrm{e}^{-b}, \quad \overline{C} = C\,\mathrm{e}^{-2b}, \quad \bar{A} = A\,\mathrm{e}^{b}, \quad \overline{B} = B. \tag{46}$$

Finally, the reflection (15) permutes the solutions (ii) and (iii) without changing C and changes the signs of K and A without changing B :

$$T_*: \quad \overline{K} = -K; \quad \text{(ii)} \longleftrightarrow \text{(iii)},\ \overline{C} = C; \quad \bar{A} = -A; \quad \overline{B} = B. \tag{47}$$

Equations (45)–(47) provide the Galois representation of the group of Lie symmetries of (11).

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References

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Focal Systems for Pfaffian Systems with Characteristics

Niky Kamran

Abstract Focal systems are a generalization to the case of Pfaffian system with characteristics of the classical notion of focal curves for first-order scalar partial differential equations. We show how focal systems can be used to prove local normal form results for second-order scalar hyperbolic equations in the plane. We also illustrate their use in integration methods for first-order equations.

1 Introduction

Given a semi-basic Pfaffian system I on a fibered manifold and an invariantly defined sub-system J of I, the focal system $F(J)$ is a Pfaffian system having the property that each of its integral curves which is transversal to the vertical distribution projects on the base manifold to a curve which lies in the envelope of a one-parameter family of J-characteristic curves. The integral curves of $F(J)$ thus generalize to the case of Pfaffian systems the classical focal curves for first-order partial differential equations [2], which appear naturally in the study of the geometric singularities of the solutions. In the general context of semi-basic Pfaffian systems, the focal curves were first introduced in a fundamental paper of R.B. Gardner on characteristics for Pfaffian systems [3]. Even though they suffer from the disadvantage of not being defined in a contact invariant way, the focal systems can nevertheless be used in the geometric study of differential equations because some of their properties turn out to be contact-invariant when the geometric context is suitably chosen [5]. They also have the advantage of being typically easier to compute than the characteristics and their invariants. Focal systems were thus used in [5] to study the contact geometry of determined systems of two first-order partial differential equations in the plane, and also to obtain a local existence theorem for the Cauchy problem in the C^∞ category.

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In this paper, we consider focal systems for the Pfaffian systems associated to some classes of first-order and second-order partial differential equations in the plane. In the first-order case, we focus our attention on the those equations for which the focal system is completely integrable. This leads to a method of integration which is presented in Theorem 2, and which bypasses the explicit use of characteristics. In the second-order case, we consider hyperbolic Monge–Ampère equations and show how the focal systems of the Monge characteristic systems can be used to characterize the contact orbit of the wave equation as well as the strata of orbits corresponding to semi-linear hyperbolic equations. These results are given in Theorems 4 and 5, and should be contrasted in their simplicity with the corresponding normal form results formulated and proved in [4] using the Monge characteristics.

Much of the content of this paper is directly inspired by the work of Robby Gardner and by numerous discussions I had the privilege of having with him over the course of our collaboration. These discussions should have led to a joint paper, but unfortunately this did not happen because of Robby's untimely death, in 1998. I would like to dedicate this paper to his memory, with my heartfelt gratitude for his friendship and for all that I learned from him.

2 Focal Systems: General Results

We assume that the reader is familiar with the basic definitions in the theory of Pfaffian systems, as presented in Chap. II of [1]. In particular we will use the standard notational convention of denoting by curly brackets the module closure of a set of differential forms over the ring of C^∞ functions and by $I^{(1)}$ the first derived system of a Pfaffian system I.

Let $\pi : B \to M$ be a fibered manifold and I be a Pfaffian system on B. We assume that the set

$$V(B) = \operatorname{Ker} \pi_\star, \tag{1}$$

of π-vertical tangent vectors defines a sub-bundle of TM, and that I is semi-basic, meaning that

$$V(B) \subset I^\perp. \tag{2}$$

Let $J \subset I$ be a Pfaffian sub-system of I. The focal system of J, denoted by $F(J)$, is the Pfaffian system defined by

$$F(J) = \{\mathcal{L}_Z\omega \,|\, \omega \in J,\ Z \in V(B)\}. \tag{3}$$

An integral curve of $F(J)$ will be called a J-focal curve. Recall [3] that to any Pfaffian sub-system J of I one associates the J-characteristic vector field system $Char(I, dJ)$ and the J-characteristic Pfaffian system $C(I, dJ)$, defined by

$$Char(I, dJ) = \{X \,|\, X \in I^\perp\ X \rfloor dJ \subset I\}, \tag{4}$$

and
$$C(I, dJ) = Char(I, dJ)^{\perp}. \tag{5}$$

The following result shows that the focal system $F(J)$ is always embedded by inclusion in the J-characteristic system $C(I, dJ)$.

Proposition 1. *If I is semi-basic for $\pi : B \to M$, then for every subsystem J of I, we have*
$$F(J) \subset C(I, dJ). \tag{6}$$

Proof. We must show that
$$Char(I, dJ) \subset F(J)^{\perp}. \tag{7}$$
Suppose that $Y \in I^{\perp}$ satisfies
$$Y \rfloor dJ \subset I, \tag{8}$$
and that $Z \in V(B)$. Then
$$< Y, Z \rfloor dJ >= - < Z, Y \rfloor dJ >, \tag{9}$$
so that Y lies in the annihilator of $F(J)$, as required. □

The general construction given above can be applied to the case of the Pfaffian systems associated to systems of differential equations. Let
$$F^a(x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{1_1 \ldots 1_r}) = 0, \quad a = 1, \ldots, n, \tag{10}$$
denote a system of n partial differential equations of order r for maps $u : \mathbf{R}^p \to \mathbf{R}^q$. We work locally and assume that the system (10) defines a smooth submanifold $i : \Sigma \to J^r(\mathbf{R}^p, \mathbf{R}^q)$ which fibers over $J^{r-1}(\mathbf{R}^p, \mathbf{R}^q)$. We let I be the Pfaffian system defined by
$$I = i^* \Omega^r(\mathbf{R}^p, \mathbf{R}^q), \tag{11}$$
where $\Omega^r(\mathbf{R}^p, \mathbf{R}^q)$ denotes the canonical contact Pfaffian system on $J^r(\mathbf{R}^p, \mathbf{R}^q)$. From [3], we know that the regular solutions of (10) are in one-to-one correspondence with the integral manifolds of I which are transversal to the fibers of the vertical distribution $V(\Sigma)$. Furthermore, is easy to verify that I is semi-basic with respect to the projection map $\pi = i \circ \pi^r_{r-1} : \Sigma \to J^{r-1}(\mathbf{R}^p, \mathbf{R}^q)$. The following theorem due to Gardner [3] shows that the characterization of focal curves in terms of envelopes holds in a fairly general context. Even though we won't make use of this result in the rest of our paper, we nevertheless quote it for the sake of completeness.

Theorem 1. *Let J be a subsystem of the Pfaffian I given by (11) and let $\gamma : \mathbf{R} \to \Sigma$ be a J-focal curve which is transversal to $V(\Sigma)$ and not J-characteristic. Then the projected focal curve $\pi \circ \gamma : \mathbf{R} \to J^{r-1}(\mathbf{R}^p, \mathbf{R}^q)$ lies in the envelope of the projection of a 1-parameter family of J-characteristic curves.*

We refer the reader to [3] for the proof of this result, which will be illustrated on a simple example in the following section.

3 Focal Systems for First-Order Partial Differential Equations in the Plane

We consider a first-order partial differential equation in the plane

$$F(x, y, z, p, q) = 0, \tag{12}$$

where $(x, y, z, p = z_x, q = z_y)$ are standard jet coordinates in $J^1(\mathbf{R}^2, \mathbf{R})$ in which the contact Pfaffian system is generated by

$$\omega = dz - pdx - qdy. \tag{13}$$

By our earlier assumptions, the locus defined in $J^1(\mathbf{R}^2, \mathbf{R})$ by (12) is the image of a four-dimensional submanifold $i : \Sigma_4 \to J^1(\mathbf{R}^2, \mathbf{R})$, which fibers over $J^0(\mathbf{R}^2, \mathbf{R})$ with a projection map π. We have

$$i_* V(\Sigma_4) = \{F_q \partial_p - F_p \partial_q\}, \tag{14}$$

so that

$$F(I) = \{i^*(dz - pdx - qdy), i^*(F_q dx - F_p dy)\}. \tag{15}$$

Furthermore, we have

$$Char(I) = \{X\}, \tag{16}$$

where

$$i_* X = F_p \partial_x + F_q \partial_y + (pF_p + qF_q)\partial_z - (F_x + pF_z)\partial_p - (F_y + qF_z)\partial_q. \tag{17}$$

We now give a characterization by differential conditions on F of the first order partial differential equation (12) which have a completely integrable focal system.

Theorem 2. *A first order partial differential equation (12) will have a completely integrable focal system $F(I)$ if and only of F satisfies*

$$F_q^2 F_{pp} - 2F_p F_q + F_p^2 F_{qq} = 0, \tag{18}$$

This condition is equivalent to the focal system $F(I)$ factoring through $J^0(\mathbf{R}^2, \mathbf{R})$.

Proof. From the Frobenius Theorem, we know that $F(I)$ will be completely integrable if and only if the conditions

$$(dp \wedge dx + dq \wedge dy) \wedge \omega \wedge (F_q dx - F_p dy) \wedge dF = 0, \tag{19}$$

and

$$(dF_p \wedge dy - dF_q \wedge dx) \wedge \omega \wedge (F_q dx - F_p dy) \wedge dF = 0, \tag{20}$$

hold. Condition (19) is equivalent to

$$\omega \in F(I)^{(1)}, \tag{21}$$

and is therefore identically satisfied. Condition (20) is equivalent to (18), as claimed. Now, from the retraction theorem for exterior differential systems [1], we know that a necessary and sufficient condition for the focal system $F(I)$ to drop to $J^0(\mathbf{R}^2, \mathbf{R})$ is that the vertical vector field system $V(\Sigma_4)$ and the Cauchy characteristic system $Char(F(I))$ be equal. Now a vector field X on Σ_4 will be an element of $Char(F(I))$ if and only if

$$X \rfloor F(I) = 0, \quad X \rfloor d\omega \in F(I), \quad X \rfloor d(F_q dx - F_p dy) = 0. \tag{22}$$

The first condition in (22) is satisfied on account of (14) and of the fact that $F(I)$ is semi-basic over $J^0(\mathbf{R}^2, \mathbf{R})$. The second condition is an identity, while the third will hold if and only if there exists a function $\lambda \in C^\infty(\Sigma_4, \mathbf{R})$ such that

$$F_q F_{pp} - F_p F_{pq} = \lambda F_p, \quad F_p F_{pq} - F_p F_{qq} = \lambda F_p. \tag{23}$$

The condition (18) follows by eliminating λ. □

We now show how the hypothesis of complete integrability of the focal system $F(I)$ gives rise to an integration method for first order partial differential equations (12).

Theorem 3. *Suppose that the first order partial differential equation (12) has a completely integrable focal system $F(I)$, with first integrals $u, v : J^0(\mathbf{R}^2, \mathbf{R}) \to \mathbf{R}$. Let $G \in C^\infty(\mathbf{R}^2, \mathbf{R})$ be such that*

$$\frac{\partial G}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial z} \neq 0. \tag{24}$$

Then the function $z(x, y)$ defined implicitly by

$$G(u(x, y, z), v(x, y, z)) = 0, \tag{25}$$

is a solution of (12).

Proof. By Theorem 2, the assumption that $F(I)$ is completely integrable is equivalent to $F(I)$ dropping to $J^0(\mathbf{R}^2, \mathbf{R})$, with first integrals $u, v : J^0(\mathbf{R}^2, \mathbf{R}) \to \mathbf{R}$. So there exist non-zero functions $a, b : \Sigma_4 \to \mathbf{R}$ such that

$$I = \{i^*\omega\} = \{adu + bdv\}. \tag{26}$$

It follows that we obtain integral manifolds of $I = \{\omega\}$ by setting

$$u = f(v), \quad \frac{b}{a} = f'(v). \tag{27}$$

Conversely, suppose that there exist functions $u, v : J^0(\mathbf{R}^2, \mathbf{R}) \to \mathbf{R}$ such that (25) defines solutions of (12) for all choices of G satisfying (24). We must show that $F(I)$ is completely integrable with first integrals given by du, dv. By our assumptions, we have

$$\{i^*\omega\} = \{G_u du + G_v dv\}, \tag{28}$$

and we know that (25) defines a map $\sigma_G : \mathbf{R}^2 \to \Sigma_4$ such that

$$\sigma_G \circ (\alpha \circ \sigma_G)^{-1} = j^1 f, \tag{29}$$

where $\alpha : \Sigma_4 \to \mathbf{R}^2$ denotes the source map, and f is a map from $\mathbf{R}^2$ to $\mathbf{R}$. We then have

$$\sigma_G^* dF = 0, \quad \sigma_G^* i^*\omega = 0, \quad \sigma_G^*((\sigma_{G*}X) \rfloor d(i^*\omega)) = 0, \tag{30}$$

for all vector fields X on the source $\mathbf{R}^2$. Since σ_G is a map of rank at most two, there must be a non-trivial linear relation of the form

$$a\omega + b dF + c((\sigma_{G*}X) \rfloor d(i^*\omega)) = 0. \tag{31}$$

Let w denote a coordinate along the fibers of Σ_4 over $J^0(\mathbf{R}^2, \mathbf{R})$. We can choose the 1-forms in (31) in such a way that $a\omega + b dF$ contains no dw differentials and $c((\sigma_{G*}X) \rfloor d(i^*\omega)$ contains no dz differential. It then follows that

$$(\sigma_{G*}X) \rfloor d\omega = e dx + g dy \tag{32}$$

with

$$\sigma_G^*(e dx + g dy) = 0, \tag{33}$$

which implies that the rank of σ_G is less than two. We conclude that $F(I)$ is indeed completely integrable, with first integrals given by u and v. By Theorem 2, these first integrals factor through $J^0(\mathbf{R}^2, \mathbf{R})$. □

We conclude this section by illustrating Theorems 1 and 3 on a very simple example, namely, the first-order partial differential equation

$$p + zq = 0. \tag{34}$$

We have

$$I = \{dz + zq dx - q dy\}, \tag{35}$$

and the focal system $F(I)$ is completely integrable

$$F(I) = \{dz + zq dx - q dy,\ z dx - dy\} = \{dz,\ d(zx - y)\}. \tag{36}$$

It is then straightforward to check that the function $z(x, y)$ defined implicitly by

$$G(z,\ zx - y) = 0, \tag{37}$$

where G is chosen as in Theorem 3 is a solution of (34). We can also verify explicitly on this example that the focal curves have the geometric interpretation given in Theorem 1. An easy calculation gives

$$V(\Sigma_4) = \{Z\}, \quad Char(I) = \{X\}, \tag{38}$$

where

$$Z = \partial_q - z\partial_p, \quad X = \partial_x + z\partial_y - pq\partial_p - q^2\partial_q. \tag{39}$$

We consider now the focal curves $\gamma(t)$ which are the integral curves of

$$Y = X + q^2 Z, \tag{40}$$

given by

$$x(t) = t + x_0,\ y(t) = z_0 t + y_0,\ z(t) = z_0,\ p(t) = -z_0 q_0,\ q(t) = q_0, \tag{41}$$

where $(x_0, y_0, z_0, -z_0 q_0, q_0)$ are the Cauchy data. We obtain a one-parameter family of characteristic curves $\phi_s(t)$ by exponentiating X off the image of $\gamma(t)$,

$$\phi(s,t) = \exp_{\gamma(t)} sX, \tag{42}$$

that is

$$\phi(s,t) = (x(s,t), y(s,t), z(s,t), p(s,t), q(s,t)), \tag{43}$$

where

$$x(s,t) = s + t + x_0,\ y(s,t) = z_0 s + z_0 t + y_0, \tag{44}$$

$$z(s,t) = z_0,\ p(s,t) = \frac{-z_0 q_0}{q_0 s + 1},\ q(s,t) = \frac{q_0}{q_0 s + 1}. \tag{45}$$

We then compute

$$\phi_* \partial_t|_{(0,t)} = (\partial_x + z\partial_y)|_{(0,t)} = Y|_{(0,t)} = (X + q^2 Z)|_{(0,t)}, \tag{46}$$

$$\phi_* \partial_s|_{(0,t)} = (\partial_x + z\partial_y + pq\partial_p + q^2\partial_q)|_{(0,t)} = X|_{(0,t)}. \tag{47}$$

The projection of the one-parameter family $\phi(s,t)$ of characteristic curves onto $J^0(\mathbf{R}^2, \mathbf{R})$ is given by

$$\psi(s,t) = (\pi \circ \phi)(s,t), \tag{48}$$

so that

$$\psi_* \partial_t|_{(0,t)} = \pi_* \circ \phi_* \partial_t|_{(0,t)} = \pi_*(X + q^2 Z)|_{(0,t)} = \pi_* X|_{0,t}, \tag{49}$$

and

$$\psi_* \partial_s|_{(0,t)} = \pi_* \circ \phi_* \partial_s|_{(0,t)} = \pi_* X|_{(0,t)}, \tag{50}$$

since Z is π-vertical. We see that the rank of ψ_* drops at $(0, t)$, so that the projection of the focal curve $\gamma(t)$ lies indeed in the envelope of the projection $\psi(s, t)$ of a one-parameter family of characteristic curves.

We conclude this section by remarking that the condition (18) for the focal system $F(I)$ to drop to $J^0(\mathbf{R}^2, \mathbf{R})$ has the obvious solutions

$$F(x, y, z, p, q) = a(x, y, z)p + b(x, y, z)q + c(x, y, z), \tag{51}$$

corresponding to sections of the affine hyperplane sub-bundle of the affine bundle $\pi_0^1 : J^1(\mathbf{R}^2, \mathbf{R}) \to J^0(\mathbf{R}^2, \mathbf{R})$. It would be interesting to obtain other solutions of (18), which is a parabolic second-order equation for fixed (x, y, z).

4 Focal Systems for Second-Order Hyperbolic Partial Differential Equations in the Plane

We now focus our attention on second-order partial differential equation in the plane

$$F(x, y, z, p, q, r, s, t) = 0, \tag{52}$$

where $p = z_x, q = z_y, r = z_{xx}, s = z_{xy}, t = z_{yy}$ are standard jet coordinates on $J^2(\mathbf{R}^2, \mathbf{R})$ in which the contact Pfaffian system is generated by

$$\theta^1 = dz - pdx - qdy, \quad \theta^2 = dp - rdx - sdy, \quad \theta^3 = dq - sdx - tdy. \tag{53}$$

We assume that the locus defined in $J^2(\mathbf{R}^2, \mathbf{R})$ by (12) is the image of a seven-dimensional submanifold $i : \Sigma_7 \to J^2(\mathbf{R}^2, \mathbf{R})$, which fibers over $J^1(\mathbf{R}^2, \mathbf{R})$. On Σ_7 we consider the Pfaffian system I defined by

$$I = \{\omega^1 := i^*\theta^1,\ \omega^2 := i^*\theta^2,\ \omega^3 := i^*\theta^3\}. \tag{54}$$

As noted in Sect. 2, the solutions of (52) are in one-to-one correspondence with the integral manifolds of I which are transversal to $V(\Sigma_7)$. We assume that (52) is everywhere hyperbolic, meaning that

$$F_r F_t - \frac{1}{4} F_s^2 < 0 \tag{55}$$

on the locus $i(\Sigma_7)$. We know from [4] that there exist locally linearly independent 1-forms $\omega^1, \pi^2, \pi^3, \omega^4, \omega^5, \omega^6, \omega^7$ such that ω^1, π^2, π^3 generate I and such that the following structure equations hold

$$d\omega^1 \equiv 0, \quad d\pi^2 \equiv \omega^4 \wedge \omega^5, \quad d\pi^3 \equiv \omega^6 \wedge \omega^7, \ mod\ I \tag{56}$$

and such that

$$V(\Sigma_7) = \{\omega^1, \pi^2, \pi^3, \omega^4, \omega^6\}. \tag{57}$$

It is also known from [3] and [4] that the Pfaffian systems

$$M_1 = \{\omega^1, \pi^2\}, \quad M_2 = \{\omega^1, \pi^3\}, \tag{58}$$

of I, which correspond to the Monge characteristics, are contact invariant sub-systems of I, geometrically defined as maximal isotropic sub-systems with respect to a certain conformal symmetric $C^\infty(\Sigma_7; \mathbf{R})$-bilinear form on I. In [4], it is shown that we have the sharp lower bounds

$$class(M_1) \geq 6, \quad class(M_2) \geq 6, \tag{59}$$

and moreover that we have $class(M_1) = 6 = class(M_2)$ if and only if (52) belongs to class of the Monge–Ampère class of equations

$$a(rt - s^2) + br + cs + dt + e = 0, \tag{60}$$

where a, b, c, d, e are functions of x, y, z, p, q. From now on we will focus our attention on hyperbolic Monge–Ampère equations, with the purpose of showing that a number of the normal form results obtained in [4] admit simpler formulations in terms of focal systems, similarly to what was shown in [5] for determined systems of two first-order partial differential equations in the plane. We should stress here that these normal form results are under local contact equivalence, where we recall that (52) and an equation

$$F(\bar{x}, \bar{y}, \bar{z}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t}) = 0, \tag{61}$$

are said to be locally contact equivalent if there exists a local diffeomorphism $\Phi : \Sigma_7 \to \bar{\Sigma}_7$ such that

$$\Phi^* \bar{I} = I. \tag{62}$$

First of all, we recall from [4] that in the Monge–Ampère case, the structure equations (56) can be put in the form

$$d\omega^1 \equiv \pi^2 \wedge \omega^4 + \pi^3 \wedge \omega^6 \quad mod\, \{\omega^1\} \tag{63}$$
$$d\pi^2 \equiv \omega^4 \wedge \omega^5 \quad mod\, \{\omega^1, \pi^2\} \tag{64}$$
$$d\pi^3 \equiv \omega^6 \wedge \omega^7 \quad mod\, \{\omega^1, \pi^3\}, \tag{65}$$

with (57) still holding. We now state and prove a proposition that gives the expressions of the focal systems of the Monge characteristic systems, as well as their first derived systems. This will then be used to establish the normal form results referred to above.

Proposition 2. *For any hyperbolic Monge–Ampère equation (60) there exists locally a coframe* $(\omega^1, \pi^2, \pi^3, \omega^4, \omega^5, \omega^6, \omega^7)$ *on* Σ_7 *such that the structure* *equations (63), (64), (65) hold and the focal systems of* $M_1 = \{\omega^1, \pi^2\}$, $M_2 = \{\omega^1, \pi^3\}$ *and their first derived systems are given by*

$$F(M_1) = \{\omega^1, \pi^2, \omega^4\}, \quad F(M_1)^{(1)} = \{\pi^2, \omega^4\} \tag{66}$$

and

$$F(M_2) = \{\omega^1, \pi^3, \omega^6\}, \quad F(M_1)^{(1)} = \{\pi^3, \omega^6\}. \tag{67}$$

Proof. We know from the structure equations (65) and (66) that

$$d\omega^1 \equiv \pi^2 \wedge \omega^4 + \pi^3 \wedge \omega^6 + \lambda \wedge \omega^1, \tag{68}$$
$$d\pi^2 \equiv \omega^4 \wedge \omega^5 + \mu \wedge \omega^1 + \nu \wedge \pi^2, \tag{69}$$

for some one-forms λ, μ, ν on Σ_7. Therefore if $Z \in V(\Sigma_7)$, then

$$Z \rfloor d\omega^1 =< Z, \lambda > \omega^1, \tag{70}$$

and

$$Z \rfloor d\pi^2 = - < Z, \omega^5 > \omega^4 + < Z, \mu > \omega^1 + < Z, \nu > \pi^2, \tag{71}$$

which shows that

$$F(M_1) \subset \{\omega^1, \pi^2, \omega^4\}. \tag{72}$$

The proof of the reverse inclusion is left as an exercise to the reader. To compute the first derived system of $F(M_1)$, we first observe that as an immediate consequence of the structure equations (63) and (64), we have

$$\omega^1 \notin F(M_1)^{(1)}, \quad \pi^2 \in F(M_1)^{(1)}. \tag{73}$$

Next, we choose $Z \in V(\Sigma_7)$ scaled in such a way that $< Z, \omega^5 >= -1$, so that

$$\mathcal{L}_Z \pi^2 = \omega^4 + f\omega^1 + g\omega^2, \tag{74}$$

for some scalar coefficients f, g. Taking the exterior derivative on both sides of (74), we obtain that

$$\mathcal{L}_Z d\pi^2 \equiv d\omega^4 \quad mod\, \{\omega^1, \pi^2, \omega^4\}, \tag{75}$$

and on the other hand by Lie differentiating both sides of the structure equation (64) with respect to Z we obtain

$$\mathcal{L}_Z d\pi^2 \equiv 0 \quad mod\, \{\omega^1, \pi^2, \omega^4\}, \tag{76}$$

since $\omega^4 \in F(M_1)$. It follows that

$$d\omega^4 \equiv 0 \quad mod\, \{\omega^1, \pi^2, \omega^4\}, \tag{77}$$

and therefore that $\omega^4 \in F(M_1)^{(1)}$, as claimed. The proof is similar for M_2. □

We are now ready to state and prove our normal form results under contact equivalence. The first result gives a characterization of the contact orbit of the wave equation in terms of focal systems.

Theorem 4. *A hyperbolic Monge–Ampère equation is locally contact equivalent to the wave equation*

$$s = 0, \tag{78}$$

if and only if the first derived systems $F(M_1)^{(1)}$ and $F(M_2)^{(1)}$ of the focal systems associated to the Monge characteristic systems are both completely integrable.

Proof. The idea of the proof of the sufficiency of the conditions stated in the theorem is to show that these guarantee that each of the M_1- and M_2-characteristic systems $C(I, dM_1)$ and $C(I, dM_2)$ has a completely integrable subsystem of rank three. This will guarantee the existence of a local contact equivalence between the given Monge–Ampère equation and the wave equation thanks to Theorem 5.15 of [4] (which was itself known to Lie, but in a different formulation). Our assumption of complete integrability implies that

$$d\pi^2 \wedge \pi^2 \wedge \omega^4 = 0, \tag{79}$$

which by the quadratic lemma in exterior algebra [1] implies that

$$d\pi^2 \wedge d\pi^2 \wedge^2 = 0, \tag{80}$$

which is equivalent to

$$class(\{\pi^2\}) = \dim C(\{\pi^2\}) = 3. \tag{81}$$

But $C(\{\pi^2\})$ is a Cartan system and is therefore completely integrable, so that

$$C(\{\pi^2\}) = \{du_1, du_2, du_3\}, \tag{82}$$

for some functionally independent functions locally defined on Σ_7. But by Proposition 1, we have an inclusion

$$C(\{\pi^2\}) \subset C(I, dM_1), \tag{83}$$

and therefore $C(I, dM_1)$ admits a three-dimensional completely integrable subsystem. Likewise $C(I, dM_1)$ admits a three-dimensional completely integrable sub-system. The conclusion follows by Theorem 5.15 of [4]. The proof of necessity is trivial since for the wave equation we have

$$\{\pi^2, \omega^4\} = \{dp, dx\}, \quad \{\pi^3, \omega^6\} = \{dq, dy\}. \tag{84}$$

□

The final case we consider is that in which neither $F(M_1)^{(1)}$ nor $F(M_2)^{(1)}$ is necessarily completely integrable, but both the second derived systems $F(M_1)^{(2)}$ and $F(M_2)^{(2)}$ are completely integrable. We shall see that this hypothesis characterizes the contact orbit of a class of quasi-linear equations (see [6] for the analogue of this result using characteristic invariants).

Theorem 5. *A hyperbolic Monge–Ampère equation is locally contact equivalent to a quasi-linear equation of the form*

$$s = f(x, y, z, p, q) \tag{85}$$

if and only if the second derived systems $F(M_1)^{(2)}$ *and* $F(M_2)^{(2)}$ *of its focal systems* $F(M_1)$ *and* $F(M_2)$ *are both completely integrable.*

Proof. In view of Theorem 4, we may assume while proving necessity that neither $F(M_1)^{(1)}$ nor $F(M_2)^{(1)}$ are completely integrable, so that the derived flags of the focal systems are given by

$$F(M_1) = \{\omega^1, \pi^2, \omega^4\}, \quad F(M)^{(1)} = \{\pi^2, \omega^4\}, \quad F(M_1)^{(2)} = \{dX\}, \tag{86}$$

and

$$F(M_2) = \{\omega^1, \pi^3, \omega^6\}, \quad F(M)^{(1)} = \{\pi^3, \omega^6\}, \quad F(M_1)^{(2)} = \{dY\}, \tag{87}$$

where $dX \wedge dY \neq 0$. From the structure equation (63), we have

$$d\omega^1 \wedge \omega^1 = \tilde{\pi}^2 \wedge dX \wedge \omega^1 + \tilde{\pi}^3 \wedge dY \wedge \omega^1. \tag{88}$$

where $\tilde{\pi}^2 \in \{\pi^2, \omega^4\}$ and $\tilde{\pi}^3 \in \{\pi^3, \omega^6\}$ are one-forms such that the structure equations (64) and (65) are valid with π^2 and π^3 replaced by $\tilde{\pi}^2$ and $\tilde{\pi}^3$. From the preceding equation it follows immediately that

$$d\omega^1 \wedge \omega^1 \wedge dX \wedge dY = 0, \tag{89}$$

so that there exist functions Z, P, Q locally defined on Σ_7 such that

$$\{\omega^1\} = \{dZ - PdX - QdY\}, \tag{90}$$

with $dZ \wedge dP \wedge dQ \wedge dX \wedge dY \neq 0$. Now, (90) implies that

$$d\omega^1 \equiv -dP \wedge dX - dQ \wedge dY \quad mod\, \{\omega^1\}, \tag{91}$$

so that

$$(\tilde{\pi}^2 - dP) \wedge dX + (\tilde{\pi}^3 - dQ) \wedge dY \equiv 0 \quad mod\, \{\omega^1\}. \tag{92}$$

From Cartan's Lemma [1], we then get

$$\tilde{\pi}^2 - dP \equiv -RdX - FdY \quad mod\,\{\omega^1\} \tag{93}$$

$$\tilde{\pi}^3 - dQ \equiv -FdX - TdY \quad mod\,\{\omega^1\}. \tag{94}$$

But we have

$$0 = d\tilde{\pi}^2 \wedge \omega^1 \wedge \tilde{\pi}^2 \wedge \tilde{\pi}^3 \wedge \omega^4 = dF \wedge dX \wedge dY \wedge dZ \wedge dP \wedge dQ, \tag{95}$$

which implies that $F = F(X, Y, Z, P, Q)$, and that the Pfaffian system I is generated by

$$dZ - PdX - QdY,\ dP - RdX - FdY,\ dQ - FdX - TdY, \tag{96}$$

as required. The proof of the converse is a direct calculation starting from (85). □

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Hamiltonian Structures for General PDEs

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Abstract We sketch out a new geometric framework to construct Hamiltonian operators for generic, non-evolutionary partial differential equations. Examples on how the formalism works are provided for the KdV equation, Camassa–Holm equation, and Kupershmidt's deformation of a bi-Hamiltonian system.

1 Introduction

In this short paper we will discuss the following question: What happens to a Hamiltonian operator of an evolution system if we change coordinates so that the system becomes non-evolution?

Using the traditional definition of a Hamiltonian structure one cannot answer this question, since the definition is tied to evolution form of the system at hand. However, first, not all equations have a natural evolution form, and, second, an evolution form of a system of equations is not unique. Let us consider some examples.

Example 1 (KdV). It is well known that the KdV equation $u_t = u_{xxx} + 6uu_x$ has two compatible Hamiltonian operators:

$$A_1 = D_x, \qquad A_2 = D_{xxx} + 4uD_x + 2u_x, \tag{1}$$

so that the equation can be written in the following ways:

$$\begin{aligned} u_t = u_{xxx} + 6uu_x &= D_x \frac{\delta}{\delta u}(u^3 - u_x^2/2) \\ &= (D_{xxx} + 4uD_x + 2u_x)\frac{\delta}{\delta u}(u^2/2), \end{aligned}$$

where $\delta/\delta u$ denotes the Euler operator (the variational derivative) and is applied to the two Hamiltonian densities.

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Let us introduce new dependent variables v and w and rewrite the KdV equation in the form

$$u_x = v, \quad v_x = w, \quad w_x = u_t - 6uv. \tag{2}$$

In the new coordinates, the KdV still has an evolutionary form, but with respect to another independent variable (x instead of t). A natural question arises then: Is the KdV equation in the form (2) Hamiltonian? An affirmative answer to this question was obtained by Tsarev in [9]. He proved that transformations of the type (2) preserve the Hamiltonian property of all evolution systems for which the Cauchy problem is solvable. Our approach is very different from Tsarev's one. Below we explain why this fact holds true for all transformations of variables and without the assumption on the Cauchy problem. We will also show how to compute the Hamiltonian structure in new coordinates. For the above example the answer is the following:

$$\begin{aligned}\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -6u \\ 0 & 6u & D_t \end{pmatrix} \frac{\delta}{\delta u}(uw - v^2/2 + 2u^3) \\ &= \begin{pmatrix} 0 & -2u & -D_t - 2v \\ 2u & D_t & -12u^2 - 2w \\ -D_t + 2v & 12u^2 + 2w & 8uD_t + 4u_t \end{pmatrix} \frac{\delta}{\delta u}(-3u^2/2 - w/2).\end{aligned} \tag{3}$$

Example 2 (Camassa–Holm equation). Camassa and Holm have written their equation $u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$ in a bi-Hamiltonian form by introducing the new variable $m = u - u_{xx}$. The equation now takes the form

$$m_t = -um_x - 2u_x m = B_1 \frac{\delta \mathcal{H}_1}{\delta m} = B_2 \frac{\delta \mathcal{H}_2}{\delta m} \tag{4}$$

with

$$\begin{aligned} B_1 &= -(mD_x + D_x m), \quad \mathcal{H}_1 = \frac{1}{2}\int mu\, dx, \\ B_2 &= D_x^3 - D_x, \quad \mathcal{H}_2 = \frac{1}{2}\int (u^3 + uu_x^2)\, dx. \end{aligned}$$

Note that $\mathcal{H}_1$ and $\mathcal{H}_2$ are viewed as functionals in m and u, but not in u solely. To get rid of m, one is forced to assume that $u = (1 - D_x^2)^{-1} m$ in the Hamiltonian densities. The use of the inverse of the operator $1 - D_x^2$ is not elegant from mathematical viewpoint. We will find a bi-Hamiltonian structure for the Camassa–Holm equation written in the initial non-evolution form and thus get rid of the term $(1 - D_x^2)^{-1}$.

Example 3 (Kupershmidt deformation). Consider a bi-Hamiltonian evolution system of equations $u_t = f(t, x, u, u_x, u_{xx}, \dots)$, u and f being vector functions, with compatible Hamiltonian operators A_1 and A_2 and a Magri hierarchy of conserved densities $H_1, H_2, \dots$

$$D_t(H_i) = 0, \qquad A_1 \frac{\delta H_i}{\delta u} = A_2 \frac{\delta H_{i+1}}{\delta u}.$$

In [8], Kupershmidt defined what he called the *nonholonomic deformation* of the above system:

$$u_t = f - A_1(w), \qquad A_2(w) = 0. \tag{5}$$

We call system (1) the *Kupershmidt deformation* of the system $u_t = f$. The motivating example of this construction is the so-called KdV6 equation (see [4])

$$u_t = u_{xxx} + 6uu_x - w_x, \quad w_{xxx} + 4uw_x + 2u_x w = 0 \tag{6}$$

which is the Kupershmidt deformation of the KdV equation. The authors of [4] have shown that the KdV6 passes the Painlevé test and conjectured that the system is integrable. Kupershmidt, in [8], found a hierarchy of conservation laws of the KdV6 as a particular case of the following general fact.

Theorem (Kupershmidt). *Let $u_t = f$ be an evolution bi-Hamiltonian system, with A_1, A_2 being the corresponding Hamiltonian operators. If this equation has a Magri hierarchy of conserved densities $\frac{dH_i}{dt} = 0$, $A_1 \frac{\delta H_i}{\delta u} = A_2 \frac{\delta H_{i+1}}{\delta u}$ then $H_1, H_2, \ldots$ are conserved densities for* (1).

Proof.

$$\frac{dH_i}{dt} = \left\langle \frac{\delta H_i}{\delta u}, f + A_1(w) \right\rangle = \left\langle -A_1 \frac{\delta H_i}{\delta u}, w \right\rangle$$
$$= \left\langle -A_2 \frac{\delta H_{i+1}}{\delta u}, w \right\rangle = \left\langle \frac{\delta H_{i+1}}{\delta u}, A_2(w) \right\rangle = 0.$$

□

Kupershmidt also conjectured that $H_1, H_2, \ldots$ commute in some sense so that the KdV6 is indeed integrable. Below we will see that this is true and, moreover, system (1) is bi-Hamiltonian.

Our framework to study Hamiltonian structures for general PDEs is the geometry of jet spaces and differential equations. We assume the reader to be familiar with the geometric approach to differential equations and hence we include only the notation and the coordinate descriptions in the next section. We refer the reader to the books [1,6] for further information.

2 Notation: Infinite Jets and Differential Equations

In what follows everything is supposed to be smooth.

We denote an infinite jet space by J^∞. This can be the space of jets of submanifolds, maps, sections of a bundle, and so on, and it is not important to us here.

Coordinates on J^∞ are x_i (independent variables, $i = 1, \dots, n$) and u^j_σ (dependent variables, $j = 1, \dots, m$, σ being multi-indices).

The formulas

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}$$

provide expressions for the total derivatives. The vector fields D_i span the Cartan distribution on J^∞. To every vector function on J^∞, there corresponds the evolutionary field

$$E_\varphi = \sum_{j,\sigma} D_\sigma(\varphi^j) \frac{\partial}{\partial u^j_\sigma}.$$

The matrix differential operator

$$\ell_f = \left\| \sum_\sigma \frac{\partial f^i}{\partial u^j_\sigma} D_\sigma \right\|.$$

is the linearization of a vector function f. It is defined by the formula $\ell_f(\varphi) = E_\varphi(f)$. The linearization is a differential operator in total derivatives; we shall call such operators *$\mathscr{C}$-differential operators*.

The coordinate expression for the adjoint $\mathscr{C}$-differential operator is

$$\Delta^* = \left\| \sum_\sigma (-1)^{|\sigma|} D_\sigma a^{ji}_\sigma \right\|$$

if $\Delta = \left\| \sum_\sigma a^{ij}_\sigma D_\sigma \right\|$.

Let $F_k(x_i, u^j_\sigma) = 0$, $k = 1, \dots\ l$, be a system of differential equations. Then the relations $F = (F_1, \dots, F_l) = 0$ together with $D_\sigma(F) = 0$ define its infinite prolongation $\mathscr{E} \subset J^\infty$. For the sake of brevity we shall call the infinite prolongation of a system of differential equations the equation. The operator $\ell_{\mathscr{E}} = \ell_F|_{\mathscr{E}}$ is the linearization of the equation $\mathscr{E}$.

In this paper, we only consider equations $\mathscr{E}$ whose linearization $\ell_{\mathscr{E}}$ is *normal* in the following sense.

Definition 1. A $\mathscr{C}$-differential operator ∇ called *normal* if the compatibility operators for both ∇ and ∇^* are trivial. In other words, if there exists a $\mathscr{C}$-differential operator Δ such that $\Delta \circ \nabla = 0$ on $\mathscr{E}$ then $\Delta = 0$ on $\mathscr{E}$ as well, and the same holds true with ∇^* instead of ∇.

An evolutionary field E_φ is a symmetry of the equation $\mathscr{E}$ if $E_\varphi(F)|_{\mathscr{E}} = \ell_{\mathscr{E}}(\varphi) = 0$. If E_φ is a symmetry then φ is said to be its generating function. We often identify symmetries with their generating functions.

A vector function $S = (S^1, \dots, S^n)$ on $\mathscr{E}$ is a conserved current if $\sum_i D_i(S^i) = 0$ on $\mathscr{E}$. A conserved current is trivial if there exist functions T_{ij} on $\mathscr{E}$ such that $S^i = \sum_{j<i} D_j(T^{ji}) - \sum_{i<j} D_j(T^{ij})$.

Conservation laws of $\mathscr{E}$ are classes of conserved currents modulo trivial ones. To every conservation law, there correspond its generating function, which is computed in the following way. If $S = (S^1, \dots, S^n)$ is a conserved current, so that $\sum_i D_i(S^i) = 0$ on $\mathscr{E}$, then there exists a $\mathscr{C}$-differential operator Δ such that $\sum_i D_i(S^i) = \Delta(F)$ on J^∞. The generating function of the conservation law is defined by $\psi = (\psi_1, \dots, \psi_m) = \Delta^*(1)$. Note that $\psi = 0$ if and only if the conserved current S is trivial. One can prove that every generating function ψ satisfies the equation $\ell^*_{\mathscr{E}}(\psi) = 0$, so that the set $\mathrm{CL}(\mathscr{E})$ of conservation laws of $\mathscr{E}$ is a subset in the kernel of $\ell^*_{\mathscr{E}}$, $\mathrm{CL}(\mathscr{E}) \subset \operatorname{Ker} \ell^*_{\mathscr{E}}$.

3 Cotangent Bundle to an Equation

Let us introduce our main hero. For every differential equation $\mathscr{E}$ we define a canonical covering $\tau^*\colon \mathscr{L}^*(\mathscr{E}) \to \mathscr{E}$, called the *$\ell^*$-covering*. The equation $\mathscr{L}^*(\mathscr{E})$ is given by the system

$$\ell^*_F(p) = 0, \qquad F = 0,$$

if $\mathscr{E}$ is given by $F = 0$. Here $p = (p^1, \dots, p^l)$ are new dependent variables, l being the number of equations $F = (F_1, \dots, F_l)$. We endow $\mathscr{L}^*(\mathscr{E})$ with the structure of a supermanifold by choosing the variables p^k to be odd. The covering τ^* is the natural projection $\tau^*\colon (u^j_\sigma, p^k_\sigma) \mapsto (u^j_\sigma)$.

Note that

$$\langle F, p \rangle = \sum_{i=1}^{l} F_i p^i \tag{7}$$

is the Lagrangian for the equation $\mathscr{L}^*(\mathscr{E})$.

It is easily shown that $\ell_{\mathscr{L}^*(\mathscr{E})}$ is normal if $\ell_{\mathscr{E}}$ is normal.

From the above definition it is not seen why we said that ℓ^*-covering is *canonical*. Indeed, the definition uses the embedding $\mathscr{E} \to J^\infty$, but later we will show that $\mathscr{L}^*(\mathscr{E})$ is independent of the choice of this embedding.

Remark 1. For an arbitrary $\mathscr{C}$-differential operator Δ one can define the Δ-covering in the same way as the ℓ^*-covering is associated with the operator $\ell^*_{\mathscr{E}}$.

The most interesting for us property of the ℓ^*-covering is given by the following theorem.

Theorem 1. *There is a natural* 1–1 *correspondence between the symmetries of $\mathscr{E}$ and the conservation laws of $\mathscr{L}^*(\mathscr{E})$ linear along the fibers of τ^*.*

The expression "linear conservation law" means that the corresponding conserved current is linear along the fibers of τ^* (i.e., linear in variables p^k). Here and below we skip the proofs that can be found in our joint paper with S. Igonin [3]. Let us nevertheless describe the correspondence stated in the theorem in terms of generating functions. If φ is a symmetry of equation $\mathscr{E}$ then there exists a $\mathscr{C}$-differential operator Δ such that $\ell_F(\varphi) = \Delta(F)$. Consider the adjoint operator Δ^*. It can be naturally identified with a fiberwise linear vector function φ_Δ on $\mathscr{L}^*(\mathscr{E})$. Then the vector function $(\varphi, \varphi_\Delta)$ is the generating function of the conservation law that corresponds to the symmetry φ.

In the geometry of differential equation it is very useful to construct an analogy with geometry of finite dimensional manifolds. We shall now use this approach to clarify the meaning of the above theorem. Let us start building our analogy with the following two rather standard correspondences (cf. [10] and references therein):

Manifold M		*PDE* $\mathscr{E}$
functions	$\longleftrightarrow$	conservation laws
vector fields	$\longleftrightarrow$	symmetries

Now, using Theorem 1, we can say that the analog of the ℓ^*-covering is a vector bundle such that vector fields on the base are in 1–1 correspondence with fiberwise linear functions on the total space of the bundle. Obviously, such a bundle is the cotangent bundle. So, the ℓ^*-covering is the cotangent bundle to an equation, and we can continue our manifold-equation dictionary:

Manifold M		*PDE* $\mathscr{E}$
functions	$\longleftrightarrow$	conservation laws
vector fields	$\longleftrightarrow$	symmetries
$T^*(M)$	$\longleftrightarrow$	$\mathscr{L}^*(\mathscr{E})$

Remark 2. This dictionary can be easily extended:

Manifold M		*PDE* $\mathscr{E}$
functions	$\longleftrightarrow$	conservation laws
vector fields	$\longleftrightarrow$	symmetries
$T^*(M)$	$\longleftrightarrow$	$\mathscr{L}^*(\mathscr{E})$
$T(M)$	$\longleftrightarrow$	$\mathscr{L}(\mathscr{E})$
De Rham complex	$\longleftrightarrow$	$E_1^{0,n-1} \to E_1^{1,n-1} \to E_1^{2,n-1} \to \cdots$

Here $\mathscr{L}(\mathscr{E})$ is the ℓ-covering (see Remark 1). The complex $E_1^{0,n-1} \to E_1^{1,n-1} \to E_1^{2,n-1} \to \cdots$ is $(n-1)$st line of the Vinogradov $\mathscr{C}$-spectral sequence (see [10] and references therein). In this paper we use only the first three entries of the dictionary.

Remark 3. In [7], Kupershmidt defined the cotangent bundle to a bundle. This construction can be identified with the ℓ^*-covering of the system

$$u_t^1 = 0, \quad u_t^2 = 0, \quad \dots \quad u_t^m = 0.$$

At this point, a natural question may arise: what is the analog of the Poisson bracket on the cotangent bundle? The answer is that the ℓ^*-covering is endowed with a canonical Poisson bracket. More precisely, since we changed the parity of fibers in the ℓ^*-covering, this bracket is a superbracket and is the analog of the Schouten bracket. We shall call it the *variational Schouten bracket*.

To define the bracket, recall that $\mathscr{L}^*(\mathscr{E})$ has the Lagrangian structure (7). Hence, by the Noether theorem there is a 1–1 correspondence between conservation laws on $\mathscr{L}^*(\mathscr{E})$ and Noether symmetries of $\mathscr{L}^*(\mathscr{E})$. If ψ is the generating function of a conservation law, then E_ψ is the corresponding Noether symmetry. The set of Noether symmetries is a Lie superalgebra with respect to the commutator, so we obtain a structure of Lie superalgebra on conservation laws on $\mathscr{L}^*(\mathscr{E})$ uniquely determined by the equality

$$E_{[\![\psi_1,\psi_2]\!]} = [E_{\psi_1}, E_{\psi_2}]. \tag{8}$$

According to our manifold-equation dictionary, conservation laws on $\mathscr{L}^*(\mathscr{E})$ correspond to functions on $T^*(M)$. The latter are skew multivectors on M (this is why we have changed the parity of fibers of the ℓ^*-covering – to get skew-symmetric multivectors). So, we shall call conservation laws on $\mathscr{L}^*(\mathscr{E})$ the *variational multivectors*. Linear conservation laws, as we saw, are vectors, bilinear ones are bivectors and so on.

The generating function of a variational k-vector is a vector function on $\mathscr{L}^*(\mathscr{E})$ which is $(k-1)$-linear along τ^*-fibers. Such a function can be identified with a $(k-1)$-linear $\mathscr{C}$-differential operator on $\mathscr{E}$. In coordinates, this correspondence boils down to the change $p_\sigma \mapsto D_\sigma$. Thus, we can (and will) identify variational multivectors to multilinear $\mathscr{C}$-differential operators.

More precisely, in the above identification we will use not operators but equivalence classes of $\mathscr{C}$-differential operators modulo operators divisible by $\ell^*_{\mathscr{E}}$. This is being done, because operators of the form $\square \circ \ell^*_{\mathscr{E}}$ correspond to trivial functions on $\mathscr{L}^*(\mathscr{E})$. But we will not change terminology, we say operator instead of the equivalence class.

For the sake of brevity and because we are interested in the Hamiltonian formalism, let us restrict ourselves to bivectors, which are identified with linear $\mathscr{C}$-differential operators. Formulas presented below for bivectors (= linear operators) can be easily generalized to multivectors (= multilinear operators).

Theorem 2. *An operator A is a variational bivector on equation $\mathscr{E}$ if and only if it satisfies the condition*

$$\ell_{\mathscr{E}} A = A^* \ell^*_{\mathscr{E}}.$$

Remark 4. If $\mathscr{E}$ is written in evolution form then the above condition implies that $A^* = -A$.

From this theorem it follows that a Hamiltonian operator A takes a generating function of a conservation law ψ to a symmetry $A(\psi)$.

This is the formula for the variational Schouten bracket of two bivectors:

$$
\begin{aligned}
[\![A_1, A_2]\!](\psi_1, \psi_2)
&= \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\
&+ \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\
&\quad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)),
\end{aligned}
$$

where $\ell_{A,\psi} = \ell_{A(\psi)} - A\ell_\psi$ and the operators B_i^* are defined by the equalities:

$$
\ell_F A_i - A_i^* \ell_F^* = B_i(F, \cdot) \quad \text{on } J^\infty,
$$
$$
B_i^*(\psi_1, \psi_2) = B_i^{*1}(\psi_1, \psi_2)|_{\mathcal{E}}.
$$

Here ^{*1} denotes that the adjoint operator is computed with respect to the first argument. The operators B_i^* are skew-symmetric and skew-adjoint in each argument. Note that if $\mathcal{E}$ is in evolution form then $B_i^*(\psi_1, \psi_2) = \ell^*_{A_i,\psi_2}(\psi_1)$.

Now we are in position to give a definition of a Hamiltonian structure for a general PDE.

Definition 2. A variational bivector A is called *Hamiltonian* if $[\![A, A]\!] = 0$.

A Hamiltonian bivector A gives rise to a Poisson bracket

$$
\{\psi_1, \psi_2\}_A = E_{A(\psi_1)}(\psi_2) + \Delta^*(\psi_2), \tag{9}
$$

where ψ_1 and ψ_2 are conservation laws of $\mathcal{E}$ and the operator Δ is defined by the relation $\ell_F(A(\psi_1)) = \Delta(F)$.

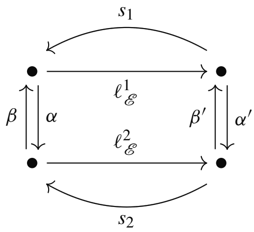
As in the evolution case, we call an equation *bi-Hamiltonian* if it possesses two Hamiltonian structures A_1 and A_2 such that $[\![A_1, A_2]\!] = 0$.

An infinite series of conservation laws $\psi_1, \psi_2, \ldots$ is called a *Magri hierarchy* if for all i we have $A_1(\psi_i) = A_2(\psi_{i+1})$. In the standard way one can show that $\{\psi_i, \psi_j\}_{A_1} = \{\psi_i, \psi_j\}_{A_2} = 0$ for all i and j.

Now let us return to the question of invariance of the ℓ^*-covering. Suppose the equation $\mathcal{E}$ under consideration is embedded in two different jet spaces

$$
\mathcal{E} \begin{array}{l} \nearrow \; J_1^\infty \\ \searrow \; J_2^\infty \end{array}
$$

We encountered an example of this situation when discussed the KdV equation, with J_1^∞ being jets with coordinates x, t and u, while J_2^∞ being jets with coordinates x, t, u, v, and w. Now, we have two linearization operators, $\ell^1_{\mathcal{E}}$ and $\ell^2_{\mathcal{E}}$, the former computed using the embedding $\mathscr{E} \to J_1^\infty$ and the latter is obtained using the embedding $\mathscr{E} \to J_2^\infty$. It is not difficult to show that these two linearization operators are related by the following diagram:



(10)

where all arrows are $\mathscr{C}$-differential operators on $\mathscr{E}$ satisfying the following relations:

$$\ell^1_{\mathscr{E}}\,\beta = \beta'\,\ell^2_{\mathscr{E}}, \quad \ell^2_{\mathscr{E}}\,\alpha = \alpha'\,\ell^1_{\mathscr{E}}, \qquad \beta\,\alpha = \mathrm{id} + s_1\,\ell^1_{\mathscr{E}}, \quad \alpha\,\beta = \mathrm{id} + s_2\,\ell^2_{\mathscr{E}}. \quad (11)$$

We use the dots $\bullet$ to avoid introducing new notations for the corresponding spaces of sections of vector bundles.

Definition 3. Two $\mathscr{C}$-differential operators Δ_1 and Δ_2 on $\mathscr{E}$ are called *equivalent* if there exist $\mathscr{C}$-differential operators α, β, α', β', s_1, and s_2 such that

$$\Delta_1\,\beta = \beta'\,\Delta_2, \quad \Delta_2\,\alpha = \alpha'\,\Delta_1, \qquad \beta\,\alpha = \mathrm{id} + s_1\,\Delta_1, \quad \alpha\,\beta = \mathrm{id} + s_2\,\Delta_2.$$

(see [2] and references therein). Thus, we can say that the linearization operators $\ell^1_{\mathscr{E}}$ and $\ell^2_{\mathscr{E}}$ are equivalent.

The following simple Lemma explains why this notion is really important.

Lemma 1. *$\mathscr{C}$-differential operators Δ_1 and Δ_2 are equivalent if and only if the Δ_1- and Δ_2-coverings are isomorphic as linear coverings.*

So, to prove that ℓ^*-covering is invariant we have to establish that the operators $\ell^{1*}_{\mathscr{E}}$ and $\ell^{2*}_{\mathscr{E}}$ are equivalent. This is implied by the following result.

Theorem 3. *If two normal operators Δ_1 and Δ_2 are equivalent then Δ_1^* is equivalent to Δ_2^*.*

Corollary 1. *The equation $\mathscr{L}^*(\mathscr{E})$ does not depend on the embedding $\mathscr{E} \to J^\infty$.*

Now, recall that bivectors were defined as conservation laws on $\mathscr{L}^*(\mathscr{E})$, while operators that correspond to them are essentially generating functions of these conservation laws. Thus, the operators depend on using an embedding $\mathscr{E} \to J^\infty$. Assume that we have two different embeddings as above, so that they give rise to two operators A^1 and A^2 that correspond to the same bivector. Here are the formulas that relate these two operators:

$$\begin{aligned} A^2 &= \alpha\,A^1\,\alpha'^*, \\ A^1 &= \beta\,A^2\,\beta'^*. \end{aligned} \qquad (12)$$

4 Examples

Let us revise the three examples from the Introduction.

Example 4 (KdV). We considered two different embeddings of the KdV equation to jets:

$$u_t - u_{xxx} - 6uu_x = 0,$$

$$\begin{pmatrix} u_x - v \\ v_x - w \\ w_x - u_t + 6uv \end{pmatrix} = 0.$$

Here are all operators of diagram (3):

$$\ell^1_{\mathscr{E}} = D_t - D_{xxx} - 6uD_x - 6u_x, \qquad \ell^2_{\mathscr{E}} = \begin{pmatrix} D_x & -1 & 0 \\ 0 & D_x & -1 \\ -D_t + 6v & 6u & D_x \end{pmatrix},$$

$$\alpha = \begin{pmatrix} 1 \\ D_x \\ D_{xx} \end{pmatrix}, \quad \alpha' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \begin{array}{l} \beta = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \\ \beta' = \begin{pmatrix} -D_{xx} - 6u & -D_x & -1 \end{pmatrix}, \end{array}$$

$$s_1 = 0, \quad s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ D_x & 1 & 0 \end{pmatrix}.$$

Formulas (12) relate Hamiltonian operators (4) and (3).

Remark 5. If we take an operator from (4) for A^1 and compute A^2 via (12) we will get an operator from (3) only up to the equivalence.

Example 5 (Camassa–Holm equation). The Camassa–Holm equation written in the usual form $u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$ has a bi-Hamiltonian structure:

$$A_1 = D_x \qquad A_2 = -D_t - uD_x + u_x.$$

If we rewrite the equation in the form

$$\begin{aligned} &m_t + um_x + 2u_x m = 0, \\ &m - u + u_{xx} = 0 \end{aligned}$$

then the bi-Hamiltonian structure takes the form

$$A'_1 = \begin{pmatrix} D_x & 0 \\ D_x - D_x^3 & 0 \end{pmatrix} \qquad A'_2 = \begin{pmatrix} 0 & -1 \\ 2mD_x + m_x & 0 \end{pmatrix}$$

Note that the operators B_1 and B_2 from Example 2 are entries (up to sign) of the matrix A_1' and A_2'. Thus we see that studying bi-Hamiltonian structure of the Camassa–Holm equation does not require the use of the $(1 - D_x^2)^{-1}$ "operator".

Example 6 (Kupershmidt deformation). Let $\mathscr{E}$ be a bi-Hamiltonian equation given by $F = 0$ and A_1 and A_2 be the Hamiltonian operators.

Definition 4. The Kupershmidt deformation $\tilde{\mathscr{E}}$ of $\mathscr{E}$ has the form

$$F + A_1^*(w) = 0, \qquad A_2^*(w) = 0,$$

where $w = (w^1, \dots, w^l)$ are new dependent variables.

Theorem 4. *The Kupershmidt deformation $\tilde{\mathscr{E}}$ is a bi-Hamiltonian system.*

The proof of this theorem consists of checking that the following two bivectors define a bi-Hamiltonian structure:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}$$

The generalization of Kupershmidt's theorem from the Introduction is the following.

Theorem 5. *If $\psi_1, \psi_2, \dots$ is a Magri hierarchy for $\mathscr{E}$ then, under some technical assumptions, $(\psi_i, -\psi_{i+1})$, $i = 1, 2, \dots,$ is a Magri hierarchy for the Kupershmidt deformation $\tilde{\mathscr{E}}$.*

Details and proofs of Theorems 4 and 5 can be found in [5].

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Point Classification of Second Order ODEs: Tresse Classification Revisited and Beyond

Boris Kruglikov

Abstract In 1896 Tresse gave a complete description of relative differential invariants for the pseudogroup action of point transformations on the second order ODEs. The purpose of this paper is to review, in light of modern geometric approach to PDEs, this classification and also discuss the role of absolute invariants and the equivalence problem.

Introduction

Second order scalar ordinary differential equations have been the classical target of investigations and source of inspiration for complicated physical models. Under contact transformations all these equations are locally equivalent, but to find such a transformation for a pair of ODEs is the same hard problem as to find a general solution, which as we know from Painlevé equations is not always possible.

Most integration methods for second order ODEs are related to another pseudogroup action – point transformations, which do not act transitively on the space of all such equations. All linear second order ODEs are point equivalent.

S. Lie noticed that ODEs linearizable via point transformations have necessarily cubic non-linearity in the first derivatives and described a general test to construct this linearization map [Lie_2]. Later R. Liouville formulated these precise conditions for linearization via differential invariants and explored them [Lio]. But it was A. Tresse who first wrote the complete set of differential invariants for general second order ODEs.

The paper [Tr_2] is a milestone in the geometric theory of differential equations, but mostly one result (linearization of S. Lie – R. Liouville – A. Tresse) from the manuscript is used nowadays. In this note we would like to revise the Tresse classification in modern terminology and provide some alternative formulations and proofs. We make relation to the equivalence problem more precise and also compare this approach with E. Cartan's equivalence method.

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This classification can illustrate the finite representation theorem for differential invariants algebra, also known as Lie–Tresse theorem. The latter in the ascending degree of generality was proven in different sources [Lie_1, Tr_1, Ov, Ku, Ol, KL_1]. In particular, the latter reference contains the full generality statement, when the pseudogroup acts on a system of differential equations $\mathcal{E} \subset J^l(\pi)$ (under regularity assumption, see also [SS]). We refer to it for details and further references and we also cite [KLV, KL_2] as a source of basic notations, methods and results.

The structure of the paper is the following. In the first section we provide a short introduction to scalar differential invariants of a pseudogroup action and recall what the algebra of relative differential invariants is. In Sect. 2 we review the results of Tresse, confirming his formulae with independent computer calculation. In Sect. 3 we complete Tresse's paper by describing the algebra of absolute invariants and proving the equivalence theorem (in [Tr_2] this was formulated via relative invariants, which makes unnecessary complications with homogeneity, and only necessity of the criterion was explained). In Sect. 4 we discuss the non-generic second order equations, which contain in particular linearizable ODEs. Section 5 is devoted to discussion of symmetric ODEs.

Finally in Appendix (written jointly with V. Lychagin) we provide another approach to the equivalence problem, based on a reduction of an infinite-dimensional pseudogroup action to a Lie group action.

1 Scalar Differential Invariants

We refer to the basics of pseudogroup actions to [Ku, KL_2], but recall the relevant theory about differential invariants (see also [Tr_1, Ol]). We'll be concerned with the infinite Lie pseudogroup $G = \text{Diff}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ with the corresponding Lie algebras sheaf (LAS) $\mathfrak{g} = \mathfrak{D}_{\text{loc}}(\mathbb{R}^2)$ of vector fields.

The action of G has the natural lift to an action on the space $J^\infty\pi$ for an appropriate vector bundle π, provided we specify a Lie algebras homomorphism $\mathfrak{g} \to \mathfrak{D}_{\text{loc}}(J^0\pi)$.[1] Then we can restrict to the action of formal LAS $J^\infty(\mathbb{R}^2, \mathbb{R}^2)$.

A function $I \in C^\infty(J^\infty\pi)$ (this means that I is a function on a finite jet space $J^k\pi$ for some $k > 1$) is called a (scalar absolute) differential invariant if it is constant along the orbits of the lift of the action of G to $J^k\pi$.

For connected groups G we have an equivalent formulation: I is an (absolute) differential invariant if the Lie derivative vanishes $L_{\hat{X}}(I) = 0$ for all vector fields X from the lifted action of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$.

Note that often functions I are defined only locally near families of orbits. Alternatively we should allow I to have meromorphic behavior over smooth functions (but we'll be writing though about local functions in what follows, which is a kind of micro-locality, i.e., locality in finite jet-spaces).

[1] In this paper $\pi = M \times \mathbb{R}$ is a trivial one-dimensional bundle over $M \simeq \mathbb{R}^3$, so $J^k\pi = J^k M$.

The space $\mathcal{I} = \{I\}$ forms an algebra with respect to usual algebraic operations of linear combinations over $\mathbb{R}$ and multiplication and also the composition $I_1, \ldots, I_s \mapsto I = F(I_1, \ldots, I_s)$ for any $F \in C^\infty_{\text{loc}}(\mathbb{R}^s, \mathbb{R})$, $s = 1, 2, \ldots$ any finite number. However even with these operations the algebra $\mathcal{I}$ is usually not locally finitely generated. Indeed, the subalgebras $\mathcal{I}_k \subset \mathcal{I}$ of order k differential invariants are finitely generated on non-singular strata with respect to the above operations, but their injective limit $\mathcal{I}$ is not.

However finite-dimensionality is restored if we add invariant derivatives, i.e., $\mathscr{C}$-vector fields $\vartheta \in C^\infty(J^\infty\pi) \otimes_{C^\infty(M)} \mathfrak{D}(M)$ commuting with the G-action on the bundle π. These operators map differential invariants to differential invariants $\vartheta : \mathcal{I}_k \to \mathcal{I}_{k+1}$.

Lie–Tresse theorem claims that the algebra of differential invariants $\mathcal{I}$ is finitely generated with respect to algebraic-functional operations and invariant derivatives.

A helpful tool on the practical way to calculate algebra $\mathcal{I}$ of invariants are relative invariants, because they often occur on the lower jet-level than absolute invariants. A function $F \in C^\infty(J^\infty\pi)$ is called a relative scalar differential invariant if the action of pseudogroup G writes

$$g^* F = \mu(g) \cdot F$$

for a certain weight, which is a smooth function $\mu : G \to C^\infty(J^\infty\pi)$, satisfying the axioms of multiplier representation (see also [FO])

$$\mu(g \cdot h) = h^*\mu(g) \cdot \mu(h), \quad \mu(e) = 1.$$

The corresponding infinitesimal analog for an action of LAS $\mathfrak{g}$ is given via a smooth map (the multiplier representation is denoted by the same letter) $\mu : \mathfrak{g} \to \mathfrak{D}(J^\infty\pi)$, which satisfies the relations

$$\mu_{[X,Y]} = L_{\hat{X}}(\mu_Y) - L_{\hat{Y}}(\mu_X), \quad \forall\, X, Y \in \mathfrak{g}.$$

Then a relative scalar invariant is a function $F \in C^\infty(J^\infty\pi)$ such that $L_{\hat{X}} I = \mu_X \cdot I$. In other words (in both cases) the equation $F = 0$ is invariant under the action.

Let $\mathfrak{M} = \{\mu_X\}$ be the space of admissible weights.[2] Denote by $\mathcal{R}^\mu$ the space of scalar relative differential invariants of weight μ. Then

$$\mathcal{R} = \bigcup_{\mu \in \mathfrak{M}} \mathcal{R}^\mu$$

is a $\mathfrak{M}$-graded module over the algebra of absolute scalar differential invariants $\mathcal{I} = \mathcal{R}^0$ corresponding to the weight $\mu = 0$ for the LAS action ($\mu = 1$ for the pseudogroup action).

[2] It is given via a certain cohomology theory, which will be considered elsewhere.

The space $\mathfrak{M}$ of weights (multipliers) is always a group, but we can transform it into a $\mathbf{k}$-vector space ($\mathbf{k} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$) by taking tensor product $\mathfrak{M} \otimes \mathbf{k}$ and considering (sometimes formal) combinations $(I_1)^{\alpha_1} \cdots (I_s)^{\alpha_s}$. Then we have:

$$\mathcal{R}^{\mu} \cdot \mathcal{R}^{\bar{\mu}} \subset \mathcal{R}^{\mu+\bar{\mu}}, \quad (\mathcal{R}^{\mu})^{\alpha} \subset \mathcal{R}^{\alpha \cdot \mu}.$$

2 Tresse Classification Revisited

We start by re-phrasing the main results of Tresse classification.[3]

2.1 Relative Differential Invariants of Second Order ODEs

The point transformation LAS $\mathfrak{D}_{\text{loc}}(J^0\mathbb{R})$, with $J^0\mathbb{R}(x) = \mathbb{R}^2(x, y)$, equals $\mathfrak{g} = \{\xi_0 = a\partial_x + b\partial_y : a = a(x, y), b = b(x, y)\}$ and it prolongs to the subalgebra

$$\mathfrak{g}_2 = \{\xi = a\partial_x + b\partial_y + A\partial_p + B\partial_u\} \subset \mathfrak{D}_{\text{loc}}(J^2\mathbb{R}), \quad J^2\mathbb{R} = \mathbb{R}^4(x, y, p, u),$$
$$A = b_x - (a_x - b_y)p - a_y p^2, \; B = B_0 + uB_1,$$
$$B_0 = b_{xx} - (a_x - 2b_y)_x p - (2a_x - b_y)_y p^2 - a_{yy} p^3, \; B_1 = -(2a_x - b_y) - 3a_y p$$

where we denote $p = y'$, $u = y''$ the jet coordinates.

Using the notations $D_x = \partial_x + p\,\partial_y$, $\varphi = (dy - p\,dx)(a\partial_x + b\partial_y) = b - p\,a$ (we'll see soon these show up naturally), these expressions can be rewritten as

$$A = D_x(\varphi), \; B_0 = D_x^2(\varphi), \; B_1 = \partial_y(\varphi) - 2D_x(a)$$

Thus the LAS $\mathfrak{h} = \mathfrak{g}_2 \subset \mathfrak{D}_{\text{loc}}(J^0\mathbb{R}^3(x, y, p))$ being given we represent a second order ODE as a surface $u = f(x, y, p)$ in $J^0\mathbb{R}^3(x, y, p) = \mathbb{R}^4(x, y, p, u)$ and k^{th} order differential invariants of this ODE are invariant functions $I \in C^{\infty}_{\text{loc}}(J^k\mathbb{R}^3)$ of the prolongation

$$\mathfrak{h}_k = \{\hat{\xi} = a\,\mathcal{D}_x + b\,\mathcal{D}_y + A\,\mathcal{D}_p + \sum_{|\sigma| \le k} \mathcal{D}_{\sigma}^{(k)}(f)\,\partial_{u_\sigma}\} \subset \mathfrak{D}(J^k\mathbb{R}^3),$$
$$f = B_0 + B_1 u - a\,u_x - b\,u_y - A\,u_p \;: \qquad \hat{\xi}(I) = 0.$$

Here $\mathcal{D}_{\sigma}^{(k)} = \mathcal{D}_{\sigma}|_{J^k}$ with $\mathcal{D}_{\sigma} = \mathcal{D}_x^l \mathcal{D}_y^m \mathcal{D}_p^n$ for $\sigma = (l \cdot 1_x + m \cdot 1_y + n \cdot 1_p)$, so that

[3] We use different notations p instead of z, u instead of ω, etc., but this is not crucial.

$$\mathcal{D}_\sigma(f) = \mathcal{D}_\sigma(B_0) + \sum \frac{|\tau|!}{\tau!}\Big(\mathcal{D}_\tau(B_1)u_{\sigma-\tau} - D_\tau(a)u_{\sigma-\tau+1_x} - \mathcal{D}_\tau(b)u_{\sigma-\tau+1_y} - \mathcal{D}_\tau(A)u_{\sigma-\tau+1_p}\Big).$$

In the above formula we used the usual partial derivatives ∂_x, etc., in the total derivative operators $\mathcal{D}_\sigma$, etc. All these operators commute.

It is more convenient, following Tresse, to use the operator $D_x = \partial_x + p\,\partial_y$ on the base instead and to form the corresponding total derivative $\hat{\mathcal{D}}_x = \mathcal{D}_x + p\mathcal{D}_y$. These operators will no longer commute and we need a better notation for the corresponding non-holonomic partial derivatives.

Denote $u^k_{lm} = \hat{\mathcal{D}}^l_x \mathcal{D}^m_y \mathcal{D}^k_p(u)$, which equals u_{lmk} mod (lower order terms). The first relative invariants calculated by Tresse have order 4 and are:

$$I = u^4, \quad H =$$
$$u^2_{20} - 4u^1_{11} + 6u_{02} + u(2u^3_{10} - 3u^2_{01}) - u^1(u^2_{10} - 4u^1_{01}) + u^3u_{10} - 3u^2u_{01} + u\cdot u\cdot u^4.$$

In this case the weights form two-dimensional lattice and the relative invariants are

$$\mathcal{R}^{r,s} = \{\psi \in C^\infty(J^\infty\mathbb{R}^3) : \hat{\xi}(\psi) = -(rD_x(a) + s\partial_y(\varphi))\psi\}.$$

Note that $\hat{\xi}(\psi) = -(w\,C^w_\xi + q\,C^q_\xi)\psi$ for $w = r$, $q = s - r$ (weight and quality in Tresse terminology). Here the coefficients can be expressed as operators of $\xi_0 = a\partial_x + b\partial_y$ and $\xi_1 = a\partial_x + b\partial_y + A\partial_p$:

$$C^w_\xi = a_x + b_y = \operatorname{div}_{\omega_0}(\xi_0) \text{ and } C^q_\xi = \partial_y(\varphi) = \tfrac{1}{2}\operatorname{div}_{\Omega_0}(\xi_1)$$

with $\omega_0 = dx \wedge dy$ the volume form on $J^0\mathbb{R}$ and $\Omega_0 = -\omega \wedge d\omega$ on $J^1\mathbb{R}$, where $\omega = dy - p\,dx$ is the standard contact form of $J^1\mathbb{R}$. These two form the base of all weights.[4]

There are relative invariant differentiations[5] (differential parameters in the classical language):

$$\Delta_p = \mathcal{D}_p + (r-s)\frac{u^5}{5u^4} : \mathcal{R}^{r,s} \to \mathcal{R}^{r-1,s+1},$$
$$\Delta_x = \hat{\mathcal{D}}_x + u\,\Delta_p + \Big((3r+2s)\Big(u^1 + \frac{3u\,u^5}{5u^4}\Big) + (2r+s)\frac{u^4_{10}}{u^4}\Big) : \mathcal{R}^{r,s} \to \mathcal{R}^{r+1,s},$$

[4] This is a result from a joint discussion with V. Lychagin. It is important since in Tresse [Tr2] this is an ad-hoc result, based on the straightforward calculations, but not fully justified. More details will appear in a separate publication.

[5] Note that they are differential operators of the first order, obtained from the base derivations via an invariant connection.

$$\Delta_y = \mathcal{D}_y + \frac{u^5}{5u^4}\,\Delta_x + \Big(2\,u^1 + \frac{u^4_{10} + u\,u^5}{u^4}\Big)\Delta_p + \Big((r+2s)\frac{u^4_{01}}{4u^4} + \\ +(3r+2s)\Big(\frac{u^2}{8} + \frac{3}{20}\frac{u^5(u^4_{10} + u\,u^5 + 2u^1u^4)}{u^4u^4}\Big)\Big) : \mathcal{R}^{r,s} \to \mathcal{R}^{r,s+1}.$$

Theorem 1 ([Tr$_2$]). *The space of relative differential invariants $\mathcal{R}$ is generated by the invariant H and differentiations Δ_x, Δ_y, Δ_p on the generic stratum.*

Notice that the latter two first order $\mathscr{C}$-differential operators have the form:

$$\Delta_x = \mathcal{D}_x + p\,\mathcal{D}_y + u\,\mathcal{D}_p + r\Big(3u^1 + 2\frac{u\,u^5 + u^4_{10}}{u^4}\Big) + s\Big(2u^1 + \frac{u\,u^5 + u^4_{10}}{u^4}\Big),$$

$$\Delta_y = \frac{u^5}{5u^4}\,\mathcal{D}_x + \Big(1 + p\frac{u^5}{5u^4}\Big)\mathcal{D}_y + \Big(2\,u^1 + \frac{5u^4_{10} + 6u\,u^5}{u^4}\Big)\mathcal{D}_p + r\Big(\frac{3u^2}{8} + \frac{u^4_{01}}{4u^4} \\ + \frac{19u^1u^5}{10u^4} + \frac{21(u\,u^5 + u^4_{01})u^5}{20\,u^4\cdot u^4}\Big) + s\Big(\frac{u^2}{4} + \frac{u^4_{01}}{2u^4} + \frac{3u^1u^5}{5u^4} + \frac{3(u\,u^5 + u^4_{01})u^5}{10\,u^4\cdot u^4}\Big),$$

and so Δ_x, Δ_y, Δ_p are linearly independent everywhere outside $I = 0$.

2.2 Specifications

Several remarks are noteworthy in relation with the theorem:

(1) The number of basic relative differential invariants of pure order k is given in the following table

k :	0	1	2	3	4	5	6	7	8	…	k	…
# :	0	0	0	0	2	3	11	17	24	…	$\frac{1}{2}(k^2-k-8)$	

The generators in order 4 are $I \in \mathcal{R}^{-2,3}$ and $H \in \mathcal{R}^{2,1}$; in order 5 are $H_{10} = \Delta_x(H) \in \mathcal{R}^{3,1}$, $H_{01} = \Delta_y(H) \in \mathcal{R}^{2,2}$ and $K = \Delta_p(H) \in \mathcal{R}^{1,2}$; in order 6 are $(H_{20}, H_{11}, H_{02}) \in \mathcal{R}^{4,1} \oplus \mathcal{R}^{3,2} \oplus \mathcal{R}^{2,3}$, $(K_{10}, K_{01}) \in \mathcal{R}^{2,2} \oplus \mathcal{R}^{1,3}$ and $\Omega^l_{ij} = u^l_{ij} +$ (lower terms for certain order on monomials) $\in \mathcal{R}^{i+2-l,j+l-1}$, $\deg \Omega^l_{ij} = i + j + l = 6$, $l > 3$:[6]

order k	basic relative differential invariants
4	$I,\ H$
5	$H_{10},\ H_{01},\ K$
6	$H_{20},\ H_{11},\ H_{02},\ K_{10},\ K_{01},\ \Omega^4_{20},\ \Omega^4_{11},\ \Omega^4_{02},\ \Omega^5_{10},\ \Omega^5_{01},\ \Omega^6$

[6] We let $H_{ij} = \Delta^i_x\Delta^j_y H$ and $K_{ij} = \Delta^i_x\Delta^j_y K$, though in [Tr$_2$] there is a difference between $\Delta_x K$ and K_{10}, $\Delta_y K$ and K_{01}. Since this only involves a linear transformation, this is possible.

Thus in ascending order k, we must add the generators I, H and then Ω_{ij}^{6-i-j}, $i+j \le 2$ (one encounters the relations $\Delta_x(I) = \Delta_y(I) = \Delta_p(I) = 0$). Invariants of order $k > 6$ are obtained via invariant derivations from the lower order.

(2) The theorem as formulated gives only generators. The relations (differential syzygies) are the following (also contained in [Tr$_2$]):

$$[\Delta_p, \Delta_x] = \Delta_y + \frac{3(3r+2s)}{5}\frac{\Omega_{10}^5}{I}$$

$$[\Delta_p, \Delta_y] = \frac{\Omega^6}{5I}\Delta_x + \frac{\Omega_{10}^5}{I}\Delta_p - \frac{3(3r+2s)}{20}\frac{\Omega_{01}^5}{I}$$

$$[\Delta_x, \Delta_y] = \frac{\Omega_{10}^5}{5I}\Delta_x + \frac{\Omega_{20}^4}{I}\Delta_p - \frac{3(3r+2s)}{4}\frac{\Omega_{11}^4}{I}$$

together with the following relations for coefficients-invariants (the first of which is just the application of the above commutator relation)

$$\Omega_{10}^5 = \frac{5I}{24H}([\Delta_p, \Delta_x]H - \Delta_y H), \qquad \Omega_{01}^5 = \frac{4}{9}(\Delta_p \Omega_{10}^5 - \Delta_x \Omega^6),$$

$$\Omega_{20}^4 = \Delta_p^2 H - \frac{\Omega^6}{5I}H, \qquad \Omega_{11}^4 = \frac{4}{3}(\Delta_p \Omega_{20}^4 - \Delta_x \Omega_{10}^5).$$

It is important that the relation for the last additional invariant of order 6

$$\Omega_{02}^4 = \frac{4}{5}(\Delta_y \Omega_{10}^5 - \Delta_x \Omega_{01}^5 + \frac{5\Omega_{20}^4 \Omega^6 + \Omega_{10}^5 \Omega_{01}^5}{5I})$$

can be considered as definition, while first additional invariant[7] of order 6

$$\Omega^6 = u^6 - \frac{6}{5}\frac{u^5 \cdot u^5}{u^4}$$

can be obtained from a higher relation via application of the relation for $[\Delta_p, \Delta_y]$ to H and K.

Thus we see that involving syzygy of higher order invariants (prolongation–projection) we can restore the invariants I, Ω_{ij}^k from H and invariant differentiations Δ_j, as the theorem claims.

(3) The theorem specifies the relative invariants only on the generic stratum. If we take the minimal number of generators $(H, \Delta_x, \Delta_y, \Delta_p)$, then this stratum is specified by a number of non-degeneracy conditions of high order.

However if we take more generators $(I, H, \Omega^6, \Delta_x, \Delta_y, \Delta_p)$, or the collection of basic invariants $(I, H, \Omega^6, \Omega_{10}^5, \dots, \Omega_{02}^4, \Delta_x, \Delta_y, \Delta_p)$ for the completeness in ascending order k, then this condition is very easy: just $I \neq 0$.

Notice that the condition $I = 0$ is important, since it describes the singular stratum (see however Sect. 4.2 where this case is handled).

[7] This invariant is important with another approach, see Appendix.

3 Classification of Second Order ODEs

While a complete classification of relative differential invariants for second order scalar ODEs was achieved by Tresse, absolute invariants are not described in [Tr_2]. They however can be easily deduced.

3.1 Dimensional Count

Let us at first count the number of absolute invariants on a generic stratum.[8] This number equals the codimension of a generic orbit in the corresponding jet-space.

Denote by $\mathcal{O}_k$ the orbit through a generic point in $J^k\mathbb{R}^3(x, y, p)$ of the pseudogroup of point transformations. Tangent to it is determined by the corresponding LAS and so we can calculate codimension of the orbit. Indeed, denoting by St_k the stabilizer of the LAS $\mathfrak{h}_k$ at the origin we get

$$\dim \mathcal{O}_k = \operatorname{codim} \text{St}_k \,.$$

To calculate the stabilizer we should adjust the normal form of the equation at the origin via a point transformation. This can be done via a projective configuration (Desargues-type) theorem of [A] (Sect. 1.6): any second order ODE $y'' = u(x, y, p)$, $p = y'$, can be transformed near a given point to

$$y'' = \alpha(x)y^2 + o(|y|^3 + |p|^3).$$

Denote by $\mathfrak{m}$ the maximal ideal at the given point (so $\mathfrak{m}^k$ is the space of functions vanishing to order k). Then we can suppose that at a given point

$$u, u_x, u_y, u_p, u_{xx}, u_{xy}, u_{xp}, u_{yp}, u_{pp} \in \mathfrak{m}.$$

Therefore the stabilizer St_k is given by the union of the following conditions on the coefficients of $\hat{\xi} \in \mathfrak{h}_k$ (equivalently on coefficients of $\xi_0 \in \mathfrak{g}$)

$$a \in \mathfrak{m}^{k-2},\ a_{yy} \in \mathfrak{m}^{k-3},\ b \in \mathfrak{m}^{k-1},\ b_{xx} \in \mathfrak{m}^k,$$
$$a_x \in \mathfrak{m}^{k-2},\ (2a_x - b_y)_y \in \mathfrak{m}^{k-2},\ (a_x - 2b_y)_x \in \mathfrak{m}^{k-1}.$$

Thus the Taylor expansion of $a = a(x, y)$ can contain only the following monomials

$$\{x^i y^j : i + j \le k - 1\},\ \{x^i y^{k-i} : i > 1\},\ \{x^i y^{k+1-i} : i > 2\}$$

[8] This count is independent of Tresse argumentation, and so together with Footnote 4 it provides a rigorous proof of the table in Sect. 2.2.

and the allowed monomials for $b = b(x, y)$ are

$$\{x^i y^j : i + j \le k\},\ \{x^i y^{k+1-i} : i \ge 1\},\ \{x^i y^{k+2-i} : i \ge 2\}.$$

This yields that $\operatorname{codim}(\mathrm{St}_k)$ equals:

$$\dim\big(\mathbb{C}[x, y]^2/\,\mathrm{St}_k\big) = \frac{k(k+1)}{2} + 2(k-1) + \frac{(k+1)(k+2)}{2} + 2(k+1) = k^2 + 6k + 1$$

and so the number u_k of the basic differential invariants of order $\le k$ is equal to

$$\begin{aligned} u_k &= \operatorname{codim} \mathcal{O}_k = \dim J^k\mathbb{R}^3 - \dim \mathcal{O}_k \\ &= 3 + \frac{(k+1)(k+2)(k+3)}{6} - (k^2 + 6k + 1) = \frac{k^3 - 25k + 18}{6}. \end{aligned}$$

As this formula indicates for $k \le 4$ the generic orbit is open, so that such stratum has no absolute invariants (however for $k = 4$ there are singular orbits, so that the relative invariants I, H appear).

In order $k = 5$ the formula yields $u_5 = 3$ differential invariants. For $k > 5$ we deduce the number of pure order k basic differential invariants:

$$u_k - u_{k-1} = \frac{k(k-1)}{2} - 4.$$

3.2 Absolute Differential Invariants

There are two ways of adjusting a basis on the lattice $\mathfrak{M}$ of weights via relative invariants. As follows from specification for $\mathbb{Z}^2$-lattice of weights from Sect. 2.2, the basic invariants are

$$J_1 = I^{-1/8} H^{3/8} \in \mathcal{R}^{1,0}, \quad J_2 = I^{1/4} H^{1/4} \in \mathcal{R}^{0,1}.$$

Another choice, which allow to avoid branching but increase the order, is

$$\tilde{J}_1 = \frac{H_{10}}{H} \in \mathcal{R}^{1,0}, \quad \tilde{J}_2 = \frac{H_{01}}{H} \in \mathcal{R}^{0,1}.$$

Then (choosing J_i or $\tilde{J}_i$) we get isomorphism for $k > 4$:

$$\mathcal{R}_k^{r,s}/\mathcal{R}_{k-1}^{r,s} \simeq \mathcal{I}_k/\mathcal{I}_{k-1}, \qquad F \mapsto F/(J_1^r J_2^s).$$

Thus with any choice the list of basic differential invariants in order 5 is

$$\bar{H}_{10} = H_{10}/(J_1^3 J_2),\ \bar{H}_{01} = H_{01}/(J_1^2 J_2^2),\ \bar{K} = K/(J_1 J_2^2)$$

and in pure order 6 is

$$\bar{H}_{20} = H_{20}/(J_1^4 J_2),\ \bar{H}_{11} = H_{11}/(J_1^3 J_2^2),\ \bar{H}_{02} = H_{02}/(J_1^2 J_2^3),$$
$$\bar{K}_{10} = K_{10}/(J_1^2 J_2^2),\ \bar{K}_{01} = K_{01}/(J_1 J_2^3),$$
$$\bar{\Omega}^4_{20} = \Omega^4_{20}/(J_2^3),\ \bar{\Omega}^4_{11} = \Omega^4_{11}/(J_1^{-1} J_2^4),\ \bar{\Omega}^4_{02} = \Omega^4_{02}/(J_1^{-2} J_2^5),$$
$$\bar{\Omega}^5_{10} = \Omega^5_{10}/(J_1^{-2} J_2^4),\ \bar{\Omega}^5_{01} = \Omega^5_{01}/(J_1^{-3} J_2^5),\ \bar{\Omega}^6 = \Omega^6/(J_1^{-4} J_2^5).$$

Higher order differential invariants can be obtained in a similar way from the basic relative invariants, but alternatively we can adjust invariant derivations by letting $\nabla_j = J_1^{\rho_j} J_2^{\sigma_j} \cdot \Delta_j|_{r=s=0}$ with a proper choice of the weights ρ_j, σ_j. Namely we let

$$\nabla_p = \frac{J_1}{J_2}\mathcal{D}_p, \qquad \nabla_x = \frac{1}{J_1}\big(\hat{\mathcal{D}}_x + u\mathcal{D}_p\big),$$
$$\nabla_y = \frac{1}{J_2}\Big(\mathcal{D}_y + \frac{u^5}{5u^4}\hat{\mathcal{D}}_x + \big(\frac{u^4_{10}}{u^4} + \frac{6u\,u^5}{5u^4} + 2u^1\big)\mathcal{D}_p\Big).$$

These form a basis of invariant derivatives over $\mathcal{I}$ and we have:

$$[\nabla_p, \nabla_x] = -\tfrac{1}{8}\bar{H}_{10}\nabla_p - \tfrac{3}{8}\bar{K}\nabla_x + \nabla_y,$$
$$[\nabla_p, \nabla_y] = (\bar{\Omega}^5_{10} - \tfrac{1}{8}\bar{H}_{01})\nabla_p + \tfrac{1}{5}\bar{\Omega}^6\nabla_x - \tfrac{1}{4}\bar{K}\nabla_y,$$
$$[\nabla_x, \nabla_y] = \bar{\Omega}^4_{20}\nabla_p + (\tfrac{1}{5}\bar{\Omega}^5_{10} + \tfrac{3}{8}\bar{H}_{01})\nabla_x - \tfrac{1}{4}\bar{H}_{10}\nabla_y.$$

The derivations and coefficients can be also expressed in terms of non-branching invariants $\tilde{J}_1 = \frac{8}{3}\nabla_x J_1$ and $\tilde{J}_2 = 4\nabla_y J_2$.

Theorem 2. *The space $\mathcal{I}$ of differential invariants is generated by the invariant derivations $\nabla_x, \nabla_y, \nabla_p$ on the generic stratum.*

Indeed, we mean here that taking coefficients of the commutators, adding their derivatives, etc., leads to a complete list of basic differential invariants.

On the other hand, if we want to list generators according to the order, so that invariant derivations only add new in the corresponding order, then we shall restrict to $\bar{H}_{10}, \bar{H}_{01}, \bar{K}$ in order 5, add $\bar{\Omega}^{6-i-j}_{ij}$ in order 6 and the rest in every order is generated from these by invariant derivations with ∇_j. The relations can be deduced from these of Sect. 2.2.

3.3 Equivalence Problem

Second order ODEs $\mathcal{E}$ can be considered as sections $\mathfrak{s}_{\mathcal{E}}$ of the bundle π, whence we can restrict any differential invariant $J \in \mathcal{I}_k$ to the equation via pull-back of the prolongation:

$$J^{\mathcal{E}} := (\mathfrak{s}^{(k)}_{\mathcal{E}})^*(J) \in C^\infty_{\rm loc}(\mathbb{R}^3(x, y, p)).$$

Consider most non-degenerate second order ODEs $\mathcal{E}$, such that $\bar{H}^{\mathcal{E}}_{10}$, $\bar{H}^{\mathcal{E}}_{01}$, $\bar{K}^{\mathcal{E}}$ are local coordinates on $\mathbb{R}^3(x, y, p)$.[9] Then the other differential invariants on the equation can be expressed as functions of these:

$$\bar{H}^{\mathcal{E}}_{ij} = \Phi^{\mathcal{E}}_{ij}(\bar{H}^{\mathcal{E}}_{10}, \bar{H}^{\mathcal{E}}_{01}, \bar{K}^{\mathcal{E}}),\ \ \bar{K}^{\mathcal{E}}_{ij} = \Psi^{\mathcal{E}}_{ij}(\bar{H}^{\mathcal{E}}_{10}, \bar{H}^{\mathcal{E}}_{01}, \bar{K}^{\mathcal{E}}),$$
$$\bar{\Omega}^{k\,\mathcal{E}}_{ij} = \Upsilon^{k\,\mathcal{E}}_{ij}(\bar{H}^{\mathcal{E}}_{10}, \bar{H}^{\mathcal{E}}_{01}, \bar{K}^{\mathcal{E}}).$$

Due to the relations above we can restrict to the following collection of functions:

$$\Phi^{\mathcal{E}}_{20}, \Phi^{\mathcal{E}}_{11}, \Phi^{\mathcal{E}}_{02}, \Psi^{\mathcal{E}}_{10}, \Psi^{\mathcal{E}}_{01}, \Upsilon^{6\,\mathcal{E}}, \Upsilon^{5\,\mathcal{E}}_{10}, \Upsilon^{4\,\mathcal{E}}_{20}, \tag{1}$$

the others being expressed through the given ones via the operators of derivations (which naturally restrict to $\mathcal{E}$ as directional derivatives).

Theorem 3. *Two generic second order differential equations $\mathcal{E}_1$, $\mathcal{E}_2$ are point equivalent iff the collections (1) of functions on $\mathbb{R}^3$ coincide.*

Proof. Necessity of the claim is obvious. Sufficiency is based on investigation of solvability of the corresponding Lie equation[10]

$$\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2) = \{[\varphi]^2_z \in J^2(\mathbb{R}^2, \mathbb{R}^2) : \varphi^{(2)}(\mathcal{E}_1 \cap \pi^{-1}_{2,0}(z)) = \mathcal{E}_2 \cap \pi^{-1}_{2,0}(\varphi(z))\}, \tag{2}$$

which has finite type. Notice that the prolongation $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)^{(k)}$ consists of the jets $[\varphi]^{k+2}_z$ such that $\varphi^{(2)}$ transforms k-jets of the equation $\mathcal{E}_1$ to the k-jets of the equation $\mathcal{E}_2$ along the whole fiber over $z \in J^0\mathbb{R} = \mathbb{R}^2(x, y)$.

Proposition 4 *Suppose that the system $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$ is formally solvable; more precisely let $\mathcal{T} \subset \mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)^{(10)} \subset J^{12}(\mathbb{R}^2, \mathbb{R}^2)$ be such a manifold that $\pi_{12}|_{\mathcal{T}}$ is a submersion onto $\mathbb{R}^2$. Then this system is locally solvable,[11] so that the equations are point equivalent, i.e., $\exists \varphi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)\ \forall z \in \mathbb{R}^2 :\ [\varphi]^2_z \in \mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$.*

Indeed, the symbol of the system $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$ (provided it is non-empty; notice that for generic $\mathcal{E}_1, \mathcal{E}_2$ it is empty) is the same as for the symmetry algebra $\mathrm{sym}(\mathcal{E})$, namely: $g_0 = T = \mathbb{R}^2$, $g_1 = T^* \otimes T$, $g_2 \subset S^2T^* \otimes T$ has codimension 4 and no (complex) characteristic covectors, so that $g_3 = g_2^{(1)} = 0$, whence $\oplus\, g_i \simeq \mathrm{sym}(y'' = 0) \simeq \mathrm{sl}_3$.

It should be noted that the first prolongation $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)^{(1)} \subset J^3(\mathbb{R}^2, \mathbb{R}^2)$ always exists and is of Frobenius type, while the next one has proper projection unless the compatibility conditions vanish.

[9] Here and in what follows one can assume (higher micro-)local treatment.

[10] It is important not to mix solvability, i.e., existence of local solutions, with compatibility [KL3], i.e., existence of solutions with all admissible Cauchy data. The latter may be cut by the compatibility conditions. This confusion occurred in the proof of Theorem 8.3 from [Y]: the Lie equation is not formally integrable except for maximally symmetric case.

[11] A regularity assumption is needed for this, which is given by the non-degeneracy condition $d\bar{H}^{\mathcal{E}}_{10} \wedge d\bar{H}^{\mathcal{E}}_{01} \wedge d\bar{K}^{\mathcal{E}} \neq 0$.

We are interested in solvability of the system, so we successively add the compatibility conditions. The first belongs to the space $H^{2,2}(\mathfrak{Lie}) \simeq \mathbb{R}^2$, but it may happen that only one of the components is non-zero (if both are zero, the system is compatible and we are done, if both are non-zero we have more equations to add and the process stops earlier). So we add this equation of the second order to the system of four equations and get a new system $\widetilde{\mathfrak{Lie}}$ of formal codim = 5.

Then we continue to add equations-compatibilities and can do it maximum $\sum \dim g_i = 8$ times, so that we get $3 + 8 = 11$-th order condition. After this we get only discrete set of possibilities for solutions and checking them we get that either we have a 12-jet solution or there do not exist solutions at all.

In these arguments we adapted dimensional count, i.e., we assumed regularity. But singularities can bring only zero measure of values (by Sard's lemma), so that our condition still works even in smooth (not only analytic) situation.

Now let us explain formal solvability for our problem. A jet $[\varphi]_z^{k+2}$ belongs to the prolongation $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)^{(k)}$ iff $\varphi^{(k+2)}$ transforms $\mathcal{E}_1^{(k)} \cap \pi_{k+2,0}^{-1}(z)$ to $\mathcal{E}_2 \cap \pi_{k+2,0}^{-1}(\varphi(z))$. For randomly chosen equations the system $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$ will be empty over any point $z \in \mathbb{R}^2$ just because none map can transform the whole fiber $\mathcal{E}_1 \cap \pi_{2,0}^{-1}(z_1)$ into another fiber $\mathcal{E}_2 \cap \pi_{2,0}^{-1}(z_2)$ (example: ODEs $y'' = f(x, y, y')$ with polynomial dependence on $p = y'$ of degrees 3 and 4).

The compatibility for the system $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$ of order k are the conditions that φ^* transforms the restricted order k differential invariants $J^{\mathcal{E}_2}$ into $J^{\mathcal{E}_1}$. Since this is possible by our assumption, we get prolongation $\mathcal{T} \subset \mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)^{(10)}$. Moreover this $\mathcal{T}$ will be a submanifold and no singularity issues arise. This yields us local point equivalence.

Remark 1 *If differential invariants $J_1 \dots J_3$ are independent on equation $\mathcal{E}$, then there is another way to define invariant derivatives [Lie$_1$, Ol, KL$_1$], so called Tresse derivatives, which in local coordinates have the form: $\hat{\partial}/\hat{\partial} J_i = \sum_j [\mathcal{D}_a(J_b)]_{ij}^{-1} \mathcal{D}_j$. In our case, when we take $\bar{H}_{10}^{\mathcal{E}}$, $\bar{H}_{01}^{\mathcal{E}}$, $\bar{K}^{\mathcal{E}}$ as coordinates on the equation, they are just $\partial/\partial \bar{H}_{10}^{\mathcal{E}}$, $\partial/\partial \bar{H}_{01}^{\mathcal{E}}$, $\partial/\partial \bar{K}^{\mathcal{E}}$, when restricted to $\mathcal{E}$.*

Another generic case is when we have three functional independent invariants among[12]

$$\bar{H}_{10}^{\mathcal{E}},\ \bar{H}_{01}^{\mathcal{E}},\ \bar{K}^{\mathcal{E}},\ \bar{H}_{20}^{\mathcal{E}},\ \bar{H}_{11}^{\mathcal{E}},\ \bar{H}_{02}^{\mathcal{E}},\ \bar{K}_{10}^{\mathcal{E}},\ \bar{K}_{01}^{\mathcal{E}},\ \bar{\Omega}^{6\,\mathcal{E}},\ \bar{\Omega}_{10}^{5\,\mathcal{E}},\ \bar{\Omega}_{20}^{4\,\mathcal{E}}. \tag{3}$$

In this case we can express the rest of invariants through the given three basic, and the classification is precisely the same as in Theorem 3.

There are other regular classes of second order ODEs (in general, equations are stratified according to functional ranks):

1. Collection (3) has precisely two functionally independent invariants,
2. Collection (3) has only one functionally independent invariant,
3. Collection (3) consists of constants.

[12] We do not know if this is realizable in other cases, than these of Theorem 3.

In cases 1 or 2 we can choose basic invariants (2 or 1 respectively – note that the space of all differential invariants, not only of collection (3), will then have functional rank 2 or 1) and express the rest through them. The functions-relations will be again the only obstructions to point equivalence.

In the latter case all differential invariants are constant on the equation $\mathcal{E}$, so for the equivalence these (finite number of) constants should coincide.

Remark 2 *Cartan's equivalence method provides a canonical frame (on some bundle over the original manifold), which yields all differential invariants but with mixture of orders. Otherwise around, given the algebra of differential invariants, we can choose $J_1, \dots, J_s$ among them, which are functionally independent on a generic (prolonged) equation. Then $dJ_1, \dots, dJ_s$ will be a canonical basis of 1-forms, which can work as a (holonomic) moving frame. Non-holonomic frames can appear upon dualizing invariant (non-Tresse) derivatives.*

Let us finally give another formulation of the equivalence theorem. We can consider collection (3) as a map $\mathbb{R}^3 \simeq \mathcal{E} \to \mathbb{R}^{11}$ by varying the point of our equation $\mathcal{E}$. Thus we get (in regular case) a submanifold of $\mathbb{R}^{11}$ of dimension 3, 2, 1 or 0 respectively. This submanifold is an invariant (and the previous formulation was only a way to describe it as a graph of a vector-function):

Theorem 5. *Two second order regular differential equations $\mathcal{E}_1$, $\mathcal{E}_2$ are point equivalent iff the corresponding submanifolds in the space of differential invariants $\mathbb{R}^{11}$ coincide.*

4 Singular Stratum: Projective Connections

On the space $J^3\mathbb{R}^3(x, y, p)$ the lifted action of the pseudogroup $\mathfrak{h}$ is transitive. But its lift to the space of 4-jets is not longer such: There are singular strata, given by the equations $I = 0$, $H = 0$. Moreover they have a singular substratum $I = H = 0$ in itself, on which the pseudogroup action is transitive, so that any equation from it is point equivalent to trivial ODE $y'' = 0$ [Lie$_2$, Lio, Tr$_1$].

In this subsection we consider the singular stratum $I = 0$.[13] It corresponds to equations of the form

$$y'' = \alpha_0(x, y) + \alpha_1(x, y)p + \alpha_2(x, y)p^2 + \alpha_3(x, y)p^3, \quad p = y'. \tag{4}$$

[13] The other stratum $H = 0$ can be treated similarly. Indeed, though the invariants I, H look quite unlike, they are proportional to self-dual and anti-self-dual components of the Fefferman metric [F] and this duality is very helpful [NS].

Note however that even though it is difficult to solve the PDE $H = 0$ without non-local transformations, some partial solutions can be found using symmetry methods. For instance, a three-dimensional family of solutions is $y'' = \varphi(p)/x$ with $\varphi''' = \dfrac{\varphi''(2\varphi - 2 - \varphi')}{\varphi(\varphi - 1)}$.

This class of equations is invariant under point transformations. Moreover it has very important geometric interpretation, namely such ODEs correspond to projective connections on two-dimensional manifolds [C]. We will indicate three different approaches to the equivalence problem.

4.1 The Original Approach of Tresse

The idea is to investigate the algebra of differential invariants, following S. Lie's method, and then to solve the equivalence problem via them. In [Tr_1] lifting the action of point transformation to the space $J^k(2,4)$ (jets of maps $(x,y)\mapsto(\alpha_0,\dots,\alpha_3)$) he counts the number of basic differential invariants of pure order k to be

$$\begin{array}{lcccccccccccc} k: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots & k & \dots \\ \#: & 0 & 0 & 0 & 0 & 6 & 8 & 10 & 12 & 14 & \dots & 2(k-1) & \end{array}$$

An independent check of this (with the same method as in our Sect. 3.1) is given in [Y].

The action of $\mathfrak{g}$ is transitive on the space of first jets and its lift is transitive on the space of second jets $J^2(2,4)$ outside the singular orbit $L_1=L_2=0$, where

$$\begin{aligned} L_1 &= -\alpha_{2xx}+2\alpha_{1xy}-3\alpha_{0yy} \\ &\qquad -3\alpha_3\alpha_{0x}+\alpha_1\alpha_{2x}-6\alpha_0\alpha_{3x}+3\alpha_2\alpha_{0y}-2\alpha_1\alpha_{1y}+3\alpha_0\alpha_{2y} \\ L_2 &= -3\alpha_{3xx}+2\alpha_{2xy}-\alpha_{1yy} \\ &\qquad -3\alpha_3\alpha_{1x}+2\alpha_2\alpha_{2x}-3\alpha_1\alpha_{3x}+6\alpha_3\alpha_{0y}-\alpha_2\alpha_{1y}+3\alpha_0\alpha_{3y}. \end{aligned}$$

These second order operators[14] were found by S. Lie [Lie_2] who showed that vanishing $L_1=L_2=0$ characterizes trivial (equivalently: linearizable) ODEs. R. Liouville [Lio] proved that the tensor

$$L=(L_1dx+L_2dy)\otimes(dx\wedge dy), \tag{5}$$

responsible for this, is an absolute differential invariant.

Further on Tresse claims that all absolute differential invariants can be expressed via L_1, L_2, but [Lio, Tr_2] do not contain these formulae. The problem was handled recently by V. Yumaguzhin [Y] (the whole set of invariants was presented, though not fully described).

Namely it was shown that the action of $\mathfrak{g}$ in $J^3(2,4)$ is transitive outside the stratum $F_3=0$, where

$$\begin{aligned} F_3 &= (L_1)^2\mathcal{D}_y(L_2)-L_1L_2(\mathcal{D}_x(L_2)+\mathcal{D}_y(L_1))+(L_2)^2\mathcal{D}_x(L_1) \\ &\qquad -(L_1)^3\alpha_3+(L_1)^2L_2\alpha_2-L_1(L_2)^2\alpha_1+(L_2)^2\alpha_0 \end{aligned}$$

[14] Corresponding to $(3k,-3h)$ in [Tr_1].

is the relative differential invariant from [Lio]. The other tensor invariants can be expressed through these. The invariant derivations are[15]

$$\nabla_1 = \frac{L_2}{(F_3)^{2/5}}\mathcal{D}_x - \frac{L_1}{(F_3)^{2/5}}\mathcal{D}_y, \quad \nabla_2 = \frac{\Psi_2}{(F_3)^{4/5}}\mathcal{D}_x - \frac{\Psi_1}{(F_3)^{2/5}}\mathcal{D}_y,$$

where

$$\Psi_1 = -L_1(L_1)_y + 4L_1(L_2)_x - 3L_2(L_1)_x - (L_1)^2\alpha_2 + 2L_1L_2\alpha_1 - 3(L_2)^2\alpha_0,$$
$$\Psi_2 = 3L_1(L_2)_y - 4L_2(L_1)_y + L_2(L_2)x - 3(L_1)^2\alpha_3 + 2L_1L_2\alpha_2 - (L_2)^2\alpha_1.$$

Now we can get two differential invariants of order 4 as the coefficients of the commutator

$$[\nabla_1, \nabla_2] = I_1\nabla_1 + I_2\nabla_2.$$

Related invariants are the following: one applies the invariant derivations ∇_i (extended to the relative invariants) to F_3 and gets another relative differential invariant of the same weight (the relation here is almost obvious since $\nabla_1 \wedge \nabla_2$ is proportional to F_3). Thus $\nabla_1(F_3)/F_3$, $\nabla_2(F_3)/F_3$ are absolute invariants.

To get four more invariants $I_3, \dots, I_6$ of order 4, consider the Lie equation, formed similar to (2) for the cubic second order ODEs (4), see (6). After a number of prolongation–projection we get a Frobenius system, and its integrability conditions yield the required differential invariants (in [Y] these are obtained in a different but seemingly equivalent way).

Now we can state that the algebra $\mathcal{I}$ is generated by the invariants $I_1, \dots, I_6$ together with the invariant derivatives ∇_1, ∇_2. An interesting problem is to describe all differential syzygies between these generators.

4.2 The Second Tresse Approach

The invariants of Sect. 2.2 are not defined on the stratum $I = 0$ due to the fact that most expressions contain I in denominator. But due to Footnote 13 the relative invariants I, H are on equal footing. And in fact Tresse in [Tr$_2$] constructs another basis of relative invariants with H in denominator (this seems to be in correspondence with duality of [NS]).

[15] The first one in the relative form was known already to Liouville [Lio]:

$$\tilde{\nabla}_1 = L_1\mathcal{D}_y - L_2\mathcal{D}_x + m(\mathcal{D}_x(L_2) - \mathcal{D}_y(L_1)) : \mathcal{R}^m \to \mathcal{R}^{m+2},$$

where $\mathcal{R}^m$ is the space of weight m relative differential invariants corresponding to the cocycle $C_\xi = \mathrm{div}_{\omega_0}(\xi)$, where $\omega_0 = dx \wedge dy$. He was very close, but did not write the second one.

Thus if we restrict this set to the stratum $I = 0$ minus the trivial equation, corresponding to $I = H = 0$, we get relative/absolute differential invariants of the ODEs (4). For instance H is proportional to $L_1 + L_2 p$, which under substitution of $p = \frac{dy}{dx}$ is proportional to the tensor L. The other invariants are rational functions in p the cubics (4), which may be taken in correspondence with the invariants of the approach from Sect. 4.1.

The proposed idea can be viewed as a change of coordinates in the algebra $\mathcal{I}$. Yet, another approach was sketched in [Tr$_1$], which can be called a non-local substitution.

Namely by a change of variables Tresse achieves $L_2 = 0$, and so brings the tensor $L_1 dx + L_2 dy$ to the form $\lambda\, dx$. Then the point transformation pseudogroup is reduced to the triangular pseudogroup $x \mapsto X(x)$, $y \mapsto Y(x, y)$, and the invariants are generated by the invariant derivatives Δ_x, Δ_y and the invariants B, C, D of orders 1, 2, 2 respectively ([Tr$_1$], Chap. III), which though do not correspond to the orders in the approach of Sect. 4.1.

4.3 Lie Equations

Let $\mathfrak{s}_\mathcal{E} : \mathbb{R}^2 \to \mathbb{R}^4$ be the map $(x, y) \mapsto (a_0, a_1, a_2, a_3)$ corresponding to a second order ODE $\mathcal{E}$ (4). With two such ODEs we relate the Lie equation on the equivalence between them:

$$\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2) = \{[\varphi]^2_z \in J^2(2,2) : \hat{\varphi}\big(\mathfrak{s}_{\mathcal{E}_1}(z)\big) = \mathfrak{s}_{\mathcal{E}_2}(\varphi(z))\}, \tag{6}$$

where $\hat{\varphi} : \mathbb{R}^2 \times \mathbb{R}^4 \to \mathbb{R}^2 \times \mathbb{R}^4$ is the lift of a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ to a map of ODEs (4). On infinitesimal level, the lift of a vector field $X = a\,\partial_x + b\,\partial_y$ is

$$\begin{aligned}\hat{X} &= a\,\partial_x + b\,\partial_y + (b_{xx} + \alpha_0(b_y - 2a_x) - \alpha_1 b_x)\,\partial_{\alpha_0}\\ &+ (2b_{xy} - a_{xx} - 3\alpha_0 a_y - \alpha_1 a_x - 2\alpha_2 b_x)\partial_{\alpha_1} + (b_{yy} - 2a_{xy} - 2\alpha_1 a_y - \alpha_2 b_y - 3\alpha_3 b_x)\partial_{\alpha_2}\\ &+ (-a_{yy} - \alpha_2 a_y + \alpha_3(a_x - 2b_y))\partial_{\alpha_3}.\end{aligned}$$

For one equation $\mathcal{E}_1 = \mathcal{E}_2$ infinitesimal version of the finite Lie equation $\mathfrak{Lie}(\mathcal{E}, \mathcal{E})$ describes the symmetry algebra (which more properly should be called a Lie equation [KSp]) $\operatorname{sym}(\mathcal{E})$: it is formed by the solutions of

$$\mathfrak{lie}(\mathcal{E}) = \{[X]^2_z \in J^2(2,2) : \hat{X} \in T_{\mathfrak{s}_\mathcal{E}(z)}[\mathfrak{s}_\mathcal{E}(\mathbb{R}^2)]\}. \tag{7}$$

The basic differential invariants of the pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ action on ODEs (4) arise as the obstruction to formal integrability of the equation $\mathfrak{lie}(\mathcal{E})$ (for the equivalence problem $\mathfrak{Lie}(\mathcal{E}_1, \mathcal{E}_2)$, but the investigation is similar). In coordinates, when the section $\mathfrak{s}_\mathcal{E}$ is given by four equations $\alpha_i - \alpha_i(x, y) = 0$, overdetermined system (6) is written as

$$\begin{aligned}
b_{xx} + \alpha_0(b_y - 2a_x) - \alpha_1 b_x &= a\,\alpha_{0x} + b\,\alpha_{0y}\\
2b_{xy} - a_{xx} - 3\alpha_0 a_y - \alpha_1 a_x - 2\alpha_2 b_x &= a\,\alpha_{1x} + b\,\alpha_{1y}\\
b_{yy} - 2a_{xy} - 2\alpha_1 a_y - \alpha_2 b_y - 3\alpha_3 b_x &= a\,\alpha_{2x} + b\,\alpha_{2y}\\
-a_{yy} - \alpha_2 a_y + \alpha_3(a_x - 2b_y) &= a\,\alpha_{3x} + b\,\alpha_{3y}
\end{aligned}$$

The symbols $g_i \subset S^i T^* \otimes T$ are: $g_0 = T = \mathbb{R}^2$, $g_1 = T^* \otimes T \simeq \mathbb{R}^4$, $g_2 \simeq \mathbb{R}^2$ and $g_{3+i} = 0$ for $i \geq 0$. The compatibility conditions belong to the Spencer cohomology group $H^{2,2}(\mathfrak{lie}) \simeq \mathbb{R}^2$: this is equivalent to the tensor L of (5). If $L = 0$, the equation is integrable[16] and the solution space is the Lie algebra sl_3.

If $L \neq 0$, the equation $\mathfrak{lie}_0 = \mathfrak{lie}(\mathcal{E})$ has prolongation–projection[17] $\mathfrak{lie}_1 = \pi_{4,1}(\mathfrak{lie}^{(2)})$ with symbols $g_0 = T$, $\bar{g}_1 \simeq \mathbb{R}^2 \subset g_1$, $g_2 \simeq \mathbb{R}^2$ and $g_{3+i} = 0$ for $i \geq 0$.

After prolongation–projection, one gets the equation $\mathfrak{lie}_2$ with symbols $g_0 = T$, $\tilde{g}_1 \simeq \mathbb{R}^1 \subset \bar{g}_1$ and $g_{2+i} = 0$ for $i \geq 0$. This equation has the following space of compatibility conditions: $H^{1,2}(\mathfrak{lie}_3) \simeq \mathbb{R}^1$. It yields the condition of the third order in α_i: $F_3 = 0$ (this, together with other invariants [R], characterizes equations with three-dimensional symmetry algebra, namely sl_2).

If $F_3 \neq 0$, then the prolongation–projection yields the equation $\mathfrak{lie}_3$ with $g_0 = T$ and $g_{1+i} = 0$ for $i \geq 0$. The compatibility conditions are given by the Frobenius theorem and this provides the basis of differential invariants.

Remarks. (1) The idea to reformulate equivalence problem via solvability of an overdetermined system appeared in S. Lie's linearization theorem, where he showed that an ODE (4) is point equivalent to the trivial equation $y'' = 0$ iff the system (see [Lie$_2$], p. 365 (we let $z = c$, $w = C$, etc.), and also [IM])

$$\frac{\partial w}{\partial x} = zw - \alpha_0\alpha_3 - \frac{1}{3}\frac{\partial \alpha_1}{\partial y} + \frac{2}{3}\frac{\partial \alpha_2}{\partial x}, \qquad \frac{\partial z}{\partial x} = z^2 - \alpha_0 w - \alpha_1 z + \frac{\partial \alpha_0}{\partial y} + \alpha_0\alpha_2,$$
$$\frac{\partial z}{\partial y} = \alpha_0\alpha_3 - zw - \frac{1}{3}\frac{\partial \alpha_2}{\partial x} + \frac{2}{3}\frac{\partial \alpha_1}{\partial y}, \qquad \frac{\partial w}{\partial y} = \alpha_2 w - w^2 + \alpha_3 z + \frac{\partial \alpha_3}{\partial x} - \alpha_1\alpha_3$$

is compatible. The compatibility conditions here are given by the Frobenius theorem: $L_1 = L_2 = 0$. In fact, the system can be transformed into a linear system,[18] which is equivalent to half of our once prolonged Lie equation $\mathfrak{lie}^{(1)}$ (Lie considers combinations of the unknown functions-component of the point transformation, that's why in the third order we get only $4 = 8/2$ equations, the second half of equations was not much used by him).

[16] Not only formally, but also locally smoothly due to the finite type of $\mathfrak{lie}$.

[17] This means that the Lie equation has the first prolongation $\mathfrak{lie}^{(1)} \subset J^3(2,2)$, but the next prolongation exists only over the jets of vector fields X, preserving the tensor L.

[18] S. Lie considers finite transformations, whence the non-linearity. A projective transformation is needed to change this into a linear system, while the infinitesimal analog – our Lie equation $\mathfrak{lie}(\mathcal{E})$ – is linear from the beginning.

(2) Cartan equivalence method can be interpreted in this language as follows. Consider the most symmetric equation (prolongation) – model $\mathcal{E}_0$ and then study the problem of equivalence between $\mathcal{E}$ and $\mathcal{E}_0$. This is done by constructing a Cartan connection, and the (normalized) curvatures provide fundamental invariants for the problem. These are essentially the same functions that appear in the prolongation–projection scheme for formal integrability of the Lie equation.

For the equivalence problem treated here, the corresponding $\mathcal{E}_0$ is $y'' = 0$ with eight-dimensional symmetry algebra sl_3, and so there is an eight-dimensional manifold with Cartan geometry [C] associated to any second order ODE (general, not necessarily cubic in p). This eight-dimensional manifold comes equipped with a connection mimicking Maurer–Cartan connection on SL_3, which measures non-flatness. It is believed all the Tresse invariants can be obtained in this way, but this claim is yet to be justified.

(3) Other ways of getting differential invariants arise from problems which have projectively invariant answers. For instance the following system arose in three independent problems:

$$u_y = P_0[u, v, w], \quad u_x + 2v_y = P_1[u, v, w],$$
$$2v_x + w_y = P_2[u, v, w], \quad w_x = P_3[u, v, w],$$

where $P_i[u, v, w]$ are linear operators of a special type, with coefficients being smooth functions in x, y. This system can be obtained similar to $\mathfrak{lie}$ from the condition of existence of Killing tensors.[19]

In [K] solvability of this system lead to an invariant characterization of Liouville metrics, in [BMM] to normal forms of metrics with transitive group of projective transformations and in [BDE] – to the condition of local metrisability of projective structures on surfaces.

All these problems have the answers (for instance, in the first mentioned paper, the number of Killing tensors of a metric), which are projective invariants. Thus they provide projective differential invariants and in turn can be expressed via any basis of them.

(4) Many papers addressed the higher-dimensional version of the same equivalence problem (which is surprisingly easier, because the Lie equation is more overdetermined). In Cartan [C] this is the study of the projective connection. Refs. [Th, Lev] address the algebra of scalar projective differential invariants.

However in neither of these approaches the Tresse method was superseded. For instance, in the latter reference even the number of differential invariants for the two-dimensional case was not determined. On the other hand, the method of Lie equations allows to obtain the algebra of projective invariants in the higher-dimensional case as well.

[19] Substitution $u = 3\xi_y, w = 3\eta_x, v = -(\xi_x + \eta_y)/2$ transforms this system to the kind $\xi_{yy} = \dots, 2\xi_{xy} - \eta_{yy} = \dots, \xi_{xx} - 2\eta_{xy} = \dots, -\eta_{xx} = \dots$, which has the same symbol as (7).

5 Application to Symmetries

At the end of [Tr_2] a classification of symmetric equations is given. It turns out that the symmetry algebra can be of dimensions 8, 3, 2, 1 or 0. This follows from the study of dependencies among differential invariants, and it is not obvious that this automatically applies to all singular strata (but it is true).

Thus if $\dim \mathrm{Sym}(\mathcal{E}) = 8$ the ODE is equivalent to the trivial $y'' = 0$. If $\dim \mathrm{Sym}(\mathcal{E}) = 3$, the normal forms are ($y' = p$):

$$y'' = p^a \qquad y'' = \frac{(cp + \sqrt{1-p^2})(1-p^2)}{x}$$
$$y'' = e^p \qquad y'' = \pm(xp - y)^3.$$

Only the last form belongs to the singular stratum $I = 0$.

Due to symmetry between I and H, there should be corresponding normal form with $H = 0$. Here one can be mislead since direct calculations shows that none has vanishing H. The reason is however that Tresse uses Lie's classification of Lie algebras representation by vector fields on the plane. For three-dimensional algebras Lie used normal forms over $\mathbb{C}$, and indeed the third normal form has $H = 0$ for the parameter $c = \pm i$. Thus over $\mathbb{R}$ the above normal forms should be extended.

As the symmetry algebra reduces to dimensions 2 we have the respective normal forms

$$y'' = \psi(p) \qquad \text{and} \qquad y'' = \psi(p)/x.$$

It is important that for singular strata the classification shall be finer. This is almost obvious for projective connections (cubic ψ), but for metric projective connections this is already substantial, see [BMM].

The case $\dim \mathrm{Sym}(\mathcal{E}) = 1$ has only one quite general form $y'' = \psi(x, p)$ with an obvious counterpart for projective connections (for the metric case see [Ma]).

Remark 3 *When the transformation pseudogroup reduces from point to fiber-preserving (triangular) transformations of $J^0\mathbb{R}(x) = \mathbb{R}^2(x, y)$, the algebra of differential invariants grows, but the symmetric cases change completely. In particular, the symmetry algebra can have dimensions 6, 3, 2, 1 or 0, see [KSh, HK].*

Not much is known about the criteria for having the prescribed dimension of the symmetry algebra, except for the corollary of Lie–Liouville–Tresse theorem: $\dim \mathrm{Sym}(\mathcal{E}) = 8$ iff $L = 0$.

In [Tr_2] the following was claimed (we translate it to the language of absolute differential invariants):

- ◇ The symmetry algebra is three-dimensional iff all the differential invariants (on the equation) are constant.
- ◇ The symmetry algebra is two-dimensional iff the space of differential invariants has functional rank 1, i.e., any two of them are functionally dependent (Jacobian vanishes).
- ◇ The symmetry algebra is one-dimensional iff the space of differential invariants has functional rank 2 (all 3×3 Jacobians vanish).

This (unproved in Tresse, but correct statement) is however inefficient, since checking all invariants is not practically possible. Here is an improvement:

Theorem 6. *The above claims hold true if we restrict to the basic absolute differential invariants of order* ≤ 6 *described in Sect. 2.2.*

What is the minimal collection of differential invariants answering the above question is seemingly unknown (except for the case $\dim \mathrm{Sym}(\mathcal{E}) = 3$ for $I = 0$ handled in [R]).

Appendix: Another Approach

B. Kruglikov and V. Lychagin

(1) Consider the stabilizer $\mathfrak{l}_8 \subset \mathfrak{g}$ of a point $(x, y) \in \mathbb{R}^2(x, y) = J^0\mathbb{R}(x)$. We can choose coordinates so that $x = y = 0$. The vector fields generating this subalgebra of vector fields of $F_{x,y} = \mathbb{R}^2(p, u) = \pi_{2,0}^{-1}(x, y)$ are

$$\mathfrak{l}_8 = \langle \partial_p,\ \partial_u,\ p\,\partial_p,\ u\,\partial_u,\ p\,\partial_u,\ p^2\partial_u,\ p^3\partial_u,\ p^2\partial_p + 3p\,u\,\partial_u \rangle$$

This is an eight-dimensional Lie algebra with Levi decomposition $\mathcal{R}_5 \ltimes \mathrm{sl}_2$, where $\mathcal{R}_5$ is the radical, which is a solvable Lie algebra with four-dimensional (commutative) nil-radical.

In $F_{x,y}$ the second order equation $\mathcal{E}$ is a curve $u = f(p)$.[20] Since in equivalence problem we can transform one base point to another by a point transformation (any point to any if the equation possesses a two-dimensional symmetry group, transitive on the base), the equivalence problem is reduced to the equivalence of curves on the plane $\mathbb{R}^2(p, u)$ with respect to the Lie group $\mathfrak{l}_8$ action.

The action lifts to the spaces $J^k\mathbb{R}(p) = \mathbb{R}^{k+2}(p, u, u_p, \dots)$ and is transitive up to third jets. The first singular orbit appears in the space $J^4\mathbb{R}(p)$ and is

$$\mathcal{S}_1 = \{G_4 = u^4 = 0\}$$

(we continue to use the same notations as above, so that $u^4 = u_{pppp}$). The next singular orbit (different from prolongation of this one) appears in the space $J^6\mathbb{R}(p)$ and is

$$\mathcal{S}_2 = \{G_6 = 5\,u^4u^6 - 6\,u^5 \cdot u^5 = 0\}.$$

Notice that the second equation belongs to the prolongation of the first: $\mathcal{S}_2 \subset \mathcal{S}_1^{(2)}$ (so it is a sub-singular orbit). The functions G_4, G_6 are relative differential invariants. In this case the weights can be chosen via cocycles $C_\xi^r = \mathrm{div}_{\omega_0}(\xi) - \frac{1}{2}\,\mathrm{div}_\Omega(\xi_1)$ and $C_\xi^s = \frac{1}{2}\,\mathrm{div}_\Omega(\xi_1)$, where $\omega_0 = dp \wedge du$, $\Omega = -\omega \wedge d\omega = dp \wedge$

[20] Depending parametrically on x, y.

$du \wedge du_p$ (for $\omega = du - u_p\,dp$) and $\xi = A\partial_p + B\partial_u$, $\xi_1 = X_f = A\partial_p + B\partial_u + (\mathcal{D}_p(B) - \partial_p(A)u_p)\partial_{u_p}$, $\hat{\xi} = X_f^{(\infty)}$ with

$$f = B - A\,u_p,\;\; A = a_0 + a_1 p + a_2 p^2,\;\; B = b_0 + b_1 p + b_2 p^2 + b_3 p^3 + b_4 u + 3a_2 p\,u.$$

Denoting $\mathcal{R}^{r,s} = \{\psi \in C^\infty(J^\infty\mathbb{R}) : \hat{\xi}(\psi) = -(r\,C_\xi^r + s\,C_\xi^s)\psi\}$, we get:[21]

$$G_4 \in \mathcal{R}^{4,-1},\;\; G_6 \in \mathcal{R}^{10,-2}.$$

The relative invariant derivative here equals

$$\Diamond_p = \mathcal{D}_p - \frac{2r+3s}{5}\frac{u^5}{u^4} : \mathcal{R}^{r,s} \to \mathcal{R}^{r+1,s}.$$

It acts trivially on G_4, but from its action on G_6 we can extract an absolute invariant. Indeed since $G_4/\sqrt{G_6} \in \mathcal{R}^{-1,0}$, we have: $\Diamond_p(G_4/\sqrt{G_6}) \in \mathcal{R}^{0,0} = \mathcal{I}$ and the latter expression is non-zero.

Actually the action of our eight-dimensional group has open orbits through generic points on $J^6\mathbb{R}(p)$ and the first absolute differential invariant appear at order 7 and equals[22]

$$I_7 = \frac{25(u^4)^2u^7 + 84(u^5)^3 - 105u^4u^5u^6}{(G_6)^{3/2}},$$

which coincides with $-10\,\Diamond_p(G_4/\sqrt{G_6})$.

In each higher order we get 1 new differential invariant. They determine Tresse derivative (see [KL$_1$]), but we can obtain the absolute invariant derivative directly:

$$\square_p = \frac{G_4}{\sqrt{G_6}}\Diamond_p\Big|_{r=s=0} = \frac{u^4}{\sqrt{5u^4u^6 - 6u^5\cdot u^5}}\,\mathcal{D}_p : \mathcal{I} \to \mathcal{I}.$$

This can be expressed via invariants of Sect. 3.2 as $\frac{1}{\sqrt{5\Omega^6}}\nabla_p$.

Thus on the generic stratum every differential invariant is (micro-locally) a function of the invariants $I_7, \square_p(I_7), \square_p^2(I_7), \dots$ of orders 7, 8, 9, ...

(2) It is easy to see that the class of cubic curves $u = Q_3(p)$ is invariant with respect to the Lie group $\mathfrak{L}_8 = \mathrm{Exp}(\mathfrak{l}_8)$ action. This four-dimensional space forms a singular orbit, on which $\mathfrak{L}_8$ acts transitively.

Another singular orbit is $\mathcal{S}_2$, which is a six-dimensional manifold of curves $u = (a_0 + a_1 p + a_2 p^2 + a_3 p^3) + b/(p-c)$, and $\mathfrak{L}_8$ acts transitively there on the generic stratum. The singular stratum is given by the equation $b = 0$ and coincides with the previous stratum $\mathcal{S}_1$.

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[21] Note that since the group is changed the grading is changed as well. In particular G_4, which formally coincides with I of Sect. 2.1, has a different grading.

[22] Superscript after brackets means the power, while the others are indices.

(3) Consider the stabilizer $\mathfrak{l}_6 = \mathrm{St}_0 \subset \mathfrak{h} = \mathfrak{g}_2$ of a point $x_2 = (x, y, p, u) \in J^0\mathbb{R}^3(x, y, p) = J^2\mathbb{R}(x)$. Since the pseudogroup $\mathfrak{g}_2$ acts transitively on $J^2\mathbb{R}(x)$, choice of x_2 is not essential, in particular we can take a coordinate representative $(x, y, 0, 0)$. This Lie algebra is generated by (prolongation of) the fields

$$\mathfrak{l}_6 = \langle p\,\partial_p,\ u\,\partial_u,\ p\,\partial_u,\ p^2\partial_u,\ p^3\partial_u,\ p^2\partial_p + 3p\,u\,\partial_u\rangle$$

and is solvable (with four-dimensional nil-radical of length 2). Thus investigation of its invariants is easier thanks to S. Lie quadrature theorem.

Moreover we can continue and take the stabilizer $\mathrm{St}_3 \subset \mathfrak{g}_5$ of a point in $J^3\mathbb{R}^3(x, y, p)$, where the action of $\mathfrak{h}$ is still transitive. This stabilizer is a trivial one-dimensional extension of the two-dimensional solvable Lie algebra.

(4) With all these approaches we get enough invariants to pursue classification in generic case (and even to deal with singular orbits). Indeed, we get a subalgebra $\mathfrak{V}$ in the algebra of all differential invariants $\mathcal{I}$, which we can restrict to the equation. Provided that there are invariants in $\mathfrak{V}$ independent, on our second order ODE $\mathcal{E}$, we solve the equivalence problem.

In general to restore the whole algebra $\mathcal{I}$ we must add to this vertical invariants algebra $\mathfrak{V}$ invariant derivatives ∇_x, ∇_y. This is similar to Liouville approach for cubic second order ODEs (retrospectively after [C] – projective connections), who found in [Lio] only a subalgebra of (relative) differential invariants (the second relevant invariant derivative $\nabla_2 = \Psi_2\mathcal{D}_x - \Psi_1\mathcal{D}_y + \dots : \mathcal{R}^r \to \mathcal{R}^{r+4}$ and the corresponding part of differential invariants was not established).

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Classification of Monge–Ampère Equations

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Abstract In the current paper we present a survey of our results on classification of the Monge–Ampère equations and operators with two independent variables. We use Lychagin's approach to such equations.

Introduction

The problem of equivalence and classification of the Monge–Ampère equations

$$Av_{xx} + 2Bv_{xy} + Cv_{yy} + D(v_{xx}v_{yy} - v_{xy}^2) + E = 0 \quad (1)$$

with two independent variables x and y goes back to Sophus Lie's papers from the 1870s and 1880s [37–39]. Here A, B, C, D, and E are functions of x, y, an unknown function $v = v(x, y)$, and its first partial derivatives v_x and v_y. We suppose that these functions are of class C^∞.

Lie had raised the following problem.

Find equivalence classes of nonlinear second-order differential equations with respect to the group of contact transformations.

The important steps in the solution of this problem were made by Darboux [5–7] and Goursat [10, 12, 13], who had basically treated the hyperbolic Monge–Ampère equations. Particularly, Goursat considered the problem of contact equivalence of Darboux integrable Monge–Ampère equations [11]. His ideas were developed by Vessiot [55]. See also [18, 22].

Lie himself had found conditions under which it is possible to transform a Monge–Ampère equation to a quasilinear one and to some linear equation with constant coefficients. But a complete proof of Lie's theorems had never been published. A problem of reducibility of hyperbolic and elliptic Monge–Ampère equations,

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whose coefficients do not depend on the variable v (such equations are called *symplectic*), to the equations with constant coefficients was solved by Lychagin and Rubtsov in 1983 [43,44]. In 1996 Tunitskii took off the above mentioned restriction and solved the problem for the general equations [54].

In 1978 Lychagin noted that the classical Monge–Ampère equations and its multidimensional analogues admit an effective description in terms of differential forms on the space of 1-jets of smooth functions [40]. His idea was fruitful, and it generated a new approach to Monge–Ampère equations.

Following this approach, the classification problems for symplectic Monge–Ampère operators and equations were considered by Kruglikov [19–21] and the author [23–26]. Classification results for multidimensional analogues of the Monge–Ampère equations (i.e., equations with n $(n > 2)$ independent variables) were obtained by Lychagin, Rubtsov and Chekalov [45], and by Banos [1,2].

A contact linearization problem for the general hyperbolic and elliptic Monge–Ampère equations was solved by the author in a series of papers [27–32, 34].

The special classes of parabolic Monge–Ampère equations were considered by Blanco, Manno and Pugliese [3].

In 1979 Morimoto obtained classification results for the Monge–Ampère equations by using the G-structure theory [46].

The Cartan method of moving coframes was applied to the problem of local equivalence of some classes of linear and nonlinear equations by Morozov [47–50] and The [52].

Ibragimov considered a classification problem for some partial classes of the Monge–Ampère equations [15, 17].

In the current paper we present a survey of our results on classification of Monge–Ampère equations and operators with two independent variables.

The paper has the following structure.

In the first two sections we describe the main ideas of Lychagin and give a short introduction to the geometry of the Monge–Ampère equations on two-dimensional manifolds. Here we follow the papers [40, 41] and the book [42].

In the third and fourth sections a decomposition of the de Rham complex is used to construct tensor invariants of hyperbolic and elliptic equations. In particular, we construct two invariant differential 2-forms for the Monge–Ampère equations. These forms are analogues of the classical Laplace and Cotton invariants, which are defined for linear equations only. Note that classical Laplace's and Cotton's invariants are coefficients in our forms [27, 30].

In the fifth section we formulate the results of Lychagin, Rubtsov, and Tunitskii on local equivalence of the Monge–Ampère equations and Monge–Ampère equations with constant coefficients [43, 54].

Sixth and seventh sections are dedicated to the solution of the contact linearization problem for hyperbolic and elliptic equations. We follow the papers [30–34].

In the eighth section we construct an absolute parallelism (e-structure) for hyperbolic and elliptic equations. This allows us to reduce an equivalence problem for equations to an equivalence problem for e-structures, whose solution is well known. Results of this section were not published earlier.

The ninth section is devoted to classification of the symplectic Monge–Ampère equations and operators of hyperbolic, elliptic, and mixed types [23–26, 28, 29].

All results are given without proofs. They can be found in the original papers.

1 Lychagin's Approach to Monge–Ampère Equations

1.1 Differential Operators

Let M be an n-dimensional smooth manifold, and let J^1M be the manifold of 1-jets of smooth functions on M.

The manifold J^1M is endowed with the natural contact structure (Cartan's distribution) (see [56])

$$\mathcal{C} : a \in J^1M \mapsto \mathcal{C}(a) \subset T_a(J^1M)$$

given by the universal differential 1-form $\mathcal{U} \in \Omega^1(J^1M)$ (Cartan's form): $\mathcal{C}(a) = \operatorname{Ker}\mathcal{U}_a$.

In the local canonical Darboux coordinates

$$(q, u, p) = (q_1, \ldots, q_n, u, p_1, \ldots, p_n)$$

on J^1M the Cartan form can be written as follows:

$$\mathcal{U} = du - pdq = du - p_1dq_1 - \cdots - p_ndq_n.$$

At each point $a \in J^1M$, the restriction of the differential of the Cartan form to the $(2n)$-dimensional space $\mathcal{C}(a)$ defines the following symplectic structure:

$$\Omega_a = d\mathcal{U}|_{\mathcal{C}(a)}.$$

Define the "nonholonomic symplectic structure" on J^1M:

$$\Omega : J^1M \ni a \mapsto \Omega_a \in \Lambda^2(\mathcal{C}(a)^*).$$

Note that Ω is not a differential form, because the 2-form Ω_a is defined on $\mathcal{C}(a)$ only.

The main idea of Lychagin [40][1] can be explained as follows:

With any differential n-form ω on J^1M, one can associate a differential operator $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$, which acts as

$$\Delta_\omega(v) = \omega|_{j_1(v)(M)}.$$

Here $v \in C^\infty(M)$ and $j_1(v)(M) \subset J^1M$ is the graph of the 1-jet of v.

[1] An explicit presentation of results of this paper one can find in [41] and [35].

This construction does not cover all nonlinear second-order differential operators, but only a certain subclass of them. This subclass is rather wide and contains all linear, quasilinear and Monge–Ampère operators.

The constructed operator Δ_ω and the equation $E_\omega = \{\Delta_\omega(v) = 0\} \subset J^2M$ are called the *Monge–Ampère operator* and the *Monge–Ampère equation*, respectively.

The following observation justifies this definition: being written in local canonical contact coordinates on J^1M, the operators Δ_ω have the same type of nonlinearity as the Monge–Ampère operators. Namely, the nonlinearity involves the determinant of the Hesse matrix and its minors. For instance, in the case $n = 2$ we get classical Monge–Ampère equations (1).

An advantage of this approach is the reduction of the order of the jet space: we use the simpler space J^1M instead of the space J^2M, where Monge–Ampère equations should be *ad hoc* as second-order partial differential equations [56].

1.2 Effective Differential Forms

The constructed map "differential forms" → "differential operators" is not a 1-to-1 map: it has a huge kernel. This kernel consists of differential forms that vanish on any integral manifold of the Cartan distribution. These forms form a graded ideal

$$I^* = \oplus_{s\geq 0} I^s, \quad I^s \subset \Omega^s(J^1M)$$

of the exterior algebra $\Omega^*(J^1M)$. Elements of the factor module

$$\Omega^s_\varepsilon(J^1M) = \Omega^s(J^1M)/I^s$$

are called *effective* s-forms. Denote by ω_ε the class in $\Omega^s_\varepsilon(J^1M)$:

$$\omega_\varepsilon = \omega \bmod I^s$$

corresponding to a differential s-form ω.

By definition, we put $\Delta_{\omega_\varepsilon} = \Delta_\omega$.

1.3 Contact Equivalence, Symmetries and Multivalued Solutions

Define actions of contact diffeomorphisms on effective n-forms. Let ϕ be a contact diffeomorphism of J^1M. Since ϕ^* saves I^n, we can define its action on $\Omega^n_\varepsilon(J^1M)$ by the following formula:

$$\phi^*(\omega_\varepsilon) = \phi^*(\omega)_\varepsilon.$$

Note, that, generally speaking, the contact diffeomorphisms do not save the Cartan form $\mathcal{U}$ and, because of this, they do not act directly to a chosen representative of an effective form.

Now we can define an action of a contact diffeomorphism on Monge–Ampère operators and equations:

$$\phi(\Delta_\omega) = \Delta_{\phi^*(\omega)_\varepsilon} \qquad \text{and} \qquad \phi(E_\omega) = E_{\phi^*(\omega)_\varepsilon}.$$

Two Monge–Ampère operators Δ_ω and Δ_θ are called *contact equivalent* if $\phi(\Delta_\omega) = \Delta_\theta$ for some contact diffeomorphism ϕ.

Similarly, two Monge–Ampère equations E_ω and E_θ are called *contact equivalent* if there exists a contact diffeomorphism ϕ such that $\phi(E_\omega) = E_\theta$.

Two Monge–Ampère operators Δ_ω and Δ_θ (equations E_ω and E_θ) are called *locally contact equivalent at a point* $a \in J^1M$ if there exists a local contact diffeomorphism ϕ saving this point and such that $\phi(\Delta_\omega) = \Delta_\theta$ ($\phi(E_\omega) = E_\theta$) in some neighborhood of a.

We reformulate a notion of symmetry in terms of effective forms. A contact diffeomorphism ϕ is called a *symmetry* of E_ω (Δ_ω) if $\phi(E_\omega) = E_\omega$ ($\phi(\Delta_\omega) = \Delta_\omega$). A contact vector field X_f is an *infinitesimal symmetry* of E_ω (Δ_ω) if its local translation group consists of symmetries of E_ω (Δ_ω).

An n-dimensional integral manifold of the Cartan distribution L is called *multivalued* solution of an equation E_ω if the restriction $\omega|_L = 0$.

Example 1. Spheres with radius 1 in the space $\mathbb{R}^3(x, y, u)$ are projections of multivalued solutions of the equation

$$\frac{v_{xx}v_{yy} - v_{xy}^2}{(1 + v_x^2 + v_y^2)^2} = 1. \tag{2}$$

to $J^0\mathbb{R}^2$.

Suppose that $n = 2$ and find a coordinate representation of effective 2-forms. For any element of the factor module Ω_ε^2, one can choose a unique representative $\omega \in \Omega^2(J^1M)$ such that $X_1 \rfloor \omega = 0$ and $\omega \wedge d\mathcal{U} = 0$. Here X_1 is a contact vector field with the generating function 1. In the local Darboux coordinates such representatives have the form

$$\begin{aligned} \omega =& E dq_1 \wedge dq_2 + B\,(dq_1 \wedge dp_1 - dq_2 \wedge dp_2) + \\ & C dq_1 \wedge dp_2 - A dq_2 \wedge dp_1 + D dp_1 \wedge dp_2, \end{aligned} \tag{3}$$

where A, B, C, D and E are smooth functions on J^1M.

We identify effective forms as elements of the factor module $\Omega_\varepsilon^2(J^1M)$ with differential forms of type (3) and also call such differential forms effective.

Remark 1. Form (3) corresponds to (1).

1.4 Symplectic Geometry of Equations

A Monge–Ampère equation E_ω (operator Δ_ω) is called *symplectic* if its coefficients do not depend on the variable v.

This means that the Lie derivative $L_{X_1}(\omega) = 0$. Here X_1 is the contact vector field with the generating function 1. In the local Darboux coordinates, $X_1 = \partial_u$.

Naturally, this property is not invariant under contact transformation of J^1M. However, if we restrict the class of contact transformations and consider only such transformations that are symmetries of the one-dimensional distribution $\mathcal{F}\langle X_1\rangle$, then we can use the cotangent bundle T^*M instead of J^1M.

Indeed, in this case $\omega = \pi^*(\widetilde{\omega})$ for some $\widetilde{\omega} \in \Omega^n(T^*M)$, where

$$\pi : J^1M \to T^*M$$

is the natural projection.

The symplectic structure on T^*M is generated by the universal form ρ: $\Omega = -d\rho$.

The class of equations, which can be reduced to symplectic ones, is wide. Indeed, suppose that an equation E_ω admits an infinitesimal symmetry X_f and $f(a) \neq 0$ in some point $a \in J^1M$. We can take X_f to X_1 in some neighborhood of a by a suitable contact transformation ϕ [35]. Then we get the symplectic equation $E_{\phi^*(\omega)}$.

Let us denote the $C^\infty(T^*M)$-module of effective s-forms on T^*M by $\Omega^s_\varepsilon(T^*M)$.

The projection $\pi(L)$ of any multivalued solution L of E_ω is an embedded Lagrange manifold such that the restriction $\widetilde{\omega}|_{\pi(L)} = 0$. Conversely, if $\widetilde{L}$ is the Lagrange manifold such that $\widetilde{\omega}|_{\widetilde{L}} = 0$, then there is a manifold $L \subset J^1M$ such that $\mathcal{U}|_L = 0$, $\widetilde{L} = \pi(L)$, and L is a multivalued solution of E_ω.

For this reason, the Lagrange manifolds $\widetilde{L} \subset T^*M$ such that $\widetilde{\omega}|_{\widetilde{L}} = 0$ are also called *multivalued solutions*.

Two symplectic Monge–Ampère operators Δ_ω and Δ_θ are called *symplectic equivalent* if $\phi(\Delta_\omega) = \Delta_\theta$ for some symplectic diffeomorphism ϕ.

Similarly, two symplectic Monge–Ampère equations E_ω and E_θ are called *symplectic equivalent* if there exists a symplectic diffeomorphism ϕ such that $\phi(E_\omega) = E_\theta$.

Two symplectic Monge–Ampère operators Δ_ω and Δ_θ (symplectic equations E_ω and E_θ) are called *locally symplectic equivalent at a point* $a \in T^*M$ if there exists a local symplectic diffeomorphism ϕ, which saves this point and such that $\phi(\Delta_\omega) = \Delta_\theta$ ($\phi(E_\omega) = E_\theta$) in some neighborhood of a.

Remark 2. For a differential n-form $\omega \in \Omega^n(T^*M)$ the Monge–Ampère operator can be defined as

$$\Delta_\omega(v) = (dv)^*(\omega),$$

where $dv : M \to T^*M$ is a natural section associated with $v \in C^\infty(M)$.

2 Geometry of Monge–Ampère Equations on Two-Dimensional Manifolds

The Monge–Ampère equations on two-dimensional manifolds possess remarkable geometric structures. We consider them for the general and symplectic equations separately.

2.1 Geometric Structures on J^1M

In what follows, we suppose that $n = 2$ and consider the classical Monge–Ampère equations (1) only.

With any effective differential 2-form ω one can associate a smooth function $\mathrm{Pf}(\omega)$ on J^1M by means of the following equality [43]:

$$\mathrm{Pf}(\omega)\Omega \wedge \Omega = \omega \wedge \omega. \tag{4}$$

This function is called the *Pfaffian* of ω.

We say that the Monge–Ampère equation E_ω is *hyperbolic, elliptic* or *parabolic* in a domain $\mathcal{D} \subset J^1M$ if the function $\mathrm{Pf}(\omega)$ is negative, positive or zero at each point of $\mathcal{D}$, respectively. If the function $\mathrm{Pf}(\omega)$ changes its sign at some points of $\mathcal{D}$, then the equation E_ω is called the *mixed type* equation.

The hyperbolic and elliptic equations are called *nondegenerate*.

An effective 2-form ω generates the "nonholonomic"[2] field of endomorphisms A_ω on the Cartan distribution by means of the formula

$$X \rfloor \omega = A_\omega X \rfloor\ \Omega, \tag{5}$$

where X is a vector field from the Cartan distribution.

The square of the operator A_ω is scalar and

$$A_\omega^2 + \mathrm{Pf}(\omega) = 0. \tag{6}$$

Moreover, A_ω is symmetric with respect to Ω, i.e.,

$$\Omega(A_\omega X, Y) = \Omega(X, A_\omega Y) \tag{7}$$

for any $X, Y \in D(\mathcal{C})$ [43]. Here $D(\mathcal{C})$ is the module of vector fields from the Cartan distribution.

The effective forms ω and $h\omega$, where h is any nonvanishing function, define the same equation. Therefore, for a nondegenerate equation E_ω the form ω can be

[2] The endomorphism $A_{\omega,a}$ is defined on $\mathcal{C}(a)$ only.

normed in such a way that $|\,\mathrm{Pf}(\omega)| = 1$. It is sufficient to replace ω by $\frac{\omega}{\sqrt{|\,\mathrm{Pf}(\omega)|}}$. By (6), the hyperbolic and elliptic equations generate the product structure $A_{\omega,a} = 1$ and the complex structure $A_{\omega,a} = -1$ on $\mathcal{C}(a)$, respectively [42].

A nondegenerate Monge–Ampère equation generates two two-dimensional distributions on J^1M, which are generated by two eigenspaces of the operator A_{ω_a} ($a \in J^1M$). These eigenspaces $\mathcal{C}_+(a)$ and $\mathcal{C}_-(a)$ correspond to the eigenvalues 1 and -1 for the hyperbolic equations or to ι and $-\iota$ for the elliptic ones, respectively. Here $\iota = \sqrt{-1}$.

The distributions $\mathcal{C}_+$ and $\mathcal{C}_-$ are called *characteristic*. They are defined up to the permutation $\mathcal{C}_+ \leftrightarrow \mathcal{C}_-$.

The characteristic distributions are real for the hyperbolic equations and complex for the elliptic ones. For elliptic equations they are complex conjugate.

The planes $\mathcal{C}_+(a)$ and $\mathcal{C}_-(a)$ are skew-orthogonal with respect to the symplectic structure Ω_a ($a \in J^1M$). On each of them the 2-form Ω_a is nondegenerate. The first derivatives of the characteristic distributions $\mathcal{C}_\pm^{(1)} = \mathcal{C}_\pm + [\mathcal{C}_\pm, \mathcal{C}_\pm]$ are three-dimensional. Their intersection $l = \mathcal{C}_+^{(1)} \cap \mathcal{C}_-^{(1)}$ is a one-dimensional distribution, which is transversal to $\mathcal{C}$ [42].

For the hyperbolic equations the tangent space T_aJ^1M splits into the direct sum

$$T_aJ^1M = \mathcal{C}_+(a) \oplus l(a) \oplus \mathcal{C}_-(a). \tag{8}$$

at each point $a \in J^1M$ [42].

For elliptic equations we get similar decomposition of the complexification of T_aJ^1M. In this case the distributions l is real also.

A nondegenerate equation is called *regular* if the derivatives $\mathcal{C}_\pm^{(k)}$ ($k = 1, 2, 3$) of the characteristic distributions are distributions also.

Hence, on J^1M any regular nondegenerate Monge–Ampère equation defines a 3-tuple almost product structure[3]

$$\mathcal{C}_+ \oplus l \oplus \mathcal{C}_-.$$

Tensor invariants of such structures were constructed in [27] (see also [35], where this structure is called $\mathcal{AP}$-structure). In Sect. 3 we use these invariants to construct tensor invariants of equations.

Let E_ω be a nondegenerate regular equation, and let $\mathcal{C}_j$ be one of the characteristic distributions. Then for the distribution we expect one of the following four cases [46]:

(1) $\mathcal{C}_j \neq \mathcal{C}_j^{(1)} = \mathcal{C}_j^{(2)}$ and $\dim \mathcal{C}_j^{(1)} = 3$.
(2) $\mathcal{C}_j \neq \mathcal{C}_j^{(1)} \neq \mathcal{C}_j^{(2)} = \mathcal{C}_j^{(3)}$ and $\dim \mathcal{C}_j^{(1)} = 3$, $\dim \mathcal{C}_j^{(2)} = 4$.

[3] A set of distributions $\mathcal{P} = (P_1, \dots, P_r)$ on a smooth manifold N is called an *n-tuple almost product structure* if at each point $a \in N$ the tangent space T_aN (for real distributions) or its complexification (for complex ones) splits into the direct sum of the subspaces $P_1(a), \dots, P_r(a)$.

(3) $\mathcal{C}_j \neq \mathcal{C}_j^{(1)} \neq \mathcal{C}_j^{(2)} \neq \mathcal{C}_j^{(3)} = TJ^1M$ and $\dim \mathcal{C}_j^{(1)} = 3$, $\dim \mathcal{C}_j^{(2)} = 4$.
(4) $\mathcal{C}_j \neq \mathcal{C}_j^{(1)} \neq \mathcal{C}_j^{(2)} = TJ^1M$ and $\dim \mathcal{C}_j^{(1)} = 3$.

For complex distributions, "dim" means a complex dimension.

We say that a Monge–Ampère equation *belongs to the class* $H_{k,l}$ $(k, l = 0, \ldots, 4; k \leq l)$ if the case (k) holds for one of $\mathcal{C}_j$ and the case (l) holds for the other. The classes $H_{k,l}$ are invariant under contact transformations.

2.2 Geometric Structures on T^*M

Consider symplectic equations (see page 228).

The cotangent bundle T^*M is a four-dimensional symplectic manifold with a structure differential 2-form Ω. A differential 2-form $\omega \in \Omega^2(T^*M)$ is effective if and only if $\omega \wedge \Omega = 0$.

Let E_ω be a symplectic equation, i.e., $\omega \in \Omega^2_\varepsilon(T^*M)$.

We can define the Pfaffian $\mathrm{Pf}(\omega)$ of the form ω and the operator A_ω by means of the same formulas (4) and (5). In this case, in contrast to a "nonholonomic" field on J^1M (see Sect. 2.1), A_ω is a field of endomorphisms on T^*M.

The operator A_ω inherits properties (6) and (7) of the "nonholonomic" field on J^1M.

A nondegenerate equation also defines an almost product structure $\mathcal{V}_+ \oplus \mathcal{V}_-$ on T^*M, where the distributions $\mathcal{V}_+$ and $\mathcal{V}_-$ are generated by the eigensubspaces of the operator A_ω. At each point the tangent space $T_a(T^*M)$ (or its complexification) splits into the direct sum

$$\mathcal{V}_+(a) \oplus \mathcal{V}_-(a) = T_a(T^*M) \quad (\text{or} \quad T_a^{\mathbb{C}}(T^*M)) \tag{9}$$

The distributions $\mathcal{V}_+$ and $\mathcal{V}_-$ are called *characteristic*.

3 Tensor Invariants of Equations

Tensor invariants of structures of a n-tuple almost product structures on smooth manifolds were constructed in [27].

Let us construct a decomposition of the de Rham complex, which is generated by an equation.

Decomposition (8) generates a decomposition of the module of exterior s-forms (or its complexification in the elliptic case). Denote the distributions $\mathcal{C}_+$, l, and $\mathcal{C}_-$ by P_1, P_2, and P_3, respectively.

Let $D(J^1M)$ be the module of vector fields on J^1M, and let D_j be the module of vector fields from the distribution P_j. Define the following submodules of $\Omega^s(J^1M)$:

$$\Omega_i^s = \{\alpha \in \Omega^s(J^1M) |\ X \rfloor \alpha = 0\ \forall\ X \in D_j,\ \ j \neq i\} \quad (i = 1, 2, 3).$$

We get the following decomposition of the module of differential s-forms on J^1M:

$$\Omega^s(J^1M) = \bigoplus_{|\mathbf{k}|=s} \Omega^{\mathbf{k}}, \tag{10}$$

where $\mathbf{k} = (k_1, k_2, k_3)$ is a multi-index, $k_i \in \{0, 1, \ldots, \dim P_i\}$, $|\mathbf{k}| = k_1 + k_2 + k_3$,

$$\Omega^{\mathbf{k}} = \left\{ \sum_{j_1+j_2+j_3=|\mathbf{k}|} \alpha_{j_1} \wedge \alpha_{j_2} \wedge \alpha_{j_3}, \text{ where } \alpha_{j_i} \in \Omega_i^{k_i} \right\} \subset \bigotimes_{i=1}^{3} \Omega_i^{k_i}.$$

The exterior differential also splits into the direct sum

$$d = \bigoplus_{|\mathbf{t}|=1} d_{\mathbf{t}},$$

where $d_{\mathbf{t}} : \Omega^{\mathbf{k}} \to \Omega^{\mathbf{k}+\mathbf{t}}$. For example,

$$\begin{aligned} \Omega^1(J^1M) =& \Omega^{100} \oplus \Omega^{010} \oplus \Omega^{001}, \\ \Omega^2(J^1M) =& \Omega^{200} \oplus \Omega^{110} \oplus \Omega^{101} \oplus \Omega^{011} \oplus \Omega^{002}. \end{aligned}$$

The first three terms of the decomposition are presented in the diagram below (see Fig. 1.).

If one of the components t_i of a multi-index $\mathbf{t}$ is negative, then the operator $d_{\mathbf{t}}$ is a $C^\infty(J^1M)$-homomorphism [27].

We get four such nontrivial homomorphisms:

$$d_{-1,1,1}, d_{1,1,-1}, d_{2,0-1}, \text{ and } d_{-1,0,2}.$$

Let $\mathbf{1}_i = (0, 1_i, 0)$ (1 is in the ith position only) be a multi-index ($i = 1, 2, 3$). The homomorphism $d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}$ ($s, j, k = 1, 2, 3$; $s \neq j, k$) can be viewed as a map

$$d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s} : D_j \times D_k \to D_s.$$

Therefore,

$$\begin{aligned} d_{2,-1,0} :& \quad D(\mathcal{C}_+) \wedge D(\mathcal{C}_+) \to D(l), \\ d_{0,-1,2} :& \quad D(\mathcal{C}_-) \wedge D(\mathcal{C}_-) \to D(l), \\ d_{-1,1,1} :& \quad D(\mathcal{C}_-) \wedge D(l) \to D(\mathcal{C}_+), \\ d_{1,1,-1} :& \quad D(\mathcal{C}_+) \wedge D(l) \to D(\mathcal{C}_-). \end{aligned}$$

Construct tensor invariants of the equation

$$\tau_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}(X, Y) = d_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}\left(\mathbf{P}_j X, \mathbf{P}_k Y\right), \tag{11}$$

where $\mathbf{P}_j : D(J^1M) \to D_j$ is the projector to the distribution P_j ($j = 1, 2, 3$).

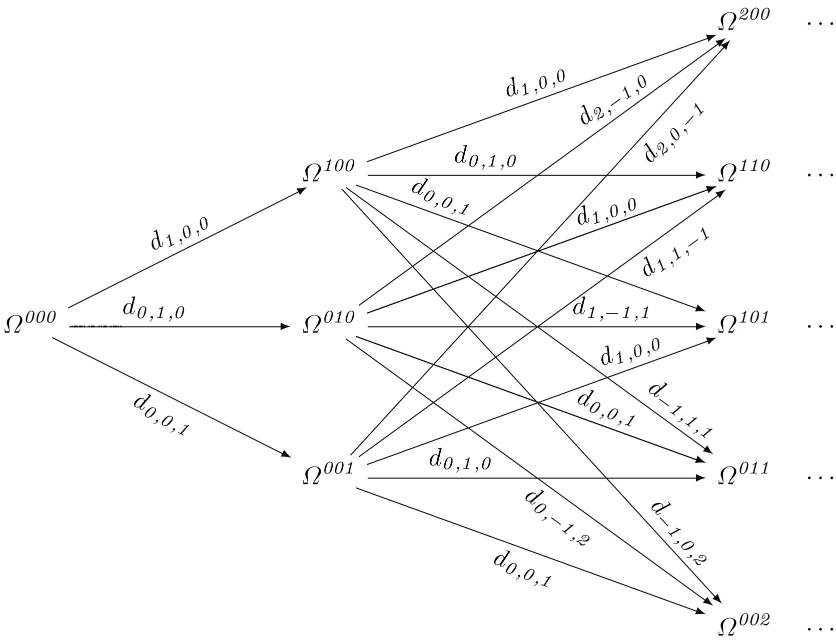

Fig. 1 Decomposition of the de Rham complex on J^1M

So, we get four tensors of (2,1)-type:

$$\tau_{2,-1,0}, \quad \tau_{0,-1,2}, \quad \tau_{-1,1,1} \quad \text{and} \quad \tau_{1,1,-1}. \tag{12}$$

The first two of them are the curvatures of the distributions $\mathcal{C}_+$ and $\mathcal{C}_-$, respectively. Note that

$$\tau_{\mathbf{1}_j+\mathbf{1}_k-\mathbf{1}_s}(X, Y) = -\mathbf{P}_s\left[\mathbf{P}_j X, \mathbf{P}_k Y\right],$$

where $j, k, s = 1, 2, 3;\ s \neq j, k$. Therefore

$$\tau_{2,-1,0} \neq 0, \quad \tau_{0,-1,2} \neq 0, \tag{13}$$

and

$$\dim \mathcal{C}_+^{(2)} = 3 + \dim \operatorname{Im} \tau_{1,1,-1}, \qquad \dim \mathcal{C}_-^{(2)} = 3 + \dim \operatorname{Im} \tau_{-1,1,1}. \tag{14}$$

Example 2. Find a coordinate representation of tensor invariants for the hyperbolic Monge–Ampère equations of the class $H_{k,t}$ $(1 \leq k \leq t \leq 2)$. Such equations are locally contact equivalent to the equations of the following type:

$$v_{xy} = f\left(x, y, v, v_x, v_y\right), \tag{15}$$

where f is some smooth function.

For this equation the invariants of tensors (12) have the form:

$$\tau_{-1,1,1} = \big(ff_{p_2p_2}dq_1 \wedge du - f_{p_2p_2}dp_2 \wedge du - p_1 f_{p_2p_2}dq_1 \wedge dp_2 - p_2 f_{p_2p_2}dq_2 \wedge dp_2 + \big(f_u - p_2 f_{p_2u} + f_{p_1} f_{p_2} - ff_{p_1p_2} - f_{q_2p_2}\big)\, dq_2 \wedge du + \big(p_1 f_u - p_1 p_2 f_{p_2u} - p_2 ff_{p_2p_2} + p_1 f_{p_1} f_{p_2} - p_1 ff_{p_1p_2} - p_1 f_{q_2p_2}\big)\, dq_1 \wedge dq_2\big) \otimes \frac{\partial}{\partial p_1},$$

$$\tau_{1,1,-1} = \big(ff_{p_1p_1}dq_2 \wedge du - f_{p_1p_1}dp_1 \wedge du - p_1 f_{p_1p_1}dq_1 \wedge dp_1 - p_2 f_{p_1p_1}dq_2 \wedge dp_1 + \big(f_u + f_{p_1} f_{p_2} - p_1 f_{p_1u} - ff_{p_1p_2} - f_{q_1p_1}\big)\, dq_1 \wedge du + \big(-p_2 f_u - p_2 f_{p_1} f_{p_2} + p_1 p_2 f_{p_1u} + p_2 ff_{p_1p_2} + p_1 ff_{p_1p_1} + p_2 f_{q_1p_1}\big)\, dq_1 \wedge dq_2\big) \otimes \frac{\partial}{\partial p_2},$$

$$\tau_{2,-1,0} = \big(dq_1 \wedge dp_1 - f_{p_2}dq_1 \wedge du + \big(p_2 f_{p_2} - f\big)\, dq_1 \wedge dq_2\big) \otimes \left(\frac{\partial}{\partial u} + f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}\right),$$

$$\tau_{0,-1,2} = \big(dq_2 \wedge dp_2 - f_{p_1}dq_2 \wedge du - \big(p_1 f_{p_1} - f\big)\, dq_1 \wedge dq_2\big) \otimes \left(\frac{\partial}{\partial u} + f_{p_2}\frac{\partial}{\partial p_1} + f_{p_1}\frac{\partial}{\partial p_2}\right).$$

4 The Laplace Forms

Define two differential 2-forms λ_- and λ_+ from the module Ω^{101} as "wedge contractions" of the tensor fields:

$$\lambda_+ = \big\langle \tau_{0,-1,2}, \tau_{1,1,-1} \big\rangle, \qquad \lambda_- = \big\langle \tau_{2,-1,0}, \tau_{-1,1,1} \big\rangle. \tag{16}$$

Here the bracket $\langle\cdot,\cdot\rangle$ is defined by the formula

$$\langle \alpha \otimes X, \beta \otimes Y \rangle = (Y \rfloor \alpha) \wedge (X \rfloor \beta)$$

for tensors $\alpha \otimes X$ and $\beta \otimes Y$.

We call forms (16) the *Laplace forms* or the *Laplace invariants* of the Monge–Ampère equations E_ω.

Clearly, the Laplace invariants are defined up to the permutation $\lambda_+ \leftrightarrow \lambda_-$.

Remark 3. For the elliptic equations the Laplace forms are complex conjugate.

Example 3. For (15), the Laplace forms have the following coordinate representation:

$$\lambda_- = f_{p_2p_2}\big(f_{p_1}dq_1 \wedge du - dq_1 \wedge dp_2\big) + \big(f_u - p_2 f_{p_2u} + f_{p_1} f_{p_2} - p_2 f_{p_1} f_{p_2p_2} - ff_{p_1p_2} - f_{q_2p_2}\big)\, dq_1 \wedge dq_2, \tag{17}$$

$$\lambda_+ = f_{p_1p_1}\left(f_{p_2}dq_2 \wedge du - dq_2 \wedge dp_1\right) + \left(-f_u + p_1 f_{p_1u} - f_{p_1}f_{p_2} + p_1 f_{p_2} f_{p_1p_1} + f f_{p_1p_2} + f_{q_1p_1}\right) dq_1 \wedge dq_2. \tag{18}$$

In particular, for the linear equation

$$v_{xy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y), \tag{19}$$

the Laplace forms are

$$\lambda_- = k dx \wedge dy \quad \text{and} \quad \lambda_+ = -h dx \wedge dy, \tag{20}$$

where

$$k = ab + c - b_y \qquad h = ab + c - a_x \tag{21}$$

are the classical Laplace invariants [9, 36]. This observation justifies our definition.

We emphasize that classical Laplace invariants (21) of (19) are not absolute invariants even with respect to transformations

$$\phi : (x, y, v) \mapsto (X(x), Y(y), A(x, y)v), \qquad (A(x, y) \neq 0) \tag{22}$$

in contrast to the forms $\lambda_\pm$.

Remark 4. A complete system of differential invariants for (19) with respect to the transformations (22) was constructed by Ibragimov [16]. He proved that all differential invariants of (19) can be obtained from functions (21).

Example 4. Equation

$$\frac{v_{xx}v_{yy} - v_{xy}^2}{(1 + v_x^2 + v_y^2)^2} = K(x, y),$$

describes surfaces $v = v(x, y)$ with the Gauss curvature $K(x, y)$. In the case $K = -1$ the above equation is hyperbolic and its Laplace forms are

$$\lambda_- = \frac{1}{2(1 + p_1^2 + p_2^2)}(dq_1 \wedge dp_2 - dq_2 \wedge dp_1 - p_2 du \wedge dp_1 + p_1 du \wedge dp_2),$$

and $\lambda_+ = -\lambda_-$.

Example 5. For the canonical form of the elliptic Monge–Ampère equations of the class $H_{k,t}$ $(1 \leq k \leq t \leq 2)$

$$v_{xx} + v_{yy} = f\left(x, y, v, v_x, v_y\right), \tag{23}$$

the Laplace forms are

$$\lambda_- = \frac{1}{16}\Big(-2\iota f_{p_1}^2 + 4\iota f_{q_2p_2} - 2\iota f_{p_2}^2 - 2f_{p_1p_2}f_{p_2}p_2 +$$
$$4\iota f_{up_2}p_2 + 4\iota f_{up_1}p_1 + 2\iota f f_{p_1p_1} + 2p_1 f_{p_1p_2}f_{p_1} -$$
$$8\iota f_u + 4\iota f_{q_1p_1} + 2\iota f f_{p_2p_2} - p_1 f_{p_1p_1}f_{p_2} +$$
$$p_2 f_{p_2p_2}f_{p_1} + f_{p_2p_2}f_{p_2}p_1 - f_{p_1p_1}f_{p_1}p_2 +$$
$$4f_{up_2}p_1 - 4f_{up_1}p_2 + 2\iota f_{p_1p_2}f_{p_1}p_2 - \iota f_{p_1p_1}f_{p_2}p_2 +$$
$$\iota p_1 f_{p_1p_1}f_{p_1} + 2\iota p_1 f_{p_1p_2}f_{p_2} + \iota p_2 f_{p_2p_2}f_{p_2} -$$
$$\iota f_{p_2p_2}f_{p_1}p_1 + 4f_{q_1p_2} - 4f_{q_2p_1}\Big)\,dq_1 \wedge dq_2 +$$
$$\frac{1}{16}\big(f_{p_1p_1}f_{p_1} + 2f_{p_1p_2}f_{p_2} - 2\iota f_{p_1p_2}f_{p_1} + \iota f_{p_1p_1}f_{p_2} -$$
$$\iota f_{p_2p_2}f_{p_2} - f_{p_2p_2}f_{p_1}\big)\,dq_1 \wedge du + \frac{1}{8}\big(-f_{p_1p_1} +$$
$$f_{p_2p_2} + 2\iota f_{p_1p_2}\big)\,dq_1 \wedge dp_1.$$

$$\lambda_+ = \frac{1}{16}\big(-2f_{p_1p_2}f_{p_2}p_2 + 2p_1 f_{p_1p_2}f_{p_1} - 4\iota f_{q_1p_1} +$$
$$2\iota f_{p_1}^2 + 8\iota f_u + 2\iota f_{p_2}^2 - 4\iota f_{q_2p_2} -$$
$$p_1 f_{p_1p_1}f_{p_2} + p_2 f_{p_2p_2}f_{p_1} + f_{p_2p_2}f_{p_2}p_1 -$$
$$f_{p_1p_1}f_{p_1}p_2 + 4f_{up_2}p_1 - 4f_{up_1}p_2 - 4\iota f_{up_2}p_2 +$$
$$4f_{q_1p_2} - 4f_{q_2p_1} - 4\iota f_{up_1}p_1 - 2\iota f f_{p_2p_2} - 2\iota f f_{p_1p_1} -$$
$$2\iota f_{p_1p_2}f_{p_1}p_2 + \iota f_{p_1p_1}f_{p_2}p_2 - \iota p_1 f_{p_1p_1}f_{p_1} -$$
$$2\iota p_1 f_{p_1p_2}f_{p_2} - \iota p_2 f_{p_2p_2}f_{p_2} +$$
$$\iota f_{p_2p_2}f_{p_1}p_1)dq_1 \wedge dq_2 + \frac{1}{16}\big(f_{p_1p_1}f_{p_1} + 2f_{p_1p_2}f_{p_2} -$$
$$f_{p_2p_2}f_{p_1} + 2\iota f_{p_1p_2}f_{p_1} - \iota f_{p_1p_1}f_{p_2} +$$
$$\iota f_{p_2p_2}f_{p_2}\big)\,dq_1 \wedge du +$$
$$\frac{1}{8}\big(-f_{p_1p_1} + f_{p_2p_2} - 2\iota f_{p_1p_2}\big)\,dq_1 \wedge dp_1.$$

Example 6 (The Morimoto equation). The Morimoto equation [46]

$$v_{xx}v_{yy} - v_{xy}^2 = \big(v - xv_x - yv_y\big)^4 \tag{24}$$

is elliptic (if $v - xv_x - yv_y \neq 0$) and it is contact equivalent to the following equation [8]:

$$v^4(v_{xx}v_{yy} - v_{xy}^2) = 1.$$

For the last equation one of the Laplace invariants is

$$\lambda_- = -\frac{1}{2u^3}(3\iota(dq_1 \wedge dq_2 + u^3(p_1 dp_2 \wedge du + p_1^2 dq_1 \wedge dp_2 - p_2(dp_1 \wedge du + p_1(dq_1 \wedge dp_1 - dq_2 \wedge dp_2) + p_2 dq_2 \wedge dp_1) + u dp_1 \wedge dp_2))).$$

The another one is complex conjugate: $\lambda_+ = \overline{\lambda_-}$.

Example 7. For the linear elliptic equation

$$v_{xx} + v_{yy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y) \tag{25}$$

the Laplace forms are

$$\lambda_\pm = \frac{1}{4}\left(b_x - a_y \pm \left(\frac{1}{2}(a^2 + b^2) + 2c - a_x - b_y\right)\iota\right) dx \wedge dy. \tag{26}$$

The functional coefficients

$$K = b_x - a_y \qquad \text{and} \qquad H = \frac{1}{2}(a^2 + b^2) + 2c - a_x - b_y \tag{27}$$

of these forms are the Cotton invariants [4].

5 Monge–Ampère Equations with Constant Coefficients

One of the Lie problem is the following:

Problem 1. Find a class of the Monge–Ampère equations that are locally contact equivalent to the Monge–Ampère equations with constant coefficients.

The following theorem of Lychagin and Rubtsov gives the solution of this problem for symplectic equations.

Theorem 1 ([43]). *A nondegenerate symplectic Monge–Ampère equation E_ω is equivalent to some Monge–Ampère equation with constant coefficients if and only if*

$$d\omega = \frac{1}{2}d\left(\ln|\mathrm{Pf}(\omega)|\right) \wedge \omega. \tag{28}$$

If the above condition holds, then the hyperbolic equation is equivalent to the wave equation $v_{xx} - v_{yy} = 0$, and the elliptic one is equivalent to the Laplace equation $v_{xx} + v_{yy} = 0$.

Remark 5. In terms of the operator A_ω ($A_\omega^2 = \pm 1$), condition (28) means that the Nijenhuis bracket of A_ω is zero. In terms of distributions, this means that the distributions $\mathcal{V}_+$ and $\mathcal{V}_-$ are completely integrable.

Tunitskii extends this result to general Monge–Ampère equations:

Theorem 2 ([54]). *A nondegenerate Monge–Ampère equation is contact equivalent to some Monge–Ampère equation with constant coefficients if and only if it belongs to the class* $H_{1,1}$.

6 Contact Linearization of the Monge–Ampère Equations

Consider the following problem for nondegenerate regular equations.

Problem 2. Find a class of the Monge–Ampère equations that are locally contact equivalent to the linear equations

$$v_{xx} \pm v_{yy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y). \tag{29}$$

A solution of the problem can be conveniently formulated in terms of the Laplace forms.

We consider three possible cases.

6.1 $\lambda_+ = \lambda_- = 0$

It is well known that if the classical Laplace invariants h and k of a linear hyperbolic equation are zero, then the equation can be reduced to the wave equation (see, for example, [51]).

Similar statement is true for the Monge–Ampère equations.

Theorem 3 ([30]). *A hyperbolic Monge–Ampère equation is locally contact equivalent to the wave equation* $v_{xy} = 0$ *if and only if its Laplace invariants are zero:* $\lambda_+ = \lambda_- = 0$.

Corollary 1. *The equation*

$$v_{xy} = f\left(x, y, v, v_x, v_y\right)$$

is locally contact equivalent to the wave equation $v_{xy} = 0$ *if and only if the function* f *has the following form:*

$$f = \varphi_y v_x + \varphi_x v_y + (\varphi_v + \Phi_v)v_x v_y + R,$$

where the function $R = R(x, y, v)$ *satisfies the following ordinary linear differential equation:*

$$R_v = (\varphi_v + \Phi_v)R + \varphi_{xy} - \varphi_x\varphi_y.$$

The above equation can be solved:

$$R = e^{\varphi+\Phi}\left(\int(\varphi_{xy}-\varphi_x\varphi_y)e^{-\varphi-\Phi}\,dv + g\right).$$

Here $\varphi = \varphi(x, y, v)$, $\Phi = \Phi(v)$, and $g = g(x, y)$ are arbitrary functions.

Theorem 4 ([30]). *An elliptic Monge–Ampère equation is locally contact equivalent to the Poisson equation $v_{xx} + v_{yy} = f(x, y)$ if and only if its Laplace invariants are zero: $\lambda_+ = \lambda_- = 0$.*

If, in addition, the coefficients of the Monge–Ampère equation are analytic functions, then the equation is locally contact equivalent to the Laplace equation $v_{xx} + v_{yy} = 0$.

6.2 $\lambda_+ \neq 0$ *and* $\lambda_- \neq 0$

Note that for the Laplace invariants of the linear equations (see (20) and (26)) the conditions

$$\lambda_+ \wedge \lambda_+ = 0, \quad \lambda_- \wedge \lambda_- = 0, \quad \lambda_+ \wedge \lambda_- = 0, \quad \text{and} \quad d\lambda_+ = d\lambda_- = 0 \tag{30}$$

hold. Hence, this is also true for the Monge–Ampère equations that are locally contact equivalent to (29).

It follows from the next theorem that conditions (30) are sufficient.

Theorem 5 ([27, 30]). *Suppose $\lambda_+ \neq 0$ and $\lambda_- \neq 0$. A nondegenerate Monge–Ampère equation is locally contact equivalent to (29) if and only if conditions (30) hold.*

Example 8 (The Hunter–Saxton equation). As an example, we consider the Hunter–Saxton equation

$$v_{tx} = v v_{xx} + \kappa u_x^2,$$

where κ is a constant. This equation is hyperbolic, and it has applications in the theory of director fields of liquid crystals [14]. For this equation,

$$\lambda_- = -dq_2 \wedge dp_1,$$
$$\lambda_+ = 2(1-\kappa)\,dq_2 \wedge dp_1.$$

Due to Theorem 5, the equation is linearizable. The corresponding linear equation is the Euler–Poisson equation (see [49])

$$v_{tx} = \frac{1}{\kappa(t+x)}v_t + \frac{2(1-\kappa)}{\kappa(t+x)}v_x - \frac{2(1-\kappa)}{(\kappa(t+x))^2}v.$$

6.3 One of the Laplace Forms Is Zero and the Another One Is Not

Due to Remark 3, this case realizes only for the hyperbolic equations.

For definiteness, suppose that $\lambda_- = 0$ and $\lambda_+ \neq 0$. We shall suppose $\lambda_+ \wedge \lambda_+ = 0$ because this condition holds for the linear equations. This means that $\lambda_+ = \eta_- \wedge \vartheta_+$, where $\eta_- \in \Omega^{001}$ and $\vartheta_+ \in \Omega^{100}$ are differential 1-forms.

Theorem 6 (see [30]). *Suppose that one of the Laplace forms is zero and the another one, say λ_+, is not. A hyperbolic Monge–Ampère equation is locally contact equivalent to a linear equation if and only if $d\lambda_+ = 0$, $\lambda_+ = \eta_- \wedge \vartheta_+$, and the distribution $\mathcal{F}\langle\vartheta_+\rangle$ is completely integrable.*

6.4 Normal Form $v_{xx} \pm v_{yy} = k(x, y)v + f(x, y)$

Sometimes in (29) the terms with v_x and v_y can be eliminated.

Theorem 7 ([34]). *A nondegenerate Monge–Ampère equation from the class $H_{2,2}$ at a point $a \in J^1M$ is locally contact equivalent to the equation*

$$v_{xx} \pm v_{yy} = k(x, y)v + f(x, y) \tag{31}$$

if and only if the conditions:

1. $\lambda_+ \wedge \lambda_- = 0$
2. $d\lambda_+ = d\lambda_- = 0$
3. $\lambda_+ + \lambda_- = 0$

hold.

For the hyperbolic equations the term $f(x, y)$ in (31) can be eliminated. For the elliptic equations with analytic coefficients this term also can be eliminated.

7 Equivalence of Monge–Ampère Equations to Linear Equations with Constant Coefficients

Consider the following problem.

Problem 3. Find a class of nondegenerate Monge–Ampère equations that are locally contact equivalent to the linear equations with constant coefficients.

All such equations belong to one of the classes $H_{1,1}$ or $H_{2,2}$. Due to Theorem 2, all equations of the first class are equivalent to the linear equations with constant coefficients. Therefore here we consider the equations of the class $H_{2,2}$ only.

The equation of the class $H_{2,2}$ of the form

$$v_{xx} \pm v_{yy} = \alpha v_x + \beta v_y + \gamma v + f(x, y) \tag{32}$$

with constant coefficients α, β, and γ is locally contact equivalent to the equation

$$v_{xx} \pm v_{yy} = \kappa v + f(x, y) \tag{33}$$

with a constant $\kappa \neq 0$ (see [9]). Therefore, we consider the problem of equivalence of the Monge–Ampère equations and (33).

The equations

$$v_{xx} - v_{yy} = v \tag{34}$$

and

$$v_{xx} + v_{yy} = \kappa v + f(x, y) \tag{35}$$

are called the *telegraph* equation and the *Helmholtz equation*, respectively.

Theorem 8 ([33, 34]). *A nondegenerate Monge–Ampère equation from the class $H_{2,2}$ is locally equivalent to (33) at a point $a \in J^1M$ if and only if its Laplace forms are*

$$\lambda_+ = \Phi(g, h)dg \wedge dh \qquad \text{and} \qquad \lambda_- = -\Phi(g, h)dg \wedge dh, \tag{36}$$

where g and h are first integrals of the distributions $C_+^{(2)}$ and $C_-^{(2)}$, and the function $\Phi(g, h)$ does not vanish at the point a and satisfies the following differential equation:

$$\Phi\Phi_{gh} - \Phi_g\Phi_h = 0. \tag{37}$$

The functions g and h are defined up to the gauge transformation $g \mapsto \eta(g), h \mapsto \zeta(h)$ and up to the permutation $g \mapsto h, h \mapsto g$. Here η and ζ are arbitrary functions such that $\eta'\zeta' \neq 0$. Equation (37) is invariant under such transformations.

Theorem 8 has the following forms for the linear equations [34, 51]. We give these forms in terms of the classical Laplace and Cotton invariants.

Theorem 9. *A hyperbolic equation of the form*

$$v_{xy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y)$$

is locally equivalent to the telegraph equation $v_{xy} = v$ at a point $a \in M$ if and only if its Laplace invariants (21) coincide in some neighborhood of a, do not vanish at this point and satisfy the following differential equation:

$$kk_{xy} - k_xk_y = 0. \tag{38}$$

Theorem 10. *The linear elliptic equation*

$$v_{xx} + v_{yy} = a(x, y)v_x + b(x, y)v_y + c(x, y)v + g(x, y) \tag{39}$$

is locally equivalent to the Helmholtz equation (35) at a point $a \in M$ if and only if the Cotton invariant $H = 0$ (see (27)) and another Cotton invariant K doesn't vanish at the point a and satisfy the following differential equation:

$$K(K_{xx} + K_{yy}) = K_x^2 + K_y^2. \tag{40}$$

8 Equivalence Problem for Nondegenerate Equations

Here we consider the following problem:

Problem 4. Find conditions under which two nondegenerate equations are locally contact equivalent.

This problem was solved in [27], where the author constructed a "nonholonomic" de Rham complex for an equation.

In the current paper we describe other approach to the problem. We use tensor invariants to construct an absolute parallelism (e-structure) associated with an equation.

Let E_ω be a nondegenerate Monge–Ampère equation. Suppose that

$$\lambda_+ \wedge \lambda_+ = 0 \qquad \text{and} \quad \lambda_- \wedge \lambda_- = 0.$$

This means that $\lambda_+ = \eta_- \wedge \vartheta_+$ and $\lambda_- = \eta_+ \wedge \vartheta_-$ for some differential 1-forms $\eta_-, \vartheta_- \in \Omega^{001}$ and $\eta_+, \vartheta_+ \in \Omega^{100}$.

The equation E_ω can be considered as the set of one-dimensional distributions $\mathcal{C}_+ \cap \operatorname{Ker} \vartheta_+$, $\mathcal{C}_+ \cap \operatorname{Ker} \eta_+$, l, $\mathcal{C}_- \cap \operatorname{Ker} \vartheta_-$, and $\mathcal{C}_- \cap \operatorname{Ker} \eta_-$.

Let P be an arbitrary vector field from the distribution $\mathcal{C}_+ \cap \operatorname{Ker} \eta_+$. The condition

$$\tau_{-1,1,1}(Z, \tau_{1,1,-1}(Z, P)) = P$$

defines the vector field $Z \in D(l)$ uniquely up to multiplication by -1.

In order to normalize the Cartan form $\mathcal{U}$, we put

$$\widetilde{\mathcal{U}}(Z) = 1.$$

The contact form $\widetilde{\mathcal{U}}$ and its exterior differential $\widetilde{\Omega} = d\widetilde{\mathcal{U}}$ are defined uniquely up to multiplication by -1.

The vector fields $P_1 \in D(\mathcal{C}_+ \cap \operatorname{Ker} \eta_+)$ and $Q_1 \in D(\mathcal{C}_- \cap \operatorname{Ker} \eta_-)$ are defined uniquely by means of the following equalities:

$$\widetilde{\mathcal{U}}([P_1, Z]) = 1 \qquad \text{and} \qquad \widetilde{\mathcal{U}}([Q_1, Z]) = 1.$$

The vector fields $P_2 \in D(\mathcal{C}_+ \cap \operatorname{Ker} \vartheta_+)$ and $Q_2 \in D(\mathcal{C}_- \cap \operatorname{Ker} \vartheta_-)$ are defined uniquely up to multiplication by -1 by means of the equalities

$$\widetilde{\Omega}(P_1, P_2) = 1 \qquad \text{and} \qquad \widetilde{\Omega}(Q_1, Q_2) = 1.$$

So, we get the following e-structure:

$$P_1,\ P_2 \in D(\mathcal{C}_+), \quad Z \in D(l), \quad \text{and} \quad Q_1,\ Q_2 \in D(\mathcal{C}_-).$$

Theorem 11. *Two nondegenerate Monge–Ampère equations are locally contact equivalent if and only if their e-structures are locally equivalent.*

Example 9. For (15) we have:

$$\begin{aligned}
\eta_+ =& dq_1, \\
\vartheta_+ =& (f_u - p_1 f_{p_2} f_{p_1p_1} - f_{q_1p_1} - p_1 f_{up_1} - f f_{p_1p_2} + f_{p_1} f_{p_2}) dq_1 - \\
& f_{p_1p_1}(p_2 f_{p_2} - f) dq_2 + f_{p_2} f_{p_1p_1} du - f_{p_1p_1} dp_1, \\
\eta_- =& dq_2, \\
\vartheta_- =& (f_u - p_2 f_{p_1} f_{p_2p_2} - f_{q_2p_2} - p_2 f_{up_2} - f f_{p_1p_2} + f_{p_1} f_{p_2}) dq_2 + \\
& f_{p_2p_2}(f - p_1 f_{p_1}) dq_1 + f_{p_1} f_{p_2p_2} du - f_{p_2p_2} dp_2.
\end{aligned}$$

The dual basis for $(\eta_+, \vartheta_+, \mathcal{U}, \eta_-, \vartheta_-)$ is the following:

$$\begin{aligned}
P_{\eta_+} =& \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial u} + \frac{1}{f_{p_1p_1}} (f_u - f_{q_1p_1} - p_1 f_{up_1} - f f_{p_1p_2} + f_{p_1} f_{p_2}) \frac{\partial}{\partial p_1} + f \frac{\partial}{\partial p_2}, \\
P_{\vartheta_+} =& -\frac{1}{f_{p_1p_1}} \frac{\partial}{\partial p_1}, \\
Q_{\eta_-} =& \frac{\partial}{\partial q_2} + p_2 \frac{\partial}{\partial u} + f \frac{\partial}{\partial p_1} + \frac{1}{f_{p_2p_2}} (f_u - p_2 f_{up_2} + f_{p_1} f_{p_2} - f_{q_2p_2} - f f_{p_1p_2}) \frac{\partial}{\partial p_2}, \\
Q_{\vartheta_-} =& -\frac{1}{f_{p_2p_2}} \frac{\partial}{\partial p_2}.
\end{aligned}$$

Suppose that

$$f_{p_1p_1} f_{p_2p_2} > 0.$$

Then

$$Z = \frac{1}{\sqrt{f_{p_1p_1} f_{p_2p_2}}} \left(\frac{\partial}{\partial u} + f_{p_2} \frac{\partial}{\partial p_1} + f_{p_1} \frac{\partial}{\partial p_2} \right)$$

and

$$\widetilde{\mathcal{U}} = \sqrt{f_{p_1p_1} f_{p_2p_2}} (du - p_1 dq_1 - p_2 dq_2).$$

Therefore we get the following normed vector fields:

$$\begin{aligned}
P_1 =& -\frac{2 f_{p_1p_1} f_{p_2p_2}}{f_{p_2p_2} f_{p_1p_1p_1} + f_{p_1p_1} f_{p_1p_2p_2}} \frac{\partial}{\partial p_1}, \\
P_2 =& \frac{f_{p_2p_2} f_{p_1p_1p_1} + f_{p_1p_1} f_{p_1p_2p_2}}{2 (f_{p_1p_1} f_{p_2p_2})^{3/2}} P_{\eta_+},
\end{aligned}$$

$$Q_1 = - \frac{2 f_{p_1 p_1} f_{p_2 p_2}}{f_{p_2 p_2} f_{p_1 p_1 p_2} + f_{p_1 p_1} f_{p_2 p_2 p_2}} \frac{\partial}{\partial p_2},$$
$$Q_2 = \frac{f_{p_2 p_2} f_{p_1 p_1 p_2} + f_{p_1 p_1} f_{p_2 p_2 p_2}}{2 (f_{p_1 p_1} f_{p_2 p_2})^{3/2}} Q_{\eta_-}.$$

9 Symplectic Equations and Operators

9.1 Tensor Invariants of Nondegenerate Symplectic Equations

Let E_ω be a symplectic Monge–Ampère equation, $\omega \in \Omega^2_\varepsilon(T^*M)$.

Formula (9) generates a decomposition of the de Rham complex (see Fig. 2).

Similar decompositions for Jacobi equations were constructed by Lychagin [42].

The operators $d_{-1,2}$ and $d_{2,-1}$ are $C^\infty(T^*M)$-homomorphisms. These operators generate tensor fields $\tau_{-1,2}$ and $\tau_{2,-1}$ on T^*M:

$$\tau_{-1,2}(X, Y) = d_{-1,2} (\mathbf{P}_- X, \mathbf{P}_- Y), \quad \tau_{2,-1}(X, Y) = d_{2,-1} (\mathbf{P}_+ X, \mathbf{P}_+ Y), \tag{41}$$

where $\mathbf{P}_\pm : D(J^1M) \to D(\mathcal{V}_\pm)$ are the projectors to the distributions $\mathcal{V}_\pm$.

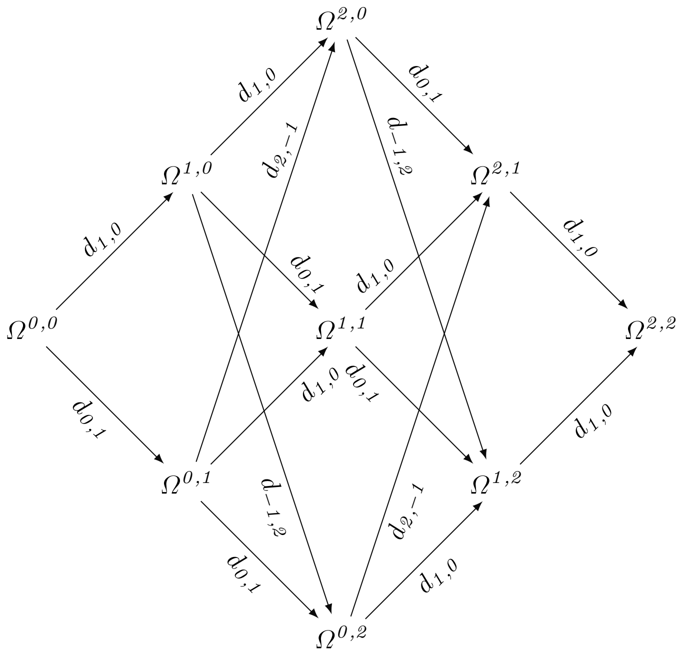

Fig. 2 Decomposition of the de Rham complex on T^*M

The distribution $\mathcal{V}_-$ ($\mathcal{V}_+$) is completely integrable if and only if $\tau_{-1,2} = 0$ ($\tau_{2,-1} = 0$).

Suppose that the tensors $\tau_{-1,2}$ and $\tau_{2,-1}$ do not vanish.

Due to Remark 5, $d\omega \neq 0$. Define the vector fields W_ω on T^*M by the following formula:

$$W_\omega \rfloor \Omega^2 = 2d\omega,$$

where $\Omega^2 = \Omega \wedge \Omega$. Applying the operator A_ω to W_ω, we get the vector field

$$V_\omega = A_\omega W_\omega.$$

In the next sections we use these vector fields to construct an absolute parallelism for a Monge–Ampère equation.

We consider the classification problem for generic symplectic Monge–Ampère equations and operators. Suppose that all the equations and operators are regular.

9.2 Absolute Parallelism for Symplectic Equations

Consider the following problem:

Problem 5. Find conditions under which two symplectic nondegenerate equations are locally symplectic equivalent.

We construct an absolute parallelism (e-structure) for a symplectic Monge–Ampère equation. Our construction does not depend on a type of an equation: it is applicable to the hyperbolic, elliptic, and mixed type equations [26].

Thus, let us consider a Monge–Ampère equation E_ω, where ω is an effective 2-form on T^*M.

Let X_ω be a Hamiltonian vector field with the generating function $F_\omega = \mathrm{Pf}(\omega)$, i.e.,

$$X_\omega \rfloor \Omega = -dF_\omega.$$

The vector field

$$Y^1_\omega = X_\omega + 2V_\omega$$

is a relative invariant of the equation with respect to multiplication ω by arbitrary nonvanishing function h: $Y^1_{h\omega} = h^2 Y^1_\omega$. Applying the operator A_ω to Y^1_ω, we get the vector field

$$Y^2_\omega = A_\omega Y^1_\omega.$$

This vector field is a relative invariant too: $Y^2_{h\omega} = h^3 Y^2_\omega$.

Example 10. For a hyperbolic equation of the form

$$v_{xy} = f(x, y, v_x, v_y)$$

the effective 2-form is

$$\omega = -2f\,dq_1 \wedge dq_2 + dq_1 \wedge dp_1 - dq_2 \wedge dp_2$$

and

$$X_\omega = 0, \quad W_\omega = 2f_{p_2}\frac{\partial}{\partial p_1} - 2f_{p_1}\frac{\partial}{\partial p_2}, \quad V_\omega = 2f_{p_2}\frac{\partial}{\partial p_1} + 2f_{p_1}\frac{\partial}{\partial p_2}.$$

Therefore

$$Y^1_\omega = 4f_{p_2}\frac{\partial}{\partial p_1} + 4f_{p_1}\frac{\partial}{\partial p_2}, \quad Y^2_\omega = 4f_{p_2}\frac{\partial}{\partial p_1} - 4f_{p_1}\frac{\partial}{\partial p_2}.$$

Example 11. For a mixed type equation of the form

$$xv_{yy} + v_{xx} = f(x, y, v_x, v_y)$$

we have:

$$Y^1_\omega = (1 + 2q_1 f_{p_1})\frac{\partial}{\partial p_1} + 2f_{p_2}\frac{\partial}{\partial p_2}, \quad Y^2_\omega = 2q_1 f_{p_2}\frac{\partial}{\partial p_1} - (1 + 2q_1 f_{p_1})\frac{\partial}{\partial p_2}.$$

Let σ^i_ω be the differential 1-form which is conjugate to the vector fields Y^i_ω $(i = 1, 2)$:

$$\sigma^i_\omega = Y^i_\omega \rfloor \Omega.$$

Suppose that the distribution $\operatorname{Ker}\sigma^1_\omega$ is not completely integrable, i.e.,

$$\sigma^1_\omega \wedge d\sigma^1_\omega \neq 0.$$

Construct the vector fields Y^3_ω and Y^4_ω:

$$Y^3_\omega \rfloor \Omega^2 = 2\sigma^1_\omega \wedge d\sigma^1_\omega, \qquad Y^4_\omega = A_\omega Y^3_\omega.$$

They are relative invariants too:

$$Y^3_{h\omega} = h^4 Y^3_\omega \quad \text{and} \quad Y^4_{h\omega} = h^5 Y^4_\omega.$$

Define two functions

$$r_\omega = \Omega(Y^1_\omega, Y^3_\omega) \qquad \text{and} \qquad s_\omega = \Omega(Y^2_\omega, Y^4_\omega).$$

Then

$$\sigma^1_\omega \wedge \sigma^2_\omega \wedge \sigma^3_\omega \wedge \sigma^4_\omega = \frac{1}{2} g_\omega \Omega^2,$$

where

$$g_\omega = F_\omega r_\omega^2 + s_\omega^2, \sigma_\omega^i = Y_\omega^i \rfloor \Omega (i = 3, 4).$$

Suppose that $s_\omega(a) \neq 0$ and $g_\omega(a) \neq 0$ for a point $a \in T^*M$. The functions r_ω, s_ω, and g_ω are relative invariants:

$$r_{h\omega} = h^6 r_\omega, \quad s_{h\omega} = h^7 s_\omega, \quad g_{h\omega} = h^{14} g_\omega.$$

Therefore, the function

$$F = F_\omega s_\omega^{-\frac{2}{7}}$$

is an absolute invariant of the equation.

The vector fields

$$X_1 = \frac{s_\omega^{\frac{2}{7}}}{g_\omega}\left(F_\omega r_\omega Y_\omega^3 + s_\omega Y_\omega^4\right),$$

$$X_2 = -s_\omega^{-\frac{2}{7}} Y_\omega^1,$$

$$X_3 = \frac{s_\omega^{\frac{3}{7}}}{g_\omega}\left(s_\omega Y_\omega^3 - r_\omega Y_\omega^4\right),$$

$$X_4 = -s_\omega^{-\frac{3}{7}} Y_\omega^2$$

are absolute invariants of the equation, and they form an e-structure.

Suppose that the set of differential 1-forms $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ forms the dual basis for the basis of vector fields $X = (X_1, X_2, X_3, X_4)$.

Normalize the form ω:

$$\vartheta = s_\omega^{-1/7}\omega.$$

This form is an absolute invariant of the equation E_ω.

Theorem 12 ([25, 26]). *Suppose that $\omega_a \neq 0$, $s_\omega(a) \neq 0$, and $g_\omega(a) \neq 0$. Then in a neighborhood of the point a we have the following representation of the forms Ω and ϑ:*

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4, \tag{42}$$
$$\vartheta = \theta_3 \wedge \theta_2 + F\theta_4 \wedge \theta_1.$$

Representation (42) allows us to reduce an equivalence problem for equations to an equivalence problem for e-structures.

Let E_1 and E_2 be two equations. Suppose a local diffeomorphism ϕ takes Θ_1 to Θ_2. By (42), ϕ is symplectic.

Note that equivalence of e-structures doesn't sufficient for equivalence of equations E_1 and E_2. Indeed, ϕ should take the function F_1 to F_2.

Let us define the first covariant derivatives $F^{(i)}$ $(i = 1, \ldots, 4)$ of the function F with respect to the local free basis Θ:

$$dF = \sum_{i=1}^{4} F^{(i)}\theta_i.$$

We can define the covariant derivatives $F^{(i_1,\ldots,i_k,i_{k+1})}$ of higher orders by induction:

$$dF^{(i_1,\ldots,i_k)} = \sum_{i=1}^{4} F^{(i_1,\ldots,i_k,i_{k+1})}\theta_i.$$

Theorem 13. *Two symplectic Monge–Ampère equations E_1 and E_2 with analytic coefficients are locally symplecticaly equivalent at a point $a \in J^1M$ if and only if their e-structures are locally equivalent and the corresponding covariant derivatives of the functions F_1 and F_2 coincide at the point a.*

9.3 Absolute Parallelism for Symplectic Operators

Consider the following problem:

Problem 6. Find conditions under which two symplectic Monge–Ampère operators are locally symplectic equivalent.

Hyperbolic Operators

Let Δ_ω be a symplectic hyperbolic Monge–Ampère operator, $\omega \in \Omega^2_\varepsilon(T^*M)$.

Using decomposition (9), we get $W_\omega = W_+ + W_-$, where $W_+ \in D(\mathcal{V}_+)$ and $W_- \in D(\mathcal{V}_-)$.

Since the distributions $\mathcal{V}_+$ and $\mathcal{V}_-$ are skew-orthogonal, the 1-forms

$$\mu_+ = W_+ \rfloor \Omega \quad \text{and} \quad \mu_- = W_- \rfloor \Omega$$

belong to $\Omega^{1,0}$ and $\Omega^{0,1}$ respectively.

The first derivatives of the characteristic distributions are generated by the 1-forms μ_+ and μ_-:

$$\mathcal{V}_+^{(1)} = \operatorname{Ker}\mu_- \quad \text{and} \quad \mathcal{V}_-^{(1)} = \operatorname{Ker}\mu_+.$$

Suppose that these distributions are completely integrable.

The two-dimensional distribution $\mathcal{V}_+^{(1)} \cap \mathcal{V}_-^{(1)}$ is generated by the vector fields W_+ and W_-.

We can define two scalar differential invariants g_+ and g_- of the differential operator by the following formula:

$$[W_+, W_-] = g_+ W_+ + g_- W_-.$$

The 1-forms μ_+ and $W_+ \rfloor d\mu_+$ are linearly dependent:

$$W_+ \rfloor d\mu_+ = g_0 \mu_+$$

for some function g_0. This function is an invariant of the differential operator. Note also that $W_- \rfloor d\mu_- = -g_0 \mu_-$,

$$d_{0,1}\mu_+ = \mu_+ \wedge \gamma_- \quad \text{and} \quad d_{1,0}\mu_- = \mu_- \wedge \gamma_+$$

for some uniquely determined differential 1-forms $\gamma_- \in \Omega^{0,1}$ and $\gamma_+ \in \Omega^{1,0}$. Hence

$$\gamma_-(W_-) = g_+ \quad \text{and} \quad \gamma_+(W_+) = -g_-.$$

Then (see [29])

$$\mu_+ \wedge \gamma_+ \wedge \mu_- \wedge \gamma_- = g_+ g_- \Omega^2.$$

Suppose that

$$g_+ g_-(a) \neq 0$$

at a point $a \in T^*M$. Then the differential 1-forms $\mu_+,\ \mu_-\ \gamma_+$, and γ_- are linearly independent in a neighborhood of the point $a \in T^*M$.

Define two vector fields X_+ and X_-:

$$X_\pm \rfloor \Omega = \gamma_\pm$$

and construct the normed basis of $D(T^*M)$:

$$X_1 = W_+, \quad X_2 = \frac{1}{g_-} X_+,$$
$$X_3 = W_-, \quad X_4 = -\frac{1}{g_+} X_-$$

and the basis of the module $\Omega^1\,(T^*M)$:

$$\theta_1 = -X_2 \rfloor \Omega, \quad \theta_2 = X_1 \rfloor \Omega,$$
$$\theta_3 = -X_4 \rfloor \Omega, \quad \theta_4 = X_3 \rfloor \Omega.$$

The bases $X = (X_1, X_2, X_3, X_4)$ and $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ are dual and define the e-structure of the operator Δ_ω in some neighborhood of the point a.

The symplectic structure Ω and the effective form ω have the following representations:

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4,$$
$$\omega = \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4.$$

Theorem 14 ([29]). *Two symplectic hyperbolic Monge–Ampère operators are locally symplecticaly equivalent if and only if their e-structures are locally equivalent.*

Remark 6. If the distributions $\mathcal{V}_+^{(1)}$ and $\mathcal{V}_-^{(1)}$ are not completely integrable, we also can construct an e-structure for Δ_ω (see [29]).

Remark 7. Similar e-structure for a Monge–Ampère operator was constructed by Kruglikov [19, 20]. To achieve this, he used the Nijenhuis tensor.

Let us construct a coordinate representation of an e-structure.

If the distributions $\mathcal{V}_+^{(1)}$ and $\mathcal{V}_-^{(1)}$ are completely integrable, then the Monge–Ampère operator is symplecticaly equivalent to the operator

$$\Delta_\omega = (v_{xy} - f(x, y, v_x, v_y))dx \wedge dy$$

with the effective form

$$\omega = -2f dq_1 \wedge dq_2 + dq_1 \wedge dp_1 - dq_2 \wedge dp_2,$$

where $f = f(q, p)$ [35].

The characteristic distributions are

$$\mathcal{V}_+ = \left\langle \frac{\partial}{\partial q_1} + f\frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_1} \right\rangle \quad \text{and} \quad \mathcal{V}_- = \left\langle \frac{\partial}{\partial q_2} + f\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2} \right\rangle,$$

The coordinate representations of the tensors $d_{2,-1}$ and $d_{-1,2}$ are

$$d_{2,-1} = f_{p_1}(dq_1 \wedge dp_1 - f dq_1 \wedge dq_2) \otimes \frac{\partial}{\partial p_2},$$
$$d_{-1,2} = f_{p_2}(dq_2 \wedge dp_2 + f dq_1 \wedge dq_2) \otimes \frac{\partial}{\partial p_1}.$$

Then

$$\mu_+ = -2f_{p_2}dq_1 \quad \text{and} \quad \mu_- = 2f_{p_1}dq_2.$$

The functional invariants are

$$g_+ = \frac{2f_{p_1}f_{p_2p_2}}{f_{p_2}}, \quad g_- = \frac{2f_{p_2}f_{p_1p_1}}{f_{p_1}}, \quad g_0 = 2f_{p_1p_2}.$$

Then we have the following e-structure:

$$
\begin{aligned}
X_1 &= 2f_{p_2}\frac{\partial}{\partial p_1},\\
X_2 &= \frac{1}{2f_{p_2}}\frac{\partial}{\partial q_1} - \frac{ff_{p_1p_2}+f_{q_1p_1}}{2f_{p_2}f_{p_1p_1}}\frac{\partial}{\partial p_1} + \frac{f}{2f_{p_2}}\frac{\partial}{\partial p_2},\\
X_3 &= -2f_{p_1}\frac{\partial}{\partial p_2},\\
X_4 &= \frac{1}{2f_{p_1}}\frac{\partial}{\partial q_2} + \frac{f}{2f_{p_1}}\frac{\partial}{\partial p_1} - \frac{ff_{p_1p_2}+f_{q_2p_2}}{2f_{p_1}f_{p_2p_2}}\frac{\partial}{\partial p_2}.
\end{aligned}
$$

Elliptic Operators

For an elliptic operator the characteristic distributions and the vector fields W_+ and W_- are complex conjugate.

Suppose that the first derivatives of the characteristic distributions are completely integrable distributions.

Let

$$Q = [W_+, W_-] = Q_+ + Q_-,$$

where $Q_\pm \in D(\mathcal{V}_\pm)$. Define a real vector field X: if $\operatorname{Re} Q_+ \neq 0$, we put $X = \operatorname{Re} Q_+$ and $X = \operatorname{Im} Q_+$ otherwise. Denote

$$Z = A_\omega X, \quad \eta = V\rfloor\Omega, \quad \xi = X\rfloor\Omega, \quad \tau = Z\rfloor\Omega.$$

Then (see [28])

$$\Omega^2(W, V, X, Z) = 2\left(v^2 + w^2\right),$$

where $W = W_\omega$, $V = V_\omega$, $v = \Omega(X, V)$ and $w = \Omega(X, W)$. Therefore the vector fields W, V, X, Z (and the differential 1-forms ξ, τ, μ, η) are linearly independent if and only if

$$v^2 + w^2 \neq 0.$$

Suppose that the above condition holds. Then the vector fields

$$
\begin{aligned}
X_1 &= -\frac{1}{v^2+w^2}(vV + wW),\\
X_2 &= X,\\
X_3 &= \frac{1}{v^2+w^2}(vW - wV),\\
X_4 &= -Z
\end{aligned}
$$

define an e-structure.

Let $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ be the dual bases for $X = (X_1, X_2, X_3, X_4)$. We get the following representations:

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4,$$
$$\omega = \theta_1 \wedge \theta_4 + \theta_2 \wedge \theta_3.$$

Theorem 15 ([28]). *Two symplectic elliptic Monge–Ampère operators are locally symplecticaly equivalent if and only if their e-structures are locally equivalent.*

Remark 8. The case when the distributions $\mathcal{V}_+^{(1)}$ and $\mathcal{V}_-^{(1)}$ are not completely integrable was considered in [28].

Mixed Type Operators

Let ω be an effective differential 2-form such that at a point $a \in T^*M$ the Pfaffian $\mathrm{Pf}(\omega)$ vanishes. Assume also that $\mathrm{Pf}(\omega) = F^n$, where n is a natural number and F is a smooth function such that its differential does not vanish at the point a: $dF_a \neq 0$.

Then in a neighborhood of the point a the surface $\{F = 0\}$ is a smooth manifold. Let $X = X_F$ be a Hamiltonian vector field with a Hamiltonian F.

Define the vector field

$$Z = A_\omega X. \tag{43}$$

Denote $\tau = Z \rfloor \Omega$, $\mu = W_\omega \rfloor \Omega$, $\eta = V_\omega \rfloor \Omega$, $v = V_\omega(F)$, and $w = W_\omega(F)$.

The table below shows the values of the 1-forms dF, τ, μ, η on the vector fields X, Z, W, V (here $W = W_\omega$, $V = V_\omega$):

	X	Z	W	V
dF	0	0	w	v
τ	0	0	$-v$	$F^n w$
μ	w	v	0	0
η	v	$-F^n w$	0	0

Note that

$$dF \wedge \tau \wedge \eta \wedge \mu = \frac{1}{2} g \Omega^2,$$

where

$$g = F^n w^2 + v^2. \tag{44}$$

Suppose that $v(a) \neq 0$. Then the vector fields

$$X_1 = \frac{1}{g}(F^n w W + v V),$$
$$X_2 = X,$$
$$X_3 = \frac{1}{g}(v W - w V)$$
$$X_4 = Z$$

define an e-structure in some neighborhood of a.

We get the following representation of the forms Ω and ω:

$$\Omega = \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4$$
$$\omega = \theta_3 \wedge \theta_2 + F^n \theta_4 \wedge \theta_1,$$

where $\Theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ is the dual bases for $X = (X_1, X_2, X_3, X_4)$.

Remark 9. If we suppose that the vector fields X_1, X_2, X_3 and X_4 generate a Lie algebra, we get eight normal forms of the Monge–Ampère operators [24].

Theorem 16 ([24]). *Two mixed type hyperbolic Monge–Ampère operators Δ_1 and Δ_2 with analytic coefficients are locally symplecticaly equivalent at a point $a \in J^1M$ if and only if their e-structures are locally equivalent, $n_1 = n_2$, and the corresponding covariant derivatives of the functions F_1 and F_2 coincide at the point a.*

9.4 The Tricomi Operator

Consider the following operator:

$$\Delta = (xv_{yy} + v_{xx} + f(x, y, v_x, v_y))dx \wedge dy. \tag{45}$$

This is a mixed type operator on the line $x = 0$. The corresponding effective differential 2-form is

$$\omega_T = q_1 dq_1 \wedge dp_2 - dq_2 \wedge dp_1 + f dq_1 \wedge dq_2.$$

Let ω be a symplectic effective differential 2-form with Pfaffian $\mathrm{Pf}(\omega) = F$. Suppose that

$$F(a) = 0 \quad \text{and} \quad dF_a \neq 0$$

at a point $a \in T^*M$. Then $\{F = 0\} \subset T^*M$ be a smooth manifold in some neighborhood of a.

Theorem 17 ([23]). *The mixed type Monge–Ampère operator Δ_ω is locally symplectic equivalent to operator (45) at the point a if and only if the vector field $Z = A_\omega X_F$ is a Hamiltonian and the point a is regular for Z, i.e., $L_Z(\Omega) = 0$ and $Z_a \neq 0$.*

Moreover, if the differential 2-form ω is closed, then Δ_ω is locally symplectic equivalent to the Tricomi operator [53]

$$\Delta_T(v) = (xv_{yy} + v_{xx} + f(x, y))dx \wedge dy. \tag{46}$$

Remark 10. For the Tricomi operator Δ_T the function g (see (44)) vanishes at the point a.

Remark 11. The mixed type operators

$$(x^n v_{yy} + v_{xx} + \alpha v_x + \beta v_y + f(x, y))dx \wedge dy$$

and

$$(v_{yy} + x^n v_{xx} + \alpha v_x + \beta v_y + f(x, y))dx \wedge dy,$$

where $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$, were considered in [23].

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On Nonabelian Theories and Abelian Differentials

A. Marshakov

Abstract I discuss integrable systems and their solutions arising in the context of supersymmetric gauge theories and topological string models. For the simplest cases these are particular singular solutions to the dispersionless KdV and Toda systems, and they produce in most straightforward way the generating functions for the Gromov–Witten classes, including well-known intersection and Hurwitz numbers, in terms of the "mirror" target-space rational complex curve. In order to generalize them to the higher genus curves, corresponding in this context to nonabelian gauge theories via the topological gauge/string duality, one has to solve a similar problem, using the Abelian differentials, generally with extra singularities at the branching points.

1 Introduction

Integrable differential equations appear in different aspects in the modern mathematical physics, but in the last years they attracted a lot of attention, due to the study of partition function in the simplest models of quantum gauge and string theories. The partition functions, which can be symbolically written as

$$\log \tau(\mathbf{t}) = \langle \exp \sum_k t_k \sigma_k \rangle_{\text{string}} \tag{1}$$

are initially defined by summing the perturbation series and even the instanton expansions. It turns out, however, that all essential information about this summation is hidden in rather simple differential equations for $\log \tau(\mathbf{t})$ and its derivatives.

In a certain sense the theoretical physicists are lucky: both for the toy-models and "physical" theories one gets the rather well-known from the applied problems

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of mathematical physics integrable systems in $1+1$ and $2+1$ dimensions. Moreover, quite often we are interested even in the solutions with the *finite* number of degrees of freedom – so called moduli of the theory (the finite-dimensional set of the *primary* operators and corresponding time-parameters, sometimes also called as "small phase space"). The geometry of these solutions involve mostly the complex curves and their Jacobians, so that Abel map plays the role of an integrating change of the variables.

Being however closely related to the algebro-geometric solutions of the KP and Toda-type integrable systems, the *string solutions* in a certain sense are absolutely new. In order to demonstrate this the best thing is to start from two well-known examples.

Example 1: The famous KdV equation $u_t + uu_x + u_{xxx} = 0$ is trivially satisfied by $u = \frac{x}{t}$: a linear potential, slowly "falling down" in time. This rather trivial solution corresponds to the pure topological gravity (simplest topological string model) or the Gromov–Witten theory of a point [1–4], with the partition function

$$\log\tau = \mathsf{F}_K(x,t) + \ldots = \frac{x^3}{6t} + \ldots \underset{t\to t+1}{=} \frac{x^3}{3!}\langle\mathbf{111}\rangle + \ldots \\ u = \frac{\partial^2 \log\tau}{\partial x^2} \tag{2}$$

producing the intersection numbers,

$$\langle\sigma_{k_1}\ldots\sigma_{k_n}\rangle = \int_{\overline{M}_{g,n}} \prod_{i=1}^{n} \psi_i^{k_i} \tag{3}$$

defined as integrals over compactified moduli spaces $\overline{M}_{g,n}$ of genus g complex curves with n punctures (the world-sheets for the n-point correlators in string theory), where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of canonical line bundle on $\overline{M}_{g,n}$ with the fiber $T^*_{\Sigma_{g,n}}(P_i)$ at i-th marked point P_i. Literally in (2) the only intersection number $\langle\mathbf{111}\rangle = 1$ is written down, corresponding to the trivial integral over $\overline{M}_{0,3} =$ point. In order to get the rest of the numbers (3), one needs to solve the full KdV hierarchy (to switch on higher flows in rest of the variables $t_1 = x, t_3 = t, t_5, t_7\ldots$ of the hierarchy) for $u = x/t$

$$\log\tau = \sum_{\{k_i\}\geq 0} \frac{t_{2k_1+1}\ldots t_{2k_n+1}}{n!}\langle\sigma_{k_1}\ldots\sigma_{k_n}\rangle\hbar^{2g-2} \tag{4}$$

In physical language one gets by (4) the generating function for the correlators of the *gravitational descendants* $\sigma_k \equiv \sigma_k(\mathbf{1})$, corresponding to the multiplication of the primary operators by k-th powers of the Chern classes. The "target-space" part of the theory of a point is trivial and contains only the unity operator $\mathbf{1}$. For convenience, the string coupling $\hbar$ is introduced in (4), with the weight, fixed by selection rule $\sum k_i = 3g - 3 + n$. For example, the contribution explicitly presented in (2) is weighted by $\hbar^{-2}$, since it comes from $\overline{M}_{0,3}$ with $g = 0$.

This extra parameter is needed, since of special interest is the quasiclassical limit $\hbar \to 0$ of the generating function, which behaves then as

$$\log \tau(\mathbf{t}) \underset{\hbar\to 0}{\sim} \frac{1}{\hbar^2}\mathcal{F}(\mathbf{t}) + O(\hbar^0) \tag{5}$$

with $\mathcal{F}$ often called as prepotential. The quasiclassical part of the expansion (4) is described in terms of dispersionless limit for the KdV equation with the Lax *function* $W = z^2 - u$ (the Lax operator after $\partial/\partial x \to \hbar\partial/\partial x \to z$), or explicitly by the formulas

$$\begin{aligned} x &= \mathrm{res}_\infty W^{-1/2} z dW \sim u \\ \frac{\partial \mathcal{F}}{\partial x} &= \mathrm{res}_\infty W^{1/2} z dW \sim u^2 \end{aligned} \tag{6}$$

so that

$$\mathcal{F} \underset{\hbar\to 0}{=} \hbar^2 \log \tau = \mathsf{F}_K(x) \sim x^3 + \dots \tag{7}$$

Formulas like (6) define the prepotential (7), the quasiclassical part of (4), also if all gravitational descendants are added, as degenerate prepotential of an almost trivial complex manifold – the target-space rational curve $W = z^2 - u$. Generally one has many variables, the residues should be replaced by the period integrals res $\to \oint$ over all nontrivial cycles, and integrability of the equations like (6) is guaranteed by the Riemann bilinear identities.

The full partition function $\tau(\mathbf{t})$ can be also restored [5] as a solution to the Virasoro constraints

$$\begin{gathered} L_n \tau = 0, \quad n \geq -1, \\ L_n = \frac{1}{2}\sum k t_k \frac{\partial}{\partial t_{k+2n}} + \frac{1}{4}\sum_{a+b=2n} \frac{\partial^2}{\partial t_a \partial t_b} + \\ +\delta_{n+1,0}\frac{t_1^2}{4} + \frac{\delta_{n,0}}{16} \end{gathered} \tag{8}$$

an infinite set of *linear* differential equations.

Example 2: Consider now the second simplest example of the Toda chain [6] (in dispersionless limit):

$$\frac{\partial^2 \mathcal{F}}{\partial t_1^2} = \exp \frac{\partial^2 \mathcal{F}}{\partial a^2} \tag{9}$$

The "stringy solution"

$$\mathcal{F} = \tfrac{1}{2} a^2 t_1 + e^{t_1} \tag{10}$$

(again for the prepotential $\mathcal{F} \underset{\hbar\to 0}{=} \hbar^2 \log \tau$) describes system of particles with the co-ordinates $a^D = \partial\mathcal{F}/\partial a = a t_1$ moving with constant velocity = number = a and t_1 is the first time of Toda chain hierarchy. These two parameters (a, t_1) replace here the only "target-space" parameter x of the KdV hierarchy, since the target space primary operators here – instead of the only **1** in the KdV example – correspond to the cohomologies of $\mathbb{P}^1$: $a \leftrightarrow \mathbf{1} \in H^0(\mathbb{P}^1)$ and $t_1 \leftrightarrow \varpi \in H^2(\mathbb{P}^1)$. The truncated generation function $\mathcal{F} \sim \langle \exp(a\mathbf{1} + t_1\varpi)\rangle$ gives rise to the deformation of the multiplication in cohomology ring: $\varpi \cdot \varpi \simeq e^{t_1}\mathbf{1}$, corresponding to the only nontrivial relation in the operator algebra of the target-space primary operators.

To restore the dependence upon the gravitational descendants $t_{k+1} \leftrightarrow \sigma_k(\varpi)$, $T_n \leftrightarrow \sigma_n(\mathbf{1})$, (in these notations $a \equiv -T_0$) one has to solve the Toda chain hierarchy, with the initial condition, corresponding to (10). Quasiclassically, for the prepotential

$$\mathcal{F} = \frac{a^2 t_1}{2} + e^{t_1} \Rightarrow \mathcal{F}(\mathbf{t}, a) \Rightarrow \mathcal{F}(\mathbf{t}, \mathbf{T}) \tag{11}$$

it can be done in two steps (certainly the half-truncated $\mathcal{F}(\mathbf{t}, a)$ and full $\mathcal{F}(\mathbf{t}, \mathbf{T})$ still satisfy the first Toda equation (9)). Solving Toda chain hierarchy in **t**-variables gives rise to a (half-truncated) Gromov–Witten theory of complex projective line $\mathbb{P}^1$

$$\log\tau = \sum_{\{k_i\}, d\geq 0} \frac{t_{k_1}\dots t_{k_n}}{n!}\langle\sigma_{k_1}(\varpi)\dots\sigma_{k_n}(\varpi)\rangle \hbar^{2g-2}q^d \tag{12}$$

where $\sigma_k(\varpi)$ are the descendants of the Kähler class ϖ, (to distinguish from $\sigma_k(\mathbf{1})$) and the correlators are now identified

$$\langle\sigma_{k_1}\dots\sigma_{k_n}\rangle = \int_{\overline{M}_{g,n}(\mathbb{P}^1,d)} \prod_{i=1}^n \psi_i^{k_i}\mathrm{ev}_i^*(\varpi) \tag{13}$$

with the integrals taken over the moduli spaces $\overline{M}_{g,n}(\mathbb{P}^1, d)$ of stable maps of degree d, and $\mathrm{ev}_i : \overline{M}_{g,n}(\mathbb{P}^1, d) \mapsto \mathbb{P}^1$ is evaluation map at the i-th marked point. The extra parameter q in (13), counting degree of the maps, or the number of instantons, can be absorbed by shift of $t_1 \to t_1 - \log q$.

The quasiclassical part of the generating function (12) is again described by a prepotential in a dual picture, sometimes also called as the "Landau–Ginzburg" approach. The Landau–Ginzburg superpotential can be now chosen as a function on cylinder

$$z = v + \Lambda\left(w + \frac{1}{w}\right) \tag{14}$$

which has also an obvious sense of the Lax function of the dispersionless Toda chain (the r.h.s. of (14) represents the three-diagonal Lax matrix of Toda chain in terms of powers of spectral parameter w). Equation (14) can be viewed as a rational curve (a cylinder), embedded in $(z, w) \subset \mathbb{C} \times \mathbb{C}^*$, and can be considered as a particular oversimplified example of the N_c-periodic Toda chain curves

$$\Lambda^{N_c}\left(w + \frac{1}{w}\right) = P_{N_c}(z) = \prod_{i=1}^{N_c}(z - v_i) \tag{15}$$

The topological type-A string theory on $\mathbb{P}^1$ is in this way dual to the $N_c = 1$ oversimplified Abelian $\mathcal{N} = 2$ supersymmetric gauge theory [7]. Solving the dispersionless Toda chain, corresponding to (14), is therefore an intermediate step towards understanding the geometry of the extended nonabelian $\mathcal{N} = 2$ supersymmetric gauge theory with the $U(N_c)$ gauge group, being associated in the Seiberg–Witten context with the families of the curves (15).

2 Topological Solution of Dispersionless Toda

The solution for the half-truncated dispersionless Toda hierarchy for $\mathcal{F}(\mathbf{t}, a)$ at $T_n = \delta_{n,1}$, or switched off gravitational descendants of unity $\{\sigma_k(\mathbf{1})\}$ is given in terms of the rational curve (14) (or the dispersionless Toda Lax operator), endowed with

$$S \underset{z\to\infty}{=} -2z(\log z - 1) + \sum_{k>0} t_k z^k + 2a \log z - \frac{\partial \mathcal{F}}{\partial a} - 2\sum_{k>0} \frac{1}{kz^k} \frac{\partial \mathcal{F}}{\partial t_k} \tag{16}$$

odd under the involution $w \leftrightarrow \frac{1}{w}$, which has the sense of the logarithm $S \sim \log \Psi$ of Ψ-function, solving the auxiliary linear problem. As always for the integrable systems, solution for the dynamical variables themselves comes from reconstructing Ψ or S. In terms of the variable w one globally writes

$$S = -2\left(z \log w + \Lambda(\log \Lambda - 1)\left(w - \frac{1}{w}\right)\right) + \sum_{k>0} t_k \Omega_k(w) + 2a \log w \tag{17}$$

fixed by asymptotic $w \underset{z\to\infty}{\sim} z$ and being odd under $w \leftrightarrow \frac{1}{w}$. Here $\Omega_k = z(w)_+^k - z(w)_-^k$, where $\pm$ stand for the strictly positive and negative parts of the Laurent polynomials (powers of z (14)) in w, e.g., $\Omega_1(w) = \Lambda\left(w - \frac{1}{w}\right)$, $\Omega_2(w) = \Lambda^2\left(w^2 - \frac{1}{w^2}\right) + 2\Lambda v\left(w - \frac{1}{w}\right)$, etc.

Expressions for v, Λ, $\mathcal{F}$ as functions of a and all times $\mathbf{t}$ are found from

$$\left.\frac{dS}{d \log w}\right|_{dz=0} = 0 \tag{18}$$

imposed at the zeroes of dS coinciding with the zeroes of dz, i.e., at the ramification points [8]. These are two algebraic equations, to be solved for the coefficients $v = v(a; \mathbf{t})$ and $\Lambda = \Lambda(a; \mathbf{t})$ of the curve (14).

Small phase space. If, for example, $t_k = 0$ for $k > 1$, it gives

$$v = a, \quad \Lambda^2 = e^{t_1} \tag{19}$$

and from the "regular tail" of the expansion (16) one reads off the prepotential (10) on the small phase space.

One can interpret this as a particular degenerate case of a general definition of prepotential of a complex curve Σ, endowed with two meromorphic differentials with the fixed periods [8], or with the generating "Seiberg–Witten" one-form dS_{SW}. The variables are generally introduced via the period integrals

$$a_i = \frac{1}{4\pi i}\oint_{A_i} dS_{SW} \tag{20}$$

over the chosen half of the cycles from $H_1(\Sigma)$, and the gradients of prepotential $\mathcal{F}$ determined

$$a_i^D = \oint_{B_i} dS_{SW} = \frac{\partial \mathcal{F}}{\partial a_i} \tag{21}$$

by the dual periods. The definition (21) is consistent due to

$$\frac{\partial a_i^D}{\partial a_j} = T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} \tag{22}$$

where the symmetricity of the r.h.s. (or of the second derivatives of the prepotential) is guaranteed by symmetricity of the period matrix of Σ. Equality (22) follows from (21) and is implied by the fact that variation of dS is holomorphic, an analog of the property (18).

The above solution for dispersionless Toda with only nonvanishing a and t_1 is just a degenerate version for this construction, where the smooth curve is replaced by a cylinder. Then

$$\begin{gathered} S = -2\left(z\log w + \Lambda(\log\Lambda - 1)\left(w - \frac{1}{w}\right)\right) + t_1\Lambda\left(w - \frac{1}{w}\right) + 2a\log w = \\ \underset{(19)}{=} -2z\log w + 2\Lambda\left(w - \tfrac{1}{w}\right) + 2a\log w \\ S_{SW} = S + 2z\log w \end{gathered} \tag{23}$$

so that

$$\frac{1}{4\pi i}\oint_{A_i} dS_{SW} = \frac{1}{2\pi i}\oint_A z\frac{dw}{w} = \mathrm{res}_{z=\infty}\, z\frac{dw}{w} = a \tag{24}$$

(dz and $\frac{dw}{w}$ are two meromorphic differentials with the fixed periods), and

$$\frac{\partial \mathcal{F}}{\partial a} \sim \int_B z\frac{dw}{w} \sim [S]_0 = at_1 \tag{25}$$

where the role of the regularized "infinite" B-period is played by the constant part of the function (17).

Higher flows. To add the higher flows, one have to introduce the generalized periods, or just the coefficients of the expansion (16), which can be denoted as

$$t_k = \frac{1}{k}\, \mathrm{res}_{P_+} z^{-k} dS = -\frac{1}{k}\, \mathrm{res}_{P_-} z^{-k} dS, \quad k > 0 \tag{26}$$

and

$$\frac{\partial \mathcal{F}}{\partial t_k} = \frac{1}{2} \mathrm{res}_{P_+} z^k dS = -\frac{1}{2} \mathrm{res}_{P_-} z^k dS, \quad k > 0 \tag{27}$$

where $z(P_+) = z(P_-) = \infty$ and these are two infinities of (14), exchanged by the involution $w \leftrightarrow \frac{1}{w}$. Equations (18) remain the same, but they cannot be solved explicitly in general.

Already adding nonvanishing t_2 their solutions [9]

$$\begin{aligned} v &= a - \frac{1}{2t_2} \mathbf{L}\left(-4t_2^2 e^{t_1+2t_2a}\right) \\ \log \Lambda^2 &= t_1 + 2t_2 a - \mathbf{L}\left(-4t_2^2 e^{t_1+2t_2a}\right) \end{aligned} \tag{28}$$

are expressed in terms of the Lambert function $\mathbf{L}(x)e^{\mathbf{L}(x)} = x$. This is nothing, but the asymptotic of the generation function for the Hurwitz numbers

$$H_{g,d} = \langle \sigma_1(\varpi)^{2g+2d-2} \rangle_{g,d} \tag{29}$$

each of them having a meaning of the number of genus g, d-sheeted covers of $\mathbb{P}^1$, with a fixed general branch divisor of degree $d \cdot \chi(\mathbb{P}^1) - \chi(\Sigma_g) = 2d + 2g - 2$, as follows from the Riemann–Hurwitz formula.

Indeed, one gets from (12), (29)

$$\mathcal{F}(a = 0, t_1, t_2 = \tfrac{1}{2}, 0, \ldots) = \sum_{d>0} \frac{H_{d,0}}{(2d-2)!} e^{dt_1} \tag{30}$$

From our solution (28)

$$\left.\frac{\partial^2 \mathcal{F}}{\partial t_1^2}\right|_{a=0, t_2=1/2} = \Lambda^2 = -\mathbf{L}\left(-e^{t_1}\right) \tag{31}$$

which produces exactly $H_{d,0} = \frac{(2d-2)!}{d!} d^{d-3}$ since the Lambert function has an expansion

$$\mathbf{L}(t) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1} t^n}{n!} = t - t^2 + \frac{3}{2} t^3 - \frac{8}{3} t^4 + \ldots \tag{32}$$

giving rise to the desired result.

3 Nekrasov Partition Function

The tau-function (12) can be in fact defined beyond quasiclassical (corresponding to $g = 0$ Gromov–Witten potential) theory [7, 10]. The definition can be written as sum over partitions: the sets of integers $\mathbf{k} = (k_1 \geq k_2 \geq \ldots \geq k_{\ell_\mathbf{k}} = 0 \geq 0 \ldots)$

$$\tau(a, \mathbf{t}) = \sum_{\mathbf{k}} \frac{\mathbf{m}_\mathbf{k}^2}{(-\hbar^2)^{|\mathbf{k}|}} e^{\frac{1}{\hbar^2} \sum_{k>0} \frac{t_k}{k+1} \mathrm{ch}_{k+1}(a,\mathbf{k},\hbar)} \sim \exp\left(\frac{1}{\hbar^2}\mathcal{F}(a, \mathbf{t}) + \ldots\right) \tag{33}$$

weighted with the squared Plancherel measure

$$\mathbf{m}_\mathbf{k} = \frac{\prod_{1\leq i<j\leq \ell_\mathbf{k}} (k_i - k_j + j - i)}{\prod_{i=1}^{\ell_\mathbf{k}} (\ell_\mathbf{k} + k_i - i)!} \sim \prod_{i<j} \frac{k_i - k_j + j - i}{j - i} \tag{34}$$

and coupled to the Toda times by the Chern polynomials

$$\left(e^{\frac{\hbar u}{2}} - e^{-\frac{\hbar u}{2}}\right) \sum_{i=1}^{\infty} e^{u(a+\hbar(\frac{1}{2}-i+k_i))} = \sum_{l=0}^{\infty} \frac{u^l}{l!} \mathrm{ch}_l(a, \mathbf{k}; \hbar) \tag{35}$$

or

$$\begin{gathered} \mathrm{ch}_0(a, \mathbf{k}) = 1, \quad \mathrm{ch}_1(a, \mathbf{k}) = a, \\ \mathrm{ch}_2(a, \mathbf{k}) = a^2 + 2\hbar^2|\mathbf{k}| \\ \mathrm{ch}_3(a, \mathbf{k}) = a^3 + 6\hbar^2 a|\mathbf{k}| + 3\hbar^3 \textstyle\sum_i k_i(k_i + 1 - 2i) \\ \ldots \end{gathered} \tag{36}$$

The expression in the r.h.s. of the last one has an easily recognizable ingredient (see, e.g., [11])

$$\sum_i k_i(k_i + 1 - 2i) = \sum_i \left((k_i - i + \tfrac{1}{2})^2 - (-i + \tfrac{1}{2})^2\right) \tag{37}$$

from the combinatorics of Hurwitz numbers – the class of a transposition, and it is exactly the element, whose coupling to $t_2 = \frac{1}{2}$ in (30) ensures appearance of the asymptotic of Hurwitz numbers via the Lambert function.

Topological gauge string duality reinterprets the sum over partitions in the expression for the exponentiated full Gromov–Witten potential (33) as summing over all instantons in the deformed four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory [7]. Expression (33) is a particular example of the Nekrasov partition function [10] for the $N_c = 1$ or deformed $U(1)$ gauge theory.

Formula (33) also states, that quasiclassics $\hbar \to 0$ of the Nekrasov partition function coincides with the genus zero Gromov–Witten potential or the Seiberg–Witten prepotential of the extended $U(1)$ theory. This equivalence leads, in particular, to the strange phenomenon – effective actions in four-dimensional supersymmetric gauge theories satisfy *the same* differential equations as partition functions of the correlators in topological strings!

Quasiclassical contribution into (33) is given by solution to extremum problem for the functional

$$\mathcal{F} = \tfrac{1}{2}\int dx f''(x) \sum_{k>0} t_k \frac{x^{k+1}}{k+1} - \tfrac{1}{2}\int_{x_1>x_2} dx_1 dx_2 f''(x_1) f''(x_2) F(x_1 - x_2) + \\ + a^D \left(a - \tfrac{1}{2}\int dx\, x f''(x)\right) + \sigma\left(1 - \tfrac{1}{2}\int dx\, f''(x)\right) \tag{38}$$

whose form is derived from the integral representation of the Chern polynomials

$$\mathrm{ch}_l(a, \mathbf{k}) = \tfrac{1}{2}\int dx\, f''_{\mathbf{k}}(x) x^l \sim \sum_{i=1}^{\infty}\left((a + \hbar(k_i - i + 1))^l - (a + \hbar(k_i - i))^l\right) \tag{39}$$

and the Plancherel measure

$$\mathbf{m}_{\mathbf{k}}^2 \sim \prod_{i,j}\left(k_i - k_j + j - i\right) = \exp\sum_{i,j}\log\left(k_i - k_j + j - i\right) \sim \\ \sim \exp\left(-\tfrac{1}{2\hbar^2}\int_{x_1>x_2} dx_1 dx_2 f''_{\mathbf{k}}(x_1) f''_{\mathbf{k}}(x_2)\gamma(x_1 - x_2; \hbar)\right) \sim \\ \underset{\hbar\to 0}{\sim} \exp\left(-\tfrac{1}{2\hbar^2}\int_{x_1>x_2} dx_1 dx_2 f''_{\mathbf{k}}(x_1) f''_{\mathbf{k}}(x_2) F(x_1 - x_2)\right) \tag{40}$$

via the second derivative of the shape function [12]

$$f''_{\mathbf{k}}(x) \sim 2\sum_{i=1}^{\infty}\left(\delta(x - a - \hbar(k_i - i + 1)) - \delta(x - a - \hbar(k_i - i))\right) \tag{41}$$

for random partitions. In (40) the kernel $\gamma(x;\hbar)$ satisfies the second order difference equation

$$\gamma\,(x + \hbar) + \gamma\,(x - \hbar) - 2\gamma\,(x) = \hbar^2\log x \tag{42}$$

and for $\hbar \to 0$ can be replaced in the main order by

$$\gamma(x;\hbar) \underset{\hbar\to 0}{\Rightarrow} F(x) = \frac{x^2}{2}\left(\log x - \frac{3}{2}\right) \tag{43}$$

or just $F''(x) = \log x$, known as perturbative prepotential for the four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory.

The shape function

$$f_{\mathbf{k}}(x) = |x - a| + \varDelta f_{Y_{\mathbf{k}}}(x) \sim \\ \sim \sum_{i=1}^{\infty}\left(|x - a - \hbar(k_i - i + 1)| - |x - a - \hbar(k_i - i)|\right) \tag{44}$$

literally corresponds to the shape of the Young diagram $Y_{\mathbf{k}}$ of partition $\mathbf{k}$, put into the right angle $|x-a|$ whose vertex is located at $x=a$ of the x-axis. The functional (38) should be computed on the extremal large partition $\mathbf{k}_*$, with the shape function $f_{\mathbf{k}_*}(x) \equiv f(x)$, found as solution for the extremal equation, following from (38). Two last terms in the r.h.s. of (38) reflect added with the Lagrange multipliers constraints for the shape function, following from (44): $f'_{\mathbf{k}}(x^+)-f'_{\mathbf{k}}(x^-)=2$ corresponding to approaching of the right angle by the shape function $f_{\mathbf{k}}(x^\pm)=|x^\pm-a|$ at certain points $x^\pm$, and location of its vertex at $a=\frac{1}{2}\int dx\, xf''_{\mathbf{k}}(x)$.

Extremizing the functional (38), one gets for $S(z)=\frac{d}{dz}\frac{\delta\mathcal{F}}{\delta f''(z)}$, or

$$S(z)=\sum_{k>0} t_k z^k - \int dx f''(x)(z-x)\left(\log(z-x)-1\right)-a^D \tag{45}$$

that its real part

$$\text{Re } S(z)=\tfrac{1}{2}\left(S(z+i0)+S(z-i0)\right)=0, \quad z\in\mathbf{I} \tag{46}$$

vanishes on the cut $\mathbf{I}$, where $\Delta f(x)\neq 0$: $x^-<x<x^+$. The asymptotic of (45) at $z\to\infty$ coincides with (16), and to construct such function, satisfying (46) one takes the double cover $y^2=(z-x^+)(z-x^-)$ of the z-plane or the cylinder (14) with $x^\pm=v\pm 2\Lambda$, and writes literally the odd under exchange of the two z-sheets expression (17), which solves (46).

The extremal shape function is found from the (17) as $f'(x)\sim$ jump $\left(\frac{dS}{dx}\right)$, as follows from the integral representation (45). It has been found in [9], for example, that if one adds nonvanishing $t_2\neq 0$ to the small phase space, the extremal shape function equals

$$f'(x)=\frac{2}{\pi}\left(\arcsin\left(\frac{x-v}{2\Lambda}\right)+2t_2\sqrt{4\Lambda^2-(x-v)^2}\right), \qquad v-2\Lambda\le x\le v+2\Lambda, \tag{47}$$

i.e., the Vershik–Kerov arcsin law is deformed by the Wigner semicircle and "renormalization" $v=a\to v(a;\mathbf{t})$ and $\Lambda=e^{t_1/2}\to\Lambda(a;\mathbf{t})$ of the parameters of the curve, solving (18).

4 The Gromov–Witten Potential of $\mathbb{P}^1$

To restore $\mathbf{T}$-dependence in the partition function $\tau(a,\mathbf{t})\to\tau(a,\mathbf{t},\mathbf{T})$ or to switch on the descendants of unity $\{\sigma_k(\mathbf{1})\}$ for $k>0$ one has to solve the Virasoro constraints

$$L_n(\mathbf{t},\mathbf{T};\partial_{\mathbf{t}},\partial_{\mathbf{T}};\partial^2_{\mathbf{t}})\tau(a,\mathbf{t},\mathbf{T})=0, \quad n\ge -1 \tag{48}$$

with the initial condition $\tau(a,\mathbf{t}) = \tau(a,\mathbf{t},\mathbf{T})|_{T_n=\delta_{n,1}}$, see [13–17]. Quasiclassically, solution to these Virasoro constraints, producing the full genus zero Gromov–Witten potential $\mathcal{F}(a,\mathbf{t},\mathbf{T})$ is described [18] by the following generalization of the formula (16)

$$S(z) \underset{z\to\infty}{=} \sum_{k>0} t_k z^k - 2\sum_{n>0} T_n z^n (\log z - c_n) + \\ +2a\log z - \frac{\partial \mathcal{F}}{\partial a} - 2\sum_{k>0} \frac{1}{kz^k}\frac{\partial \mathcal{F}}{\partial t_k} \tag{49}$$

($c_k = \sum_{i=1}^k \frac{1}{i}$ are harmonic numbers), which defines $\mathcal{F}$ when constructed globally on (14), odd under the involution $w \leftrightarrow \frac{1}{w}$.

To do this, one needs just to substitute again $z^k \to \Omega_k(w) = z(w)_+^k - z(w)_-^k$ and

$$z^k(\log z - c_k) \to H_k(z,w) = z^k \log w + \sum_{j=1}^{k} C_j^{(k)}\Omega_j(w) \tag{50}$$

for the polynomial $\mathbf{t}$-flows and logarithmic $\mathbf{T}$-flows. The coefficients $C_j^{(k)}$ in the r.h.s. of (50) are fixed by asymptotic at $z\to\infty$ (see [18] for details). The $\mathbf{T}$-dependence of $\mathcal{F}(a,\mathbf{t},\mathbf{T})$ is given quasiclassically

$$\left.\frac{\partial \mathcal{F}}{\partial T_n}\right|_{\mathbf{t}} = (-)^n n!\,(S_n)_0 \tag{51}$$

(inspired by K. Saito formula [19]), where

$$\frac{d^n S_n}{dz^n} = S, \quad n \geq 0 \tag{52}$$

or S_n is the n-th primitive (odd under $w \leftrightarrow \frac{1}{w}$). Certainly, the particular $n = 0$ case of the formula (51) coincides with (25), since the variable a corresponds to the primary $\sigma_0(\mathbf{1}) \equiv \mathbf{1}$ operator. For $n = 1$ formula (51) gives rise to

$$\mathcal{F}(t_1,a,T_1) = \frac{a^2 t_1}{2T_1} + T_1^2 \exp\frac{t_1}{T_1} \tag{53}$$

which at $T_1 = 1$ obviously coincides with (10), while at $T_1 \to \infty$ gives

$$\mathcal{F}(t_1,a,T_1) \underset{T_1\to\infty}{\sim} \ldots \mathsf{F}_K(t_1+a,T_1) + \mathsf{F}_K(t_1-a,T_1) + \ldots, \tag{54}$$

i.e., from what we started in (2), (7) – a linear in x solution to the KdV equation $u(x,T_1) \sim \frac{x}{T_1}$.

The $\mathbf{t}$-dependence of the quasiclassical tau-function $\mathcal{F}(\mathbf{t},\mathbf{T})$ is governed by dispersionless Toda hierarchy, while the $\mathbf{T}$-dependence can be encoded in terms of so called extended Toda hierarchy [17, 20], which is a special reduction [16] of the two-dimensional Toda lattice.

5 Nonabelian Theory and Abelian Integrals

Topological gauge/string duality claims in particular, that the truncated Gromov–Witten genus zero potential $\mathcal{F}(a,\mathbf{t})$ coincides with a particular oversimplified $N_c = 1$ case of the (extended) Seiberg–Witten $\mathcal{N} = 2$ supersymmetric gauge theory. Moreover, it turns out that quasiclassical solution to the nonabelian theory with the gauge group $U(N_c)$ can be obtained almost in the same way [9] – extremizing the functional (38), with the "slight modification" of the constraint part of the problem. Instead of a single constraint $\frac{1}{2}\int dx\, xf''(x) = a$ one has now to impose a set of N_c similar constraints

$$\frac{1}{2}\int_{\mathbf{I}_i} dx\, xf''(x) = a_i, \quad i = 1,\dots,N_c \tag{55}$$

for the shape function $f(x)$, corresponding to N_c-tuples of partitions [12], located at $a_1,\dots,a_{N_c}$ correspondingly. These constraints are taken into account, just as in (38), with the Lagrange multipliers $a_1^D,\dots,a_{N_c}^D$.

To solve the extremal equations under the set of new constraints, one now takes the double cover of z-plane with N_c-cuts $\{\mathbf{I}_j : x_j^- < z < x_j^+\}$

$$y^2 = \prod_{i=1}^{N_c}(z-x_i^+)(z-x_i^-) \tag{56}$$

or the hyperelliptic curve of genus N_c-1, and constructs S, odd under the involution $y \leftrightarrow -y$ pure imaginary on the set $\mathbf{I}$ of N_c cuts $\mathbf{I} = \cup_{j=1}^{N_c}\mathbf{I}_j$.

Such nonabelian extended $U(N_c)$ $\mathcal{N} = 2$ supersymmetric gauge theory is solved in terms of the Abelian differentials. The functional equation (46) is now solved by the differential of

$$\Phi(z) = \frac{dS}{dz} = \sum_{k>0} kt_k z^{k-1} - \frac{1}{2}\int dx f''(x)\log(z-x) \tag{57}$$

(remember that the shape function is restored from the jump $f'(x) \sim$ jump $\Phi(x)$). On hyperelliptic curve (56), one writes for $d\Phi$

$$d\Phi = \pm\frac{\phi(z)dz}{y} = \pm\frac{\phi(z)dz}{\sqrt{\prod_{i=1}^{N_c}(z-x_i^+)(z-x_i^-)}} \tag{58}$$

where the numerator $\phi(z)$ is totally fixed by asymptotics and the periods

$$\oint_{A_j} d\Phi = -2\pi i\int_{\mathbf{I}_j} f''(x)dx = -2\pi i\left(f'(x_j^+) - f'(x_j^-)\right) = -4\pi i \tag{59}$$

If all $t_k = 0$, for $k > 1$, $t_1 = \log \Lambda^{N_c}$ (of course still $T_n = \delta_{n,1}$!) at the vicinity of the points $P_\pm$, where $z(P_\pm) = \infty$, one finds

$$\Phi \underset{P \to P_\pm}{=} \mp 2N_c \log z \pm 2N_c \log \Lambda + O(z^{-1}) \tag{60}$$

and there exists a meromorphic function $w = \Lambda^{N_c} \exp(-\Phi)$, satisfying (15). At $N_c = 1$ we come back to the curve (14) (the Lax operator of dispersionless Toda chain).

If however the higher $t_k \neq 0$ are nonvanishing, $\exp(-\Phi)$ has an essential singularity and cannot be described algebraically. Implicitly it is still fixed by asymptotics

$$d\Phi \underset{z \to \infty}{\sim} \sum_{k>1} k(k-1) t_k z^{k-2} + \dots \tag{61}$$

and the period constraints (59), implying in particular

$$\delta(dS) = \delta(\Phi dz) \underset{z \to x_j^\pm}{=} \frac{-\phi(x_j^\pm)\delta x_j^\pm}{\prod_k' \sqrt{(x_j^\pm - x_k^+)(x_j^\pm - x_k^-)}} \frac{dz}{\sqrt{z - x_j^\pm}} + \dots \\ \simeq \text{holomorphic} \tag{62}$$

the general analog of (18).

The A-period constraints together with the asymptotics (61) fix completely the form of the differential $d\Phi$. Vanishing of the B-periods together with the constraint (60) impose N_c constraints for the $2N_c$ ramification points of the curve (56). Additional N_c variables are "eaten" by the Seiberg–Witten periods

$$a_i = \frac{1}{4\pi i} \oint_{A_i} z d\Phi, \quad i = 1, \dots, N_c \tag{63}$$

where the extra dependent period is also included, an alternative option is to fix the residue at infinity a, as in (24). The dual to (63) B-periods define the prepotential

$$a_i^D = \tfrac{1}{2} \oint_{B_i} z d\Phi = \frac{\partial \mathcal{F}}{\partial a_i}, \quad i = 1, \dots, N_c \tag{64}$$

and, as in the $U(1)$ case, the $\mathbf{t}$-derivatives are determined by the residue formulas

$$\frac{\partial \mathcal{F}}{\partial t_k} = -\frac{1}{k+1} \text{res}_{P_+} \left(z^{k+1} d\Phi \right), \quad k > 0 \tag{65}$$

but now on the curve (56). Integrability condition (22) for the gradients (64) is ensured by the symmetricity of the period matrix of (56), or more generally, by the Riemann bilinear identities for Abelian differentials.

If higher $T_n \neq 0$ are nonvanishing, say we consider N descendants of unity to be switched on for $n = 1, \ldots, N$,[1] only the $(N+1)$-th derivative of (49)

$$d\Phi^{(N-1)} = d\left(\frac{d^N S}{dz^N}\right) \tag{66}$$

can be decomposed over the basis of single-valued Abelian differentials. It is totally determined by singularities at $z(P_\pm) = \infty$ and at the branch points $\{x_j\}$, $j = 1, \ldots, 2N_c$ of the curve (56), where it acquires the extra poles. In fact $\Phi', \ldots, \Phi^{(N-1)}$ are regular, if being considered as $2-, \ldots, N-$ differentials on the curve (56).

To construct (66) explicitly one writes

$$d\Phi^{(N-1)} = \frac{\phi(z)dz}{y} + \frac{dz}{y}\sum_{j=1}^{2N_c}\sum_{k=1}^{N-1}\left(\frac{q_j^k}{(z-x_j)^k}\right) \tag{67}$$

fix the periods of $d\Phi^{(N-1)}, d\Phi^{(N-2)}, \ldots, d\Phi', d\Phi$ by $2N_c \cdot N$ constraints, ending up, therefore with

$$(2N+1)N_c - 2N_c \cdot N = N_c \tag{68}$$

variables, to be absorbed by the generalized Seiberg–Witten periods

$$a_j = \frac{1}{4\pi i}\oint_{A_j} \frac{z^N}{N!} d\Phi^{(N-1)}, \quad j = 1, \ldots, N_c \tag{69}$$

and define the prepotential by

$$a_j^D = \tfrac{1}{2}\oint_{B_j} \frac{z^N}{N!} d\Phi^{(N-1)} = \frac{\partial \mathcal{F}}{\partial a_j}, \quad j = 1, \ldots, N_c \tag{70}$$

The generalized Seiberg–Witten form is now the N-tuple Legendre transform of S-function (49) (certainly a multivalued Abelian integral on the curve (56)).

6 Different Functional Formulations

In the perturbative limit $\Lambda \to 0$ the cuts of the curve (56) shrink to the points $z = v_j$, $j = 1, \ldots, N_c$ and the curve becomes rational, possibly parameterized as

$$w_{\text{pert}} = P_{N_c}(z) = \prod_{i=1}^{N_c}(z - v_i) \tag{71}$$

[1] The "minimal" theory has $T_n = \delta_{n,1}$ and $\mathcal{F} = \mathcal{F}(a, \mathbf{t})$; $T_1 = 1$ corresponds to the condensate $\langle \sigma_1(\varpi) \rangle \neq 0$.

This curve is endowed with two polynomials (of arbitrary power): $\mathbf{t}(z) \equiv \sum_{k>0} t_k z^k$ and $T(x) \equiv \sum_{n>0} T_n x^n$. The S-functions is computed explicitly and reads

$$S(z) = \mathbf{t}'(z) - 2\sum_{j=1}^{N_c} \sigma(z; v_j), \tag{72}$$

i.e., is defined in terms of the function

$$\sigma(z; x) = \sum_{k>0} \frac{T^{(k)}(x)}{k!}(z-x)^k(\log(z-x) - c_k) \tag{73}$$

where the sum is finite, if restricted to the N-th class of backgrounds, with only $T_1, \ldots, T_N \neq 0$.

The perturbative prepotential is defined by

$$a_i^D = S(v_i) = \frac{\partial \mathcal{F}_{\text{pert}}}{\partial a_i} \tag{74}$$

and this formula gives rise to the following explicit expression

$$\begin{gathered} \mathcal{F}_{\text{pert}}(a_1, \ldots, a_{N_c}; \mathbf{t}, \mathbf{T}) = \sum_{j=1}^{N_c} F_{UV}(a_j; \mathbf{t}, \mathbf{T}) + \sum_{i \neq j} F(a_i, a_j; \mathbf{T}) \\ a_j = T(v_j), \quad j = 1, \ldots, N_c \end{gathered} \tag{75}$$

with

$$\begin{gathered} F_{UV}(x) \equiv F_{UV}(x; \mathbf{t}, \mathbf{T}) = \int_0^x \mathbf{t}'(\mathsf{x}) dT(\mathsf{x}) \\ \frac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2; \mathbf{T}) = T'(x_1)T'(x_2)\log(x_1 - x_2) \end{gathered} \tag{76}$$

If $T_n = \delta_{n,1}$ one gets from (76)

$$F_{UV}(x)|_{T_n=\delta_{n,1}} = \sum_{k>0} t_k \frac{x^{k+1}}{k+1} \tag{77}$$

which is the ultraviolet prepotential for the switched off descendants of unity, except for the condensate of $\langle \sigma_1(\mathbf{1}) \rangle$, and

$$F(x_1, x_2; \mathbf{T})|_{T_n=\delta_{n,1}} = F(x_1 - x_2) \tag{78}$$

where the r.h.s. depends only upon the difference of the arguments, is $\mathbf{T}$-independent and reproduces the constant kernel from the r.h.s. of (43). Generally, switching on the descendants of unity one induces a reparametrization from $a = v$ to $a = T(v) = \sum_n T_n v^n$ which also results in (76).

The perturbative prepotential (75) defines therefore the general form of the functional $\mathcal{F}(a, \mathbf{t}, \mathbf{T})$ given by

$$\mathcal{F} = -\tfrac{1}{2}\int_{x_1>x_2} dx_1 dx_2 f''(x_1) f''(x_2) F(x_1, x_2; \mathbf{T}) + \int dx f''(x) F_{UV}(x; \mathbf{t}, \mathbf{T}) + \\ + \textstyle\sum_i a_i^D \left(a_i - \tfrac{1}{2}\int dx\, x f''(x)\right) + \sigma\left(1 - \tfrac{1}{2}\int dx\, f''(x)\right) \tag{79}$$

The functional (79) can be treated, except for one point we specially pay attention to below, just in the same way as the functional (38) at $T_n = \delta_{n,1}$. For example, since the kernel (the second formula from (76)) is $\mathbf{t}$-independent, one finds that

$$\frac{\partial \mathcal{F}}{\partial t_k} = \int dx f''(x) \frac{\partial F_{UV}(x; \mathbf{t}, \mathbf{T})}{\partial t_k} \tag{80}$$

are still given by the "regular tail" of the expansion of

$$S(z) = \mathbf{t}'(z) - a^D - \int dx f''(x) \sigma(z; x) \tag{81}$$

However, a problem with the functional (79) is in computing the derivatives $\frac{\partial \mathcal{F}}{\partial T_n}$, since the kernel $F(x_1, x_2; \mathbf{T})$ is $\mathbf{T}$-dependent. Fortunately, there exists an equivalent alternative functional formulation, related to (79) by an integral transform $\int dx \rho(x) g(x) = \int dx f''(x) \hat{D}_{N-1}(x) g(x)$

$$\mathcal{F} = \mathcal{F}_N[\rho] = \frac{1}{2}\int dx \rho(x) \mathbf{t}_N(x) + \frac{(-)^N}{(2N)!}\int_{dx_1 dx_2} \rho(x_1)\rho(x_2) H_{2N}^{(+)}(x_1 - x_2) + \\ + \sum_{n=0}^{N} \sigma_n \left(T_n - \frac{(-)^{n-1}}{2}\int dx \frac{x^{N-n}}{(N-n)!}\rho(x)\right) \tag{82}$$

where $\mathbf{t}_N(x) = \sum_{k>0} t_k \frac{x^{k+N}}{(k+1)\ldots(k+N)}$, the kernel

$$\frac{(-)^{N-1}}{(2N)!} H_{2N}^{(+)}(x) = \frac{1}{(2N)!} x^{2N} (\log x - c_{2N})$$

is $\mathbf{T}$-independent, and

$$\hat{D}_{N-1}(x) = T'(x)\frac{d^{N-1}}{dx^{N-1}} - T''(x)\frac{d^{N-2}}{dx^{N-2}} + \ldots + (-)^{N-1} T^{(N)} \tag{83}$$

is the $(N-1)$-th order differential operator.

A nontrivial consequence of the alternative functional formulation (82) is the "multiple-primitives" Saito formula (51)

$$\frac{\partial \mathcal{F}}{\partial T_n} = \sigma_n = (-)^n n!\, (S_n)_0\,, \quad n = 0, \ldots, N \tag{84}$$

where the right equality expresses the Lagrange multipliers in (82) in terms of the constant parts of

$$S_{N-1}(z) = \mathbf{t}_{N-1}(z) - \frac{(-)^{N-1}}{(2N-1)!}\int dx\rho(x)H^{(+)}_{2N-1}(z-x) + \sum_{n=0}^{N-1}\sigma_n\frac{(-)^n z^{N-n-1}}{(N-n-1)!}$$
$$\vdots$$
$$S(z) = \mathbf{t}'(z) - \tfrac{(-)^{N-1}}{N!}\int dx\rho(x)H^{(+)}_N(z-x) + \sigma_0 \quad (85)$$

a sequence of functions vanishing on the cut.

7 Conclusion

I have tried to demonstrate in these notes, that instead of solving complicated problems of mathematical physics directly one can sometimes solve instead the simple differential equations. The desired solutions for the purposes of topological string and gauge theories are very strange from conventional point of view. Nevertheless, they are still related to geometry of complex curves.

The simplified $N_c = 1$ extended Seiberg–Witten theory is dual to the topological string, or the Gromov–Witten theory of projective line $\mathbb{P}^1$. Such Abelian $N_c = 1$ theory is solved completely in terms of dispersionless extended Toda chain hierarchy. To extend these results to the nonabelian theory one needs, however, to use the technique of constructing quasiclassical tau-functions in terms of Abelian integrals on hyperelliptic curve of arbitrary genus $N_c - 1$. We have demonstrated, how it can be performed explicitly for the Toda chain solution, and presented the implicit formulation for generic case. It is important to point out, that extension of the Seiberg–Witten theory, similar to switching on all gravitational descendants, requires extension of the basis of Abelian differentials, which should contain necessarily the differentials with extra points in the branching points of the curve.

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Geometric Aspects of the Quantization of a Rigid Body

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Abstract In this paper we review our results on the quantization of a rigid body. The fact that the configuration space is not simply connected yields two inequivalent quantizations. One of the quantizations allows us to recover classically double-valued wave functions as single valued sections of a non-trivial complex line bundle. This reopens the problem of a physical interpretation of these wave functions.

1 Introduction

The idea of writing quantum mechanics in a coordinate-free way circulated among physicists and mathematicians as a natural consequence of the general relativity principle. One of the main features of quantum mechanics is that it must contain, according to Dirac's ideas, a correspondence with classical mechanics. Having symplectic mechanics at hand, it was natural to formulate a correspondence principle between classical symplectic mechanics and quantum mechanics that associates a self adjoint operator on a Hilbert space with every quantizable classical observable [13, 21]. This is the heart of what has been called the Geometric Quantization (GQ for short).

The above theory proved to be useful in some physically simple situations, but showed to have a number of drawbacks, discussed in detail in Sect. 2.

The aim of this paper is to discuss some features of a recent geometric approach to quantum theory, the Covariant Quantum Mechanics (CQM for short). The CQM (introduced by Jadczyk and Modugno [10] and further developed in [1, 11, 12, 15–17, 19, 23, 25]) has two distinguished features with respect to GQ: on one hand, it is simpler, because it deals only with quantum particles in a given gravitational and electromagnetic field, so loosing the generality of GQ; on the other hand, it is more complete, because it naturally incorporates time in a covariant way.

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In Sect. 3 we describe the main features of CQM. In particular we will see that its greater conceptual simplicity implies weaker existence conditions than those of GQ [25]. Moreover, the algebra of quantizable observables is naturally selected from the geometric structures of the theory itself [10,15,19]. Finally, it can be proved [11] that the energy operator is characterized as the unique second order covariant operator on the appropriate space. This implies that all possible non-linear modifications of the Schrödinger equation are not invariant with respect to time-dependent changes of coordinates.

In Sect. 4 we will focus on the quantum theory of a rigid body in the framework of CQM. We remark that the main application of this theory is quantum mechanics of molecule [9]. Indeed, molecules also have vibrational motions, but they hold at a much higher energy than rotational and translational motions, and can be dealt with separately.

We show that there are two possible choices for the quantum structure. For one of them wave functions are sections of a trivial bundle, whereas for the other one they are sections of a non-trivial bundle. Accordingly, there are two energy operators, each of which operates on sections of one of the two bundles. We recall that the spectrum of the energy operator represents the physically allowed values of the corresponding quantum observable (see [4], for example). We computed the spectrum of the energy operator in several situations [zero electromagnetic field (free rigid body), magnetic monopole, constant electric field (Stark effect)], obtaining two families of eigenfunctions and eigenvalues. One of them corresponds to the non-trivial bundle, and is parametrized by half-integers.

The above solutions were discovered in the very beginning of the development of quantum mechanics (see, e.g., [3]), but were classically discarded due to "lack of continuity" (see, e.g., [4, 14]). Indeed, using Euler angles as coordinates on the rotational part of the rigid body configuration space, these solutions turn out to be double-valued functions. However this is not fully true, because sections of non-trivial bundles *are* indeed continuous and single-valued.

In a sense, our results show that those sections exist due to a geometric phase effect in quantum mechanics of rigid bodies which is analogous to the Aharonov–Bohm effect. Although experiments seem to show no evidence of these solutions in nature, other reasons than continuity should be found in order to justify the fact that they do not play any role.

2 Geometric Quantization

In this section we will briefly recall the GQ setting.

The classical setting in GQ is based on a symplectic manifold (M, ω) with Hamiltonian H. The manifold M models the classical phase space, and the classical observables are real functions on M.

The quantum setting in GQ is based on a Hilbert space $\mathscr{H}$ of quantum states. The quantization is a linear map

$$\mathscr{Q}\colon \mathscr{O} \subset \mathscr{C}^\infty(M) \to \mathrm{Herm}(\mathscr{H})$$

fulfilling

$$[\mathscr{Q}(f), \mathscr{Q}(g)] = -i\hbar \mathscr{Q}(\{f, g\}), \quad \mathscr{Q}(1) = \mathrm{Id}_{\mathscr{H}}.$$

Here, $\mathrm{Herm}(\mathscr{H})$ is the set of Hermitian operators on $\mathscr{H}$. Note that quantization is always defined on a subspace $\mathscr{O} \subset \mathscr{C}^\infty(M)$. This is due to some physical restrictions. For instance, if $M = T^*P$ and P is the configuration space of a particle, then quantizing all of $\mathscr{C}^\infty(T^*P)$ would imply the possibility of localizing any observable around a point with arbitrary precision, which is forbidden by Heisenberg's uncertainty principle. Another problem is the irreducibility of the representation map $\mathscr{Q}$, which is sometimes broken by the full set of observables $\mathscr{C}^\infty(M)$ (Groenewold–Van Hove's no-go theorem, [7]).

The problem of constructing a quantum theory from the classical setting is solved as follows. First of all we require the existence of *pre-quantization structures*, i.e.:

- A complex Hermitian line bundle $L \to M$, whose sections $\psi\colon M \to L$ are interpreted as wave functions
- A Hermitian connection ∇ on $L \to M$ such that its curvature $R[\nabla]$ fulfills the equation $R[\nabla] = i\frac{1}{\hbar}\omega \otimes \mathrm{Id}_L$

The existence of such structures implies that M and ω have to satisfy certain topological conditions (Kostant–Souriau theorem), namely:

$$\Big[\frac{1}{h}\omega\Big] \in i(H^2(M, \mathbb{Z})) \subset H^2(M, \mathbb{R}),$$

where i is the map induced in cohomology by the inclusion $i\colon \mathbb{Z} \hookrightarrow \mathbb{R}$. If the above condition is fulfilled then:

- $i^{-1}([\omega]) \subset H^2(M, \mathbb{Z})$ parametrizes line bundles.
- $H^1(E, \mathbb{R})\big/H^1(E, \mathbb{Z})$ parametrizes connections which satisfy the above condition on the curvature.

Summarizing, by a well-known theorem of algebraic topology, pre-quantization structures are parametrized by $H^1(M, U(1))$.

The Hilbert space of quantum states is then defined as the L^2-completion of the space of compactly supported wave functions. For $f \in \mathscr{O} \subset \mathscr{C}^\infty(M)$ the Hamiltonian vector field $X_f\colon M \to TM$ is lifted to a ∇-horizontal vector field $\tilde{X}_f\colon L \to TL$. The pre-quantization maps any observable $f \in \mathscr{C}^\infty(M)$ to the operator $\mathscr{Q}(f)$ defined by

$$\mathscr{Q}(f)(\psi) := i\hbar \tilde{X}_f \cdot \psi.$$

It remains to define the subset $\mathscr{O}$. This is usually accomplished by choosing a polarization in M, i.e., a Lagrangian subbundle $P \subset TM$ with further hypotheses (like Frobenius integrability, see [26] for example). Then, the elements of $\mathscr{O}$

are functions which are constant along the leaves of the polarization, and the corresponding Hilbert space is constructed from compactly supported wave functions $\psi : M \to L$ which are covariantly constant along the polarization P:

$$\nabla|_P \psi = 0.$$

The fact that not all symplectic manifolds admit polarizations amounts to imposing stronger topological conditions on M and P. See [20, 26] for more details.

It may happen that $H \notin \mathscr{O}$. In such a case a problem for quantizing the energy arises; this is usually solved by means of the Blattner–Kostant–Sternberg method [20]. This is equivalent to defining a (trivial) bundle of Hilbert spaces $\mathscr{H} \times \mathbb{R} \to \mathbb{R}$ and considering the flow of the Hamiltonian vector field X_H as a time-dependent family of bundle automorphism. The quantization of H is then achieved as the time derivative at $t = 0$ of the above family of operators. It has been recently shown [6] that the flow of X_H can also be interpreted as the parallel transport of a connection on the Hilbert bundle. Hence deriving the flow of X_H produces the covariant derivative associated with the connection.

3 Covariant Quantum Mechanics

In CQM "covariance" is regarded as explicit independence of fundamental laws with respect not only to observers and coordinates but also to units of measurement as well.

The classical framework for one particle of mass $m \neq 0$ and charge q is represented by a fibred manifold $t : E \to T$, where T is a one-dimensional affine space modeling time, and E is an $n+1$-dimensional manifold modeling spacetime. A motion is a section $s : T \to E$. The classical phase space is the first jet space $J^1 E$. An observer is a section $o : E \to J^1 E$. We use local coordinates (x^0) on T, (x^0, x^i) on E and the induced coordinates (x^0, x^i, x^i_0) on $J^1 E$.

We postulate the following geometric structures:

- A spacelike Riemannian metric g on E, i.e., a Riemannian metric on the fibres of spacetime
- A connection Γ on $TE \to E$, representing the gravitational field, which is compatible with the fibring t and the metric g
- A closed two-form F on E, representing the electromagnetic field

Thus, Γ is determined by g only partially, due to the degeneracy of the metric along "horizontal" directions.

The above structures can be naturally encoded into a 2-form Ω on $J^1 E$

$$\Omega \equiv \Omega(g, \Gamma, F) = \Omega(g, \Gamma) + \frac{q}{2m} F, \tag{1}$$

where $\Omega(g, \Gamma)$ is induced by g, Γ and the contact structure of J^1E via an algebraic operation. Its coordinate expression is

$$\Omega(g, \Gamma) = g_{ij}(dx_0^i - (\Gamma^i_{\lambda h} x_0^h + \Gamma^i_{\lambda 0}) dx^\lambda) \wedge (dx_0^i - x_0^i dx^0) \tag{2}$$

(the index λ runs from 0 to n). Conservation laws of classical mechanics require that Ω be closed; indeed, later this property is also a necessary consistency condition for the quantum theory. The closure of Ω turns out to be equivalent to a certain symmetry property of the curvature tensor of Γ. It can be proved that $dt \wedge \Omega \wedge \Omega \wedge \Omega \neq 0$. Thus, (Ω, dt, J^1E) is a *cosymplectic manifold*[1] (see, e.g., [2] for more details).

Note that the cosymplectic form Ω encodes all dynamical structures. This is an important difference between CQM and GQ. In particular, it can be proved (see, e.g., [19]) that Ω admits "horizontal" potentials Θ, i.e., potentials valued in T^*E. Thus, by choosing an observer o, we can write a potential Θ of Ω as

$$\Theta = -H + P = -\Big(\frac{1}{2} m g_{ij} x_0^i x_0^j - A_0\Big) dx^0 + (m g_{ij} x_0^j + A_i) dx^i, \tag{3}$$

where H is the observed Hamiltonian, P is the observed momentum and $A_0 dx^0 + A_i dx^i$ is the observed potential of both the gravitational and the electromagnetic fields.

In this framework we can develop a Hamiltonian stuff including non standard results. In particular, the phase functions $f : J^1E \to \mathbb{R}$ can be lifted to phase vector fields $X_f : J^1E \to TJ^1E$. Even more, these vector fields X_f are projectable to vector fields of spacetime if and only if the phase functions f are second order polynomials in the velocities whose leading coefficients are proportional to g through a real function f^0 of spacetime, i.e., if and only if their coordinate expression is of the type

$$f = f^0 g_{ij} x_0^i x_0^j + f^i g_{ij} x_0^j + \breve{f}, \qquad \text{with} \qquad f^0, f^i, \breve{f} \colon E \to \mathbb{R}. \tag{4}$$

Indeed, these "special quadratic phase functions" constitute a Lie algebra, which is different from the Poisson Lie algebra [12]. This Lie algebra includes energy, momentum and position functions and treats them on the same footing.

Quantum structures are postulated in a way which is partially similar to that of GQ.

The starting assumption of CQM is a *quantum bundle* defined as a complex line bundle $L \to E$. Then, CQM postulates a Hermitian connection ∇ on the pullback L^1 of the quantum bundle over the phase space J^1E, which fulfills two conditions:

- The curvature of ∇ is proportional to Ω according to the equality

$$R[\nabla] = i\frac{m}{\hbar}\Omega \otimes \mathrm{Id}_{L^1}.$$

[1] This definition is due to A. Lichnerowicz.

- The covariant differential of a quantum section ψ is "horizontal", i.e., valued in T^*E. This property of ∇ is allowed by the property of Ω to admit horizontal potentials. Indeed, the quantum connection C can be regarded as a *distinguished* family of Hermitian connections of the quantum bundle parametrized by the observers $o : E \to J^1E$.

Thus, there are two main differences of the postulates of CQM with respect to GQ. In CQM the line bundle is assumed to be based on spacetime E and not on the phase space J^1E. On the other hand, CQM needs to assume the quantum connection ∇ on the bundle L^1 in order to link ∇ with Ω, which lives on the phase space J_1E. Clearly, in CQM the topological conditions of Konstant–Souriau's theorem have to be fulfilled on E.

In CQM all further geometric quantum structures and the quantum dynamics are derived from the quantum connection by means of a covariant procedure. The requirement of covariance leads us to a method of projectability in order to get rid of observers (which are encoded in the quantum connection); in a sense, this method replaces successfully the search for polarizations of GQ.

The Schrödinger operator S can be derived from the quantum connection ∇ by several geometric methods implementing the criterion of projectability and even more it is uniquely determined by the requirement of covariance [11]. In coordinates we obtain

$$\begin{aligned} S(\psi) = \Big(\frac{\partial}{\partial x^0} - iA_0 + \frac{1}{2\sqrt{|g|}}\frac{\partial\sqrt{|g|}}{\partial x^0} - i\frac{k}{2}r \\ - i\frac{\hbar}{2m}g^{hk}\big((\frac{\partial}{\partial x^h} - iA_h)(\frac{\partial}{\partial x^k} - iA_k) + \Gamma^l_{hk}(\frac{\partial}{\partial x^l} - iA_l)\big)\Big)\psi, \end{aligned} \tag{5}$$

where r is the scalar curvature of Γ and k is a constant. We stress that if we release the hypothesis of invariance with respect to units of measurement (for instance, by assuming a distinguished length), then further terms are allowed in the expression of the Schrödinger operator; for instance terms proportional to $|\psi|^2$ may appear, so yielding well known non-linear generalizations of the Schrödinger operator.

Also the quantizable observables and the corresponding quantum operators can be achieved by means of the projectability criterion. One starts by classifying the projectable Hermitian vector fields of the quantum bundle. It can be proved that these vector fields constitute a Lie algebra which is naturally isomorphic to the Lie algebra of special quadratic phase functions, according to the formula [12]

$$f \mapsto \tilde{X}_f = f^0\,\partial_0 - f^i\,\partial_i + i(f^0\,A_0 - f^i\,A_i + \check{f}) \otimes \mathrm{Id}_L$$

Then, we obtain an injective Lie algebra morphism between the Lie algebra of special quadratic phase functions and the Lie algebra of operators acting on the quantum sections, according to the equality

$$\mathcal{Q}(f)(\psi) := i\hbar\tilde{X}_f \cdot \psi.$$

Indeed, the above results can be applied to energy, momentum and position functions on the same footing. We stress that they are obtained with no further topological conditions on E and that they naturally include the so-called metaplectic correction [20, 26].

Next, the Hilbert bundle $\mathscr{H} \to T$ over time is defined as the L^2-completion of the space of quantum sections $\psi: E \to L$ with spacelike compact support. Each section $\hat{\psi} : T \to \mathscr{H}$ can be regarded as a section $\psi : E \to L$ of the quantum bundle. The quantum states are described by the sections $\hat{\psi}$ of the Hilbert bundle. Moreover, the Schrödinger operator can be regarded as a connection of this infinite dimensional bundle.

Eventually, we can associate a symmetric operator $\hat{f}$ acting on the sections $\hat{\psi}$ of the Hilbert bundle with each special quadratic phase function f by means of the equality

$$\hat{f}(\psi) = (\mathscr{Q}(f) - i f^0 S)(\psi)\,.$$

This is the quantization procedure of CQM, which deals with all quantizable functions (including energy) on the same footing.

4 Rigid Body

Following [5, 17, 18], we treat the classical mechanics of a system of n particles by representing this system as a single particle moving in a higher dimensional spacetime which fulfill the same properties postulated for the standard spacetime. Then we define the rigidity constraint and study its main properties. For this purpose we postulate a flat spacetime.

More precisely, we require E to be an affine four-dimensional space, $t : E \to T$ to be an affine surjective map and g to be a Euclidean metric on $S = \operatorname{Ker} Dt$. Note that VE is naturally isomorphic to $E \times S$. We choose $\varGamma$ to be the natural flat connection on E, and we can consider different examples of electromagnetic field F on E.

The configuration space for a system of n particles is then

$$E_n = E \underset{T}{\times} \cdots \underset{T}{\times} E \to T.$$

This is endowed with the natural flat connection $\varGamma_n$ induced by $\varGamma$ and by the product electromagnetic field $F \times \cdots \times F$ (n times). Analogously, we introduce the vector space $S_n = S \times \cdots \times S$. If the n particles have masses $m_1, \ldots, m_n$, then we define the metric on S_n, or inertia tensor, as

$$I = \mu_1 g + \cdots + \mu_n g,$$

where $\mu_i = m_i/m$ and $m = \varSigma_i m_i$. The above data fulfill the classical axioms of CQM, hence produce a cosymplectic form $\varOmega_n$ which turns out to be exact, due to the topological triviality of E.

The constraint of rigidity is then defined by

$$R = \{(e_1, \ldots, e_n) \in E_n;\ \|e_i - e_j\| = l_{ij}, i \neq j\},$$

where l_{ij} are positive numbers fulfilling $l_{ij} = l_{ji}$ and $l_{ij} \leq l_{ik} + l_{kj}$.

It can be proved [5, 17, 18] that R is diffeomorphic either to $E \times O(3)$, $E \times SO(3)$ or $E \times S^2$. In all three cases E is the space of center of mass configurations, and the second factor is the space of relative configurations. Intuitively, relative configurations can be thought of as if particles either "fill" the space, lie in a plane, or are aligned. From now on we only consider the case where R is diffeomorphic either to $E \times O(3)$ or to $E \times SO(3)$. Moreover, the former case can be reduced to the latter because any of the two connected components of $E \times O(3)$ is diffeomorphic to $E \times SO(3)$, and motions starting in one of the two connected components remain there forever.

The natural inclusion $R \hookrightarrow E_n$ allows us to define, by pullback, a connection Γ_r and an "electromagnetic field" F_r on R. It can be proved that these constrained data fulfill the classical axioms of CQM. Moreover, the induced form Ω_r turns out to be exact, hence the quantum structure postulated by CQM exists.

Both the configuration space E_n and the rigidity constraint fulfill the same axioms as the classical one-particle theory. For this reason the CQM machinery can be applied, and a quantum theory for the rigid body can be formulated.

Let us compute all possible inequivalent quantum structures on R. Observe that $H^1(SO(3), U(1)) = \mathbb{Z}_2$. Then we have the following theorem [24]; see also [17,22].

Theorem 1. *There are two inequivalent quantum structures:*

$$L^+ = R \times \mathbb{C} \to R, \qquad L^- \not\simeq R \times \mathbb{C} \to R$$

Both L^+ and L^- admit a unique flat Hermitian connection, that can be naturally deformed with dynamical terms in order to obtain the quantum connections ∇^+ and ∇^-:

It is interesting to observe that the above line bundles (as well as their flat connections) are obtained as vector bundles which are associated with the $\mathbb{Z}_2$-principal bundle $SU(2) \to SO(3)$ by means of the two representations of $\mathbb{Z}_2$ into $\mathbb{C}$.

We stress that the above two quantum structures give rise to two different energy operators with two different spectra. We computed spectra in several examples in [17,22]; here we will only sketch some results in simple cases.

It is worth to remark that we only compute *rotational* spectra. This means that we only consider rotations of a rigid body around its center of mass, dropping the center-of-mass component of the energy operator. This idea is physically justified by remembering that the most important application of our model is to the study of quantum dynamics of molecule. In that case rotational and translational phenomena are located on very different energy sectors, and the translational spectrum of molecule yields a negligible continuum infrared component [9]. In mathematical terms, we will only compute spectra on the subspace of sections of L^+, L^- which are constant on the center of mass space.

Moreover, we distinguish between three types of rigid body. In fact $SO(3)$ is a Lie group endowed with the left-invariant metric I and the standard bi-invariant Killing metric k. Hence, I can be diagonalized with respect to k. The rigid body is said to be *spherical* if all the three eigenvalues are equal, *symmetric*, or a *top*, if two eigenvalues are equal, *asymmetric* if all the eigenvalues are different. All cases exist in molecular dynamics, e.g., CH_4 is a spherical molecule, NH_3 is a symmetric molecule, ...

We have the energy operators $\hat{H}^+$, $\hat{H}^-$ acting respectively on sections of $L^+ \to R$ and $L^- \to R$:

$$\hat{H}^+(\psi^+) = \frac{1}{2}(\Delta^+ + A_0 + kr)(\psi^+), \quad \hat{H}^-(\psi^-) = \frac{1}{2}(\Delta^- + A_0 + kr)(\psi^-), \quad (6)$$

where $\Delta^\pm$ is Bochner Laplacian of $\nabla^\pm$.

Theorem 2 ([22]; see also [17]). *In the free (i.e., $F = 0$) spherical case the spectrum of $S^\pm$ is the set*

$$E_j^\pm = \frac{\hbar^2}{2I} j(j+1) + k\frac{3\hbar^2}{4I}$$

where:

- E_j^+ *is parametrized by* $j \in \mathbb{Z}$.
- E_j^- *is parametrized by* $j + 1/2 \in \mathbb{Z}$.

(in other words, j is half integer in the latter case).

Note that $SO(3)$ has constant scalar curvature. This implies that scalar curvature contributes to the spectrum through an overall shift.

Now, choose a splitting $E \simeq T \times P$, where P is a three-dimensional affine space, and let $o \in P$. A magnetic monopole field is a closed 2-form B on P which is invariant with respect to rotations about o. This means that B is proportional to the volume form on the unit sphere with scaling factor given by the magnetic charge . A magnetic monopole B induces a left-invariant 2-form B on $SO(3)$. Let $q = \sum_i q_i r_i / \|r_i\|$ be the center of charge of the rigid body.

Theorem 3 ([22]). *In the spherical case, if $F = B$, then the spectrum of $S^\pm$ is the set*

$$E_{j,l}^\pm = \frac{\hbar^2}{2I} j(j+1) - \hbar\nu\frac{\|q\|}{I} l + \nu^2\frac{\|q\|^2}{2I} + k\frac{3\hbar^2}{4I}$$

where ν is the magnetic charge of the monopole and:

- $E_{j,l}^+$ *is parametrized by* $j, l \in \mathbb{Z}$, $-j \leq l \leq j$.
- $E_{j,l}^-$ *is parametrized by* $j + 1/2, l + 1/2 \in \mathbb{Z}$, $-j \leq l \leq j$.

Note that the existence conditions of quantum structures imply that the magnetic charge ν is quantized.

Other examples of spectral computations have been considered so far:

- Energy spectra for the top and the asymmetric rigid body have been computed in [22] both with $F = 0$ and with a magnetic monopole field. The case of a linear rotor, i.e., $R \simeq E \times S^2$, has also been computed (just as an example, CO_2 is a linear molecule).
- We have considered the Stark effect in [17]. Assume a constant (spacelike) electric field **E**. The component of F along $SO(3)$ has potential $A_0 = \frac{1}{m}\mathbf{E} \cdot \mu$, where $\mu = \sum_i q_i r_i$ is the dipole momentum. The spectrum of the energy operator can be computed with the same techniques as in [8], yielding another family of solutions on the non-trivial bundle that are parametrized by half integers.

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Shooting for the Eight: A Topological Existence Proof for a Figure-Eight Orbit of the Three-Body Problem

Richard Moeckel

Abstract A topological existence proof is given for a figure-eight periodic solution of the equal mass three-body problem. The proof is based on the construction of a Wazewski set W in the phase space. The figure-eight solution is then found by a kind of shooting argument in which symmetrical initial conditions entering W are followed under the flow until they exit W. A linking argument shows that the image of the symmetrical entrance states under this flow map must intersect an appropriate set of symmetrical exit states.

1 Introduction

The goal of this paper is to give a topological proof of existence of a figure-eight type periodic solution of the equal-mass three-body problem. A variational existence proof for such an orbit was given by Chenciner and Montgomery [5]. The orbit is such that the three masses chase one another around a single figure-eight shaped curve (see Fig. 1). It has zero angular momentum and, by adjusting the size and speed, may be given an arbitrary negative energy.

In Chenciner and Montgomery's proof, a 12-fold symmetry plays an important role and the same is true here. The orbit can be parametrized in such a way that it begins at the Eulerian central configuration with one mass at the midpoint of the other two, for example with mass m_3 between m_1, m_2. During the first 12th of the period the masses move to an isosceles configuration with m_1 on the axis of symmetry (see Fig. 1). Now the rest of the orbit can be obtained by reflection, translation and permutation of the masses. For example, during the next 12th of the period, the masses return to an Eulerian configuration, but this time with m_2 in the middle. The motion during the second 12th of the period is obtained from the motion in the first 12th by permuting m_2 and m_3, reversing time, and reflecting in the long axis of the eight.

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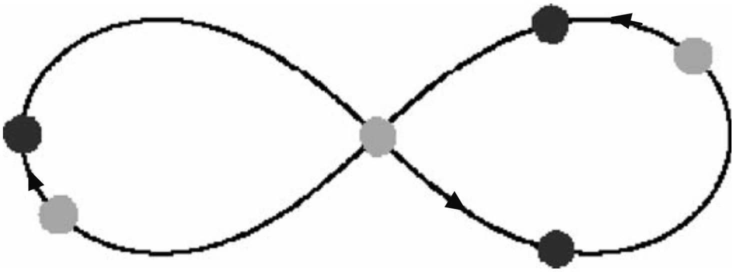

Fig. 1 Figure-eight orbit. In the first 12th of the period, the configuration changes from collinear (*light gray dots*) to isosceles (*dark gray dots*)

The method of proof used here is a variation of an idea used by Conley in the restricted three-body problem [2]. In Conley's paper, the retrograde lunar orbit of Hill is shown to exist for a wide range of values of the Jacobi constant. After regularizing double collisions, the problem becomes one of finding a solution of a system of second-order differential equations in the plane which moves from the positive x-axis to the positive y-axis across the first quadrant and meets both axes orthogonally.

Such a solution is found by a shooting argument. Points starting orthogonal to the x-axis are followed until one of two exit conditions holds – they either hit the positive y-axis or their velocity vectors become horizontal. As the initial point along the x-axis varies, the final behavior changes from hitting the y-axis with nonzero slope to having a horizontal velocity vector before hitting the positive y-axis. Somewhere in between, there must be a point whose velocity becomes horizontal exactly when it reaches the positive y-axis and this gives the desired periodic solution.

The main difficulty in this approach is to show that the solutions really arrive at one of the two kinds of final states, and that the final state depends continuously on the initial condition. For this, Conley constructed an isolating block. Isolating blocks were developed by Conley and Easton as a way to defining a topological index for invariant sets [3, 4, 6]. Among their useful properties is the fact for initial conditions which leave the isolating block, the amount of time required to leave depends continuously on initial conditions. It follows that the location of the exit point also varies continuously. In [2] Conley constructs an isolating block in a manifold of fixed Jacobi constant and uses it to justify the shooting argument outlined above.

In this paper, a figure-eight orbit will be found by shooting from the Eulerian configuration to the isosceles configurations. More precisely, initial conditions in phase space orthogonal to the set whose configuration is Eulerian will be followed under the flow and shown to meet the set whose configuration is isosceles orthogonally. This involves a higher-dimensional version of the usual shooting argument. It turns out that topologically, the problem is to show that a two-dimensional surface of initial conditions can be followed under the flow to meet another two-dimensional surface inside a four-dimensional ambient space. The proof uses a linking argument. Another example of a multidimensional shooting argument based on isolating blocks and linking can be found in [1].

The concept of isolating block is related to earlier ideas of Wazewski [11]. It is possible to get the crucial property of continuous exit times under weaker assumptions than are needed for the topological index theory. For example, whereas isolating blocks are always compact, Wazewski sets need not be. In this paper the proof will be based on the construction of a Wazewski set, rather than an isolating block.

A computer-assisted existence proof based on shooting can be found in [7]. In that paper, the computations which are carried out near the actual figure-eight orbit suffice to establish continuity of an appropriate local Poincaré map and to find the required symmetric orbit. The method used in the present paper provides a continuous flow-defined map defined on a large region of phase space and does not rely in an essential way on numerical computations (although some numerical evaluations of elementary functions and their definite integrals are used for convenience). On the other hand the localized approach of [7] yields some additional properties of the orbit.

Since existence of a figure-eight solution was already known, some justification for a new proof may be required. The proof presented here seems interesting for several reasons. First it connects the figure-eight with a number of well-known features of the three-body problem, including double and triple collisions, central configurations and homothetic solutions. These are used in studying the boundary behavior of the map in the shooting argument (particularly in the proof of Lemma 5). Second it locates a figure-eight solution in a smaller region of phase space than the variational proof – namely, in the Wazewski set $\mathcal{W}$ used for the proof. This shows that the solution moves monotonically in shape space, a fact which is not obvious from the other proof. Finally, there may be topological proofs similar to this one for the existence of other symmetric periodic solutions which have been discovered numerically but for which the variational methods fail.

2 Equation of Motion and Reduction

Consider the planar three-body problem with equal masses $m_1 = m_2 = m_3 = 1$. Let the positions be $q_i \in \mathbf{R}^2$ and the velocities be $v_i = \dot{q}_i \in \mathbf{R}^2$. The Newton's laws of motion are the Euler–Lagrange equation of the Lagrangian

$$L = \tfrac{1}{2}K + U \tag{1}$$

where

$$\begin{aligned} K &= |v_1|^2 + |v_2|^2 + |v_3|^2 \\ U &= \frac{1}{r_{12}} + \frac{1}{r_{13}} + \frac{1}{r_{23}}. \end{aligned} \tag{2}$$

Here $r_{ij} = |q_i - q_j|$ denotes the distance between the i-th and j-th masses. The total energy of the system is constant:

$$\tfrac{1}{2}K - U = h.$$

Assume without loss of generality that total momentum is zero and that the center of mass is at the origin, i.e.,

$$v_1 + v_2 + v_3 = q_1 + q_2 + q_3 = 0.$$

Introduce Jacobi variables

$$\xi_1 = q_2 - q_1 \qquad \xi_2 = q_3 - \tfrac{1}{2}(q_1 + q_2)$$

and their velocities $\eta_i = \dot{\xi}_i$. Then the equations of motion are given by a Lagrangian of the same form (1) where now

$$\begin{aligned} K &= \tfrac{1}{2}|\eta_1|^2 + \tfrac{2}{3}|\eta_2|^2 \\ U &= \frac{1}{r_{12}} + \frac{1}{r_{13}} + \frac{1}{r_{23}}. \end{aligned} \tag{3}$$

The mutual distances are given by

$$\begin{aligned} r_{12} &= |\xi_1| \\ r_{13} &= |\xi_2 + \tfrac{1}{2}\xi_1| \\ r_{23} &= |\xi_2 - \tfrac{1}{2}\xi_1|. \end{aligned} \tag{4}$$

The use of Jacobi coordinates eliminates the translational symmetry of the problem and reduces the number of degrees of freedom from 6 to 4. The next step is the elimination of the rotational symmetry to reduce from 4 to 3 degrees of freedom. This is accomplished by fixing the angular momentum and working in a quotient space. When the angular momentum is zero, there is a particularly elegant way to accomplish this reduction. The discussion below follows Montgomery [9].

Define new variables r, w_1, w_2, w_3 via:

$$\begin{aligned} r^2 &= \tfrac{1}{2}|\xi_1|^2 + \tfrac{2}{3}|\xi_2|^2 \\ w_1 &= \tfrac{1}{4}|\xi_1|^2 - \tfrac{1}{3}|\xi_2|^2 \\ w_2 + i\, w_3 &= \tfrac{1}{\sqrt{3}}\, \xi_1 \bar{\xi}_2 \end{aligned} \tag{5}$$

where in the last line, vectors in $\mathbf{R}^2$ are identified with complex numbers. It is easy to check that the new variables satisfy the relation:

$$w_1^2 + w_2^2 + w_3^2 = \tfrac{1}{4} I^2 = \tfrac{1}{4} r^4.$$

The quantity $I = r^2$ is the moment of inertia, a convenient measure of the overall size of the triangle formed by the three bodies. The shape of the triangle can be represented by the normalized vector $S = 2w/r^2$ which lies in the unit sphere $\mathbf{S}^2$, which will be called the shape sphere.

Using the fact that the angular momentum is zero, it can be shown [9] that the variables r, S satisfy the Euler–Lagrange equations of a reduced Lagrangian of the form

$$L = \tfrac{1}{2}K + \tfrac{1}{r}W(S) \tag{6}$$

where:

$$\begin{aligned} K &= \dot{r}^2 + \tfrac{1}{4}r^2|\dot{S}|^2 \\ W &= \frac{1}{\rho_{12}} + \frac{1}{\rho_{13}} + \frac{1}{\rho_{23}}. \end{aligned} \tag{7}$$

Here $|\_|$ is the Euclidean metric on the unit sphere and $\rho_{ij} = r_{ij}/r$ is the normalized interparticle distance, i.e., the interparticle distance after scaling the configuration to have moment of inertia 1. After some computation using (4) and (5) one finds

$$\begin{aligned} \rho_{12}^2 &= 1 + s_1 \\ \rho_{13}^2 &= 1 - \tfrac{1}{2}s_1 + \tfrac{\sqrt{3}}{2}s_2 \\ \rho_{23}^2 &= 1 - \tfrac{1}{2}s_1 - \tfrac{\sqrt{3}}{2}s_2 \end{aligned} \tag{8}$$

where $S = (s_1, s_2, s_3)$.

Figure 2 shows a contour plot of the shape potential W. The equator of the shape sphere is represented by $s_3 = w_3 = 0$ which means that the vectors ξ_1, ξ_2 are

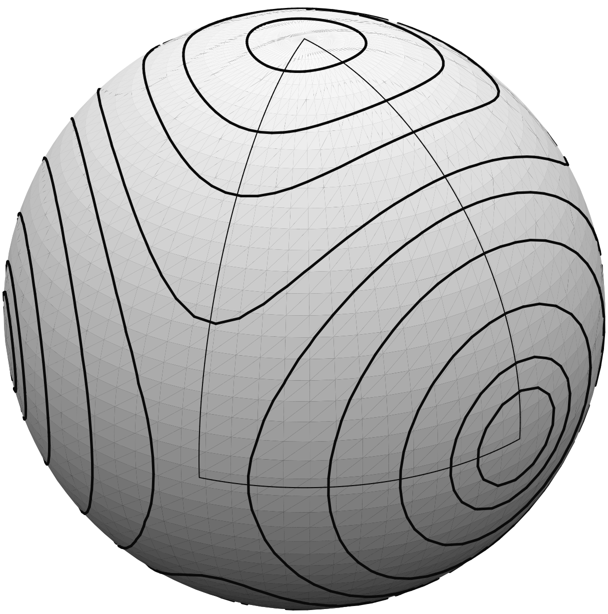

Fig. 2 Contour plot of the shape potential W and the fundamental region $\mathcal{F}$. The *top corner* of $\mathcal{F}$ (the north pole) is the equilateral triangle configuration l. The *bottom edge* (on the equator) consists of collinear configurations. The *left endpoint* of this edge is the Eulerian central configuration e while the *right endpoint* is the double collision singularity

parallel and so the three bodies are collinear. There are three saddle points of W on the equator corresponding to the three collinear central configurations. For example, at the point $S = (1, 0, 0)$ the scaled distances are $\rho_{12} = \sqrt{2}$ and $\rho_{13} = \rho_{23} = 1/\sqrt{2}$ so mass m_3 is at the midpoint of the line segment between masses m_1, m_2. This shape is the collinear central configuration in the equal mass case. In addition to the saddle points, there are three singular points of W along the equator at the binary collision shapes where $\rho_{ij} = 0$. One has $\rho_{12} = 0$ at $S = (-1, 0, 0)$ while $\rho_{13} = 0$ and $\rho_{23} = 0$ at $S = (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ and $S = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ respectively. The north and south poles of the sphere represent the two equilateral triangle shapes, which are also central configurations.

Because of the equal masses the shape potential W has a 12-fold symmetry. In fact, a triangular region $\mathcal{F}$ as in Fig. 2 can serve as a fundamental region in the sense that the shape sphere can be tiled by 12 reflected and rotated copies of $\mathcal{F}$ while preserving W. The fundamental region $\mathcal{F}$ will be the spherical triangle with vertices at $S = (1, 0, 0)$, the collinear central configuration, $S = (0, 0, 1)$, one of the equilateral central configurations, and $S = (\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$, the double collision where $\rho_{23} = 0$. The edges of $\mathcal{F}$ consist of two meridional arcs and an arc of the equator. The left meridian in Fig. 2 consists of isosceles configurations with m_3 on the axis of symmetry. Similarly the right meridian is made up of isosceles configurations with m_1 on the axis of symmetry. The equatorial arc on the bottom represents collinear configurations.

In Chenciner and Montgomery's original existence proof for the figure-eight orbit, this symmetry plays an important role. One may parametrize their orbit in such a way that its projection to the shape sphere begins at the collinear central configuration $S = (1, 0, 0)$ (the lower left vertex of $\mathcal{F}$). Then, in the first $1/12$ of its period, it moves across the fundamental region $\mathcal{F}$ to meet the right isosceles meridian orthogonally. The rest of the orbit can be obtained by symmetry. The proof presented here will show by a topological shooting argument that there is an orbit segment whose projection to the shape sphere crosses $\mathcal{F}$ in this way and which can be continued by the 12-fold symmetry to a periodic orbit.

Two different systems of coordinates will be used on the shape sphere. The first set of *stereographic* coordinates are derived by stereographic projection from the south pole. With $S = (s_1, s_2, s_3)$ as above, let $z = (z_1, z_2)$ where

$$z_1 = \frac{s_1}{1 + s_3} \qquad z_2 = \frac{s_2}{1 + s_3} \qquad \zeta_1 = \dot{z}_1 \qquad \zeta_2 = \dot{z}_2.$$

The inverse formulas are

$$s_1 = \frac{2z_1}{1 + |z|^2} \qquad s_2 = \frac{2z_2}{1 + |z|^2} \qquad s_3 = \frac{1 - |z|^2}{1 + |z|^2}.$$

The Lagrangian is (6) where now

$$K = \dot{r}^2 + \frac{r^2(\dot{z}_1^2 + \dot{z}_2^2)}{(1 + |z|^2)^2}$$

$$
\begin{aligned}
W &= \frac{1}{\rho_{12}} + \frac{1}{\rho_{13}} + \frac{1}{\rho_{23}} \\
\rho_{12}^2 &= \frac{(z_1+1)^2 + z_2^2}{1+|z|^2} \\
\rho_{13}^2 &= \frac{(z_1-\frac{1}{2})^2 + (z_2+\frac{\sqrt{3}}{2})^2}{1+|z|^2} \\
\rho_{23}^2 &= \frac{(z_1-\frac{1}{2})^2 + (z_2-\frac{\sqrt{3}}{2})^2}{1+|z|^2}.
\end{aligned}
\tag{9}
$$

The other set of *stereographic polar* coordinates are the polar coordinates in the z-plane. Let $z = (s\cos\theta, s\sin\theta)$ where $s = |z|$. Then the Lagrangian is (6) with

$$
\begin{aligned}
K &= \dot{r}^2 + \frac{r^2(\dot{s}^2 + s^2\dot{\theta}^2)}{(1+s^2)^2} \\
W &= \frac{1}{\rho_{12}} + \frac{1}{\rho_{13}} + \frac{1}{\rho_{23}} \\
\rho_{12}^2 &= \frac{1+s^2+2s\cos\theta}{1+s^2} \\
\rho_{13}^2 &= \frac{1+s^2-s\cos\theta+\sqrt{3}s\sin\theta}{1+s^2} \\
\rho_{23}^2 &= \frac{1+s^2-s\cos\theta-\sqrt{3}s\sin\theta}{1+s^2}.
\end{aligned}
\tag{10}
$$

These Lagrangian systems represent the zero angular momentum three-body problem reduced to three degrees of freedom by elimination of all the symmetries. The differential equation are the usual second-order Euler–Lagrange equations which can be written as a first-order system in phase space. One final improvement is to blow-up the triple collision singularity at $r = 0$ by introducing the time rescaling $' = r^{\frac{3}{2}}\,\dot{}$ and the variable $v = r'/r$ [8]. The result is the following first-order system:

$$
\begin{aligned}
r' &= vr \\
v' &= W(z) - \tfrac{1}{2}v^2 + 2rh \\
z_1' &= \zeta_1 \\
z_2' &= \zeta_2 \\
\zeta_1' &= (1+|z|^2)^2\, W_{z_1} - \tfrac{1}{2}v\zeta_1 + \frac{2}{1+|z|^2}\Big(|\zeta|^2 z_1 - 2\zeta_2(z_1\zeta_2 - z_2\zeta_1)\Big) \\
\zeta_2' &= (1+|z|^2)^2\, W_{z_2} - \tfrac{1}{2}v\zeta_2 + \frac{2}{1+|z|^2}\Big(|\zeta|^2 z_2 + 2\zeta_1(z_1\zeta_2 - z_2\zeta_1)\Big).
\end{aligned}
\tag{11}
$$

with energy equation:

$$\tfrac{1}{2}v^2 + \tfrac{1}{2}\frac{|\zeta|^2}{(1+|z|^2)^2} - W(z) = rh \tag{12}$$

Note that $\{r = 0\}$ is now an invariant set for the flow, called the triple collision manifold. The differential equations are still singular due to the double collisions when $\rho_{ij} = 0$. However, in the argument below, it is possible to avoid these.

The corresponding equations in stereographic polar coordinates are:

$$\begin{aligned}
r' &= vr \\
v' &= W(s,\theta) - \tfrac{1}{2}v^2 + 2rh \\
s' &= \sigma \\
\theta' &= \tau \\
\sigma' &= (1+s^2)^2\, W_s - \tfrac{1}{2}v\sigma + \frac{2s\sigma^2}{1+s^2} + \frac{s(1-s^2)\tau^2}{1+s^2} \\
s^2\tau' &= (1+s^2)^2\, W_\theta - \tfrac{1}{2}vs^2\tau - \frac{2s\sigma\tau}{1+s^2}.
\end{aligned} \tag{13}$$

with energy equation:

$$\tfrac{1}{2}v^2 + \tfrac{1}{2}\frac{\sigma^2 + s^2\tau^2}{(1+s^2)^2} - W(s,\theta) = rh \tag{14}$$

Figures 3 and 4 show the potential W in stereographic and stereographic polar coordinates, respectively.

3 A Wazewski Set

Consider the system (11) on a manifold of fixed energy $h < 0$. Due to scaling symmetry, one may assume without loss of generality that $h = -1$. The energy equation (12) shows that the energy manifold is five-dimensional and is a graph over its projection to the (v, z, ζ)-space. The goal of this section is to construct a Wazewski set in this five-dimensional manifold.

A Wazewski set for a flow $\phi_t(x)$ on a topological space X is a subset $\mathcal{W} \subset X$ satisfying technical hypotheses which guarantee that the time required to exit $\mathcal{W}$ depends continuously on initial conditions [3, 11]. To formulate these hypotheses, let $\mathcal{W}^0$ be the set of points in $\mathcal{W}$ which eventually leave $\mathcal{W}$ in forward time, and let $\mathcal{E}$ be the set of points which exit immediately:

$$\begin{aligned}
\mathcal{W}^0 &= \{x \in \mathcal{W} : \exists t > 0, \phi_t(x) \notin \mathcal{W}\} \\
\mathcal{E} &= \{x \in \mathcal{W} : \forall t > 0, \phi_{[0,t)}(x) \not\subset \mathcal{W}\}.
\end{aligned}$$

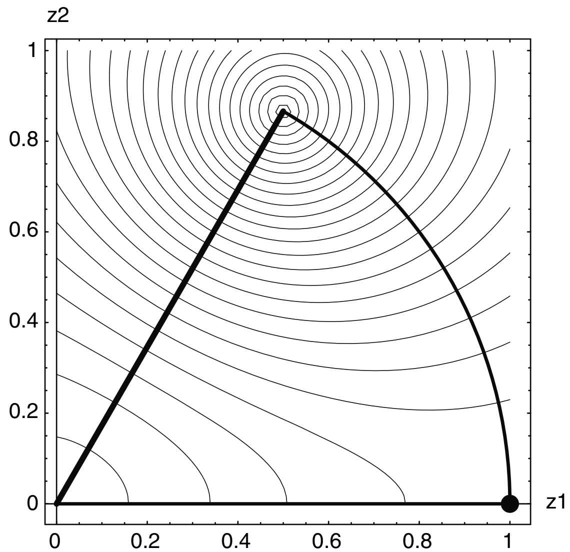

Fig. 3 Contour plot of the shape potential W and the fundamental region $\mathcal{F}$ in stereographic coordinates. The *dot* represents collinear central configuration and the *opposite bold edge* of $\mathcal{F}$ represents the isosceles configurations which will enter into the shooting argument. The *top corner* of $\mathcal{F}$ is the double collision point $z = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

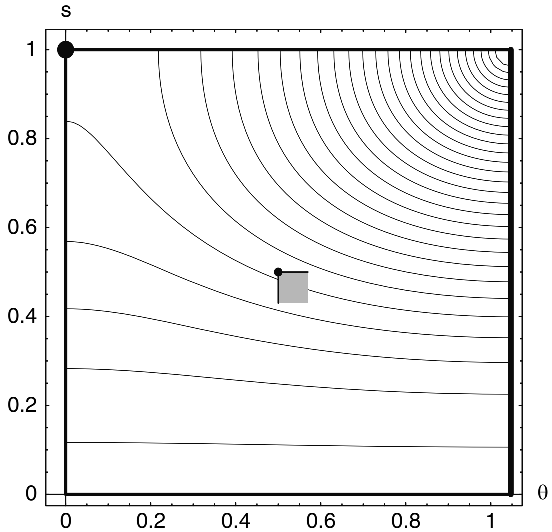

Fig. 4 Contour plot of the shape potential W and the fundamental region $\mathcal{F}$ in stereographic polar coordinates. The fundamental region $\mathcal{F}$ is the square $0 \leq \theta \leq \pi/3$, $0 \leq s \leq 1$. The south-east velocity cone field defining the Wazewski set is shown in *gray*

Clearly, $\mathcal{E} \subset \mathcal{W}^0$. Given $x \in \mathcal{W}^0$ define the *exit time*

$$\tau(x) = \sup\{t \geq 0 : \phi_{[0,t]}(x) \subset \mathcal{W}\}.$$

Note that $\tau(x) = 0$ if and only if $x \in \mathcal{E}$.

The appropriate hypotheses which guarantee continuity of τ are [3]:

(a) If $x \in \mathcal{W}$ and $\phi_{[0,t]}(x) \subset \overline{\mathcal{W}}$, then $\phi_{[0,t]}(x) \subset \mathcal{W}$.
(b) $\mathcal{E}$ is a relatively closed subset of $\mathcal{W}^0$.

The choice of the set $\mathcal{W}$ is motivated by the shooting argument outlined above. Recall the spherical triangle $\mathcal{F}$, the fundamental region for the potential. Roughly speaking, the Wazewski set consists of the points in phase space whose shapes lie in $\mathcal{F}$ and whose velocities lie in a certain cone. In stereographic polar coordinates, the velocity cone is given by

$$s' = \sigma \leq 0 \qquad \theta' = \tau \geq 0.$$

A typical element of the is cone field is shaded gray in Fig. 4. Thus the solutions in $\mathcal{W}$ will be moving south-east across the figure. Because of the polar coordinate singularity at $s = 0$, the actual definition of $\mathcal{W}$ will be given in ordinary stereographic coordinates, where the velocity conditions become

$$\begin{array}{lll} z \cdot \zeta = z_1\zeta_1 + z_2\zeta_2 \leq 0 & z \wedge \zeta = z_1\zeta_2 - z_2\zeta_1 \geq 0 & \text{if } (z_1, z_2) \neq (0,0) \\ (1,0) \cdot \zeta = \zeta_1 \leq 0 & (1, \sqrt{3}) \wedge \zeta = \zeta_2 - \sqrt{3}\zeta_1 \geq 0 & \text{if } (z_1, z_2) = (0,0). \end{array} \tag{15}$$

This includes a special definition for the origin, which is taken to be the union of all of the velocity cones at nearby points $z \in \mathcal{F}$, $z \neq (0,0)$. Let

$$\overline{\mathcal{W}} = \{(r, v, z, \zeta) : (12) \text{ holds}, r \geq 0, z \in \mathcal{F}, \zeta \text{ satisfies } (15)\}. \tag{16}$$

Because of the way the cone at $z = (0,0)$ was handled, $\overline{\mathcal{W}}$ is a closed subset of $\mathbf{R}^6$.

For technical reasons, certain parts of the set $\overline{\mathcal{W}}$ must be excluded from the Wazewski set. First, there is the double collision singularity at the vertex $z = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let

$$\mathcal{D} = \{(r, v, z, \zeta) : z = (\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})\} = \{(r, v, s, \theta, \sigma, \tau) : s = 1, \theta = \tfrac{\pi}{3}\}.$$

Next, there is the collinear invariant manifold, which is given by

$$\mathcal{C} = \{(r, v, z, \zeta) : |z| = 1, z_1\zeta_1 + z_2\zeta_2 = 0\} = \{(r, v, s, \theta, \sigma, \tau) : s = 1, s' = \sigma = 0\}.$$

On the one hand, these orbits will have double collisions; on the other hand, their exiting behavior is somewhat different from that of the nearby, non-collinear orbits. Finally there is the set

$$\mathcal{I}_1 = \{(r, v, z, \zeta) : z_2 - \sqrt{3}z_1 = \zeta_2 - \sqrt{3}\zeta_1 = 0\}$$
$$\mathcal{I}_1 \cap \{s > 0\} = \{(r, v, s, \theta, \sigma, \tau) : 0 < s \le 1, \theta = \tfrac{\pi}{3}, \tau = 0\}.$$

This is an invariant manifold of isosceles motions (with mass m_1 moving on the symmetry axis of the isosceles triangle). There is another invariant isosceles manifold $\mathcal{I}_3$ given by $\theta = \tau = 0$ which will play a role later on. However, it is convenient and feasible to keep $\mathcal{I}_3$ in the Wazewski set.

With these definitions in place, the Wazewski set is

$$\mathcal{W} = \overline{\mathcal{W}} \setminus \mathcal{C} \cup \mathcal{D} \cup \mathcal{I}_1.$$

Note that $\overline{\mathcal{W}}$ is the closure of $\mathcal{W}$ in $\mathbf{R}^6$ as the notation suggests. Also $\mathcal{W}$ is open in $\overline{\mathcal{W}}$.

The rest of this section is devoted to proving

Theorem 1. *$\mathcal{W}$ is a Wazewski set for the flow on the constant energy manifold.*

First, note that any initial condition in $\mathcal{W}$ defines a solution which continues to exist as long as it remains in $\mathcal{W}$. To see this recall that it is a well-known property of the three-body problem that a solution which fails to exist for all time must end in collision [10]. Since we have blown-up the triple collision, the only possibility for a singularity is the double collision at $(s, \theta) = (1, \frac{\pi}{3})$. If the initial value $s_0 < 1$ then since $s(t)$ is non-increasing for orbits in $\mathcal{W}$, it is impossible to reach the double collision. Similarly if $s_0 = 1$ but $\sigma_0 < 0$, then $s(t) < 1$ for all $t > 0$ and again, collision cannot occur. The only remaining case, $s_0 = 1, \sigma_0 = 0$ is excluded as a point of $\mathcal{C}$.

To verify Wazewski property (a), one must check that whenever $x \in \mathcal{W}$ and the closed orbit segment $\phi_{[0,t]}(x)$ is contained in the closure $\overline{\mathcal{W}}$, then $\phi_{[0,t]}(x)$ is actually contained in $\mathcal{W}$ itself. Assume for the sake of contradiction, that t is the smallest time such $\phi_t(x) \in \overline{\mathcal{W}} \setminus \mathcal{W}$. In particular, $t > 0$ since $x \in \mathcal{W}$ by assumption. Since $\overline{\mathcal{W}} \setminus \mathcal{W} \subset \mathcal{C} \cup \mathcal{D} \cup \mathcal{I}_1$, it suffices to show that an orbit segment beginning in $\mathcal{W}$ cannot enter $\mathcal{C} \cup \mathcal{D} \cup \mathcal{I}_1$. Now the argument of the last paragraph shows that for the orbit segment under consideration, $s(t) < 1$ so $\phi_t(x) \notin \mathcal{C} \cup \mathcal{D}$. Also, $\phi_t(x) \notin \mathcal{I}_1$ since $x \notin \mathcal{I}_1$ and $\mathcal{I}_1$ is an invariant set. Thus property (a) holds for $\mathcal{W}$.

To check property (b), one must first identify the subsets $\mathcal{W}^0, \mathcal{E}$. It turns out that all of the solutions beginning in $\mathcal{W}$ eventually leave, except those which converge to one of the equilibrium points on the triple collision manifold. There are exactly four equilibrium points in $\mathcal{W}$. In (r, v, z, ζ) coordinates, they are

$$E_\pm = (0, \pm\sqrt{2W(e)}, e, 0) \qquad L_\pm = (0, \pm\sqrt{2W(l)}, l, 0)$$

where e, l are the Eulerian and Lagrangian central configurations, with z-coordinates:

$$e = (1, 0) \qquad l = (0, 0).$$

These are just the critical points of the potential $W(z)$ in $\mathcal{F}$ (see Fig. 3). It is known that these are hyperbolic equilibria for the differential equation (11). It turns out that parts of the stable manifolds of $L_\pm$ lie in $\mathcal{W}$ and the corresponding solutions may never exit. Define the immediate stable manifolds:

$$W_0^s(L_\pm) = \{x \in \mathcal{W} : \phi_t(x) \in \mathcal{W} \text{ for all } t \geq 0, \phi_t(x) \to L_\pm \text{ as } t \to \infty\}.$$

Then

Lemma 1. $\mathcal{W}^0 = \mathcal{W} \setminus W_0^s(L_\pm)$.

Proof. Consider a solution $\phi_t(x)$ which remains in $\mathcal{W}$ for all $t \geq 0$. On the energy manifold, one can use (12) to eliminate r from the differential equations to obtain a five-dimensional differential equation for (v, z, ζ) or $(v, s, \theta, \sigma, \tau)$. The monotonicity of $s(t)$ and $\theta(t)$ and the compactness of $\mathcal{F}$ imply that $z_\infty = \lim_{t\to\infty} z(t)$ exists and is a point of $\mathcal{F}$. Since $s(t) < 1$ for $t > 0$ and is non-increasing, it follows that $|z_\infty| < 1$. In particular, z_∞ cannot be the Eulerian central configuration or the double collision.

The energy equation gives a bound

$$v^2 + \frac{|\zeta|^2}{(1+|z|^2)^2} \leq 2W(z).$$

Thus the omega limit set $\omega(x)$ is nonempty and is contained in the compact set

$$z = z_\infty \qquad v^2 + \frac{|\zeta|^2}{(1+|z_\infty|^2)^2} \leq 2W(z_\infty).$$

Any invariant subset of this form must have $\zeta = \zeta' = 0$ and then (11) shows that $W_{z_1} = W_{z_2} = 0$, i.e., z_∞ is a critical point of the shape potential. The only critical point with $|z_\infty| < 1$ is the Lagrangian central configuration l so $z_\infty = l$.

Now the line segment $z = l, \zeta = 0, v^2 \leq 2W(l)$ is the Lagrangian homothetic solution connecting the restpoint L_+ to L_-. It follows that $\phi_t(x)$ must converge to one of these restpoints, as claimed. □

To find the immediate exit set $\mathcal{E}$ one must examine the boundary points of $\mathcal{W}$. It turns out that there are essentially just two ways to exit $\mathcal{W}$ – either the orbit reaches an isosceles configuration (with m_1 on the symmetry axis) or the stereographic polar radius stops decreasing and begins to increase again. Corresponding to these two modes of exiting, it is convenient to distinguish two subsets of the boundary. Let $x = (r, v, z, \zeta)$ and let

$$\begin{aligned}\mathcal{B}_1 =& \{x \in \mathcal{W} : z_2 - \sqrt{3}z_1 = 0\} \\ \mathcal{B}_2 =& \{x \in \mathcal{W} : z_1\zeta_1 + z_2\zeta_2 = 0\}.\end{aligned}$$

The parts of these sets with $|z| > 0$ can be described more intuitively in stereographic polar coordinates $x = (r, v, s, \theta, \sigma, \tau)$ as

$$\begin{aligned}\mathcal{B}_1 \cap \{s > 0\} &= \{x \in \mathcal{W} : \theta = \tfrac{\pi}{3}\} \\ \mathcal{B}_2 \cap \{s > 0\} &= \{x \in \mathcal{W} : s' = \sigma = 0\}.\end{aligned}$$

When $z = (0, 0)$ the conditions on the velocities reduce to those required for membership in $\mathcal{W}$. These definitions make it clear that $\mathcal{B}_1$ and $\mathcal{B}_2$ are relatively closed subsets of $\mathcal{W}$, hence also of $\mathcal{W}_0$. So the following lemma completes the verification of property (b) and the proof the Theorem 1.

Lemma 2. *The immediate exit set of $\mathcal{W}$ is $\mathcal{E} = \mathcal{B}_1 \cup \mathcal{B}_2$.*

Proof. By definition, $\mathcal{B}_1, \mathcal{B}_2$ each contain $\{x \in \mathcal{W} : z = (0, 0)\}$. Note that for any such x, the shape velocity $\zeta \neq (0, 0)$ since $z = \zeta = 0$ is part of the isosceles invariant manifold $\mathcal{I}_1$ which was excluded from $\mathcal{W}$. Furthermore, by definition of the velocity cone when $z = (0, 0)$, ζ points out of $\mathcal{F}$ and so these points x are indeed in $\mathcal{E}$. For the rest of the proof one may assume $z \neq (0, 0)$.

Let $x = (r, v, z, \zeta) \in \mathcal{B}_1$ with $z \neq (0, 0)$. Using stereographic polar coordinates, one has $\theta = \frac{\pi}{3}$ and since the isosceles set $\mathcal{I}_1$ has been excluded $\theta' = \tau > 0$. Thus x is an immediate exit point and $\mathcal{B}_1 \subset \mathcal{E}$.

Next consider a point $x = (r, v, s, \theta, \sigma, \tau) \in \mathcal{B}_2$ with $s > 0$. Since $\sigma = 0$, (13) gives

$$\sigma' = (1 + s^2)^2 W_s + \frac{s(1 - s^2)\tau^2}{1 + s^2} \geq (1 + s^2)^2 W_s.$$

Now it is easy to check that $W_s \geq 0$ in $\mathcal{F}$ with equality only if $s = 0, 1$. One has $s > 0$ by assumption and $s < 1$ since the set $s = 1, \sigma = 0$ is the excluded collinear invariant manifold $\mathcal{C}$. Therefore $\sigma' > 0$ for all points of $\mathcal{B}_2$ with $s > 0$. This shows $\mathcal{B}_2 \subset \mathcal{E}$.

To complete the proof, it remains to show that there are no other immediate exit points. Suppose, for the sake of contradiction, that $x \in \mathcal{W}$ is an immediate exit point which is not in $\mathcal{B}_1 \cup \mathcal{B}_2$. Only the case $s > 0$ needs to be considered since the points with $s = 0$ are in $\mathcal{B}_1 \cup \mathcal{B}_2$. By definition of $\mathcal{E}$ there is a sequence of times $t_n > 0$, $t_n \to 0$ such that $\phi_{t_n}(x) \notin \mathcal{W}$. By Wazewski property (a), one may assume $\phi_{t_n}(x) \notin \overline{\mathcal{W}}$.

From the definition (16), there are four other conceivable ways to exit $\overline{\mathcal{W}}$, each of which must be ruled out. First, one could exit by having $r = 0$ but $r(t_n) < 0$ for a sequence of positive times $t_n \to 0$. However, this is impossible because $\{r = 0\}$ is the invariant triple collision manifold.

Next, one might have $s = 1$ and $s(t_n) > 1$ for small positive times as above. Clearly this implies $s' = \sigma \geq 0$, but all points $x \in \mathcal{W}$ with $s = 1$ have $\sigma < 0$ so this is also impossible.

If $\theta = 0$ and $\theta(t_n) < 0$, one would have $\theta' = \tau = 0$. However, the equations $\theta = \tau = 0$ define the invariant manifold of isosceles orbits $\mathcal{I}_3$ (with m_3 on the axis of symmetry). Therefore $\theta(t)$ remains constant rather than decreasing.

Finally, if $\tau = 0$ and $\tau(t_n) < 0$, then it must be that $s^2\tau' = (1 + s^2)^2 W_\theta \leq 0$. Now it can be shown that $W_\theta \geq 0$ in $\mathcal{F}$ with equality only when $\theta = 0, \pi/3$. The case $\theta = \tau = 0$ has just been discussed while $\theta = \pi/3$, $\tau = 0$ defines the invariant isosceles manifold $\mathcal{I}_1$ which has been excluded from $\mathcal{W}$. This completes the proof. □

4 The Shooting Argument

The idea is to find the first 12th of the figure-eight orbit by shooting from the collinear central configuration with m_3 in the middle to the set of isosceles configurations with m_1 on the symmetry axis (see Fig. 1). A certain set of initial conditions $\mathcal{S}_0$ with $z = (1, 0)$, the collinear central configuration, will be followed under the flow until they exit $\mathcal{W}$ and it will be shown that at least one of the resulting exit points lies in a certain target set $\mathcal{T}$ with isosceles configurations, $z_2 = \sqrt{3}z_1$. In Fig. 3, the projection of the first 12th of the orbit will move from the dot at one vertex of the fundamental triangle to the opposite edge. In Fig. 4, it will move the upper left corner to the right edge. Moreover, appropriate orthogonality conditions on the velocities must be met.

More precisely, let

$$\mathcal{S} = \{(r, v, z, \zeta) \in \mathcal{W} : z = (1, 0), v = 0\}$$
$$\mathcal{T} = \{(r, v, z, \zeta) \in \mathcal{W} : z_2 - \sqrt{3}z_1 = \zeta_1 + \sqrt{3}\zeta_2 = v = 0\}.$$

Using stereographic polar coordinates one has

$$\mathcal{S} = \{(r, v, s, \theta, \sigma, \tau) \in \mathcal{W} : s = 1, \theta = v = 0\}$$
$$\mathcal{T} \cap \{s > 0\} = \{(r, v, s, \theta, \sigma, \tau) \in \mathcal{W} : \theta = \tfrac{\pi}{3}, \sigma = v = 0\}.$$

Imposing the equations $v = 0$ in $\mathcal{S}$ and $\sigma = v = 0$ in $\mathcal{T}$ guarantees that any orbit segment beginning in $\mathcal{S}$ and ending in $\mathcal{T}$ can be extended using the 12-fold symmetry to a periodic solution. The proof actually uses a proper subset $\mathcal{S}_0 \subset \mathcal{S}$ which will be described later.

It is possible to visualize $\mathcal{S}, \mathcal{T}$ and to see how $\mathcal{T}$ is embedded in the immediate exit set $\mathcal{E}$. There are six variables, but r will always be eliminated using the energy relation. Since points of $\mathcal{S}$ all have the same shape, it is only necessary to draw the velocity variables (v, ζ_1, ζ_2) or (v, σ, τ). Using the polar variables, (14) shows that the projection of the energy manifold is the solid region where

$$v^2 + \frac{\sigma^2 + s^2\tau^2}{(1 + s^2)^2} \leq 2W(s, \theta).$$

For $(s, \theta) = (1, 0)$ this is a spheroid. The inequalities $\sigma \leq 0, \tau \geq 0$ defining the velocity cone cut out a quarter of the spheroid, as shown in Fig. 5. The set $\mathcal{S}$ is the slice $v = 0$. Thus $\mathcal{S}$ is a two-dimensional surface. It is a quarter circle, homeomorphic to a disk.

The initial set $\mathcal{S}$ will be shrunk to a more manageable subset $\mathcal{S}_0$ as shown in Fig. 5. Let

$$\mathcal{S}_0 = \{(r, v, s, \theta, \sigma, \tau) \in \mathcal{W} : s = 1, \theta = v = 0, \sigma^2 + \tau^2 \geq c_1, \sigma + c_2\tau \leq 0\} \subset \mathcal{S}.$$

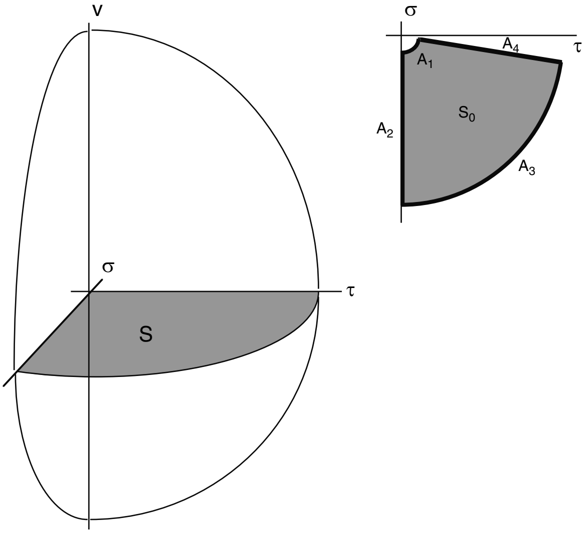

Fig. 5 Velocity fiber over the Eulerian central configuration. $\mathcal{S}_0 \subset \mathcal{S}$ is the initial set for the shooting argument

This moves the initial set away from the Eulerian homothetic orbit at $\sigma = \tau = 0$ and away from the collinear invariant manifold $\mathcal{C}$ (where $s = 1, \sigma = 0$) which was deleted from $\mathcal{W}$. For later reference the four boundary arcs of $\mathcal{S}_0$ have been labeled $\mathcal{A}_1, \ldots \mathcal{A}_4$ in the figure.

The target set $\mathcal{T}$ is also homeomorphic to a two-dimensional disk. It will be important for the proof to know how $\mathcal{T}$ is embedded in the immediate exit set $\mathcal{E}$. Since the latter is four-dimensional, a projection which preserves the relevant topological properties will be used. Since the variable τ plays a lesser role in the argument, the projections will be along this axis.

First consider the set $\mathcal{B}_1$ which forms part of the immediate exit set. Since $\theta = \frac{\pi}{3}$ is constant, one can use variables (v, s, σ, τ) to parametrize it. Projecting to (v, s, σ) space, the energy relation gives

$$v^2 + \frac{\sigma^2}{(1+s^2)^2} \leq 2W(s, \tfrac{\pi}{3}).$$

The part of this solid with $\sigma \leq 0$ is shown in Fig. 6. The outer surface of the solid corresponds to $\tau = 0$. Since one also has $\theta = \frac{\pi}{3}$, this surface is part of the isosceles manifold $\mathcal{I}_1$ which is not part of $\mathcal{W}$. Similarly, $s = 1, \theta = \frac{\pi}{3}$ is the double collision configuration which has been deleted. Over each remaining point of the solid, one should imagine a line segment of τ-values. In this picture, the projection of the target set $\mathcal{T}$ appears as the line segment $v = \sigma = 0, 0 \leq s < 1$. When $s = 0$ there is an extra flap extending away from the solid due to the larger velocity cone used there.

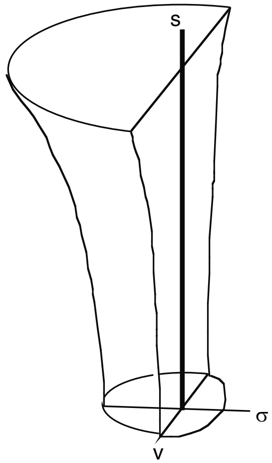

Fig. 6 Projection of $\mathcal{B}_1$ which forms half of the immediate exit set $\mathcal{E}$. The s-axis runs vertically from $s = 0$ to $s = 1$. The diameter (σ, v)-directions approaches ∞ as $s \to 1$. The target set T is shown as a *bold line segment*

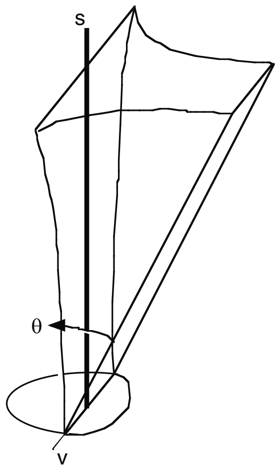

Fig. 7 Projection of $\mathcal{B}_2$ which forms the other half of the immediate exit set $\mathcal{E}$. The variables in the projection are v and two shape variables $(z_1, z_2$ or $s, \theta)$ lying in the fundamental triangle $\mathcal{F}$. The target set T is shown as a *bold line segment*

There is a similar picture showing the projection of $\mathcal{B}_2$ (where $\sigma = 0$) to the (v, z) space (see Fig. 6 where the shape is labeled by (s, θ) instead of z). Over each point $z \in \mathcal{F}$ there is a line segment of v-values: $-\sqrt{2W(z)} \le v \le \sqrt{2W(z)}$. This time, the outer surface where $\tau = 0$ is not deleted, except for the curve where it intersects the plane $z_2 = \sqrt{3}z_1$ (i.e., $\theta = \frac{\pi}{3}$). Of course over each (v, z) in the solid there is a line segment of τ-values which has been projected out. In Fig. 7, the projection of the target $\mathcal{T}$ is the line segment $v = 0$, $z_2 = \sqrt{3}z_1$.

Finally one can identify the plane where $\sigma = 0$ in $\mathcal{B}_1$ with the plane where $\theta = \frac{\pi}{3}$ in $\mathcal{B}_2$ to obtain a projection of the entire immediate exit set $\mathcal{E}$ and the target

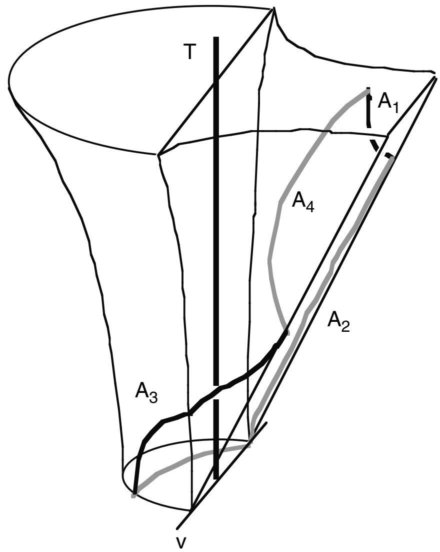

Fig. 8 Projection of the immediate exit set $\mathcal{E}$ together with the target set T and the images under the continuous map F of the four boundary arcs, $\mathcal{A}_i$, of $\mathcal{S}_0$. The images of the four arcs (shown alternately in *gray* and *black*) link the target set

(Fig. 8). Note that the target set T becomes a line segment through the middle of the solid region. In the actual, four-dimensional figure, T is a two-dimensional disk embedded in the four-dimensional immediate exit set $\mathcal{E}$. But the figure correctly depicts the essential topological relationship between T and $\mathcal{E}$ in the sense that the projection from the pair of spaces $(\mathcal{E}, T)$ onto their images in Fig. 8 is a homotopy equivalence. This follows easily by contracting the line segments of τ values which have been projected out.

The rest of the argument will be carried out in a series of lemmas to be proved in the next section. First it will be shown that almost all of the initial points in $\mathcal{S}_0$ eventually leave $\mathcal{W}$ and so can be followed forward under the flow to the immediate exit set $\mathcal{E}$.

Lemma 3. *The initial set $\mathcal{S}_0 \subset \mathcal{W}_0$ except for those points on the isosceles edge $\theta = \tau = 0$ ($\mathcal{A}_2$ in Fig. 5) which are also in $W^s_0(L_-)$.*

Since $\mathcal{W}$ is a Wazewski set, there is a continuous map $F : \mathcal{W}_0 \to \mathcal{E}$ taking initial points to their exit states. This will be defined on most of $\mathcal{S}_0$. The next lemma shows that it extends to all of $\mathcal{S}_0$.

Lemma 4. *The restriction of the flow defined map across $\mathcal{W}$ can be extended to a continuous map $F : \mathcal{S}_0 \to \overline{\mathcal{E}}$ by setting $F(x) = L_-$ for any $x \in W^s_0(L_-)$.*

The main point of the shooting proof is to show that $F(\mathcal{S}_0) \cap \mathcal{T} \neq \emptyset$. This will be done by examining the behavior of F on the boundary of $\mathcal{S}_0$. This naturally divides into four arcs, $\mathcal{A}_1, \ldots, \mathcal{A}_4$ as shown in Fig. 5. More precisely, using stereographic polar coordinates $x = (r, v, s, \theta, \sigma, \tau)$

$$\begin{aligned}
\mathcal{A}_1 &= \{x \in \mathcal{S}_0 : \sigma^2 + \tau^2 = c_1\} \\
\mathcal{A}_2 &= \{x \in \mathcal{S}_0 : \tau = 0\} \\
\mathcal{A}_3 &= \{x \in \mathcal{S}_0 : r = 0\} \\
\mathcal{A}_4 &= \{x \in \mathcal{S}_0 : \sigma + c_2\tau = 0\}
\end{aligned}$$

Lemma 5. *The exit behaviors of the arcs $\mathcal{A}_i$ are as follows. Points in $\mathcal{A}_1$ exit in $\mathcal{B}_2$ with $v < 0$. Points in $\mathcal{A}_2$ exit either in $\mathcal{B}_1$ at the equilateral configuration $z = 0$ or in $\mathcal{B}_2$ with $v < 0$ and $\theta = 0$. Points in $\mathcal{A}_3$ exit with $r = 0$, $v > 0$. Finally, points in $\mathcal{A}_4$ exit in $\mathcal{B}_2$.*

Using these lemmas, it follows that the projection of the image under F of the boundary of $\mathcal{S}_0$ appears as shown in Fig. 8. In particular, the image is a closed curve which links the projection of the target set $\mathcal{T}$. Since the projection is a homotopy equivalence, it follows that $F(\mathcal{S}_0) \cap \mathcal{T} \neq \emptyset$ as required.

Using the 12-fold symmetry one can now construct a periodic solution of the reduced differential equations. It follows as in Chenciner and Montgomery's original proof that this orbit is actually periodic for the unreduced system, i.e., the planar three-body problem. This proves the main result:

Theorem 2. *For the equal mass three-body problem with zero angular momentum and arbitrary negative energy, there exists a figure-eight periodic solution of the following type. During the first 12th of the period, the three masses move from the collinear central configuration with mass m_3 between m_1, m_2 to an isosceles configuration with m_1 on the symmetry axis. The projection of this orbit segment to the shape sphere lies in the fundamental triangle $\mathcal{F}$ and has the property that the stereographic polar radius s is monotonically decreasing while the angle θ is monotonically increasing. The rest of the orbit is obtained by symmetry.*

5 Proofs of the Lemmas

This section contains the proofs of the Lemmas 3–5.

Proof (Proof of Lemma 3). It must be shown that, with the indicated exceptions, most of the points in the initial set $\mathcal{S}_0$ lie in $\mathcal{W}_0$, i.e., they eventually exit $\mathcal{W}$. By Lemma 1 any points which do not exit must lie in one of the local stable manifolds $W^s_0(L_\pm)$.

First it will be shown that $\mathcal{S}_0 \cap W^s_0(L_+) = \emptyset$. It is well-known from studies of the triple collision manifold that $W^s(L_+) \subset \{r = 0\}$. Since $\{r = 0\}$ is an invariant set for the flow, it suffices to show that if $x \in \mathcal{S}_0$ is an initial point with $r = 0$ then $x \notin W^s_0(L_+)$. Since initially $v = 0$ and since $v = \sqrt{2W(l)} = \sqrt{6}$ at L_+, it will be enough to show that the change in v along the orbit $\phi_t(x)$ satisfies $\Delta v < \sqrt{6}$. This will be done by comparing the change in v to the arclength of the projection of the solution to shape space.

For solutions with $r = 0$ we have $v' = W - \frac{1}{2}v^2 \geq 0$. On the other hand, the length of the velocity of the projected curve with respect to the kinetic energy metric satisfies

$$(\lambda')^2 = \frac{|z'|^2}{(1+|z|^2)^2} = 2W - v^2$$

where we use λ to denote arclength. Hence

$$v' = \sqrt{\tfrac{1}{2}W - \tfrac{1}{4}v^2}\,\sqrt{2W - v^2} \leq \sqrt{\tfrac{1}{2}W}\,\lambda'$$

and

$$\Delta v \leq \int_\gamma \sqrt{\tfrac{1}{2}W}\,d\lambda$$

where γ is the projected curve in the fundamental region $\mathcal{F}$.

To estimate this integral note that in stereographic polar coordinates, one has

$$d\lambda = \frac{\sqrt{ds^2 + s^2\,d\theta^2}}{1+s^2} \leq \frac{|ds|}{1+s^2} + \frac{s|d\theta|}{1+s^2} \leq \frac{|ds|}{1+s^2} + \frac{|d\theta|}{2}.$$

Along γ, s and θ are both monotonic, with s decreasing from 1 to 0 and θ increasing from 0 to at most $\frac{\pi}{3}$. Also, the shape potential $W(s,\theta)$ satisfies $W_s \geq 0$ and $W_\theta \geq 0$ in $\mathcal{F}$ so increasing s to 1 or θ to $\frac{\pi}{3}$ only increases W. From these observations it follows that $\Delta v \leq I_1 + I_2$ where

$$I_1 = \int_0^1 \sqrt{\tfrac{1}{2}W(s,\tfrac{\pi}{3})}\,\frac{ds}{1+s^2} \qquad I_2 = \frac{1}{2}\int_0^{\frac{\pi}{3}} \sqrt{\tfrac{1}{2}W(1,\theta)}\,d\theta.$$

Using (10) one can approximate these integrals numerically with the results

$$I_1 \approx 1.34227 \qquad I_2 \approx 1.02795.$$

Hence $\Delta v \leq I_1 + I_2 \approx 2.37022 < \sqrt{6} \approx 2.44949$ as required.

It remains to show that $\mathcal{S}_0 \cap W_0^s(L_-)$ is contained in the isosceles invariant set $\mathcal{I}_3$ where $\theta = \tau = 0$. To see this, consider $x \notin \mathcal{I}_3$. Then either $\theta > 0$ or $\theta = 0$, $\tau > 0$. In either case, one has $\theta(t) > 0$ for small positive times t.

In $\mathcal{W}$, one has $\theta' = \tau \geq 0$. So orbits in $W_0^s(L_-)$ approach L_- in such at way that $\theta(t)$ has a limit and of course $s(t)$ decreases monotonically to 0. The differential equations (13) give

$$\tau' = \frac{(1+s^2)^2}{s^2}\,W_\theta - \tfrac{1}{2}v\tau - \frac{2\sigma\tau}{s(1+s^2)}.$$

It can be shown that $W_\theta \geq 0$ in $\mathcal{F}$ with equality only along the isosceles lines $\theta = 0, \frac{\pi}{3}$. First of all, this shows that it is not possible for a non-isosceles orbit in $W_0^s(L_-)$ to approach L_- with $\theta(t)$ constant and $\tau(t)$ identically 0. But then $\theta(t)$ must converge to its limit strictly from below. In that case, there would have to be a sequence of times $t_n \to \infty$ with $\theta'(t_n) = \tau(t_n) > 0$, $\theta''(t_n) = \tau'(t_n) < 0$. But since,

$v(t_n) \to -\sqrt{6}$ the differential equation shows that $\tau(t_n) > 0$ implies $\tau'(t_n) > 0$, a contradiction. □

Proof (Proof of Lemma 4). Consider a point $x \in W^s_0(L_-)$. It was shown in Lemma 3 that such a point must lie in the isosceles invariant manifold $\mathcal{I}_3$ but this will not be needed here. Note that the equilibrium point L_- lies on a corner of $\overline{\mathcal{W}}$ where the boundary manifolds $r = 0, \theta = \frac{\pi}{3}, \sigma = 0, \tau = 0$ come together (L_- itself is not part of $\mathcal{W}$ since it lies is $\mathcal{I}_1$).

The main point is that the local punctured unstable manifold $W^u(L_-) \setminus L_-$ lies outside of $\overline{\mathcal{W}}$. Indeed, a computation using (v, z, ζ) as local coordinates shows that the unstable eigenspace consists of all vectors of the form

$$(\delta v, \delta z, \delta \zeta) = (0, u, ku)$$

with $u \in \mathbf{R}^2$, and $k = \frac{1}{2}\sqrt{21 + 3\sqrt{13}} > 0$. On this subspace the quadratic form $z \cdot \zeta$ is positive definite whereas, by definition, $z \cdot \zeta \leq 0$ in $\overline{\mathcal{W}}$.

Now consider a point $y \in \mathcal{W}_0$ near the point $x \in W^s_0(L_-)$. Let a small neighborhood $\mathcal{U}$ of L_- be given and let $T > 0$ be such that $\phi_t(x) \in \mathcal{U}$ for $t \geq T$. Now the immediate exit set $\mathcal{E}$ is closed in $\mathcal{W}_0$ and it is disjoint from the compact orbit segment $\phi_{[0,T]}(x)$. It follows that if $y \in \mathcal{W}_0$ is sufficiently close to x, it cannot exit $\mathcal{W}$ before entering $\mathcal{U}$. On the other hand, after entering $\mathcal{U}$, the lambda lemma shows that $\phi_t(y)$ passes close to L_- and emerges from $\mathcal{U}$ close to $W^u(L_-)$. Since $W^u(L_-) \setminus L_-$ lies outside of $\overline{\mathcal{W}}$, it follows that $\phi_t(y)$ exits $\mathcal{W}$ somewhere in $\mathcal{U}$, i.e., close to L_-. This means that defining $F(x) = L_-$ is a continuous extension as required. □

Proof (Proof of Lemma 5). Each of the arcs $\mathcal{A}_i$ will be considered in turn. The arc $\mathcal{A}_1$ is the subset of the quarter circle $\mathcal{S}$ defined by $\sigma^2 + \tau^2 = c_1$ where $c_1 > 0$ is a small constant. Note that the center of the quarter circle, given by $\sigma = \tau = 0$, is the collinear homothetic solution. The behavior of the solutions $\phi_t(x)$, $x \in \mathcal{A}_1$ will be well approximated by the variational equations along the homothetic orbit, provided c_1 is sufficiently small and only a bounded interval of time is involved. The collinear homothetic orbit has $s = 1, \theta = \sigma = \tau = 0$. The other variables are given by:

$$r(t) = W_0 \operatorname{sech}^2(\sqrt{\tfrac{W_0}{2}}t) \qquad v(t) = -\sqrt{2W_0}\tanh(\sqrt{\tfrac{W_0}{2}}t)$$

where $W_0 = W(1, 0) = \frac{5}{\sqrt{2}}$ is the value of the potential at the central configuration. Note that $v(t)$ decreases monotonically from 0 to $-\sqrt{2W_0}$.

The variational equations along this solution are quite simple. The equations involving the variations in the variables s, σ are decoupled from the others:

$$\begin{aligned} \delta s' &= \delta\sigma \\ \delta\sigma' &= 4W_{ss}\delta s - \tfrac{1}{2}v(t)\delta\sigma. \end{aligned} \tag{17}$$

Moreover, on the homothetic orbit $W_{ss} = -\frac{7}{4\sqrt{2}}$.

Let $x = (r, v, s, \theta, \sigma, \tau) \in \mathcal{A}_1$. Then $s = 1, v = \theta = 0, \sigma^2 + \tau^2 = c_1$ and $\sigma + c_2\tau \leq 0$, where $c_2 > 0$ is another constant to be chosen later. It follows that the values of σ are all small and negative. It will now be shown, that for any solution of the variational equations with $\delta s(0) = 0$ and $\delta\sigma(0) < 0$, $\delta\sigma(t)$ reaches 0 after a uniformly bounded time. It follows that if c_1 is sufficiently small, the solution $\phi_t(x)$ reaches a state with $\sigma(t) = 0$ before $\theta(t)$ can reach $\frac{\pi}{3}$ (so the exit is in $\mathcal{B}_2$). Moreover, the value of $v(t)$ will be negative, as is true for the homothetic solution.

Consider a nonzero solution of the linear differential equation (17). The goal is to show that any solution beginning on the negative $\delta\sigma$ axis passes clockwise through the third quadrant in the $(\delta s, \delta\sigma)$-plane and reaches the negative δs axis. To this end, define an angle $\gamma \in [0, \frac{\pi}{2}]$ such that $\tan\gamma = \frac{k\delta s}{\delta\sigma}$ where $k^2 = \frac{7}{\sqrt{2}}$, $k > 0$. To show that this increases from 0 to $\frac{\pi}{2}$ one computes

$$\gamma'(t) = k + \tfrac{1}{2}v(t)\frac{k\delta s\delta\sigma}{k^2\delta s^2 + \delta\sigma^2} \geq k - \tfrac{1}{4}|v(t)| \geq k - \tfrac{1}{4}W_0.$$

It is easy to check that $k - \frac{1}{4}W_0 > 1.55$ and it follows that $\gamma(t)$ reaches $\frac{\pi}{2}$ before time $t = \frac{\pi}{3.1}$ as claimed.

Next, consider $x \in \mathcal{A}_2$. Note that $\mathcal{A}_2$ is contained in the invariant set $\mathcal{I}_3$ of isosceles solutions with m_3 on the axis of symmetry. Some of the points of $\mathcal{A}_2$ may lie in $W_0^s(L_-)$ and therefore never exit $\mathcal{W}$, but Lemma 4 provides a continuous extension of the exit map F. Assuming $\phi_t(x)$ does exit $\mathcal{W}$, it must do so through $\mathcal{E} = \mathcal{B}_1 \cup \mathcal{B}_2$ so it suffices to show that exiting through $\mathcal{B}_2$ with $v \geq 0$ is impossible. In other words, if the exit occurs with $\sigma(t) = 0$ then $v(t) < 0$. To prove this, another lemma will be used.

Lemma 6. *There is a function of the form $F = v + f(s)\sigma$ where $f \geq 0$ such that the set $\{F \leq 0\}$ is strictly positively invariant relative to $\mathcal{W} \cap \mathcal{I}_3$, i.e., if $F(x) \leq 0$ for $x \in \mathcal{W} \cap \mathcal{I}_3$ then $F(\phi_t(x)) < 0$ holds for all $t > 0$ as long as the orbit remains in $\mathcal{W} \cap \mathcal{I}_3$.*

This will be proved later on. Note that for any initial condition in $\mathcal{A}_2$, one has $v = 0$ and $\sigma \leq 0$ and so $F \leq 0$. It follows that $F < 0$ at the exit point, i.e., $v < -f\sigma$. If $\sigma = 0$ this implies $v < 0$ as required.

The third arc, $\mathcal{A}_3$, lies entirely in the triple collision manifold $\{r = 0\}$. This is an invariant set, so the equation $r(t) = 0$ continues to hold for all t. In addition, the differential equation for v' on the collision manifold reduces to

$$v' = W(s, \theta) - \tfrac{1}{2}v^2 = \tfrac{1}{2}\frac{\sigma^2 + s^2\tau^2}{(1 + s^2)^2} \geq 0.$$

Initially this is strictly positive and it follows that the exit must occur with $v(t) > 0$ as claimed.

Finally, consider the arc $\mathcal{A}_4$ given by $\sigma = -c_2\tau$. For $c_2 > 0$ small, this is close to the invariant collinear set $\mathcal{C}$ which has been deleted from $\mathcal{W}$. It must be shown that if $x \in \mathcal{A}_4$ then $\phi_t(x)$ exits with $\sigma(t) = 0$ rather than with $\theta(t) = \frac{\pi}{3}$. This will be done by using the variational equations along solutions in $\mathcal{C}$. As before, the equations involving the variations in the variables s, σ are decoupled from the others. This time one finds

$$\begin{aligned}\delta s' &= \delta\sigma \\ \delta\sigma' &= 4W_{ss}\delta s - \tau^2\delta s - \tfrac{1}{2}v(t)\delta\sigma.\end{aligned} \tag{18}$$

but W_{ss} is no longer constant. In fact $W_{ss}(1,\theta) \le W_{ss}(1,0) = -\frac{7}{8}\sqrt{2} \approx -1.237$. Furthermore, $W_{ss}(1,\theta) \to -\infty$ as $\theta \to \frac{\pi}{3}$ and this will be the main point of the argument.

Define a different angle $\omega(t)$ in the $(\delta s, \delta\sigma)$-plane such that $\tan\omega = \frac{\delta\sigma}{\delta s}$. Then

$$\omega'(t) = \frac{(-4W_{ss}+\tau^2)\delta s^2+\delta\sigma^2+\frac{1}{2}v(t)\delta s\delta\sigma}{\delta s^2+\delta\sigma^2} \ge \frac{-4W_{ss}\delta s^2-\sqrt{W/2}\delta s\delta\sigma+\delta\sigma^2}{\delta s^2+\delta\sigma^2}$$

where the energy bound $v^2 \le 2W(t)$ has been used.

Write the quadratic form in the numerator of $\omega'(t)$ as $A\delta s^2 + 2B\delta s\delta\sigma + C\delta\sigma^2$ where

$$A = -W_{ss}(1,\theta) \qquad B = \sqrt{W/8} \qquad C = 1.$$

Then $A > 0, C > 0$ and $\Delta = AC - B^2 = -W_{ss}(1,\theta) - \frac{1}{8}W(1,\theta)$. It is not too difficult to show that $\Delta > 0$ for $0 \le \theta < \frac{\pi}{3}$. In fact, it is increasing in θ achieving the minimum value $\Delta = \frac{9}{8}\sqrt{2}$ when $\theta = 0$ and $\Delta \to \infty$ as $\theta \to \frac{\pi}{3}$. It follows that the angle $\omega(t)$ is strictly increasing.

For solutions $\phi_t(x)$, $x \in \mathcal{A}_4$, $\theta(t) > 0$ for $t > 0$ and $\theta(t)$ is increasing. Meanwhile the behavior of $s(t), \sigma(t)$ is well-approximated by that of $\delta s(t), \delta\sigma(t)$ obeying (18). It will be shown that the angle $\omega(t)$ reaches $\frac{\pi}{2}$ before $\theta(t)$ reaches $\frac{\pi}{3} - \delta$ where $\delta > 0$ is a small constant. In fact, the change in $\omega(t)$ as $\theta(t)$ varies over $[\frac{\pi}{3} - 2\delta, \frac{\pi}{3} - \delta]$ will be seen to exceed $\frac{\pi}{2}$.

To see this first note that $\theta'(t) = \tau(t) \le \sqrt{2W(1,\theta)}$. Therefore

$$\frac{d\omega}{d\theta} \ge \frac{A\delta s^2 + 2B\delta s\delta\sigma + C\delta\sigma^2}{\sqrt{2W}(\delta s^2 + \delta\sigma^2)}. \tag{19}$$

This has a positive lower bound $\frac{d\omega}{d\theta} \ge d_1 > 0$ for $0 \le \theta < \frac{\pi}{3}$. Therefore, by the time $\theta = 1$, one has $\omega(t) \ge d_1$. This gives a positive lower bound

$$\frac{\delta s^2}{\delta s^2 + \delta\sigma^2} \ge \sin^2 d_1$$

valid for $\theta \ge 1$. Using this in (19) gives

$$\frac{d\omega}{d\theta} \ge \frac{A\sin^2 d_1 - (2B + C)}{\sqrt{2W}} \ge \frac{A\sin^2 d_1}{\sqrt{2W}} - k_1$$

where k_1 is a constant (it turns out that one could even use $k_1 = 1$). Now $A = -4W_{ss}(1,\theta)$ has a pole of order 3 at $\theta = \frac{\pi}{3}$ whereas $W(1,\theta)$ has a simple pole. It follows that there is a constant $k_2 > 0$ such that

$$\frac{d\omega}{d\theta} \geq k_2\delta^{-5/2} - k_1$$

holds for $\theta \in [\frac{\pi}{3} - 2\delta, \frac{\pi}{3} - \delta]$ for δ small enough. Hence the change in $\omega(t)$ as $\theta(t)$ varies over this interval tends to ∞ like $\delta^{-3/2}$ as $\delta \to 0$. If δ is small enough the change in ω will be at least $\frac{\pi}{2}$ as required. □

Proof (Proof of Lemma 6). The lemma refers only to orbits in the isosceles manifold $\mathcal{I}_3$, where $\theta = \tau = 0$. It will be convenient to replace the variable s by a new variable ψ such that

$$s = \frac{\cos\psi}{1 + \sin\psi}.$$

ψ is one of the ordinary spherical coordinates on the shape sphere. In addition, let $\alpha = \psi'$. Then the differential equations can be written

$$\begin{aligned} r' &= vr \\ v' &= W(\psi) - \tfrac{1}{2}v^2 + \tfrac{1}{4}\alpha^2 \\ \psi' &= \alpha \\ \alpha' &= 4\,W_\psi - \tfrac{1}{2}v\alpha. \end{aligned} \tag{20}$$

with energy equation:

$$\tfrac{1}{2}v^2 + \tfrac{1}{8}\alpha^2 - W(\psi) = rh \tag{21}$$

and

$$W(\psi) = \frac{1}{\sqrt{1+\cos\psi}} + \frac{2}{\sqrt{1 - \frac{1}{2}\cos\psi}}.$$

Using these coordinates, define $F = v - \frac{1}{4}\psi\alpha$. Since σ and α are related by $\sigma = -\frac{\alpha}{1+\sin\psi}$, this can be written in the form $F = v + f\,\sigma$ with $f \geq 0$. It suffices to show that for all $x = (r, v, \psi, \alpha) \in \mathcal{W} \cap \mathcal{I}_3$ such that $F(x) = 0$, the time derivative along the flow satisfies $F'(x) < 0$. Calculating F' from the differential equations and using $F = 0$ to replace v by $\frac{1}{4}\psi\alpha$ gives

$$F'|_{F=0} = \frac{\psi^2}{16}\alpha^2 - W - \psi W_\psi.$$

With $v = \frac{1}{4}\psi\alpha$, the energy equation gives

$$\alpha^2 = \frac{16}{(4+\psi^2)}$$

and so

$$F'|_{F=0} \leq \frac{\psi^2}{4+\psi^2} - W(\psi) - \psi W_\psi.$$

It can be shown that this function is strictly negative for $0 \leq \psi \leq \frac{\pi}{2}$ as required. □

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Compatible Poisson Brackets, Quadratic Poisson Algebras and Classical r-Matrices

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Abstract We show that for a general quadratic Poisson bracket it is possible to define a lot of associated linear Poisson brackets: linearizations of the initial bracket in the neighborhood of special points. We prove that the constructed linear Poisson brackets are always compatible with the initial quadratic Poisson bracket.

We apply the obtained results to the cases of the standard quadratic r-matrix bracket and to classical "twisted reflection algebra" brackets. In the first case we obtain that there exists only one non-equivalent linearization: the standard linear r-matrix bracket and recover well-known result that the standard quadratic and linear r-matrix brackets are compatible. We show that there are a lot of non-equivalent linearizations of the classical twisted Reflection Equation Algebra bracket and all of them are compatible with the initial quadratic bracket.

1 Introduction

The theory of compatible Poisson brackets (or so-called bihamiltonian theory) has appeared almost 30 years ago [1]. Magri had observed the following highly non-trivial fact: a linear combination of two Poisson brackets (or two Poisson tensors) is not always again a Poisson but demands an additional compatibility condition (the annihilation of mutual Schouten brackets for two Poisson tensors). The theory of such compatible brackets (or in other terminology of linear Poisson pencils) was developed later in many papers (see for example [2–6]). Compatibility conditions of Lie-Poisson brackets on finite-dimensional spaces was recently systematically studied in [7] in the assumption that one of the brackets is semisimple.

In the present paper we study linear Poisson pencils of quadratic and linear Poisson brackets. Although there exists a lot of examples of such compatible brackets in literature (see for example [4, 14–16]), there is no (up to our knowledge) a general compatibility theory for such the brackets .

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The central idea in this paper is the idea of a linearization which comes from the Poisson–Lie group theory. In more detail, we show that it is possible to construct compatible linear and quadratic Poisson brackets by a linearization of the quadratic bracket in the neighborhood of zeros of their right-hand sides. For each of such zero it is possible to define a linear Poisson bracket which, as we will show, is automatically compatible with the initial quadratic one. We consider the application of this result to the theory of integrable systems in a spirit of the Lenard–Magri scheme [1]. In particular, we obtain a set of commuting functions with respect to both linear and quadratic brackets (we call them "integrals" or "hamiltonians") starting from polynomial Casimirs. It is necessary to emphasize that different choice of zeros of the quadratic Poisson structure may produce non-equivalent linearizations and non-equivalent sets of commuting hamiltonians. We show that the hamiltonian dynamics with respect to one of these hamiltonians of degree k and quadratic bracket can be re-written in the terms of hamiltonian dynamics with respect to a hamiltonian of the degree $k + 1$ and the corresponding linear bracket. The last statement has a conceptual importance from the point of view of classical dynamical systems: it means that on the classical level the quadratic Poisson bracket produce the same integrable dynamics as its linearization.

We apply the obtained results to a case of the standard quadratic r-matrix bracket and after that to the classical "twisted Reflection Equation Algebra" brackets. We consider spectral parameter-dependent case, though the same results evidently holds true for the spectral parameter-independent situation. The geometry of compatible linear and quadratic brackets associated with a constant r-matrix was considered in [8]. The case of constant Reflection Equation Algebra bracket was extensively studied in [9].

In the case of the standard quadratic r-matrix bracket [10, 11] (in the present paper for the sake of simplicity we consider only the case of $gl(n)$-valued T-matrices):

$$\{T_1(\lambda), T_2(\mu)\}_2 = [r_{12}(\lambda - \mu), T_1(\lambda)T_2(\mu)], \tag{1}$$

we show that its linearizations in the neighborhood of its different zeros produce the standard linear r-matrix bracket [10, 11]:

$$\{T_1(\lambda), T_2(\mu)\}_1 = [r_{12}(\lambda - \mu), T_1(\lambda) + T_2(\mu)], \tag{2}$$

or isomorphic to it brackets. In this way we obtain that brackets (1) and (2) are compatible for the arbitrary choice of the r-matrix $r(\lambda - \mu)$ satisfying classical Yang–Baxter equation [10, 12, 13] and for the arbitrary choice of the "monodromy" matrices $T(\lambda)$ satisfying these brackets. The compatibility holds true both for special matrices $T(\lambda)$ that define structure of the finite dimensional quadratic algebra (e.g. those with simple poles in some fixed set of points) and for general meromorphic $T(\lambda)$ that define a structure of a infinite dimensional quadratic algebra. This result generalizes a set of recent results [14–16] about the compatibility of the linear and quadratic r-matrix brackets for the cases of the Belavins classical elliptic r-matrix [17] and monodromy matrices $T(\lambda)$ possessing simple poles. In the present paper we show that $d(\lambda) = \det T(\lambda)$ is a generating function of the Casimir functions of the brackets (1) for any classical r-matrix $r(\lambda - \mu)$ taking the values in $sl(n) \otimes sl(n)$. This fact is certainly known but its rigorous proof seems to be absent in literature. We produce generating functions of commutative integrals of the both of the above brackets applying argument-shift method to the function $d(\lambda)$. We show that generating functions of integrals of order k produce the same integrable dynamics with respect to the quadratic brackets as generating functions of order $k+1$ with respect to the linear bracket.

The main class of our examples is connected with the classical twisted reflection algebra:

$$\{\mathcal{T}_1(\lambda), \mathcal{T}_2(\mu)\}_2 = r_{12}(\lambda - \mu)\mathcal{T}_1(\lambda)\mathcal{T}_2(\mu) - \mathcal{T}_1(\lambda) r_{12}^{\sigma_1}(-\lambda - \mu)\mathcal{T}_2(\mu) - \\ - \mathcal{T}_2(\mu) r_{12}^{\sigma_2}(\lambda + \mu)\mathcal{T}_1(\lambda) + \mathcal{T}_1(\lambda)\mathcal{T}_2(\mu) r_{12}^{\sigma_1\sigma_2}(\mu - \lambda) \quad (3)$$

where σ is a second order automorphism of $gl(n)$. (In in the case $\sigma = 1$ brackets (3) coincide with the classical limits of standard Reflection Equation algebras [18, 19].) We show that there are non-equivalent "linearizations" of these brackets determined by the non-dynamical matrices $K(\lambda)$, where $K(\lambda)$ satisfies the following equation:

$$r_{12}(\lambda - \mu)K_1(\lambda)K_2(\mu) - K_1(\lambda) r_{12}^{\sigma_1}(-\lambda - \mu)K_2(\mu) = \\ = K_2(\mu) r_{12}^{\sigma_2}(\lambda + \mu)K_1(\lambda) - K_1(\lambda)K_2(\mu) r_{12}^{\sigma_1\sigma_2}(\mu - \lambda). \quad (4)$$

These linear brackets have in terms of new variables $\mathcal{T}^K(\lambda) \equiv \mathcal{T}(\lambda)K^{-1}(\lambda)$ the following explicit form:

$$\{\mathcal{T}_1^K(\lambda), \mathcal{T}_2^K(\mu)\}_1 = [r_{12}(\lambda - \mu) - K_2(\mu) r_{12}^{\sigma_2}(\lambda + \mu)K_2^{-1}(\mu), \mathcal{T}_1^K(\lambda)] - \\ - [r_{21}(\mu - \lambda) - K_1(\lambda) r_{21}^{\sigma_1}(\lambda + \mu)K_1^{-1}(\lambda), \mathcal{T}_2^K(\mu)]. \quad (5)$$

All linearizations corresponding to different $K(\lambda)$ are compatible with the initial twisted reflection algebra bracket (3). In the case of $\mathfrak{g} = sl(2)$, elliptic r-matrix $r(\lambda - \mu)$ of Sklyanin [10], trivial automorphism σ and $K(\lambda) = 1$ such compatibility was observed in the first time in [15]. In the general case our result seems to be new.

Let us also note that the bracket (5) is an example of the linear Poisson bracket governed by the non-skew symmetric r-matrices $r(\lambda, \mu)$ [20–22]. In our case we have:

$$r(\lambda, \mu) = r^{\sigma,K}(\lambda, \mu) \equiv r_{12}(\lambda - \mu) - K_2(\mu) r_{12}^{\sigma_2}(\lambda + \mu)K_2^{-1}(\mu). \quad (6)$$

The classical r-matrix $r^{\sigma,K}(\lambda, \mu)$ may be obtained also from some other considerations [23].

Let us emphasize that each solution of (4) produces its own set of commuting with respect to the bracket (3) functions. This gives one possibility to obtain a lot of different commuting families of functions with respect to the same bracket (3). We consider in detail the families of solutions of (4) and the families of the corresponding linear structures given by the r-matrix (6) in the simplest case of $\mathfrak{g} = sl(2)$ and elliptic r-matrix of Sklyanin [10]. We also show that $D(\lambda) = \det\mathcal{T}(\lambda)$ is a generating function of the Casimir functions of the brackets (3) for any classical r-matrix $r(\lambda - \mu)$. We produce generating functions of commutative integrals of the above brackets applying argument-shift method to the function $D(\lambda)$. We show that generating functions of the integrals of order k produce the same integrable dynamics with respect to the quadratic bracket (3) as generating functions of order $k+1$ with respect to the linear bracket (5).

The structure of the paper is the following: in Sect. 2 we consider general theory of consistent quadratic and linear Poisson brackets. In Sect. 3 we apply the results of Sect. 2 to the quadratic brackets (1) and (3) investigate their linearizations and corresponding algebras of integrals.

2 Compatible Brackets and Linearization: General Case

Let M be a (finite or infinite-)dimensional manifold. Let $\{T_i | i \in I\}$ (where I is some finite set in the finite-dimensional case, or $\mathbf{Z}$ in the infinite-dimensional case) be the coordinate functions in some local chart on this manifold. Let us consider a quadratic Poisson bracket on M, admitting in these coordinates the following explicit form:

$$\{T_i, T_j\}_2 = \sum_{k,l \in I} C_{ij}^{kl} T_k T_l. \tag{7}$$

Here the tensor C_{ij}^{kl} is assumed to be such that the sum (in infinite-dimensional case) is correctly defined. In this paper we will not study the global aspects of the manifold M geometry, preferring to work in the fixed local chart.

We will now linearize this quadratic bracket. In other words we will obtain a linear Poisson bracket which is in a natural way associated with the bracket (7) and is defined on the tangent space V in some fixed point $c \in M$ with the coordinates c_i. In more detail, let us consider in the same local card some curve $T_i(\eta)$ such that the following decomposition

$$T_i = c_i + \eta L_i + o(\eta) \tag{8}$$

holds true in the vicinity of the point $T_i = c_i$. Here L_i are the coordinates in the tangent space.

Remark 1. The idea of linearization had come form the theory of Poisson–Lie groups, where T_i are considered to be components of the group element and L_i are considered to be components of the corresponding element of the Lie algebra. The expansion (8) in this case is simply expansion of the exponent in the Taylor power series. We will return to this case in the further examples.

It occurred, that not for all the points $c \in M$ or, equivalently, not for all parameters c_i such an expansion and "linearized" Poisson bracket exist. The following Proposition holds true:

Proposition 2.1 *The expansion (8) define in a correct way a linear Poisson bracket on V if and only if constants c_i satisfy the following condition:*

$$\sum_{k,l \in I} C_{ij}^{kl} c_k c_l = 0, \forall i, j \in I, \tag{9}$$

i.e. lie on zeros cone of the quadratic Poisson structure (7). The corresponding linear Poisson structure has in this case the following form:

$$\{L_i, L_j\}_1 = \sum_{k,l \in I} C_{ij}^{kl} (c_k L_l + c_l L_k). \tag{10}$$

Proof. In order to obtain a correct linearization we will consider instead the bracket (7) an equivalent (proportional) bracket :

$$\{T_i, T_j\}_2' = \eta \sum_{k,l \in I} C_{ij}^{kl} T_k T_l. \tag{11}$$

Let us substitute expansion (8) in this Poisson bracket. Taking into account that c_i are complex or real numbers constant lying in the kernel of the Poisson brackets we obtain:

$$\eta^2 \{L_i, L_j\} + o(\eta^2) = \eta \sum_{k,l \in I} C_{ij}^{kl} c_k c_l + \eta^2 \sum_{k,l \in I} C_{ij}^{kl} (c_k L_l + c_l L_k) + o(\eta^2).$$

Comparing the coefficients near powers of η in the both sides of this expression we obtain the statement of the Proposition.

Now, having defined the linear Poisson structure on the tangent space V, we can (using the diffeomorphism of the tangent space in a point of the local chart to the local chart itself) consider the bracket (7) also to be defined on the space V, i.e. on the coordinate functions L_i or vise verse bracket (10) to be defined on the coordinate functions T_i.

In the subsequent we will use the following standard definition [1]:

Definition. Two poisson brackets $\{\ ,\ \}_2$ and $\{\ ,\ \}_1$ are called to be compatible if their linear combination is again a Poisson bracket.

The existence of the compatible Poisson bracket is important for the corresponding theory of integrable systems. The theory of compatible Poisson brackets was conceived in the paper [1] and developed in papers [2–7]. Using compatible Poisson brackets it is possible to construct the algebra of mutually Poisson-commutative functions – "integrals of motion" of an integrable hamiltonian system (see, for example, Theorem 2.1 below).

The following important Proposition holds true:

Proposition 2.2 *The quadratic order Poisson bracket (7) is always compatible with its linearization – the linear Poisson bracket (10).*

Remark 2. Note that it is possible to define a linearization – the corresponding linear Poisson bracket for an arbitrary polynomial bracket of the degree n. But in the case $n > 2$ the linearized bracket will not, generally speaking, be compatible with the initial polynomial Poisson bracket.

Proof. The statement of the Proposition follows from the combination of the previous Proposition and trick with the shift of the argument [24]. Let us consider shift of the local coordinates T_i: $T_i \to T_i + \eta c_i$, where c_i are some constants and substitute it in the bracket (7). Using the fact that c_i are constants we will obtain the following expression:

$$\{T_i, T_j\}_\eta \equiv \{T_i + \eta c_i, T_j + \eta c_j\}_2 = \sum_{k,l \in I} \left(C_{ij}^{kl} T_k T_l + \eta C_{ij}^{kl} (c_k T_l + c_l T_k) + \eta^2 C_{ij}^{kl} c_k c_l \right). \tag{12}$$

Let now constants c_i satisfy conditions (9) (this is needed to the very existence of the linearization (10)). In this case we will have that the last summand in the expression (12) vanishes and we obtain the following expression:

$$\{T_i, T_j\}_\eta = \{T_i, T_j\}_2 + \eta \{T_i, T_j\}_1,$$

where $\{T_i, T_j\}_1 \equiv \sum_{k,l \in I} C_{ij}^{kl}(c_k T_l + c_l T_k)$ is "linearized" bracket (10) and $\{T_i, T_j\}_2 = \sum_{k,l \in I} C_{ij}^{kl} T_k T_l$ is the bracket (7). By other words, this means that linear combination of the brackets $\{T_i, T_j\}_2$ and $\{T_i, T_j\}_1$ is again Poisson bracket, i.e. these brackets are compatible.

Proposition is proved.

This Proposition is a generalization of the well-known in the quantum group theory fact about the compatibility of the "first and second Sklyanin brackets" [10]. In the next subsections we will consider several examples of this sort.

But at first, let us remind the reason of the importance of theory of compatible Poisson brackets (Lenard–Magri scheme) in the theory of classical integrable systems [1]. We will need the following version of this scheme of the construction of the set of mutually commuting functions starting from the Casimir functions of one of the compatible brackets:

Theorem 2.1 *Let $I^n(T)$, $I^m(T)$ be homogeneous polynomial invariants of the Poisson structure $\{\ ,\ \}_2$ of the degree n and m correspondingly. Let C satisfy conditions (10). Then the functions $I^n_k(T, C)$, $I^m_k(T, C)$ obtained via the decomposition of the function $I^n(T + \eta C)$ and $I^m(T + \eta C)$ in the degrees of the parameter η:*

$$I^n(T + \eta C) = \sum_{k=0}^{n} \eta^k I^n_k(T, C), \ \ I^m(T + \eta C) = \sum_{k=0}^{m} \eta^k I^m_k(T, C)$$

constitute commuting family with respect to the brackets $\{\ ,\ \}_2$ *and* $\{\ ,\ \}_1$*:*

$$\{I^n_k(T,C), I^m_l(T,C)\}_2 = \{I^n_k(T,C), I^m_l(T,C)\}_1 = 0.$$

Proof. In order to prove this Proposition let us note that $\{I^n(T+\eta C), F(T+\eta C)\}_\eta = 0$ for arbitrary function F due to the fact that $I^n(T+\eta C)$ is a Casimir function of the bracket $\{\ ,\ \}_\eta$. Putting $F(T) \equiv I^m(T+(\eta'-\eta)C)$ we obtain the following equality:

$$\{I^n(T+\eta C), I^m(T+\eta' C)\}_\eta = 0. \tag{13}$$

In the analogous way one obtains the equality

$$\{I^n(T+\eta C), I^m(T+\eta' C)\}_{\eta'} = 0. \tag{14}$$

From these identities, decomposing the functions $I^n(T+\eta C)$ and $I^m(T+\eta' C)$ in the powers of parameters η and η' and using the fact that $\{\ ,\ \}_\eta = \{\ ,\ \}_2 + \eta\{\ ,\ \}_1$ we obtain the following equalities:

$$\{I^n_k(T,C), I^m_l(T,C)\}_2 + \{I^n_{k-1}(T,C), I^m_l(T,C)\}_1 = 0. \tag{15a}$$

$$\{I^n_k(T,C), I^m_l(T,C)\}_2 + \{I^n_k(T,C), I^m_{l-1}(T,C)\}_1 = 0, \tag{15b}$$

which are coefficients near the powers $\eta^k\eta'^l$ in these expansions. Taking into account that $I^n_n(T,C) = I^n(C)$, $I^m_m(T,C) = I^m(C)$ are constants with respect to the both brackets, i.e.:

$$\{I^n(C), I^m_l(T,C)\}_2 = \{I^n(C), I^m_l(T,C)\}_1 = 0$$
$$\{I^n_k(T,C), I^m(C)\}_2 = \{I^n_k(T,C), I^m(C)\}_1 = 0$$

and substituting this into (15) we obtain the following equalities:

$$\{I^n_{n-1}(T,C), I^m_l(T,C)\}_1 = 0, l \geq 0, \tag{16a}$$

$$\{I^n_k(T,C), I^m_{m-1}(T,C)\}_1 = 0, k \geq 0. \tag{16b}$$

Making use of the equality (16a) and substituting it into the equality (15b), putting there $k = n-1$ we obtain that $\{I^n_{n-1}(T,C), I^m_l(T,C)\}_2 = 0, l > 0$. In a similar way using the equality (16b), substituting it into the equality (15a), putting there $l = m-1$ we obtain that $\{I^m_{m-1}(T,C), I^n_k(T,C)\}_2 = 0, k > 0$. Taking into account that $I^n_0(T,C) \equiv I^n(T)$, $I^m_0(T,C) \equiv I^m(T)$ are Casimir functions of the brackets $\{\ ,\ \}_2$ we obtain also that $\{I^n_{n-1}(T,C), I^m_0(T,C)\}_2 = 0$, $\{I^m_{m-1}(T,C), I^n_0(T,C)\}_2 = 0$. Hence we have proved that

$$\{I^n_{n-1}(T,C), I^m_l(T,C)\}_2 = \{I^n_{n-1}(T,C), I^m_l(T,C)\}_1 = 0, l \in \overline{0,m};$$
$$\{I^m_{m-1}(T,C), I^n_k(T,C)\}_2 = \{I^m_{m-1}(T,C), I^n_k(T,C)\}_1 = 0, k \in \overline{0,n}.$$

Proceeding inductively we at last obtain that

$$\{I^n_{n-k}(T,C), I^m_l(T,C)\}_1 = \{I^n_k(T,C), I^m_{m-l}(T,C)\}_1 = 0,$$
$$\{I^n_{n-k}(T,C), I^m_l(T,C)\}_2 = \{I^n_k(T,C), I^m_{m-l}(T,C)\}_2 = 0.$$

for $k \in \overline{0,n}, l \in \overline{0,m}$. Theorem is proved.

Remark 3. Note, that each point C satisfying conditions (10) provides in the framework of our construction its own algebra of Poisson-commuting (with respect to the brackets $\{\ ,\ \}_i, i \in 1, 2)$ integrals.

The following proposition (a kind of "Lenard recursive relation" [1]) is a consequence of the fact the constructed Poisson brackets $\{\ ,\ \}_i, i \in 1, 2$ are compatible:

Proposition 2.3 *The hamiltonian equation of motion with respect to the bracket* $\{\ ,\ \}_2$ *with the hamiltonian function* $I^n_k(T,C)$, *(*$k < n$ *and is fixed) coincides with the hamiltonian equation of motion with respect to the brackets* $\{\ ,\ \}_1$ *with the hamiltonian function* $-I^n_{k-1}(T,C)$.

Proof. In order to prove this Proposition, let us consider the following Poisson bracket:

$$\{T_i + \eta C_i, I^n(T+\eta C)\}_\eta = \{T_i, I^n(T+\eta C)\}_\eta = \{T_i, I^n(T+\eta C)\}_2 +$$
$$+\eta\{T_i, I^n(T+\eta C)\}_1 = \sum_{k=0}^{n} \eta^k \{T_i, I^n_k(T,C)\}_2 + \sum_{k=1}^{n+1} \eta^k \{T_i, I^n_{k-1}(T,C)\}_1 = 0,$$

where we have used that the function $I^n(T+\eta C)$ is a Casimir of the bracket $\{\ ,\ \}_\eta$ and C_i are constants with respect to this bracket. Due to the fact that this equality holds true for all degrees of the parameter η, we obtain for $0 < k < n$:

$$\{T_i, I^n_k(T,C)\}_2 + \{T_i, I^n_{k-1}(T,C)\}_1 = 0.$$

The hamiltonian equations of motion with respect to the hamiltonian H and brackets $\{\ ,\ \}_k$ are standardly defined as follows:

$$\frac{dT_i}{dt} = \{T_i, H\}_k$$

we obtain the statement of the proposition.

Remark 4. The above Proposition has a conceptual importance for the theory of classical dynamical systems. Indeed, it means that the integrable dynamical systems constructed with the help of quadratic Poisson brackets and the hamiltonian obtained with the help of the "argument shift method" as in the Theorem 2.1 may be interpreted as hamiltonian equations of motion with respect to the corresponding "linearized" Poisson bracket and the other "hamiltonian" obtained by the same "argument shift". Thus, on the level of classical integrable dynamical systems, it is sufficient to consider only linear Poisson brackets, because the corresponding quadratic bracket produces the same integrable dynamics.

3 Quadratic Algebras and Poisson–Lie Groups

In this section we will illustrate the results of the previous section on the examples of the quadratic Poisson algebra related to the RTT and to the Reflection Equation Algebras which are the main objects of interest of the present paper. In some cases the obtained results are well-known, in other cases they seemed to be new. For the sake of simplicity and in order to avoid global geometrical and topological subtleties we will restrict ourselves to the case of the Lie group $GL(n)$ and the corresponding Lie algebra $gl(n)$.

3.1 First and Second Sklaynin Brackets

Compatibility

Let us consider the reductive Lie algebra $gl(n)$ and its simple subalgebra $sl(n)$. Let us consider a meromorphic function of two complex variables $r_{12}(\lambda-\mu)$ taking values in $sl(n)\otimes sl(n)$ satisfying the usual classical Yang–Baxter equation [10,12,13]:

$$[r_{12}(\lambda-\mu), r_{13}(\lambda-\nu)] = [r_{23}(\mu-\nu), r_{12}(\lambda-\mu) + r_{13}(\lambda-\nu)], \qquad (17)$$

where $r_{12}(\lambda-\mu)$, considered as a matrix acting in the space $sl(n)$, is nondegenerate.

Remark 5. It is possible also to consider nondegenerate $gl(n)$-valued r-matrices

$$r'(\lambda,\mu) \equiv f(\lambda,\mu)\cdot 1\otimes 1 + r(\lambda-\mu),$$

where $r(\lambda-\mu) \in sl(n)\otimes sl(n)$, $f(\lambda,\mu)$ is an arbitrary function of two complex variables, but they will lead to the same Poisson structures as $r(\lambda-\mu)$ and, that is why, we will restrict ourselves to the consideration of $sl(n)$-valued r-matrices.

The important fact [13] is that from the classical Yang–Baxter equation follows the skew-symmetry of the r-matrix $r_{12}(\lambda,\mu)$:

$$r_{12}(\lambda-\mu) = -r_{21}(\mu-\lambda).$$

This gives a possibility to define, using $r_{12}(\lambda,\mu)$, so-called Sklyanin quadratic bracket [10]:

$$\{T_1(\lambda), T_2(\mu)\}_2 = [r_{12}(\lambda-\mu), T_1(\lambda)T_2(\mu)], \qquad (18)$$

where $T_1(\lambda) \equiv T(\lambda) \otimes 1$, $T_2(\mu) \equiv 1 \otimes T(\mu)$, $T(\lambda)$ takes values in a classical matrix Lie group $GL(n)$, and is a meromorphic function of spectral parameter λ. Due to the fact, that all our consideration will have a local character, we will, slightly abusing the language, consider $T(\lambda)$ to be an element of $gl(n)$.

Remark 6. Generally speaking, the bracket (18) is defined in the infinite-dimensional space of meromorphic functions of λ where it defines a structure of an infinite-dimensional quadratic algebra. It also defines the structure of finite-dimensional quadratic algebras in special subspaces of meromorphic functions. For example, in the case of $sl(2)$ elliptic r-matrix and matrices $T(\lambda)$ possessing one simple pole at $\lambda = 0$ it defines the Sklyanin algebra [25] or "many-poled" Sklyanin algebras in the cases of matrices $T(\lambda)$ with many simple poles [15].

We will not need to write more explicitly the commutation relations (18) leaving them in the "convoluted" form. Let us now consider the expansion (8):

$$T(\lambda) = C(\lambda) + \eta L(\lambda) + o(\eta),$$

where $C \in GL(n)$. It gives us the following linearization:

$$\{L_1(\lambda), L_2(\mu)\}_1 = [r_{12}(\lambda - \mu), L(\lambda) \otimes C(\mu) + C(\lambda) \otimes L(\mu)]. \tag{19}$$

Here the "initial point" C in this decomposition satisfies the following condition:

$$[r_{12}(\lambda, \mu), C(\lambda) \otimes C(\mu)] = 0. \tag{20}$$

Remark 7. Note that the linearization condition (20) has the same form for all infinite-dimensional quadratic structures defined by the (18) and for all finite-dimensional substructures (like the Sklyanin algebra [25], the many-poled Sklyanin algebra [15], etc.).

In the special case, when $C(\lambda)$ coincides with a unit element of the group $GL(n)$ ($C(\lambda) = 1$), we obtain from (21) the standard "second" Sklyanin bracket:

$$\{L_1(\lambda), L_2(\mu)\}_1 = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)]. \tag{21}$$

Hence we have recovered the well-known fact [11] that "first" and "second Sklyanin brackets" are compatible.

It is easy to show that the bracket (19) is equivalent to the bracket (21). Indeed, making a change of variables: $L(\lambda) = L^C(\lambda)C(\lambda)$ and using the condition (19) we obtain that the bracket (19) pass to the bracket (21). Hence, in this class of examples one does not come to the different compatible structures choosing different initial points for linearization. The situation is different in the case of other quadratic algebras, the examples of which we will consider in the next subsections. As we will see the choice of the "initial point" in the neighborhood of which we linearize Poisson bracket may lead to different Lie algebraic structures.

Algebra of Integrals

Let us now consider an impact of compatibility of quadratic and linear Sklyanin brackets in the corresponding theory of integrable systems. We will show, in particular, that commutativity of generating functions of the classical integrals $tr(T(\lambda))^k$ (or $tr(L(\lambda))^k$) with respect to the both linear and second order Poisson brackets may be derived using only the theory of compatible Poisson brackets.

In order to use the theory of bihamiltonian systems we have to describe Casimir functions of the quadratic Sklyanin brackets on $gl(n)$. The following result is true:

Theorem 3.1 *Let $d(\lambda)$ be a determinant of the $gl(n)$-valued monodromy matrix $T(\lambda)$. Then $d(\lambda)$ is a generating function of the Casimir functions of the bracket (18) for all types of "monodromy" matrices $T(\lambda)$ satisfying the brackets (18) and for all types of $sl(n)\otimes sl(n)$-valued r-matrices.*

Proof. In order to prove the theorem we have to show that $\{d(\lambda), T_{ij}(\mu)\}_2 = 0$, where the bracket $\{\ ,\ \}_2$ has the form (18). Let us notice at first that $trL(\lambda)$ is a Casimir function of the bracket (21) for any $sl(n)$-valued r-matrix $r_{12}(\lambda,\mu)$. Indeed:

$$\{tr_1(L_1(\lambda)), L_2(\mu)\} = tr_1([r_{12}(\lambda-\mu), L_1(\lambda)]) + [tr_1(r_{12}(\lambda-\mu)), L_2(\mu)] = 0.$$

The first summand is equal to zero as a trace of commutator and the second one is equal to zero because $tr_1(r_{12}(\lambda-\mu)) = 0$ due to the fact that our r-matrix is $sl(n)$-valued.

Now let us make the following change of variables:

$$T(\lambda) = \exp L(\lambda). \tag{22}$$

Keeping in mind the well-known relation of determinants and traces we obtain:

$$d(\lambda) \equiv detT(\lambda) = \exp trL(\lambda). \tag{23}$$

If the change of variables (22) had been Poisson,the equality (23) would have been sufficient for the proof of the theorem. But, unfortunately, it is not Poisson. Nevertheless the change of variables (22) will be still useful. Indeed, let us explicitly calculate expressions $\{\exp L_1(\lambda), \exp L_2(\mu)\}_2$ and $\{\exp L_1(\lambda), \exp L_2(\mu)\}_1$. We have:

$$\{\exp L_1(\lambda), \exp L_2(\mu)\}_2 = [r_{12}(\lambda-\mu), \exp L_1(\lambda)\exp L_2(\mu)] =$$
$$= \sum_{k,l=0}^{\infty} \frac{1}{k!l!}[r_{12}(\lambda-\mu), L_1^k(\lambda)L_2^l(\mu)]. \tag{24}$$

Hereafter it is implied that $L_1^0(\lambda) \equiv 1$. On the other hand we obtain:

$$\{\exp L_1(\lambda), \exp L_2(\mu)\}_1 = \sum_{k,l=0}^{\infty} \frac{1}{k!l!}\{L_1^k(\lambda), L_2^l(\mu)\}_1. \tag{25}$$

By the recursion applied first to k and then to l we obtain the following equality:

$$\{L_1^k(\lambda), L_2^l(\mu)\}_1 = (\sum_{i=0}^{k-1} L_1(\lambda)^i [r_{12}(\lambda-\mu), L_2^l(\mu)] L_1(\lambda)^{k-1-i} + \sum_{j=0}^{l-1} L_2(\mu)^j [r_{12}(\lambda-\mu), L_1^k(\lambda)] L_2(\mu)^{l-1-j}). \quad (26)$$

Now we will rewrite the formula (26) in the following way:

$$\frac{1}{k!l!}\{L_1^k(\lambda), L_2^l(\mu)\}_1 =$$
$$\frac{1}{2(k-1)!l!}(L_1(\lambda)^{k-1}[r_{12}(\lambda-\mu), L_2^l(\mu)] + [r_{12}(\lambda-\mu), L_2^l(\mu)]L_1(\lambda)^{k-1}) +$$
$$\frac{1}{2k!(l-1)!}([r_{12}(\lambda-\mu), L_1^k(\lambda)]L_2(\mu)^{l-1} + L_2(\mu)^{l-1}[r_{12}(\lambda-\mu), L_1^k(\lambda)]) +$$
$$\frac{1}{2k!l!}(\sum_{i=0}^{k-1}[L_1(\lambda)^i, [[r_{12}(\lambda-\mu), L_2^l(\mu)], L_1(\lambda)^{k-1-i}]] +$$
$$\sum_{j=0}^{l-1}[[L_2(\mu)^j, [r_{12}(\lambda-\mu), L_1^k(\lambda)]], L_2(\mu)^{l-1-j}]).$$

Using this equality, the Leibniz rule for the commutator, the fact that $[L_1^k(\lambda), L_2^l(\mu)] = 0$ and renaming the indices $k-1 \to k$, when summing the first two summands of the right-hand side of this equality with respect to k, l and $l-1 \to l$ when summing the next two summands of the right-hand side of this equality with respect to k, l, it is easy to show that

$$\sum_{k,l=0}^{\infty} \frac{1}{k!l!}\{L_1^k(\lambda), L_2^l(\mu)\}_1 = \sum_{k,l=0}^{\infty} \frac{1}{k!l!}[r_{12}(\lambda-\mu), L_1^k(\lambda)L_2^l(\mu)] +$$
$$\sum_{k,l=0}^{\infty} \frac{1}{2k!l!}(\sum_{i=0}^{k-1}[L_1(\lambda)^i, [[r_{12}(\lambda-\mu), L_2^l(\mu)], L_1(\lambda)^{k-1-i}]] +$$
$$\sum_{j=0}^{l-1}[[L_2(\mu)^j, [r_{12}(\lambda-\mu), L_1^k(\lambda)]], L_2(\mu)^{l-1-j}]).$$

By other words we have obtained that:

$$\{\exp L_1(\lambda), \exp L_2(\mu)\}_2 = \{\exp L_1(\lambda), \exp L_2(\mu)\}_1 - \widetilde{X}_{12}(L), \quad \text{where}$$

$$\widetilde{X}_{12}(L) \equiv \sum_{k,l=0}^{\infty} \frac{1}{2k!l!}\Big(\sum_{i=0}^{k-1}[L_1(\lambda)^i, [[r_{12}(\lambda-\mu), L_2^l(\mu)], L_1(\lambda)^{k-i}]]+ \\ +\sum_{j=0}^{l-1}[[L_2(\mu)^j, [r_{12}(\lambda-\mu), L_1^k(\lambda)]], L_2(\mu)^{l-1-j}]\Big).$$

Now, we remark, that using the Jacobi identity and the fact that $[L_1^k(\lambda), L_2^j(\mu)] = 0$ the expression $\sum_{j=0}^{l-1}[[L_2(\mu)^j, [r_{12}(\lambda - \mu), L_1^k(\lambda)]], L_2(\mu)^{l-1-j}])$ can be further transformed in the following form: $\sum_{j=0}^{l-1}[L_1(\lambda)^k, [[r_{12}(\lambda-\mu), L_2^j(\mu)], L_2(\mu)^{l-1-j}]]$. Hence, we can re-write the expression for $\widetilde{X}_{12}(L)$ as:

$$\widetilde{X}_{12}(L) = \sum_{m=0}^{\infty}[L_1^m(\lambda), X_{12}^{(m)}(L)]. \tag{27}$$

Now, let us consider the expression

$$\{d(\lambda), T_2(\mu)\}_2 \equiv \sum_{k,l=1}^{n}\{d(\lambda), T_{kl}(\mu)\}_2 X_{kl} = \\ = \sum_{i,j,k,l=1}^{n} \frac{\partial d(\lambda)}{\partial T_{ij}(\lambda)}\{T_{ij}(\lambda), T_{kl}(\mu)\}_2 X_{kl} = \langle \nabla_1 d(\lambda), \{T_1(\lambda), T_2(\mu)\}_2\rangle_1,$$

where $\nabla_1 d(\lambda) \equiv \sum_{i,j=1}^{n} \frac{\partial d(\lambda)}{\partial T_{ij}(\lambda)} X_{ji} \otimes 1$, and $\langle\ ,\ \rangle_1$ means the scalar product in the first factor of the tensor product $gl(n) \otimes gl(n)$. As it was shown above, we can re-write this as follows:

$$\langle \nabla_1 d(\lambda), \{T_1(\lambda), T_2(\mu)\}_2\rangle_1 = \langle \nabla_1 d(\lambda), \{T_1(\lambda), T_2(\mu)\}_1\rangle_1 - \langle \nabla_1 d(\lambda), \widetilde{X}_{12}(L)\rangle_1, \tag{28}$$

where $\{T_1(\lambda), T_2(\mu)\}_1 \equiv \{\exp L_1(\lambda), \exp L_2(\mu)\}_1$. On the other hand first summand in (28) can be written in the following form:

$$\langle \nabla_1 d(\lambda), \{T_1(\lambda), T_2(\mu)\}_1\rangle_1 = \{d(\lambda), T_2(\mu)\}_1 = \{\exp trL_1(\lambda), \exp L_2(\mu)\}_1.$$

Taking into account that $tr L(\lambda)$ is a Casimir function of the brackets $\{\ ,\ \}_1$ we obtain that $\{\exp tr L_1(\lambda), \exp L_2(\mu)\}_1 = 0$.

Let us consider the second summand in the equality (28). Let us take into account that $\frac{1}{d(\lambda)}\frac{\partial d(\lambda)}{\partial T_{ij}(\lambda)} = (T^{-1}(\lambda))_{ji}$ and, hence, $\nabla_1 d(\lambda) = d(\lambda)T_1^{-1}(\lambda)$. That is why we obtain:

$$\langle \frac{\nabla_1 d(\lambda)}{d(\lambda)}, \widetilde{X}_{12}(L)\rangle_1 = \langle T_1^{-1}(\lambda), \widetilde{X}_{12}(L)\rangle_1 = \sum_{m=0}^{\infty} \langle T_1(\lambda)^{-1}, [L_1^m(\lambda), X_{12}^{(m)}(L)]\rangle_1.$$

On the other hand

$$\langle T_1(\lambda)^{-1}, [L_1^m(\lambda), X_{12}^{(m)}(L)]\rangle_1 = \langle [T_1(\lambda)^{-1}, L_1^m(\lambda)], X_{12}^{(m)}(L)\rangle_1,$$

where we have used the ad-invariance of the pairing $\langle\ ,\ \rangle_1$. Now using the fact that $T_1(\lambda)^{-1} = \exp(-L_1(\lambda))$ and the evident fact that $[\exp(-L_1(\lambda)), L_1^m(\lambda)] = 0$ we obtain that the second summand in (28) is also zero.

Theorem is proved.

Remark 8. The theorem above is a generalization of the well-known for the concrete r-matrices fact that $\det T(\lambda)$ is a Casimir function of the quadratic r-matrix bracket onto the case of the arbitrary r-matrices (including all trigonometric r-matrices of Belavin and Drienfield [13], non-standard rational r-matrices of Stolin [26], etc.).

From the Theorems 3.1 and 2.1 follows the next:

Corollary 3.1 *Denote by $d_k(T(\lambda))$ the coefficients in the decomposition: $Det(T(\lambda) + \eta I) = \sum_{k=0}^{n} d_k(T(\lambda))\eta^k$. Then $d_k(T(\lambda))$ are generators of an Abelian subalgebra with respect to the both linear and quadratic r-matrix brackets (21) and (18), i.e.:*

$$\{d_k(T(\lambda)), d_l(T(\mu))\}_1 = \{d_k(T(\lambda)), d_l(T(\mu))\}_2 = 0.$$

Proof. In order to prove this corollary it is enough to use the Theorem 2.1 applied for the two polynomial Casimir functions $d(T(\lambda))$ and $d(T(\mu))$ of the bracket $\{\ ,\ \}_2$.

Remark 9. Functions $\{d_k(T(\lambda))\}$ define the same algebra of integrals as functions $\{trT^l(\lambda)\}$ and are connected with them by the well-known Newton formulas. Nevertheless, it is nice to notice that they could be obtained from the consideration of bihamiltonity and Casimir element $d(\lambda)$ only. This may have deep meaning, because in the quantum case operators $\{tr\hat{T}^l(\lambda)\}$ stops to be commutative, but quantum analogues of the functions $\{d_k(T(\lambda))\}$ still are [27, 28].

Remark 10. It is easy to see that integrals obtained by the shift on another matrix $C \in Gl(n)$ are equivalent to the integrals obtained by the shift on the unit matrix. Indeed, it is evident that $Det(T(\lambda) + \eta C(\lambda)) = DetC Det(T^C(\lambda) + \eta I)$, where $T^C(\lambda) = T(\lambda)C^{-1}(\lambda)$. But, due to the fact that $[r_{12}(\lambda,\mu), C(\lambda)\otimes C(\mu)] = [r_{12}(\lambda,\mu), C^{-1}(\lambda)\otimes C^{-1}(\mu)] = 0$ elements $T^C(\lambda)$ define the same algebra (both with respect to the quadratic and the linear brackets) as elements $T(\lambda)$ in the case $C(\lambda) = 1$.

Now, let us consider what the condition of compatibility of brackets (21) and (18) yields from the point of view of the corresponding dynamics. For this purpose, in principle, we need to decompose the generating functions $d_k(T(\lambda))$ with respect to the spectral parameter λ. Using the fact that $T(\lambda)$ is meromorphic functions this can always be done using the decomposition in Laurent power series. In the special cases the other decompositions may be also used [14]. In the present paper we will not do these decompositions explicitly leaving all the expressions in the "convoluted" form of the generating functions. Using the Proposition 2.3 we obtain the following:

Proposition 3.1 *Hamiltonian equations of motion with respect to the bracket* $\{\ ,\ \}_2$ *with a hamiltonian being one of the functions obtained by a decomposition of function* $d_k(T(\lambda)), k < n$*, coincide with the hamiltonian equations of motion with respect to the brackets* $\{\ ,\ \}_1$ *with the hamiltonian being the corresponding function obtained by the decomposition of function* $-d_{k-1}(T(\lambda))$*, i.e. on the level of generating functions the following equality is true:*

$$\{d_k(T(\lambda)), T(\mu)\}_2 = -\{d_{k-1}(T(\lambda)), T(\mu)\}_1.$$

In the case $k = n - 1$ this Proposition implies the following important corollary:

Corollary 3.2 *Hamiltonian equations of motion with respect to the bracket* $\{\ ,\ \}_2$ *with a hamiltonian being one of the functions obtained by a decomposition of function* $trT(\lambda)$ *coincide with the hamiltonian equations of motion with respect to the brackets* $\{\ ,\ \}_1$ *with the hamiltonian being the corresponding function obtained by the decomposition of the function* $\frac{1}{2}tr(T^2(\lambda))$*, i.e. on the level of generating functions the following equality holds true:*

$$\{tr(T(\lambda)), T(\mu)\}_2 = \frac{1}{2}\{tr(T^2(\lambda)), T(\mu)\}_1.$$

Proof. In order to derive this statement it is sufficient to put in the previous proposition $k = n - 1$, use Newton identities and the fact that function $trT(\lambda)$ is a Casimir function of the linear r-matrix bracket $\{\ ,\ \}_1$.

3.2 "Twisted Reflection Equation Algebra" and Its Linearizations

Compatible Brackets

Let, as in the previous subsection, $r(\lambda - \mu)$ be a $sl(n) \otimes sl(n)$-valued solution of the classical Yang–Baxter equation (17), and $\mathcal{T}(\lambda)$ be a meromorphic function of spectral parameter λ taking the values in a classical matrix Lie group $GL(n)$.

Let σ be some automorphism of $gl(n)$ (and $sl(n)$). Let us consider the space of functions $\mathcal{T}(\lambda)$ the following quadratic brackets:

$$\begin{aligned}\{\mathcal{T}_1(\lambda), \mathcal{T}_2(\mu)\}_2 = {} & r_{12}(\lambda - \mu)\mathcal{T}_1(\lambda)\mathcal{T}_2(\mu) - \mathcal{T}_1(\lambda)r_{12}^{\sigma_1}(-\lambda - \mu)\mathcal{T}_2(\mu) - \\ & - \mathcal{T}_2(\mu)r_{12}^{\sigma_2}(\lambda + \mu)\mathcal{T}_1(\lambda) + \mathcal{T}_1(\lambda)\mathcal{T}_2(\mu)r_{12}^{\sigma_1\sigma_2}(\mu - \lambda), \quad (29)\end{aligned}$$

where $r_{12}^{\sigma_1}(\lambda) = \sigma \otimes 1 \cdot r_{12}(\lambda)$, $r_{12}^{\sigma_2}(\lambda) = 1 \otimes \sigma \cdot r_{12}(\lambda)$, etc. By direct and tedious calculation one can show that (29) is indeed a Poisson bracket, i.e. that it satisfies Jacobi condition for the case of arbitrary automorphism σ and an arbitrary solution $r_{12}(\lambda - \mu)$ of the classical Yang–Baxter equation (see also Remark 14).

One may consider the bracket (29) defined on the infinite-dimensional space of meromorphic functions of λ like in the case of previous subsection. We will call the corresponding algebra a "twisted classical Reflection Equation Algebra". The bracket (29) may be also considered in some special finite-dimensional subspaces of the space of meromorphic functions where it defines a structure of a finite-dimensional quadratic algebra (see for example [16]).

Remark 11. Due to the known fact that all automorphisms of $gl(n)$ are either internal or have a second order we may write that $\sigma = Ad_{K_0}\sigma_0$, where $K_0 \in GL(n)$ and σ_0 is an involutive external automorphism of $gl(n)$ or $\sigma_0 = 1$. All external automorphisms of $gl(n)$ as a Lie algebra are can be written as minus anti-automorphism of $gl(n)$ considered as an associative algebra, and we may write symbolically that $\sigma_0(X) = -X^t$ where upper superscript t denotes anti-automorphism of $gl(n)$ (in particular the ordinary transposition). Moreover by substitution $\mathcal{T}(\lambda) \to \mathcal{T}(\lambda)K_0^{-1}$ we can get rid of Ad_{K_0} and to consider hereafter only the case $\sigma = \sigma_0$.

In the simplest case $\sigma = 1$, using the skew-symmetry of $r(\lambda - \mu)$, we obtain the "classical Reflection Equation Algebra" which is a classical limit of the standard quantum Reflection Equation Algebra [10, 19]:

$$\{\mathcal{T}_1(\lambda), \mathcal{T}_2(\mu)\}_2 = r_{12}(\lambda - \mu)\mathcal{T}_1(\lambda)\mathcal{T}_2(\mu) + \mathcal{T}_1(\lambda)r_{21}(\lambda + \mu)\mathcal{T}_2(\mu) - \\ - \mathcal{T}_2(\mu)r_{12}(\lambda + \mu)\mathcal{T}_1(\lambda) - \mathcal{T}_1(\lambda)\mathcal{T}_2(\mu)r_{21}(\lambda - \mu). \quad (30)$$

Let us linearize the quadratic bracket (29) it in the neighborhood of some point $\mathcal{T}(\lambda) = K(\lambda)$. The sufficient condition of an existence of such a linearization is the requirement (9), which acquires in the case at hand the following form:

$$r_{12}(\lambda - \mu)K_1(\lambda)K_2(\mu) + K_1(\lambda)K_2(\mu)r_{12}^{\sigma_1\sigma_2}(\mu - \lambda) = \\ = K_2(\mu)r_{12}^{\sigma_2}(\lambda + \mu)K_1(\lambda) + K_1(\lambda)r_{12}^{\sigma_1}(-\lambda - \mu)K_2(\mu). \quad (31)$$

Remark 12. For the case $\sigma = 1$ and r matrices $r_{12}(\lambda - \mu)$ satisfying a symmetry condition:

$$r_{12}(\lambda - \mu) = r_{21}(\lambda - \mu), \quad (32)$$

the simplest possible choice $K(\lambda) = 1$ is a solution of (31). In $sl(2)$ case the condition (32) is true, for example, for the classical rational r-matrix of Yang, for standard trigonometric r-matrix and elliptic r-matrix. It is not true for non-standard rational r-matrices of Stolin. In the case $sl(n)$, $n > 2$ condition (32) is not true nor for standard trigonometric r-matrices neither for $sl(n)$ elliptic r-matrices. Moreover, as we will show in the example below, even for the r-matrices for which condition (32) is satisfied there are a lot of other (except $K = 1$) solutions of (31).

Using nondegenerate matrices $K(\lambda)$ satisfying condition (31) it is possible to introduce the corresponding linear bracket (10) which will have the following explicit form:

$$\{\mathcal{L}_1(\lambda), \mathcal{L}_2(\mu)\}_1 = r_{12}(\lambda-\mu)K_1(\lambda)\mathcal{L}_2(\mu) + r_{12}(\lambda-\mu)\mathcal{L}_1(\lambda)K_2(\mu) -$$
$$K_1(\lambda)r_{12}^{\sigma_1}(-\lambda-\mu)\mathcal{L}_2(\mu) - \mathcal{L}_2(\mu)r_{12}^{\sigma_2}(\lambda+\mu)K_1(\lambda) + K_1(\lambda)\mathcal{L}_2(\mu)r_{12}^{\sigma_1\sigma_2}(\mu-\lambda) -$$
$$-\mathcal{L}_1(\lambda)r_{12}^{\sigma_1}(-\lambda-\mu)K_2(\mu) - K_2(\mu)r_{12}^{\sigma_2}(\lambda+\mu)\mathcal{L}_1(\lambda) - \mathcal{L}_1(\lambda)K_2(\mu)r_{12}^{\sigma_1\sigma_2}(\mu-\lambda). \quad (33)$$

Despite its complicated form, bracket (33) may be substantially simplified. Indeed, after the replacement of the variables: $\mathcal{L}(\lambda) = \mathcal{L}^K(\lambda)K(\lambda)$, usage of the equality (31) and skew symmetry of $r_{12}(\lambda-\mu)$ it can be re-written in the following simple form:

$$\{\mathcal{L}_1^K(\lambda), \mathcal{L}_2^K(\mu)\}_1 = [r_{12}(\lambda-\mu) - K_2(\mu)r_{12}^{\sigma_2}(\lambda+\mu)K_2^{-1}(\mu), \mathcal{L}_1^K(\lambda)] -$$
$$- [r_{21}(\mu-\lambda) - K_1(\lambda)r_{21}^{\sigma_1}(\lambda+\mu)K_1^{-1}(\lambda), \mathcal{L}_2^K(\mu)]. \quad (34)$$

Linear Poisson bracket (34) is a particular example of linear Poisson brackets governed by the non-skew symmetric r-matrices $r_{12}(\lambda,\mu)$:

$$\{L_1(\lambda), L_2(\mu)\}_1 = [r_{12}(\lambda,\mu), L_1(\lambda)] - [r_{21}(\mu,\lambda), L_2(\mu)], \quad (35)$$

where $r_{12}(\lambda,\mu)$ is a solution of "generalized" classical Yang–Baxter equation [20–22]:

$$[r_{12}(\lambda,\mu), r_{13}(\lambda,\nu)] = [r_{23}(\mu,\nu), r_{12}(\lambda,\mu)] + [r_{13}(\lambda,\nu), r_{32}(\nu,\mu)].$$

As it is evident from the very definition, in our case:

$$r_{12}(\lambda,\mu) = r_{12}^{\sigma,K}(\lambda,\mu) \equiv r_{12}(\lambda-\mu) - K_2(\mu)r_{12}^{\sigma_2}(\lambda+\mu)K_2^{-1}(\mu). \quad (36)$$

Remark 13. The classical r-matrix (36) can be rewritten in the form:

$$r_{12}^{\sigma(\mu)}(\lambda,\mu) = r_{12}(\lambda-\mu) - \sigma_2(\mu)\cdot r_{12}(\lambda+\mu), \quad (37)$$

where $\sigma_2(\mu) \equiv Ad_{K_2(\mu)}\cdot\sigma_2$. The r-matrix (37) may be obtained also from the other considerations without appealing to the "twisted" quadratic Poisson brackets (29) (see [23]).

Hence, we see that in this case the resulting classical r-matrices $r_{12}^{\sigma,K}(\lambda,\mu)$ and corresponding linear Lie-algebraic structures substantially depend on the initial point $K(\lambda)$ around which we linearize our quadratic Poisson bracket. For the fixed r-matrix $r(\lambda-\mu)$ there may be many (even parametric families!) of such the points $K(\lambda)$ and all of the corresponding linear Poisson structures are compatible with the quadratic Poisson algebra (29). In order to illustrate this, we will consider the following example of such the points $K(\lambda)$ and their r-matrices $r_{12}^{\sigma,K}(\lambda,\mu)$:

Example 1. Let $\mathfrak{g} = sl(2)$ and $\sigma = 1$. Let X_i, $i \in \overline{1,3}$ be the orthonormal basis in $sl(2) \simeq so(3)$ with the commutation relations:

$$[X_i, X_j] = \epsilon_{ijk} X_k.$$

Let us consider the classical elliptic r-matrix of Sklyanin [18]:

$$r(\lambda - \mu) = \sum_{k=1}^{3} r_k(\lambda - \mu) X_k \otimes X_k, \tag{38}$$

where $r_k(\lambda)$ are expressed via Jacobi functions:

$$r_1(\lambda) = \frac{1}{sn(\lambda)},\ r_2(u) = \frac{dn(\lambda)}{sn(\lambda)},\ r_3(u) = \frac{cn(\lambda)}{sn(\lambda)}. \tag{39}$$

It is easy to see that $r_{12}(\lambda - \mu) = -r_{12}(\lambda - \mu)$ due to the fact that functions $r_k(\lambda)$ are odd.

Let us now introduce the standard "root" basis in $sl(2)$: $X_0 = iX_3$, $X_\pm = i(X_1 \pm iX_2)$, with the standard commutation relations:

$$[X_0, X_\pm] = \pm X_\pm, [X_+, X_-] = 2X_0.$$

In this basis we have that the skew-symmetric elliptic r-matrix acquires the following form:

$$r(\lambda - \mu) = r_0(\lambda - \mu) X_0 \otimes X_0 + r_+(\lambda - \mu)(X_+ \otimes X_- + X_- \otimes X_+) + \\ + r_-(\lambda - \mu)(X_+ \otimes X_+ + X_- \otimes X_-), \tag{40}$$

where $r_0(\lambda) = r_3(\lambda)$, $r_+(\lambda) = \frac{1}{4}(r_2(\lambda) + r_1(\lambda))$, $r_-(\lambda) = \frac{1}{4}(r_2(\lambda) - r_1(\lambda))$ and $r_i(\lambda)$ are defined using the formula (39).

Using the addition laws for the Jacobi functions [29] it is possible to prove that the matrix

$$K(\lambda) \equiv K(\lambda, \xi) = \mathrm{diag}(k_1(\lambda), k_2(\lambda)) \equiv \mathrm{diag}(\frac{sn(\lambda)}{cn(\lambda)} + \frac{sn(\xi)}{cn(\xi)}, -\frac{sn(\lambda)}{cn(\lambda)} + \frac{sn(\xi)}{cn(\xi)})$$

satisfies (31) for arbitrary complex parameter ξ. This permits one to define a new non-skew symmetric elliptic r-matrix using the formula (36). It has the following form:

$$r^K(\lambda, \mu) = r_0^K(\lambda, \mu) X_0 \otimes X_0 + r_{+-}^K(\lambda, \mu) X_+ \otimes X_- + r_{-+}^K(\lambda, \mu) X_- \otimes X_+ + \\ + r_{++}^K(\lambda, \mu) X_+ \otimes X_+ + r_{--}^K(\lambda, \mu) X_- \otimes X_-, \tag{41}$$

where the correspondent coefficients are given by the following formulas:

$$
\begin{aligned}
r^K_{+-}(\lambda,\mu) &= r_+(\lambda-\mu) - \frac{k_2(\mu)}{k_1(\mu)} r_+(\lambda+\mu),\\
r^K_{-+}(\lambda,\mu) &= r_+(\lambda-\mu) - \frac{k_1(\mu)}{k_2(\mu)} r_+(\lambda+\mu),\\
r^K_{++}(\lambda,\mu) &= r_-(\lambda-\mu) - \frac{k_1(\mu)}{k_2(\mu)} r_-(\lambda+\mu),\\
r^K_{--}(\lambda,\mu) &= r_-(\lambda-\mu) - \frac{k_2(\mu)}{k_1(\mu)} r_-(\lambda+\mu),\\
r^K_0(\lambda,\mu) &= r_0(\lambda-\mu) - r_0(\lambda+\mu).
\end{aligned}
$$

Algebra of Integrals

Let us now consider the impact that has a compatibility of the second and linear Poisson brackets on the corresponding theory of the integrable systems. We will show, in particular, that commutativity of the generating functions of the classical integrals with respect to the both linear and quadratic Poisson brackets may be shown using theory of compatible Poisson brackets only.

The following theorem holds true:

Theorem 3.2 *Let $D(\lambda)$ be a determinant of the $gl(n)$-valued "monodromy" matrix $\mathcal{T}(\lambda)$ satisfying the brackets (29). Then $D(\lambda)$ is a generating function of the Casimir function of the bracket (29) for all types of the nondegenerated r-matrices $r(\lambda - \mu) \in sl(n) \otimes sl(n)$.*

Proof. For the proof of the Theorem we will need the following Lemma that provides a classical $K(\lambda) \neq 1$ analogue of the known formula [18] that connect the "twisted classical Reflection Equation Algebra" with the corresponding "classical" limit of quantum group:

Lemma 3.1 *Let matrix $T(\lambda)$ satisfy the Poisson brackets (18) and constant matrix $K(\lambda)$ satisfy condition (31). Then:*

(1) If $\sigma(X) = X$ then the matrix

$$
\mathcal{T}(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda) \tag{42}
$$

satisfy the Poisson brackets (29).

(2) If $\sigma(X) = -X^t$, where superscript "t" denotes the anti-automorphism of $gl(n)$ as associative algebra, then the matrix

$$
\mathcal{T}(\lambda) = T(\lambda)K(\lambda)T^t(-\lambda) \tag{43}
$$

satisfy the Poisson brackets (29).

Proof. In the case (1), using classical Yang–Baxter equations (18), it is easy to derive the following equalities:

$$\{T_1^{-1}(-\lambda), T_2^{-1}(-\mu)\}_2 = -[r_{12}(\mu-\lambda), T_1^{-1}(-\lambda)T_2^{-1}(-\mu)], \tag{44a}$$

$$\{T_1(\lambda), T_2^{-1}(-\mu)\}_2 = T_1(\lambda)r_{12}(\lambda+\mu)T_2^{-1}(-\mu) - T_2^{-1}(-\mu)r_{12}(\lambda+\mu)T_1(\lambda), \tag{44b}$$

$$\{T_1^{-1}(-\lambda), T_2(\mu)\}_2 = -T_1^{-1}(-\lambda)r_{12}(-\lambda-\mu)T_2(\mu)+T_2(\mu)r_{12}(-\lambda-\mu)T_1^{-1}(-\lambda). \tag{44c}$$

In the case (2), in the analogous way, using classical Yang–Baxter equations (18) and the property of σ to be minus anti-automorphism of $gl(n)$ it is straightforward to obtain that:

$$\{T_1^{t_1}(-\lambda), T_2^{t_2}(-\mu)\}_2 = -[r_{12}^{\sigma_1\sigma_2}(\mu-\lambda), T_1^{t_1}(-\lambda)T_2^{t_2}(-\mu)], \tag{45a}$$

$$\{T_1(\lambda), T_2^{t_2}(-\mu)\}_2 = T_1(\lambda)r_{12}^{\sigma_2}(\lambda+\mu)T_2^{t_2}(-\mu)-T_2^{t_2}(-\mu)r_{12}^{\sigma_2}(\lambda+\mu)T_1(\lambda), \tag{45b}$$

$$\{T_1^{t_1}(-\lambda), T_2(\mu)\}_2 = -T_1^{t_1}(-\lambda)r_{12}^{\sigma_1}(-\lambda-\mu)T_2(\mu) + T_2(\mu)r_{12}^{\sigma_1}(-\lambda-\mu)T_1^{t_1}(-\lambda). \tag{45c}$$

Now the Lemma is proved by direct calculation, using the Leibnitz rule for the Poisson bracket, relations (18), (44), (45) and (31).

In order to prove the theorem it is necessary to show that $\{D(\lambda), \mathcal{T}(\mu)\}_2 = 0$. By virtue of the multiplicative properties of the determinant and its invariance with respect to the antiautomorphism t we obtain that $D(\lambda) = k(\lambda)d(\lambda)d^{-1}(-\lambda)$ (if $\sigma = 1$) or $D(\lambda) = k(\lambda)d(\lambda)d(-\lambda)$ (if σ is minus antiautomorphism of $gl(n)$). On the other hand, using the fact that $d(\lambda)$ is a generating function of the Casimir functions of the bracket (18) we obtain:

$$\{d(\pm\lambda), T(\mu)\}_2 = \{d(\pm\lambda), T^{-1}(-\mu)\}_2 = \\ = \{d^{-1}(-\lambda), T(\mu)\}_2 = \{d^{-1}(-\lambda), T^{-1}(-\mu)\}_2 = 0.$$

Keeping in mind that $K(\lambda)$ is a constant (with respect to the Poisson bracket) matrix and making use of the Lemma 3.1 we obtain that $\{D(\lambda), \mathcal{T}(\mu)\}_2 = 0$.

Theorem is proved.

Remark 14. Due to the Remark 11 the Lemma 3.1 may be viewed as a proof (an alternative to the direct calculational one) of the fact that formula (29) correctly defines Poisson bracket.

From the Theorems 3.2 and 2.1 follows the next statement:

Proposition 3.2 *Denote by $D_k(\mathcal{T}(\lambda), K(\lambda))$ the coefficients of the decomposition: $Det(\mathcal{T}(\lambda) + \eta K(\lambda)) = k(\lambda) \sum_{k=0}^{n} D_k(\mathcal{T}(\lambda), K(\lambda))\eta^k$, where $k(\lambda) \equiv \det K(\lambda)$. Then $D_k(\mathcal{T}(\lambda), K(\lambda))$ are generators of an Abelian subalgebra with respect to the linear and quadratic r-matrix brackets (33) and (29), i.e.:*

$$\{D_k(\mathcal{T}(\lambda), K(\lambda)), D_l(\mathcal{T}(\mu), K(\mu))\}_2 = \{D_k(\mathcal{T}(\lambda), K(\lambda)), D_l(\mathcal{T}(\mu), K(\mu))\}_1 = 0.$$

Remark 15. The set of functions $\{D_k(\mathcal{T}(\lambda), K(\lambda))\}$ defines the same algebra of integrals as the set of functions $\{tr(\mathcal{T}(\lambda)K^{-1}(\lambda))^k\}$ and they are connected by the well-known Newton formulas. The commutativity of functions from the last set may be also proved directly using the condition (31), explicit form of brackets (29) and brackets (33).

Remark 16. Note, that we have constructed different sets of functions $\{D_k(\mathcal{T}(\lambda), K(\lambda))\}$ corresponding to different elements $K(\lambda)$ commuting with respect to the same quadratic brackets (29)! Commutativity of each of these sets of functions is closely connected with the existence of the corresponding linearization of the bracket (29) in the neighborhood of the point $K(\lambda)$.

In order to identify the hamiltonian flows with respect to the different brackets we need to decompose the generating functions $D_k(\mathcal{T}(\lambda), K(\lambda))$ with respect to the spectral parameter λ. Using the fact that $\mathcal{T}(\lambda)$ is meromorphic functions this can always be done using the decomposition in Laurent power series. In the special cases [15] one can also use the other decompositions. Nevertheless, we will not do these decompositions explicitly, leaving all the expressions in the "convoluted" form of the generating functions.

Using the Proposition 2.3 we obtain the following Proposition:

Proposition 3.3 *Hamiltonian equations of motion with respect to the bracket (29) with a hamiltonian being one of the functions obtained by a decomposition of the function $D_k(\mathcal{T}(\lambda), K(\lambda))$, $k < n$, coincide with the hamiltonian equations of motion with respect to the brackets (33) with the hamiltonian being the corresponding function obtained by the decomposition of the function $-D_{k-1}(\mathcal{T}(\lambda), K(\lambda))$, i.e. on the level of generating functions the following equality is true:*

$$\{D_k(\mathcal{T}(\lambda), K(\lambda)), \mathcal{T}(\mu)\}_2 = -\{D_{k-1}(\mathcal{T}(\lambda), K(\lambda)), \mathcal{T}(\mu)\}_1.$$

In the case $k = n - 1$ this Proposition implies the following Corollary:

Corollary 3.3 *Hamiltonian equations of motion with respect to the bracket (29) with a hamiltonian being one of the functions obtained by a decomposition of the function $tr(\mathcal{T}(\lambda)K^{-1}(\lambda))$ coincide with the hamiltonian equations of motion with respect to the brackets (33) with the hamiltonian being the corresponding function* *obtained by the decomposition of the function* $\frac{1}{2}tr(\mathcal{T}(\lambda)K^{-1}(\lambda))^2$*, i.e. on the level of generating functions the following equality is true:*

$$\{tr(\mathcal{T}(\lambda)K^{-1}(\lambda)), \mathcal{T}(\mu)\}_2 = \frac{1}{2}\{tr(\mathcal{T}(\lambda)K^{-1}(\lambda))^2, \mathcal{T}(\mu)\}_1.$$

Proof. In order to derive this statement it is sufficient to put in the previous proposition $k = n - 1$, use Newton identities and the fact that function $tr(\mathcal{T}(\lambda)K^{-1}(\lambda))$ is a Casimir function of the linear r-matrix bracket (33).

4 Conclusion and Discussion

In the present paper we have shown that for a general quadratic Poisson bracket it is possible to define many associated linear Poisson brackets – its linearizations in the neighborhood of special points. We prove that the constructed linear Poisson brackets are always compatible with the initial quadratic Poisson bracket. Using the famous Lenard–Magri scheme we obtain mutually commuting with respect to the both brackets "integrals" starting from Casimir functions of the initial quadratic brackets. We show, that the hamiltonian dynamics with respect to one of these hamiltonians of degree k and quadratic bracket can be re-written in terms of hamiltonian dynamics with respect to the corresponding linear bracket and the other one of these hamiltonians of the degree $k + 1$.

We apply the obtained results to the cases of the standard quadratic r-matrix bracket and classical "twisted reflection algebra" bracket. We show that in the last case there are a lot of non-equivalent linearizations of the classical twisted reflection algebra bracket and all of them are compatible with initial quadratic bracket. In the both cases we show that generating functions of the classical integrals may be obtained using the decomposition of the "shifted" Casimir function (determinant of the monodromy matrix) of the corresponding quadratic Poisson bracket. In the first case this fact may be viewed as kind of classical explanation of the trick with the "quantum argument shift" of [27] (if the classical r-matrix is rational) and one more "classical" argument for the support of the hypothesis of [30] in the case of general classical r-matrix. In the second case it may give a hint how to quantize "higher Gaudin hamiltonians" associated with the non-skew-symmetric r-matrix $r^{\sigma,K}(\lambda, \mu)$ (see [23]).

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Contact Geometry of Second Order I

Keizo Yamaguchi

Abstract Classical theory for systems of the first order partial differential equations for a scalar function can be rephrased as the submanifold theory of contact manifolds (geometric first order jet spaces). In the same spirit, we will develop the geometric theory of systems of partial differential equations of second order for a scalar function as the *Contact Geometry of Second Order*, following E. Cartan. We will formulate the submanifold theory of second order jet spaces as the geometry of PD manifolds $(R;\, D^1,\, D^2)$ of second order. Moreover we will establish the First Reduction Theorem for $(R;\, D^1,\, D^2)$ admitting non-trivial Cauchy characteristic systems. By utilizing Parabolic Geometry, we will give, directly or combined with reduction theorems, several classes of systems of partial differential equations of second order, for which the model equation of each class admits the Lie algebra of infinitesimal contact transformations, which is finite dimensional and simple.

1 Introduction

In [5] and [6], E. Cartan studied involutive systems of second order partial differential equations for a scalar function with two or three independent variables, following the tradition of the geometric theory of partial differential equations developed by Monge, Jacobi, Lie, Darboux, Goursat and others. In fact he investigated the contact equivalence and the integration problems of such involutive systems of second order. In this course, he found out the link between the contact equivalence of involutive systems of second order and the geometry of differential systems (Pfaffian systems) on five-dimensional spaces.

The main purpose of the present paper is to reformulate his study as the *Contact Geometry of Second Order*. As is well known, the classical theory of systems of the first order partial differential equations for a scalar function can be rephrased as

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the submanifold theory of contact manifolds. In this spirit, we formulate the submanifold theory of second order contact manifolds as the geometry of *PD* (partial differential) manifolds $(R;\, D^1, D^2)$ of second order, where R is a manifold together with a pair of differential systems D^1 and D^2 satisfying the appropriate conditions ([18], [23] see Sect. 4).

By Bäcklund Theorem (see Sect. 2.2), the symbols of second order equations become the first invariants under contact transformations. In fact, in [6], E. Cartan first classified involutive symbols algebraically and wrote the structure equations of such involutive systems of second order with three independent variables. To capture good classes of second order equations, we cannot pursue this line in general (see the discussion in Sect. 3.3). Our guiding principle in this paper is to utilize Parabolic Geometries, directly or combined with the reduction procedures, to find good classes of *PD* manifolds of second order. Here the *Parabolic Geometry* is a geometry modeled after the homogeneous space G/G', where G is a (semi-)simple Lie group and G' is a parabolic subgroup of G (cf. [1]). Precisely, in this paper, we mean, by a Parabolic Geometry, the *Geometry associated with the Simple Graded Lie Algebra* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ in the sense of N. Tanaka [16], where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$ is the Lie algebra of G'. As for the reduction procedures, we will establish the First Reduction Theorem for *PD* manifolds admitting non-trivial Cauchy characteristic systems in Sect. 4 and will treat the second reduction procedures (two step reductions) as Part II in the sequel to this paper.

Now let us proceed to describe the contents of each sections. In Sect. 2, we will recall the geometric (Grassmannian) construction of Jet spaces. Especially, starting from a contact manifold (J, C), we will construct Lagrange–Grassmann bundle $L(J)$ together with the canonical system E on $L(J)$, which is the geometric second order jet space with the contact differential system. Bäcklund Theorem tells us that an isomorphism Φ of $(L(J), E)$, i.e., a diffeomorphism Φ of $L(J)$ preserving E, coincides with the lift φ_* of the induced contact transformation φ of (J, C). We will also prepare basics of differential systems, especially the Tanaka Theory of (linear) differential systems, e.g., the symbol algebra $\mathfrak{m}(x)$ of a regular differential system (M, D) at $x \in M$ and the notion of the algebraic prolongation of $\mathfrak{m}(x)$. In Sect. 3, we will discuss the symbol algebra $\mathfrak{s}(v) = \bigoplus_{p=-1}^{-3} \mathfrak{s}_p(v)$ of second order equations $R \subset L(J)$ at $v \in R$ as the first invariants in the Contact Geometry of Second Order. We will consider submanifolds $R \subset L(J)$ which contain no equations of the first order. In particular we will study the properties of differential systems D^1 and D^2 on R, where D^i is the restriction to R of the canonical system C^i on $L(J)$. Here $C^2 = E$ and C^1 is the lift of C and coincides with the derived system ∂E of E. Abstracting the properties of these differential systems and utilizing Realization Lemma, we will formulate the submanifold theory of second order contact manifolds in Sect. 4 as the geometry of *PD* manifolds of second order. Moreover we will establish the First Reduction Theorem for PD manifold $(R;\, D^1, D^2)$ admitting non-trivial Cauchy characteristic system $\mathrm{Ch}\,(D^2)$, which reduces the equivalence of $(R;\, D^1, D^2)$ to that of (X, D), where $X = R/\mathrm{Ch}\,(D^2)$, $\rho_*^{-1}(D) = D^2$ and $\rho : R \to X$ is the projection. In Sect. 5, we will first prepare the notation for the simple graded Lie algebras $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ and state the Prolongation

Theorem, which especially clarifies when $\mathfrak{g}$ coincides with the Lie algebra of infinitesimal automorphisms of the standard (model) differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ constructed group-theoretically from $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ (see Sect. 2 for the notion of the standard differential system of type $\mathfrak{m}$). Then we will exhibit Parabolic Geometries, which directly correspond to the geometry of *PD* manifolds of second order. Namely we will exhibit the cases when $\mathfrak{m} = \bigoplus_{p=-1}^{-3} \mathfrak{g}_p$ represents the symbol algebra $\mathfrak{s} = \bigoplus_{p=-1}^{-3} \mathfrak{s}_p$ of PD manifold (R, D^2) of second order such that $D^1 = \partial D^2$. Finally in Sect. 6, we will discuss one of the Typical classes of involutive systems of second order as the application of the First Reduction Theorem and will show several examples of Parabolic Geometries (X, D), which are linked to the geometry of *PD* manifolds $(R; D^1, D^2)$ of second order through the First Reduction Theorem. In these cases the system $(R; D^1, D^2)$ of second order equations is characterized by the symbol algebra $\mathfrak{m}$ of (X, D).

This paper constitutes the extended version of the lecture at the Abel Sympozium 2008 at Tromso and also the extended version of our previous paper [23]. In fact, the full proofs concerning the First Reduction Theorem, which were formulated in [23], are given in Sect. 4 and the materials in Sects. 4.3 and 4.4 are added. In Sect. 6, from these added view points, we will discuss the Typical class of involutive system of type $\mathfrak{f}^3(r)$ and G_2-Geometry cases, and give new examples of Parabolic Geometries.

2 Geometry of Jet Spaces

We will recall the geometric construction of Jet spaces and fix our notations for the basic notion for differential systems, following [22] and [24].

2.1 Spaces of Contact Elements

Let us start with the construction of the space $J(M, n)$ of contact elements to M: Let M be a (real or complex) manifold of dimension $m+n$. Fixing the number n, we consider the space of n-dimensional *contact elements* to M, i.e., the *Grassmannian bundle* over M consisting of all n-dimensional contact elements to M;

$$J(M, n) = \bigcup_{x \in M} J_x \xrightarrow{\pi} M,$$

where $J_x = \mathrm{Gr}(T_x(M), n)$ is the Grassmann manifold of all n-dimensional subspaces of the tangent space $T_x(M)$ to M at x. Each element $u \in J(M, n)$ is a linear subspace of $T_x(M)$ of codimension m, where $x = \pi(u)$. Hence we have a differential system C of codimension m on $J(M, n)$ by putting:

$$C(u) = \pi_*^{-1}(u) \subset T_u(J(M, n)) \xrightarrow{\pi_*} T_x(M).$$

for each $u \in J(M,n)$. C is called the *Canonical System* on $J(M,n)$. Introducing the *inhomogeneous Grassmann coordinate* $(x_1, \cdots, x_n, z^1, \cdots, z^m, p_1^1, \cdots, p_n^m)$ of $J(M,n)$ around $u_o \in J(M,n)$, C is defined by;

$$C = \{\varpi^1 = \cdots = \varpi^m = 0\},$$

where

$$\varpi^\alpha = dz^\alpha - \sum_{i=1}^{n} p_i^\alpha \, dx_i, \qquad (\alpha = 1, \cdots, m).$$

Here $(x_1, \cdots, x_n, z^1, \cdots, z^m)$ is a coordinate system of M around $x_o = \pi(u_o)$ such that $dx_1 \wedge \cdots \wedge dx_n \mid_{u_o} \neq 0$. Coordinate functions p_i^α are introduced by

$$dz^\alpha \mid_u = \sum_{i=1}^{n} p_i^\alpha(u) \, dx_i \mid_u.$$

$(J(M,n), C)$ is the (geometric) 1-jet space for n-dimensional submanifolds in M. Let M, $\hat{M}$ be manifolds (of dimension $m+n$) and $\varphi : M \to \hat{M}$ be a diffeomorphism between them. Then φ induces the *isomorphism* $\varphi_* : (J(M,n), C) \to (J(\hat{M},n), \hat{C})$, i.e., the differential map $\varphi_* : J(M,n) \to J(\hat{M},n)$ is a diffeomorphism sending C onto $\hat{C}$.

2.2 Second Order Contact Manifolds

Let J be a manifold and C be a (linear) differential system on J of codimension 1. Namely C is a subbundle of $T(J)$ of codimension 1. Thus, locally at each point u of J, there exists a 1-form ϖ defined around $u \in J$ such that

$$C = \{\varpi = 0\}.$$

Then (J, C) is called a *contact manifold* if $\varpi \wedge (d\,\varpi)^n$ forms a volume element of J. This condition is equivalent to the following conditions (1), (2) or (3):

(1) The restriction $d\varpi \mid_C$ of $d\varpi$ to $C(u)$ is non-degenerate at each point $u \in J$.
(2) There exists a coframe $\{\varpi, \omega_1, \ldots, \omega_n, \pi_1, \ldots, \pi_n\}$ defined around $u \in J$ such that the following holds;

$$d\varpi \equiv \omega_1 \wedge \pi_1 + \cdots + \omega_n \wedge \pi_n \qquad (\text{mod} \quad \varpi)$$

(3) The Cauchy characteristic system $\mathrm{Ch}\,(C)$ of C is trivial (see Sect. 2.3 below).

By the *Darboux Theorem*, a contact manifold (J, C) of dimension $2n+1$ can be regarded locally as a space of 1-jets for one unknown function. Namely, at each point of (J, C), there exists a canonical coordinate system $(x_1, \ldots, x_n, z, p_1, \ldots, p_n)$ such that

$$C = \{dz - \sum_{i=1}^{n} p_i\, dx_i = 0\}.$$

Starting from a contact manifold (J, C), we can construct the geometric second order jet space $(L(J), E)$ as follows: We consider the *Lagrange–Grassmann bundle* $L(J)$ over J consisting of all n-dimensional integral elements of (J, C);

$$L(J) = \bigcup_{u \in J} L_u \xrightarrow{\pi} J,$$

where L_u is the Grassmann manifolds of all lagrangian (or *legendrian*) subspaces of the symplectic vector space $(C(u), d\varpi)$. Here ϖ is a local contact form on J. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J), \qquad \text{for} \quad v \in L(J).$$

Let us fix a point $v_o \in L(J)$. Starting from a canonical coordinate system $(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ of (J, C) around $u_o = \pi(v_o)$ such that $dx_1 \wedge \cdots \wedge dx_n \mid_{v_o} \neq 0$, we can introduce a coordinate system (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ by defining coordinate functions p_{ij} as follows;

$$dp_i \mid_v = \sum_{i=1}^{n} p_{ij}(v) dx_j \mid_v .$$

Then, since $v \in C(u)$, we have $dz \mid_v = \sum_{i=1}^{n} p_i(u) dx \mid_v$ and, since $d\varpi \mid_v = 0$, we get $p_{ij} = p_{ji}$.

Thus E is defined on this canonical coordinate system by

$$E = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\},$$

where

$$\varpi = dz - \sum_{i=1}^{n} p_i\, dx_i, \qquad \text{and} \qquad \varpi_i = dp_i - \sum_{j=1}^{n} p_{ij}\, dx_j \qquad \text{for} \quad i = 1, \cdots, n.$$

Let (J, C), $(\hat{J}, \hat{C})$ be contact manifolds of dimension $2n + 1$ and $\varphi : (J, C) \to (\hat{J}, \hat{C})$ be a contact diffeomorphism between them. Then φ induces an isomorphism $\varphi_* : (L(J), E) \to (L(\hat{J}), \hat{E})$. Conversely we have (cf. Theorem 3.2 [18], [20])

Theorem 1. (Bäcklund) *Let (J, C) and $(\hat{J}, \hat{C})$ be contact manifolds of dimension $2n + 1$. Then, for an isomorphism $\Phi : (L(J), E) \to (L(\hat{J}), \hat{E})$, there exists a contact diffeomorphism $\varphi : (J, C) \to (\hat{J}, \hat{C})$ such that $\Phi = \varphi_*$.*

2.3 Derived Systems and Cauchy Characteristic Systems

Now we prepare basic notions for *(linear) differential systems* (or Pfaffian systems). By a (linear) differential system (M, D), we mean a subbundle D of the tangent bundle $T(M)$ of a manifold M of dimension d. Locally D is defined by 1-forms $\omega_1, \ldots, \omega_{d-r}$ such that $\omega_1 \wedge \cdots \wedge \omega_{d-r} \neq 0$ at each point, where r is the rank of D;

$$D = \{ \omega_1 = \cdots = \omega_{d-r} = 0 \}.$$

For two differential systems (M, D) and $(\hat{M}, \hat{D})$, a diffeomorphism φ of M onto $\hat{M}$ is called an *isomorphism* of (M, D) onto $(\hat{M}, \hat{D})$ if the differential map φ_* of φ sends D onto $\hat{D}$.

For a non-integrable differential system D, we consider the *Derived System* ∂D of D, which is defined, in terms of sections, by

$$\partial \mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}].$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D.

Furthermore the *Cauchy Characteristic System* $\mathrm{Ch}\,(D)$ of (M, D) is defined at each point $x \in M$ by

$$\mathrm{Ch}\,(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \quad (\mathrm{mod}\ \omega_1, \ldots, \omega_s) \quad \text{for } i = 1, \ldots, s\},$$

where $\rfloor$ denotes the interior multiplication, i.e., $X \rfloor d\omega(Y) = d\omega(X, Y)$ and $s = d - r$. When $\mathrm{Ch}\,(D)$ is a differential system (i.e., has constant rank), it is always completely integrable.

Moreover *Higher Derived Systems* $\partial^k D$ are usually defined successively (cf. [4]) by

$$\partial^k D = \partial(\partial^{k-1} D),$$

where we put $\partial^0 D = D$ for convention.

On the other hand we define the k-th *Weak Derived System* $\partial^{(k)} D$ of D inductively by

$$\partial^{(k)} \mathcal{D} = \partial^{(k-1)} \mathcal{D} + [\mathcal{D}, \partial^{(k-1)} \mathcal{D}],$$

where $\partial^{(0)} D = D$ and $\partial^{(k)} \mathcal{D}$ denotes the space of sections of $\partial^{(k)} D$.

2.4 Review of Tanaka Theory

A differential system (M, D) is called *regular*, if $D^{-(k+1)} = \partial^{(k)} D$ are subbundles of $T(M)$ for every integer $k \geqq 1$. For a regular differential system (M, D), we have ([14], Proposition 1.1)

$(S1)$ *There exists a unique integer* $\mu > 0$ *such that, for all* $k \geqq \mu$,

$$D^{-k} = \cdots = D^{-\mu} \supsetneqq D^{-\mu+1} \supsetneqq \cdots \supsetneqq D^{-2} \supsetneqq D^{-1} = D,$$

$(S2)$ $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ *for all* $p, q < 0$.

where $\mathcal{D}^p$ denotes the space of sections of D^p. $(S2)$ implies subbundles D^p define a filtration on M.

Let (M, D) be a regular differential system such that $T(M) = D^{-\mu}$. As a first invariant for non-integrable differential systems, we now define the *symbol algebra* $\mathfrak{m}(x)$ *associated with a differential system* (M, D) at $x \in M$, which was introduced by N. Tanaka [14].

We put $\mathfrak{g}_{-1}(x) = D^{-1}(x)$, $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$ $(p < -1)$ and

$$\mathfrak{m}(x) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x).$$

Let π_p be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$. Then, for $X \in \mathfrak{g}_p(x)$ and $Y \in \mathfrak{g}_q(x)$, the bracket product $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined by

$$[X, Y] = \pi_{p+q}([\tilde{X}, \tilde{Y}]_x),$$

where $\tilde{X}$ and $\tilde{Y}$ are any element of $\mathcal{D}^p$ and $\mathcal{D}^q$ respectively such that $\pi_p(\tilde{X}_x) = X$ and $\pi_q(\tilde{Y}_x) = Y$.

Endowed with this bracket operation, by $(S2)$ above, $\mathfrak{m}(x)$ becomes a nilpotent graded Lie algebra such that $\dim \mathfrak{m}(x) = \dim M$ and satisfies

$$\mathfrak{g}_p(x) = [\mathfrak{g}_{p+1}(x), \mathfrak{g}_{-1}(x)] \qquad \text{for } p < -1.$$

We call $\mathfrak{m}(x)$ the *Symbol Algebra of* (M, D) at $x \in M$.

Furthermore, let $\mathfrak{m}$ be a FGLA (fundamental graded Lie algebra) of μ-th kind, that is,

$$\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$$

is a nilpotent graded Lie algebra such that

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \qquad \text{for } p < -1.$$

Then (M, D) is called of type $\mathfrak{m}$ if the symbol algebra $\mathfrak{m}(x)$ is isomorphic to $\mathfrak{m}$ at each $x \in M$.

Conversely, given a FGLA $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$, we can construct a model differential system of type $\mathfrak{m}$ as follows: Let $M(\mathfrak{m})$ be the simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ defines a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$. By definition of symbol algebras, it is easy to see that $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is a regular differential system of type $\mathfrak{m}$. $(M(\mathfrak{m}), D_{\mathfrak{m}})$ is called the *Standard Differential System of Type* $\mathfrak{m}$. The Lie algebra of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be calculated algebraically as the *Prolongation* $\mathfrak{g}(\mathfrak{m})$ *of* $\mathfrak{m}$ ([13], cf. [22]).

In fact, let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a fundamental graded Lie algebra of μ-th kind defined over a field $\mathbb{K}$. Here $\mathbb{K}$ denotes the field of real numbers $\mathbb{R}$ or that of complex numbers $\mathbb{C}$. We put

$$\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p(\mathfrak{m}),$$

where $\mathfrak{g}_p(\mathfrak{m}) = \mathfrak{g}_p$ for $p < 0$, $\mathfrak{g}_0(\mathfrak{m})$ is the Lie algebra of all (gradation preserving) derivations of graded Lie algebra $\mathfrak{m}$ and $\mathfrak{g}_k(\mathfrak{m})$ is defined inductively by the following for $k \geqq 1$;

$$\mathfrak{g}_k(\mathfrak{m}) = \{\, u \in \bigoplus_{p<0} \mathfrak{g}_{p+k} \otimes \mathfrak{g}_p^* \mid u([Y, Z]) = [u(Y), Z] - [u(Z), Y] \,\}.$$

Thus, as a vector space over $\mathbb{K}$, $\mathfrak{g}_k(\mathfrak{m})$ is a linear subspace of $\mathrm{End}\,(\mathfrak{m}, \mathfrak{m}^k) = \mathfrak{m}^k \otimes \mathfrak{m}^*$, where $\mathfrak{m}^k = \mathfrak{m} \oplus \mathfrak{g}_0(\mathfrak{m}) \oplus \cdots \oplus \mathfrak{g}_{k-1}(\mathfrak{m})$. The bracket operation of $\mathfrak{g}(\mathfrak{m})$ is given accordingly (see [13], [22] for detail).

The structure of the Lie algebra $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ of all infinitesimal automorphisms of $(M(\mathfrak{m}), D_{\mathfrak{m}})$ can be described by $\mathfrak{g}(\mathfrak{m})$. Especially $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ is isomorphic to $\mathfrak{g}(\mathfrak{m})$, when $\mathfrak{g}(\mathfrak{m})$ is finite dimensional.

Let $\mathfrak{g}_0$ be a subalgebra of $\mathfrak{g}_0(\mathfrak{m})$. We define a subspace $\mathfrak{g}_k$ of $\mathfrak{g}_k(\mathfrak{m})$ for $k \geqq 1$ inductively by

$$\mathfrak{g}_k = \{\, u \in \mathfrak{g}_k(\mathfrak{m}) \mid [u, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1} \,\}.$$

Then, putting

$$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0) = \mathfrak{m} \oplus \bigoplus_{k \geqq 0} \mathfrak{g}_k,$$

we see, with the generating condition of $\mathfrak{m}$, that $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$. $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

We will recall in Sect. 5.1 when $\mathfrak{g}(\mathfrak{m})$ or $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$ becomes finite dimensional and simple.

2.5 Symbol Algebra of $(L(J), E)$

As an example to calculate symbol algebras, let us show that $(L(J), E)$ is a regular differential system of type $\mathfrak{c}^2(n)$:

$$\mathfrak{c}^2(n) = \mathfrak{c}_{-3} \oplus \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-3} = \mathbb{R}$, $\mathfrak{c}_{-2} = V^*$ and $\mathfrak{c}_{-1} = V \oplus S^2(V^*)$. Here V is a vector space of dimension n and the bracket product of $\mathfrak{c}^2(n)$ is defined accordingly through the pairing between V and V^* such that V and $S^2(V^*)$ are both abelian subspaces of $\mathfrak{c}_{-1}$. This fact can be checked as follows: Let us take a canonical coordinate system (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ of $(L(J), E)$. Then we have a coframe $\{\varpi, \varpi_i, dx_i, dp_{ij}\}$ $(1 \leqq i \leqq j \leqq n)$ at each point in this coordinate neighborhood, where $\varpi = dz - \sum_{i=1}^n p_i\, dx_i$, $\varpi_i = dp_i - \sum_{j=1}^n p_{ij}\, dx_j$ $(i = 1, \cdots, n)$. Now take the dual frame $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial p_i}, \frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}}\}$, of this coframe, where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + p_i \frac{\partial}{\partial z} + \sum_{j=1}^n p_{ij} \frac{\partial}{\partial p_j}$$

is the classical notation. Notice that $\{\frac{d}{dx_i}, \frac{\partial}{\partial p_{ij}}\}$ $(1 \leqq i \leqq j \leqq n)$ forms a free basis of $\Gamma(E)$. Then we have

$$\left[\frac{\partial}{\partial p_{ii}}, \frac{d}{dx_j}\right] = \delta^i_j \frac{\partial}{\partial p_i}, \quad \left[\frac{\partial}{\partial p_{ij}}, \frac{d}{dx_k}\right] = \delta^i_k \frac{\partial}{\partial p_i} + \delta^j_k \frac{\partial}{\partial p_j} \quad \text{for} \quad i \neq j,$$

$$\left[\frac{\partial}{\partial p_i}, \frac{d}{dx_j}\right] = \delta^i_j \frac{\partial}{\partial z}, \qquad \left[\frac{d}{dx_i}, \frac{d}{dx_j}\right] = 0.$$

It follows that $T(L(J)) = \partial^{(2)}E$ and the derived system ∂E of E satisfies the following:

$$\partial E = \{\varpi = 0\} = \pi_*^{-1}C, \qquad \mathrm{Ch}\,(\partial E) = \mathrm{Ker}\,\pi_*.$$

These facts provide the proof of Theorem 1 (cf. Theorem 3.2 [18]).

Moreover, in terms of the defining 1-forms of E and ∂E around $v \in L(J)$, the structure of the symbol algebra $\mathfrak{c}^2(n)$ can be described by the following Structure Equation of E. Cartan [5, 6];

$$d\varpi \equiv \omega_1 \wedge \varpi_1 + \cdots + \omega_n \wedge \varpi_n \quad (\text{mod} \quad \varpi,\ \varpi_i \wedge \varpi_j (1 \leqq i \leqq j \leqq n))$$

$$\begin{cases} d\varpi_1 \equiv \omega_1 \wedge \pi_{11} + \cdots + \omega_n \wedge \pi_{1n} \\ \qquad \cdots \\ d\varpi_n \equiv \omega_1 \wedge \pi_{n1} + \cdots + \omega_n \wedge \pi_{nn} \end{cases} \qquad (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_n)$$

where $\partial E = \{\varpi = 0\}$, $E = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}$ and $\{\varpi, \varpi_i, \omega_i, \pi_{ij} \quad (1 \leqq i \leqq j \leqq n)\}$ forms a coframe around $v \in L(J)$. Here we understand that $\pi_{ij} = \pi_{ji}$.

Similarly we see that $(J(M, n), C)$ is a regular differential system of type $\mathfrak{c}^1(n, m)$:

$$\mathfrak{c}^1(n, m) = \mathfrak{c}_{-2} \oplus \mathfrak{c}_{-1},$$

where $\mathfrak{c}_{-2} = W$ and $\mathfrak{c}_{-1} = V \oplus W \otimes V^*$ for vector spaces V and W of dimension n and m respectively, and the bracket product of $\mathfrak{c}^1(n, m)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes V^*$ are both abelian subspaces of $\mathfrak{c}_{-1}$.

3 Symbols of Second Order Equations

In view of the Bäcklund Theorem, we will discuss the symbols of second order equations as the first invariants in the Contact Geometry of Second Order.

3.1 Symbol Algebras

Let R be a submanifold of $L(J)$ satisfying the following condition:

$$(R.0) \qquad p : R \to J;\ \text{submersion},$$

where $p = \pi\ |_R$ and $\pi : L(J) \to J$ is the projection. This condition implies that the system of equations R of second order contains no equations of first order. We have two differential systems $C^1 = \partial E$ and $C^2 = E$ on $L(J)$. We denote by D^1 and D^2 those differential systems on R obtained by restricting C^1 and C^2 to R. Moreover we denote by the same symbols those 1-forms obtained by restricting the defining 1-forms $\{\varpi, \varpi_1, \cdots, \varpi_n\}$ of the canonical system E to R, where $\varpi = dz - \sum_{i=1}^{n} p_i dx_i$, and $\varpi_i = dp_i - \sum_{j=1}^{n} p_{ij} dx_j$ $(i = 1, \ldots, n)$. Then it follows from $(R.0)$ that these 1-forms are independent at each point on R and that

$$D^1 = \{\varpi = 0\}, \qquad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}.$$

Thus D^1 and D^2 are subbundles of $T(R)$ such that $\partial D^2 \subset D^1$. Hence subbundles D^2, D^1 and $T(R)$ define a filtration on R. Namely, putting $D^{-1} = D^2$, $D^{-2} = D^1$, $D^p = T(R)$ for $p \leqq -3$, we have

$$[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q} \qquad \text{for} \quad p, q < 0,$$

where $\mathcal{D}^p = \Gamma(D^p)$.

Now we define the *Symbol Algebra* $\mathfrak{s}(v)$ of R at $v \in R$ by

$$\mathfrak{s}(v) = \mathfrak{s}_{-3}(v) \oplus \mathfrak{s}_{-2}(v) \oplus \mathfrak{s}_{-1}(v),$$

where $\mathfrak{s}_{-3}(v) = T_v(R)/D^1(v)$, $\mathfrak{s}_{-2}(v) = D^1(v)/D^2(v)$ and $\mathfrak{s}_{-1}(v) = D^2(v)$. The bracket operation in $\mathfrak{s}(v)$ is defined, similarly as in Sect. 2.3, as follows: For $X \in \mathfrak{s}_p(v)$ and $Y \in \mathfrak{s}_q(v)$, let us take $\tilde{X} \in \mathcal{D}^p$ and $\tilde{Y} \in \mathcal{D}^q$ such that $X = \pi_p((\tilde{X})_v)$ and $Y = \pi_q((\tilde{Y})_v)$,where $\pi_p : D^p(v) \to \mathfrak{s}_p(v)$ is the projection. Then the bracket product is defined by

$$[X, Y] = \pi_{p+q}([\tilde{X}, \tilde{Y}]_v) \in \mathfrak{s}_{p+q}(v).$$

The bracket product $[X, Y]$ is well-defined for $X \in \mathfrak{s}_p(v)$ and $Y \in \mathfrak{s}_q(v)$, i.e., is independent of the choice of $\tilde{X}$ and $\tilde{Y}$. In fact, in our case, this can be shown as follows: The defining 1-forms $\varpi, \varpi_1, \ldots, \varpi_n$ for D^1 and D^2 actually define a basis $\{A\}$ of $\mathfrak{s}_{-3}(v)$ and $\{B_1, \ldots, B_n\}$ of $\mathfrak{s}_{-2}(v)$ such that $\varpi(\tilde{A}) = 1, \pi_{-3}(\tilde{A}) = A$, $\varpi_i(\tilde{B}_j) = \delta^i_j, \pi_{-2}(\tilde{B}_i) = B_i$ and $\tilde{B}_i \in D^1(v)$. Then, for $X_1, X_2 \in \mathfrak{s}_{-1}(v) = D^2(v)$, we calculate

$$d\varpi_i(X_1, X_2) = \tilde{X}_1(\varpi_i(\tilde{X}_2)) - \tilde{X}_2(\varpi_i(\tilde{X}_1)) - \varpi_i([\tilde{X}_1, \tilde{X}_2]) = -\varpi_i([\tilde{X}_1, \tilde{X}_2]).$$

Thus, putting $\beta_i = -d\varpi_i(X_1, X_2)$, we get

$$[X_1, X_2] = \beta_1 B_1 + \cdots + \beta_n B_n \in \mathfrak{s}_{-2}(v).$$

For $X \in \mathfrak{s}_{-1}(v)$ and $Y \in \mathfrak{s}_{-2}(v)$, we calculate

$$d\varpi(X, \tilde{Y}_v) = \tilde{X}_v(\varpi(\tilde{Y})) - \tilde{Y}_v(\varpi(\tilde{X})) - \varpi([\tilde{X}, \tilde{Y}]_v) = -\varpi([\tilde{X}, \tilde{Y}]_v).$$

Similarly we have $d\varpi(X_1, X_2) = 0$ for $X_1, X_2 \in \mathfrak{s}_{-1}(v)$. Thus $d\varpi(X, \tilde{Y}_v)$ depends only on $X \in \mathfrak{s}_{-1}(v)$ and $Y \in \mathfrak{s}_{-2}(v)$. Hence, putting $\alpha = -d\varpi(X, \tilde{Y}_v)$, we have

$$[X, Y] = \alpha A \in \mathfrak{s}_{-3}(v).$$

Moreover it follows that, for $X \in \mathfrak{s}_{-1}(v)$,

$$X \rfloor d\varpi(Y) = 0 \quad \text{for } \forall Y \in D^1(v) \quad \text{if and only if } \ [X, \mathfrak{s}_{-2}(v)] = 0.$$

Hence, from $\mathrm{Ch}\,(D^1) = \mathrm{Ker}\, p_* \subset D^2$, we have, putting $\mathfrak{f}(v) = \mathrm{Ch}\,(D^1)(v)$,

$$\mathfrak{f}(v) = \{X \in \mathfrak{s}_{-1}(v) \mid [X, \mathfrak{s}_{-2}(v)] = 0\}.$$

$\mathfrak{f}(v)$ is a subspace of $\mathfrak{s}_{-1}(v)$ of codimension n.

By the description of the bracket operation in $\mathfrak{s}(v)$ above, since ϖ and $\varpi_1, \ldots, \varpi_n$ are the restriction of defining 1-forms of $C^1 = \partial E$ and $C^2 = E$ on $L(J)$, we immediately see that ϖ and $\varpi_1, \ldots, \varpi_n$, at the same time, define bases of $\mathfrak{g}_{-3}(v)$ and $\mathfrak{g}_{-2}(v)$ of the symbol algebra $\mathfrak{m}(v) = \mathfrak{g}_{-3}(v) \oplus \mathfrak{g}_{-2}(v) \oplus \mathfrak{g}_{-1}(v)$ ($\cong \mathfrak{c}^2(n)$) of $(L(J), E)$ at $v \in L(J)$ so that $\mathfrak{s}(v)$ is a graded subalgebra of $\mathfrak{m}(v)$ satisfying $\mathfrak{s}_{-3}(v) = \mathfrak{g}_{-3}(v)$, $\mathfrak{s}_{-2}(v) = \mathfrak{g}_{-2}(v)$ and $\mathfrak{f}(v) = T_v(R) \cap \mathrm{Ch}\,(C^1)(v)$.

Now we consider the following compatibility condition for R:

$$(C) \qquad p^{(1)} : R^{(1)} \to R \textit{ is onto.}$$

where $R^{(1)}$ is the first prolongation of R. Namely we assume that there exists an n-dimensional integral element V of (R, D^2) at each $v \in R$ such that

$$\mathfrak{s}_{-1}(v) = V \oplus \mathfrak{f}(v).$$

V is an abelian subalgebra in $\mathfrak{s}(v)$. By fixing a basis of $\mathfrak{s}_{-3}(v)$, $\mathfrak{s}_{-3}(v)$ is identified with $\mathbb{R}$ and, through $[,] : \mathfrak{s}_{-2}(v) \times \mathfrak{s}_{-1}(v) \to \mathfrak{s}_{-3}(v) \cong \mathbb{R}$, $\mathfrak{s}_{-2}(v)$ is identified with

V^*, since $V \cap \mathfrak{f}(v) = \{0\}$ and $\mathfrak{f}(v) = \{X \in \mathfrak{s}_{-1}(v) \mid [X, \mathfrak{s}_{-2}(v)] = 0\}$. Moreover we have a map $\mu : \mathfrak{f}(v) \to S^2(V^*)$ defined by

$$\mu(f)(v_1, v_2) = [[f, v_1], v_2] \in \mathfrak{s}_{-3}(v) \cong \mathbb{R} \qquad \text{for } f \in \mathfrak{f}(v).$$

Here $\mu(f)(v_1, v_2) = \mu(f)(v_2, v_1)$ follows from $[v_1, v_2] = 0$ and the Jacobi identity of $\mathfrak{s}(v)$. We can check the injectivity of μ as follows: If $\mu(f) = 0$, we have $[f, v_1] = 0$ for $\forall v_1 \in V$ by $\mathfrak{s}_{-2}(v) \cong V^*$. Then we have $[f, \mathfrak{s}_{-1}(v)] = 0$, since $\mathfrak{f}(v)$ is abelian, which implies $f \in \mathfrak{f}(v) \cap \mathrm{Ch}\,(D^2)(v)$. From $\mathrm{Ch}\,(D^1)(v) \cap \mathrm{Ch}\,(D^2)(v) = 0$ (see Sect. 4.1), we obtain $f = 0$.

Hence, by fixing a basis of $\mathfrak{s}_{-3}(v)$ and the brackets in $\mathfrak{s}(v)$, we obtain

$$\mathfrak{s}_{-3}(v) \cong \mathbb{R}, \quad \mathfrak{s}_{-2}(v) \cong V^*, \quad \mathfrak{s}_{-1}(v) = V \oplus \mathfrak{f}(v) \quad \text{and} \quad \mathfrak{f}(v) \subset S^2(V^*).$$

Thus $\mathfrak{f}(v) \subset S^2(V^*)$ is the first invariant of R under contact transformation. We will first examine $\mathfrak{f}(v) \subset S^2(V^*)$ in the case $\dim V = 2$ in the next section.

3.2 Case $n = 2$

When $\dim V = 2$, we have $\dim S^2(V^*) = 3$. Through the natural pairing of $S^2(V)$ and $S^2(V^*)$ as subspaces of $V \otimes V$ and $V^* \otimes V^*$, we identify $S^2(V)$ with the dual space of $S^2(V^*)$. For a basis $\{e_1, e_2\}$ of V, we have a basis $\{e_1^* \circledcirc e_1^*, 2e_1^* \circledcirc e_2^*, e_2^* \circledcirc e_2^*\}$ of $S^2(V^*)$ and its dual basis $\{e_1 \circledcirc e_1, e_1 \circledcirc e_2, e_2 \circledcirc e_2\}$ of $S^2(V)$, where $\{e_1^*, e_2^*\}$ is the dual basis of $\{e_1.e_2\}$ and $e_i \circledcirc e_j = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$. For a subspace $\mathfrak{f}$ of $S^2(V^*)$, we denote by $\mathfrak{f}^\perp$ the annihilator of $\mathfrak{f}$ in $S^2(V)$.

Now let us classify subspaces $\mathfrak{f}$ of $S^2(V^*)$ under the action of $GL(V)$:

(1) codim $\mathfrak{f} = 1$
In this case $\dim \mathfrak{f}^\perp = 1$. Hence we can classify a generator f of $\mathfrak{f}^\perp$ as a quadratic form and obtain the following classification into three cases, i.e., there exists a basis of V such that

$$\mathfrak{f}^\perp = \begin{cases} \langle \{e_1 \circledcirc e_1\} \rangle, \\ \langle \{e_1 \circledcirc e_2\} \rangle, \\ (\langle \{e_1 \circledcirc e_1 + e_2 \circledcirc e_2\} \rangle) \end{cases}$$

The third case occurs when we classify over $\mathbb{R}$. Here f is of rank 1, rank 2 (indefinite) and rank 2 (definite) respectively.

(2) codim $\mathfrak{f} = 2$
In this case $\dim \mathfrak{f} = 1$. Hence, similarly as above, we have the following classification into three cases, i.e., there exists a basis of V such that

$$\mathfrak{f} = \begin{cases} \langle\{e_2^* \circledcirc e_2^*\}\rangle, \\ \langle\{e_1^* \circledcirc e_2^*\}\rangle, \\ (\langle\{e_1^* \circledcirc e_1^* + e_2^* \circledcirc e_2^*\}\rangle), \end{cases}$$

Thus, dually, we have

$$\mathfrak{f}^{\perp} = \begin{cases} \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2\}\rangle, \\ \langle\{e_1 \circledcirc e_1, e_2 \circledcirc e_2\}\rangle, \\ (\langle\{e_1 \circledcirc e_2, e_1 \circledcirc e_1 - e_2 \circledcirc e_2\}\rangle) \end{cases}$$

The third case occurs when we classify over $\mathbb{R}$. We note here that, for the prolongation $\mathfrak{f}^{(1)} = \mathfrak{f} \otimes V^* \cap S^3(V^*)$, we have $(\mathfrak{f}^{(1)})^{\perp} = \langle\{e_1 \circledcirc e_1 \circledcirc e_1, e_1 \circledcirc e_1 \circledcirc e_2, e_1 \circledcirc e_2 \circledcirc e_2, e_2 \circledcirc e_2 \circledcirc e_2\}\rangle$ for the second and third cases (see Sect. 3.3). Namely $\mathfrak{f}^{(1)} = \{0\}$ for the second and third cases, whereas the first case is involutive.

We can classify the symbol algebra $\mathfrak{s}(v)$ of R at $v \in R$ according to the above classification for $\mathfrak{f}(v) \subset S^2(V^*)$ under the condition (C). In the case codim $R = 1$, R is called *parabolic, hyperbolic* and *elliptic* at v according as f is of rank 1, rank 2 (indefinite) and rank 2 (definite) respectively, where f is a generator of $(\mathfrak{f}(v))^{\perp} \subset S^2(V)$ (see Sect. 3.3).

Now we assume the regularity for the symbol algebras. Namely assume that symbol algebras $\mathfrak{s}(v)$ of R are locally isomorphic to the fixed symbol $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$ where $\mathfrak{s}_{-3} = \mathbb{R}$, $\mathfrak{s}_{-2} = V^*$ and $\mathfrak{s}_{-1} = V \oplus \mathfrak{f}$ for the fixed $\mathfrak{f} \subset S^2(V^*)$. Then, for example, the Structure Equation reads as follows:

(a) $\mathfrak{f}^{\perp} = \langle\{e_1 \circledcirc e_2\}\rangle$

$$\begin{cases} d\varpi \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 & (\text{mod} \quad \varpi) \\ d\varpi_1 \equiv \omega_1 \wedge \pi_{11} & (\text{mod} \quad \varpi, \varpi_1, \varpi_2) \\ d\varpi_2 \equiv \omega_2 \wedge \pi_{22} & (\text{mod} \quad \varpi, \varpi_1, \varpi_2) \end{cases}$$

(b) $\mathfrak{f}^{\perp} = \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2\}\rangle$

$$\begin{cases} d\varpi \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 & (\text{mod} \quad \varpi) \\ d\varpi_1 \equiv 0 & (\text{mod} \quad \varpi, \varpi_1, \varpi_2) \\ d\varpi_2 \equiv \omega_2 \wedge \pi_{22} & (\text{mod} \quad \varpi, \varpi_1, \varpi_2) \end{cases}$$

(c) $\mathfrak{f}^{\perp} = \langle\{e_1 \circledcirc e_1, e_2 \circledcirc e_2\}\rangle$

$$\begin{cases} d\varpi \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 & (\text{mod} \quad \varpi) \\ d\varpi_1 \equiv \omega_2 \wedge \pi_{12} & (\text{mod} \quad \varpi, \varpi_1, \varpi_2) \\ d\varpi_2 \equiv \omega_1 \wedge \pi_{21} & (\text{mod} \quad \varpi, \varpi_1, \varpi_2) \end{cases}$$

Thus we see, from (b), that R admits a one-dimensional Cauchy characteristic system in case $\mathfrak{f}^{\perp} = \langle\{e_1 \circledcirc e_1, e_1 \circledcirc e_2\}\rangle$, i.e., in case R is of codimension 2 and $\mathfrak{f}$ is involutive. In fact, in [5], E. Cartan characterized overdetermined involutive system R by the condition that R admits a one-dimensional Cauchy characteristic system.

We will discuss the case when R admits a non-trivial Cauchy characteristic system in Sect. 4.2 in general. We will encounter the case (3) in Sect. 5.3 as an example of Parabolic Geometry associated with $\mathfrak{sl}(4)$.

3.3 Involutive Symbols

In general we will consider the case when $\dim V = n$. We identify $S^2(V)$ with the dual space of $S^2(V^*)$ through the natural pairing of $S^2(V)$ and $S^2(V^*)$ as subspaces of $V \otimes V$ and $V^* \otimes V^*$. Then, for a basis $\{e_1, \ldots, e_n\}$ of V, we have a basis $\{e_1^* \circledcirc e_1^*, \ldots, e_n^* \circledcirc e_n^*, 2e_i^* \circledcirc e_j^* (1 \leqq i < j \leqq n)\}$ of $S^2(V^*)$ and its dual basis $\{e_i \circledcirc e_j (1 \leqq i \leqq j \leqq n)\}$ of $S^2(V)$, where $\{e_1^*, \ldots, e_n^*\}$ is the dual basis of V^*.

Here we note the adjoint map $\sigma^*(v) : S^k(V) \to S^{k+1}(V)$ of the interior multiplication $\sigma(v) : S^{k+1}(V^*) \to S^k(V^*)$ by a vector $v \in V$; $\sigma(v)(f) = v \rfloor f$, i.e., $\sigma(v)(f)(v_1, \ldots, v_k) = f(v, v_1, \ldots, v_k)$ for $f \in S^{k+1}(V^*)$, is given by

$$\sigma^*(v)(a) = v \circledcirc a \qquad \text{for} \quad a \in S^k(V).$$

Hence, for a subspace $\mathfrak{f} \subset S^2(V^*)$ such that $\mathfrak{f}^{\perp} = \langle\{f_1, \ldots, f_s\}\rangle$, the first prolongation $\mathfrak{f}^{(1)} = \mathfrak{f} \otimes V^* \cap S^3(V^*)$ is given by $(\mathfrak{f}^{(1)})^{\perp} = \langle\{e_i \circledcirc f_j \mid 1 \leqq i \leqq n, 1 \leqq j \leqq s\}\rangle$.

As in Sect. 3.2, we can classify codimension 1 subspace $\mathfrak{f} \subset S^2(V^*)$ as follows; In this case, we can classify a generator f of $\mathfrak{f}^{\perp}$ as a quadratic form and obtain

$$f = e_1 \circledcirc e_1 \pm e_2 \circledcirc e_2 \pm \cdots \pm e_r \circledcirc e_r,$$

for a basis $\{e_1, \ldots, e_n\}$ of V, where r is the rank of f and we have a index when we classify over $\mathbb{R}$. In each case, $\{e_n, \ldots, e_1\}$ forms a regular basis for $\mathfrak{f}$ and the Cartan characters are given by $s_i = n - i + 1$ for $i = 1, \ldots, n-1$ and $s_n = 0$. $\mathfrak{f}$ is always involutive [10].

For a single equation of second order

$$R = \{F(x_1, \ldots, x_n, z, p_1, \ldots, p_n, p_{11}, \ldots, p_{nn}) = 0\} \subset L(J),$$

we observe the following: From Sect. 2.5, we calculate

$$\left[\left[\frac{\partial}{\partial p_{ii}}, X\right], X\right] = v_i^2 \frac{\partial}{\partial z}, \quad \left[\left[\frac{\partial}{\partial p_{ij}}, X\right], X\right] = 2\, v_i \cdot v_j \frac{\partial}{\partial z} \quad (i \neq j),$$

for $X = \sum_{i=1}^{n} v_i \frac{d}{dx_i}$. Thus $\frac{\partial}{\partial p_{ii}}$ is identified with $e_i^* \circledcirc e_i^*$ in $S^2(V^*)$ and $\frac{\partial}{\partial p_{ij}}$ is identified with $2\, e_i^* \circledcirc e_j^*$ in $S^2(V^*)$, where $\{e_i = \frac{d}{dx_i} (i = 1, \ldots, n)\}$ forms a basis of V. Then, from $\mathfrak{f}(v) = T_v(R) \cap \mathrm{Ch}\,(C^2)(v)$ at $v \in R$ and

$$dF \equiv \sum_{1 \leqq i \leqq j \leqq n} \frac{\partial F}{\partial p_{ij}} dp_{ij} \quad (\mathrm{mod} \quad \varpi, \varpi_1, \ldots, \varpi_n, dx_1, \ldots, dx_n),$$

we see that $(\mathfrak{f}(v))^{\perp}$ is generated by

$$f = \sum_{1 \leqq i \leqq j \leqq n} \frac{\partial F}{\partial p_{ij}} (v)\, e_i \circledcirc e_j.$$

Next we consider the case when codim $\mathfrak{f} = 2$. In this case the involutiveness becomes rather a restrictive condition and, in fact, we have (cf. [6] and [19]).

Proposition 1. *Let $\mathfrak{f}$ be a subspace of $S^2(V^*)$ of codimension 2. Then $\mathfrak{f}$ is involutive if and only if there exists a basis $\{e_1, \ldots, e_n\}$ of V such that the annihilator $\mathfrak{f}^{\perp}$ of $\mathfrak{f}$ in $S^2(V)$ is generated by $e_1 \circledcirc e_2$ and $e_1 \circledcirc e_3$ or by $e_1 \circledcirc e_1$ and $e_1 \circledcirc e_2$.*

Proof. Fist we observe that $S^2(V^*)$ is a involutive subspace of $V^* \otimes V^*$ with the Cartan characters $\sigma_i = n - i + 1$ for $i = 1, \ldots, n$. Let $\mathfrak{f}$ be a subspace of $S^2(V^*)$ of codimension 2 and let $s_1, \ldots s_n$ be the Cartan characters of $\mathfrak{f}$. Then we have $s_i \leqq \sigma_i$ for $i = 1, \ldots, n$ and

$$\dim \mathfrak{f} = s_1 + \cdots + s_n = \sigma_1 + \cdots + \sigma_n - 2.$$

Since the Cartan characters have the property $s_1 \geqq \cdots \geqq s_n$, if $s_{i_o} = \sigma_{i_o} - 2$ for some $i_o \in \{1, \ldots, n\}$, it follows $\mathfrak{s}_{i_o+1} = \sigma_{i_o+1} = n - i_o > s_{i_o} = \sigma_{i_o} - 2 = n - i_o - 1$, which is a contradiction. Hence there exist j and k $(1 \leqq j < k \leqq n)$ such that $s_j = \sigma_j - 1$ and $s_k = \sigma_k - 1$.

Now assume that $\mathfrak{f}$ is involutive. Then, from $\dim \mathfrak{f}^{(1)} = s_1 + 2s_2 + \cdots + ns_n$, we obtain

$$\mathrm{codim}\ \mathfrak{f}^{(1)} = \sum_{i=1}^{n} i(\sigma_i - s_i) = j + k \leqq (n-1) + n = 2n - 1.$$

This implies that the generator $\{e_1 \circledcirc f, \ldots, e_n \circledcirc f, e_1 \circledcirc g, \ldots, e_n \circledcirc g\}$ of $(\mathfrak{f}^{(1)})^{\perp}$ in $S^3(V)$ are linearly dependent, where f and $g \in S^2(V)$ are the generator of $\mathfrak{f}^{\perp}$. Namely there exist v_1 and $v_2 \in V$ such that

$$v_1 \circledcirc f - v_2 \circledcirc g = 0.$$

Here v_1 and v_2 are linearly independent since f and g are independent. Hence there exists $v \in V$ such that $f = v_2 \circledcirc v$ and $g = v_1 \circledcirc v$. In case $\{v_1, v_2, v\}$ are independent, there exists a basis $\{e_1, \ldots, e_n\}$ of V such that $f = e_1 \circledcirc e_2$ and $g = e_1 \circledcirc e_3$. In case $\{v_1, v_2, v\}$ are dependent, there exists a basis of V such that $\mathfrak{f}^\perp$ is generated by $e_1 \circledcirc e_1$ and $e_1 \circledcirc e_2$. Consequently, in these cases, we have $s_i = n - i + 1$ for $1 \leqq i \leqq n-2$, $s_{n-1} = 1$ and $s_n = 0$. □

In [6], E. Cartan, in fact, first classified involutive subspaces $\mathfrak{f} \subset S^2(V^*)$ when $\dim V = 3$ and immediately wrote the Structure Equation for each involutive system in this case.

However we cannot pursue this line in general by the following facts. By counting the dimensions, we see that the dimension of $\mathrm{Gr}(S^2(V^*), r)$ exceeds the dimension of $GL(V)$ for $n \geqq 4$ and $r, s \geqq 2$, where $r + s = \dim \mathrm{Gr}(S^2(V^*), r)$. Hence we will have a functional moduli if we try to classify r dimensional subspaces $\mathfrak{f}$ in $S^2(V^*)$ for $n \geqq 4$ and $r, s \geqq 2$. We suspect this phenomena even if we assume the involutiveness of $\mathfrak{f}$.

Thus we need other guide lines to proceed. In this paper, after preparing the structure theory for submanifolds in $L(J)$ in Sect. 4, we will utilize Parabolic Geometries to find good classes of second order equations in Sects. 5 and 6.

3.4 Typical Symbols

We exhibit here typical examples of involutive symbols in $S^2(V^*)$, which are the only invariants of the corresponding involutive systems of second order, which were found in [21]. Namely we describe here the involutive subspaces $\mathfrak{f}^1(r)$, $\mathfrak{f}^2(r)$ and $\mathfrak{f}^3(r)$ of $S^2(V^*)$ which have the following property: Let R be an involutive systems of second order, which is regular of type $\mathfrak{f}$, i.e., R satisfies the condition (C) such that the symbol $\mathfrak{f}(v)$ at each point $v \in R$ is isomorphic to $\mathfrak{f} \subset S^2(V^*)$, where $\mathfrak{f}$ is one of $\mathfrak{f}^1(r)$, $\mathfrak{f}^2(r)$ or $\mathfrak{f}^3(r)$. Then, as in the case of the system of first order partial differential equations of one dependent variable, R can be transformed to the model linear equation by a contact transformation (cf. [21]):

(1) $\mathfrak{f}^1(r) \subset S^2(V^*) \quad (2 \leqq r \leqq n-2) \qquad V = V_r \oplus V_s$

$$(\mathfrak{f}^1(r))^\perp = \langle \{e_i \circledcirc e_\alpha \mid 1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n\} \rangle = V_r \otimes_S V_s,$$

When $\mathfrak{f} = \mathfrak{f}^1(r)$ (see [21] Sect. 2), there exists a canonical coordinate system (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ of $L(J)$ such that

$$R = \{\frac{\partial^2 z}{\partial x_i \partial x_\alpha} = 0 \quad (1 \leqq i \leqq r, r+1 \leqq \alpha \leqq n)\}.$$

(2) $\mathfrak{f}^2(r) \subset S^2(V^*) \quad (r \geqq 2)$

$$(\mathfrak{f}^2(r))^\perp = \langle \{e_i \circledcirc e_j \mid 1 \leqq i \leqq j \leqq r\} \rangle = S^2(V_r),$$

When $\mathfrak{f} = \mathfrak{f}^2(r)$ (see [21] Sect. 3), there exists a canonical coordinate system (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ of $L(J)$ such that

$$R = \{\frac{\partial^2 z}{\partial x_i \partial x_j} = 0 \quad (1 \leqq i \leqq j \leqq r)\}.$$

(3) $\mathfrak{f}^3(r) \subset S^2(V^*) \quad (r \leqq n-2)$

$$(\mathfrak{f}^3(r))^\perp = \langle\{e_i \odot e_a \mid 1 \leqq i \leqq r, 1 \leqq a \leqq n\}\rangle = V_r \otimes_S V,$$

When $\mathfrak{f} = \mathfrak{f}^3(r)$ (see [21] Sect. 4), there exists a canonical coordinate system (x_i, z, p_i, p_{ij}) $(1 \leqq i \leqq j \leqq n)$ of $L(J)$ such that

$$R = \{\frac{\partial^2 z}{\partial x_i \partial x_a} = 0 \quad (1 \leqq i \leqq r, 1 \leqq a \leqq n)\}.$$

Here $\{e_1, \dots, e_n\}$ is a basis of V, $V_r = \langle\{e_1, \dots, e_r\}\rangle$ and $V_s = \langle\{e_{r+1}, \dots, e_n\}\rangle$.

We need Reduction Theorems to explain why second order equations with these symbols have the property that their symbols are the only invariants under contact transformations. We will explain this fact for the type $\mathfrak{f}^3(r)$ in Sect. 6.1 by utilizing the First Reduction Theorem in Sect. 4. The other cases will be explained by utilizing the two step reduction procedure in Part II.

4 *PD* Manifolds of Second Order

We will here formulate the submanifold theory for $(L(J), E)$ as the geometry of *PD* manifolds of second order [18] and discuss the First Reduction Theorem.

4.1 Realization Theorem

Let R be a submanifold of $L(J)$ satisfying the following condition:

$$(R.0) \qquad p : R \to J \text{ ; submersion,}$$

where $p = \pi \mid_R$ and $\pi : L(J) \to J$ is the projection. Let D^1 and D^2 be differential systems on R obtained by restricting $C^1 = \partial E$ and $C^2 = E$ to R. Moreover we denote by the same symbols those 1-forms obtained by restricting the defining 1-forms $\{\varpi, \varpi_1, \cdots, \varpi_n\}$ of the canonical system E to R. Then it follows from $(R.0)$ that these 1-forms are independent at each point on R and that

$$D^1 = \{\varpi = 0\}, \qquad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}.$$

In fact $(R; D^1, D^2)$ further satisfies the following conditions:

$(R.1)$ *D^1 and D^2 are differential systems of codimension* 1 *and* $n+1$ *respectively.*

$(R.2)$ $\partial D^2 \subset D^1$.

$(R.3)$ $\mathrm{Ch}\,(D^1)$ *is a subbundle of D^2 of codimension n.*

$(R.4)$ $\mathrm{Ch}\,(D^1)(v) \cap \mathrm{Ch}\,(D^2)(v) = \{0\}$ *at each* $v \in R$.

Here $(R.2)$ follows from $d\varpi \equiv 0 \pmod{\varpi, \varpi_1, \ldots, \varpi_n}$. $(R.3)$ follows from $\mathrm{Ch}\,(D^1) = \mathrm{Ker}\, p_* = \{\, dz = dx_1 = \cdots = dx_n = dp_1 = \cdots = dp_n = 0\}$. Moreover the last condition follows easily from the Realization Lemma below.

Conversely these four conditions characterize submanifolds in $L(J)$ satisfying $(R.0)$. To see this, we first recall the following *Realization Lemma*, which characterize submanifolds of $(J(M,n), C)$.

Realization Lemma. *Let R and M be manifolds. Assume that the quadruple (R, D, p, M) satisfies the following conditions:*

(1) *p is a map of R into M of constant rank.*
(2) *D is a differential system on R such that $F = \mathrm{Ker}\ p_*$ is a subbundle of D of codimension n.*

Then there exists a unique map ψ of R into $J(M,n)$ satisfying $p = \pi \cdot \psi$ and $D = \psi_*^{-1}(C)$, where C is the canonical differential system on $J(M,n)$ and $\pi : J(M,n) \to M$ is the projection. Furthermore, let v be any point of R. Then ψ is in fact defined by

$$\psi(v) = p_*(D(v)) \qquad \text{as a point of } Gr\ (T_{p(v)}(M)),$$

and satisfies

$$\mathrm{Ker}\,(\psi_*)_v = F(v) \cap Ch\,(D)(v).$$

where $Ch\,(D)$ is the Cauchy Characteristic System of D.

For the proof, see Lemma 1.5 [18].

In view of this Lemma, we call the triplet $(R; D^1, D^2)$ of a manifold and two differential systems on it a *PD manifold of second order* if these satisfy the above four conditions $(R.1)$ to $(R.4)$. Here we note, by $(R.2)$, subbundles D^2, D^1 and $T(R)$ define a filtration on R. Hence we can form the symbol algebra $\mathfrak{s}(v) = \mathfrak{s}_{-3}(v) \oplus \mathfrak{s}_{-2}(v) \oplus \mathfrak{s}_{-1}(v)$ of $(R; D^1, D^2)$ at $v \in R$ as in Sect. 3.1.

We have the (local) Realization Theorem for *PD* manifolds as follows: From conditions $(R.1)$ and $(R.3)$, it follows that the codimension of the foliation defined by the completely integrable system $\mathrm{Ch}\,(D^1)$ is $2n+1$. Assume that R is regular with respect to $\mathrm{Ch}\,(D^1)$, i.e., the space $J = R/\mathrm{Ch}\,(D^1)$ of leaves of this foliation is a manifold of dimension $2n+1$ such that each fibre of the projection $p : R \to J = R/\mathrm{Ch}\,(D^1)$ is connected and p is a submersion. Then D^1 drops down to J. Namely there exists a differential system C on J of codimension 1 such that $D^1 = p_*^{-1}(C)$. From $\mathrm{Ch}\,(C) = \{0\}$, (J, C) becomes a contact manifold of dimension

$2n + 1$. Conditions $(R.1)$ and $(R.2)$ guarantees that the image of the following map ι is a legendrian subspace of (J, C):

$$\iota(v) = p_*(D^2(v)) \subset C(u), \qquad u = p(v).$$

Finally the condition $(R.4)$ shows that $\iota : R \to L(J)$ is an immersion by Realization Lemma for (R, D^2, p, J). Furthermore we have (Corollary 5.4 [18])

Theorem 2. *Let $(R; D^1, D^2)$ and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ be PD manifolds of second order. Assume that R and $\hat{R}$ are regular with respect to $Ch\,(D^1)$ and $Ch\,(\hat{D}^1)$ respectively. Let (J, C) and $(\hat{J}, \hat{C})$ be the associated contact manifolds. Then an isomorphism $\Phi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ induces a contact diffeomorphism $\varphi : (J, C) \to (\hat{J}, \hat{C})$ such that the following commutes;*

$$\begin{array}{ccc} R & \xrightarrow{\ \iota\ } & L(J) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \hat{R} & \xrightarrow{\ \hat{\iota}\ } & L(\hat{J}) \end{array}$$

Proof. Since Φ is an isomorphism of (R, D^1, D^2) onto $(\hat{R}, \hat{D}^1, \hat{D}^2)$, we have $\Phi_*(D^1) = \hat{D}^1$. Hence we get $\Phi_*(\mathrm{Ch}\,(D^1)) = \mathrm{Ch}\,(\hat{D}^1)$. Therefore, since $\mathrm{Ch}\,(D^1) = \mathrm{Ker}\, p_*$ and each fibre of $p : R \to J$ is connected, Φ is fibre-preserving and induces a unique diffeomorphism φ of J onto $\hat{J}$ such that $\hat{p} \cdot \Phi = \varphi \cdot p$. By $D^1 = (p_*)^{-1}(C)$ and $\hat{D}^1 = (\hat{p}_*)^{-1}(\hat{C})$, φ is a contact diffeomorphism of J onto $\hat{J}$, Put $\bar{\iota} = (\varphi_*)^{-1} \cdot \hat{\iota} \cdot \Phi$. Then it is easy to see that $\bar{\iota}$ is a map of R into $L(J)$ satisfying $\pi \cdot \bar{\iota} = p$ and $D^2 = (\bar{\iota}_*)^{-1}(C^2)$. Therefore by the uniqueness of the canonical immersion ι of R into $L(J)$, we obtain $\iota = \bar{\iota}$, i.e., $\hat{\iota} \cdot \Phi = \varphi_* \cdot \iota$. □

By this theorem, the submanifold theory for $(L(J), E)$ is reformulated as the geometry of *PD* manifolds of second order.

4.2 First Reduction Theorem

When $D^1 = \partial D^2$ holds for a *PD* manifold $(R; D^1, D^2)$ of second order, the geometry of $(R; D^1, D^2)$ reduces to that of (R, D^2) and the Tanaka theory is directly applicable to this case. We will treat this case as Parabolic Geometries associated with *PD* manifolds of second order in Sect. 5. Concerning about this situation, we will show the following proposition under the compatibility condition (C) :

$$(C) \qquad p^{(1)} : R^{(1)} \to R \textit{ is onto}.$$

where $R^{(1)}$ is the first prolongation of $(R; D^1, D^2)$, i.e.,

$$R^{(1)} = \{n\text{-dim. } \textit{integral elements} \text{ of } (R, D^2),\ \text{transversal to } F = \mathrm{Ker}\, p_*\} \subset J(R, n),$$

(cf. Proposition 5.11 [18]).

Proposition 2. *Let* $(R;\, D^1, D^2)$ *be a PD manifold of second order satisfying the condition* (C) *above. Then the following equality holds at each point* v *of* R*:*

$$\dim D^1(v) - \dim \partial D^2(v) = \dim Ch\,(D^2)(v).$$

In particular $D^1 = \partial D^2$ *holds if and only if* $Ch\,(D^2) = \{0\}$.

Proof. By the condition (C), there exists an n-dimensional integral element V of (R, D^2) at v such that

$$D^2(v) = \mathfrak{s}_{-1}(v) = V \oplus \mathfrak{f}(v).$$

In terms of the symbol algebra $\mathfrak{s}(v) = \mathfrak{s}_{-3}(v) \oplus \mathfrak{s}_{-2}(v) \oplus \mathfrak{s}_{-1}(v)$, we have

$$\mathrm{Ch}\,(D^2)(v) = \{X \in \mathfrak{s}_{-1}(v) \mid [X, \mathfrak{s}_{-1}(v)] = 0\}$$

For a vector $X = w_X + f_X \in \mathrm{Ch}\,(D^2)(v)$, where $w_X \in V$ and $f_X \in \mathfrak{f}(v)$, $[X, V] = 0$ forces $f_X = 0$, that is, $\mathrm{Ch}\,(D^2)(v) \subset V \subset \mathfrak{s}_{-1}(v)$. We put $\mathcal{C}(v) = \mathrm{Ch}\,(D^2)(v) \subset V$. Then we have

$$\mathcal{C}(v) = \{w \in V \mid w \rfloor f = 0 \quad \text{for all } f \in \mathfrak{f}(v)\} \subset V.$$

Thus $\mathcal{C}(v) = E$ is the largest subspace of V such that $\mathfrak{f}(v) \subset S^2(E^{\perp})$. On the other hand, let us consider the derived system ∂D^2 in terms of the symbol algebra $\mathfrak{s}(v)$. From $[\mathfrak{s}_{-1}(v), \mathfrak{s}_{-1}(v)] = [V, \mathfrak{f}(v)]$, putting $\mathcal{D}(v) = \pi_2(\partial D^2(v))$, we have

$$\mathcal{D}(v) = \{z \rfloor f \mid z \in V,\, f \in \mathfrak{f}(v)\} \subset V^*,$$

under the identification $\mathfrak{s}_{-2}(v) \cong V^*$, where $\pi_2 : D^1(v) \to \mathfrak{s}_{-2}(v)$ is the projection. Thus w belongs to the annihilator of $\mathcal{D}(v)$ iff $\langle w, z \rfloor f \rangle = 0$ for all $z \in V$ and all $f \in \mathfrak{f}(v)$, hence iff $w \rfloor f = 0$ for all $f \in \mathfrak{f}(v)$. This implies $\mathcal{D}(v) = (\mathcal{C}(v))^{\perp}$, which completes the proof of Proposition. □

When a *PD* manifold $(R;\, D^1, D^2)$ admits a non-trivial Cauchy characteristics, i.e., when $\mathrm{rank}\,\mathrm{Ch}\,(D^2) > 0$, the geometry of $(R;\, D^1, D^2)$ is further reducible to the geometry of a single differential system. Here we will be concerned with the local equivalence of $(R;\, D^1, D^2)$, hence we may assume that R is regular with respect to $\mathrm{Ch}\,(D^2)$, i.e., the leaf space $X = R/\mathrm{Ch}\,(D^2)$ is a manifold such that the projection $\rho : R \to X$ is a submersion and there exists a differential system D on X satisfying $D^2 = \rho_*^{-1}(D)$. Then the local equivalence of $(R;\, D^1, D^2)$ is further reducible to that of (X, D) as in the following: We assume that $(R;\, D^1, D^2)$ satisfies the condition (C) above and $\mathrm{Ch}\,(D^2)$ is a subbundle of rank r $(0 < r < n)$. Then, by Proposition 2, ∂D^2 is a subbundle of D^1 of codimension r.

By the information of the symbol algebra $\mathfrak{s}(v)$ of $(R;\, D^1, D^2)$ at $v \in R$, we can find locally independent 1-forms $\varpi, \varpi_i, \omega_i$ $(i = 1, \ldots, n)$ around v such that

$$D^1 = \{\varpi = 0\},$$

$$\partial D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\}, \quad D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}$$

$$\begin{cases} d\varpi \equiv \omega_1 \wedge \varpi_1 + \cdots + \omega_n \wedge \varpi_n (\text{mod} \quad \varpi) \\ d\varpi_i \equiv 0 \qquad (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_n) \qquad i = 1, \ldots, r \\ d\varpi_\alpha \not\equiv 0 \qquad (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_n) \qquad \alpha = r+1, \ldots, n \end{cases}$$

Thus we have

$$\begin{cases} d\varpi \equiv \omega_{r+1} \wedge \varpi_{r+1} + \cdots + \omega_n \wedge \varpi_n (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_r) \\ d\varpi_i \equiv \pi_i^{r+1} \wedge \varpi_{r+1} + \cdots + \pi_i^n \wedge \varpi_n (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_r) \quad i = 1, \ldots, r \end{cases}$$

for some 1-forms π_i^α. This shows that the Cartan rank of $(X, \partial D)$ (see [4] II Sect. 4) equals to $s = n - r$ at $x = \rho(v) \in X$, which gives us a necessary condition for a differential system $(X.D)$ to be obtained from a *PD* manifold $(R; D^1, D^2)$ as $X = R/\mathrm{Ch}\,(D^2)$.

From (X, D), at least locally, we can reconstruct the *PD* manifold $(R; D^1, D^2)$ as follows. First let us consider the collection $P(X)$ of hyperplanes v in each tangent space $T_x(X)$ at $x \in X$ which contains the fibre $\partial D(x)$ of the derived system ∂D of D.

$$P(X) = \bigcup_{x \in X} P_x \subset J(X, m-1),$$

$$P_x = \{v \in \mathrm{Gr}(T_x(X), m-1) \mid v \supset \partial D(x)\} \cong \mathbb{P}(T_x(X)/\partial D(x)) = \mathbb{P}^r,$$

where $m = \dim X$ and $r = \mathrm{rank}\,\mathrm{Ch}\,(D^2)$. Moreover D_X^1 is the canonical system obtained by the Grassmannian construction and D_X^2 is the lift of D. Precisely, D_X^1 and D_X^2 are given by

$$D_X^1(v) = \nu_*^{-1}(v) \supset D_X^2(v) = \nu_*^{-1}(D(x)),$$

for each $v \in P(X)$ and $x = \nu(v)$, where $\nu : P(X) \to X$ is the projection. Then we have a map κ of R into $P(X)$ given by

$$\kappa(v) = \rho_*(D^1(v)) \subset T_x(X),$$

for each $v \in R$ and $x = \rho(v)$. By Realization Lemma for (R, D^1, ρ, X), κ is a map of constant rank such that

$$\mathrm{Ker}\ \kappa_* = \mathrm{Ch}\,(D^1) \cap \mathrm{Ker}\,\rho_* = \mathrm{Ch}\,(D^1) \cap \mathrm{Ch}\,(D^2) = \{0\}.$$

Thus κ is an immersion and, by a dimension count, in fact, a local diffeomorphism of R into $P(X)$ such that

$$\kappa_*(D^1) = D_X^1 \qquad \text{and} \qquad \kappa_*(D^2) = D_X^2.$$

Namely $\kappa : (R, D^1, D^2) \to (P(X), D^1_X, D^2_X)$ is a local isomorphism of *PD* manifolds of second order. Thus $(R; D^1, D^2)$ is reconstructed from (X, D), at least locally, as a part of $(P(X); D^1_X, D^2_X)$. (Precisely, in general, $(P(X), D^1_X, D^2_X)$ becomes a *PD* manifold on an open subset. See Proposition 3.) By the construction of $(P(X); D^1_X, D^2_X)$, an isomorphism of (X, D) naturally lifts to an isomorphism of $(P(X); D^1_X, D^2_X)$.

Summarizing the above consideration, we obtain the following First Reduction Theorem for *PD* manifolds admitting non-trivial Cauchy characteristics.

Theorem 3. *Let (R, D^1, D^2) and $(\hat{R}; \hat{D}^1, \hat{D}^2)$ be PD manifolds satisfying the condition (C) such that $Ch\,(D^2)$ and $Ch\,(\hat{D}^2)$ are subbundles of rank r $(0 < r < n)$. Assume that R and $\hat{R}$ are regular with respect to $Ch\,(D^2)$ and $Ch\,(\hat{D}^2)$ respectively. Let (X, D) and $(\hat{X}, \hat{D})$ be the leaf spaces, where $X = R/Ch\,(D^2)$ and $\hat{X} = \hat{R}/Ch\,(\hat{D}^2)$. Let us fix points $v_o \in R$ and $\hat{v}_o \in \hat{R}$ and put $x_o = \rho(v_o)$ and $\hat{x}_o = \hat{\rho}(\hat{v}_o)$. Then a local isomorphism $\psi : (R; D^1, D^2) \to (\hat{R}; \hat{D}^1, \hat{D}^2)$ such that $\psi(v_o) = \hat{v}_o$ induces a local isomorphism $\varphi : (X, D) \to (\hat{X}, \hat{D})$ such that $\varphi(x_o) = \hat{x}_o$ and $\varphi_*(\kappa(x_o)) = \hat{\kappa}(\hat{x}_o)$, and vice versa.*

4.3 Construction of $(R(X); D^1_X, D^2_X)$

Now we will characterize differential systems (X, D), which are obtained by the First Reduction Theorem from *PD* manifolds $(R; D^1, D^2)$ as $X = R/\mathrm{Ch}\,(D^2)$. We already saw that the necessary condition for (X, D) is that ∂D is of Cartan rank $s = n - r$. We will show that this condition is also sufficient.

Let (X, D) be a differential system satisfying the following conditions:

$(X.1)$ *D is a differential system of codimension $n + 1$ such that $Ch\,(D)$ is trivial.*
$(X.2)$ *∂D is a differential system of codimension $r + 1$.*
$(X.3)$ *∂D is of Cartan rank $s = n - r$.*

Under the conditions $(X.1)$ and $(X.2)$, Cartan rank of ∂D is less than or equal to s (see the proof of Proposition 3 below). Thus $(X.3)$ is a nondegeneracy condition for (X, D).

We form the weak symbol algebra $\mathfrak{t}(x)$ of (X, D) at $x \in X$ as follows: Put $\mathfrak{t}_{-3}(x) = T_x(X)/\partial D(x)$, $\mathfrak{t}_{-2}(x) = \partial D(x)/D(x)$ and $\mathfrak{t}_{-1}(x) = D(x)$. The subbundles D, ∂D and $T(X)$ give a filtration on X. Hence as in the symbol algebra of PD manifolds, we can introduce the Lie brackets in

$$\mathfrak{t}(x) = \mathfrak{t}_{-3}(x) \oplus \mathfrak{t}_{-2}(x) \oplus \mathfrak{t}_{-1}(x),$$

so that $\mathfrak{t}(x)$ becomes a graded Lie algebra.

Now let us consider the collection $P(X)$ of hyperplanes v in each tangent space $T_x(X)$ at $x \in X$ which contains the fibre $\partial D(x)$ of the derived system ∂D of D.

$$P(X) = \bigcup_{x \in X} P_x \subset J(X, m - 1),$$

$$P_x = \{v \in \mathrm{Gr}(T_x(X), m-1) \mid v \supset \partial D(x)\} \cong \mathbb{P}(T_x(X)/\partial D(x)) = \mathbb{P}^r,$$

where $m = \dim X$ and $r + 1 = \operatorname{codim} \partial D$. Moreover D^1_X is the canonical system obtained by the Grassmannian construction and D^2_X is the lift of D. In fact, D^1_X and D^2_X are given by

$$D^1_X(v) = \nu_*^{-1}(v) \supset D^2_X(v) = \nu_*^{-1}(D(x)),$$

for each $v \in P(X)$ and $x = \nu(v)$, where $\nu : P(X) \to X$ is the projection. For a point $v \in P(X)$, we define the *symbol subspace* $\hat{\mathfrak{f}}(v)$ of $D(x)$ by

$$\hat{\mathfrak{f}}(v) = \{X \in \mathfrak{t}_{-1}(x) \mid [X, \mathfrak{t}_{-2}(x)] \subset \hat{v}\},$$

where $x = \nu(v) \in X$, $\hat{v} = p_{-3}(v) \subset \mathfrak{t}_{-3}(x)$ and $p_{-3} : T_x(X) \to \mathfrak{t}_{-3} = T_x(X)/\partial D(x)$ is the projection. We put

$$R(X) = \{v \in P(X) \mid \operatorname{codim} \hat{\mathfrak{f}}(v) = s\}$$

($R(X)$ is an open subset of $P(X)$ under the condition $(X.3)$. See below). We denote the restrictions of differential systems D^1_X and D^2_X of $P(X)$ to $R(X)$ by the same symbols.

Then we have

Proposition 3. $(R(X), D^1_X, D^2_X)$ *is a PD manifold of second order.*

Proof. By $(X.1)$ and the construction of $P(X)$, it follows that D^1_X and D^2_X are differential systems on $P(X)$ of codimension 1 and $n+1$ respectively. Moreover $\partial D^2_X \subset D^1_X$ holds on $P(X)$ by construction. Since $P(X)$ is a submanifold of $J(X, m-1)$, Realization Lemma for $(P(X); D^1_X, \nu, X)$ implies

$$\operatorname{Ker} \bar{\iota}_* = \operatorname{Ker} \nu_* \cap \operatorname{Ch}(D^1_X) = \operatorname{Ch}(D^1_X) \cap \operatorname{Ch}(D^2_X) = \{0\},$$

where $\bar{\iota} : P(X) \to J(X, m-1)$ is the inclusion. Thus it remains to show that $R(X)$ is an open subset of $P(X)$ under the condition $(X.3)$ and $\operatorname{Ch}(D^1_X)$ is a subbundle of D^2_X of codimension n on $R(X)$.

For this purpose, let us take a point $v_o \in P(X)$. We can find locally independent 1-forms $\varpi_0, \ldots, \varpi_r, \pi_1, \ldots, \pi_s$ on a neighborhood U of $x_o = \nu(v_o) \in X$ such that

$$\partial D = \{\varpi_0 = \cdots = \varpi_r = 0\}, \qquad D = \{\varpi_0 = \cdots = \varpi_r = \pi_1 = \cdots = \pi_s = 0\}.$$

Here we may assume that $v_o = \{\varpi_0 = 0\} \subset T_{x_o}(X)$. Then we have

$$d\varpi_i \equiv 0 \pmod{\varpi_0, \ldots, \varpi_r, \pi_1, \ldots, \pi_s} \qquad \text{for } i = 0, \ldots, r.$$

Hence there exist 1-forms β_i^α such that

$$d\varpi_i \equiv \beta_i^1 \wedge \pi_1 + \cdots + \beta_i^s \wedge \pi_s \qquad (\text{mod} \quad \varpi_0, \ldots, \varpi_r) \qquad \text{for } i = 0, \ldots, r.$$

Thus the Cartan rank of ∂D is less than or equal to s.

Now let us consider

$$\varpi = \varpi_0 + \lambda_1 \varpi_1 + \cdots + \lambda_r \varpi_r$$

on U. Namely we consider a point $v \in P(X)$ such that $v = \{\varpi = 0\} \subset T_x(X)$, where $x = \nu(v) \in U$. Here $(\lambda_1, \ldots, \lambda_r)$ constitutes an inhomogeneous coordinate of the fibres of $\nu : P(X) \to X$. Denoting the pullback on $P(X)$ of 1-forms on X by the same symbol, we have

$$D_X^1 = \{\varpi = 0\},$$

and

$$d\varpi = d\varpi_0 + \sum_{i=1}^r \lambda_i d\varpi_i + \sum_{i=1}^r d\lambda_i \wedge \varpi_i.$$

on $\nu^{-1}(U)$. From $d\varpi_i \equiv \sum_{\alpha=1}^s \beta_i^\alpha \wedge \pi_\alpha + \sum_{j=1}^r \gamma_i^j \wedge \varpi_j \pmod{\varpi}$ for $i = 0, \ldots, r$, we calculate

$$d\varpi \equiv \sum_{\alpha=1}^s (\beta_0^\alpha + \sum_{i=1}^r \lambda_i \beta_i^\alpha) \wedge \pi_\alpha + \sum_{i=1}^r (d\lambda_i + \gamma_0^i + \sum_{j=1}^r \lambda_j \gamma_j^i) \wedge \varpi_i \quad (\text{mod} \quad \varpi).$$

For a vector $Y \in \partial D(x)$, we have

$$Y \rfloor d\varpi \equiv -\sum_{\alpha=1}^s \pi_\alpha(Y)(\beta_0^\alpha + \sum_{i=1}^r \lambda_i \beta_i^\alpha) \qquad (\text{mod} \quad \varpi_0, \ldots, \varpi_r, \pi_1, \ldots, \pi_s).$$

By the definition of brackets in the weak symbol algebra $\mathfrak{t}(x)$, it follows that

$$\hat{\mathfrak{f}}(v) = \{X \in D(x) \mid (\beta_0^\alpha + \sum_{i=1}^r \lambda_i \beta_i^\alpha)(X) = 0 \quad \text{for } \alpha = 1, \ldots, s\}.$$

Hence we see that codim $\hat{\mathfrak{f}}(v) = s$ if and only if $\{\beta_0^\alpha + \sum_{i=1}^r \lambda_i \beta_i^\alpha\}_{\alpha=1}^s$ are independent (mod $\varpi_0, \ldots, \varpi_r, \pi_1, \ldots, \pi_s$) at $x \in X$. Thus $R(X)$ is a non-empty open subset of $P(X)$ under the condition $(X.3)$. Moreover we have, at each $v \in R(X)$,

$$\text{Ch}\,(D_X^1)(v) = \{\varpi = \pi_\alpha = \beta_0^\alpha + \sum_{i=1}^r \lambda_i \beta_i^\alpha = \varpi_i = d\lambda_i + \gamma_0^i + \sum_{j=1}^r \lambda_j \gamma_j^i = 0$$
$$(i = 1, \ldots, r, \;\; \alpha = 1, \ldots, s)\,\}.$$

Therefore $\mathrm{Ch}\,(D_X^1)$ is a subbundle of D_X^2 of codimension n on $R(X)$, which completes the proof of Proposition. □

4.4 Symbol Subspaces

We will consider the relation between the symbol subspaces $\mathfrak{f}(v) = \mathrm{Ch}\,(D_X^1)(v)$ and $\hat{\mathfrak{f}}(v) \in D(x)$ for a point $v \in R(X)$ and $x = \nu(v) \in X$.

First observe that there exists an integral element V of $(R(X), D_X^2)$ at v such that

$$\mathfrak{s}_{-1}(v) = V \oplus \mathfrak{f}(v)(= D_X^2(v)),$$

if and only if there exists an integral element W of (X, D) at x such that

$$\mathfrak{t}_{-1}(x) = W \oplus \hat{\mathfrak{f}}(v)(= D(x)).$$

In fact $W = \nu_*(V)$ and $V = \nu_*^{-1}(W)$, where $\nu_* : D_X^2(v) \to D(x)$ is onto, $\mathrm{Ker}\,\nu_* = \mathrm{Ch}\,(D_X^2)(v)$ and $\nu_* : \mathfrak{f}(v) \to \hat{\mathfrak{f}}(v)$ is a linear isomorphism.

Put $\hat{\mathfrak{t}}_{-3}(v) = \mathfrak{t}_{-3}(x)/\hat{v}$, and let us consider

$$\hat{\mathfrak{t}}(v) = \hat{\mathfrak{t}}_{-3}(v) \oplus \mathfrak{t}_{-2}(x) \oplus \mathfrak{t}_{-1}(x).$$

$\hat{\mathfrak{t}}(v)$ is a quotient algebra of $\mathfrak{t}(x)$ and $\hat{\mathfrak{f}}(v) = \{X \in \mathfrak{t}_{-1}(x) \mid [X, \mathfrak{t}_{-2}(x)] = 0\}$ in $\hat{\mathfrak{t}}(v)$. Let us fix a basis of $\hat{\mathfrak{t}}_{-3}(v)$. Then, as in the proof of Proposition 3, the basis of $\mathfrak{s}_{-3}(v)$ is fixed. Hence we have $\mathfrak{s}_{-3}(v) \cong \mathbb{R}$, $\mathfrak{s}_{-2}(v) \cong V^*$ and $\mathfrak{f}(v) \subset S^2(V^*)$. Moreover we have $\hat{\mathfrak{t}}_{-3}(v) \cong \mathbb{R}$, $\mathfrak{t}_{-2}(x) \cong W^*$ and the map $\mu : \hat{\mathfrak{f}}(v) \to S^2(W^*)$ is defined by

$$\mu(\tilde{f})(w_1, w_2) = [[\tilde{f}, w_1], w_2] \in \mathbb{R} \cong \hat{\mathfrak{t}}_{-3}(v) \qquad \text{for } \tilde{f} \in \hat{\mathfrak{f}}(v).$$

Here we note that there exists a unique $f \in \mathfrak{f}(v)$ such that $\tilde{f} = \nu_*(f)$ and we obtain, by $\mathfrak{f}(v) \subset S^2(E^\perp)$ and the definition of the brackets of $\mathfrak{s}(v)$ and $\hat{\mathfrak{t}}(v)$,

$$[f, v_1] = \kappa^*([\tilde{f}, w_1]) \subset V^*,$$

where $w_1 = \kappa(v_1)$, $\kappa^* : W^* \to V^*$, $\kappa = \nu_* : V \to W$ and $\mathrm{Ker}\,\kappa = E$, $\kappa^*(W^*) = E^\perp$. Moreover we have

$$[[f, v_1], v_2] = [[\tilde{f}, w_1], w_2] \qquad \text{for } w_2 = \kappa(v_2).$$

Namely μ is injective and $\kappa_2^*(\hat{\mathfrak{f}}(v)) = \mathfrak{f}(v) \subset S^2(E^\perp) \subset S^2(V^*)$ where $\kappa_2^* : S^2(W^*) \to S^2(V^*)$ is induced from $\kappa^* : W^* \to V^*$.

Next we consider the algebraic prolongation $\mathfrak{f}(v)^{(1)} \subset S^3(V^*)$ of $\mathfrak{f}(v) \subset S^2(V^*)$,

$$\mathfrak{f}(v)^{(1)} = \mathfrak{f}(v) \otimes V^* \cap S^3(V^*).$$

Then, since $\mathfrak{f}(v) \subset S^2(E^\perp)$, we observe $\mathfrak{f}(v)^{(1)} \subset S^3(E^\perp)$. Therefore, for the prolongation $\hat{\mathfrak{f}}(v)^{(1)} = \hat{\mathfrak{f}}(v) \otimes W^* \cap S^3(W^*)$ of $\hat{\mathfrak{f}}(v) \subset S^2(W^*)$, we get

$$\kappa_3^*(\hat{\mathfrak{f}}(v)^{(1)}) = \mathfrak{f}(v)^{(1)} \subset S^3(E^\perp).$$

where $\kappa_3^* : S^3(W^*) \to S^3(V^*)$ is induced from $\kappa^* : W^* \to V^*$. Repeatedly we obtain $\kappa_{k+2}^*(\hat{\mathfrak{f}}(v)^{(k)}) = \mathfrak{f}(v)^{(k)} \subset S^{k+2}(E^\perp)$ for the higher prolongations.

Moreover, from $\mathfrak{f}(v) \subset S^2(E^\perp)$, we also observe that, for a regular basis $\{w_1, \dots, w_s\}$ of W for $\hat{\mathfrak{f}}(v) \subset S^2(W^*)$, we obtain the regular basis $\{v_1, \dots, v_s, v_{s+1}, \dots, v_n\}$ of V for $\mathfrak{f}(v) \subset S^2(V^*)$ by taking $v_i \in V$ such that $\kappa(v_i) = w_i$ $(i = 1, \dots, s)$ and adding a basis $\{v_{s+1}, \dots, v_n\}$ of $E = \operatorname{Ker}\kappa$. In fact, in this case, we have

$$\kappa_2^*(\hat{\mathfrak{f}}_k) = \mathfrak{f}_k \quad \text{for } k = 1, \dots, s \quad \text{and } \mathfrak{f}_k = 0 \quad \text{for } k = s, \dots, n.$$

where $\hat{\mathfrak{f}}_k = \{\tilde{f} \in \hat{\mathfrak{f}}(v) \mid w_1 \rfloor \tilde{f} = \cdots = w_k \rfloor \tilde{f} = 0\}$ and $\mathfrak{f}_k = \{f \in \mathfrak{f}(v) \mid v_1 \rfloor f = \cdots = v_k \rfloor f = 0\}$.

Summarizing the above discussion, we obtain

Proposition 4. *Notations being as above:*

(1) $\mathfrak{f}(v) \subset S^2(V^*)$ *is involutive if and only if* $\hat{\mathfrak{f}}(v) \subset S^2(W^*)$ *is involutive.*
(2) $\mathfrak{f}(v) \subset S^2(V^*)$ *is of finite type if and only if* $\hat{\mathfrak{f}}(v) \subset S^2(W^*)$ *is of finite type.*

5 Parabolic Geometries Associated with *PD*-Manifolds of Second Order

We will here exhibit Parabolic Geometries which directly correspond to the geometry of *PD* manifolds of second order, following [22] and [26].

5.1 Differential Systems Associated with Simple Graded Lie Algebras (Parabolic Geometries)

We first recall basic materials for simple graded Lie algebras over $\mathbb{C}$ and state the Prolongation Theorem. We will work mainly over $\mathbb{C}$ in this section for the sake of simplicity.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a simple root system $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Φ of $\mathfrak{g}$ relative to $\mathfrak{h}$. Then every $\alpha \in \Phi$ is an (all non-negative or all non-positive) integer coefficient linear combination of elements of Δ and we have the root space decomposition of $\mathfrak{g}$;

$$\mathfrak{g} = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha},$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [h, X] = \alpha(h)X \quad \text{for } h \in \mathfrak{h}\}$ is (one-dimensional) root space (corresponding to $\alpha \in \Phi$) and Φ^+ denotes the set of positive roots [8].

Now let us take a nonempty subset Δ_1 of Δ. Then Δ_1 defines the partition of Φ^+ as in the following and induces the gradation of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ as follows:

$$\Phi^+ = \cup_{p\geqq 0}\Phi_p^+, \qquad \Phi_p^+ = \{\alpha = \sum_{i=1}^{\ell} n_i\alpha_i \mid \sum_{\alpha_i\in\Delta_1} n_i = p\},$$

$$\mathfrak{g}_p = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi_0^+} \mathfrak{g}_{-\alpha}, \quad \mathfrak{g}_{-p} = \bigoplus_{\alpha\in\Phi_p^+} \mathfrak{g}_{-\alpha},$$

$$[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q} \qquad \text{for} \quad p, q \in \mathbb{Z}.$$

Moreover the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ satisfies the following generating condition:

$$\mathfrak{g}_p = [\mathfrak{g}_{p+1}, \mathfrak{g}_{-1}] \quad \text{for} \quad p < -1$$

We denote the simple graded Lie algebra $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ obtained from Δ_1 in this manner by (X_ℓ, Δ_1), when $\mathfrak{g}$ is a simple Lie algebra of type X_ℓ. Here X_ℓ stands for the Dynkin diagram of $\mathfrak{g}$ representing Δ and Δ_1 is a subset of vertices of X_ℓ. Moreover we have

$$\mu = \sum_{\alpha_i\in\Delta_1} n_i(\theta),$$

where $\theta = \sum_{i=1}^{\ell} n_i(\theta)\,\alpha_i$ is the highest root of Φ^+.

Conversely we have (Theorem 3.12 [22])

Theorem A. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition. Let X_ℓ be the Dynkin diagram of $\mathfrak{g}$. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic to a graded Lie algebra (X_ℓ, Δ_1) for some $\Delta_1 \subset \Delta$. Moreover (X_ℓ, Δ_1) and (X_ℓ, Δ_1') are isomorphic if and only if there exists a diagram automorphism ϕ of X_ℓ such that $\phi(\Delta_1) = \Delta_1'$.*

In the real case, we can utilize the Satake diagram of $\mathfrak{g}$ to describe gradations of $\mathfrak{g}$ (Theorem 3.12 [22]).

By Theorem A, the classification of the gradation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $\mathfrak{g}$ satisfying the generating condition coincides with that of parabolic subalgebras $\mathfrak{g}' = \bigoplus_{p\geqq 0} \mathfrak{g}_p$

of $\mathfrak{g}$. Accordingly, to each simple graded Lie algebra (X_ℓ, Δ_1), there corresponds a unique R-space $M_\mathfrak{g} = G/G'$ (compact simply connected homogeneous complex manifold) (see [22] Sect. 4.1 for detail). Furthermore, when $\mu \geqq 2$, there exists the G-invariant differential system $D_\mathfrak{g}$ on $M_\mathfrak{g}$, which is induced from $\mathfrak{g}_{-1}$, and the standard differential system $(M(\mathfrak{m}), D_\mathfrak{m})$ of type $\mathfrak{m}$ becomes an open submanifold of $(M_\mathfrak{g}, D_\mathfrak{g})$. For the Lie algebras of all infinitesimal automorphisms of $(M_\mathfrak{g}, D_\mathfrak{g})$, hence of $(M(\mathfrak{m}), D_\mathfrak{m})$, we have the following *Prolongation Theorem* (Theorem 5.2 [22]).

Prolongation Theorem. *Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{C}$ satisfying the generating condition. Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases:*

(1) $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *is of depth* 1.
(2) $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$ *is a contact gradation.*
(3) $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic to* $(A_\ell, \{\alpha_1, \alpha_i\})$ $(1 < i < \ell)$ *or* $(C_\ell, \{\alpha_1, \alpha_\ell\})$.

Furthermore $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ *except when $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is isomorphic to* $(A_\ell, \{\alpha_1\})$ *or* $(C_\ell, \{\alpha_1\})$.

Here R-spaces corresponding to the above exceptions (1), (2) and (3) are as follows: (1) correspond to compact irreducible hermitian symmetric spaces. (2) correspond to contact manifolds of Boothby type (Standard contact manifolds), which exist uniquely for each simple Lie algebra other than $\mathfrak{sl}(2, \mathbb{C})$(see Sect. 5.2 below). In case of (3), $(J(\mathbb{P}^\ell, i), C)$ corresponds to $(A_\ell, \{\alpha_1, \alpha_i\})$ and $(L(\mathbb{P}^{2\ell-1}), E)$ corresponds to $(C_\ell, \{\alpha_1, \alpha_\ell\})$ $(1 < i < \ell)$, where $\mathbb{P}^\ell$ denotes the ℓ-dimensional complex projective space and $\mathbb{P}^{2\ell-1}$ is the Standard contact manifold of type C_ℓ corresponding to $(C_\ell, \{\alpha_1\})$. Here we note that R-spaces corresponding to (2) and (3) are all Jet spaces of the first or second order.

For the real version of this theorem, we refer the reader to Theorem 5.3 [22].

Now the *Parabolic Geometry* is a geometry modeled after the homogeneous space G/G', where G is a (semi-)simple Lie group and G' is a parabolic subgroup of G (cf. [1]). Precisely, in this paper, we mean, by a Parabolic Geometry, the *Geometry associated with the Simple Graded Lie Algebra* in the sense of N. Tanaka [16].

In fact, let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a simple graded Lie algebra over $\mathbb{R}$ satisfying the generating condition. Let M be a manifold with a $G_0^\sharp$-structure of type $\mathfrak{m}$ in the sense of [16] (for the precise definition, see Sect. 2 of [16]). In [16], [15], under the assumption that $\mathfrak{g}$ is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$, N. Tanaka constructed the *Normal Cartan Connection* (P, ω) *of Type* $\mathfrak{g}$ over M, which settles the equivalence problem for the $G_0^\sharp$-structure of type $\mathfrak{m}$ in the following sense: Let M and $\hat{M}$ be two manifolds with $G_0^\sharp$-structures of type $\mathfrak{m}$. Let (P, ω) and $(\hat{P}, \hat{\omega})$ be the normal connections of type $\mathfrak{g}$ over M and $\hat{M}$ respectively. Then a diffeomorphism φ of M onto $\hat{M}$ preserving the $G_0^\sharp$-structures lifts uniquely to an isomorphism $\varphi^\sharp$ of (P, ω) onto $(\hat{P}, \hat{\omega})$ and vice versa ([16], Theorem 2.7).

Here we note that, if $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$, a $G_0^\sharp$-structure on M is nothing but a regular differential system of type $\mathfrak{m}$ (see [16, Sect. 2.2]). Thus a Parabolic

Geometry modeled after G/G' is the geometry of *PD* manifold of second order with the symbol algebra $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$, if $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ and $\mathfrak{m}$ is isomorphic to $\mathfrak{s}$.

Hence, among simple graded Lie algebras $\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p \cong (X_\ell, \Delta_1)$, we will seek those algebras such that $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is isomorphic to the symbol algebra of *PD* manifolds of second order. Thus a necessary condition for this is $\mu = 3$ and $\dim \mathfrak{g}_{-3} = 1$. Then, by the above construction of (X_ℓ, Δ_1), $\mathfrak{g}_3$ should be the highest root space. This forces $\Delta_\theta \subset \Delta_1$ where (X_ℓ, Δ_θ) is the (standard) contact gradation of $\mathfrak{g}$ (see Sect. 5.2 below). These two conditions confine the possibility of (X_ℓ, Δ_1). In fact, a simple graded Lie algebra (X_ℓ, Δ_1), which satisfies both $\mu = 3$ and $\Delta_\theta \subset \Delta_1$ is one of the following: $(A_\ell, \{\alpha_1, \alpha_i, \alpha_\ell\})$ $(1 < i \leqq [\frac{\ell+1}{2}])$, $(B_\ell, \{\alpha_1, \alpha_2\})$, $(C_\ell, \{\alpha_1, \alpha_\ell\})$, $(D_\ell, \{\alpha_1, \alpha_2\})$, $(D_\ell, \{\alpha_2, \alpha_\ell\})$, $(E_6, \{\alpha_1, \alpha_2\})$ and $(E_7, \{\alpha_1, \alpha_7\})$ up to conjugacy.

In fact, as we will see in Sects. 5.3 and 5.4, these simple graded Lie algebras (X_ℓ, Δ_1) represent the Parabolic Geometries of *PD* manifolds of second order (of finite type), except for $(C_\ell, \{\alpha_1, \alpha_\ell\})$, which is one of the exception in Prolongation Theorem and represents the Parabolic Geometry of third order equations of finite type (cf. [26] Sect. 3 Case (4)).

Furthermore, in Sects. 6.2 and 6.3, by utilizing the First Reduction Theorem, we will see more examples of Parabolic Geometry, which is associated with the geometry of (X, D) in Sect. 4.3.

5.2 Standard Contact Manifolds

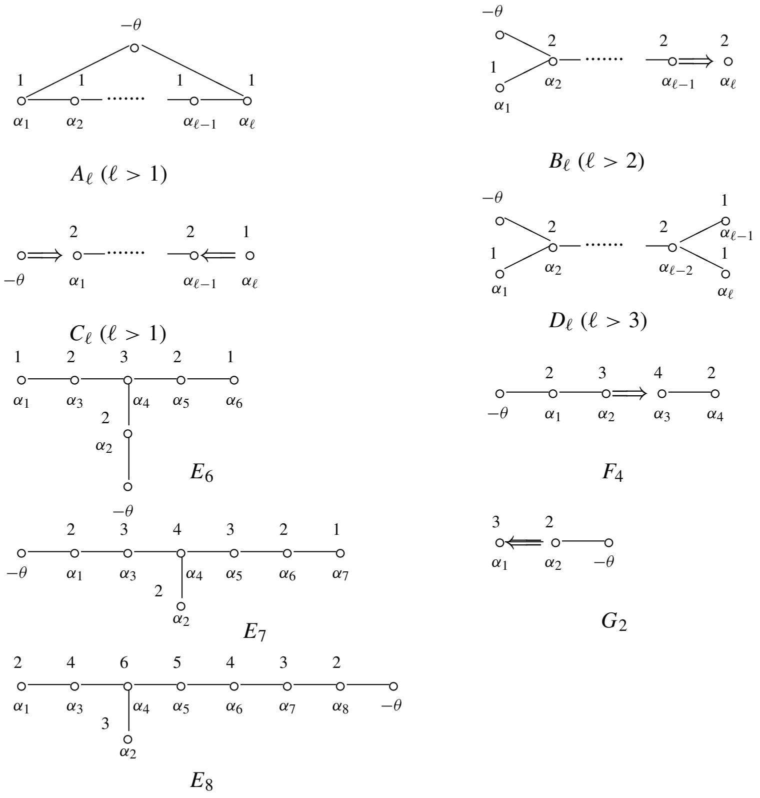
Each simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has the highest root θ. Let Δ_θ denote the subset of Δ consisting of all vertices which are connected to $-\theta$ in the Extended Dynkin diagram of X_ℓ $(\ell \geqq 2)$. This subset Δ_θ of Δ, by the construction in Sect. 5.1, defines a gradation (or a partition of Φ^+), which distinguishes the highest root θ. Then, this gradation (X_ℓ, Δ_θ) turns out to be a contact gradation, which is unique up to conjugacy (Theorem 4.1 [22]). Explicitly we have $\Delta_\theta = \{\alpha_1, \alpha_\ell\}$ for A_ℓ type and $\Delta_\theta = \{\alpha_\theta\}$ for other types. Here $\alpha_\theta = \alpha_2, \alpha_1, \alpha_2$ for B_ℓ, C_ℓ, D_ℓ types respectively and $\alpha_\theta = \alpha_2, \alpha_1, \alpha_8, \alpha_1, \alpha_2$ for E_6, E_7, E_8, F_4, G_2 types respectively.

Moreover we have the adjoint (or equivalently coadjoint) representation, which has θ as the highest weight. The R-space $J_\mathfrak{g}$ corresponding to (X_ℓ, Δ_θ) can be obtained as the projectiviation of the (co-)adjoint orbit of G passing through the root vector of θ. By this construction, $J_\mathfrak{g}$ has the natural contact structure $C_\mathfrak{g}$ induced from the symplectic structure as the coadjoint orbit, which corresponds to the contact gradation (X_ℓ, Δ_θ) (cf. [22], Sect. 4). Standard contact manifolds $(J_\mathfrak{g}, C_\mathfrak{g})$ were first found by Boothby [2] as compact simply connected homogeneous complex contact manifolds.

For the explicit description of the standard contact manifolds of the classical type, we refer the reader to Sect. 4.3 [22].

In Sects. 5.3 and 5.4, the model equation $(R_\mathfrak{g}, D^2_\mathfrak{g})$ can be realized as a R-space orbit in $L(J_\mathfrak{g})$.

Extended Dynkin Diagrams with the coefficient of Highest Root (cf. [3])



5.3 Classical Type Examples

We will describe here the symbol algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ of each (X_ℓ, Δ_1) of the classical type and give the model equations of second order. We refer the reader to Sect. 3 of [26] for the detailed description of the symbol algebras in matrices form. In this and next subsections, we will discuss in the complex analytic or the real C^∞ category depending on whether $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

(1) *Case of* $(A_\ell, \{\alpha_1, \alpha_{i+1}, \alpha_\ell\})$ $(1 < i + 1 \leqq [\frac{\ell+1}{2}])$.

We have the following matrix representation of $(A_\ell, \{\alpha_1, \alpha_{i+1}, \alpha_\ell\})$:

$$\mathfrak{sl}\,(\ell + 1, \mathbb{K}) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3.$$

where the gradation is given by subdividing matrices as follows;

$$
\begin{aligned}
\mathfrak{g}_{-3} &= \left\{ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ a&0&0&0 \end{pmatrix} \middle| \quad a \in \mathbb{K} \right\} \cong \mathbb{K},\\
\mathfrak{g}_{-2} &= \left\{ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ -\xi_2&0&0&0\\ 0&{}^t\xi_1&0&0 \end{pmatrix} \middle| \quad \xi_1 \in \mathbb{K}^i \quad \xi_2 \in \mathbb{K}^j \right\},\\
\mathfrak{g}_{-1} &= V \oplus \mathfrak{f},\\
\mathfrak{f} &= \left\{ \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&A&0&0\\ 0&0&0&0 \end{pmatrix} \middle| \quad A \in M(j,i) \right\},\\
V &= \left\{ \begin{pmatrix} 0&0&0&0\\ x_1&0&0&0\\ 0&0&0&0\\ 0&0&{}^tx_2&0 \end{pmatrix} \middle| \quad x_1 \in \mathbb{K}^i, \quad x_2 \in \mathbb{K}^j \right\} \cong \mathbb{K}^{\ell-1},\\
\mathfrak{g}_0 &= \left\{ \begin{pmatrix} b&0&0&0\\ 0&B&0&0\\ 0&0&C&0\\ 0&0&0&c \end{pmatrix} \middle| \quad \begin{array}{r} b,c \in \mathbb{K},\ B \in \mathfrak{gl}\,(i,\mathbb{K}),\ C \in \mathfrak{gl}\,(j,\mathbb{K}),\\ b+c+\mathrm{tr}B+\mathrm{tr}C = 0 \end{array} \right\}\\
\mathfrak{g}_k &= \{ {}^tX \mid X \in \mathfrak{g}_{-k} \}, (k = 1,2,3),
\end{aligned}
$$

where $i + j = \ell - 1$. Then we have

$$
\begin{aligned}
\mathfrak{m} &= \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus (V \oplus \mathfrak{f})\\
&= \left\{ \begin{pmatrix} 0&0&0&0\\ x_1&0&0&0\\ -\xi_2&A&0&0\\ a&{}^t\xi_1&{}^tx_2&0 \end{pmatrix} = \hat{a} + \check{\xi} + \hat{x} + \hat{A} \middle| \begin{array}{c} \xi = \begin{pmatrix} \xi_1\\ \xi_2 \end{pmatrix}, x = \begin{pmatrix} x_1\\ x_2 \end{pmatrix} \in \mathbb{K}^{\ell-1},\\ a \in \mathbb{K},\ A \in M(j,i) \end{array} \right\}.
\end{aligned}
$$

By calculating $[\check{\xi}, \hat{x}]$ and $[[\hat{A}, \hat{x}], \hat{x}]$, we have

$$
[\check{\xi}, \hat{x}] = \widehat{({}^t\xi x)},\ \ [[\hat{A}, \hat{x}], \hat{x}] = (-\widehat{2^t x_2 A} x_1), \quad 2^t x_2 A x_1 = ({}^tx_1, {}^tx_2) \begin{pmatrix} 0 & {}^tA\\ A & 0 \end{pmatrix} \begin{pmatrix} x_1\\ x_2 \end{pmatrix}
$$

Thus we have $\mathfrak{g}_{-2} \cong V^*$ and $\mathfrak{f} = \langle \{ e_k^* \circledcirc e_\alpha^* (1 \leqq k \leqq i, i+1 \leqq \alpha \leqq \ell - 1) \} \rangle \subset S^2(V^*)$ for a basis $\{e_1, \ldots, e_{\ell-1}\}$ of V. This implies that the model equation of the second order is given by

$$\frac{\partial^2 z}{\partial x_k \partial x_l} = \frac{\partial^2 z}{\partial x_\alpha \partial x_\beta} = 0 \quad \text{for } 1 \leqq k, l \leqq i \quad \text{and } i+1 \leqq \alpha, \beta \leqq \ell - 1,$$

where z is dependent variable and $x_1, \dots, x_{\ell-1}$ are independent variables.

(2) *Case of* $(B_\ell, \{\alpha_1, \alpha_2\})$ $(\ell \geqq 3)$, $(D_\ell, \{\alpha_1, \alpha_2\})$ $(\ell \geqq 4)$.

Similarly in this case, we have $\dim \mathfrak{f} = 1$ and $\mathfrak{f} = \langle \{e_1^* \circledcirc e_1^* + \cdots + e_n^* \circledcirc e_n^*\} \rangle$ for a basis $\{e_1, \dots, e_n\}$ of V, where $n = 2\ell - 3$ in case of B_ℓ and $n = 2\ell - 4$ in case of D_ℓ. Thus we have the following model equation of the second order:

$$\frac{\partial^2 z}{\partial x_p \partial x_q} = \delta_{pq} \frac{\partial^2 z}{\partial^2 x_1} \qquad \text{for} \quad 1 \leqq p, q \leqq n,$$

where z is dependent variable and $x_1, \dots, x_n$ are independent variables. This equation is the embedding equation as a hypersurface for the quadric Q^n. For the explicit matrix description of the gradation, we refer the reader to Case (5) in Sect. 3 of [26].

(3) *Case of* $(D_\ell, \{\alpha_2, \alpha_\ell\})$ $(\ell \geqq 4)$.

Similarly in this case, we have $\dim \mathfrak{f} = \frac{1}{2}(\ell - 1)(\ell - 2)$ and

$$\mathfrak{f} = \langle \{ \sum_{p,q=1}^{\ell-1} a_{pq} (e_p^1)^* \circledcirc (e_q^2)^* \mid A = (a_{pq}) \in \mathfrak{o}(\ell - 1) \} \rangle$$

for a basis $\{e_1^1, \dots, e_{\ell-1}^1, e_1^2, \dots, e_{\ell-1}^2\}$ of V. Thus we have the following model equation of the second order:

$$\frac{\partial^2 z}{\partial x_p^i \partial x_q^j} + \frac{\partial^2 z}{\partial x_q^i \partial x_p^j} = 0 \qquad \text{for} \quad 1 \leqq i, j \leqq 2, \quad 1 \leqq p < q \leqq \ell - 1,$$

where z is dependent variable and $x_1^1, \dots, x_{\ell-1}^1, x_1^2, \dots, x_{\ell-1}^2$ are independent variables. This equation is the Plücker embedding equation for the Grassmann manifold $\mathrm{Gr}(\ell + 1, 2)$ (see [11] Sect. 3). For the explicit matrix description of the gradation, we refer the reader to Case (10) in Sect. 3 of [26].

5.4 *Exceptional Type Examples*

We here only describe the model equation $(R_\mathfrak{g}, D_\mathfrak{g}^2)$ of second order in the form of the standard differential system of type $\mathfrak{m}$. We refer the reader to Sect. 4 of [26] for the detailed description of the symbol algebras by the use of the Chevalley basis of $\mathfrak{g}$.

(1) *Case of* $(E_6, \{\alpha_1, \alpha_2\})$.

The symbol algebra $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is described as follows:

$$\mathfrak{g}_{-3} \cong \mathbb{K}, \;\; \mathfrak{g}_{-2} \cong V^*, \;\; \mathfrak{g}_{-1} = V \oplus \mathfrak{f}, \;\; \mathfrak{f} \subset S^2(V^*) \quad \text{and } \dim V = 10, \dim \mathfrak{f} = 5.$$

Here the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case is given by

$$D_{\mathfrak{m}} = \{\varpi = \varpi_1 = \varpi_2 = \cdots = \varpi_{10} = 0\},$$

where

$$\begin{aligned}
\varpi &= dz - p_1dx_1 - \cdots - p_{10}dx_{10}, \\
\varpi_1 &= dp_1 + q_5dx_8 + q_4dx_9 + q_3dx_{10}, & \varpi_2 &= dp_2 - q_5dx_5 - q_4dx_7 + q_2dx_{10}, \\
\varpi_3 &= dp_3 + q_5dx_4 + q_4dx_6 + q_1dx_{10}, & \varpi_4 &= dp_4 + q_5dx_3 - q_3dx_7 - q_2dx_9, \\
\varpi_5 &= dp_5 - q_5dx_2 + q_3dx_6 - q_1dx_9, & \varpi_6 &= dp_6 + q_4dx_3 + q_3dx_5 + q_2dx_8, \\
\varpi_7 &= dp_7 - q_4dx_2 - q_3dx_4 + q_1dx_8, & \varpi_8 &= dp_8 + q_5dx_1 + q_2dx_6 + q_1dx_7, \\
\varpi_9 &= dp_9 + q_4dx_1 - q_2dx_4 - q_1dx_5 & \varpi_{10} &= dp_{10} + q_3dx_1 + q_2dx_2 + q_1dx_3.
\end{aligned}$$

Here $(x_1, \ldots, x_{10}, z, p_1, \ldots, p_{10}, q_1, \ldots, q_5)$ is a coordinate system of $M(\mathfrak{m}) \cong \mathbb{K}^{26}$.

(2) *Case of* $(E_7, \{\alpha_1, \alpha_7\})$.

The symbol algebra $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ is described as follows:

$$\mathfrak{g}_{-3} \cong \mathbb{K}, \;\; \mathfrak{g}_{-2} \cong V^*, \;\; \mathfrak{g}_{-1} = V \oplus \mathfrak{f}, \;\; \mathfrak{f} \subset S^2(V^*) \quad \text{and } \dim V = 16, \dim \mathfrak{f} = 10.$$

Here the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case is given by

$$D_{\mathfrak{m}} = \{\varpi = \varpi_1 = \varpi_2 = \cdots = \varpi_{16} = 0\},$$

where

$$\begin{aligned}
\varpi &= dz - p_1dx_1 - \cdots - p_{16}dx_{16}, \\
\varpi_1 &= dp_1 + q_{10}dx_{11} + q_9dx_{12} + q_8dx_{14} + q_7dx_{15} + q_5dx_{16}, \\
\varpi_2 &= dp_2 - q_{10}dx_9 - q_9dx_{10} - q_8dx_{13} + q_6dx_{15} + q_4dx_{16}, \\
\varpi_3 &= dp_3 + q_{10}dx_6 + q_9dx_8 - q_7dx_{13} - q_6dx_{14} + q_3dx_{16}, \\
\varpi_4 &= dp_4 - q_{10}dx_5 - q_9dx_7 - q_5dx_{13} - q_4dx_{14} - q_3dx_{15}, \\
\varpi_5 &= dp_5 - q_{10}dx_4 + q_8dx_8 + q_7dx_{10} + q_6dx_{12} + q_2dx_{16}, \\
\varpi_6 &= dp_6 + q_{10}dx_3 - q_8dx_7 + q_5dx_{10} + q_4dx_{12} - q_2dx_{15}, \\
\varpi_7 &= dp_7 - q_9dx_4 - q_8dx_6 - q_7dx_9 - q_6dx_{11} + q_1dx_{16}, \\
\varpi_8 &= dp_8 + q_9dx_3 + q_8dx_5 - q_5dx_9 - q_4dx_{11} - q_1dx_{15}, \\
\varpi_9 &= dp_9 - q_{10}dx_2 - q_7dx_7 - q_5dx_8 + q_3dx_{12} + q_2dx_{14},
\end{aligned}$$

$$
\begin{aligned}
\varpi_{10} &= dp_{10} - q_9 dx_2 + q_7 dx_5 + q_5 dx_6 - q_3 dx_{11} + q_1 dx_{14}, \\
\varpi_{11} &= dp_{11} + q_{10} dx_1 - q_6 dx_7 - q_4 dx_8 - q_3 dx_{10} - q_2 dx_{13}, \\
\varpi_{12} &= dp_{12} + q_9 dx_1 + q_6 dx_5 + q_4 dx_6 + q_3 dx_9 - q_1 dx_{13}, \\
\varpi_{13} &= dp_{13} - q_8 dx_2 - q_7 dx_3 - q_5 dx_4 - q_2 dx_{11} - q_1 dx_{12}, \\
\varpi_{14} &= dp_{14} + q_8 dx_1 - q_6 dx_3 - q_4 dx_4 + q_2 dx_9 + q_1 dx_{10}, \\
\varpi_{15} &= dp_{15} + q_7 dx_1 + q_6 dx_2 - q_3 dx_4 - q_2 dx_6 - q_1 dx_8, \\
\varpi_{16} &= dp_{16} + q_5 dx_1 + q_4 dx_2 + q_3 dx_3 + q_2 dx_5 + q_1 dx_7.
\end{aligned}
$$

Here $(x_1, \ldots, x_{16}, z, p_1, \ldots, p_{16}, q_1, \ldots, q_{10})$ is a coordinate system of $M(\mathfrak{m}) \cong \mathbb{K}^{43}$.

The model linear equations in Sect. 5.3 (2), (3) and Sect. 5.4 (1) and (2) appeared as the embedding equations of the corresponding symmetric spaces into the projective spaces in [12], [11] and [7] (see [11] Sect. 1). Except for Sect. 5.3 (2), these equations have rigidity properties. As is pointed out in Sect. 5 of [26], by the vanishing of the second cohomology (cf. [9], Theorems 2.7 and 2.9 [16], Proposition 5.5 [22]), we observe that Parabolic Geometries associated with $(D_\ell, \{\alpha_2, \alpha_\ell\})$, $(E_6, \{\alpha_1, \alpha_2\})$ and $(E_7, \{\alpha_1, \alpha_7\})$ have no local invariant. Thus the model second order equations of these cases is solely characterized by their symbols $\mathfrak{f} \subset S^2(V^*)$ under the condition (C), as in the case of Typical involutive symbols in Sect. 2.4 (cf. [25]).

6 Examples of First Reduction Theorem

Utilizing the First Reduction Theorem, we will discuss the Typical class of type $\mathfrak{f}^3(r)$ and exhibit several examples of *PD* manifolds of second order given through Parabolic Geometries on $(X.D)$.

6.1 Typical Class of Type $\mathfrak{f}^3(r)$

Let $(R; D^1, D^2)$ be a *PD* manifold of second order satisfying the condition (C), which is regular of type $\mathfrak{f}^3(r)$ $(r \leqq n-2)$. Namely $(R; D^1, D^2)$ is a *PD* manifold of second order such that symbol algebra $\mathfrak{s}(v)$ at each point $v \in R$ is isomorphic to $\mathfrak{s} = \mathfrak{s}_{-3} \oplus \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1}$ where

$$
\mathfrak{s}_{-3} = \mathbb{R}, \quad \mathfrak{s}_{-2} = V^* \quad \text{and} \quad \mathfrak{s}_{-1} = V \oplus \mathfrak{f}^3(r).
$$

Then, by Lemma 1.2 [21], there exists a coframe $\{\varpi, \varpi_a, \omega_a, \pi_{\alpha\beta} (1 \leqq a \leqq n, r+1 \leqq \alpha, \beta \leqq n)\}$ on a neighborhood U of each point $v \in R$ such that $D^1 = \{\varpi = 0\}$, $D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}$ and that the following equalities hold:

$$
d\varpi \equiv \omega_1 \wedge \varpi_1 + \cdots + \omega_n \wedge \varpi_n \qquad (\text{mod} \quad \varpi),
$$

$$\begin{cases} d\varpi_i \equiv \quad 0 & (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_n), \\ d\varpi_\alpha \equiv \quad \omega_{r+1} \wedge \pi_{\alpha r+1} + \cdots + \omega_n \wedge \pi_{\alpha n} & (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_n). \end{cases}$$

for $1 \leqq i \leqq r$ and $r+1 \leqq \alpha \leqq n$ and $\pi_{\alpha\beta} = \pi_{\beta\alpha}$. Thus we see that Ch (D^2) is a subbundle of D^2 of rank r. A coframe $\{\varpi, \varpi_a, \omega_a, \pi_{\alpha\beta}(1 \leqq a \leqq n, r+1 \leqq \alpha, \beta \leqq n)\}$ on U satisfying $D^1 = \{\varpi = 0\}$ and $D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}$ is called adapted if it satisfies the above structure equations. Then we have (Lemma 4.1 [21])

Lemma A. *Let $(R;\ D^1,\ D^2)$ be as above and $r \leqq n-2$. Then there exists an adapted coframe on U such that the following equalities hold:*

$$d\varpi_i \equiv 0 \quad (\text{mod} \quad \varpi_1, \ldots, \varpi_r) \quad \textit{for} \;\; i = 1, \ldots, r.$$

By this lemma, for $\partial D^2 = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\}$, we have

$$\begin{cases} d\varpi \;\equiv \omega_{r+1} \wedge \varpi_{r+1} + \cdots + \omega_n \wedge \varpi_n & (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_r), \\ d\varpi_i \equiv \quad 0 & (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_r). \end{cases}$$

Hence $\partial^2 D^2 = \partial^{(2)} D^2 = \{\varpi_1 = \cdots = \varpi_r = 0\}$ and $B = \partial^2 D^2$ is completely integrable.

Now we assume that R is regular with respect to Ch (D^2), i.e., the leaf space $X = R/\text{Ch}\,(D^2)$ is a manifold such that the projection $\rho : R \to X$ is a submersion and there exists differential system D on X satisfying $D^2 = \rho_*^{-1}(D)$. Then, from the above information, we see that there exists a coframe $\{\varpi, \varpi_1, \ldots, \varpi_n, \omega_{r+1}, \ldots, \omega_n, \pi_{\alpha\beta}(r+1 \leqq \alpha \leqq \beta \leqq n)\}$ on a neighborhood of each $x \in X$ such that

$$D = \{\varpi = \varpi_1 = \cdots = \varpi_n = 0\}, \partial D = \{\varpi = \varpi_1 = \cdots = \varpi_r = 0\},$$

$$\partial^2 D = \partial^{(2)} D = \{\varpi_1 = \cdots = \varpi_r = 0\},$$

and that

$$\begin{cases} d\varpi_i \equiv \quad 0 & (\text{mod} \quad \varpi_1, \ldots, \varpi_r) \\ d\varpi \;\equiv \omega_{r+1} \wedge \varpi_{r+1} + \cdots + \omega_n \wedge \varpi_n & (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_r), \\ d\varpi_\alpha \equiv \omega_{r+1} \wedge \pi_{\alpha r+1} + \cdots + \omega_n \wedge \pi_{\alpha n} & (\text{mod} \quad \varpi, \varpi_1, \ldots, \varpi_n). \end{cases}$$

for $1 \leqq i \leqq r$ and $r+1 \leqq \alpha \leqq n$ and $\pi_{\alpha\beta} = \pi_{\beta\alpha}$. Thus $B = \partial^2 D$ is completely integrable. Let $p_1, \ldots, p_r$ be the independent first integral of B around $x \in X$. Then we obtain

$$D = \{\varpi = \varpi_{r+1} = \cdots = \varpi_n = dp_1 = \cdots = dp_r = 0\},$$

$$\begin{cases} d\varpi \equiv \omega_{r+1} \wedge \varpi_{r+1} + \cdots + \omega_n \wedge \varpi_n \pmod{\varpi, dp_1, \ldots, dp_r}, \\ d\varpi_\alpha \equiv \omega_{r+1} \wedge \pi_{\alpha r+1} + \cdots + \omega_n \wedge \pi_{\alpha n} \pmod{\varpi, \varpi_{r+1}, \ldots, \varpi_n, dp_1, \ldots, dp_r}. \end{cases}$$

for $r + 1 \leqq \alpha \leqq n$.

Namely (X, D) is a *parameterized second order contact manifold*. Hence, by the Darboux theorem, we obtain a coordinate system $(x_\alpha, z, p_i, p_\alpha, p_{\alpha\beta})$ $(1 \leqq i \leqq r, r + 1 \leqq \alpha, \leqq \beta \leqq n)$ around $x \in X$ such that (see [21] Sect. 4.2 and [24] Sect. 1.4)

$$D = \{\hat{\varpi} = \hat{\varpi}_{r+1} = \cdots = \hat{\varpi}_n = dp_1 = \cdots = dp_r = 0\},$$

where

$$\hat{\varpi} = dz - \sum_{\alpha=r+1}^{n} p_\alpha dx_\alpha, \quad \hat{\varpi}_\alpha = dp_\alpha - \sum_{\beta=r+1}^{n} p_{\alpha\beta} dx_\beta.$$

We refer the reader to Sect. 4.2 [21] for the detail to obtain a canonical coordinate system of $(R; D^1, D^2)$.

Moreover we observe that (X, D) satisfies three conditions in Sect. 4.3,

$$R(X) = \{v \in P(X) \mid v \not\supset \partial^2 D(x) \mid x = \nu(v)\}.$$

and $\hat{\mathfrak{f}}(v) = \mathrm{Ch}\,(\partial D)(x) (\cong S^2((V_s)^*)) \subset D(x)$ for $v \in R(X)$. Thus, by Proposition 4, R is involutive, because $S^2((V_s)^*)$ is involutive.

6.2 G_2-Geometry

Let (X_ℓ, Δ_θ) be the (standard) contact gradation. Then we have $\Delta_\theta = \{\alpha_\theta\}$ except for A_ℓ type (see Sect. 5.2). As we observed in Sect. 6.3 in [23], for the exceptional simple Lie algebras, there exists, without exception, a unique simple root α_G next to α_θ such that the coefficient of α_G in the highest root is 3. We will consider simple graded Lie algebras $(X_\ell, \{\alpha_G\})$ of depth 3 and will show that regular differential systems of these types satisfy the conditions $(X.1)$ to $(X.3)$ in Sect. 4.3.

Explicitly we will here consider the following simple graded Lie algebras of depth 3: $(G_2, \{\alpha_1\})$, $(F_4, \{\alpha_2\})$, $(E_6, \{\alpha_4\})$, $(E_7, \{\alpha_3\})$, $(E_8, \{\alpha_7\})$, $(B_\ell, \{\alpha_1, \alpha_3\})$ $(\ell \geqq 3)$, $(D_\ell, \{\alpha_1, \alpha_3\})$ $(\ell \geqq 5)$ and $(D_4, \{\alpha_1, \alpha_3, \alpha_4\})$. These graded Lie algebras have the common feature with $(G_2, \{\alpha_1\})$ as follows: In these cases, $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ satisfies $\dim \mathfrak{g}_{-3} = 2$ and $\dim \mathfrak{g}_{-1} = 2 \dim \mathfrak{g}_{-2}$. Moreover, in the description of the gradation in terms of the root space decomposition in Sect. 5.1, we have $\Phi_3^+ = \{\theta, \theta - \alpha_\theta\}$ such that the coefficient of α_θ in each $\beta \in \Phi_2^+$ is 1 and Φ_1^+ consists of roots $\theta - \beta, \theta - \alpha_\theta - \beta$ for each $\beta \in \Phi_2^+$. Hence, ignoring the bracket product in $\mathfrak{g}_{-1}$, we can describe the bracket products of other part of $\mathfrak{m}$, in terms of paring, by

$$\mathfrak{g}_{-3} = W, \qquad \mathfrak{g}_{-2} = V \qquad \text{and} \qquad \mathfrak{g}_{-1} = W \otimes V^*,$$

where $\dim W = 2$.

Thus let (X, D) be a regular differential system of type $\mathfrak{m}$, where $\mathfrak{m}$ is the negative part of one of the above graded Lie algebras. Then $(X, \partial D)$ is a regular differential system of type $\mathfrak{c}^1(s,2)$. Namely, there exists a coframe $\{\varpi_1, \varpi_2, \pi_1, \dots, \pi_s, \pi_1^1, \dots, \pi_1^s, \pi_2^1, \dots, \pi_2^s\}$ around $x \in X$ such that

$$\partial D = \{\varpi_1 = \varpi_2 = 0\},$$

and

$$\begin{cases} d\varpi_1 \equiv \pi_1^1 \wedge \pi_1 + \cdots + \pi_1^s \wedge \pi_s \pmod{\varpi_1, \varpi_2} \\ \\ d\varpi_2 \equiv \pi_2^1 \wedge \pi_1 + \cdots + \pi_2^s \wedge \pi_s \pmod{\varpi_1, \varpi_2} \end{cases}$$

Thus (X, D) satisfies the conditions $(X.1)$ to $(X.3)$ in Sect. 4.3.

Now, putting $\varpi = \lambda_1\varpi_1 + \lambda_2\varpi_2$, we consider a point $v \in P(X)$ such that $v = \{\varpi = 0\} \subset T_x(X)$, where $x = \nu(v)$. Then, for $(\lambda_1, \lambda_2) \neq (0,0)$, $\{\lambda_1\pi_1^i + \lambda_2\pi_2^i (i = 1, \dots, s)\}$ are linearly independent $(\bmod\ \varpi_1, \varpi_2, \pi_1, \dots, \pi_s)$. Thus $\hat{\mathfrak{f}}(v) \subset D(x)$ is of codimension s at each $v \in P(X)$ (see the proof of Proposition 3). Hence we obtain $R(X) = P(X)$ in this case, i.e., $R(X)$ is a $\mathbb{P}^1$-bundle over X.

In fact, when (X, D) is the model space $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $(X_\ell, \{\alpha_G\})$, $R(X)$ can be identified with the model space $(R_{\mathfrak{g}}, E_{\mathfrak{g}})$ of type $(X_\ell, \{\alpha_\theta, \alpha_G\})$ as follows (here, we understand α_G denotes two simple roots α_1 and α_3 in case of BD_ℓ types and three simple roots α_1,α_3 and α_4 in case of D_4): Let $(J_{\mathfrak{g}}, C_{\mathfrak{g}})$ be the standard contact manifold of type $(X_\ell, \{\alpha_\theta\})$. Then we have the double fibration:

$$\begin{array}{ccc} R_{\mathfrak{g}} & \xrightarrow{\pi_c} & J_{\mathfrak{g}} \\ {\scriptstyle \pi_g}\downarrow & & \\ M_{\mathfrak{g}} & & \end{array}$$

Here $(X_\ell, \{\alpha_\theta, \alpha_G\})$ is a graded Lie algebra of depth 5 and satisfies the following: $\dim \check{\mathfrak{g}}_{-5} = \dim \check{\mathfrak{g}}_{-4} = 1$, $\dim \check{\mathfrak{g}}_{-3} = \dim \check{\mathfrak{g}}_{-2} = s$ and $\dim \check{\mathfrak{g}}_{-1} = s + 1$. In fact, comparing with the gradation of $(X_\ell, \{\alpha_G\})$, we have $\check{\Phi}_5^+ = \{\theta\}$, $\check{\Phi}_4^+ = \{\theta - \alpha_\theta\}$, $\check{\Phi}_3^+ = \Phi_2^+$, $\check{\Phi}_2^+$ consists of roots $\theta - \beta$ for each $\beta \in \check{\Phi}_3^+$ and $\check{\Phi}_1^+$ consists of roots α_θ and $\theta - \alpha_\theta - \beta$ for each $\beta \in \check{\Phi}_3^+$. Thus we see that $\partial^{(3)} E_{\mathfrak{g}} = (\pi_c)_*^{-1}(C_{\mathfrak{g}})$, $\partial^{(2)} E_{\mathfrak{g}} = (\pi_g)_*^{-1}(\partial D_{\mathfrak{g}})$ and $\partial E_{\mathfrak{g}} = (\pi_g)_*^{-1}(D_g)$. We put $D^1 = \partial^{(3)} E_{\mathfrak{g}}$ and $D^2 = \partial E_{\mathfrak{g}}$. Then $(R_{\mathfrak{g}}; D^1, D^2)$ is a *PD* manifold of second order. In fact, we have an isomorphism of $(R_{\mathfrak{g}}; D^1, D^2)$ onto $(R(M_{\mathfrak{g}}); D^1_{M_{\mathfrak{g}}}, D^2_{M_{\mathfrak{g}}})$ by the Realization Lemma for $(R_{\mathfrak{g}}, D^1, \pi_g, M_{\mathfrak{g}})$ and an embedding of $R_{\mathfrak{g}}$ into $L(J_{\mathfrak{g}})$ by the Realization Lemma for $(R_{\mathfrak{g}}, D^2, \pi_c, J_{\mathfrak{g}})$. Thus $R_{\mathfrak{g}}$ is identified with a R-space orbit in $L(J_{\mathfrak{g}})$.

Now we will calculate the symbol of $(R(X); D^1_X, D^2_X)$ by utilizing the model *PD* manifold $(R_{\mathfrak{g}}; D^1, D^2)$ of second order, especially when $(X_\ell, \{\alpha_\theta, \alpha_G\})$ is of BD_ℓ types. Let us describe the gradation of $(BD_\ell, \{\alpha_1, \alpha_2, \alpha_3\})$ or $(D_4, \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ in matrices form as follows: First we describe

$$\mathfrak{o}(k+6) = \{X \in \mathfrak{gl}\,(k+6, \mathbb{K}) \mid {}^tXJ + JX = 0\},$$

where

$$J = \begin{pmatrix} 0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&I_k&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0 \end{pmatrix} \in \mathfrak{gl}\,(k+6, \mathbb{K}), \quad I_k = (\delta_{ij}) \in \mathfrak{gl}\,(k, \mathbb{K}).$$

Here $I_k \in \mathfrak{gl}\,(k, \mathbb{K})$ is the unit matrix and the gradation is given again by subdividing matrices as follows:

$$\check{\mathfrak{g}}_{-5} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ y&0&0&0&0&0&0 \\ 0&-y&0&0&0&0&0 \end{pmatrix} \right\}, \; \check{\mathfrak{g}}_{-4} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ \xi_0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&-\xi_0&0&0&0&0 \end{pmatrix} \right\}, \quad y, \xi_0 \in \mathbb{K}$$

$$\check{\mathfrak{g}}_{-3} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ \xi&0&0&0&0&0&0 \\ 0&\xi_1&0&0&0&0&0 \\ 0&0&-\xi_1&0&0&0&0 \\ 0&0&0&-{}^t\xi&0&0&0 \end{pmatrix} \middle| \; \xi \in \mathbb{K}^k, \; \xi_1 \in \mathbb{K} \right\},$$

$$\check{\mathfrak{g}}_{-2} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ x_1&0&0&0&0&0&0 \\ 0&x&0&0&0&0&0 \\ 0&0&0&0&0&0&0 \\ 0&0&0&-{}^tx&0&0&0 \\ 0&0&0&0&-x_1&0&0 \end{pmatrix} = \hat{x} + \hat{x}_1 \middle| \; x \in \mathbb{K}^k, \; x_1 \in \mathbb{K} \right\},$$

$$\check{\mathfrak{g}}_{-1} = \left\{ \begin{pmatrix} 0&0&0&0&0&0&0 \\ a_1&0&0&0&0&0&0 \\ 0&x_0&0&0&0&0&0 \\ 0&0&a&0&0&0&0 \\ 0&0&0&-{}^ta&0&0&0 \\ 0&0&0&0&-x_0&0&0 \\ 0&0&0&0&0&-a_1&0 \end{pmatrix} = \hat{x}_0 + \hat{a}_1 + \hat{a} \middle| \; \begin{aligned} x_0, a_1 \in \mathbb{K}, \\ a \in \mathbb{K}^k \end{aligned} \right\},$$

$$\check{\mathfrak{g}}_0 = \left\{ \left. \begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -b \end{pmatrix} \right| \; b, c, e \in \mathbb{K}, \; B \in \mathfrak{o}(k) \right\}$$

$$\check{\mathfrak{g}}_\ell = \{ {}^t X \mid X \in \mathfrak{g}_{-\ell} \}, (\ell = 1, 2, 3, 4, 5),$$

Then, for $X = \hat{x} + \hat{x}_1 + \hat{x}_0$ and $A = \hat{a} + \hat{a}_1$, we calculate

$$[[A, X], X] = (2x_1 {}^t\widehat{ax - a}_1 {}^t xx) \in \check{\mathfrak{g}}_{-5}$$

Thus we obtain

$$\mathfrak{f} = \langle \{ 2e_1^* \odot e_2^*, \ldots, 2e_1^* \odot e_{k+1}^*, e_2^* \odot e_2^* + \cdots + e_{k+1}^* \odot e_{k+1}^* \rangle \subset S^2(E^\perp),$$

where $\{e_0, e_1, \ldots, e_{k+1}\}$ is a basis of V and $E = \langle \{e_0\} \rangle$ is the Cauchy characteristic direction. In case $k = 1$, $\mathfrak{f} = \langle \{2e_1^* \odot e_2^*, e_2^* \odot e_2^*\} \rangle$ is an involutive subspace of $S^2(E^\perp)$. Hence $(R_\mathfrak{g}; D^1, D^2)$ is involutive when $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is of type $(B_3, \{\alpha_1, \alpha_2, \alpha_3\})$. In case $k > 1$, we have

$$\begin{aligned} \mathfrak{f}^\perp = \langle \{ & e_1 \odot e_1, e_i \odot e_j (2 \leqq i < j \leqq k+1), \\ & e_2 \odot e_2 - e_{k+1} \odot e_{k+1}, \ldots, e_k \odot e_k - e_{k+1} \odot e_{k+1} \} \rangle \subset S^2(W), \end{aligned}$$

where $W = \langle \{e_1, \ldots, e_{k+1}\} \rangle$. Then we see that $(\mathfrak{f}^{(1)})^\perp$ contains every $e_i \odot e_i \odot e_i$ for $i = 1, \ldots, k+1$, which implies $\mathfrak{f}$ is of finite type. We can also check that exceptional cases other than G_2 are of finite type by utilizing R-space orbit $(R_\mathfrak{g}; D^1, D^2)$, whereas $\mathfrak{f} = S^2(W^*)$ is involutive in case of G_2, where $\dim W = 1$. We will discuss these cases in a uniform way in other occasion.

6.3 Other Examples

We exhibit here two other examples of simple graded Lie algebras $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ of depth 3, such that regular differential systems (X, D) of type $\mathfrak{m}$ satisfy the conditions $(X.1)$ to $(X.3)$ in Sect. 4.3, where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$.

The first example is of type $(C_\ell, \{\alpha_2, \alpha_\ell\})$. Here we have $\dim \mathfrak{g}_{-3} = 3$, $\dim \mathfrak{g}_{-2} = 2(\ell - 2)$ and $\dim \mathfrak{g}_{-1} = 2(\ell-2) + \frac{1}{2}(\ell-2)(\ell-1)$. Utilizing the calculation in Case (3) of Sect. 3 in [26], we have the following description of the standard differential system $(M_\mathfrak{m}, D_\mathfrak{m})$ of type $\mathfrak{m}$ in this case:

$$D_\mathfrak{m} = \{ \varpi_0 = \varpi_1 = \varpi_2 = \pi_1^1 = \cdots = \pi_1^{\ell-2} = \pi_2^1 = \cdots = \pi_2^{\ell-2} = 0 \},$$

where

$$\begin{cases} \varpi_0 = d\,y_0 - \sum_{\alpha=1}^{\ell-2} \xi_1^\alpha dx_2^\alpha - \sum_{\alpha=1}^{\ell-2} \xi_2^\alpha dx_1^\alpha, \\ \varpi_1 = d\,y_1 - 2\sum_{\alpha=1}^{\ell-2} \xi_1^\alpha dx_1^\alpha, \quad \varpi_2 = d\,y_2 - 2\sum_{\alpha=1}^{\ell-2} \xi_2^\alpha dx_2^\alpha, \\ \pi_p^\alpha = d\xi_p^\alpha - \sum_{\beta=1}^{\ell-2} a_{\alpha\beta} dx_p^\beta \qquad (p = 1, 2 \quad \alpha = 1, \ldots, \ell - 2), \end{cases}$$

Here we put $y_{12} = y_0$, $y_{11} = y_1$, $y_{22} = y_2$ and $a_{\alpha\beta} = a_{\beta\alpha}$ for $1 \leqq \alpha, \beta \leqq \ell - 2$. Let (X, D) be a regular differential system of type $\mathfrak{m}$. Then we have the structure equation of $(X, \partial D)$ as follows;

$$\begin{cases} d\varpi_0 \equiv \omega_2^1 \wedge \pi_1^1 + \cdots + \omega_2^{\ell-2} \wedge \pi_1^{\ell-2} + \omega_1^1 \wedge \pi_2^1 + \cdots + \omega_1^{\ell-2} \wedge \pi_2^{\ell-2} \\ d\varpi_1 \equiv 2\,\omega_1^1 \wedge \pi_1^1 + \cdots + 2\,\omega_1^{\ell-2} \wedge \pi_1^{\ell-2} \qquad\qquad (\text{mod} \quad \varpi_0, \varpi_1, \varpi_2) \\ d\varpi_2 \equiv 2\,\omega_2^1 \wedge \pi_2^1 + \cdots + 2\,\omega_2^{\ell-2} \wedge \pi_2^{\ell-2} \end{cases}$$

Now, putting $\varpi = \lambda_0\varpi_0 + \lambda_1\varpi_1 + \lambda_2\varpi_2$, we consider a point $v \in P(X)$ such that $v = \{\varpi = 0\} \subset T_x(X)$, where $x = \nu(v)$. Then, from

$$d\varpi \equiv \sum_{\alpha=1}^{\ell-2} (\lambda_0\omega_2^\alpha + 2\,\lambda_1\omega_1^\alpha) \wedge \pi_1^\alpha + \sum_{\alpha=1}^{\ell-2} (\lambda_0\omega_1^\alpha + 2\,\lambda_2\omega_2^\alpha) \wedge \pi_2^\alpha, \ \ (\text{ mod } \varpi_0, \varpi_1, \varpi_2),$$

and

$$(\lambda_0\omega_2^\alpha + 2\,\lambda_1\omega_1^\alpha) \wedge (\lambda_0\omega_1^\alpha + 2\,\lambda_2\omega_2^\alpha) = (4\lambda_1\lambda_2 - \lambda_0^2)\omega_1^\alpha \wedge \omega_2^\alpha,$$

we see that (X, D) satisfies the condition $(X.1)$ to $(X.3)$ in Sect. 4.3 and we obtain

$$R(X) = \{v \in P(X) \mid 4\lambda_1\lambda_2 - \lambda_0^2 \neq 0\},$$

where $(\lambda_0, \lambda_1, \lambda_2)$ is the homogeneous coordinate of the fibre $\nu : P(X) \to X$.

By the calculation in Case (3) of Sect. 3 in [26], we have $y_{pq} = 2 \sum_{\alpha,\beta=1}^{\ell-2} a_{\alpha\beta} x_p^\alpha x_q^\beta$ for $1 \leqq p \leqq q \leqq 2$ and $a_{\alpha\beta} = a_{\beta\alpha}$ $(1 \leqq \alpha, \beta \leqq \ell - 2)$ so that

$$\lambda_0 y_0 + \lambda_1 y_1 + \lambda_2 y_2 = 2 \sum_{\alpha,\beta=1}^{\ell-2} a_{\alpha\beta}(\lambda_0 x_1^\alpha x_2^\beta + \lambda_1 x_1^\alpha x_1^\beta + \lambda_2 x_2^\alpha x_2^\beta).$$

Hence, from

$$\sum_{\alpha,\beta=1}^{\ell-2} a_{\alpha\beta}(\lambda_0 x_1^\alpha x_2^\beta + \lambda_1 x_1^\alpha x_1^\beta + \lambda_2 x_2^\alpha x_2^\beta) = \sum_{\alpha=1}^{\ell-2} a_{\alpha\alpha}(\lambda_0 x_1^\alpha x_2^\alpha + \lambda_1 x_1^\alpha x_1^\alpha + \lambda_2 x_2^\alpha x_2^\alpha)$$
$$+ \sum_{\alpha<\beta} a_{\alpha\beta}(\lambda_0(x_1^\alpha x_2^\beta + x_1^\beta x_2^\alpha) + 2\,\lambda_1 x_1^\alpha x_1^\beta + 2\,\lambda_2 x_2^\alpha x_2^\beta),$$

we have

$$\begin{aligned}\hat{\mathfrak{f}}(v) = \langle\{&\lambda_0(e_1^\alpha)^* \circledcirc (e_2^\alpha)^* + \lambda_1(e_1^\alpha)^* \circledcirc (e_1^\alpha)^* + \lambda_2(e_2^\alpha)^* \circledcirc (e_2^\alpha)^* (1 \leqq \alpha \leqq \ell-2),\\ &\lambda_0((e_1^\alpha)^* \circledcirc (e_2^\beta)^* + (e_1^\beta)^* \circledcirc (e_2^\alpha)^*) + 2\,\lambda_1(e_1^\alpha)^* \circledcirc (e_1^\beta)^* + 2\,\lambda_2(e_2^\alpha)^* \circledcirc (e_2^\beta)^*\\ &(1 \leqq \alpha < \beta \leqq \ell-2)\}\rangle \subset S^2(W^*),\end{aligned}$$

where $W = \langle\{e_1^1, \ldots, e_1^{\ell-2}, e_2^1, \ldots, e_2^{\ell-2}\}\rangle$. Thus, assuming $\lambda_0 \neq 0$, we get

$$\begin{aligned}\hat{\mathfrak{f}}(v)^\perp = \langle\{&\lambda_0 e_1^\alpha \circledcirc e_1^\alpha - 2\,\lambda_1 e_1^\alpha \circledcirc e_2^\alpha, \lambda_0 e_2^\alpha \circledcirc e_2^\alpha - 2\,\lambda_2 e_1^\alpha \circledcirc e_2^\alpha (1 \leqq \alpha \leqq \ell-2),\\ &e_1^\alpha \circledcirc e_2^\beta - e_1^\beta \circledcirc e_2^\alpha, \lambda_0 e_1^\alpha \circledcirc e_1^\beta - 2\,\lambda_1 e_1^\alpha \circledcirc e_2^\beta, \lambda_0 e_2^\alpha \circledcirc e_2^\beta - 2\,\lambda_2 e_1^\alpha \circledcirc e_2^\beta,\\ &(1 \leqq \alpha < \beta \leqq \ell-2)\}\rangle \subset S^2(W),\end{aligned}$$

Then, from the first two generator of $\hat{\mathfrak{f}}(v)^\perp$, we see that $\hat{\mathfrak{f}}^{(1)}(v)^\perp$ contains $e_1^\alpha \circledcirc e_1^\alpha \circledcirc e_2^\alpha$ and $e_1^\alpha \circledcirc e_2^\alpha \circledcirc e_2^\alpha$. Moreover it follows that $\hat{\mathfrak{f}}^{(1)}(v)^\perp$ contains every $e_1^\alpha \circledcirc e_1^\alpha \circledcirc e_1^\alpha$ and $e_2^\alpha \circledcirc e_2^\alpha \circledcirc e_2^\alpha$ for $\alpha = 1, \ldots, \ell-2$, which implies that $\hat{\mathfrak{f}}(v)$ is of finite type.

Now let us construct the model equation of second order from the coordinate description of the standard differential system $(M_{\mathfrak{m}}, D_{\mathfrak{m}})$. We calculate

$$\begin{aligned}\varpi &= \varpi_0 + \lambda_1\varpi_1 + \lambda_2\varpi_2\\ &= dy_0 + \lambda_1 dy_1 + \lambda_2 dy_2 - \sum_{\alpha=1}^{\ell-2}(\xi_2^\alpha + 2\,\lambda_1\xi_1^\alpha)dx_1^\alpha - \sum_{\alpha=1}^{\ell-2}(\xi_1^\alpha + 2\,\lambda_2\xi_2^\alpha)dx_2^\alpha\\ &= d(y_0 + \lambda_1 y_1 + \lambda_2 y_2) - y_1 d\lambda_1 - y_2 d\lambda_2\\ &\qquad - \sum_{\alpha=1}^{\ell-2}(\xi_2^\alpha + 2\,\lambda_1\xi_1^\alpha)dx_1^\alpha - \sum_{\alpha=1}^{\ell-2}(\xi_1^\alpha + 2\,\lambda_2\xi_2^\alpha)dx_2^\alpha\end{aligned}$$

Thus we put

$$\begin{cases} z = y_0 + \lambda_1 y_1 + \lambda_2 y_2, \quad x_1^0 = \lambda_1, \quad x_2^0 = \lambda_2, \quad p_1^0 = y_1, \quad p_2^0 = y_2,\\ p_1^\alpha = \xi_2^\alpha + 2\,\lambda_1\xi_1^\alpha, \quad p_2^\alpha = \xi_1^\alpha + 2\,\lambda_2\xi_2^\alpha \quad (\alpha = 1, \ldots, \ell-2).\end{cases}$$

Then we have

$$\xi_1^\alpha = \frac{p_2^\alpha - 2\,x_2^0 p_1^\alpha}{1-4x_1^0x_2^0}, \quad \xi_2^\alpha = \frac{p_1^\alpha - 2\,x_1^0 p_2^\alpha}{1-4x_1^0x_2^0},$$

and

$$\begin{aligned}
\varpi &= dz - \sum_{\alpha=0}^{\ell-2} p_1^\alpha dx_1^\alpha - \sum_{\alpha=0}^{\ell-2} p_2^\alpha dx_2^\alpha,\\
\pi_1^0 &= dp_1^0 - 2\sum_{\alpha=1}^{\ell-2} \xi_1^\alpha dx_1^\alpha, \qquad \pi_2^0 = dp_2^0 - 2\sum_{\alpha=1}^{\ell-2} \xi_2^\alpha dx_2^\alpha,\\
\pi_1^\alpha &= dp_1^\alpha - 2\,\xi_1^\alpha dx_1^0 - 2\sum_{\beta=1}^{\ell-2} \lambda_1 a_{\alpha\beta} dx_1^\beta - \sum_{\beta=1}^{\ell-2} a_{\alpha\beta} dx_2^\beta,\\
\pi_2^\alpha &= dp_2^\alpha - 2\,\xi_2^\alpha dx_2^0 - \sum_{\beta=1}^{\ell-2} a_{\alpha\beta} dx_1^\beta - 2\sum_{\beta=1}^{\ell-2} \lambda_2 a_{\alpha\beta} dx_2^\beta,
\end{aligned}$$

where $\pi_1^0 = \varpi_1$, $\pi_2^0 = \varpi_2$, $\pi_1^\alpha = \varpi_2^\alpha + 2\,\lambda_1\varpi_1^\alpha$ and $\pi_2^\alpha = \varpi_1^\alpha + 2\,\lambda_2\varpi_2^\alpha$. Hence we obtain the following model equation of second order:

$$\begin{aligned}
&\frac{\partial^2 z}{\partial x_1^0 \partial x_1^0} = \frac{\partial^2 z}{\partial x_2^0 \partial x_2^0} = \frac{\partial^2 z}{\partial x_1^0 \partial x_2^0} = 0, \qquad \frac{\partial^2 z}{\partial x_1^0 \partial x_2^\alpha} = \frac{\partial^2 z}{\partial x_2^0 \partial x_1^\alpha} = 0,\\
&\frac{\partial^2 z}{\partial x_1^0 \partial x_1^\alpha} = \frac{2}{1-4x_1^0x_2^0}\left(\frac{\partial z}{\partial x_2^\alpha} - 2\,x_2^0\frac{\partial z}{\partial x_1^\alpha}\right) \quad (= 2\,\xi_1^\alpha),\\
&\frac{\partial^2 z}{\partial x_2^0 \partial x_2^\alpha} = \frac{2}{1-4x_1^0x_2^0}\left(\frac{\partial z}{\partial x_1^\alpha} - 2\,x_1^0\frac{\partial z}{\partial x_2^\alpha}\right) \quad (= 2\,\xi_2^\alpha),\\
&\frac{\partial^2 z}{\partial x_1^\alpha \partial x_1^\beta} = 2\,x_1^0 \frac{\partial^2 z}{\partial x_2^\alpha \partial x_1^\beta}, \qquad \frac{\partial^2 z}{\partial x_2^\alpha \partial x_2^\beta} = 2\,x_2^0 \frac{\partial^2 z}{\partial x_2^\alpha \partial x_1^\beta},\\
&\frac{\partial^2 z}{\partial x_2^\alpha \partial x_1^\beta} = \frac{\partial^2 z}{\partial x_2^\beta \partial x_1^\alpha} \quad (= a_{\alpha\beta}) \qquad (1 \leqq \alpha \leqq \beta \leqq \ell-2).
\end{aligned}$$

Our second example is of type $(E_7, \{\alpha_6, \alpha_7\})$. Here we have $\dim \mathfrak{g}_{-3} = 10$, $\dim \mathfrak{g}_{-2} = 16$ and $\dim \mathfrak{g}_{-1} = 16 + 1$. Utilizing the calculation in Case (4) of Sect. 4 in [26], we have the following description of the standard differential system $(M_{\mathfrak{m}}, D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case:

$$D_{\mathfrak{m}} = \{\varpi_1 = \cdots = \varpi_{10} = \pi_1 = \cdots = \pi_{16} = 0\},$$

where

$$\begin{aligned}
\varpi_1 = dy_1 - p_{11}dx_1 + p_9 dx_2 - p_6 dx_3 + p_5 dx_4&\\
+ p_4 dx_5 - p_3 dx_6 + p_2 dx_9 - p_1 dx_{11},&
\end{aligned}$$

$$
\begin{aligned}
\varpi_2 &= dy_2 - p_{13}dx_1 + p_{10}dx_2 - p_8dx_3 + p_7dx_4 \\
&\qquad + p_4dx_7 - p_3dx_8 + p_2dx_{10} - p_1dx_{13}, \\
\varpi_3 &= dy_3 - p_{14}dx_1 + p_{12}dx_2 - p_8dx_5 + p_7dx_6 \\
&\qquad + p_6dx_7 - p_5dx_8 + p_2dx_{12} - p_1dx_{14}, \\
\varpi_4 &= dy_4 - p_{15}dx_1 + p_{12}dx_3 - p_{10}dx_5 + p_9dx_7 \\
&\qquad + p_7dx_9 - p_5dx_{10} + p_3dx_{12} - p_1dx_{15}, \\
\varpi_5 &= dy_5 - p_{15}dx_2 + p_{14}dx_3 - p_{13}dx_5 + p_{11}dx_7 \\
&\qquad + p_7dx_{11} - p_5dx_{13} + p_3dx_{14} - p_2dx_{15}, \\
\varpi_6 &= dy_6 - p_{16}dx_1 + p_{12}dx_4 - p_{10}dx_6 + p_9dx_8 \\
&\qquad + p_8dx_9 - p_6dx_{10} + p_4dx_{12} - p_1dx_{16}, \\
\varpi_7 &= dy_7 - p_{16}dx_2 + p_{14}dx_4 - p_{13}dx_6 + p_{11}dx_8 \\
&\qquad + p_8dx_{11} - p_6dx_{13} + p_4dx_{14} - p_2dx_{16}, \\
\varpi_8 &= dy_8 - p_{16}dx_3 + p_{15}dx_4 - p_{13}dx_9 + p_{11}dx_{10} \\
&\qquad + p_{10}dx_{11} - p_9dx_{13} + p_4dx_{15} - p_3dx_{16}, \\
\varpi_9 &= dy_9 - p_{16}dx_5 + p_{15}dx_6 - p_{14}dx_9 + p_{12}dx_{11} \\
&\qquad + p_{11}dx_{12} - p_9dx_{14} + p_6dx_{15} - p_5dx_{16}, \\
\varpi_{10} &= dy_{10} - p_{16}dx_7 + p_{15}dx_8 - p_{14}dx_{10} + p_{13}dx_{12} \\
&\qquad + p_{12}dx_{13} - p_{10}dx_{14} + p_8dx_{15} - p_7dx_{16}, \\
\pi_i &= dp_i - a dx_i \qquad (i = 1, 2, \ldots, 16).
\end{aligned}
$$

Let (X, D) be a regular differential system of type $\mathfrak{m}$. Then we have the structure equation of $(X, \partial D)$ as follows;

$$
\begin{aligned}
d\varpi_1 &\equiv \omega_1 \wedge \pi_{11} - \omega_2 \wedge \pi_9 + \omega_3 \wedge \pi_6 - \omega_4 \wedge \pi_5 \\
&\qquad - \omega_5 \wedge \pi_4 + \omega_6 \wedge \pi_3 - \omega_9 \wedge \pi_2 + \omega_{11} \wedge \pi_1, \\
d\varpi_2 &\equiv \omega_1 \wedge \pi_{13} - \omega_2 \wedge \pi_{10} + \omega_3 \wedge \pi_8 - \omega_4 \wedge \pi_7 \\
&\qquad - \omega_7 \wedge \pi_4 + \omega_8 \wedge \pi_3 - \omega_{10} \wedge \pi_2 + \omega_{13} \wedge \pi_1, \\
d\varpi_3 &\equiv \omega_1 \wedge \pi_{14} - \omega_2 \wedge \pi_{12} + \omega_5 \wedge \pi_8 - \omega_6 \wedge \pi_7 \\
&\qquad - \omega_7 \wedge \pi_6 + \omega_8 \wedge \pi_5 - \omega_{12} \wedge \pi_2 + \omega_{14} \wedge \pi_1, \\
d\varpi_4 &\equiv \omega_1 \wedge \pi_{15} - \omega_3 \wedge \pi_{12} + \omega_5 \wedge \pi_{10} - \omega_7 \wedge \pi_9 \\
&\qquad - \omega_9 \wedge \pi_7 + \omega_{10} \wedge \pi_5 - \omega_{12} \wedge \pi_3 + \omega_{15} \wedge \pi_1, \\
d\varpi_5 &\equiv \omega_2 \wedge \pi_{15} - \omega_3 \wedge \pi_{14} + \omega_5 \wedge \pi_{13} - \omega_7 \wedge \pi_{11} \\
&\qquad - \omega_{11} \wedge \pi_7 + \omega_{13} \wedge \pi_5 - \omega_{14} \wedge \pi_3 + \omega_{15} \wedge \pi_2, \\
d\varpi_6 &\equiv \omega_1 \wedge \pi_{16} - \omega_4 \wedge \pi_{12} + \omega_6 \wedge \pi_{10} - \omega_8 \wedge \pi_9 \\
&\qquad - \omega_9 \wedge \pi_8 + \omega_{10} \wedge \pi_6 - \omega_{12} \wedge \pi_4 + \omega_{16} \wedge \pi_1, \\
d\varpi_7 &\equiv \omega_2 \wedge \pi_{16} - \omega_4 \wedge \pi_{14} + \omega_6 \wedge \pi_{13} - \omega_8 \wedge \pi_{11} \\
&\qquad - \omega_{11} \wedge \pi_8 + \omega_{13} \wedge \pi_6 - \omega_{14} \wedge \pi_4 + \omega_{16} \wedge \pi_2,
\end{aligned}
$$

$$d\varpi_8 \equiv \omega_3 \wedge \pi_{16} - \omega_4 \wedge \pi_{15} + \omega_9 \wedge \pi_{13} - \omega_{10} \wedge \pi_{11} \\ - \omega_{11} \wedge \pi_{10} + \omega_{13} \wedge \pi_9 - \omega_{15} \wedge \pi_4 + \omega_{16} \wedge \pi_3,$$
$$d\varpi_9 \equiv \omega_5 \wedge \pi_{16} - \omega_6 \wedge \pi_{15} + \omega_9 \wedge \pi_{14} - \omega_{11} \wedge \pi_{12} \\ - \omega_{12} \wedge \pi_{11} + \omega_{14} \wedge \pi_9 - \omega_{15} \wedge \pi_6 + \omega_{16} \wedge \pi_5,$$
$$d\varpi_{10} \equiv \omega_7 \wedge \pi_{16} - \omega_8 \wedge \pi_{15} + \omega_{10} \wedge \pi_{14} - \omega_{12} \wedge \pi_{13} \\ - \omega_{13} \wedge \pi_{12} + \omega_{14} \wedge \pi_{10} - \omega_{15} \wedge \pi_8 + \omega_{16} \wedge \pi_7,$$

$(\text{mod} \quad \varpi_1, \ldots, \varpi_{10})$.

Now, putting $\varpi = \varpi_1 + \sum_{i=2}^{10} \lambda_i \varpi_i$, we consider a point $v \in P(X)$ such that $v = \{\varpi = 0\} \subset T_x(X)$, where $x = \nu(v)$. Then we have

$$d\varpi \equiv \check{\omega}_1 \wedge \pi_1 + \cdots + \check{\omega}_{16} \wedge \pi_{16} \qquad (\text{mod} \quad \varpi_1, \ldots, \varpi_{10}),$$

where

$$\begin{aligned}
\check{\omega}_1 &= \omega_{11} + \lambda_2\omega_{13} + \lambda_3\omega_{14} + \lambda_4\omega_{15} + \lambda_6\omega_{16},\\
\check{\omega}_2 &= -\omega_9 - \lambda_2\omega_{10} - \lambda_3\omega_{12} + \lambda_5\omega_{15} + \lambda_7\omega_{16},\\
\check{\omega}_3 &= \omega_6 + \lambda_2\omega_8 - \lambda_4\omega_{12} - \lambda_5\omega_{14} + \lambda_8\omega_{16},\\
\check{\omega}_4 &= -\omega_5 - \lambda_2\omega_7 - \lambda_6\omega_{12} - \lambda_7\omega_{14} - \lambda_8\omega_{15},\\
\check{\omega}_5 &= -\omega_4 + \lambda_3\omega_8 + \lambda_4\omega_{10} + \lambda_5\omega_{13} + \lambda_9\omega_{16},\\
\check{\omega}_6 &= \omega_3 - \lambda_3\omega_7 + \lambda_6\omega_{10} + \lambda_7\omega_{13} - \lambda_9\omega_{15},\\
\check{\omega}_7 &= -\lambda_2\omega_4 - \lambda_3\omega_6 - \lambda_4\omega_9 - \lambda_5\omega_{11} + \lambda_{10}\omega_{16},\\
\check{\omega}_8 &= \lambda_2\omega_3 + \lambda_3\omega_5 - \lambda_6\omega_9 - \lambda_7\omega_{11} - \lambda_{10}\omega_{15},\\
\check{\omega}_9 &= -\omega_2 - \lambda_4\omega_7 - \lambda_6\omega_8 + \lambda_8\omega_{13} + \lambda_9\omega_{14},\\
\check{\omega}_{10} &= -\lambda_2\omega_2 + \lambda_4\omega_5 + \lambda_6\omega_6 - \lambda_8\omega_{11} + \lambda_{10}\omega_{14},\\
\check{\omega}_{11} &= \omega_1 - \lambda_5\omega_7 - \lambda_7\omega_8 - \lambda_8\omega_{10} - \lambda_9\omega_{12},\\
\check{\omega}_{12} &= -\lambda_3\omega_2 - \lambda_4\omega_3 - \lambda_6\omega_4 - \lambda_9\omega_{11} - \lambda_{10}\omega_{13},\\
\check{\omega}_{13} &= \lambda_2\omega_1 + \lambda_5\omega_5 + \lambda_7\omega_6 + \lambda_8\omega_9 - \lambda_{10}\omega_{12},\\
\check{\omega}_{14} &= \lambda_3\omega_1 - \lambda_5\omega_3 - \lambda_7\omega_4 + \lambda_9\omega_9 + \lambda_{10}\omega_{10},\\
\check{\omega}_{15} &= \lambda_4\omega_1 + \lambda_5\omega_2 - \lambda_8\omega_4 - \lambda_9\omega_6 - \lambda_{10}\omega_8,\\
\check{\omega}_{16} &= \lambda_6\omega_1 + \lambda_7\omega_2 + \lambda_8\omega_3 + \lambda_9\omega_5 + \lambda_{10}\omega_7.
\end{aligned}$$

We calculate

$$\begin{aligned}
\check{\omega}_7 &= -\lambda_5\check{\omega}_1 + \lambda_4\check{\omega}_2 - \lambda_3\check{\omega}_3 + \lambda_2\check{\omega}_5 + \lambda\omega_{16},\\
\check{\omega}_8 &= -\lambda_7\check{\omega}_1 + \lambda_6\check{\omega}_2 - \lambda_3\check{\omega}_4 + \lambda_2\check{\omega}_6 - \lambda\omega_{15},\\
\check{\omega}_{10} &= -\lambda_8\check{\omega}_1 + \lambda_6\check{\omega}_3 - \lambda_4\check{\omega}_4 + \lambda_2\check{\omega}_9 + \lambda\omega_{14},\\
\check{\omega}_{12} &= -\lambda_9\check{\omega}_1 + \lambda_6\check{\omega}_5 - \lambda_4\check{\omega}_6 + \lambda_3\check{\omega}_9 - \lambda\omega_{13},\\
\check{\omega}_{13} &= -\lambda_8\check{\omega}_2 + \lambda_7\check{\omega}_3 - \lambda_5\check{\omega}_4 + \lambda_2\check{\omega}_{11} - \lambda\omega_{12},\\
\check{\omega}_{14} &= -\lambda_9\check{\omega}_2 + \lambda_7\check{\omega}_5 - \lambda_5\check{\omega}_6 + \lambda_3\check{\omega}_{11} + \lambda\omega_{10},
\end{aligned}$$

$$\check{\omega}_{15} = -\lambda_9\check{\omega}_3 + \lambda_8\check{\omega}_5 - \lambda_5\check{\omega}_9 + \lambda_4\check{\omega}_{11} - \lambda\omega_8,$$
$$\check{\omega}_{16} = -\lambda_9\check{\omega}_4 + \lambda_8\check{\omega}_6 - \lambda_7\check{\omega}_9 + \lambda_6\check{\omega}_{11} + \lambda\omega_7,$$

where $\lambda = \lambda_{10} - \lambda_2\lambda_9 + \lambda_3\lambda_8 - \lambda_4\lambda_7 + \lambda_5\lambda_6$. Thus we see that (X, D) satisfies the condition $(X.1)$ to $(X.3)$ in Sect. 4.3 and we obtain

$$R(X) = \{v \in P(X) \mid \lambda \neq 0\},$$

where $(\lambda_2, \ldots, \lambda_{10})$ is the inhomogeneous coordinate of the fibre $\nu : P(X) \to X$. Moreover we have

$$\hat{\mathfrak{f}}(v) = \{X \in \mathfrak{g}_{-1}(x) \mid [X, \mathfrak{g}_{-2}(x)] = 0\} \quad \text{for each } v \in R(X), x = \nu(v).$$

and $\dim \hat{\mathfrak{f}}(v) = 1$. Utilizing the calculation in Case (4) of Sect. 4 in [26], we have

$$\begin{aligned} f = \ & (e_1^* \circledcirc e_{11}^* - e_2^* \circledcirc e_9^* + e_3^* \circledcirc e_6^* - e_4^* \circledcirc e_5^*) \\ & + \lambda_2(e_1^* \circledcirc e_{13}^* - e_2^* \circledcirc e_{10}^* + e_3^* \circledcirc e_8^* - e_4^* \circledcirc e_7^*) \\ & + \lambda_3(e_1^* \circledcirc e_{14}^* - e_2^* \circledcirc e_{12}^* + e_5^* \circledcirc e_8^* - e_6^* \circledcirc e_7^*) \\ & + \lambda_4(e_1^* \circledcirc e_{15}^* - e_3^* \circledcirc e_{12}^* + e_5^* \circledcirc e_{10}^* - e_7^* \circledcirc e_9^*) \\ & + \lambda_5(e_2^* \circledcirc e_{15}^* - e_3^* \circledcirc e_{14}^* + e_5^* \circledcirc e_{13}^* - e_7^* \circledcirc e_{11}^*) \\ & + \lambda_6(e_1^* \circledcirc e_{16}^* - e_4^* \circledcirc e_{12}^* + e_6^* \circledcirc e_{10}^* - e_8^* \circledcirc e_9^*) \\ & + \lambda_7(e_2^* \circledcirc e_{16}^* - e_4^* \circledcirc e_{14}^* + e_6^* \circledcirc e_{13}^* - e_8^* \circledcirc e_{11}^*) \\ & + \lambda_8(e_3^* \circledcirc e_{16}^* - e_4^* \circledcirc e_{15}^* + e_9^* \circledcirc e_{13}^* - e_{10}^* \circledcirc e_{11}^*) \\ & + \lambda_9(e_5^* \circledcirc e_{16}^* - e_6^* \circledcirc e_{15}^* + e_9^* \circledcirc e_{14}^* - e_{11}^* \circledcirc e_{12}^*) \\ & + \lambda_{10}(e_7^* \circledcirc e_{16}^* - e_8^* \circledcirc e_{15}^* + e_{10}^* \circledcirc e_{14}^* - e_{12}^* \circledcirc e_{13}^*) \end{aligned}$$

for the generator f of $\hat{\mathfrak{f}}(v)$. Then we calculate

$$\begin{aligned} f = \alpha_1 \circledcirc \alpha_{11} - \alpha_2 \circledcirc \alpha_9 + \alpha_3 \circledcirc \alpha_6 - \alpha_4 \circledcirc \alpha_5 \\ + \lambda(e_7^* \circledcirc e_{16}^* - e_8^* \circledcirc e_{15}^* + e_{10}^* \circledcirc e_{14}^* - e_{12}^* \circledcirc e_{13}^*), \end{aligned}$$

where $\lambda = \lambda_{10} - \lambda_2\lambda_9 + \lambda_3\lambda_8 - \lambda_4\lambda_7 + \lambda_5\lambda_6$ and

$$\begin{aligned} \alpha_1 &= e_{11}^* + \lambda_2 e_{13}^* + \lambda_3 e_{14}^* + \lambda_4 e_{15}^* + \lambda_6 e_{16}^*, \\ \alpha_2 &= -e_9^* - \lambda_2 e_{10}^* - \lambda_3 e_{12}^* + \lambda_5 e_{15}^* + \lambda_7 e_{16}^*, \\ \alpha_3 &= e_6^* + \lambda_2 e_8^* - \lambda_4 e_{12}^* - \lambda_5 e_{14}^* + \lambda_8 e_{16}^*, \\ \alpha_4 &= -e_5^* - \lambda_2 e_7^* - \lambda_6 e_{12}^* - \lambda_7 e_{14}^* - \lambda_8 e_{15}^*, \\ \alpha_5 &= -e_4^* + \lambda_3 e_8^* + \lambda_4 e_{10}^* + \lambda_5 e_{13}^* + \lambda_9 e_{16}^*, \\ \alpha_6 &= e_3^* - \lambda_3 e_7^* + \lambda_6 e_{10}^* + \lambda_7 e_{13}^* - \lambda_9 e_{15}^*, \\ \alpha_9 &= -e_2^* - \lambda_4 e_7^* - \lambda_6 e_8^* + \lambda_8 e_{13}^* + \lambda_9 e_{14}^*, \\ \alpha_{11} &= e_1^* - \lambda_5 e_7^* - \lambda_7 e_8^* - \lambda_8 e_{10}^* - \lambda_9 e_{12}^*, \end{aligned}$$

Thus f is a non-degenerate quadratic form in $S^2(W^*)$, where $W = \langle\{e_1, \ldots, e_{16}\}\rangle$. Hence $\hat{\mathfrak{f}}(v)$ is of finite type (see Case (5) of Sect. 3 in [26]).

Now let us construct the model equation of second order from the coordinate description of the standard differential system $(M_{\mathfrak{m}}, D_{\mathfrak{m}})$. We calculate

$$\begin{aligned}
\varpi &= \varpi_1 + \lambda_2\varpi_2 + \cdots + \lambda_{10}\varpi_{10} \\
&= dy_1 + \lambda_2 dy_2 + \cdots + \lambda_{10} dy_{10} - \hat{p}_1 dx_1 - \cdots - \hat{p}_{16} dx_{16} \\
&= d(y_1 + \lambda_2 y_2 + \cdots + \lambda_{10} y_{10}) - y_2 d\lambda_2 - \cdots - y_{10} d\lambda_{10} \\
&\qquad\qquad\qquad\qquad - \hat{p}_1 dx_1 - \cdots - \hat{p}_{16} dx_{16}, \\
&= dz - \hat{p}_1 dx_1 - \cdots - \hat{p}_{16} dx_{16} - \hat{p}_{17} dx_{17} - \cdots - \hat{p}_{25} dx_{25},
\end{aligned}$$

where we put

$$\begin{aligned}
z &= y_1 + \lambda_2 y_2 + \cdots + \lambda_{10} y_{10}, \\
\hat{p}_1 &= p_{11} + \lambda_2 p_{13} + \lambda_3 p_{14} + \lambda_4 p_{15} + \lambda_6 p_{16}, \\
\hat{p}_2 &= -p_9 - \lambda_2 p_{10} - \lambda_3 p_{12} + \lambda_5 p_{15} + \lambda_7 p_{16}, \\
\hat{p}_3 &= p_6 + \lambda_2 p_8 - \lambda_4 p_{12} - \lambda_5 p_{14} + \lambda_8 p_{16}, \\
\hat{p}_4 &= -p_5 - \lambda_2 p_7 - \lambda_6 p_{12} - \lambda_7 p_{14} - \lambda_8 p_{15}, \\
\hat{p}_5 &= -p_4 + \lambda_3 p_8 + \lambda_4 p_{10} + \lambda_5 p_{13} + \lambda_9 p_{16}, \\
\hat{p}_6 &= p_3 - \lambda_3 p_7 + \lambda_6 p_{10} + \lambda_7 p_{13} - \lambda_9 p_{15}, \\
\hat{p}_7 &= -\lambda_2 p_4 - \lambda_3 p_6 - \lambda_4 p_9 - \lambda_5 p_{11} + \lambda_{10} p_{16}, \\
\hat{p}_8 &= \lambda_2 p_3 + \lambda_3 p_5 - \lambda_6 p_9 - \lambda_7 p_{11} - \lambda_{10} p_{15}, \\
\hat{p}_9 &= -p_2 - \lambda_4 p_7 - \lambda_6 p_8 + \lambda_8 p_{13} + \lambda_9 p_{14}, \\
\hat{p}_{10} &= -\lambda_2 p_2 + \lambda_4 p_5 + \lambda_6 p_6 - \lambda_8 p_{11} + \lambda_{10} p_{14}, \\
\hat{p}_{11} &= p_1 - \lambda_5 p_7 - \lambda_7 p_8 - \lambda_8 p_{10} - \lambda_9 p_{12}, \\
\hat{p}_{12} &= -\lambda_3 p_2 - \lambda_4 p_3 - \lambda_6 p_4 - \lambda_9 p_{11} - \lambda_{10} p_{13}, \\
\hat{p}_{13} &= \lambda_2 p_1 + \lambda_5 p_5 + \lambda_7 p_6 + \lambda_8 p_9 - \lambda_{10} p_{12}, \\
\hat{p}_{14} &= \lambda_3 p_1 - \lambda_5 p_3 - \lambda_7 p_4 + \lambda_9 p_9 + \lambda_{10} p_{10}, \\
\hat{p}_{15} &= \lambda_4 p_1 + \lambda_5 p_2 - \lambda_8 p_4 - \lambda_9 p_6 - \lambda_{10} p_8, \\
\hat{p}_{16} &= \lambda_6 p_1 + \lambda_7 p_2 + \lambda_8 p_3 + \lambda_9 p_5 + \lambda_{10} p_7, \\
\hat{p}_{17} &= y_2, \ldots, \hat{p}_{\alpha+15} = y_\alpha, \ldots, \hat{p}_{25} = y_{10}, \\
x_{17} &= \lambda_2, \ldots, x_{\alpha+15} = \lambda_\alpha, \ldots, x_{25} = \lambda_{10}.
\end{aligned}$$

Then we have, for $(R(M_{\mathfrak{m}}); D^1_{M_{\mathfrak{m}}}, D^2_{M_{\mathfrak{m}}})$,

$$D^1_{M_{\mathfrak{m}}} = \{\varpi = 0\}, \quad D^2_{M_{\mathfrak{m}}} = \{\varpi_k = \pi_i = 0 \quad (1 \leqq k \leqq 10, 1 \leqq i \leqq 16)\},$$

where we denote the pullback on $R(M_{\mathfrak{m}})$ of 1-forms on $M_{\mathfrak{m}}$ by the same symbol. By taking the exterior derivatives of both sides of the above defining equations for $\hat{p}_i$ $(i = 1, \ldots, 16)$, we put

$$\begin{aligned}
\hat{\pi}_1 &= d\hat{p}_1 - a(dx_{11} + x_{17}dx_{13} + x_{18}dx_{14} + x_{19}dx_{15} + x_{21}dx_{16}) \\
&\qquad - p_{13}dx_{17} - p_{14}dx_{18} - p_{15}dx_{19} - p_{16}dx_{21}, \\
\hat{\pi}_2 &= d\hat{p}_2 - a(-dx_9 - x_{17}dx_{10} - x_{18}dx_{12} + x_{20}dx_{15} + x_{22}dx_{16}) \\
&\qquad + p_{10}dx_{17} + p_{12}dx_{18} - p_{15}dx_{20} - p_{16}dx_{22}, \\
\hat{\pi}_3 &= d\hat{p}_3 - a(dx_6 + x_{17}dx_8 - x_{19}dx_{12} - x_{20}dx_{14} + x_{23}dx_{16}) \\
&\qquad - p_8dx_{17} + p_{12}dx_{19} + p_{14}dx_{20} - p_{16}dx_{23}, \\
\hat{\pi}_4 &= d\hat{p}_4 - a(-dx_5 - x_{17}dx_7 - x_{21}dx_{12} - x_{22}dx_{14} - x_{23}dx_{15}) \\
&\qquad + p_7dx_{17} + p_{12}dx_{21} + p_{14}dx_{22} + p_{15}dx_{23}, \\
\hat{\pi}_5 &= d\hat{p}_5 - a(-dx_4 + x_{18}dx_8 + x_{19}dx_{10} + x_{20}dx_{13} + x_{24}dx_{16}) \\
&\qquad - p_8dx_{18} - p_{10}dx_{19} - p_{13}dx_{20} - p_{16}dx_{24}, \\
\hat{\pi}_6 &= d\hat{p}_6 - a(dx_3 - x_{18}dx_7 + x_{21}dx_{10} + x_{22}dx_{13} - x_{24}dx_{15}) \\
&\qquad + p_7dx_{18} - p_{10}dx_{21} - p_{13}dx_{22} + p_{15}dx_{24}, \\
\hat{\pi}_7 &= d\hat{p}_7 - a(-x_{17}dx_4 - x_{18}dx_6 - x_{19}dx_9 - x_{20}dx_{11} + x_{25}dx_{16}) \\
&\qquad + p_4dx_{17} + p_6dx_{18} + p_9dx_{19} + p_{11}dx_{20} - p_{16}dx_{25}, \\
\hat{\pi}_8 &= d\hat{p}_8 - a(x_{17}dx_3 + x_{18}dx_5 - x_{21}dx_9 - x_{22}dx_{11} - x_{25}dx_{15}) \\
&\qquad - p_3dx_{17} - p_5dx_{18} + p_9dx_{21} + p_{11}dx_{22} + p_{15}dx_{25}, \\
\hat{\pi}_9 &= d\hat{p}_9 - a(-dx_2 - x_{19}dx_7 - x_{21}dx_8 + x_{23}dx_{13} + x_{24}dx_{14}) \\
&\qquad + p_7dx_{19} + p_8dx_{21} - p_{13}dx_{23} - p_{14}dx_{24}, \\
\hat{\pi}_{10} &= d\hat{p}_{10} - a(-x_{17}dx_2 + x_{19}dx_5 + x_{21}dx_6 - x_{23}dx_{11} + x_{25}dx_{14}) \\
&\qquad + p_2dx_{17} - p_5dx_{19} - p_6dx_{21} + p_{11}dx_{23} - p_{14}dx_{25}, \\
\hat{\pi}_{11} &= d\hat{p}_{11} - a(dx_1 - x_{20}dx_7 - x_{22}dx_8 - x_{23}dx_{10} - x_{24}dx_{12}) \\
&\qquad + p_7dx_{20} + p_8dx_{22} + p_{10}dx_{23} + p_{12}dx_{24}, \\
\hat{\pi}_{12} &= d\hat{p}_{12} - a(-x_{18}dx_2 - x_{19}dx_3 - x_{21}dx_4 - x_{24}dx_{11} - x_{25}dx_{13}) \\
&\qquad + p_2dx_{18} + p_3dx_{19} + p_4dx_{21} + p_{11}dx_{24} + p_{13}dx_{25}, \\
\hat{\pi}_{13} &= d\hat{p}_{13} - a(x_{17}dx_1 + x_{20}dx_5 + x_{22}dx_6 + x_{23}dx_9 - x_{25}dx_{12}) \\
&\qquad - p_1dx_{17} - p_5dx_{20} - p_6dx_{22} - p_9dx_{23} + p_{12}dx_{25}, \\
\hat{\pi}_{14} &= d\hat{p}_{14} - a(x_{18}dx_1 - x_{20}dx_3 - x_{22}dx_4 + x_{24}dx_9 + x_{25}dx_{10}) \\
&\qquad - p_1dx_{18} + p_3dx_{20} + p_4dx_{22} - p_9dx_{24} - p_{10}dx_{25}, \\
\hat{\pi}_{15} &= d\hat{p}_{15} - a(x_{19}dx_1 + x_{20}dx_2 - x_{23}dx_4 - x_{24}dx_6 - x_{25}dx_8) \\
&\qquad - p_1dx_{19} - p_2dx_{20} + p_4dx_{23} + p_6dx_{24} + p_8dx_{25}, \\
\hat{\pi}_{16} &= d\hat{p}_{16} - a(x_{21}dx_1 + x_{22}dx_2 + x_{23}dx_3 + x_{24}dx_5 + x_{25}dx_7) \\
&\qquad - p_1dx_{21} - p_2dx_{22} - p_3dx_{23} - p_5dx_{24} - p_7dx_{25}.
\end{aligned}$$

Then we see that $\{\hat{\pi}_1, \ldots, \hat{\pi}_{16}\}$ can be written as the linear combinations of $\{\pi_1, \ldots, \pi_{16}\}$ with the same coefficients (in λ's) such that $\{\hat{p}_1, \ldots, \hat{p}_{16}\}$ are written as the linear combination of $\{p_1, \ldots, p_{16}\}$ as in the above equations. Hence we have

$$D^1_{M_{\mathrm{m}}} = \{\varpi = o\}, \quad D^2_{M_{\mathrm{m}}} = \{\varpi = \hat{\pi}_1 = \cdots = \hat{\pi}_{16} = \hat{\pi}_{17} = \cdots = \hat{\pi}_{25} = 0\},$$

where $\hat{\pi}_{15+\alpha} = \varpi_\alpha$ $(2 \leqq \alpha \leqq 10)$ are written as follows:

$$\begin{aligned}
\hat{\pi}_{17} &= d\hat{p}_{17} - p_{13}dx_1 + p_{10}dx_2 - p_8dx_3 + p_7dx_4 \\
&\qquad + p_4dx_7 - p_3dx_8 + p_2dx_{10} - p_1dx_{13}, \\
\hat{\pi}_{18} &= d\hat{p}_{18} - p_{14}dx_1 + p_{12}dx_2 - p_8dx_5 + p_7dx_6 \\
&\qquad + p_6dx_7 - p_5dx_8 + p_2dx_{12} - p_1dx_{14}, \\
\hat{\pi}_{19} &= d\hat{p}_{19} - p_{15}dx_1 + p_{12}dx_3 - p_{10}dx_5 + p_9dx_7 \\
&\qquad + p_7dx_9 - p_5dx_{10} + p_3dx_{12} - p_1dx_{15}, \\
\hat{\pi}_{20} &= d\hat{p}_{20} - p_{15}dx_2 + p_{14}dx_3 - p_{13}dx_5 + p_{11}dx_7 \\
&\qquad + p_7dx_{11} - p_5dx_{13} + p_3dx_{14} - p_2dx_{15}, \\
\hat{\pi}_{21} &= d\hat{p}_{21} - p_{16}dx_1 + p_{12}dx_4 - p_{10}dx_6 + p_9dx_8 \\
&\qquad + p_8dx_9 - p_6dx_{10} + p_4dx_{12} - p_1dx_{16}, \\
\hat{\pi}_{22} &= d\hat{p}_{22} - p_{16}dx_2 + p_{14}dx_4 - p_{13}dx_6 + p_{11}dx_8 \\
&\qquad + p_8dx_{11} - p_6dx_{13} + p_4dx_{14} - p_2dx_{16}, \\
\hat{\pi}_{23} &= d\hat{p}_{23} - p_{16}dx_3 + p_{15}dx_4 - p_{13}dx_9 + p_{11}dx_{10} \\
&\qquad + p_{10}dx_{11} - p_9dx_{13} + p_4dx_{15} - p_3dx_{16}, \\
\hat{\pi}_{24} &= d\hat{p}_{24} - p_{16}dx_5 + p_{15}dx_6 - p_{14}dx_9 + p_{12}dx_{11} \\
&\qquad + p_{11}dx_{12} - p_9dx_{14} + p_6dx_{15} - p_5dx_{16}, \\
\hat{\pi}_{25} &= d\hat{p}_{25} - p_{16}dx_7 + p_{15}dx_8 - p_{14}dx_{10} + p_{13}dx_{12} \\
&\qquad + p_{12}dx_{13} - p_{10}dx_{14} + p_8dx_{15} - p_7dx_{16}.
\end{aligned}$$

Moreover we calculate

$$\begin{aligned}
\lambda p_{16} &= x_{20}\hat{p}_1 - x_{19}\hat{p}_2 + x_{18}\hat{p}_3 - x_{17}\hat{p}_5 + \hat{p}_7, \\
\lambda p_{15} &= -x_{22}\hat{p}_1 + x_{21}\hat{p}_2 - x_{18}\hat{p}_4 + x_{17}\hat{p}_6 - \hat{p}_8, \\
\lambda p_{14} &= x_{23}\hat{p}_1 - x_{21}\hat{p}_3 + x_{19}\hat{p}_4 - x_{17}\hat{p}_9 + \hat{p}_{10}, \\
\lambda p_{13} &= -x_{24}\hat{p}_1 + x_{21}\hat{p}_5 - x_{19}\hat{p}_6 + x_{18}\hat{p}_9 - \hat{p}_{12}, \\
\lambda p_{12} &= -x_{23}\hat{p}_2 + x_{22}\hat{p}_3 - x_{20}\hat{p}_4 + x_{17}\hat{p}_{11} - \hat{p}_{13}, \\
\lambda p_{11} &= x_{25}\hat{p}_1 - x_{21}\hat{p}_7 + x_{19}\hat{p}_8 - x_{18}\hat{p}_{10} + x_{17}\hat{p}_{12}, \\
\lambda p_{10} &= x_{24}\hat{p}_2 - x_{22}\hat{p}_5 + x_{20}\hat{p}_6 - x_{18}\hat{p}_{11} + \hat{p}_{14}, \\
\lambda p_9 &= -x_{25}\hat{p}_2 + x_{22}\hat{p}_7 - x_{20}\hat{p}_8 + x_{18}\hat{p}_{13} - x_{17}\hat{p}_{14}, \\
\lambda p_8 &= -x_{24}\hat{p}_3 + x_{23}\hat{p}_5 - x_{20}\hat{p}_9 + x_{19}\hat{p}_{11} - \hat{p}_{15}, \\
\lambda p_7 &= x_{24}\hat{p}_4 - x_{23}\hat{p}_6 + x_{22}\hat{p}_9 - x_{21}\hat{p}_{11} + \hat{p}_{16}, \\
\lambda p_6 &= x_{25}\hat{p}_3 - x_{23}\hat{p}_7 + x_{20}\hat{p}_{10} - x_{19}\hat{p}_{13} + x_{17}\hat{p}_{15}, \\
\lambda p_5 &= -x_{25}\hat{p}_4 + x_{23}\hat{p}_8 - x_{22}\hat{p}_{10} + x_{21}\hat{p}_{13} - x_{17}\hat{p}_{16}, \\
\lambda p_4 &= -x_{25}\hat{p}_5 + x_{24}\hat{p}_7 - x_{20}\hat{p}_{12} + x_{19}\hat{p}_{14} - x_{18}\hat{p}_{15},
\end{aligned}$$

$$\lambda p_3 = x_{25}\hat{p}_6 - x_{24}\hat{p}_8 + x_{22}\hat{p}_{12} - x_{21}\hat{p}_{14} + x_{18}\hat{p}_{16},$$
$$\lambda p_2 = -x_{25}\hat{p}_9 + x_{24}\hat{p}_{10} - x_{23}\hat{p}_{12} + x_{21}\hat{p}_{15} - x_{19}\hat{p}_{16},$$
$$\lambda p_1 = x_{25}\hat{p}_{11} - x_{24}\hat{p}_{13} + x_{23}\hat{p}_{14} - x_{22}\hat{p}_{15} + x_{20}\hat{p}_{16},$$

where $\lambda = x_{25} - x_{17}x_{24} + x_{18}x_{23} - x_{19}x_{22} + x_{20}x_{21}$. Thus we obtain the following model equation of second order:

$$\frac{\partial^2 z}{\partial x_1 \partial x_{11}} = -\frac{\partial^2 z}{\partial x_2 \partial x_9} = \frac{\partial^2 z}{\partial x_3 \partial x_6} = -\frac{\partial^2 z}{\partial x_4 \partial x_5} \quad (= a)$$
$$\frac{\partial^2 z}{\partial x_1 \partial x_{13}} = -\frac{\partial^2 z}{\partial x_2 \partial x_{10}} = \frac{\partial^2 z}{\partial x_3 \partial x_8} = -\frac{\partial^2 z}{\partial x_4 \partial x_7} = x_{17}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_1 \partial x_{14}} = -\frac{\partial^2 z}{\partial x_2 \partial x_{12}} = \frac{\partial^2 z}{\partial x_5 \partial x_8} = -\frac{\partial^2 z}{\partial x_6 \partial x_7} = x_{18}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_1 \partial x_{15}} = -\frac{\partial^2 z}{\partial x_3 \partial x_{12}} = \frac{\partial^2 z}{\partial x_5 \partial x_{10}} = -\frac{\partial^2 z}{\partial x_7 \partial x_9} = x_{19}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_2 \partial x_{15}} = -\frac{\partial^2 z}{\partial x_3 \partial x_{14}} = \frac{\partial^2 z}{\partial x_5 \partial x_{13}} = -\frac{\partial^2 z}{\partial x_7 \partial x_{11}} = x_{20}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_1 \partial x_{16}} = -\frac{\partial^2 z}{\partial x_4 \partial x_{12}} = \frac{\partial^2 z}{\partial x_6 \partial x_{10}} = -\frac{\partial^2 z}{\partial x_8 \partial x_9} = x_{21}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_2 \partial x_{16}} = -\frac{\partial^2 z}{\partial x_4 \partial x_{14}} = \frac{\partial^2 z}{\partial x_6 \partial x_{13}} = -\frac{\partial^2 z}{\partial x_8 \partial x_{11}} = x_{22}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_3 \partial x_{16}} = -\frac{\partial^2 z}{\partial x_4 \partial x_{15}} = \frac{\partial^2 z}{\partial x_9 \partial x_{13}} = -\frac{\partial^2 z}{\partial x_{10} \partial x_{11}} = x_{23}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_5 \partial x_{16}} = -\frac{\partial^2 z}{\partial x_6 \partial x_{15}} = \frac{\partial^2 z}{\partial x_9 \partial x_{14}} = -\frac{\partial^2 z}{\partial x_{11} \partial x_{12}} = x_{24}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_7 \partial x_{16}} = -\frac{\partial^2 z}{\partial x_8 \partial x_{15}} = \frac{\partial^2 z}{\partial x_{10} \partial x_{14}} = -\frac{\partial^2 z}{\partial x_{12} \partial x_{13}} = x_{25}\,\frac{\partial^2 z}{\partial x_1 \partial x_{11}}$$
$$\frac{\partial^2 z}{\partial x_{13} \partial x_{17}} = \frac{\partial^2 z}{\partial x_{14} \partial x_{18}} = \frac{\partial^2 z}{\partial x_{15} \partial x_{19}} = \frac{\partial^2 z}{\partial x_{16} \partial x_{21}} \quad (= p_1)$$
$$= \lambda^{-1}\left(x_{25}\frac{\partial z}{\partial x_{11}} - x_{24}\frac{\partial z}{\partial x_{13}} + x_{23}\frac{\partial z}{\partial x_{14}} - x_{22}\frac{\partial z}{\partial x_{15}} + x_{20}\frac{\partial z}{\partial x_{16}}\right),$$
$$\frac{\partial^2 z}{\partial x_{10} \partial x_{17}} = \frac{\partial^2 z}{\partial x_{12} \partial x_{18}} = -\frac{\partial^2 z}{\partial x_{15} \partial x_{20}} = -\frac{\partial^2 z}{\partial x_{16} \partial x_{22}} \quad (= -p_2)$$
$$= \lambda^{-1}\left(x_{25}\frac{\partial z}{\partial x_9} - x_{24}\frac{\partial z}{\partial x_{10}} + x_{23}\frac{\partial z}{\partial x_{12}} - x_{21}\frac{\partial z}{\partial x_{15}} + x_{19}\frac{\partial z}{\partial x_{16}}\right),$$
$$\frac{\partial^2 z}{\partial x_8 \partial x_{17}} = -\frac{\partial^2 z}{\partial x_{12} \partial x_{19}} = -\frac{\partial^2 z}{\partial x_{14} \partial x_{20}} = \frac{\partial^2 z}{\partial x_{16} \partial x_{23}} \quad (= p_3)$$
$$= \lambda^{-1}\left(x_{25}\frac{\partial z}{\partial x_6} - x_{24}\frac{\partial z}{\partial x_8} + x_{22}\frac{\partial z}{\partial x_{12}} - x_{21}\frac{\partial z}{\partial x_{14}} + x_{18}\frac{\partial z}{\partial x_{16}}\right),$$

$$\frac{\partial^2 z}{\partial x_7 \partial x_{17}} = \frac{\partial^2 z}{\partial x_{12} \partial x_{21}} = \frac{\partial^2 z}{\partial x_{14} \partial x_{22}} = \frac{\partial^2 z}{\partial x_{15} \partial x_{23}} \quad (= -p_4)$$
$$= \lambda^{-1}(x_{25}\frac{\partial z}{\partial x_5} - x_{24}\frac{\partial z}{\partial x_7} + x_{20}\frac{\partial z}{\partial x_{12}} - x_{19}\frac{\partial z}{\partial x_{14}} + x_{18}\frac{\partial z}{\partial x_{15}}),$$

$$\frac{\partial^2 z}{\partial x_8 \partial x_{18}} = \frac{\partial^2 z}{\partial x_{10} \partial x_{19}} = \frac{\partial^2 z}{\partial x_{13} \partial x_{20}} = \frac{\partial^2 z}{\partial x_{16} \partial x_{24}} \quad (= p_5)$$
$$= \lambda^{-1}(-x_{25}\frac{\partial z}{\partial x_4} + x_{23}\frac{\partial z}{\partial x_8} - x_{22}\frac{\partial z}{\partial x_{10}} + x_{21}\frac{\partial z}{\partial x_{13}} - x_{17}\frac{\partial z}{\partial x_{16}}),$$

$$-\frac{\partial^2 z}{\partial x_7 \partial x_{18}} = \frac{\partial^2 z}{\partial x_{10} \partial x_{21}} = \frac{\partial^2 z}{\partial x_{13} \partial x_{22}} = -\frac{\partial^2 z}{\partial x_{15} \partial x_{24}} \quad (= p_6)$$
$$= \lambda^{-1}(x_{25}\frac{\partial z}{\partial x_3} - x_{23}\frac{\partial z}{\partial x_7} + x_{20}\frac{\partial z}{\partial x_{10}} - x_{19}\frac{\partial z}{\partial x_{13}} + x_{17}\frac{\partial z}{\partial x_{15}}),$$

$$-\frac{\partial^2 z}{\partial x_4 \partial x_{17}} = -\frac{\partial^2 z}{\partial x_6 \partial x_{18}} = -\frac{\partial^2 z}{\partial x_9 \partial x_{19}} = -\frac{\partial^2 z}{\partial x_{11} \partial x_{20}} = \frac{\partial^2 z}{\partial x_{16} \partial x_{25}} \quad (= p_7)$$
$$= \lambda^{-1}(x_{24}\frac{\partial z}{\partial x_4} - x_{23}\frac{\partial z}{\partial x_6} + x_{22}\frac{\partial z}{\partial x_9} - x_{21}\frac{\partial z}{\partial x_{11}} + \frac{\partial z}{\partial x_{16}}),$$

$$-\frac{\partial^2 z}{\partial x_3 \partial x_{17}} = -\frac{\partial^2 z}{\partial x_5 \partial x_{18}} = \frac{\partial^2 z}{\partial x_9 \partial x_{21}} = \frac{\partial^2 z}{\partial x_{11} \partial x_{22}} = \frac{\partial^2 z}{\partial x_{15} \partial x_{25}} \quad (= -p_8)$$
$$= \lambda^{-1}(x_{24}\frac{\partial z}{\partial x_3} - x_{23}\frac{\partial z}{\partial x_5} + x_{20}\frac{\partial z}{\partial x_9} - x_{19}\frac{\partial z}{\partial x_{11}} + \frac{\partial z}{\partial x_{15}}),$$

$$\frac{\partial^2 z}{\partial x_7 \partial x_{19}} = \frac{\partial^2 z}{\partial x_8 \partial x_{21}} = -\frac{\partial^2 z}{\partial x_{13} \partial x_{23}} = -\frac{\partial^2 z}{\partial x_{14} \partial x_{24}} \quad (= -p_9)$$
$$= \lambda^{-1}(x_{25}\frac{\partial z}{\partial x_2} - x_{22}\frac{\partial z}{\partial x_7} + x_{20}\frac{\partial z}{\partial x_8} - x_{18}\frac{\partial z}{\partial x_{13}} + x_{17}\frac{\partial z}{\partial x_{14}}),$$

$$-\frac{\partial^2 z}{\partial x_2 \partial x_{17}} = \frac{\partial^2 z}{\partial x_5 \partial x_{19}} = \frac{\partial^2 z}{\partial x_6 \partial x_{21}} = -\frac{\partial^2 z}{\partial x_{11} \partial x_{23}} = \frac{\partial^2 z}{\partial x_{14} \partial x_{25}} \quad (= p_{10})$$
$$= \lambda^{-1}(x_{24}\frac{\partial z}{\partial x_2} - x_{22}\frac{\partial z}{\partial x_5} + x_{20}\frac{\partial z}{\partial x_6} - x_{18}\frac{\partial z}{\partial x_{11}} + \frac{\partial z}{\partial x_{14}}),$$

$$-\frac{\partial^2 z}{\partial x_7 \partial x_{20}} = -\frac{\partial^2 z}{\partial x_8 \partial x_{22}} = -\frac{\partial^2 z}{\partial x_{10} \partial x_{23}} = -\frac{\partial^2 z}{\partial x_{12} \partial x_{24}} \quad (= p_{11})$$
$$= \lambda^{-1}(x_{25}\frac{\partial z}{\partial x_1} - x_{21}\frac{\partial z}{\partial x_7} + x_{19}\frac{\partial z}{\partial x_8} - x_{18}\frac{\partial z}{\partial x_{10}} + x_{17}\frac{\partial z}{\partial x_{12}}),$$

$$\frac{\partial^2 z}{\partial x_2 \partial x_{18}} = \frac{\partial^2 z}{\partial x_3 \partial x_{19}} = \frac{\partial^2 z}{\partial x_4 \partial x_{21}} = \frac{\partial^2 z}{\partial x_{11} \partial x_{24}} = \frac{\partial^2 z}{\partial x_{13} \partial x_{25}} \quad (= -p_{12})$$
$$= \lambda^{-1}(x_{23}\frac{\partial z}{\partial x_2} - x_{22}\frac{\partial z}{\partial x_3} + x_{20}\frac{\partial z}{\partial x_4} - x_{17}\frac{\partial z}{\partial x_{11}} + \frac{\partial z}{\partial x_{13}}),$$

$$-\frac{\partial^2 z}{\partial x_1 \partial x_{17}} = -\frac{\partial^2 z}{\partial x_5 \partial x_{20}} = -\frac{\partial^2 z}{\partial x_6 \partial x_{22}} = -\frac{\partial^2 z}{\partial x_9 \partial x_{23}} = \frac{\partial^2 z}{\partial x_{12} \partial x_{25}} \quad (= -p_{13})$$
$$= \lambda^{-1}(x_{24}\frac{\partial z}{\partial x_1} - x_{21}\frac{\partial z}{\partial x_5} + x_{19}\frac{\partial z}{\partial x_6} - x_{18}\frac{\partial z}{\partial x_9} + \frac{\partial z}{\partial x_{12}}),$$

$$\frac{\partial^2 z}{\partial x_1 \partial x_{18}} = -\frac{\partial^2 z}{\partial x_3 \partial x_{20}} = -\frac{\partial^2 z}{\partial x_4 \partial x_{22}} = \frac{\partial^2 z}{\partial x_9 \partial x_{24}} = \frac{\partial^2 z}{\partial x_{10} \partial x_{25}} \quad (= p_{14})$$
$$= \lambda^{-1}(x_{23}\frac{\partial z}{\partial x_1} - x_{21}\frac{\partial z}{\partial x_3} + x_{19}\frac{\partial z}{\partial x_4} - x_{17}\frac{\partial z}{\partial x_9} + \frac{\partial z}{\partial x_{10}}),$$
$$-\frac{\partial^2 z}{\partial x_1 \partial x_{19}} = -\frac{\partial^2 z}{\partial x_2 \partial x_{20}} = \frac{\partial^2 z}{\partial x_4 \partial x_{23}} = \frac{\partial^2 z}{\partial x_6 \partial x_{24}} = \frac{\partial^2 z}{\partial x_8 \partial x_{25}} \quad (= -p_{15})$$
$$= \lambda^{-1}(x_{22}\frac{\partial z}{\partial x_1} - x_{21}\frac{\partial z}{\partial x_2} + x_{18}\frac{\partial z}{\partial x_4} - x_{17}\frac{\partial z}{\partial x_6} + \frac{\partial z}{\partial x_8}),$$
$$\frac{\partial^2 z}{\partial x_1 \partial x_{21}} = \frac{\partial^2 z}{\partial x_2 \partial x_{22}} = \frac{\partial^2 z}{\partial x_3 \partial x_{23}} = \frac{\partial^2 z}{\partial x_5 \partial x_{24}} = \frac{\partial^2 z}{\partial x_7 \partial x_{25}} \quad (= p_{16})$$
$$= \lambda^{-1}(x_{20}\frac{\partial z}{\partial x_1} - x_{19}\frac{\partial z}{\partial x_2} + x_{18}\frac{\partial z}{\partial x_3} - x_{17}\frac{\partial z}{\partial x_5} + \frac{\partial z}{\partial x_7}),$$
$$\frac{\partial^2 z}{\partial x_i \partial x_j} = 0 \quad \text{otherwise.}$$

One can check that, among simple graded Lie algebras (of depth 3) of class (D) in Sect. 5 of [26], regular differential systems (X, D) of type $\mathfrak{m}$ satisfy the condition $(X.1)$ to $(X.3)$ in Sect. 4.3 when $\mathfrak{m}$ is the negative part of one of the simple graded Lie algebras $(C_\ell, \{\alpha_i, \alpha_\ell\})$ for $i = 2, \ldots, \ell - 1$, $(D_\ell, \{\alpha_i, \alpha_\ell\})$ $(2 < i < \ell - 1)$, when i is even, or $(E_7, \{\alpha_6.\alpha_7\})$, whereas the condition $(X.3)$ is not satisfied by the other cases.

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