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Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations

International Workshop, York University, Canada, August 4 – 8, 2008

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Editors:

Bert-Wolfgang Schulze
Institut für Mathematik
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam
Germany
e-mail: schulze@math.uni-potsdam.de

M. W. Wong
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada
e-mail: mwwong@mathstat.yorku.ca

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Contents

Preface

The *International Workshop on Pseudo-Differential Operators: Complex Analysis and Partial Differential Equations* was held at York University on August 4–8, 2008. The first phase of the workshop on August 4–5 consisted of a mini-course on pseudo-differential operators and boundary value problems given by Professor Bert-Wolfgang Schulze of Universität Potsdam for graduate students and post-docs. This was followed on August 6–8 by a conference emphasizing boundary value problems; explicit formulas in complex analysis and partial differential equations; pseudo-differential operators and calculi; analysis on the Heisenberg group and sub-Riemannian geometry; and Fourier analysis with applications in time-frequency analysis and imaging.

The role of complex analysis in the development of pseudo-differential operators can best be seen in the context of the well-known Cauchy kernel and the related Poisson kernel in, respectively, the Cauchy integral formula and the Poisson integral formula in the complex plane $\mathbb{C}$. These formulas are instrumental in solving boundary value problems for the Cauchy-Riemann operator $\overline{\partial}$ and the Laplacian Δ on specific domains with the unit disk and its biholomorphic companion, i.e., the upper half-plane, as paradigm models. The corresponding problems in several complex variables can be formulated in the context of the unit disk in $\mathbb{C}^n$, which may be the unit polydisk or the unit ball in $\mathbb{C}^n$. Analogues of the Cauchy kernel and the Poisson kernel and their ramifications to express solutions of boundary value problems in several complex variables can be looked upon as singular integral operators, which are de facto equivalent manifestations of pseudo-differential operators. It is the vision that bringing together experts in explicit formulas for boundary value problems in complex analysis working with *kernels* and specialists in pseudo-differential operators working with *symbols* should build synergy between the two groups. The functional analysis and real-life applications of pseudo-differential operators are always among top priorities in our agenda and these are well represented in the workshop and in this volume.

Operator Theory:
Advances and Applications, Vol. 205, 1–50

Boundary Value Problems with the Transmission Property

B.-W. Schulze

Abstract. We give a survey on the calculus of (pseudo-differential) boundary value problems with the transmision property at the boundary, and ellipticity in the Shapiro–Lopatinskij sense. Apart from the original results of the work of Boutet de Monvel we present an approach based on the ideas of the edge calculus. In a final section we introduce symbols with the anti-transmission property.

Mathematics Subject Classification (2000). 35J40, 58J32, 58J40.

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1. Introduction

Boundary value problems (BVPs) for elliptic (pseudo-) differential operators have attracted mathematicians and physicists during all periods of modern analysis. While the definition of ellipticity of an operator on an open (smooth) manifold is very simple, such a notion in connection with a (smooth or non-smooth) boundary is much less evident. During the past few years the interest in BVPs increased again considerably, motivated by new applications and also by unsolved problems in the frame of the structural understanding of ellipticity in new situations. Several classical periods of the development created deep and beautiful ideas, for instance, in connection with function theory, potential theory, with boundary operators satisfying the complementing condition, cf. Agmon, Douglas, Nirenberg [1], or pseudo-differential theories from Vishik and Eskin [29], Eskin [7], Boutet de Monvel [4]. Other branches of the development concern ellipticity with global projection conditions (analogues of Atiyah, Patodi, Singer conditions, cf. [3]), or elliptic theories on manifolds with geometric singularities, cf. the author's papers [18] or [19].

After all that it is not easy to imagine how many basic and interesting problems remained open. A part of the new developments is connected with the analysis on configurations with singularities that includes boundary value problems. In that context it seems to be desirable to see the pseudo-differential machinery of Boutet de Monvel and also of Vishik and Eskin from an alternative viewpoint, using the achievements of the cone and edge pseudo-differential calculus as is pointed out in [16], [21], and in the author's joint paper with Seiler [25], see also the monographs [22], or those jointly with Egorov [6], Kapanadze [12], Harutyunyan [10].

Our exposition just intends to emphasize such an approach, here mainly focused on operators with the transmission property at the boundary from the work of Boutet de Monvel. We also introduce symbols with the anti-transmission property at the boundary. Together with those with the transmission property they span the space of all (classical) symbols that are smooth up to the boundary. A pseudo-differential calculus for such general symbols needs more tools from the edge algebra than developed here.

The present paper is the elaborated version of introductory lectures, given during the International Workshop on Pseudo-Differential Operators, Complex Analysis and Partial Differential Equations at York University on August 4–8, 2008, in Toronto.

2. Interior and Boundary Symbols for Differential Operators

Let X be a C^∞ manifold with boundary $Y = \partial X$. Moreover, let $2X$ be the double, defined by gluing together two copies $X_\pm$ of X to a C^∞ manifold along the common boundary Y. Let us fix a Riemannian metric on $2X$ and consider Y in the induced metric. There is then a tubular neighbourhood of Y in $2X$ that can be identified with $Y \times [-1, 1]$, with a splitting of variables $x = (y, t)$, where t is the variable normal to the boundary and $y \in Y$. We assume that (y, t) belongs to $X =: X_+$ for $0 \leq t \leq 1$ and to X_- for $-1 \leq t \leq 0$.

If M is a C^∞ manifold (with or without boundary), by $\mathrm{Diff}^\mu(M)$ we denote the set of all differential operators of order μ on M with smooth coefficients (smooth up to the boundary when $\partial M \neq \emptyset$).

Local descriptions near Y will refer to charts

$$\chi : U \to \Omega \times \mathbb{R}$$

for open $U \subseteq 2X$, $U \cap Y \neq \emptyset$, and open $\Omega \subseteq \mathbb{R}^{n-1}$, and induced charts

$$\chi : U \cap Y \to \Omega$$

on Y and

$$\chi_\pm : U_\pm := U \cap X_\pm \to \Omega \times \overline{\mathbb{R}}_\pm$$

on $X_\pm$ near the boundary. Concerning the transition maps $\Omega \times \mathbb{R} \to \widetilde{\Omega} \times \mathbb{R}$, $(y, t) \to (\tilde{y}, \tilde{t})$, for simplicity we assume that the normal variable remains unchanged near the boundary, i.e., $t = \tilde{t}$ for $|t|$ sufficiently small. The map $y \to \tilde{y}$ corresponds to a diffeomorphism $\Omega \to \widetilde{\Omega}$.

Let $A \in \mathrm{Diff}^{\mu}(X)$, $B_j \in \mathrm{Diff}^{\mu_j}(V_+)$, $V_+ := V \cap X$, $j = 1, \ldots, N$, for some $N \in \mathbb{N}$, and set

$$Tu := (B_j u|_Y)_{j=1,\ldots,N}.$$

Then the equations

$$Au = f \text{ in int}X, \quad Tu = g \text{ on } Y \tag{2.1}$$

represent a boundary value problem for A. Consider for the moment functions in $C^\infty(X)$; then (2.1) can be regarded as a continuous operator

$$\mathcal{A} = \begin{pmatrix} A \\ T \end{pmatrix} : C^\infty(X) \to \begin{matrix} C^\infty(X) \\ \oplus \\ C^\infty(Y, \mathbb{C}^N) \end{matrix}. \tag{2.2}$$

If X is compact, we have the standard Sobolev spaces $H^s(2X)$ on $2X$ and

$$H^s(\mathrm{int}X) := H^s(2X)|_{\mathrm{int}X},$$

$s \in \mathbb{R}$. Then (2.2) extends to continuous operators

$$\mathcal{A} : H^s(\mathrm{int}X) \to \begin{matrix} H^{s-\mu}(\mathrm{int}X) \\ \oplus \\ \oplus_{j=1}^N H^{s-\mu_j-\frac{1}{2}}(Y) \end{matrix} \tag{2.3}$$

for all $s > \max\{\mu_j + \frac{1}{2} : j = 1, \ldots, N\}$.

We will give a survey on elliptic boundary value problems (BVPs), starting from (2.2), and we ask to what extent we may expect a pseudo-differential calculus (an algebra) that contains the operators (2.2) together with the parametrices of elliptic elements. First we have to explain what we understand by ellipticity of a boundary value problem.

In contrast to the notion of ellipticity of a differential operator (or a, say, classical pseudo-differential operator) A on an open C^∞ manifold M, in the case of a manifold with boundary we have from the very beginning a variety of choices.

For an open C^∞ manifold M denote by $L^\mu_{\mathrm{cl}}(M)$ the space of all classical pseudo-differential operators of order $\mu \in \mathbb{R}$ on M. An operator $A \in L^\mu_{\mathrm{cl}}(M)$ is called elliptic if its homogeneous principal symbol $\sigma_\psi(A)(x,\xi)$ of order μ never vanishes on $T^*M \setminus 0$ (the cotangent bundle of M minus the zero section). The union of spaces $L^\mu_{\mathrm{cl}}(M)$ over $\mu \in \mathbb{R}$ is closed under the construction of parametrices of elliptic elements, to be more precise, every elliptic $A \in L^\mu_{\mathrm{cl}}(M)$ has a (properly supported) parametrix $P \in L^{-\mu}_{\mathrm{cl}}(M)$ such that $1 - PA, 1 - AP \in L^{-\infty}(M)$ (here and in future by 1 we often denote identity operators). The space $L^{-\infty}(M)$ can be identified with $C^\infty(M \times M)$ via a fixed Riemannian metric on M.

Let us recall the well-known fact that when M is compact and closed, the ellipticity of $A \in L^\mu_{\mathrm{cl}}(M)$ is equivalent to the property that

$$A : H^s(M) \to H^{s-\mu}(M) \tag{2.4}$$

is a Fredholm operator for some $s = s_0 \in \mathbb{R}$. Moreover, from the Fredholm property of (2.4) for $s = s_0$ it follows that (2.4) is Fredholm for all $s \in \mathbb{R}$. In addition it is known that $L^\mu_{\rm cl}(M)$ for every $\mu \in \mathbb{R}$ contains so-called order reducing operators, i.e., elliptic operators R^μ that induce isomorphisms,

$$R^\mu : H^s(M) \to H^{s-\mu}(M) \tag{2.5}$$

for all $s \in \mathbb{R}$; then $(R^\mu)^{-1} \in L^{-\mu}_{\rm cl}(M)$ is again order reducing (of opposite order). Below we shall establish more tools on pseudo-differential operators.

Let us now return to BVPs of the form (2.3), where X is a compact manifold with smooth boundary Y.

Writing our differential operator A in local coordinates $x \in \Omega \times \overline{\mathbb{R}}_+$ near the boundary as

$$A = \sum_{|\alpha| \le \mu} a_\alpha(x) D^\alpha_x,$$

$a_\alpha \in C^\infty(\Omega \times \overline{\mathbb{R}}_+)$, we define

$$\sigma_\psi(A)(x, \xi) = \sum_{|\alpha| = \mu} a_\alpha(x) \xi^\alpha, \tag{2.6}$$

$(x, \xi) \in T^*(\Omega \times \overline{\mathbb{R}}_+) \setminus 0$, and observe the homogeneity

$$\sigma_\psi(A)(x, \lambda\xi) = \lambda^\mu \sigma_\psi(A)(x, \xi),\ \lambda \in \mathbb{R}_+.$$

Let $x = (y, t)$, $\xi = (\eta, \tau)$, and set

$$\sigma_\partial(A)(y, \eta) = \sum_{|\alpha| = \mu} a_\alpha(y, 0)(\eta, D_t)^\alpha, \tag{2.7}$$

where $(\eta, D_t)^\alpha = \eta^{\alpha'} D_t^{\alpha''}$ for $\alpha = (\alpha', \alpha'') \in \mathbb{N}^n$, $(y, \eta) \in T^*\Omega \setminus 0$, or, equivalently, $\sigma_\partial(A)(y, \eta) = \sigma_\psi(A)(y, 0, \eta, D_t)$. The expression (2.7) represents a family of continuous operators

$$\sigma_\partial(A) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+),\ s \in \mathbb{R}, \tag{2.8}$$

called the (homogeneous principal) boundary symbol of A.

Let $H^s(\mathbb{R}_+)$ be endowed with the strongly continuous group $\kappa = \{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ of isomorphisms

$$\kappa_\lambda : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+), (\kappa_\lambda u)(t) = \lambda^{1/2} u(\lambda t),\ \lambda \in \mathbb{R}_+.$$

Then we obtain the following kind of homogeneity of the boundary symbol,

$$\sigma_\partial(A)(y, \lambda\eta) = \lambda^\mu \kappa_\lambda \sigma_\partial(A)(y, \eta) \kappa_\lambda^{-1},\ \lambda \in \mathbb{R}_+. \tag{2.9}$$

Homogeneity in that sense will also referred to as twisted homogeneity (of order μ).

It makes sense also to define the (homogeneous principal) boundary symbol of the trace operator $T = {}^{\rm t}(T_1, \ldots, T_N)$, by

$$\sigma_\partial(T_j)(y, \eta)u := \sigma_\psi(B_j)(y, 0, \eta, D_t)u|_{t=0}, \tag{2.10}$$

$u \in H^s(\mathbb{R}_+)$, $s > \max\{\mu_j + \frac{1}{2} : j = 1, \dots, N\}$ where $\sigma_\psi(B_j)(x, \xi)$ is the homogeneous principal symbol of the operator B_j, and (2.10) is interpreted as a family of operators

$$\sigma_\partial(T_j)(y, \eta) : H^s(\mathbb{R}_+) \to \mathbb{C},$$

$(y, \eta) \in T^*\Omega \setminus 0$. The boundary symbol (2.10) is homogeneous in the following sense:

$$\sigma_\partial(T_j)(y, \lambda\eta) = \lambda^{\mu_j + \frac{1}{2}} \sigma_\partial(T_j)(y, \eta)\kappa_\lambda^{-1}, \ \lambda \in \mathbb{R}_+. \tag{2.11}$$

It is often convenient to compose (2.2) from the left by an operator

$$\mathrm{diag}(1, R_1, \dots, R_N) \tag{2.12}$$

where $R_j \in L_{\mathrm{cl}}^{\mu - (\mu_j + \frac{1}{2})}(Y)$ is an order reducing operator on the boundary in the above-mentioned sense and to pass to a modified operator

$$\begin{pmatrix} A \\ {}^{\mathrm{t}}(R_1T_1, \dots, R_NT_N) \end{pmatrix} : H^s(\mathrm{int}X) \to \begin{matrix} H^{s-\mu}(\mathrm{int}X) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{C}^N) \end{matrix},$$

related to the former one by a trivial pseudo-differential reduction of orders on the boundary. This is formally a little easier (later on we admit such trace operators anyway). Instead of (2.11) we then obtain

$$\sigma_\partial(R_jT_j)(y, \lambda\eta) = \lambda^\mu \sigma_\partial(R_jT_j)(y, \eta)\kappa_\lambda^{-1}, \ \lambda \in \mathbb{R}_+, \tag{2.13}$$

where

$$\sigma_\partial(R_jT_j)(y, \eta) = \sigma_\psi(R_j)(y, \eta)\sigma_\partial(T_j)(y, \eta)$$

with $\sigma_\partial(R_j)(y, \eta)$ being the homogeneous principal symbol of R_j of order $\mu - (\mu_j + \frac{1}{2})$ as a classical pseudo-differential operator on the boundary.

Let us now explain the role of the trace operators in connection with the ellipticity of a boundary value problem. We call the pair

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$$

the principal symbol of $\mathcal{A}$, consisting of the (principal) interior symbol $\sigma_\psi(\mathcal{A}) := \sigma_\psi(A)$ and the (principal) boundary symbol $\sigma_\partial(\mathcal{A}) := {}^{\mathrm{t}}(\sigma_\partial(A), \sigma_\partial(T_1), \dots, \sigma_\partial(T_N))$ of $\mathcal{A}$,

$$\sigma_\partial(\mathcal{A}) : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^N \end{matrix}.$$

Ellipticity of $\mathcal{A}$ requires the bijectivity of both components on $T^*X \setminus 0$ and $T^*Y \setminus 0$, respectively, the latter as an operator function for $s - \mu > -\frac{1}{2}$. Since the operators $\sigma_\partial(T_j)(y, \eta)$ are of finite rank, $\sigma_\partial(A)(y, \eta)$ has to be a family of Fredholm operators. The following lemma shows that this is an automatic consequence of the ellipticity of A with respect to σ_ψ.

Lemma 2.1. *Let A be an elliptic differential operator; then*

$$\sigma_\partial(A)(y,\eta) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$$

is a surjective family of Fredholm operators for every real $s > \mu - \frac{1}{2}$, and the kernel $\ker \sigma_\partial(A)(y,\eta)$ *is a finite-dimensional subspace of $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$ which is independent of s. Moreover,* $\dim\ker \sigma_\partial(A)(y,\eta) = \dim\ker \sigma_\partial(A)(y,\eta/|\eta|)$ *for all $(y,\eta) \in T^*Y \setminus 0$.*

Proof. Set for the moment $a(\tau) := \sigma_\psi(A)(y,0,\eta,\tau)$ with frozen variables (y,η), $\eta \neq 0$. Then $\sigma_\partial(\mathcal{A}) = a(D_t)$ can be written as $\mathrm{op}^+(a) := \mathrm{r}^+\mathrm{op}_t(a)\mathrm{e}^+$ for the operator of extension e^+ of functions by 0 to $t < 0$ and r^+ the restriction to $t > 0$, and $\mathrm{op}_t(\cdot)$ is the pseudo-differential operator on $\mathbb{R}$ with the symbol $a(\tau)$, i.e., $\mathrm{op}_t(a)u(t) = \iint e^{i(t-t')\tau} a(\tau)u(t')dt'đ\tau$, $đ\tau = (2\pi)^{-1}d\tau$. Then $\mathrm{op}^+(a^{-1})$ is a right inverse of $\mathrm{op}^+(a)$, since $\mathrm{op}^+(a)\mathrm{op}^+(a^{-1}) = \mathrm{op}^+(aa^{-1}) + \mathrm{r}^+\mathrm{op}(a)\mathrm{e}^-\mathrm{op}^+(a^{-1}) = 1$, because of $\mathrm{r}^+\mathrm{op}(a)\mathrm{e}^- = 0$. This shows the surjectivity of $\mathrm{op}^+(a)$. The fact that solutions u of the homogeneous equation $a(D_t)u = 0$ form a finite-dimensional subspace of $\mathcal{S}(\overline{\mathbb{R}}_+)$ is standard. However, we will show those things below once again independently, cf. Theorem 3.29 below. The last assertion follows from the homogeneity (2.9). □

Example. Let $A = \Delta$ be the Laplacian, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ in local variables. Then $\sigma_\partial(\Delta) = -|\xi|^2$, and

$$\sigma_\partial(\Delta)(\eta) = -|\eta|^2 + D_t^2 : H^s(\mathbb{R}_+) \to H^{s-2}(\mathbb{R}_+).$$

We have

$$\ker \sigma_\partial(\Delta)(\eta) = \{ce^{-|\eta|t} : c \in \mathbb{C}\},$$

i.e., $\dim\ker \sigma_\partial(\Delta)(\eta) = 1$ for all $\eta \neq 0$ and all $s > 3/2$.

Remark 2.2. The operators $T_k : H^s(X) \to H^{s-k-1/2}(Y)$, locally near Y defined by

$$T_k u := D_t^k u|_{t=0}, \;\; k \in \mathbb{N},$$

have the boundary symbols

$$\sigma_\partial(T_k)u = D_t^k u|_{t=0}, \quad \sigma_\partial(T_k)(\eta) : H^s(\mathbb{R}_+) \to \mathbb{C}$$

and are (although they are independent of η) of homogeneity $k + \frac{1}{2}$, i.e.,

$$\sigma_\partial(T_k)(\lambda\eta)u = \lambda^{k+\frac{1}{2}}\sigma_\partial(T_k)(\eta)\kappa_\lambda^{-1}u, \;\; \lambda \in \mathbb{R}_+.$$

Moreover, as we see from Lemma 2.1 together with Lemma 2.3 below, the column matrix

$$\begin{pmatrix} \sigma_\partial(\Delta)(\eta) \\ \sigma_\partial(T_k)(\eta) \end{pmatrix} : H^s(\mathbb{R}_+) \to \begin{array}{c} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{array} \tag{2.14}$$

is an isomorphism for every $\eta \neq 0$, $s > \max\{\frac{3}{2}, k + \frac{1}{2}\}$; this is true for every $k \in \mathbb{N}$. Observe that T_0 represents Dirichlet and T_1 Neumann conditions.

In other words, the boundary symbol $\sigma_\partial(T_k)$ fills up the Fredholm operators $\sigma_\partial(\Delta)(\eta) : H^s(\mathbb{R}_+) \to H^{s-2}(\mathbb{R}_+)$, $s > 3/2$, to a family of isomorphisms (2.14). In this way we have examples of so-called elliptic BVPs, namely,

$$\mathcal{A}_k := \begin{pmatrix} \Delta \\ T_k \end{pmatrix} : H^s(X) \to \begin{matrix} H^{s-2}(X) \\ \oplus \\ H^{s-k-\frac{1}{2}}(Y) \end{matrix} \tag{2.15}$$

for every $k \in \mathbb{N}$. In connection with such constructions it is useful to recall the following simple algebraic result.

Lemma 2.3. *Let $H, \widetilde{H}, \widetilde{L}$ be Hilbert spaces and $a : H \to \widetilde{H}, b : H \to \widetilde{L}$ linear continuous operators. Then the column matrix operator $\boldsymbol{a} := \binom{a}{b} : H \to \begin{matrix} \widetilde{H} \\ \oplus \\ \widetilde{L} \end{matrix}$ is an isomorphism if and only if $a : H \to \widetilde{H}$ is surjective, and $b : H \to \widetilde{L}$ restricts to an isomorphism $b|_{\ker a} : \ker a \to \widetilde{L}$.*

Proof. Let $a : H \to \widetilde{H}$ be surjective, and $b_0 := b|_{\ker a} : \ker a \to \widetilde{L}$ an isomorphism. Then $\boldsymbol{a}$ is obviously surjective. Moreover, $\boldsymbol{a}u = 0$ implies $u \in \ker a$ and $b_0 u = 0$; then, since b_0 is an isomorphism it follows that $u = 0$. Thus $\boldsymbol{a}$ is injective and hence an isomorphism. Conversely, assume that $\boldsymbol{a}$ is an isomorphism. The surjectivity of $\boldsymbol{a}$ implies that $a : H \to \widetilde{H}$, $b : H \to \widetilde{L}$ are both surjective. In particular, if H_1 denotes the orthogonal complement of $\ker a_1$ in H, we obtain an isomorphism $a_1 := a|_{H_1} : H_1 \to \widetilde{H}$, and $\boldsymbol{a}$ can be written as a block matrix

$$\boldsymbol{a} = \begin{pmatrix} a_1 & 0 \\ b_1 & b_0 \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ \ker a \end{matrix} \to \begin{matrix} \widetilde{H} \\ \oplus \\ \widetilde{L} \end{matrix}$$

for $b_1 := b|_{H_1}$. It remains to show that $b_0 : \ker a \to \widetilde{L}$ is an isomorphism. The operator $\begin{pmatrix} a_1^{-1} & 0 \\ -b_1 a_1^{-1} & 1 \end{pmatrix} : \begin{matrix} \widetilde{H} \\ \oplus \\ \widetilde{L} \end{matrix} \to \begin{matrix} H_1 \\ \oplus \\ \widetilde{L} \end{matrix}$ is an isomorphism, and we have

$$\begin{pmatrix} a_1^{-1} & 0 \\ -b_1 a_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ b_1 & b_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b_0 \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ \ker a \end{matrix} \to \begin{matrix} H_1 \\ \oplus \\ \widetilde{L} \end{matrix}.$$

Therefore, since both factors on the left-hand side are isomorphisms, it follows that also $b_0 : \ker a \to \ker a$ is an isomorphism. $\square$

This shows us the meaning of the above-mentioned N, the number of trace operators which turns the boundary symbol

$$\sigma_\partial(\mathcal{A})(y, \eta) : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^N \end{matrix} \tag{2.16}$$

to a family of isomorphisms. According to Lemmas 2.1 and 2.3, for an elliptic differential operator A we have $N = \dim\ker\sigma_\partial(A)(y,\eta)$; this number is required to be independent of y and $\eta \neq 0$. As is well-known, if A is of order $2m$ and admits boundary operators $\{B_1, \dots, B_m\}$ satisfying the so-called complementing condition with respect to A, then (for $m = N$) that property holds (cf. Agmon, Douglis, Nirenberg [1], Lions, Magenes [13]).

Definition 2.4. The block matrix operator $\mathcal{A} = {}^{\mathrm{t}}(A \quad T)$ is said to be a (Shapiro–Lopatinskij) elliptic boundary value problem for the elliptic differential operator A if the boundary symbol (2.16) is a family of isomorphisms for any sufficiently large s, for all $(y,\eta) \in T^*Y \setminus 0$. We also talk about Shapiro–Lopatinskij trace (or boundary) conditions for the operator A.

Remark 2.5. Observe that not every elliptic differential operator A admits Shapiro–Lopatinskij elliptic trace conditions. The simplest example is the Cauchy-Riemann operator $\overline{\partial}$ in the complex plane. More general examples are Dirac operators in even dimensions, and other important geometric operators. We will return later on to this discussion in the context of the Atiyah–Bott obstruction for the existence of Shapiro–Lopatinskij elliptic conditions.

If we ask for an algebra of BVPs a first essential formal problem is that column matrices cannot be composed with each other in a reasonable manner. However, we extend the notion "algebra" and talk about block matrix operators where the algebraic operations are carried out only under natural conditions, namely, addition when the matrices have the same number of rows and columns and multiplication when the number of rows and columns in the middle fit together. For instance, if we consider the Dirichlet problem $\mathcal{A}_0$ for the Laplacian, cf. the formula (2.15) for $k = 0$, we have invertibility of

$$\mathcal{A}_0 : C^\infty(X) \to \begin{array}{c} C^\infty(X) \\ \oplus \\ C^\infty(Y) \end{array}.$$

Denoting by $\mathcal{P} := (P_0 \quad K_0)$ the inverse of $\mathcal{A}_0$ (which belongs to the pseudo-differential operator calculus to be discussed here), then we have two kinds of compositions, namely, for $A = \Delta$,

$$\mathcal{A}_0\mathcal{P}_0 = \begin{pmatrix} A \\ T_0 \end{pmatrix} (P_0 \quad K_0) = \begin{pmatrix} AP_0 & AK_0 \\ T_0P_0 & T_0K_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.17}$$

and

$$(P_0 \quad K_0) \begin{pmatrix} A \\ T_0 \end{pmatrix} = P_0A + K_0T_0 = 1. \tag{2.18}$$

It also makes sense to consider

$$\mathcal{A}_k\mathcal{P}_0 = \begin{pmatrix} 1 & 0 \\ T_kP_0 & T_kK_0 \end{pmatrix} \tag{2.19}$$

for every $k \in \mathbb{N}$. The lower right corner of the latter matrix has the meaning of the reduction of the boundary condition T_k to the boundary (by means of the

Dirichlet problem). It turns out that $T_k K_0$ is a classical elliptic pseudo-differential operator of order k on the boundary. Its symbol will be computed in the following section, cf. the formula (3.28).

3. Inverses of Boundary Symbols

Let us first recall that the construction of a parametrix of an elliptic operator $A \in L^{\mu}_{\mathrm{cl}}(M)$ on an open C^{∞} manifold M can be started by inverting the homogeneous principal symbol and forming a $B \in L^{-\mu}_{\mathrm{cl}}(M)$ such that $\sigma_{\psi}(B) = \sigma_{\psi}^{-1}(A)$ (B is obtained via an operator convention). In a second step we form

$$1 - BA = C \in L^{-1}_{\mathrm{cl}}(M)$$

(everything in the frame of properly supported pseudo-differential operators), then we pass to a $D \in L^{-1}_{\mathrm{cl}}(M)$ such that $(1 + D)(1 - C) = 1 \bmod L^{-\infty}(M)$. Such a D can be found as an asymptotic sum $\sum_{j=1}^{\infty} C^j$, and $P = (1 + D)B$ is then a left parametrix of A. (For future references we call the latter procedure a formal Neumann series argument.) In an analogous manner we find a right parametrix, and then a simple algebraic consideration shows that P is a two-sided parametrix.

These arguments are based on the following properties of (classical) pseudo-differential operators:

1. every pseudo-differential operator has a properly supported representative modulo a smoothing operator;
2. any sequence of operators of order $\mu - j$, $j \in \mathbb{N}$, has an asymptotic sum, uniquely determined modulo a smoothing operator;
3. there is a symbolic map that assigns the unique principal symbol of an operator; the algebraic operations between operators are compatible with those for associated principal symbols (in particular, the principal symbol of a composition is equal to the composition (product) of the principal symbols);
4. every smooth homogeneous function of order μ on $T^*M \setminus 0$ is the principal symbol of an associated pseudo-differential operator of order μ (i.e., there is an operator convention that is right inverse of the principal symbolic map of 3.);
5. an operator of order μ with vanishing principal symbol is of order $\mu - 1$.

It turns out that boundary value problems as in Section 2 can be completed to a graded algebra of 2×2 block matrix operators with a two-component principal symbolic hierarchy $\sigma = (\sigma_{\psi}, \sigma_{\partial})$, where analogues of the properties 1.–5. hold. Such an algebra has been introduced by Boutet de Monvel [4], and we discuss here (among other things) some elements of that calculus.

The first essential point is to analyse the nature of inverses of bijective boundary symbols. Since such inverses are computed (y, η)-wise for $(y, \eta) \in T^*Y \setminus 0$ we first freeze those variables and look at operators on $\mathbb{R}_+$. Let us consider a classical symbol $a(\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R})$, $\mu \in \mathbb{R}$. Examples of such symbols are

$$l^{\mu}_{\pm}(\tau) := (1 \pm i\tau)^{\mu}. \tag{3.1}$$

Let us set

$$\mathrm{op}^+(a)u(t) = \mathrm{r}^+\mathrm{op}(a)(\mathrm{e}^+u)(t), \tag{3.2}$$

for every $u \in H^s(\mathbb{R}_+)$ $(= H^s(\mathbb{R})|_{\mathbb{R}_+})$, $s > -\frac{1}{2}$, where $\mathrm{e}^+u \in \mathcal{S}'(\mathbb{R})$ is the distribution obtained by extending u by zero to $\mathbb{R}_-$, i.e.,

$$\mathrm{e}^+u(t) = u(t) \text{ for } t > 0, \ \mathrm{e}^+u(t) = 0 \text{ for } t < 0.$$

Moreover r^+ is the operator of restriction from $\mathbb{R}$ to $\mathbb{R}_+$, and

$$\mathrm{op}(a)v(t) = \iint e^{i(t-t')\tau}a(\tau)u(t')dt'đ\tau,$$

$đ\tau = (2\pi)^{-1}d\tau$. In an analogous manner we define the extension e^- by zero from $\mathbb{R}_-$ to $\mathbb{R}$ and the restriction r^- from $\mathbb{R}$ to $\mathbb{R}_-$. The operator (3.2) defines a linear map

$$\mathrm{op}^+(a) : H^s(\mathbb{R}_+) \to \mathcal{S}'(\mathbb{R}_+)$$

for every $s > -\frac{1}{2}$, $\mathcal{S}'(\mathbb{R}_+) := \mathcal{S}'(\mathbb{R})|_{\mathbb{R}_+}$. As is well known (cf. [7]), in some cases $\mathrm{op}^+(\cdot)$ induces a continuous operator

$$\mathrm{op}^+(a) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+) \tag{3.3}$$

for every $s > -1/2$, namely, when $a(\tau)$ is a so-called minus-symbol.

Let $\mathcal{A}(U)$, $U \subseteq \mathbb{C}$ open, denote the space of all holomorphic functions in U, and set $\mathbb{C}_\pm := \{z = \tau + i\beta : \beta \gtrless 0\}$. Then $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ is said to be a minus-symbol if $a(\tau)$ has an extension to a function in $\mathcal{A}(\mathbb{C}_+) \cap C^\infty(\overline{\mathbb{C}}_+)$ such that

$$|a(z)| \leq c(1 + |z|^2)^{\mu/2} \tag{3.4}$$

for all $z \in \overline{\mathbb{C}}_+$, for some constant $c > 0$. By a plus-symbol of order μ we understand an element $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ that extends to a function in $\mathcal{A}(\mathbb{C}_-) \cap C^\infty(\overline{\mathbb{C}}_-)$ such that the estimates (3.4) hold for all $z \in \overline{\mathbb{C}}_-$. For $s \in \mathbb{R}$ we have a relation similar to (3.3) when we replace e^+ by a continuous extension operator $\mathrm{e}^+_s : H^s(\mathbb{R}_+) \to H^s(\mathbb{R})$ with $\mathrm{e}^+_s u|_{\mathbb{R}_+} = u$; then in the minus-case the latter map is independent of the choice of e^+_s.

If $a(\tau)$ is a plus-symbol of order μ and $H^s_0(\overline{\mathbb{R}}_+) := \{u \in H^s(\mathbb{R}) : u = 0 \text{ on } \mathbb{R}_-\}$, then

$$\mathrm{op}^+(a) : H^s_0(\mathbb{R}_+) \to H^{s-\mu}_0(\mathbb{R}_+) \tag{3.5}$$

is continuous for every $s \in \mathbb{R}$. Concerning a proof of the continuity of (3.3) and (3.5), see [7, Lemma 4.6 and Theorem 4.4], (cf. also [10, Section 4.1.2]). Moreover, for an arbitrary $p(\tau) \in S^\nu_{\mathrm{cl}}(\mathbb{R})$, $\nu \in \mathbb{R}$, we have

$$\mathrm{op}^+(ap) = \mathrm{op}^+(a)\mathrm{op}^+(p) \tag{3.6}$$

when $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ is a minus-symbol (since $\mathrm{r}^+\mathrm{op}(a)\mathrm{e}^- = 0$) and

$$\mathrm{op}^+(pa) = \mathrm{op}^+(p)\mathrm{op}^+(a) \tag{3.7}$$

when $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ is a plus-symbol (since $\mathrm{r}^-\mathrm{op}(a)\mathrm{e}^+ = 0$).

Example. A polynomial in τ is both a minus- and a plus-symbol.

Remark 3.1. The function $l_-^\mu(\tau) = (1 - i\tau)^\mu$ is a minus-symbol of order $\mu \in \mathbb{R}$, and

$$\mathrm{op}^+(l_-^\mu) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$$

is an isomorphism for every $s \in \mathbb{R}$, $s > \max\{-\frac{1}{2}, \mu - \frac{1}{2}\}$, where $(\mathrm{op}^+(l_-^\mu))^{-1} = \mathrm{op}^+(l_-^{-\mu})$. Moreover, $l_+^\mu(\tau) = (1 + i\tau)^\mu$ is a plus-symbol of order $\mu \in \mathbb{R}$, and

$$\mathrm{op}^+(l_+^\mu) : H_0^s(\mathbb{R}_+) \to H_0^{s-\mu}(\mathbb{R}_+)$$

is an isomorphism for every $s \in \mathbb{R}$ where $(\mathrm{op}^+(l_+^\mu))^{-1} = \mathrm{op}^+(l_+^{-\mu})$.

A classical symbol $a(\tau) \in S_{\mathrm{cl}}^\mu(\mathbb{R})$ has an asymptotic expansion

$$a(\tau) \sim \sum_{j=0}^{\infty} a_j^\pm (i\tau)^{\mu-j} \text{ for } \tau \to \pm\infty \tag{3.8}$$

for unique coefficients $a_j^\pm \in \mathbb{C}$ (the imaginary unit $i = \sqrt{-1}$ is taken for convenience; powers are defined as $(i\tau)^\nu = e^{\nu \log(i\tau)}$ with the principal branch of the logarithm).

If $\chi(\tau) \in C^\infty(\mathbb{R})$ is an excision function in τ (i.e., $\chi(\tau) = 0$ for $|\tau| < c_0$, $\chi(\tau) = 1$ for $|\tau| > c_1$, for some $0 < c_0 < c_1$), then we have

$$a(\tau) \sim \sum_{j=0}^{\infty} \chi(\tau) a_{(\mu-j)}(\tau) \tag{3.9}$$

for

$$a_{(\mu-j)}(\tau) = \{a_j^+ \theta^+(\tau) + a_j^- \theta^-(\tau)\}(i\tau)^{\mu-j}, \tag{3.10}$$

with θ^+ being the characteristic function of the $\pm$ half-axis in τ, where (3.9) has the meaning of an asymptotic expansion of symbols, $\chi(\tau)a_{(\mu-j)}(\tau) \in S_{\mathrm{cl}}^{\mu-j}(\mathbb{R})$.

Definition 3.2. A symbol $a(\tau) \in S_{\mathrm{cl}}^\mu(\mathbb{R})$ for $\mu \in \mathbb{Z}$ has the transmission property if

$$a_j^+ = a_j^- \text{ for all } j \in \mathbb{N}. \tag{3.11}$$

Let $S_{\mathrm{tr}}^\mu(\mathbb{R})$ denote the space of all symbols in $S_{\mathrm{cl}}^\mu(\mathbb{R})$ with the transmission property.

Remark 3.3. A symbol $a(\tau) \in S_{\mathrm{cl}}^\mu(\mathbb{R})$ has the transmission property exactly when

$$a_{(\mu-j)}(\tau) = (-1)^{\mu-j} a_{(\mu-j)}(-\tau) \tag{3.12}$$

for all $\tau \in \mathbb{R} \setminus \{0\}$ and all $j \in \mathbb{N}$.

In fact, the transmission property means that $a_{(\mu-j)}(\tau) = c_j (i\tau)^{\mu-j}$ for $c_j := a_j^+ = a_j^-$ for all $j \in \mathbb{N}$, and this shows the relation (3.12). Conversely from (3.12) we deduce

$$\begin{aligned}\{a_j^+\theta^+(\tau) + a_j^-\theta^-(\tau)\}(i\tau)^{\mu-j} &= (-1)^{\mu-j}\{a_j^+\theta^+(-\tau) + a_j^-\theta^-(-\tau)\}(-i\tau)^{\mu-j} \\ &= \{a_j^+\theta^+(-\tau) + a_j^-\theta^-(-\tau)\}(i\tau)^{\mu-j}\end{aligned}$$

for all $\tau \neq 0$, which implies $a_j^+(\theta^+(\tau) - \theta^+(-\tau)) = a_j^-(\theta^-(-\tau) - \theta^-(\tau))$. For $\tau > 0$ we have $\theta^+(-\tau) = \theta^-(\tau) = 0$ and $\theta^+(\tau) = \theta^-(-\tau) = 1$ which yields $a_j^+ = a_j^-$.

Remark 3.4. The space $S^{\mu}_{\mathrm{cl}}(\mathbb{R})$ is a nuclear Fréchet space in a natural way, and $S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ is a closed subspace in the induced topology.

Example. 1. Every polynomial in τ has the transmission property;
2. the τ-wise product of two symbols with the transmission property has again the transmission property;
3. If $a \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ and $a^{+}_{0} = a^{-}_{0} \neq 0$, then it follows that $\chi(\tau)a^{-1}(\tau) \in S^{-\mu}_{\mathrm{tr}}(\mathbb{R})$ for a suitable excision function $\chi(\tau)$. If in addition $a(\tau) \neq 0$ for all $\tau \in \mathbb{R}$, then $a^{-1}(\tau) \in S^{-\mu}_{\mathrm{tr}}(\mathbb{R})$.

In particular, the symbols (3.1) for $\mu \in \mathbb{Z}$ have the transmission property.

Remark 3.5. The multiplication of symbols by $l^{-\mu}_{+}(\tau)$ (or $l^{-\mu}_{-}(\tau)$) induces an isomorphism
$$S^{\mu}_{\mathrm{tr}}(\mathbb{R}) \to S^{0}_{\mathrm{tr}}(\mathbb{R}).$$

Remark 3.6. Let $a(\tau) \in S^{0}_{\mathrm{cl}}(\mathbb{R})$, and form the bounded set $L(a) := \{a(\tau) \in \mathbb{C} : \tau \in \mathbb{R}\}$ which is a smooth curve (with admitted self-intersections) and end points $a^{\pm}_{0} = a(\pm\infty)$. Then we have $a(\tau) \in S^{0}_{\mathrm{tr}}(\mathbb{R})$ if and only if $L(a)$ is a closed curve which is smooth including $a^{+}_{0} = a^{-}_{0}$.

Remark 3.7. Every symbol $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ can be written in the form
$$a(\tau) = p(\tau) + b(\tau)$$
where $p(\tau)$ is a polynomial in τ of order μ (only relevant for $\mu \geq 0$) and $b(\tau) \in S^{-1}_{\mathrm{tr}}(\mathbb{R})$.

In fact, this is an evident consequence of Definition 3.2.

Proposition 3.8. *Let $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$; then for every $N \in \mathbb{N}$ there is a minus-symbol $m_N(\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R})$ and a plus-symbol $p_N(\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R})$ such that $a(\tau) - m_N(\tau) \in S^{-(N+1)}_{\mathrm{cl}}(\mathbb{R})$ and $a(\tau) - p_N(\tau) \in S^{-(N+1)}_{\mathrm{cl}}(\mathbb{R})$.*

Proof. Since a polynomial in τ is a plus- and a minus-symbol it suffices to assume $\mu = -1$. By definition there are constants a_j such that for any fixed excision function $\chi(\tau)$
$$a(\tau) = \chi(\tau)\sum_{j=1}^{N} a_j(i\tau)^{-j} + r_N(\tau) \tag{3.13}$$
where $r_N(\tau) \in S^{-(N+1)}_{\mathrm{cl}}(\mathbb{R})$. The relation $\frac{1}{i\tau} = -\frac{1}{1-i\tau} + \frac{1}{i\tau}\frac{1}{1-i\tau}$ can be iterated, and we obtain $\frac{1}{i\tau} = -\frac{1}{1-i\tau} + \{-\frac{1}{1-i\tau} + \frac{1}{i\tau}\frac{1}{1-i\tau}\}\frac{1}{1-i\tau} = -\frac{1}{1-i\tau} - \frac{1}{(1-i\tau)^2} + \frac{1}{i\tau}\frac{1}{(1-i\tau)^2} = \ldots = -\sum_{k=1}^{N}\frac{1}{(1-i\tau)^k} + \frac{1}{i\tau}\frac{1}{(1-i\tau)^N}$. This yields $(i\tau)^{-j} = \left(-\sum_{k=1}^{N}(1-i\tau)^{-k}\right)^j + r_{j,N}(\tau)$ for every $j \in \mathbb{N}\setminus\{0\}$ where $\chi(\tau)r_{j,N}(\tau) \in S^{-(N+1)}_{\mathrm{cl}}(\mathbb{R})$ for every excision function $\chi(\tau)$. Thus, setting $m_{j,N}(\tau) := a_j\left(-\sum_{k=1}^{N}(1-i\tau)^{-k}\right)^j$ we obtain
$$\chi(\tau)a_j(i\tau)^{-j} = m_{j,N}(\tau) + \chi(\tau)a_j r_{j,N}(\tau) \tag{3.14}$$

modulo a symbol in $S^{-\infty}(\mathbb{R})$, where $m_{j,N}(\tau)$ is a minus-symbol, cf. Remark 3.1. Then from (3.13) we obtain the first assertion, for $m_N(\tau) = \sum_{j=1}^{N} m_{j,N}(\tau)$. Moreover, writing $\frac{1}{i\tau} = \frac{1}{1+i\tau} + \frac{1}{i\tau}\frac{1}{1+i\tau} = \sum_{k=1}^{N} \frac{1}{(1+i\tau)^k} + \frac{1}{i\tau}\frac{1}{(1+i\tau)^N}$ we obtain a plus-symbol $p_N(\tau) := \sum_{j=1}^{N} p_{j,N}(\tau)$, $p_{j,N}(\tau) := a_j \big(\sum_{k=1}^{N}(1+i\tau)^{-k}\big)^j$ with the desired property. □

Corollary 3.9. *Let $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$; then $\mathrm{op}^+(a)$ induces a continuous operator*

$$\mathrm{op}^+(a) : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$$

for every real $s > -\frac{1}{2}$.

Proof. Let us write $\mathrm{op}(a) = \mathrm{op}(m_N) + \mathrm{op}(c_N)$ where, according to Proposition 3.8, m_N is a minus-symbol of order μ, and $c_N \in S_{\mathrm{cl}}^{-(N+1)}(\mathbb{R})$. Then we have

$$\mathrm{op}^+(a) = \mathrm{op}^+(m_N) + \mathrm{op}^+(c_N). \tag{3.15}$$

We observed before that $\mathrm{op}^+(m_N)$ has the desired mapping property. Let us now assume $s \in (-\frac{1}{2}, 0]$. We employ the known fact that for those s we have $\mathrm{e}^+ H^s(\mathbb{R}_+) = H^s_0(\mathbb{R}_+)$. As noted before we have an isomorphism

$$\mathrm{op}(l^s_+) : H^s_0(\mathbb{R}_+) \to L^2(\mathbb{R}_+) = H^0_0(\mathbb{R}_+)$$

with the inverse $\mathrm{op}^+(l_+^{-s})$. Moreover, using the relation (3.7) we have

$$\mathrm{op}^+(c_N) = \mathrm{op}^+(c_N l_+^{-s})\mathrm{op}^+(l^s_+)$$

where $(c_N l_+^{-s})(\tau) \in S_{\mathrm{cl}}^{-(N+1)-s}(\mathbb{R})$. Thus it remains to verify that $\mathrm{op}^+(c_N l_+^{-s}) : L^2(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$ is continuous. However, when N is large enough, we have the continuity

$$\mathrm{op}^+(c_N l_+^{-s}) : L^2(\mathbb{R}_+) \to H^{N+1+s}(\mathbb{R}_+).$$

Thus for N so large that $N + 1 \geq -\mu$ we obviously obtain the desired continuity. Finally for $s \geq 0$ it suffices to employ the continuous embedding $\mathrm{e}^+ H^s(\mathbb{R}_+) \hookrightarrow L^2(\mathbb{R})$, i.e., we can argue similarly as before and obtain the continuity $\mathrm{op}^+(c_N) : L^2(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$ for $N + 1 > s - \mu$. □

Proposition 3.10. *Every symbol $a(\tau) \in S^{-1}_{\mathrm{tr}}(\mathbb{R})$ can be written in the form*

$$a(\tau) = a_+(\tau) + a_-(\tau) \tag{3.16}$$

for uniquely determined

$$a_+(\tau) \in F_{t\to\tau}\big(\mathrm{e}^+\mathcal{S}(\overline{\mathbb{R}}_+)\big), \quad a_-(\tau) \in F_{t\to\tau}\big(\mathrm{e}^-\mathcal{S}(\overline{\mathbb{R}}_-)\big)$$

which are plus/minus symbols in $S^{-1}_{\mathrm{tr}}(\mathbb{R})$.

Concerning a proof of Proposition 3.10, see [15, Section 2.1.1.1].

Corollary 3.11. *Let $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$; then $\mathrm{op}^+(a)$ induces a continuous operator*

$$\mathrm{op}^+(a) : \mathcal{S}(\overline{\mathbb{R}}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+).$$

Proof. For $\mu \in \mathbb{N}$ the symbol $a(\tau)$ is equal to a polynomial in τ of order μ, modulo a symbol in $S^{-1}_{\mathrm{tr}}(\mathbb{R})$. Thus without loss of generality we assume $\mu = -1$. The Fourier transform $F = F_{t\to\tau}$ induces a continuous operator

$$F : \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) \to \mathcal{S}^{-1}_{\mathrm{tr}}(\mathbb{R}). \tag{3.17}$$

Moreover, the multiplication between symbols with the transmission property is bilinear continuous. In particular, the composition of (3.17) with the multiplication by the symbol (3.16) gives us a continuous operator

$$a(\tau)F : \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) \to S^{-1}_{\mathrm{tr}}(\mathbb{R}).$$

Finally $F^{-1} : S^{-1}_{\mathrm{tr}}(\mathbb{R}) \to \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) + \mathrm{e}^{-}\mathcal{S}(\overline{\mathbb{R}}_{-})$ is a topological isomorphism (the sum on the right-hand side is direct) and

$$\mathrm{r}^{+} : \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) + \mathrm{e}^{-}\mathcal{S}(\overline{\mathbb{R}}_{-}) \to \mathcal{S}(\overline{\mathbb{R}}_{+})$$

is obviously continuous. Thus $\mathrm{op}^{+}(a) = \mathrm{r}^{+}F^{-1}a(\tau)F\mathrm{e}^{+}$ is a composition of continuous operators. □

Proposition 3.12. *Let $a(\tau) \in S^{0}_{\mathrm{tr}}(\mathbb{R})$; then the adjoint of*

$$\mathrm{op}^{+}(a) : L^2(\mathbb{R}_{+}) \to L^2(\mathbb{R}_{+})$$

with respect to the $L^2(\mathbb{R}_{+})$-scalar product has the form $\mathrm{op}^{+}(\overline{a})$ for the complex conjugate $\overline{a}(\tau) \in S^{0}_{\mathrm{tr}}(\mathbb{R})$.

Proof. The computation is completely elementary. □

Proposition 3.13. *Let $a(\tau) \in S^{0}_{\mathrm{cl}}(\mathbb{R})$ be a symbol with the transmission property, let $\varepsilon : \mathbb{R}_{\pm} \to \mathbb{R}_{\mp}$ be defined by $\varepsilon(t) = -t$, and $\varepsilon^{*} : L^2(\mathbb{R}_{\pm}) \to L^2(\mathbb{R}_{\mp})$ the corresponding function pull back. Then*

$$\mathrm{r}^{+}\mathrm{op}(a)\mathrm{e}^{-}\varepsilon^{*},\ \varepsilon^{*}\mathrm{r}^{-}\mathrm{op}(a)\mathrm{e}^{+} : L^2(\mathbb{R}_{+}) \to L^2(\mathbb{R}_{+}) \tag{3.18}$$

induce continuous operators $L^2(\mathbb{R}_{+}) \to \mathcal{S}(\overline{\mathbb{R}}_{+})$.

Proof. If $a(\tau)$ is a constant, both operators are zero. Therefore, it suffices to assume $a(\tau) \in S^{-1}_{\mathrm{cl}}(\mathbb{R})$. By virtue of the identity,

$$\mathrm{r}^{+}\mathrm{op}(a) = \mathrm{op}^{+}((l^{N}_{-})(l^{-N}_{-}))\mathrm{r}^{+}\mathrm{op}(a) = \mathrm{op}^{+}(l^{N}_{-})\mathrm{r}^{+}\mathrm{op}(l^{-N}_{-}a)$$

for any $N \in \mathbb{Z}$ (cf. the relation (3.6) taking into account that $l^{\pm N}_{-}(\tau)$ are minus-symbols); we may even consider the symbol $l^{-N}_{-}(\tau)a(\tau) \in S^{-(N+1)}_{\mathrm{cl}}(\mathbb{R})$ rather than $a(\tau)$, for any $N > 1$, since $\mathrm{op}^{+}(l^{N}_{-}) : \mathcal{S}(\overline{\mathbb{R}}_{+}) \to \mathcal{S}(\overline{\mathbb{R}}_{+})$ is continuous, cf. Corollary 3.11. In other words, let $a(\tau) \in S^{-2}_{\mathrm{cl}}(\mathbb{R})$; then

$$\begin{aligned}\mathrm{r}^{+}\mathrm{op}(a)\mathrm{e}^{-}\varepsilon^{*}v(t) &= \mathrm{r}^{+}\int_{\mathbb{R}}\int_{0}^{\infty} e^{i(t+t')\tau}a(\tau)v(-t')dt'đ\tau \\ &= \mathrm{r}^{+}\int_{0}^{\infty}\Big\{\int e^{i(t+t')\tau}a(\tau)đ\tau\Big\}v(-t')dt'.\end{aligned}$$

By virtue of Proposition 3.10 we have

$$\int e^{ir\tau}a(\tau)đ\tau \in \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+}) + \mathrm{e}^{-}\mathcal{S}(\overline{\mathbb{R}}_{-})$$

with $r \in \mathbb{R}$ being the variable on the right-hand side. Since r has the meaning of $t + t'$ for $t > 0$, $t' > 0$, we obtain

$$\mathrm{r}^{+}\mathrm{op}(a)\mathrm{e}^{-}\varepsilon^{*}v(t) = \mathrm{r}^{+}\int_{0}^{\infty} f(t+t')v(-t')dt' \tag{3.19}$$

for some $f(r) \in \mathrm{e}^{+}\mathcal{S}(\overline{\mathbb{R}}_{+})$. It remains to observe that the right-hand side of (3.19) represents a continuous operator $L^2(\mathbb{R}_+) \to \mathcal{S}(\overline{\mathbb{R}}_{+})$. The second operator in (3.18) can be treated in an analogous manner. □

Corollary 3.14. *Let g denote one of the operators in* (3.18)*, and let g^* be its adjoint in $L^2(\mathbb{R}_+)$. Then g and g^* induce continuous operators*

$$g, g^{*} : L^2(\mathbb{R}_+) \to \mathcal{S}(\overline{\mathbb{R}}_{+}). \tag{3.20}$$

Proof. The assertion for g is contained in Proposition 3.13. Moreover, because of $(\mathrm{r}^{+}\mathrm{op}(a)\mathrm{e}^{-}\varepsilon^{*})^{*} = \varepsilon^{*}\mathrm{r}^{-}\mathrm{op}(\overline{a})\mathrm{e}^{+}$ by Proposition 3.13 we also obtain the result for g^*. □

Remark 3.15. It can be proved that an operator $g \in \mathcal{L}(L^2(\mathbb{R}_+))$ that defines continuous operators (3.20) can be represented in the form

$$gu(t) = \int_{0}^{\infty} c(t,t')u(t')dt'$$

for some $c(t,t') \in \mathcal{S}(\overline{\mathbb{R}}_{+} \times \overline{\mathbb{R}}_{+})$ $(= \mathcal{S}(\mathbb{R}\times\mathbb{R})\big|_{\overline{\mathbb{R}}_{+}\times\overline{\mathbb{R}}_{+}})$, see [10, Theorem 2.4.87].

Definition 3.16.
1. An operator $g \in \mathcal{L}(L^2(\mathbb{R}_+))$ which induces continuous operators (3.20) is called a Green operator of type 0. Let $\Gamma^0(\mathbb{R}_+)$ denote the space of those operators.
2. An operator of the form $\sum_{j=0}^{d} g_j\partial_t^j$ for $g_j \in \Gamma^0(\mathbb{R}_+)$, $d \in \mathbb{N}$, is called a Green operator of type d. Let $\Gamma^d(\mathbb{R}_+)$ denote the space of those operators.

Remark 3.17. Any $g \in \Gamma^d(\mathbb{R}_+)$ induces a compact operator

$$g : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$$

for every $s \in \mathbb{R}$, $s > d - \frac{1}{2}$. Moreover g induces a continuous operator

$$g : H^s(\mathbb{R}_+) \to \mathcal{S}(\overline{\mathbb{R}}_{+}) \tag{3.21}$$

for those s.

In fact, $\partial_t^j : H^s(\mathbb{R}_+) \to H^{s-j}(\mathbb{R}_+)$ is continuous for every $j \in \mathbb{N}$ as well as $g_0 : H^{s-j}(\mathbb{R}_+) \to \mathcal{S}(\overline{\mathbb{R}}_{+})$ when $s - j > -\frac{1}{2}$, $g_0 \in \Gamma^0(\mathbb{R}_+)$.

Lemma 3.18. *Let $g \in \Gamma^0(\mathbb{R}_+)$, and let $1 + g : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ be an invertible operator. Then there is an $h \in \Gamma^0(\mathbb{R}_+)$ such that $(1+g)^{-1} = 1 + h$.*

Proof. Let $b := (1+g)^{-1}$ which belongs to $\mathcal{L}(L^2(\mathbb{R}_+))$. Writing $b = 1 + h$ for $h := b-1$ we obtain $(1+g)(1+h) = 1$, i.e., $h+g+gh = 0$. This yields $h = -g(1+h)$, and hence $h : L^2(\mathbb{R}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$ is continuous, cf. Definition 3.16 (i). Moreover $(1+g^*)(1+h^*) = 1$ yields $g^* + h^* + g^*h^* = 0$, i.e. $h^* = -g^*(1+h^*)$ which shows again the continuity $h^*\colon L^2(\mathbb{R}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$. In other words, $h \in \Gamma^0(\mathbb{R}_+)$. □

Corollary 3.19. *Let $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R}), b(\tau) \in S^{\nu}_{\mathrm{tr}}(\mathbb{R})$; then*

$$\mathrm{op}^+(a)\mathrm{op}^+(b) = \mathrm{op}^+(ab) + g \tag{3.22}$$

for some $g \in \Gamma^0(\mathbb{R}_+)$.

Proof. For $\mu = \nu = 0$ we have

$$\mathrm{op}^+(a)\mathrm{op}^+(b) = \mathrm{r}^+\mathrm{op}(a)\mathrm{e}^+\mathrm{r}^+\mathrm{op}(b)\mathrm{e}^+ = \mathrm{r}^+\mathrm{op}(a)\mathrm{op}(b)\mathrm{e}^+ + \mathrm{r}^+\mathrm{op}(a)\vartheta_-\mathrm{op}(b)\mathrm{e}^+$$

for the characteristic function ϑ_- of $\mathbb{R}_-$. Since

$$\mathrm{r}^+\mathrm{op}(a)\vartheta_-\mathrm{op}(b)\mathrm{e}^+ = (\mathrm{r}^+\mathrm{op}(a)\mathrm{e}^-\varepsilon^*)(\varepsilon^*\mathrm{r}^-\mathrm{op}(b)\mathrm{e}^+) =: g$$

and the factors in the middle are Green operators of type zero, cf. Proposition 3.13, we obtain $g \in \Gamma^0(\mathbb{R}_+)$, since $\Gamma^0(\mathbb{R}_+)$ is closed under compositions.

It remains to consider $\mu \in \mathbb{N}$ or $\nu \in \mathbb{N}$. In this case we write

$$a(\tau) = a_0(\tau) + p(\tau),\ b(\tau) = b_0(\tau) + q(\tau)$$

for $a_0, b_0 \in S^{-1}_{\mathrm{tr}}(\mathbb{R})$ and polynomials p and q of degree μ and ν, respectively. Since polynomials are minus- and plus-symbols at the same time we have

$$\mathrm{op}^+(p)\mathrm{op}^+(b_0) = \mathrm{op}^+(pb_0),\ \mathrm{op}^+(a_0)\mathrm{op}^+(q) = \mathrm{op}^+(a_0q),$$

i.e., when we define g by $\mathrm{op}^+(a_0)\mathrm{op}^+(b_0) = \mathrm{op}^+(a_0b_0) + g$ (according to the first part of the proof) we obtain

$$\mathrm{op}^+(a)\mathrm{op}^+(b) = (\mathrm{op}^+(a_0)+\mathrm{op}^+(p))(\mathrm{op}^+(b_0)+\mathrm{op}^+(q)) = \mathrm{op}^+((a_0+p)(b_0+q))+g.$$

□

More generally we have the following composition property.

Theorem 3.20. *Let $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$, $b(\tau) \in S^{\nu}_{\mathrm{tr}}(\mathbb{R})$, and $g \in \Gamma^d(\mathbb{R}_+)$, $h \in \Gamma^e(\mathbb{R}_+)$. Then*

$$(\mathrm{op}^+(a) + g)(\mathrm{op}^+(b) + h) = \mathrm{op}^+(ab) + k$$

for a certain $k \in \Gamma^{\max\{\nu+d,e\}}(\mathbb{R}_+)$.

Proof. By virtue of Corollary 3.19 it remains to discuss the compositions

$$\mathrm{op}^+(a)h,\ g\mathrm{op}^+(b),\ \text{and } gh.$$

It is evident that $\mathrm{op}^+(a)h \in \Gamma^e(\mathbb{R}_+)$ and $gh \in \Gamma^e(\mathbb{R}_+)$. For the operator in the middle we write

$$g\mathrm{op}^+(b) = g(\mathrm{op}^+(b_0) + \mathrm{op}^+(p))$$

where $b_0 \in S_{\mathrm{tr}}^{-1}(\mathbb{R})$ and p is a polynomial in τ of order ν (which vanishes for $\nu \leq -1$). It is clear that $g\mathrm{op}^+(p) \in \Gamma^{\nu+d}(\mathbb{R}_+)$. What concerns $g\mathrm{op}^+(b_0)$ it suffices to assume $g = g_0 D_t^j$ for any $0 \leq j \leq d$. Since τ^j is a minus-symbol we have

$$g\mathrm{op}^+(b_0) = g_0\mathrm{op}^+(b_j) \text{ for } b_j(\tau) = \tau^j b_0(\tau) \in S_{\mathrm{tr}}^{j-1}(\mathbb{R}).$$

Thus, writing $b_j(\tau) = c_j(\tau) + q_{j-1}(\tau)$ for a polynomial $q_{j-1}(\tau)$ in τ of degree $j-1$ (when $j-1 \geq 0$) and some $c_j \in S_{\mathrm{tr}}^{-1}(\mathbb{R})$ it follows that

$$g\mathrm{op}^+(b_0) = g_0\mathrm{op}^+(c_j) + g_0\mathrm{op}^+(q_{j-1}).$$

The second summand on the right obviously belongs to $\Gamma^{j-1}(\mathbb{R}_+)$ for $j \geq 1$ while the first one belongs to $\Gamma^0(\mathbb{R}_+)$ which follows from the continuity $\mathrm{op}^+(c_j) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ and $g_0 : L^2(\mathbb{R}_+) \to \mathcal{S}(\overline{\mathbb{R}}_+)$ and an analogous conclusion for the adjoints. □

Remark 3.21. As a special case of Theorem 3.20 for $a(\tau) \in S_{\mathrm{tr}}^{\mu}(\mathbb{R})$, $g \in \Gamma^d(\mathbb{R}_+)$, we obtain that $(\mathrm{op}^+(a) + g)\mathrm{op}^+(l_-^{-N}) = \mathrm{op}^+(al_-^{-N}) + k$ for $k \in \Gamma^0(\mathbb{R}_+)$ when $-N + d \leq 0$.

Let us now turn to 2×2 block matrices of operators with upper left corners of the form

$$\begin{pmatrix} \mathrm{op}^+(a) + g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix} \tag{3.23}$$

for arbitrary $a(\tau) \in S_{\mathrm{tr}}^{\mu}(\mathbb{R})$, $\mu \in \mathbb{Z}$, $g_{11} \in \Gamma^d(\mathbb{R}_+)$, $d \in \mathbb{N}$, $s > d - \frac{1}{2}$, $g_{22} \in \mathbb{C}$,

$$g_{21}u(t) = \sum_{l=0}^{d} g_{21,l}\partial_t^l u(t)\; u \in H^s(\mathbb{R}_+) \tag{3.24}$$

for $g_{21,l}v(t) := \int_0^\infty f_{21,l}(t)v(t)dt$, $f_{21,l} \in \mathcal{S}(\overline{\mathbb{R}}_+)$, $l = 0, \ldots, d$, and

$$g_{12}c := cf(t),\; c \in \mathbb{C}, \tag{3.25}$$

for some $f \in \mathcal{S}(\overline{\mathbb{R}}_+)$. An operator of the form (3.24) is called a trace operator of type d, and (3.25) a potential operator (for the boundary symbolic calculus of operators with the transmission property at the boundary).

In a similar manner we define analogues of (3.23) where $\mathbb{C}$ on the left is replaced by $\mathbb{C}^{j_-}$ and on the right by $\mathbb{C}^{j_+}$ for certain $j_-, j_+ \in \mathbb{N}$ (if one of the dimensions is zero, then we have row or column matrices which are admitted as well). Let $\mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; j_-, j_+)$ denote the space of such block matrices. Moreover, let $\mathcal{B}_G^d(\overline{\mathbb{R}}_+; j_-, j_+)$ be the subspace of operators (3.23) defined by $a \equiv 0$.

Thus $\mathcal{B}_G^d(\overline{\mathbb{R}}_+; 0, 0) = \Gamma^d(\mathbb{R}_+)$; in future we also write $\mathcal{B}_G^d(\overline{\mathbb{R}}_+)$ rather than $\Gamma^d(\mathbb{R}_+)$.

Remark 3.22. More generally we have $\mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; \boldsymbol{v})$ for $\boldsymbol{v} = (k, l; j_-, j_+)$, defined to be the space of 2×2 block matrices where the upper left corner itself is an $l \times k$ matrix of operators as in the upper left corner of (3.23), while g_{12} is a $j_- \times k$ matrix of potential operators, etc. For every fixed $\mu \in \mathbb{Z}$, $d \in \mathbb{N}$, the space of such matrices is a (nuclear) Fréchet space in a natural way. The future homogeneous boundary symbols of BVPs are symbols in (y,η) with values in such spaces.

Remark 3.23. It can easily be proved, cf. [22, Proposition 4.1.46], that every $g \in \Gamma^d(\mathbb{R}_+)$ for $d > 0$ has a unique representation

$$g = g_0 + \sum_{j=0}^{d-1} k_j \circ \mathrm{r}' D_t^j$$

for $g_0 \in \Gamma^0(\mathbb{R}_+)$, potential operators k_j, and $\mathrm{r}'u := u(0)$. Similarly, a trace operator b of type $d > 0$ can uniquely be written as

$$b = b_0 + \sum_{j=0}^{d-1} c_j \circ \mathrm{r}' D_t^j$$

for a trace operator b_0 of type 0 and constants c_j.

Example. The operator

$$\begin{pmatrix} \mathrm{op}^+(-|\eta|^2 - \tau^2) \\ \mathrm{r}' D_t^k \end{pmatrix} : H^s(\mathbb{R}_+) \to \begin{matrix} H^{s-2}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}, \qquad (3.26)$$

$\mathrm{r}'v = v|_{t=0}$, belongs to $\mathcal{B}^{2,k+1}(\overline{\mathbb{R}}_+; 0, 1)$, $k \in \mathbb{N}$, and (3.26) is an isomorphism for every $\eta \in \mathbb{R}^{n-1} \setminus \{0\}$, $s > \max\{\frac{3}{2}, k + \frac{1}{2}\}$. The operator family (3.26) for $k = 0$ is just the boundary symbol of the Dirichlet problem for the Laplace equation and for $k = 1$ of the Neumann problem.

The inverse of (3.26) for $k = 0$ is explicitly computed in [12, Section 3.3.4]. Setting $a(\eta,\tau) := -|\eta|^2 - |\tau|^2$ the result is

$$\begin{pmatrix} \mathrm{op}^+(a)(\eta) \\ \mathrm{r}' \end{pmatrix}^{-1} = \begin{pmatrix} -\mathrm{op}^+(l_+^{-1})(\eta)\mathrm{op}^+(l_-^{-1})(\eta) & d(\eta) \end{pmatrix}$$

for $l_\pm(\eta) := |\eta| \pm i\tau$ and a potential operator $d(\eta)$ defined by $d(\eta) : c \to ce^{-|\eta|t}$, $c \in \mathbb{C}$. By virtue of Corollary 3.19 we have

$$-\mathrm{op}^+(l_+^{-1})(\eta)\mathrm{op}^+(l_-^{-1})(\eta) = \mathrm{op}^+(a^{-1})(\eta) + g(\eta)$$

for a Green operator family $g(\eta)$ of type 0. Note that $g(\eta)$ is just the homogeneous boundary symbol of the well-known Green's function of the Dirichlet problem for the Laplacian (twisted homogeneous of order -2).

It is now easy also to compute the inverses of (3.26) for arbitrary $k \in \mathbb{N}$, especially, of the boundary symbol of the Neumann problem. In fact, similarly as (2.19), now on the level of boundary symbols, we have

$$\begin{pmatrix} \mathrm{op}^+(a)(\eta) \\ \mathrm{r}' D_t^k \end{pmatrix} \begin{pmatrix} p(\eta) & d(\eta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b(\eta) & q_k(\eta) \end{pmatrix} \qquad (3.27)$$

for $p(\eta) := \mathrm{op}^+(a^{-1})(\eta) + g(\eta)$, $b(\eta) := \mathrm{r}' D_t^k(\mathrm{op}^+(a^{-1})(\eta) + g(\eta))$, $g_k(\eta) := \mathrm{r}' D_t^k d(\eta)$. We have

$$\mathrm{r}' D_t^k d(\eta) = D_t^k e^{-|\eta| t}\big|_{t=0} = (i|\eta|)^k \tag{3.28}$$

which is just the homogeneous principal symbol of the elliptic operator $T_k K_0 \in L^k_{\mathrm{cl}}(Y)$ occurring in the lower right corner of the operator (3.27). Thus

$$\begin{aligned}\begin{pmatrix} \mathrm{op}^+(a)(\eta) \\ \mathrm{r}' D_t^k \end{pmatrix}^{-1} &= \begin{pmatrix} p(\eta) & d(\eta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b(\eta) & q_k(\eta) \end{pmatrix}^{-1} \\ &= \begin{pmatrix} p(\eta) - d(\eta) q_k^{-1}(\eta) b(\eta) & d(\eta) q^{-1}(\eta) \end{pmatrix}.\end{aligned}$$

General compositions of boundary symbols are studied in Theorem 3.26 below.

Remark 3.24. It is interesting to consider elliptic boundary value problems for the elliptic operator $T_k K_0$ on a smooth submanifold of Y with boundary Z. This makes sense, for instance, when we reduce the Zaremba problem for Δ (defined by jumping conditions from Dirichlet to Neumann along Z) to Y. Then a basic difficulty is that $T_k K_0$ fails to have the transmission property at Z, cf. Definition 4.11 below, unless k is even. Mixed problems (i.e., with jumping boundary conditions) belong to the motivation to study BVPs for operators without the transmission property. Another (possibly even stronger) motivation is the similarity between mixed and (specific) edge problems.

Theorem 3.25. *We have*

$$\boldsymbol{a} \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; j_0, j_+), \boldsymbol{b} \in \mathcal{B}^{\nu,e}(\overline{\mathbb{R}}_+; j_-, j_0) \Rightarrow \boldsymbol{ab} \in \mathcal{B}^{\mu+\nu, \max\{\nu+d, e\}}(\overline{\mathbb{R}}_+; j_-, j_+),$$

and $(\boldsymbol{a}, \boldsymbol{b}) \to \boldsymbol{ab}$ *defines a bilinear continuous map*

$$\mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; j_0, j_+) \times \mathcal{B}^{\nu,e}(\overline{\mathbb{R}}_+; j_-, j_0) \to \mathcal{B}^{\mu+\nu, \max\{\nu+d, e\}}(\overline{\mathbb{R}}_+; j_-, j_+)$$

between the respective Fréchet spaces.

Proof. The result for the composition of upper left corners is contained in Theorem 3.20. The proof for the remaining entries is straightforward and left to the reader. □

Theorem 3.26. *Let* $\boldsymbol{a} \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+; j_-, j_+)$, *and define the adjoint* $\boldsymbol{a}^*$ *by*

$$(\boldsymbol{a}u, v)_{L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_+}} = (u, \boldsymbol{a}^* v)_{L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}}$$

for all $u \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}$, $v \in L^2(\mathbb{R}_+) \oplus \mathbb{C}^{j_+}$. *Then we have* $\boldsymbol{a}^* \in \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+; j_+, j_-)$, *and* $\boldsymbol{a} \to \boldsymbol{a}^*$ *defines an (antilinear), continuous map*

$$\mathcal{B}^{0,0}(\overline{\mathbb{R}}_+; j_-, j_+) \to \mathcal{B}^{0,0}(\overline{\mathbb{R}}_+; j_+, j_-).$$

Proof. The result for the upper left corner follows from Proposition 3.12, together with Corollary 3.14. The proof for the remaining entries is straightforward and left to the reader. □

Definition 3.27. A symbol $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ is called elliptic (of order μ) if $a(\tau) \neq 0$ for all $\tau \in \mathbb{R}$, and if $a_0 (= a_0^- = a_0^+)$ does not vanish (cf. the notation in (3.8)). Moreover, we call an $\boldsymbol{a} \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; j_-, j_+)$ elliptic if the symbol $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ in the upper left corner of (3.23) is elliptic.

Theorem 3.28. *Let $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ be elliptic, and $g \in \Gamma^d(\mathbb{R}_+)$; then*

$$\boldsymbol{a} := \mathrm{op}^+(a) + g : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+) \tag{3.29}$$

is a Fredholm operator for every $s > \max\{\mu, d\} - \frac{1}{2}$, and $\boldsymbol{p} := \mathrm{op}^+(a^{-1})$ is a parametrix of $\boldsymbol{a}$.

Proof. Because of the assumption on s the operator

$$\mathrm{op}^+(a^{-1}) : H^{s-\mu}(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$$

is continuous. From Corollary 3.19 and Theorem 3.20 we have

$$\mathrm{op}^+(a^{-1})\{\mathrm{op}^+(a) + g\} = \mathrm{op}^+(aa^{-1}) + k = 1 + k \tag{3.30}$$

where $k = h + \mathrm{op}^+(a^{-1})g$ for an $h \in \Gamma^0(\mathbb{R}_+)$ and $\mathrm{op}^+(a^{-1})g \in \Gamma^d(\mathbb{R}_+)$. Thus since

$$k : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$$

is compact for $s > d - \frac{1}{2}$, cf. Remark 3.17, the operator $\mathrm{op}^+(a^{-1})$ is a left parametrix. In a similar manner we obtain that $\mathrm{op}^+(a^{-1})$ is a right parametrix. In fact, we have to compute

$$\{\mathrm{op}^+(a) + g\}\mathrm{op}^+(a^{-1}) = \mathrm{op}^+(aa^{-1}) + k = 1 + k$$

where $k = h + g\mathrm{op}^+(a^{-1})$, for a $h \in \Gamma^0(\mathbb{R}_+)$, and $g\mathrm{op}^+(a^{-1}) \in \Gamma^{\max\{-\mu+d,0\}}(\mathbb{R}_+)$.

This can be applied to functions in $H^{s-\mu}(\mathbb{R}_+)$ when s satisfies the conditions $s - \mu > -\frac{1}{2}$ and $s - \mu > \max\{-\mu + d, 0\} - \frac{1}{2}$. In the case $\max\{-\mu + d, 0\} = 0$ the latter is the same as the first condition while for $\max\{-\mu + d, 0\} = -\mu + d \geq 0$ the condition is $s - \mu > -\mu + d - \frac{1}{2}$, i.e., $s > d - \frac{1}{2}$. For s it follows altogether $s > \max\{\mu, d\} - \frac{1}{2}$, and we can apply again Remark 3.17. □

Theorem 3.29. *Let $a(\tau) \in S^{\mu}_{\mathrm{tr}}(\mathbb{R})$ be elliptic, and $g \in \Gamma^d(\mathbb{R}_+)$. Then $V := \ker(\mathrm{op}^+(a) + g)$ is a finite-dimensional subspace of $\mathcal{S}(\overline{\mathbb{R}}_+)$, and there is a finite-dimensional subspace $W \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ such that*

$$\mathrm{im}(\mathrm{op}^+(a) + g) + W = H^{s-\mu}(\mathbb{R}_+). \tag{3.31}$$

This is true for all real $s > \max\{\mu, d\} - \frac{1}{2}$ with the same spaces V and W. It follows that $\mathrm{ind}(\mathrm{op}^+(a) + g)$ *is independent of s.*

Proof. Let us set $\boldsymbol{a} := \mathrm{op}^+(a) + g$ and assume $u \in H^s(\mathbb{R}_+)$, $\boldsymbol{a}u = 0$. Then from the relation (3.30) it follows that $(1 + k)u = 0$, i.e., $u = -ku$, which implies $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$, cf. the formula (3.21). In other words, $V = \ker \boldsymbol{a} \subseteq \ker(1 + k)$ is a finite-dimensional subspace of $\mathcal{S}(\overline{\mathbb{R}}_+)$, independent of $s = \max\{\mu, d\} - \frac{1}{2}$. In the case $d = 0$, $\mu \leq 0$ we can do the same for the formal adjoint $\boldsymbol{a}^*$, and we may set $W = \ker \boldsymbol{a}^*$ which is a finite-dimensional subspace of $\mathcal{S}(\overline{\mathbb{R}}_+)$ independent of s.

To find W in general we set $\boldsymbol{a} := \mathrm{op}^+(a) + g$ which we check as an operator $\boldsymbol{a} : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$ for $s > \max\{\mu, d\} - \frac{1}{2}$. Let us set $\boldsymbol{l}_-^N := \mathrm{op}^+(l_-^N)$ for any $N \in \mathbb{Z}$. In particular, for $N := \max\{\mu, d\}$ we have

$$\boldsymbol{a} = \boldsymbol{a}\boldsymbol{l}_-^N = \boldsymbol{a}_0\boldsymbol{l}_-^N : H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+) \tag{3.32}$$

where $\boldsymbol{a}_0 = \mathrm{op}^+(al_-^{-N}) + k$ for some $k \in \Gamma^0(\mathbb{R}_+)$, i.e., $\boldsymbol{a}_0 \in \mathcal{B}^{\nu,0}(\overline{\mathbb{R}}_+)$ for $\nu = \mu - N \leq 0$. Then $\boldsymbol{a}$ can be regarded as a chain of operators

$$\boldsymbol{a} : H^s(\mathbb{R}_+) \to H^{s-N}(\mathbb{R}_+) \to H^{s-N-\nu}(\mathbb{R}_+) = H^{s-\mu}(\mathbb{R}_+)$$

where the first one, namely, $\boldsymbol{l}_-^N$ is an isomorphism where $s - N \geq -\frac{1}{2}$, and the second one $\boldsymbol{a}_0$ is elliptic of order ν. For the latter we apply the first part of the proof, i.e., we find a finite-dimensional $W \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ such that $\mathrm{im}\boldsymbol{a}_0 + W = H^{s-\mu}(\mathbb{R}_+)$. This entails $\mathrm{im}\,\boldsymbol{a} + W = H^{s-\mu}(\mathbb{R}_+)$, since $\boldsymbol{a} = \boldsymbol{a}_0\boldsymbol{l}_-^N$. □

Proposition 3.30. *Let $\boldsymbol{a} = \mathrm{op}^+(a) + g \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+)$ where $a(\tau) \in S^\mu_{\mathrm{tr}}(\mathbb{R})$ is elliptic of order μ. Moreover, let $W \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ be a finite-dimensional subspace, and $k : \mathbb{C}^j \to W$ a linear map. Then*

$$\begin{pmatrix} u \\ c \end{pmatrix} \in \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix}, \quad \boldsymbol{a}u + kc = 0 \tag{3.33}$$

for any $s > \max\{\mu, d\} - \frac{1}{2}$ implies $u \in \mathcal{S}(\overline{\mathbb{R}}_+)$, and the space of all solutions of (3.33) is a finite-dimensional subspace of $H^s(\mathbb{R}_+) \oplus \mathbb{C}^j$, independent of s.

Proof. First observe that $(\boldsymbol{a} \quad k)$ is a Fredholm operator

$$(\boldsymbol{a} \quad k) : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix} \to H^{s-\mu}(\mathbb{R}_+). \tag{3.34}$$

Then, analogously as in the proof of Theorem 3.29 we pass to the operator

$$\begin{pmatrix} \boldsymbol{p} \\ 0 \end{pmatrix}(\boldsymbol{a} \quad k) = \begin{pmatrix} \boldsymbol{pa} & \boldsymbol{p}k \\ 0 & 0 \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix} \to \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix}$$

for $\boldsymbol{p} := \mathrm{op}^+(a^{-1})$. The composition $l := \boldsymbol{p}k$ is a potential operator, and we have $\boldsymbol{pa} = 1 + h$ for an operator $h \in \Gamma^d(\mathbb{R}_+)$. The kernel of (3.34) is contained in the kernel of $(1 + h \quad l)$. The kernel of $(1 + h \quad l)$ consists of all ${}^{\mathrm{t}}(u \quad c)$ such that $(1 + h)u + lc = 0$, i.e., $u = -hu + lc \in \mathcal{S}(\overline{\mathbb{R}}_+)$. □

Proposition 3.31. *Let $\mathrm{op}^+(a) + g \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+)$ be elliptic of order μ. Then there exists a 2×2 block matrix operator*

$$\boldsymbol{a} = \begin{pmatrix} \mathrm{op}^+(a) + g & k \\ b & q \end{pmatrix} : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix} \tag{3.35}$$

for a trace operator b, a potential operator k and a $j_+ \times j_-$ matrix q, such that (3.35) *is an isomorphism for all $s > \max\{\mu, d\} - \frac{1}{2}$, and we have*

$$\operatorname{ind}(\operatorname{op}^+(a) + g) = \operatorname{ind}\operatorname{op}^+(a) = j_+ - j_-. \tag{3.36}$$

The operator (3.35) *is an isomorphism if and only if*

$$\boldsymbol{a} : \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \to \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix} \tag{3.37}$$

is an isomorphism.

Proof. Applying Theorem 3.29 we find a finite-dimensional subspace $W \subset \mathcal{S}(\overline{\mathbb{R}}_+)$ such that (3.31) holds for all $s > \max\{\mu, d\} - \frac{1}{2}$. Choose any $j_- \in \mathbb{N}$, $j_- \geq \dim W$, and a linear surjective map $k : \mathbb{C}^j \to W$. Then

$$(\operatorname{op}^+(a) + g \quad k) : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \to H^{s-\mu}(\mathbb{R}_+)$$

is obvisously surjective for all s. By virtue of Proposition 3.30 its kernel V is a subspace of ${}^{\mathrm{t}}(\mathcal{S}(\overline{\mathbb{R}}_+) \oplus \mathbb{C}^{j_-})$ of finite dimension j_+. Choosing an isomorphism

$$(b \quad q) : V \to \mathbb{C}^{j_+}$$

it suffices to extend b to a trace operator $b : H^s(\mathbb{R}_+) \to \mathbb{C}^{j_+}$ (for simplicity denoted by the same letter). Then, according to Lemma 2.3 we obtain an isomorphism (3.35). □

Theorem 3.32. *Let $\boldsymbol{a} \in \mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; j_-, j_+)$ be given as in* (3.35)*, let the upper left corner be elliptic in the sense of Definition* 3.27*, and assume that $\boldsymbol{a}$ defines an isomorphism* (3.37)*. Then we have $\boldsymbol{a}^{-1} \in \mathcal{B}^{-\mu,(d-\mu)^+}(\overline{\mathbb{R}}_+; j_+, j_-)$ where $\nu^+ := \max\{\nu, 0\}$.*

Proof. By virtue of Theorem 3.28 the operator (3.29) is Fredholm where $\operatorname{op}^+(a^{-1})$ is a parametrix. According to Proposition 3.31 there is a 2×2 block matrix isomorphism of the form

$$\boldsymbol{p} := \begin{pmatrix} \operatorname{op}^+(a^{-1}) & h \\ c & r \end{pmatrix} : \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{g_+} \end{matrix} \to \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{g_-} \end{matrix}$$

for a suitable trace operator c of type 0 and a potential operator h. Since $\operatorname{op}^+(a^{-1})$ is a parametrix of $\operatorname{op}^+(a)$, cf. Theorem 3.28, we have $\operatorname{ind}\operatorname{op}^+(a^{-1}) = -\operatorname{ind}\operatorname{op}^+(a)$ $= j_- - j_+$ and from (3.36)

$$\operatorname{ind}\operatorname{op}^+(a^{-1}) = g_- - g_+ = j_- - j_+.$$

In the case $N := g_- - j_- \in \mathbb{N}$ which implies $g_+ - j_+ = N$ we pass from $\boldsymbol{a}$ to $\boldsymbol{a} \oplus \mathrm{id}_{\mathbb{C}^N}$ which is again an isomorphism with (j_-, j_+) replaced by (g_-, g_+). On the other hand, when $N := j_- - g_- \in \mathbb{N}$ where $j_+ - g_+ = N$, from $\boldsymbol{p}$ we pass to $\boldsymbol{p} \oplus \mathrm{id}_{\mathbb{C}^N}$ which is an isomorphism with (g_-, g_+) replaced by (j_-, j_+). In any case, to find $\boldsymbol{a}^{-1}$ it suffices to assume that $j_- = g_-$, $j_+ = g_+$. Now the composition $\boldsymbol{ap}$ is of the form

$$\boldsymbol{ap} = \begin{pmatrix} 1 + g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} : \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix} \to \begin{matrix} \mathcal{S}(\overline{\mathbb{R}}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix}$$

for a $\boldsymbol{g} = (g_{ij})_{i,j=1,2} \in \mathcal{B}_G^{(d-\mu)^+}(\overline{\mathbb{R}}_+; j_+, j_+)$. By virtue of Lemma 3.33 below we have

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{g} \right)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{l} \tag{3.38}$$

for an $\boldsymbol{l} \in \mathcal{B}_G^{(d-\mu)^+}(\overline{\mathbb{R}}_+; j_+, j_+)$. Then Theorem 3.25 gives us

$$\boldsymbol{a}^{-1} = \boldsymbol{p} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{l} \right) \in \mathcal{B}^{-\mu,(d-\mu)^+}(\overline{\mathbb{R}}_+; j_+, j_-). \qquad \square$$

Lemma 3.33. *Let $\boldsymbol{g} \in \mathcal{B}_G^d(\overline{\mathbb{R}}_+; j, j)$, and assume that*

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{g} : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix} \to \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^j \end{matrix} \tag{3.39}$$

is invertible for any $s > d - \frac{1}{2}$. Then the inverse of (3.39) *has the form* $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \boldsymbol{l}$ *for some $\boldsymbol{l} \in \mathcal{B}_G^d(\overline{\mathbb{R}}_+; j, j)$.*

Proof. For convenience we set $\boldsymbol{g} = \begin{pmatrix} G & K \\ T & Q \end{pmatrix}$. Then, in particular, Q is a $j \times j$ matrix. Since isomorphisms in a Hilbert space form an open set, a small perturbation of Q allows us to pass to an invertible operator $\begin{pmatrix} 1+G & K \\ T & R \end{pmatrix}$ where R is an invertible $j \times j$ matrix. Assume that we have computed $\begin{pmatrix} 1+G & K \\ T & R \end{pmatrix}^{-1}$. Then we have $\begin{pmatrix} 1+G & K \\ T & R \end{pmatrix}^{-1} \begin{pmatrix} 1+G & K \\ T & Q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ D & J \end{pmatrix}$ which is again invertible; this entails the invertibility of J. We obtain

$$\begin{pmatrix} 1+G & K \\ T & Q \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -J^{-1}D & J^{-1} \end{pmatrix} \begin{pmatrix} 1+G & K \\ T & R \end{pmatrix}^{-1}. \tag{3.40}$$

Thus it remains to characterise the second factor on the right of (3.40). The identity

$$\begin{pmatrix} 1 & -KR^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+G & K \\ T & R \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -R^{-1}T & R^{-1} \end{pmatrix} = \begin{pmatrix} 1+C & 0 \\ 0 & 1 \end{pmatrix}$$

for $C := G - KR^{-1}T$ shows that the operator $1 + C$ is invertible, and it follows that

$$\begin{pmatrix} 1+G & K \\ T & R \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -R^{-1}T & R^{-1} \end{pmatrix} \begin{pmatrix} (1+C)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -KR^{-1} \\ 0 & 1 \end{pmatrix}.$$

This reduces the task to the computation of $(1 + C)^{-1}$.

The operator $C \in \mathcal{B}^d_G(\overline{\mathbb{R}}_+)$ can be written in the form $C = C_0 + \sum_{j=0}^{d-1} K_j T_j$ for a $C_0 \in \mathcal{B}^0_G(\overline{\mathbb{R}}_+)$, potential operators K_j and trace operators $T_j := \mathrm{r}' D_t^j$, cf. Remark 3.23. Since C_0 is compact in Sobolev spaces, we have $\operatorname{ind}(1 + C_0) = 0$. Because of the nature of $V := \ker(1+C_0)$ and $W = \operatorname{coker}(1+C_0)$ (which are of the same dimension l) there is a trace operator B of type 0 and a potential operator D which induces isomorphisms

$$B = {}^{\mathrm{t}}(B_1, \dots, B_l) : V \to \mathbb{C}^l, \quad D = (D_1, \dots, D_l) : \mathbb{C}^l \to W$$

such that

$$1 + C_0 + DB : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+)$$

is an isomorphism. Note that $C_1 := C_0 + DB \in \mathcal{B}^0_G(\overline{\mathbb{R}}_+)$. We obtain

$$1 + C = 1 + C_1 - \sum_{k=1}^{d+l} D_k B_k \tag{3.41}$$

for $D_{l+j+1} = -K_j$, $B_{l+j+1} = T_j$ for $j = 0, \dots, d-1$. Now we employ the fact that there is a $C_2 \in \mathcal{B}^0_G(\overline{\mathbb{R}}_+)$ such that $1 + C_2 = (1 + C_1)^{-1}$, say, as an operator $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, cf. Lemma 3.18. In order to characterise $(1 + C)^{-1}$ we form

$$(1 + C_2)(1 + C) = 1 + (1 + C_2) \sum_{k=1}^{d+l} D_k B_k = 1 + \sum_{k=1}^{d+l} M_k B_k = 1 + \mathcal{M}\mathcal{B}$$

for $M_k = (1 + C_2)D_k$, $\mathcal{M} := (M_1, \dots, M_{d+l})$, $\mathcal{B} := {}^{\mathrm{t}}(B_1, \dots, B_{d+l})$. This reduces the task to invert the operator $1 + C$ to the inversion of

$$1 + \mathcal{M}\mathcal{B} : H^s(\mathbb{R}_+) \to H^s(\mathbb{R}_+).$$

With the operators $\mathcal{M}$ and $\mathcal{B}$ we can also associate the operator

$$1 + \mathcal{B}\mathcal{M} : \mathbb{C}^{l+d} \to \mathbb{C}^{l+d}, \; 1 := \mathrm{id}_{\mathbb{C}^{l+d}}.$$

Now we verify that $1 + \mathcal{M}\mathcal{B}$ is invertible if and only if $1 + \mathcal{B}\mathcal{M}$ is invertible. In fact, setting

$$\mathfrak{M} := \begin{pmatrix} 1 & \mathcal{M} \\ 0 & 1 \end{pmatrix}, \mathfrak{B} := \begin{pmatrix} 1 & 0 \\ -\mathcal{B} & 1 \end{pmatrix}, \mathfrak{F} := \begin{pmatrix} 1 & -\mathcal{M} \\ \mathcal{B} & 1 \end{pmatrix},$$

it follows that

$$\mathfrak{M}\mathfrak{F}\mathfrak{B} = \begin{pmatrix} 1 + \mathcal{M}\mathcal{B} & 0 \\ 0 & 1 \end{pmatrix}, \mathfrak{B}\mathfrak{F}\mathfrak{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \mathcal{B}\mathcal{M} \end{pmatrix}$$

which gives us the desired equivalence. At the same time we see that

$$(1 + \mathcal{M}\mathcal{B})^{-1} = 1 - \mathcal{M}(1 + \mathcal{B}\mathcal{M})^{-1}\mathcal{B}$$

which is of the form $1 + G_1$ for a $G_1 \in \mathcal{B}_G^d(\overline{\mathbb{R}}_+)$. Thus

$$(1 + C)^{-1} = (1 + C_1)(1 + G_1) = 1 + C_1 + G_1 + C_1G_1$$

where $C_1 + G_1 + C_1G_1 \in \mathcal{B}_G^d(\overline{\mathbb{R}}_+)$. □

Remark 3.34. Theorem 3.32 easily extends to $\mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; (k, k; j_-, j_+))$ for arbitrary $k, j_-, j_+ \in \mathbb{N}$ (cf. Remark 3.22). The technique for the proof which mainly employs compositions of some operators also shows that the inverse continuously depends on the given operator $\boldsymbol{a}$.

4. Pseudo-Differential Boundary Value Problems

We develop basics on pseudo-differential BVPs with the transmission property at the boundary. Other material may be found in the author's joint monographs with Rempel [15], with Kapanadze [12], or with Harutyunyan [10], and in the monograph of Grubb [8]. The ideas here are related to the calculus on manifolds with edges. Let us first consider operators in local coordinates $x = (y, t) \in \mathbb{R}^{n-1} \times \overline{\mathbb{R}}_+$. The operator convention refers to the embedding of $\overline{\mathbb{R}}^n_+$ into the ambient space $\mathbb{R}^n$. Therefore, we first look at operators

$$\mathrm{Op}_x(p)u(x) = \iint e^{i(x-x')\xi} p(x, \xi)u(x')dx'đ\xi. \tag{4.1}$$

Here p belongs to Hörmander's symbol classes. Let $S^\mu(U \times \mathbb{R}^n)$ for $\mu \in \mathbb{R}$ and $U \subseteq \mathbb{R}^m$ open denote the set of all $p \in C^\infty(U \times \mathbb{R}^n)$ such that

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \le c\langle\xi\rangle^{\mu-|\beta|}$$

for all $(x, \xi) \in K \times \mathbb{R}^n$, $K \Subset U$, and all $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, for constants $c = c(\alpha, \beta, K) > 0$. We will freeley employ various standard properties such as asymptotic expansions, etc., developed in textbooks on pseudo-differential operators. The subspace $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^n)$ of classical symbols is defined by asymptotic expansions

$$p(x, \xi) \sim \sum_{j=0}^{\infty} \chi(\xi)p_{(\mu-j)}(x, \xi)$$

where $p_{(\mu-j)}(x, \xi) \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$, $p_{(\mu-j)}(x, \lambda\xi) = \lambda^{\mu-j}p_{(\mu-j)}(x, \xi)$ for all $\lambda \in \mathbb{R}_+$, and χ is any excision function. If some assertion is valid for the classical and the general case we also write $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n)$. Recall that the spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n)$ are Fréchet in a natural way. It is then obvious that $S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)$ (the space of x-independent elements) is closed in $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n)$, and that

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^n) = C^\infty(U, S^\mu_{(\mathrm{cl})}(\mathbb{R}^n)). \tag{4.2}$$

In order to illustrate some consequences of the presence of a boundary, here $t = 0$, we rephrase (4.1) in anisotropic form, by carrying out the action first in t and then in y. It will be not essential that y varies in $\mathbb{R}^{n-1}$; we often assume $y \in \Omega$ for an open set $\Omega \subseteq \mathbb{R}^{n-1}$. Moreover, for simplicity, we first consider a

t-independent symbol, i.e., $p(y,\eta,\tau) \in S^\mu(\Omega \times \mathbb{R}^{n-1}_\eta \times \mathbb{R}_\tau)$. We form $\mathrm{Op}_t(p)(y,\eta) : H^s(\mathbb{R}) \to H^{s-\mu}(\mathbb{R})$ as an operator family parametrised by $(y,\eta) \in \Omega \times \mathbb{R}^{n-1}$ and then

$$\mathrm{Op}_x(p) = \mathrm{Op}_y(\mathrm{Op}_t(p))$$

where $\mathrm{Op}_t(p)(y,\eta)$ is regarded as an operator-valued symbol in the variables and covariables (y,η).

In order to formulate the latter aspect in a more precise manner we fix a group $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : H^s(\mathbb{R}) \to H^s(\mathbb{R})$ by setting $(\kappa_\lambda u)(t) = \lambda^{1/2}u(\lambda t)$, $\lambda \in \mathbb{R}_+$. Then a simple computation shows the identity

$$\kappa^{-1}_{\langle\eta\rangle}\mathrm{Op}_t(p)(y,\eta)\kappa_{\langle\eta\rangle} = \mathrm{Op}_t(p_\eta)(y,\eta) \tag{4.3}$$

for

$$p_\eta(y,\eta,\tau) = p(y,\eta\langle\tau\rangle,\tau). \tag{4.4}$$

Using the symbolic estimates for p, especially, $|p(y,\eta,\tau)| \le c\langle\eta,\tau\rangle^\mu$ for all $(y,\eta,\tau) \in K \times \mathbb{R}^n$, $K \Subset \Omega$, and constants $c(K) > 0$, it follows that

$$|p(y,\eta\langle\tau\rangle,\tau)| \le c\langle\eta\rangle^\mu\langle\tau\rangle^\mu, \tag{4.5}$$

taking into account the relation $\langle\eta\langle\tau\rangle,\tau\rangle = \langle\eta\rangle\langle\tau\rangle$.

Lemma 4.1. *Under the above assumptions we have*

$$\|\kappa^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta \mathrm{Op}_t(p)(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H^s(\mathbb{R}),H^{s-\mu+|\beta|}(\mathbb{R}))} \le c\langle\eta\rangle^{\mu-|\beta|} \tag{4.6}$$

for all $(y,\eta) \in K \times \mathbb{R}^{n-1}$, $K \Subset \Omega$, *and all* $\alpha,\beta \in \mathbb{N}^{n-1}$, *and every* $s \in \mathbb{R}$, *for constants* $c = c(\alpha,\beta,K,s) > 0$.

Proof. Let first $\alpha = \beta = 0$, and set $a(y,\eta) := \mathrm{Op}_t(p)(y,\eta)$. Then the relation (4.3) together with the estimate (4.5) yields

$$\begin{aligned}\|\kappa^{-1}_{\langle\eta\rangle}a(y,\eta)\kappa_{\langle\eta\rangle}u\|^2_{H^{s-\mu}(\mathbb{R})} &= \int\langle\tau\rangle^{2(s-\mu)}|p(y,\eta\langle\tau\rangle,\tau)\hat{u}(\tau)|^2 d\tau \\ &\le \sup_{\tau\in\mathbb{R},y\in K}\langle\tau\rangle^{-2\mu}|p(y,\eta\langle\tau\rangle,\tau)|^2\int\langle\tau\rangle^{2s}|\hat{u}(\tau)|^2d\tau \le c\langle\eta\rangle^{2\mu}\|u\|^2_{H^s(\mathbb{R})}.\end{aligned}$$

This implies (4.6) for $\alpha = \beta = 0$. The assertion for arbitrary α,β follows in an analogous manner, using $D^\alpha_y D^\beta_\eta p(y,\eta,\tau) \in S^{\mu-|\beta|}(U \times \mathbb{R}^n)$. □

Remark 4.2. Lemma 4.1 remains true in analogous form under the assumption $p(y,t,\eta,\tau) \in S^\mu(\Omega \times \mathbb{R} \times \mathbb{R}^n_{\eta,\tau})$ when p is independent of t for $|t| >$ const for a constant > 0 (and also under certain weaker assumptions with respect to $|t| \to \infty$).

Definition 4.3. 1. By a group action on a Hilbert space H we understand a strongly continuous group $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ of isomorphisms $\kappa_\lambda : H \to H$, such that $\kappa_{\lambda\lambda'} = \kappa_\lambda\kappa_{\lambda'}$ for all $\lambda,\lambda' \in \mathbb{R}_+$ (strongly continuous means that $\kappa_\lambda h \in C(\mathbb{R}_+, H)$ for every $h \in H$).

2. Let H and $\widetilde{H}$ be Hilbert spaces with group actions κ and $\tilde{\kappa}$, respectively. Then $S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ for $\Omega \subseteq \mathbb{R}^p$ open, $\mu \in \mathbb{R}$, is defined to be the set of all $a(y,\eta) \in C^\infty(\Omega \times \mathbb{R}^q, \mathcal{L}(H, \widetilde{H}))$ such that
$$\|\tilde{\kappa}^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta a(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H,\widetilde{H})} \leq c\langle\eta\rangle^{\mu-|\beta|}$$
for all $(y,\eta) \in K \times \mathbb{R}^q$, $K \Subset \Omega$, and all $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, for constants $c = c(\alpha,\beta,K) > 0$.
3. The space $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ of classical elements is the set of all $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$ such that there are functions $a_{(\mu-j)}(y,\eta) \in C^\infty(\Omega \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(H, \widetilde{H}))$, $j \in \mathbb{N}$, with $a_{(\mu-j)}(y,\lambda\eta) = \lambda^{\mu-j}\tilde{\kappa}_\lambda a_{(\mu-j)}(y,\eta)\kappa^{-1}_\lambda$ for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$, with
$$a(y,\eta) - \sum_{j=0}^{N} \chi(\eta) a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(\Omega \times \mathbb{R}^q; H, \widetilde{H})$$
for every $N \in \mathbb{N}$ and any excision function $\chi(\eta)$.

Example. 1. For $p(y,\eta,\tau) \in S^\mu(\Omega \times \mathbb{R}^n)$ and $a(y,\eta) = \mathrm{Op}_t(p)(y,\eta)$ we have $a(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))$ for every $s \in \mathbb{R}$.
2. For $p(y,t,\eta,\tau) \in S^\mu(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ under the assumption of Remark 4.2 we have $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^{n-1}; H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))$ for every $s \in \mathbb{R}$.

Remark 4.4. Observe that in the latter Example we did not exhaust the full information of (4.6) with respect to s. In fact, differentiation in η gives us better smoothness in the image spaces. For our purposes it suffices to fix the Hilbert spaces H and $\widetilde{H}$; in applications it will be clear anyway to what extent we can say more when those spaces run over scales of spaces, parametrised by s.

Parallel to the spaces of operator-valued symbols we have vector-valued analogues of Sobolev spaces.

Definition 4.5. Let H be a Hilbert space with group action $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. Then $\mathcal{W}^s(\mathbb{R}^q, H)$ for $s \in \mathbb{R}$ is defined to be the completion of $\mathcal{S}(\mathbb{R}^q, H)$ with respect to the norm $\|\langle\eta\rangle^s \kappa^{-1}_{\langle\eta\rangle}\hat{u}(\eta)\|_{L^2(\mathbb{R}^q,H)}$.

The space $\mathcal{W}^s(\mathbb{R}^q, H)$ is contained in $\mathcal{S}'(\mathbb{R}^q, H) = \mathcal{L}(\mathcal{S}(\mathbb{R}^q), H)$. For every open $\Omega \subseteq \mathbb{R}^q$ we define $\mathcal{W}^s_{\mathrm{comp}}(\Omega, H)$ to be the set of all $u \in \mathcal{W}^s(\mathbb{R}^q, H)$ with compact support and $\mathcal{W}^s_{\mathrm{loc}}(\Omega, H) \subset \mathcal{D}'(\Omega, H) = \mathcal{L}(C^\infty_0(\Omega), H)$ by $\varphi u \in \mathcal{W}^s_{\mathrm{comp}}(\Omega, H)$ for every $\varphi \in C^\infty_0(\Omega)$.

Example. 1. Let $H := H^s(\mathbb{R}^m)$, $(\kappa_\lambda u)(x) := \lambda^{m/2} u(\lambda x)$ for $\lambda \in \mathbb{R}_+$. Then for every $s \in \mathbb{R}$ we have
$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^m)) = H^s(\mathbb{R}^q \times \mathbb{R}^m).$$
2. Let $H := H^s(\mathbb{R}_+)$, $(\kappa_\lambda u)(t) = \lambda^{1/2} u(\lambda t)$, $\lambda \in \mathbb{R}_+$. Then for every $s \in \mathbb{R}$ we have
$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}^q \times \mathbb{R}_+).$$

Remark 4.6. The notion of group actions also makes sense for Fréchet spaces that are written as projective limits of Hilbert spaces. An example is the Schwartz space

$$\mathcal{S}(\mathbb{R}^m) = \varprojlim_{j\in\mathbb{N}} \langle x\rangle^{-j} H^j(\mathbb{R}^m)$$

with κ_λ being defined as in the above example. Then there are natural extensions of Definitions 4.3 and 4.5 as well as comp/loc spaces to the case of Fréchet spaces with group action (for more details cf. also [20], [22]).

Theorem 4.7. *Let H and $\widetilde{H}$ be Hilbert (Fréchet) spaces with group action and $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; H, \widetilde{H})$. Then $\mathrm{Op}_y(a) : C_0^\infty(\Omega, H) \to C^\infty(\Omega, \widetilde{H})$ extends to a continuous operator*

$$\mathrm{Op}_y(a) : \mathcal{W}^s_{\mathrm{comp}}(\Omega, H) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \widetilde{H}) \tag{4.7}$$

for every $s \in \mathbb{R}$. If $a(y,\eta) \in S^\mu(\mathbb{R}^q \times \mathbb{R}^q; H, \widetilde{H})$ is independent of y for $|y| \geq$ const > 0, then we obtain a continuous operator

$$\mathrm{Op}_y(a) : \mathcal{W}^s(\mathbb{R}^q, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \widetilde{H}) \tag{4.8}$$

for every $s \in \mathbb{R}$.

Remark 4.8. The continuity of (4.8) can be proved under much more general assumptions on $a(y,\eta)$ than in Theorem 4.7, see, for instance, [20] or [28].

Let us now turn to what we did at the beginning of this section.
For $p(y,t,\eta,\tau) \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^n)$ we have

$$\mathrm{Op}_t(p)(y,\eta) \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^s(\mathbb{R}), H^{s-\mu}(\mathbb{R}))$$

when p satisfies the assumption of Remark 4.2. For our purposes it suffices to assume that p is a classical symbol of order $\mu \in \mathbb{Z}$, and independent of (y,t) for $|y,t| \geq$ const for some constant > 0.

In a theory of elliptic boundary value problems that relies on standard Sobolev spaces $H^s(\mathbb{R}^n_+) = H^s(\mathbb{R}^n)|_{\mathbb{R}^n_+}$ we should possess the continuity of

$$\mathrm{Op}^+(p) = \mathrm{r}^+\mathrm{Op}(p)\mathrm{e}^+ : H^s(\mathbb{R}^n_+) \to H^{s-\mu}(\mathbb{R}^n_+) \tag{4.9}$$

for $s > -1/2$, similarly as in Corollary 3.9; here e^+ is the operator of extension by zero from $\mathbb{R}^n_+$ to $\mathbb{R}^n$, and r^+ the restriction to $\mathbb{R}^n_+$ (analogously we have the extension and restriction operators e^- and r^-, respectively). It turns out that the continuity of (4.9) requires certain very restrictive assumptions on the symbol p. For instance, for $p(x,\xi) = \chi(\xi)|\xi|$ where χ is some excision function, the operator (4.9) will not be continuous for all $s > -\frac{1}{2}$.

According to Theorem 4.7 for the continuity in Sobolev spaces it suffices to know that

$$\mathrm{op}^+(p)(y,\eta) \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)) \tag{4.10}$$

for $s > -\frac{1}{2}$, i.e.,

$$\|\kappa^{-1}_{\langle\eta\rangle}\{D^\alpha_y D^\beta_\eta \mathrm{op}^+(p)(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H^s(\mathbb{R}_+),H^{s-\mu}(\mathbb{R}_+))} \leq c\langle\eta\rangle^{\mu-|\beta|} \tag{4.11}$$

for all $(y,\eta) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, and all α, β, for $c = c(\alpha, \beta, K, s) > 0$.

Moreover, it is desirable to have

$$\mathrm{op}^+(p)(y,\eta) \in S^\mu(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{4.12}$$

In order to illustrate the effect for the moment we consider the case that p is independent of y and t. To obtain (4.11) we assume

$$\tilde{p}(\eta,\tau) := p(\eta\langle\tau\rangle,\tau) \in S^\mu(\mathbb{R}^q, S^\mu_{\mathrm{tr}}(\mathbb{R})). \tag{4.13}$$

The notation $S^\mu(\mathbb{R}^q, E)$ for a Fréchet space E with the semi-norm system $(\pi_k)_{k\in\mathbb{N}}$ means the set of all $a(\eta) \in C^\infty(\mathbb{R}^q, E)$ such that

$$\pi_k(D^\beta_\eta a(\eta)) \le c\langle\eta\rangle^{\mu-|\beta|}$$

for all $\eta \in \mathbb{R}^q$, $\beta \in \mathbb{N}^q$, $k \in \mathbb{N}$, for constants $c = c(\beta, k) > 0$.

Lemma 4.9. *Let E and F be Fréchet spaces with the semi-norm systems $(\pi_j)_{j\in\mathbb{N}}$ and $(\sigma_j)_{j\in\mathbb{N}}$, respectively, and let $B : E \to F$ be a continuous operator. Then*

$$T_B : C^\infty(\mathbb{R}^q, E) \to C^\infty(\mathbb{R}^q, F) \tag{4.14}$$

defined by the composition $a : \mathbb{R}^q \to E$ and $B : E \to F$ induces a continuous operator

$$T_B : S^\mu(\mathbb{R}^q, E) \to S^\mu(\mathbb{R}^q, F) \tag{4.15}$$

for every $\mu \in \mathbb{R}$.

Proof. Without loss of generality we assume $\sigma_{j+1}(\cdot) \ge \sigma_j(\cdot)$ and $\pi_{j+1}(\cdot) \ge \pi_j(\cdot)$ for all j. Then continuity of B means that for every $k \in \mathbb{N}$ there is a $j \in \mathbb{N}$ such that $\sigma_k(Bu) \le c\pi_j(u)$ for all $u \in E$, for some $c > 0$. Analogously, the continuity of (4.15) means that for every $k \in \mathbb{N}$, $\beta \in \mathbb{N}^q$, there are $j, N \in \mathbb{N}$ such that

$$\sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{\mu+|\beta|}\sigma_k(D^\beta_\eta T_B a(\eta)) \le c \sup_{\substack{\eta\in\mathbb{R}^q\\ |\alpha|\le N}} \langle\eta\rangle^{-\mu+|\alpha|}\pi_j(D^\alpha_\eta a(\eta)) \tag{4.16}$$

for some $c>0$. Since $T_B a(\eta) = (Ba)(\eta)$ with pointwise composition and $D^\alpha_\eta(Ba)(\eta) = B(D^\alpha_\eta a)(\eta)$ it follows that

$$\sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|}\sigma_k(D^\beta_\eta T_B a(\eta)) \le c \sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu+|\beta|}\pi_j(D^\alpha_\eta a(\eta))$$

which implies (4.16). □

Lemma 4.10. *Let $p(y,\eta,\tau) \in C^\infty(\mathbb{R}^{n-1}, S^\mu_{\mathrm{cl}}(\mathbb{R}^n))$ and*

$$\tilde{p}(y,\eta,\tau) \in C^\infty(\mathbb{R}^{n-1}, S^\mu(\mathbb{R}^{n-1}, S^\mu_{\mathrm{tr}}(\mathbb{R}))).$$

Then we have the relations (4.10) for $s > -\frac{1}{2}$, and (4.12).

Proof. For (4.10) we have to verify the estimates (4.11). Let first $\alpha = \beta = 0$. For simplicity let p be independent of y. An analogue of the relations (4.4) gives us

$$\kappa^{-1}_{\langle\eta\rangle}\mathrm{op}^+(p)(\eta)\kappa_{\langle\eta\rangle} = \mathrm{op}^+(\tilde{p})(\eta). \tag{4.17}$$

The operation $\mathrm{op}^+(\cdot)$ induces a continuous operator

$$\mathrm{op}^+(\cdot) : S^\mu_{\mathrm{tr}}(\mathbb{R}) \to \mathcal{L}(H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+))$$

for every $s > -\frac{1}{2}$. That means, for every s there is a semi-norm π_j from the Fréchet topology of $S^\mu_{\mathrm{tr}}(\mathbb{R})$ such that

$$\|\mathrm{op}^+(a)\|_{\mathcal{L}(H^s(\mathbb{R}_+),H^{s-\mu}(\mathbb{R}_+))} \leq c\pi_j(a)$$

for every $a \in S^\mu_{\mathrm{tr}}(\mathbb{R})$. Thus, for $E = S^\mu_{\mathrm{tr}}(\mathbb{R})$, $\widetilde{F} = \mathcal{L}(H^s(\mathbb{R}_+),\ H^{s-\mu}(\mathbb{R}_+))$, from Lemma 4.9 it follows that

$$\sup\langle\eta\rangle^{-\mu}\|\mathrm{op}^+(\tilde{p})(\eta)\|_{\mathcal{L}(H^s(\mathbb{R}_+),H^{s-\mu}(\mathbb{R}_+))} \leq c\sup\langle\eta\rangle^{-\mu}\pi_j(\tilde{p}(\eta,\cdot)) < \infty,$$

i.e., using (4.17), that

$$\|\kappa^{-1}_{\langle\eta\rangle}\mathrm{op}^+(p)(\eta)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(H^s(\mathbb{R}_+),H^{s-\mu}(\mathbb{R}_+))} \leq c\langle\eta\rangle^\mu.$$

In a similar manner we can proceed with the derivatives $D^\beta_\eta p(\eta,\tau)$ for every $\beta \in \mathbb{N}^{n-1}$.

The proof (4.12) is straightforward as well and left to the reader. □

Definition 4.11. A symbol $p(y,t,\eta,\tau) \in S^\mu_{\mathrm{cl}}(\Omega_y \times \mathbb{R} \times \mathbb{R}^n_{\eta,\tau})$ for $\mu \in \mathbb{Z}$ is said to have the transmission property at $t = 0$ if the homogeneous components $p_{(\mu-j)}$ of p satisfy the conditions

$$D^\alpha_{y,t}D^\beta_{\eta,\tau}\{p_{(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j}p_{(\mu-j)}(y,t,-\eta,-\tau)\} = 0 \tag{4.18}$$

on the set $\{(y,t,\eta,\tau) \in \Omega \times \mathbb{R} \times \mathbb{R}^n : y \in \Omega,\ t = 0,\ \eta = 0,\ \tau \in \mathbb{R} \setminus \{0\}\}$ of non-vanishing conormal vectors over the boundary, for all $\alpha, \beta \in \mathbb{N}^n$, $j \in \mathbb{N}$. Let $S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ denote the space of all symbols of that kind. Moreover, set

$$S^\mu_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_\pm \times \mathbb{R}^n) := \{p|_{\Omega\times\overline{\mathbb{R}}_\pm\times\mathbb{R}^n} : p \in S^\mu_{\mathrm{tr}}(\Omega \times \mathbb{R} \times \mathbb{R}^n)\}.$$

Since the transmission property is a local condition near $t = 0$ it can easily be extended to symbols in an arbitrary open set $U \subseteq \mathbb{R}^n$ intersecting $\{t = 0\}$. (It is clear that it suffices to ask (4.18) only for all $\alpha = (0,\dots,\alpha_n)$, $\beta = (\beta_1,\dots,\beta_{n-1},0)$).

Operators with symbols with the transmission property in connection with boundary value problems (and also transmission problems) have been studied by many outhors, first of all Boutet de Monvel [5], [4], Eskin [7], and later on Myshkis [14], Rempel and Schulze [15], Grubb [8], [9], and many others. One of the main motivations was to find a framework to express parametrices of elliptic boundary value problems for differential operators and to prove an analogue of the Atiyah–Singer index theorem. In this connection it appeared not too perturbing that generically symbols (that are smooth up to the boundary) have not the transmission property at the boundary. We will return to more general symbols below.

The first important aspect is that a pseudo-differential theory of boundary value problems concerns continuous operators (4.9) (and analogously on manifolds with smooth boundary). Another essential point is to understand the behaviour of such operators under compositions.

Proposition 4.12. *For every* $p(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ *we have*

$$\tilde{p}(y,t,\eta,\tau) := p(y,t,\eta\langle\tau\rangle,\tau) \in C^{\infty}(\Omega \times \overline{\mathbb{R}}_+, S^{\mu}(\mathbb{R}^{n-1}, S^{\mu}_{\mathrm{tr}}(\mathbb{R}))).$$

The simple proof is left to the reader.

In the local analysis of BVPs it suffices to assume that the involved symbols are independent of t for large t.

Proposition 4.13. *For every* $p(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ *which is independent of t for large t we have*

$$\mathrm{op}^+(p)(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; H^s(\mathbb{R}_+), H^{s-\mu}(\mathbb{R}_+)) \tag{4.19}$$

for every $s > -\frac{1}{2}$, *and*

$$\mathrm{op}^+(p)(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q; \mathcal{S}(\overline{\mathbb{R}}_+), \mathcal{S}(\overline{\mathbb{R}}_+)). \tag{4.20}$$

The t-independent case is contained in Lemma 4.10. After that the proof in general is straightforward.

Theorem 4.7 together with (4.19) entails the continuity of

$$\mathrm{Op}^+(p) = \mathrm{Op}_y(\mathrm{op}^+(p)) : H^s_{[\mathrm{comp})}(\Omega \times \mathbb{R}_+) \to H^{s-\mu}_{[\mathrm{loc})}(\Omega \times \mathbb{R}_+); \tag{4.21}$$

here $H^s_{[\mathrm{comp})/[\mathrm{loc})}(\Omega \times \mathbb{R}_+) = \mathcal{W}^s_{\mathrm{comp}/\mathrm{loc}}(\Omega, H^s(\mathbb{R}_+))$, cf. also Example 4 (ii). Let us now give a motivation of the conditions (4.18) in Definition 4.11. First it is evident that when p is a polynomial in ξ, the homogeneous components $p_{(\mu-j)}$ of order $\mu - j$, $j = 0, \ldots, \mu$, satisfy the relations (4.18). For instance, we have in this case

$$p_{(\mu)}(y,t,\lambda\eta,\lambda\tau) = \lambda^{\mu} p_{(\mu)}(y,t,\eta,\tau) \tag{4.22}$$

for every $\lambda \in \mathbb{R}$, not only for $\lambda \in \mathbb{R}_+$, and hence,

$$p_{(\mu)}(y,t,\eta,\tau) = (-1)^{\mu} p_{(\mu)}(y,t,-\eta,-\tau),$$

even for all (y,t,η,τ).

If $p(x,\xi)$ is elliptic of order μ, then the Leibniz inverse which belongs to $S^{-\mu}_{\mathrm{cl}}(\Omega \times \mathbb{R} \times \mathbb{R}^n_{\xi})$ satisfies those conditions as well with respect to the order $-\mu$.

The behaviour of operators under compositions locally near the boundary can be reduced to the composition of operators with operator-valued symbols, modulo smoothing operators. In general, if $H, \widetilde{H}$, and $\widetilde{\widetilde{H}}$ are Hilbert spaces with group actions $\kappa = \{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, $\tilde{\kappa} = \{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, and $\tilde{\tilde{\kappa}} = \{\tilde{\tilde{\kappa}}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively, and

$$a(y,\eta) \in S^{\mu}(\Omega \times \mathbb{R}^q, \widetilde{H}, \widetilde{\widetilde{H}}),\ \tilde{a}(y,\eta) \in S^{\tilde{\mu}}(\Omega \times \mathbb{R}^q; H, \widetilde{H}),$$

for simplicity, with compact support with respect to y, then we can form

$$\mathrm{Op}_y(a)\mathrm{Op}_y(\tilde{a}) = \mathrm{Op}(a\#\tilde{a})$$

with the Leibniz product $a\#\tilde{a}(y,\eta) \in S^{\mu+\tilde{\mu}}(\Omega \times \mathbb{R}^q; H, \widetilde{\widetilde{H}})$ that can be computed by an operator-valued analogue of the respective oscillatory integral expression in

Kumano-go's formalism.This entails an asymptotic expansion

$$a\#\tilde{a}(y,\eta) \sim \sum_{\alpha\in\mathbb{N}^q} \frac{1}{\alpha!}(\partial_\eta^\alpha a(y,\eta))D_y^\alpha \tilde{a}(y,\eta),$$

$\partial_\eta^\alpha := \partial^{\alpha_1}/\partial y_1^{\alpha_1}\dots\partial^{\alpha_q}/\partial y_q^{\alpha_q}$.

If we apply this to the case

$$a(y,\eta) = \mathrm{op}^+(p)(y,\eta),\ \tilde{a}(y,\eta) = \mathrm{op}^+(\tilde{p})(y,\eta)$$

for symbols $p(x,\xi) \in S_{\mathrm{tr}}^\mu(\Omega\times\mathbb{R}\times\mathbb{R}^n)$, $\tilde{p}(x,\xi) \in S_{\mathrm{tr}}^{\tilde{\mu}}(\Omega\times\mathbb{R}\times\mathbb{R}^n)$ (say, under the simplifying condition of compact support in (y,t)), then we have to understand the compositions

$$(\partial_\eta^\alpha \mathrm{op}^+(p)(y,\eta))D_y^\alpha \mathrm{op}^+(\tilde{p})(y,\eta) = \mathrm{op}^+(\partial_\eta^\alpha p)(y,\eta)\mathrm{op}^+(D_y^\alpha \tilde{p})(y,\eta).$$

Since $\mu,\tilde{\mu}\in\mathbb{Z}$ are arbitrary, and $\partial_\eta^\alpha p \in S_{\mathrm{tr}}^{\mu-|\alpha|}$, $D_y^\alpha\tilde{p}\in S_{\mathrm{tr}}^{\tilde{\mu}}$, we may consider, for instance, the case $\alpha = 0$. From the information of Section 3 we know that

$$\mathrm{op}^+(p)(y,\eta)\mathrm{op}^+(\tilde{p})(y,\eta) = \mathrm{op}^+(p\#_t\tilde{p})(y,\eta) + g(y,\eta)$$

where $p\#_t\tilde{p}$ is the Leibniz product between p and $\tilde{p}$ with respect to the t-variable, and $g(y,\eta)$ is a family of operators in $\Gamma^0(\mathbb{R}_+)$.

More precisely, the operator families $g(y,\eta)$ are Green symbols in the following sense.

Definition 4.14.

1. An operator-valued symbol $g(y,\eta)$ belongs to $\mathcal{R}_G^{\mu,0}(\Omega\times\mathbb{R}^{n-1})$ if
$$g(y,\eta), g^*(y,\eta) \in S_{\mathrm{cl}}^\mu(\Omega\times\mathbb{R}^{n-1}; L^2(\mathbb{R}_+), \mathcal{S}(\overline{\mathbb{R}}_+)).$$
Here $g^*(y,\eta)$ is the (y,η)-wise $L^2(\mathbb{R}_+)$-adjoint. Elements of $\mathcal{R}_G^{\mu,0}(\Omega\times\mathbb{R}^{n-1})$ are called Green symbols of type 0.
2. An operator family $g(y,\eta)$ belongs to $\mathcal{R}_G^{\mu,d}(\Omega\times\mathbb{R}^{n-1})$, then called a Green symbol of type $d\in\mathbb{N}$, if
$$g(y,\eta) = \sum_{j=0}^d g_j(y,\eta)\partial_t^j$$
for $g_j(y,\eta)\in\mathcal{R}_G^{\mu-j,0}(\Omega\times\mathbb{R}^{n-1})$, $j = 0,\dots,d$.

Similarly as (3.23) we also define 2×2 block matrices

$$\boldsymbol{a}(y,\eta) := \begin{pmatrix} \mathrm{op}^+(p)(y,\eta) + g_{11}(y,\eta) & g_{12}(y,\eta) \\ g_{21}(y,\eta) & g_{22}(y,\eta)\end{pmatrix} : \begin{matrix} H^s(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_-}\end{matrix} \to \begin{matrix} H^{s-\mu}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_+}\end{matrix} \tag{4.23}$$

for arbitrary $p(y,t,\eta,\tau)\in S_{\mathrm{tr}}^\mu(\Omega\times\mathbb{R}\times\mathbb{R}_{\eta,\tau}^n)$ (independent of t for $|t| > \mathrm{const}$), and $g_{11}(y,\eta)\in\mathcal{R}_G^{\mu,d}(\Omega\times\mathbb{R}^q)$, $s > d-\frac{1}{2}$, while (say, for the case $j_- = j_+ = 1$)

$$g_{12}(y,\eta)\in S_{\mathrm{cl}}^\mu(\Omega\times\mathbb{R}^{n-1};\mathbb{C},\mathcal{S}(\overline{\mathbb{R}}_+))$$

(with $\mathbb{C}$ being endowed with the trivial group action),

$$g_{21}(y,\eta)u(t)=\sum_{l=0}^{d}g_{21,l}(y,\eta)\partial_t^l u(t)$$

for $g_{21,l}^*(y,\eta)\in S_{\mathrm{cl}}^{\mu-l}(\Omega\times\mathbb{R}^{n-1};\mathbb{C},\mathcal{S}(\overline{\mathbb{R}}_+))$ with the (y,η)-wise adjoint in the following sense:

$$(g_{21,l}(y,\eta)v,c)_{\mathbb{C}}=(v,g_{21,l}^*(y,\eta)c)_{L^2(\mathbb{R}_+)},$$

for arbitrary $v\in L^2(\mathbb{R}_+)$, $c\in\mathbb{C}$, and $g_{22}(y,\eta)\in S_{\mathrm{cl}}^{\mu}(\Omega\times\mathbb{R}^{n-1})$.

The definition for arbitrary $j_\pm$ is analogous. We call $g_{21}(y,\eta)$ a trace symbol of order $d\in\mathbb{N}$ and $g_{12}(y,\eta)$ a potential symbol.

From the definition it follows altogether that

$$\boldsymbol{a}(y,\eta)\in S^{\mu}(\Omega\times\mathbb{R}^{n-1};H^s(\mathbb{R}_+)\oplus\mathbb{C}^{j_-},H^{s-\mu}(\mathbb{R}_+)\oplus\mathbb{C}^{j_+}) \tag{4.24}$$

for all $s>d-\frac{1}{2}$. For $\boldsymbol{g}(y,\eta):=(g_{ij}(y,\eta))_{i,j=1,2}$ we have

$$\boldsymbol{g}(y,\eta)\in S_{\mathrm{cl}}^{\mu}(\Omega\times\mathbb{R}^{n-1};H^s(\mathbb{R}_+)\oplus\mathbb{C}^{j_-},\mathcal{S}(\overline{\mathbb{R}}_+)\oplus\mathbb{C}^{j_+}) \tag{4.25}$$

for $s>d-\frac{1}{2}$. Let $\mathcal{R}_G^{\mu,d}(\Omega\times\mathbb{R}^{n-1};j_-,j_+)$ denote the set of all such $\boldsymbol{g}(y,\eta)$. Moreover, let $\mathcal{R}^{\mu,d}(\Omega\times\mathbb{R}^{n-1};j_-,j_+)$ denote the set of all symbols $\boldsymbol{a}(y,\eta)$ of the form (4.23).

Now let X be a C^∞ manifold with boundary Y. Define $\mathcal{B}^{-\infty,d}(X;j_-,j_+)$ to be the space of smoothing operators of type d. For simplicity let again $j_-=j_+=1$ (the general case is analogous).

Based on the Riemannian metrics on X and $Y=\partial X$ we identify the spaces $C^\infty(X\times X),C^\infty(X\times Y)$, etc., with corresponding integral operators with such kernels, for instance, $u\to\int_X c(x,x')u(x')dx'$ and $v\to\int_Y k(x,y')v(y')dy'$ for $c(x,x')\in C^\infty(X\times X)$ and $k(x,y')\in C^\infty(X\times Y)$, respectively. Let $\mathcal{B}^{-\infty,0}(X;1,1)$ denote the space of all operators

$$\mathcal{C}=(C_{ij})_{i,j=1,2}:\begin{array}{c}C_0^\infty(X)\\ \oplus\\ C_0^\infty(Y)\end{array}\to\begin{array}{c}C^\infty(X)\\ \oplus\\ C^\infty(Y)\end{array}$$

such that C_{11} has a kernel in $C^\infty(X\times X),C_{12}$ a kernel in $C^\infty(X\times Y),C_{21}$ a kernel in $C^\infty(Y\times X)$ and C_{22} a kernel in $C^\infty(Y\times Y)$. Moreover, by $\mathcal{B}^{-\infty,d}(X;1,1)$ for $d\in\mathbb{N}$ we denote the space of all 2×2 block matrix operators $\mathcal{C}$ where C_{12} and C_{22} are as before but

$$C_{11}=\sum_{l=0}^{d}C_{11,l}D^l,\ C_{21}=\sum_{l=0}^{d}C_{21,l}D^l$$

for $C_{11,l}$ and $C_{21,l}$ as in the case $d=0$ and a first-order differential operator D on X that is close to Y equal to ∂_t, the differentiation in normal direction. In an analogous manner we define $\mathcal{B}^{-\infty,d}(X;j_-,j_+)$ for arbitrary $j_\pm\in\mathbb{N}$.

Let us fix a collar neighbourhood V of Y in X, let $(U_\iota)_{\iota\in I}$ be a locally finite open covering of V, and let $\chi_\iota:U_\iota\to\overline{\mathbb{R}}_+^n$ be charts, $\iota\in I$. Those induce charts

$\chi'_\iota : U_\iota \cap Y \to \mathbb{R}^{n-1}$ on Y. For every $\boldsymbol{a}_\iota(y,\eta) \in \mathcal{R}^{\mu,d}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; j_-, j_+)$ we have an operator $\mathrm{Op}_y(\boldsymbol{a}_\iota)$, and we form the pull-back $\mathrm{diag}(\chi_*^{-1}, \chi_*'^{-1})\mathrm{Op}_y(\boldsymbol{a}_\iota)$ which is a 2×2 block matrix operator over U_ι. Let us fix a system of functions $\varphi_\iota \in C_0^\infty(U_\iota)$ such that $\sum_{\iota\in I} \varphi_\iota \equiv 1$ near Y, set $\varphi_\iota := \varphi_\iota|_Y$, moreover, choose $\psi_\iota \in C_0^\infty(U_\iota)$ that are equal to 1 on $\mathrm{supp}\varphi_\iota$, set $\psi'_\iota = \psi_\iota|_Y$, and form

$$\mathcal{A}_\iota := \mathrm{diag}(\varphi_\iota, \varphi'_\iota)\mathrm{diag}(\chi_*^{-1}, \chi_*'^{-1})\mathrm{Op}_y(\boldsymbol{a}_\iota)\mathrm{diag}(\psi_\iota, \psi'_\iota). \tag{4.26}$$

Moreover, choose functions $\sigma, \tilde{\sigma}, \tilde{\tilde{\sigma}} \in C_0^\infty(V)$, that are equal to 1 close to Y, such that $\tilde{\sigma} = 1$ on $\mathrm{supp}\,\sigma$, and $\sigma \equiv 1$ on $\mathrm{supp}\,\tilde{\tilde{\sigma}}$.

Definition 4.15. Let $\mathcal{B}^{\mu,d}(X; j_-, j_+)$ for $\mu \in \mathbb{Z}$, $d \in N$; denote by $\mathcal{A}$ the space of all operators

$$\mathcal{A} = (A_{ij})_{i,j=1,2} : \begin{array}{c} C_0^\infty(X) \\ \oplus \\ C^\infty(Y, \mathbb{C}^{j_-}) \end{array} \to \begin{array}{c} C^\infty(X) \\ \oplus \\ C^\infty(Y, \mathbb{C}^{j_+}) \end{array}$$

of the form

$$\mathcal{A} = \mathrm{diag}(\sigma, 1)\sum_{\iota\in I} \mathcal{A}_\iota \mathrm{diag}(\tilde{\sigma}, 1) + \mathrm{diag}((1-\sigma)A(1-\tilde{\tilde{\sigma}}), 0) + \mathcal{C} \tag{4.27}$$

for arbitrary operators $\mathcal{A}_\iota$ as in (4.26), $A \in L^\mu_{\mathrm{cl}}(\mathrm{int}X)$, and $\mathcal{C} \in \mathcal{B}^{-\infty,d}(X; j_-, j_+)$.

The definition applies in particular to $X = \overline{\mathbb{R}}^n_+$ with the variables $x = (y,t)$. In this case the shape of the operators is easier, since the sum on the right-hand side of (4.27) can be replaced by

$$\mathrm{diag}(\sigma, 1)\mathrm{Op}(\boldsymbol{a})\mathrm{diag}(\tilde{\sigma}, 1) \tag{4.28}$$

for an $\boldsymbol{a}(y,\eta) \in \mathcal{R}^{\mu,d}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; j_-, j_+)$.

Let us define the principal symbolic structure

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}))$$

consisting of the interior and the boundary symbol $\sigma_\psi(\mathcal{A})$ and $\sigma_\partial(\mathcal{A})$, respectively.

The upper left corner A_{11} of an operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; j_-, j_+)$ belongs to $L^\mu_{\mathrm{cl}}(\mathrm{int}X)$, and we simply define $\sigma_\psi(\mathcal{A})$ for $(x,\xi) \in T^*X \setminus 0$ as the homogeneous principal symbol of A of order μ in the standard sense (here we take into account that the symbols are smooth up to the boundary). What concerns the boundary symbol we first look at the situation of the half-space, cf. (4.28). In this case we define

$$\sigma_\partial(\mathcal{A})(y,\eta) := \sigma_\partial(\boldsymbol{a})(y,\eta)$$

for $(y,\eta) \in T^*\mathbb{R}^{n-1} \setminus 0$ by

$$\sigma_\partial(\boldsymbol{a})(y,\eta) := \mathrm{diag}(\mathrm{op}^+(p_{(\mu)}|_{t=0})(y,\eta), 0) + \sigma_\partial(g)(y,\eta) \tag{4.29}$$

where $p_{(\mu)}(y,t,\eta,\tau)$ is the homogeneous principal symbol of $p(y,t,\eta,\tau)$, and $\sigma_\partial(g)(y,\eta)$ is the homogeneous principal symbol of (4.25) as a classical operator-valued symbol. Together with

$$\mathrm{op}^+(p_{(\mu)}|_{t=0})(y,\lambda\eta) = \kappa_\lambda \mathrm{op}^+(p_{(\mu)}|_{t=0})(y,\eta)\kappa_\lambda^{-1}$$

for $\lambda \in \mathbb{R}_+$ we obtain

$$\sigma_\partial(\mathcal{A})(y, \lambda\eta) = \lambda^\mu \mathrm{diag}(\kappa_\lambda, 1)\sigma_\partial(\mathcal{A})(y, \eta)\mathrm{diag}(\kappa_\lambda^{-1}, 1)$$

for all $\lambda \in \mathbb{R}_+$.

The construction of the operator spaces $\mathcal{B}^{\mu,d}(X; j_-, j_+)$ in terms of local representations and subsequent pull-backs to the manifold is possible because of natural invariance properties under coordinate changes. The same is true of the principal symbols, and then we obtain, in particular, also an invariantly defined principal boundary symbol on a manifold with boundary, using the local descriptions (4.29). In other words, we have

$$\sigma_\psi(\mathcal{A}) \in C^\infty(T^*X \setminus 0), \sigma_\partial(\mathcal{A}) \in C^\infty(T^*Y \setminus 0, \mathcal{L}(H^s(\mathbb{R}_+) \oplus \mathbb{C}^{j_-}, H^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C}^{j_+})). \tag{4.30}$$

In many contexts it is adequate to admit operators between sections of smooth complex vector bundles E, F on X and J_-, J_+ on Y, respectively,

$$\mathcal{A} : \begin{matrix} C_0^\infty(X, E) \\ \oplus \\ C_0^\infty(Y, J_-) \end{matrix} \rightarrow \begin{matrix} C^\infty(X, F) \\ \oplus \\ C^\infty(Y, J_-) \end{matrix}. \tag{4.31}$$

The generalisation of the scalar case in the upper left corner to systems and then to the case of bundles, and E, F of the other entries from trivial to general vector bundles J_-, J_+, is straightforward and left to the reader.

If M is a C^∞ manifold, by $\mathrm{Vect}(M)$ we denote the set of all smooth complex vector bundles over M. If M is C^∞ with boundary, then we assume that every $E \in \mathrm{Vect}(M)$ is the restriction of some $\widetilde{E} \in \mathrm{Vect}(2M)$ to M. Then there is a standard definition of Sobolev spaces of distributional sections in $E \in \mathrm{Vect}(M)$ in comp/loc-version denoted by

$$H^s_{\mathrm{comp/loc}}(M, E), \quad s \in \mathbb{R},$$

when M is an open manifold. If M is compact, then we simply write $H^s(M, E)$. Moreover, if M is C^∞ with boundary, we define

$$H^s_{[\mathrm{comp}/[\mathrm{loc})}(\mathrm{int}M, E) := H^s_{\mathrm{comp/loc}}(2M, \widetilde{E})\big|_{\mathrm{int}M}.$$

For the vector bundles $E, F \in \mathrm{Vect}(X)$, $J_-, J_+ \in \mathrm{Vect}(Y)$, in (4.31) we write $\boldsymbol{v} := (E, F; J_-, J_+)$ and denote by $\mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ the set of all operators (4.31).

From the vector-valued analogue of (4.24) together with Theorem 4.7 and corresponding invariance properties we obtain that every $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; , \boldsymbol{v})$ induces continuous operators

$$\mathcal{A} : \begin{matrix} H^s_{[\mathrm{comp})}(\mathrm{int}X, E) \\ \oplus \\ H^s_{\mathrm{comp}}(Y, J_-) \end{matrix} \rightarrow \begin{matrix} H^{s-\mu}_{[\mathrm{loc})}(\mathrm{int}X, F) \\ \oplus \\ H^{s-\mu}_{\mathrm{loc}}(Y, J_+) \end{matrix}$$

for all real $s > d - \frac{1}{2}$. In particular, if X is compact, we have

$$\mathcal{A} : \begin{matrix} H^s(\mathrm{int}(X, E) \\ \oplus \\ H^s(Y, J_-) \end{matrix} \rightarrow \begin{matrix} H^{s-\mu}(\mathrm{int}X, F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{matrix} . \tag{4.32}$$

The pair of principal symbols $\sigma = (\sigma_\psi, \sigma_\partial)$ in this case means

$$\sigma_\psi(\mathcal{A})(x, \xi) : \pi_X^* E \rightarrow \pi_X^* F$$

with the pull-back π_X^* of bundles under the canonical projection $\pi_X : T^*X \backslash 0 \rightarrow X$, and

$$\sigma_\partial(\mathcal{A})(y, \eta) : \pi_Y^* \begin{pmatrix} H^s(\mathbb{R}_+) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \rightarrow \pi_Y^* \begin{pmatrix} H^{s-\mu}(\mathbb{R}_+) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}$$

for $E' := E|_Y$, $F' := F|_Y$ and the canonical projection $\pi_Y : T^*Y \setminus 0 \rightarrow Y$, for $s > d - \frac{1}{2}$. Alternatively we may consider

$$\sigma_\partial(\mathcal{A})(y, \eta) : \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \rightarrow \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes F' \\ \oplus \\ J_+ \end{pmatrix} .$$

In the following we often discuss operators in the set-up of $\mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, though the reader who is mainly interested in the analytical details may consider the case $\mathcal{B}^{\mu,d}(X; j_-, j_+)$ which corresponds to the trivial bundles $E = X \times \mathbb{C}$, $F = X \times \mathbb{C}$ and $J_\pm = Y \times \mathbb{C}^{j_\pm}$, respectively.

Remark 4.16. Let X be compact. Then $\sigma(\mathcal{A}) = 0$ implies that (4.32) is a compact operator for every $s > d - \frac{1}{2}$.

5. Ellipticity of Boundary Value Problems

We now turn to the ellipticity of BVPs, more precisely, to the Shapiro–Lepatinskij ellipticity. For elliptic operators there is also another kind of ellipticity of boundary conditions, known in special cases, as conditions of Atiyah–Patodi–Singer type ("APS-conditions"), and in general as global projection conditions. While not every elliptic operator on a C^∞ manifold X with boundary admits Shapiro–Lopatinskij elliptic boundary conditions, there are always global projection conditions (when X is compact), see [24] where both concepts are unified to an operator algebra, containing also Boutet de Monvel's calculus. Let $L^\mu_{\mathrm{tr}}(X; E, F)$ for $E, F \in \mathrm{Vect}(X)$ denote the set of all operators $A = \mathrm{r}^+ \widetilde{A} \mathrm{e}^+$, $\widetilde{A} \in L^\mu_{\mathrm{tr}}(2X; \widetilde{E}, \widetilde{F})$, with $L^\mu_{\mathrm{tr}}(2X; \widetilde{E}, \widetilde{F})$ being the space of classical pseudo-differential operators on the double $2X$, referring to $\widetilde{E}, \widetilde{F} \in \mathrm{Vect}(2X)$ with $E = \widetilde{E}|_X$, $F = \widetilde{F}|_X$, and with the transmission property at $Y = \partial X$.

For convenience we assume that Y is compact. The nature of elliptic boundary conditions for an elliptic operator $A + G \in \mathcal{B}^{\mu,d}(X; E, F)$ (i.e., for elliptic

$A \in L^{\mu}_{\mathrm{tr}}(X; E, F)$, $\mu \in \mathbb{Z}$, and a Green operator G on X of order μ and type d) depends on the principal boundary symbol of A,

$$\sigma_{\partial}(A)(y,\eta) : \pi_Y^* H^s(\mathbb{R}_+) \otimes E' \to \pi_Y^* H^{s-\mu}(\mathbb{R}_+) \otimes F' \tag{5.1}$$

for any fixed $s > \max\{\mu, d - \frac{1}{2}\}$, but not so much on

$$\sigma_{\partial}(G)(y,\eta) : \pi_Y^* H^s(\mathbb{R}_+) \otimes E' \to \pi_Y^* H^{s-\mu}(\mathbb{R}_+) \otimes F'. \tag{5.2}$$

(5.2) is a family of compact operators that cannot affect the possibility to pose Shapiro–Lopatinskij elliptic conditions for the operator A.

Definition 5.1. An operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$ for $\boldsymbol{v} = (E, F; J_-, J_+)$ is called elliptic if both the principal interior symbol

$$\sigma_{\psi}(\mathcal{A}) : \pi_X^* E \to \pi_X^* F, \tag{5.3}$$

$\pi_X : T^*X \setminus 0 \to X$, and the principal boundary symbol

$$\sigma_{\partial}(\mathcal{A}) : \pi_Y^* \begin{pmatrix} H^s(\mathbb{R}_+) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} H^{s-\mu}(\mathbb{R}_+) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}, \tag{5.4}$$

$\pi_Y : T^*Y \setminus 0 \to Y$, define isomorphisms.

The second condition is just what we call Shapiro–Lopatinskij ellipticity. The smoothness $s > \max\{\mu, d\} - \frac{1}{2}$ is fixed, but the choice is unessential. The bijectivity of $\sigma_{\partial}(\mathcal{A})$ holds if and only if its restriction to Schwartz functions in the upper left corner induces an isomorphism

$$\sigma_{\partial}(\mathcal{A}) : \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes E' \\ \oplus \\ J_- \end{pmatrix} \to \pi_Y^* \begin{pmatrix} \mathcal{S}(\overline{\mathbb{R}}_+) \otimes F' \\ \oplus \\ J_+ \end{pmatrix}.$$

If $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J_-, J_+)$, is elliptic in the sense of Definition 5.1, then for the pair of inverses $\sigma_{\psi}^{-1}(\mathcal{A})$ and $\sigma_{\partial}^{-1}(\mathcal{A})$ we find an operator $\mathcal{A}^{(-1)} \in \mathcal{B}^{-\mu,(d-\mu)^+}(X; \boldsymbol{v}^{-1})$, $\boldsymbol{v}^{-1} = (F, E; J_+, J_-)$, $(d-\mu)^+ = \max\{d-\mu, 0\}$, such that $\sigma_{\psi}(\mathcal{A}^{(-1)}) = \sigma_{\psi}^{-1}(\mathcal{A})$, $\sigma_{\partial}(\mathcal{A}^{(-1)}) = \sigma_{\partial}^{-1}(\mathcal{A})$. This is a consequence of a more general operator convention to find operators for a prescribed pair of principal symbols (those can be described independently of the operator level, similarly as in the case of classical pseudo-differential operators on an open manifold).

In that case we have compact remainders

$$\mathcal{G} := 1 - \mathcal{A}^{(-1)}\mathcal{A} \in \mathcal{B}^{-1,\max\{\mu,d\}}(X; (E, E; J_-, J_-)), \tag{5.5}$$

$$\mathcal{D} := 1 - \mathcal{A}\mathcal{A}^{(-1)} \in \mathcal{B}^{-1,(d-\mu)^+}(X; (F, F; J_+, J_+)) \tag{5.6}$$

in the respective Sobolev spaces, since $\sigma(\mathcal{G}) = 0$, $\sigma(\mathcal{D}) = 0$, (with σ referring to order 0). This shows the first part of the following theorem.

Theorem 5.2. *Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ be elliptic; then*

$$\mathcal{A} : \begin{array}{c} H^s(\mathrm{int}X, E) \\ \oplus \\ H^s(Y, J_-) \end{array} \to \begin{array}{c} H^{s-\mu}(\mathrm{int}X, F) \\ \oplus \\ H^{s-\mu}(Y, J_+) \end{array} \tag{5.7}$$

is a Fredholm operator for every $s > \max\{\mu, d\} - \frac{1}{2}$. Conversely, if (5.7) is Fredholm for some $s = s_0 > \max\{\mu, d\} - \frac{1}{2}$, then $\mathcal{A}$ is elliptic which entails the Fredholm property for all $s > \max\{\mu, d\} - \frac{1}{2}$.

The second part of the latter theorem requires arguments that are omitted here; details may be found in [15].

Remark 5.3.
1. If $\mathcal{A} \in \mathcal{B}^{\mu,d}(X, \boldsymbol{v})$ is elliptic, then there is a parametrix $\mathcal{A}^{(-1)} \in \mathcal{B}^{-\mu,(d-\mu)^+}(X;\boldsymbol{v}^{-1})$ which means that the above-mentioned remainders $\mathcal{G}$ and $\mathcal{D}$ belong to $\mathcal{B}^{-\infty,\max\{\mu,d\}}$ and $\mathcal{B}^{-\infty,(d-\mu)^+}$, respectively.
2. Let $\mathcal{A} \in \mathcal{B}^{\mu,d}(X;\boldsymbol{v})$ be an operator such that (5.7) is an isomorphism for some $s = s_0 > \max\{\mu, d\} - \frac{1}{2}$. Then (5.7) is an isomorphism for all $s > \max\{\mu, d\} - \frac{1}{2}$, and for the inverse (which is a special parametrix of $\mathcal{A}$) we have $\mathcal{A}^{-1} \in \mathcal{B}^{-\mu,(d-\mu)^+}(X;\boldsymbol{v}^{-1})$.

In fact, 1. can be obtained by improving $\mathcal{A}^{(-1)}$ of (5.5), (5.6) by applying a formal Neumann series argument. The property 2. is a consequence of the second assertion of Theorem 5.2 (more details may be found in [17]).

Example. Let $A = \Delta$, the Laplacian on X (with respect to a Riemannian metric), moreover, let $T_0 u := u|_Y$. Then, for every order-reducing isomorphism $R \in L^{3/2}_{\mathrm{cl}}(Y)$ on the boundary we have

$$\begin{pmatrix} \Delta \\ T \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \Delta \\ T_0 \end{pmatrix} \in \mathcal{B}^{2,0}(X; 1, 1; 0, 1)$$

where 1 on the right-hand side stands for trivial bundles of fibre dimension 1 over X and Y, respectively, and we have

$$\begin{pmatrix} \Delta \\ T \end{pmatrix}^{-1} \in \mathcal{B}^{-2,0}(X; 1, 1; 1, 0).$$

Let us now discuss the nature of Shapiro–Lopatinskij ellipticity in more detail. A closer look at (5.1) reveals some interesting structures that are useful also to understand the difference to ellipticity with global projection conditions, mentioned at the beginning of this section.

Consider an operator $A \in \mathcal{B}^{\mu,0}(X; E, F)$ (i.e., A is of the type of an upper left corner in the 2×2 block matrix set-up) satisfying the ellipticity condition (5.3). Then (5.1) is a family of Fredholm operators, where $\dim\ker \sigma_\partial(A)(y,\eta)$ and $\dim\operatorname{coker} \sigma_\partial(A)(y,\eta)$ are independent of $s > \max\{\mu, d\} - \frac{1}{2}$. The same is true of

$$\sigma_\partial(\mathcal{A})(y,\eta)_{1,1} = \sigma_\partial(A)(y,\eta) + \sigma_\partial(G)(y,\eta). \tag{5.8}$$

If (5.4) is a family of isomorphisms then the role of the additional entries $(\sigma_\partial(\mathcal{A})(y,\eta))_{i,j}$ for $i+j>2$ is to fill up (5.8) to a family of isomorphisms. However, many operators $A \in \mathcal{B}^{\mu,0}(X;E,F)$ that are elliptic with respect to $\sigma_\psi(\cdot)$ do not admit such families of block matrix isomorphisms.

As noted before, an example is the Cauchy–Riemann operator in a smooth bounded domain in $\mathbb{C}$ which is elliptic of order 1. Other examples are Dirac operators in even dimensions.

In order to illustrate the phenomenon in general we recall a few notions from K-theory which are connected with the index of families of Fredholm operators parametrised by a compact topological space. In the present case we consider (5.8) for $(y,\eta) \in S^*Y$, the unit cosphere bundle induced, by $T^*Y \setminus 0$. Observe that by virtue of the homogeneity

$$(\sigma_\partial(\mathcal{A})(y,\lambda\eta))_{1,1} = \lambda^\mu \kappa_\lambda(\sigma_\partial(\mathcal{A})(y,\eta)_{1,1})\kappa_\lambda^{-1},$$

the values of $\sigma_\partial(\mathcal{A})(y,\eta)_{1,1}$ for all $(y,\eta) \in T^*Y \setminus 0$ are determined by those for $(y,\eta) \in S^*Y$. The compact topological spaces that we have in mind here are S^*Y and Y, respectively (we discuss the case that X is a smooth manifold with compact boundary Y).

First, on a compact topological space M (connected, to simplify matters) we have the set $\mathrm{Vect}(M)$ of (locally trivial) continuous complex vector bundles on M. In the case of a C^∞ manifold M we may (and will) take smooth complex vector bundles. Roughly speaking, continuous vector bundles over M are topological spaces which are disjoint unions $E = \bigcup_{x\in M} E_x$ of fibres E_x that are vector spaces isomorphic to $\mathbb{C}^k$ for some $k \in N$, and every point $x_0 \in M$ has a neighbourhood U such that $E|_U = \bigcup_{x\in U} E_x$ is homeomorphic to $U \times \mathbb{C}^k$ where this homorphism is fibrewise an isomorphism and commutes with the canonical projections $p : E \to M$, $e_x \to x$ for $e_x \in E_x$, and $q : U \times \mathbb{C}^k \to U$, $(x,v) \to x$ for $v \in \mathbb{C}^k$. An example is $E = M \times \mathbb{C}^k$ which is a so-called trivial vector bundle. Thus a part of the general definition requires $E|_U$ to be isomorphic to a trivial bundle which is just the meaning of “locally trivial”. We do not repeat here everything on vector bundles such as what is a vector bundle isomorphism $\cong$, but the notion directly comes from vector space isomorphisms, now parametrised by $x \in M$. More generally, we have vector bundle morphisms which are fibrewise vector space homomorphisms. Moreover, we have a natural notion of a direct sum $E \oplus F$ for $E, F \in \mathrm{Vect}(M)$, fibrewise defined by $E_x \oplus F_x$, $x \in M$.

Similarly we can form tensor products $E \otimes F$ by taking fibrewise tensor products $E_x \otimes F_x$, $x \in M$.

The K-group $K(M)$ over M is defined as the set of equivalence classes of pairs $(E,F) \in \mathrm{Vect}(M) \times \mathrm{Vect}(M)$ where

$$(E,F) \sim (\widetilde{E},\widetilde{F})$$

means that there is a $G \in \mathrm{Vect}(M)$ such that $E \oplus \widetilde{F} \oplus G \cong F \oplus \widetilde{E} \oplus G$. The equivalence class represented by (E,F) is denoted by $[E]-[F]$. The structure of

$K(M)$ of a commutative group comes from the direct sum, namely,

$$([E_1]-[F_1])+([E_2]-[F_2]) := [E_1\oplus E_2]-[F_1\oplus F_2].$$

Note that the tensor product between bundles turns $K(M)$ even to a commutative ring.

Moreover, recall that when $f: M\to N$ is a continuous map, we have the bundle pull back $E\to f^*E$ for $E\in \mathrm{Vect}(N)$ and a resulting $f^*E\in\mathrm{Vect}(M)$. This gives rise to a homomorphism

$$f^*: K(N)\to K(M)$$

defined by $f^*([E]-[F])=[f^*E]-[f^*F]$.

An example is $M:=S^*Y$, $N=Y$, with the canonical projection,

$$\pi_1: S^*Y\to Y,\ \pi_1(y,\eta)=y. \tag{5.9}$$

(Non-trivial) vector bundles may appear in connection with elliptic boundary value problems, or, more generally, with families of Fredholm operators. The latter ones give rise to an equivalent definition of $K(M)$. The construction is closely related to the task to find entries $\sigma_\partial(\mathcal{A})(y,\eta)_{i,j}$ for $i,j=1,2$, $i+j>2$, for a given $\sigma_\partial(\mathcal{A})(y,\eta)_{1,1}$ that complete the latter Fredholm family to a family of isomorphisms, cf. (5.4). The general construction is as follows.

By $\mathcal{F}(H,\widetilde{H})$ for Hilbert spaces $H,\widetilde{H}$ we denote the set of all Fredholm operators $H\to\widetilde{H}$. Recall that $\mathcal{F}(H,\widetilde{H})$ is open in $\mathcal{L}(H,\widetilde{H})$, the space of all linear continuous operators in the operator norm topology.

Lemma 5.4. *Let $a\in C(M,\mathcal{F}(H,\widetilde{H}))$, and assume that $a(x): H\to\widetilde{H}$ is surjective for every $x\in M$. Then the family of kernels*

$$\ker_M a := \{\ker a(x) : x\in M\}$$

has the structure of a (continuous) vector bundle over M.

Proof. Let $\pi: H\to\ker a(x_1)$ be the orthogonal projection to $\ker a(x_1)$ for any fixed $x_1\in M$. Then the family of continuous operators

$$\begin{pmatrix} a(x)\\ \pi(x_1)\end{pmatrix}: H\to \begin{matrix}\widetilde{H}\\ \oplus\\ \ker a(x_1)\end{matrix} \tag{5.10}$$

is an isomorphism at $x=x_1$ and hence for all x in an open neighbourhood U of x_1. Therefore, by virtue of Lemma 2.3 the operator $\pi(x_1)$ induces isomorphisms $\pi(x_1):\ker a(x)\to\ker a(x_1)$ for all $x\in U$. This gives us a continuous family of maps

$$\{\ker a(x): x\in U\}\to U\times\ker a(x_1)$$

which is just the desired trivialisation when we identify $\ker a(x_1)$ with $\mathbb{C}^k$ for $k=\dim\ker a(x_1)$. □

Lemma 5.5. *For every $a \in C(M, \mathcal{F}(H, \widetilde{H}))$ there exists a $j_- \in \mathbb{N}$ and a linear operator* $\mathrm{ker} : \mathbb{C}^{j_-} \to \widetilde{H}$ *such that*

$$\begin{pmatrix} a(x) & k \end{pmatrix} : \begin{matrix} H \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \to \widetilde{H} \tag{5.11}$$

is surjective for every $x \in M$.

Proof. For every $x_1 \in M$ there exists a finite-dimensional subspace $W_1 \subset \widetilde{H}$ and an isomorphism $k_1 : \mathbb{C}^{j_1} \to W_1$ for $j_1 = \dim W_1$ such that

$$\begin{pmatrix} a(x) & k_1 \end{pmatrix} : \begin{matrix} H \\ \oplus \\ \mathbb{C}^{j_1} \end{matrix} \to \widetilde{H} \tag{5.12}$$

is surjective for $x = x_1$. Then (5.12) is surjective for all $x \in U_1$ for some open neighbourhood U_1 of x_1. Those neighbourhoods, parametrised by $x_1 \in M$, form an open covering of M. Since M is compact, there are finitely many points $x_1, \ldots, x_N \in M$ such that $M = \bigcup_{l=1}^{N} U_l$ for the respective U_l. Choosing operators k_l analogously as in (5.12) for every $1 \leq l \leq 1$, with dimensions j_l rather than j_1, we obtain the assertion for $k := (k_1, \ldots, k_N)$, and $j_- := \sum_{l=1}^{N}$. □

Proposition 5.6. *For every $a \in C(M, \mathcal{F}(H, \widetilde{H}))$ there exist vector bundles $J_-, J_+ \in$* $\mathrm{Vect}(M)$ *and a continuous family of isomorphisms*

$$\boldsymbol{a} := \begin{pmatrix} a(x) & k(x) \\ t(x) & q(x) \end{pmatrix} : \begin{matrix} H \\ \oplus \\ J_{-,x} \end{matrix} \to \begin{matrix} \widetilde{H} \\ \oplus \\ J_{+,x} \end{matrix}, \quad x \in M. \tag{5.13}$$

Proof. Choose $k = k(x)$ as in Lemma 5.5 for the trivial bundle $J_- = M \times \mathbb{C}^{j_-}$. Then applying Lemma 5.4 to the Fredholm family (5.11) we obtain that

$$\mathrm{ker}_M \begin{pmatrix} a(x) & k \end{pmatrix}$$

is a finite-dimensional subbundle of ${}^{\mathrm{t}}(H \oplus \mathbb{C}^{j_-})$, isomorphic to J_+ for some $J_+ \in \mathrm{Vect}(M)$. Choosing a bundle isomorphism

$$b_0 : \mathrm{ker}_M \begin{pmatrix} a(x) & k \end{pmatrix} \to J_+$$

and setting $b := b_0 \circ \pi(x)$ for the family of orthogonal projections $\pi(x) : {}^{\mathrm{t}}(H \oplus \mathbb{C}^{j_-}) \to \mathrm{ker} \begin{pmatrix} a(x) & k \end{pmatrix}$ we obtain our result when we set

$$t(x) := b(x)|_M, \; q(x) := b(x)|_{\mathbb{C}^{j_-}}. \qquad \square$$

Definition 5.7. For $a \in C(M, \mathcal{F}(H, \widetilde{H}))$ and any choice of (5.13) we set

$$\mathrm{ind}_M a := [J_+] - [J_-], \tag{5.14}$$

called the K-theoretic index of the Fredholm family a.

It can be proved, cf. [10], that $\mathrm{ind}_M a$ only depends on a but not on the specific choice of the family of isomorphisms (5.13).

In particular, we obtain the same $\mathrm{ind}_M a$ when we replace (5.13) by isomorphisms of the kind

$$\begin{pmatrix} \boldsymbol{a}(x) & h(x) \\ b(x) & d(x) \end{pmatrix} : \begin{matrix} H \oplus J_{-,x} \\ \oplus \\ L_{-,x} \end{matrix} \to \begin{matrix} \widetilde{H} \oplus J_{+,x} \\ \oplus \\ L_{+,x} \end{matrix}$$

for some $L_-, L_+ \in \mathrm{Vect}(M)$. Moreover, if $c \in C(M, \mathcal{L}(H, \widetilde{H}))$ is a family of compact operators, then

$$\mathrm{ind}_M(a + c) = \mathrm{ind}_M a.$$

The map $\mathrm{ind}_M : C(M, \mathcal{F}(H, \widetilde{H})) \to K(M)$ is surjective and induces a map only depending on the homotopy classes of Fredholm families. This gives rise to an equivalent definition of $K(M)$, cf. Jänich [11].

Let X be compact, $E, F \in \mathrm{Vect}(X)$, and let $A \in \mathcal{B}^{\mu,d}(X; (E, F; 0, 0))$ be elliptic with respect to σ_ψ (cf. the first condition of Definition 5.1). Then the restriction of $\sigma_\partial(A)(y, \eta)$ to S^*Y (denoted briefly again by $\sigma_\partial(A)(y, \eta)$) gives us a family of Fredholm operators

$$\sigma_\partial(A)(y, \eta) : H^s(\mathbb{R}_+) \otimes E' \to H^{s-\mu}(\mathbb{R}_+) \otimes F', \tag{5.15}$$

$s > \max\{\mu, d\} - \frac{1}{2}$, parametrised by $(y, \eta) \in S^*Y$. Therefore, we obtain an index element

$$\mathrm{ind}_{S^*Y} \sigma_\partial(A) \in K(S^*Y)$$

(which is independent of s). The following theorem was first formulated in the case of differential operators in the paper [2] by Atiyah and Bott, and then for pseudo-differential operators with the transmission property at the boundary in [4] by Boutet de Monvel, cf. also [24]. An analogue for edge operators may be found in [20], cf. also the author's joint papers [26], [27], with Seiler, and the references there.

Theorem 5.8. *A σ_ψ-elliptic operator $A \in \mathcal{B}^{\mu,d}(X; (E, F; 0, 0))$ can be completed by additional entries to a $(\sigma_\psi, \sigma_\partial)$-elliptic 2×2 block matrix operator*

$$\mathcal{A} \in \mathcal{B}^{\mu,d}(X; (E, F; J_-, J_+))$$

for suitable $J_-, J_+ \in \mathrm{Vect}(Y)$ with A in the upper left corner if and only if

$$\mathrm{ind}_{S^*Y} \sigma_\partial(A) \in \pi_1^* K(Y) \tag{5.16}$$

(*cf. the notation* (5.9)).

Proof. The condition (5.16) is necessary, since the Shapiro–Lopatinskij ellipticity means that (5.4) is a family of isomorphisms and hence, by virtue of (5.14),

$$\mathrm{ind}_{S^*Y} \sigma_\partial(A) = [\pi_1^* J_+] - [\pi_1^* J_-].$$

Conversely, the condition (5.16) allows us to construct a block matrix family of isomorphisms of the kind (5.13) with $\sigma_\partial(A)$ in the upper left corner and vector bundles over S^*Y that are pull backs of vector bundles over Y. The construction for every $(y,\eta) \in S^*Y$ is practically the same at that in the proof of Proposition 5.6. In addition we guarantee that the resulting block matrix operators locally belong to $\mathcal{B}^{\mu,d}(\overline{\mathbb{R}}_+; k, k; j_-, j_+)$ and smoothly depend on (y,η), for $k = \dim E_y = \dim F_y$, $j_\pm = \dim J_{\pm,y}$. The corresponding operator functions $k(y,\eta)$, $t(y,\eta)$ and $q(y,\eta)$ can be extended from S^*Y to $T^*Y \setminus 0$ by κ_λ-homogeneity of order μ. This can be done in terms of principal parts of symbols belonging to (4.25). Then applying an operator convention which assigns to such principal symbols associated operators gives us the additional entries. □

Remark 5.9. The proof of Theorem 5.8 shows how we can find (in principle all) Shapiro–Lopatinskij elliptic boundary value problems $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; \boldsymbol{v})$, $\boldsymbol{v} = (E, F; J_-, J_+)$, for any given σ_ψ-elliptic operator $A \in \mathcal{B}^{\mu,d}(X; (E, F; 0, 0))$ provided that the topological condition (5.16) is satisfied. It turns out that, from the point of view of the associated Fredholm indices, for every two such $\mathcal{A}_1, \mathcal{A}_2$ with the same upper left corner we can construct an elliptic operator R on the boundary such that

$$\mathrm{ind}\mathcal{A}_1 - \mathrm{ind}\mathcal{A}_2 = \mathrm{ind}R.$$

The latter relation is known as the Agranovich–Dynin formula (see also [4] and [15]). The proof is close to what we did in (2.19) modified for general 2×2 matrices rather than column matrices, cf. [15, Section 3.2.1.3.].

It may happen that $\sigma_\partial(A)(y,\eta)$ is a family of isomorphisms (5.15), i.e., that for the ellipticity of A with respect to σ_ψ and σ_∂ no additional entries are necessary. For instance, consider the symbol

$$r_-^\mu(\eta,\tau) := \big(\varphi(\frac{\tau}{C\langle\eta\rangle})\langle\eta\rangle - i\tau\big)^\mu$$

for some fixed $\varphi(t) \in \mathcal{S}(\mathbb{R})$ such that $\varphi(0) = 1$ and $\mathrm{supp}F^{-1}\varphi \subset \mathbb{R}_-$, for instance, $\varphi(\tau) := c^{-1}\int_{-\infty}^0 e^{-it\tau}\psi(t)dt$ for some $\psi \in C_0^\infty(\mathbb{R}_-)$ where $c := \int_{-\infty}^0 \psi(t)dt \neq 0$. Then, if $C > 0$ is a sufficiently large constant, we have $r_-^\mu(\eta,\tau) \in S_{\mathrm{tr}}^\mu(\mathbb{R}^n)$, and $r_-^\mu(\eta,\tau)$ is elliptic of order μ. This symbol can be smoothly connected with $\langle\eta,\tau\rangle^\mu$ far from $t = 0$ by forming $r_-^{\mu\omega(t)}(\eta,\tau)\langle\eta,\tau\rangle^{\mu(1-\omega(t))}$ for a real-valued $\omega \in C_0^\infty(\mathbb{R})$ such that $\omega \equiv 1$ in a neighbourhood of $t = 0$. Then, if we interpret $t \in \overline{\mathbb{R}}_+$ as the inner normal of a collar neighbourhood of Y in X there is obviously a σ_ψ-elliptic operator R_-^μ on X with such amplitude functions near the boundary, and $\sigma_\partial(R_-^\mu)$ has the desired property, indeed. A similar construction is possible in the vector bundle set-up, which gives us such an operator $R_{-,E}^\mu \in \mathcal{B}^{\mu,d}(X; (E, E; 0, 0))$,

$$R_{-,E}^\mu : H^s(\mathrm{int}X, E) \to H^{s-\mu}(\mathrm{int}X, E). \tag{5.17}$$

In addition the operator convention can be chosen in such a way that (5.17) is an isomorphism for every $s > \max\{\mu, d\} - \frac{1}{2}$. More details on such constructions may be found in the paper [9] of Grubb, see also the author's joint monograph with Harutyuyan [10, Section 4.1].

Using the fact that there are also order-reducing operators of any order on the boundary (which is a compact C^∞ manifold, cf. the formulas (2.5), (2.12)) we can compose any $(\sigma_\psi, \sigma_\partial)$-elliptic operator $\mathcal{A} \in \mathcal{B}^{\mu,d}(X; (E, F; J_-, J_+))$ by diagonal matrices of order reductions to a $(\sigma_\psi, \sigma_\partial)$-elliptic operator $\mathcal{A}_0 \in \mathcal{B}^{0,0}(X; (E, F; J_-, J_+))$. For many purposes it is convenient to deal with operators of order and type zero, and we will assume that for a while, in order to illustrate other interesting aspects of elliptic pseudo-differential boundary value problems.

Let us set

$$\Xi^* := S^*X|_Y \cup N^* \tag{5.18}$$

with $S^*X|_Y$ denoting the restriction of the unit cosphere bundle to the boundary, and

$$N^* = \{(y, 0, 0, \tau) \in T^*X|_Y : -1 \le \tau \le 1\}$$

which refers to the splitting of variables $x = (y, t)$ near the boundary. The interval bundle N^* is trivial and its fibres $N^*_y = \{(y, 0, 0, \tau) : -1 \le \tau \le 1\}$ connect the south poles ($\tau = -1$) with the north poles ($\tau = +1$) of $S^*X|_y$, $y \in Y$. In other words, Ξ^* is a kind of cage with bars N^*_y, called the conormal cage. Let

$$\pi_{\mathrm{c}} : \Xi^* \to Y$$

denote the canonical projection.

Remark 5.10. Let $A \in \mathcal{B}^{0,0}(X; (E, F; 0, 0))$ be σ_ψ-elliptic; then $\sigma_\psi(A)X|_Y$ extends to an isomorphism

$$\sigma'_\psi(A) : \pi^*_{\mathrm{c}} E' \to \pi^*_{\mathrm{c}} F'. \tag{5.19}$$

In fact, $\sigma_\psi(A)(y, 0, \eta, \tau) : E_y \to F_y$ is a family of isomorphisms for all $(y, 0, \eta, \tau) \in S^*X|_Y$. By virtue of the transmission property we have

$$\sigma_\psi(A)(y, 0, 0 - 1) = \sigma_\psi(A)(y, 0, 0, +1). \tag{5.20}$$

The principal symbol $\sigma_\psi(A)(x, \xi)$ is altogether (positively) homogeneous of order zero in $\xi \neq 0$; in particular, we have

$$\begin{aligned} \sigma_\psi(A)(y, 0, 0, \tau) &= \sigma_\psi(A)(y, 0, 0, -1) \text{ for all } \tau < 0, \\ \sigma_\psi(A)(y, 0, 0, \tau) &= \sigma_\psi(A)(y, 0, 0, +1) \text{ for all } \tau > 0. \end{aligned}$$

Now the relation (5.20) shows that $\sigma_\psi(A)(y, 0, 0, \tau)$ does not depend on $\tau \neq 0$, and hence it extends to N^*_y when we define

$$\sigma''_\psi(A)(y) := \sigma_\psi(A)(y, 0, 0, 0) := \sigma_\psi(A)(y, 0, 0, 1).$$

We obtain an isomorphism

$$\sigma''_\psi(A) : E' \to F'. \tag{5.21}$$

Let us now return to operators on the half-axis

$$\mathrm{op}^+(\sigma_\psi(A)|_{t=0})(y, \eta) : L^2(\mathbb{R}_+) \otimes E'_y \to L^2(\mathbb{R}_+) \otimes F'_y$$

parametrised by $(y, \eta) \in S^*Y$. By virtue of (5.21) we may replace F' by E'. As usual we interpret $\bigcup_{y \in Y} L^2(\mathbb{R}_+) \otimes E'_y$ as $L^2(\mathbb{R}_+) \otimes E'$ which is a Hilbert space bundle over Y (by Kuiper's theorem it is trivial). Set $a(y,\eta,\tau) := \sigma_\psi(A)(y,0,\eta,\tau)$ which is a family of isomorphisms

$$a(y,\eta,\tau) : E'_y \to E'_y,$$

$(y,\eta,\tau) \in S^*X|_Y$. By virtue of the homogeneity we have $a(y,\lambda\eta,\lambda\tau) = a(y,\eta,\tau)$ for all $\lambda \in \mathbb{R}_+$, in particular,

$$a(y, \frac{\eta}{|\tau|}, \frac{\tau}{|\tau|}) = a(y,\eta,\tau)$$

for all $\tau \neq 0$. Thus (5.20) gives us

$$\lim_{\tau\to-\infty} a(y,\eta,\tau) = a(y,0,-1) = a(y,0,+1) = \lim_{\tau\to+\infty} a(y,\eta,\tau).$$

This fits to the picture of symbols with the transmission property in τ described in Section 3. In other words, we have

$$a(y,\eta,\tau) \in S^0_{\mathrm{tr}}(\mathbb{R}) \otimes \mathrm{Iso}(E_y, E_y)$$

for every fixed $(y,\eta) \in S^*Y$ (here $\mathrm{Iso}(\cdot,\cdot)$ means the space of isomorphisms between the vector spaces in parenthesis). The operators

$$\mathrm{op}^+(a)(y,\eta) : L^2(\mathbb{R}_+) \otimes E_y \to L^2(\mathbb{R}_+) \otimes E_y$$

are Fredholm and their pointwise index is equal to the winding number of the curve

$$L(a) := \{\det a(y,\eta,\tau) : \tau \in \mathbb{R}\} \subset \mathbb{C}.$$

This is a useful information for the construction of extra trace and potential conditions in an elliptic BVP. It would be optimal to know the dimensions of kernel and cokernel; of course, those are not necessarily constant in y.

6. The Anti-Transmission Property

In this section we return to scalar symbols (for simplicity). Recall that the transmission property of a symbol $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ means the condition (3.11). In general, the curve

$$L(a) = \{a(\tau) \in \mathbb{C} : \tau \in \mathbb{R}\} \tag{6.1}$$

is not closed. Let $a(y,t,\eta,\tau) \in S^0_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n_{\eta,\tau})$ be an elliptic symbol, $a_{(0)}$ its homogeneous principal part, and $a(\tau) := a_{(0)}(y,0,\eta,\tau)$ for fixed $(y,\eta) \in T^*\Omega \setminus 0$. Then, similarly as in elliptic BVPs with the transmission property, a task is to find a bijective 2×2 block matrix

$$\boldsymbol{a} = \begin{pmatrix} \mathrm{op}^+(a) & k \\ b & q \end{pmatrix} : \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_-} \end{matrix} \to \begin{matrix} L^2(\mathbb{R}_+) \\ \oplus \\ \mathbb{C}^{j_+} \end{matrix}$$

for suitable $j_\pm \in \mathbb{N}$. This is possible if and only if

$$\mathrm{op}^+(a) : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+) \tag{6.2}$$

is a Fredholm operator. Set

$$M(a) := \{z \in \mathbb{C} : z = (1-\lambda)a_0^+ + \lambda a_0^-,\ 0 \le \lambda \le 1\}. \tag{6.3}$$

The following result is well-known.

Theorem 6.1. *The operator* (6.2) *is Fredholm if and only if*

$$L(a) \cup M(a) \subset \mathbb{C} \setminus \{0\}. \tag{6.4}$$

A proof of the Fredholm property of (6.2) under the condition (6.4) is given in Eskin's book [7]; it is also noted there that (6.1) is necessary. Details of that part of the proof may be found in [16].

Corollary 6.2. *Let $a(\tau) \in S^0_{\mathrm{cl}}(\mathbb{R})$ be elliptic in the sense $L(a) \subset \mathbb{C}\setminus\{0\}$. Then* (6.2) *is a Fredholm operator if and only if*

$$0 \notin M(a). \tag{6.5}$$

The union

$$C(a) := L(a) \cup M(a)$$

is a continuous and piecewise smooth curve which can be represented as the image under a continuous map $\gamma : [0,1] \to \mathbb{C}$. If (6.4) holds, we have a winding number wind $C(a)$, and there is the well-known relation

$$\operatorname{ind} \operatorname{op}^+(a) = \text{ wind } C(a).$$

Observe that

$$a_0^- = -a_0^+ \Rightarrow 0 \in M(a),$$

i.e., the operator (6.2) cannot be Fredholm in this case.

Definition 6.3. A symbol $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ for $\mu \in \mathbb{Z}$ is said to have the anti-transmission property if the coefficients $a_j^\pm$ in the asymptotic expansion (3.8) satisfy the condition

$$a_j^+ = -a_j^- \text{ for all } j \in \mathbb{N}. \tag{6.6}$$

Let $S^\mu_{-\mathrm{tr}}(\mathbb{R})$ denote the space of all symbols with the anti-transmission property.

Note that (6.6) is just the opposite of (3.11).

Proposition 6.4. *Every $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ can be written in the form*

$$a(\tau) = \frac{1}{2}(a_{\mathrm{tr}}(\tau) + a_{-\mathrm{tr}}(\tau)) + c(\tau) \tag{6.7}$$

for suitable $a_{\mathrm{tr}}(\tau) \in S^\mu_{\mathrm{tr}}(\mathbb{R})$, $a_{-\mathrm{tr}}(\tau) \in S^\mu_{-\mathrm{tr}}(\mathbb{R})$, $c(\tau) \in \mathcal{S}(\mathbb{R})$.

Proof. Similarly as (3.7), (3.8) we form a symbol

$$b(\tau) \sim \sum_{j=0}^{\infty} \chi(\tau)(a_j^- \theta^+(\tau) + a_j^+ \theta^-(\tau))(i\tau)^{\mu-j}$$

belonging to $S^\mu_{\mathrm{cl}}(\mathbb{R})$, where $\chi(\tau)$ is some excision function. Then we obviously have $a_{\mathrm{tr}}(\tau) := a(\tau) + b(\tau) \in S^\mu_{\mathrm{tr}}(\mathbb{R})$, $a_{-\mathrm{tr}}(\tau) := a(\tau) - b(\tau) \in S^\mu_{-\mathrm{tr}}(\mathbb{R})$, and we obtain the relation (6.7). □

Remark 6.5. A symbol $a(\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R})$ has the anti-transmission property exactly when

$$a_{(\mu-j)}(\tau) = (-1)^{\mu-j+1} a_{(\mu-j)}(-\tau) \tag{6.8}$$

for all $\tau \in \mathbb{R} \setminus \{0\}$ and all $j \in \mathbb{N}$.

In fact, the anti-transmission property means that

$$a_{(\mu-j)}(\tau) = (c_j\theta^+(\tau) - c_j\theta^-(\tau))(i\tau)^{\mu-j}$$

for constants $c_j := a_j^+ \in \mathbb{C}$. This yields the relation

$$\begin{aligned} a_{(\mu-j)}(-\tau) &= (c_j\theta^+(-\tau) - c_j\theta^-(-\tau))(-i\tau)^{\mu-j} \\ &= (-1)^{\mu-j}(c_j\theta^-(\tau) - c_j\theta^+(\tau)) = (-1)^{\mu-j+1} a_{(\mu-j)}(\tau), \end{aligned}$$

using $\theta^+(-\tau) = \theta^-(\tau), \theta^-(-\tau) = \theta^+(\tau)$.

Conversely, from (6.8) we obtain

$$\begin{aligned} \{a_j^+\theta^+(\tau) + a_j^-\theta^-(\tau)\}(i\tau)^{\mu-j} &= (-1)^{\mu-j+1}\{a_j^+\theta^+(-\tau) + a_j^-\theta^-(-\tau)\}(-i\tau)^{\mu-j} \\ &= \{\theta^- a_j^+\theta^-(\tau) + a_j^-\theta^+(\tau)\}(i\tau)^{\mu-j}. \end{aligned}$$

This gives us $a_j^+ = -a_j^-$ which are the conditions of Definition 6.3.

Observe that there is also a higher-dimensional analogue of Definition 4.11 for symbols $p(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}_{y,\tau})$ where instead of (4.18) we ask

$$D^{\alpha}_{y,t} D^{\beta}_{\eta,\tau}\{p_{(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j+1} p_{(\mu-j)}(y,t,-\eta,-\tau)\} = 0$$

on $\{(y,t,\eta,\tau) : y \in \Omega,\ t = 0,\ \eta = 0,\ \tau \in \mathbb{R} \setminus \{0\}\}$ for all α, β, j. This gives us the symbol class $S^{\mu}_{-\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$. There is then a higher-dimensional analogue of Proposition 6.4.

In fact, let $a(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ be arbitrary, and define the homogeneous components

$$a_{\mathrm{tr},(\mu-j)}(y,t,\eta,\tau) := a_{(\mu-j)}(y,t,\eta,\tau) + (-1)^{\mu-j} a_{(\mu-j)}(y,t,-\eta,-\tau)$$

and

$$a_{-\mathrm{tr},(\mu-j)}(y,t,\eta,\tau) := a_{(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j} a_{(\mu-j)}(y,t,-\eta,-\tau)$$

for all j and $(y,t,\eta,\tau) \in \Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \setminus \{0\}$. Then we have

$$\begin{aligned} &a_{\mathrm{tr},(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j} a_{\mathrm{tr},(\mu-j)}(y,t,-\eta,-\tau) \\ &\quad = a_{(\mu-j)}(y,t,\eta,\tau) + (-1)^{\mu-j} a_{(\mu-j)}(y,t,-\eta,-\tau) \\ &\qquad - (-1)^{\mu-j}\{a_{(\mu-j)}(y,t,-\eta-\tau) + (-1)^{\mu-j} a_{(\mu-j)}(y,t,\eta,\tau)\} = 0, \end{aligned}$$

and

$$\begin{aligned} &a_{-\mathrm{tr},(\mu-j)}(y,t,\eta,\tau) + (-1)^{\mu-j} a_{-\mathrm{tr},(\mu-j)}(y,t,-\eta,-\tau) \\ &\quad = a_{(\mu-j)}(y,t,\eta,\tau) - (-1)^{\mu-j} a_{(\mu-j)}(y,t,-\eta,-\tau) \\ &\qquad + (-1)^{\mu-j}\{a_{(\mu-j)}(y,t,-\eta,-\tau) - (-1)^{\mu-j} a_{(\mu-j)}(y,t,\eta,\tau)\} = 0 \end{aligned}$$

for all j and $(y,t,\eta,\tau) \in \Omega \times \overline{\mathbb{R}}_+ \times (\mathbb{R}^n \setminus \{0\})$. In other words, if we define

$$a_{\mathrm{tr}}(y,t,\eta,\tau) \sim \sum_{j=0}^{\infty} \chi(\eta,\tau) a_{\mathrm{tr},(\mu-j)}(y,t,\eta,\tau),$$

$$a_{-\mathrm{tr}}(y,t,\eta,\tau) \sim \sum_{j=0}^{\infty} \chi(\eta,\tau) a_{-\mathrm{tr},(\mu-j)}(y,t,\eta,\tau),$$

then a_{tr} has the transmission property, $a_{-\mathrm{tr}}$ the anti-transmission property; here $\chi(\eta,\tau)$ is any excision function. Thus we have proved the following result.

Proposition 6.6. *Every symbol* $a(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{cl}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ *can be written in the form*

$$a(y,t,\eta,\tau) = \frac{1}{2}\{a_{\mathrm{tr}}(y,t,\eta,\tau) + a_{-\mathrm{tr}}(y,t,\eta,\tau)\} + c(y,t,\eta,\tau)$$

for symbols $a_{\mathrm{tr}}(y,t,\eta,\tau) \in S^{\mu}_{\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$, $a_{-\mathrm{tr}}(y,t,\eta,\tau) \in S^{\mu}_{-\mathrm{tr}}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ (*uniquely determined* $\operatorname{mod} S^{-\infty}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$), $c(y,t,\eta,\tau) \in S^{-\infty}(\Omega \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$.

The role of those considerations here is not to really carry out a calculus of BVPs having the anti-transmission property. As noted at the beginning such a calculus is possible indeed, however, based on tools from the cone and edge calculus that go beyond the scope of this exposition. Let us only mention that for such a program we need to reorganise both the symbolic structure and the operator conventions of our operators as well as the spaces that substitute the standard Sobolev spaces. Details on the new boundary symbolic calculus for zero order operators on the half-axis may be found in [7], and in [21]. Concerning the cone and edge calculus in general, cf. [20], [22], [23].

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B.-W. Schulze
Universität Potsdam
Institut für Mathematik
Am Neuen Palais 10
14469 Potsdam
Germany
e-mail: `schulze@math.uni-potsdam.de`

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Spectral Invariance of SG Pseudo-Differential Operators on $L^p(\mathbb{R}^n)$

Aparajita Dasgupta and M. W. Wong

Abstract. We prove the spectral invariance of SG pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, by using the equivalence of ellipticity and Fredholmness of SG pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. A key ingredient in the proof is the spectral invariance of SG pseudo-differential operators on $L^2(\mathbb{R}^n)$.

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1. SG Pseudo-Differential Operators

Let us first recall that for $m_1, m_2 \in (-\infty, \infty)$, S^{m_1,m_2} is the set of all functions σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|x|)^{m_2-|\alpha|}(1+|\xi|)^{m_1-|\beta|}, \quad x,\xi \in \mathbb{R}^n.$$

We call any function σ in S^{m_1,m_2} a SG symbol of order m_1, m_2. It is clear that if $\sigma \in S^{m_1,m_2}$ and $m_2 \le 0$, then $\sigma \in S^{m_1}$, where S^{m_1} is the class of classical symbols studied extensively in the book [7] by Wong.

Let $\sigma \in S^{m_1,m_2}$. Then we define the SG pseudo-differential operator T_σ with symbol σ by

$$(T_\sigma \varphi)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi)\hat{\varphi}(\xi)\, d\xi, \quad x \in \mathbb{R}^n,$$

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for all functions φ in $\mathcal{S}$, where

$$\hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \varphi(x)\, dx, \quad \xi \in \mathbb{R}^n.$$

As a note on terminology, SG pseudo-differential operators are pseudo-differential operators with symbols of global type and they are also called by Schulze [6] pseudo-differential operators with conical exit at infinity. It can be proved easily that $T_\sigma : \mathcal{S} \to \mathcal{S}$ is a continuous linear mapping.

The domain of a SG pseudo-differential operator can be extended from the Schwartz space $\mathcal{S}$ to the space $\mathcal{S}'$ of all tempered distributions by means of the formal adjoint. It can then be checked easily that $T_\sigma : \mathcal{S}' \to \mathcal{S}'$ is a continuous linear mapping. In fact, we have the following theorem.

Theorem 1.1. *Let $\sigma \in S^{0,0}$. Then $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bounded linear operator for $1 < p < \infty$.*

Theorem 1.1 is an easy consequence of the L^p-boundedness of pseudo-differential operators with symbols in S^0 as given in Theorem 10.7 of the book [7].

Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$. Then σ is said to be elliptic if there exist positive constants C and R such that

$$|\sigma(x,\xi)| \geq C(1+|x|)^{m_2}(1+|\xi|)^{m_1}, \quad |x|^2 + |\xi|^2 \geq R.$$

The following theorem is contained in Theorem 1.4.36 of the book [6] by Schulze.

Theorem 1.2. *Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$, be elliptic. Then there exists a symbol τ in $S^{-m_1,-m_2}$ such that*

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where R and S are infinitely smoothing in the sense that they are SG pseudo-differential operators with symbols in $\cap_{k_1,k_2\in\mathbb{R}} S^{k_1,k_2}$.

The SG pseudo-differential operator T_τ is known as a parametrix of T_σ.

For $s_1, s_2 \in (-\infty, \infty)$, we let J_{s_1,s_2} be the Bessel potential of order s_1, s_2 defined by

$$J_{s_1,s_2} = T_{\sigma_{s_1,s_2}},$$

where

$$\sigma_{s_1,s_2}(x,\xi) = (1+|x|^2)^{-s_2}(1+|\xi|^2)^{-s_1}, \quad x, \xi \in \mathbb{R}^n.$$

For $1 < p < \infty$ and $-\infty < s_1, s_2 < \infty$, we define the L^p-Sobolev space $H^{s_1,s_2,p}$ of order s_1, s_2 by

$$H^{s_1,s_2,p} = \{u \in \mathcal{S}' : J_{-s_1,-s_2} u \in L^p(\mathbb{R}^n)\}.$$

Then $H^{s_1,s_2,p}$ is a Banach space in which the norm $\| \, \|_{s_1,s_2,p}$ is given by

$$\|u\|_{s_1,s_2,p} = \|J_{-s_1,-s_2} u\|_{L^p(\mathbb{R}^n)}, \quad u \in H^{s_1,s_2,p},$$

where $\| \|_{L^p(\mathbb{R}^n)}$ is the norm in $L^p(\mathbb{R}^n)$.

We have the following Sobolev embedding theorem, which is proved in [3].

Theorem 1.3. *Let $s_1, s_2, t_1, t_2 \in (-\infty, \infty)$ be such that $s_1 \leq t_1$ and $s_2 \leq t_2$. Then $H^{t_1,t_2,p} \subseteq H^{s_1,s_2,p}$ and the inclusion $i : H^{t_1,t_2,p} \hookrightarrow H^{s_1,s_2,p}$ is a bounded linear operator. Moreover, if $s_1 < t_1$ and $s_2 < t_2$, then the inclusion $i : H^{t_1,t_2,p} \hookrightarrow H^{s_1,s_2,p}$ is a compact operator.*

Theorem 1.1 can now be put in the following more general setting.

Theorem 1.4. *Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$. Then for $1 < p < \infty$ and $-\infty < s_1, s_2 < \infty$, $T_\sigma : H^{s_1,s_2,p} \to H^{s_1-m_1,s_2-m_2,p}$ is a bounded linear operator.*

Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$. Then T_σ is a linear operator from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ with dense domain $\mathcal{S}$. We can then introduce the minimal operator $T_{\sigma,0}$ as its smallest closed extension and the maximal operator $T_{\sigma,1}$ as the largest closed extension in the sense that if B is any closed extension of T_σ such that $\mathcal{S}$ is contained in the domain of the true adjoint B^t of B, then $T_{\sigma,1}$ is an extension of B. In fact, if σ is elliptic, then

$$T_{\sigma,0} = T_{\sigma,1}$$

and

$$\mathcal{D}(T_{\sigma,0}) = H^{m_1,m_2,p}.$$

Applying the compact Sobolev embedding to the infinitely smoothing remainders R and S in Theorem 1.2, we have the following result in [3].

Theorem 1.5. *Let $\sigma \in S^{m_1,m_2}$, $m_1, m_2 > 0$, be elliptic. Then for $1 < p < \infty$, $T_{\sigma,0}$ is a Fredholm operator on $L^p(\mathbb{R}^n)$ with domain $H^{m_1,m_2,p}$. Furthermore, if $\sigma \in S^{0,0}$ is elliptic, then the bounded linear operator $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is Fredholm.*

The following result due to Dasgupta [2] tells us that the converse is also true.

Theorem 1.6. *Let $\sigma \in S^{m_1,m_2}$, $-\infty < m_1, m_2 < \infty$, be such that $T_\sigma : H^{s_1,s_2,p} \to H^{s_1-m_1,s_2-m_2,p}$ is a Fredholm operator, where $1 < p < \infty$. Then σ is elliptic.*

All the results hitherto described are contained in the Ph.D. dissertation [2] of Dasgupta.

The aim of this paper is to use the equivalence of ellipticity and Fredhomness of SG pseudo-differential operators to prove the spectral invariance of these operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, to the effect that if $\sigma \in S^{0,0}$ is such that $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is invertible, then $T_\sigma^{-1} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is also a pseudo-differential operator with symbol in $S^{0,0}$. A proof given in this paper is based on the L^2 spectral invariance of these results first proved by Grieme [4], and is recalled in Section 2 for the sake of making the paper self-contained and the proof more widely disseminated. The L^p spectral invariance is proved in Section 3 by a descent of the problem to that of the L^2 invariance.

2. L^2 Spectral Invariance

Theorem 2.1. *Let $\sigma \in S^{0,0}$ be such that the pseudo-differential operator $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible. Then $T_\sigma^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is also a pseudo-differential operator with symbol in $S^{0,0}$.*

Proof. Since $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible, it follows that $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Fredholm with zero index. So, by Theorem 1.6, σ is an elliptic symbol in $S^{0,0}$. Thus, by Theorem 1.2, there exists a symbol τ in $S^{0,0}$ such that

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where R and S are infinitely smoothing in the sense that they are pseudo-differential operators with symbols in $\cap_{k_1,k_2\in\mathbb{R}} S^{k_1,k_2}$, and by Theorems 1.3 and 1.4, R and S are compact operators on $L^2(\mathbb{R}^n)$. Now, T_τ is Fredholm and hence by Theorem 1.6, T_τ is elliptic. Also,

$$i(T_\tau) + i(T_\sigma) = i(T_\tau T_\sigma) = i(I + R) = 0 \;\Rightarrow\; i(T_\tau) = 0.$$

To see that the null space $N_{L^2}(T_\tau)$ of $T_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a subspace of $\mathcal{S}$, let $u \in N_{L^2}(T_\tau)$. Then

$$T_\tau u = 0 \;\Rightarrow\; T_\sigma T_\tau u = 0 \;\Rightarrow\; (I + S)u = 0 \;\Rightarrow\; u = -Su.$$

Since $\cap_{s_1,s_2\in\mathbb{R}} H^{s_1,s_2,2} = \mathcal{S}$, it follows that

$$u = -Su \in \mathcal{S}.$$

Similarly, the null space $N_{L^2}(T_\tau^t)$ of the true adjoint $T_\tau^t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ of $T_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is also a subspace of $\mathcal{S}$. Now, we write

$$L^2(\mathbb{R}^n) = N_{L^2}(T_\tau) \oplus N_{L^2}(T_\tau)^\perp$$

and

$$L^2(\mathbb{R}^n) = N_{L^2}(T_\tau^t) \oplus R_{L^2}(T_\tau),$$

where $R_{L^2}(T_\tau)$ is the range of $T_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Let $P = iF\pi$, where π is the projection of $L^2(\mathbb{R}^n)$ onto $N_{L^2}(T_\tau)$, F is an isomorphism of $N_{L^2}(T_\tau)$ onto $N_{L^2}(T_\tau^t)$ and i is the inclusion of $N_{L^2}(T_\tau^t)$ into $L^2(\mathbb{R}^n)$. To wit, the figure

$$\begin{array}{l} L^2(\mathbb{R}^n) \\ \quad\downarrow \pi \\ N_{L^2}(T_\tau) \oplus N_{L^2}(T_\tau)^\perp \\ \quad\updownarrow F \\ N_{L^2}(T_\tau^t) \oplus R_{L^2}(T_\tau) \\ \quad\downarrow i \\ L^2(\mathbb{R}^n) \end{array}$$

best illustrates the situation. Then $T_\tau + P : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bijective parametrix of T_σ. Therefore, without loss of generality, we may assume that the parametrix $T_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bijective. So, $I + R$ is bijective. In fact,

$$(I + R)^{-1} = I + K,$$

where K is infinitely smoothing. Indeed, there exists a bounded linear operator $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that

$$(I + R)(I + K) = I.$$

So,

$$K = -R - RK : L^2(\mathbb{R}^n) \to \mathcal{S}.$$

Also,

$$K^* = -R^* - R^*K^* : L^2(\mathbb{R}^n) \to \mathcal{S}.$$

So, by Theorem 2.4.80 in [5], the kernel of $K : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Schwartz function on $\mathbb{R}^n \times \mathbb{R}^n$. Thus,

$$T_\sigma^{-1} T_\tau^{-1} = I + K$$

or equivalently

$$T_\sigma^{-1} = (I + K) T_\tau$$

and this completes the proof. □

3. L^p Spectral Invariance

Theorem 3.1. *Let $\sigma \in S^{0,0}$ be such that the pseudo-differential operator $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is invertible, where $1 < p < \infty$. Then $T_\sigma^{-1} : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is also a pseudo-differential operator with symbol in $S^{0,0}$.*

Proof. Since $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is invertible, it follows that $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is Fredholm with zero index. So, by Theorem 1.6, σ is an elliptic symbol in $S^{0,0}$. Thus, by Theorem 1.5, $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is Fredholm. So, there exists a symbol τ in $S^{0,0}$ such that

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where R and S are infinitely smoothing. We first show that $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is injective. To this end, let $u \in L^2(\mathbb{R}^n)$ be such that $T_\sigma u = 0$. Then

$$T_\tau T_\sigma u = 0 \;\Rightarrow\; (I + R)u = 0 \;\Rightarrow\; u = -Ru \in \mathcal{S}. \tag{3.1}$$

So, u is also in the null space of $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, which is injective. Thus, $u = 0$, i.e., $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is injective. So,

$$N_{L^2}(T_\sigma) = \{0\},$$

where $N_{L^2}(T_\sigma)$ is the null space of $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Next, we want to show that $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is surjective. To do this, let u be a function in the null space $N_{L^2}(T_\sigma^t)$ of the true adjoint $T_\sigma^t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ of $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. Then

$$T_\sigma^t u = 0 \ \Rightarrow \ T_{\sigma^*} u = 0,$$

where σ^* is in $S^{0,0}$ and is the symbol of the formal adjoint of T_σ. Since σ^* is elliptic, we can use a left parametrix of T_{σ^*} as in (3.1) to conclude that $u = 0$. This proves that $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is surjective and hence bijective. So, by the L^2 spectral invariance, the inverse $T_\sigma^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the pseudo-differential operator $T_\omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where $\omega \in S^{0,0}$. So,

$$T_\sigma^{-1}\varphi = T_\omega \varphi, \quad \varphi \in \mathcal{S}.$$

Since $\mathcal{S}$ is dense in $L^p(\mathbb{R}^n)$, it follows that $T_\sigma^{-1} = T_\omega$ on $L^p(\mathbb{R}^n)$ and the theorem is proved. □

4. Conclusions

Ellipticity is a condition on the operators at infinity in which the lower-order terms cannot be neglected. Fredholmness measures the almost invertibility of the operators and depends very much on the space on which the operators live. For a good theory of pseudo-differential operators on a given function space, an important ingredient is the equivalence of ellipticity and Fredholmness, which can then be used to establish the spectral invariance.

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Aparajita Dasgupta
Department of Mathematics
Indian Institute of Science
Bangalore-560012
India
e-mail: adgupta@math.iisc.ernet.in

M. W. Wong
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada
e-mail: mwwong@mathstat.yorku.ca

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Edge-Degenerate Families of Pseudo-Differential Operators on an Infinite Cylinder

J. Abed and B.-W. Schulze

Abstract. We establish a parameter-dependent pseudo-differential calculus on an infinite cylinder, regarded as a manifold with conical exits to infinity. The parameters are involved in edge-degenerate form, and we formulate the operators in terms of operator-valued amplitude functions.

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1. Introduction

The analysis of (pseudo-) differential operators on a manifold (stratified space) with higher polyhedral singularities employs to a large extent parameter-dependent families of operators on a (in general singular) base X of a cone, where the parameters (ρ, η) have the meaning of covariables in cone axis and edge direction, respectively. These covariables appear in edge-degenerate form, i.e., in the combination $(r\rho, r\eta)$ where $r \in \mathbb{R}_+$ is the axial variable of the cone $X^\Delta = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ with base X. For operators on a wedge $X^\Delta \times \Omega \ni (\cdot, y)$ it is essential to understand the structure of what we call edge symbols, together with associated weighted distributions on the cone. It is natural to split up the investigation into a part for $r \to 0$, i.e., close to the tip of the cone and a part for $r \to \infty$, the conical exit to infinity. For higher corner theories it is desirable to do that in an axiomatic manner, i.e., to point out those structures which make the calculus iterative. For the case $r \to 0$ the authors developed in [1] such an axiomatic approach. What concerns $r \to \infty$ it seems to be advisable first to concentrate on the case when the base X is smooth and compact. It turns out that the edge-degeneration of symbols in a calculus on $\mathbb{R}$ up to infinity causes a highly non-standard behaviour with respect to symbolic rules for operator-valued symbols (to be invented in the right manner). This has to be analysed first, where the approach should rely on the principles and key properties that are essential for the iteration. The goal of our paper is just to develop some crucial steps in that sense.

To be more precise, we show (here for a smooth compact manifold as the base of a cone) how very simple and general phenomena on the norm-growth of parameter-dependent pseudo-differential operators in the sense of Theorem 2.1 are sufficient to induce the essential properties of a calculus on the manifold $X^{\asymp} \cong \mathbb{R} \times X$ with conical exits $r \to \pm\infty$. In other words, knowing a suitable variant of Theorem 2.1 for a singular (compact) manifold, we can expect essentially the same things on the respective singular cylinder. Details in that case go beyond the scope of the present paper; let us only note that the article [1] just contains an analogue of Theorem 2.1 for a base manifold with edge. We introduce here a pseudo-differential calculus in spaces $H^{s;g}_{\mathrm{cone}}(X^{\asymp})$ in a self-contained manner, including those spaces themselves. In the smooth case there is, of course, also a completely independent approach, usually organised without parameter η, well-known under the key-words operators on manifolds with conical exit to infinity, here realised on such a manifold $X^{\asymp}$ modelled on an infinite cylinder. Concerning the generalities we refer to Shubin [14], Parenti [9], Cordes [5], or to the corresponding sections in [11]. There are also several papers for singular X, cf., for instance, [12], or [3], [4], but those are based on more direct information from corner-degenerate symbols. We think that the present idea admits to manage the iterative process for higher singularities in a more transparent way. Let us finally note that motivations and examples may be found already in Rempel and Schulze [10]; since then many authors contributed to the general concepts of pseudo-differential calculi for conical points or edges, see, in particular, the references in [6].

2. A New Class of Operator-Valued Symbols

2.1. Edge-Degenerate Families on a Smooth Compact Manifold

Edge-degenerate families of pseudo-differential operators occur in connection with the edge symbols of operators of the form

$$A = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y)\,(-r\partial_r)^j\,(rD_y)^\alpha \tag{2.1}$$

with coefficients $a_{j\alpha} \in C^\infty\big(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X)\big)$ for an open set $\Omega \subseteq \mathbb{R}^q_y$; here X is a smooth compact manifold, and $\mathrm{Diff}^\nu(X)$ is the space of all differential operators of order ν on X with smooth coefficients. The analysis of such edge-degenerate operators is crucial for understanding the solvability of elliptic equations on spaces with polyhedral singularities, (cf. [13], [2], or [8]). Apart from the standard homogeneous principal symbol of (2.1) which is a C^∞ function on $T^*(\mathbb{R}_+ \times X \times \Omega) \setminus 0$, we have the so-called principal edge symbol

$$\sigma_\wedge(A) := r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(0,y)\,(-r\partial_r)^j\,(r\eta)^\alpha \tag{2.2}$$

parametrised by $(y, \eta) \in T^*\Omega \setminus 0$ and with values in Fuchs type differential operators on the open infinite stretched cone $X^\wedge := \mathbb{R}_+ \times X$ with base X. For the construction of parametrices of A (in the elliptic case) we need to understand, in particular, the nature of parameter-dependent parametrices of operator families

$$r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(0, y)(-ir\rho)^j (r\eta)^\alpha \tag{2.3}$$

on $\mathbb{R}_+ \times X$ for $r \to \infty$. We often set $\tilde{\rho} = r\rho$, $\tilde{\eta} = r\eta$. If A is edge-degenerate elliptic (cf. [11], [6]) it turns out that $\sum_{j+|\alpha|\leq\mu} a_{j\alpha}(0, y)(-i\tilde{\rho})^j (\tilde{\eta})^\alpha$ is parameter-dependent elliptic on X with parameters $(\tilde{\rho}, \tilde{\eta}) \in \mathbb{R}^{1+q}$, for every fixed $y \in \Omega$. Let $L^\mu_{\mathrm{cl}}(X; \mathbb{R}^l)$ denote the set of all parameter-dependent classical pseudo-differential operators of order $\mu \in \mathbb{R}$ on the manifold X, with parameters $\lambda \in \mathbb{R}^l$, $l \in \mathbb{N}$. That means, the amplitude functions $a(x, \xi, \lambda)$ in local coordinates $x \in \mathbb{R}^n$ on X are classical symbols of order μ in (ξ, λ). The space $L^{-\infty}(X; \mathbb{R}^l)$ of parameter-dependent smoothing operators is defined via kernels in $\mathcal{S}\big(\mathbb{R}^l, C^\infty(X \times X)\big)$ (a fixed Riemannian metric on X admits to identify $C^\infty(X \times X)$ with corresponding integral operators).

For future references we state and prove a standard property of parameter-dependent operators.

Theorem 2.1. *Let M be a closed compact C^∞ manifold and $A(\lambda) \in L^\mu_{\mathrm{cl}}(M; \mathbb{R}^l)$ a parameter-dependent family of order μ, and let $\nu \geq \mu$. Then there is a constant $c = c(s, \mu, \nu) > 0$ such that*

$$\|A(\lambda)\|_{\mathcal{L}(H^s(M), H^{s-\nu}(M))} \leq c\langle\lambda\rangle^{\max\{\mu, \mu-\nu\}}. \tag{2.4}$$

In particular, for $\mu \leq 0$, $\nu = 0$ we have

$$\|A\|_{\mathcal{L}(H^s(M), H^s(M))} \leq c\langle\lambda\rangle^\mu. \tag{2.5}$$

Moreover, for every $s', s'' \in \mathbb{R}$ and every $N \in \mathbb{N}$ there exists a $\mu(N) \in \mathbb{R}$ such that for every $\mu \leq \mu(N)$, $k := \mu(N) - \mu$, and $A(\lambda) \in L^\mu_{\mathrm{cl}}(M; \mathbb{R}^l)$ we have

$$\|A\|_{\mathcal{L}(H^{s'}(M), H^{s''}(M))} \leq c\langle\lambda\rangle^{-N-k} \tag{2.6}$$

for all $\lambda \in \mathbb{R}^l$, and a constant $c = c(s', s'', \mu, N, k) > 0$.

Proof. In this proof we write $\|\cdot\|_{s',s''} = \|\cdot\|_{\mathcal{L}(H^{s'}(M), H^{s''}(M))}$. The estimates (2.4) and (2.5) are standard. Concerning (2.6) we first observe that we have to choose μ so small that $A(\lambda) : H^{s'}(M) \to H^{s''}(M)$ is continuous. This is the case when $s'' \leq s' - \mu$, i.e., $\mu \leq s' - s''$. Let $R^{s''-s'}(\lambda) \in L^{s''-s'}_{\mathrm{cl}}(M, \mathbb{R}^l)$ be an order-reducing family with the inverse $R^{s'-s''}(\lambda) \in L^{s'-s''}_{\mathrm{cl}}(M, \mathbb{R}^l)$. Then we have

$$R^{s''-s'}(\lambda) : H^{s''}(M) \to H^{s'}(M),$$

i.e., $R^{s''-s'}(\lambda)A(\lambda) : H^{s'}(M) \to H^{s'}(M)$. The estimate (2.5) gives us

$$\|R^{s''-s'}(\lambda)A(\lambda)\|_{s',s'} \leq c\langle\lambda\rangle^{\mu+(s''-s')}$$

for $\mu \le s' - s''$. Moreover, (2.4) yields $\|R^{s'-s''}(\lambda)\|_{s',s''} \le c\langle\lambda\rangle^{s'-s''}$. Thus

$$\begin{aligned}\|A(\lambda)\|_{s',s''} &= \|R^{s'-s''}(\lambda)R^{s''-s'}(\lambda)A(\lambda)\|_{s',s''}\\ &\le \|R^{s'-s''}(\lambda)\|_{s',s''}\|R^{s''-s'}(\lambda)A(\lambda)\|_{s',s'} \le c\langle\lambda\rangle^{(s'-s'')+\mu+(s''-s')} = c\langle\lambda\rangle^{\mu}.\end{aligned}$$

In other words, when we choose $\mu(N)$ in such a way that $\mu \le s' - s''$, and $\mu(N) \le -N$, then (2.6) is satisfied. In addition, if we take $\mu = \mu(N) - k$ for some $k \ge 0$ then (2.6) follows in general. □

Corollary 2.2. *Let $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(M; \mathbb{R}^l)$, and assume that the estimate*

$$\|A(\lambda)\|_{s',s''} \le c\langle\lambda\rangle^{-N}$$

is true for given $s', s'' \in \mathbb{N}$ and some N. Then we have

$$\|D^{\alpha}_{\lambda}A(\lambda)\|_{s',s''} \le c\langle\lambda\rangle^{-N-\alpha}$$

for every $\alpha \in \mathbb{N}^l$.

Since we are interested in families for $r \to \infty$ it will be convenient to ignore the specific edge-degenerate behaviour for $r \to 0$ and to consider the cylinder $\mathbb{R}\times X$ rather than $\mathbb{R}_+ \times X$. Far from $r = \pm\infty$ our calculus will be as usual; therefore, for convenience, we fix a strictly positive function $r \to [r]$ in $C^{\infty}(\mathbb{R})$ such that $[r] = |r|$ for $|r| > R$ for some $R > 0$. The operator-valued amplitude functions in our calculus on $\mathbb{R} \times X$ are operator families of the form

$$a(r, \rho, \eta) = \tilde{a}(r, [r]\rho, [r]\eta)$$

where $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in C^{\infty}\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. In addition it will be important to specify the dependence of the latter function for large $|r|$. In other words, the crucial definition is as follows.

Definition 2.3. (i) Let E be a Fréchet space with the (countable) system of semi-norms $(\pi_j)_{j\in\mathbb{N}}$; then $S^{\nu}(\mathbb{R}, E)$, $\nu \in \mathbb{R}$, is defined to be the set of all $a(r) \in C^{\infty}(\mathbb{R}, E)$ such that

$$\pi_j\left(D^k_r a(r)\right) \le c[r]^{\nu-k}$$

for all $r \in \mathbb{R}, k \in \mathbb{N}$, with constants $c = c(k, j) > 0$,

(ii) $\boldsymbol{S}^{\mu,\nu}$ for $\mu, \nu \in \mathbb{R}$ denotes be the set of all operator families

$$a(r, \rho, \eta) = \tilde{a}(r, [r]\rho, [r]\eta)$$

for $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in S^{\nu}\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$ (topologised by the natural nuclear Fréchet topology of the space $L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$).

We first establish some properties of $\boldsymbol{S}^{\mu,\nu}$ that play a role in our calculus.

Proposition 2.4. (i) *$\varphi(r) \in S^{\sigma}(\mathbb{R})$, $a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$ implies $\varphi(r)a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\sigma+\nu}$.*

(ii) *For every $k, l \in \mathbb{N}$ we have*

$$a \in \boldsymbol{S}^{\mu,\nu} \;\Rightarrow\; \partial^l_r a \in \boldsymbol{S}^{\mu,\nu-l},\; \partial^k_{\rho} a \in \boldsymbol{S}^{\mu-k,\nu+k},\; \partial^k_{\eta} a \in \boldsymbol{S}^{\mu-k,\nu+k}.$$

(iii) $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$, $b(r,\rho,\eta) \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}}$ *implies* $a(r,\rho,\eta)b(r,\rho,\eta) \in \boldsymbol{S}^{\mu+\tilde{\mu},\nu+\tilde{\nu}}$.

Proof. (i) is evident. (ii) For simplicity we assume $q = 1$ and compute

$$\partial_r \tilde{a}(r,[r]\rho,[r]\eta) = \big((\partial_r + [r]'\rho\partial_{\tilde{\rho}} + [r]'\eta\partial_{\tilde{\eta}})\tilde{a}\big)(r,[r]\rho,[r]\eta)$$

where $[r]' := \partial_r[r]$. Since $\tilde{\rho}\tilde{a}(r,\tilde{\rho},\tilde{\eta})$, $\tilde{\eta}\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu(\mathbb{R}, L^{\mu+1}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}}))$, and $\partial_{\tilde{\rho}}\tilde{a}, \partial_{\tilde{\eta}}\tilde{a} \in S^\nu(\mathbb{R}, L^{\mu-1}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}))$, we obtain

$$\partial_r \tilde{a}(r,[r]\rho,[r]\eta) = \big((\partial_r + ([r]'/[r])[r]\rho\partial_{\tilde{\rho}} + ([r]'/[r])[r]\eta\partial_{\tilde{\eta}})\tilde{a}\big)(r,[r]\rho,[r]\eta) \in \boldsymbol{S}^{\mu,\nu-1}.$$

It follows that $\partial_r^l a \in \boldsymbol{S}^{\mu,\nu-l}$ for all $l \in \mathbb{N}$. Moreover, we have

$$\partial_\rho \tilde{a}(r,[r]\rho,[r]\eta) = [r](\partial_{\tilde{\rho}}\tilde{a})(r,[r]\rho,[r]\eta)$$

which gives us $\partial_\rho a \in \boldsymbol{S}^{\mu-1,\nu+1}$, and, by iteration, $\partial_\rho^k a \in \boldsymbol{S}^{\mu-k,\nu+k}$. In a similar manner we can argue for the η-derivatives.

(iii) By definition we have

$$a(r,\rho,\eta) = \tilde{a}(r,[r]\rho,[r]\eta),\ \ b(r,\rho,\eta) = \tilde{b}(r,[r]\rho,[r]\eta)$$

for $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X,\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$, $\tilde{b}(r,\tilde{\rho},\tilde{\eta}) \in S^{\tilde{\nu}}\big(\mathbb{R}, L^{\tilde{\mu}}_{\mathrm{cl}}(X,\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. Then the assertion is a consequence of the relation

$$(\tilde{a}\tilde{b})(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu+\tilde{\nu}}\big(\mathbb{R}, L^{\mu+\tilde{\mu}}_{\mathrm{cl}}(X,\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$ □

Corollary 2.5. *For $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$, $b(r,\rho,\eta) \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}}$ for every $k \in \mathbb{N}$ we have*

$$\partial_\rho^k a(r,\rho,\eta) D_r^k b(r,\rho,\eta) \in \boldsymbol{S}^{\mu+\tilde{\mu}-k,\nu+\tilde{\nu}}.$$

Remark 2.6. (i) Let $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R})$ be strictly positive functions such that $\varphi_j(r) = |r|$ for $|r| \geq c_j$ for some $c_j > 0$, $j = 1,2$. Then we have

$$\boldsymbol{S}^{\mu,\nu} = \Big\{a(r,\varphi_1(r)\rho,\varphi_2(r)\eta) : a(r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)\Big\};$$

(ii) $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ implies $a(\lambda r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ for every $\lambda \in \mathbb{R}_+$.

Proof. (i) We can write

$$a(r,\varphi_1(r)\rho,\varphi_2(r)\eta) = a(r,\psi_1(r)[r]\rho,\psi_2(r)[r]\eta)$$

for $\psi_j(r) \in C^\infty(\mathbb{R})$, $\psi_j(r) = 1$ for $|r| > c$ for some $c > 0$, $j = 1,2$. Then it suffices to verify that

$$a(r,\psi_1(r)\tilde{\rho},\psi_2(r)\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big);$$

however, this is straightforward.

(ii) It is evident that $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$ implies $\tilde{a}(\lambda r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. Therefore, it suffices to show $\tilde{a}(r,[\lambda r]\rho,[\lambda r]\eta) \in \boldsymbol{S}^{\mu,\nu}$. Let us write

$$\tilde{a}(r,[\lambda r]\rho,[\lambda r]\eta) = \tilde{a}(r,\varphi_\lambda(r)[r]\rho,\varphi_\lambda(r)[r]\eta)$$

for $\varphi_\lambda(r) := [\lambda r]/[r]$. We have $\varphi_\lambda(r) = \lambda$ for $|r| > c$ for a constant $c > 0$, i.e., $\varphi_\lambda(r) - \lambda \in C_0^\infty(\mathbb{R})$. Thus there is an r-excision function $\chi(r)$ (i.e., $\chi \in C^\infty(\mathbb{R})$, $\chi(r) = 0$ for $|r| \le c_0$, $\chi(r) = 1$ for $|r| \ge c_1$ for certain $0 < c_0 < c_1$) such that

$$\chi(r)\tilde{a}(r, [\lambda r]\rho, [\lambda r]\eta) = \chi(r)\tilde{a}(r, [r]\lambda\rho, [r]\lambda\eta),$$

which belongs to $\boldsymbol{S}^{\mu,\nu}$. It remains to characterise $\big(1-\chi(r)\big)\tilde{a}(r, \varphi_\lambda(r)[r]\rho, \varphi_\lambda(r)[r]\eta)$ which vanishes for $|r| \ge c_1$, and a simple calculation shows

$$\big(1 - \chi(r)\big)\tilde{a}(r, \varphi_\lambda(r)\tilde{\rho}, \varphi_\lambda(r)\tilde{\eta}) \in C_0^\infty\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big),$$

which is contained in $\boldsymbol{S}^{\mu;-\infty}$. □

Proposition 2.7. *Let $\tilde{a}_j(r, \tilde{\rho}, \tilde{\eta}) \in S^\nu(\mathbb{R}, L^{\mu-j}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}))$, $j \in \mathbb{N}$, be an arbitrary sequence, $\mu, \nu \in \mathbb{R}$ fixed. Then there is an $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in S^\nu(\mathbb{R}, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q}))$ such that*

$$a - \sum_{j=0}^{N} a_j \in S^\nu(\mathbb{R}, L^{\mu-(N+1)}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}))$$

for every $N \in \mathbb{N}$, and a is unique modulo $S^\nu\big(\mathbb{R}, L^{-\infty}_{\mathrm{cl}}(X; \mathbb{R}^{1+q})\big)$.

Proof. The proof is similar to the standard one on asymptotic summation of symbols. We can find an asymptotic sum as a convergent series

$$\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) = \sum_{j=0}^{\infty} \chi\left((\tilde{\rho}, \tilde{\eta})/c_j\right) \tilde{a}_j(r, \tilde{\rho}, \tilde{\eta})$$

for some excision function χ in $\mathbb{R}^{1+q}$, with a sequence $c_j > 0$, $c_j \to \infty$ as $j \to \infty$ so fast, that $\sum_{j=N+1}^{\infty} \chi\left((\tilde{\rho}, \tilde{\eta})/c_j\right) \tilde{a}(r, \tilde{\rho}, \tilde{\eta})$ converges in $S^\nu(\mathbb{R}, L^{\mu-(N+1)}_{\mathrm{cl}})$ for every N. □

2.2. Continuity in Schwartz Spaces

Theorem 2.8. *Let $p(r, \rho, \eta) = \tilde{p}(r, [r]\rho, [r]\eta)$, $\tilde{p}(r, \tilde{\rho}, \tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\, \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$, i.e., $p(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$. Then $\mathrm{Op}_r(p)(\eta)$ induces a family of continuous operators*

$$\mathrm{Op}_r(p)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$$

for every $\eta \neq 0$.

Proof. We have

$$\mathrm{Op}_r(p)(\eta)u(r) = \int e^{ir\rho} p(r, \rho, \eta)\hat{u}(\rho) đ\rho,$$

first for $u \in C_0^\infty\big(\mathbb{R}, C^\infty(X)\big)$. In the space $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$ we have the semi-norm system

$$\pi_{m,s}(u) = \max_{\alpha+\beta \le m} \sup_{r \in \mathbb{R}} \|[r]^\alpha \partial_r^\beta u(r)\|_{H^s(X)}$$

for $m \in \mathbb{N}$, $s \in \mathbb{Z}$, which defines the Fréchet topology of $\mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$.

If necessary we indicate the variable r, i.e., write $\pi_{m,s;r}$ rather than $\pi_{m,s}$. The Fourier transform $\mathcal{F}_{r\to\rho}$ induces an isomorphism

$$\mathcal{F} : \mathcal{S}\big(\mathbb{R}_r, H^s(X)\big) \to \mathcal{S}\big(\mathbb{R}_\rho, H^s(X)\big)$$

for every s. For every $m \in \mathbb{N}$ there exists a $C > 0$ such that

$$\pi_{m,s;\rho}(\mathcal{F}u) \leq C\pi_{m+2,s;r}(u) \tag{2.7}$$

for all $u \in \mathcal{S}(\mathbb{R}, H^s(X))$ (see [7, Chapter 1] for scalar functions; the case of functions with values in a Hilbert space is completely analogous). We have to show that for every $\tilde{m} \in \mathbb{N}$, $\tilde{s} \in \mathbb{Z}$ there exist $m \in \mathbb{N}$, $s \in \mathbb{Z}$, such that

$$\pi_{\tilde{m},\tilde{s}}\big((\mathrm{Op}(p)u)(r)\big) \leq c\pi_{m,s}(u) \tag{2.8}$$

for all $u \in \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$, for some $c = c(\tilde{m}, \tilde{s}) > 0$. According to Proposition 2.10 below we write the operator $\mathrm{Op}(p)(\eta)$ in the form

$$\mathrm{Op}_r(p)(\eta)\circ\langle r\rangle^{-M}\circ\langle r\rangle^{M} = \langle r\rangle^{-M}\mathrm{Op}_r(b_{MN})(\eta)\circ\langle r\rangle^{M} + \mathrm{Op}_r(d_{MN})(\eta)\circ\langle r\rangle^{M} \tag{2.9}$$

for a symbol $b_{MN}(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$ and a remainder $d_{MN}(r, \rho, \eta)$ satisfying estimates analogously as (2.17).

We have

$$\begin{aligned}
\|\mathrm{Op}_r(p)(\eta)u(r)\|_{H^{\tilde{s}}(X)} &= \Big\| \int e^{ir\rho} p(r, \rho, \eta)\hat{u}(\rho) đ\rho \Big\|_{H^{\tilde{s}}(X)} \\
&\leq \Big\| \int e^{ir\rho}\langle r\rangle^{-M} b_{MN}(r, \rho, \eta)\big(\langle r\rangle^{M}u\big)^{\wedge}(\rho) đ\rho\Big\|_{H^{\tilde{s}}(X)} \\
&\quad + \|\mathrm{Op}_r(d_{MN})(\eta)(\langle r\rangle^{M}u(r))\|_{H^{\tilde{s}}(X)}.
\end{aligned} \tag{2.10}$$

For the first term on the right of (2.10) we obtain for $s := \tilde{s} + \mu$ and arbitrary $\widetilde{M} \in \mathbb{N}$

$$\begin{aligned}
&\Big\| \int e^{ir\rho}\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M} b_{MN}(r, \rho, \eta)\langle\rho\rangle^{\widetilde{M}}\big(\langle r\rangle^{M}u\big)^{\wedge}(\rho) đ\rho\Big\|_{H^{\tilde{s}}(X)} \\
&\leq \int \|\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M} b_{MN}(r, \rho, \eta)\langle\rho\rangle^{\widetilde{M}}\big(\langle r\rangle^{M}u\big)^{\wedge}(\rho)\|_{H^{\tilde{s}}(X)} đ\rho \\
&\leq c \sup_{(r,\rho)\in\mathbb{R}^2} \langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r, \rho, \eta)\|_{\mathcal{L}(H^s(X), H^{\tilde{s}}(X))} \\
&\quad \times \int \langle\rho\rangle^{\widetilde{M}}\|\big(\langle r\rangle^{M}u\big)^{\wedge}(\rho)\|_{H^s(X)} đ\rho.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\int \langle\rho\rangle^{\widetilde{M}}\|\big(\langle r\rangle^{M}u\big)^{\wedge}(\rho)\|_{H^s(X)} đ\rho &\leq \sup_{\rho\in\mathbb{R}} \langle\rho\rangle^{\widetilde{M}+2}\|\big(\langle r\rangle^{M}u\big)^{\wedge}(\rho)\|_{H^s(X)} \int \langle\rho\rangle^{-2} đ\rho \\
&\leq c\pi_{\widetilde{M}+2,s;\rho}\big(\big(\langle r\rangle^{M}u\big)^{\wedge}(\rho)\big) \leq \pi_{\widetilde{M}+4,s;r}(\langle r\rangle^{M}u)) \leq c\pi_{M+\widetilde{M}+4,s;r}(u).
\end{aligned}$$

Here we employed the estimate (2.7). Thus (2.10) yields

$$\begin{aligned}
\pi_{0,\tilde{s}}(\mathrm{Op}(p)(\eta)u) &\leq c \sup_{(r,\rho)\in\mathbb{R}^2} \langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r, \rho, \eta)\|_{\mathcal{L}(H^s(X), H^{\tilde{s}}(X))} \\
&\quad \times \pi_{M+\widetilde{M}+4,s;r}(u) + \pi_{0,\tilde{s}}(\mathrm{Op}_r(d_{MN})(\eta)(\langle r\rangle^{M}u)).
\end{aligned} \tag{2.11}$$

The factor $c\sup_{(r,\rho)\in\mathbb{R}^2}\langle\rho\rangle^{-\widetilde{M}}\langle r\rangle^{-M}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))}$ is finite when we choose M so large that $\nu - M \le 0$ and $\widetilde{M}$ so large that

$$\sup_{\rho\in\mathbb{R}}\langle\rho\rangle^{-\widetilde{M}}\|b_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^s(X),H^{\tilde{s}}(X))} < \infty.$$

Next we consider the second term on the right-hand side of (2.11). We have

$$\begin{aligned}
&\|\mathrm{Op}_r(d_{MN})(\eta)\langle r\rangle^M u(r)\|_{H^{\tilde{s}}(X)}\\
&= \left\|\int e^{ir\rho}\langle\rho\rangle^{-M}d_{MN}(r,\rho,\eta)\langle\rho\rangle^M(\langle r\rangle^M u)^\wedge(\rho)d\!\!\!^{-}\rho\right\|_{H^{\tilde{s}}(X)}\\
&\le \int \|\langle\rho\rangle^{-M}d_{MN}(r,\rho,\eta)\langle\rho\rangle^M\big(\langle r\rangle^M u\big)^\wedge(\rho)\|_{H^{\tilde{s}}(X)}d\!\!\!^{-}\rho\\
&\le \int \sup_{(r,\rho)\in\mathbb{R}^2}\|\langle\rho\rangle^{-M}d_{MN}(r,\rho,\eta)\|_{\mathcal{L}(H^{\tilde{s}}(X),H^s(X))}\|\langle\rho\rangle^M\big(\langle r\rangle^M u\big)^\wedge(\rho)\|_{H^s(X)}d\!\!\!^{-}\rho.
\end{aligned}$$

From the analogue of the estimate (2.17) for $d_{MN}(r,\rho,\eta)$ we see that for N sufficiently large it follows that the right-hand side of the latter expression can be estimated by

$$\begin{aligned}
c\int \|\langle\rho\rangle^M\big(\langle r\rangle^M u\big)^\wedge(\rho)\|_{H^s(X)}d\!\!\!^{-}\rho &\le \sup_{\rho\in\mathbb{R}}\langle\rho\rangle^{M+2}\|\big(\langle r\rangle^M u\big)^\wedge(\rho)\|_{H^s(X)}\int\langle\rho\rangle^{-2}d\!\!\!^{-}\rho\\
&\le c\pi_{2M+2,s;\rho}(\hat{u}(\rho)) \le c\pi_{2M+4,s;r}(u).
\end{aligned}$$

In other words we proved that

$$\pi_{0,\tilde{s}}(\mathrm{Op}(p)(\eta)u) \le c\{\pi_{M+\widetilde{M}+4,s}(u) + \pi_{2M+4,s}(u)\} \le c\pi_{L,s}(u) \tag{2.12}$$

for $s = \tilde{s} + \mu$, $L := \max\{M + \widetilde{M} + 4, 2M + 4\}$. Now we write

$$\begin{aligned}
\partial_r\mathrm{Op}(p)(\eta)u(r) &= \int e^{ir\rho}\partial_r p(r,\rho,\eta)\hat{u}(\rho)d\!\!\!^{-}\rho, + \int e^{ir\rho}p(r,\rho,\eta)(\partial_r u)^\wedge(\rho)d\!\!\!^{-}\rho\\
r\mathrm{Op}(p)(\eta)u(r) &= \int e^{ir\rho}(i\partial_\rho p(r,\rho,\eta))\hat{u}(\rho)d\!\!\!^{-}\rho + \int e^{ir\rho}ip(r,\rho,\eta)\partial_\rho\hat{u}(\rho)d\!\!\!^{-}\rho.
\end{aligned}$$

From Proposition 2.4 we have

$$\partial_r p(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu-1},\ i\partial_\rho p(r,\rho,\eta) \in \boldsymbol{S}^{\mu-1,\nu+1}.$$

Since the estimate (2.12) is true for elements in the respective symbol classes of arbitrary order, it follows altogether the estimate (2.8) for every $\tilde{m}\in\mathbb{N}$, $\tilde{s}\in\mathbb{Z}$ and suitable m, s. □

2.3. Leibniz Products and Remainder Estimates

Let $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu(\mathbb{R}, L^\mu_{\mathrm{cl}})$, $\tilde{b}(r,\tilde{\rho},\tilde{\eta}) \in S^{\tilde{\nu}}(\mathbb{R}, L^{\tilde{\mu}}_{\mathrm{cl}})$ where $L^\mu_{\mathrm{cl}} = L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. The operator functions

$$a(r,\rho,\eta) := \tilde{a}(r,[r]\rho,[r]\eta),\ b(r,\rho,\eta) := \tilde{b}(r,[r]\rho,[r]\eta)$$

will be interpreted as amplitude functions of a pseudo-differential calculus on $\mathbb{R}$ containing η as a parameter (below we assume $\eta \neq 0$). We intend to apply an analogue of Kumano-go's technique [7] and form the oscillatory integral

$$a\#b(r,\rho,\eta) = \iint e^{-it\tau} a(r,\rho+\tau,\eta) b(r+t,\rho,\eta) dt đ\tau \tag{2.13}$$

which has the meaning of a Leibniz product, associated with the composition of operators. The rule

$$\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) = \mathrm{Op}_r(a\#b)(\eta) \tag{2.14}$$

for $\eta \neq 0$ will be justified afterwards. Similarly as in [7], applying Taylor's formula, the function $a\#b$ can be decomposed in the form

$$a\#b(r,\rho,\eta) = \sum_{k=0}^{N} \frac{1}{k!} \partial_\rho^k a(r,\rho,\eta) D_r^k b(r,\rho,\eta) + r_N(r,\rho,\eta) \tag{2.15}$$

for

$$r_N(r,\rho,\eta) = \frac{1}{N!} \iint e^{-it\tau} \{ \int_0^1 (1-\theta)^N (\partial_\rho^{N+1} a)(r,\rho+\theta\tau,\eta) d\theta \} \times (D_r^{N+1} b)(r+t,\rho,\eta) dt đ\tau. \tag{2.16}$$

By virtue of Corollary 2.5 we have $\frac{1}{k!}\partial_\rho^k a(r,\rho,\eta) D_r^k b(r,\rho,\eta) =: c_k(r,\rho,\eta)$ for $c_k(r,\rho,\eta) = \tilde{c}_k(r,[r]\rho,[r]\eta)$, $\tilde{c}_k(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu+\tilde{\nu}}(\mathbb{R}, L_{\mathrm{cl}}^{\mu+\tilde{\mu}-k})$. Let us now characterise the remainder.

Lemma 2.9. *For every $s', s'' \in \mathbb{R}$, $l, m, k \in \mathbb{N}$, there is an $N \in \mathbb{N}$ such that*

$$\| D_r^i D_\rho^j r_N(r,\rho,\eta) \|_{s',s''} \leq c \langle \rho \rangle^{-k} \langle r \rangle^{-l} \langle \eta \rangle^{-m} \tag{2.17}$$

for all $(r,\rho) \in \mathbb{R}^2$, $|\eta| \geq \varepsilon > 0$, $i, j \in \mathbb{N}$, for some constant $c = c(s', s'', k, l, m, N, \varepsilon) > 0$, here $\| \cdot \|_{s',s''} = \| \cdot \|_{\mathcal{L}(H^{s'}(X), H^{s''}(X))}$.

Proof. Let us write $\boldsymbol{S}^{\mu,\nu} := \{ \tilde{a}(r,[r]\rho,[r]\eta) : \tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu(\mathbb{R}, L_{\mathrm{cl}}^\mu) \}$. By virtue of Proposition 2.4 we have

$$\partial_\rho^k \tilde{a}(r,[r]\rho,[r]\eta) \in \boldsymbol{S}^{\mu-k,\nu+k}, \partial_r^k \tilde{b}(r,[r]\rho,[r]\eta) \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}-k}$$

for every k. Let us set

$$\tilde{a}_{N+1}(r,[r]\rho+[r]\theta\tau,[r]\eta) := (\partial_\rho^{N+1} a)(r,\rho+\theta\tau,\eta),$$
$$\tilde{b}_{N+1}(r+t,[r+t]\rho,[r+t]\eta) := (D_r^{N+1} b)(r+t,\rho,\eta).$$

By virtue of Theorem 2.1 for every $s_0, s'' \in \mathbb{R}$ and every M there exists a $\mu(M)$ such that for every $p(\tilde{\rho},\tilde{\eta}) \in L_{\mathrm{cl}}^\mu(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$, $\mu \leq \mu(M)$, we have

$$\| p(\tilde{\rho},\tilde{\eta}) \|_{s_0,s''} \leq c \langle \tilde{\rho},\tilde{\eta} \rangle^{-M} \tag{2.18}$$

for all $(\tilde{\rho},\tilde{\eta}) \in \mathbb{R}^{1+q}$, $c = c(s_0, s'', \mu, M) > 0$. Moreover, for every $s', s_0 \in \mathbb{R}$ there exists a $B \in \mathbb{R}$ such that $\| p(\tilde{\rho},\tilde{\eta}) \|_{s',s_0} \leq c \langle \tilde{\rho},\tilde{\eta} \rangle^B$ for all $(\tilde{\rho},\tilde{\eta}) \in \mathbb{R}^{1+q}$, $c = c(s', s_0, \mu) > 0$. We apply this for $\tilde{a}_{N+1}(r,\tilde{\rho},\tilde{\eta})$ and $\tilde{b}_{N+1}(r,\tilde{\rho},\tilde{\eta})$, combined with the dependence on $r \in \mathbb{R}$ as a symbol in this variable. In other words, we have the estimates

$$\|\tilde{a}_{N+1}(r,\tilde{\rho},\tilde{\eta})\|_{s_0,s''} \le c\,\langle r\rangle^{\nu+(N+1)}\langle\tilde{\rho},\tilde{\eta}\rangle^{-M}, \tag{2.19}$$

$$\|\tilde{b}_{N+1}(r,\tilde{\rho},\tilde{\eta})\|_{s',s_0} \le c\langle r\rangle^{\tilde{\nu}-(N+1)}\langle\tilde{\rho},\tilde{\eta}\rangle^{B}; \tag{2.20}$$

here we applied the above-mentioned result to $\tilde{a}_{N+1}$ for the pair (s_0, s'') for N sufficiently large, and for $\tilde{b}_{N+1}$ the second estimate for (s', s_0) with some exponent B. Let us take $s_0 := s' - \tilde{\mu}$; then we can set $B = \max\{\tilde{\mu}, 0\}$. The remainder (2.16) is regularised as an oscillatory integral in (t,τ), i.e., we may write

$$\begin{aligned} r_N(r,\rho,\eta) = \frac{1}{N!}\iint e^{-it\tau}\langle t\rangle^{-2L}(1-\partial_\tau^2)^L\langle\tau\rangle^{-2K}(1-\partial_t^2)^K \\ \times\Big\{\int_0^1(1-\theta)^N\tilde{a}_{N+1}(r,[r]\rho+[r]\theta\tau,[r]\eta)d\theta\Big\}\tilde{b}_{N+1}(r+t,[r+t]\rho,[r+t]\eta)dt\bar{d}\tau \end{aligned} \tag{2.21}$$

for sufficiently large L, K. For simplicity from now on we assume $q = 1$; the considerations for the general case are completely analogous. Then we have for every $l \le L$

$$\partial_\tau^{2l}\tilde{a}_{N+1}(r,[r]\rho+[r]\theta\tau,[r]\eta) = \big(\partial_{\tilde{\rho}}^{2l}\tilde{a}_{N+1}\big)(r,[r]\rho+[r]\theta\tau,[r]\eta)([r]\theta)^{2l},$$

and for every $k \le K$

$$\begin{aligned} \partial_t^{2k}\,\tilde{b}_{N+1}(r+t,[r+t]\rho,[r+t]\eta) &= \big(\partial_t^{2k}\tilde{b}_{N+1}\big)(r+t,[r+t]\rho,[r+t]\eta) \\ &+\big(\partial_{\tilde{\rho}}^{2k}\tilde{b}_{N+1}\big)(r+t,[r+t]\rho,[r+t]\eta)(\rho\partial_t[r+t])^{2k} \\ &+\big(\partial_{\tilde{\eta}}^{2k}\tilde{b}_{N+1}\big)(r+t,[r+t]\rho,[r+t]\eta)(\eta\partial_t[r+t])^{2k}+R, \end{aligned}$$

where R denotes several mixed derivatives. From (2.19) we have

$$\|\partial_\tau^{2l}\tilde{a}_{N+1}(r,[r]\varrho+r[\theta]\tau,[r]\eta)\|_{s_0,s''} \le c\langle r\rangle^{\nu+(N+1)}\langle[r]\varrho+[r]\theta\tau,[r]\eta\rangle^{-M-2l}([r]\theta)^{2l}, \tag{2.22}$$

see Corollary 2.2, and (2.20) gives us

$$\|(\partial_t^{2k}\tilde{b}_{N+1})(r+t,[r+t]\varrho,[r+t]\eta)\|_{s',s_0} \le c\langle r+t\rangle^{\tilde{\nu}-(N+1)}\langle[r+t]\varrho,[r+t]\eta\rangle^{B} \tag{2.23}$$

(where we take N so large that $\tilde{\nu}-(N+1)\le 0$), and

$$\begin{aligned} &\|\big(\partial_{\tilde{\varrho}}^{2k}\tilde{b}_{N+1}\big)(r+t,[r+t]\varrho,[r+t]\eta)(\varrho\partial_t[r+t])^{2k}\|_{s',s_0} \\ &\quad\le c\langle r+t\rangle^{\tilde{\nu}-(N+1)}\langle[r+t]\varrho,[r+t]\eta\rangle^{B-2k}|\varrho\partial_t[r+t]|^{2k}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} &\|\big(\partial_{\tilde{\eta}}^{2k}\tilde{b}_{N+1}\big)(r+t,[r+t]\varrho,[r+t]\eta)(\eta\partial_t[r+t])^{2k}\|_{s',s_0} \\ &\quad\le c\langle r+t\rangle^{\tilde{\nu}-(N+1)}\langle[r+t]\varrho,[r+t]\eta\rangle^{B-2k}|\eta\partial_t[r+t]|^{2k}. \end{aligned} \tag{2.25}$$

The above-mentioned mixed derivatives admit similar estimates (in fact, better ones; so we concentrate on those contributed by (2.22), (2.23), (2.24), (2.25)).

We now derive an estimate for $\|r_N(r,\varrho,\eta)\|_{s',s''}$. Using the relation (2.21) we have

$$\|r_N(r,\varrho,\eta)\|_{s',s''} \leq \iint\int_0^1 \|\langle t\rangle^{-2L}(1-\partial_\tau^2)^L\langle\tau\rangle^{-2K}(1-\partial_t^2)^K \\ \times (1-\theta)^N \tilde{a}_{N+1}(r,[r]\varrho+[r]\theta\tau,[r]\eta)\tilde{b}_{N+1}(r+t,[r+t]\varrho,[r+t]\eta)\|_{s',s''}d\theta dt đ\tau.$$

The operator norm under the integral can be estimated by expressions of the kind

$$I := c\langle r\rangle^{\nu+(N+1)}\langle r+t\rangle^{\tilde{\nu}-(N+1)}\langle t\rangle^{-2L}\langle\tau\rangle^{-2K}\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M-2l}([r]\theta)^{2l} \\ \times \langle[r+t]\rho,[r+t]\eta\rangle^{B}\big\{1+\langle[r+t]\rho,[r+t]\eta\rangle^{-2k}(|\rho|^{2k}+|\eta|^{2k})|(\partial_t[r+t])^{2k}|\big\},$$

$l\leq L$, $k\leq K$, plus terms from R of a similar character. We have, using Peetre's inequality,

$$\langle r\rangle^{\nu+(N+1)}\langle r+t\rangle^{\tilde{\nu}-(N+1)} \leq \langle r\rangle^{\nu+\tilde{\nu}}\langle t\rangle^{|\tilde{\nu}-(N+1)|}.$$

Moreover, we have $\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-2l}([r]\theta)^{2l} \leq c\langle[r]\eta\rangle^{-2l}[r]^{2l} \leq c$ for $|\eta|\geq\varepsilon>0$ (as always, c denotes different constants), and

$$\langle[r+t]\rho,[r+t]\eta\rangle^{-2k}(|\rho|^{2k}+|\eta|^{2k})|(\partial_t[r+t])^{2k}| \\ \leq\ c\big\{\langle[r+t]\rho\rangle^{-2k}([r+t]|\rho|)^{2k}+\langle[r+t]\eta\rangle^{-2k}([r+t]|\eta|)^{2k}\big\}[r+t]^{-2k}\leq c,$$

using $|(\partial_t[r+t])^{2k}|\leq c$, $[r+t]^{-2k}\leq c$ for all $r,t\in\mathbb{R}$ and $|\zeta|\leq c\langle\zeta\rangle$ for every ζ in $\mathbb{R}^d$. This yields

$$I\leq c\langle r\rangle^{\nu+\tilde{\nu}}\langle t\rangle^{|\tilde{\nu}-(N+1)|}\langle t\rangle^{-2L}\langle\tau\rangle^{-2K}\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M}\langle[r+t]\rho,[r+t]\eta\rangle^{B}.$$

Writing $M=M'+M''$ for suitable $M',M''\geq 0$ to be fixed later on, we have

$$\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M} = \langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M'}\langle[r]\rho+[r]\theta\tau,[r]\eta\rangle^{-M''} \\ \leq c\langle[r]\eta\rangle^{-M'}\langle[r]\rho,[r]\eta\rangle^{-M''}\langle[r]\theta\tau\rangle^{M''}\leq c\langle[r]\eta\rangle^{-M'}\langle[r]\rho\rangle^{-M''}\langle[r]\theta\tau\rangle^{M''}.$$

We applied once again Peetre's inequality which gives us also

$$\langle[r+t]\rho,[r+t]\eta\rangle^{B}\leq c\langle[r+t]\rho\rangle^{B}\langle[r+t]\eta\rangle^{B}$$

since $B\geq 0$. Thus

$$I\leq\ c\langle r\rangle^{\nu+\tilde{\nu}}\langle t\rangle^{|\tilde{\nu}-(N+1)|-2L}\langle\tau\rangle^{-2K}\langle[r]\theta\tau\rangle^{M''} \\ \times\langle[r+t]\rho\rangle^{B}\langle[r]\rho\rangle^{-M''}\langle[r+t]\eta\rangle^{B}\langle[r]\eta\rangle^{-M'}.$$

Let us show

$$\langle t\rangle^{-B}\langle[r+t]\rho\rangle^{B}\langle[r]\rho\rangle^{-B}\leq c.$$

In fact, this is evident in the regions $|r|\leq C$, $|t|\leq C$ or $|r|\geq C$, $|t|\leq C$ for some $C>0$. For $|r|\leq C$, $|t|\geq C$ the estimate essentially follows from $1+t^2\rho^2\leq(1+t^2)(1+\rho^2)$. For $|r|\geq C$, $|t|\geq C$, $[r+t]\leq C$ the estimate is evident as well.

It remains the case $|r| \geq C$, $|t| \geq C$, $[r+t] \geq C$, where the estimate follows (for $C \geq 1$ so large that $[r+t] = |r+t|$, $[r] = |r|$) from

$$\langle t\rangle^{-2}\langle [r+t]\rho\rangle^{2}\langle [r]\rho\rangle^{-2} = \frac{1+|r+t|^2|\rho|^2}{(1+|t|^2)(1+|r\rho|^2)} \leq \frac{1+|r\rho|^2+2|rt\rho|^2|t\rho|^2}{1+|t|^2+|r\rho|^2+|rt\rho|^2}$$
$$\leq c\frac{1+|r\rho|^2+|t\rho|^2+2|rt\rho|^2}{1+|r\rho|^2+|t\rho|^2} \leq c\Big(1+\frac{2|rt\rho|^2}{1+|r\rho|^2+|t\rho|^2}\Big) \leq \text{const}.$$

Here we employed $|rt\rho|^2 \geq |t\rho|^2$ for $|r| \geq C \geq 1$ and

$$\frac{|rt\rho|^2}{1+|r\rho|^2+|t\rho|^2} \leq \frac{r^2t^2}{r^2+t^2} = \frac{r^2}{r^2+t^2}\frac{t^2}{r^2+t^2} \leq \text{const}.$$

Analogously we have $\langle t\rangle^{-B}\langle [r+t]\eta\rangle^{B}\langle [r]\eta\rangle^{-B} \leq c$. This gives us the estimate

$$I \leq c\langle r\rangle^{\nu+\tilde{\nu}}\langle t\rangle^{|\tilde{\nu}-(N+1)|-2L+2B}\langle\tau\rangle^{-2K}\langle [r]\theta\tau\rangle^{M''}\langle [r]\rho\rangle^{B-M''}\langle [r]\eta\rangle^{B-M'}.$$

Finally, using $\langle\tau\rangle^{-M''}\langle r\rangle^{-M''}\langle [r]\theta\tau\rangle^{M''} \leq c$ for all $0 \leq \theta \leq 1$ and all r, τ, we obtain

$$I \leq c\langle r\rangle^{\nu+\tilde{\nu}+M''}\langle t\rangle^{|\tilde{\nu}-(N+1)|-2L+2B}\langle\tau\rangle^{-2K+M''}\langle [r]\rho\rangle^{B-M''}\langle [r]\eta\rangle^{B-M'}$$

for all $r, t \in \mathbb{R}$, $\rho, \tau \in \mathbb{R}$, $0 \leq \theta \leq 1$. Choosing K and L so large that

$$-2K+M'' < -1, |\tilde{\nu}-(N+1)|+2B-2L < -1,$$

it follows that $\|r_N(r,\rho,\eta)\|_{s',s''} \leq c\langle r\rangle^{\nu+\tilde{\nu}+M''}\langle [r]\eta\rangle^{B-M'}\langle\rho\rangle^{B-M''}$ for $\eta \neq 0$ using that $\langle [r]\rho\rangle^{B-M''} \leq c\langle\rho\rangle^{B-M''}$ for $B-M'' \leq 0$. Let us now show that for $B-M' \leq 0$,

$$\langle [r]\eta\rangle^{B-M'} \leq c[r]^{B-M'}\langle\eta\rangle^{B-M'} \tag{2.26}$$

for all $|\eta| \geq \varepsilon > 0$ and some $c = c(\varepsilon) > 0$. In fact, we have

$$\frac{[r]^2\langle\eta\rangle^2}{1+|[r]\eta|^2} = \frac{[r]^2}{1+|[r]\eta|^2}\frac{\langle\eta\rangle^2}{1+|[r]\eta|^2} \leq c\frac{1}{[r]^{-2}+|\eta|^2}\frac{1}{|\eta|^{-2}+[r]^{-2}} \leq c,$$

i.e., $(1+|[r]\eta|^2)^{-1} \leq c[r]^{-2}\langle\eta\rangle^{-2}$ which entails the estimate (2.26). It follows that

$$\|r_N(r,\rho,\eta)\|_{s',s'} \leq c\langle r\rangle^{\nu+\tilde{\nu}+M''+B-M'}\langle\rho\rangle^{B-M''}\langle\eta\rangle^{B-M'}.$$

Now B is fixed, and M, M'' can be chosen independently so large that

$$B-M'' \leq -k,\ B-M' \leq -m,\ \nu+\tilde{\nu}+M''+B-M' \leq -l.$$

Therefore, we proved that for every $s', s'' \in \mathbb{R}$ and $k, l, m \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that

$$\|r_N(r,\rho,\eta)\|_{s',s''} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m} \tag{2.27}$$

for all $(r,\rho) \in \mathbb{R}^2$, $|\eta| \geq \varepsilon > 0$. In an analogous manner we can show the estimates (2.17) for all i, j. □

Proposition 2.10. *For every* $a(r,\rho,\eta) \in S^{\mu,\nu}$ *and* $\varphi(r) = [r]^{\tilde{\nu}}$ *(which belongs to* $\boldsymbol{S}^{0,\tilde{\nu}}$*) for every* $\eta \neq 0$ *we have (as operators* $\mathrm{Op}_r(\tilde{a}(r,[r]\rho,[r]\eta)) : C_0^\infty(\mathbb{R}, C^\infty(X)) \to C^\infty(\mathbb{R}, C^\infty(X))$*)*

$$\mathrm{Op}_r(a)(\eta)\circ\varphi = \varphi\circ\mathrm{Op}_r(b)(\eta)+d(\eta) \tag{2.28}$$

for some $b(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ *and a remainder* $d(\eta) = \mathrm{Op}_r(r_N)(\eta)$ *which is an operator function* $r_N(r,\rho,\eta) \in C^\infty\big(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^q_\eta, \mathcal{L}\big(H^{s'}(X), H^{s''}(X)\big)\big)$ *for every given* s', s'' *and sufficiently large* $N = N(s',s'') \in \mathbb{N}$*, satisfying the estimates* (2.17) *for all* $(r,\rho) \in \mathbb{R}^2$ *and all* $|\eta| \geq \varepsilon > 0$*.*

Proof. We apply the relation (2.15) to the case $b(r,\rho,\eta) = \varphi(r)$ and obtain

$$\mathrm{Op}(a)\circ\varphi = \mathrm{Op}(a\#\varphi) = \sum_{k=0}^{N} \mathrm{Op}\big(\frac{1}{k!}\partial_\rho^k a(r,\rho,\eta) D_r^k \varphi(r)\big) + \mathrm{Op}(r_N).$$

According to Corollary 2.5 we can form the product $\big(\partial_\rho^k a D_r^k \varphi\big)(r,[r]\rho,[r]\eta)$ with $\big(\partial_\rho^k a D_r \varphi\big)(r,\tilde\rho,\tilde\eta) \in S^{\nu+\tilde\nu}(\mathbb{R}, L_{\mathrm{cl}}^{\mu-k}(X;\mathbb{R}^{1+q}))$. There is then an element $\tilde c_N(r,\tilde\rho,\tilde\eta) \in S^{\nu+\tilde\nu}(\mathbb{R}, L_{\mathrm{cl}}^{\mu-(N+1)}(X;\mathbb{R}^{1+q}))$ which is the asymptotic sum of the symbols

$$\frac{1}{k!}\big(\partial_\rho^k a D_r^k \varphi\big)(r,\tilde\rho,\tilde\eta),\ \ k \geq N+1.$$

Writing $c_N(r,\rho,\eta) = \tilde c_N(r,[r]\rho,[r]\eta)$ we obtain

$$a\#b(r,\rho,\eta) = p_N(r,\rho,\eta) + d_N(r,\rho,\eta)$$

for $p_N(r,\rho,\eta) = \tilde p_N(r,[r]\rho,[r]\eta)$,

$$p_N(r,\rho,\eta) = \sum_{k=0}^{N} \frac{1}{k!}\partial_\rho^k a(r,\rho,\eta) D_r^k\varphi(r) + c_N(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu+\tilde\nu},$$

and $d_N(r,\rho,\eta) = r_N(r,\rho,\eta) - c_N(r,\rho,\eta)$. Now $r_N(r,\rho,\eta)$ satisfies the desired estimates. Similarly as in connection with (2.18) for every $s', s'' \in \mathbb{R}$ and $M \in \mathbb{N}$ we find an $N \in \mathbb{N}$ sufficiently large such that

$$\|c_N(r,\tilde\rho,\tilde\eta)\|_{s',s''} \leq c\langle r\rangle^{\nu+\tilde\nu}\langle\tilde\rho,\tilde\eta\rangle^{-4M}.$$

This entails

$$\|c_N(r,[r]\rho,[r]\eta)\|_{s',s''} \leq c\langle r\rangle^{\nu+\tilde\nu}\langle[r]\rho,[r]\eta\rangle^{-4M}$$

for all r,ρ,η. Now

$$\begin{aligned}\langle[r]\rho,[r]\eta\rangle^{-4} &= [r]^{-4}\left(\frac{1}{\frac{1+[r]^2\rho^2+[r]^2\eta^2}{[r]^2}}\right)^2 \\ &= [r]^{-4}\frac{1}{[r]^{-2}+\rho^2+\eta^2}\,\frac{1}{[r]^{-2}+\rho^2+\eta^2} \leq c[r]^{-4}\langle\rho\rangle^{-2}\langle\eta\rangle^{-2}\end{aligned}$$

for $|\eta| \geq \varepsilon > 0$, for a constant $c = c(\varepsilon) > 0$. We thus obtain

$$\|c_N(r,[r]\rho,[r]\eta)\|_{s',s''} \leq c\langle r\rangle^{\nu+\tilde\nu-4M}\langle\rho\rangle^{-2M}\langle\eta\rangle^{-2M}.$$

This completes the proof since M is arbitrary. $\square$

Let us now return to the interpretation of (2.13) as the left symbol of a composition of operators. From Theorem 2.8 we know that

$$\mathrm{Op}_r(a)(\eta), \mathrm{Op}_r(b)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$$

are continuous operators. Thus also $\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta)$ is continuous between the Schwartz spaces. This shows, in particular, that the oscillatory integral techniques of [7] also apply for our (here operator-valued) amplitude functions, and we obtain the relation (2.14).

Let $A(\eta) = \mathrm{Op}_r(a)(\eta)$ for

$$a(r,\rho,\eta) := \tilde{a}(r,[r]\rho,[r]\eta),\ \ \tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S\nu\big(\mathbb{R}, L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$

Then we form the formal adjoint $A^*(\eta)$ with respect to the $L^2(\mathbb{R}\times X)$-scalar product, according to

$$\big(A(\eta)u,v\big)_{L^2(\mathbb{R}\times X)} = \big(u, A^*(\eta)v\big)_{L^2(\mathbb{R}\times X)}$$

for all $u,v \in \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$. As usual we obtain

$$A^*(\eta)v(r') = \mathrm{Op}_{r'}(a^*)(\eta)v(r')$$

for the right symbol $a^*(r',\rho,\eta) = \bar{a}(r',\rho,\eta) = \bar{\tilde{a}}(r',[r']\rho,[r']\eta)$. Similarly as before we can prove that

$$\mathrm{Op}_{r'}(a^*)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big)$$

is continuous for every $\eta \neq 0$. Thus by duality it follows that

$$\mathrm{Op}_r(a)(\eta) : \mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big) \to \mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big) \tag{2.29}$$

is continuous for every $\eta \neq 0$. Note here that $f \in \mathcal{E}'(X) \Leftrightarrow f \in H^s(X)$ for some real $s \in \mathbb{R}$; then $\mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big)$ means the inductive limit of the spaces $\mathcal{L}\big(\mathcal{S}(\mathbb{R}), H^s(X)\big)$ over $s \in \mathbb{R}$.

Lemma 2.11. *For every $s', s'' \in \mathbb{R}$ and $l,m,k \in \mathbb{N}$ there exists a real $\mu(s',s'',k,l,m)$ such that for every $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$, $\nu \in \mathbb{R}$ we have*

$$\|a(r,\rho,\eta)\|_{s',s''} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m}$$

$(r,\rho) \in \mathbb{R}^2$, whenever $\mu \leq \mu(s',s'',k,l,m)$, $|\eta| \geq \varepsilon > 0$.

Proof. The proof is straightforward, using Theorem 2.1, more precisely, writing $a(r,\rho,\eta) = \tilde{a}(r,[r]\rho,[r]\eta)$, we have the estimate

$$\|\tilde{a}(r,\tilde{\rho},\tilde{\eta})\|_{s',s''} \leq c\langle r\rangle^{\nu}\langle\tilde{\rho},\tilde{\eta}\rangle^{-N}$$

for every fixed $N \in \mathbb{N}$ when μ is chosen sufficiently negative (depending on N), uniformly in $r \in \mathbb{R}$. Then, similarly as in the proof of Lemma 2.9, we obtain for suitable N and given k,l,m that $\langle[r]\rho,[r]\eta\rangle^{-N} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l-\nu}\langle\eta\rangle^{-m}$ for $|\eta| \geq \varepsilon > 0$. □

Corollary 2.12. *Let $a(r,\rho,\eta) \in \boldsymbol{S}^{-\infty,\nu}$ $\Big(:= \bigcap_{\mu\in\mathbb{R}} \boldsymbol{S}^{\mu,\nu}\Big)$. Then for every $s',s'' \in \mathbb{R}$, $l,m,k \in \mathbb{N}$ we have*

$$\|D^i_r D^j_\rho a(r,\rho,\eta)\|_{s',s''} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m}$$

for all $(r,\rho) \in \mathbb{R}^2$, $|\eta| \geq \varepsilon > 0$, $i,j \in \mathbb{N}$, for some constants $c = c(s',s'',k,l,m,\varepsilon) > 0$.

Proposition 2.13. *The kernels* $c(r, r', \eta)$ *of operators* $\mathrm{Op}_r(a)(\eta)$ *for* $a \in \boldsymbol{S}^{-\infty,\nu}$, $\nu \in \mathbb{R}$, *belong to*

$$C^\infty\big(\mathbb{R}^q \setminus \{0\}, \mathcal{S}\big(\mathbb{R} \times \mathbb{R}, \mathcal{L}(H^{s'}(X), H^{s''}(X))\big)\big) \tag{2.30}$$

and are strongly decreasing in η *for* $|\eta| \geq \varepsilon > 0$ *together with all* η*-derivatives; more precisely, we have*

$$\sup \|\langle\eta\rangle^\alpha D_\eta^\beta \langle r, r'\rangle^\sigma D_{r,r'}^\tau c(r, r', \eta)\|_{s',s''} < \infty \tag{2.31}$$

for every $\alpha, \beta \in \mathbb{N}^q$, $\sigma, \tau \in \mathbb{R}^2$ *with* sup *being taken over all* $|\eta| \geq \varepsilon > 0$, $(r, r') \in \mathbb{R}$.

Proof. If we show the result for $\nu = 0$ from Proposition 2.4 it follows immediately for all ν. Write $a(r, \rho, \eta) = \tilde{a}(r, [r]\rho, [r]\eta)$ for $\tilde{a}(r, \tilde{\rho}, \tilde{\eta}) \in S^0\big(\mathbb{R}, L^{-\infty}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. Then we have

$$\|D^\gamma_{\tilde{\rho},\tilde{\eta}} \tilde{a}(r, \tilde{\rho}, \tilde{\eta})\|_{s',s''} \leq c\langle\tilde{\rho}, \tilde{\eta}\rangle^{-N}$$

for every $s', s'' \in \mathbb{R}$, $\gamma \in \mathbb{N}^{1+q}$, $N \in \mathbb{N}$. This gives us easily

$$\|D_r^i D_\rho^j D_\eta^\alpha a(r, \rho, \eta)\|_{s',s''} \leq c\langle\rho\rangle^{-k}\langle r\rangle^{-l}\langle\eta\rangle^{-m}$$

for every $k, l, m \in \mathbb{N}$ for a sufficiently large N, $|\eta| \geq \varepsilon > 0$. Now the kernel of $\mathrm{Op}_r(a)(\eta)$ has the form

$$\int e^{i(r-r')\rho} a(r, \rho, \eta) đ\rho = \int e^{i(r-r')\rho}\big(1 + |r - r'|^2\big)^{-M}(1 - \Delta_\rho)^M a(r, \rho, \eta) đ\rho \tag{2.32}$$

for every sufficiently large M. This implies

$$\begin{aligned} \left\| \int e^{i(r-r')\rho} a(r, \rho, \eta) đ\rho \right\|_{s',s''} &\leq \int \|\big(1 + |r - r'|^2\big)^{-M}(1 - \Delta_\rho)^M a(r, \rho, \eta)\|_{s',s''} đ\rho \\ &\leq c\big(1 + |r - r'|^2\big)^{-M}\langle r\rangle^{-l}\langle\eta\rangle^{-m} \int \langle\rho\rangle^{-k} đ\rho \\ &\leq c\big(1 + |r - r'|^2\big)^{-M}\langle r\rangle^{-l}\langle\eta\rangle^{-m} \end{aligned}$$

for $k \geq 2$. In a similar manner we can treat the (r, r')-derivatives of the kernel and which completes the proof. □

Definition 2.14. (i) Let $\boldsymbol{L}^{-\infty,-\infty}(X^\asymp; \mathbb{R}^q \setminus \{0\})$ denote the space of all operators with kernels $c(r, r', \eta)$ as in Peoposition 2.13. Moreover, for purposes below, let $\boldsymbol{L}^{-\infty,-\infty}(X^\asymp)$ denote the space of all operators with kernels

$$c(r, r') \in \bigcap_{s',s'' \in \mathbb{R}} \mathcal{S}\big(\mathbb{R} \times \mathbb{R}, \mathcal{L}\big(H^{s'}(X), H^{s''}(X)\big)\big).$$

(ii) Let $\boldsymbol{L}^{\mu,\nu}(X^\asymp; \mathbb{R}^q \setminus \{0\})$ denote the space of all operators of the form

$$A(\eta) = \mathrm{Op}_r(a)(\eta) + C(\eta)$$

depending on the parameter $\eta \in \mathbb{R}^q \setminus \{0\}$, for arbitrary $a(r, \rho, \eta) \in \boldsymbol{S}^{\mu,\nu}$ and operators $C(\eta)$ with kernels $c(r, r', \eta)$ as in Proposition 2.13.

Theorem 2.15. *For every* $\tilde{p}(r,\tilde{\rho},\tilde{\eta}) \in S^0\big(\mathbb{R}, L^s_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$*,* $s \le 0$*, and* $p(r,\rho,\eta) = \tilde{p}(r,[r]\rho,[r]\eta)$*, the operator*

$$\mathrm{Op}_r(p)(\eta) : L^2(\mathbb{R}\times X) \to L^2(\mathbb{R}\times X) \tag{2.33}$$

is continuous for every $\eta \in \mathbb{R}^q \setminus \{0\}$*, and we have*

$$\|\mathrm{Op}_r(p)(\eta)\|_{\mathcal{L}(L^2(\mathbb{R}\times X))} \le c\langle\eta\rangle^s \tag{2.34}$$

for all $|\eta| \ge \varepsilon, \varepsilon > 0$ *and a constant* $c = c(\varepsilon) > 0$*.*

Proof. For the continuity (2.33) and the estimate (2.34) we apply a version of the Calderón–Vaillancourt theorem which states that if H is a Hilbert space and $a(r,\rho) \in C^\infty\big(\mathbb{R}\times\mathbb{R},\mathcal{L}(H)\big)$ a symbol satisfying the estimate

$$\pi(a) := \sup_{\substack{k,l=0,1\\ (r,\rho)\in\mathbb{R}^2}} \|D^l_r D^k_\rho a(r,\rho)\|_{\mathcal{L}(H)} < \infty \tag{2.35}$$

the operator

$$\mathrm{Op}_r(a) : L^2(\mathbb{R},H) \to L^2(\mathbb{R},H)$$

is continuous, where

$$\|\mathrm{Op}_r(a)\|_{\mathcal{L}(L^2(\mathbb{R},H))} \le c\pi(a)$$

for a constant $c > 0$. In the present case we have

$$a(r,\rho) = p(r,[r]\rho,[r]\eta) \tag{2.36}$$

where $\eta \ne 0$ appears as an extra parameter. It is evident that the right-hand side of (2.36) belongs to $C^\infty\big(\mathbb{R}\times\mathbb{R}\times\mathbb{R}^q,\mathcal{L}(L^2(X))\big)$. From the assumption on $\tilde{p}(r,\tilde{\rho},\tilde{\eta})$ we have

$$\sup_{r\in\mathbb{R}} \|\tilde{p}(r,\tilde{\rho},\tilde{\eta})\|_{\mathcal{L}(L^2(X))} \le c\langle\tilde{\rho},\tilde{\eta}\rangle^s \tag{2.37}$$

for all $(\tilde{\rho},\tilde{\eta}) \in \mathbb{R}^{1+q}$ and some $c > 0$. In fact, when $\tilde{p}$ is independent of r the latter estimate corresponds to (2.4) for $s = \nu = 0$ and $\mu = s \le 0$. In the r-dependent case the operator norms that play a role in Theorem 2.1 are uniformly bounded in $r \in \mathbb{R}$, since $\tilde{p}(r,\tilde{\rho},\tilde{\eta})$ is a symbol of order 0 in r with values in $L^s_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$. For (2.35) we first check the case $l = k = 0$. We have

$$\sup_{(r,\rho)\in\mathbb{R}^2} \langle [r]\rho,[r]\eta\rangle^s \le c\langle\eta\rangle^s \tag{2.38}$$

for all $|\eta| \ge \varepsilon > 0$ and some $c = c(\varepsilon) > 0$. Thus (2.37) gives us

$$\sup_{(r,\rho)\in\mathbb{R}^2} \|\tilde{p}(r,[r]\rho,[r]\eta)\|_{\mathcal{L}(L^2(X))} \le c\langle\eta\rangle^s$$

for such a $c(\varepsilon) > 0$. Assume now for simplicity $q = 1$ (The general case is analogous). For the first-order derivatives of $\tilde{p}(r,[r]\rho,[r]\eta)$ in r we have

$$\partial_r\tilde{p}(r,[r]\rho,[r]\eta) = (\partial_r\tilde{p})(r,[r]\rho,[r]\eta) + [r]'(\rho\partial_{\tilde{\rho}} + \eta\partial_{\tilde{\eta}})\tilde{p}(r,[r]\rho,[r]\eta) \tag{2.39}$$

for $[r]' = \frac{d}{dr}[r]$. For the derivatives of $\tilde{p}$ with respect to $\tilde{\rho}, \tilde{\eta}$ we employ that $\partial_{\tilde{\rho}}\tilde{p}(r,\tilde{\rho},\tilde{\eta}), \partial_{\tilde{\eta}}\tilde{p}(r,\tilde{\rho},\tilde{\eta}) \in S^0\big(\mathbb{R}, L^{s-1}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$. Thus, similarly as before we obtain

$$\|\partial^{\alpha}_{\tilde{\rho},\tilde{\eta}}\tilde{p}(r,\tilde{\rho},\tilde{\eta})\|_{\mathcal{L}(L^2(X))} \leq c\langle\tilde{\rho},\tilde{\eta}\rangle^{s-1}$$

for any $\alpha \in \mathbb{N}^2$, $|\alpha| = 1$. This gives us for the summand on the right of (2.39) that

$$\begin{aligned}&\sup_{(r,\rho)\in\mathbb{R}^2} \|[r]^{-1}[r]'([r]\rho\partial_{\tilde{\rho}} + [r]\eta\partial_{\tilde{\eta}})p(r,[r]\rho,[r]\eta)\|_{\mathcal{L}(L^2(X))}\\ &\leq \sup[r]^{-1}|[r]\rho + [r]\eta|\langle r\rho, r\eta\rangle^{s-1}\\ &\leq c\langle\eta\rangle^s \sup[r]^{-1}|[r]\rho,[r]\eta|\langle r\rho, r\eta\rangle^{-1} \leq c\langle\eta\rangle^s.\end{aligned}$$

Here we employed (2.38). For the derivative of $p(r,[r]\rho,[r]\eta)$ in ρ we have

$$\begin{aligned}\sup\|\partial_\rho\tilde{p}(r,[r]\rho,[r]\eta)\|_{\mathcal{L}(L^2(X))} &= \sup\|[r](\partial_{\tilde{\rho}}\tilde{p})(r,[r]\rho,[r]\eta)\|_{\mathcal{L}(L^2(X))}\\ &\leq c\sup[r]\langle[r]\rho,[r]\eta\rangle^{s-1} \leq c\langle\eta\rangle^s\end{aligned}$$

for all $|\eta| \geq \varepsilon > 0$. This gives altogether the estimate (2.34). □

Theorem 2.16. *Let $a \in \boldsymbol{S}^{\mu,\nu}, b \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}}$; then we have*

$$\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) \in \boldsymbol{L}^{\mu+\tilde{\mu},\nu+\tilde{\nu}}(X^{\asymp};\mathbb{R}^q\setminus\{0\}).$$

Proof. According to (2.14) the composition can be expressed by $a\#b$, given by the formula (2.13). By virtue of Corollary 2.5 we have

$$\frac{1}{k!}\partial^k_\rho a(r,\rho,\eta)D^k_r b(r,\rho,\eta) \in \boldsymbol{S}^{\mu+\tilde{\mu}-k,\nu+\tilde{\nu}},$$

i.e., this symbol has the form $c_k(r,\rho,\eta) = \tilde{c}_k(r,[r]\rho,[r]\eta)$ for some

$$\tilde{c}_k(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu+\tilde{\nu}}\big(\mathbb{R}, L^{\mu+\tilde{\mu}-k}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$

Applying Propositon 2.7 we form the asymptotic sum

$$\sum_{k=0}^{\infty}\tilde{c}_k(r,\tilde{\rho},\tilde{\eta}) \sim \tilde{c}(r,\tilde{\rho},\tilde{\eta}) \in S^{\nu+\tilde{\nu}}\big(\mathbb{R}, L^{\mu+\tilde{\mu}}_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big).$$

Setting $c(r,\rho,\eta) = \tilde{c}(r,[r]\rho,[r]\eta)$ from (2.15) we obtain

$$\mathrm{Op}_r(a\#b)(\eta) = \mathrm{Op}_r(c)(\eta) + \mathrm{Op}_r\left(\sum_{k=0}^{N} c_k - c\right)(\eta) + \mathrm{Op}_r(r_N)(\eta)$$

mod $\boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q\setminus\{0\})$, where $\left(\sum_{k=0}^N c_k - c\right)(r,\rho,\eta) \in \boldsymbol{S}^{\mu+\tilde{\mu}-(N+1),\nu}$. Since this is true for every $N \in \mathbb{N}$, Lemma 2.11 gives us the right remainder estimate also for $\|\mathrm{Op}_r\left(\sum_{k=0}^N c_k - c\right)\|_{s',s''}$, and it follows altogether that the kernel of $\mathrm{Op}_r(a\#b)(\eta) - \mathrm{Op}_r(c)(\eta)$ has finite semi-norms (2.31) as indicated in Proposition 2.13 for arbitrary $\alpha,\beta \in \mathbb{N}^q$, $\sigma,\tau \in \mathbb{R}^2$, $s',s'' \in \mathbb{R}$, $|\eta| \geq \varepsilon > 0$. □

Theorem 2.17. *Let $\tilde{p}(\tilde{\rho},\tilde{\eta}) \in L^s_{\mathrm{cl}}(X;\mathbb{R}^{1+q})$ be parameter-dependent elliptic of order $s \in \mathbb{R}$, and set $p(r,\rho,\eta) = \tilde{p}([r]\rho,[r]\eta)$. Then there exists a $C > 0$ such that for every $|\eta| \geq C$ the operator*

$$[r]^{-s}\mathrm{Op}_r(p)(\eta) : \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \to \mathcal{S}\big(\mathbb{R}, C^\infty(X)\big) \tag{2.40}$$

extends to an injective operator

$$[r]^{-s}\mathrm{Op}_r(p)(\eta) : L^2(\mathbb{R}\times X) \to \mathcal{S}'\big(\mathbb{R}, \mathcal{E}'(X)\big). \tag{2.41}$$

More precisely, if we consider $[r]^{-s}\mathrm{Op}_r(p)(\eta)$ as an operator

$$[r]^{-s}\mathrm{Op}_r(p)(\eta) : L^2(\mathbb{R}\times X) \to \mathcal{L}\big(\langle r\rangle^{-g}H^l(\mathbb{R}), H^t(X)\big) \tag{2.42}$$

which is continuous for some $t \in \mathbb{R}$ and all $g, l \in \mathbb{R}$, then it is injective.

Proof. First, according to (2.29) there is a t such that (2.42) is continuous for all $g, l \in \mathbb{R}$. For the injectivity we show that the operator hast a left inverse. This will be approximated by $\mathrm{Op}_r(a)$ for

$$a(r,\rho,\eta) := [r]^s\tilde{p}^{(-1)}([r]\rho,[r]\eta) \tag{2.43}$$

where $\tilde{p}^{(-1)}(\tilde{\rho},\tilde{\eta}) \in L^{-s}_{\mathrm{cl}}(X;\mathbb{R}^{1+q})$ is a parameter-dependent parametrix of $\tilde{p}(\tilde{\rho},\tilde{\eta})$. Setting

$$b(r,\rho,\eta) := [r]^{-s}\tilde{p}([r]\rho,[r]\eta) \tag{2.44}$$

we can write the composition of the associated pseudo-differential operators in r for every $N \in \mathbb{N}$ in the form

$$\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) = \mathrm{Op}_r(a\#b)(\eta) = \mathrm{Op}_r(1 + c_N(r,\rho,\eta) + r_N(r,\rho,\eta)) \tag{2.45}$$

for $c_N(r,\rho,\eta) = \sum_{k=1}^N \frac{1}{k!}\partial_\rho^k a(r,\rho,\eta)D_r^k b(r,\rho,\eta)$ which has the form $c_N(r,\rho,\eta) = \tilde{c}_N(r,[r]\rho,[r]\eta)$ for some $\tilde{c}_N(r,\tilde{\rho},\tilde{\eta}) \in S^0\big(\mathbb{R}, L^{-1}_{\mathrm{cl}}(X;\mathbb{R}^{1+q})\big)$. Moreover, the remainder r_N is as in (2.16). From Theorem 2.15 for $s = -1$ we know that

$$\|\mathrm{Op}_r(c_N)(\eta)\|_{\mathcal{L}(L^2(\mathbb{R}\times X))} \leq c\langle\eta\rangle^{-1}$$

for $|\eta| > \varepsilon$. Moreover, Lemma 2.9, applied to $s' = s'' = 0$ together with an operator-valued version of the Calderón–Vaillancourt theorem, gives us

$$\|\mathrm{Op}_r(r_N)(\eta)\|_{\mathcal{L}(L^2(\mathbb{R}\times X))} \leq c\langle\eta\rangle^{-1}$$

for sufficiently large N. Thus for every $|\eta|$ sufficiently large the operator on the right of (2.45) is invertible in $L^2(\mathbb{R}\times X)$, i.e., $\mathrm{Op}_r(b)(\eta)$ has a left inverse which implies the injectivity. □

Corollary 2.18. *Let $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$, i.e., $a(r,\rho,\eta) = \tilde{a}(r,[r]\rho,[r]\eta)$ for $\tilde{a}(r,\tilde{\rho},\tilde{\eta}) \in S^\nu\big(\mathbb{R}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})\big)$, and let $\eta \neq 0$. Then*

$$\mathrm{Op}_r(a)(\eta) : \mathcal{S}(\mathbb{R}\times X) \to \mathcal{S}(\mathbb{R}\times X)$$

extends to a continuous operator

$$\mathrm{Op}_r(a)(\eta) : \langle r\rangle^m H^s(\mathbb{R}, H^t(X)) \to \langle r\rangle^{\tilde{m}} H^{\tilde{s}}(\mathbb{R}, H^{\tilde{t}}(X))$$

for every $\tilde{m}, \tilde{s}, \tilde{t} \in \mathbb{R}$ *and suitable* $m, s, t \in \mathbb{R}$*. If* $\tilde{a}(r, \tilde{\rho}, \tilde{\eta})$ *is parameter-dependent elliptic for every* $r \in \mathbb{R}$ *then there is a* $C > 0$ *such that*

$$\mathrm{Op}_r(a)(\eta) : \langle r\rangle^m H^s(\mathbb{R}, H^t(X)) \to \mathcal{S}'(\mathbb{R}, \mathcal{E}'(X))$$

is injective for every $m, s, t \in \mathbb{R}$ *and* $|\eta| \geq C$*.*

Theorem 2.19. $A \in \boldsymbol{L}^{\mu,\nu}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$*,* $B \in \boldsymbol{L}^{\tilde{\mu},\tilde{\nu}}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$ *implies* $AB \in \boldsymbol{L}^{\mu+\tilde{\mu},\nu+\tilde{\nu}}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$*.*

Proof. Let us write

$$A(\eta) = \mathrm{Op}_r(a)(\eta) + C_1(\eta), \;\; B(\eta) = \mathrm{Op}_r(b)(\eta) + C_2(\eta)$$

for $a \in \boldsymbol{S}^{\mu,\nu}$, $b \in \boldsymbol{S}^{\tilde{\mu},\tilde{\nu}}$, and $C_1(\eta), C_2(\eta) \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$. Then we have

$$AB = \mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) + C_1(\eta)\mathrm{Op}_r(b)(\eta) + \mathrm{Op}_r(a)(\eta)C_2(\eta) + C_1(\eta)C_2(\eta).$$

Theorem 2.16 yields that $\mathrm{Op}_r(a)(\eta)\mathrm{Op}_r(b)(\eta) \in \boldsymbol{L}^{\mu+\tilde{\mu},\nu+\tilde{\nu}}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$. Moreover, the composition of smoothing families is again smoothing. It remains to show that $C_1(\eta)\mathrm{Op}_r(b)(\eta), \mathrm{Op}_r(a)(\eta)C_2(\eta) \in \boldsymbol{L}^{\mu+\tilde{\mu},\nu+\tilde{\nu}}(X^{\asymp}; \mathbb{R}^q \setminus \{0\})$. However, this is a simple consequence of Corollary 2.18. □

3. Parameter-Dependent Operators on an Infinite Cylinder

3.1. Weighted Cylindrical Spaces

Definition 3.1. Let $\tilde{p}(\tilde{\rho}, \tilde{\eta}) \in L^s_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ be as in Theorem 2.17. Then $H^{s,g}_{\mathrm{cone}}(X^{\asymp})$ for $s, g \in \mathbb{R}$ is defined to be the completion of $\mathcal{S}(\mathbb{R} \times X)$ with respect to the norm

$$\|[r]^{-s+g}\mathrm{Op}_r(p)(\eta^1)u\|_{\mathcal{L}(L^2(\mathbb{R}\times X))}$$

for any fixed $\eta^1 \in \mathbb{R}^q$, $|\eta^1| \geq C$ for some $C > 0$ sufficiently large.

Setting $p^{s,g}(r, \rho, \eta) := [r]^{-s+g}\tilde{p}([r]\rho, [r]\eta)$, from Definition 3.1 it follows that

$$\mathrm{Op}(p^{s,g})(\eta^1) : \mathcal{S}(\mathbb{R} \times X) \to \mathcal{S}(\mathbb{R} \times X)$$

extends to a continuous operator

$$\mathrm{Op}(p^{s,g})(\eta^1) : H^{s,g}_{\mathrm{cone}}(X^{\asymp}) \to L^2(\mathbb{R} \times X). \tag{3.1}$$

Theorem 3.2. *The operator* (3.1) *is an isomorphism for every fixed* $s, g \in \mathbb{R}$ *and* $|\eta^1|$ *sufficiently large.*

Proof. We show the invertibility by verifying that there is a right and a left inverse. By notation we have $p^{s,g}(r, \rho, \eta) = [r]^{-s+g}\tilde{p}([r]\rho, [r]\eta) \in \boldsymbol{S}^{s,-s+g}$. The operator family $\tilde{p}(\tilde{\rho}, \tilde{\eta}) \in L^s_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ is invertible for large $|\tilde{\rho}, \tilde{\eta}| \geq C$ for some $C > 0$. There exists a parameter-dependent parametrix $\tilde{p}^{(-1)}(\tilde{\rho}, \tilde{\eta}) \in L^{-s}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})$ such that $\tilde{p}^{(-1)}(\tilde{\rho}, \tilde{\eta}) = \tilde{p}^{-1}(\tilde{\rho}, \tilde{\eta})$ for $|\tilde{\rho}, \tilde{\eta}| \geq C$. Let us set

$$p^{-s,-g}(r, \rho, \eta) := [r]^{s-g}\tilde{p}^{(-1)}([r]\rho, [r]\eta) \in \boldsymbol{S}^{-s,s-g},$$

and $P^{s,g}(\eta) := \mathrm{Op}(p^{s,g})(\eta)$, $P^{-s,-g}(\eta) := \mathrm{Op}(p^{-s,-g})(\eta)$. Then we have

$$P^{s,g}(\eta)P^{-s,-g}(\eta) = 1 + \mathrm{Op}(c_N)(\eta) + R_N(\eta) \tag{3.2}$$

for some $c_N(r,\rho,\eta) \in \boldsymbol{S}^{-1,0}$ and a remainder $R_N(\eta) = \mathrm{Op}(r_N)(\eta)$ where r_N is as in Lemma 2.9. We have $\mathrm{Op}(c_N)(\eta) \to 0$ and $R_N(\eta) \to 0$ in $\mathcal{L}(L^2(\mathbb{R}\times X))$ as $|\eta| \to \infty$; the first property is a consequence of Theorem 2.15, the second one of the estimate (2.17). Thus (3.2) shows that $P^{s,g}(\eta)$ has a right inverse for $|\eta|$ sufficiently large. Such considerations remain true when we interchange the role of s,g and $-s,-g$. In other words, we also have

$$P^{-s,-g}(\eta)P^{s,g}(\eta) = 1 + \mathrm{Op}(\tilde{c}_N)(\eta) + \widetilde{R}_N(\eta)$$

where $\mathrm{Op}(\tilde{c}_N)(\eta)$ and $\widetilde{R}_N(\eta)$ are of analogous behaviour as before. This shows that $P^{s,g}(\eta)$ has a left inverse for large $|\eta|$, and we obtain altogether that (3.1) is an isomorphism for $\eta = \eta^1$, $|\eta^1|$ sufficiently large. □

3.2. Elements of the Calculus

The results of Section 2.3 show the behaviour of compositions of parameter-dependent families $\mathrm{Op}(a)(\eta)$ for $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ and $\eta \neq 0$, first on $\mathcal{S}(\mathbb{R}\times X)$. In particular, it can be proved that, when we concentrate, for instance, on the case $s' = s'' = 0$, invertible operators of the form $1 + K : L^2(\mathbb{R}\times X) \to L^2(\mathbb{R}\times X)$, for $K \in \boldsymbol{L}^{-\infty,-\infty}(X^{\asymp};\mathbb{R}^q \setminus \{0\})$, can be written in the form $1 + L$ where L is again an operator of such a smoothing behaviour. Moreover, there are other (more or less standard) constructions that are immediate by the results of Section 2. For instance, if we look at $c(r,\rho,\eta) \in \boldsymbol{S}^{-1,0}$ in the relation (3.2), by a formal Neumann series argument we find a $d(r,\rho,\eta) \in \boldsymbol{S}^{-1,0}$ such that

$$\big(1 + \mathrm{Op}(c)\big)\big(1 + \mathrm{Op}(d)\big) = 1 + \mathrm{Op}(r_N)$$

for every $N \in \mathbb{N}$ with a remainder r_N which is again as in Lemma 2.9.

Theorem 3.3. *Let $a(r,\rho,\eta) \in \boldsymbol{S}^{\mu,\nu}$ and $|\eta| \neq 0$. Then*

$$\mathrm{Op}(a)(\eta) : \mathcal{S}(\mathbb{R}\times X) \to \mathcal{S}(\mathbb{R}\times X)$$

extends to a continuous operator

$$\mathrm{Op}(a)(\eta) : H^{s,g}_{\mathrm{cone}}(X^{\asymp}) \to H^{s-\mu,g-\nu}_{\mathrm{cone}}(X^{\asymp}) \tag{3.3}$$

for every $s,g \in \mathbb{R}$.

Proof. Let $u \in \mathcal{S}(\mathbb{R}\times X)$, and set $\|\cdot\|_{s,g} := \|\cdot\|_{H^{s,g}_{\mathrm{cone}}(X^{\asymp})}$, in particular, $\|\cdot\|_{0,0} = \|\cdot\|_{L^2(\mathbb{R}\times X)}$. By definition we have $\|u\|_{s,g} = \|\mathrm{Op}(p^{s,g})(\eta^1)\|_{0,0}$. Thus

$$\begin{aligned}\|\mathrm{Op}(a)(\eta)u\|_{s-\mu,g-\nu} &= \|\mathrm{Op}(p^{s-\mu,g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)u\|_{0,0}\\ &= \|\mathrm{Op}(p^{s-\mu,g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)\mathrm{Op}(p^{s,g})^{-1}(\eta^1)\mathrm{Op}(p^{s,g})(\eta^1)u\|_{0,0} \le c\|u\|_{s,g},\end{aligned}$$

$c := \|\mathrm{Op}(p^{s-\mu,g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)\mathrm{Op}(p^{s,g})^{-1}(\eta^1)\|_{\mathcal{L}(L^2(\mathbb{R}\times X))}$. It remains to prove that c is a finite constant. This is completely straightforward when we replace $\mathrm{Op}(p^{s,g})^{-1}(\eta^1)$ by $\mathrm{Op}(p^{-s,-g})(\eta^1)$; in that case we have

$$\mathrm{Op}(p^{s-\mu,g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)\mathrm{Op}(p^{-s,-g})(\eta^1)=\mathrm{Op}\big(p^{s-\mu,g-\nu}(\cdot,\eta^1)a(\cdot,\eta)p^{-s,-g}(\cdot,\eta^1)\big)$$

(where $\cdot$ stands for r,ρ) modulo a remainder of the form $\mathrm{Op}(c)+R_N$ and $\mathrm{Op}(c)$ is bounded in $L^2(\mathbb{R}\times X)$ for similar reasons as in Theorem 2.15 and the boundedness of R_N in $L^2(\mathbb{R}\times X)$ is clear anyway. In general, $\mathrm{Op}(p^{s,g})^{-1}(\eta^1)$ has the form $\mathrm{Op}(p^{-s,-g})(\eta^1)+C_N(\eta^1)+R_N(\eta^1)$ for $C_N(\eta^1)=\mathrm{Op}(c_N(\cdot,\eta^1))$ and a remainder $R_N(\eta^1)\in \boldsymbol{L}^{-\infty,-\infty}$ while $c_N(\cdot,\eta)$ belongs to $\boldsymbol{S}^{-s-1,-g}$. Then, compared with the first step of the proof, we obtain extra terms, namely,

$$\mathrm{Op}(p^{s-\mu,g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)\mathrm{Op}(c_N)(\eta^1), \tag{3.4}$$

$$\mathrm{Op}(p^{s-\mu,g-\nu})(\eta^1)\mathrm{Op}(a)(\eta)R_N(\eta^1) \tag{3.5}$$

which have to be bounded in $L^2(\mathbb{R}\times X)$. The arguments for (3.4) are of the same structure as those at the beginning of the proof (the order of c_N is even better than before), while for the composition (3.5) we apply another remark from the preceding section, namely, that smoothing operators composed by other operators of the calculus give rise to smoothing operators again. □

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J. Abed and B.-W. Schulze
Universität Potsdam
Institut für Mathematik
Am Neuen Palais 10
14469 Potsdam
Germany
e-mail: abed@math.uni-potsdam.de
schulze@math.uni-potsdam.de

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Global Regularity and Stability in S-Spaces for Classes of Degenerate Shubin Operators

Todor Gramchev, Stevan Pilipović and Luigi Rodino

Abstract. We study the uniform regularity and the decay at infinity for anisotropic tensor products of Shubin-type differential operators as well as for degenerate harmonic oscillators. As applications of our general results we obtain new theorems for global hypoellipticity for classes of degenerate operators in inductive and projective Gelfand–Shilov spaces.

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1. Introduction

The main goal of this paper is to derive reductions to normal forms of (tensor products of) degenerate harmonic oscillators in $\mathbb{R}^2$ of the type

$$-\Delta + \frac{\tau^2}{4}(x_1^2 + x_2^2) + \tau(x_2 D_{x_1} - x_1 D_{x_2}), \qquad x = (x_1, x_2) \in \mathbb{R}^2,\ \tau \in \mathbb{R} \setminus \{0\}$$

and their extensions to products of powers of harmonic oscillators by means of symplectic transformations. Secondly, we study anisotropic tensor products of harmonic oscillators in suitable S-type spaces defined by Fourier expansions using the Hermite functions. Finally, as an application, we derive new global regularity and solvability results for products of degnerate harmonics oscillators with lower-order perturbations.

We recall (cf. M. Shubin [13]) that the globally elliptic Shubin operators generalize the Schrödinger harmonic oscillator operator

$$H = -\Delta + |x|^2, \tag{1.1}$$

appearing in quantum mechanics. The spectrum of H in $L^2(\mathbb{R}^n)$ is discrete with eigenvalues $\lambda = \lambda_k = \sum_{j=1}^n (2k_j + 1)$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ while eigenfunctions are the Hermite functions

$$u(x) = H_k(x) = \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi} k_j!} P_{k_j}(x_j) \exp(-|x|^2/2), \tag{1.2}$$

where $P_r(t)$ stands for the r-th Hermite polynomial.

We recall that $f \in S_\nu^\mu(\mathbb{R}^n)$, $\mu > 0$, $\nu > 0$, $\mu + \nu \geq 1$, iff $f \in C^\infty(\mathbb{R}^n)$ and there exist $C > 0$, $\varepsilon > 0$ such that

$$\left|\partial_x^\beta f(x)\right| \leq C^{|\beta|+1}(\beta!)^\mu e^{-\varepsilon|x|^{1/\nu}} \tag{1.3}$$

for all $x \in \mathbb{R}^n$, $\beta \in \mathbb{Z}_+^n$ or, equivalently, one can find $C > 0$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial_x^\alpha f(x)| \leq C^{|\alpha|+|\beta|+1}(\alpha!)^\mu(\beta!)^\nu, \qquad \alpha, \beta \in \mathbb{Z}_+^n. \tag{1.4}$$

This definition gives the smallest Gelfand–Shilov-type space $S_{1/2}^{1/2}(\mathbb{R}^n)$. Beurling-type spaces of ultradifferentiable functions denoted by Σ_μ^σ, $\sigma > 0$, $\mu > 0$, $\sigma + \mu > 1$, called here as projective-type Gelfand–Shilov spaces, are defined similarly. Especially if $\sigma = \mu = 1/2$, then $\Sigma_{1/2}^{1/2} = \{0\}$. Note that $S_{1/2}^{1/2} \subset \Sigma_\sigma^\sigma \subset S_\sigma^\sigma, \sigma > 1/2$.

Recently, M. Cappiello, T. Gramchev and L. Rodino [2] have proved the S_ν^μ-regularity of eigenfunctions to Shubin-type partial differential operators in $\mathbb{R}^n$,

$$P = \sum_{|\alpha|+|\beta|\leq m} c_{\alpha\beta} x^\beta D_x^\alpha, \tag{1.5}$$

where m is a positive integer, provided P is globally elliptic, namely, there exist $C > 0$ and $R > 0$ such that

$$\left|\sum_{|\alpha|+|\beta|\leq m} c_{\alpha\beta} x^\beta \xi^\alpha\right| \geq C(1 + |x|^2 + |\xi|^2)^{m/2}, \qquad |x| + |\xi| \geq R. \tag{1.6}$$

Global ellipticity in the previous sense implies both local regularity and asymptotic decay of the solutions, namely we have the following basic result (see for example M. Shubin [13]): $Pu = f \in \mathcal{S}(\mathbb{R}^n)$ for $u \in \mathcal{S}'(\mathbb{R}^n)$ implies actually $u \in \mathcal{S}(\mathbb{R}^n)$. The main theorem in [2] states that P is also globally hypoelliptic in the Gelfand–Shilov spaces $S_\nu^\mu(\mathbb{R}^n)$, $\mu \geq 1/2$, $\nu \geq 1/2$.

We will consider the symmetric Gelfand–Shilov spaces $S_\mu^\mu, (\mathbb{R}^n)$, $\mu \geq 1/2$ and $\Sigma_\mu^\mu(\mathbb{R}^n)$, $\mu > 1/2$.

Our aim is to weaken the global ellipticity condition (1.6) for linear operators with polynomial coefficients.

Let $\tau \in \mathbb{R} \setminus 0$. We consider the degenerate operators considered by Wong [14] in $\mathbb{R}^2$ (for $\tau = 1$)

$$\begin{aligned} W^\tau &= D_{x_1}^2 + D_{x_2}^2 + \tau^2 \frac{1}{4}(x_1^2 + x_2^2) + \tau x_2 D_{x_1} - \tau x_1 D_{x_2} \\ &= (D_{x_1} + \tau \frac{x_2}{2})^2 + (D_{x_2} - \tau \frac{x_1}{2})^2, \qquad D_{x_k} = i^{-1}\partial_{x_k}. \end{aligned} \tag{1.7}$$

In particular, M.W. Wong [14] has shown that $W = W^1$ is globally hypoelliptic, namely

$$u \in \mathcal{S}'(\mathbb{R}^2),\ Wu \in \mathcal{S}(\mathbb{R}^2) \ \Rightarrow\ u \in \mathcal{S}(\mathbb{R}^2). \tag{1.8}$$

Later on, A. Dasgupta and M.W. Wong [3] established global hypoellipticity for W_τ in the critical inductive Gelfand–Shilov class $S^{1/2}_{1/2}(\mathbb{R}^2)$.

Given $f \in \mathcal{S}(\mathbb{R}^n)$ or $f \in \mathcal{S}'(\mathbb{R}^n)$ we write

$$f = \sum_{k\in\mathbb{Z}^n_+} f_k H_k(x), \qquad f_k = \langle f, H_k\rangle. \tag{1.9}$$

If $n \geq 2$, using the standard lexicographical order we can rewrite $H_k(x)$, $k \in \mathbb{Z}^n_+$ as $\widetilde{H}_j(x)$, $j = 0, 1, \dots$, and (1.9) becomes

$$f = \sum_{j=0}^{\infty} \tilde{f}_j \widetilde{H}_j(x), \qquad \tilde{f}_j = \langle f, \widetilde{H}_j\rangle. \tag{1.10}$$

We define the following spaces of sequences (see A. Avantaggiati [1], H. Holden, B. Øksendal, J. Ubøe and T. Zhang [6], M. Langenbruch [8], Z. Lozanov Crvenković, D. Perisić and M. Tasković [10], B.S. Mitjagin [9], S. Pilipovi ć [11], [12] and the references therein).

$$\ell_{\mathcal{S}}(\mathbb{Z}^n_+) = \{\{v_k\}_{k\in\mathbb{Z}^n_+} : \sup_{k\in\mathbb{Z}^n_+} (|k|^N |v_k|) < +\infty, \text{for all } N > 0\}, \tag{1.11}$$

$$\ell_{\mathcal{S}^\mu_\mu}(\mathbb{Z}^n_+) = \{\{v_k\}_{k\in\mathbb{Z}^n_+} : \sup_{k\in\mathbb{Z}^n_+} (\exp(\delta|k|^{1/(2\mu)} |v_k|) < +\infty, \text{for some } \delta > 0\}, \tag{1.12}$$

$\mu \geq 1/2$, and

$$\ell_{\Sigma^\mu_\mu}(\mathbb{Z}^n_+) = \{\{v_k\}_{k\in\mathbb{Z}^n_+} : \sup_{k\in\mathbb{Z}^n_+} (\exp(\delta|k|^{1/(2\mu)} |v_k|) < +\infty, \text{for all } \delta > 0\}, \tag{1.13}$$

$\mu > 1/2$.

The next assertion is a particular case of more general theorems contained in the aforementioned papers.

Theorem 1.1. *The Hermite expansion is an isomorphism between sequence spaces quoted above and the corresponding Schwartz and Gelfand–Shilov-type spaces (written in the indices).*

2. FIO Reduction of W^τ

In [5] we established a connection between W^τ and the one-dimensional harmonic oscillator through Hermite expansions. Here we propose an alternative approach, in terms of Fourier integral operators. As before, assume, $\tau \in \mathbb{R}\setminus\{0\}$. We observe that in fact the operator W^τ is globally reducible via a symplectic transformation to the one-dimensional harmonic oscillator, namely:

We define an explicit symplectic transformation

$$\varkappa \quad : \quad \mathbb{R}^2_y \times \mathbb{R}^2_\eta \to \mathbb{R}^2_x \times \mathbb{R}^2_\xi \tag{2.1}$$

defined by the generating function

$$\varphi_\tau(x,\eta) = |\tau|(\eta_1\eta_2 + \frac{x_1x_2}{2} + x_2\eta_1 + x_1\eta_2) \tag{2.2}$$

via

$$y_j = \partial_{\eta_j}\varphi_\tau(x,\eta), \qquad j = 1,2 \tag{2.3}$$

$$\xi_j = \partial_{x_j}\varphi_\tau(x,\eta), \qquad j = 1,2, \tag{2.4}$$

which leads to explicit formulas for $\varkappa$

$$y_1 = |\tau|x_2 + |\tau|\eta_2, \tag{2.5}$$

$$y_2 = |\tau|x_1 + |\tau|\eta_1 \tag{2.6}$$

$$\eta_1 = \xi_2/|\tau| - \frac{x_1}{2}, \tag{2.7}$$

$$\eta_2 = \xi_1/|\tau| - \frac{x_2}{2}. \tag{2.8}$$

Then we have:

Theorem 2.1.

$$\varkappa^* W^\tau(y,\eta) = \begin{cases} \tau^2\eta_1^2 + y_1^2 & \text{if} \quad \tau > 0, \\ \tau^2\eta_2^2 + y_2^2 & \text{if} \quad \tau < 0. \end{cases} \tag{2.9}$$

Moreover one can write (2.9) *as an Egorov conjugation-type theorem via a globally defined FIO:*

$$J^{-1} \circ W^\tau \circ J = \begin{cases} \tau^2 D^2_{y_1} + y_1^2 & \text{if} \quad \tau > 0, \\ \tau^2 D^2_{y_2} + y_2^2 & \text{if} \quad \tau < 0. \end{cases} \tag{2.10}$$

where J is the FIO with a phase function defined in a canonical way by $\varkappa$

$$\begin{aligned} Jv(x) &= \int_{\mathbb{R}^2} \exp(i\varphi_\tau(x,\eta))\hat{v}(\eta)\, \bar{d}\eta \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \exp(i\Phi_\tau(x,y,\eta))v(y)dy\, \bar{d}\eta \end{aligned} \tag{2.11}$$

with

$$\Phi_\tau(x,y,\eta) = \varphi_\tau(x,\eta) - y\eta. \tag{2.12}$$

Moreover, J is an automorphism of

$$S(\mathbb{R}^2),\ S^\mu_\mu(\mathbb{R}^2),\ \mu \geq 1/2,\ \text{and}\ \Sigma^{1/2}_{1/2}(\mathbb{R}^2),\ \mu > 1/2. \tag{2.13}$$

Proof of Theorem 2.1. The first part of the statement can be seen as consequence of the general results of L. Hörmander [7] concerning FIOs corresponding to linear symplectic transformations. We have in particular that J is an automorphism of

$\mathcal{S}(\mathbb{R}^2)$. To be definite, we provide here a direct computation for (2.10). In fact, after an integration by parts in sense of oscillatory integrals, we have for $\tau > 0$

$$
\begin{aligned}
(D_{x_1} + \tau\frac{x_2}{2})Jv(x) &= \int_{\mathbb{R}^2} (D_{x_1} + \tau\frac{x_2}{2})(\exp(i\varphi_\tau(x,\eta)))\hat{v}(\eta)\, \bar{d}\eta \\
&= \int_{\mathbb{R}^2} \exp(i\varphi_\tau(x,\eta))(\tau\eta_2 + \tau x_2))\hat{v}(\eta)\, \bar{d}\eta \\
&= \int_{\mathbb{R}^2} D_{\eta_1}(\exp(i\varphi_\tau(x,\eta))))\hat{v}(\eta)\, \bar{d}\eta \\
&= -\int_{\mathbb{R}^2} \exp(i\varphi_\tau(x,\eta))D_{\eta_1}\hat{v}(\eta)\, \bar{d}\eta \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \exp(i\Phi_\tau(x,y,\eta))y_1 v(y) dy\, \bar{d}\eta \\
&= J(y_1 v(y))
\end{aligned} \tag{2.14}
$$

which yields

$$
J^{-1} \circ (D_{x_1} + \tau\frac{x_2}{2})^2 \circ Jv(y) = y_1^2 v(y) \tag{2.15}
$$

We get immediately that

$$
\begin{aligned}
(D_{x_2} - \tau\frac{x_1}{2})Jv(x) &= \int_{\mathbb{R}^2} (D_{x_2} - \tau\frac{x_1}{2})(\exp(i\varphi_\tau(x,\eta)))\hat{v}(\eta)\, \bar{d}\eta \\
&= \int_{\mathbb{R}^2} \exp(i\varphi_\tau(x,\eta))\eta_1\hat{v}(\eta)\, \bar{d}\eta \\
&= J(\tau D_{y_1} v(y))
\end{aligned} \tag{2.16}
$$

which yields

$$
J^{-1} \circ (D_{x_2} - \tau\frac{x_1}{2})^2 \circ Jv(x) = \tau^2 D_{y_1}^2 v(y). \tag{2.17}
$$

Combining (2.15) and (2.17) we get the first part of (2.10) for $\tau > 0$. Similarly we argue in the case $\tau < 0$.

Next, we deal with the automorphism property of J. First we note that

$$
J = e^{ix_1x_2/2}H,
$$

where

$$
Hv(x) = (2\pi)^{-n}\int_{\mathbb{R}^2} e^{i\eta_1\eta_2 + x_1\eta_2 + x_2\eta_1}\hat{v}(\eta_1,\eta_2)d\eta_1 d\eta_2.
$$

Note that

$$
\mathcal{S}_\mu^\mu \ni v \mapsto e^{ix_1x_2/2}v \text{ and } \Sigma_\mu^\mu \ni v \mapsto e^{ix_1x_2/2}v
$$

are automorphisms on $\mathcal{S}_\mu^\mu, \mu \geq 1/2$ and $\Sigma_\mu^\mu, \mu > 1/2$. So it is enough to prove that H is an automorphism on quoted spaces.

Denote by G the mapping $\hat{v}(\eta_1,\eta_2) \mapsto \hat{v}(\eta_1,\eta_2)e^{i\eta_1\eta_2}$. As above one can simply conclude that this is an automorphism on $\mathcal{S}_\mu^\mu$, $\mu \geq 1/2$ and Σ_μ^μ, $\mu > 1/2$, respectively. Now we see that

$$
Hv(x_1,x_2) = (2\pi)^{-n}\mathcal{F}^{-1}(G(\mathcal{F}v(\eta_2,\eta_1)))(x_1,x_2),
$$

and since the Fourier transformation is an automorphism, it follows that H is an automorphism on $\mathcal{S}^\mu_\mu$, $\mu \geq 1/2$ and Σ^μ_μ, $\mu > 1/2$, respectively and the assertion for J is proved. □

Remark 2.2. By the last part of Theorem 2.1 we reduce the problem of the global hypoellipticity and the global solvability in the Schwartz class and the Gelfand–Shilov spaces for an operator M to the operator $J^{-1} \circ M \circ J$, in particular for W^τ to $\tau^2 D^2_{y_1} + y_1^2$ for $\tau > 0$. We used this idea in [5], where J was expressed in terms of Hermite expansions, to recapture the results of A. Dasgupta and M.W. Wong [3] and M.W. Wong [14] for the global hypoellipticity of W^τ. In the present paper we shall go further, applying the same argument to higher order degenerate Shubin operators, see Section 3.

3. Perturbations of Tensor Products of Harmonic Oscillators

We considered in [5] the operator

$$P = P_1^{m_1} \otimes P_2^{m_2}, \tag{3.1}$$

where

$$P_i = x_i^2 - \frac{\partial^2}{\partial x_i^2}, i = 1, 2 \tag{3.2}$$

This operator is not hypo-elliptic in the sense of [13], Definition 25.1.

We consider operators Q with polynomial coefficients of the type

$$Q(x, D) = \sum_{\alpha+\beta\leq 2m_0} q_{\alpha\beta} x^\alpha D_x^\beta, \tag{3.3}$$

where

$$m_0 = \min\{m_1, m_2\}. \tag{3.4}$$

We investigate the perturbation

$$P_\delta(x, D)u = P(x, D)u + \delta Q(x, D)u = f, \tag{3.5}$$

where $\delta \in \mathbb{R}$ is a parameter.

Theorem 3.1. *There exists $\delta_0 > 0$ such that the equation* (3.5) *is globally solvable and globally hypoelliptic in $\mathcal{S}(\mathbb{R}^2)$, $S^\mu_\mu(\mathbb{R}^2)$ (respectively, $\Sigma^\mu_\mu(\mathbb{R}^2)$) with $\mu \geq 1/2$ (respectively, $\mu > 1/2$) provided $|\delta| < \delta_0$.*

Proof. Recall, for $t \in \mathbb{R}$ and $n \in \mathbb{Z}_+$,

$$\frac{d}{dt} H_n(t) = \sqrt{\frac{n}{2}} H_{n-1}(t) - \sqrt{\frac{n+1}{2}} H_{n+1}(t), \tag{3.6}$$

$$t H_n(x) = \sqrt{\frac{n}{2}} H_{n-1}(t) + \sqrt{\frac{n+1}{2}} H_{n+1}(t). \tag{3.7}$$

($H_i = 0$ if $i < 0$.) We develop u and f into Hermite series

$$u = \sum_{(i_1,i_2)\in\mathbb{Z}_+^2} \tilde{u}_{(i_1,i_2)} H_{i_1}(x_1)H_{i_2}(x_2), f = \sum_{i_1,i_2\in\mathbb{Z}_+^2} \tilde{f}_{(i_1,i_2)} H_{i_1}(x_1)H_{i_2}(x_2).$$

Performing

$$x_1^{\alpha_1} x_2^{\alpha_2} \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} H_{i_1}(x_1)H_{i_2}(x_2) =: H_{(i_1,i_2)}^{(\alpha_1,\alpha_2);(\beta_1,\beta_2)}(x_1,x_2)$$

we obtain that the term above can be written as

$$\sum_{p=-\alpha_1-\beta_1}^{\alpha_1+\beta_1} \sum_{q=-\alpha_2-\beta_2}^{\alpha_2+\beta_2} c_{p,q}^{i_1,i_2} u_{i_1+p,i_2+q} H_{i_1+p}(x_1)H_{i_2+q}(x_2), \tag{3.8}$$

for some $c_{p,q}^{i_1,i_2} \in \mathbb{R}$, $p = -\alpha_1-\beta_1, \ldots, \alpha_1+\beta_1$, $q = -\alpha_2-\beta_2, \ldots, \alpha_2+\beta_2$. Then, by the change $i_1 + p \to i_1$, $i_2 + q \to i_2$ it follows that

$$\sum_{p=-\alpha_1-\beta_1}^{\alpha_1+\beta_1} \sum_{q=-\alpha_2-\beta_2}^{\alpha_2+\beta_2} c_{p,q}^{i_1-p,i_2-q} \tilde{u}_{i_1,i_2} H_{i_1}(x_1)H_{i_2}(x_2), \tag{3.9}$$

where $c_{p,q}^{i_1-p,i_2-q} = 0$ if some of the indices $i_1 - p$ or $i_2 - q$ is negative. We denote by $d_{(i_1,i_2)}$ the sum in front of $\tilde{u}_{i_1,i_2}$ on the left-hand side of (3.9) and obtain

$$\sum_{(i_1,i_2)\in\mathbb{Z}_+^2} d_{(i_1,i_2)} \tilde{u}_{(i_1,i_2)} H_{i_1}(x_1)H_{i_2}(x_2) = \sum_{(i_1,i_2)\in\mathbb{Z}_+^2} \tilde{f}_{(i_1,i_2)} H_{i_1}(x_1)H_{i_2}(x_2). \tag{3.10}$$

We will estimate $d_{(i_1,i_2)}$. By the use of (3.6), (3.7) it follows that

$$d_{(i_1,i_2)} = O(|i_1+i_2|^{\alpha_1+\beta_1+\alpha_2+\beta_2}), \qquad (i_1,i_2) \in \mathbb{Z}_+^2.$$

Thus, coming back to equation (3.5), we have

$$\sum_{(i_1,i_2)\in\mathbb{Z}_+^2} d_{(i_1,i_2)} \tilde{u}_{(i_1,i_2)} H_{i_1}(x_1)H_{i_2}(x_2) = \sum_{(i_1,i_2)\in\mathbb{Z}_+^2} \tilde{f}_{(i_1,i_2)} H_{i_1}(x_1)H_{i_2}(x_2). \tag{3.11}$$

This implies that for some sufficiently small $\delta > 0$ and $\varepsilon > 0$,

$$(2i_1+1)^{m_1}(2i_2+1)^{m_2} + \delta \sum_{|i_1+i_2|\le 2m_0} O(|i_1+i_2|^{m_0}) \tag{3.12}$$
$$= (2i_1+1)^{m_1}(2i_2+1)^{m_2}(1-\epsilon), (i_1,i_2) \in \mathbb{Z}_+^2.$$

Thus by (3.12), we can solve (3.10) so that

$$\tilde{u}_{(i_1,i_2)} = \tilde{f}_{(i_1,i_2)}/d_{(i_1,i_2)}, \qquad (i_1,i_2) \in \mathbb{Z}_+^2,$$

and again by the use of (3.12), we prove both assertions of the theorem. Actually, by Theorem 1.1,

$$|u_{(i_1,i_2)}| \exp(\delta|i_1+i_2|^{1/(2\mu)}) = \exp(\delta|i_1+i_2|^{1/(2\mu)}) |\tilde{f}_{(i_1,i_2)}/d_{(i_1,i_2)}| < C, \ (i_1,i_2) \in \mathbb{Z}_+^2$$

for some δ, respectively, for all δ, which implies that u belongs to $\mathcal{S}_\mu^\mu$, $\mu \ge 1/2$, respectively Σ_μ^μ, $\mu > 1/2$. □

4. Applications

Consider the operator W^τ in (1.7) for $\tau = \pm 1$; to be definite

$$W = W^1 = D_{x_1}^2 + D_{x_2}^2 + \frac{1}{4}(x_1^2 + x_2^2) + x_2 D_{x_1} - x_1 D_{x_2} \tag{4.1}$$

$$\widetilde{W} = W^{-1} = D_{x_1}^2 + D_{x_2}^2 + \frac{1}{4}(x_1^2 + x_2^2) - x_2 D_{x_1} + x_1 D_{x_2} \tag{4.2}$$

Set

$$M = W^{m_1}\widetilde{W}m_2 + \delta \sum_{\alpha+\beta\le m_0} c_{\alpha\beta} W^\alpha \widetilde{W}^\beta, \tag{4.3}$$

where m_0 satisfies (3.4), $c_{\alpha\beta} \in \mathbb{C}$ and $\delta \in \mathbb{C}$ is a parameter.

Theorem 4.1. *There exists $\delta_0 > 0$ such that the equation*

$$Mu(x) = f(x), \qquad x \in \mathbb{R}^2$$

is globally solvable and globally hypoelliptic in $\mathcal{S}(\mathbb{R}^2)$, $S_\mu^\mu(\mathbb{R}^2)$ (respectively, $\Sigma_\mu^\mu(\mathbb{R}^2)$) with $\mu \ge 1/2$ (respectively, $\mu > 1/2$) provided $|\delta| < \delta_0$.

Proof. We apply first Remark 2.2, so that we are reduced to the study the global hypoellipticity and the global solvability for $J \circ M \circ J^{-1}$, where J is fixe now with $|\tau| = 1$ in (2.2). We then have from (2.10)

$$J^{-1} \circ W \circ J = D_{y_1}^2 + y_1^2, \tag{4.4}$$

$$J^{-1} \circ \widetilde{W} \circ J = D_{y_2}^2 + y_2^2. \tag{4.5}$$

On the other hand,

$$\begin{aligned} J^{-1} \circ M \circ J &= (J^{-1} \circ W \circ J)^{m_1} \circ (J^{-1} \circ \widetilde{W} \circ J)^{m_2} \\ &\quad + \delta \sum_{\alpha+\beta\le m_0} c_{\alpha\beta} (J^{-1} \circ W \circ J)^\alpha (J^{-1} \circ \widetilde{W} \circ J)^\beta. \end{aligned} \tag{4.6}$$

Plugging (4.5) and (4.5) into (4.6) we obtain

$$J^{-1} \circ M \circ J = P + \delta Q, \tag{4.7}$$

where, returning to the write $x = (x_1, x_2)$ for the variables, P is given by (3.1) and $Q = Q(x, D)$ by (3.3) for suitable constants $q_{\alpha\beta}$. From Theorem 2.1 we get the conclusion. □

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Todor Gramchev
Dipartimento di Matematica e Informatica
Università di Cagliari
Via Ospedale 72
09124 Cagliari
Italy
e-mail: `todor@unica.it`

Stevan Pilipović
Institute of Mathematics
University of Novi Sad
Trg D. Obradovića 4
21000 Novi Sad
Serbia
e-mail: `stevan.pilipovic@im.ns.acad.yu`

Luigi Rodino
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10
10123 Torino
Italy
e-mail: luigi.rodino@unito.it

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Weyl's Lemma and Converse Mean Value for Dunkl Operators

M. Maslouhi and R. Daher

Abstract. We give a version of Weyl's lemma for the Dunkl Laplacian and apply this result to characterize Dunkl harmonic functions in a class of tempered distribution by invariance under Dunkl convolution with suitable kernels.

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1. Introduction

Let $\langle , \rangle$ be the euclidean scalar product on $\mathbb{R}^d$ and $\| \, \|$ the associated norm. The form $\langle , \rangle$ is extended in a natural way to a bilinear form in $\mathbb{C}^d$ again denoted by $\langle , \rangle$.

We recall that for $\alpha \in \mathbb{R}^d$ the reflection σ_α with respect to the hyperplane H_α, orthogonal to α is given for $x \in \mathbb{R}^d$ by

$$\sigma_\alpha(x) = x - \frac{2\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha.$$

We fix a root system R in $\mathbb{R}^d \setminus \{0\}$. This is a finite subset of $\mathbb{R}^d$ that satisfies $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. Let G be the reflection subgroup of the orthogonal group $O(n)$ generated by the reflections σ_α, $\alpha \in R$. The action of G on functions $f : \mathbb{R}^d \to \mathbb{C}$ is given for $g \in G$ by $R(g)f(x) = f(gx), x \in \mathbb{R}^d$. We fix a positive root system $R_+ = \{\alpha \in R : \langle \alpha, \beta\rangle > 0\}$ for some $\beta \in R$. A G-invariant function $k : R \to \mathbb{C}$ is called a multiplicity function. If $\xi \in \mathbb{R}^d$, the Dunkl operator T_ξ associated to the group G and multiplicity function k is given for $f \in \mathcal{C}^1(\mathbb{R}^d)$ by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\langle \alpha, \xi\rangle \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \tag{1.1}$$

where ∂_ξ is the directional derivative in the direction ξ.

The properties of these operators can be found in [2].

The Dunkl Laplacian Δ_k on $\mathbb{R}^d$ is given by $\Delta_k := \sum_{j=1}^d T_{e_j}^2$, where $(e_1, \ldots, e_d)$ is the standard canonical basis of $\mathbb{R}^d$.

A twice differentiable function f on a G-invariant open set Ω of $\mathbb{R}^d$ is said to be Dunkl harmonic (D-harmonic) if it is annihilated by Δ_k.

In this paper we assume that $k \geq 0$. We consider the weight function

$$\omega_k(x) = \prod_{\alpha \in R+} |\langle \alpha, x\rangle|^{2k(\alpha)};$$

ω_k is G-invariant and homogeneous of degree 2γ where

$$\gamma := \sum_{\alpha \in R_+} k(\alpha).$$

We let η be the normalized surface measure on the unit sphere $\mathbb{S}$ in $\mathbb{R}^d$ and set

$$d\eta_k(y) := \omega_k(x) d\eta(y).$$

Then η_k is a G-invariant measure on $\mathbb{S}$, we let $d_k := \eta_k(\mathbb{S})$. We will also set

$$d\omega_k(z) := \omega_k(z) dz$$

where dz is the Lebesgue measure in $\mathbb{R}^d$.

Throughout the paper $r\mathbb{B}$ is the euclidean ball in $\mathbb{R}^d$ centered at 0 with radius $r > 0$, and we write $\mathbb{B}$ when $r = 1$.

Finally, we use the classic notations $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$ to denote the space of smooth compactly supported functions and the space of Schwartz functions respetively.

The Dunkl intertwining operator V_k is defined in [2] on polynomials f by

$$T_\xi V_k f = V_k \partial_\xi f \quad \text{and} \quad V_k 1 = 1.$$

It was shown [11] to extend as topological isomorphism to the space $\mathcal{C}^\infty(\mathbb{R}^d)$ of smooth functions on $\mathbb{R}^d$ onto itself. Moreover, there is a unique family (μ_x) of probability measures [7], with support in the convex hull of the set $\{gx : g \in G\}$, satisfying

$$V_k(f)(x) = \int_{\mathbb{R}^d} f(z) d\mu_x(z), \quad x \in \mathbb{R}^d \tag{1.2}$$

We set

$$E_k(x, y) := V_k\left(e^{\langle \cdot, y\rangle}\right)(x) = \int_{\mathbb{R}^d} e^{\langle z, y\rangle} d\mu_x(z), \quad x \in \mathbb{R}^d,\ y \in \mathbb{C}^d.$$

For all $y \in \mathbb{R}^d$, the function $f : x \mapsto E_k(x, y)$ is the unique solution of the system

$$f(0) = 1, \quad T_\xi f(x) = \langle \xi, y\rangle f(x)$$

for all $\xi \in \mathbb{R}^d$. E_k is called the generalized exponential or the Dunkl kernel.

A thorough study of this kernel can be found in [1], we give here some of its basic properties needed for the sequel.

Proposition 1.1. *Let $g \in G$, $z, w \in \mathbb{C}^d$ and $\lambda \in \mathbb{C}$. Then*

1. $E_k(z, 0) = 1$.
2. $E_k(z, \omega) = E_k(\omega, z)$.
3. $E_k(g\,z, g\,\omega) = E_k(z, \omega)$ *and* $E_k(\lambda\,z, \omega) = E_k(z, \lambda\,\omega)$.
4. *For all $\nu \in \mathbb{N}^d$, $x \in \mathbb{R}^d$, $z \in \mathbb{C}^d$, we have*
$$|D_z^\nu E_k(x, z)| \leq \|x\|^{|\nu|}\, exp(\|x\|\,\|Re(z)\|)$$
where
$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \ldots \partial z_n^{\nu_n}}, \quad |\nu| = \nu_1 + \cdots + \nu_n.$$
In particular
$$|E_k(i\,x, y)| \leq 1$$
for all $x, y \in \mathbb{R}^d$.

The Dunkl transform is defined by

$$\mathcal{F}_k(f)(\xi) = c_k^{-1} \int_{\mathbb{R}^d} f(x) E_k(-i\xi, x) d\omega_k(x), \quad f \in L^1(\mathbb{R}^d, d\omega_k) \tag{1.3}$$

where the constant c_k is given by

$$c_k = \int_{\mathbb{R}^d} e^{-\|z\|^2/2} d\omega_k(z).$$

The Dunkl transform shares several properties with its conterpart in the classical case (that is k=0), in particular it has been shown [1] that $\mathcal{F}_k$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ into itself. The inverse transform is given by

$$\mathcal{F}_k^{-1}(f)(x) = c_k^{-1} \int_{\mathbb{R}^d} f(\xi) E_k(i\xi, x) d\omega_k(\xi) \tag{1.4}$$

and

$$\mathcal{F}_k\left(T_{e_j} f\right)(\xi) = i\xi_j \mathcal{F}_k(f)(\xi), \quad j = 1, \ldots, d. \tag{1.5}$$

A usefull value of this transform is obtained for the function

$$\psi_t(x) = e^{-t\|x\|^2}, \quad t > 0,\ x \in \mathbb{R}^d,$$

namely, we have

$$\mathcal{F}_k(\psi_t)(\xi) = \frac{1}{(2t)^{d/2+\gamma}} e^{-\|\xi\|^2/4t}, \quad \xi \in \mathbb{R}^d. \tag{1.6}$$

We let V_k^{-1} denote the inverse of V_k. The Dunkl translation operator τ_x is defined by

$$\tau_x(f)(y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (V_k^{-1} f)(t + u) d\mu_x(t) d\mu_y(u), \quad y \in \mathbb{R}^d.$$

where the measure μ_x is defined by (1.2). Note that if $k = 0$, then τ_x reduces to the classical translation operator. The following properties are proven in [9].

Proposition 1.2. *For all $x, y \in \mathbb{R}^d$ and $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ we have:*

1. $\tau_x f(y) = \tau_y f(x)$.

2. $\tau_x f(0) = f(x)$.
3. *If* $f \in \mathcal{D}(\mathbb{R}^d)$ *with* $\operatorname{supp}(f) \subset r\mathbb{B}$, *then* $\tau_x f \in \mathcal{D}(\mathbb{R}^d)$ *and*
$$\operatorname{supp}(\tau_x f) \subset (r + \|x\|)\mathbb{B}.$$
4. *For all* $j = 1, \dots, d$,
$$T_{e_j}^y(\tau_x f)(y) = \tau_x(T_j f)(y) = T_{e_j}^x(\tau_y f)(x).$$
5. *If* $f \in \mathcal{S}(\mathbb{R}^d)$, *then* $\tau_x f \in \mathcal{S}(\mathbb{R}^d)$. *Moreover we have*
$$\tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^d} \mathcal{F}_k(f)(z) E_k(iy, z) E_k(ix, z) \omega_k(z) dz.$$
6. *If* $f, g \in \mathcal{S}(\mathbb{R}^d)$, *then*
$$\int_{\mathbb{R}^d} (\tau_x f)(z) g(z) \omega_k(z) dz = \int_{\mathbb{R}^d} f(z)(\tau_{-x} g)(z) \omega_k(z) dz.$$

Note that in particular if $f \in \mathcal{S}(\mathbb{R}^d)$ is even, we have
$$\tau_{-x} f(y) = \tau_x f(-y) = \tau_{-y} f(x).$$

The generalized convolution for Dunkl operators is defined by means of the Dunkl translation and given by
$$f *_k g(x) = \int_{\mathbb{R}^d} f(y) \tau_x(g)(-y) \omega_k(y) dy. \tag{1.7}$$
As in the classical case, we have
$$\mathcal{F}_k(f *_k g) = c_k \mathcal{F}_k(f) \mathcal{F}_k(g) \tag{1.8}$$
whenever $f, g \in \mathcal{S}(\mathbb{R}^d)$.

Appealing [4] (see also [5]), D-harmonic functions u in $\mathbb{R}^d$ are characterized by the mean value property
$$u(x) = \frac{1}{d_k} \int_S (\tau_x u)(\rho y) d\eta_k(y), \quad x \in \mathbb{R}^d,\ \rho \geq 0.$$

Let $\sigma \in L^1(\mathbb{R}^d, d\omega_k)$ be radial with $\sigma(x) = F(\|x\|)$, $x \in \mathbb{R}^d$, and consider a D-harmonic function u in $\mathbb{R}^d$ such that $\sigma *_k u$ is defined. Then
$$\begin{aligned}
\sigma *_k u(x) &= \int_{\mathbb{R}^d} \sigma(z) \tau_x(u)(z) d\omega_k(z) \\
&= \int_0^{+\infty} r^{d+2\gamma-1} F(r) \left(\int_{\mathbb{S}} \tau_x(u)(ry) d\eta_k(y) \right) dr \\
&= d_k u(x) \int_0^{+\infty} r^{d+2\gamma-1} F(r) dr \\
&= u(x) \int_{\mathbb{R}^d} \sigma(z) d\omega_k(z).
\end{aligned}$$

Thus if we choose σ such that $\int_{\mathbb{R}^d} \sigma(z)d\omega_k(z) = 1$, that is $c_k\mathcal{F}_k(\sigma)(0) = 1$, then u solves the equation

$$\sigma *_k u = u. \tag{1.9}$$

The aim of this paper is to characterize D-harmonic functions as solutions of (1.9) in suitable spaces and for suitable kernels σ.

We point out that a version of (1.9) in the classical case has been studied in different contexts and by different approaches, see [3] and the references therein.

2. Weyl's Lemma for the Dunkl Laplacian

In the classical case, Weyl's lemma asserts that if u is a function satisfying

$$\Delta u = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d),$$

then $u \in \mathcal{C}^\infty(\mathbb{R}^d)$ and $\Delta u = 0$ in $\mathbb{R}^d$. In this section we state a version of Weyl's lemma for the Dunkl Laplacian which is a fundamental ingredient for the next section.

Following [10], if $u \in \mathcal{D}'(\mathbb{R}^d)$ we define for $j = 1, \ldots, d$,

$$\langle T_j u, \varphi\rangle = -\langle u, T_j\varphi\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}^d)$$

where $T_j = T_{e_j}$ is defined by (1.1).

Theorem 2.1. *Let $u : \mathbb{R}^d \to \mathbb{C}$ be locally bounded such that*

$$\Delta_k(\omega_k u) = 0 \quad \textit{in} \quad \mathcal{D}'(\mathbb{R}^d).$$

Then there exists v D-harmonic in $\mathbb{R}^d$ such that $u = v$ a.e. in $\mathbb{R}^d$.

Proof. Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ radial non-negative such that

$$\text{supp}(\varphi) \subset \overline{\mathbb{B}} \quad \text{and} \quad \int_{\mathbb{R}^d} \varphi(z)d\omega_k(z) = 1.$$

For $\varepsilon > 0$ set

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^{2\gamma+d}}\varphi\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d$$

and

$$u_\varepsilon(x) = \int_{\mathbb{R}^d} u(z)\tau_{-x}(\varphi_\varepsilon)(z)d\omega_k(z). \tag{2.1}$$

From [9] we have

$$\text{supp}\tau_{-x}\varphi_\varepsilon \subset (1 + \|x\|)\mathbb{B} \quad \text{for} \quad 0 < \varepsilon < 1$$

which, together with Proposition 1.2, easily lead to $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^d)$ for all $\varepsilon > 0$ and

$$\begin{aligned}(\Delta_k^x u_\varepsilon)(x) &= \int_{\mathbb{R}^d} u(z)\Delta_k^x(\tau_{-z}\varphi_\varepsilon)(x)d\omega_k(z)\\ &= \int_{\mathbb{R}^d} u(z)\tau_{-x}(\Delta_k\varphi_\varepsilon)(z)d\omega_k(z)\\ &= \int_{\mathbb{R}^d} u(z)\Delta_k^z(\tau_{-x}\varphi_\varepsilon)(z)d\omega_k(z)\\ &= \langle \Delta_k(\omega_k u), \tau_{-x}\varphi_\varepsilon\rangle\\ &= 0\end{aligned}$$

for $\tau_x\varphi_\varepsilon \in \mathcal{D}(\mathbb{R}^d)$ and $\Delta_k(\omega_k u) = 0$ in $\mathcal{D}'(\mathbb{R}^d)$. Alternatively, for $f \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathrm{supp}(f) \subset r\mathbb{B}$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} u_\varepsilon(x)f(x)d\omega_k(x) &= \int_{\mathbb{R}^d} f(x)\left(\int_{\mathbb{R}^d} u(z)\tau_{-x}(\varphi_\varepsilon)(z)d\omega_k(z)\right)d\omega_k(x)\\ &= \int_{r\mathbb{B}} f(x)\left(\int_{(1+r)\mathbb{B}} u(z)\tau_{-x}(\varphi_\varepsilon)(z)d\omega_k(z)\right)d\omega_k(x)\\ &= \int_{(1+r)\mathbb{B}} u(z)\left(\int_{r\mathbb{B}} f(x)\tau_{-z}(\varphi_\varepsilon)(x)d\omega_k(x)\right)d\omega_k(z).\end{aligned}$$

Set

$$h_\varepsilon(z) = \int_{B_r} f(x)\tau_{-z}(\varphi_\varepsilon)(x)d\omega_k(x).$$

From [6, Theorem 5.1] we have $\tau_{-z}(\varphi_\varepsilon) \geq 0$ for all $z \in \mathbb{R}^d$ and $\varepsilon > 0$, hence by [8, Theorem 3.4] we get

$$\begin{aligned}|h_\varepsilon(z)| &\leq \|f\|_\infty \int_{\mathbb{R}^d} \tau_{-z}(\varphi_\varepsilon)(x)d\omega_k(x)\\ &= \|f\|_\infty \int_{\mathbb{R}^d} \varphi_\varepsilon(x)d\omega_k(x)\\ &= \|f\|_\infty\end{aligned}$$

for all $\varepsilon > 0$. On the other hand, we have

$$\begin{aligned}\int_{r\mathbb{B}} f(x)\tau_{-z}(\varphi_\varepsilon)(x)d\omega_k(x) &= \int_{\mathbb{R}^d} f(x)\tau_{-z}(\varphi_\varepsilon)(x)d\omega_k(x)\\ &= \int_{\mathbb{R}^d} \tau_z(f)(x)\varphi_\varepsilon(x)d\omega_k(x)\\ &= \int_{\mathbb{R}^d} \tau_z(f)(\varepsilon x)\varphi(x)d\omega_k(x).\end{aligned}$$

Since

$$|\tau_z(f)(\varepsilon x)| \leq \|\mathcal{F}_k(f)\|_{L^1(\mathbb{R}^d, d\omega_k)}$$

for all $\varepsilon > 0$ and all $x \in \mathbb{R}^d$, the dominated convergence theorem leads now to

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} u_\varepsilon(x) f(x) d\omega_k(x) = \int_{(1+r)\mathbb{B}} u(z) f(z) d\omega_k(z) = \int_{\mathbb{R}^d} u(z) f(z) d\omega_k(z)$$

which shows that $\omega_k u_\varepsilon$ converges to $\omega_k u$ in $\mathcal{D}'(\mathbb{R}^d)$ as ε tends to 0.

Now we will show that the sequence (u_ε) is uniformly bounded in each compact subset of $\mathbb{R}^d$ which will terminates the proof by use of [4, Theorem B].

Fix $r > 0$ and let $C(r) > 0$ such that

$$|u(z)| \leq C(r) \quad \text{for all} \quad z \in (1+r)\mathbb{B}.$$

Arguing as before, we have for all $x \in r\mathbb{B}$ and all $\varepsilon > 0$

$$\begin{aligned} |u_\varepsilon(x)| &\leq \int_{(1+r)\mathbb{B}} |u(z)| \, \tau_{-x}(\varphi_\varepsilon)(z) d\omega_k(z) \\ &\leq C(r) \int_{(1+r)\mathbb{B}} \tau_{-x}(\varphi_\varepsilon)(z) d\omega_k(x) \\ &= C(r) \int_{\mathbb{R}^d} \tau_{-z}(\varphi_\varepsilon)(x) d\omega_k(x) \\ &= C(r) \int_{\mathbb{R}^d} \varphi_\varepsilon(x) d\omega_k(x) \\ &= C(r). \end{aligned}$$

This shows that the sequence (u_ε) is uniformly bounded in each compact subset of $\mathbb{R}^d$. Hence from [4, Theorem B], and up to a subsequence, the sequence u_ε converges uniformly in each compact subset of $\mathbb{R}^d$ to a function v D-harmonic in $\mathbb{R}^d$, we conclude then that

$$u = v \quad \text{in} \quad \mathbb{R}^d \setminus \left(\cup_{\alpha \in R_+} H_\alpha\right)$$

where H_α is the hyperplane orthogonal to α, whence our proclaim. □

3. Application: Converse Mean Value

We have seen in Section 1 that the mean value property of D-harmonic functions leads to an invariance of Dunkl convolution with suitable kernels. In this section we will use the results of Section 2 to establish a converse of this fact for a class of tempered distribution.

To simplify matters, let us denote by $\mathcal{S}$ the space of Schwartz functions on $\mathbb{R}^d$, that is, $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$.

Lemma 3.1. *Let $\sigma \in L^1(\mathbb{R}^d, d\omega_k)$ and define the operator $K_\sigma : f \mapsto \sigma *_k f$. Then we have*

$$(I - K_\sigma)\mathcal{S} = \Delta_k \mathcal{S} \tag{3.1}$$

if and only if

$$1 - c_k \mathcal{F}_k(\sigma)(\xi) = \|\xi\|^2 \theta(\xi), \quad \xi \in \mathbb{R}^d \tag{3.2}$$

where θ is a function satisfying

$$\theta\mathcal{S} \subset \mathcal{S} \quad \textit{and} \quad \frac{1}{\theta}\mathcal{S} \subset \mathcal{S}. \tag{3.3}$$

Proof. Let θ be given by (3.2) and (3.3) and consider $v \in \mathcal{S}$. Define φ by

$$\varphi := \theta\mathcal{F}_k(v),$$

then $\varphi \in \mathcal{S}$ and

$$\mathcal{F}_k((I - K_\sigma)v) = \|\xi\|^2 \varphi = \mathcal{F}_k(\Delta_k\psi)$$

where $\psi = -\mathcal{F}_k^{-1}(\varphi)$. Hence $(I - K_\sigma)\mathcal{S} \subset \Delta_k\mathcal{S}$.

Now consider $\varphi \in \mathcal{S}$, we are looking for $v \in \mathcal{S}$ such that $(I - K_\sigma)v = \Delta_k\varphi$. Since $\mathcal{F}_k$ is a homeomorphism from $\mathcal{S}$ into itself, the latter equation equivalents to find $v \in \mathcal{S}$ such that

$$(1 - c_k\mathcal{F}_k(\sigma)) \times v = \|\xi\|^2 \mathcal{F}_k(\varphi);$$

using (3.2) this reduces to

$$v = \frac{1}{\theta}\mathcal{F}_k(\varphi).$$

Thus the whole task is to show that

$$\frac{1}{\theta}\mathcal{F}_k(\varphi) \in \mathcal{S}$$

which is guaranteed by (3.3).

For the converse, set

$$\theta(\xi) = \frac{1 - c_k\mathcal{F}_k(\sigma)(\xi)}{\|\xi\|^2}, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

It suffices to show that θ and $\frac{1}{\theta}$ extend to a smooth functions in $\mathbb{R}^d$, which is possible view of (3.1) and since there exists $\varphi \in \mathcal{S}$ such that $(\mathcal{F}_k(\varphi))_{|r\mathbb{B}} \equiv 1$ for a given $r > 0$. □

We point out that the sufficient part of Lemma 3.1 is established for a special value of σ in [3] with the setting $k = 0$.

Proposition 3.3 will give concrete example of such a kernel σ.

For $\sigma \in \mathcal{S}(\mathbb{R}^d)$ and a tempered distribution u the tempered distribution $\sigma *_k u$ is defined by

$$\langle \sigma *_k u, \psi \rangle := \langle u, \sigma *_k \psi \rangle, \quad \psi \in \mathcal{S}(\mathbb{R}^d).$$

Note that if $u \in L_k^1(\mathbb{R}^d)$, then $\sigma *_k u$ is a function.

Theorem 3.2. *Fix $\sigma \in \mathcal{S}(\mathbb{R}^d)$ satisfying (3.2) and (3.3) and let $u : \mathbb{R}^d \to \mathbb{C}$ be locally bounded such that*

$$\sigma *_k (\omega_k u) = \omega_k u \quad \textit{in} \quad \mathcal{S}'(\mathbb{R}^d).$$

Then there exists v D-harmonic in $\mathbb{R}^d$ such that $u = v$ a.e in $\mathbb{R}^d$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$. From Lemma 3.1 there exists $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\begin{aligned}\langle \Delta_k(\omega_k u), \varphi\rangle &= \langle \omega_k u, \Delta_k \varphi\rangle \\ &= \langle \omega_k u, \psi\rangle - \langle \omega_k u, \sigma *_k \psi\rangle \\ &= \langle \omega_k u, \psi\rangle - \langle \sigma *_k (\omega_k u), \psi\rangle \\ &= 0;\end{aligned}$$

hence $\Delta_k(\omega_k u) = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ and Theorem 2.1 completes the proof. □

Proposition 3.3. *Let $u : \mathbb{R}^d \to \mathbb{C}$ be locally bounded such that*

$$e^{-\frac{\|x\|^2}{2}} *_k (\omega_k u) = \omega_k u \quad \textit{in} \quad \mathcal{S}'(\mathbb{R}^d).$$

Then there exists v D-harmonic in $\mathbb{R}^d$ such that $u = v$ a.e in $\mathbb{R}^d$.

Proof. Let f be the function defined by

$$f(x) = e^{\frac{-\|x\|^2}{2}}, \quad x \in \mathbb{R}^d. \tag{3.4}$$

From (1.6) we have

$$\mathcal{F}_k(f)(\xi) = e^{-\frac{\|\xi\|^2}{2}}, \quad \xi \in \mathbb{R}^d.$$

By Theorem 3.2, all what we have to do is to show that f satisfies (3.2) and (3.3). For this purpose, set

$$\theta(\xi) = \frac{1 - e^{-\frac{\|\xi\|^2}{2}}}{\|\xi\|^2}, \quad \xi \in \mathbb{R}^d.$$

It is clear that θ is well defined and of class $\mathcal{C}^\infty$ on $\mathbb{R}^d$. Moreover, we have

$$\theta(\xi) = \frac{1}{2}\int_0^1 e^{-t\frac{\|\xi\|^2}{2}}\,dt \tag{3.5}$$

for all $\xi \in \mathbb{R}^d$, hence $D^\nu \theta$ is of polynomial growth for all $\nu \in \mathbb{N}_0^d$, where

$$D^\nu = \frac{\partial^{\nu_1}}{\partial \xi_1^{\nu_1}} \cdots \frac{\partial^{\nu_d}}{\partial \xi_d^{\nu_d}}, \quad \nu = (\nu_1, \ldots, \nu_d)$$

whence $\theta\mathcal{S} \subset \mathcal{S}$. On the other hand, we have $\theta(\xi) > 0$ for all $\xi \in \mathbb{R}^d$ and it is not hard to see that $D^\nu\left(\frac{1}{\theta}\right)$ is a linear combinaison of expressions of the form

$$\frac{D^{\mu_1}\theta(\xi) \times \cdots \times D^{\mu_m}\theta(\xi)}{\theta^{m+1}(\xi)}, \quad m = 0, 1, 2, \ldots,$$

and $\mu_1, \mu_2, \ldots$ are some multi-indices in $\mathbb{N}^d$. Thus since the derivatives of θ are of polynomial growth, to show that $\frac{1}{\theta}\mathcal{S} \subset \mathcal{S}$ it suffices to show that $\frac{1}{\theta}$ is of polynomial growth.

By continuity argument, there exists $C_1 > 0$ such that

$$\frac{1}{|\theta(\xi)|} = \frac{\|\xi\|^2}{1 - e^{-\frac{\|\xi\|^2}{2}}} \le C_1 \quad \text{for all} \quad \xi \in \mathbb{B}.$$

Next, since the map $t \mapsto \frac{1}{1-e^{-t}}$ is non-increasing in $(0,+\infty)$, we see that we may write

$$\frac{1}{|\theta(\xi)|} \leq C_2 \|\xi\|^2 \quad \text{for} \quad \|\xi\| \geq 1$$

for some positive constant C_2 and we are done. □

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M. Maslouhi
CPGE Rabat 10000
Morocco
e-mail: mostafa.maslouhi@gmail.com

R. Daher
Université Hassan II
Ain Chock, Casablanca 20153
Morocco
e-mail: ra_daher@yahoo.fr

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Dirichlet Problems for Inhomogeneous Complex Mixed-Partial Differential Equations of Higher Order in the Unit Disc: New View

H. Begehr, Zhihua Du and Ning Wang

Abstract. In this paper, we discuss some Dirichlet problems for inhomogeneous complex mixed-partial differential equations of higher order in the unit disc. Using higher-order Pompeiu operators $T_{m,n}$, we give some special solutions for the inhomogeneous equations. The solutions of homogeneous equations are given on the basis of decompositions of polyanalytic and polyharmonic functions. Combining the solutions of the homogeneous equations and special solutions, we obtain all solutions of the inhomogeneous equations.

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1. Introduction

Recently, a large number of investigations on various boundary value problems (simply, BVPs) for polyanalytic functions, metaanalytic functions have widely been published, refer to papers [10, 15, 16, 18, 29, 30] and references there. However, the investigations on Dirichlet problems for polyharmonic functions (simply, PHD problems) in the case of the unit disc just appeared in recent two years [7, 8, 11]. All of these works are based on two kinds of methods: one is called iterating method by making use of so-called poly-Cauchy operator [10, 15], the other is

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called reflection method in terms of Schwarz symmetric extension principle and decompositions for polyanalytic and polyharmonic functions due to Begehr, Du and Wang [8, 15]. In [15], Du and Wang established a beautiful decomposition theorem for polyanalytic functions such that BVPs for polyanalytic functions can be easily transformed to BVPs for analytic functions while the theory of the latter is completely developed [20, 23, 24]. Further, in [8], Begehr, Du and Wang also obtained a decomposition theorem for polyharmonic functions by the decomposition theorem for polyanalytic functions. In fact, these decomposition theorems have appeared in the book [2] of Balk in some implicit forms. Just using the decomposition theorem, in [8], Begehr, Du and Wang studied the Dirichlet problem for polyharmonic functions in the unit disc by reflection method. They found that the problem is uniquely solvable and the solution is closely connected with a sequence of kernel functions with some elegant properties. However, explicit expressions for all kernel functions are not yet attained although the kernel functions exist and satisfy certain inductive relations.

In [17], Du, Guo and Wang established a new decomposition theorem for the polyharmonic functions in a simply connected (bounded or unbounded) domain of the complex plane which is a natural extension of Goursat decomposition theorem for biharmonic functions [21]. Using the decomposition theorem in the case of the unit disc, they gave a unified expression for the kernel functions appearing in [8] which are expressed in terms of some vertical sums with nice structure. Then the PHD problems in the unit disc are completely solved. Furthermore, they have considered some Dirichlet problems for homogeneous complex mixed-partial differential equations of higher order and obtained the complete solutions in terms of the decompositions of polyanalytic and polyharmonic functions.

For inhomogeneous equations, the corresponding problems have been little investigated [12, 22], and it is the purpose of the present paper to obtain some results in this direction. In [12], since the explicit expressions of kernel functions are unknown, in fact, Begehr and Wang only solved a Dirichlet problem for inhomogeneous triharmonic equations in the unit disc although the general solution for the polyharmonic case is indicated by a final remark. In [22], Kumar and Prakash consider the same equations appearing in the present paper with different boundary conditions using another method. In the present paper, we first apply the differentiability of higher-order Pompeiu operators introduced by Begehr and Hile [10] to get special solutions for the inhomogeneous equations. Further, we use the known results of homogeneous equations [8, 17] due to the explicit expressions of kernel functions [17] which are obtained by the decompositions of polyanalytic and polyharmonic functions, and the continuity of the higher-order Pompeiu operators. Combining the special solutions and homogeneous solutions, we obtain the solutions of Dirichlet problems for the inhomogeneous equations under some suitable conditions of solvability. It is a new view to solve Dirichlet problems for inhomogeneous equations which is different from the usual method depending on the higher-order Green functions [6] whose explicit expressions are unknown except for some lower orders up to now. It is more interesting that the view appearing here to solve a Dirichlet problem for inhomogeneous higher-order complex partial differential equations is similar to the one usually used in linear algebra to solve an inhomogeneous system of linear equations. In many aspects, the present paper is related to [17].

As in [17], in what follows, we always use polyharmonic operators $(\partial_z\partial_{\overline{z}})^n$ $(n \geq 1)$ to define polyharmonic functions, where $\partial_{\overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ is the Cauchy-Riemann operator and $\partial_z = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ is its adjoint operator. In particular, $\partial_z\partial_{\overline{z}}$ is the harmonic operator and $(\partial_z\partial_{\overline{z}})^2$ is the biharmonic operator. In addition, the functions are complex except for some special statements about real functions.

2. Decompositions of Functions

As in [17], in what follows, we always suppose that Ω is a simply connected (bounded or unbounded) domain in the complex plane with smooth boundary $\partial\Omega$. If a real valued function $f \in C^{2n}(\Omega)$ satisfies polyharmonic equation $(\partial_z\partial_{\overline{z}})^n f = 0$ in Ω, then f is called an n-harmonic function in Ω, concisely, a polyharmonic function. As usual, if $f \in C^n(\Omega)$ satisfies the polyanalytic equation $\partial_{\overline{z}}^n f = 0$ in Ω, then f is called an n-analytic function in Ω, concisely, a polyanalytic function [2]. The set of polyanalytic (polyharmonic) functions of order n in Ω is simply denoted by $H_n(\Omega)$ $(Har_n(\Omega))$. Especially, $H_1(\Omega)$ $(Har_1(\Omega))$ is the set of all analytic (harmonic) functions in Ω. Sometimes we need to consider $Har_n^{\mathbb{C}}(\Omega) = \{f + ig : f, g \in Har_n(\Omega)\}$ consisting of all complex polyharmonic functions of order n in Ω.

In addition, we introduce the function spaces $H^j_{1,z_0}(\Omega) = \{\,\varphi \in H_1(\Omega) : \varphi^{(k)}(z_0) = 0,\ z_0 \in \Omega,\ 0 \leq k < j\,\}$ and $\Pi^j_{1,z_0}(\Omega) = \{\, ic(z - z_0)^j : c \in \mathbb{R},\ z, z_0 \in \Omega\,\}$, where $\mathbb{R}$ denotes the set of all real numbers and $j = 0, 1, 2, \ldots$. Obviously, for $j > 1$, $H^j_{1,z_0}(\Omega)$ is the set of all analytic functions which have a zero of order at least j at $z_0 \in \Omega$ whereas $H^0_{1,z_0}(\Omega) = H_1(\Omega)$. Of course, $\Pi^j_{1,z_0}(\Omega) \subset H^j_{1,z_0}(\Omega) \subset H_1(\Omega)$. If $\varphi, \widetilde{\varphi} \in H^j_{1,z_0}(\Omega)$ and $\varphi - \widetilde{\varphi} \in \Pi^j_{1,z_0}(\Omega)$, then we say that φ and $\widetilde{\varphi}$ are equivalent and write that $\varphi \sim_j \widetilde{\varphi}$. Moreover, define $\sim = \cup_j \sim_j$, that is, $f \sim g$ if $f \sim_j g$ for some $j \in \mathbb{N}$. Especially, for example, $0 \sim_j ic(z - z_0)^j$ for any nonzero $c \in \mathbb{R}$.

With these preliminaries, the following decomposition fact for polyharmonic functions holds.

Theorem 2.1 (see [17]). *Let Ω be a simply connected (bounded or unbounded) domain in the complex plane with smooth boundary $\partial\Omega$. If $f \in Har_n(\Omega)$, then for any $z_0 \in \Omega$, there exist functions $f_j \in H^j_{1,z_0}(\Omega)$, $j = 0, 1, \ldots, n-1$ such that*

$$f(z) = 2\Re\Big\{\sum_{j=0}^{n-1}(\overline{z} - \overline{z}_0)^j f_j(z)\Big\}, \quad z \in \Omega, \tag{2.1}$$

where $\Re$ denotes the real part. The above decomposition expression of f is unique in the sense of the equivalence relation $\sim$, more precisely, $\sim_j$ for f_j. That is, if (2.1) *also holds for $\hat{f}_j \in H^j_{1,z_0}(\Omega)$, $j = 0, 1, \ldots, n-1$, then $\hat{f}_j \sim_j f_j$, $j = 0, 1, \ldots, n-1$.*

Remark 2.1. As $n = 2$, Theorem 2.1 is the well-known result for real biharmonic funcions due to Goursat in [21] (also see [28]). So (2.1) is called Goursat decomposition form and the function f_j in (2.1) is called the analytic jth decomposition component of f (see [17]).

By Theorem 2.1, we also have

Corollary 2.1 (see [17]). *Let the sequence of functions $\{f_n\}$ defined in Ω satisfy*

1. f_1 *is a harmonic function in* Ω, *i.e.*, $f_1 \in Har_1(\Omega)$;
2. $(\partial_z \partial_{\overline{z}}) f_n = f_{n-1}$ *in* Ω *for* $n > 1$.

Then $f_n \in Har_n(\Omega)$ *for* $n > 1$, *and*

$$\partial_z f_{n,j} = j^{-1} f_{n-1,j-1}, \quad 1 \leq j \leq n-1, \tag{2.2}$$

where $f_{n,j}$ is the analytic jth decomposition component of the n-harmonic function f_n. (2.2) holds in the sense of the equivalence relation $\sim$. More precisely, $\sim_j$ for $f_{n,j}$ and $\sim_{j-1}$ for $f_{n-1,j-1}$, $j = 1, 2, \ldots, n-1$.

For polyanalytic functions, we have the following decomposition theorem.

Theorem 2.2 (see [15]). *Let $f \in H_n(\Omega)$, then for any $z_0 \in \Omega$,*

$$f(z) = \sum_{j=0}^{n-1} (\overline{z} - \overline{z}_0)^j f_j(z), \ z \in \Omega, \tag{2.3}$$

where $f_j \in H_1(\Omega)$, $j = 0, 1, \ldots, n-1$. The decomposition (2.3) is unique. f_j is called the analytic jth decomposition component of f.

As in [8], $\overline{H}_n(\Omega)$ denotes the set of all functions satisfying $\partial_z^n f = 0$. Since $\overline{\partial_z f} = \partial_{\overline{z}} \overline{f}$, similarly or directly following from Theorem 2.2, we also get

Theorem 2.3 (see [17]). *Let $f \in \overline{H}_n(\Omega)$, then for any $z_0 \in \Omega$,*

$$f(z) = \sum_{j=0}^{n-1} (z - z_0)^j \overline{f_j(z)}, \ z \in \Omega, \tag{2.4}$$

where $f_j \in H_1(\Omega)$, $j = 0, 1, \ldots, n-1$. The decomposition (2.4) is unique. f_j is called the analytic jth decomposition component of f.

Let Π_n denote the set of all complex polynomials of degree at most n, we define another equivalence relation $\backsim_n$ as follows:

$$\text{If } f - g \in \Pi_n \text{ for } f, g \in H_1(\Omega), \text{ then } f \backsim_n g.$$

In addition, we set $\backsim = \cup_n \backsim_n$, that is, $f \backsim g$ if $f \backsim_n g$ for some $n \in \mathbb{N}$.

Let $M_{m,n}(\Omega) = \{ f \in C^{m+n}(\Omega) : (\partial_z^m \partial_{\overline{z}}^n) f(z) = 0, \ z \in \Omega \}$, especially, $M_{0,n}(\Omega) = H_n(\Omega)$ and $M_{n,0}(\Omega) = \overline{H}_n(\Omega)$ as well as $M_{n,n}(\Omega) = Har_n^{\mathbb{C}}(\Omega)$.

By the above theorems, we have (see [17]):

Theorem 2.4 (Harmonic Decomposition). *If $f \in M_{m,n}(\Omega)$, where $m, n > 1$ and $m \neq n$, then for any $z_0 \in \Omega$,*

1. *as* $m > n$,

$$f(z) = 2\Re\Big\{\sum_{k=0}^{n-1}(\overline{z}-\overline{z}_0)^k\varphi_k(z)\Big\} + 2i\Re\Big\{\sum_{k=0}^{n-1}(\overline{z}-\overline{z}_0)^k\widehat{\varphi}_k(z)\Big\}$$
$$+ (z-z_0)^n\sum_{l=0}^{m-n-1}\frac{l!}{(n+l)!}(z-z_0)^l\overline{\widetilde{\varphi}_l(z)}, \ \ z\in\Omega, \tag{2.5}$$

where $\varphi_k, \widehat{\varphi}_k \in H^k_{1,z_0}(\Omega)$ *and* $\widetilde{\varphi}_l \in H_1(\Omega)$;

2. *as* $m < n$,

$$f(z) = 2\Re\Big\{\sum_{s=0}^{m-1}(\overline{z}-\overline{z}_0)^s\psi_s(z)\Big\} + 2i\Re\Big\{\sum_{s=0}^{m-1}(\overline{z}-\overline{z}_0)^s\widehat{\psi}_s(z)\Big\}$$
$$+ (\overline{z}-\overline{z}_0)^m\sum_{t=0}^{n-m-1}\frac{t!}{(m+t)!}(\overline{z}-\overline{z}_0)^t\widetilde{\psi}_t(z), \ \ z\in\Omega, \tag{2.6}$$

where $\psi_s, \widehat{\psi}_s \in H^s_{1,z_0}(\Omega)$ *and* $\widetilde{\psi}_t \in H_1(\Omega)$.

(2.5) *and* (2.6) *are unique in the sense of equivalence relations* $\sim$ *and* $\backsim$, *more precisely,* $\sim_k$ *for* $\varphi_k, \widehat{\varphi}_k$ *and* $\sim_s$ *for* $\psi_s, \widehat{\psi}_s$ *whereas* $\backsim_{n-1}$ *for all* $\widetilde{\varphi}_l$ *and* $\backsim_{m-1}$ *for all* $\widetilde{\psi}_t$.

Theorem 2.5 (Canonical Decomposition). *If* $f \in M_{m,n}(\Omega)$, *where* $m, n > 1$ *and* $m \neq n$, *then for any* $z_0 \in \Omega$,

$$f(z) = \sum_{p=0}^{n-1}(\overline{z}-\overline{z}_0)^p\mu_p(z) + \sum_{q=0}^{m-1}(z-z_0)^q\overline{\nu_q(z)} \tag{2.7}$$

where $\mu_p, \nu_q \in H_1(\Omega)$. (2.7) *is unique in the sense of equivalence relation* $\backsim$ *for* μ_p *and* ν_q. *More precisely,* $\backsim_{m-1}(\backsim_{n-1})$ *for* μ_p *while* $\nu_q(\mu_p)$ *is unique,* $p = 0, 1, \ldots, n-1$, $q = 0, 1, \ldots, m-1$.

Remark 2.2. As in [17], all above theorems can be simplified as follows:

$$Har_n(\Omega) = 2\Re\Big\{\sum_{j=0}^{n-1}\oplus(\overline{z}-\overline{z}_0)^j\,(H/\Pi)^j_{1,z_0}(\Omega)\Big\}, \tag{2.8}$$

where $(H/\Pi)^j_{1,z_0}(\Omega)$ denotes the set of all equivalence classes about $\sim_j$, $j = 0, 1, \ldots, n-1$ and $\sum_{j=0}^{n-1}\oplus a_j := a_0 \oplus a_1 \oplus \cdots \oplus a_{n-1}$ which denotes the direct sum of $a_0, a_1, \ldots, a_{n-1}$.

$$H_n(\Omega) = \sum_{j=0}^{n-1}\oplus(\overline{z}-\overline{z}_0)^j\,H_1(\Omega). \tag{2.9}$$

$$\overline{H}_n(\Omega) = \sum_{j=0}^{n-1}\oplus(z-z_0)^j\,\overline{H}_1(\Omega). \tag{2.10}$$

$$M_{m,n}(\Omega) = Har_n^{\mathbb{C}}(\Omega) \oplus (z - z_0)^n \overline{H}_{m-n}(\Omega) \ (m > n). \tag{2.11}$$

$$M_{m,n}(\Omega) = Har_m^{\mathbb{C}}(\Omega) \oplus (\overline{z} - \overline{z}_0)^m H_{n-m}(\Omega) \ (m < n). \tag{2.12}$$

$$M_{m,n}(\Omega) = H_n(\Omega) \oplus \overline{H}_m(\Omega). \tag{2.13}$$

All the decompositions (2.8)–(2.13) are understood in the sense of the equivalence relations $\sim$ and $\backsim$.

3. Higher-Order Pompeiu Operators

In [27], Vekua systematically studied the so-called Pompeiu operators. They are defined as

$$T_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{\zeta - z}\, d\xi d\eta, \tag{3.1}$$

$$\overline{T}_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{\overline{\zeta} - \overline{z}}\, d\xi d\eta, \tag{3.2}$$

and the so-called Π and $\overline{\Pi}$ operators defined as the Cauchy principle value integrals

$$\Pi_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{(\zeta - z)^2}\, d\xi d\eta, \tag{3.3}$$

$$\overline{\Pi}_{\mathcal{D}}w(z) = -\frac{1}{\pi}\int\int_{\mathcal{D}} \frac{w(\zeta)}{(\overline{\zeta} - \overline{z})^2}\, d\xi d\eta, \tag{3.4}$$

where $\mathcal{D}$ is a domain in the complex plane, w is a suitable complex valued function defined in $\mathcal{D}$.

Now it is well known that the operators T and Π play an important role to solve various linear or nonlinear boundary value problems for first and second-order complex partial differential equations. So it happens that the operators T and Π have elegant properties such as continuity, differentiability, even unitarity in L^2 when w is in some certain function spaces. For example, one of the famous properties of T is its differentiability in the Sobolev sense as follows:

$$\partial_{\overline{z}} T_{\mathcal{D}}w(z) = w(z), \ \partial_z T_{\mathcal{D}}w(z) = \Pi_{\mathcal{D}}w(z). \tag{3.5}$$

In [10], Begehr and Hile introduced kernel functions

$$K_{m,n}(z) = \begin{cases} \frac{(-m)!(-1)^m}{(n-1)!\pi} z^{m-1}\overline{z}^{n-1}, & m \le 0; \\ \frac{(-n)!(-1)^n}{(m-1)!\pi} z^{m-1}\overline{z}^{n-1}, & n \le 0; \\ \frac{1}{(m-1)!(n-1)!\pi} z^{m-1}\overline{z}^{n-1}\big[\log|z|^2 - \sum_{k=1}^{m-1}\frac{1}{k} - \sum_{l=1}^{n-1}\frac{1}{l}\big], & m,n \ge 1, \end{cases} \tag{3.6}$$

where m,n are integers with $m+n \ge 0$ but $(m,n) \ne (0,0)$.

Using the above kernel functions, they defined a hierarchy of integral operators, more precisely,

$$T_{m,n,\mathcal{D}}w(z) = \int\int_{\mathcal{D}} K_{m,n}(z-\zeta)w(\zeta)\, d\xi d\eta. \tag{3.7}$$

Obviously,

$$T_{0,1,\mathcal{D}} = T_{\mathcal{D}},\ T_{1,0,\mathcal{D}} = \overline{T}_{\mathcal{D}}, \tag{3.8}$$

and

$$T_{-1,1,\mathcal{D}} = \Pi_{\mathcal{D}},\ T_{1,-1,\mathcal{D}} = \overline{\Pi}_{\mathcal{D}}. \tag{3.9}$$

Operators $T_{m,n,\mathcal{D}}$ are seen as the higher-order analogues of the operator $T_{\mathcal{D}}$, by comparing their properties such as Lebesgue integrability, continuity and differentiability and so on. They are called higher-order Pompeiu operators. The following properties of $T_{m,n,\mathcal{D}}$ are needed in the sequel. They are partial results from [10].

Theorem 3.1 (see [10]). *Let $\mathcal{D}$ be a bounded domain, suppose $m+n \geq 1$ and $w \in L^p(\mathcal{D})$, $p > 2$, then $T_{m,n,\mathcal{D}}w(z)$ exists as a Lebesgue integral for all z in $\mathbb{C}$, $T_{m,n,\mathcal{D}}$ is continuous in $\mathbb{C}$. Especially, $T_{m,n,\mathcal{D}}$ is locally Hölder continuous in $\mathbb{C}$, more precisely, for $|z_1|, |z_2| \leq R$ with any $R > 0$,*

$$|T_{m,n,\mathcal{D}}w(z_1) - T_{m,n,\mathcal{D}}w(z_2)| \leq \begin{cases} M_1|z_1 - z_2|, & m+n \geq 2, \\ M_2|z_1 - z_2|^{(p-2)/p}, & m+n = 1, \end{cases} \tag{3.10}$$

where the constants M_1, M_2 depend only on $m, n, p, \mathcal{D}, R$. Moreover, in $\mathbb{C}$, there are the Sobolev derivatives

$$\partial_z T_{m,n,\mathcal{D}}w(z) = T_{m-1,n,\mathcal{D}}w(z),\ \ \partial_{\overline{z}} T_{m,n,\mathcal{D}}w(z) = T_{m,n-1,\mathcal{D}}w(z) \tag{3.11}$$

and

$$\partial_z T_{1,0,\mathcal{D}}w(z) = \partial_{\overline{z}} T_{0,1,\mathcal{D}}w(z) = w(z). \tag{3.12}$$

4. Higher-Order Poisson Kernels and Homogeneous Equations

In the present section, we discussed three kinds of Dirichlet boundary value problems for homogeneous complex mixed-partial differential equations of higher order in the unit disc. In the next section, we discuss some corresponding problems for inhomogeneous equations. All results are sketched here. Their detailed proofs can be found in [17].

We begin with the PHD problems in the unit disc. Let $\Omega = \mathbb{D}$ which is the unit disc in the complex plane, $\partial\mathbb{D}$ is its boundary, i.e., the unit circle in the complex plane.

PHD problem in the unit disc: find a function $w \in Har_n^{\mathbb{C}}(\mathbb{D})$ satisfying the Dirichlet-type boundary conditions

$$[(\partial_z \partial_{\overline{z}})^j w]^+(t) = \gamma_j(t),\ t \in \partial\mathbb{D},\ 0 \leq j < n, \tag{4.1}$$

where $\gamma_j \in C(\partial\mathbb{D})$ which denotes the set of all complex continuous functions on $\partial\mathbb{D}$ for $0 \leq j < n$.

The solution of the PHD problem (4.1) is connected with a sequence $\{\, g_n(z,\tau)\,\}_{n=1}^{\infty}$ of real-valued functions of two variables defined on $\mathbb{D}\times\partial\mathbb{D}$ which are called higher-order Poisson kernel functions since they are polyharmonic analogues of the classical Poisson kernel. They have the following properties:

1. $g_n(z,\tau)\in C^{2n}(\mathbb{D})$ as a function of z with fixed $\tau\in\partial\mathbb{D}$ and $g_n(z,\tau), \partial_z g_n(z,\tau)$, $\partial_{\overline{z}} g_n(z,\tau)\in C(\mathbb{D}\times\partial\mathbb{D})$, $n=1,2,\dots$;
2. $(\partial_z\partial_{\overline{z}})g_1(z,\tau)=0$ and $(\partial_z\partial_{\overline{z}})g_n(z,\tau)=g_{n-1}(z,\tau)$ for $n>1$;
3. $\lim_{z\to t,\,|t|=1,\,|z|<1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_1(z,\tau)\frac{\mathrm{d}\tau}{\tau}=\gamma(t)$ for any $\gamma\in C(\partial\mathbb{D})$;
4. $\lim_{z\to t,\,|t|=1,\,|z|<1}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_2(z,\tau)\frac{\mathrm{d}\tau}{\tau}=0$ for any $\gamma\in C(\partial\mathbb{D})$;
5. $\lim_{z\to t,\,|t|=1,\,|z|<1} g_n(z,\tau)=0$ for $n>2$.

In what follows, we will give explicit expressions for all higher-order Poisson kernels. To do so, we need:

Lemma 4.1. *Let Ω_1 be a domain and Ω_2 be a compact set in the complex plane, $\Omega_1\cap\Omega_2=\emptyset$, $g(z,\xi)$ be a continuous function defined in $\Omega_1\times\Omega_2$ such that $g(z,\xi)\in H_1(\Omega_1)$ as a function of z with fixed $\xi\in\Omega_2$. For any fixed $z_0\in\Omega_1$, take $D_{z_0,R}=\{z:0<|z-z_0|<R\}\subset\Omega_1$ and define*

$$F_z(z_0,\xi)=\frac{g(z,\xi)-g(z_0,\xi)}{z-z_0},\ \ \xi\in\Omega_2 \tag{4.2}$$

and

$$G_z(z_0,\xi)=\frac{g(z,\xi)-g(z_0,\xi)}{\overline{z}-\overline{z}_0},\ \ \xi\in\Omega_2 \tag{4.3}$$

with fixed $z\in D_{z_0,R/2}$. Then $F_z(z_0,\cdot)$, $G_z(z_0,\cdot)\in L(\Omega_2)$.

Proof. Since $g(z,\xi)\in H_1(\Omega_1)$ with respect to z for fixed $\xi\in\Omega_2$, by the Cauchy integral formula, for fixed $\xi\in\Omega_2$,

$$g(z_0,\xi)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=R}\frac{g(\zeta,\xi)}{\zeta-z_0}\,\mathrm{d}\zeta$$

and

$$g(z,\xi)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=R}\frac{g(\zeta,\xi)}{\zeta-z}\,\mathrm{d}\zeta.$$

Thus

$$F_z(z_0,\xi)=\frac{1}{2\pi i}\int_{|\zeta-z_0|=R}\frac{g(\zeta,\xi)}{(\zeta-z)(\zeta-z_0)}\,\mathrm{d}\zeta.$$

So

$$\begin{aligned}\int_{\Omega_2}|F_z(z_0,\xi)|\mathrm{d}\nu(\xi)&=\int_{\Omega_2}\Big|\frac{1}{2\pi i}\int_{|\zeta-z_0|=R}\frac{g(\zeta,\xi)}{(\zeta-z)(\zeta-z_0)}\,\mathrm{d}\zeta\Big|\mathrm{d}\nu(\xi)\\&\le\frac{1}{2\pi i}\int_{|\zeta-z_0|=R}\frac{|\tilde{g}(\zeta)|}{|(\zeta-z)|}\,\frac{\mathrm{d}\zeta}{\zeta-z_0}\\&\le\frac{2\sup|\tilde{g}(\zeta)|}{R},\end{aligned}$$

where ν is the Lebesgue measure on Ω_2, $\tilde{g}(\zeta) = \int_{\Omega_2} |g(\zeta,\xi)| \mathrm{d}\nu(\xi)$ is bounded on $\{\zeta : |\zeta - z_0| = R\}$ since $g(z,\xi) \in C(\Omega_1 \times \Omega_2)$ and Ω_2 is compact. That is to say $F_z(z_0,\cdot) \in L(\Omega_2)$. Note that

$$G_z(z_0,\xi) = \frac{z - z_0}{\overline{z} - \overline{z}_0} F_z(z_0,\xi),$$

therefore $G_z(z_0,\cdot) \in L(\Omega_2)$. □

By Theorem 2.1, Corollary 2.1, Lemma 4.1 and the above properties of higher-order Poisson kernels, we have

Theorem 4.1 (see [17]). *If $\{ g_n(z,\tau) \}_{n=1}^{\infty}$ is a sequence of kernel functions defined on $\mathbb{D} \times \partial\mathbb{D}$, i.e., $\{ g_n(z,\tau) \}_{n=1}^{\infty}$ fulfills the above properties 1–5, then, for $n > 1$, there exist functions $g_{n,0}(z,\tau), g_{n,1}(z,\tau), \ldots, g_{n,n-1}(z,\tau)$ defined on $\mathbb{D} \times \partial\mathbb{D}$ such that*

$$g_n(z,\tau) = 2\Re\Big\{ \sum_{j=0}^{n-1} \overline{z}^j g_{n,j}(z,\tau) \Big\}, \ z \in \mathbb{D}, \ \tau \in \partial\mathbb{D} \tag{4.4}$$

with

$$\partial_z g_{n,j}(z,\tau) = j^{-1} g_{n-1,j-1}(z,\tau) \tag{4.5}$$

for $1 \le j \le n-1$ and

$$\partial_z^k g_{n,j}(0,\tau) = 0 \tag{4.6}$$

for $0 \le k \le j-1$ with respect to $\tau \in \partial\mathbb{D}$ as well as

$$g_{n,0}(z,\tau) = -\sum_{j=1}^{n-1} z^{-j} g_{n,j}(z,\tau). \tag{4.7}$$

However,

$$g_1(z,\tau) = \frac{1}{1 - z\overline{\tau}} + \frac{1}{1 - \overline{z}\tau} - 1 \tag{4.8}$$

is the Poisson kernel. Such a sequence $\{ g_n(z,\tau) \}_{n=1}^{\infty}$ is unique. Moreover, the decomposition components $g_{n,j}(z,\tau) \in C(\mathbb{D} \times \partial\mathbb{D})$ satisfy $g_{n,j}(\cdot,\tau) \in H^j_{1,0}(\mathbb{D})$ for fixed $\tau \in \partial\mathbb{D}$ and $\partial_z g_{n,j}(z,\tau) \in C(\mathbb{D} \times \partial\mathbb{D})$, $n = 1, 2, \ldots,$ $j = 0, 1, \ldots, n-1$.

By Theorem 4.1, we can get the explicit unified expressions of $g_n(z,\tau)$ (see [17]). To do so, we introduce a vertical sum

$$\sum \begin{cases} a_1 \\ a_2 \\ \vdots \\ a_n \end{cases} =: a_1 + a_2 + \cdots + a_n; \tag{4.9}$$

then, in general, we have

Theorem 4.2. *For $1 \le j \le n-5$, let $W_{n,j}(z,\tau)$ be a vertical sum of the form*

$$
\sum\left\{\begin{array}{l}
\sum\left\{\begin{array}{l}
\cdots\sum\left\{\begin{array}{l}
\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j}(k+1)\cdots(k+j-1)}+\frac{1}{j!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 2!}\right]\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 3!}\right]
\end{array}\right.\\
-\frac{1}{2!}\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 2!}\right]
\end{array}\right.\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 3!}\right]\\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 4!}\right]
\end{array}\right.\\
\qquad\vdots\\
\frac{(-1)^{n-j-4}}{(n-j-3)!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{4}(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{1}{j!\cdot(n-j-3)!}\right]\\
\frac{(-1)^{n-j-3}}{(n-j-2)!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{3}(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-3)}+\frac{1}{j!\cdot(n-j-2)!}\right]
\end{array}\right.\\
-\frac{1}{2!}\sum\left\{\begin{array}{l}
\cdots\sum\left\{\begin{array}{l}
\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 2!}\right]\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 3!}\right]
\end{array}\right.\\
-\frac{1}{2!}\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-3}(k+1)^4\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 2!\cdot 2!}\right]
\end{array}\right.\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 3!}\right]\\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)}+\frac{1}{j!\cdot 2!\cdot 4!}\right]
\end{array}\right.\\
\qquad\vdots\\
\frac{(-1)^{n-j-4}}{(n-j-3)!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{4}(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{1}{j!\cdot 2!\cdot(n-j-3)!}\right]
\end{array}\right.\\
\qquad\vdots\\
\qquad\vdots
\end{array}\right.
$$

$$
\sum\left\{\begin{array}{l}
\vdots \\
\vdots \\
\frac{(-1)^{n-j-6}}{(n-j-5)!}\sum\left\{\begin{array}{l}
\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^6(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!\cdot 2!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!\cdot 2!}\right] \\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^4(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!\cdot 3!}\right]
\end{array}\right. \\
-\frac{1}{2!}\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^5(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!\cdot 2!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^4(k+1)^4\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!\cdot 2!\cdot 2!}\right]
\end{array}\right. \\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^4(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!\cdot 3!}\right] \\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^3(k+1)^3(k+2)^3(k+3)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-6)}+\frac{1}{j!\cdot(n-j-5)!\cdot 4!}\right]
\end{array}\right. \\
\frac{(-1)^{n-j-5}}{(n-j-4)!}\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^5(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{1}{j!\cdot(n-j-4)!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^4(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{1}{j!\cdot(n-j-4)!\cdot 2!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^4(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{1}{j!\cdot(n-j-4)!\cdot 2!}\right] \\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^3(k+1)^3(k+2)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-5)}+\frac{1}{j!\cdot(n-j-4)!\cdot 3!}\right]
\end{array}\right. \\
\frac{(-1)^{n-j-4}}{(n-j-3)!}\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^4(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{1}{j!\cdot(n-j-3)!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^3(k+1)^3\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{1}{j!\cdot(n-j-3)!\cdot 2!}\right]
\end{array}\right. \\
\frac{(-1)^{n-j-3}}{(n-j-2)!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^3(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-3)}+\frac{1}{j!\cdot(n-j-2)!}\right] \\
\frac{(-1)^{n-j-2}}{(n-j-1)!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^2(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-2)}+\frac{1}{j!\cdot(n-j-1)!}\right]
\end{array}\right. \tag{4.10}
$$

with $d_{k-1}(z,\tau)=(z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}$, *and let*

$$
W_{n,n-4}(z,\tau)=\sum\left\{\begin{array}{l}
\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^4(k+1)(k+2)\cdots(k+n-5)}+\frac{1}{(n-4)!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^3(k+1)^2(k+2)\cdots(k+n-5)}+\frac{1}{(n-4)!\cdot 2!}\right] \\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^3(k+1)^2(k+2)\cdots(k+n-5)}+\frac{1}{(n-4)!\cdot 2!}\right] \\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^2(k+1)^2(k+2)^2\cdots(k+n-5)}+\frac{1}{(n-4)!\cdot 3!}\right]
\end{array}\right., \tag{4.11}
$$

$$W_{n,n-3}(z,\tau) = \sum \left\{ \begin{array}{c} \left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^3(k+1)(k+2)\cdots(k+n-4)} + \frac{1}{(n-3)!} \right] \\ -\frac{1}{2!}\left[\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^2(k+1)^2(k+2)\cdots(k+n-4)} + \frac{1}{(n-3)!\cdot 2!} \right] \end{array} \right., \tag{4.12}$$

$$W_{n,n-2}(z,\tau) = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^2(k+1)(k+2)\cdots(k+n-3)} + \frac{1}{(n-2)!}, \tag{4.13}$$

$$W_{n,n-1}(z,\tau) = \sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k(k+1)(k+2)\cdots(k+n-2)} + \frac{1}{(n-1)!}. \tag{4.14}$$

If $\{ g_n(z,\tau) \}_{n=1}^{\infty}$ *is a sequence of higher-order Poisson kernels defined on* $\mathbb{D} \times \partial\mathbb{D}$*, then*

$$g_n(z,\tau) = D_1(z,\tau) + D_2(z,\tau) + \cdots + D_{n-1}(z,\tau), \tag{4.15}$$

where $D_j(z,\tau) = (-1)^{n-j}\frac{1-|z|^{2j}}{j!}W_{n,j}(z,\tau)$, $j = 1,2,\ldots,n-1$*. In all above formulae, by convention,* $\prod_{\ell=\imath}^{\jmath}(k+\ell) = 1$ *as* $\imath > \jmath$*.*

Remark 4.1. Carefully observing all above vertical sums $W_{n,j}(z,\tau)$, $j = 1,2,\ldots,$ $n-1$, one may find that the vertical sums take on some structural orderlines. More precisely, there is a distinct circulatory structure of the vertical sum

$$\frac{(-1)^{p-1}}{p!} \sum \left\{ \begin{array}{c} \frac{(-1)^{q-1}}{q!}\alpha \sum \left\{ \begin{array}{c} \varepsilon \\ -\frac{1}{2!}\zeta \\ -\frac{1}{2!}\zeta \\ \frac{1}{3!}\varsigma \end{array} \right. \\ \frac{(-1)^q}{(q+1)!}\beta \sum \left\{ \begin{array}{c} \varpi \\ -\frac{1}{2!}\omega \end{array} \right. \\ \frac{(-1)^{q+1}}{(q+2)!}\mu\gamma \\ \frac{(-1)^{q+2}}{(q+3)!}\nu\delta \end{array} \right., \tag{4.16}$$

where α,β,μ,ν are 1 or 0, all of which are nonzero or only one of which is nonzero, the latter only happens when $j = n-4, n-3, n-2, n-1$, $1 \le p \le n-4$ and $0 \le q \le n-4$. However, $\varepsilon,\zeta,\varsigma,\varpi,\omega,\gamma,\delta$ are sums of the form

$$\sum_{k=2}^{\infty} \frac{d_{k-1}(z,\tau)}{k^{m_1}(k+1)^{m_2}(k+2)^{m_3}\cdots(k+n-2)^{m_{n-1}}} + \frac{1}{\vartheta}, \tag{4.17}$$

where $m_1, m_2, \ldots, m_{n-1}$ are nonnegative integers satisfying

$$m_1 \ge m_2 \ge \cdots \ge m_{n-1} \ge 0 \text{ and } m_1 + m_2 + \cdots + m_{n-1} = n-1, \tag{4.18}$$

whereas ϑ is a product of some factorials which takes on some evident regularity, i.e., ϑ is the product of $j!$ and all denominators of the coefficients appearing before the vertical sum symbols and the sum which it belongs to. Moreover, when $\alpha = \beta = \gamma = \delta = 1$, the multiplicities have the following sequential properties:

(1) From ε to ζ and ϖ to ω, m_1 decreases by 1 whereas m_2 simultaneously increase by 1.

(2) From ζ to ς, m_1 decreases by 1 whereas m_3 simultaneously increase by 1.
(3) From ε to ϖ, ϖ to γ and γ to δ, m_1 decreases by 1 for each step whereas m_{q+1}, m_{q+2} and m_{q+3} sequentially increases by 1.

It must be noted that the new multiplicities also satisfy (4.18) all the same. In addition, for $W_{n,j}(z,\tau)$, there are $n-j-1$ vertical sums as its summands in the outmost vertical sum. From the top down, these vertical sums respectively have 2^{n-j-3}, 2^{n-j-4}, ..., 2, 1, 1 summands of the form as (4.17). The above property (3) holds for the variance of the multiplicites about the first summand of the form as (4.17) between two adjacent vertical sums and the coefficients appearing before the sum symbols are in turn 1, $-\frac{1}{2!}$, ..., $\frac{(-1)^{n-j-3}}{(n-j-2)!}$, $\frac{(-1)^{n-j-2}}{(n-j-1)!}$. Interestingly, any one of these vertical sums has similar structure and properties as the outmost vertical sum.

Just because of the above sequential properties of the multiplicities and the nice circulatory structure, we can sequentially define $W_{n,j}(z,\tau)$ as the vertical sum (4.10) only from the first summand $\sum_{k=2}^{\infty}\frac{d_{k-1}(z,\tau)}{k^{n-j}(k+1)\cdots(k+j-1)}+\frac{1}{j!}$. A nice example is $g_5(z,\tau)$ of vertical form which can be found in [17].

With the higher-order Poisson kernels, the PHD problem is uniquely solvable. To do so, we need the following lemmas, one of which is about another property of the higher-order Poisson kernels.

Lemma 4.2 (Differentiability of Integral). *Let $\{\,g_n(z,\tau)\,\}_{n=1}^{\infty}$ be the sequence of higher-order Poisson kernels, then for any $\gamma\in C(\partial\mathbb{D})$,*

$$(\partial_z\partial_{\overline{z}})\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_n(z,\tau)\frac{\mathrm{d}\tau}{\tau}\Big]=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_{n-1}(z,\tau)\frac{\mathrm{d}\tau}{\tau},\ n=2,3,\ldots. \tag{4.19}$$

Proof. For any fixed $z\in\mathbb{D}$, choose an arbitrary sequence $\{z_l\}$ such that $z_l\neq z$ for any l and $z_l\to z$ as $l\to\infty$. Define

$$Z_l(z,\tau)=\frac{g_n(z_l,\tau)-g_n(z,\tau)}{z_l-z} \tag{4.20}$$

for fixed l. Obviously, $Z_l(z,\tau)\in C(\partial\mathbb{D})\subset L(\mathbb{D})$ with respect to τ and

$$\lim_{l\to\infty}Z_l(z,\tau)=\partial_z g_n(z,\tau). \tag{4.21}$$

In addition, by the decomposition (4.4) of $g_n(z,\tau)$ and Lemma 4.1 with $\Omega_1=\mathbb{D}$ and $\Omega_2=\partial\mathbb{D}$, it is easy to see that $Z_l(z,\cdot)\in L(\partial\mathbb{D})$. Note the continuity of $\partial_z g_n(z,\tau)$, by the dominated convergence theorem,

$$\lim_{l\to\infty}\frac{1}{z_l-z}\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_n(z_l,\tau)\frac{d\tau}{\tau}-\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_n(z,\tau)\frac{d\tau}{\tau}\Big]$$
$$=\lim_{l\to\infty}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)\frac{g_n(z_l,\tau)-g_n(z,\tau)}{z_l-z}\frac{d\tau}{\tau}$$
$$=\lim_{l\to\infty}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)Z_l(z,\tau)\frac{d\tau}{\tau}$$
$$=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)\partial_z g_n(z,\tau)\frac{d\tau}{\tau}.$$

Because of the arbitrariness of $\{z_l\}$, therefore in view of the Heine principle,

$$\partial_z\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)g_n(z,\tau)\frac{d\tau}{\tau}\Big]=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)\partial_z g_n(z,\tau)\frac{d\tau}{\tau}. \tag{4.22}$$

Further, similarly define

$$H_l(z,\tau)=\frac{\partial_z g_n(z_l,\tau)-\partial_z g_n(z,\tau)}{\overline{z}_l-\overline{z}}, \tag{4.23}$$

again by (4.4), Lemma 4.1, the dominated convergence theorem and the Heine principle,

$$\partial_{\overline{z}}\Big[\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)\partial_z g_n(z,\tau)\frac{d\tau}{\tau}\Big]=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma(\tau)\partial_{\overline{z}}[\partial_z g_n(z,\tau)]\frac{d\tau}{\tau}. \tag{4.24}$$

So (4.19) follows from the last two equalities and the induction property of the higher-order Poisson kernels. □

Lemma 4.3. *If $\varphi\in H_1(\mathbb{D})$ and $\frac{\partial\varphi}{\partial z}\in C(\overline{\mathbb{D}})$, then $\varphi\in C(\overline{\mathbb{D}})$.*

Proof. It immediately follows from

$$\varphi(z)=\int_0^z\frac{\partial\varphi}{\partial z}(\zeta)d\zeta-\varphi(0),\ z\in\mathbb{D}.$$ □

Theorem 4.3. *The PHD problem* (4.1) *is solvable and its unique solution is*

$$w(z)=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_{k-1}(\tau)g_k(z,\tau)\frac{d\tau}{\tau},\ z\in\mathbb{D}, \tag{4.25}$$

where $g_k(z,\tau)\,(1\le k\le n)$ is the kth Poisson kernel given by (4.15).

Proof. At first, we show that (4.25) is a solution. By Lemma 4.2 and the induction property of the higher-order Poisson kernels, using the operators $(\partial_z\partial_{\overline{z}})^j$, $j=1,2,\ldots,n-1$ to act on two sides of (4.25), we get

$$(\partial_z\partial_{\overline{z}})^j w(z)=\sum_{k=j+1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_{k-1}(\tau)g_{k-j}(z,\tau)\frac{d\tau}{\tau}. \tag{4.26}$$

Thus

$$[\partial_z\partial_{\overline{z}})^j w]^+(t)=\gamma_j(t),\ t\in\mathbb{D},\ 0\le j<n \tag{4.27}$$

follows from (4.26) and the other properties of the higher-order Poisson kernels, i.e., (4.25) is a solution.

Next, we turn to the uniqueness of (4.25). To do so, we must show that (4.1) only has zero as its solution when all $\gamma_j = 0$ on $\partial\mathbb{D}$. It is enough to consider $w \in Har_n(\mathbb{D})$ for this case. Since $w \in Har_n(\mathbb{D})$, by Theorem 2.1, there exist some functions $w_j \in H^j_{1,0}(\mathbb{D})$, $j = 0, 1, \ldots, n-1$ such that

$$w(z) = 2\Re\Big\{\sum_{j=0}^{n-1} \overline{z}^j w_j(z)\Big\}, \quad z \in \mathbb{D}. \tag{4.28}$$

Applying the operators $(\partial_z\partial_{\overline{z}})^j$, $j = 1, 2, \ldots, n-1$ to both sides of (4.28), we have

$$(\partial_z\partial_{\overline{z}})^j w(z) = 2\Re\Big\{\sum_{k=j}^{n-1} \frac{k!}{(k-j)!}\overline{z}^{k-j}\partial_z^j w_k(z)\Big\}, \quad z \in \mathbb{D}. \tag{4.29}$$

By (4.29), Lemma 4.3 and the boundary value conditions of (4.1) with $\gamma_j = 0$,

$$\Re[\partial_z^j w_j(t)] = 0, \ \ t \in \partial\mathbb{D}, \ \ 0 \le j \le n-1. \tag{4.30}$$

So it is easy to get $w_j \in \Pi^j_{1,0}(\mathbb{D})$ from the last equality and then $w = 0$. □

Remark 4.2. In [8], Begehr, Du and Wang only considered the PHD problem (4.1) with Hölder continuous but not continuous boundary conditions. So it happens since they solve the problem by reflection method which transfers the problem to the classical Riemann jump problems for analytic functions. However, the Hölder continuity is necessary for the latter considering the singular integrals on the unit circle. In [12], to solve the same problem when $n = 3$, Begehr and Wang used a new approach which transfers the problem to the classical Schwarz problem for analytic functions in the unit disc. So the Hölder continuity is weakened to the condition of continuity. In fact, with continuous boundary conditions discussed in the last theorem, the unique solvability of PHD problem (4.1) obviously follows from the properties of the higher-order Poisson kernels $g_n(z,\tau)$ by induction.

Next we consider two kinds of Dirichlet type boundary value problems for functions in $M_{m,n}(\mathbb{D})$, one of which is of the form: find a function $L(z) \in M_{m,n}(\mathbb{D})$ $(m > n)$ satisfying the boundary conditions

$$[(\partial_z\partial_{\overline{z}})^j L]^+(t) = \gamma_j(t), \ 0 \le j < n \ \text{ and } \ [\partial_z^{n+k}\partial_{\overline{z}}^n L]^+(t) = \sigma_k(t), \ 0 \le k < m-n, \tag{4.31}$$

where $t \in \partial\mathbb{D}$, $\gamma_j, \sigma_k \in C(\partial\mathbb{D})$ for $0 \le j < n, \ 0 \le k < m-n$.

The other is to find a function $N(z) \in M_{m,n}(\mathbb{D})$ which fulfills the boundary conditions

$$[(\partial_z^m\partial_{\overline{z}}^j)N]^+(t) = \chi_j(t), \ 0 \le j < n \ \text{ and } \ [\partial_z^k\partial_{\overline{z}}^n N]^+(t) = \lambda_k(t), \ 0 \le k < m, \tag{4.32}$$

where $t \in \partial\mathbb{D}$, $\chi_j, \lambda_k \in C(\partial\mathbb{D})$ for $0 \le j < n, \ 0 \le k < m$.

For these problems, by Theorems 2.4–2.5, Theorem 4.3 and the classical Dirichlet problem for analytic functions [3], we have (see [17]):

Theorem 4.4. *Set*

$$A(t)=\begin{pmatrix} n! & (n+1)!t & \cdots & \frac{(m-2)!}{(m-n-2)!}t^{m-n-2} & \frac{(m-1)!}{(m-n-1)!}t^{m-n-1}\\ 0 & (n+1)! & \cdots & \frac{(m-2)!}{(m-n-3)!}t^{m-n-3} & \frac{(m-1)!}{(m-n-2)!}t^{m-n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (m-2)! & (m-1)!t\\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix}, \tag{4.33}$$

$$a(t)=\begin{pmatrix} \sigma_0(t)\\ \sigma_1(t)\\ \vdots\\ \sigma_{m-n-2}(t)\\ \sigma_{m-n-1}(t) \end{pmatrix}, \tag{4.34}$$

$$\Xi_l(z)=\frac{1}{n!(n+1)!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{\overline{\det(A_l(\tau))}}{\tau-z}\mathrm{d}\tau, \tag{4.35}$$

and

$$\widetilde{\varphi}_l(z)=\int_0^z\int_0^{\zeta_{n-1}}\cdots\int_0^{\zeta_1}\Xi_l(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1}+\pi_l(z), \tag{4.36}$$

where $t\in\partial\mathbb{D}$, $\pi_l\in\Pi_{n-1}$, *the matrix* $A_l(t)$ *is given by replacing the* l*th column of* $A(t)$ *by* $a(t)$, $0\le l\le m-n-1$. *Then*

$$\begin{aligned} L(z)=&\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}g_k(z,\tau)\Big[\gamma_{k-1}(\tau)-\sum_{l=0}^{m-n-1}\tfrac{(n+l)!}{(n+l-k+1)!}\tau^{n+l-k+1}\overline{\partial_z^{k-1}\widetilde{\varphi}_l(\tau)}\Big]\frac{\mathrm{d}\tau}{\tau}\\ &+z^n\sum_{l=0}^{m-n-1}z^l\overline{\widetilde{\varphi}_l(z)} \end{aligned} \tag{4.37}$$

are all solutions of the problem (4.31) *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\frac{z\det A_l(\tau)}{\tau-z}\frac{\mathrm{d}\tau}{\tau}=0,\ z\in\mathbb{D}, \tag{4.38}$$

where $g_k(z,\tau)\,(1\le k\le n)$ *are the former* n *higher-order Poisson kernels.*

Theorem 4.5. *Set*

$$B(t)=\begin{pmatrix} 1 & \overline{t} & \overline{t}^2 & \cdots & \overline{t}^{n-1}\\ 0 & 1 & 2\overline{t} & \cdots & (n-1)\overline{t}^{n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (n-2)! & (n-1)!\overline{t}\\ 0 & 0 & \cdots & 0 & (n-1)! \end{pmatrix}, \tag{4.39}$$

$$C(t) = \begin{pmatrix} 1 & t & t^2 & \cdots & t^{m-1} \\ 0 & 1 & 2t & \cdots & (m-1)t^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (m-2)! & (m-1)!t \\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix}, \tag{4.40}$$

$$b(t) = \begin{pmatrix} \chi_0(t) \\ \chi_1(t) \\ \vdots \\ \chi_{n-1}(t) \end{pmatrix}, \quad c(t) = \begin{pmatrix} \lambda_0(t) \\ \lambda_1(t) \\ \vdots \\ \lambda_{m-1}(t) \end{pmatrix} \tag{4.41}$$

and

$$\Theta_p(z) = \frac{1}{1!2!\cdots(n-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\det B_p(\tau)}{\tau - z}\,\mathrm{d}\tau, \tag{4.42}$$

$$\Lambda_q(z) = \frac{1}{1!2!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{\det C_q(\tau)}}{\tau - z}\,\mathrm{d}\tau, \tag{4.43}$$

as well as

$$\mu_p(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Theta_p(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{m-1} + \kappa_p(z), \tag{4.44}$$

$$\nu_q(z) = \int_0^z \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_1} \Lambda_q(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1} + \xi_q(z), \tag{4.45}$$

where $t \in \partial\mathbb{D}$, $\kappa_p \in \Pi_{m-1}$, $\xi_q \in \Pi_{n-1}$, the matrices $B_p(t), C_q(t)$ are respectively given by replacing the pth, qth column by $b(t)$, $c(t)$, $0 \le p \le n-1$, $0 \le q \le m-1$. Then

$$N(z) = \sum_{p=0}^{n-1} \overline{z}^p \mu_p(z) + \sum_{q=0}^{m-1} z^q \overline{\nu_q(z)} \tag{4.46}$$

are all solutions of the problem (4.32) *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\overline{\det B_p(\tau)}}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0, \quad \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\det C_q(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0, \tag{4.47}$$

in which $z \in \mathbb{D}$.

Remark 4.3. It must be noted that the solutions of Dirichlet problems (4.31) and (4.32) are not unique even if the conditions (4.38) and (4.47) are fulfilled. That is, the polynomials in (4.36), (4.44) and (4.45) can be arbitrarily chosen. It follows from an easy fact that the difference of two analytic functions in the unit disc $\mathbb{D}$ is a polynomial of order at most $n-1$ if their derivatives of order n have the same continuous boundary values on the unit circle $\partial\mathbb{D}$. Respectively, using the harmonic decomposition (2.5) and canonical decomposition (2.7) to solve Dirichlet problems (4.31) and (4.32), we will encounter this easy fact so that the polynomials π_l, κ_p

and ξ_q naturally appear in (4.36), (4.44) and (4.45); then the solutions are not unique. Similarly, we can also consider another Dirichlet type problem as follows,

$$[(\partial_z\partial_{\overline{z}})^j R]^+(t)=\rho_j(t),\ 0\le j<m \ \text{ and } \ [\partial_z^m\partial_{\overline{z}}^{m+k}R]^+(t)=\varrho_k(t),\ 0\le k<n-m, \tag{4.48}$$

where $t\in\partial\mathbb{D}$, $\rho_j,\varrho_k\in C(\partial\mathbb{D})$ for $0\le j<m$, $0\le k<n-m$ and the object function $R(z)\in M_{m,n}(\mathbb{D})$ $(m<n)$. In the exactly same way for the problem (4.31), by Theorem 2.4, Theorem 4.3 and the classical Dirichlet problem for analytic functions, it is easy to get all solutions of the problem (4.48) under some suitable conditions.

5. Inhomogeneous Equations

In the present section, we consider the corresponding Dirichlet problems discussed in the last section for inhomogeneous equations. According to the results of the last section for homogeneous equations, the key is to find some special solutions for the inhomogeneous equations. By Theorem 3.1, this is no problem under suitably assumable conditions. As mentioned in the introduction of [10], we will find that higher-order Pompeiu operators $T_{m,n}$ are useful in the study of boundary value problems for higher-order complex partial differential equations.

Let $f\in L^p(\mathbb{D})$, $p>2$, by Theorem 3.1, we get

$$\partial_z^k\partial_{\overline{z}}^l T_{m,n,\mathbb{D}}f(z)=T_{m-k,n-l,\mathbb{D}}f(z),\quad 0\le k+l\le m+n \tag{5.1}$$

in the Sobolev sense. Moreover,

$$T_{m-k,n-l,\mathbb{D}}f(z)\in H_{loc}(\mathbb{C})\subset C(\mathbb{C}),\ \text{ as } 1\le k+l<m+n, \tag{5.2}$$

where $H_{loc}(\mathbb{C})$ denotes the set of all locally Hölder continuous functions in $\mathbb{C}$.

Noting (5.1), we know that $w(z)=T_{m,n,\mathbb{D}}f(z)$ is a weak solution of the inhomogeneous equations

$$(\partial_z^m\partial_{\overline{z}}^n)w(z)=f(z),\ \ z\in\mathbb{D},\ f\in L^p(\mathbb{D}),\ p>2. \tag{5.3}$$

First, we consider the so-called Dirichlet problem for the inhomogeneous polyharmonic equations [12]:

$$\begin{cases}(\partial_z\partial_{\overline{z}})^n w(z)=f(z), & z\in\mathbb{D}, f\in L^p(\mathbb{D}),\, p>2,\\ (\partial_z\partial_{\overline{z}})^k w(\tau)=\gamma_k(\tau), & \tau\in\partial\mathbb{D},\gamma_k\in C(\partial\mathbb{D}),\, 0\le k\le n-1.\end{cases} \tag{5.4}$$

By Theorem 4.3, (5.1) and (5.2), we have

Theorem 5.1. *The problem* (5.4) *is solvable and its unique solution is*

$$w(z)=T_{n,n,\mathbb{D}}f(z)+\sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}[\gamma_{k-1}(\tau)-T_{n+1-k,n+1-k,\mathbb{D}}f(\tau)]g_k(z,\tau)\frac{d\tau}{\tau}, \tag{5.5}$$

where $z\in\mathbb{D}$, $T_{l,l,\mathbb{D}}$ $(1\le l\le n)$ are the higher-order Pompeiu operators, $g_k(z,\tau)$ $(1\le k\le n)$ are the former n higher-order Poisson kernel functions.

Proof. Note that by (5.1) and (5.2), the problem (5.4) is equivalent to the PHD problem of simplified form

$$\begin{cases} w - T_{n,n,\mathbb{D}}f \in Har_n^{\mathbb{C}}(\mathbb{D}), \quad f \in L^p(\mathbb{D}),\, p > 2, \\ (\partial_z \partial_{\overline{z}})^k [w - T_{n,n,\mathbb{D}}f] = \gamma_k - T_{n-k,n-k,\mathbb{D}}f,\ \gamma_k \in C(\partial\mathbb{D}),\, 0 \le k \le n-1. \end{cases} \tag{5.6}$$

So it is obvious that Theorem 5.1 follows from Theorem 4.3. □

Noting (3.6) and (3.7), by Theorem 4.2, we can give the explicit expressions of the double integrals in (5.5). To do so, we need some lemmas as follows.

Lemma 5.1.

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \tau^k \frac{\mathrm{d}\tau}{\tau} = \delta_{k0}, \quad k \in \mathbb{Z}, \tag{5.7}$$

where δ_{k0} is the Kronecker sign and $\mathbb{Z}$ is the set of all integers.

Proof. It is obvious since $\tau = e^{i\theta},\ \theta \in [0, 2\pi)$. □

Lemma 5.2.

$$|\tau - \zeta|^{2n} = \sum_{p,q=0}^{n} \binom{n}{p}\binom{n}{q} \zeta^p \overline{\zeta}^q \overline{\tau}^p \tau^q, \quad \tau \in \partial\mathbb{D}, \zeta \in \mathbb{D}, n \in \mathbb{Z}_+, \tag{5.8}$$

where $\mathbb{Z}_+$ is the set of all positive integers.

Proof. (5.8) follows from the fact that $|\tau - \zeta|^2 = |1 - \overline{\tau}\zeta|^2 = (1 - \overline{\tau}\zeta)(1 - \tau\overline{\zeta}),\ \tau \in \partial\mathbb{D}$. □

Lemma 5.3.

$$\log|\tau - \zeta|^2 = -\sum_{s=1}^{\infty} s^{-1}[(\overline{\tau}\zeta)^s + (\tau\overline{\zeta})^s],\ \tau \in \partial\mathbb{D}, \zeta \in \mathbb{D}. \tag{5.9}$$

Proof. Since $\tau \in \partial\mathbb{D}$, $\zeta \in \mathbb{D}$, therefore

$$\begin{aligned} \log|\tau - \zeta|^2 &= \log|1 - \overline{\tau}\zeta|^2 \\ &= \log(1 - \overline{\tau}\zeta)(1 - \tau\overline{\zeta}) \\ &= \log(1 - \overline{\tau}\zeta) + \log(1 - \tau\overline{\zeta}) \\ &= -\sum_{s=1}^{\infty} s^{-1}[(\overline{\tau}\zeta)^s + (\tau\overline{\zeta})^s]. \end{aligned} \tag{5.10}$$

The last equality follows from the fact that $\log(1 - x) = -\sum_{s=1}^{\infty} \frac{x^s}{s},\ |x| < 1$. □

Theorem 5.2. *Suppose that $m, n \in \mathbb{Z}_+$, for $1 \le j \le n - 5$, let $N_{m,n,j}(z, \zeta)$ be a vertical sum of the following form:*

$$
\sum\begin{cases}
\sum\begin{cases}
\cdots\sum\begin{cases}
\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j}(k+1)\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 3!}\right]
\end{cases}\\
-\frac{1}{2!}\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]
\end{cases}\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 3!}\right]\\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 4!}\right]
\end{cases}\\
\vdots\\
\frac{(-1)^{n-j-4}}{(n-j-3)!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-3)!}\right]\\
\frac{(-1)^{n-j-3}}{(n-j-2)!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-3)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-2)!}\right]
\end{cases}\\
-\frac{1}{2!}\sum\begin{cases}
\cdots\sum\begin{cases}
\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-1}(k+1)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 3!}\right]
\end{cases}\\
-\frac{1}{2!}\sum\begin{cases}
\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^3\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!}\right]\\
-\frac{1}{2!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-3}(k+1)^4\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 2!\cdot 2!}\right]
\end{cases}\\
\frac{1}{3!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-2}(k+1)^2(k+2)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 3!}\right]\\
-\frac{1}{4!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^{n-j-3}(k+1)^2(k+2)^2(k+3)^2\cdots(k+j-1)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot 4!}\right]
\end{cases}\\
\vdots\\
\frac{(-1)^{n-j-4}}{(n-j-3)!}\left[\sum_{k=2}^{\infty}\frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^2\cdots(k+j-1)^2(k+j)\cdots(k+n-j-4)}+\frac{\Delta_{m,0}(z,\zeta)}{j!\cdot 2!\cdot(n-j-3)!}\right]
\end{cases}\\
\vdots\\
\vdots
\end{cases}
$$

$$
\sum \begin{cases}
\vdots \\
\vdots \\
\dfrac{(-1)^{n-j-6}}{(n-j-5)!} \sum \begin{cases}
\sum \begin{cases}
\left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^6(k+1)^2 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!} \right] \\
\frac{1}{3!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3(k+2)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 3!} \right]
\end{cases} \\
-\frac{1}{2!} \sum \begin{cases}
\left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^4 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 2!\cdot 2!} \right]
\end{cases} \\
\frac{1}{3!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3(k+2)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 3!} \right] \\
-\frac{1}{4!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^3(k+2)^3(k+3)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-6)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-5)!\cdot 4!} \right]
\end{cases} \\
\dfrac{(-1)^{n-j-5}}{(n-j-4)!} \sum \begin{cases}
\left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^5(k+1)^2 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!\cdot 2!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!\cdot 2!} \right] \\
\frac{1}{3!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^3(k+2)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-5)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-4)!\cdot 3!} \right]
\end{cases} \\
\dfrac{(-1)^{n-j-4}}{(n-j-3)!} \sum \begin{cases}
\left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)^2 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-4)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-3)!} \right] \\
-\frac{1}{2!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^3 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-4)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-3)!\cdot 2!} \right]
\end{cases} \\
\dfrac{(-1)^{n-j-3}}{(n-j-2)!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^2 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-3)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-2)!} \right] \\
\dfrac{(-1)^{n-j-2}}{(n-j-1)!} \left[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)^2 \cdots (k+j-1)^2(k+j)\cdots(k+n-j-2)} + \frac{\Delta_{m,0}(z,\zeta)}{j!\cdot(n-j-1)!} \right]
\end{cases}
\tag{5.11}
$$

and let

$$N_{m,n,n-4}(z,\zeta) = \sum \begin{cases} \Big[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^4(k+1)(k+2)\cdots(k+n-5)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-4)!}\Big] \\ -\frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^2(k+2)\cdots(k+n-5)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-4)!\cdot 2!}\Big] \\ -\frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)^2(k+2)\cdots(k+n-5)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-4)!\cdot 2!}\Big] \\ \frac{1}{3!}\Big[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)^2(k+2)^2\cdots(k+n-5)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-4)!\cdot 3!}\Big] \end{cases}, \quad (5.12)$$

$$N_{m,n,n-3}(z,\zeta) = \sum \begin{cases} \Big[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^3(k+1)(k+2)\cdots(k+n-4)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-3)!}\Big] \\ -\frac{1}{2!}\Big[\sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)^2(k+2)\cdots(k+n-4)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-3)!\cdot 2!}\Big] \end{cases}, \quad (5.13)$$

$$N_{m,n,n-2}(z,\zeta) = \sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k^2(k+1)(k+2)\cdots(k+n-3)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-2)!}, \quad (5.14)$$

$$N_{m,n,n-1}(z,\zeta) = \sum_{k=2}^{\infty} \frac{\Delta_{m,k-1}(z,\zeta)}{k(k+1)(k+2)\cdots(k+n-2)} + \frac{\Delta_{m,0}(z,\zeta)}{(n-1)!}, \quad (5.15)$$

where

$$\begin{aligned} \Delta_{m,\ell}(z,\zeta) = -\frac{1}{[(m-1)!]^2\pi}\Big\{ & \sum_{\substack{1\le s<\infty \\ 0\le p,q\le m-1 \\ p=q+s+\ell}} \binom{m-1}{p}\binom{m-1}{q} s^{-1} \\ & \cdot [\overline{\zeta}^p \zeta^{q+s} z^\ell + \zeta^p \overline{\zeta}^{q+s}\overline{z}^\ell] \\ & + \sum_{\substack{1\le l\le m-1 \\ 0\le p,q\le m-1 \\ p=q+\ell}} \binom{m-1}{p}\binom{m-1}{q} l^{-1} \\ & \cdot 2[\overline{\zeta}^p \zeta^q z^\ell + \zeta^p \overline{\zeta}^q \overline{z}^\ell]\Big\}, \end{aligned} \quad (5.16)$$

$\ell = 0, 1, 2, \ldots$. *Moreover,* $G_{m,n}(z,\zeta) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} K_{m,m}(\tau-\zeta) g_n(z,\tau)\,\frac{d\tau}{\tau}$, $g_n(z,\tau)$ *is the nth higher-order Poisson kernel, then*

$$G_{m,n}(z,\zeta) = D_{m,1}(z,\zeta) + D_{m,2}(z,\zeta) + \cdots + D_{m,n-1}(z,\zeta), \quad (5.17)$$

where $D_{m,j}(z,\zeta) = (-1)^{n-j}\frac{1-|z|^{2j}}{j!} N_{m,n,j}(z,\zeta)$, $j = 1, 2, \ldots, n-1$. *In all above formulae, by convention,* $\prod_{\ell=\imath}^{\jmath}(k+\ell) = 1$ *as* $\imath > \jmath$.

Proof. By (3.6),

$$K_{m,m}(\tau-\zeta) = \frac{1}{[(m-1)!]^2\pi}|\tau-\zeta|^{2(m-1)}\Big[\log|\tau-\zeta|^2 - 2\sum_{l=1}^{m-1}\frac{1}{l}\Big]. \quad (5.18)$$

Noting (4.10)-(4.14), in order to get $G_{m,n}(z,\zeta)$, the key is to obtain

$$\Delta_{m,k-1}(z,\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}K_{m,m}(\tau-\zeta)d_{k-1}(z,\tau)\frac{\mathrm{d}\tau}{\tau},\ k\geq 2 \tag{5.19}$$

in which $d_{k-1}(z,\tau)=(z\overline{\tau})^{k-1}+(\overline{z}\tau)^{k-1}$ and

$$\Delta_{m,0}(z,\zeta)=\frac{1}{2\pi i}\int_{\partial\mathbb{D}}K_{m,m}(\tau-\zeta)\frac{\mathrm{d}\tau}{\tau}. \tag{5.20}$$

By Lemmas 5.2–5.3,

$$|\tau-\zeta|^{2(m-1)}\log|\tau-\zeta|^2=-\sum_{\substack{1\leq s<\infty\\ 0\leq p,q\leq m-1}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}\cdot[\zeta^{p+s}\overline{\zeta}^q\overline{\tau}^{p+s}\tau^q+\zeta^p\overline{\zeta}^{q+s}\overline{\tau}^p\tau^{q+s}], \tag{5.21}$$

$$|\tau-\zeta|^{2(m-1)}d_{k-1}(z,\tau)=\sum_{0\leq p,q\leq m-1}\binom{m-1}{p}\binom{m-1}{q}\cdot[\zeta^p\overline{\zeta}^q z^{k-1}\overline{\tau}^{p+k-1}\tau^q+\zeta^p\overline{\zeta}^q\overline{z}^{k-1}\overline{\tau}^p\tau^{q+k-1}] \tag{5.22}$$

and

$$|\tau-\zeta|^{2(m-1)}\log|\tau-\zeta|^2 d_{k-1}(z,\tau)=-\sum_{\substack{1\leq s<\infty\\ 0\leq p,q\leq m-1}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}\cdot[\zeta^{p+s}\overline{\zeta}^q z^{k-1}\overline{\tau}^{p+s+k-1}\tau^q+\zeta^p\overline{\zeta}^{q+s}\overline{z}^{k-1}\overline{\tau}^p\tau^{q+s+k-1}]. \tag{5.23}$$

Applying (5.18), (5.22)–(5.23), by Lemma 5.1, we have

$$\begin{aligned}\Delta_{m,k-1}(z,\zeta)=-\frac{1}{[(m-1)!]^2\pi}\Big\{&\sum_{\substack{1\leq s<\infty\\ 0\leq p,q\leq m-1\\ p=q+s+k-1}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}\\ &\cdot[\overline{\zeta}^p\zeta^{q+s}z^{k-1}+\zeta^p\overline{\zeta}^{q+s}\overline{z}^{k-1}]\\ &+\sum_{\substack{1\leq l\leq m-1\\ 0\leq p,q\leq m-1\\ p=q+k-1}}\binom{m-1}{p}\binom{m-1}{q}l^{-1}\\ &\cdot 2[\overline{\zeta}^p\zeta^q z^{k-1}+\zeta^p\overline{\zeta}^q\overline{z}^{k-1}]\Big\}.\end{aligned} \tag{5.24}$$

Applying (5.8), (5.18) and (5.21), by Lemma 5.1, we get

$$\Delta_{m,0}(z,\zeta) = -\frac{1}{[(m-1)!]^2\pi}\Big\{\sum_{\substack{1\le s<\infty\\0\le p,q\le m-1\\p=q+s}}\binom{m-1}{p}\binom{m-1}{q}s^{-1}$$

$$\cdot[\overline{\zeta}^p\zeta^{q+s}+\zeta^p\overline{\zeta}^{q+s}]$$

$$+\sum_{\substack{1\le l\le m-1\\0\le p\le m-1}}4\Big[\binom{m-1}{p}\Big]^2 l^{-1}|\zeta|^{2p}\Big\}. \tag{5.25}$$

Thus we complete the proof of this theorem. □

Remark 5.1. By Theorem 5.2, applying $G_{m,n}(z,\zeta)$, we can rewrite the unique solution of the problem (5.4) as

$$w(z) = \sum_{k=1}^{n}\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_{k-1}(\tau)g_k(z,\tau)\frac{d\tau}{\tau}$$

$$+\int_{\mathbb{D}}f(\zeta)\Big\{K_{n,n}(z-\zeta)+\sum_{k=1}^{n}G_{n+1-k,k}(z,\zeta)\Big\}d\xi d\eta. \tag{5.26}$$

Similarly, the double integrals appearing in what follows can easily be given in terms of $G_{m,n}(z,\zeta)$. To avoid technical difficulty, we will not repeat them again in the sequel.

Next, we consider two kinds of Dirichlet problems for the higher-order inhomogeneous complex mixed-partial differential equations of simplified form:

$$\begin{cases}(\partial_z^m\partial_{\overline{z}}^n)w = f, \ \ f\in L^p(\mathbb{D}), p>2, m>n,\\(\partial_z\partial_{\overline{z}})^j w = \gamma_j, \ \ \gamma_j\in C(\partial\mathbb{D}), 0\le j<n,\\(\partial_z^{n+k}\partial_{\overline{z}}^n)w = \sigma_k, \ \ \sigma_k\in C(\partial\mathbb{D}), 0\le k<m-n\end{cases} \tag{5.27}$$

and

$$\begin{cases}(\partial_z^m\partial_{\overline{z}}^n)w = f, \ \ f\in L^p(\mathbb{D}), p>2,\\(\partial_z^m\partial_{\overline{z}}^j)w = \chi_j, \ \ \chi_j\in C(\partial\mathbb{D}), 0\le j<n,\\(\partial_z^k\partial_{\overline{z}}^n)w = \lambda_k, \ \ \lambda_k\in C(\partial\mathbb{D}), 0\le k<m.\end{cases} \tag{5.28}$$

By Theorems 4.4–4.5, (5.1) and (5.2), we have:

Theorem 5.3. *Set*

$$A(t) = \begin{pmatrix} n! & (n+1)!t & \cdots & \frac{(m-2)!}{(m-n-2)!}t^{m-n-2} & \frac{(m-1)!}{(m-n-1)!}t^{m-n-1}\\ 0 & (n+1)! & \cdots & \frac{(m-2)!}{(m-n-3)!}t^{m-n-3} & \frac{(m-1)!}{(m-n-2)!}t^{m-n-2}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & (m-2)! & (m-1)!t\\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix}, \tag{5.29}$$

$$a^*(t) = \begin{pmatrix} \sigma_0(t) - T_{m-n,0,\mathbb{D}}f(t) \\ \sigma_1(t) - T_{m-n-1,0,\mathbb{D}}f(t) \\ \vdots \\ \sigma_{m-n-2}(t) - T_{2,0,\mathbb{D}}f(t) \\ \sigma_{m-n-1}(t) - T_{1,0,\mathbb{D}}f(t) \end{pmatrix}, \tag{5.30}$$

$$\Xi_l^*(z) = \frac{1}{n!(n+1)!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{\det(A_l^*(\tau))}}{\tau - z}\mathrm{d}\tau, \tag{5.31}$$

and

$$\widetilde{\varphi}_l^*(z) = \int_0^z \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_1} \Xi_l^*(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1} + \pi_l^*(z), \tag{5.32}$$

where $t \in \partial\mathbb{D}$, $\pi_l^* \in \Pi_{n-1}$, *the matrix* $A_l^*(t)$ *is given by replacing the* l*th column of* $A(t)$ *by* $a^*(t)$, $0 \le l \le m-n-1$. *Then*

$$w(z) = \sum_{k=1}^{n} \frac{1}{2\pi i}\int_{\partial\mathbb{D}} g_k(z,\tau)\Big[\gamma_{k-1}^*(\tau) - \sum_{l=0}^{m-n-1} \tfrac{(n+l)!}{(n+l-k+1)!}\tau^{n+l-k+1}\overline{\partial_z^{k-1}\widetilde{\varphi}_l^*(\tau)}\Big]\frac{\mathrm{d}\tau}{\tau}$$
$$+ z^n \sum_{l=0}^{m-n-1} z^l\overline{\widetilde{\varphi}_l^*(z)} + T_{m,n,\mathbb{D}}f(z) \tag{5.33}$$

are all solutions of the problem (5.27) *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\det A_l^*(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0, \ z \in \mathbb{D}, \tag{5.34}$$

where $\gamma_{k-1}^*(\tau) = \gamma_{k-1}(\tau) - T_{m+1-k,n+1-k,\mathbb{D}}f(\tau)$, $g_k(z,\tau)\,(1 \le k \le n)$ *are the former* n *higher-order Poisson kernels.*

Theorem 5.4. *Set*

$$B(t) = \begin{pmatrix} 1 & \overline{t} & \overline{t}^2 & \cdots & \overline{t}^{n-1} \\ 0 & 1 & 2\overline{t} & \cdots & (n-1)\overline{t}^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-2)! & (n-1)!\overline{t} \\ 0 & 0 & \cdots & 0 & (n-1)! \end{pmatrix}, \tag{5.35}$$

$$C(t) = \begin{pmatrix} 1 & t & t^2 & \cdots & t^{m-1} \\ 0 & 1 & 2t & \cdots & (m-1)t^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (m-2)! & (m-1)!t \\ 0 & 0 & \cdots & 0 & (m-1)! \end{pmatrix}, \tag{5.36}$$

$$b^*(t) = \begin{pmatrix} \chi_0(t) - T_{0,n,\mathbb{D}}f(t) \\ \chi_1(t) - T_{0,n-1,\mathbb{D}}f(t) \\ \vdots \\ \chi_{n-1}(t) - T_{0,1,\mathbb{D}}f(t) \end{pmatrix}, \quad c^*(t) = \begin{pmatrix} \lambda_0(t) - T_{m,0,\mathbb{D}}f(t) \\ \lambda_1(t) - T_{m-1,0,\mathbb{D}}f(t) \\ \vdots \\ \lambda_{m-1}(t) - T_{1,0,\mathbb{D}}f(t) \end{pmatrix} \tag{5.37}$$

and

$$\Theta_p^*(z) = \frac{1}{1!2!\cdots(n-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\det B_p^*(\tau)}{\tau - z}\,\mathrm{d}\tau, \tag{5.38}$$

$$\Lambda_q^*(z) = \frac{1}{1!2!\cdots(m-1)!}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{\det C_q^*(\tau)}}{\tau - z}\,\mathrm{d}\tau, \tag{5.39}$$

as well as

$$\mu_p^*(z) = \int_0^z \int_0^{\zeta_{m-1}} \cdots \int_0^{\zeta_1} \Theta_p^*(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{m-1} + \kappa_p^*(z), \tag{5.40}$$

$$\nu_q^*(z) = \int_0^z \int_0^{\zeta_{n-1}} \cdots \int_0^{\zeta_1} \Lambda_q^*(\zeta)\mathrm{d}\zeta\mathrm{d}\zeta_1\cdots\mathrm{d}\zeta_{n-1} + \xi_q^*(z), \tag{5.41}$$

where $t \in \partial\mathbb{D}$, $\kappa_p^* \in \Pi_{m-1}$, $\xi_q^* \in \Pi_{n-1}$, matrices $B_p^*(t), C_q^*(t)$ are respectively given by replacing the pth, qth column of $B(t), C(t)$ by $b^*(t)$, $c^*(t)$, $0 \le p \le n-1$, $0 \le q \le m-1$. Then

$$w(z) = T_{m,n,\mathbb{D}}f(z) + \sum_{p=0}^{n-1} \overline{z}^p \mu_p(z) + \sum_{q=0}^{m-1} z^q \overline{\nu_q(z)} \tag{5.42}$$

are all solutions of the problem (5.28) *if and only if*

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\overline{\det B_p^*(\tau)}}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0, \quad \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{z\det C_q^*(\tau)}{\tau - z}\frac{\mathrm{d}\tau}{\tau} = 0, \tag{5.43}$$

in which $z \in \mathbb{D}$.

Proofs of Theorems 5.3–5.4. From (5.1) and (5.2), the problems (5.27) and (5.28) are respectively equivalent to the following ones:

$$\begin{cases} w - T_{m,n,\mathbb{D}}f \in M_{m,n}(\mathbb{D}), \ \ f \in L^p(\mathbb{D}), p > 2, m > n, \\ (\partial_z\partial_{\overline{z}})^j[w - T_{m,n,\mathbb{D}}f] = \gamma_j - T_{m-j,n-j,\mathbb{D}}f, \ \gamma_j \in C(\partial\mathbb{D}), 0 \le j < n, \\ (\partial_z^{n+k}\partial_{\overline{z}}^n)[w - T_{m,n,\mathbb{D}}f] = \sigma_k - T_{m-n-k,0,\mathbb{D}}f, \ \sigma_k \in C(\partial\mathbb{D}), 0 \le k < m-n \end{cases} \tag{5.44}$$

and

$$\begin{cases} w - T_{m,n,\mathbb{D}}f \in M_{m,n}(\mathbb{D}), \ \ f \in L^p(\mathbb{D}), p > 2, \\ (\partial_z^m\partial_{\overline{z}}^j)[w - T_{m,n,\mathbb{D}}f] = \chi_j - T_{0,n-j,\mathbb{D}}f, \ \chi_j \in C(\partial\mathbb{D}), 0 \le j < n, \\ (\partial_z^k\partial_{\overline{z}}^n)[w - T_{m,n,\mathbb{D}}f] = \lambda_k - T_{m-k,0,\mathbb{D}}f, \ \lambda_k \in C(\partial\mathbb{D}), 0 \le k < m. \end{cases} \tag{5.45}$$

So, by Theorems 4.4–4.5, we complete the proofs of Theorems 5.3–5.4. □

Remark 5.2. All and the same, we can consider the following Dirichlet problem for inhomogeneous complex mixed-partial differential equations:

$$\begin{cases} (\partial_z^m\partial_{\overline{z}}^n)w = f, \ \ f \in L^p(\mathbb{D}), p > 2, m < n, \\ (\partial_z\partial_{\overline{z}})^j w = \rho_j, \ \rho_j \in C(\partial\mathbb{D}), 0 \le j < m, \\ (\partial_z^m\partial_{\overline{z}}^{m+k})w = \varrho_k, \ \varrho_k \in C(\partial\mathbb{D}), 0 \le k < n-m. \end{cases} \tag{5.46}$$

Noting Remark 4.3, similar as Theorem 5.3, it is easy to solve the problem (5.46).

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H. Begehr
Institute of Mathematics
Freie Universität Berlin
Arnimallee 3
14195 Berlin
Germany
e-mail: `begehrh@zedat.fu-berlin.de`

Zhihua Du
Department of Mathematics
Jinan University
Guangzhou 510632
China
e-mail: `tzhdu@jnu.edu.cn`

Ning Wang
School of Mathematics and Statistics
Wuhan University
Wuhan 430072
China
e-mail: `wninglily@yahoo.com.cn`

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Dirichlet Problems for the Generalized n-Poisson Equation

Ü. Aksoy and A.O. Çelebi

Abstract. Polyharmonic hybrid Green functions, obtained by convoluting polyharmonic Green and Almansi Green functions, are taken as kernels to define a hierarchy of integral operators. They are used to investigate the solvability of some types of Dirichlet problems for linear complex partial differential equations with leading term as the polyharmonic operator.

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1. Introduction

The Dirichlet problem is one of the basic boundary value problems in complex analysis. This type of problems are investigated in many articles for homogeneous and inhomogeneous Cauchy–Riemann equations, for higher-order Poisson equations in the unit disc of the complex plane [9, 10, 3, 13, 12, 8, 6, 11], in the half plane [7] and in the circular rings [4, 15]. In this work, some types of Dirichlet problems are considered for the inhomogeneous linear complex partial differential equations in which leading terms are the polyharmonic operators; we call them as the generalized n-Poisson equations. In Section 2 and 3, we review the harmonic, biharmonic and polyharmonic Green and hybrid Green functions with their properties and corresponding Dirichlet problems for complex model equations. In Section 4, we introduce a new class of singular integral operators and derive their properties. These properties are employed in Section 5 to obtain the corresponding singular integral equations for the hybrid Dirichlet problems of the generalized n-Poisson equation. Afterwards we discuss the solvability of the problems using Fredholm alternative.

2. Preliminaries

In this section, we review the harmonic and biharmonic Green functions with their properties and the related Dirichlet problems for Poisson and bi-Poisson equation.

In the unit disc $\mathbb{D}$ of the complex plane, the harmonic Green function is defined as

$$G_1(z,\zeta) = \log \left| \frac{1-z\bar{\zeta}}{\zeta - z} \right|^2 .$$

The properties [10, 11] of the harmonic Green function are given by

- $G_1(z,\zeta)$ is harmonic in $\mathbb{D}\backslash\{\zeta\}$ for any $\zeta \in \mathbb{D}$,
- $G_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in $z \in \mathbb{D}$ for any $\zeta \in \mathbb{D}$,
- $G_1(z,\zeta) = 0$ on $\partial\mathbb{D}$ for any $\zeta \in \mathbb{D}$.

It is a symmetric function, i.e., $G_1(z,\zeta) = G_1(\zeta, z)$ holds, [9]. $G_1(z,\zeta)$ is related to the following Dirichlet problem for Poisson equation [10].

Theorem 2.1. *The Dirichlet problem*

$$w_{z\bar{z}} = f \text{ in } \mathbb{D}, \; w = \gamma \text{ on } \partial\mathbb{D}, \; f \in L^1(\mathbb{D}) \cap C(\mathbb{D}), \; \gamma \in C(\partial\mathbb{D})$$

is uniquely solvable. The solution is

$$w(z) = -\frac{1}{4\pi i} \int_{\mathbb{D}} \partial_{\nu_\zeta} G_1(z,\zeta)\gamma(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\iint_{\mathbb{D}} G_1(z,\zeta) f(\zeta) d\xi d\eta.$$

A biharmonic Green function is obtained explicitly by convoluting the harmonic Green functions, [9, 10].

$$\begin{aligned} G_2(z,\zeta) &= -\frac{1}{\pi}\iint_{\mathbb{D}} G_1(z,\tilde{\zeta}) G_1(\tilde{\zeta},\zeta) d\tilde{\xi} d\tilde{\eta}, \\ &= |\zeta - z|^2 \log \left| \frac{1-z\bar{\zeta}}{\zeta - z}\right|^2 + (1-|z|^2)(1-|\zeta|^2)\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1-z\bar{\zeta})}{\bar{z}\zeta}\right]. \end{aligned}$$

It satisfies

$$\partial_z\partial_{\bar{z}} G_2(z,\zeta) = G_1(z,\zeta) \text{ in } \mathbb{D}, \; G_2 = 0 \;, \; \partial_z\partial_{\bar{z}} G_2(z,\zeta) = 0 \text{ on } \partial\mathbb{D} \text{ for } \zeta \in \mathbb{D}.$$

It is related to the following problem for the bi-Poisson equation [11].

Theorem 2.2. *The Dirichlet problem*

$$(\partial_z \partial\bar{z})^2 w = f \text{ in } \mathbb{D}, w = \gamma_0, w_{z\bar{z}} = \gamma_2 \text{ on } \partial\mathbb{D}$$

is uniquely solvable for $f \in L^1(\mathbb{D}) \cap C(\mathbb{D}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D})$ by

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} [g_1(z,\zeta)\gamma_0(\zeta) + \hat{g}_2(z,\zeta)\gamma_2(\zeta)]\frac{d\zeta}{\zeta} - \frac{1}{\pi}\iint_{\mathbb{D}} G_2(z,\zeta) f(\zeta) d\xi d\eta$$

where

$$g_1(z,\zeta) = \frac{1}{1-z\bar{\zeta}} + \frac{1}{1-\bar{z}\zeta} - 1 = \frac{1-|z|^2|\zeta|^2}{|1-z\bar{\zeta}|^2}$$

is the Poisson kernel for $\mathbb{D}$ *and*

$$\hat{g}_2(z,\zeta) = -\frac{1}{\pi}\int\limits_{\mathbb{D}} G_1(z,\tilde{\zeta})g_1(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta} = (1-|z|^2)[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta} + 1].$$

Another bi-harmonic Green function [1] for $\mathbb{D}$ is

$$\tilde{G}_2(z,\zeta) = |\zeta - z|^2 \log\left|\frac{1-z\bar{\zeta}}{\zeta - z}\right|^2 - (1-|z|^2)(1-|\zeta|^2) \tag{2.1}$$

satisfying [10]

- $\partial_z\partial_{\bar{z}}\tilde{G}_2(z,\zeta) = G_1(z,\zeta) - g_1(z,\zeta)(1-|\zeta|^2)$ in $\mathbb{D}$ for $\zeta \in \mathbb{D}$,
- $\tilde{G}_2(z,\zeta) = 0,\ \partial_{\nu_z} G_2(z,\zeta) = 0$ on $\partial\mathbb{D}$ for $\zeta \in \mathbb{D}$.

$\tilde{G}_2$ is related to the following type of Dirichlet problem for the bi-Poisson equation [9].

Theorem 2.3. *The Dirichlet problem*

$$(\partial_z\partial_{\bar{z}})^2 w = f \text{ in } \mathbb{D}, \quad w = \gamma_0, \quad \partial_\nu w = \gamma_1 \text{ on } \mathbb{D}$$

is uniquely solvable for $f \in L^1(\mathbb{D}) \cap C(\mathbb{D}), \gamma_0 \in C^2(\partial\mathbb{D}), \gamma_1 \in C^1(\partial\mathbb{D})$ *by*

$$\begin{aligned} w(z) = {} & \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\Big[g_1(z,\zeta)(1+|z|^2) + g_2(z,\zeta)(1-|z|^2)\Big]\gamma_0(\zeta)\frac{d\zeta}{\zeta} \\ & - \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} g_1(z,\zeta)(1-|z|^2)\gamma_1(\zeta)\frac{d\zeta}{\zeta} \\ & - \frac{1}{\pi}\iint_{\mathbb{D}} \tilde{G}_2(z,\zeta)f(\zeta)d\xi d\eta \end{aligned}$$

where

$$g_2(z,\zeta) = \frac{1}{(1-z\bar{\zeta})^2} + \frac{1}{(1-\bar{z}\zeta)^2} - 1\ .$$

3. Review of Polyharmonic Green Functions and Dirichlet Problems for the n-Poisson Equation

In this section we will review the generalizations of the properties given in the above section. A polyharmonic Green function G_n is given iteratively by

$$G_n(z,\zeta) = -\frac{1}{\pi}\iint_{\mathbb{D}} G_1(z,\tilde{\zeta})G_{n-1}(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

for $n \geq 2$. It has the properties [3]

- $G_n(z,\zeta)$ is polyharmonic of order n in $\mathbb{D}\backslash\{\zeta\}$ for any $\zeta \in \mathbb{D}$,
- $G_n(z,\zeta) + \dfrac{|\zeta - z|^{2(n-1)}}{(n-1)!^2}\log|\zeta - z|^2$ is polyharmonic of order n for $z \in \mathbb{D}$ for any $\zeta \in \mathbb{D}$,
- $(\partial_z\partial_{\bar{z}})^\mu G_n(z,\zeta) = 0$ for $0 \leq \mu \leq n-1$ on $\partial\mathbb{D}$ for any $\zeta \in \mathbb{D}$,

- $G_n(z,\zeta) = G_n(\zeta, z)$, for any $z, \zeta \in \mathbb{D}$.

These functions are related to the following n-Dirichlet problem for the higher-order Poisson equation [3].

Theorem 3.1. *The Dirichlet problem*

$$(\partial_z\partial_{\bar z})^n w = f \quad \text{in} \quad \mathbb{D}, \; (\partial_z\partial_{\bar z})^\mu w = \gamma_\mu, \; 0 \le \mu \le n-1 \quad \text{on} \quad \partial\mathbb{D}$$

with $f \in L^1(\mathbb{D}) \cap C(\mathbb{D}), \gamma_\mu \in C(\partial\mathbb{D}),\; 0 \le \mu \le n-1$ *is uniquely solvable. The solution is*

$$w(z) = -\sum_{\mu=1}^{n} \frac{1}{4\pi i} \int_{\partial\mathbb{D}} \partial_{\nu_\zeta} G_\mu(z,\zeta)\gamma_{\mu-1}(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\iint_{\mathbb{D}} G_n(z,\zeta) f(\zeta) d\xi d\eta.$$

The generalization of (2.1) is the Green–Almansi function $\tilde G_n$ given by

$$\begin{aligned}\tilde G_n(z,\zeta) = {} & \frac{|\zeta - z|^{2(n-1)}}{(n-1)!^2} \log\left|\frac{1-z\bar\zeta}{\zeta - z}\right|^2 \\ & - \sum_{\mu=1}^{n-1} \frac{1}{\mu(n-1)!^2} |\zeta - z|^{2(n-1-\mu)}(1-|z|^2)^\mu(1-|\zeta|^2)^\mu \end{aligned} \tag{3.1}$$

with the properties [3]

- $\tilde G_n(z,\zeta)$ is polyharmonic of order n in $\mathbb{D}\backslash\{\zeta\}$ for any $\zeta \in \mathbb{D}$,
- $\tilde G_n(z,\zeta) - \dfrac{|\zeta - z|^{2(n-1)}}{(n-1)!^2} \log|\zeta - z|^2$ is polyharmonic of order n for $z \in \mathbb{D}$ for any $\zeta \in \mathbb{D}$,
- $(\partial_z\partial_{\bar z})^\mu \tilde G_n(z,\zeta) = 0$ for $0 \le 2\mu \le n-1$ on $\partial\mathbb{D}$ for any $\zeta \in \mathbb{D}$,
- $\partial_{\nu_z}(\partial_z\partial_{\bar z})^\mu \tilde G_n(z,\zeta) = 0$ for $0 \le 2\mu \le n-2$ on $\partial\mathbb{D}$ for any $\zeta \in \mathbb{D}$,
- $\tilde G_n(z,\zeta) = \tilde G_n(\zeta, z)$ for any $z, \zeta \in \mathbb{D}$,
- $\tilde G_1(z,\zeta) = G_1(z,\zeta)$.

$\tilde G_n(z,\zeta)$ is used to solve the following n-Dirichlet problem for the n-Poisson equation [3].

Theorem 3.2. *The Dirichlet problem*

$$(\partial_z\partial_{\bar z})^n w = f \quad \text{in} \quad \mathbb{D},$$

subject to conditions

$$(\partial_z\partial_{\bar z})^\mu w = \gamma_\mu, \; 0 \le 2\mu \le n-1 \quad \partial_{\nu_z}(\partial_z\partial_{\bar z})^\mu w = \hat\gamma_\mu, \; 0 \le 2\mu \le n-2 \quad \text{on} \quad \partial\mathbb{D}$$

for $f \in L^1(\mathbb{D}) \cap C(\mathbb{D})$, $\gamma_\mu \in C^{n-2\mu}(\partial\mathbb{D})$, $0 \le 2\mu \le n-1$, $\hat{\gamma}_\mu \in C^{n-1-2\mu}(\partial\mathbb{D})$, $0 \le 2\mu \le n-2$ *is uniquely solvable. The solution is*

$$\begin{aligned} w(z) = &-\sum_{\mu=0}^{[\frac{n}{2}]-1} \frac{1}{4\pi i} \int_{\partial\mathbb{D}} \partial_{\nu_\zeta}(\partial_\zeta\partial_{\bar\zeta})^{n-\mu-1}\tilde{G}_n(z,\zeta)\gamma_\mu(\zeta)\frac{d\zeta}{\zeta} \\ &+ \sum_{\mu=0}^{[\frac{n-1}{2}]} \frac{1}{4\pi i} \int_{\partial\mathbb{D}} (\partial_\zeta\partial_{\bar\zeta})^{n-\mu-1}\tilde{G}_n(z,\zeta)\hat{\gamma}_\mu(\zeta)\frac{d\zeta}{\zeta} \\ &- \frac{1}{\pi}\iint_{\mathbb{D}} \tilde{G}_n(z,\zeta) f(\zeta) d\xi d\eta. \end{aligned}$$

The Green-m-Green Almansi-n function $G_{m,n}(z,\zeta)$ for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ (which is also called a polyharmonic hybrid Green function) is defined by the convolution of G_m and $\tilde{G}_n$ as

$$G_{m,n}(z,\zeta) = -\frac{1}{\pi}\iint_{\mathbb{D}} G_m(z,\tilde{\zeta})\tilde{G}_n(\tilde{\zeta},\zeta) d\tilde{\xi} d\tilde{\eta}\ .$$

Note that, $G_{m,1}(z,\zeta) = G_{m+1}(z,\zeta)$ for $m \in \mathbb{N}$ and we take $G_{0,n}(z,\zeta) = \tilde{G}_n(z,\zeta)$ for $n \in \mathbb{N}$. Thus, $G_{0,1}(z,\zeta) = G_1(z,\zeta)$. $G_{m,n}(z,\zeta)$ has the following properties:

- $G_{m,n}(z,\zeta)$ is polyharmonic of order $m+n$ in $\mathbb{D}\backslash\{\zeta\}$ for any $\zeta \in \mathbb{D}$,
- $G_{m,n}(z,\zeta) - \dfrac{|\zeta - z|^{2(m+n-1)}}{(m+n-1)!^2} \log|\zeta - z|^2$ is polyharmonic of order $m+n$ for $z \in \mathbb{D}$ for any $\zeta \in \mathbb{D}$,
- $(\partial_z\partial_{\bar z})^t G_{m,n}(z,\zeta) = G_{m-t,n}(z,\zeta)$ in $\mathbb{D}$ if $t \le m$,
- $(\partial_z\partial_{\bar z})^\mu G_{m,n}(z,\zeta) = 0$ for $0 \le \mu \le m-1$ on $\partial\mathbb{D}$,
  $(\partial_z\partial_{\bar z})^{\mu+m} G_{m,n}(z,\zeta) = 0$ for $0 \le \mu \le [\frac{n-1}{2}]$ on $\partial\mathbb{D}$,
  $\partial_{\nu_z}(\partial_z\partial_{\bar z})^{\mu+m} G_{m,n}(z,\zeta) = 0$ for $0 \le \mu \le [\frac{n}{2}]-1$ on $\partial\mathbb{D}$ for any $\zeta \in \mathbb{D}$,
- $(\partial_\zeta\partial_{\bar\zeta})^n G_{m,n}(z,\zeta) = G_m(z,\zeta)$ for any $z \in \mathbb{D}$,
- $(\partial_\zeta\partial_{\bar\zeta})^\mu G_{m,n}(z,\zeta) = 0$ for $0 \le \mu \le [\frac{n-1}{2}]$ on $\partial\mathbb{D}$,
  $\partial_{\nu_\zeta}(\partial_\zeta\partial_{\bar\zeta})^\mu G_{m,n}(z,\zeta) = 0$ for $0 \le \mu \le [\frac{n}{2}]-1$ on $\partial\mathbb{D}$,
  $(\partial_\zeta\partial_{\bar\zeta})^{\mu+n} G_{m,n}(z,\zeta) = 0$ for $0 \le \mu \le m-1$ on $\partial\mathbb{D}$ for any $z \in \mathbb{D}$.

It can be easily seen that $G_{m,n}(z,\zeta)$ is not symmetric in its variables and is employed in the following (m,n)-type Dirichlet problem.

Theorem 3.3. *The* (m,n)*-Dirichlet problem*

$$(\partial_z\partial_{\bar z})^n w = f \quad \text{in} \quad \mathbb{D},$$

$$(\partial_z\partial_{\bar z})^\mu w = 0,\ 0 \le \mu \le m-1 \quad \text{on} \quad \partial\mathbb{D},$$

$$(\partial_z\partial_{\bar z})^{\mu+m} w = 0,\ 0 \le 2\mu \le n-m-1 \quad \text{on} \quad \partial\mathbb{D},$$

$$\partial_{\nu_z}(\partial_z\partial_{\bar z})^{\mu+m} w = 0,\ 0 \le 2\mu \le n-m-2 \quad \text{on} \quad \partial\mathbb{D}$$

for $f \in L^1(\mathbb{D}) \cap C(\mathbb{D})$, *is uniquely solvable. The solution is*

$$w(z) = -\frac{1}{\pi} \iint_{\mathbb{D}} G_{m,n-m}(z,\zeta) f(\zeta) d\xi d\eta.$$

Note. This type of a theorem may also be given in the case of nonzero boundary conditions but computations take very large space.

4. A Class of Integral Operators Related to Dirichlet Problems

In this section, using $G_{m,n}(z,\zeta)$ and its derivatives with respect to z and $\bar{z}$ as the kernels, we define a class of integral operators related to (m,n)-type Dirichlet problems.

Definition 4.1. For $m,k,l \in \mathbb{N}_0$, $n \in \mathbb{N}$ with $(k,l) \neq (m+n, m+n)$ and $k+l \leq 2(m+n)$, we define

$$G^{k,l}_{m,n} f(z) = -\frac{1}{\pi} \iint_{\mathbb{D}} \partial_z^k \partial_{\bar{z}}^l G_{m,n}(z,\zeta) f(\zeta) d\xi d\eta$$

for a suitable complex-valued function f given in $\mathbb{D}$.

It is easy to observe that the operators $G^{k,l}_{m,n}$ are weakly singular for $k+l < 2(m+n)$ and strongly singular for $k+l = 2(m+n)$. Using the above definition we can obtain the following operators by some particular choices of m,n,k and l:

$$G^{0,0}_{0,1} f(z) = -\frac{1}{\pi} \iint_{\mathbb{D}} G_1(z,\zeta) f(\zeta) d\xi d\eta = -\frac{1}{\pi} \iint_{\mathbb{D}} \log \left| \frac{1-z\bar{\zeta}}{\zeta - z} \right|^2 f(\zeta) d\xi d\eta \ ,$$

$$\begin{aligned} G^{1,0}_{0,1} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} \partial_z G_1(z,\zeta) f(\zeta) d\xi d\eta \\ &= -\frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{1}{(\zeta - z)} - \frac{\bar{\zeta}}{(1 - z\bar{\zeta})} \right) f(\zeta) d\xi d\eta \ , \end{aligned}$$

$$\begin{aligned} G^{2,0}_{0,1} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} \partial_z^2 G_1(z,\zeta) f(\zeta) d\xi d\eta \\ &= -\frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{1}{(\zeta - z)^2} - \frac{\bar{\zeta}^2}{(1 - z\bar{\zeta})^2} \right) f(\zeta) d\xi d\eta \ . \end{aligned}$$

Thus, $G^{0,0}_{0,1}$, $G^{1,0}_{0,1}$ and $G^{2,0}_{0,1}$ are the operators Π_0, Π_1 and Π_2 which are investigated by Vekua [16]. It can be shown that these operators satisfy

$$\partial_z G^{0,0}_{0,1} f = G^{1,0}_{0,1} f \ \text{ and } \ \partial_z^2 G^{0,0}_{0,1} f = G^{2,0}_{0,1} f \tag{4.1}$$

for $f \in L^p(\mathbb{D})$, $p > 2$, in Sobolev's sense.

4.1. Properties of the Operators $G^{k,l}_{m,n}f$

The following lemmas and corollary will be proved by the use of the techniques given in [2] and the properties of the operators having the harmonic Green functions discussed in [16]. For the sake of completeness, we will sketch the proofs. First, we will give some properties of the operators $G^{0,0}_{0,1}$, $G^{1,0}_{0,1}$ and $G^{2,0}_{0,1}$.

The following lemma is proved in [16, p. 337–340].

Lemma 4.2. *For $f \in L^p(\mathbb{D})$ where $p > 2$,*

$$\left| G^{k,0}_{0,1} f(z) \right| \le C(k,p) \|f\|_{L^p(\mathbb{D})} \tag{4.2}$$

for $k = 0, 1$,

$$\left| G^{k,0}_{0,1} f(z_1) - G^{k,l}_{0,1} f(z_2) \right| \le C(k,p) \|f\|_{L^p(\mathbb{D})} \begin{cases} |z_1 - z_2|^{(p-2)/p} & \text{if } k = 1, \\ |z_1 - z_2| & \text{if } k = 0, \end{cases} \tag{4.3}$$

for z_1, $z_2 \in \mathbb{D}$ and

$$\|G^{2,0}_{0,1} f\|_{L^p(\mathbb{D})} \le C(p) \|f\|_{L^p(\mathbb{D})} \tag{4.4}$$

for $p > 1$. Moreover

$$\|G^{2,0}_{0,1} f\|_{L^2(\mathbb{D})} \le \|f\|_{L^2(\mathbb{D})} \tag{4.5}$$

holds.

Now we give a property of $G^{k,l}_{m,n}$ to simplify the discussions.

Lemma 4.3. *For $f \in L^p(\mathbb{D})$,*

$$G^{k,l}_{m,n} f(z) = \begin{cases} G^{k-l,0}_{m-l,n} f(z), & k \ge l, \\ G^{0,l-k}_{m,n-k} f(z), & k < l, \end{cases}$$

for a suitable p. Moreover

$$G^{k,l}_{m,n} f(z) = \overline{G^{l,k}_{m,n} \overline{f(z)}} =: \overline{G^{k,l}_{m,n}} f(z) \ . \tag{4.6}$$

Proof. For $k \ge l$, the polyharmonic hybrid Green function $G_{m,n}(z,\zeta)$ satisfies

$$\begin{aligned} \partial_z^k \partial_{\bar z}^l G_{m,n}(z,\zeta) &= \partial_z^{k-l} \partial_{\bar z}^l \partial_z^l G_{m,n}(z,\zeta) \\ &= \partial_z^{k-l} G_{m-l,n}(z,\zeta). \end{aligned}$$

Therefore, using (4.1) we have $G^{k,l}_{m,n} f(z) = G^{k-l,0}_{m-l,n} f(z)$.

For $k < l$, the similar arguments apply. The relation

$$\overline{\partial_z^k \partial_{\bar z}^l G_{m,n}(z,\zeta)} = \partial_z^l \partial_{\bar z}^k G_{m,n}(z,\zeta)$$

proves (4.6). □

From now on, we will give the properties of $G^{k,l}_{m,n}$ for $l = 0$, $0 \le k \le 2(m+n)$ without loss of generality. Using Lemma 4.3, similar properties can be obtained for the operators $G^{k,l}_{m,n}$ with $l \ne 0$.

Lemma 4.4. *For $f \in L^p(\mathbb{D})$, $p > 2$*

$$G^{0,0}_{m,n} f(z) = G^{0,0}_{0,1}(G^{0,0}_{m-1,n}) f(z) \, , \tag{4.7}$$

$$G^{1,0}_{m,n} f(z) = \partial_z G^{0,0}_{m,n} f(z) = G^{1,0}_{0,1}(G^{0,0}_{m-1,n}) f(z) \, , \tag{4.8}$$

$$G^{2,0}_{m,n} f(z) = \partial_z^2 G^{0,0}_{m,n} f(z) = G^{2,0}_{0,1}(G^{0,0}_{m-1,n}) f(z) \tag{4.9}$$

hold.

Proof. The operator $G^{0,0}_{m,n}$ is given by

$$\begin{aligned} G^{0,0}_{m,n} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} G_{m,n}(z,\zeta) f(\zeta) d\xi d\eta \\ &= -\frac{1}{\pi} \iint_{\mathbb{D}} \left(-\frac{1}{\pi} \iint_{\mathbb{D}} G_{0,1}(z,\tilde{\zeta}) G_{m-1,n}(\tilde{\zeta},\zeta) d\tilde{\xi} d\tilde{\eta} \right) f(\zeta) d\xi d\eta \\ &= G^{0,0}_{0,1}(G^{0,0}_{m-1,n}) f(z). \end{aligned}$$

Thus inductively we get (4.7). To prove (4.8) and (4.9) we use the fact that $G^{0,0}_{0,1} f$ has generalized derivatives given by (4.1). □

Lemma 4.5. *For $k \in \mathbb{N}$, if $f \in W^{k,p}(\mathbb{D})$, then*

$$\partial_z^{k-1} G^{2,0}_{0,1} f(z) = G^{1,0}_{0,1}((D - D_*)^k f(z)) \tag{4.10}$$

and $\partial_z^{k-1} G^{2,0}_{0,1}$ is in $L^p(\mathbb{D})$ where $Df(z) = \partial_z f(z)$, $D_* f(z) = \partial_{\bar{z}}(\bar{z}^2 f(z))$.

Proof. We can rewrite $G^{2,0}_{0,1} f$ as

$$\begin{aligned} G^{2,0}_{0,1} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{1}{(\zeta - z)^2} - \frac{\bar{\zeta}^2}{(1 - z\bar{\zeta})^2} \right) f(\zeta) d\xi d\eta \\ &= -\frac{1}{\pi} \iint_{\mathbb{D}} \left[-\frac{\partial}{\partial \zeta} \left(\left(\frac{1}{\zeta - z} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}} \right) f(\zeta) \right) \right. \\ &\quad + \frac{\partial}{\partial \bar{\zeta}} \left(\left(\frac{1}{\zeta - z} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}} \right) \bar{\zeta}^2 f(\zeta) \right) + \left(\frac{1}{\zeta - z} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}} \right) \frac{\partial f}{\partial \zeta} \\ &\quad \left. - \left(\frac{1}{\zeta - z} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}} \right) \frac{\partial}{\partial \bar{\zeta}} (\bar{\zeta}^2 f(\zeta)) \right] d\xi d\eta \end{aligned}$$

It follows from Green's theorem and $-\bar{\zeta} d\zeta = \zeta d\bar{\zeta}$ that

$$\begin{aligned} G^{2,0}_{0,1} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{1}{\zeta - z} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}} \right) \frac{\partial f}{\partial \zeta} d\xi d\eta \\ &\quad + \frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{1}{\zeta - z} - \frac{\bar{\zeta}}{1 - z\bar{\zeta}} \right) \frac{\partial (\bar{\zeta}^2 f)}{\partial \bar{\zeta}} d\xi d\eta \end{aligned}$$

or

$$G_{0,1}^{2,0}f(z) = G_{0,1}^{1,0}((D - D_*)f(z)) \tag{4.11}$$

that corresponds to the case $k = 1$. Using (4.11) we have

$$\begin{aligned}\partial_z G_{0,1}^{2,0}f(z) &= \partial_z G_{0,1}^{1,0}((D - D_*)f(z)) \\ &= G_{0,1}^{1,0}((D - D_*)^2 f(z))\end{aligned}$$

and differentiating iteratively, we get (4.10). □

Corollary 4.6. *If*

$$f \in \begin{cases} L^p(\mathbb{D}), & 1 \le k \le 2m-1, \\ W^{k+1-2m,p}(\mathbb{D}), & k \ge 2m, \end{cases}$$

then

$$G_{m,n}^{k,0}f(z) = G_{0,1}^{1,0}((D - D_*)^{k-1} G_{m-1,n}^{0,0} f(z)) \tag{4.12}$$

and $G_{m,n}^{k,0} f \in L^p(\mathbb{D})$ *hold.*

Proof. By Lemma 4.4 we have

$$G_{m,n}^{k,0}f(z) = \partial_z^k G_{m,n}^{0,0} f(z) = \partial_z^k G_{0,1}^{0,0}(G_{m-1,n}^{0,0})f(z)\ .$$

Then by Lemma 4.5

$$\begin{aligned}G_{m,n}^{k,0}f(z) &= \partial_z^{k-2}(\partial_z^2 G_{0,1}^{0,0}(G_{m-1,n}^{0,0})f(z) \\ &= \partial_z^{k-2} G_{0,1}^{2,0}(G_{m-1,n}^{0,0})f(z) \\ &= G_{0,1}^{1,0}((D - D_*)^{k-1} G_{m-1,n}^{0,0} f(z))\end{aligned}$$

is obtained since $G_{m-1,n}^{1,0} f(z) \in W^{k-1,p}(\mathbb{D})$ holds with $1 \le k \le 2m-1$ for $f \in L^p(\mathbb{D})$. In the case of $k \ge 2m$, $G_{m-1,n}^{1,0} f(z) \in W^{k-1,p}(\mathbb{D})$ holds if $f \in W^{k+1-2m,p}(\mathbb{D})$. □

The following theorems give the boundedness, uniform continuity and L^p boundedness of the operators $G_{m,n}^{k,l}$, respectively.

Theorem 4.7. *Let* $f \in L^p(\mathbb{D})$, $p > 2$ *and* $k + l < 2(m+n)$. *Then,*

$$\left|G_{m,n}^{k,l} f(z)\right| \le C \|f\|_{L^p(\mathbb{D})} \tag{4.13}$$

for $z \in \mathbb{D}$.

Proof. We prove this property for the operators $G_{m,n}^{k,0}$ for $k \le 2(m+n) - 1$. The case $m = 0, n = 1$ is proved in Lemma 4.2. For $n > 1$ and $k = 0$,

$$|G_{m,n}^{0,0} f(z)| = |G_{0,1}^{0,0}(G_{m-1,n}^{0,0} f)| \le C \|f\|_{L^p(\mathbb{D})}$$

is obtained by Lemma 4.2. If $1 \le k \le 2(m+n) - 1$, we have

$$|G_{m,n}^{k,0} f(z)| = |G_{0,1}^{1,0}((D - D_*)^{k-1} G_{m-1,n}^{1,0} f(z))| \le C \|f\|_{L^p(\mathbb{D})}$$

by iterative use of Lemma 4.5 and Corollary 4.6. □

Theorem 4.8. *Let $f \in L^p(\mathbb{D})$, $p > 2$ and $k + l < 2(m+n)$. Then for z_1, $z_2 \in \mathbb{D}$,*

$$\left|G^{k,l}_{m,n}f(z_1) - G^{k,l}_{m,n}f(z_2)\right| \le C\|f\|_{L^p(\mathbb{D})}\begin{cases} |z_1 - z_2|^{(p-2)/p} & \text{if } k+l = 2(m+n)-1 \\ |z_1 - z_2| & \text{otherwise.}\end{cases} \tag{4.14}$$

Proof. For $n > 1$ and $0 \le k + l \le 2(m+n) - 2$,

$$\partial_z G^{k,l}_{m,n}f(z) = G^{k+1,l}_{m,n}f(z)$$

and

$$\partial_{\bar z} G^{k,l}_{m,n}f(z) = G^{k,l+1}_{m,n}f(z)$$

are bounded in $\mathbb{D}$ by Theorem 4.7. Then using the mean value theorem, the result is achieved. For the case $k + l = 2(m+n) - 1$, using Corollary 4.6, we write

$$G^{m+n-1,0}_{m,n}f(z) = G^{1,0}_{0,1}((D - D_*)^{m+n-2}G^{1,0}_{m-1,n}f(z)) ,$$

and the result follows from Lemma 4.2 and (4.10). □

Theorem 4.9. *If $k + l = 2(m+n)$, then $G^{k,l}_{m,n}f \in L^p(\mathbb{D})$ and*

$$\|G^{k,l}_{m,n}f\|_{L^p(\mathbb{D})} \le C_p\|f\|_{L^p(\mathbb{D})} \tag{4.15}$$

for $f \in L^p(\mathbb{D})$ with $p > 1$. Particularly,

$$\|G^{m+2,m}_{m,1}f\|_{L^2(\mathbb{D})} = \|G^{m,m+2}_{m,1}f\|_{L^2(\mathbb{D})} \le \|f\|_{L^2(\mathbb{D})} . \tag{4.16}$$

Proof. (4.16) can be obtained by use of Lemma 4.2 and Lemma 4.3 iteratively. We need to prove (4.15) for the operator $G^{m+n,0}_{m,n}$ for $n > 1$. In this case, by Corollary 4.6, we have

$$\begin{aligned} G^{m+n,0}_{m,n}f(z) &= \partial_z G^{1,0}_{0,1}((D - D_*)^{m+n-2}G^{1,0}_{m-1,n}f(z)) \\ &= G^{2,0}_{0,1}((D - D_*)^{m+n-2}G^{1,0}_{m-1,n}f(z)) . \end{aligned} \tag{4.17}$$

To get the result, we use (4.17) with the L^p boundedness of $G^{1,0}_{0,1}$ and $G^{2,0}_{0,1}$. □

5. Dirichlet Problem for the Generalized Higher-Order Poisson Equation

In this section, using the hierarchy of integral operators defined in the previous section we transform the (m,n)-Dirichlet problems for the generalized n-Poisson equation into singular integral equations. Solvability of the problems are investigated through these singular integral equations by use of Fredholm theory.

We will consider the generalized higher-order Poisson equation

$$\begin{aligned} (\partial_z\partial_{\bar z})^n w + \sum_{\substack{k+l=2n \\ (k,l)\ne(n,n)}} (q^{(1)}_{kl}(z)\partial^k_z\partial^l_{\bar z}w + q^{(2)}_{kl}(z)\partial^l_z\partial^k_{\bar z}\overline{w}) \\ + \sum_{0\le k+l<2n} (a_{kl}(z)\partial^k_z\partial^l_{\bar z}w + b_{kl}(z)\partial^l_z\partial^k_{\bar z}\overline{w}) = f(z) \quad \text{in } \mathbb{D} \end{aligned} \tag{5.1}$$

where

$$a_{kl}, b_{kl}, f \in L^p(\mathbb{D}), \tag{5.2}$$

and $q_{kl}^{(1)}$ and $q_{kl}^{(2)}$, are measurable bounded functions subject to

$$\sum_{\substack{k+l=2r\\(k,l)\neq(r,r)}} (|q_{kl}^{(1)}(z)| + |q_{kl}^{(2)}(z)|) \leq q_0 < 1 \,. \tag{5.3}$$

Now, the following problem is posed.

Dirichlet-(m,n) Problem. *Find $w \in W^{2n,p}(\mathbb{D})$ as a solution to equation* (5.1) *satisfying the Dirichlet condition*

$$(\partial_z \partial_{\bar{z}})^{\mu} w = 0, \; 0 \leq \mu \leq m-1 \;\text{ on }\; \partial\mathbb{D}, \tag{5.4}$$

$$(\partial_z \partial_{\bar{z}})^{\mu+m} w = 0, \; 0 \leq 2\mu \leq n-m-1 \;\text{ on }\; \partial\mathbb{D}, \tag{5.5}$$

$$\partial_{\nu_z} (\partial_z \partial_{\bar{z}})^{\mu+m} w = 0, \; 0 \leq 2\mu \leq n-m-2 \;\text{ on }\; \partial\mathbb{D} \,. \tag{5.6}$$

Solvability of the problem is given depending on the values of p. First we take the case of $p > 2$.

Theorem 5.1. *The equation* (5.1) *with the conditions* (5.4), (5.5) *and* (5.6) *is solvable if*

$$q_0 \max_{k+l=2n} \|G_{m,n-m}^{k,l}\|_{L^p(\mathbb{D})} \leq 1 \tag{5.7}$$

and a solution is of the form $w(z) = G_{m,n-m}^{0,0} g(z)$ where $g \in L^p(\mathbb{D})$, $p > 2$, is a solution of the singular integral equation

$$(I + D + K) g = f \tag{5.8}$$

where

$$Dg = \sum_{\substack{k+l=2n\\(k,l)\neq(n,n)}} (q_{kl}^{(1)}(z) G_{m,n-m}^{k,l} g + q_{kl}^{(2)}(z) \overline{G_{m,n-m}^{k,l} g}) \,,$$

$$Kg = \sum_{0 \leq k+l < 2n} (a_{kl}(z) G_{m,n-m}^{k,l} g + b_{kl}(z) \overline{G_{m,n-m}^{k,l} g}) \,.$$

Proof. The operator $I+D$ is an invertible operator if the condition (5.7) is satisfied, and Theorems 4.7, 4.8 and the Arzela–Ascoli theorem imply that K is a compact operator in $L^p(\mathbb{D})$. Thus the Fredholm alternative applies to the singular integral equation (5.8). □

Now we will take the case $0 < p-2 < \epsilon$. We decompose the strongly singular integral operator D as

$$\begin{aligned} Dg = & \sum_{\substack{k+l=2n\\(k,l)\neq(n,n)}} (q_{kl}^{(1)}(z)G_{m,n-m}^{k,l}g + q_{kl}^{(2)}(z)\overline{G_{m,n-m}^{k,l}g}) \\ = & \sum_{\substack{k+l=2(m+1)\\k=m or l=m}} (q_{kl}^{(1)}G_{m,1}^{k,l}g + q_{kl}^{(2)}\overline{G_{m,1}^{k,l}g}) + \sum_{\substack{k+l=2(m+1)\\k\neq m,\ l\neq m}} (q_{kl}^{(1)}G_{m,1}^{k,l}g + q_{kl}^{(2)}\overline{G_{m,1}^{k,l}g}) \\ & + \sum_{\substack{k+l=2n\\n>m+1}} (q_{kl}^{(1)}G_{m,n-m}^{k,l}g + q_{kl}^{(2)}\overline{G_{m,n-m}^{k,l}g}) \\ := & D_1g + D_2g + D_3g. \end{aligned}$$

By Theorem 4.9, D_1 is an operator in L^2 and its norm is less than or equal to 1 and thus $I + D_1$ is an invertible operator for $2 < p < 2+\epsilon$. The operator $D_2 + D_3$ is bounded, thus using bounded index stability theorem [14] the following result can be stated.

Theorem 5.2. *The equation* (5.1) *with the conditions* (5.4), (5.5) *and* (5.6) *is solvable if*

$$q_0 \max_{k+l=2n} \|G_{m,n-m}^{k,l}\|_{L^p(\mathbb{D})} \|(I+D_1)^{-1} - K_1\|_{L^p(\mathbb{D})} < 1 \tag{5.9}$$

holds for some K_1 which is a compact operator in $L^p(\mathbb{D})$, $0 < p-2 < \epsilon$ and a solution is of the form $w(z) = G_{m,n-m}^{0,0}g(z)$ where $g \in L^p(\mathbb{D})$, $0 < p-2 < \epsilon$, is a solution of the singular integral equation

$$(I + D_1 + D_2 + D_3 + K)g = f. \tag{5.10}$$

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Ü. Aksoy
Department of Mathematics
Atilim University
06836 Ankara
Turkey
e-mail: `uaksoy@atilim.edu.tr`

A.O. Çelebi
Department of Mathematics
Yeditepe University
Istanbul
Turkey
e-mail: `acelebi@yeditepe.edu.tr`

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Schwarz, Riemann, Riemann–Hilbert Problems and Their Connections in Polydomains

Alip Mohammed

Abstract. This paper presents results on some boundary value problems for holomorphic functions of several complex variables in polydomains. The Cauchy kernel is one of the significant tools for solving the Riemann and the Riemann–Hilbert boundary value problems for holomorphic functions as well as for establishment of the connection between them. For polydomains, the Cauchy kernel is modified in such a way that it corresponds to a certain symmetry of the boundary values of holomorphic functions in polydomains. This symmetry is lost if the classical counterpart of the one-dimensional form of the Cauchy kernel is applied. The general integral representation formulas for the functions, holomorphic in polydomains, the solvability conditions and the solutions of the corresponding Schwarz problems are given explicitly. A necessary and sufficient condition for the boundary values of a holomorphic function for arbitrary polydomains is given and an exact, yet compact way of notation for holomorphic functions in arbitrary polydomains is introduced and applied. The Riemann jump problem and the Riemann–Hilbert problem are solved for holomorphic functions of several complex variables with the unit torus as the jump manifold. The higher-dimensional Plemelj–Sokhotzki formula for holomorphic functions in polydomains is established. The canonical functions of the Riemann problem for polydomains are represented and applied in order to construct solutions for both of the homogeneous and inhomogeneous problems. For all three boundary value problems, well-posed formulations are given which does not demand more solvability conditions than in the one variable case. The connection between the Riemann and the Riemann–Hilbert problem for polydomains is proven. Thus contrary to earlier research the results are similar to the respective ones for just one variable.

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1. Introduction

The motivation behind the study of the Riemann and Riemann–Hilbert problems in higher-dimensional polydomains comes from both the theoretical significance and the numerous applications of their one-dimensional analogue from crack problems in engineering [18, 20] to analysis of Markov processes with a two-dimensional state space in queueing system theory [8].

The Riemann and Riemann–Hilbert problems are of interest not only for theoretical reasons but also with respect to applications. On this topic a great deal of research has been done and rich results are achieved in the plane case [11, 27, 5, 19]. They lead to the development of new promising techniques for the analysis of a large class of problems [8, 18, 20]. The Riemann–Hilbert problem has been the solution provider for a vast array of problems in mathematics, mathematical physics and applied mathematics [10]. Moreover the results of the Riemann and the Riemann–Hilbert problems in truly higher-dimensional polydomains can be directly applied to the analysis of Markov processes with a higher-dimensional state space (higher than two) due to independence of the variables of polydomains.

It is well known that the polydisc and the ball in the higher-dimensional space are typical different natural extensions of the disc in the complex plane. Problems of the ball are well studied, but problems of polydomains are almost untouched due to geometrical complexity and some special properties of the polydomain in the higher-dimensional space [3]. We know that holomorphic functions are very important for boundary value problems and that a holomorphic function in a polydomain can be fully determined by the values not on the whole boundary but by the values just on the characteristic boundary [15]. As the boundary of a ball – the sphere divides the space $\mathbb{C}^n (n > 1)$ into two parts just like in the one variable case, the characteristic boundary of polydomains – the torus, the essential boundary or the Shilov boundary – divides the space $\mathbb{C}^n (n > 1)$ into 2^n parts and contrary to the fact that the variables of the ball are dependent, the variables of polydomains are independent. Thus problems for polydomains turn out to be more interesting and applicable. However in order to solve the Riemann and the Riemann–Hilbert problems, we have to solve the Schwarz problem for holomorphic functions of polydomains. So first we address some issues on the Schwarz problem [6, 24].

In the one variable case there are several equivalent definitions of holomorphic functions. The most important ones are via power series and via the Cauchy integral. In higher-dimensional spaces, at least for polydomains, most studies about holomorphic functions start from defining a holomorphic function by a Cauchy integral, simply applying the original form of one-dimensional Cauchy kernel [7] and [17]. These considerations may be proper for many cases. However, some artificial assumptions on the form of the holomorphic functions have to be made when the connection between the Riemann–Hilbert problem and the Riemann problem is considered even for the simplest cases. The reason is that the symmetry of the boundary values of holomorphic functions of respective pairs of polydomains is lost if the original one-dimensional form of the Cauchy kernel is applied. Taking this fact into account another approach is applied. Power series are taken as the starting point and proper Cauchy kernels are derived through a careful factorization of the boundary values of holomorphic functions on the essential boundary. For this purpose the Wiener algebra is introduced and temporarily it is assumed that the functions on the essential boundary belong to the Wiener algebra. Having the Cauchy kernel established, the Wiener algebra is not needed for the solution of the Schwarz problem. Another issue for holomorphic functions of polydomains was to find a precise and yet a compact way of notation.

The Schwarz problem in the unit polydiscs was considered by [3]. However, about the holomorphic functions of the other polydomains of $C^n(n > 1)$ nothing was known in the literature until [6, 24]. Its study provides vital information for discussions of all kinds of boundary value problems of polydomains.

Resolving the issues about the Schwarz problem, the Riemann and the Riemann–Hilbert problems can be addressed. About the Riemann problem for polydisc and polydomains there are some known results [14, 2, 17, 7, 28] and recent ones [9, 22].

In the higher-dimensional space, in general, the zero sets of holomorphic functions of several complex variables can be connected and thus the index method which was vital in the one variable case is questionable to apply. There was also not any convincible higher-dimensional analogue of the Plemelj–Sokhotzki formula for holomorphic functions in polydomains which is fundamental for finding solutions of the one-dimensional problem. To develop the higher-dimensional Plemelj–Sokhotzki formula for holomorphic functions in polydomains, the existing one-dimensional theorem is far from being satisfactory. More deeper inside knowledge is needed. Because of these difficulties, although there were some papers about the Shilov boundary related special inhomogeneous Riemann problem in $\mathbb{C}^n(n > 1)$ [13], [17], [7], no one had given a solution which is constructed by the canonical function for true higher-dimensional torus domains, except [12] and [14] for bi-disc domains. However, the latter results have been found incorrect or inadequate [2]. Only in [22], the higher-dimensional Plemelj–Sokhotzki formula for polydomains is established and a solution which is constructed by the canonical function is provided. Among the previous studies there was no one which could work for solving the corresponding homogeneous problem. The reason is that every attempt was based on the ready form of the one-dimensional Cauchy kernel, the one-dimensional Plemelj–Sokhotzki formula and the one-dimensional Noether condition. They were repeatedly applied for the problem variable by variable. These techniques work well for one inner and its outer domain in the plane. To apply these techniques for polydomains the problem was considered variable by variable so that one inner and its outer domain are always available. Thus because of the Noether condition for holomorphic functions of the outer domain in the plane, holomorphic functions of some polydomains get more strict restrictions [13] than necessary and adequate [24, 6].

The problem in the one variable case essentially is about one pair of holomorphic functions, i.e., about one pair of domains. In the case of a torus there are pairs of domains which could be identified neither as definitive inner nor outer domains, every pair has nothing to do with the others [24]–[25]. In this sense previous studies have treated an holomorphic function of a polydomain also in other irrelevant polydomains. Additionally in order to obtain values of a holomorphic function of a polydomain, its values on the whole boundary of the domain were needed [17], [7] which is contradictory to the statement that a holomorphic function in a polydomain can be fully determined by the boundary values of this function on the essential boundary [15]. The viewpoint of the previous studies is always one of the n variables rather than one pair of the 2^n polydomains in $\mathbb{C}^n$, except [22].

About the Riemann–Hilbert problem for polydomains there was nothing known, except for the polydisc $\mathbb{D}^n$ [3, 1]. But the latter studies are just for one pair of all polydomains, for the rest the Riemann–Hilbert problem remained open [23]. Only in a recent relevant paper [9] some kind of special Rieman and Riemann–Hilbert problems for holomorphic functions were treated from a new perspective. Necessary and sufficient conditions for the existence of finitely linearly independent solutions and finitely many solvability conditions were derived, solutions however were not provided. They for the Rieman problem and the Riemann–Hilbert problem can be seen as a special subject for the considerations of [22, 23].

For formulation of both the Rieman problem and the Riemann–Hilbert problem, there are essentially two different ways. We show that only one of the problems is essential in each set and we solve only the essential ones.

One remarkable fact which deserves significant attention is the natural connection between the Riemann problem and the Riemann–Hilbert problem. This connection had never been established and proven anywhere for higher-dimensional space, although in the one variable case it is well established and proven [5, 11]. To fulfill the gap, applying Fourier method and analyzing structures of boundary values of holomorphic functions, some complementary concepts about the Noether condition [13], for higher-dimensional polydomains have been clarified and so the Cauchy kernel is modified [24, 6], which forced previous discussions [14, 2] to make some compromises on the properties of holomorphic functions in general and so lead to some artificial assumptions. The rearranged form of boundary values of holomorphic functions in polydomains by the modified Cauchy kernel [24, 6] and the well-posed formulation of the Riemann and the Riemann–Hilbert problems turn out to be the key factors for the establishment of the connection between the two problems for polydomains. Thus another main contribution of [23] is to establish and prove the connection for polydomains.

The described one-dimensional problems are well studied and there are numerous results. The following are the most important ones among the known results [11, 27, 5, 19]. For different boundary value problems for holomorphic functions in the unit disc see [26].

2. The Schwarz Problem

About the Schwarz problem for holomorphic functions in polydiscs and polydomains some full scale studies are conducted by [3, 4] and [6, 24] which provided not only solvability conditions but also explicit solutions. Its study provides vital information for discussions of all kinds of boundary value problems of polydomains.

2.1. The Formulation of the Problem for $\mathbb{C}^2$

In the case of unit circle in $\mathbb{C}$, there is one boundary, only one inside and only one outside domain. As it is still the case for the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, for the unit polydisc the situation is quite different. The torus has more domains than simply pure inner and pure outer domains, i.e., the torus has some domains which is neither a pure inner nor a pure outer domain [25] . For $\mathbb{C}^2$ we have four different domains divided by the torus and we can find four holomorphic functions in the respective domains for the given value on torus. For a given value, a holomorphic function that can be defined in a respective torus domain has nothing in common with the holomorphic functions defined in the other respective polydomains, except with the holomorphic function defined in the totally opposed polydomain, see [24, 6].

For the reason of decomposition we need to define a set of complex-valued functions:

$$W(\partial\mathbb{D},\mathbb{C}) = \left\{ f \middle| f(\zeta) = \sum_{-\infty}^{+\infty} a_k \zeta^k, \quad \zeta \in \partial\mathbb{D}, \quad \| f \|_W := \sum_{-\infty}^{+\infty} |a_k| < \infty \right\}$$

which is called the one-dimensional *Wiener algebra* [21] and is simply denoted by W^1 . By the Weierstrass theorem the Fourier series of functions from the Wiener algebra are also uniformly convergent. Because of the independence of the variables of polydomains on the torus $\partial_0\mathbb{D}^n (n > 1)$, we have the Wiener algebra for torus as

$$W^n = \left\{ f \middle| f(z) = \sum_{-\infty}^{+\infty} a_\kappa \zeta^\kappa, \quad \zeta \in \partial_0\mathbb{D}^n, \quad \| f \|_{W^n} := \sum_{-\infty}^{+\infty} |a_\kappa| < \infty \right\}.$$

For the sake of simplicity the Wiener algebra is applied as the function space in some cases to highlight the essence of the higher-dimensional problem without being lost in technical detail.

The discussions on the Riemann and on the Riemann–Hilbert problems and on their connection [22, 23] moreover are not restricted to the Wiener algebra as it was done to the Riemann problem by[28], but only to the Hölder function space $C^\alpha(\partial_0\mathbb{D}^n,\mathbb{C})$ with $0 < \alpha < 1$. However according to the Bernstein theorem $C^\alpha(\partial_0\mathbb{D}^n,\mathbb{C})$ turns out to be the Wiener algebra for $\alpha > 1/2$ [16].

Let $\gamma \in W(\partial_0\mathbb{D}^2;\mathbb{R})$. Then

$$\gamma(\zeta_1,\zeta_2) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} a_{k_1,k_2} \zeta_1^{k_1} \zeta_2^{k_2},$$

$$a_{k_1,k_2} = \frac{1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta_1,\zeta_2)\zeta_1^{-k_1}\zeta_2^{-k_2}\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2},$$
$$a_{-k_1,-k_2} = \overline{a_{k_1,k_2}}\ ,\ k_1,k_2\in\mathbb{Z}\ ,\ (\zeta_1,\zeta_2)\in\partial_0\mathbb{D}^2. \tag{2.1}$$

Then for $(\zeta_1,\zeta_2)\in\partial_0\mathbb{D}^2$ we have

$$\gamma(\zeta_1,\zeta_2) = \sum_{\substack{k_1=0\\k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty}a_{k_1,k_2}\zeta_1^{k_1}\zeta_2^{k_2} + \overline{\sum_{\substack{k_1=0\\k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty}a_{k_1,k_2}\zeta_1^{k_1}\zeta_2^{k_2}} + a_{0,0}$$
$$+\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty}a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2} + \overline{\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty}a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2}}$$
$$= 2\mathrm{Re}\Big\{\sum_{\substack{k_1=0\\k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty}a_{k_1,k_2}\zeta_1^{k_1}\zeta_2^{k_2}\Big\} + a_{0,0} + 2\mathrm{Re}\Big\{\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty}a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2}\Big\}\,.$$

Obviously, $\gamma(\zeta)$ can be decomposed as the boundary values of two harmonic functions – one in $\mathbb{D}^2=\mathbb{D}^+\times\mathbb{D}^+$ ($\mathbb{D}^{-2}=\mathbb{D}^-\times\mathbb{D}^-$) and one in $\mathbb{D}^{+-}=\mathbb{D}^+\times\mathbb{D}^-=\{(z_1,z_2):|z_1|<1,|z_2|>1\}$ ($\mathbb{D}^{-+}=\mathbb{D}^-\times\mathbb{D}^+=\{(z_1,z_2):|z_1|>1,|z_2|<1\}$) respectively.

As we have seen, a given real function $\gamma(\zeta)$ on $\partial_0\mathbb{D}^2$ is not always the real part of boundary values of a harmonic function in $\mathbb{D}^2$, as it was in the one-dimensional case. It is if and only if

$$\mathrm{Re}\Big\{\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty}a_{k_1,-k_2}\zeta_1^{k_1}\zeta_2^{-k_2}\Big\} \ = \ 0\ , \quad (\zeta_1,\zeta_2)\in\partial_0\mathbb{D}^2.$$

This is the reason why solvability conditions occur for the Schwarz problem for holomorphic functions in polydomains.

From the decomposed boundary values which are uniformly and absolutely convergent, we have the respective holomorphic functions

$$\sum_{k_1=0}^{+\infty}\sum_{k_2=0}^{+\infty}a_{k_1,k_2}z_1^{k_1}z_2^{k_2} = \frac{1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta)\frac{\zeta}{\zeta-z}\frac{d\zeta}{\zeta} =: \phi^{++}(z),$$

$$\sum_{\substack{k_1=0\\k_1+k_2>0}}^{+\infty}\sum_{k_2=0}^{+\infty}a_{-k_1,-k_2}z_1^{-k_1}z_2^{-k_2} = \frac{1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta)\Big[\frac{z}{z-\zeta}-1\Big]\frac{d\zeta}{\zeta} =: \phi^{--}(z),$$

$$\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty}a_{k_1,-k_2}z_1^{k_1}z_2^{-k_2} = \frac{-1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta)\frac{z_1}{\zeta_1-z_1}\frac{\zeta_2}{\zeta_2-z_2}\frac{d\zeta}{\zeta} =: \phi^{+-}(z),$$

$$\sum_{k_1=1}^{+\infty}\sum_{k_2=1}^{+\infty} a_{-k_1,k_2} z_1^{-k_1} z_2^{k_2} = \frac{-1}{(2\pi i)^2}\int_{\partial_0\mathbb{D}^2}\gamma(\zeta)\frac{\zeta_1}{\zeta_1-z_1}\frac{z_2}{\zeta_2-z_2}\frac{d\zeta}{\zeta} =: \phi^{-+}(z),$$

for z in, respectively, $\mathbb{D}^2$, $\mathbb{D}^{-2}$, $\mathbb{D}^{+-}$ and $\mathbb{D}^{-+}$. It is evident that the driven kernel above is different than the original form of the one-dimensional Cauchy kernel. However the boundary values of holomorphic functions driven by this new kernel posses certain symmetry.

Let us denote the space of boundary values of functions, holomorphic in $\mathbb{D}^{\chi_1,\chi_2}$, by $\partial\mathcal{H}^{\chi_1,\chi_2}$ and harmonic in $\mathbb{D}^{\chi_1,\chi_2}$ by $\mathcal{BH}^{\chi_1,\chi_2}$ where $\chi_1,\chi_2 \in \{+,-\}$. Clearly $\mathcal{BH}^{\chi_1,\chi_2} = \partial\mathcal{H}^{\chi_1,\chi_2} \oplus \partial\mathcal{H}^{-\chi_1,-\chi_2}$ and $\mathcal{BH}^{\chi_1,\chi_2} = \mathcal{BH}^{-\chi_1,-\chi_2}$. Throughout the paper we need the values of functions on $\partial_0\mathbb{D}^2$ to be at least Hölder continuous, so the corresponding holomorphic function in $\mathbb{D}^{\chi_1,\chi_2}$ has the same Hölder continuity in the closure of $\mathbb{D}^{\chi_1,\chi_2}$, due to the independence of the variables of torus, details can be seen in [5] for the case of one dimension.

2.2. The Problem Formulations for $\mathbb{C}^n$

Although one can describe holomorphic functions in the two-dimensional case very easily, it would not be very convenient to do the same in a higher-dimensional space. One has to find a better way of description. For this reason an exact and yet compact notation is introduced in [6, 24].

Definition 2.1. Let $\chi = (\chi_1,\dots,\chi_n)$ be a multi-sign, satisfying

$$\chi_1,\dots,\chi_n \in \{+,-\},\ 0\le\nu\le n\ ,1\le\sigma_1<\cdots<\sigma_\nu\le n\ ,$$
$$1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n\ ,\ \{\sigma_1,\dots,\sigma_n\}=\{1,\dots,n\}\ ,\chi_{\sigma_1}=-\ ,\dots,\chi_{\sigma_\nu}=-\ ,$$
$$\chi_{\sigma_{\nu+1}}=+\ ,\dots,\chi_{\sigma_n}=+\ ,\chi(\nu)=\chi_{\sigma_1\cdots\sigma_\nu}(\nu)\ ,$$

where ν gives the number of minus $(-)$ signs and the indices $\sigma_1,\dots,\sigma_\nu$ show the position of these minus sign components.

$\chi(\nu)$ obviously has $(n-\nu)$ plus $(+)$ sign components at the positions $\sigma_{\nu+1},\dots,\sigma_n$. In addition $\chi(\nu)=\chi_{\sigma_1\cdots\sigma_\nu}(\nu)=-\chi_{\rho_1\cdots\rho_{n-\nu}}(n-\nu)=-\chi(n-\nu)$, for $0\le\nu\le n$ and $\{\rho_1,\dots,\rho_{n-\nu}\}=\{1,\dots,n\}\backslash\{\sigma_1\cdots\sigma_\nu\}=\{\sigma_{\nu+1}\cdots\sigma_n\}$, when treating $\chi(\nu)$ as a vector.

For convenience we denote $\mathbb{D}^-_{\sigma_1}\times\cdots\times\mathbb{D}^-_{\sigma_\nu}\times\mathbb{D}^+_{\sigma_{\nu+1}}\times\cdots\times\mathbb{D}^+_{\sigma_n}$ as $\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ and $\mathbb{D}^+_{\sigma_1}\times\cdots\times\mathbb{D}^+_{\sigma_\nu}\times\mathbb{D}^-_{\sigma_{\nu+1}}\times\cdots\times\mathbb{D}^-_{\sigma_n}$ as $\mathbb{D}^{-\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$.

Actually $\chi_{\sigma_1\cdots\sigma_\nu}(\nu)$, $0\le\nu\le n$, can be understood as signs of vertices of the n-dimensional cube $\big[-1,+1\big]^n$. In the case $n=2$ the signs $(+,+),(+,-),(-,+),(-,-)$ correspond to the signs of the vertices $(1,1),(1,-1),(-1,1),(-1,-1)$ of the unit square. Therefore we denote χ^* as the vertices of the $\big[-1,+1\big]^n$ cube, while χ represents the respective multi-sign.

Let $\varphi \in W(\partial_0\mathbb{D}^n,\mathbb{R})$. Then φ can be represented as

$$\varphi(\eta)=\sum_{\kappa\in\mathbb{Z}^n}\alpha_\kappa\eta^\kappa\ ,\quad \alpha_\kappa=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\varphi(\zeta)\zeta^{-\kappa}\frac{d\zeta}{\zeta}\ ,\quad \overline{\alpha}_\kappa=\alpha_{-\kappa},\quad \kappa\in\mathbb{Z}^n, \tag{2.2}$$

where $\mathbb{Z}$ is the set of all integers. This Fourier series is absolutely and uniformly convergent to $\varphi(\eta)$, $\eta \in \partial_0\mathbb{D}^n$, because of $\varphi \in W(\partial_0\mathbb{D}^n,\mathbb{R})$ and it can be decomposed into 2^n parts:

$$\begin{cases} \Big[\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}\zeta_t^{-k_t}+1\Big)-1\Big]\alpha_{-k_1,\dots,-k_n} = \sum_{|\kappa|>0\ ,\ \kappa\in\mathbb{Z}_+^n}\alpha_{-\kappa}\zeta^{-\kappa} =: (-1)^n\phi^{\chi(n)}(\zeta), \\ \prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(\zeta_t^{\chi_t^*})^{k_t}+\delta_t^{\chi}\Big)\alpha_{\chi_1^*k_1,\dots,\chi_n^*k_n} =: (-1)^\nu\phi^{\chi(\nu)}(\zeta),\ 0\le\nu<n, \end{cases} \tag{2.3}$$

for all $\zeta \in \partial_0\mathbb{D}^n$, where

$$\delta_t^{\chi_t} = \frac{|\chi_t^*+\chi_{t^*+1}^*|}{2}\ ,\quad 1\le t\le n\ ,\quad t^* = t \bmod(n)\ .$$

Remark 2.2. If the δ_k is treated as numbers, then there is an interesting fact

$$1 = \frac{|\chi_{t+1}^*+1|}{2}+\frac{|\chi_{t+1}^*-1|}{2} =: \delta_t^+ + \delta_t^-,\ 1\le t\le n-1\ ,$$

$$1 = \frac{|\chi_1^*+1|}{2}+\frac{|\chi_1^*-1|}{2} =: \delta_n^+ + \delta_n^-$$

However throughout our paper we interpret δ_h $(1\le h\le n)$ as components of an n-dimensional tuple. Any element of the tuple is composed of exactly just n components, including some δ_k or a_t $(k+t=n,\ 0\le k,t\le n\ )$. The dimension of this kind of tuples n must satisfy $n\ge 2$. Any of δ_k alone does not make any sense, unless it comes with the other $n-1$ components together of an element of the set of tuples.

Clearly $\phi^{\chi(\nu)}(\zeta)$ in (2.3) can be seen as the boundary value of a holomorphic function $\phi^{\chi(\nu)}(z)$ in $\mathbb{D}^{\chi(\nu)}$.

Uniqueness of the decomposition of (2.2) as (2.3) can be proven by the following lemma from [6, 24].

Lemma 2.3.

$$\prod_{t=1}^{n}\Big(a_t+\overline{a_t}+1\Big)+1 = \prod_{t=1}^{n}\Big[\Big(a_t+\delta_t^+\Big)+\Big(\overline{a}_t+\delta_t^-\Big)\Big]$$
$$= \sum_{\nu=0}^{n}\sum_{\substack{1\le\sigma_1<\cdots<\sigma_\nu\le n\\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}}\prod_{t=1}^{\nu}\Big(\overline{a}_{\sigma_t}+\delta_{\sigma_t}^-\Big)\prod_{t=\nu+1}^{n}\Big(a_{\sigma_t}+\delta_{\sigma_t}^+\Big)$$

for $a_t\in\mathbb{C}$, $1\le t\le n$, *where* $cd\{\sigma_1,\dots,\sigma_\nu,\sigma_{\nu+1},\dots,\sigma_n\}=n$,

$$\chi_{\sigma_1}^* = \cdots = \chi_{\sigma_\nu}^* = -1\ ,\quad \chi_{\sigma_{\nu+1}}^* = \cdots = \chi_{\sigma_n}^* = +1\ .$$

By the decomposition (2.3) of boundary values, an arbitrary holomorphic function $\phi^{\chi(\nu)}(z)$ in $\mathbb{D}^{\chi(\nu)}$ with boundary values in W^n and continuous on $\partial_0\mathbb{D}^n$, without loss of generality, can be expressed as

$$\begin{cases} \prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(z_t^{k_t})^{\chi_t^*}+\delta_t^{\chi_t}\Big)\alpha_{\chi_1^*k_1,\dots,\chi_n^*k_n} =: (-1)^{\nu}\phi^{\chi(\nu)}(z),\, 0\le\nu<n, z\in\mathbb{D}^{\chi(\nu)},\\ \Big[\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}z_t^{-k_t}+1\Big)-1\Big]\alpha_{-k_1,\dots,-k_n} =: (-1)^n\phi^{\chi(n)}(z)\,, \quad z\in\mathbb{D}^{-n}\,, \end{cases} \tag{2.4}$$

and they converge absolutely and uniformly even on $\partial_0\mathbb{D}^n$ [6, 24]. From this expression the corresponding Cauchy kernel can be established and the Wiener algebra from now on is not any more needed.

Thus for $\varphi\in C^{\alpha}(\partial_0\mathbb{D}^n,\mathbb{C})$ with $0<\alpha<1$, instead of $\varphi\in W(\partial_0\mathbb{D}^n,\mathbb{C})$, the Cauchy integral can be defined as

$$\phi^{\chi(\nu)}(z)=\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\varphi(\zeta)C(z,\zeta)\frac{d\zeta}{\zeta} \tag{2.5}$$

where

$$C(z,\zeta)=\begin{cases} (-1)^{\nu}\prod_{k=1}^{n}\Big[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\Big], & 0\le\nu\le n-1,\ z\in\mathbb{D}^{\chi(\nu)},\\ (-1)^n\Big[\frac{z}{z-\zeta}-1\Big], & \nu=n,\ z\in\mathbb{D}^{-n},\\ \frac{\zeta}{\zeta-z}, & z\in\partial_0\mathbb{D}^n. \end{cases}$$

Obviously by the decomposition (2.3) of boundary values, all the corresponding holomorphic functions can be represented as (2.5). We call (2.5) *the Cauchy integral for polydomains.*

The holomorphic functions defined by (2.4) can be obtained from (2.5) in the corresponding polydomains and their boundary values (2.3) can also be given by $\phi^{\chi(\nu)}(\zeta):=\lim_{\substack{z\to\zeta\in\partial_0\mathbb{D}^n\\ z\in\mathbb{D}^{\chi(\nu)}}}\phi(z)$.

Interestingly

$$\overline{\prod_{t=1}^{n}\Big(a_t^{\chi_t}+\delta_t^{\chi}\Big)}=\prod_{t=1}^{n}\Big(a_t^{-\chi_t}+\delta_t^{\chi}\Big).$$

If $\varphi(\eta)$ is real and $\varphi(0)=0$, then

$$\overline{(-1)^{\nu}\phi^{\chi(\nu)}(\zeta)} := \overline{\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(\zeta_t^{k_t})^{\chi_t^*}+\delta_t^{\chi},\Big)\alpha_{\chi_1^*k_1,\dots,\chi_n^*k_n}}$$
$$=\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}(\zeta_t^{k_t})^{-\chi_t^*}+\delta_t^{\chi}\Big)\alpha_{-\chi_1^*k_1,\dots,-\chi_n^*k_n}=(-1)^{\nu}\phi^{-\chi(\nu)}(\zeta),$$

$\zeta\in\partial_0\mathbb{D}^n$, holds for $0\le\nu\le n$, and $\phi^{\chi(\nu)}(\zeta)$ can be seen as the reflection of $\phi^{-\chi(\nu)}(\zeta)$ with respect to $\partial_0\mathbb{D}^n$. This property of boundary values turns out to be very useful for solving the Riemann problem [22] as well as for proving the connection between the Riemann and the Riemann–Hilbert problems [23] without imposing artificial restrictions on holomorphic functions of polydomains.

The function $\phi^{\chi(\nu)}(z)$ defined by (2.5) has the following property.

Let k be a fixed integer in $\{1,\dots,n\}$, $\chi(\nu)$ be a fixed sign and $z\in\mathbb{D}^{\chi(\nu)}$.

$$\begin{cases}\phi^{\chi(\nu)}(z)\Big|_{z_{k^*}=\infty}=0\quad, & \text{for } k^*\in\{\sigma_1,\dots,\sigma_\nu\} \text{ and } k^*+1\in\{\sigma_{\nu+1},\dots,\sigma_n\},\\ \phi^{\chi(\nu)}(z)\Big|_{z_{k^*}=0}=0\quad, & \text{for } k^*\in\{\sigma_{\nu+1},\dots,\sigma_n\} \text{ and } k^*+1\in\{\sigma_1,\dots,\sigma_\nu\}.\end{cases}\tag{2.6}$$

It is known that if ϕ^+ , ϕ^- are boundary values of a holomorphic function in $\mathbb{D}^+$, $\mathbb{D}^-$ respectively and are continuous on $\partial\mathbb{D}$, then

$$(a)\qquad \frac{1}{(2\pi i)^n}\int_{\partial\mathbb{D}}\phi^+(\zeta)\frac{\zeta/z}{1-\zeta/z}\frac{d\zeta}{\zeta}\ =\ 0\,,\quad z\in\mathbb{D}^-\,,$$

$$(b)\qquad \frac{1}{(2\pi i)^n}\int_{\partial\mathbb{D}}\phi^-(\zeta)\frac{z/\zeta}{1-z/\zeta}\frac{d\zeta}{\zeta}\ =\ 0\,,\quad z\in\mathbb{D}^+\,.$$

Repeatedly applying these two formulas leads to the following result which will be useful in the sequel.

Lemma 2.4. *Let $\phi^{\chi(\nu)}(\zeta)$ be boundary values of a function, holomorphic in $\mathbb{D}^{\chi(\nu)}$ and continuous on $\mathbb{D}^{\chi(\nu)}\cup\partial_0\mathbb{D}^n$. Then*

$$\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)C(z,\zeta)\frac{d\zeta}{\zeta}=0,\quad z\in\mathbb{D}^{\chi^0(\mu)},\ \chi(\nu)\ne\chi^0(\mu),\quad 0\le\nu,\mu\le n,\tag{2.7}$$

where $C(\zeta,z)$ and $\phi^{\chi(\nu)}(\zeta)$ are defined as in (2.5).

The sense of Lemma 2.4 is that by decomposing the boundary values as in (2.3) and defining the kernel as in (2.5), the kernel of the domain $\mathbb{D}^{\chi(\nu)}$ produces a nontrivial result only for the domain $\mathbb{D}^{\chi(\nu)}$ with the boundary values of a function, holomorphic in $\mathbb{D}^{\chi(\nu)}$.

2.3. The Schwarz Problem for Polydomains

By the Cauchy integral (2.5) it is trivial to get the following lemma.

Lemma 2.5. *Let* $\phi^{\chi(\nu)}$ *be holomorphic in* $\mathbb{D}^{\chi(\nu)}$ *and continuous on* $\mathbb{D}^{\chi(\nu)}\cup\partial_0\mathbb{D}^n$ *. Then*

$$\phi^{\chi(0)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left(\mathrm{Re}\phi^{\chi(0)}(\zeta)\right)\left[2\frac{1}{1-z/\zeta}-1\right]\frac{d\zeta}{\zeta} + iIm\phi^{\chi(0)}(0)\ ,\ z\in\mathbb{D}^n, \tag{2.8}$$

$$\mathrm{Re}\phi^{\chi(0)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left(\mathrm{Re}\phi^{\chi(0)}(\zeta)\right)\prod_{k=1}^{n}\mathrm{Re}\frac{\zeta_k+z_k}{\zeta_k-z_k}\frac{d\zeta}{\zeta}\ ,\ z\in\mathbb{D}^n, \tag{2.9}$$

$$\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left(2\mathrm{Re}\phi^{\chi(n)}(\zeta)\right)\left[\frac{1}{1-\zeta/z}-1\right]\frac{d\zeta}{\zeta},\quad z\in(\mathbb{D}^-)^n, \tag{2.10}$$

$$\mathrm{Re}\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left(\mathrm{Re}\phi^{\chi(n)}(\zeta)\right)\left[\prod_{k=1}^{n}\mathrm{Re}\frac{z_k+\zeta_k}{z_k-\zeta_k}-1\right]\frac{d\zeta}{\zeta}\ ,\quad z\in(\mathbb{D}^-)^n, \tag{2.11}$$

$$\phi^{\chi(\nu)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left(2\mathrm{Re}\phi^{\chi(\nu)}(\zeta)\right)\prod_{k=1}^{n}\left[\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\right]\frac{d\zeta}{\zeta},$$
$$0<\nu<n\ ,\quad z\in\mathbb{D}^{\chi(\nu)}, \tag{2.12}$$

$$\mathrm{Re}\phi^{\chi(\nu)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\left(\mathrm{Re}\phi^{\chi(\nu)}(\zeta)\right)\prod_{k=1}^{n}\left[2\mathrm{Re}\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\right]\frac{d\zeta}{\zeta},$$
$$0<\nu<n\ ,\quad z\in\mathbb{D}^{\chi(\nu)}. \tag{2.13}$$

Applying Lemma 2.4 and Lemma 2.5 it is easy to obtain the following theorem.

Theorem 2.6. *Let* ν $(0\le\nu\le n)$ *and* $\sigma_1,\ldots,\sigma_\nu(1\le\sigma_1<\cdots<\sigma_\nu\le n)$ *be fixed,* $\gamma\in C^\alpha(\partial_0\mathbb{D}^n;\mathbb{R})$ *with* $0<\alpha<1$ *and satisfying*

$$\gamma(\zeta) = \mathcal{P}_{\mathcal{BH}^{\chi(\nu)}}[\gamma(\zeta)], \tag{2.14}$$

where $\mathcal{P}_{\mathcal{BH}^{\chi(\nu)}}[\gamma(\zeta)]$ *is the projection of* $\gamma(\zeta)$ *on* $\mathcal{BH}^{\chi(\nu)}$*. Then for the fixed* ν *and* $\sigma_1,\ldots,\sigma_\nu$ *we have*

$$\phi^{\chi(0)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}\gamma(\zeta)\left[2\frac{1}{1-z/\zeta}-1\right]\frac{d\zeta}{\zeta}+iC_1\ ,\quad z\in\mathbb{D}^n\ , \tag{2.15}$$

$$\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}2\gamma(\zeta)\left[\frac{1}{1-\zeta/z}-1\right]\frac{d\zeta}{\zeta},\quad z\in(\mathbb{D}^-)^n\ , \tag{2.16}$$

$$\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{D}^n}2\gamma(\zeta)\prod_{k=1}^{n}\left(\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1-(z_k\zeta_k^{-1})^{\chi_k^*}}+\delta_k^{\chi_k}\right)\frac{d\zeta}{\zeta}\ ,$$
$$z\in\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}\ ,\ 0<\nu<n \tag{2.17}$$

are holomorphic functions in respective domains with arbitrary real C_1 *and satisfy*

$$\mathrm{Re}\phi^{\pm\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) = \gamma(\zeta), \quad \zeta \in \partial_0\mathbb{D}^n. \tag{2.18}$$

The condition (2.14) *is not only sufficient but also necessary.*

2.4. Well-Posed Formulation of the Schwarz Problem

2.4.1. Plane Case. In the case of the unit disc the given real value on the unit circle is enough to determine the real part of one holomorphic function in the unit disc or in the outer domain of the unit disc. In another word, alone with the given real values on the circle, one can determine one holomorphic function in the unit disc and one other holomorphic function outside the disc. To determine a holomorphic function in or outside the unit disc it is necessary and sufficient to have its real part on the unit circle.

2.4.2. Higher-Dimensional Space. From two dimensions on we have two possible interpretation of the unit disc on the plane: unit ball and unit polydisc. The boundary of the unit ball, the sphere divides the whole space only in two parts. In this sense the unit ball is not very different from the unit disc. But for the polydisc the situation is very different. The boundary of the unit polydisc can be defined in two different ways: the whole boundary or the characteristic boundary–torus. This makes the essential difference. Since holomorphic functions in polydiscs can be described completely by their values just on the essential boundary $\partial_0\mathbb{D}^n$, it is enough to restrict the boundary to the essential boundary–torus [15].

It is well known that all the problems polydomains are accompanied always with some solvability conditions due to the fact that the given values on the torus have more components than necessary components for the problem under consideration. This phenomenon appears because the torus has more domains then simply pure inner and pure outer domains, i.e., the torus has some domains which are neither pure inner nor pure outer domains. In this sense investigating the holomorphic functions in other polydomains has major impact on all kinds of polydomain-related problem solving. By having the properly defined form of holomorphic functions for every polydomain it is obvious that concerning only one special domain of the torus is always accompanied with some solvability conditions. These solvability conditions are seen as natural phenomena for the torus. However this can be understood also as ill-posed formulation of the original problem – we have usually more information than we need and less equations than necessary. Taking into account that the original problem was established for half of the space by the given values on the circle (the other half can be obtained by the given value too), if we formulate the problem exactly for half of the polydomains (for half space) by the given values on the torus, then no solvability conditions could appear. Now it is clear that if we want to get a holomorphic function for a very tiny part of the space, one polydomain, by the given values on the torus, the other non-relevant part of the given values has to vanish and so we have solvability conditions. If we consider more domains of a torus we would have less solvability conditions. If we consider half or more of the torus domains, then we have no solvability conditions. Thus we can have a well-posed analogue of the Schwarz problem for the polydomains which is originally well defined for the circle in the plane.

Before we give the well-posed or modified definition of the Schwarz problem, we define some sets. Let

$I^* = \{\chi^* \mid \chi^* = (\chi_1^*, \dots, \chi_n^*)\}$ be the set of vertices of the $[-1, +1]^n$ cube.

For every element $\chi_1^* \in I^*$ there is one and only one element $\chi_2^* \in I^*$ so that $\chi_1^* = -\chi_2^*$. Denote $I_+^* = \{\chi^* \mid \chi^* = (+1, \chi_2^*, \dots, \chi_n^*) \in I^*\}$. Clearly I_+^* contains exactly half of the elements of I^* and has no any reflected element.

Respectively we denote $I = \{\chi \mid \chi = (\chi_1, \dots, \chi_n)$ sign of the vertices $\chi^* \in I^*\}$ and $I_+ = \{\chi \mid \chi = (+, \chi_2, \dots, \chi_n) \in I\}$.

Now we give our modified well-posed definition of the Schwarz problem for the torus.

The Modified Problem Let $\gamma \in C^\alpha(\partial_0\mathbb{D}^n; \mathbb{R})$ with $0 < \alpha < 1$. Find a holomorphic function $\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)$ in $\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ for $\chi_{\sigma_1\cdots\sigma_\nu} \in I_+$ so that

$$\sum_{\nu=0}^{\left[\frac{n}{2}\right]} \sum_{\substack{2\le\sigma_1<\cdots<\sigma_\nu\le n \\ 1\le\sigma_{\nu+1}<\cdots<\sigma_n\le n}} \operatorname{Re} \phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) = \gamma(\zeta), \quad \zeta \in \partial_0\mathbb{D}^n. \tag{2.19}$$

On the basis of Theorem 2.6 one can easily obtain the following conclusion.

2.5. The Schwarz Problem Without Solvability Conditions

Theorem 2.7. *Let* $\gamma \in C^\alpha(\partial_0\mathbb{D}^n; \mathbb{R})$ *with* $0 < \alpha < 1$. *Then*

$$\phi^{\chi(0)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} \gamma(\zeta) \left[2\frac{1}{1 - z/\zeta} - 1\right] \frac{d\zeta}{\zeta} + iC_1 , \quad z \in \mathbb{D}^n ,$$

$$\phi^{\chi(n)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} 2\gamma(\zeta) \left[\frac{1}{1 - \zeta/z} - 1\right] \frac{d\zeta}{\zeta}, \quad z \in (\mathbb{D}^-)^n , \tag{2.20}$$

$$\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0\mathbb{D}^n} 2\gamma(\zeta) \prod_{k=1}^n \left(\frac{(z_k\zeta_k^{-1})^{\chi_k^*}}{1 - (z_k\zeta_k^{-1})^{\chi_k^*}} + \delta_k^{\chi_k} \right) \frac{d\zeta}{\zeta} ,$$

$$z \in \mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)} , \ 0 < \nu < n, \tag{2.21}$$

are holomorphic functions in respective domains with arbitrary real C_1 *and satisfying* (2.19).

Evidently condition (2.19) is always satisfied and therefore it is not a solvability condition.

2.6. A Necessary and Sufficient Condition for the Boundary Values of Holomorphic Functions on the Torus Domains

Sometimes some simple checking methods of the boundary values of holomorphic functions are needed. Having the structures of holomorphic functions in arbitrary torus domain is fixed, we may be interested to solve problems in this domain. Then surely we are confronted with the boundary values of the holomorphic functions in these polydomains.

In order to know whether $\phi^{-}(\zeta) \in \partial\mathcal{H}(\mathbb{D}^{-})$ it is enough to know if $\overline{\phi^{-}}(\zeta) \in \partial\mathcal{H}(\mathbb{D})$ and $\phi^{-}(\infty) = 0$. This idea can be applied to check boundary values of holomorphic functions in arbitrary polydomains. However we need to introduce a slightly modified version of complex conjugate.

Let $\varphi \in C^{\alpha}(\partial_0\mathbb{D}^n;\mathbb{C})$ with $0 < \alpha < 1$ and $\phi^{\chi(\nu)}$ be a holomorphic function in $\mathbb{D}^{\chi(\nu)}$ which has the boundary values defined as in (2.3) for the given function φ .

We define the *boundary partial conjugate* of $\phi^{\chi(\nu)}$ as follows.

$$\mathcal{C}_{\zeta}\left[\phi^{\chi(n)}(\zeta)\right] := \overline{\left[\phi^{\chi(n)}(\zeta)\right]}_{\zeta} := \overline{\left[\prod_{t=1}^{n}\left(1+\sum_{k_t=1}^{+\infty}\zeta_t^{-k_t}\right)\right.}\overline{\alpha}_{-k_1,\ldots,-k_n} - \overline{\alpha}_{0,\ldots,0}\Bigg]$$
$$= \left[\prod_{t=1}^{n}\left(1+\sum_{k_t=1}^{+\infty}\zeta_t^{k_t}\right)\overline{\alpha}_{-k_1,\ldots,-k_n} - \overline{\alpha}_{0,\ldots,0}\right], \quad \zeta \in \partial_0\mathbb{D}^n; \quad (2.22)$$

$$\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right] =: \overline{\left[\phi^{\chi(\nu)}(\zeta)\right]}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}$$
$$:= \overline{\left[\prod_{t=1}^{\nu}\left(\delta_{\sigma_t}^{-}+\sum_{k_{\sigma_t}=1}^{+\infty}\zeta_{\sigma_t}^{-k_{\sigma_t}}\right)\right.}\prod_{t=\nu+1}^{k}\left(\delta_{\sigma_t}^{+}+\sum_{k_{\sigma_t}=1}^{+\infty}\zeta_{\sigma_t}^{k_{\sigma_t}}\right)\alpha_{\chi_1^* k_1,\ldots,\chi_n^* k_n}\Bigg]$$
$$= \left[\prod_{t=1}^{n}\left(\delta_t^{\chi_t}+\sum_{k_t=1}^{+\infty}\zeta_t^{k_t}\right)\alpha_{\chi_1^* k_1,\ldots,\chi_n^* k_n}\right], \quad 0 < \nu < n\ ,\ \zeta \in \partial_0\mathbb{D}^n. \quad (2.23)$$

We call $\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right]$ the *boundary partial conjugate* of $\phi^{\chi(\nu)}$. Obviously $\phi^{\chi(\nu)}(\zeta) \in \partial\mathcal{H}(\mathbb{D}^{\chi(\nu)})$ $(0 < \nu \le n)$ is equivalent to $\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right] \in \partial\mathcal{H}(\mathbb{D}^n)$ $(0 < \nu \le n)$ if the function $\phi^{\chi(\nu)}(\zeta)$ satisfies condition (2.6). Thus we have

Theorem 2.8. *Let* $\phi^{\chi(\nu)} \in C^{\alpha}(\partial_0\mathbb{D}^n,\mathbb{C})$ *with* $0 < \alpha < 1$ *and continuous in* $\mathbb{D}^{\chi(\nu)}$. *Suppose* $\phi^{\chi(\nu)}$ *satisfies condition* (2.6) *and* $\phi^{\chi(n)}(\infty) = 0$ *for* $\nu = n$. *Then* $\mathcal{C}_{\zeta_{\sigma_1}\cdots\,\zeta_{\sigma_\nu}}\left[\phi^{\chi(\nu)}(\zeta)\right] \in \partial\mathcal{H}(\mathbb{D}^n)$ *is the necessary and sufficient condition for* $\phi^{\chi(\nu)}(\zeta) \in \partial\mathcal{H}(\mathbb{D}^{\chi(\nu)})$.

So by Theorem 2.8 we can check boundary values of holomorphic functions in any polydomains.

3. The Riemann Problem

For the Riemann problem of polydomains the Plemelj–Sokhotkzi formula is the number one challenging obstacle to overcome and at the same time it is also decisive for the solution of the problem. The difficulty lies on the deceptive form of the one-dimensional Plemelj–Sokhotkzi formula. From the two dimension on the original formula fails, one has to find another equivalent form of the one-dimensional formula which is extendable to higher-dimensional one.

3.1. The Plemelj–Sokhotkzi Formula

We define

$$\begin{cases} \Big[\prod_{t=1}^{n}\Big(\sum_{k_t=1}^{+\infty}\zeta_t^{-k_t}-1\Big)-(-1)^n\Big]\alpha_{-k_1,\dots,-k_n} =: \phi_*^{\chi(n)}(\zeta)\ , \quad \zeta\in\partial_0\mathbb{D}^n, \\ \prod_{t=1}^{\nu}\Big(\sum_{k_{\sigma_t}=1}^{+\infty}\zeta_{\sigma_t}^{-k_{\sigma_t}}-\delta_{\sigma_t}^-\Big)\prod_{t=\nu+1}^{n}\Big(\sum_{k_{\sigma_t}=1}^{+\infty}\zeta_{\sigma_t}^{k_{\sigma_t}}+\delta_{\sigma_t}^+\Big)\alpha_{\chi_1^*k_1,\dots,\chi_n^*k_n} =: \phi_*^{\chi(\nu)}(\zeta)\ , \end{cases} \tag{3.1}$$

$$0\le\nu<n\ ,\ \zeta\in\partial_0\mathbb{D}^n,$$

as *boundary integral conjugates* of (2.3) . Evidently for $\eta\in\partial_0\mathbb{D}^n$

$$\frac{1}{(\pi i)^n}\int\limits_{\partial_0\mathbb{D}^n}\phi^{\chi(\nu)}(\zeta)\frac{d\zeta}{\zeta-\eta}=\phi_*^{\chi(\nu)}(\eta)\ \ ,\ \frac{1}{(\pi i)^n}\int\limits_{\partial_0\mathbb{D}^n}\phi_*^{\chi(\nu)}(\zeta)\frac{d\zeta}{\zeta-\eta}=\phi^{\chi(\nu)}(\eta). \tag{3.2}$$

From (3.1) and (2.3) it clear that

$$\phi^{\chi(n)}(\zeta)\ne\phi_*^{\chi(n)}(\zeta)\text{ for }n\ge 2. \tag{3.3}$$

However this is the remarkable reason why it is not possible to get Plemelj-Sokhotkzi formula in the one-dimensional original form. Surprisingly

$$\phi^+(\zeta)=\phi_*^+(\zeta),\quad \phi^-(\zeta)=\phi_*^-(\zeta).$$

Now from this relation we see that second part of the Plemelj-Sokhotkzi formula is indeed about boundary integral conjugates. The first part of the formula is obvious if the given function φ on the torus and the boundary values of holomorphic functions $\phi+,\phi-$ defined by (2.5) are represented as series.

Paying attention to (2.2) and (2.3), applying Lemma 2.3, (3.1), (2.5) and taking (3.1) into account the next result is evident.

Theorem 3.1 (Plemelj–Sokhotzki Formula for Torus Domains). *Under the condition* $\varphi\in C^\alpha(\partial_0\mathbb{D}^n,\mathbb{C})$ *with* $0<\alpha<1$ *, the boundary values of the function* $\phi^{\chi(\nu)}$ *which is holomorphic in* $\mathbb{D}^{\chi(\nu)}$ *and defined as in* (2.5) *satisfy*

$$(-1)^\nu\phi^{\chi(\nu)}(\zeta)+(-1)^{n-\nu}\phi^{-\chi(\nu)}(\zeta)=\varphi^{\chi(\nu)}(\zeta)+\varphi^{-\chi(\nu)}(\zeta)\ ,\quad \zeta\in\partial_0\mathbb{D}^n\ , \tag{3.4}$$

$$2^n \phi(\zeta) = \sum_{\chi(\nu)} \phi_*^{\chi(\nu)}(\zeta) \ , \qquad \zeta \in \partial_0 \mathbb{D}^n \ , \tag{3.5}$$

where $\varphi^{\chi(\nu)}(\zeta) \in \partial \mathcal{H}^{\chi(\nu)}$ *, can be given by Fourier series for* $\varphi(\zeta)$ *on* $\partial_0 \mathbb{D}^n$ *, i.e.,*

$$\varphi^{\chi(\nu)}(\zeta) \ + \ \varphi^{-\chi(\nu)}(\zeta) = \varphi(\zeta)\Big|_{\partial\mathcal{H}^{\chi(\nu)} \oplus \partial\mathcal{H}^{-\chi(\nu)}} \ , \quad \zeta \in \partial_0 \mathbb{D}^n \ .$$

Further $\phi_*^{\chi(\nu)}(\zeta)$ *is the boundary integral conjugate of* $\phi^{\chi(\nu)}(\zeta)$ *defined in* (3.1). *The summation over* $\chi(\nu)$ *actually runs over all* σ'*s, see Lemma* 2.3.

3.2. The Formulation of the Riemann Problem

We introduce here only two different formulations of the Riemann problem with projection coefficient and its main result, details can be found in [22]. It is well known that there is a connection between the Riemann–Hilbert problem and the Riemann problem. For the Riemann problem there are two kind of formulations [22]. Thus also for the Riemann–Hilbert problem, two different formulations can be given.

The Riemann Problem RI(p) *(with projection coefficient).* Let $G, g \in C^\alpha(\partial_0 \mathbb{D}^n, \mathbb{C})$, $0 < \alpha < 1$. Find holomorphic functions $\phi^{\chi(\nu)}(\zeta)$ in $\mathbb{D}^{\chi(\nu)}, 0 \le \nu \le n$, such that

$$\sum_{\chi(\nu)} G^{\chi(\nu)}(\zeta)\phi^{\chi(\nu)}(\zeta) = g(\zeta) \quad , \quad \zeta \in \partial_0 \mathbb{D}^n, \tag{3.6}$$

where $G^{\chi(\nu)}(\zeta) = \mathcal{P}_{\mathcal{BH}^{\chi(\nu)}}[G(\zeta)]$ with $G^{\chi(\nu)}(\zeta) \neq 0$, $\zeta \in \partial_0 \mathbb{D}^n$.

The Riemann Problem RII(p) *(with projection coefficient).* Let $G, g \in C^\alpha(\partial_0 \mathbb{D}^n, \mathbb{C})$, $0 < \alpha < 1$. For a fixed $0 \le \nu \le n$ find functions $\phi^{\chi(\nu)}$, $\phi^{-\chi(\nu)}$ holomorphic in $\mathbb{D}^{\chi(\nu)}$, $\mathbb{D}^{-\chi(\nu)}$ respectively, such that

$$\phi^{\chi(\nu)}(\zeta) + \phi^{-\chi(\nu)}(\zeta) G^{\chi(\nu)}(\zeta) = g^{\chi(\nu)}(\zeta) \ , \quad \zeta \in \partial_0 \mathbb{D}^n, \tag{3.7}$$

where $G^{\chi(\nu)}(\zeta) = \mathcal{P}_{\mathcal{BH}^{\chi(\nu)}}[G(\zeta)]$ with $G^{\chi(\nu)}(\zeta) \neq 0$, $\zeta \in \partial_0 \mathbb{D}^n$ and $g^{\chi(\nu)}(\zeta) = \mathcal{P}_{\mathcal{BH}^{\chi(\nu)}}[g(\zeta)]$.

Now every function in equation (3.7) belongs to the same space $\mathcal{BH}^{\chi(\nu)}$ just like in the one variable case. Thus for solving equation (3.7) we do not need any restrictions.

The starting point or subject of this formulation is *the single space of boundary values of harmonic functions on a pair of polydomains* but *not a single variable* as it was the case for considerations about the Riemann problem for polydomains [12, 13, 14, 17, 2], only exceptions are [22, 1] and [3]. In [3, 1] the Riemann–Hilbert problem was considered for a single torus domain $\mathbb{D}^n$. Results for $\mathbb{D}^{-n}$ can be achieved similarly.

3.3. The Homogeneous Riemann Problem

Lemma 3.2. *The homogeneous problem* (3.7) *with* $g^{\chi(\nu)} = 0$ *is nontrivially solvable if and only if*

$$\operatorname{sign}[K(\chi(\nu))] = \chi(\nu) \tag{3.8}$$

holds for $K(\chi(\nu)) = (-K_{\sigma_1}, \dots, -K_{\sigma_\nu}, K_{\sigma_{\nu+1}}, \dots, K_{\sigma_n})$ *with* $K_{\sigma_\tau} \geq 0$ *for* $1 \leq \tau \leq n$, *i.e., the sign of the domain is the same as the sign of the index* $K(\chi(\nu))$, *where*

$$K_{\sigma_i} := \left| \frac{1}{(2\pi i)} \int\limits_{\partial \mathbb{D}_{\sigma_i}} d \log\big(- G^{\chi(\nu)}(\zeta)\big) \right| \in \mathbb{N} \cup \{0\}.$$

For $K(\chi(\nu))$, *which satisfies* (4.14), *the homogeneous problem* (3.7) *with* $g^{\chi(\nu)} = 0$ *has* $|K(\chi(\nu))| + 1$ *linearly independent solutions*

$$\begin{cases} \phi^{\chi(\nu)}(z) = z_{\sigma_1}^{-k_{\sigma_1}} \cdots z_{\sigma_\nu}^{-k_{\sigma_\nu}} z_{\sigma_{\nu+1}}^{k_{\sigma_{\nu+1}}} \cdots z_{\sigma_n}^{k_{\sigma_n}} e^{\gamma^{\chi(\nu)}(z)}, \\ \phi^{-\chi(\nu)}(z) = z_{\sigma_1}^{K_{\sigma_1} - k_{\sigma_1}} \cdots z_{\sigma_\nu}^{K_{\sigma_\nu} - k_{\sigma_\nu}} z_{\sigma_{\nu+1}}^{k_{\sigma_{\nu+1}} - K_{\sigma_{\nu+1}}} \cdots z_{\sigma_n}^{k_{\sigma_n} - K_{\sigma_n}} e^{\gamma^{-\chi(\nu)}(z)}, \end{cases} \tag{3.9}$$

for z *in, respectively,* $\mathbb{D}^{\chi(\nu)}$ *and* $\mathbb{D}^{-\chi(\nu)}$, *where* $0 \leq k_{\sigma_\tau} \leq K_{\sigma_\tau}$, $1 \leq \tau \leq n$, *and*

$$\gamma^{\pm\chi(\nu)}(z) := \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 \mathbb{D}^n} \log\big\{\zeta^{-K(\chi(\nu))}\big(- G^{\chi(\nu)}(\zeta)\big)\big\} C(\zeta, z) \frac{d\zeta}{\zeta}, \quad z \in \mathbb{D}^{\pm\chi(\nu)}.$$

The simple canonical function of the Riemann problem is

$$\begin{cases} X_0^{\chi(\nu)}(z) = e^{\gamma^{\chi(\nu)}(z)}, & z \in \mathbb{D}^{\chi(\nu)}, \\ X_0^{-\chi(\nu)}(z) = e^{\gamma^{-\chi(\nu)}(z)}, & z \in \mathbb{D}^{-\chi(\nu)}, \end{cases} \tag{3.10}$$

and the general solution to the homogeneous problem is

$$\begin{cases} \phi^{\chi(\nu)}(z) = P^{\chi(\nu)}_{K^+(\chi(\nu))}(z) X_0^{\chi(\nu)}(z), & z \in \mathbb{D}^{\chi(\nu)}, \\ \phi^{-\chi(\nu)}(z) = P^{-\chi(\nu)}_{K^+(\chi(\nu))}(z) X_0^{-\chi(\nu)}(z), & z \in \mathbb{D}^{-\chi(\nu)}, \end{cases} \tag{3.11}$$

where $P^{\chi(\nu)}_{K^+(\chi(\nu))}(z)$ *is a polynomial of* $z \in \mathbb{D}^{\chi(\nu)}$ *with degree up to* $K^+(\chi(\nu))$ *with arbitrary coefficients and* $K^+(\chi(\nu)) = (K_{\sigma_1}, \dots, K_{\sigma_\nu}, K_{\sigma_{\nu+1}}, \dots, K_{\sigma_n})$.

Remark 3.3. There is an equivalent, more classical but less straightforward formulation of (3.10–3.11) as follows.

The canonical function is

$$\begin{cases} X^{\chi(\nu)}(z) = e^{\gamma^{\chi(\nu)}(z)}, & z \in \mathbb{D}^{\chi(\nu)}, \\ X^{-\chi(\nu)}(z) = z^{-K(\chi(\nu))} e^{\gamma^{-\chi(\nu)}(z)}, & z \in \mathbb{D}^{-\chi(\nu)}. \end{cases} \tag{3.12}$$

The general solution to the homogeneous problem is

$$\begin{cases} \phi^{\chi(\nu)}(z) = P_{K(\chi(\nu))}(z) X^{\chi(\nu)}(z), & z \in \mathbb{D}^{\chi(\nu)}, \\ \phi^{-\chi(\nu)}(z) = P_{K(\chi(\nu))}(z) X^{-\chi(\nu)}(z), & z \in \mathbb{D}^{-\chi(\nu)}, \end{cases} \tag{3.13}$$

where $P_{K(\chi(\nu))}(z)$ is a polynomial of $z \in \mathbb{D}^{\chi(\nu)} = \mathbb{D}^-_{\sigma_1} \times \cdots \times \mathbb{D}^-_{\sigma_\nu} \times \mathbb{D}^+_{\sigma_{\nu+1}} \times \cdots \times \mathbb{D}^+_{\sigma_n}$ with degree up to $K\chi(\nu)$ with arbitrary coefficients.

3.4. The Inhomogeneous Riemann Problem

Lemma 3.4. *If the sign of the index $[K(\chi(\nu))]$ of $G^{\chi(\nu)}(\zeta)$ in (3.7) is exactly the same as $\chi(\nu)$, the solution to the problem can be given by*

$$\phi^{\pm\chi(\nu)}(z) = X^{\pm\chi(\nu)}(z)\Big[\psi^{\pm\chi(\nu)}(z) + P^{\pm\chi(\nu)}_{K^+(\chi(\nu))}(z)\Big] , \quad z \in \mathbb{D}^{\pm\chi(\nu)} . \tag{3.14}$$

If the sign $[K(\chi(\nu))]$ has $\tau+\mu$ $(0 \le \tau \le \nu,\ 0 \le \mu \le n-\nu,\ 0 < \mu+\tau \le n)$ opposite components compared with $\chi(\nu)$ (i.e., $K_{\sigma_i} < 0$ $(1 \le i \le \tau \le \nu)$, $K_{\sigma_{\nu+j}} < 0$ $(1 \le j \le \mu \le n-\nu)$ and the remaining $K'_{\sigma_i}s$ are nonnegative), the solvability condition

$$\frac{1}{(2\pi i)^n} \int\limits_{\partial_0 \mathbb{D}^n} \frac{g^{\chi(\nu)}(\zeta)}{X^{\chi(\nu)}(\zeta)} \prod_{\alpha=1}^{\tau} \overline{\zeta}_{\sigma_\alpha}^{k_{\sigma_\alpha}} \prod_{\beta=\tau+1}^{\nu} \overline{\zeta}_{\sigma_\beta}^{k_{\sigma_\beta}} \prod_{j=\nu+1}^{\nu+\mu} \zeta_{\sigma_j}^{k_{\sigma_j}} \prod_{\theta=\nu+\mu+1}^{n} \zeta_{\sigma_\theta}^{k_{\sigma_\theta}} \frac{d\zeta}{\zeta} = 0 \tag{3.15}$$

with

$$0 \le k_{\sigma_\alpha} \le -K_{\sigma_\alpha}\ (1 \le \alpha \le \tau),\ 0 \le k_{\sigma_j} \le -K_{\sigma_j}\ (\nu+1 \le j \le \nu+\mu) ,$$

$$0 \le \sum_{\alpha=1}^{\tau} k_{\sigma_\alpha} + \sum_{j=\nu+1}^{\nu+\mu} k_{\sigma_j} \le -\sum_{\alpha=1}^{\tau} K_{\sigma_\alpha} - \sum_{j=\nu+1}^{\nu+\mu} K_{\sigma_j} - 1 , \quad k_{\sigma_\beta} ,\ k_{\sigma_\theta} \in \mathbb{Z}_+$$

must be satisfied. Then the solution is

$$\phi^{\pm\chi(\nu)}(z) = X^{\pm\chi(\nu)}(z)\psi^{\pm\chi(\nu)}(z) , \qquad z \in \mathbb{D}^{\pm\chi(\nu)}, \tag{3.16}$$

where

$$\psi^{\pm\chi(\nu)}(z) = \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 \mathbb{D}^n} \Big[g^{\chi(\nu)}(\zeta) \Big/ X^{\chi(\nu)}(\zeta)\Big] C(\zeta, z) \frac{d\zeta}{\zeta} , \ z \in \mathbb{D}^{\pm\chi(\nu)} . \tag{3.17}$$

Remark 3.5. Condition (3.15) can be represented as

$$\Big[g^{\chi(\nu)}(\zeta) \Big/ X^{\chi(\nu)}(\zeta)\Big] \in \zeta_{\sigma_1}^{K_{\sigma_1}} \cdots \zeta_{\sigma_\tau}^{K_{\sigma_\tau}} \zeta_{\sigma_{\nu+1}}^{-K_{\sigma_{\nu+1}}} \cdots \zeta_{\sigma_{\nu+\mu}}^{-K_{\sigma_{\nu+\mu}}} \mathcal{BH}^{\chi_{\sigma_1 \cdots \sigma_\nu}(\nu)}. \tag{3.18}$$

4. The Riemann–Hilbert Problem

Applying the results of the previous sections the Riemann–Hilbert problem for polydomains can be solved and its connection to the Riemann problem can be established. For this reason the problem itself has to be well formulated and some additional known facts have to be applied.

For the proof of the Riemann- Hilbert problem, some equivalent forms of a property of holomorphic functions of the unit disc are needed.

Lemma 4.1. *Let*

$$f(\zeta^{\chi^*}) = \sum_{k=1}^{\infty} f_{\chi^* k}\zeta^{\chi^* k} + \delta^{\chi} f_0 \in \mathcal{H}^{\chi}, \quad \zeta \in \partial\mathbb{D}.$$

Assume $K \in \mathbb{Z}$ and $K < 0$. Then

$$f(z^{\chi^*}) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta^{\chi^*})\left[\frac{(z\zeta^{-1})^{\chi^*}}{1-(z\zeta^{-1})^{\chi^*}} + \delta^{\chi}\right]\frac{d\zeta}{\zeta}, \quad z \in \mathbb{D}^{\chi}$$

and for $\delta^{\chi} \in \{0,1\}$ the following are equivalent:

- $\zeta^{\chi^* K} f(\zeta^{\chi^*}) = F(\zeta^{\chi^*}) \in \mathcal{H}^{\chi}$,
- $f_{\chi^* k} = 0$ for $1-\delta^{\chi} \le k \le -K-\delta^{\chi}$,
- $f(z^{\chi^*}) = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\mathbb{D}} f(\zeta^{\chi^*})(z\zeta^{-1})^{-\chi^* K}\left[\frac{(z\zeta^{-1})^{\chi^*}}{1-(z\zeta^{-1})^{\chi^*}} + \delta^{\chi}\right]\frac{d\zeta}{\zeta}, \quad z \in \mathbb{D}^{\chi}.$

4.1. The Well-Posed Formulation of the Riemann–Hilbert Problem for Polydomains

Prior to [23], the Riemann–Hilbert problem for a polydisc $\mathbb{D}^n$ was studied by [3, 1] and a solution is provided.

The **Riemann–Hilbert Problem** (*for a polydisc* $\mathbb{D}^n$) is to find a holomorphic function ϕ in $\mathbb{D}^n$ such that

$$\mathrm{Re}\left\{\overline{\lambda(\zeta)}\phi(\zeta)\right\} = \varphi(\zeta)\ , \quad \zeta \in \partial_0\mathbb{D}^n, \tag{4.1}$$

for $\varphi \in C(\partial_0\mathbb{D}^n, \mathbb{R}),\ \lambda \in C(\partial_0\mathbb{D}^n, \mathbb{C}),\ |\lambda(\zeta)| = 1,\ \zeta \in \partial_0\mathbb{D}^n$.

Lemma 4.2. *Let $\lambda \in C(\partial_0\mathbb{D}^n;\mathbb{C})$, $|\lambda(\zeta)| = 1$ for $\zeta \in \partial_0\mathbb{D}^n$, $\kappa = \mathrm{ind}\ \lambda$, and $\varphi \in C(\partial_0\mathbb{D}^n;\mathbb{R})$ satisfying*

$$\arg\{\zeta^{-\kappa}\lambda(\zeta)\}\ \in \mathcal{BH}^n \tag{4.2}$$

and

$$e^{Im\,\gamma(\zeta)}\varphi(\zeta) \in \mathcal{BH}^n. \tag{4.3}$$

Moreover, if $\kappa = (\kappa_1,\dots,\kappa_n)$, $\kappa_{\lambda_\sigma} < 0 \le \kappa_{\lambda_\rho}$ for $\sigma \in \{1,\dots,\mu\}$, $\rho \in \{\mu{+}1,\dots,n\}$ where μ, $1 \le \mu \le n$, is fixed and $\{\lambda_1,\dots,\lambda_n\} = \{1,\dots,n\}$ satisfying $\lambda_1 < \lambda_2 < \dots < \lambda_\mu$, $\lambda_{\mu+1} < \lambda_{\mu+2} < \dots < \lambda_n$,

$$e^{Im\,\gamma(\zeta)}\varphi(\zeta) = \zeta_{\lambda_1}^{-\kappa_{\lambda_1}}\cdots\zeta_{\lambda_\mu}^{-\kappa_{\lambda_\mu}}\psi,\ \psi \in \mathcal{BH}^n \tag{4.4}$$

with

$$\gamma(z) := \frac{1}{(2\pi i)^n}\int\limits_{\partial_0\mathbb{D}^n} \arg\{\zeta^{-\kappa}\lambda(\zeta)\}\Big(\frac{2\zeta}{\zeta - z} - 1\Big)\frac{d\zeta}{\zeta}\,. \tag{4.5}$$

For $\mu = 0$, condition (4.5) *coincides with condition* (4.2). *Then problem* (4.1) *is solvable. The solution is given by*

$$w(z) = e^{i\gamma(z)}\Big[\frac{z^{\kappa}}{(2\pi i)^n}\int\limits_{\partial_0\mathbb{D}^n} e^{Im\,\gamma(\zeta)}\varphi(\zeta)\Big(\frac{2\zeta}{\zeta - z} - 1\Big)\frac{d\zeta}{\zeta} + \sum_{\nu=0}^{2\kappa} a_\nu z^\nu\Big] \tag{4.6}$$

with arbitrary complex constants a_ν, $0 \le \nu \le 2\kappa$, *satisfying*

$$a_\nu + \overline{a_{2\kappa-\nu}} = 0 \quad \textit{for} \quad 0 \le \nu \le \kappa\,,$$

if $\mu = 0$. *For* $1 \le \mu \le n$ *the solution is*

$$w(z) = \frac{e^{i\gamma(z)}}{(2\pi i)^n} z_{\lambda_{\mu+1}}^{\kappa_{\lambda_{\mu+1}}} \cdots z_{\lambda_n}^{\kappa_{\lambda_n}} \int\limits_{\partial_0 \mathbb{D}^n} 2e^{Im\,\gamma(\zeta)} \varphi(\zeta) \zeta_{\lambda_1}^{\kappa_{\lambda_1}} \cdots \zeta_{\lambda_\mu}^{\kappa_{\lambda_\mu}} \prod_{\sigma=1}^{n} \frac{1}{1 - z_\sigma \overline{\zeta}_\sigma} \frac{d\zeta_\sigma}{\zeta_\sigma}\,. \tag{4.7}$$

In the case $1 \le \mu \le n$ the homogeneous problem $\varphi = 0$ is only trivially solvable.

The Riemann–Hilbert Problem RH(Ip). Let $\lambda \in C^\alpha(\partial_0\mathbb{D}^n, \mathbb{C})$, $\varphi \in C^\alpha(\partial_0\mathbb{D}^n, \mathbb{R})$ with $0 < \alpha < 1$. Find a function $\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)$, holomorphic in $\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$, for $0 \le \nu \le \left[\frac{n}{2}\right]$, $1 \le \sigma_1 \le \cdots \le \sigma_\nu \le n$, such that

$$\sum_{\chi(\nu)} \mathrm{Re}\left\{\overline{\lambda_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)}\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\right\} = \varphi(\zeta)\,, \qquad \zeta \in \partial_0\mathbb{D}^n, \tag{4.8}$$

where $\lambda_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) := \mathcal{P}_{\mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}}\left[\lambda(\zeta)\right]$ and $|\lambda_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)| = 1$ on $\zeta \in \partial_0\mathbb{D}^n$.

Now every function in equation (4.8) belongs to the same space $\mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ which is actually the necessary condition for solvability of (4.8). This condition is automatically satisfied in the one variable case. Now for solving equation (4.8) we do not need any restrictions and this means we got a well-posed formulation of the Riemann–Hilbert problem for the higher-dimensional torus.

By projecting equation (4.8) with respective function spaces $\mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ we can reduce the problem to

The Riemann–Hilbert Problem RH(IIp). Let $\lambda \in C^\alpha(\partial_0\mathbb{D}^n, \mathbb{C})$, $\varphi \in C^\alpha(\partial_0\mathbb{D}^n, \mathbb{R})$ with $0 < \alpha < 1$. Find a function $\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)$, holomorphic in $\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ for the given ν and $\sigma_1\cdots\sigma_\nu$ such that

$$\mathrm{Re}\left\{\overline{\lambda_p^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)}\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\right\} = \varphi_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\,, \qquad \zeta \in \partial_0\mathbb{D}^n \tag{4.9}$$

where $\varphi_p^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) = \mathcal{P}_{\mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}}\left[\varphi(\zeta)\right]$, $\lambda_p^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) := \mathcal{P}_{\mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}}\left[\lambda(\zeta)\right]$ and $|\lambda_p^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)| = 1$ on $\zeta \in \partial_0\mathbb{D}^n$.

What remained is to solve equation (4.9), since all the other forms can be reduced to this equation.

4.2. Solution of the Problem

Lemma 4.3. *The general solution to the special Riemann–Hilbert problem for functions, holomorphic in* $\mathbb{D}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ *of the form*

$$\mathrm{Re}\left\{\overline{\zeta}^{K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))}\varphi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\right\} = 0\,, \qquad \zeta \in \partial_0\mathbb{D}^n \tag{4.10}$$

for the multi-index $K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu)) = (-K_{\sigma_1},\dots,-K_{\sigma_\nu},K_{\sigma_{\nu+1}},\dots,K_{\sigma_n})$, $0 \le K_{\sigma_\nu}$, $0 \le \nu \le n$, *is*

$$\begin{aligned}\varphi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) &= \sum_{\mu_{\sigma_1}=0}^{2K_{\sigma_1}} \cdots \sum_{\mu_{\sigma_\nu}=0}^{2K_{\sigma_\nu}} \sum_{\mu_{\sigma_{\nu+1}}=0}^{2K_{\sigma_{\nu+1}}} \cdots \sum_{\mu_{\sigma_n}=0}^{2K_{\sigma_n}} \alpha_\mu z_{\sigma_1}^{-\mu_{\sigma_1}} \cdots z_{\sigma_\nu}^{-\mu_{\sigma_\nu}} z_{\sigma_{\nu+1}}^{\mu_{\sigma_{\nu+1}}} \cdots z_{\sigma_n}^{\mu_{\sigma_n}} \\ &=: P_{2K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) \;\; \textit{with} \;\; \alpha_\mu + \overline{\alpha}_{[2K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))-\mu]} = 0 \quad (4.11)\end{aligned}$$

for $0 \le \mu \le 2K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))$, *where*

$$K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu)) = (K_{\sigma_1},\dots,K_{\sigma_\nu},K_{\sigma_{\nu+1}},\dots,K_{\sigma_n}).$$

Lemma 4.3 becomes actually Lemma 5.16 in [3] for $\nu = 0$. The proof for $\nu \ne 0$ is trivial, so is omitted.

Theorem 4.4. *Let the assumptions of* RH(IIp) *hold and let* $K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu)) = \operatorname{ind} \lambda_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}$ *satisfy*

$$\arg\{\zeta^{-K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))}\lambda_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\} \in \mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)} \tag{4.12}$$

and

$$e^{Im\,\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)}\varphi_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) \in \mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}. \tag{4.13}$$

We suppose that for the case the sign of the index is not the same as the sign of the domain, i.e.,

$$\operatorname{sign}[K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))] \ne \chi_{\sigma_1\cdots\sigma_\nu}(\nu) \quad \textit{for fixed}\, \nu \tag{4.14}$$

where index

$$\begin{aligned}&K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu)) \\ &= (-K_{\sigma_1},\dots,-K_{\sigma_\mu},\ -K_{\sigma_{\mu+1}},\dots,-K_{\sigma_\nu},K_{\sigma_{\nu+1}},\dots,K_{\sigma_{\nu+\lambda}},K_{\sigma_{\nu+\lambda+1}},\dots,K_{\sigma_n})\end{aligned}$$

with

$$K_{\sigma_1} \le 0,\dots,K_{\sigma_\mu} \le 0,\;\; K_{\sigma_{\mu+1}} \ge 0,\dots,K_{\sigma_\nu} \ge 0\,, \quad 0 \le \mu \le \nu,$$

$$K_{\sigma_{\nu+1}} \le 0,\dots,K_{\sigma_{\nu+\lambda}} \le 0, \quad K_{\sigma_{\nu+\lambda+1}} \ge 0,\dots,K_{\sigma_n} \ge 0\,, \quad 0 \le \lambda \le n-\nu,$$

solvability condition

$$e^{Im\,\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)}\varphi_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta) \in \zeta_{\sigma_1}^{K_{\sigma_1}} \cdots \zeta_{\sigma_\mu}^{K_{\sigma_\mu}} \zeta_{\sigma_{\nu+1}}^{-K_{\sigma_{\nu+1}}} \cdots \zeta_{\sigma_{\nu+\lambda}}^{-K_{\sigma_{\nu+\lambda}}} \mathcal{BH}^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)} \tag{4.15}$$

is satisfied with

$$\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) := \frac{2}{(2\pi i)^n} \int\limits_{\partial_0\mathbb{D}^n} \arg\{\zeta^{-K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))}\lambda_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\}C(\zeta,z)\frac{d\zeta}{\zeta} \tag{4.16}$$

where

$$K_{\sigma_\tau} := \frac{1}{2\pi i} \int\limits_{\partial\mathbb{D}_{\sigma_\tau}} d\log\big(\lambda_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\big) \in \mathbb{Z}\,, \quad 1 \le \tau \le n\,.$$

Then RH(IIp) *is solvable. For the case* $\mu = \lambda = 0$*, i.e., the sign of the index is the same as the sign of the domain, condition* (4.15) *becomes* (4.13) *and the solution is given by*

$$\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) = e^{i\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z)}\left[z^{K(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))}\psi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z)+P^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}_{2K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))}(z)\right] \tag{4.17}$$

with arbitrary complex constants α_κ, $0 \le \kappa \le 2K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))$ *and coefficients of polynomial* $P^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}_{2K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))}(z)$ *satisfying*

$$\alpha_\kappa + \overline{\alpha}_{[2K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu))-\kappa]} = 0 \quad \textit{for} \quad 0 \le \kappa \le K^+(\chi_{\sigma_1\cdots\sigma_\nu}(\nu)),$$

where

$$\psi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) = \frac{2}{(2\pi i)^n}\int\limits_{\partial_0\mathbb{D}^n} e^{Im\,\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)}\varphi_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)C(\zeta,z)\frac{d\zeta}{\zeta}. \tag{4.18}$$

For $1 \le \mu+\lambda \le n$*, the solution is*

$$\phi^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z) = e^{i\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(z)}\prod_{\rho=\mu+1}^{\nu} z_{\sigma_\rho}^{-K_{\sigma_\rho}}\prod_{\tau=\mu+\lambda+1}^{n} z_{\sigma_\tau}^{K_{\sigma_\tau}}$$
$$\times\frac{2}{(2\pi i)^n}\int\limits_{\partial_0\mathbb{D}^n} e^{Im\,\gamma^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)}\varphi_s^{\chi_{\sigma_1\cdots\sigma_\nu}(\nu)}(\zeta)\prod_{\rho=1}^{\mu}\zeta_{\sigma_\rho}^{-K_{\sigma_\rho}}\prod_{\tau=\nu+1}^{\nu+\lambda}\zeta_{\sigma_\rho}^{K_{\sigma_\rho}}C(\zeta,z)\frac{d\zeta}{\zeta}. \tag{4.19}$$

In the case $1 \le \mu+\lambda \le n$ *the homogeneous problem* $\varphi = 0$ *is only trivially solvable.*

The theorem can be proven applying (4.3), (4.4) and Lemmas 3.2–3.4 and 4.1.

The corresponding homogeneous problem is only trivially solvable.

5. The Connection

We have mentioned that for holomorphic functions defined by the modified Cauchy kernel (2.5) the relationship

$$\overline{(-1)^\nu\phi^{\chi(\nu)}(\zeta)} = (-1)^\nu\phi^{-\chi(\nu)}(\zeta)\ ,\ \zeta\in\partial_0\mathbb{D}^n,\ 0\le\nu\le n,$$

holds for $\varphi(\eta)$ real on $\partial_0\mathbb{D}^n$ and $\varphi(0) = 0$ (without $\varphi(0) = 0$ we have one free parameter to fix) . Therefore with the transformation from (3.7) to (4.9) we do not need to impose any restriction on the form of holomorphic functions, i.e., we don't have to narrow types of holomorphic functions to get the transformation as all known studies have to do if they try to establish the connection.

Theorem 5.1. *The solution* (3.14), (3.16) *to the Riemann problem* (3.7) *with*

$$-G^{\chi(\nu)}(\zeta) = \frac{\lambda^{\chi(\nu)}(\zeta)}{\overline{\lambda^{\chi(\nu)}(\zeta)}}\ , \qquad g^{\chi(\nu)}(\zeta) = \frac{2\varphi^{\chi(\nu)}(\zeta)}{\overline{\lambda^{\chi(\nu)}(\zeta)}}$$

is a solution (4.17)*,* (4.19) *to the Riemann–Hilbert problem* (4.9)*, if some free complex parameters are chosen properly.*

The connection is proven in two cases separately: $\text{sign}[K(\chi(\nu))] = \text{sign}[\chi(\nu)]$ and $\text{sign}[K(\chi(\nu))] \neq \text{sign}[\chi(\nu)]$.

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Alip Mohammed
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada
e-mail: alipm@mathstat.yorku.ca

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L^p-Boundedness of Multilinear Pseudo-Differential Operators

Viorel Catană, Shahla Molahajloo and M.W. Wong

Abstract. Results on the L^p-boundedness of multilinear pseudo-differential operators are given. The proofs are based on elementary estimates on the multilinear Rihaczek transforms, the multilinear Wigner transforms and the multilinear Weyl transforms.

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1. Introduction

For functions f and g in $L^2(\mathbb{R}^n)$, a well-known distribution of f and g is the Wigner transform $W(f,g)$ given by

$$W(f,g)(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot y} f\left(x + \frac{y}{2}\right) \overline{g\left(x - \frac{y}{2}\right)}\, dy \tag{1.1}$$

for all x and ξ in $\mathbb{R}^n$. It is easy to check that $W(f,g)$ is a function in the Schwartz space $\mathcal{S}(\mathbb{R}^{2n})$ if f and g are functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Closely related to the Wigner transform is the Weyl transform, which is also known as the pseudo-differential operator of the Weyl type. To wit, let σ be a tempered distribution on $\mathbb{R}^{2n}$. Then the Weyl transform W_σ corresponding to the symbol σ is the mapping from $\mathcal{S}(\mathbb{R}^n)$ into the space $\mathcal{S}'(\mathbb{R}^n)$ of all tempered distributions on $\mathbb{R}^n$ given by

$$(W_\sigma f)(g) = (2\pi)^{-n/2} \sigma(W(f,\overline{g})), \quad f,g \in \mathcal{S}(\mathbb{R}^n).$$

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The notation that we use in this paper is that for a tempered distribution u on $\mathbb{R}^n$ that is also a tempered function,

$$u(\varphi) = \int_{\mathbb{R}^n} u(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

If σ is a symbol in $\mathcal{S}(\mathbb{R}^{2n})$, then for all functions f in $\mathcal{S}(\mathbb{R}^n)$, $W_\sigma f$ is also a function in $\mathcal{S}(\mathbb{R}^n)$ and is given by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi) W(f,g)(x,\xi)\,dx\,d\xi \tag{1.2}$$

for all functions g in $\mathcal{S}(\mathbb{R}^n)$, where $(\,,\,)_{L^2(\mathbb{R}^n)}$ is the inner product in $L^2(\mathbb{R}^n)$. (In this paper, we denote by $(F,G)_{L^2(\mathbb{R}^N)}$ the integral $\int_{\mathbb{R}^N} F(x)\overline{G(x)}\,dx$ for all measurable functions F and G in $\mathbb{R}^N$ such that the integral exists, and we denote by $\|\,\|_{L^p(\mathbb{R}^N)}$ the norm in $L^p(\mathbb{R}^N)$, $1 \le p \le \infty$.) Moreover, it is shown on page 44 of [17] that

$$(W_\sigma f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{\sigma}(w,v)(\rho(w,v)f)(x)\,dw\,dv, \quad x \in \mathbb{R}^n, \tag{1.3}$$

where

$$(\rho(w,v)f)(x) = e^{iw\cdot x + \frac{1}{2}iw\cdot v} f(x+v), \quad x \in \mathbb{R}^n, \tag{1.4}$$

and the Fourier transform $\wedge$ is taken to be the one given by

$$\hat{F}(\xi) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} F(x)\,dx, \quad \xi \in \mathbb{R}^N,$$

for all functions F in $L^1(\mathbb{R}^N)$. The boundedness of Weyl transforms W_σ on $L^p(\mathbb{R}^n)$ when $\sigma \in L^q(\mathbb{R}^{2n})$ has been studied in [17, 19], where $1 \le p, q \le \infty$.

While it is well known that Weyl transforms are based on Wigner transforms, it is relatively recent that the genesis of pseudo-differential operators, first studied systematically in [12], is explored in the context of time-frequency analysis using the Rihaczek transforms [2, 3, 4, 6, 7, 8, 9, 13, 14].

Let us recall that for functions f and g in $L^2(\mathbb{R}^n)$, the Rihaczek transform $R(f,g)$ of f and g is defined by

$$R(f,g)(x,\xi) = e^{ix\cdot\xi}\hat{f}(\xi)\overline{g(x)}, \quad x,\xi \in \mathbb{R}^n.$$

In order to see how pseudo-differential operators are related to the Rihaczek transforms, we first recall that for m in $(-\infty,\infty)$, S^m is the set of all C^∞ functions σ on $\mathbb{R}^{2n}$ such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ for which

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \quad x,\xi \in \mathbb{R}^n.$$

Let $\sigma \in S^m$. Then the pseudo-differential operator T_σ corresponding to the symbol σ is defined by

$$(T_\sigma f)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\,d\xi, \quad x \in \mathbb{R}^n, \tag{1.5}$$

for all functions f in $\mathcal{S}(\mathbb{R}^n)$. It can be proved that T_σ maps $\mathcal{S}(\mathbb{R}^n)$ continuously into $\mathcal{S}(\mathbb{R}^n)$. See, for instance, the books [10] and [18] for expositions of pseudo-differential operators.

Let $\sigma \in \mathcal{S}(\mathbb{R}^{2n})$. Then for all functions f and g in $\mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}(T_\sigma f, g)_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (T_\sigma f)(x)\overline{g(x)}\,dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\overline{g(x)}\,dx\,d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\overline{g(x)}\,dx\,d\xi \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi)R(f,g)(x,\xi)\,dx\,d\xi.\end{aligned}$$

Thus, in view of (1.2), the Rihaczek transform plays the role of the Wigner transform in the genesis of pseudo-differential operators and it allows us to define pseudo-differential operators corresponding to symbols in $\mathcal{S}'(\mathbb{R}^{2n})$. Indeed, let $\sigma \in \mathcal{S}'(\mathbb{R}^{2n})$. Then the pseudo-differential operator T_σ corresponding to the symbol σ is defined by

$$(T_\sigma f)(g) = (2\pi)^{-n/2}\sigma(R(f,\overline{g})), \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

The aim of this paper is to study the L^p-boundedness of multilinear pseudo-differential operators using multilinear Rihaczek transforms, multilinear Wigner transforms and multilinear Weyl transforms. To fix the notation and terminology in this paper, we say that for $1 \le p_1, p_2, \ldots, p_m, q \le \infty$, a multilinear mapping $T : \prod_{j=1}^m L^{p_j}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is said to be a bounded multilinear operator if there exists a positive constant C such that

$$\|Tf\|_{L^q(\mathbb{R}^n)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \tag{1.6}$$

for all $f = (f_1, f_2, \ldots, f_m)$ in $\prod_{j=1}^m L^{p_j}(\mathbb{R}^n)$. The infimum of all the constants C for which the inequality (1.6) is valid is denoted by $\|T\|_{B(\prod_{j=1}^m L^{p_j}(\mathbb{R}^n), L^q(\mathbb{R}^n))}$.

In Section 2, multilinear Rihaczek transforms and the corresponding multilinear pseudo-differential operators are given. In Section 3, we give the Moyal identity for the multilinear Rihaczek transform and use it to establish the L^2-boundedness of multilinear pseudo-differential operators with L^2-symbols. The L^p-boundedness of multilinear pseudo-differential operators with L^p-symbols, $1 \le p \le 2$, is proved in Section 4 using some elementary estimates for the multilinear Rihaczek transform and the Hausdorff–Young inequality for the Fourier transform. In Section 5, we give a self-contained account of multilinear Wigner transforms and multilinear Weyl transforms based on the notation and contents of the book [17]. We give only results that are needed for a more in-depth study of the L^p-boundedness of multilinear pseudo-differential operators in this paper. The basic connection between pseudo-differential operators and Weyl transforms is extended to the multilinear case in Section 6. In Section 7, we introduce a family of spaces $L^p_\mu(\mathbb{R}^{(m+1)n})$, $1 \le p \le \infty$, that incorporate the basic connection into the spaces $L^p_*(\mathbb{R}^{(m+1)n})$ first studied in Chapter 14 of [17]. The L^p-boundedness of pseudo-differential operators with symbols in $L^1_\mu(\mathbb{R}^{(m+1)n})$, $1 \le p < \infty$, is given.

Related works on multilinear pseudo-differential operators on modulation spaces can be found in [2, 3, 4]. Results on multilinear localization operators can be found in [9].

We end this section with some notation that is convenient for the multilinear Fourier analysis throughout this paper. Points in $\mathbb{R}^{nm}$ are denoted by $x = (x_1, x_2, \ldots, x_m)$, where $x_1, x_2, \ldots, x_m$ are points in $\mathbb{R}^n$. For all points $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ in $\mathbb{R}^{nm}$, the inner product $x \cdot y$ of x and y is given by

$$x \cdot y = \sum_{j=1}^{m} x_j \cdot y_j,$$

where $x_j \cdot y_j$ is the ordinary Euclidean inner product of x_j and y_j in $\mathbb{R}^n$. The sum $|x|$ of x is given by

$$|x| = \sum_{j=1}^{m} x_j.$$

We also denote by dx the Lebesgue measure $dx_1\, dx_2 \cdots dx_m$.

2. Multilinear Rihaczek Transforms and Multilinear Pseudo-Differential Operators

The multilinear Rihaczek transform $R(f,g)$ of $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$ and g in $\mathcal{S}(\mathbb{R}^n)$ is defined by

$$R(f,g)(x,\xi) = e^{ix\cdot|\xi|} \prod_{j=1}^{m} \widehat{f_j}(\xi_j)\overline{g(x)}$$

for all x in $\mathbb{R}^n$ and ξ in $\mathbb{R}^{nm}$. It is clear that $R(f,g)$ is a Schwartz function on $\mathbb{R}^{(m+1)n}$.

To simplify the notation, we recall that the tensor product $\otimes_{j=1}^m f_j$ of the measurable functions $f_1, f_2, \ldots, f_m$ on $\mathbb{R}^n$ is the function on $\mathbb{R}^{nm}$ defined by

$$\left(\otimes_{j=1}^m f_j\right)(x) = f_1(x_1) f_2(x_2) \cdots f_m(x_m)$$

for all $x = (x_1, x_2, \ldots, x_m)$ in $\mathbb{R}^{nm}$.

Now, the Rihaczek transform $R(f,g)$ of f in $L^2(\mathbb{R}^n)^m$ and g in $L^2(\mathbb{R}^n)$ is the function on $\mathbb{R}^{(m+1)n}$ given by

$$R(f,g)(x,\xi) = e^{ix\cdot|\xi|} \left(\otimes_{j=1}^m \widehat{f_j}\right)(\xi)\overline{g(x)}$$

for all x in $\mathbb{R}^n$ and ξ in $\mathbb{R}^{nm}$.

Let $\sigma \in \mathcal{S}(\mathbb{R}^{(m+1)n})$. Then for all $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$, we define $T_\sigma f$ to be the function on $\mathbb{R}^n$ by

$$(T_\sigma f)(x) = (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} e^{ix\cdot|\xi|} \sigma(x,\xi) \left(\otimes_{j=1}^m \widehat{f_j}\right)(\xi)\, d\xi, \quad x \in \mathbb{R}^n.$$

So, for all g in $\mathcal{S}(\mathbb{R}^n)$, we get

$$\begin{aligned}(T_\sigma f, g)_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (T_\sigma f)(x)\overline{g(x)} dx \\ &= (2\pi)^{-mn/2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{nm}} e^{ix\cdot|\xi|} \sigma(x,\xi) \left(\otimes_{j=1}^m \widehat{f_j}\right)(\xi)\overline{g(x)} d\xi\, dx \\ &= (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} \sigma(x,\xi) e^{ix\cdot|\xi|} \left(\otimes_{j=1}^m \widehat{f_j}\right)(\xi)\overline{g(x)} dx\, d\xi \\ &= (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} \sigma(x,\xi) R(f,g)(x,\xi)\, dx\, d\xi. \end{aligned} \tag{2.1}$$

Now, let σ be a tempered distribution on $\mathbb{R}^{(m+1)n}$, *i.e.*, $\sigma \in \mathcal{S}'(\mathbb{R}^{(m+1)n})$. Then the pseudo-differential operator T_σ corresponding to the symbol σ is defined on $\mathcal{S}(\mathbb{R}^n)^m$ by

$$(T_\sigma f)(g) = (2\pi)^{-mn/2} \sigma(R(f, \overline{g}))$$

for all f in $\mathcal{S}(\mathbb{R}^n)^m$ and g in $\mathcal{S}(\mathbb{R}^n)$. It can be proved easily that $T_\sigma f$ is a tempered distribution on $\mathbb{R}^n$.

3. The Moyal Identity and L^2-Boundedness

The following theorem gives the Moyal identity for the Rihaczek transform. The Moyal identity, which dates back to Moyal [11] for linear Wigner transforms, is in fact a Plancherel formula.

Theorem 3.1. *For all $f = (f_1, f_2, \ldots, f_m)$ and $h = (h_1, h_2, \ldots, h_m)$ in $L^2(\mathbb{R}^n)^m$, and all g_1 and g_2 in $L^2(\mathbb{R}^n)$,*

$$(R(f, g_1), R(h, g_2))_{L^2(\mathbb{R}^{(m+1)n})} = \prod_{j=1}^m (f_j, h_j)_{L^2(\mathbb{R}^n)} \overline{(g_1, g_2)_{L^2(\mathbb{R}^n)}}.$$

Proof. By the definition of the Rihaczek transform, the Plancherel formula for the Fourier transform and Fubini's theorem, we obtain

$$\begin{aligned}&(R(f, g_1), R(h, g_2))_{L^2(\mathbb{R}^{(m+1)n})} \\ &= \int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} e^{ix\cdot|\xi|} \left(\otimes_{j=1}^m \widehat{f_j}\right)(\xi)\overline{g_1(x)} e^{-ix\cdot|\xi|} \overline{\left(\otimes_{j=1}^m \widehat{h_j}\right)(\xi) g_2(x)}\, dx\, d\xi \\ &= \int_{\mathbb{R}^{nm}} \prod_{j=1}^m \widehat{f_j}(\xi_j)\overline{\widehat{h_j}(\xi_j)}\, d\xi \overline{\int_{\mathbb{R}^n} g_1(x)\overline{g_2(x)}\, dx}\end{aligned}$$

$$= \prod_{j=1}^{m} (\widehat{f_j}, \widehat{h_j})_{L^2(\mathbb{R}^n)} \overline{(g_1, g_2)_{L^2(\mathbb{R}^n)}}$$

$$= \prod_{j=1}^{m} (f_j, h_j)_{L^2(\mathbb{R}^n)} \overline{(g_1, g_2)_{L^2(\mathbb{R}^n)}}. \qquad \square$$

An immediate corollary of Theorem 3.1 is the L^2-boundedness of multilinear pseudo-differential operators with L^2-symbols. More precisely, we have the following theorem.

Theorem 3.2. *Let $\sigma \in L^2(\mathbb{R}^{(m+1)n})$. Then $T_\sigma : L^2(\mathbb{R}^n)^m \to L^2(\mathbb{R}^n)$ is a bounded multilinear operator and*

$$\|T_\sigma\|_{B(L^2(\mathbb{R}^n)^m, L^2(\mathbb{R}^n))} \le (2\pi)^{-mn/2} \|\sigma\|_{L^2(\mathbb{R}^{(m+1)n})}.$$

Proof. Let $f = (f_1, f_2, \dots, f_m) \in \mathcal{S}(\mathbb{R}^n)^m$ and let $g \in \mathcal{S}(\mathbb{R}^n)$. Then by (2.1), Schwarz' inequality and the Moyal identity for the Rihaczek transform,

$$\begin{aligned} |(T_\sigma f, g)_{L^2(\mathbb{R}^n)}| &\le (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} |\sigma(x,\xi)|\, |R(f,g)(x,\xi)|\, dx\, d\xi \\ &\le (2\pi)^{-mn/2} \|\sigma\|_{L^2(\mathbb{R}^{(m+1)n})} \|R(f,g)\|_{L^2(\mathbb{R}^{(m+1)n})} \\ &= (2\pi)^{-mn/2} \|\sigma\|_{L^2(\mathbb{R}^{(m+1)n})} \prod_{j=1}^{m} \|f_j\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

The proof is then complete by a density argument. $\square$

4. L^p-Boundedness, $1 \le p \le 2$

We give in this section a result on the L^p-boundedness of multilinear pseudo-differential operators with L^p-symbols. For this, we need the Hausdorff–Young inequality, which states that for $1 \le p \le 2$, there exists a positive constant C_p such that

$$\|\hat{\varphi}\|_{L^{p'}(\mathbb{R}^n)} \le C_p \|\varphi\|_{L^p(\mathbb{R}^n)}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

where p' is the conjugate index of p and C_p is the norm of the bounded linear operator $\wedge : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$. The classical estimate for C_p is obtained by means of the boundedness of $\wedge$ from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, the unitarity of $\wedge$ on $L^2(\mathbb{R}^n)$ and an interpolation, and is given by

$$C_p \le (2\pi)^{n\left(\frac{1}{2} - \frac{1}{p}\right)}.$$

See, for instance, Chapter IV of [16]. A significant improvement obtained in [1] is that

$$C_p = \left\{ \frac{\left(\frac{p}{2\pi}\right)^{1/p}}{\left(\frac{p'}{2\pi}\right)^{1/p'}} \right\}^{n/2}.$$

Theorem 4.1. *Let $\sigma \in L^p(\mathbb{R}^{(m+1)n})$, $1 \le p \le 2$. Then $T_\sigma : L^p(\mathbb{R}^n)^m \to L^p(\mathbb{R}^n)$ is a bounded multilinear operator and*

$$\|T_\sigma\|_{B(L^p(\mathbb{R}^n)^m, L^p(\mathbb{R}^n))} \le (2\pi)^{-mn/2} C_p^m \|\sigma\|_{L^p(\mathbb{R}^{(m+1)n})}.$$

Proof. Let $f = (f_1, f_2, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n)^m$ and let $g \in L^{p'}(\mathbb{R}^n)$. Then, by (2.1), Hölder's inequality, Fubini's theorem and the Hausdorff–Young inequality,

$$\begin{aligned}
&|(T_\sigma f, g)_{L^2(\mathbb{R}^n)}| \\
&\le (2\pi)^{-mn/2} \|\sigma\|_{L^p(\mathbb{R}^{(m+1)n})} \left(\int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} \prod_{j=1}^m |\widehat{f}_j(\xi_j)|^{p'} |g(x)|^{p'} \, dx \, d\xi \right)^{1/p'} \\
&= (2\pi)^{-mn/2} \|\sigma\|_{L^p(\mathbb{R}^{(m+1)n})} \prod_{j=1}^m \|\widehat{f}_j\|_{L^{p'}(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)} \\
&\le (2\pi)^{-mn/2} \|\sigma\|_{L^p(\mathbb{R}^{(m+1)n})} C_p^m \prod_{j=1}^m \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

Since the dual space of $L^p(\mathbb{R}^n)$ is $L^{p'}(\mathbb{R}^n)$, the theorem is proved. □

5. Multilinear Wigner Transforms and Multilinear Weyl Transforms

Let $f = (f_1, f_2, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n)^m$, $v = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^{nm}$ and $w \in \mathbb{R}^n$. Then we define the function $\rho(w, v)f$ on $\mathbb{R}^n$ by

$$(\rho(w, v)f)(x) = e^{iw\cdot x + \frac{1}{2m} iw\cdot|v|} \prod_{j=1}^m f_j(x + v_j), \quad x \in \mathbb{R}^n.$$

Now, let $f = (f_1, f_2, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n)^m$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Then we define the multilinear Fourier–Wigner transform $V(f, g)$ of f and g to be the function on $\mathbb{R}^{(m+1)n}$ by

$$V(f, g)(w, v) = (2\pi)^{-mn/2} (\rho(w, v)f, g)_{L^2(\mathbb{R}^n)}$$

for all w in $\mathbb{R}^n$ and all $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^{nm}$. It is easy to see that the mapping $V : \mathcal{S}(\mathbb{R}^n)^{m+1} \to \mathcal{S}(\mathbb{R}^{(m+1)n})$ is multilinear with respect to the "first" variable f and antilinear with respect to the "second" variable g.

The following result is an analog of Proposition 2.1 in [17] for multilinear Fourier–Wigner transforms.

Proposition 5.1. *For $1 \le p_1, p_2, \ldots, p_m \le \infty$ with*

$$\sum_{j=1}^m \frac{1}{p_j} \le 1,$$

let p_0 be given by

$$\frac{1}{p_0} = \sum_{j=1}^{m} \frac{1}{p_j}.$$

Then for all points $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^{nm}$ and all points w in $\mathbb{R}^n$, $\rho(w, v) : \prod_{j=1}^m L^{p_j}(\mathbb{R}^n) \to L^{p_0}(\mathbb{R}^n)$ is a surjective and bounded multilinear operator and

$$\|\rho(w, v)\|_{B\left(\prod_{j=1}^m L^{p_j}(\mathbb{R}^n), L^{p_0}(\mathbb{R}^n)\right)} \le 1. \tag{5.1}$$

Proof. Let $f = (f_1, f_2, \ldots, f_m) \in \prod_{j=1}^m L^{p_j}(\mathbb{R}^n)$ and let $g \in L^{p_0'}(\mathbb{R}^n)$, where p_0' is the conjugate index of p_0. Then by Hölder's inequality and the generalized Hölder's inequality,

$$\begin{aligned} |(\rho(w, v)f, g)_{L^2(\mathbb{R}^n)}| &\le \int_{\mathbb{R}^n} \prod_{j=1}^m |f_j(x + v_j)|\, |g(x)|\, dx \\ &\le \left\| \prod_{j=1}^m f_j(\cdot - v_j) \right\|_{p_0} \|g\|_{p_0'} \\ &\le \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}\, \|g\|_{L^{p_0'}(\mathbb{R}^n)}. \end{aligned}$$

This proves that $\rho(w, v) : \prod_{j=1}^m L^{p_j}(\mathbb{R}^n) \to L^{p_0}(\mathbb{R}^n)$ is a bounded multilinear operator and the inequality (5.1) is satisfied. Now, let $g \in L^{p_0}(\mathbb{R}^n)$. For $j = 1, 2, \ldots, m$, let f_j be the function on $\mathbb{R}^n$ defined by

$$f_j(x) = e^{-iw\cdot\frac{x}{m} + \frac{1}{2m} iw\cdot v_j} (g(x - v_j))^{p_0/p_j}, \quad x \in \mathbb{R}^n.$$

Then it is obvious that $f_j \in L^{p_j}(\mathbb{R}^n)$ for $j = 1, 2, \ldots, m$. By the definition of $\rho(w, v)$, we get for $f = (f_1, f_2, \ldots, f_m)$,

$$(\rho(w, v)f)(x) = e^{iw\cdot x + \frac{1}{2m} iw\cdot|v|} \prod_{j=1}^m e^{-i\frac{w}{m}\cdot(x+v_j) + \frac{1}{2m} iw\cdot v_j} (g(x))^{p_0/p_j} = g(x)$$

for all x in $\mathbb{R}^n$. Therefore $\rho(w, v) : \prod_{j=1}^m L^{p_j}(\mathbb{R}^n) \to L^{p_0}(\mathbb{R}^n)$ is surjective. □

Theorem 5.2. *Let $f = (f_1, f_2, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n)^m$ and let $g \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$V(f, g)(w, v) = (2\pi)^{-mn/2} \int_{\mathbb{R}^n} e^{iy\cdot w} \prod_{j=1}^m f_j\left(y + v_j - \frac{1}{2m}|v|\right) \overline{g\left(y - \frac{1}{2m}|v|\right)}\, dy$$

for all w in $\mathbb{R}^n$ and all $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^{nm}$.

Proof. By the definition of the multilinear Fourier–Wigner transform, we get

$$\begin{aligned} V(f, g)(w, v) &= (2\pi)^{-mn/2} (\rho(w, v)f, g)_{L^2(\mathbb{R}^n)} \\ &= (2\pi)^{-mn/2} \int_{\mathbb{R}^n} e^{iw\cdot x + \frac{1}{2m} iw\cdot|v|} \prod_{j=1}^m f_j(x + v_j) \overline{g(x)}\, dx \end{aligned}$$

for all w in $\mathbb{R}^n$ and v in $\mathbb{R}^{nm}$. The theorem is proved if we let

$$x = y - \frac{1}{2m}|v|. \qquad \square$$

We can now introduce the Wigner transform $W(f,g)$ of $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$ and g in $\mathcal{S}(\mathbb{R}^n)$ defined by

$$W(f,g) = V(f,g)^\wedge.$$

The following theorem gives another explicit formula for $W(f,g)$.

Theorem 5.3. *Let $f = (f_1, f_2, \ldots, f_m) \in \mathcal{S}(\mathbb{R}^n)^m$ and let $g \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$W(f,g)(x,\xi) = (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} e^{-iv\cdot\xi} \prod_{j=1}^m f_j\left(x + v_j - \frac{1}{2m}|v|\right) \overline{g\left(x - \frac{1}{2m}|v|\right)}\, dv$$

for all x in $\mathbb{R}^n$ and ξ in $\mathbb{R}^{nm}$.

Proof. For every positive number ε, we define the function $I_\varepsilon : \mathbb{R}^{(m+1)n} \to \mathbb{C}$ by

$$I_\varepsilon(x,\xi) = \int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} e^{-ix\cdot w - iv\cdot\xi} e^{-\varepsilon^2|w|^2/2} V(f,g)(w,v)\, dw\, dv$$

for all x in $\mathbb{R}^n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ in $\mathbb{R}^{nm}$. Using Fubini's theorem and the fact that

$$(e^{-|\cdot|^2/2})^\wedge(w) = e^{-|w|^2/2}, \quad w \in \mathbb{R}^n,$$

we get

$$\begin{aligned}
&I_\varepsilon(x,\xi)\\
&= \int_{\mathbb{R}^{(m+1)n}} e^{-ix\cdot w - iv\cdot\xi - \varepsilon^2|w|^2/2} \left[(2\pi)^{-mn/2} \int_{\mathbb{R}^n} e^{iy\cdot w} F(y,v)\overline{G(y,v)} dy\right] dw\, dv\\
&= (2\pi)^{-mn/2} \int_{\mathbb{R}^{(m+1)n}} e^{-iv\cdot\xi} \left[\int_{\mathbb{R}^n} e^{-i(x-y)\cdot w - \varepsilon^2|w|^2/2} dw\right] F(y,v)\overline{G(y,v)} dy\, dv\\
&= (2\pi)^{-(m-1)n/2} \int_{\mathbb{R}^{nm}} e^{-iv\cdot\xi} \left[\int_{\mathbb{R}^n} \varepsilon^{-n} e^{-|x-y|^2/(2\varepsilon^2)} F(y,v)\overline{G(y,v)}\, dy\right] dv,
\end{aligned}$$

where

$$F(y,v) = \prod_{j=1}^m f_j\left(y + v_j - \frac{1}{2m}|v|\right)$$

and

$$G(y,v) = g\left(y - \frac{1}{2m}|v|\right)$$

for all y in $\mathbb{R}^n$ and v in $\mathbb{R}^{nm}$.

Now, letting $\varepsilon \to 0$, we get by Lebesgue's dominated convergence theorem the asserted formula. $\square$

Let $\sigma \in \mathcal{S}(\mathbb{R}^{(m+1)n})$. Then for all $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$ and all g in $\mathcal{S}(\mathbb{R}^n)$, we define the multilinear Weyl transform $W_\sigma f$ of f corresponding to the symbol σ on $\mathbb{R}^n$ by

$$(W_\sigma f, g)_{L^2(\mathbb{R}^n)} = (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} \sigma(x,\xi) W(f,g)(x,\xi)\, dx\, d\xi$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. Thus, for every tempered distribution σ on $\mathbb{R}^{(m+1)n}$, we define the Weyl transform $W_\sigma f$ of $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$ by

$$(W_\sigma f)(g) = (2\pi)^{-mn/2} \sigma(W(f, \overline{g}))$$

for all g in $\mathcal{S}(\mathbb{R}^n)$.

The following alternative formulas for the multilinear Weyl transform are useful to us.

Proposition 5.4. *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{(m+1)n})$. Then for all $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$ and all g in $\mathcal{S}(\mathbb{R}^n)$,*

$$(W_\sigma f)(g) = (2\pi)^{-mn/2} \hat{\sigma}(V(f, \overline{g})).$$

As a consequence of Proposition 5.4 and the definition of the Fourier–Wigner transform, we get the following multilinear analog of the formula (1.3).

Corollary 5.5. *Let $\sigma \in \mathcal{S}(\mathbb{R}^{(m+1)n})$. Then for all $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$,*

$$(W_\sigma f)(x) = (2\pi)^{-mn} \int_{\mathbb{R}^{nm}} \int_{\mathbb{R}^n} \hat{\sigma}(w,v)(\rho(w,v)f)(x)\, dw\, dv$$

for all x in $\mathbb{R}^n$.

6. A Basic Connection

The connection between the multilinear pseudo-differential operators and multilinear Weyl transforms is provided by the following formula.

Theorem 6.1. *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{(m+1)n})$. Then*

$$T_\sigma = W_\tau,$$

where

$$\hat{\tau}(w,v) = (2\pi)^{(m-1)n/2} e^{-\frac{1}{2m} i w \cdot |v|} \hat{\sigma}(w,v)$$

for all w in $\mathbb{R}^n$ and $v = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^{nm}$.

Proof. We first suppose that $\sigma \in \mathcal{S}(\mathbb{R}^{(m+1)n})$. Then using the definition of the multilinear pseudo-differential operator T_σ, Fubini's theorem and the Fourier inversion formula, we get for all x in $\mathbb{R}^n$,

$$
\begin{aligned}
(T_\sigma f)(x) &= (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} e^{ix\cdot|\xi|}\sigma(x,\xi)\left(\otimes_{j=1}^m \widehat{f}_j\right)(\xi_j)\,d\xi \\
&= (2\pi)^{-mn} \int_{\mathbb{R}^{nm}}\int_{\mathbb{R}^{nm}} e^{i\sum_{j=1}^m (x-y_j)\cdot\xi_j}\sigma(x,\xi)\prod_{j=1}^m f_j(y_j)\,dy\,d\xi \\
&= (2\pi)^{-mn} \int_{\mathbb{R}^{nm}}\int_{\mathbb{R}^{nm}} e^{-iv\cdot\xi}\sigma(x,\xi)\prod_{j=1}^m f_j(x+v_j)\,dv\,d\xi \\
&= (2\pi)^{-mn/2} \int_{\mathbb{R}^{nm}} (\mathcal{F}_2\sigma)(x,v)\prod_{j=1}^m f_j(x+v_j)\,dv \\
&= (2\pi)^{-(m+1)n/2} \int_{\mathbb{R}^{nm}}\int_{\mathbb{R}^{n}} e^{iw\cdot x}(\mathcal{F}_1\mathcal{F}_2\sigma)(w,v)\prod_{j=1}^m f_j(x+v_j)\,dw\,dv \\
&= (2\pi)^{-(m+1)n/2} \int_{\mathbb{R}^{nm}}\int_{\mathbb{R}^{n}} e^{iw\cdot x}\hat{\sigma}(w,v)\prod_{j=1}^m f_j(x+v_j)\,dw\,dv,
\end{aligned}
$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ denote, respectively, the Fourier transforms with respect to the "first" and "second" variables. Thus, for all $f = (f_1, f_2, \ldots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$, we get by the definition of $\rho(w, v)$,

$$
\begin{aligned}
&(T_\sigma f)(x) \\
&= (2\pi)^{-(m+1)n/2} \int_{\mathbb{R}^{nm}}\int_{\mathbb{R}^{n}} e^{-\frac{1}{2m}iw\cdot|v|}\hat{\sigma}(w,v)e^{iw\cdot x+\frac{1}{2m}iw\cdot|v|}\prod_{j=1}^m f_j(x+v_j)\,dw\,dv \\
&= (2\pi)^{-(m+1n)/2} \int_{\mathbb{R}^{nm}}\int_{\mathbb{R}^{n}} e^{-\frac{1}{2m}iw\cdot|v|}\hat{\sigma}(w,v)(\rho(w,v)f)(x)\,dw\,dv
\end{aligned}
$$

for all x in $\mathbb{R}^n$. So, by Corollary 5.5, we get

$$T_\sigma = W_\tau,$$

and we can then complete the proof by a limiting argument. □

7. L^p-Boundedness, $1 \le p < \infty$

Let us begin with a recall of the space $L^p_*(\mathbb{R}^{(m+1)n})$, $1 \le p \le \infty$, introduced in Chapter 14 of the book [17] and defined by

$$L^p_*(\mathbb{R}^{(m+1)n}) = \{\sigma \in L^p(\mathbb{R}^{(m+1)n}) : \hat{\sigma} \in L^{p'}(\mathbb{R}^{(m+1)n})\}.$$

It follows from the Hausdorff–Young inequality that for $1 \le p \le 2$,

$$L^p_*(\mathbb{R}^{(m+1)n}) = L^p(\mathbb{R}^{(m+1)n}).$$

Now, for $1 \le p \le \infty$, we let $L^p_\mu(\mathbb{R}^{(m+1)n})$ be the subspace of $L^p(\mathbb{R}^{(m+1)n})$ defined by

$$L^p_\mu(\mathbb{R}^{(m+1)n}) = \{\sigma \in L^p(\mathbb{R}^{(m+1)n}) : \mathcal{F}^{-1}\mu\mathcal{F}\sigma \in L^p_*(\mathbb{R}^{(m+1)n})\},$$

where $\mathcal{F}$, $\mathcal{F}^{-1}$ and μ are, respectively, the Fourier transform, the inverse Fourier transform and the multiplication operator by the function

$$\mu(w,v) = (2\pi)^{(m-1)n/2} e^{-i\frac{1}{2m}w\cdot|v|}$$

for all w in $\mathbb{R}^n$ and all $v = (v_1, v_2, \dots, v_m)$ in $\mathbb{R}^{nm}$. Then we have the following result on the L^p-boundedness of multilinear pseudo-differential operators.

Theorem 7.1. *Let $\sigma \in L^1_\mu(\mathbb{R}^{(m+1)n})$. Then for $1 \le p < \infty$, $T_\sigma : L^p(\mathbb{R}^n)^m \to L^p(\mathbb{R}^n)$ is a bounded multilinear operator and*

$$\|T_\sigma\|_{B(L^p(\mathbb{R}^n)^m, L^p(\mathbb{R}^n))} \le (2\pi)^{-mn}\Omega_{m,n,p}\|\sigma\|_{L^1_\mu(\mathbb{R}^{(m+1)n})},$$

where

$$\|\sigma\|_{L^1_\mu(\mathbb{R}^{(m+1)n})} = \|\mathcal{F}^{-1}\mu\mathcal{F}\sigma\|_{L^1(\mathbb{R}^{(m+1)n})}$$

and

$$\Omega_{m,n,p} = (2m)^{mn}(2m-1)^{mn/p}.$$

Proof. Using the basic connection in the preceding section, we get

$$T_\sigma = W_\tau,$$

where $\tau = \mathcal{F}^{-1}\mu\mathcal{F}\sigma$. So, for all $f = (f_1, f_2, \dots, f_m)$ in $\mathcal{S}(\mathbb{R}^n)^m$ and all g in $L^{p'}(\mathbb{R}^n)$, we get

$$\begin{aligned}(T_\sigma f, g)_{L^2(\mathbb{R}^n)} &= (W_\tau f, g)_{L^2(\mathbb{R}^n)} \\ &= (2\pi)^{-mn/2}\int_{\mathbb{R}^{nm}}\int_{\mathbb{R}^n} \tau(x,\xi) W(f,g)(x,\xi)\,dx\,d\xi.\end{aligned}$$

Using Theorem 5.3 for the Wigner transform and Hölder's inequality, we get

$$\|W(f,g)\|_{L^\infty(\mathbb{R}^{(m+1)n})} \le (2\pi)^{-mn/2}\Omega_{m,n,p}\prod_{j=1}^m \|f_j\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}.$$

So,

$$\begin{aligned}&|(T_\sigma f, g)_{L^2(\mathbb{R}^n)}| \\ &\le (2\pi)^{-mn/2}\|\tau\|_{L^1(\mathbb{R}^{(m+1)n})}\|W(f,g)\|_{L^\infty(\mathbb{R}^{(m+1)n})} \\ &\le (2\pi)^{-mn/2}(2\pi)^{-mn/2}\Omega_{m,n,p}\|\tau\|_{L^1(\mathbb{R}^{(m+1)n})}\prod_{j=1}^m \|f_j\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}. \\ &= (2\pi)^{-mn}\Omega_{m,n,p}\|\sigma\|_{L^1_\mu(\mathbb{R}^{(m+1)n})}\prod_{j=1}^m \|f_j\|_{L^p(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

Since $L^{p'}(\mathbb{R}^n)$ is the dual of $L^p(\mathbb{R}^n)$, we complete the proof. □

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Viorel Catană
Department of Mathematics
University Polytechnica of Bucharest
Splaiul Independenţei 313
060041, Bucharest
Romania
e-mail: catana_viorel@yahoo.co.uk

Shahla Molahajloo and M.W. Wong
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada
e-mail: smollaha@mathstat.yorku.ca
mwwong@mathstat.yorku.ca

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A Trace Formula for Nuclear Operators on L^p

Julio Delgado

Abstract. We establish a trace for nuclear operators on L^p, this trace generalize a formula already known in the L^2 case. To prove this we first show a characterization of nuclear operators in the L^p setting. As a corollary a formula for the trace of pseudo-differential operators on L^p is obtained.

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1. Introduction

Let T be a compact operator on a complex Hilbert space $\mathbf{H}$. It is well known that we can diagonalize the positive operator T^*T by an orthonormal sequence $(\psi_n)_n$ of eigenvectors with the corresponding eigenvalues $\mu_n > 0$. Define $\lambda_n = \sqrt{\mu_n}$ and $\phi_n = \lambda_n^{-1} T\psi_n$. Since $ker(T^*T) = ker(T)$, we obtain

$$cl\, span(\psi_n)_n = ker(T^*T)^{\perp} = ker(T).$$

Now we can represent the operator T as

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \phi_n$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbf{H}$. The numbers λ_n are the eigenvalues of $|T|$ and are called the singular values of T. T is called a *trace class* operator if the singular values are summable. If the singular values are square-summable, T is called a *Hilbert-Schmidt* operator. Every trace class operator is a Hilbert-Schmidt operator. We are interested in integral operators. In the $L^2(\Omega, \mu)$ case, T is a Hilbert-Schmidt operator if and only if T can be represented by a kernel in $k(x, y)$ in $L^2(\Omega \times \Omega, \mu \otimes \mu)$. In the setting of Banach spaces the concept of trace class

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operators may be generalized as follows. Let E and F be two Banach spaces. A linear operator T from E to F is called *nuclear* if there exists sequences (x'_n) in E' and (y_n) in F so that

$$Ax = \sum_n \langle x, x'_n \rangle \, y_n \ \text{ and } \ \sum_n \|x'_n\|_{E'} \|y_n\|_F < \infty.$$

This definition coincides with the concept of trace class operators in the setting of Hilbert spaces ($E = F = \mathbf{H}$). We recall that in the general context of Banach spaces the trace of a nuclear operator $T : E \longrightarrow E$ is defined by

$$tr\, T = \sum_{n=1}^{\infty} x'_n(y_n),$$

where $T = \sum_{n=1}^{\infty} x'_n \otimes y_n$ is a representation of T. It can be shown that this definition is independent of the representation. For a treatment on traces, see [5], [6] or [8]. Our goal consists in obtaining a generalization for L^p spaces of a trace proved by Chris Brislawn (cf. [1]) in the L^2 case. As a consequence a trace for pseudo-differential operators on L^p is established.

We begin by studying nuclear operators in the context of L^p spaces.

2. A Characterization of Nuclear Operators on $L^p(\mu)$, $1 \le p < \infty$

It is convenient to begin considering finite measures. The following lemma gives a kernel for nuclear operators in this setting and its fundamental properties. Throughout this paper we consider $1 \le p_1, p_2 < \infty$ and q_1, q_2 such that $\frac{1}{p_i} + \frac{1}{q_i} = 1\, (i = 1, 2)$.

Lemma 2.1. *Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be two finite measure spaces. Let $f \in L^{p_1}(\mu_1)$, and $(g_n)_n, (h_n)_n$ be two sequences in $L^{p_2}(\mu_2)$ and $L^{q_1}(\mu_1)$, respectively, so that $\sum_{n=1}^{\infty} \|g_n\|_{L^{p_2}} \|h_n\|_{L^{q_1}} < \infty$. Then*

(a) $\lim_n \sum_{j=1}^{n} g_j(x)h_j(y)$ *is finite for almost every* (x, y)*, and* $\sum_{j=1}^{\infty} g_j(x)h_j(y)$ *is absolutely convergent for almost every* (x, y).

(b) $k \in L^1(\mu_2 \otimes \mu_1)$*, where* $k(x, y) = \sum_{j=1}^{\infty} g_j(x)h_j(y)$.

(c) *If* $k_n(x, y) = \sum_{j=1}^{n} g_j(x)h_j(y)$ *then* $\|k_n - k\|_{L^1} \to 0$.

(d) $$\lim_n \int \left(\sum_{j=1}^{n} g_j(x)h_j(y) \right) f(y) d\mu_1(y) = \int \left(\sum_{n=1}^{\infty} g_n(x)h_n(y) \right) f(y) d\mu_1(y),$$

for a.e x.

Proof. In order to simplify some notation we write

$$L^{p_1} = L^{p_1}(\mu_1),\ L^{p_2} = L^{p_2}(\mu_2),\ L^{q_1} = L^{q_1}(\mu_1),\ L^{q_2} = L^{q_2}(\mu_2).$$

Let $k_n(x,y) = \sum_{j=1}^{n} g_j(x)h_j(y)f(y)$. Applying the Hölder inequality we have that

$$\begin{aligned}\int\!\!\int |k_n(x,y)|d\mu_2(x)d\mu_1(y) &\leq \int\!\!\int \sum_{j=1}^{n} |g_j(x)h_j(y)f(y)|d\mu_2(x)d\mu_1(y)\\ &\leq \sum_{j=1}^{n} \int |g_j(x)|d\mu_2(x) \int |h_j(y)||f(y)|d\mu_1(y)\\ &\leq (\mu_2(\Omega_2))^{\frac{1}{q_2}} \|f\|_{L^{p_1}} \sum_{j=1}^{n} \|g_j\|_{L^{p_2}} \|h_j\|_{L^{q_1}}\\ &\leq M < \infty \text{ for all } n.\end{aligned}$$

Thus $\|k_n\|_{L^1} \leq M$ for all n. On the other hand, the sequence (s_n) with $s_n(x,y) = \sum_{j=1}^{n} |g_j(x)h_j(y)f(y)|$, is increasing in $L^1(\mu_2 \otimes \mu_1)$ and verifies

$$\sup_n \int\!\!\int |s_n(x,y)|d\mu(x)d\mu(y) < M.$$

Using Levi's monotone convergence theorem the limit $s(x,y) = \lim_n s_n(x,y)$ exists and it is finite for almost every (x,y). Moreover $s \in L^1(\mu_2 \otimes \mu_1)$; choosing $f = 1$ and from the fact that $|k(x,y)| \leq s(x,y)$ we deduce (a) and (b). Part (c) can be deduced using Lebesgue's dominated convergence theorem applied to the sequence (k_n) dominated by $s(x,y)$. For the part (d) we observe that letting $k_n(x,y) = \sum_{j=1}^{n} g_j(x)h_j(y)f(y)$, we have $|k_n(x,y)| \leq s(x,y)$ for all n and every (x,y). From the fact that $s \in L^1(\mu_2 \otimes \mu_1)$ we obtain that $s(x,\cdot) \in L^1(\mu_2)$ for a.e x. Then by Lebesgue's dominated convergence theorem we get (d). □

Theorem 2.2 (Characterization of Nuclear Operators on the L^p Spaces for Finite Measures). *Let $(\Omega_i, \mathcal{M}_i, \mu_i)\,(i = 1,2)$ be two measure spaces with finite measures μ_1 and μ_2, respectively. Then, T is a nuclear operator from $L^{p_1}(\mu_1)$ to $L^{p_2}(\mu_2)$ if and only if there is a sequence (g_n) in $L^{p_2}(\mu_2)$, and a sequence (h_n) in $L^{q_1}(\mu_1)$ such that $\sum_{n=1}^{\infty} \|g_n\|_{L^{p_2}(\mu_2)}\|h_n\|_{L^{q_1}(\mu_1)} < \infty$, and for all $f \in L^{p_1}(\mu_1)$*

$$Tf(x) = \int \left(\sum_{n=1}^{\infty} g_n(x)h_n(y)\right) f(y)d\mu_1(y), \text{ for a.e. } x.$$

Proof. Let be T a nuclear operator from L^{p_1} to L^{p_2}. Then there are sequences (g_n) in $L^{p_2}(\mu_2)$, (h_n) in $L^{q_1}(\mu_1)$ so that $\sum\limits_{n=1}^{\infty}\|g_n\|_{L^{p_2}}\|h_n\|_{L^{q_1}}<\infty$ and

$$Tf=\sum_n \langle f,h_n\rangle\, g_n.$$

Now

$$Tf=\sum_n \langle f,h_n\rangle\, g_n=\sum_n\left(\int h_n(y)f(y)d\mu_1(y)\right)g_n\,,$$

where the sums converges in the $L^{p_2}(\mu_2)$ norm. There exists two subsequences $(\tilde{g}_n)$ and $(\tilde{h}_n)$ of (g_n) and (h_n), respectively, so that

$$(Tf)(x)=\sum_n\left\langle f,\tilde{h}_n\right\rangle\tilde{g}_n(x)=\sum_n\left(\int \tilde{h}_n(y)f(y)d\mu_1(y)\right)\tilde{g}_n(x),\ \text{ for a.e. } x.$$

Since the pair $\left((\tilde{g}_n),(\tilde{h}_n)\right)$ satisfies

$$\sum_{n=1}^{\infty}\|\tilde{g}_n\|_{L^{p_2}}\|\tilde{h}_n\|_{L^{q_1}}<\infty\,,$$

applying Lemma 2.1 (d), it follows that

$$\begin{aligned}\sum_n\left(\int \tilde{h}_n(y)f(y)d\mu_1(y)\right)\tilde{g}_n(x)&=\lim_n\sum_{j=1}^{n}\left(\int \tilde{h}_j(y)f(y)d\mu_1(y)\right)\tilde{g}_j(x)\\&=\lim_n\int\left(\sum_{j=1}^{n}\tilde{g}_j(x)\tilde{h}_j(y)f(y)\right)d\mu_1(y)\\&=\int\left(\sum_{n=1}^{\infty}\tilde{g}_n(x)\tilde{h}_n(y)\right)f(y)d\mu_1(y)\,,\ \text{ for a.e. } x.\end{aligned}$$

Conversely, assume that there exists sequences $(g_n)_n$ in $L^{p_2}(\mu_2)$, and $(h_n)_n$ in $L^{q_1}(\mu_1)$ so that $\sum\limits_{n=1}^{\infty}\|g_n\|_{L^{p_2}}\|h_n\|_{L^{q_1}}<\infty$, and for all $f\in L^{p_1}$

$$Tf(x)=\int\left(\sum_{n=1}^{\infty}g_n(x)h_n(y)\right)f(y)d\mu_1(y)\,,\quad\text{for a.e. } x.$$

By Lemma 2.1 (d) we have

$$\begin{aligned}\int\left(\sum_{n=1}^{\infty}g_n(x)h_n(y)\right)f(y)d\mu_1(y)&=\lim_n\int\left(\sum_{j=1}^{n}g_j(x)h_j(y)f(y)\right)d\mu_1(y)\\&=\lim_n\sum_{j=1}^{n}\left(\int h_j(y)f(y)d\mu_1(y)\right)g_j(x)\end{aligned}$$

$$= \sum_n \left(\int h_n(y) f(y) d\mu_1(y) \right) g_n(x)$$
$$= \sum_n \langle f, h_n \rangle g_n(x) = (Tf)(x), \text{ for a.e. } x.$$

To establish that $Tf = \sum_n \langle f, h_n \rangle g_n$ in $L^{p_2}(\mu_2)$ we let $s_n = \sum_{j=1}^{n} < f, h_j > g_j$, then $(s_n)_n$ is a sequence in $L^{p_2}(\mu_2)$ and

$$|s_n(x)| \leq \|f\|_{L^{p_1}} \sum_{j=1}^{n} \|h_j\|_{L^{q_1}} |g_j(x)|$$
$$\leq \|f\|_{L^{p_1}} \sum_{j=1}^{\infty} \|h_j\|_{L^{q_1}} |g_j(x)| = \gamma(x), \text{ for all } n.$$

Furthermore, γ is well defined and $\gamma \in L^{p_2}(\mu_2)$ since it is the increasing limit of the sequence $(\gamma_n)_n = (\|f\|_{L^{p_1}} \sum_{j=1}^{n} \|h_j\|_{L^{q_1}} |g_j(x)|)_n$ of L^{p_2} functions and

$$\|\gamma_n\|_{L^{p_2}} \leq \|f\|_{L^{p_1}} \sum_{j=1}^{\infty} \|h_j\|_{L^{q_1}} \|g_j\|_{L^{p_2}} < M.$$

By the monotone convergence theorem of B. Levi we see that $\gamma \in L^{p_2}(\mu_2)$. Finally, applying the Lebesgue dominated convergence theorem we deduce that $s_n \to Tf$ in $L^{p_2}(\mu_2)$. □

We give now a generalization for $\sigma-$finite measures of the characterization 2.2. We begin by generalize the parts (a) and (d) of lemma 2.1 which are essentials for the proof of theorem 2.2.

Lemma 2.3. *Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be two measure spaces with σ-finite measures μ_1 and μ_2, respectively; $f \in L^{p_1}(\mu_1)$, and $(g_n)_n, (h_n)_n$ sequences in $L^{p_2}(\mu_2)$ and $L^{q_1}(\mu_1)$, respectively, such that $\sum_{n=1}^{\infty} \|g_n\|_{L^{p_2}} \|h_n\|_{L^{q_1}} < \infty$. Then the parts* (a) *and* (d) *in Lemma* 2.1 *hold.*

Proof. (a) Since μ_1, μ_2 are σ-finite measures, there exists two sequences $(\Omega_1^k)_k$ and $(\Omega_2^l)_l$ of subsets of Ω_1 and Ω_2, respectively, such that $\mu_1(\Omega_1^k) < \infty$, $\mu_2(\Omega_2^l) < \infty$ and $\bigcup_k \Omega_1^k = \Omega_1$, $\bigcup_l \Omega_2^l = \Omega_2$. Consider the finite measure spaces $(\Omega_1^k, \mathcal{M}_1^k, \mu_1^k)$ and $(\Omega_2^l, \mathcal{M}_2^l, \mu_2^l)$ that we obtain restricting Ω_1 to Ω_1^k, and Ω_2 to Ω_2^l for every k, and restricting the functions g_n to Ω_2^l, and h_n to Ω_1^k. Then, for all k, l

$$\sum_{n=1}^{\infty} \|g_n\|_{L^p(\mu_2^l)} \|h_n\|_{L^q(\mu_1^k)} < \infty.$$

By Lemma 2.1 (a) we have that the series $\sum_{j=1}^{\infty} g_j(x)h_j(y)$ converges absolutely for a.e. $(x,y) \in \Omega_2^l \times \Omega_1^k$. Hence $\sum_{j=1}^{\infty} g_j(x)h_j(y)$ converges absolutely for almost every $(x,y) \in \Omega \times \Omega$. This proves part (a).

From part (a) we know that the series $\sum_{j=1}^{\infty} g_j(x)h_j(y)f(y)$ converges absolutely for a.e. (x,y), part (d) follows from Lebesgue's dominated convergence theorem applied as in the reciprocal part of the proof of Theorem 2.2 (see the use of γ_n, and γ). □

Notice that we cannot conclude that $k(x,y) \in L^1(\mu_2 \otimes \mu_1)$ as in the case of finite measures. Take for example $\Omega_1 = \Omega_2 = \mathbb{R}^n$, and $\mu_1, \mu_2 = \lambda$, the Lebesgue measure, then using the fact that $q_2 > 1$, we define $k(x,y) = g(x)h(y)$, with $g \in L^{p_2}(\lambda) \setminus \{0\}, h \in L^{q_1}(\lambda) \setminus L^1(\lambda)$. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |k(x,y)| d\lambda(x) d\lambda(y) = \int_{\mathbb{R}^n} |g(x)| d\lambda \int_{\mathbb{R}^n} |h(y)| d\lambda = \infty.$$

We are now ready to formulate the characterization of nuclear operators on $L^p(\mu)$ for σ-finite measures as a consequence of the above lemma and the proof is similar to that of Theorem 2.2.

Theorem 2.4 (Characterization of Nuclear Operators on L^p for σ-Finite Measures). *Let $(\Omega_i, \mathcal{M}_i, \mu_i)\,(i = 1,2)$ be measure spaces with σ-finite measures μ_1 and μ_2, respectively. Then, T is a nuclear operator from $L^{p_1}(\mu_1)$ to $L^{p_2} = L^{p_2}(\mu_2)$ if and only if there exists sequences $(g_n)_n$ in $L^{p_2}(\mu_2)$, $(h_n)_n$ in $L^{q_1}(\mu_1)$ such that $\sum_{n=1}^{\infty} \|g_n\|_{L^{p_2}} \|h_n\|_{L^{q_1}} < \infty$, and for all $f \in L^{p_1}$*

$$Tf(x) = \int \left(\sum_{n=1}^{\infty} g_n(x)h_n(y) \right) f(y) d\mu_1(y), \text{ for a.e. } x.$$

Proof. The proof follows the same lines of the proof of Theorem 2.2, just substitute the references to part (d) of Lemma 2.1 with Lemma 2.3. □

3. Calculus of the Trace on $L^p(\mu)$

We will apply the characterization 2.4 to the calculus of the trace of a nuclear operator from $L^p(\mu)$ to $L^p(\mu)$. We begin by showing a simple trace formula on $L^p(\mu)$ for σ-finite measures that will be useful to establish our main formula.

Lemma 3.1. *Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measure space. If T is a nuclear operator from $L^p(\mu)$ to $L^p(\mu)$, then the kernel given by Theorem 2.4 is integrable on the* *diagonal, i.e.,* $k(x,x) = \sum_{j=1}^{\infty} g_j(x)h_j(x)$ *is finite for a.e.* x, $k(x,x) \in L^1(\mu)$, *and we have*

$$tr\, T = \int_{\Omega} k(x,x)d\mu(x).$$

Proof. If T is a nuclear operator, we can write for f in $L^p(\mu)$

$$Tf(x) = \int \left(\sum_{n=1}^{\infty} g_n(x)h_n(y) \right) f(y)d\mu(y), \text{ for a.e. } x;$$

with g_n, h_n as in Theorem 2.4, and $f \in L^p(\mu)$. The kernel

$$k(x,y) = \sum_{n=1}^{\infty} g_n(x)h_n(y)$$

satisfies $k(x,x) \in L^1(\mu)$. In fact, letting $k_n(x,x) = \sum_{j=1}^{n} g_j(x)h_j(x)$, we have for all n that

$$\begin{aligned}
\int_{\Omega} |k_n(x,x)|d\mu(x) &\leq \sum_{j=1}^{n} \int |g_j(x)||h_j(x)|d\mu(x) \\
&\leq \sum_{j=1}^{n} \|g_n\|_{L^p} \|h_n\|_{L^q} \\
&\leq \sum_{j=1}^{\infty} \|g_n\|_{L^p} \|h_n\|_{L^q} \\
&< \infty.
\end{aligned}$$

Therefore, $k(x,x) = \sum_{j=1}^{\infty} g_j(x)h_j(x)$ exists, it is finite for a.e. x, and $k(x,x) \in L^1$.

Now we may compute the trace of T. Using similar arguments already employed in this work, it is now clear that

$$\sum_{n=1}^{\infty} \int g_n(x)h_n(x)d\mu(x) = \int \sum_{n=1}^{\infty} g_n(x)h_n(x)d\mu(x).$$

Then

$$\begin{aligned}
tr\, T &= \sum_{n=1}^{\infty} \int g_n(x)h_n(x)d\mu(x) \\
&= \int \sum_{n=1}^{\infty} g_n(x)h_n(x)d\mu(x) \\
&= \int_{\Omega} k(x,x)d\mu(x)\,.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

Remark 3.2. Note that if μ is a σ-finite measure on Borel sets and $\alpha(x,y)$ is a continuous kernel of a nuclear operator T from $L^p(\mu)$ to $L^p(\mu)$, and $k(x,y)$ is the kernel given by Theorem 2.4, then $\alpha(x,y) = k(x,y)$ for a.e. (x,y), but the trace formula in Lemma 3.1 for α is not deducible from the corresponding formula for $k(x,y)$. Suitable tools of harmonic analysis will permit to overcome this problem and lead to our main result (Theorem 3.8).

In order to obtain our main theorem which consists in a trace formula known in the L^2 case and proved by Brislawn (cf. [1]), we will need some basic tools of harmonic analysis which are essentials to establish some L^p bounds and properties of convergence. Standard references on these results are [9] and [10]. We will consider the particular case of the euclidean space $\mathbb{R}^n$ endowed with the Lebesgue measure denoted by $|\cdot|$. The L^2 case was proved for more general measures by Brislawn in [2], we also hope to study this problem on L^p in a next work.

Let C_r be the n-dimensional cube of radius r centered at the origin in $\mathbb{R}^n$, and let $C_r(x)$ be the translate cube centered at $x \in \mathbb{R}^n$:

$$C_r = [-r,r]^n,\ C_r(x) = x + C_r.$$

In order to study some properties of convergence for certain operators we consider an averaging process. Let A_r be the linear operator that averages a function $f \in L^1_{loc}(\mathbb{R}^n)$ over cubes of radius r:

$$A_r f(x) = \frac{1}{|C_r(x)|}\int\limits_{C_r(x)} f(t)dt = \frac{1}{|C_r|}\int\limits_{C_r} f(x+t)dt. \tag{3.1}$$

For $r > 0$ fixed, $A_r f(x)$ is a continuous function of x, and for each $x \in \mathbb{R}^n$, $A_r f(x)$ is a continuous function of $r \in (0,\infty)$. We recall that the Hardy–Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|C_r|}\int\limits_{C_r} |f(x+t)|dt. \tag{3.2}$$

$Mf(x)$ is well defined for all $x \in \mathbb{R}^n$. The Hardy–Littlewood maximal theorem states that if $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, then Mf is finite almost everywhere, and if $1 < p \le \infty$, then

$$\|Mf\|_p \le C_p \|f\|_p, \tag{3.3}$$

where C_p is a constant depending only on p and the dimension n. By the Lebesgue differentiation Theorem , if $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r\to 0} A_r f(x) = f(x),\ \text{for a.e. } x, \tag{3.4}$$

For $r > 0$, we have

$$|A_r f(x)| \le Mf(x) \tag{3.5}$$

for all $x \in \mathbb{R}^n$, so if $1 < p \le \infty$, the Hardy–Littlewood maximal theorem implies that A_r is a bounded linear operator on $L^p(\mathbb{R}^n)$. At every point x at which the limit (3.4) holds, we may extend $A_r f(x)$ to a continuous function of $r \in [0, \infty)$ by defining $A_0 f(x) = f(x)$, and the bound (3.5) then holds when $r = 0$. We can now define

$$\tilde{f}(x) = \lim_{r \to 0} A_r f(x); \tag{3.6}$$

then, $\tilde{f}$ exists almost everywhere and $\tilde{f}(x) = f(x)$ for a.e. x, and $\tilde{f}$ agrees with f at each point of continuity.

The operator A_r possesses also the following properties of convergence:

Lemma 3.3. *Let $r > 0$, $1 \le p \le \infty$; if $f_n \to f$ in $L^p(\mathbb{R}^n)$, then $A_r f_n \to A_r f$ uniformly.*

Remark 3.4. The lemma above is valid on $L^p(\Omega)$ for a measurable subset Ω, identifying a function f in $L^p(\Omega)$ with a function equals to zero on $\mathbb{R}^n - \Omega$.

Corollary 3.5. *If $\sum f_n$ converges to f in L^p-norm, then $\sum A_r f_n$ converges uniformly to $A_r f$.*

We will use superscripts on the operators A_r or M to indicate the dimension in which averages are being taken. For instance, Definition (3.1) for averages of functions $f \in L^1_{loc}(\mathbb{R}^{2n})$ becomes

$$A_r^{(2n)} f(x, y) = \frac{1}{|C_r|^2} \int\limits_{C_r} \int\limits_{C_r} f(x + s, y + t) ds dt. \tag{3.7}$$

The operator A_r satisfies the following fundamental property of multiplicativity on the tensorial product $L^p \otimes L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$:

Lemma 3.6. *Let $g \in L^p(\mathbb{R}^n)$ and $h \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$A_r^{(2n)}(g \otimes h)(x, y) = A_r^{(n)} g(x) A_r^{(n)} h(y),$$

where $(g \otimes h)(x, y) = g(x)h(y)$.

As a consequence we have:

Lemma 3.7. *Let $g \in L^p(\mathbb{R}^n)$ and $h \in L^q(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$M^{(2n)}(g \otimes h)(x, y) \le M^{(n)} g(x) M^{(n)} h(y).$$

We dispose also of the subadditivity as a direct consequence of the definition of the Hardy–Littlewood maximal function

$$M(g + h)(x) \le M g(x) + M\, h(x); \;\; g, h \in L^p. \tag{3.8}$$

Let $K(x, y) \in L^1_{loc}(\mathbb{R}^{2n})$, then $\tilde{K}(x, y)$ is defined as in (3.6) by

$$\tilde{K}(x, y) = \lim_{r \to 0} A_r^{(2n)} K(x, y).$$

The function $K(x, y)$ is defined for almost every (x, y).

We now consider a nuclear operator T from L^p to L^p and a representation

$$k(x,y) = \sum_{j=1}^{\infty} g_j(x)h_j(y),$$

as in Theorem 2.4 with $g_j \in L^p$, $h_j \in L^q$ $(j = 1, 2, \dots)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\sum \|g_j\|_p \|h_j\|_q < \infty$.

We recall that from Lemma 2.1 we are authorized to say that $k(x,y)$ is locally integrable on $\mathbb{R}^{2n}$. To see this, it is sufficient to consider the restriction of T to a space of L^p functions defined on a finite measure set Ω.

Let (β_j) be a sequence of functions in $L^p(\mathbb{R}^n)$, we will say that x is a *regular point* of the sequence (β_j) if for all $j \in \mathbb{N}$

$$\lim_{r \to 0} A_r \beta_j(x) = \beta_j(x).$$

Since almost every point is regular for (g_j) and almost every point is regular for (h_j), then almost every $x \in \mathbb{R}^n$ is regular for the sequence (β_j) where $\beta_j(x) = g_j(x)h_j(x)$.

From Lemma 3.3, Remark 3.4 and Corollary 3.5 we obtain using the fact that $k(x,y) \in L^1_{loc}(\mathbb{R}^{2n})$,

$$A_r^{(2n)} k(x,y) = \sum_{j=1}^{\infty} A_r^{(n)} g_j(x) A_r^{(n)} h_j(y) \tag{3.9}$$

for all (x,y). Indeed the convergence is uniform on compact sets.

We are now in a position to establish a trace formula for nuclear operators on L^p, the L^2 case corresponding to Theorem 3.1 proved in [1]. Here is our main result.

Theorem 3.8 (Main Theorem). *Let $T : L^p \longrightarrow L^p$ $(1 < p < \infty)$ be a nuclear operator with a kernel $k(x,y)$ as in Theorem 2.4. Then $M^{(2n)}k(x,x) \in L^1(\mathbb{R}^n)$, $\tilde{k}(x,x) = k(x,x)$ for almost every x and consequently*

$$tr\, T = \int_{\mathbb{R}^n} \tilde{k}(x,x)dx. \tag{3.10}$$

Proof. We show first that $M^{(2n)}k(x,x) \in L^1(\mathbb{R}^n)$. Applying the subadditivity , the submultiplicativity and the boundedness of the Hardy–Littlewood maximal function on L^p and L^q, and the Hölder inequality we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} M^{(2n)} k(x,x)dx &= \int_{\mathbb{R}^n} \left(\sum_{j=1}^{\infty} M^{(n)} g_j(x) M^{(n)} h_j(x) \right) dx \\ &= \sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^n} M^{(n)} g_j(x) M^{(n)} h_j(x) dx \right)\end{aligned}$$

$$\leq C \sum_{j=1}^{\infty} \|g_j\|_{L^p} \|h_j\|_{L^q}$$
$$< \infty .$$

This proves the first assertion and the fact that the sum

$$\sum_{j=1}^{\infty} M^{(n)} g_j(x) M^{(n)} h_j(x)$$

is finite for almost every x. From Lemma 3.1, $k(x,x)$ is finite for almost every x. Now we consider the sums

$$\sum_{j=1}^{\infty} g_j(x) h_j(x),\ \sum_{j=1}^{\infty} M^{(n)} g_j(x) M^{(n)} h_j(x).$$

We choose a conull set of regular points $\Gamma \subset \mathbb{R}^n$ so that for all $x \in \Gamma$ both of the above series are finite. The $A_r^{(n)} g_j(x)$, $A_r^{(n)} h_j(x)$ are continuous functions of $r \in [0,\infty]$ for every $x \in \Gamma$ and all j. Using the fact that,

$$|A_r^{(n)} g_j(x)||A_r^{(n)} h_j(x)| \leq |M^{(n)} g_j(x)||M^{(n)} h_j(x)|$$

for $x \in \Gamma$, $r \in [0,\infty]$ and for all j, the series

$$\sum_{j=1}^{\infty} |A_r^{(n)} g_j(x)||A_r^{(n)} h_j(x)|$$

converges absolutely and uniformly with respect to $r \in [0,\infty]$. Now, by (3.9) we have for every $r > 0$ that

$$A_r^{(2n)} k(x,x) = \sum_{j=1}^{\infty} A_r^{(n)} g_j(x) A_r^{(n)} h_j(x).$$

Hence, letting $r \to 0$ we obtain for each $x \in \Gamma$ that

$$\begin{aligned}
\tilde{k}(x,x) &= \lim_{r\to 0} A_r^{(2n)} (\sum_{j=1}^{\infty} g_j(x) h_j(x)) \\
&= \sum_{j=1}^{\infty} \lim_{r\to 0} A_r^{(2n)} (g_j(x) h_j(x)) \\
&= \sum_{j=1}^{\infty} \lim_{r\to 0} A_r^{(n)} g_j(x) A_r^{(n)} h_j(x) \\
&= \sum_{j=1}^{\infty} \tilde{g}_j(x) \tilde{h}_j(x) \\
&= \sum_{j=1}^{\infty} g_j(x) h_j(x) = k(x,x).
\end{aligned}$$

Applying Lemma 3.1 we have

$$tr\,T = \int_{\mathbb{R}^n} \tilde{k}(x,x)dx. \qquad \square$$

Remark 3.9. Theorem 3.8 is also true for a σ-finite borel measure μ on a second countable topological space X. In order to obtain this generalization, the classical maximal function is replaced by the martingale maximal function and the essential properties that we need to generalize the theorem up to the $L^p(\mu)$ case are also true. The $L^2(\mu)$ case was considered by Brislawn in [2], and for the properties of the martingale maximal function see [3].

Corollary 3.10. *Let T be a nuclear operator on $L^p(\mathbb{R}^n)$. Let k be as in Theorem* 2.4 *and suppose that $\alpha(x,y)$ is a measurable function defined almost everywhere on $\mathbb{R}^n \times \mathbb{R}^n$ so that $\alpha(x,y) = k(x,y)$ for a.e. (x,y). Then α is integrable on the diagonal and*

$$tr\,T = \int_{\mathbb{R}^n} \tilde{\alpha}(x,x)dx.$$

Consequently, if α is a continuous, then

$$tr\,T = \int_{\mathbb{R}^n} \alpha(x,x)dx.$$

Proof. The first formula is valid because one has equality almost everywhere on $\mathbb{R}^n \times \mathbb{R}^n$ for α and k implies $\tilde{k}(x,x) = \tilde{\alpha}(x,x)$ for all x. Then $\tilde{\alpha}$ is integrable on the diagonal and the formula follows. The second formula is obtained immediately. $\square$

As an application of the above corollary we obtain the following formula for pseudo-differential operators. This result was established in the L^2 case by Du and Wong for Weyl transforms under a similar hypotheses (cf. [4]). In the following corollary the symbol is considered as a smooth symbol.

Corollary 3.11. *Let $\sigma(x,\xi)$ a symbol of a nuclear pseudo-differential operator T on $L^p(\mathbb{R}^n)$. Suppose that there exists a function g in $L^1(\mathbb{R}^n)$ with $|\sigma(x,\cdot)| \leq g(\cdot)$ for every x. Then*

$$tr\,T = (2\pi)^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi)d\xi dx.$$

Proof. Using the fact that $\sigma(x,\cdot) \in L^1(\mathbb{R}^n)$ for every x, we obtain the well-known kernel for functions in the Schwartz class $\mathcal{S}$ given by

$$\alpha(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}\sigma(x,\xi)d\xi.$$

Moreover, α is continuous and the density of $\mathcal{S}$ in $L^p(\mathbb{R}^n)$ implies that $\alpha(x,y)$ agrees with $k(x,y)$ almost everywhere, where k is the kernel as in Theorem 2.4.

By Corollary 3.10 α is integrable on the diagonal and

$$tr\, T = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(x, \xi) d\xi dx. \qquad \square$$

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Julio Delgado
Universidad del Valle
Calle 13 100-00
Cali-Colombia
e-mail: `delgado@math.jussieu.fr`

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Products of Two-Wavelet Multipliers and Their Traces

Viorel Catană

Abstract. Following Wong's point of view in his book [12] (see Chapter 21) we give in this paper two formulas for the product of two two-wavelet multipliers $\psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\psi T_\tau \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where σ and τ are functions in $L^2(\mathbb{R}^n)$ and φ and ψ are any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. We also give a trace formula and an upper bound estimate on the trace class norm for such a product. Moreover we find sharp estimates on the norm in the trace class of two-wavelet multipliers $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in terms of the symbols σ and the admissible wavelets φ and ψ and also we give an inequality about products of positive trace class one-wavelet multipliers. Finally, we give an example of a two-wavelet multiplier which extends Wong's result concerning the Landau-Pollak-Slepian operator from the one-wavelet case to the two-wavelet case (see Chapter 20 in the book [12] by Wong).

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1. Introduction

Let $\pi : \mathbb{R}^n \to B\left(L^2(\mathbb{R}^n)\right)$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$(\pi(\xi)u)(x) = e^{ix\xi}u(x), \quad x, \xi \in \mathbb{R}^n,$$

for all functions u in $L^2(\mathbb{R}^n)$. The following two results can be found in the paper [11] by Wong and Zhang.

Theorem 1.1. *Let $\sigma \in L^\infty(\mathbb{R}^n)$, and let φ and ψ be any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. If for all functions u in S, we* *define* $P_{\sigma,\varphi,\psi}u$ *by*

$$(P_{\sigma,\varphi,\psi}u, v) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)(u, \pi(\xi)\varphi)(\pi(\xi)\psi, v)\, d\xi \tag{1.1}$$

for all functions v *in* S*, then*

$$(P_{\sigma,\varphi,\psi}u, v) = ((\psi T_\sigma \overline{\varphi})u, v), \quad u, v \in S. \tag{1.2}$$

Theorem 1.2. *Let* $\sigma \in L^1(\mathbb{R}^n)$*, and let* φ *and* ψ *be any functions in* $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *such that* $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$*. If for any function* $u \in L^2(\mathbb{R}^n)$ *we define* $P_{\sigma,\varphi,\psi}u$ *by* (1.1) *for all functions* v *in* $L^2(\mathbb{R}^n)$*, then* $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a bounded linear operator and*

$$\|P_{\sigma,\varphi,\psi}\|_* \le (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)} \tag{1.3}$$

where $\|P_{\sigma,\varphi,\psi}\|_*$ *is the operator norm of* $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$*.*

To define the linear operator $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where $\sigma \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and φ and ψ are any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$, we need the following theorem.

Theorem 1.3. *Let* $\sigma \in L^p(\mathbb{R}^n)$*,* $1 < p < \infty$ *and let* φ *and* ψ *be any functions in* $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *such that* $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$*. Then, there exists a unique bounded linear operator* $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that*

$$\|P_{\sigma,\varphi,\psi}\|_* \le (2\pi)^{-n/p} \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{1/p'} \|\psi\|_{L^\infty(\mathbb{R}^n)}^{1/p'} \|\sigma\|_{L^p(\mathbb{R}^n)} \tag{1.4}$$

and for all functions u *and* v *in* $L^2(\mathbb{R}^n)$*,* $(P_{\sigma,\varphi,\psi}u, v)$ *is given by* (1.1)*, for all simple functions* σ *on* $\mathbb{R}^n$ *for which the Lebesgue measure of the set* $\{\xi \in \mathbb{R}^n; \sigma(\xi) \neq 0\}$ *is finite.*

Proof. Using Plancherel's theorem and the fact that

$$(\pi(\xi)\varphi)^\wedge = T_{-\xi}\widehat{\varphi}, \quad \xi \in \mathbb{R}^n,$$

where

$$(T_{-\xi}f)(x) = f(x - \xi), \quad x \in \mathbb{R}^n,$$

for all measurable functions f on $\mathbb{R}^n$, we get

$$(u, \pi(\xi)\varphi) = \left(\widehat{u} * \overline{\widehat{\varphi}}\right)(\xi) \tag{1.5}$$

and

$$(\pi(\xi)\psi, v) = \overline{(\widehat{v} * \overline{\widehat{\psi}})}(\xi), \tag{1.6}$$

for all ξ in $\mathbb{R}^n$, where

$$\left(\widehat{f} * \overline{\widehat{\varphi}}\right)(\xi) = \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta)\overline{\widehat{\varphi}}(\eta)\, d\eta, \quad \xi \in \mathbb{R}^n$$

and

$$\left(\widehat{f} * \overline{\widehat{\psi}}\right)(\xi) = \int_{\mathbb{R}^n} \widehat{f}(\xi - \eta)\,\overline{\widehat{\psi}}(\eta)\,d\eta, \quad \xi \in \mathbb{R}^n,$$

for all functions f in $S(\mathbb{R}^n)$. Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then by (1.1), (1.5), (1.6), the definition of π and the fact that

$$(f\chi)\widehat{\ } = (2\pi)^{-n/2}\left(\widehat{f} * \widehat{\chi}\right),\ f \in \mathcal{S}(\mathbb{R}^n),\ \chi \in L^2(\mathbb{R}^n) \tag{1.7}$$

we get

$$|(P_{\sigma,\varphi,\psi}u, v)| \le (2\pi)^{-n}\,\|\sigma\|_{L^\infty(\mathbb{R}^n)}\left\|\widehat{u} * \overline{\widehat{\varphi}}\right\|_{L^2(\mathbb{R}^n)}\left\|\widehat{v} * \overline{\widehat{\psi}}\right\|_{L^2(\mathbb{R}^n)} \tag{1.8}$$

for all functions u and v in $L^2(\mathbb{R}^n)$. Using (1.7)–(1.8) and Plancherel's theorem, we get

$$\begin{aligned} |(P_{\sigma,\varphi,\psi}u, v)| &\le \|\sigma\|_{L^\infty(\mathbb{R}^n)}\,\|u\varphi\|_{L^2(\mathbb{R}^n)}\,\|v\psi\|_{L^2(\mathbb{R}^n)} \\ &\le \|\sigma\|_{L^\infty(\mathbb{R}^n)}\,\|\varphi\|_{L^\infty(\mathbb{R}^n)}\,\|\psi\|_{L^\infty(\mathbb{R}^n)}\,\|u\|_{L^2(\mathbb{R}^n)}\,\|v\|_{L^2(\mathbb{R}^n)} \end{aligned} \tag{1.9}$$

for all functions u and v in $L^2(\mathbb{R}^n)$. So, by (1.9)

$$\|P_{\sigma,\varphi,\psi}\|_* \le \|\varphi\|_{L^\infty(\mathbb{R}^n)}\,\|\psi\|_{L^\infty(\mathbb{R}^n)}\,\|\sigma\|_{L^\infty(\mathbb{R}^n)}, \quad \sigma \in L^\infty(\mathbb{R}^n). \tag{1.10}$$

Thus by Theorem 1.2, (1.10) and the Riesz-Thorin interpolation theorem (see e.g. Theorem 12.4 in the book [12] by Wong), the proof is complete. □

Remark 1.1. We call the bounded linear operator $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in Theorem 1.3 a two-wavelet multiplier corresponding to the symbol $\sigma \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ and the two admissible wavelets φ and ψ. Thus we can define the linear operator $\psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where $\sigma \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ and φ and ψ are any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$, by

$$(P_{\sigma,\varphi,\psi}u, v) = ((\psi T_\sigma \overline{\varphi})\,u, v)\,, u, v \in L^2(\mathbb{R}^n)$$

(for more details see the paper [11] by Wong and Zhang).

2. The Main Results

Let the point $z = q + ip$ in $\mathbb{C}^n$, which we identify with $\mathbb{R}^n \times \mathbb{R}^n$, and we define the symplectic form on $\mathbb{C}^n$ by $[z, w] = 2\,\mathrm{Im}\,(z \cdot \overline{w})$, $z, w \in \mathbb{C}^n$, where

$$z = (z_1, \ldots, z_n)\,, w = (w_1, \ldots, w_n)$$

and

$$z \cdot \overline{w} = \sum_{j=1}^{n} z_j \cdot \overline{w}_j.$$

For two measurable functions f and g on $\mathbb{C}^n$ and for any fixed real λ, we define the twisted convolution $f *_\lambda g$ by

$$(f *_\lambda g)(z) = \int_{\mathbb{C}^n} f(z - w)\,g(w)\,e^{i\lambda[z,w]}dw, \quad z \in \mathbb{C}^n \tag{2.1}$$

(for more details see the paper [5] by Grossmann, Loupias and Stein, the book [12] by Wong and the paper [9] by Toft).

The following result, will be useful to us and can be found in the paper [5] by Grossmann, Loupias and Stein.

Theorem 2.1. *Let σ and τ be any functions in $L^2(\mathbb{C}^n)$. Then the product of the Weyl transforms $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $W_\tau : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the Weyl transform $W_\omega : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where ω is the function in $L^2(\mathbb{C}^n)$ given by*

$$\widehat{\omega} = (2\pi)^{-n} \left(\widehat{\sigma} *_{1/4} \widehat{\tau}\right). \tag{2.2}$$

Let $\sigma \in L^1\left(\mathbb{R}^{2n}\right) \cup L^2\left(\mathbb{R}^{2n}\right)$. Then the Weyl transform associated to the symbol σ is the bounded linear operator $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by

$$(W_\sigma f, g) = (2\pi)^{-n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma(x,\xi)\, W(f,g)(x,\xi)\, dx d\xi,$$

for all f and g in $L^2(\mathbb{R}^n)$, where $(,)$ is the inner product in $L^2(\mathbb{R}^n)$ and $W(f,g)$ is the Wigner transform of f and g defined by

$$W(f,g)(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi p} f\left(x + \frac{p}{2}\right) \overline{g\left(x - \frac{p}{2}\right)} dp,$$

for all x and ξ in $\mathbb{R}^n$.

The function $V(f,g)$ on $\mathbb{R}^n \times \mathbb{R}^n$ defined by $\widehat{V(f,g)}(x,\xi) = W(f,g)(x,\xi)$, $x, \xi \in \mathbb{R}^n$ is called the Fourier–Wigner transforms of f and g. This transform will be used in the proof of Theorem 2.4. To give a first formula for the product of two, two-wavelet multipliers we need the following result.

Theorem 2.2. *Let $\sigma \in L^2(\mathbb{R}^n)$ and let φ and ψ be any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Then the two-wavelet multiplier $\psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the Weyl transform $W_{\sigma_{\varphi,\psi}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ where*

$$\sigma_{\varphi,\psi}(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} W(\psi,\varphi)(x,\xi-\eta)\, \sigma(\eta)\, d\eta. \tag{2.3}$$

Proof. We begin with the observation that for all functions f in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ (in particular $f \in S(\mathbb{R}^n)$),

$$\begin{aligned}((\psi T_\sigma \overline{\varphi}) f)(x) &= (2\pi)^{-n/2} \psi(x) \left(\overset{\vee}{\sigma} * \overline{\varphi} f\right)(x) \\ &= (2\pi)^{-n/2} \psi(x) \int_{\mathbb{R}^n} \overset{\vee}{\sigma}(x-y)\, \overline{\varphi}(y)\, f(y)\, dy \\ &= \int_{\mathbb{R}^n} h(x,y)\, f(y)\, dy, \end{aligned} \tag{2.4}$$

for all $x \in \mathbb{R}^n$, where

$$h(x,y) = (2\pi)^{-n/2}\,\psi(x)\,\overset{\vee}{\sigma}(x-y)\,\overline{\varphi}(y), \quad x,y \in \mathbb{R}^n. \tag{2.5}$$

Here the symbol $\vee$ denotes the inverse Fourier transform.

Now, by (2.5), Fubini's theorem and Planchrel's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |h(x,y)|^2\,dxdy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\psi(x)|^2\,|\overset{\vee}{\sigma}(x-y)|^2\,|\varphi(y)|^2\,dxdy \\
&= (2\pi)^{-n}\int_{\mathbb{R}^n} |\varphi(y)|^2 \left(\int_{\mathbb{R}^n} |\psi(x)|^2 \left|\overset{\vee}{\sigma}(x-y)\right|^2 dx\right) dy \\
&\le (2\pi)^{-n}\,\|\psi\|^2_{L^\infty(\mathbb{R}^n)} \left(\int_{\mathbb{R}^n} |\varphi(y)|^2\,dy\right) \cdot ||\overset{\vee}{\sigma}||^2_{L^2(\mathbb{R}^n)} \\
&= (2\pi)^{-n}\,\|\psi\|^2_{L^\infty(\mathbb{R}^n)} \cdot \|\varphi\|^2_{L^2(\mathbb{R}^n)}\,\|\sigma\|^2_{L^2(\mathbb{R}^n)} < \infty.
\end{aligned} \tag{2.6}$$

So, by (2.4)–(2.6), $\psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert–Schmidt operator with kernel h. To complete the proof of Theorem 2.2. it is necessary to use the following result obtained by Pool in [8] (see also Chapter 6 in the book [10] by Wong).

Proposition 2.3. *Let h be a function $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then the Hilbert–Schmidt operator corresponding to the kernel h is necessarily of the form $W_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, for some σ in $L^2(\mathbb{R}^{2n})$. More precisely,*

$$\sigma = (2\pi)^{n/2}\,\mathcal{F}_2 T h, \tag{2.7}$$

where $\mathcal{F}_2$ is the Fourier transform on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ with the respect to the second variable and $T : L^2(\mathbb{R}^n \times \mathbb{R}^n) \to L^2(\mathbb{R}^n \times \mathbb{R}^n)$ is the linear operator defined by

$$(Tf)(x,y) = f\left(x + \frac{y}{2}, x - \frac{y}{2}\right), \quad x,y \in \mathbb{R}^n, \tag{2.8}$$

for all functions f in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

Thus, by Proposition 2.3, the operator $\psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the Weyl transform $W_{\sigma_{\varphi,\psi}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ where

$$\sigma_{\varphi,\psi}(x,\xi) = (2\pi)^{n/2}\,(\mathcal{F}_2 T h)(x,\xi), \quad x,\xi \in \mathbb{R}^n. \tag{2.9}$$

But, by (2.8) and (2.5)

$$(Th)(x,y) = h\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = (2\pi)^{-n/2}\,\psi\left(x + \frac{y}{2}\right)\overset{\vee}{\sigma}(y)\,\overline{\varphi}\left(x - \frac{y}{2}\right),$$

$x, y \in \mathbb{R}^n$ and hence by (1.7) and the definition of the Fourier–Wigner transform, we get

$$\begin{aligned}(\mathcal{F}_2 Th)(x,\xi) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-iy\xi} \psi\left(x+\frac{y}{2}\right) \overset{\vee}{\sigma}(y)\, \overline{\varphi}\left(x-\frac{y}{2}\right) dy \\ &= (2\pi)^{-n/2} \left\{\left[\psi\left(x+\frac{\cdot}{2}\right)\overline{\varphi}\left(x-\frac{\cdot}{2}\right)\right]\overset{\vee}{\sigma}\right\}^{\widehat{}}(\xi) \\ &= (2\pi)^{-n} \left(W(\psi,\varphi)(x,\cdot) * \sigma\right)(\xi) \qquad (2.10)\end{aligned}$$

for all x and ξ in $\mathbb{R}^n$. Hence by (2.9) and (2.10), the proof is complete. □

The following theorem give us a first formula for the product of two two-wavelet multipliers.

Theorem 2.4. *Let σ and τ be functions in $L^2(\mathbb{R}^n)$ and let φ and ψ be any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Then the product of two-wavelet multipliers $\psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\psi T_\tau \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the Weyl transform $W_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, and λ is the function in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ given by*

$$\widehat{\lambda} = (2\pi)^{-n} \left(\widehat{\sigma}_{\varphi,\psi} *_{1/4} \widehat{\tau}_{\varphi,\psi}\right), \qquad (2.11)$$

where $\sigma_{\varphi,\psi}$ and $\tau_{\varphi,\psi}$ are defined by (2.3).

Theorem 2.4 is an immediate consequence of Theorem 2.1. and Theorem 2.2.

We can now give a second formula for the product of two two-wavelet multipliers.

Theorem 2.5. *Let σ and τ be functions in $L^2(\mathbb{R}^n)$ and let φ and ψ be any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Then the product of the two-wavelet multipliers $\psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $\psi T_\tau \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the linear operator $\psi W_\lambda \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ where $W_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the Weyl transform associated to λ and*

$$\lambda(x,\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} W(\sigma,\overline{\tau})(\xi, y-x)\, \psi(y)\, \overline{\varphi}(y)\, dy \qquad (2.12)$$

for all x and ξ in $\mathbb{R}^n$.

The proof of Theorem 2.5 is similar as the proof of Theorem 21.2 in the book [12] by Wong, so we omit that.

Remark 2.1. If $\varphi = \psi$, then we recover from Theorems 2.2., 2.4. and 2.5. respectively, the Theorems 21.5, 21.6 and 21.2 in the book [12] by Wong.

Theorem 2.6. *Let σ and τ be functions in $L^2(\mathbb{R}^n)$ and let φ and ψ be any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Then the bounded* *linear operator* $P_{\sigma,\varphi,\psi}P_{\tau,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *is a trace class operator, and the trace of this is equal to*

$$\operatorname{tr}(P_{\sigma,\varphi,\psi}P_{\tau,\varphi,\psi}) = (2\pi)^{-n} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \sigma_{\varphi,\psi}(x,\xi)\,\tau_{\varphi,\psi}(x,\xi)\,dxd\xi.$$

Moreover,

$$\begin{aligned}&\|P_{\sigma,\varphi,\psi}P_{\tau,\varphi,\psi}\|_{S_1}\\ &\le (2\pi)^{-n}\|\sigma_{\varphi,\psi}\|_{L^2(\mathbb{R}^{2n})}\|\tau_{\varphi,\psi}\|_{L^2(\mathbb{R}^{2n})}\\ &\le (2\pi)^{-n}\left\{\min\left(\|\varphi\|_{L^\infty(\mathbb{R}^n)},\|\psi\|_{L^\infty(\mathbb{R}^n)}\right)\right\}^2\|\sigma\|_{L^2(\mathbb{R}^n)}\|\tau\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Proof. By Theorem 2.4 we know that the product of the two-wavelet multipliers $P_{\sigma,\varphi,\psi} = \psi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and $P_{\tau,\varphi,\psi} = \psi T_\tau \overline{\varphi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is the same as the Weyl transform $W_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, where λ is the function in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ given by (2.11).

Consequently, by Theorem 6.1 and Theorem 7.1 in the paper [14] by Wong we get respectively that W_λ is in the trace class S_1,

$$\|W_\lambda\|_{S_1} \le (2\pi)^{-n}\|a\|_{L^2(\mathbb{R}^{2n})}\|b\|_{L^2(\mathbb{R}^{2n})}$$

and

$$\operatorname{tr}(W_\lambda) = (2\pi)^{-n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} a(x,\xi)\,b(x,\xi)\,dxd\xi$$

if $\widehat{\lambda} = (2\pi)^{-n}\left(\widehat{a} *_{1/4} \widehat{b}\right)$, where a and b are in $L^2\left(\mathbb{R}^{2n}\right)$.

By Proposition 2.3, (2.5), (2.9), Fubini's theorem and Plancherel's theorem,

$$\begin{aligned}\|\sigma_{\varphi,\psi}\|^2_{L^2(\mathbb{R}^{2n})} &= (2\pi)^n\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|(\mathcal{F}_2 Th)(x,\xi)|^2\,dxd\xi\\ &= (2\pi)^n\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|Th(x,y)|^2\,dxdy\\ &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\left|\psi\left(x+\frac{y}{2}\right)\right|^2\left|\overset{\vee}{\sigma}(y)\right|^2\left|\varphi\left(x-\frac{y}{2}\right)\right|^2 dxdy\\ &\le \|\psi\|^2_{L^\infty(\mathbb{R}^n)}\|\sigma\|^2_{L^2(\mathbb{R}^n)}\|\varphi\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

and analogously

$$\|\sigma_{\varphi,\psi}\|^2_{L^2(\mathbb{R}^{2n})} \le \|\varphi\|^2_{L^\infty(\mathbb{R}^n)}\|\sigma\|^2_{L^2(\mathbb{R}^n)}\|\psi\|^2_{L^2(\mathbb{R}^n)}.$$

By the preceding estimates, we get

$$\|\sigma_{\varphi,\psi}\|_{L^2(\mathbb{R}^{2n})} \le \min\left(\|\varphi\|_{L^\infty(\mathbb{R}^n)},\|\psi\|_{L^\infty(\mathbb{R}^n)}\right)\|\sigma\|_{L^2(\mathbb{R}^n)}.$$

In the same manner we write

$$\|\tau_{\varphi,\psi}\|_{L^2(\mathbb{R}^{2n})} \le \min\left(\|\varphi\|_{L^\infty(\mathbb{R}^n)},\|\psi\|_{L^\infty(\mathbb{R}^n)}\right)\|\tau\|_{L^2(\mathbb{R}^n)}.$$

Thus, Theorem 2.6 is an immediate consequence of the above statements and estimates. □

Theorem 2.6 admits the following generalization.

Theorem 2.7. *Let $\sigma_1, \ldots, \sigma_m, m \geq 2$ be m functions in $L^2(\mathbb{R}^n)$ and let φ and ψ be any functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Then the product of the two-wavelet multipliers $P_{\sigma_1,\varphi,\psi}, \ldots, P_{\sigma_m,\varphi,\psi}$ is a trace class operator, which trace is given by*

$$\operatorname{tr}\left(P_{\sigma_1,\varphi,\psi} \cdot \cdots \cdot P_{\sigma_m,\varphi,\psi}\right) = (2\pi)^{-n(m-1)} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \sigma_{1,\varphi,\psi}(x,\xi)$$
$$\times \left(\widehat{\sigma}_{2,\varphi,\psi} *_{1/4} \cdots \left(\widehat{\sigma}_{m-1,\varphi,\psi} *_{1/4} \widehat{\sigma}_{m,\varphi,\psi}\right) \cdots\right)^{\vee}(x,\xi)\, dx d\xi,$$

where $(\cdots)^{\vee}$ denotes the inverse Fourier transform of $(\cdots)$. Moreover,

$$\begin{aligned}&\|P_{\sigma_1,\varphi,\psi} \cdot \cdots \cdot P_{\sigma_m,\varphi,\psi}\|_{S_1} \leq (2\pi)^{-nm/2} \|\sigma_{1,\varphi,\psi}\|_{L^2(\mathbb{R}^{2n})} \cdots \|\sigma_{m,\varphi,\psi}\|_{L^2(\mathbb{R}^{2n})}\\ &\leq (2\pi)^{-nm/2} \left\{\min\left(\|\varphi\|_{L^\infty(\mathbb{R}^n)}, \|\psi\|_{L^\infty(\mathbb{R}^n)}\right)\right\}^m \|\sigma_1\|_{L^2(\mathbb{R}^n)} \cdots \|\sigma_m\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

The proof of Theorem 2.7. follows immediately by induction on m, Theorem 4.3 in the paper [14] by Wong and Theorem 11.2 in the book [4] by Gohberg, Goldberg and Krupnik.

3. Trace Class Norm Inequalities for Two-Wavelet Multipliers

In this paragraph we give sharp estimates on the norms in the trace class of two-wavelet multipliers $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ in terms of the symbol σ and the admissible wavelets φ and ψ.

To this end let us define the function $\widetilde{\sigma} : \mathbb{R}^n \to \mathbb{C}$ by

$$\widetilde{\sigma}(\xi) = \left(P_{\sigma,\varphi,\psi}\pi(\xi)\varphi, \pi(\xi)\psi\right), \quad \xi \in \mathbb{R}^n. \tag{3.1}$$

We first recall the following result (see Theorem 1.1 in the paper [11] by Wong and Zhang) which we need in the sequel.

Theorem 3.1. *Let $\varphi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = 1$. Then for all functions u and v in the Schwartz space S,*

$$(2\pi)^{-n} \int_{\mathbb{R}^n} (u, \pi(\xi)\varphi)(\pi(\xi)\varphi, v)\, d\xi = (\varphi u, \varphi v), \tag{3.2}$$

where $(,)$ is the inner product in $L^2(\mathbb{R}^n)$.

Proposition 3.2. *Let $\sigma \in L^1(\mathbb{R}^n)$ and let $\varphi, \psi \in L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Then $\widetilde{\sigma} \in L^1(\mathbb{R}^n)$ and*

$$\|\widetilde{\sigma}\|_{L^1(\mathbb{R}^n)} \leq \frac{\|\varphi\|^4_{L^4(\mathbb{R}^n)} + \|\psi\|^4_{L^4(\mathbb{R}^n)}}{2} \|\sigma\|_{L^1(\mathbb{R}^n)}. \tag{3.3}$$

Proof. By the definition of $\pi : \mathbb{R}^n \to B\left(L^2\left(\mathbb{R}^n\right)\right)$, (1.1), (3.1), Fubini's theorem, Schwartz' inequality and (3.2), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\widetilde{\sigma}\left(\xi\right)\right| d\xi \qquad (3.4)\\
&\leq (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left|\sigma\left(\eta\right)\right| \left|\left(\pi\left(\xi\right)\varphi, \pi\left(\eta\right)\varphi\right)\right| \left|\left(\pi\left(\eta\right)\psi, \pi\left(\xi\right)\psi\right)\right| d\eta \right) d\xi\\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \left|\sigma\left(\eta\right)\right| \left(\int_{\mathbb{R}^n} \left|\left(\pi\left(\xi\right)\varphi, \pi\left(\eta\right)\varphi\right)\right| \left|\left(\pi\left(\eta\right)\psi, \pi\left(\xi\right)\psi\right)\right| d\xi \right) d\eta\\
&\leq (2\pi)^{-n} \cdot 2^{-1} \int_{\mathbb{R}^n} \left|\sigma\left(\eta\right)\right| \left(\int_{\mathbb{R}^n} (\left|\left(\pi\left(\xi\right)\varphi, \pi\left(\eta\right)\varphi\right)\right|^2 + \left|\left(\pi\left(\eta\right)\psi, \pi\left(\xi\right)\psi\right)\right|^2) d\xi \right) d\eta\\
&= (2\pi)^{-n} \cdot 2^{-1} \int_{\mathbb{R}^n} \left|\sigma\left(\eta\right)\right| \left(\left\|\varphi\pi\left(\eta\right)\varphi\right\|^2_{L^2(\mathbb{R}^n)} + \left\|\psi\pi\left(\eta\right)\psi\right\|^2_{L^2(\mathbb{R}^n)} \right) d\eta. \qquad (3.5)
\end{aligned}$$

Using the definition of $\pi : \mathbb{R}^n \to B\left(L^2\left(\mathbb{R}^n\right)\right)$, we get

$$\left\|\varphi\pi\left(\eta\right)\varphi\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\varphi\left(x\right)\right|^4 dx = \left\|\varphi\right\|^4_{L^4(\mathbb{R}^n)} \qquad (3.6)$$

and

$$\left\|\psi\pi\left(\eta\right)\psi\right\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\psi\left(x\right)\right|^4 dx = \left\|\psi\right\|^4_{L^4(\mathbb{R}^n)}. \qquad (3.7)$$

Thus, by (3.4)–(3.7), inequality (3.3) follows. □

Proposition 3.3. *Let* $\sigma \in L^1\left(\mathbb{R}^n\right)$*, and let* $\varphi, \psi \in L^2\left(\mathbb{R}^n\right) \cap L^4\left(\mathbb{R}^n\right)$ *be such that* $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$*. Then*

$$\int_{\mathbb{R}^n} \widetilde{\sigma}\left(\xi\right) d\xi = \left(\left|\psi\right|^2, \left|\varphi\right|^2\right) \int_{\mathbb{R}^n} \sigma\left(\eta\right) d\eta. \qquad (3.8)$$

Proof. By (3.1), Fubini's theorem and Plancharel's theorem, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \widetilde{\sigma}\left(\xi\right) d\xi &= (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \sigma\left(\eta\right) \left(\pi\left(\xi\right)\varphi, \pi\left(\eta\right)\varphi\right) \left(\pi\left(\eta\right)\psi, \pi\left(\xi\right)\psi\right) d\eta \right) d\xi\\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma\left(\eta\right) \left(\int_{\mathbb{R}^n} \left(\pi\left(\xi\right)\varphi, \pi\left(\eta\right)\varphi\right) \left(\pi\left(\eta\right)\psi, \pi\left(\xi\right)\psi\right) d\xi \right) d\eta\\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma\left(\eta\right) \left(\int_{\mathbb{R}^n} \left(\pi\left(\xi\right)\varphi, \varphi\right) \left(\psi, \pi\left(\xi\right)\psi\right) d\xi \right) d\eta
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \sigma(\eta) \left(\int_{\mathbb{R}^n} \widehat{|\psi|^2}(\xi) \; \overline{\widehat{|\varphi|^2}}(\xi)\, d\xi \right) d\eta$$

$$= \left(\int_{\mathbb{R}^n} \sigma(\eta)\, d\eta \right) \left(|\psi|^2, |\varphi|^2 \right). \tag{3.9}$$

We have used for the last but one inequality the following relations:

$$(\pi(\xi)\varphi, \varphi) = \overline{(\varphi, \pi(\xi)\varphi)} = \overline{\left(\widehat{\varphi} * \widehat{\overline{\varphi}}\right)}(\xi) = (2\pi)^{n/2}\, \overline{\widehat{\varphi\overline{\varphi}}}(\xi) = (2\pi)^{n/2}\, \overline{\widehat{|\varphi|^2}}(\xi),$$

$$(\psi, \pi(\xi)\psi) = \left(\widehat{\psi} * \widehat{\overline{\psi}}\right)(\xi) = (2\pi)^{n/2}\, \widehat{\psi\overline{\psi}}(\xi) = (2\pi)^{n/2}\, \widehat{|\psi|^2}(\xi). \qquad \square$$

Remark 3.1. If $\varphi = \psi$, then from Proposition 3.2 and Proposition 3.3. we find Proposition 2.1. and Proposition 2.2 respectively in the paper [13] by Wong and Zhang.

Theorem 3.4. *Let $\sigma \in L^1(\mathbb{R}^n)$ and let $\varphi, \psi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$. Then*

$$(2\pi)^{-n} \cdot 2 \left(\|\varphi\|^2_{L^\infty(\mathbb{R}^n)} + \|\psi\|^2_{L^\infty(\mathbb{R}^n)} \right)^{-1} \|\widetilde{\sigma}\|_{L^1(\mathbb{R}^n)}$$
$$\leq \|P_{\sigma,\varphi,\psi}\|_{S_1} \leq (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)} \tag{3.10}$$

Proof. By Theorem 3.1 in the paper [11] by Wong and Zhang the two-wavelet multiplier $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is in the trace class S_1. We write

$$P_{\sigma,\varphi,\psi} = V\, |P_{\sigma,\varphi,\psi}|$$

for the polar form of the bounded linear operator $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ on the complex Hilbert space $L^2(\mathbb{R}^n)$, where $|P_{\sigma,\varphi,\psi}| = \left(P^*_{\sigma,\varphi,\psi} P_{\sigma,\varphi,\psi}\right)^{1/2}$, $V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$,

$$\|Vf\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}, \quad f \in N(P_{\sigma,\varphi,\psi})^\perp$$

and

$$V(f) = 0, f \in N(P_{\sigma,\varphi,\psi}),$$

where $N(P_{\sigma,\varphi,\psi})$ is the kernel of $P_{\sigma,\varphi,\psi}$ and $N(P_{\sigma,\varphi,\psi})^\perp$ is the orthogonal complement of $N(P_{\sigma,\varphi,\psi})$. Let $\{\varphi_k : k = 1, 2, \dots\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$ consisting of eigenvectors of $|P_{\sigma,\varphi,\psi}|$ and let $s_k(P_{\sigma,\varphi,\psi})$ be the eigenvalues of $|P_{\sigma,\varphi,\psi}|$ corresponding to $\varphi_k, k = 1, 2, \dots$. For $k = 1, 2, \dots$ let

$$\psi_k = V\varphi_k.$$

We choose the orthonormal basis $\{\varphi_k : k = 1, 2, \dots\}$ such that $\{\psi_k : k = 1, 2, \dots\}$ is an orthonormal set in $L^2(\mathbb{R}^n)$ and to have

$$P_{\sigma,\varphi,\psi} f = \sum_{k=1}^{\infty} s_k(P_{\sigma,\varphi,\psi})(f, \varphi_k)\, \psi_k, \quad f \in L^2(\mathbb{R}^n), \tag{3.11}$$

where the convergence of the series is understood to be in $L^2(\mathbb{R}^n)$ (see Theorem 2.2 in the book [12] by Wong).

By (3.11)

$$\sum_{j=1}^{\infty} (P_{\sigma,\varphi,\psi}\varphi_j, \psi_j) = \sum_{j=1}^{\infty} s_j (P_{\sigma,\varphi,\psi}). \tag{3.12}$$

So, by (3.12) we get

$$\|P_{\sigma,\varphi,\psi}\|_{S_1} = \sum_{j=1}^{\infty} (P_{\sigma,\varphi,\psi}\varphi_j, \psi_j). \tag{3.13}$$

Thus, by (3.13), Fubini's theorem, the Parseval's identity, the Bessel inequality, the Schwartz' inequality and $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n)} = 1$, we get

$$\begin{aligned}
&\|P_{\sigma,\varphi,\psi}\|_{S_1} &(3.14)\\
&= \sum_{k=1}^{\infty} (P_{\sigma,\varphi,\psi}\varphi_k, \psi_k)\\
&= \sum_{k=1}^{\infty} (2\pi)^{-n} \left| \int_{\mathbb{R}^n} \sigma(\xi) (\varphi_k, \pi(\xi)\varphi) (\pi(\xi)\psi, \psi_k)\, d\xi \right|\\
&\le \sum_{k=1}^{\infty} (2\pi)^{-n} \int_{\mathbb{R}^n} |\sigma(\xi)| \, |(\varphi_k, \pi(\xi)\varphi)| \, |(\pi(\xi)\psi, \psi_k)|\, d\xi\\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} |\sigma(\xi)| \sum_{k=1}^{\infty} |(\varphi_k, \pi(\xi)\varphi)| \, |(\pi(\xi)\psi, \psi_k)|\, d\xi\\
&\le (2\pi)^{-n} \int_{\mathbb{R}^n} |\sigma(\xi)| \left(\sum_{k=1}^{\infty} |(\varphi_k, \pi(\xi)\varphi)|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |(\pi(\xi)\psi, \psi_k)|^2 \right)^{1/2} d\xi\\
&\le (2\pi)^{-n} \int_{\mathbb{R}^n} |\sigma(\xi)| \, \|\pi(\xi)\varphi\|_{L^2(\mathbb{R}^n)} \|\pi(\xi)\psi\|_{L^2(\mathbb{R}^n)}\, d\xi\\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} |\sigma(\xi)|\, d\xi\\
&= (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)}. &(3.15)
\end{aligned}$$

Using (3.1), (3.11) and the Schwartz inequality, we get

$$
\begin{aligned}
|\widetilde{\sigma}(\xi)| &= |(P_{\sigma,\varphi,\psi}\pi(\xi)\varphi, \pi(\xi)\psi)| \\
&= \left|\sum_{k=1}^{\infty} s_k(P_{\sigma,\varphi,\psi})(\pi(\xi)\varphi,\varphi_k)(\psi_k,\pi(\xi)\psi)\right| \\
&\leq \frac{1}{2}\sum_{k=1}^{\infty} s_k(P_{\sigma,\varphi,\psi})\left(|\pi(\xi)\varphi,\varphi_k)|^2 + |(\psi_k,\pi(\xi)\psi)|^2\right), \qquad (3.16)
\end{aligned}
$$

for all ξ in $\mathbb{R}^n$. Thus, by (3.16), Theorem 3.1, Fubini's theorem and $\|\varphi_k\|_{L^2(\mathbb{R}^n)} = \|\psi_k\|_{L^2(\mathbb{R}^n)} = 1, k = 1, 2, \dots$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\widetilde{\sigma}(\xi)|\, d\xi \\
&\leq \frac{1}{2}\sum_{k=1}^{\infty} s_k(P_{\sigma,\varphi,\psi})\left(\int_{\mathbb{R}^n} |(\pi(\xi)\varphi,\varphi_k)|^2 d\xi + \int_{\mathbb{R}^n} |(\psi_k,\pi(\xi)\psi)|^2 d\xi\right) \\
&= \frac{1}{2}\sum_{k=1}^{\infty} s_k(P_{\sigma,\varphi,\psi})(2\pi)^n\left(\|\varphi\varphi_k\|^2_{L^2(\mathbb{R}^n)} + \|\psi\psi_k\|^2_{L^2(\mathbb{R}^n)}\right). \qquad (3.17)
\end{aligned}
$$

For $k = 1, 2, \dots$

$$
\|\varphi\varphi_k\|_{L^2(\mathbb{R}^n)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^n)} \cdot \|\varphi_k\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{L^\infty(\mathbb{R}^n)}, \qquad (3.18)
$$

$$
\|\psi\psi_k\|_{L^2(\mathbb{R}^n)} \leq \|\psi\|_{L^\infty(\mathbb{R}^n)} \cdot \|\psi_k\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^\infty(\mathbb{R}^n)}. \qquad (3.19)
$$

By (3.17)–(3.19), we get

$$
\int_{\mathbb{R}^n} |\widetilde{\sigma}(\xi)|\, d\xi \leq (2\pi)^n \|P_{\sigma,\varphi,\psi}\|_{S_1} \frac{1}{2}\left(\|\varphi\|^2_{L^\infty(\mathbb{R}^n)} + \|\psi\|^2_{L^\infty(\mathbb{R}^n)}\right)
$$

and hence

$$
(2\pi)^{-n} \cdot 2\left(\|\varphi\|^2_{L^\infty(\mathbb{R}^n)} + \|\psi\|^2_{L^\infty(\mathbb{R}^n)}\right)^{-1} \|\widetilde{\sigma}\|_{L^1(\mathbb{R}^n)} \leq \|P_{\sigma,\varphi,\psi}\|_{S_1}. \qquad (3.20)
$$

So, by (3.15) and (3.20) the proof is complete. □

Remark 3.2. If $\varphi = \psi$ and φ is a function in $L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{L^4(\mathbb{R}^n)} = \|\varphi\|_{L^\infty(\mathbb{R}^n)} = 1$, and σ is a real-valued and non-negative function in $L^1(\mathbb{R}^n)$, then $P_{\sigma,\varphi,\psi} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a positive operator, and by Remark 2.4 in the paper [13] by Wong and Zhang, we get

$$
(2\pi)^{-n} \|\widetilde{\sigma}\|_{L^1(\mathbb{R}^n)} = \|P_{\sigma,\varphi,\psi}\|_{S_1} = (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)}.
$$

Thus the estimates in Theorem 3.4 are sharp.

Remark 3.3. If $\varphi = \psi$, we have from Theorem 3.4, Theorem 2.3 in the paper [13] by Wong and Zhang.

Now we state a result concerning the trace of products of one-wavelet multipliers.

Theorem 3.5. *Let σ and τ be any real-valued and non-negative functions in $L^1(\mathbb{R}^n)$ and let φ be a function in $L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{L^4(\mathbb{R}^n)} = \|\varphi\|_{L^\infty(\mathbb{R}^n)} = 1$. Then the one-wavelet multipliers $P_{\sigma,\varphi} = \varphi T_\sigma \overline{\varphi}$, $P_{\tau,\varphi} = \varphi T_\tau \overline{\varphi}$ are positive trace class operators and*

$$\left\| (P_{\sigma,\varphi} P_{\tau,\varphi})^k \right\|_{S_1} = \operatorname{tr}(P_{\sigma,\varphi} P_{\tau,\varphi})^k \leq (\operatorname{tr} P_{\sigma,\varphi})^k (\operatorname{tr} P_{\tau,\varphi})^k = \|P_{\sigma,\varphi}\|_{S_1}^k \|P_{\tau,\varphi}\|_{S_1}^k$$
$$= (2\pi)^{-2nk} \|\sigma\|_{L^1(\mathbb{R}^n)}^k \|\tau\|_{L^1(\mathbb{R}^n)}^k,$$

for all natural numbers k.

Proof. By Theorem 1 in the paper [7] by Liu we know that if A, B are in the trace class S_1 and are positive operators, then

$$\operatorname{tr}(AB)^k \leq (\operatorname{tr} A)^k (\operatorname{tr} B)^k,$$

for all natural numbers k.

So, if we take $A = P_{\sigma,\varphi}, B = P_{\tau,\varphi}$ and we invoke the Remark 3.2, the proof is complete. □

Remark 3.4. If we take $k = 1$ in Theorem 3.5, we get

$$\|P_{\sigma,\varphi} P_{\tau,\varphi}\|_{S_1} = \operatorname{tr}(P_{\sigma,\varphi} P_{\tau,\varphi}) \leq \operatorname{tr}(P_{\sigma,\varphi}) \operatorname{tr}(P_{\tau,\varphi}) = (2\pi)^{-2n} \|\sigma\|_{L^1(\mathbb{R}^n)} \|\tau\|_{L^1(\mathbb{R}^n)}.$$

In addition, if we suppose that $\sigma, \tau \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then we get

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma_\varphi(x,\xi) \tau_\varphi(x,\xi) \, dx d\xi \leq (2\pi)^{-n} \|\sigma\|_{L^1(\mathbb{R}^n)} \|\tau\|_{L^1(\mathbb{R}^n)}.$$

(See Theorems 21.5 and 21.6 in the book [12] by Wong and Theorem 7.1 in the paper [14] by Wong.)

4. The Generalized Landau–Pollak–Slepian Operator

Wong showed in Chapter 20 of his book [12] that the Landau–Pollak–Slepian operator arising in signal analysis is an one-wavelet multiplier. Following Wong's point of view we define a bounded linear operator from $L^2(\mathbb{R}^n)$ into itself. We show that this operator is a two-wavelet operator and is the analogue in the two-wavelet case of the Landau–Pollak–Slepian operator from the one-wavelet case. From this reason we call it the generalized Landau–Pollak–Slepian operator.

Let Ω_1, Ω_2, T be positive numbers. Then we define the linear operators $P_{\Omega_1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $P_{\Omega_2} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $Q_T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, by

$$(P_{\Omega_1} f)^\wedge(\xi) = \begin{cases} \widehat{f}(\xi), & |\xi| \leq \Omega_1 \\ 0, & |\xi| > \Omega_1, \end{cases} \tag{4.1}$$

$$(P_{\Omega_2} f)^\wedge(\xi) = \begin{cases} \widehat{f}(\xi), & |\xi| \leq \Omega_2 \\ 0, & |\xi| > \Omega_2, \end{cases} \tag{4.2}$$

$$(Q_T f)(x) = \begin{cases} f(x), & |x| \le T \\ \\ 0, & |x| > T. \end{cases} \tag{4.3}$$

Proposition 4.1. $P_{\Omega_1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $P_{\Omega_2} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $Q_T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *are self-adjoint projections.*

The proof of Proposition 4.1 is similar as the proof of Proposition 20.1 in the book [12] by Wong, so we omit that.

Using the fact that $P_{\Omega_1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $P_{\Omega_2} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $Q_T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are self-adjoint and that the last operator is a projection, we get

$$\begin{aligned} &\sup\left\{\frac{(Q_T P_{\Omega_1} f,\ Q_T P_{\Omega_2} f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} : f \in L^2(\mathbb{R}^n),\ f \ne 0\right\} \\ &= \sup\left\{\frac{(P_{\Omega_2} Q_T P_{\Omega_1} f, f)_{L^2(\mathbb{R}^n)}}{\|f\|^2_{L^2(\mathbb{R}^n)}} : f \in L^2(\mathbb{R}^n),\ f \ne 0\right\} \\ &= \sup\left\{(P_{\Omega_2} Q_T P_{\Omega_1} f, f)_{L^2(\mathbb{R}^n)} : f \in L^2(\mathbb{R}^n),\ \|f\|_{L^2(\mathbb{R}^n)} = 1\right\} \\ &= \|P_{\Omega_2} Q_T P_{\Omega_1}\|_{B(L^2(\mathbb{R}^n))}. \end{aligned} \tag{4.4}$$

In the last equality we use the fact that $P_{\Omega_2} Q_T P_{\Omega_1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is self-adjoint.

The bounded linear operator $P_{\Omega_2} Q_T P_{\Omega_1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, that it has appeared in the context of time and band-limited signals can be called the generalized Landau–Pollak–Slepian operator.

We can show that the generalized Landau–Pollak–Slepian operator is in fact a two-wavelet multiplier. To this end let us denote by B_Ω the ball in $\mathbb{R}^n$ with center at the origin and radius Ω.

Theorem 4.2. *Let φ and ψ be the functions on $\mathbb{R}^n$ defined by*

$$\varphi(x) = \begin{cases} \dfrac{1}{\sqrt{\mu(B_{\Omega_1})}}, & |x| \le \Omega_1 \\ 0 & |x| > \Omega_1, \end{cases} \tag{4.5}$$

$$\psi(x) = \begin{cases} \dfrac{1}{\sqrt{\mu(B_{\Omega_2})}}, & |x| \le \Omega_2 \\ 0 & |x| > \Omega_1, \end{cases} \tag{4.6}$$

where $\mu(B_{\Omega_1})$, $\mu(B_{\Omega_2})$ are the volume of B_{Ω_1}, B_{Ω_2} respectively and let σ be the characteristic function on B_T, i.e.,

$$\sigma(\xi) = \begin{cases} 1, & |\xi| \le T \\ 0, & |\xi| > T. \end{cases} \tag{4.7}$$

Then the generalized Landau–Pollak–Slepian operator $P_{\Omega_2} Q_T P_{\Omega_1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is unitary equivalent to a scalar multiple of the two-wavelet multiplier

$\psi T_\sigma \varphi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. *In fact*

$$P_{\Omega_2} Q_T P_{\Omega_1} = \sqrt{\mu(B_{\Omega_1})\mu(B_{\Omega_2})}\mathcal{F}^{-1}(\psi T_\sigma \varphi)\mathcal{F}. \tag{4.8}$$

Proof. First, let us observe that φ and ψ are functions in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} \|\varphi\|_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\varphi(x)|^2 \, dx = \frac{1}{\mu(B_{\Omega_1})} \int_{B_{\Omega_1}} dx = 1, \\ \|\psi\|_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\psi(x)|^2 \, dx = \frac{1}{\mu(B_{\Omega_2})} \int_{B_{\Omega_2}} dx = 1. \end{aligned} \tag{4.9}$$

So,

$$((\psi T_\sigma \varphi) u, v)_{L^2(\mathbb{R}^n)} = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi) (u, \pi(\xi)\varphi)_{L^2(\mathbb{R}^n)} (\pi(\xi)\psi, v)_{L^2(\mathbb{R}^n)} \, d\xi, \tag{4.10}$$

for all functions u and v in $\mathcal{S}$. Next we can write

$$\begin{aligned} (u, \pi(\xi)\varphi) &= \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) u(x) \, dx \\ &= \frac{1}{\sqrt{\mu(B_{\Omega_1})}} \int_{B_{\Omega_1}} e^{-ix\xi} u(x) \, dx, \quad u \in \mathcal{S}. \end{aligned} \tag{4.11}$$

By (4.1)

$$\left(P_{\Omega_1} \overset{\vee}{u}\right)^\wedge (x) = \begin{cases} u(x), & |x| \le \Omega_1 \\ 0 & |x| > \Omega_1, \end{cases} \tag{4.12}$$

for all functions u in $\mathcal{S}$, where $\overset{\vee}{u}$ is the inverse Fourier transform of u. So, by (4.12) and Fourier inversion formula, we get

$$\begin{aligned} (u, \pi(\xi)\varphi) &= \frac{1}{\sqrt{\mu(B_{\Omega_1})}} \int_{\mathbb{R}^n} e^{-ix\xi} \left(P_{\Omega_1} \overset{\vee}{u}\right)^\wedge (x) \, dx \\ &= \frac{(2\pi)^{n/2}}{\sqrt{\mu(B_{\Omega_1})}} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{+ix(-\xi)} \left(P_{\Omega_1} \overset{\vee}{u}\right)^\wedge (x) \, dx \\ &= \frac{(2\pi)^{n/2}}{\sqrt{\mu(B_{\Omega_1})}} \left(P_{\Omega_1} \overset{\vee}{u}\right)(-\xi), \quad \xi \in \mathbb{R}^n, \end{aligned} \tag{4.13}$$

for all functions u in $\mathcal{S}$. Similarly, we get

$$\begin{aligned} (\pi(\xi)\psi, v) &= \overline{(v, \pi(\xi)\psi)} \\ &= \frac{(2\pi)^{n/2}}{\sqrt{\mu(B_{\Omega_2})}} \overline{\left(P_{\Omega_2} \overset{\vee}{v}\right)}(-\xi). \end{aligned} \tag{4.14}$$

So, by (4.3), (4.7), (4.10), (4.13), (4.14), Plancherel's theorem and the fact that $P_{\Omega_2} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is self-adjoint,

$$((\psi T_\sigma \varphi) u, v)_{L^2(\mathbb{R}^n)} = \frac{1}{\sqrt{\mu(B_{\Omega_1}) \mu(B_{\Omega_2})}} \int_{\mathbb{R}^n} \sigma(\xi) \left(P_{\Omega_1} \overset{\vee}{u}\right)(\xi) \overline{\left(P_{\Omega_2} \overset{\vee}{v}\right)(\xi)} d\xi \tag{4.15}$$

$$= \frac{1}{\sqrt{\mu(B_{\Omega_1}) \mu(B_{\Omega_2})}} \int_{B_T} \left(P_{\Omega_1} \overset{\vee}{u}\right)(\xi) \overline{\left(P_{\Omega_2} \overset{\vee}{v}\right)(\xi)} d\xi \tag{4.16}$$

$$= \frac{1}{\sqrt{\mu(B_{\Omega_1}) \mu(B_{\Omega_2})}} \int_{\mathbb{R}^n} \left(Q_T P_{\Omega_1} \overset{\vee}{u}\right)(\xi) \overline{\left(P_{\Omega_2} \overset{\vee}{v}\right)(\xi)} d\xi \tag{4.17}$$

$$= \frac{1}{\sqrt{\mu(B_{\Omega_1}) \mu(B_{\Omega_2})}} \left(Q_T P_{\Omega_1} \overset{\vee}{u}, P_{\Omega_2} \overset{\vee}{v}\right)_{L^2(\mathbb{R}^n)} \tag{4.18}$$

$$= \frac{1}{\sqrt{\mu(B_{\Omega_1}) \mu(B_{\Omega_2})}} \left(P_{\Omega_2} Q_T P_{\Omega_1} \overset{\vee}{u}, \overset{\vee}{v}\right)_{L^2(\mathbb{R}^n)} \tag{4.19}$$

$$= \frac{1}{\sqrt{\mu(B_{\Omega_1}) \mu(B_{\Omega_2})}} \left(\mathcal{F} P_{\Omega_2} Q_T P_{\Omega_1} \mathcal{F}^{-1} u, v\right)_{L^2(\mathbb{R}^n)} \tag{4.20}$$

for all functions u and v in $\mathcal{S}$ and hence the proof is complete. □

The next theorem gives a formula for the trace of the generalized Landau–Pollak–Slepian operator $P_{\Omega_2} Q_T P_{\Omega_1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Theorem 4.3. $\operatorname{tr}(P_{\Omega_2} Q_T P_{\Omega_1}) = \left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^{-2} \left\{\frac{T \min(\Omega_1, \Omega_2)}{2}\right\}^n.$

Theorem 4.3 is an immediate consequence of (4.5)–(4.7), the fact that

$$\operatorname{tr}(\psi T_\sigma \varphi) = (2\pi)^{-n} (\psi, \varphi)_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^n} \sigma(\xi)\, d\xi, \tag{4.21}$$

(see Theorem 3.2 in the paper [11] by Wong and Zhang), Theorem 4.2 and the statement that the volume of the ball in $\mathbb{R}^n$ with radius r is equal to $\frac{\pi^{n/2} r^{n/2}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}$.

Remark 4.1. If $\Omega_1 = \Omega_2$, then by Theorems 4.2 and 4.3, we deduce Theorems 20.2 and 20.3, respectively of the Chapter 20 in the book [12] by Wong.

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Viorel Catană
University Politehnica of Bucharest
Department of Mathematics I
Splaiul Independenţei 313
060042, Bucharest
Romania
e-mail: catana_viorel@yahoo.co.uk

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Pseudo-Differential Operators on $\mathbb{Z}$

Shahla Molahajloo

Abstract. A necessary and sufficient condition is imposed on the symbols $\sigma : \mathbb{Z}\times\mathbb{S}^1 \to \mathbb{C}$ to guarantee that the corresponding pseudo-differential operators $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ are Hilbert–Schmidt. A special sufficient condition on the symbols $\sigma : \mathbb{Z}\times\mathbb{S}^1 \to \mathbb{C}$ for the corresponding pseudo-differential operators $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ to be bounded is given. Sufficient conditions are given on the symbols $\sigma : \mathbb{Z}\times\mathbb{S}^1 \to \mathbb{C}$ to ensure the boundedness and compactness of the corresponding pseudo-differential operators $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ for $1 \le p < \infty$. Norm estimates for the pseudo-differential operators T_σ are given in terms of the symbols σ. The almost diagonalization of the pseudo-differential operators is then shown to follow from the sufficient condition for the L^p-boundedness.

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1. Introduction

Let $a \in L^2(\mathbb{Z})$. Then the Fourier transform $\mathcal{F}_\mathbb{Z} a$ of a is the function on the unit circle $\mathbb{S}^1$ centered at the origin defined by

$$(\mathcal{F}_\mathbb{Z} a)(\theta) = \sum_{n\in\mathbb{Z}} a(n)e^{-in\theta}, \quad \theta \in [-\pi, \pi].$$

It is well known that $\mathcal{F}_\mathbb{Z} a \in L^2(\mathbb{S}^1)$ and the Plancherel formula for Fourier series gives

$$\sum_{n\in\mathbb{Z}} |a(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |(\mathcal{F}_\mathbb{Z} a)(\theta)|^2 d\theta.$$

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The Fourier inversion formula for Fourier series gives

$$a(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}(\mathcal{F}_{\mathbb{Z}}a)(\theta)\,d\theta, \quad n \in \mathbb{Z}.$$

Let $\sigma : \mathbb{Z} \times \mathbb{S}^1 \to \mathbb{C}$ be a measurable function. Then for every sequence a in $L^2(\mathbb{Z})$, we define the sequence $T_\sigma a$ by

$$(T_\sigma a)(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}}a)(\theta)\,d\theta, \quad n \in \mathbb{Z}.$$

T_σ is called the pseudo-differential operator on $\mathbb{Z}$ corresponding to the symbol σ whenever the integral exists for all n in $\mathbb{Z}$. It is the natural analog on $\mathbb{Z}$ of the standard pseudo-differential operators on $\mathbb{R}^n$ explained in, e.g., [8]. The closely related papers [1, 2, 3, 5, 6] contain results about pseudo-differential operators on $\mathbb{S}^1$.

In this paper we give conditions on the symbols σ to ensure the boundedness and compactness of the corresponding pseudo-differential operators $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$, $1 \le p < \infty$. Norm estimates and the almost diagonalization for these pseudo-differential operators are given.

In Section 2 we give a necessary and sufficient condition on σ for $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ to be a Hilbert–Schmidt operator. In Section 3, an elegantly simple sufficient condition for L^2-boundedness is first given. This is then followed by a sufficient condition on σ for $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ to be a bounded linear operator for $1 \le p < \infty$. A result on L^p-compactness is also given. In Section 4 we give the matrix A_σ of the pseudo-differential operator T_σ and show that it is almost diagonal in a sense to be made precise.

2. Hilbert–Schmidt Operators

Let A be a bounded linear operator on a complex and separable Hilbert space X in which the norm is denoted by $\| \, \|$. Then $A : X \to X$ is a Hilbert–Schmidt operator if and only if there exists an orthonormal basis $\{\varphi_k\}_{k=1}^\infty$ for X such that

$$\sum_{k=1}^\infty \|A\varphi_k\|_X^2 < \infty.$$

If $A : X \to X$ is a Hilbert–Schmidt operator, then the Hilbert–Schmidt norm $\|A\|_{HS}$ of A is given by

$$\|A\|_{HS}^2 = \sum_{k=1}^\infty \|A\varphi_k\|_X^2,$$

where $\{\varphi_k\}_{k=1}^\infty$ is any orthonormal basis for X.

Theorem 2.1. *The pseudo-differential operator $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a Hilbert–Schmidt operator $\Leftrightarrow$ $\sigma \in L^2(\mathbb{Z} \times \mathbb{S}^1)$. Moreover, if $T_\sigma : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is a Hilbert–Schmidt operator, then*

$$\|T_\sigma\|_{HS} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}.$$

Proof. The starting point is the standard orthonormal basis $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ for $L^2(\mathbb{Z})$ given by

$$\varepsilon_k(n) = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

For $k \in \mathbb{Z}$, we get

$$(\mathcal{F}_{\mathbb{Z}}\varepsilon_k)(\theta) = \sum_{n\in\mathbb{Z}} \varepsilon_k(n)e^{-in\theta} = e^{-ik\theta}, \quad \theta \in [-\pi, \pi],$$

and hence

$$\begin{aligned}(T_\sigma\varepsilon_k)(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}}\varepsilon_k)(\theta)\,d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(k-n)\theta}\sigma(n,\theta)\,d\theta \\ &= (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, k-n)\end{aligned}$$

for all $n \in \mathbb{Z}$, where $(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,\cdot)$ is the Fourier transform of the function $\sigma(n,\cdot)$ on $\mathbb{S}^1$ given by

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-im\theta}\sigma(n,\theta)\,d\theta, \quad m, n \in \mathbb{Z}.$$

So, using Fubini's theorem and the Plancherel formula for Fourier series, we get

$$\begin{aligned}\|T_\sigma\|_{HS}^2 &= \sum_{k\in\mathbb{Z}} \|T_\sigma\varepsilon_k\|_{L^2(\mathbb{Z})}^2 \\ &= \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,k-n)|^2 \\ &= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,k-n)|^2 \\ &= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,k)|^2 \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi} |\sigma(n,\theta)|^2 d\theta \\ &= \frac{1}{2\pi}\|\sigma\|_{L^2(\mathbb{Z}\times\mathbb{S}^1)}^2\end{aligned}$$

and this completes the proof. □

3. L^p-Boundedness and L^p-Compactness, $1 \leq p < \infty$

We begin with a simple and elegant result on the L^2-boundedness of pseudo-differential operators on $\mathbb{Z}$.

Theorem 3.1. *Let σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$ such that there exists a function $w\in L^2(\mathbb{Z})$ for which*

$$|\sigma(n,\theta)|\le|w(n)|$$

for all $n\in\mathbb{Z}$ and almost all θ in $[-\pi,\pi]$. Then $T_\sigma: L^2(\mathbb{Z})\to L^2(\mathbb{Z})$ is a bounded linear operator. Furthermore,

$$\|T_\sigma\|_{B(L^2(\mathbb{Z}))}\le\|w\|_{L^2(\mathbb{Z})},$$

where $\|T_\sigma\|_{B(L^2(\mathbb{Z}))}$ is the norm of the bounded linear operator $T_\sigma: L^2(\mathbb{Z})\to L^2(\mathbb{Z})$.

Proof. Let $a\in L^1(\mathbb{Z})$. Then by the Schwarz inequality and the Plancherel formula,

$$\begin{aligned}\|T_\sigma a\|^2_{L^2(\mathbb{Z})}&=\frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}}\left|\int_{-\pi}^{\pi}e^{in\theta}\sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}}a)(\theta)\,d\theta\right|^2\\&\le\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{-\pi}^{\pi}|\sigma(n,\theta)|^2|(\mathcal{F}_{\mathbb{Z}}a)(\theta)|^2d\theta\\&\le\frac{1}{2\pi}\sum_{n\in\mathbb{Z}}|w(n)|^2\int_{-\pi}^{\pi}|(\mathcal{F}_{\mathbb{Z}}a)(\theta)|^2d\theta\\&=\|w\|^2_{L^2(\mathbb{Z})}\|a\|^2_{L^2(\mathbb{Z})},\end{aligned}$$

and a density argument to the effect that $L^1(\mathbb{Z})$ is dense in $L^2(\mathbb{Z})$ completes the proof. □

The next theorem gives a single sufficient condition on the symbols σ for the corresponding pseudo-differential operators $T_\sigma: L^p(\mathbb{Z})\to L^p(\mathbb{Z})$ to be bounded for $1\le p<\infty$.

Theorem 3.2. *Let σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$ such that we can find a positive constant C and a function w in $L^1(\mathbb{Z})$ for which*

$$|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)|\le C|w(m)|,\quad m,n\in\mathbb{Z}.$$

Then the pseudo-differential operator $T_\sigma: L^p(\mathbb{Z})\to L^p(\mathbb{Z})$ is a bounded linear operator for $1\le p<\infty$. Furthermore,

$$\|T_\sigma\|_{B(L^p(\mathbb{Z}))}\le C\|w\|_{L^1(\mathbb{Z})},$$

where $\|T_\sigma\|_{B(L^p(\mathbb{Z}))}$ is the norm of the bounded linear operator $T_\sigma: L^p(\mathbb{Z})\to L^p(\mathbb{Z})$.

Proof. Let $a \in L^1(\mathbb{Z})$. Then for all $n \in \mathbb{Z}$, we get

$$\begin{aligned}
(T_\sigma a)(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\sigma(n,\theta)(\mathcal{F}_{\mathbb{Z}}a)(\theta)\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\theta}\sigma(n,\theta)\left(\sum_{m\in\mathbb{Z}} a(m)e^{-im\theta}\right)d\theta \\
&= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} a(m)\int_{-\pi}^{\pi} e^{-i(m-n)\theta}\sigma(n,\theta)\,d\theta \\
&= \sum_{m\in\mathbb{Z}} (\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,n-m)a(m) \\
&= ((\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,\cdot) * a)(n), \qquad (3.1)
\end{aligned}$$

where

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,m) = (\mathcal{F}_{\mathbb{S}^1}\sigma)(n,-m), \quad m,n \in \mathbb{Z}.$$

So,

$$\begin{aligned}
\|T_\sigma a\|^p_{L^p(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} |((\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,\cdot) * a)(n)|^p \\
&\le \sum_{n\in\mathbb{Z}} (|(\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,\cdot)| * |a|)(n))^p \\
&\le C^p \sum_{n\in\mathbb{Z}} |(|w| * |a|)(n)|^p. \qquad (3.2)
\end{aligned}$$

Thus, by Young's inequality,

$$\|T_\sigma a\|^p_{L^p(\mathbb{Z})} \le C^p \|w\|^p_{L^1(\mathbb{Z})}\|a\|^p_{L^p(\mathbb{Z})},$$

which is equivalent to

$$\|T_\sigma\|_{B(L^p(\mathbb{Z}))} \le C\|w\|_{L^1(\mathbb{Z})}.$$

The proof can then be completed using the density of $L^1(\mathbb{Z})$ in $L^p(\mathbb{Z})$ for $1 \le p < \infty$. □

The hypothesis in Theorem 3.1 can be thought of as a Lipschitz condition Λ_α, $\alpha > \frac{1}{2}$, of the symbol σ on the unit circle $\mathbb{S}^1$ and, furthermore, the Lipschitz condition is uniform with respect to n in $\mathbb{Z}$. See Bernstein's theorem and other conditions in this connection in Section 3 in Chapter VI of the book [10] by Zygmund. The very mild condition on the L^p-boundedness of pseudo-differential operators on $\mathbb{Z}$ is dramatically different from the condition for L^p-boundedness of pseudo-differential operators on $\mathbb{R}^n$ in which derivatives with respect to the configuration variables and the dual variables are essential. See Chapter 10 of [8] for boundedness of pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

The following theorem is a result on L^p-compactness.

Theorem 3.3. *Let σ be a measurable function on $\mathbb{Z}\times\mathbb{S}^1$ such that we can find a positive function C on $\mathbb{Z}$ and a function w in $L^1(\mathbb{Z})$ for which*

$$|(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m)| \le C(n)|w(m)|, \quad m,n\in\mathbb{Z},$$

and

$$\lim_{|n|\to\infty} C(n) = 0.$$

Then the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a compact operator for $1 \le p < \infty$.

Proof. For every positive integer N, we define the symbol σ_N on $\mathbb{Z}\times\mathbb{S}^1$ by

$$\sigma_N(n,\theta) = \begin{cases} \sigma(n,\theta), & |n|\le N,\\ 0, & |n|>N. \end{cases}$$

Now, by (3.1), we get for all $a \in L^p(\mathbb{Z})$,

$$(T_{\sigma_N}a)(n) = \begin{cases} ((\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,\cdot) * a)(n), & |n|\le N,\\ 0, & |n|>N. \end{cases}$$

Therefore the range of $T_{\sigma_N} : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is finite-dimensional, i.e., $T_{\sigma_N} : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is a finite-rank operator. Let ε be a positive number. Then there exists a positive integer N_0 such that

$$|C(n)| < \varepsilon$$

whenever $|n| > N_0$. So, as in the derivation of (3.2), we get for $N > N_0$,

$$\begin{aligned}
\|(T_\sigma - T_{\sigma_N})a\|^p_{L^p(\mathbb{Z})} &= \sum_{n\in\mathbb{Z}} |((\mathcal{F}_{\mathbb{S}^1}(\sigma-\sigma_N))^{\sim}(n,\cdot) * a)(n)|^p \\
&= \sum_{|n|>N} |((\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,\cdot) * a)(n)|^p \\
&\le \sum_{|n|>N} (|(\mathcal{F}_{\mathbb{S}^1}\sigma)^{\sim}(n,\cdot)| * |a|)(n)|^p \\
&\le \sum_{|n|>N} C(n)^p |(|w| * |a|)(n)|^p \\
&\le \varepsilon^p \sum_{|n|>N} |(|w| * |a|)(n)|^p.
\end{aligned}$$

By Young's inequality, we get for $N > N_0$,

$$\|(T_\sigma - T_{\sigma_N})a\|^p_{L^p(\mathbb{Z})} \le \varepsilon^p \|w\|^p_{L^1(\mathbb{Z})} \|a\|^p_{L^p(\mathbb{Z})}.$$

Hence for $N > N_0$,

$$\|T_\sigma - T_{\sigma_N}\|_{B(L^p(\mathbb{Z}))} \le \varepsilon \|w\|_{L^1(\mathbb{Z})}.$$

So, $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is the limit in norm of a sequence of compact operators on $L^p(\mathbb{Z})$ and hence must be compact. □

4. Almost Diagonalization

Theorem 4.1. *Let σ be a symbol satisfying the hypotheses of Theorem* 3.2. *Then for $1 \le p < \infty$, the matrix A_σ of the pseudo-differential operator $T_\sigma : L^p(\mathbb{Z}) \to L^p(\mathbb{Z})$ is given by*

$$A_\sigma = [\sigma_{nk}]_{n,k\in\mathbb{Z}},$$

where

$$\sigma_{nk} = (\mathcal{F}_{\mathbb{S}^1}\sigma)(n, k-n).$$

Furthermore, the matrix A_σ is almost diagonal in the sense that

$$|\sigma_{nk}| \le C|w(k-n)|, \quad n,k \in \mathbb{Z}.$$

Remark 4.2. Since $w \in L^1(\mathbb{Z})$, it follows that, roughly speaking,

$$w(m) = O(|m|^{-(1+\alpha)})$$

as $|m| \to \infty$, where α is a positive number. So, the entry σ_{nk} in the n^{th} row and the k^{th} column of the matrix A_σ decays in such a way that

$$|\sigma_{nk}| = O(|k-n|^{-(1+\alpha)})$$

as $|k-n| \to \infty$. In other words, the off-diagonal entries in A_σ are small and the matrix A_σ can be seen as almost diagonal. This fact is very useful for the numerical analysis of pseudo-differential operators on $\mathbb{Z}$. See [7] for the numerical analysis of pseudo-differential operators and related topics.

Proof of Theorem 4.1. By (3.1), we get for all $n \in \mathbb{Z}$,

$$(T_\sigma a)(n) = (\mathcal{F}_{\mathbb{S}^1}\sigma)^\sim(n,\cdot) * a)(n) = \sum_{k\in\mathbb{Z}} (\mathcal{F}_{\mathbb{S}^1}\sigma)^\sim(n, n-k)a(k).$$

So, $T_\sigma a$ is the same as the product $A_\sigma a$ of the matrices A_σ and a. □

We give a numerical example to illustrate the almost diagonalization.

Example 4.3. Let

$$\sigma(n,\theta) = \left(n+\frac{1}{2}\right)^{-2} \sum_{k\in\mathbb{Z}} e^{ik\theta}\left(k+\frac{1}{2}\right)^{-2}, \quad n \in \mathbb{Z},\, \theta \in \mathbb{S}^1.$$

Then

$$\sigma_{nk} = \left(n+\frac{1}{2}\right)^{-2}\left(k-n+\frac{1}{2}\right)^{-2}, \quad k,n \in \mathbb{Z}.$$

Computing the 7×7 matrix $A_\sigma = [\sigma_{nk}]_{-3\le k,n\le 3}$ numerically, we get the following matrix in which the entries are generated by MATLAB.

$$\begin{bmatrix} 0.6400 & 0.0711 & 0.0256 & 0.0131 & 0.0079 & 0.0053 & 0.0038 \\ 1.7778 & 1.7778 & 0.1975 & 0.0711 & 0.0363 & 0.0219 & 0.0147 \\ 1.7778 & 16.000 & 16.000 & 1.7778 & 0.6400 & 0.3265 & 0.1975 \\ 0.6400 & 1.7778 & 16.000 & 16.000 & 1.7778 & 0.6400 & 0.3265 \\ 0.0363 & 0.0711 & 0.1975 & 1.7778 & 1.7778 & 0.1975 & 0.0711 \\ 0.0079 & 0.0131 & 0.0256 & 0.0711 & 0.6400 & 0.6400 & 0.0711 \\ 0.0027 & 0.0040 & 0.0067 & 0.0131 & 0.0363 & 0.3265 & 0.3265 \end{bmatrix}.$$

Example 4.4. As another example with a closed formula for the symbol, we let

$$\sigma(n,\theta) = e^{-n^2}\theta^2/2, \quad n\in\mathbb{Z},\, \theta\in\mathbb{S}^1.$$

Then

$$(\mathcal{F}_{\mathbb{S}^1}\sigma)(n,m) = \begin{cases} (-1)^m e^{-n^2}/m^2, & m\neq 0, \\ \pi^2 e^{-n^2}/6, & m=0, \end{cases}$$

and hence

$$\sigma_{nk} = \begin{cases} (-1)^{k-n} e^{-n^2}/(k-n)^2, & k\neq n, \\ \pi^2 e^{-n^2}/6, & k=n. \end{cases}$$

The 7×7 matrix $A_\sigma = [\sigma_{nk}]_{-3\le k,n\le 3}$ is given numerically by

$$\begin{bmatrix} 0.0002 & -0.0001 & 0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.0183 & 0.0301 & -0.0183 & 0.0046 & -0.0020 & 0.0011 & -0.0007 \\ 0.0920 & -0.3679 & 0.6051 & -0.3679 & 0.0920 & -0.0409 & 0.0230 \\ -0.1111 & 0.2500 & -1.0000 & 1.6449 & -1.0000 & 0.2500 & -0.1111 \\ 0.0230 & -0.0409 & 0.0920 & -0.3679 & 0.6051 & -0.3679 & 0.0920 \\ -0.0007 & 0.0011 & -0.0020 & 0.0046 & -0.0183 & 0.0301 & -0.0183 \\ 0.0000 & -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0001 & 0.0002 \end{bmatrix}.$$

Remark 4.5. Almost diagonalization of wavelet multipliers using Weyl–Heisenberg frames can be found in [9], and almost diagonalization of Fourier integral operators using Gabor frames can be found in [4].

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Shahla Molahajloo
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada
e-mail: `smollaha@mathstat.yorku.ca`

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Pseudo-Differential Operators with Symbols in Modulation Spaces

Joachim Toft

Abstract. We establish continuity results for pseudo-differential operators with symbols in modulation spaces. Especially it follows from our general result that if $a \in W^{\infty,1}(\mathbf{R}^{2d})$, then the pseudo-differential operator $a(x, D)$ is continuous from $M^{\infty,1}(\mathbf{R}^d)$ to $W^{\infty,1}(\mathbf{R}^d)$. If instead $a \in W^{1,\infty}(\mathbf{R}^{2d})$, then it folllows that $a(x, D)$ is continuous from $M^{1,\infty}(\mathbf{R}^d)$ to $W^{1,\infty}(\mathbf{R}^d)$.

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1. Introduction

In this paper we continue the discussions from [34–37] concerning continuity properties for pseudo-differential operators in background of modulation space theory. More precisely, we establish continuity properties of pseudo-differential operators with symbols in modulation spaces, when acting on (other) modulation spaces. These investigations are based on an important result by Cordero and Okoudjou in [1] concerning mapping properties of short-time Fourier transforms on modulation spaces.

The (classical) modulation spaces $M^{p,q}$, $p, q \in [1, \infty]$, as introduced by Feichtinger in [4], consist of all tempered distributions whose short-time Fourier transforms (STFT) have finite mixed $L^{p,q}$ norm. (Cf. [7] and the references therein for an updated describtion of modulation spaces.) It follows that the parameters p and q to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}$. The theory of modulation spaces was developed further and generalized in [5, 6, 8–10, 13], where Feichtinger and Gröchenig established the theory of coorbit spaces. In particular, the modulation spaces $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$, where ω denotes a weight function on phase (or time-frequency shift) space, appears as the set of tempered (ultra-) distributions whose STFT belong to the weighted and mixed Lebesgue space $L^{p,q}_{1,(\omega)}$ and $L^{p,q}_{2,(\omega)}$ respectively. (See Section 2 for strict definitions.) By choosing the weight ω in appropriate ways, the space $W^{p,q}_{(\omega)}$ becomes a Wiener amalgam space, introduced in [2] by Feichtinger.

A major idea behind the design of these spaces was to find useful Banach spaces, which are defined in a way similar to Besov and Triebel–Lizorkin spaces, in the sense of replacing the dyadic decomposition on the Fourier transform side, characteristic to Besov and Triebel–Lizorkin spaces, with a *uniform* decomposition. From the construction of these spaces, it turns out that modulation spaces of the form $M^{p,q}_{(\omega)}$ and Besov spaces in some sense are rather similar, and sharp embeddings between these spaces can be found in [34,36], which are improvements of certain embeddings in [12]. (See also [28] for verification of the sharpness.) In the same way it follows that modulation spaces of the form $W^{p,q}_{(\omega)}$ and Triebel–Lizorkin spaces are rather similar.

During the last 15 years many results have been proved which confirm the usefulness of the modulation spaces and their Fourier transforms in time-frequency analysis, where they occur naturally. For example, in [10, 14, 19], it is shown that all such spaces admit reconstructible sequence space representations using Gabor frames.

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators. In fact, in [27], Sjöstrand introduced the modulation space $M^{\infty,1}$, which contains non-smooth functions, as a symbol class and proved that $M^{\infty,1}$ corresponds to an algebra of operators which are bounded on L^2.

Gröchenig and Heil thereafter proved in [14,16] that each operator with symbol in $M^{\infty,1}$ is continuous on all modulation spaces $M^{p,q}$, $p,q \in [1,\infty]$. This extends Sjöstrand's result since $M^{2,2} = L^2$. Some generalizations to operators with symbols in general unweighted modulation spaces were obtained in [17, 34], and in [35, 37] some further extensions involving weighted modulation spaces are presented.

Here Theorem 4.2 in [37] seems to be one of the most general results. It asserts the following. Assume that $p, q, p_j, q_j \in [1,\infty]$, $t \in \mathbf{R}$, and ω, ω_j are appropriate and satisfy

$$\frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{q_1} - \frac{1}{q_2} = 1 - \frac{1}{p} - \frac{1}{q}, \qquad q \le p_2, q_2 \le p$$

and

$$\frac{\omega_2(x - ty, \xi + (1-t)\eta)}{\omega_1(x + (1-t)y, \xi - t\eta)} \le C\omega(x, \xi, \eta, y)$$

for some constant C which is independent of $x, y, \xi, \eta \in \mathbf{R}^d$. Then Theorem 4.2 in [37] asserts that each symbol a in the modulation space $M^{p,q}_{(\omega)}(\mathbf{R}^{2d})$ gives rise to a continuous pseudo-differential operator $a_t(x, D)$ from $M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^d)$ to $M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d)$.

Modulation spaces in pseudo-differential calculus is currently an active field of research (see, e. g., [15, 17, 18, 20, 21, 23–25, 28–30, 33, 34, 37]).

In Section 3 we prove a related result, where some of the $M^{p,q}_{(\omega)}$ spaces here above are replaced by $W^{q,p}_{(\omega)}$ spaces. More precisely, for trivial weights, Theorem 3.1 asserts that if $a \in W^{q,p}(\mathbf{R}^{2d})$, then $a(x,D)$ from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ is uniquely extendable to a continuous mapping from $M^{p',q'}(\mathbf{R}^d)$ to $W^{q,p}(\mathbf{R}^d)$. Note here that by the continuity property here above for pseudo-differential operators with symbols in $M^{\infty,1}$, it follows that

$$a \in M^{\infty,1}(\mathbf{R}^{2d}) \quad \Longrightarrow \quad a(x,D) : M^{\infty,1}(\mathbf{R}^d) \to M^{\infty,1}(\mathbf{R}^d) \tag{1.1}$$

is continuous, while it follows from Theorem 3.1 that

$$a \in W^{\infty,1}(\mathbf{R}^{2d}) \quad \Longrightarrow \quad a(x,D) : M^{\infty,1}(\mathbf{R}^d) \to W^{\infty,1}(\mathbf{R}^d). \tag{1.2}$$

Here we remark that $M^{\infty,1}$ is contained in $W^{\infty,1}$. Therefore, when passing from (1.1) to (1.2), the symbol classes and the images are increased from $M^{\infty,1}$, while the domain for the operators are the same.

We also remark that for each choice of $p, q \in [1,\infty]$, we are able to establish "narrow" continuity results for pseudo-differential operators with symbols in $W^{q,p}$, while symbols of the form $M^{p,q}$ with the additional condition $q \le p$ are considered in [14, 16, 34, 35, 37].

2. Preliminaries

In this section we recall some notations and basic results. The proofs are in general omitted.

We start by discussing appropriate conditions for the involved weight functions. Assume that ω and v are positive and measureable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x+y) \le C\omega(x)v(y) \tag{2.1}$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. If v in (2.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbf{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $\varphi \in \mathscr{S}'(\mathbf{R}^d)$ be fixed, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the short-time Fourier transform $V_\varphi f(x,\xi)$ of f with respect to the *window function* φ is the tempered distribution on $\mathbf{R}^{2d}$ which is defined by

$$V_\varphi f(x,\xi) \equiv \mathscr{F}(f\,\overline{\varphi(\,\cdot\,-x)}(\xi).$$

If $f,\varphi \in \mathscr{S}(\mathbf{R}^d)$, then it follows that

$$V_\varphi f(x,\xi) = (2\pi)^{-d/2} \int f(y)\overline{\varphi(y-x)}e^{-i\langle y,\xi\rangle}\,dy.$$

Next we recall some properties on modulation spaces and their Fourier transforms. Assume that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and that $p,q \in [1,\infty]$. Then the mixed Lebesgue space $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{1,(\omega)}} < \infty$, and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{2,(\omega)}} < \infty$. Here

$$\|F\|_{L^{p,q}_{1,(\omega)}} = \Big(\int \Big(\int |F(x,\xi)\omega(x,\xi)|^p\,dx \Big)^{q/p} d\xi \Big)^{1/q},$$

and

$$\|F\|_{L^{p,q}_{2,(\omega)}} = \Big(\int \Big(\int |F(x,\xi)\omega(x,\xi)|^q\,d\xi \Big)^{p/q} dx \Big)^{1/p},$$

with obvious modifications when $p=\infty$ or $q=\infty$.

Assume that $p,q\in[1,\infty]$, $\omega\in\mathscr{P}(\mathbf{R}^{2d})$ and $\varphi\in\mathscr{S}(\mathbf{R}^d)\backslash 0$ are fixed. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the Banach space which consists of all $f\in\mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\varphi f\|_{L^{p,q}_{1,(\omega)}} < \infty. \tag{2.2}$$

The modulation space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the Banach space which consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\varphi f\|_{L^{p,q}_{2,(\omega)}} < \infty. \tag{2.3}$$

The definitions of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are independent of the choice of φ and different φ gives rise to equivalent norms. (See Proposition 2.1 below.) From the fact that

$$V_{\widehat\varphi}\widehat f(\xi,-x) = e^{i\langle x,\xi\rangle}V_{\check\varphi}f(x,\xi), \qquad \check\varphi(x)=\varphi(-x),$$

it follows that

$$f\in W^{q,p}_{(\omega)}(\mathbf{R}^d) \quad\Longleftrightarrow\quad \widehat f\in M^{p,q}_{(\omega_0)}(\mathbf{R}^d), \qquad \omega_0(\xi,-x)=\omega(x,\xi).$$

For conveniency we set $M^p_{(\omega)} = M^{p,p}_{(\omega)}$, which coincides with $W^p_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $W^{p,q} = W^{p,q}_{(\omega)}$ if $\omega\equiv 1$. If ω is given by $\omega(x,\xi)=\omega_1(x)\omega_2(\xi)$, for some $\omega_1,\omega_2\in\mathscr{P}(\mathbf{R}^d)$, then $W^{p,q}_{(\omega)}$ is a Wiener amalgam space, introduced by Feichtinger in [2].

The proof of the following proposition is omitted, since the results can be found in [3, 4, 8–10, 14, 34–37]. Here and in what follows, $p'\in[1,\infty]$ denotes the conjugate exponent of $p\in[1,\infty]$, i.e., $1/p+1/p'=1$ should be fulfilled.

Proposition 2.1. *Assume that $p,q,p_j,q_j \in [1,\infty]$ for $j = 1,2$, and $\omega,\omega_1,\omega_2,v \in \mathscr{P}(\mathbf{R}^{2d})$ are such that ω is v-moderate and $\omega_2 \le C\omega_1$ for some constant $C > 0$. Then the following are true:*

(1) *if $\varphi \in M^1_{(v)}(\mathbf{R}^d) \setminus 0$, then $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if (2.2) holds, i. e., $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is independent of the choice of φ. Moreover, $M^{p,q}_{(\omega)}$ is a Banach space under the norm in (2.2) and different choices of φ give rise to equivalent norms;*

(2) *if $p_1 \le p_2$ and $q_1 \le q_2$, then*
$$\mathscr{S}(\mathbf{R}^d) \hookrightarrow M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^n) \hookrightarrow M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^d) \hookrightarrow \mathscr{S}'(\mathbf{R}^d);$$

(3) *the L^2 product $(\,\cdot\,,\,\cdot\,)$ on $\mathscr{S}$ extends to a continuous map from $M^{p,q}_{(\omega)}(\mathbf{R}^n) \times M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$ to $\mathbf{C}$. On the other hand, if $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $\|f\| = \sup|(f,g)|$, where the supremum is taken over all $g \in \mathscr{S}(\mathbf{R}^d)$ such that $\|g\|_{M^{p',q'}_{(1/\omega)}} \le 1$, then $\|\cdot\|$ and $\|\cdot\|_{M^{p,q}_{(\omega)}}$ are equivalent norms;*

(4) *if $p,q < \infty$, then $\mathscr{S}(\mathbf{R}^d)$ is dense in $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and the dual space of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$, through the form $(\,\cdot\,,\,\cdot\,)_{L^2}$. Moreover, $\mathscr{S}(\mathbf{R}^d)$ is weakly dense in $M^{\infty}_{(\omega)}(\mathbf{R}^d)$.*

Similar facts hold if the $M^{p,q}_{(\omega)}$ spaces are replaced by $W^{p,q}_{(\omega)}$ spaces.

Proposition 2.1 (1) allows us be rather vague concerning the choice of $\varphi \in M^1_{(v)} \setminus 0$ in (2.2) and (2.3). For example, if $C > 0$ is a constant and $\mathscr{A}$ is a subset of $\mathscr{S}'$, then $\|f\|_{W^{p,q}_{(\omega)}} \le C$ for every $f \in \mathscr{A}$, means that the inequality holds for some choice of $\varphi \in M^1_{(v)} \setminus 0$ and every $f \in \mathscr{A}$. Evidently, a similar inequality is true for any other choice of $\varphi \in M^1_{(v)} \setminus 0$, with a suitable constant, larger than C if necessary.

In the following remark we list some other properties for modulation spaces. Here and in what follows we let $\langle x\rangle = (1+|x|^2)^{1/2}$, when $x \in \mathbf{R}^d$.

Remark 2.2. Assume that $p,p_1,p_2,q,q_1,q_2 \in [1,\infty]$ are such that
$$q_1 \le \min(p,p'), \quad q_2 \ge \max(p,p'), \quad p_1 \le \min(q,q'), \quad p_2 \ge \max(q,q'),$$
and that $\omega, v \in \mathscr{P}(\mathbf{R}^{2d})$ are such that ω is v-moderate. Then the following is true:

(1) if $p \le q$, then $W^{p,q}_{(\omega)}(\mathbf{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and if $p \ge q$, then $M^{p,q}_{(\omega)}(\mathbf{R}^d) \subseteq W^{p,q}_{(\omega)}(\mathbf{R}^d)$. Furthermore, if $\omega(x,\xi) = \omega(x)$, then
$$M^{p,q_1}_{(\omega)}(\mathbf{R}^d) \subseteq W^{p,q_1}_{(\omega)}(\mathbf{R}^d) \subseteq L^p_{(\omega)}(\mathbf{R}^d) \subseteq W^{p,q_2}_{(\omega)}(\mathbf{R}^d) \subseteq M^{p,q_2}_{(\omega)}(\mathbf{R}^d).$$
In particular, $M^2_{(\omega)} = W^2_{(\omega)} = L^2_{(\omega)}$. If instead $\omega(x,\xi) = \omega(\xi)$, then
$$W^{p_1,q}_{(\omega)}(\mathbf{R}^d) \subseteq M^{p_1,q}_{(\omega)}(\mathbf{R}^d) \subseteq \mathscr{F}L^q_{(\omega)}(\mathbf{R}^d) \subseteq M^{p_2,q}_{(\omega)}(\mathbf{R}^d) \subseteq W^{p_2,q}_{(\omega)}(\mathbf{R}^d).$$

Here $\mathscr{F}L^q_{(\omega_0)}(\mathbf{R}^d)$ consists of all $f\in\mathscr{S}'(\mathbf{R}^d)$ such that

$$\|\widehat{f}\,\omega_0\|_{L^q}<\infty;$$

(2) if $\omega(x,\xi)=\omega(x)$, then the following conditions are equivalent:
- $M^{p,q}_{(\omega)}(\mathbf{R}^d)\subseteq C(\mathbf{R}^d)$;
- $W^{p,q}_{(\omega)}(\mathbf{R}^d)\subseteq C(\mathbf{R}^d)$;
- $q=1$.

(3) $M^{1,\infty}(\mathbf{R}^d)$ and $W^{1,\infty}(\mathbf{R}^d)$ are convolution algebras. If $C_B'(\mathbf{R}^d)$ is the set of all measures on $\mathbf{R}^d$ with bounded mass, then

$$C_B'(\mathbf{R}^d)\subseteq W^{1,\infty}(\mathbf{R}^d)\subseteq M^{1,\infty}(\mathbf{R}^d);$$

(4) if $x_0\in\mathbf{R}^d$ is fixed and $\omega_0(\xi)=\omega(x_0,\xi)$, then

$$M^{p,q}_{(\omega)}\cap\mathscr{E}'=W^{p,q}_{(\omega)}\cap\mathscr{E}'=\mathscr{F}L^q_{(\omega_0)}\cap\mathscr{E}';$$

(5) if $\omega(x,\xi)=\omega_0(\xi,-x)$, then the Fourier transform on $\mathscr{S}'(\mathbf{R}^d)$ restricts to a homeomorphism from $M^p_{(\omega)}(\mathbf{R}^d)$ to $M^p_{(\omega_0)}(\mathbf{R}^d)$. In particular, if $\omega=\omega_0$, then $M^p_{(\omega)}$ is invariant under the Fourier transform. Similar facts hold for partial Fourier transforms;

(6) for each $x,\xi\in\mathbf{R}^d$ we have

$$\|e^{i\langle\,\cdot\,,\xi\rangle}f(\,\cdot\,-x)\|_{M^{p,q}_{(\omega)}}\le Cv(x,\xi)\|f\|_{M^{p,q}_{(\omega)}},$$

and

$$\|e^{i\langle\,\cdot\,,\xi\rangle}f(\,\cdot\,-x)\|_{W^{p,q}_{(\omega)}}\le Cv(x,\xi)\|f\|_{W^{p,q}_{(\omega)}}$$

for some constant C which is independent of $f\in\mathscr{S}'(\mathbf{R}^d)$;

(7) if $\tilde\omega(x,\xi)=\omega(x,-\xi)$, then $f\in M^{p,q}_{(\omega)}$ if and only if $\overline{f}\in M^{p,q}_{(\tilde\omega)}$;

(8) if $s\in\mathbf{R}$ and $\omega(x,\xi)=\langle\xi\rangle^s$, then $M^2_{(\omega)}=W^2_{(\omega)}$ agrees with H^2_s, the Sobolev space of distributions with s derivatives in L^2. That is, H^2_s consists of all $f\in\mathscr{S}'$ such that $\mathscr{F}^{-1}(\langle\,\cdot\,\rangle^s\widehat{f})\in L^2$.

(See, e. g., [3, 4, 8–10, 14, 26, 34–37].)

Next we recall some facts in Chapter XVIII in [22] concerning pseudo-differential operators. Assume that $a\in\mathscr{S}(\mathbf{R}^{2d})$, and that $t\in\mathbf{R}$ is fixed. Then the pseudo-differential operator $a_t(x,D)$ in

$$\begin{aligned}(a_t(x,D)f)(x)&=(\mathrm{Op}_t(a)f)(x)\\&=(2\pi)^{-d}\iint a((1-t)x+ty,\xi)f(y)e^{i\langle x-y,\xi\rangle}\,dyd\xi.\end{aligned}\tag{2.4}$$

is a linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$. For general $a\in\mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $a_t(x,D)$ is defined as the continuous operator from

$\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x+ty, y-x), \tag{2.5}$$

where $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x,y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and $F(x,y) \mapsto F((1-t)x+ty, y-x)$ are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. Moreover, it agrees with the operator in (2.4) when $a \in \mathscr{S}(\mathbf{R}^{2d})$. If $t=0$, then $a_t(x,D)$ agrees with the Kohn–Nirenberg representation $a(x,D)$. If instead $t=1/2$, then $a_t(x,D)$ is the Weyl operator $a^w(x,D)$ of a.

We also need some facts in Section 2 in [37] on narrow convergence. For any $f \in \mathscr{S}'(\mathbf{R}^d)$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\varphi \in \mathscr{S}(\mathbf{R}^d)$ and $p \in [1,\infty]$, we set

$$H_{f,\omega,p}(\xi) = \Big(\int_{\mathbf{R}^d} |V_\varphi f(x,\xi)\omega(x,\xi)|^p \, dx \Big)^{1/p}.$$

Definition 2.3. Assume that $f, f_j \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$, $j = 1,2,\ldots$. Then f_j is said to converge *narrowly* to f (with respect to $p,q \in [1,\infty]$, $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$), if the following conditions are satisfied:

1. $f_j \to f$ in $\mathscr{S}'(\mathbf{R}^d)$ as j turns to ∞;
2. $H_{f_j,\omega,p}(\xi) \to H_{f,\omega,p}(\xi)$ in $L^q(\mathbf{R}^d)$ as j turns to ∞.

Remark 2.4. Assume that $f, f_1, f_2, \ldots \in \mathscr{S}'(\mathbf{R}^d)$ satisfy (1) in Definition 2.3, and assume that $\xi \in \mathbf{R}^d$. Then it follows from Fatou's lemma that

$$\liminf_{j\to\infty} H_{f_j,\omega,p}(\xi) \geq H_{f,\omega,p}(\xi) \quad \text{and} \quad \liminf_{j\to\infty} \|f_j\|_{M^{p,q}_{(\omega)}} \geq \|f\|_{M^{p,q}_{(\omega)}}.$$

The following proposition is important to us later on. We omit the proof since the result is a restatement of Proposition 2.3 in [37].

Proposition 2.5. *Assume that $p,q \in [1,\infty]$ with $q < \infty$ and that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. Then $C_0^\infty(\mathbf{R}^d)$ is dense in $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ with respect to the narrow convergence.*

3. Pseudo-Differential Operators with Symbols in Modulation Spaces

In this section we discuss continuity of pseudo-differential operators with symbols in modulation spaces of the form $W^{p,q}_{(\omega)}$, when acting between modulation spaces.

The main result is the following theorem. Here the involved weight functions should fulfill

$$\frac{\omega_2(x,\xi+\eta)}{\omega_1(x+y,\xi)} \leq C\omega(x,\xi,\eta,y). \tag{3.1}$$

Theorem 3.1. *Assume that $p,q \in [1,\infty]$, $\omega_1,\omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ and $\omega \in \mathscr{P}(\mathbf{R}^{4d})$ are such that* (3.1) *holds for some constant C which is independent of $x,y,\xi,\eta \in \mathbf{R}^d$. Also assume that $a \in W^{q,p}_{(\omega)}(\mathbf{R}^{2d})$. Then the definition of $a(x,D)$ from $\mathscr{S}(\mathbf{R}^d)$ to*

$\mathscr{S}'(\mathbf{R}^d)$ extends uniquely to a continuous mapping from $M^{p',q'}_{(\omega_1)}(\mathbf{R}^d)$ to $W^{q,p}_{(\omega_2)}(\mathbf{R}^d)$. Furthermore, it holds that

$$\|a(x,D)f\|_{W^{q,p}_{(\omega_2)}} \le C\|a\|_{W^{q,p}_{(\omega)}}\|f\|_{M^{p',q'}_{(\omega_1)}}$$

for some constant C which is independent of $f \in M^{p',q'}_{(\omega_1)}(\mathbf{R}^d)$ and $a \in W^{q,p}_{(\omega)}(\mathbf{R}^{2d})$.

The proof is based on duality, using the fact that if $a \in \mathscr{S}'(\mathbf{R}^{2d})$, $f,g \in \mathscr{S}(\mathbf{R}^d)$ and T is the operator, defined by

$$(T\psi)(x,\xi) = \psi(\xi,-x) \tag{3.2}$$

when $\psi \in \mathscr{S}'(\mathbf{R}^{2d})$, then

$$(a(x,D)f,g)_{L^2(\mathbf{R}^d)} = (2\pi)^{-d/2}(T(\mathscr{F}a),V_f g)_{L^2(\mathbf{R}^{2d})}, \tag{3.3}$$

by Fourier's inversion formula.

For the proof of Theorem 3.1 we shall combine (3.3) with the following weighted version of Proposition 3.3 in [1].

Proposition 3.2. *Assume that $f_1, f_2 \in \mathscr{S}'(\mathbf{R}^d)$, $p,q \in [1,\infty]$, $\omega_0 \in \mathscr{P}(\mathbf{R}^{4d})$ and $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$. Also assume that $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbf{R}^d)$, and let $\Psi = V_{\varphi_1}\varphi_2$. Then the following is true:*

1. *if*
$$\omega_0(x,\xi,\eta,y) \le C\omega_1(-x-y,\eta)\omega_2(-y,\xi+\eta) \tag{3.4}$$
for some constant C, then
$$\|V_\Psi(V_{f_1}f_2)\|_{L^{p,q}_{1,(\omega_0)}} \le C\|V_{\varphi_1}f_1\|_{L^{p,q}_{1,(\omega_1)}}\|V_{\varphi_2}f_2\|_{L^{q,p}_{2,(\omega_2)}};$$
2. *if*
$$\omega_1(-x-y,\eta)\omega_2(-y,\xi+\eta) \le C\omega_0(x,\xi,\eta,y) \tag{3.5}$$
for some constant C, then
$$\|V_{\varphi_1}f_1\|_{L^{p,q}_{1,(\omega_1)}}\|V_{\varphi_2}f_2\|_{L^{q,p}_{2,(\omega_2)}} \le C\|V_\Psi(V_{f_1}f_2)\|_{L^{p,q}_{1,(\omega_0)}};$$
3. *if (3.4) and (3.5) hold for some constant C, then $f_1 \in M^{p,q}_{(\omega_1)}(\mathbf{R}^d)$ and $f_2 \in W^{q,p}_{(\omega_2)}(\mathbf{R}^d)$, if and only if $V_{f_1}f_2 \in M^{p,q}_{(\omega_0)}(\mathbf{R}^{2d})$, and*
$$C^{-1}\|V_{f_1}f_2\|_{M^{p,q}_{(\omega_0)}} \le \|f_1\|_{M^{p,q}_{(\omega_1)}}\|f_2\|_{W^{q,p}_{(\omega_2)}} \le C\|V_{f_1}f_2\|_{M^{p,q}_{(\omega_0)}},$$
for some constant C which is independent of f_1 and f_2.

Proof. It suffices to prove (1) and (2), and then we prove only (1), since (2) follows by similar arguments. We shall mainly follow the proof of Proposition 3.3 in [1], and then we only prove the result in the case $p < \infty$ and $q < \infty$. The small modifications when $p = \infty$ or $q = \infty$ are left for the reader.

By Fourier's inversion formula we have

$$|V_\varphi f_1(-x-y,\eta)V_\varphi f_2(-y,\xi+\eta)| = |V_\Psi(V_{f_1}f_2)(x,\xi,\eta,y)|$$

(cf., e. g., [1, 11, 14, 31, 32]). Hence, if

$$F_1(x,\xi) = V_{\varphi_1} f_1(x,\xi)\omega_1(x,\xi) \quad \text{and} \quad F_2(x,\xi) = V_{\varphi_2} f_2(x,\xi)\omega_2(x,\xi),$$

then we get

$$\begin{aligned}
&\|V_\Psi(V_{f_1} f_2)\|^q_{L^{p,q}_{1,(\omega_0)}} \\
&\qquad = \iint_{\mathbf{R}^{2d}} \Big(\iint_{\mathbf{R}^{2d}} |V_\Psi(V_{f_1} f_2)(x,\xi,\eta,y)\omega_0(x,\xi,\eta,y)|^p \, dxd\xi \Big)^{q/p} dyd\eta \\
&\qquad\quad \le C^q \iint_{\mathbf{R}^{2d}} \Big(\iint_{\mathbf{R}^{2d}} |F_1(-x-y,\eta)F_2(-y,\xi+\eta))|^p \, dxd\xi \Big)^{q/p} dyd\eta.
\end{aligned}$$

By taking $-y$, $\xi+\eta$, $-x-y$ and η as new variables of integration, we obtain

$$\begin{aligned}
&\|V_\Psi(V_{f_1} f_2)\|^q_{L^{p,q}_{1,(\omega_0)}} \\
&\quad \le C^q \Big(\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} |F_1(x,\eta)|^p \, dx \Big)^{q/p} d\eta \Big) \Big(\int_{\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} |F_2(y,\xi)|^p \, d\xi \Big)^{q/p} dy \Big) \\
&\qquad\qquad = C^q \|V_{\varphi_1} f_1\|^q_{L^{p,q}_{1,(\omega_1)}} \|V_{\varphi_2} f_2\|^q_{L^{q,p}_{2,(\omega_2)}}.
\end{aligned}$$

This proves the assertion. □

Proof of Theorem 3.1. We may assume that (3.1) holds for $C = 1$ and with equality. We start to prove the result when $1 < p$ and $1 < q$. Let

$$\omega_0(x,\xi,\eta,y) = \omega(-y,\eta,\xi,-x)^{-1},$$

and assume that $a \in W^{q,p}_{(\omega)}(\mathbf{R}^{2d})$ and $f, g \in \mathscr{S}(\mathbf{R}^d)$. Then $a(x,D)f$ makes sense as an element in $\mathscr{S}'(\mathbf{R}^d)$.

By Proposition 3.2 we get

$$\|V_f g\|_{M^{p',q'}_{(\omega_0)}} \le C\|f\|_{M^{p',q'}_{(\omega_1)}} \|g\|_{W^{q',p'}_{(\omega_2^{-1})}}. \tag{3.6}$$

Furthermore, if T is the same as in (3.2), then it follows by Fourier's inversion formula that

$$(V_\varphi(T\widehat{a}))(x,\xi,\eta,y) = e^{-i(\langle x,\eta\rangle + \langle y,\xi\rangle)}(V_{T\widehat{\varphi}} a)(-y,\eta,\xi,-x).$$

This gives

$$|(V_\varphi(T\widehat{a}))(x,\xi,\eta,y)\omega_0(x,\xi,\eta,y)^{-1}| = |(V_{\varphi_1} a)(-y,\eta,\xi,-x)\omega(-y,\eta,\xi,-x)|,$$

when $\varphi_1 = T\widehat{\varphi}$. Hence, by applying the $L^{p,q}_1$ norm we obtain $\|T\widehat{a}\|_{M^{p,q}_{(\omega_0^{-1})}} = \|a\|_{W^{q,p}_{(\omega)}}$.

It now follows from (12) and (3.6) that

$$
\begin{aligned}
|(a(x,D)f,g)| = (2\pi)^{-d/2}|(T\widehat{a},V_{\overline{g}}\overline{f})| &\le C_1\|T\widehat{a}\|_{M^{p,q}_{(\omega_0^{-1})}}\|V_f g\|_{M^{p',q'}_{(\omega_0)}} \\
&\le C_2\|a\|_{W^{q,p}_{(\omega)}}\|f\|_{M^{p',q'}_{(\omega_1)}}\|g\|_{W^{q',p'}_{(\omega_2^{-1})}}. \qquad (3.7)
\end{aligned}
$$

The result now follows by the facts that $\mathscr{S}(\mathbf{R}^d)$ is dense in $M^{p',q'}_{(\omega_1)}(\mathbf{R}^d)$, and that the dual of $W^{q',p'}_{(\omega_2^{-1})}$ is $W^{q,p}_{(\omega_2)}$ when $p,q>1$.

If instead $p=1$ and $q<\infty$, or $q=1$ and $p<\infty$, then we assume that $f\in M^{p',q'}_{(\omega_1)}$ and $a\in\mathscr{S}(\mathbf{R}^{2d})$. Then $a(x,D)f$ makes sense as an element in $\mathscr{S}(\mathbf{R}^d)$, and from the first part of the proof it follows that (3.7) still holds. The result now follows by duality and the fact that $\mathscr{S}(\mathbf{R}^{2d})$ is dense in $W^{q,p}_{(\omega)}(\mathbf{R}^{2d})$ for such choices of p and q.

It remains to consider the cases $p=q'=1$ and $p=q'=\infty$. In this case, the result follows by using the fact that $\mathscr{S}$ is dense in $M^{\infty,1}_{(\omega)}$ respect to the narrow convergence, and that $\mathscr{S}$ is dense in $W^{1,\infty}_{(\omega)}$ on the Fourier transform side with respect to the narrow convergence. The proof is complete. □

By interchanging the roles of p and q and choosing

$$
t=0,\quad p_1=p',\quad q_1=q',\quad p_2=q\quad\text{and}\quad q_2=p
$$

we note that Theorem 4.2 in [37] looks rather similar to Theorem 3.1. In fact, for these choices of parameters, Theorem 4.2 in [37] takes the following form.

Theorem 3.3. *Assume that $p,q\in[1,\infty]$, $\omega_1,\omega_2\in\mathscr{P}(\mathbf{R}^{2d})$ and $\omega\in\mathscr{P}(\mathbf{R}^{4d})$ are such that $q\le p$ and (3.1) holds for some constant C which is independent of $x,y,\xi,\eta\in\mathbf{R}^d$. Also assume that $a\in M^{q,p}_{(\omega)}(\mathbf{R}^{2d})$. Then the definition of $a(x,D)$ from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ extends uniquely to a continuous mapping from $M^{p',q'}_{(\omega_1)}(\mathbf{R}^d)$ to $M^{q,p}_{(\omega_2)}(\mathbf{R}^d)$. Furthermore, it holds that*

$$
\|a(x,D)f\|_{M^{q,p}_{(\omega_2)}}\le C\|a\|_{M^{q,p}_{(\omega)}}\|f\|_{M^{p',q'}_{(\omega_1)}}
$$

for some constant C which is independent of $f\in M^{p',q'}_{(\omega_1)}(\mathbf{R}^d)$ and $a\in M^{q,p}_{(\omega)}(\mathbf{R}^{2d})$.

Here we note that the hypothesis in Theorem 3.3 also contains the assumption $q\le p$, which is absent in the hypothesis in Theorem 3.1. (Cf. the introduction.)

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Joachim Toft
Department of Mathematics and Systems Engineering
Växjö University
Sweden
e-mail: `joachim.toft@vxu.se`

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Phase-Space Differential Equations for Modes

Leon Cohen

Abstract. We discuss how to transform linear partial differential equations into phase space equations. We give a number of examples and argue that phase space equations are more revealing than the original equations. Recently, phase space methods have been applied to the standard mode solution of differential equations and using this method new approximations have been derived that are better than the stationary phase approximation. The approximation methods apply to dispersion relations that exhibit propagation and attenuation. In this paper we derive the phase space differential equations that the approximations satisfy and also derive an exact phase space differential equation for a mode. By comparing the two we show that the approximations neglect higher-order derivatives in the phase space distribution.

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1. Introduction

It has been recognized for over seventy years that transforming functions into the phase space of two non-commuting variables offers considerable insight into the nature of the function, and also has many practical applications [6, 1, 2]. If the function is governed by a differential equation, then the common procedure is solving the differential equation and then calculating the phase space distribution function for the solution. Our aim here is to show that considerable advantages are gained if one transforms the original differential equation into a phase space differential equation. The advantages are manyfold, both from gaining insight into the nature of differential equations and also in devising practical methods of solution. In this introduction we heuristically motivate some of the ideas and also present some of the fundamental issues.

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1.1. What is Phase Space?

Historically, phase space arose in mechanics when it was realized that considerable simplifications occur when one combines the individual description of position and velocity into a joint position-velocity space. Many of the fundamental equations of physics are phase space equations, among the most famous being the Liouville, Boltzmann, and Focker–Planck equations. However, since the introduction of the Wigner distribution, the concept has been extended to variables which are non-commuting [13]. The two most common pair variables are time/frequency and position/wave-number. The reason we say that they are non commuting is that when the quantities are represented by operators the operators do not commute [2].

In this paper we deal mostly with position/wave-number phase space. Consider a function of position and time $u(x,t)$ which may be the solution of a partial differential equation. We define the Fourier transform, $S(k,t)$, for the position variable by[1]

$$S(k,t) = \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-ikx}\, dx, \tag{1}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,t)\, e^{ikx}\, dk. \tag{2}$$

The variable k is called wave-number or spatial-frequency. In these equations t is a passive parameter. From a physical point of view we note that the interpretation of $|\,u(x,t)\,|^2$ and $|\,S(k,t)\,|^2$ are:

$$|\,u(x,t)\,|^2 = \text{intensity/energy per unit } x \text{ at time } t,$$

$$|\,S(k,t)\,|^2 = \text{spectral intensity/energy per unit } k \text{ at time } t.$$

The fundamental idea is to find a joint distribution or representation that involves both variables, x and k, in a combined way, and that in some sense correlates the two quantities. There have been many such joint representations proposed. In this paper we use the Wigner distribution defined by

$$W_u(x,k,t) = \frac{1}{2\pi} \int u^*(x - \tfrac{1}{2}\tau, t)\, u(x + \tfrac{1}{2}\tau, t)\, e^{-i\tau k} d\tau \tag{3}$$

which in terms of the spectrum is given by

$$W_u(x,k,t) = \frac{1}{2\pi} \int S^*(k + \tfrac{1}{2}\theta, t)\, S(k - \tfrac{1}{2}\theta, t)\, e^{-i\theta x} d\theta. \tag{4}$$

The fundamental requirement of a joint distribution is that it satisfies

$$\int W_u(x,k,t)\, dk = |\,u(x,t)\,|^2, \tag{5}$$

$$\int W_u(x,k,t)\, dx = |\,S(k,t)\,|^2, \tag{6}$$

[1]All integrals go from -∞ to ∞ unless otherwise noted.

and indeed the Wigner distribution satisfies these "marginal conditions". The reason these are called marginal conditions is because they would be the marginal conditions if the Wigner distribution is considered to be a joint probability distribution of x and k, and where the individual probability distributions are $|\, u(x,t)\,|^2$ and $|\, S(k,t)\,|^2$ respectively. Therefore one can think of the Wigner distribution as a description that combines position and wave number, although it can not be interpreted as a probability distribution in the strict sense, since the Wigner distribution is not manifestly positive. In a similar fashion one can fix x in the function $u(x,t)$ and consider the Wigner distribution of time and frequency

$$W(t,\omega,x)=\frac{1}{2\pi}\int u^*(x,t-\tfrac{1}{2}\tau)\,u(x,t+\tfrac{1}{2}\tau)\,e^{-i\tau\omega}\,d\tau \tag{7}$$

where now x acts as the passive variable. In this paper we will deal mostly with Eq. (3) but the results are easily transcribed for Eq. (7).

1.2. Differential Equations and Phase-Space Distributions

To motivate our approach consider the following two differential equations

$$\frac{\partial\psi}{\partial t}=ia\frac{\partial^2\psi}{\partial x^2};\quad \frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2} \tag{8}$$

where both a and D are real. The first is the Schrödinger free particle equation (with: $a=\hbar/2m$) and the second is the classic diffusion equation, where D is the diffusion coefficient. These equations look similar with the only difference being the i in front of a, yet they have dramatically different type of solutions. Often one tries to understand the differences by attempting to set $D=ia$. We argue that an effective way to understand the equations and the differences is to transform the equations into phase space [3, 4, 5]. As we will show, the phase space equations for the Wigner distribution are

$$\frac{\partial W_\psi}{\partial t}=-2ka\frac{\partial W_\psi}{\partial x}\qquad \text{corresponds to:}\quad \frac{\partial\psi}{\partial t}=ia\frac{\partial^2\psi}{\partial x^2}, \tag{9}$$

$$\frac{\partial W_u}{\partial t}=\frac{D}{2}\frac{\partial^2 W_u}{\partial x^2}-2Dk^2W_u\qquad \text{corresponds to:}\quad \frac{\partial u}{\partial t}=D\frac{\partial^2 u}{\partial x^2}. \tag{10}$$

Now we see that in phase space there is a dramatic difference between the two and hence one would certainly not expect similar solutions. Furthermore, each term can be interpreted and gets reflected in a direct way in the solution of both the original equation and the Wigner distribution equation. In addition to showing that transforming equations into phase space has interesting mathematical issues and exposes the nature of the equation in a simple way, we will also show that they lead to new methods of solution and approximation.

1.3. Wave Equations, Modes, Group Velocity, the Stationary Phase Approximation and the Phase-Space Approximation

Of particular interest are partial differential equations that exhibit wave-like behavior. This has led to many concepts such as modes, dispersion relation, group velocity, among others [9, 11, 12]. These ideas will be reviewed in Section 3. One of the major results due to Kelvin is the stationary phase approximation. Loughlin and Cohen have recently shown that by considering the problem in phase space new approximation methods can be obtained that are more accurate [7, 8]. It is one of the aims of this paper to understand the connection between these approximation methods and the partial differential equation method discussed above.

2. Transforming Differential Equations into Phase Space

While our main emphasis is partial differential equations, it is advantageous to first formulate the ordinary differential equation case. Consider the differential equation

$$a_n \frac{d^n x(t)}{dt^n} + \ldots + a_1 \frac{dx(t)}{dt} + a_0\, x(t) = f(t) \tag{11}$$

where $f(t)$ is called the driving term. To obtain the differential equation satisfied by the Wigner distribution one rewrites Eq. (11) as

$$P(D)\, x(t) = f(t) \tag{12}$$

where

$$P(D) = a_n D^n + \ldots + a_1 D + a_0 \tag{13}$$

and $D = \frac{d}{dt}$. If we take the time-frequency distribution to be the Wigner spectrum, Eq. (7), then the differential equation governing $W_x(t,\omega)$ is [3, 4, 5]

$$P^* \left(\frac{1}{2} \frac{\partial}{\partial t} - i\omega \right) P \left(\frac{1}{2} \frac{\partial}{\partial t} + i\omega \right) W_x(t,\omega) = W_f(t,\omega) \tag{14}$$

where $W_f(t,\omega)$ is the Wigner distribution of the driving force. The notation P^* indicates complex conjugation of the polynomial coefficients $a_0,\ a_1, \ldots a_n$ only, and not of the arguments.

For the sake of completenes, we briefly outline the derivation of Eq. (14). It is convenient to define the cross Wigner distribution of two functions $x(t)$ and $y(t)$,

$$W_{x,y}(t,\omega) = \frac{1}{2\pi} \int x^*(t - \tfrac{1}{2}\tau)\, y(t + \tfrac{1}{2}\tau)\, e^{-i\tau\omega}\, d\tau \tag{15}$$

and we note that

$$W^*_{x,y}(t,\omega) = W_{y,x}(t,\omega). \tag{16}$$

The cross Wigner distribution of the functions $x(t), y(t)$, is linear in either one as long as the two functions are not the same. Now consider the cross Wigner distribution of $f(t)$ and $P_n(D)x(t)$,

$$W_{f,P_n(D)x} = \sum_{i=0}^{n} a_i W_{f,x^{(i)}} \tag{17}$$

where $x^{(i)}(t)$ is the n-th derivative of $x(t)$. By direct calculation one can establish in general

$$W_{x,\dot{y}} = \left(\frac{1}{2}\frac{\partial}{\partial t} + i\omega\right) W_{x,y}, \tag{18}$$

$$W_{\dot{x},y} = \left(\frac{1}{2}\frac{\partial}{\partial t} - i\omega\right) W_{x,y}, \tag{19}$$

and hence

$$W_{f,P_n(D)x} = \sum_{i=0}^{n} a_i \left(\frac{1}{2}\frac{\partial}{\partial t} + i\omega\right)^n W_{f,x} = P_n\left(\frac{1}{2}\frac{\partial}{\partial t} + i\omega\right) W_{f,x}. \tag{20}$$

Now take the cross Wigner distribution of f with respect to both sides of Eq. (12) to obtain

$$P_n\left(\frac{1}{2}\frac{\partial}{\partial t} + i\omega\right) W_{f,x} = W_{f,f}. \tag{21}$$

Taking the complex conjugate of both sides and using Eq. (16) we have,

$$P_n^*\left(\frac{1}{2}\frac{\partial}{\partial t} - i\omega\right) W_{x,f} = W_{f,f}. \tag{22}$$

The same approach also leads to

$$W_{x,P_n(D)x} = W_{x,f} \tag{23}$$

and

$$P_n\left(\frac{1}{2}\frac{\partial}{\partial t} + i\omega\right) W_{x,x} = W_{x,f}. \tag{24}$$

Now operate on both sides of Eq. (24) with $P_n^*\left(\frac{1}{2}\frac{\partial}{\partial t} - i\omega\right)$,

$$P_n^*\left(\frac{1}{2}\frac{\partial}{\partial t} - i\omega\right) P_n\left(\frac{1}{2}\frac{\partial}{\partial t} + i\omega\right) W_{x,x} = P_n^*\left(\frac{1}{2}\frac{\partial}{\partial t} - i\omega\right) W_{x,f} = W_{f,f} \tag{25}$$

which is Eq. (14).

We now address partial differential equations of the form[2]

$$\sum_{l=0}^{M} b_l(x,t)\frac{\partial^l}{\partial t^l}u(x,t) + \sum_{n=0}^{N} a_n(x,t)\frac{\partial^n}{\partial x^n}u(x,t) = 0 \tag{26}$$

In general it is not possible to find a differential equation for the Wigner distribution corresponding to Eq. (26); however, often one can do so, as will be discussed. However, one can always find the differential equation for the four dimensional Wigner distribution defined by

$$Z_u(x,k,t,\,\omega) = \frac{1}{(2\pi)^2}\int u^*(x - \tfrac{1}{2}\tau_x, t - \tfrac{1}{2}\tau)u(x + \tfrac{1}{2}\tau_x, t + \tfrac{1}{2}\tau)e^{-i\tau\omega - i\tau_x k}d\tau d\tau_x, \tag{27}$$

[2]We do not consider a driving term because it does not enter in our discussions in this paper, however that has been done [3, 4, 5].

and from Z one can get the standard Wigner distribution because

$$W_u(x,k,t) = \int Z_u(x,k,t,\,\omega)d\omega, \tag{28}$$

$$W_u(t,\,\omega,x) = \int Z_u(x,k,t,\,\omega)dk. \tag{29}$$

We note that W_u and Z_u are manifestly real. The differential equation satisfied by $Z_u(x,k,t,\,\omega)$ corresponding to $u(x,t)$ governed by Eq. (26) is [3, 4, 5]

$$\sum_{n=0}^{N}\left[a_n(\mathcal{F}_x,\mathcal{F}_t)\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^n\right]Z = \sum_{l=0}^{M}\left[b_l(\mathcal{F}_x,\mathcal{F}_t)\left(\frac{1}{2}\frac{\partial}{\partial t}+i\omega\right)^l\right]Z_u \tag{30}$$

where

$$\mathcal{F}_x = x-\frac{1}{2i}\frac{\partial}{\partial k}; \quad \mathcal{F}_t = t-\frac{1}{2i}\frac{\partial}{\partial\,\omega}. \tag{31}$$

There are many special cases of importance where considerable simplification occurs; we now give some that are relevant to our subsequent considerations.

Constant coefficients. If we take the coefficients to be constant, then

$$\sum_{n=0}^{N}\left[a_n\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^n\right]Z_u = \sum_{l=0}^{M}\left[b_l\left(\frac{1}{2}\frac{\partial}{\partial t}+i\omega\right)^l\right]Z_u. \tag{32}$$

It is often helpful to take the complex conjugate of this equation and subtract and add it from Eq. (32). Keeping in mind that Z is real one obtains

$$\sum_{n=0}^{N}\left[a_n\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^n \pm a_n^*\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^n\right]Z_u$$
$$=\sum_{l=0}^{M}\left[b_l\left(\frac{1}{2}\frac{\partial}{\partial t}+i\omega\right)^l \pm b_l^*\left(\frac{1}{2}\frac{\partial}{\partial t}-i\omega\right)^l\right]Z_u. \tag{33}$$

Real coefficients. If the coefficients are real, then

$$\sum_{n=0}^{N}a_n\left[\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^n \pm \left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^n\right]Z_u$$
$$=\sum_{l=0}^{M}b_l\left[\left(\frac{1}{2}\frac{\partial}{\partial t}+i\omega\right)^l \pm \left(\frac{1}{2}\frac{\partial}{\partial t}-i\omega\right)^l\right]Z_u. \tag{34}$$

In manipulating the above equations the following relations are useful:

$$\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)+\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)=\frac{\partial}{\partial x}, \tag{35}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)-\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)=2ik, \tag{36}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^2+\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^2=\frac{1}{2}\frac{\partial^2}{\partial x^2}-2k^2, \tag{37}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^2-\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^2=2ik\frac{\partial}{\partial x}, \tag{38}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^3+\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^3=\frac{1}{4}\frac{\partial^3}{\partial x^3}-3k^2\frac{\partial}{\partial x}, \tag{39}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^3-\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^3=-2ik^3+\frac{3}{2}ik\frac{\partial^2}{\partial x^2}, \tag{40}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^4-\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^4=-4ik^3\frac{\partial}{\partial x}+ik\frac{\partial^3}{\partial x^3}. \tag{41}$$

When can one obtain an equation for the standard Wigner distribution? If we integrate both sides of any of the above equations for Z, then the equation can be transformed into an equation for W if the operators do not depend on ω. We do not discuss the details as to when that can be done in general but we now give a number of examples.

Take, for example, Eq. (34) and let us assume that all b's are zero except for b_1; we let $b_1 = b$. By integrating both sides with respect to ω we have

$$\sum_{k=0}^{N} a_k\left[\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^k+\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^k\right]W_u \tag{42}$$

$$=b\int\left[\left(\frac{1}{2}\frac{\partial}{\partial t}+i\omega\right)^1+\left(\frac{1}{2}\frac{\partial}{\partial t}-i\omega\right)^1\right]Z_u d\omega=b\int\left[\frac{\partial}{\partial t}\right]Z_u d\omega \tag{43}$$

and therefore

$$\sum_{n=0}^{N} a_n\left[\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^n+\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^n\right]W_u=b\frac{\partial W_u}{\partial t}. \tag{44}$$

This worked because both sides resulted in operators independent of ω. There is one further case which is of particular interest. Taking b_1 to be pure complex, setting $b_1 = ib'$, and taking the minus sign in Eq. (33) one obtains

$$\sum_{n=0}^{N} a_n\left[\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^n-\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^n\right]W_u=ib'\frac{\partial}{\partial t}W_u. \tag{45}$$

It is of interest to write down the case where we go up to third order in the left-hand side but with $a_0 = 0$. Corresponding to Eq. (45) and Eq. (46) we respectively have

$$\left[a_1\frac{\partial}{\partial x}+a_2\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}-2k^2\right)+a_3(\frac{1}{4}\frac{\partial^3}{\partial x^3}-3k^2\frac{\partial}{\partial x})\right]W_u=b\frac{\partial}{\partial t}W_u, \tag{46}$$

$$\left[2ka_1+2a_2k\frac{\partial}{\partial x}+a_3\left(-2k^3+\frac{3}{2}k\frac{\partial^2}{\partial x^2}\right)\right]W_u=b'\frac{\partial}{\partial t}W_u, \tag{47}$$

which can be rewritten as follows,

$$\left[-2k^2a_2+\left(a_1-3k^2a_3\right)\frac{\partial}{\partial x}+\frac{a_2}{2}\frac{\partial^2}{\partial x^2}+\frac{a_3}{4}\frac{\partial^3}{\partial x^3}\right]W_u=b\frac{\partial}{\partial t}W_u, \tag{48}$$

$$\left[2ka_1+2a_2k\frac{\partial}{\partial x}+a_3\left(-2k^3+\frac{3}{2}k\frac{\partial^2}{\partial x^2}\right)\right]W_u=b'\frac{\partial}{\partial t}W_u. \tag{49}$$

Quantum-like case. There is one more special case which is of interest and that is when $a_0(x,t)$ is a function of x only. Also, take $b_1=ib'$ and set $a_0(x)=V(x)$. The resulting equation is

$$V\left(x-\frac{1}{2i}\frac{\partial}{\partial k}\right)-V^*\left(x+\frac{1}{2i}\frac{\partial}{\partial k}\right)+\sum_{n=1}^{N}a_n\left[\left(\frac{1}{2}\frac{\partial}{\partial x}+ik\right)^n-\left(\frac{1}{2}\frac{\partial}{\partial x}-ik\right)^n\right]W_u$$
$$=ib'\frac{\partial}{\partial t}W_u \tag{50}$$

which up to a_3 gives

$$\left[i\left\{V^*\left(x+\frac{1}{2i}\frac{\partial}{\partial k}\right)-V\left(x-\frac{1}{2i}\frac{\partial}{\partial k}\right)\right\}+2ka_1\right.$$
$$\left.+\,2a_2k\frac{\partial}{\partial x}+a_3\left(-2k^3+\frac{3}{2}k\frac{\partial^2}{\partial x^2}\right)\right]W_u=b'\frac{\partial}{\partial t}W_u. \tag{51}$$

Furthermore, for the standard quantum case where V is real, $a_1=0$ and $a_2=-1$ one obtains

$$\left[i\left\{V^*\left(x+\frac{1}{2i}\frac{\partial}{\partial k}\right)-V\left(x-\frac{1}{2i}\frac{\partial}{\partial k}\right)\right\}-2k\frac{\partial}{\partial x}\right]W_u=\frac{\partial}{\partial t}W_u. \tag{52}$$

This equation was previously derived by Wigner and Moyal [13, 10].

2.1. Examples

2.1.1. Schrödinger Free Particle Equation. The Schrödinger equation for a free particle is

$$i\frac{\partial\psi}{\partial t}=-a\frac{\partial^2\psi}{\partial x^2}. \tag{53}$$

Using Eq. (47) and taking $b'=1$ and $a_1=-a$ one gets

$$\frac{\partial W_\psi}{\partial t}=-2kaW_\psi \tag{54}$$

a result first obtained by Wigner. The solution of this equation is

$$W_\psi(x,k,t)=W_\psi(x-2akt,k,0). \tag{55}$$

2.1.2. Diffusion Equation with Drift. Consider the diffusion equation with drift,

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + D\frac{\partial^2 u}{\partial x^2}. \tag{56}$$

We could take the drift term to be zero because the solution with drift can easily be obtained from the solution without drift by letting $x \to x - ct$ in the solution without drift. Nonetheless we prefer to study directly Eq. (56). Using Eq. (48) with $a_1 = -c$, $a_2 = D$, and $b_1 = 1$ we have that the respective Wigner equation of motion is

$$\frac{\partial W_u}{\partial t} = -c\frac{\partial W_u}{\partial x} + \frac{D}{2}\frac{\partial^2 W_u}{\partial x^2} - 2Dk^2 W_u. \tag{57}$$

The general solution is

$$W_u(x,k,t) = \frac{1}{\sqrt{2\pi Dt}} e^{-2Dk^2 t} \int \exp\left[-\frac{(x' - x + ct)^2}{2Dt}\right] W_u(x',k,0)dx'. \tag{58}$$

2.1.3. Modified Diffusion Equation. We consider

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + iE\frac{\partial^2 u}{\partial x^2} + D\frac{\partial^2 u}{\partial x^2}. \tag{59}$$

It involves both wave-like behavior and attenuation: the second term on the right gives wave-like motion and the third term gives attenuation. To obtain the Wigner distribution we use Eq. (48) to obtain

$$\frac{\partial W_u}{\partial t} = -(c + 2kE)\frac{\partial W_u}{\partial x} + \frac{D}{2}\frac{\partial^2 W_u}{\partial x^2} - 2Dk^2 W_u. \tag{60}$$

2.1.4. Linearized KdV Equation. The equation is

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} \tag{61}$$

and using Eq. (48) we have that

$$\left[-(\alpha + 3\beta k^2)\frac{\partial}{\partial x} + \frac{\beta}{4}\frac{\partial^3}{\partial x^3}\right] W_u = \frac{\partial}{\partial t} W_u. \tag{62}$$

3. Phase-Space Approximation

The concept of modes, dispersion relation, group velocity, and related concepts arose from the study of partial differential equations whose solutions exhibit wave-like behavior. In particular, most of the historical development has concentrated on the constant coefficient case,

$$\sum_{n=0}^{N} a_n \frac{\partial^n u}{\partial t^n} = \sum_{n=0}^{M} b_n \frac{\partial^n u}{\partial x^n}. \tag{63}$$

While the free space wave equation was derived by Euler and d'Alembert in the middle of the 18th century, the general approach, the concept of modes, dispersion relations, and the solution by Fourier methods were developed in the late 19th century by Rayleigh and Kelvin, who devised the fundamental ideas and also found approximations. Eq. (63) may be solved by the classical Fourier method by substituting $e^{ikx-i\omega t}$ into Eq. (63) to give [12, 11, 9]

$$\sum_{n=0}^{N} a_n(-i\omega)^n = \sum_{n=0}^{M} b_n(ik)^n \tag{64}$$

which is an algebraic equation between k and ω. One can now solve for k in terms of ω or ω in terms of k, and which is done depends on the type of initial conditions. Generally speaking, there are two types of initial conditions corresponding to two distinct physical situations. First, is when $u(x,0)$ is given and the second is when we are given $u(0,t)$. An example of the first case is plucking a string and letting it go at time zero. Examples of the second case are sonar, radar, speech production, and fiber optics, because in those cases we produce a signal as a function of time and the place of production, x, is fixed. Here we mostly consider the first case and as is standard we write [3]

$$\omega = \omega(k). \tag{65}$$

This is called the dispersion relation and there can be many solutions and each solution is called a mode. The solution for each mode is [12, 11]

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int S(k,0)\, e^{-i\omega(k)t+ikx}\; dk \tag{66}$$

where $S(k,0)$ is the initial spatial spectrum, obtained from the initial pulse by way of

$$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0)\; e^{-ikx}\, dx. \tag{67}$$

If we define

$$S(k,t) = S(k,0)\, e^{-i\omega(k)t}, \tag{68}$$

then $u(x,t)$ and $S(k,t)$ form Fourier transform pairs for any t,

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int S(k,t)\, e^{ikx}\, dk, \tag{69}$$

$$S(k,t) \;=\; \frac{1}{\sqrt{2\pi}} \int u(x,t)\, e^{-ikx}\, dx. \tag{70}$$

In general ω is complex and we write

$$\omega = \omega_R + i\omega_I. \tag{71}$$

A central idea is the concept of group velocity defined as the derivative of the real part of ω

$$v(k) = \omega_R'(k). \tag{72}$$

[3]The ω defined here should not be confused with the ω used in Section 2..

3.1. Wigner Distribution Approximation

As mentioned in the introduction the standard approximation method is the stationary phase approximation [12, 11]. Recently, Loughlin and Cohen have obtained a phase space approximation which is an improvement in that it approaches the stationary phase approximate for large distances and times but is also accurate for small distances and times [7, 8]. The approximation is for the Wigner distribution, and we call it $W_a(x,k,t)$. The approximation is

$$W_a(x,k,t) = e^{2\omega_I(k)t} W(x - v(k)t, k, 0) \approx W(x,k,t) \tag{73}$$

where $W_a(x,k,t)$ is the approximate Wigner distribution at time t and $W(x,k,0)$ is the exact Wigner distribution at time zero. The approximation is easy to apply since one just substitutes $x - v(k)t$ for x in the initial Wigner distribution and multiplies by $e^{2\omega_I(k)t}$. It has been shown that this is a good approximation and moreover it is very revealing. It shows that each phase space point evolves (approximately) with constant velocity given by the group velocity. It also explicitly shows damping and it is clear that for $\omega_I(k) \le 0$ we have decay and for $\omega_I(k) > 0$ we have exponential growth.

A further approximation, $W_{a2}(x,k,t)$, is

$$W_{a2}(x,k,t) = \sqrt{\frac{1}{\pi|\omega_I^{(2)}(k)|t}} e^{2\omega_I(k)t} \int W(x',k,0) \exp\left[-\frac{(x - x' - v(k)t)^2}{|\omega_I^{(2)}(k)|t}\right] dx' \tag{74}$$

where it has been assumed that $\omega_I^{(2)}(k) < 0$. We mention that this approximation reverts to the first approximation when $|\omega_I^{(2)}(k)|t$ is very small. That is the case since

$$\lim_{|\omega_I^{(2)}(k)|t \to 0} \sqrt{\frac{1}{\pi|\omega_I^{(2)}(k)|t}} e^{2\omega_I(k)t} \exp\left[-\frac{(x - x' - v_g(k)t)^2}{|\omega_I^{(2)}(k)|t}\right] \to \delta(x - x' - v(k)t) \tag{75}$$

in which case

$$W_{a2}(x,k,t) \to e^{2\omega_I(k)t} W(x - x' - v(k)t, k, 0) = W_a(x,k,t). \tag{76}$$

The main aim of this paper is to attempt to understand these approximations from the point of view of the phase space differential equations described in Section 2. Toward that end we now consider what differential equations these approximations satisfy. Subsequently we derive the exact differential equations for a mode.

3.2. Differential Equations for the Approximations

For the first approximation given by Eq. (73) we have

$$\frac{\partial W_a}{\partial t} = 2\omega_I(k) W_a - v(k) e^{2\omega_I t} \frac{\partial}{\partial x} W(x - v(k)t, k, 0), \tag{77}$$

$$\frac{\partial W_a}{\partial x} = e^{2\omega_I t} \frac{\partial}{\partial x} W(x - v(k)t, k, 0). \tag{78}$$

Substituting $\frac{\partial W_a}{\partial x}$ from Eq. (78) into Eq. (77) one obtains

$$\frac{\partial W_a}{\partial t} = 2\omega_I(k)W_a - v(k)\frac{\partial W_a}{\partial x}. \tag{79}$$

For the second approximation, Eq. (74), the same approach leads to

$$\frac{\partial W_{a2}}{\partial t} = 2\omega_I W_{a2} - v(k)\frac{\partial W_{a2}}{\partial x} + \frac{|\omega_I^{(2)}(k)|}{4}\frac{\partial^2 W_{a2}}{\partial x^2}. \tag{80}$$

3.3. Examples

3.3.1. Schrödinger Free Particle Equation. The equation is

$$\frac{\partial \psi}{\partial t} = ia\frac{\partial^2 \psi}{\partial x^2} \tag{81}$$

and it is readily verified that the dispersion relation is

$$\omega = ak^2 \tag{82}$$

and

$$\omega_R = ak^2; \quad v(k) = 2ak; \quad \omega_I = 0; \quad \omega_I^{(2)}(k) = 0. \tag{83}$$

The first approximation then yields

$$W_a(x,k,t) \approx W(x - 2akt, k, 0) \tag{84}$$

and the differential equation is

$$\frac{\partial W_a}{\partial t} = -2ak\frac{\partial W_a}{\partial x}. \tag{85}$$

The second approximation gives the same answer. This is expected since for this case the first approximation gives the exact answer.

3.3.2. Diffusion Equation with Drift. The equation is

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + D\frac{\partial^2 u}{\partial x^2} \tag{86}$$

which gives

$$\omega = ck - iDk^2 \tag{87}$$

and

$$\omega_R = ck; \quad v(k) = c; \quad \omega_I = -Dk^2; \quad \omega_I^{(2)}(k) = -2D. \tag{88}$$

3.3.3. Modified Diffusion Equation with Drift. We consider the equation

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + iE\frac{\partial^2 u}{\partial x^2} + D\frac{\partial^2 u}{\partial x^2} \tag{89}$$

which gives the dispersion relation

$$\omega = ck + Ek^2 - iDk^2 \tag{90}$$

and hence

$$\omega_R = ck + Ek^2; \quad v(k) = c + 2Ek; \quad \omega_I = -Dk^2; \quad \omega_I^{(2)}(k) = -2D. \tag{91}$$

The first approximation gives

$$W_a(x,k,t) \approx e^{-Dk^2 t} W(x - ct - 2Ekt, k, 0) \tag{92}$$

and the second gives

$$W_{a2}(x,k,t) \approx \sqrt{\frac{1}{2\pi Dt}} e^{-Dk^2 t} \int W(x',k,0) \exp\left[-\frac{(x - x' - (c+2Ek)t)^2}{2Dt}\right] dx'. \tag{93}$$

The corresponding differential equations, respectively, are

$$\frac{\partial W_a}{\partial t} = -2Dk^2 W_a - (c+2Ek)\frac{\partial W_a}{\partial x}, \tag{94}$$

and

$$\frac{\partial W_{a2}}{\partial t} = -2Dk^2 W_{a2} - (c+2Ek)\frac{\partial W_{a2}}{\partial x} + \frac{D}{2}\frac{\partial^2 W_{a2}}{\partial x^2}. \tag{95}$$

3.3.4. Linearized KdV Equation. The equation is

$$\frac{\partial u}{\partial t} = -\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} \tag{96}$$

and the dispersion relation is then

$$\omega = \alpha k + \beta k^3 \tag{97}$$

and hence

$$\omega_R = \alpha k + \beta k^3; \quad v(k) = \alpha + 3\beta k^2; \quad \omega_I = 0; \quad \omega_I^{(2)}(k) = 0.$$

Therefore

$$W_a(x,k,t) \approx W(x - (\alpha + 3\beta k^2)t, k, 0) \tag{98}$$

and

$$\frac{\partial W_a}{\partial t} = -(\alpha + 3\beta k^2)\frac{\partial W_a}{\partial x}. \tag{99}$$

Note that if we compare this with the exact equation for this case, then what the approximation does is leave out the third derivative term.

4. Exact Differential Equation for a Mode

We now obtain the differential equation satisfied by a mode. Differentiating Eq. (66) with respect to time and multiplying by i we have

$$i\frac{\partial}{\partial t}u(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int \omega(k)S(k,0)\,e^{-i\omega(k)t}\,e^{ikx}\;dk. \tag{100}$$

This is an integral equation, but it can be converted to a differential equation of infinite order,

$$i\frac{\partial}{\partial t}u(x,t) \;= \omega\left(\frac{1}{i}\frac{\partial}{\partial x}\right)u(x,t). \tag{101}$$

Also, one can readily obtain the differential equation for the spatial spectrum, $S(k,t)$,

$$i\frac{\partial}{\partial t}S(k,t) = \omega(k)S(k,t). \tag{102}$$

Any number of methods can be used to derive the Wigner distribution corresponding to Eq. (101). The result is

$$i\frac{\partial}{\partial t}W(x,k,t) = \left[\omega\left(k+\frac{1}{2i}\frac{\partial}{\partial x}\right) - \omega^*\left(k-\frac{1}{2i}\frac{\partial}{\partial x}\right)\right]W(x,k,t).$$

In terms of real and imaginary parts we write

$$\begin{aligned} i\frac{\partial}{\partial t}W(x,k,t) = {}& \left[\omega_R\left(k+\frac{1}{2i}\frac{\partial}{\partial x}\right) - \omega_R\left(k-\frac{1}{2i}\frac{\partial}{\partial x}\right)\right]W(x,k,t) \\ & + i\left[\omega_I\left(k+\frac{1}{2i}\frac{\partial}{\partial x}\right) + \omega_I\left(k-\frac{1}{2i}\frac{\partial}{\partial x}\right)\right]W(x,k,t). \end{aligned} \tag{103}$$

We now expand ω_R and ω_I in a Taylor series, which after some manipulation results in

$$\begin{aligned} \frac{\partial}{\partial t}W(x,k,t) = {}& \left[\sum_{n=0}^{\infty}\frac{\omega_R^{(2n+1)}(k)}{(2n+1)!}\frac{(-1)^{n+1}}{2^{2n}}\left(\frac{\partial}{\partial x}\right)^{2n+1}\right. \\ & \left. + \sum_{n=0}^{\infty}\frac{\omega_I^{(2n)}(k)}{(2n)!}\frac{(-1)^{n}}{2^{2n-1}}\left(\frac{\partial}{\partial x}\right)^{2n}\right]W. \end{aligned} \tag{104}$$

If we keep terms up to third order in the derivatives with respect to x we have

$$\begin{aligned} \frac{\partial}{\partial t}W(x,k,t) \sim {}& \left[2\omega_I(k) - v(k)\left(\frac{\partial}{\partial x}\right) - \frac{1}{4}\omega_I^{(2)}(k)\frac{\partial^2}{\partial x^2}\right. \\ & \left. + \frac{1}{24}\omega_R^{(3)}(k)\left(\frac{\partial}{\partial x}\right)^3 \ldots\right]W(x,k,t). \end{aligned} \tag{105}$$

5. Comparison of Exact and Approximate Wigner Distributions

For clarity we repeat the first and second differential equation for the two approximations, respectively,

$$\frac{\partial W_a}{\partial t} = 2\omega_I(k)W_a - v(k)\frac{\partial W_a}{\partial x}, \tag{106}$$

$$\frac{\partial W_{a2}}{\partial t} = 2\omega_I W_{a2} - v(k)\frac{\partial W_{a2}}{\partial x} + \frac{|\omega_I^{(2)}(k)|}{4}\frac{\partial^2 W_{a2}}{\partial x^2}. \tag{107}$$

Since we consider cases where $\omega_I^{(2)}(k) \le 0$, we write the exact equation for a mode, Eq. (105), as

$$\frac{\partial}{\partial t}W(x,k,t) = \left[2\omega_I(k) - v(k)\frac{\partial}{\partial x}. + \frac{1}{4}|\omega_I^{(2)}(k)|\frac{\partial^2}{\partial x^2} + \frac{1}{24}\omega_R^{(3)}(k)\left(\frac{\partial}{\partial x}\right)^3 \dots\right]W. \tag{108}$$

Notice that the first approximation equation can be obtained from the second and the exact one by neglecting derivatives above the first. Similarly, the second approximation can be obtained from the exact equation by neglecting derivatives above the second. Also, for each of the examples considered this behavior is the case. Although we have not given a proof it seems that successive approximations are obtained to a given order by simply neglecting higher derivatives. We are currently investigating these methods for the more general case where the coefficients of the differential equation are functions of space and time, that is, Eq. (26).

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Leon Cohen
Department of Physics and Astronomy
City University of New York
Hunter College
695 Park Ave.
New York, NY 10021
USA
e-mail: leon.cohen@hunter.cuny.edu

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Two-Window Spectrograms and Their Integrals

Paolo Boggiatto, Giuseppe De Donno and Alessandro Oliaro

Abstract. We analyze in this paper some basic properties of two-window spectrograms, introduced in a previous work. This is achieved by the analysis of their kernel, in view of their immersion in the Cohen class of time-frequency representations. Further we introduce weighted averages of two-window spectrograms depending on varying window functions. We show that these new integrated representations improve some features of both the classical Rihaczek representation and the two-window spectrogram which in turns can be viewed as limit cases of them.

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1. Introduction

The need of visualizing how the energy of a signal is spread in the time-frequency plane has led in the literature to the definition of a considerable amount of time-frequency representations. Generally they are sesquilinear forms $Q : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$, where, given a signal f, the function $Q(f,f)(x,\omega)$, or for short $Q(f)(x,\omega)$, plays a role of *energy density*. It corresponds to the density of mass in classical mechanics or to the probability distribution in statistics, with the important difference that in signal theory there is not a univocal choice of which representation has to be used, each having advantages and disadvantages. The situation is of remarkable importance in time-frequency analysis and is well illustrated in a number works, see for example [4], [5], [6], [7], [9], [11], [12], [13] for detailed presentations of these topics.

The most basic considerations in order to construct a time-frequency representation are those which lie behind the definition of the Rihaczek representation. Let us consider the model case of a signal $f(x)$ having a frequency ω_0 in the time interval $[a,b]$, i.e., let $f(x) = \chi_{[a,b]}\Theta(x)$, where $\Theta(x)$ is one of the functions $\sin(2\pi\omega_0 x)$, $\cos(2\pi\omega_0 x)$, or $e^{2\pi i\omega_0 x}$, and $\chi_{[a,b]}(x)$ is the characteristic function of the interval $[a,b]$. Then $f(x)$ will be different from the null function on $[a,b]$ and $\hat{f}(\omega)$ will be different from the null function on a neighborhood of ω_0. It follows trivially that $Q(x,\omega) = f(x)\hat{f}(\omega)$ is a function on the time-frequency plane which is different from the null function on a neighborhood of the segment $\{\omega = \omega_0, x \in [a,b]\}$ and is (substantially) zero elsewhere. Intertwining the variables by a multiplication with a complex exponential will guarantee that the marginal conditions are satisfied (see, e.g., [3], [5] and Section 2), yielding the classical Rihaczek representation

$$R(f)(x,\omega) = e^{-2\pi ix\omega} f(x)\overline{\hat{f}(\omega)}. \tag{1.1}$$

As we shall show in Section 3, though very simple in its construction and showing a good behavior in the model case of the signal above, this representation presents however very disturbing interference patterns as soon as it is applied on multi-component signals, i.e., practically all real signals.

A different approach leads to the definition of the two-window spectrogram $Sp_{\phi,\psi}$. It is based on the *Gabor transform*

$$V_\phi(f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi it\omega} f(t)\overline{\phi(t-x)}\, dt \tag{1.2}$$

where, before taking the Fourier transform, the signal f is "cut" by a multiplication with time translations of the *window* function $\phi(x)$, which in the most generality can be supposed to be a tempered distribution (see, e.g., [5], [10], [13] and [17] for a general setting on groups). The *two window spectrogram*, presented in [2]–[3] (called there *generalized spectrogram*) is defined by

$$Sp_{\phi,\psi}(f)(x,w) = Sp_{\phi,\psi}(f,f)(x,w) = V_\phi f(x,w)\overline{V_\psi f}(x,w). \tag{1.3}$$

Of course for $\phi = \psi$ it reduces to the classical spectrogram

$$Sp_\phi(f)(x,\omega) = |V_\phi(f)(x,\omega)|^2. \tag{1.4}$$

In the case of the two-window spectrogram the problem of the interference is very reduced but this is "paid" in terms of the support property which is not anymore satisfied (see Section 2).

A third very important representation is the *Wigner distribution* (see, e.g., [8], [16]) defined as

$$\mathrm{Wig}(f)(x,\omega) = \mathrm{Wig}(f,f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi it\omega} f(x+t/2)\overline{f(x-t/2)}\, dt. \tag{1.5}$$

Besides the relevance it has in itself, it is the basis of the *Cohen class* (see Section 2) whose general properties will be widely used in this paper.

Given two representations, a usual way to construct better representations is to take the average of them. Quite informally speaking, if both representations signalize the presence of true frequencies accompanied with some undesirable "artifacts", then the result will in general yield even better indication where the true frequencies are present and show some reduced and more spread artifacts.

Starting from this idea, similarly to the construction presented in [1] for Wigner functions, we introduce in this paper a new type of representation, where we substitute a simple average of two time-frequency forms with a weighted integral over a path of parameterized representations.

More precisely we consider the case of parameterized windows ϕ_λ and ψ_λ, where we let one window approach a constant function and the other one the Dirac distribution δ, as λ goes to $+\infty$. This improves the behavior with respect to the support property which is not satisfied by spectrograms, but at the same time it increases the interferences phenomena. For ϕ and ψ fixed window functions, if not differently specified, we shall take

$$\phi_\lambda(x) = \lambda^{d/2}\phi(\sqrt{\lambda}x), \qquad \psi_\lambda(x) = \psi(x/\sqrt{\lambda}), \tag{1.6}$$

so that $\|\phi_\lambda\|_{L^1} = \|\phi\|_{L^1}$ for every λ.

We consider then weighted integrals of these forms, i.e., a representation of the type

$$S^\theta_{\phi,\psi} = \int_1^{+\infty} \theta(\lambda)\ Sp_{\phi_\lambda,\psi_\lambda}\, d\lambda, \tag{1.7}$$

where $\theta(\lambda)$ is a fixed real function integrable on $[1, +\infty]$.

We shall call *integrated spectrograms* the representations $S^\theta_{\phi,\psi}$ in (1.7). The choice of the function $\theta(\lambda)$ clearly determines for which values of the parameter λ the corresponding two-window spectrograms should have more "weight" in the representation. A priori any choice of $\theta(\lambda)$ leads to a well-defined representation, however in Section 5 we show that for a suitable choice of $\theta(\lambda)$ conservation of energy is satisfied by $S^\theta_{\phi,\psi}$. Further we shall show that both the classical Rihaczek representation and the two-window spectrogram $Sp_{\phi,\psi}$ can be viewed as limit cases of (1.7) when $\theta(\lambda)$ pointwise approaches specific limit functions.

On the other hand the representations $S^\theta_{\phi,\psi}$ show, for many intermediate choices of θ, a better behavior both with respect to the Rihaczek and the two-window spectrogram, reducing almost to null the interferences phenomena of the Rihaczek and improving the support localization of the spectrogram, the behavior is also showed to be better than a simple average of the Rihaczek and spectrogram representations.

For different types of integrated representation based on the Wigner function instead of spectrograms and for their relations with pseudo-differential operators see [1], for numerical implementations and applications, in particular to seismic waves, see [14].

The paper is organized as follows. We begin in Section 2 by considering the two-window spectrogram $Sp_{\phi,\psi}$ as a member of the Cohen class and deduce from its Cohen kernel some basic properties such as reality, marginals, energy conservation and support property. We also obtain a result of approximation of Cohen class representations with L^2 kernels by finite sums of two-window spectrograms.

In Section 3 we study the integrated spectrogram (1.7) and we motivate our definition by testing, for a specific choice of $\theta(\lambda)$, the behavior of $S^\theta_{\phi,\psi}$ on a standard signal, comparing the result with that of the Rihaczek and the two-window spectrogram as well as the average of them.

A precise functional setting in which the integrated spectrogram acts as bounded sesquilinear map is developed in Section 4, where a version of the uncertainty principle for this representation is also obtained.

In Section 5, after we have showed that the integrated spectrogram belongs to the Cohen class, we study some of its basic properties using results from Section 2. We conclude then showing that, as mentioned above, the Rihaczek representation and the two-window spectrogram are limit cases of the integrated spectrogram.

2. Two-Window Spectrogram and Cohen Kernel

The purpose of this section is to study some of the basic properties of the two-window spectrogram from its expression as an element of the *Cohen class.* This is a very general class of time-frequency representations, introduced by L. Cohen, see [5], and widely studied since the 1970s. It can be defined as the set of representations of the form

$$C(f) = \sigma * \mathrm{Wig}(f) \tag{2.1}$$

where, in our context, σ will be supposed to be a tempered distribution in $\mathcal{S}'(\mathbb{R}^{2d})$ and will be called *Cohen kernel.* The wide possibility of choice of the Cohen kernel permits to cover most time-frequency representations.

The following relationship between the Wigner distribution and the two window spectrogram holds (see [2]):

$$Sp_{\phi,\psi}(f,g)(x,w) = \mathrm{Wig}(\tilde{\psi},\tilde{\phi}) * \mathrm{Wig}(f,g)(x,w), \tag{2.2}$$

where $\tilde{\phi}(s) := \phi(-s)$ and $\tilde{\psi}(s) := \psi(-s)$. Equality (2.2), valid in suitable functional settings, for example when $f, g, \phi, \psi \in \mathcal{S}(\mathbb{R}^d)$, gives us the expression of the two-window spectrogram as an element of the Cohen class, where we have as Cohen kernel $\sigma = \mathrm{Wig}(\tilde{\psi},\tilde{\phi})$.

As showed in [5], Chapter 9, many of the more basic properties of a representation can be deduced from corresponding properties of the inverse Fourier transform $\mathcal{F}^{-1}(\sigma)$ of the Cohen kernel. It will therefore useful to have an explicit expression for it in the case of the two-window spectrogram.

Proposition 1. *The two-window spectrogram can be written as*

$$Sp_{\phi,\psi}(f,g)(x,w) = \sigma * \mathrm{Wig}(f,g)(x,w), \tag{2.3}$$

where

$$\mathcal{F}^{-1}(\sigma)(u,t) = A(\tilde{\psi},\tilde{\phi})(u,t) \tag{2.4}$$

and $A(\tilde{\psi},\tilde{\phi})(u,t)$ *is the* ambiguity function (*see* [5]) *defined as*

$$A(\tilde{\psi},\tilde{\phi})(u,t) = \int_{\mathbb{R}^d} e^{2\pi i u x} \tilde{\psi}(x+t/2)\overline{\tilde{\phi}(x-t/2)}\, dx.$$

Proof. The assertion follows immediately from (2.2) as we have

$$\sigma(x,\omega) = \mathrm{Wig}(\tilde{\psi},\tilde{\phi})(x,\omega) = \mathcal{F}_{t\to\omega}(\tilde{\psi}(x+t/2)\overline{\tilde{\phi}(x-t/2)}) \qquad \square$$

We recall now some of the most basic "desirable" properties of a time-frequency representation; for detailed discussion of these properties, see, for example, [4], [11].

Definiton 2 (Marginals). A time-frequency representation Q is said to satisfy the *Marginal distributions condition* if for every $f \in L^2(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} Q(f)(x,\omega)dx = |\hat{f}(\omega)|^2 \qquad \text{and} \qquad \int_{\mathbb{R}^d} Q(f)(x,\omega)d\omega = |f(x)|^2.$$

Definiton 3 (Support property). Let $H(\text{supp } f)$ be the convex hull of supp f and $H(\text{supp } \hat{f})$ that of supp $\hat{f}$. Let Π_x and Π_ω be the orthogonal projections on the first and the second factors in $\mathbb{R}^d_x \times \mathbb{R}^d_\omega$ respectively. A representation Q is said to enjoy the "support property" if $\Pi_x \text{supp } Q(f) \subseteq H(\text{supp } f)$ and $\Pi_\omega \text{supp } Q(f) \subseteq H(\text{supp } \hat{f})$.

Definiton 4 (Energy conservation). A time-frequency representation Q is said to satisfy *conservation of energy* if for $f \in L^2(\mathbb{R}^d)$:

$$\|Q(f)\|_{L^2(\mathbb{R}^{2d})} = \|f\|^2_{L^2(\mathbb{R}^d)}.$$

We recall now how these properties are related to the Cohen kernel of a representation, see [5] for references.

Proposition 5. *The following holds for a generic representation* $Q = \sigma * \mathrm{Wig}$ *in the Cohen class:*

a) *The time* marginal condition *is satisfied if and only if* $\mathcal{F}^{-1}(\sigma)(u,0) = 1$.
b) *The frequency* marginal condition *is satisfied if and only if* $\mathcal{F}^{-1}(\sigma)(0,t) = 1$.
c) *The* conservation of energy *is satisfied if and only if* $\mathcal{F}^{-1}(\sigma)(0,0) = 1$.
d) *The representation is real if and only if* $(\widetilde{\mathcal{F}^{-1}(\sigma)}) = \overline{\mathcal{F}^{-1}(\sigma)}$.

We pass now to examine the consequences that the previous general properties have in the special case of the two-window spectrogram.

Proposition 6. *The two-window spectrogram* $Sp_{\phi,\psi}(f,g)(x,w)$ *satisfies the time marginal condition if and only if* $\psi\overline{\phi} = \delta$.

Proof. From Propositions 1 and 5 (a) the thesis holds if and only if

$$1 = \int_{\mathbb{R}^d} e^{2\pi iux}\tilde{\psi}(x)\overline{\tilde{\phi}(x)}\,dx = \mathcal{F}^{-1}(\tilde{\psi}\overline{\tilde{\phi}})$$

which means $\tilde{\psi}\overline{\tilde{\phi}} = \delta$, i.e., $\psi\overline{\phi} = \delta$. $\square$

Proposition 7. *The two-window spectrogram* $Sp_{\phi,\psi}(f,g)(x,w)$ *satisfies the frequency marginal conditions if and only if* $\hat{\psi}\overline{\hat{\phi}} = \delta$.

Proof. From Propositions 1 and 5 (b) the thesis holds if and only if

$$1 = \int_{\mathbb{R}^d} \tilde{\psi}(x+t/2)\overline{\tilde{\phi}(x-t/2)}\, dx = \int_{\mathbb{R}^d} \tilde{\psi}(y)\overline{\tilde{\phi}(y-t)}\, dy = \tilde{\psi} * \overline{\phi}$$

which means $\hat{\tilde{\psi}}\hat{\overline{\tilde{\phi}}} = \delta$, i.e., $\hat{\psi}\hat{\overline{\phi}} = \delta$. □

Proposition 8. *The two-window spectrogram $Sp_{\phi,\psi}(f,g)(x,w)$ satisfies conservation of energy if and only if $(\psi,\phi) = 1$.*

Proof. From Propositions 1 and 5 (c) the thesis holds if and only if

$$1 = \int_{\mathbb{R}^d} \tilde{\psi}(x)\overline{\tilde{\phi}(x)}\, dx = (\tilde{\psi}, \tilde{\phi}) = (\psi, \phi). \qquad \square$$

Proposition 9. *The two-window spectrogram $Sp_{\phi,\psi}(f,g)(x,w)$ is real if and only if $\psi = C\phi$ for some $C \in \mathbb{R}$.*

Proof. From Propositions 1 and 5 (d) the thesis holds if and only if

$$A(\tilde{\psi}, \tilde{\phi})(u,t) = A(\tilde{\phi}, \tilde{\psi})(u,t),$$

which means

$$\mathcal{F}^{-1}_{t\to\omega}\Big(\tilde{\psi}(x+t/2)\overline{\tilde{\phi}(x-t/2)} - \tilde{\phi}(x+t/2)\overline{\tilde{\psi}(x-t/2)}\Big)(x,\omega) = 0$$

for every $(x,\omega) \in \mathbb{R}^{2d}$. As the Fourier transform is a bijection on $\mathcal{S}'(\mathbb{R}^{2d})$, this is equivalent, after a change of variables in $\mathbb{R}^{2d}$, to the condition

$$\psi(X)\overline{\phi(Y)} = \phi(X)\overline{\psi(Y)} \tag{2.5}$$

for every $(X,Y) \in \mathbb{R}^{2d}$. Suppose now that ψ, ψ are in $\mathcal{S}(\mathbb{R}^d)$, and set for example $Y = Y_0$ with $\phi(Y_o) \neq 0$ (such a Y_0 exists as ϕ can not be identically null), then (2.5) implies

$$\psi(X) = C\phi(X)$$

for a suitable complex constant $C \neq 0$ and for all $X \in \mathbb{R}^d$. Substituting in (2.5) we have

$$C\phi(X)\overline{\phi(Y)} = \phi(X)\overline{C\phi(Y)} \tag{2.6}$$

which implies that C is real, so that is must be $\psi = C\phi$. □

An L^p functional frame for the two-window spectrograms was studied in [3] (see Theorem 4.1). For completeness we recall here the result.

Proposition 10. *Let us fix $p_j, p'_j, q_j \in [1,\infty]$, $j = 1,2$, with $\frac{1}{p_j} + \frac{1}{p'_j} = 1$ and $q_j \geq \max\{p_j, p'_j\}$. If $f \in L^{p_1}$, $\phi \in L^{p'_1}$, $g \in L^{p_2}$, $\psi \in L^{p'_2}$ and $p = \frac{q_1 q_2}{q_1+q_2}$, then the following estimate for the two-window spectrogram holds:*

$$\int\int_{\mathbb{R}^{2d}} |V_\psi g\overline{V_\phi f}(x,\omega)|^p\, dxdw \leq (\prod_{j=1}^{2} Q_j P_j)^{dp}(\|f\|_{p_1}\|g\|_{p_2}\|\phi\|_{p'_1}\|\psi\|_{p'_2})^p$$

where

$$Q_j = q_j^{-\frac{1}{q_j}}(q_j-2)^{\frac{2-q_j}{2q_j}}$$
$$P_j = (p_j-1)^{\frac{1-p_j}{2p_j}} p_j^{\frac{1}{q_j}} (q_j(p_j-1)-p_j)^{\frac{q_j(p_j-1)-p_j}{2p_jq_j}} (q_j-p_j)^{\frac{q_j-p_j}{2p_jq_j}}. \tag{2.7}$$

A natural question concerning two-window spectrograms is to define in some sense "how large" this subclass of the Cohen class is. In view of (2.1) and (2.2) this is equivalent to a characterization of the range in $\mathcal{S}'(\mathbb{R}^{2d})$ of the Wigner representations, which is a difficult question that we do not address here in its generality. We remark however that some interesting informations can easily be obtained from well-known facts about the range of the Wigner representation. Namely, if $Q = F * \mathrm{Wig}$ is a representation in the Cohen class with kernel $F \in L^2(\mathbb{R}^{2d})$, then Q is not a (possibly two-window) spectrogram in the following cases:

– F is not continuous
– F does not vanish at infinity
– F is positive but is not a translation and/or dilation of a gaussian.

These assertions are actually immediate consequences of the fact that, for $f, g \in L^2(\mathbb{R}^d)$, $\mathrm{Wig}(f,g)$ is a continuous function vanishing at infinity and, from Hudson Theorem (see [8], [15]), it is positive only on generalized gaussians and in this case is itself a translated/dilated gaussian.

Instead of trying to detect explicitly which representations in the Cohen class are expressible as spectrograms, we consider next the problem of the approximation of Cohen representations through spectrograms. We show that finite sums of two-window spectrograms with $L^2(\mathbb{R}^d)$ windows are dense in the space of Cohen class representations $Q = F*\mathrm{Wig}$ with kernel $F \in L^2(\mathbb{R}^{2d})$. More precisely we formulate the result as follows.

Proposition 11. *Let $Q = F*\mathrm{Wig}$ be a representation in the Cohen class with kernel $F \in L^2(\mathbb{R}^{2d})$ and suppose that $1/p - 1/q = 1/2$ with $p, q \in [1, +\infty]$. Then for every $\epsilon > 0$ there exist a finite number of functions $h_j, k_j \in L^2(\mathbb{R}^d)$, $j = 1, \ldots, N$ such that*

$$\|Q(f,g) - \sum_{j=1}^{N} Sp_{h_j,k_j}(f,g)\|_{L^q(\mathbb{R}^{2d})} < \epsilon \tag{2.8}$$

for all $f, g \in S'(\mathbb{R}^d)$ such that $\|\mathrm{Wig}(f,g)\|_{L^p} \leq 1$.

Proof. First of all let us observe that, using Young's inequality, the condition on p and q implies

$$\|Q(f,g)\|_{L^q(\mathbb{R}^{2d})} \leq \|F\|_{L^2(\mathbb{R}^{2d})} \|\mathrm{Wig}(f,g)\|_{L^p(\mathbb{R}^{2d})}$$

and also from Young's inequality, for every couple of functions $h, k \in L^2(\mathbb{R}^d)$, we have

$$\begin{aligned}\|Sp_{h,k}(f,g)\|_{L^q(\mathbb{R}^{2d})} &= \|\mathrm{Wig}(f,g) * \mathrm{Wig}(\tilde{k},\tilde{h})\|_{L^q(\mathbb{R}^{2d})} \\ &\leq \|\mathrm{Wig}(f,g)\|_{L^p(\mathbb{R}^{2d})} \|\mathrm{Wig}(\tilde{k},\tilde{h})\|_{L^2(\mathbb{R}^{2d})},\end{aligned} \tag{2.9}$$

which means that, under our hypothesis, the left-hand side of (2.8) is well defined.

Let h, k be in $L^2(\mathbb{R}^d)$ and observe that the Wigner representation can be decomposed in the following way:

$$\mathrm{Wig}(h,k)(x,\omega) = \mathcal{F}_2(T(h\otimes\overline{k}))(x,\omega) \tag{2.10}$$

where

$$h\otimes k(x,t) = h(x)k(t),$$

$$T : F(x,t) \in L^2(\mathbb{R}^{2d}) \longrightarrow TF(x,t) = F(x+t/2, x-t/2) \in L^2(\mathbb{R}^{2d}),$$

$\mathcal{F}_2$ = partial Fourier transform with respect to t of functions $F(x,t) \in L^2(\mathbb{R}^{2d})$.

Suppose now that $F \in L^2(\mathbb{R}^{2d})$. As T and $\mathcal{F}_2$ are isometries of $L^2(\mathbb{R}^{2d})$, then $T^{-1}\mathcal{F}_2^{-1}F \in L^2(\mathbb{R}^{2d})$.

By the density in $L^2(\mathbb{R}^{2d})$ of finite sums of tensor products $(h_j\otimes k_j)(x,t) = h_j(x)k_j(t)$ with $h_j, k_j \in L^2(\mathbb{R}^d)$, we have that for every $\epsilon > 0$ there exists a finite sum

$$\sum_{j=1}^N h_j\otimes k_j, \qquad h_j, k_j \in L^2(\mathbb{R}^d),$$

such that

$$\|T^{-1}\mathcal{F}_2^{-1}F - \sum_{j=1}^N h_j\otimes k_j\|_{L^2(\mathbb{R}^{2d})} \le \epsilon.$$

Using the fact that T and $\mathcal{F}_2$ are isometries of $L^2(\mathbb{R}^{2d})$ it follows that

$$\begin{aligned}\|F - \sum_{j=1}^N \mathrm{Wig}(h_j,k_j)\|_{L^2(\mathbb{R}^{2d})} &= \|F - \sum_{j=1}^N \mathcal{F}_2 T(h_j\otimes k_j)\|_{L^2(\mathbb{R}^{2d})} \\ = \|F - \mathcal{F}_2 T\Big(\sum_{j=1}^N h_j\otimes k_j\Big)\|_{L^2(\mathbb{R}^{2d})} &= \|T^{-1}\mathcal{F}_2^{-1}F - \sum_{j=1}^N h_j\otimes k_j\|_{L^2(\mathbb{R}^{2d})} \le \epsilon.\end{aligned} \tag{2.11}$$

Thus F can be approximated in $L^2(\mathbb{R}^{2d})$ by finite sums of Wigner functions.

Let us consider now the representation $Q = F * \mathrm{Wig}$ applied to signals f, g such that $\|\mathrm{Wig}(f,g)\|_{L^p(\mathbb{R}^d)} \le 1$ with $1/p - 1/q = 1/2$. From (2.11) and Young's inequality, we have

$$\begin{aligned}&\|Q(f,g) - \sum_{j=1}^N Sp_{\tilde{k}_j,\tilde{h}_j}(f,g)\|_{L^q(\mathbb{R}^{2d})} \\ &= \|F * \mathrm{Wig}(f,g) - \sum_{j=1}^N \mathrm{Wig}(f,g) * \mathrm{Wig}(h_j,k_j)\|_{L^q(\mathbb{R}^{2d})} \\ &= \|\mathrm{Wig}(f,g) * \Big(F - \sum_{j=1}^N \mathrm{Wig}(h_j,k_j)\Big)\|_{L^q(\mathbb{R}^{2d})}\end{aligned} \tag{2.12}$$

$$= \|\mathrm{Wig}(f,g)\|_{L^p(\mathbb{R}^{2d})} \ \ \|F - \sum_{j=1}^{N} \mathrm{Wig}(h_j,k_j)\|_{L^2(\mathbb{R}^{2d})}$$
$$\leq \epsilon \ \|\mathrm{Wig}(f,g)\|_{L^p(\mathbb{R}^{2d})} \leq \epsilon$$

which proves the thesis. □

The previous result shows that $Q(f,g)$ can be arbitrarily well approximated in the $L^q(\mathbb{R}^{2d})$ norm by finite sums of two-window spectrograms with $L^2(\mathbb{R}^d)$ windows, uniformly with respect to the signals f, g such that $\|\mathrm{Wig}(f,g)\|_{L^p(\mathbb{R}^{2d})}$ is bounded, where $1/p - 1/q = 1/2$. The most significant case of the previous property is clearly the case $p = 2$, $q = \infty$, for which we reformulate the result in a specific corollary.

Corollary 12. *Let $Q = F * \mathrm{Wig}$ be a representation in the Cohen class with kernel $F \in L^2(\mathbb{R}^{2d})$. Then $Q(f,g)$ can be uniformly approximated on $\mathbb{R}^{2d}$ by a finite sum of two-window spectrograms $\sum_{j=1}^{N} Sp_{h_j,k_j}(f,g)$ with arbitrary small error, where $h_j, k_j \in L^2(\mathbb{R}^{2d})$, $j = 1,\ldots,N$, are independent of f, g such that $\|f\|_{L^2(\mathbb{R}^d)}, \|g\|_{L^2(\mathbb{R}^d)} \leq 1$.*

Proof. We recall that from the well-known Moyal formula the Wigner representation defines a bounded map

$$\mathrm{Wig} : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^{2d}). \tag{2.13}$$

We have therefore that $\|f\|_{L^2(\mathbb{R}^d)}, \|g\|_{L^2(\mathbb{R}^d)} \leq 1$ implies that the hypothesis of Proposition 11 are satisfied with $p = 2$, $q = +\infty$. The thesis follows then as particular case of Proposition 11. □

Remark 13. From the proof of Proposition 11 we have incidentally showed that the Wigner representation as map between L^2 spaces of type (2.13) has range with dense span. This fact, relying on the density of the span of tensor products, as well as Proposition 11, turn out to be a specific property of two-window spectrograms, which does not hold for the usual (one-window) spectrograms.

3. Integrated Spectrogram: Motivations

The basic idea underlying our definition of the integrated spectrogram introduced in (1.7) is to construct a family of representations which should be intermediate between the two-window spectrogram and the Rihaczek representation, improving in some sense both of them. As briefly mentioned in the introduction, spectrograms do not satisfy the support property (see Definition 3).

Actually for the two-window spectrogram, though arbitrary good localization both with respect to time and frequency can be obtain, a "spreading effect" can not be avoided, as expressed by the following property (see [2]):

Proposition 14. *Let Π_x and Π_ω be projections as in Definition 3, then*

i) $\Pi_x\left(\mathrm{supp}\ Sp_{\phi,\psi}(f)\right) \subset \overline{(\mathrm{supp}\ f + \mathrm{supp}\ \phi)} \cap \overline{(\mathrm{supp}\ f + \mathrm{supp}\ \psi)}$;

ii) $\Pi_\omega\left(\text{supp } Sp_{\phi,\psi}(f)\right) \subset \overline{(\text{supp } \hat{f} + \text{supp } \hat{\phi})} \cap \overline{(\text{supp } \hat{f} + \text{supp } \hat{\psi})}$.

Also for the integrated spectrogram the support property does not hold, however its behavior with respect to this feature turns up to be quite better than that of the spectrogram. In Section 5 we shall make these assertion precise by proving a specific property on the support of the integrated spectrogram.

On the other hand the Rihaczek representation clearly satisfies the support property (Definition 3) but shows remarkable interference patterns even for simple signals. Although the interference behavior of a representation is not easily quantified in terms of mathematical theorems, it can however be well recognized by testing the representation on standard signals. Consider a fixed signal containing the frequency $\omega = 2$ in the time interval $[-6,-2]$ and the frequency $\omega = 3$ in the time interval $[2,6]$, more precisely let $f(x) = e^{4\pi ix}\chi_{[-6,-2]}(x) + e^{6\pi ix}\chi_{[2,6]}(x)$, where $\chi_E(x)$ is the characteristic function of the set E. Figure 2 shows the spectrogram $Sp_\phi(f)$, with gaussian window $\phi(x) = e^{-\pi x^2}$, whereas Figure 1 shows the Rihaczek representations $R(f)$. As we can see, the figures confirm the different behavior that we have described above.

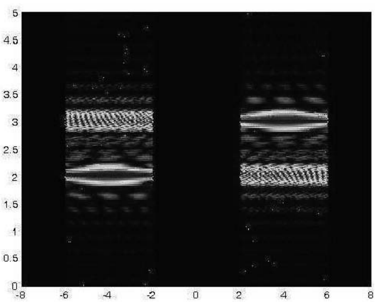

FIGURE 1.
Rihaczek representation $R(f)$

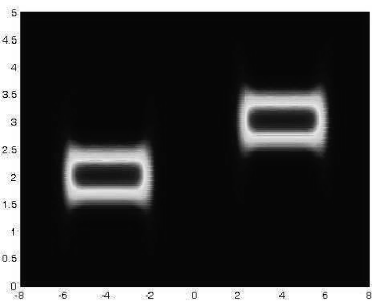

FIGURE 2.
Two-window spectrogram $Sp_{\phi,\psi}(f)$

A slightly better result can be obtained by taking the average $1/2(Sp_\phi(f) + R(f))$ of the two representations, which is a practice of common use in the applications. However, in this case, as in many others, the improvement is not really appreciable and the interferences are only slightly reduced, as Figure 3 shows.

Figure 4 shows finally the application to the same signal of the integrated spectrogram $S^\theta_{\phi,\phi}(f)$, with $\theta(\lambda) = \frac{\chi_{[1,80]}(\lambda)}{80}\sqrt{\frac{1+\lambda^2}{\lambda^2}}$ and ϕ as in (1.6) (the choice of $\theta(\lambda)$ and ϕ will be justified in Section 5). As we can see, the interference artifacts practically disappear, and at the same time one obtains a level of the time-frequency localization which considerably improves that of the spectrogram and almost equals that of the Rihaczek representation.

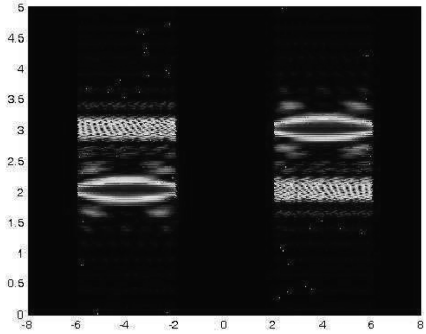

FIGURE 3. Average $\frac{1}{2}(Sp_{\phi,\psi}(f) + R(f))$

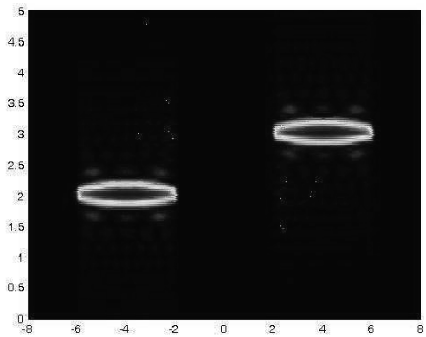

FIGURE 4. Integrated spectrogram $S^{\theta}_{\phi,\psi}$

The pictures of this section are obtained with Matlab R2006a. In Figure 4 the integral over λ contained in the integrated spectrogram has been approximated with 80 steps.

4. Boundedness of the Integrated Spectrogram

In the next part of the paper we analyze some properties of the integrated spectrogram. In this section we consider its boundedness in the frame of L^p spaces and we prove a corresponding uncertainty principle. The following lemma is a simple computation that will be useful later on.

Lemma 15. *Let us suppose that* $\phi \in L^{r_1}(\mathbb{R}^d)$, $\psi \in L^{r_2}(\mathbb{R}^d)$, $\lambda > 1$. *We define* $\phi_\lambda(t) = \lambda^{d/2}\phi(\sqrt{\lambda}t)$ *and* $\psi_\lambda(t) = \psi\big(\frac{1}{\sqrt{\lambda}}t\big)$. *Then we have*

$$\|\phi_\lambda\|_{L^{r_1}(\mathbb{R}^d)} = \lambda^{d/2}\lambda^{-\frac{d}{2r_1}}\|\phi\|_{L^{r_1}(\mathbb{R}^d)}, \qquad \|\psi_\lambda\|_{L^{r_2}(\mathbb{R}^d)} = \lambda^{\frac{d}{2r_2}}\|\psi\|_{L^{r_2}(\mathbb{R}^d)}.$$

We then have the following continuity result of the integrated spectrogram.

Theorem 16. *Let us consider* ϕ_λ *and* ψ_λ *as in Lemma* 15. *We fix* p_1, p_2 *with* $1 \le p_1, p_2 \le +\infty$, *and* p *of the form* $p = \frac{q_1 q_2}{q_1+q_2}$, *where* q_1, q_2 *satisfy* $q_j \ge \max\{p_j, p_j'\}$ *for* $j = 1, 2$; p_j' *is as usual the conjugate of* p_j, *i.e.,* $\frac{1}{p_j} + \frac{1}{p_j'} = 1$. *We suppose moreover that the function* θ *satisfies*

$$\int_1^{+\infty} \left|\lambda^{d/2}\theta(\lambda)\right| \lambda^{\frac{d}{2p_2'} - \frac{d}{2p_1'}}\, d\lambda < +\infty. \tag{4.1}$$

Then for every $\phi \in L^{p_1'}(\mathbb{R}^d)$ *and* $\psi \in L^{p_2'}(\mathbb{R}^d)$, $S_{\phi,\psi}$ *is a bounded operator on* $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ *and for all* $(f, g) \in L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ *we have*

$$\|S^\theta_{\phi,\psi}(f, g)\|_{L^p(\mathbb{R}^{2d})} \le C\|f\|_{L^{p_1}(\mathbb{R}^d)}\|g\|_{L^{p_2}(\mathbb{R}^d)}\|\phi\|_{L^{p_1'}(\mathbb{R}^d)}\|\psi\|_{L^{p_2'}(\mathbb{R}^d)}, \tag{4.2}$$

with

$$C = (Q_1P_1Q_2P_2)^d \int_1^{+\infty} \left|\lambda^{d/2}\theta(\lambda)\right| \lambda^{\frac{d}{2p_2'} - \frac{d}{2p_1'}}\, d\lambda; \tag{4.3}$$

the constants Q_j *and* P_j, *for* $j = 1, 2$, *depend only on* p_j *and* q_j *and are of the form*

$$Q_j = q_j^{-\frac{j}{q_j}} (q_j - 2)^{\frac{2-q_j}{2q_j}},$$
$$P_j = (p_j - 1)^{\frac{1-p_j}{2p_j}} p_j^{\frac{1}{q_j}} \left(q_j(p_j - 1) - p_j\right)^{\frac{q_j(p_j-1)-p_j}{2p_jq_j}} (q_j - p_j)^{\frac{q_j-p_j}{2p_jq_j}}.$$

Proof. By definition of integrated spectrogram we obtain

$$\left\|S^\theta_{\phi,\psi}(f, g)\right\|_{L^p} \le \int_1^{+\infty} |\theta(\lambda)|\, \|Sp_{\phi_\lambda,\psi_\lambda}(f, g)\|_{L^p}\, d\lambda,$$

and since $\phi_\lambda \in L^{p_1'}$, $\psi_\lambda \in L^{p_2'}$ for every $\lambda \ge 1$, we can apply the boundedness result of Proposition 10 for the two-window spectrogram, obtaining

$$\left\|S^\theta_{\phi,\psi}(f, g)\right\|_{L^p} \le (Q_1P_1Q_2P_2)^d \int_1^{+\infty} |\theta(\lambda)|\ \|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|\phi_\lambda\|_{L^{p_1'}}\|\psi_\lambda\|_{L^{p_2'}}\, d\lambda.$$

We then get the conclusion by applying Lemma 15 with $r_1 = p_1'$ and $r_2 = p_2'$. □

From the previous theorem we can obtain a corresponding uncertainty principle for the integrated spectrogram.

Theorem 17. *Let us suppose that* $f \in L^{p_1}(\mathbb{R}^d)$, $g \in L^{p_2}(\mathbb{R}^d)$, $\phi \in L^{p_1'}(\mathbb{R}^d)$, $\psi \in L^{p_2'}(\mathbb{R}^d)$ *and that* (4.1) *is satisfied. Let* $U \subset \mathbb{R}^{2d}$ *and* $\epsilon \ge 0$ *satisfy*

$$\int_U |S^\theta_{\phi,\psi}(f, g)|\, dx\, d\omega > (1 - \epsilon)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|\phi\|_{L^{p_1'}}\|\psi\|_{L^{p_2'}}. \tag{4.4}$$

Then

$$\mu(U) > \left(\frac{1-\epsilon}{Q_1P_1Q_2P_2}\right)^{\frac{p}{p-1}} \left(\int_1^{+\infty} \left|\lambda^{d/2}\theta(\lambda)\right| \lambda^{\frac{d}{2p_2'} - \frac{d}{2p_1'}}\, d\lambda\right)^{\frac{1-p}{p}}$$

for every p *satisfying the hypotheses of Theorem* 16, *where* $\mu(U)$ *is the Lebesgue measure of* U *and* Q_j, P_j *are the constants defined in Theorem* 16.

Proof. We can limit ourselves to the case when U has finite measure, otherwise the conclusion is trivial. In this case, by the Hölder inequality we have immediately

$$\int_U |S^\theta_{\phi,\psi}(f,g)|\,dx\,d\omega \le \|S^\theta_{\phi,\psi}(f,g)\|_{L^p(U)}\|1\|_{L^{p'}(U)} \le \|S^\theta_{\phi,\psi}(f,g)\|_{L^p(\mathbb{R}^{2d})}\big(\mu(U)\big)^{\frac{1}{p'}}$$

for every $p \ge 1$. We can then choose p satisfying the hypotheses of Theorem 16, and from (4.4), (4.2) we get

$$\begin{aligned}(1-\epsilon)\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|\phi\|_{L^{p_1'}}\|\psi\|_{L^{p_2'}} &< \int_U |S^\theta_{\phi,\psi}(f,g)|\,dx\,d\omega\\ &\le C\big(\mu(U)\big)^{\frac{1}{p'}}\|f\|_{L^{p_1}}\|g\|_{L^{p_2}}\|\phi\|_{L^{p_1'}}\|\psi\|_{L^{p_2'}},\end{aligned}$$

where C is given by (4.3). Since $p' = \frac{p}{p-1}$ we have $\mu(U) > \big(\frac{1-\epsilon}{C}\big)^{\frac{p}{p-1}}$ and so by (4.3) we obtain the desired estimate. □

5. Basic Properties of the Integrated Spectrogram

In this section we analyze some properties of the integrated spectrogram (1.7), starting from a result on its support. As already mentioned in Section 3 the two-window spectrogram $Sp_{\phi,\psi}(f)$ does not satisfy the support property, but we anyway have a control on the orthogonal projections of supp $Sp_{\phi,\psi}(f)$ on the x and ω-axes. We can prove for the integrated spectrogram an analogous result. In the next proposition the functions ϕ_λ, ψ_λ are not necessarily of the form $\phi_\lambda(x) = \lambda^{d/2}\phi(\sqrt{\lambda}x)$, $\psi_\lambda = \psi\big(\frac{1}{\sqrt{\lambda}}x\big)$ for fixed ϕ,ψ, as before, but can be two arbitrary families of windows depending on λ; with abuse of notation we shall call again $S^\theta_{\phi,\psi}$ the corresponding integrated spectrogram.

Proposition 18. *Fix $\phi_\lambda,\psi_\lambda, f \in \mathcal{S}(\mathbb{R}^d)$, and θ in such a way that the integrated spectrogram $S^\theta_{\phi,\psi}(f) = S^\theta_{\phi,\psi}(f,f)$ is well defined.*

(i) *Let $B_f \subset \mathbb{R}^d$ be a closed set satisfying $\big(\text{supp } f + \text{supp } \phi_\lambda\big) \cap \big(\text{supp } f + \text{supp } \psi_\lambda\big) \subset B_f$ for almost every $\lambda \in \text{supp } \theta$. Then we have*

$$\Pi_x\big(\text{supp } S^\theta_{\phi,\psi}(f)\big) \subset B_f.$$

(ii) *Let $C_f \subset \mathbb{R}^d$ be a closed set satisfying $\big(\text{supp } \hat{f} + \text{supp } \hat{\phi}_\lambda\big) \cap \big(\text{supp } \hat{f} + \text{supp } \hat{\psi}_\lambda\big) \subset C_f$ for almost every $\lambda \in \text{supp } \theta$. Then we have*

$$\Pi_\omega\big(\text{supp } S^\theta_{\phi,\psi}(f)\big) \subset C_f.$$

Proof. Regarding (i) we observe that, from Proposition 14, we obtain

$$\Pi_x\big(\text{supp } Sp_{\phi_\lambda,\psi_\lambda}(f)\big) \subset \big(\overline{\text{supp } f + \text{supp } \phi_\lambda}\big)\cap\big(\overline{\text{supp } f + \text{supp } \psi_\lambda}\big). \tag{5.1}$$

Let us fix now $\tilde{x} \notin B_f$; then there exists a neighborhood $U_{\tilde{x}}$ of $\tilde{x}$ such that $U_{\tilde{x}}$ has empty intersection with B_f, and so by hypothesis $U_{\tilde{x}}$ has empty intersection with $\big(\text{supp } f + \text{supp } \phi_\lambda\big) \cap \big(\text{supp } f + \text{supp } \psi_\lambda\big)$ for almost every $\lambda \in \text{supp } \theta$. By (5.1) we obtain that $(U_{\tilde{x}} \times \mathbb{R}^d_\omega)\cap\text{supp } Sp_{\phi_\lambda,\psi_\lambda}(f)$ is empty for almost every $\lambda \in \text{supp } \theta$, which means that $Sp_{\phi_\lambda,\psi_\lambda}(f)(x,\omega) \equiv 0$ for $(x,\omega) \in U_{\tilde{x}} \times \mathbb{R}^d_\omega$ and for almost every $\lambda \in \operatorname{supp} \theta$. This implies that in the set $U_{\tilde{x}} \times \mathbb{R}^d_\omega$

$$S^\theta_{\phi,\psi}(f)(x,\omega) \equiv 0,$$

and then in particular $\tilde{x} \notin \Pi_x\big(\operatorname{supp} S^\theta_{\phi,\psi}(f)\big)$, which proves (i). Concerning (ii) it follows from the inclusion

$$\Pi_\omega\big(\operatorname{supp} Sp_{\phi_\lambda,\psi_\lambda}(f)\big) \subset \big(\overline{\operatorname{supp} \hat{f} + \operatorname{supp} \hat{\psi}_\lambda}\big) \cap \big(\overline{\operatorname{supp} \hat{f} + \operatorname{supp} \hat{\phi}_\lambda}\big)$$

and the same arguments as above. □

We want now to study other properties of the integrated spectrogram, similarly to the case of the two-window spectrogram analyzed in Section 2. To this aim, from now on we shall fix a particular form of the windows ϕ_λ and ψ_λ: indeed, as pointed out in the introduction, the idea is to parameterize the windows in such a way that one window tends to the Dirac δ distribution and the other one tends to the function identically 1 when $\lambda \to +\infty$, in such a way that we shall obtain a 'path' described by $Sp_{\phi_\lambda,\psi_\lambda}(f,g)$ from spectrogram to Rihaczek. From now on we fix $\phi(t) = \psi(t) = e^{-\pi t^2}$, and

$$\phi_\lambda(t) = \lambda^{d/2} e^{-\pi\lambda t^2}, \quad \psi_\lambda(t) = \hat{\phi}_\lambda(t) = e^{-\pi\frac{1}{\lambda}t^2}. \tag{5.2}$$

At first we want to show that the integrated spectrogram $S^\theta_{\phi,\phi}$ belongs to the Cohen class, and we give an explicit expression of its Cohen kernel. To this aim, we shall fix the functions f and g in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ (we can anyway extend the result to more general signals by standard density arguments).

Proposition 19. *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$, $\phi_\lambda, \psi_\lambda$ as in (5.2) and $\theta \in L^1([1,+\infty])$. Then*

$$S^\theta_{\phi,\phi}(f,g) = \sigma * \operatorname{Wig}(f,g),$$

where the inverse Fourier transform of the Cohen kernel σ is given by

$$\mathcal{F}^{-1}(\sigma)(u,t) = \int_1^{+\infty} \theta(\lambda)\left(\frac{\lambda^2}{1+\lambda^2}\right)^{d/2} e^{-\pi\frac{\lambda}{1+\lambda^2}(u^2+t^2)} e^{-\pi i\frac{1-\lambda^2}{1+\lambda^2}ut}\, d\lambda. \tag{5.3}$$

Proof. From (2.2) we have that

$$\begin{aligned} S^\theta_{\phi,\phi}(f,g) &= \int_1^{+\infty} \theta(\lambda) Sp_{\phi_\lambda,\psi_\lambda}(f,g)\, d\lambda \\ &= \int_1^{+\infty} \theta(\lambda) \operatorname{Wig}(\tilde{\psi}_\lambda, \tilde{\phi}_\lambda) * \operatorname{Wig}(f,g)\, d\lambda \\ &= \left(\int_1^{+\infty} \theta(\lambda) \operatorname{Wig}(\tilde{\psi}_\lambda, \tilde{\phi}_\lambda)\, d\lambda\right) * \operatorname{Wig}(f,g), \end{aligned}$$

where the convolution is in the (x,ω)-variables. Then the inverse Fourier transform of the Cohen kernel of the integrated spectrogram is given by

$$\mathcal{F}^{-1}(\sigma) = \mathcal{F}^{-1}\left(\int_1^{+\infty} \theta(\lambda) \operatorname{Wig}(\tilde{\psi}_\lambda, \tilde{\phi}_\lambda)\, d\lambda\right).$$

Since $\mathcal{F}^{-1}$ acts in the (x,ω)-variables, by interchanging the order of integration and using Proposition 1 we obtain

$$\mathcal{F}^{-1}(\sigma)(u,t) = \int_1^{+\infty} \theta(\lambda) A(\tilde{\psi}_\lambda, \tilde{\phi}_\lambda)(u,t)\, d\lambda. \tag{5.4}$$

We need then to compute the ambiguity function of the windows (5.2). We have:

$$\begin{aligned} A(\tilde{\psi}_\lambda, \tilde{\phi}_\lambda)(u,t) &= \int e^{2\pi i u x} \tilde{\psi}(x+t/2)\overline{\tilde{\phi}(x-t/2)}\, dx \\ &= \lambda^{d/2} \int e^{2\pi i u x} e^{-\pi \frac{1}{\lambda}(x+\frac{t}{2})^2} e^{-\pi\lambda(x-\frac{t}{2})^2}\, dx \\ &= \lambda^{d/2} e^{-\pi \frac{1+\lambda^2}{4\lambda} t^2} \int e^{2\pi i u x} e^{-\pi(\frac{1+\lambda^2}{\lambda}x^2 + \frac{1-\lambda^2}{\lambda} x t)}\, dx. \end{aligned}$$

By the change of variables $x = \sqrt{\frac{\lambda}{1+\lambda^2}}\, y - \frac{1-\lambda^2}{2(1+\lambda^2)} t$ we get

$$\begin{aligned} A(\tilde{\psi}_\lambda, \tilde{\phi}_\lambda)(u,t) &= \lambda^{d/2} \Big(\frac{\lambda}{1+\lambda^2}\Big)^{d/2} e^{-\pi\frac{1+\lambda^2}{4\lambda}t^2} e^{\pi \frac{(1-\lambda^2)^2}{4\lambda(1+\lambda^2)} t^2} e^{-2\pi i u \frac{1-\lambda^2}{2(1+\lambda^2)} t} \\ &\quad \cdot \int e^{2\pi i u \sqrt{\frac{\lambda}{1+\lambda^2}} y} e^{-\pi y^2}\, dy \\ &= \Big(\frac{\lambda^2}{1+\lambda^2}\Big)^{d/2} e^{-\pi \frac{\lambda}{1+\lambda^2} t^2} e^{-\pi\frac{\lambda}{1+\lambda^2} u^2} e^{-\pi i \frac{1-\lambda^2}{1+\lambda^2} u t}. \end{aligned}$$

Now by this last equality and (5.4) we have (5.3). We observe that $\mathcal{F}^{-1}(\sigma) \in L^\infty(\mathbb{R}^{2d})$, since

$$\big|\mathcal{F}^{-1}(\sigma)(u,t)\big| \le \int_1^{+\infty} |\theta(\lambda)| \Big|A(\tilde{\psi}_\lambda, \tilde{\phi}_\lambda)(u,t)\Big|\, d\lambda \le \int_1^{+\infty} |\theta(\lambda)|\, d\lambda = \|\theta\|_{L^1} < \infty.$$

Then $\mathcal{F}^{-1}(\sigma) \in \mathcal{S}'(\mathbb{R}^{2d})$, and so $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$, which ensures us that $S^\theta_{\phi,\phi}(f,g)$ is a well defined element of the Cohen class. □

We want to use now the explicit expression of the inverse Fourier transform of the kernel of $S^\theta_{\phi,\phi}$ in order to understand if marginal conditions, conservation of energy, and reality of the representation are satisfied.

Proposition 20. *Let us consider the integrated spectrogram with windows ϕ_λ and ψ_λ as in* (5.2).

(i) $S^\theta_{\phi,\phi}(f,g)$ *does not satisfy the marginals, for any choice of the function θ.*

(ii) $S^\theta_{\phi,\phi}(f,g)$ *satisfies the conservation of energy for every function θ such that*

$$\int_1^{+\infty} \theta(\lambda) \Big(\frac{\lambda^2}{1+\lambda^2}\Big)^{d/2} d\lambda = 1.$$

(iii) $S^\theta_{\phi,\phi}(f,g)$ *is not real.*

Proof. (i) We know from Proposition 5 that the time marginal condition is satisfied if and only if $\mathcal{F}^{-1}(\sigma)(u,0) = 1$; from Proposition 19 we get

$$\mathcal{F}^{-1}(\sigma)(u,0) = \int_1^{+\infty} \theta(\lambda)\left(\frac{\lambda^2}{1+\lambda^2}\right)^{d/2} e^{-\pi\frac{\lambda}{1+\lambda^2}u^2}\, d\lambda,$$

that cannot be independent of the u-variable, for any choice of the function $\theta(\lambda)$ (not identically zero). In the same way we can prove that the frequency marginal condition is not fulfilled, since it is satisfied if and only if $\mathcal{F}^{-1}(\sigma)(0,t) = 1$.

(ii) Concerning the conservation of the energy, we know from Proposition 5 that it is satisfied if and only if $\mathcal{F}^{-1}(\sigma)(0,0) = 1$, and since

$$\mathcal{F}^{-1}(\sigma)(0,0) = \int_1^{+\infty} \theta(\lambda)\left(\frac{\lambda^2}{1+\lambda^2}\right)^{d/2} d\lambda$$

the conclusion holds.

(iii) As in the previous cases, the fact that a representation is real can be deduced directly from the inverse Fourier transform of its kernel. From Proposition 5 we have to check if $\mathcal{F}^{-1}(\sigma)(u,t) = \overline{\mathcal{F}^{-1}(\sigma)(-u,-t)}$, which is in general not true, as we can deduce from (5.3). □

Remark 21. A simple example of a class of functions θ for which the corresponding Integrated spectrogram satisfies the conservation of energy is given by

$$\theta_M(\lambda) = \frac{1}{M-1}\left(\frac{\lambda^2}{1+\lambda^2}\right)^{-d/2} \chi_{[1,M]}(\lambda) \tag{5.5}$$

for every $M > 1$, where $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$.

We consider now the integrated spectrogram with windows as in (5.2) and function θ_M as in (5.5), in such a way that we have representations satisfying the conservation of the energy. We want to show that for M between 1 and $+\infty$ the corresponding $S^{\theta_M}_{\phi,\phi}(f,g)$ describes a path between the classical spectrogram and the Rihaczek, which can therefore be seen as limit cases of integrated spectrograms.

Proposition 22. *Let us fix the windows* ϕ_λ, ψ_λ *as in* (5.2), *and* $\theta_M(\lambda)$ *as in* (5.5). *Then the integrated spectrogram* $S^{\theta_M}_{\phi,\phi}(f,g)$ *satisfies the conservation of the energy and moreover, for every* $f,g \in \mathcal{S}(\mathbb{R}^d)$ *we have:*

(i) $S^{\theta_M}_{\phi,\phi}(f,g)(x,\omega) \to Sp_\phi(f,g)(x,\omega)$ *for every* $(x,\omega) \in \mathbb{R}^{2d}$ *as* $M \to 1$ *(where* $\phi(t) = 2^{d/4}e^{-\pi t^2}$*);*

(ii) $S^{\theta_M}_{\phi,\phi}(f,g)(x,\omega) \to R(f,g)(x,\omega)$ *for every* $(x,\omega) \in \mathbb{R}^{2d}$ *as* $M \to +\infty$.

Proof. We already mentioned that $S^{\theta_M}_{\phi,\phi}(f,g)$ satisfies the conservation of the energy in Remark 21, the proof is a straightforward computation. Let σ_M be the Cohen kernel of $S^{\theta_M}_{\phi,\phi}$. We have then

$$S^{\theta_M}_{\phi,\phi}(f,g)(x,\omega) = \sigma_M * Wig(f,g) = \mathcal{F}\Big(\mathcal{F}^{-1}(\sigma_M)A(f,g)\Big), \tag{5.6}$$

where $A(f,g)$ is the ambiguity function. From (5.3) and (5.5) we have that

$$\mathcal{F}^{-1}(\sigma_M)(u,t) = \frac{1}{M-1}\int_1^M e^{-\pi\frac{\lambda}{1+\lambda^2}(u^2+t^2)} e^{-\pi i\frac{1-\lambda^2}{1+\lambda^2}ut}\,d\lambda,$$

and then

$$\left|\mathcal{F}^{-1}(\sigma_M)A(f,g)\right| \le \frac{1}{M-1}\int_1^M e^{-\pi\frac{\lambda}{1+\lambda^2}(u^2+t^2)}\,d\lambda\,|A(f,g)| \le |A(f,g)|,$$

for every $M > 1$. Taking the limits for $M \to 1$ and for $M \to \infty$ in (5.6), since $A(f,g) \in \mathcal{S}(\mathbb{R}^{2d}) \hookrightarrow L^1(\mathbb{R}^{2d})$, by the Dominated Convergence Theorem we can interchange the limits with the Fourier transform. This means that in order to get (i) and (ii) it is enough to prove the convergence of the inverse Fourier transform of the corresponding kernels. The conclusion then follows immediately from De L'Hospital's rule and the fundamental theorem of calculus. For every $(x,\omega) \in \mathbb{R}^{2d}$ we have indeed

$$\lim_{M\to 1}\mathcal{F}^{-1}(\sigma_M) = \lim_{M\to 1} e^{-\pi\frac{M}{1+M^2}(u^2+t^2)} e^{-\pi i\frac{1-M^2}{1+M^2}ut} = e^{-\frac{\pi}{2}(u^2+t^2)},$$

which coincides with the inverse Fourier transform of the Cohen kernel σ of $Sp_\phi(f,g)$, since from Proposition 1 we have

$$\mathcal{F}^{-1}(\sigma)(u,t) = A(\tilde\phi,\tilde\phi)(u,t) = 2^{d/2}\int e^{2\pi iux} e^{-\pi(x+\frac{t}{2})^2} e^{-\pi(x-\frac{t}{2})^2}\,dx = e^{-\frac{\pi}{2}(u^2+t^2)}.$$

Concerning (ii), we obtain in the same way

$$\lim_{M\to+\infty}\mathcal{F}^{-1}(\sigma_M) = \lim_{M\to+\infty} e^{-\pi\frac{M}{1+M^2}(u^2+t^2)} e^{-\pi i\frac{1-M^2}{1+M^2}ut} = e^{\pi iut},$$

that is the inverse Fourier transform of the Cohen kernel of the Rihaczek representation, see [1]. The proof is then complete. □

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Paolo Boggiatto, Giuseppe De Donno and Alessandro Oliaro
Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10
10123 Torino
Italy
e-mail: paolo.boggiatto@unito.it
giuseppe.dedonno@unito.it
alessandro.oliaro@unito.it

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Time-Time Distributions for Discrete Wavelet Transforms

C. R. Pinnegar, H. Khosravani and P. Federico

Abstract. The short-time Fourier transform has an easily defined time-domain counterpart: a set of windowed time series, each one corresponding to a specific window position. Considered collectively, these constitute a time-time distribution, since the window position gives a second time variable. Multiresolution time-time distributions can also be defined. The only such distribution that has been investigated thus far, the TT-transform, is the time-domain counterpart of a continuous wavelet transform. In this short paper, we describe a new method of calculating time-time distributions for discrete wavelet transforms, and present two examples.

Mathematics Subject Classification (2000). Primary 65T60; Secondary 47G30.

Keywords. Short-time Fourier transform, time-time distribution, TT-transform, wavelet transform.

1. Introduction

Time-time distributions [7, 8] are the time-domain counterparts of time-frequency distributions [1]. Perhaps the simplest example of a time-time distribution is the counterpart of the discrete short-time Fourier transform (STFT). This is obtained by applying an inverse discrete Fourier transform to both sides of the discrete STFT definition [4],

$$V_g f[\tau, k] = \sum_{t=0}^{N-1} f[t]\ g[t-\tau]\ e^{-2\pi ikt/N}, \tag{1}$$

resulting in the equivalent expression

$$f[t]\ g[t-\tau] = \sum_{k=-N/2}^{N/2-1} V_g f[\tau, k]\ e^{2\pi ikt/N}. \tag{2}$$

In (1) and (2), f is a function of discrete time t, $V_g f$ is its discrete STFT, k denotes frequency, N is the number of samples in f, and $g[t-\tau]$ is a positive window that has appreciable amplitude only near $t = \tau$. When considered at all τ, the LHS of (2) gives a suite of windowed time series that can be plotted as a two-dimensional function of τ and t, as in Fig. 1a; thus the term "time-time distribution".

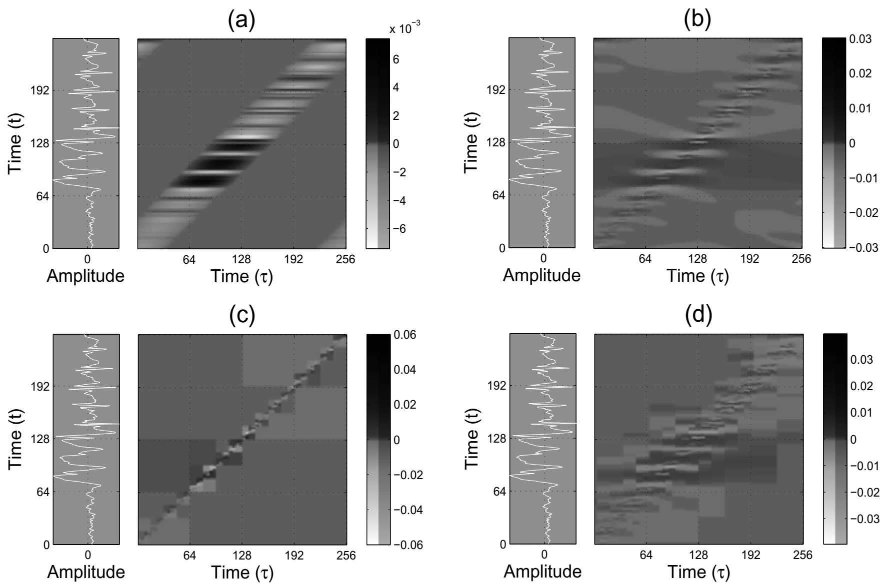

FIGURE 1. A test time series (an epileptiform EEG recording), plotted next to the time-time distributions obtained from it using (a) the STFT, (b) the S-transform, (c) the Haar wavelet transform, and (d) the Daubechies-3 wavelet transform. Since the last three examples are obtained from multiresolution/multiscale transforms, they exhibit stronger concentration of higher frequencies near $t = \tau$. In this and subsequent figures, the original time series is plotted as white on grey.

In (1), the same $g[t-\tau]$ acts on all the frequencies that make up $f[t]$, so both high and low frequencies have identical tapers away from $t = \tau$. Multiresolution transforms, which localize different frequencies around $t = \tau$ in different ways, can also yield time-time distributions. These are more complicated than (2), because they can exhibit differential concentration of different frequencies around $t = \tau$.

An example is the TT-transform [7], whose definition is

$$TTf[\tau,t] = \sum_{k=-N/2}^{N/2-1} Sf[\tau,k]\; e^{2\pi ikt/N}. \tag{3}$$

In (3), Sf is the discrete S-transform [9] of f, a multiresolution transform that is the discrete Fourier transform pair of TTf,

$$Sf[\tau,k] = \sum_{t=0}^{N-1} f[t] \left\{ \frac{|k|}{\sqrt{2\pi}N} e^{\frac{-k^2[t-\tau]^2}{2N^2}} \right\} e^{-2\pi ikt/N}. \tag{4}$$

Comparing (1) with (4), we can see that the S-transform is similar to an STFT, but with a scalable window that becomes narrower at higher frequencies, reminiscent of wavelets. In fact the S-transform is closely associated with the Morlet continuous wavelet transform [3]. The frequency scaling of the S-transform window causes TTf to concentrate higher frequencies more strongly around $t = \tau$ than lower frequencies. An example of this is shown in Fig. 1b. Note that $f[t]\; g[t-\tau]$ and TTf play similar roles in (2) and (3). In the same way that columns of $f[t]\; g[t-\tau]$ through specific values of τ give windowed time series that describe how the discrete STFT "perceives" f, the columns of TTf give local time series (the discrete Fourier transform pairs of local spectra) that give some idea of how f is perceived by the discrete S-transform at any particular τ. Several additional properties of the TT-transform, including the inverse formula

$$f[t] = \sum_{\tau=0}^{N-1} TTf[\tau,t], \tag{5}$$

are described in more detail in [6, 7, 8].

To date, time-time counterparts of discrete wavelet transforms (DWTs) have not been defined. In this short note, we describe a simple way of doing this, and present two examples. These new distributions provide insight into how time series are perceived by DWTs.

2. Time-Time Distribution for the Discrete Wavelet Transform

Suppose that $\Psi_j[t]$ is a discrete wavelet frame (see [5] for a detailed description), where j is a combined time-scale index. Then the wavelet coefficients of $f[t]$, denoted Cf_j, are obtained from

$$Cf_j = \sum_{t=0}^{N-1} f[t]\; \Psi_j^*[t], \tag{6}$$

where the asterisk denotes complex conjugation. The inverse operation of (6) is

$$f[t] = \sum_{j=0}^{J-1} Cf_j\; \Phi_j[t], \tag{7}$$

where the $\Phi_j[t]$ are the J members of the dual frame; these are defined so that

$$\sum_{t=0}^{N-1} \Phi_j[t]\ \Psi_m^*[t] = \delta_{j,m}, \tag{8}$$

where $\delta_{j,m}$ denotes the Kronecker delta.

We now define a set of functions $X_j[t]$, each of which will eventually be associated with the $\Phi_j[t]$ that has the same j value. The $X_j[t]$ are required to satisfy

$$\sum_{t=0}^{N-1} X_j[t] = 1. \tag{9}$$

Renaming the summation variable of (9) to τ, and including (9) in (7), gives

$$f[t] = \sum_{j=0}^{J-1} Cf_j\ \Phi_j[t] \sum_{\tau=0}^{N-1} X_j[\tau]. \tag{10}$$

Rearranging the order of summation, we obtain an expression that is very similar to (5),

$$f[t] = \sum_{\tau=0}^{N-1} TT_D f[\tau, t], \tag{11}$$

where $TT_D f$ is our new time-time distribution,

$$TT_D f[\tau, t] = \sum_{j=0}^{J-1} Cf_j\ \Phi_j[t]\ X_j[\tau]. \tag{12}$$

It then remains to define an appropriate X_j. In doing so we need to bear in mind that each X_j determines which local time series of $TT_D f[\tau, t]$ contain contributions from $\Phi_j[t]$. One simple approach is to reason that, for any given point τ_0 on the τ-axis, the local time series $TT_D f[\tau_0, t]$ should be a superposition only of the $\Phi_j[t]$ whose corresponding $\Psi_j[t]$ contain τ_0 within their support. Since, in (6), the wavelet coefficients are obtained from the inner product of $h[t]$ and $\Psi_j[t]$ without any other weighting function being involved, we define $X_j[t]$ to be an equal partition of unity over the support of $\Psi_j[t]$, via

$$\begin{aligned} X_j[t] &= 1/N_j,\ t \in \sup\{\Psi_j[t]\}, \\ &= 0,\ t \notin \sup\{\Psi_j[t]\}, \end{aligned} \tag{13}$$

where N_j is the number of samples in $\sup\{\Psi_j[t]\}$.

Two examples of $TT_D f$ are shown in Figs. 1c–d, one for the Haar wavelet transform ($N = 256$, level 7), and the other for the Daubechies-3 wavelet transform ($N = 256$, level 5). The Haar time-time distribution shows most clearly how the X_j function distributes wavelet amplitude across the time-time plane; the Daubechies-3 time-time distribution has an appearance intermediate between the Haar example and the TT-transform in Fig. 1b.

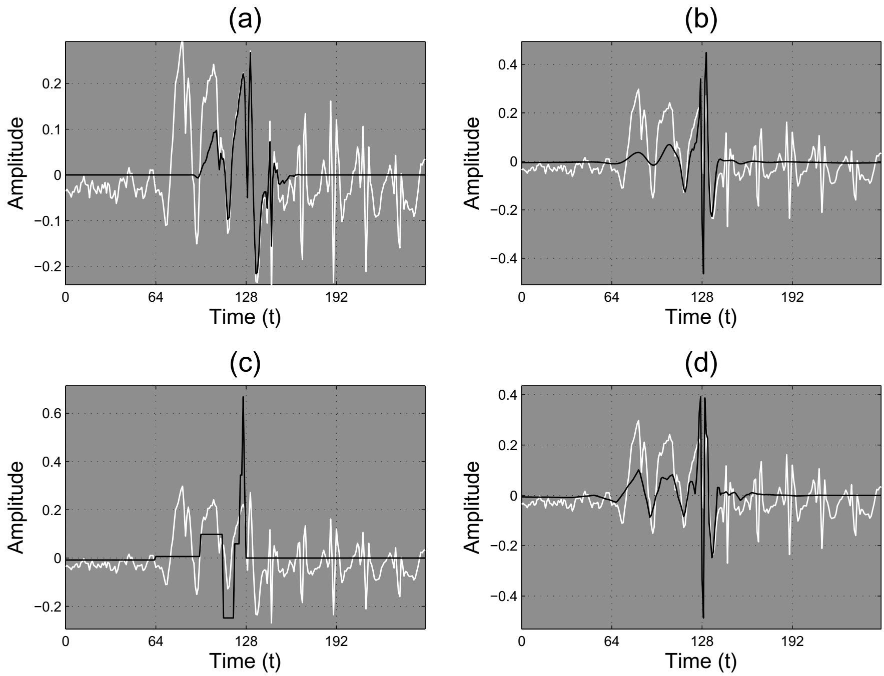

FIGURE 2. (Black lines) Local time series, each obtained from the corresponding distribution in Fig. 1 at $\tau = 127$. (White lines) The original time series from Fig. 1, for comparison. In subplots (b–d) the increased concentration of higher frequencies around $t = \tau$ is evident. In each subplot, the amplitudes of the local time series have been multiplied by 40, to facilitate comparison with the original time series. In this and subsequent figures, all time series derived from the TT-transform are plotted as black on grey, while the original time series (from which the TT-transform was obtained) is plotted as white on grey.

In Figs. 2a–d, four local time series, obtained from Figs. 1a–d at $\tau = 127$, are compared with the original time series (for clarity, the amplitudes of the local time series have been multiplied by 40). The increased concentration of higher frequencies near $t = \tau$ can be seen in Figs. 2b–d, but not in Fig. 2a because all frequencies have the same taper in the discrete STFT. Note that, in Fig. 2c, the local time series of the Haar wavelet has zero value at $t \geq 128$, since the frame members that contain $t = 127$ in their support have zero amplitude at $t \geq 128$.

Currently, time-time distributions are essentially a solution looking for a problem; the only application to date appears in [2]. They may yet prove useful, though, because some unusual filters can be designed by modifying (11). Two examples, both obtained using the $TT_D f$ from Fig. 1c, are shown below. In the first example (Fig. 3a) the filtered time series $\tilde{f}$ has been calculated from

$$\tilde{f}_a[t] = \sum_{\tau=0}^{N-1} TT_D f[\tau, t]\ H(t-\tau), \tag{14}$$

where H is the Heaviside step function, here defined as

$$\begin{aligned} H(x) &= 1,\ x \geq 0, \\ &= 0,\ x < 0. \end{aligned} \tag{15}$$

This sets $TT_D f$ to zero at $t < \tau$. Fig. 3a also shows the resulting $\tilde{f}_a[t]$, along with $f[t]$ for comparison. It is difficult to judge whether this result has any physical significance, but, in a sense, (14) gives the leading or "anticausal" part of the time series.

A more interesting example is Fig. 3b, for which all the negative values of $TT_D f$ have been set to zero, via

$$\tilde{f}_b[t] = \sum_{\tau=0}^{N-1} TT_D f[\tau, t]\ H(TT_D f[\tau, t]). \tag{16}$$

The resulting "positive time series" $\tilde{f}_b$ is different from the positive part of f, because the stronger concentration of high frequencies close to $t = \tau$ allows some high-frequency peaks whose maximum values are slightly negative on f to have positive-valued peaks on $TT_D f$, and thereby on $\tilde{f}_b$. This is particularly noticeable at $t < 60$.

3. Conclusions

We have presented a definition of time-time distributions for discrete wavelet transforms. These distributions give a quantitative time-domain description of how DWTs localize the scale content of a time series around specific points in time. The broader utility and faster computational time of the DWT over the previously defined S-transform and TT-transform may open new avenues for applications of time-time techniques.

Appendix: MATLAB Algorithm

The following algorithm (which requires the MATLAB Wavelet Toolbox) produces time-time distributions similar to those shown in Figs. 1c,d:

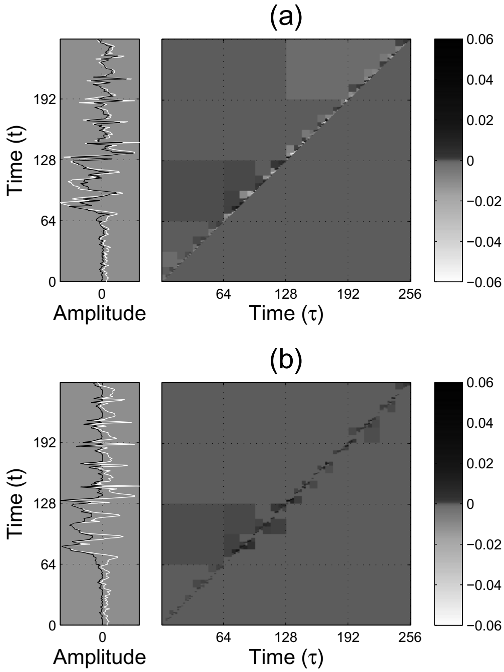

FIGURE 3. A test time series (white lines), plotted next to the time-time distributions that are obtained from its Haar wavelet transform by (a) setting the amplitude to zero at $t < \tau$, (b) setting all negative amplitudes to zero. The filtered time series that are obtained by substituting these time-time distributions in place of $TT_D f$ in (11) are also shown (black lines).

```
N=length(f); wave='db3';
dwtmode('zpd'); maxlev=5;
[Cf Cinfo]=wavedec(f,maxlev,wave);
ind=sort([1; cumsum(Cinfo(1:end-1)); ...
             cumsum(Cinfo(1:end-2))+1]);
X=zeros(N); phi=X; TT_Df=X;
for wscale=1:length(ind)/2;
  for j=ind(2*wscale-1):ind(2*wscale);
    phi(:,j)=waverec([zeros(j-1,1); 1; ...
      zeros(length(Cf)-j,1)], ...
      Cinfo,wave);
    X(j,:)=(phi(:,j)~=0).';
```

```
    X(j,:)=X(j,:)/sum(X(j,:));
    TT_Df=TT_Df+Cf(j)*phi(:,j)*X(j,:);
  end
end
```

In the program, `f` is a 256x1 input vector that must be predefined, `'db3'` denotes the Daubechies-3 DWT, and `Cf(j)`, `X(j,:)`, `phi(:,j)` and `TT_Df` denote Cf_j, $X_j[t]$, $\Phi_j[t]$ and TT_Df. To obtain the Haar result, `'db3'` must be replaced with `'db1'` and `maxlev` set equal to 7. The reader can verify that `sum(TT_Df,2)` is equal to `f`, as in equation (5). The modified versions of TT_Df from Fig. 3 are obtained from `TT_Df.*tril(ones(N))` and `TT_Df.*(TT_Df>=0)`.

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C. R. Pinnegar, H. Khosravani and P. Federico
Department of Clinical Neurosciences
University of Calgary
Foothills Medical Centre
Calgary, Alberta T2N 2T9
Canada
e-mail: pinnegar@ucalgary.ca
houman@ucalgary.ca
pfederic@ucalgary.ca

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The Stockwell Transform in Studying the Dynamics of Brain Functions

Cheng Liu, William Gaetz and Hongmei Zhu

Abstract. The dynamics of brain functional activities make time-frequency analysis a powerful tool in revealing its neuronal mechanisms. In this paper, we extend the definition of several widely used measures in spectral analysis, including the power spectral density function, coherence function and phase-locking value, from the classic Fourier domain to the time-frequency plane using the Stockwell transform. The comparisons between the Stockwell-based measures and the Morlet wavelet-based measures are addressed from both theoretical and numerical perspectives. The Stockwell approach has advantages over the Morlet wavelet approach in terms of easy interpretation and fast computation. A magnetoencephalography study using the Stockwell analysis reveals interesting temporal interaction between contralateral and ipsilateral motor cortices under the multi-source interference task.

Mathematics Subject Classification (2000). Primary 62M15, 65R10, 92C55; Secondary 42C40, 47G10.

Keywords. Stockwell transforms, Morlet wavelet transforms, power density function, coherence function, phase-locking value, dynamics of brain functions.

1. Introduction

Brain activities are characterized by multiple oscillators from different frequency bands [4], thus making spectral analysis a popular tool for non-invasively investigating the mechanism of brain functions [23]. Widely used measures, such as the power spectral density function, coherence function and phase-locking value, are traditionally defined through the Fourier transform, which relies on the assumption that signals are stationary (i.e., their spectral characteristics do not change over time) [1, 19]. However, brain functional activities are dynamic and transient, especially those associating with cognitive and behavioral events. This implies that non-stationarity is the rule rather than the exception in neural information processing. Therefore, temporal information missed by Fourier analysis must be addressed in order to understand brain functionality.

Two well-developed integral transforms providing good solutions for the non-stationarity problem are the Gabor transform and the wavelet transform. The Gabor transform $V_g s$ of a signal $s \in L^2(\mathcal{R})$ provides a time-frequency representation of s by applying the Fourier transform to the signal localized by a Gaussian window g that translates over time. The Gabor transform is mathematically defined as

$$(V_g s)(\tau, f) = \int_{-\infty}^{\infty} e^{-2\pi i t f} \overline{g(t-\tau)} s(t) dt, \quad \tau, f \in \mathcal{R}, \tag{1}$$

where $\overline{g}$ represents the complex conjugate of g. The Gabor transform is known as a special class of the short-time Fourier transform with a Gaussian window. A more in-depth discussion about the Gabor transform can be found in Chapter 3 of [13]. Time-frequency analysis based on the Gabor transform is intuitive, but limited by the fixed time and frequency resolution.

The wavelet transform overcomes such limitation with its multi-resolution characteristic [7]. The wavelet transform projects a signal on a family of wavelets obtained by scaling and shifting a mother wavelet ψ. More precisely, the wavelet transform of a signal s is defined as

$$(W_\psi s)(\tau, \alpha) = \frac{1}{\sqrt{|\alpha|}} \int_{-\infty}^{\infty} \overline{\psi\left(\frac{t-\tau}{\alpha}\right)} s(t) dt, \quad \tau \in \mathcal{R}, \quad \alpha \in \mathcal{R} \setminus \{0\}. \tag{2}$$

The multi-resolution analysis provided by the wavelet transform gives a more accurate assessment of the local features of a signal. With different choices of mother wavelets, the wavelet transform exhibits various features and is versatile for different applications. For the time-frequency analysis, the wavelet transform with a Morlet mother wavelet is plausible in terms of the intimate relation between the scaling variable and Fourier frequency. More specifically, the complex Morlet mother wavelet [12] is given by

$$\varphi^{\nu_0}(t) = \frac{1}{\sqrt{2\pi}} e^{2\pi i \nu_0 t} e^{-\frac{t^2}{2}}, \tag{3}$$

where ν_0 is a non-dimensional frequency. The scaling variable α can be easily mapped to a Fourier frequency f by $f = \frac{\nu_0}{\alpha}$. Because the Morlet wavelet transform (MWT) is a multi-resolution representation and has the scale factors that can be directly converted to Fourier frequencies, it has become more popular in various applications. As a consequence, commonly used measures in brain functionality studies were extended from the Fourier domain to the time-frequency plane through the MWT. Thus, there is an increasing interest in applying the MWT to process brain signals in neuroscience [16,17,22]. Nevertheless, the complex mathematics behind the MWT may create some obstacles for applied researchers. Moreover, studies of brain functionality often involve intensive computations. Due to the lack of fast algorithms, the MWT is therefore limited in use for practical applications.

The search for optimal data representations leads to new developments in time-frequency analysis and results in new integral transforms combining the merits of the Gabor transform and the wavelet transform. As a hybrid of the Gabor transform and wavelet transform, the Stockwell transform (ST) proposed in 1996 by geophysicists [27] uses frequency-dependent Gaussian window width to provide a multi-resolution time-frequency representation of a signal. The ST has gained popularity in the signal processing community because of its easy interpretation and fast computation [11, 24, 28].

As a result of the intimate relation to the Fourier framework, measures in brain functionality studies can be straightforwardly extended to the time-frequency plane using the ST. Although the MWT extension and the ST extension produce similar measures, the Stockwell approach is more suitable for analyzing real brain signals because of its easy interpretation and fast computation. This paper aims to address this issue from both theoretical and numerical perspectives. The relationship between the ST and the MWT is reviewed in Section 2. In Section 3, we present three commonly used measures in the study of brain functions: the power spectral density function, the coherence function and the phase-locking value. We then redefine these measures in the time-frequency plane using the ST. The similarities between the Stockwell approach and Morlet wavelet approach will be discussed. In Section 4.1, we compare the performance of the Stockwell approach to that of the Morlet wavelet approach using numerical simulations. An application of the Stockwell approach to the magnetoencephalography (MEG) signals is presented in Section 4.2.

2. The Stockwell Transform and the Morlet Wavelet Transform

Let $s \in L^2(\mathcal{R})$. Then, the Stockwell transform Ss is defined as an integral operator

$$(Ss)(\tau, f) = \frac{|f|}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(\tau-t)^2 f^2}{2}} e^{-2\pi i \tau f} s(t)dt, \quad \tau, f \in \mathcal{R}. \tag{4}$$

With a nonzero frequency f, the ST can be equivalently expressed in the Fourier domain

$$(Ss)(\tau, f) = e^{-2\pi i f \tau} \int_{-\infty}^{\infty} e^{2\pi i \tau \alpha} e^{-\frac{2\pi^2(\alpha - f)^2}{f^2}} \hat{s}(\alpha)d\alpha, \tag{5}$$

where $\hat{s}$ is the Fourier spectrum of s. The width of the Gaussian window in (5) is scaled according to the inverse of frequency f. Hence, the ST provides better frequency resolutions at low frequencies and better time resolutions at high frequencies, analogous to the wavelet transform. Equation (5) also indicates that the ST can be considered as the Fourier multiplier $\mathcal{F}^{-1} e^{-\frac{2\pi^2(\cdot - f)^2}{f^2}} \mathcal{F}$ followed by a modulation. Thus, the ST can be implemented efficiently using the fast Fourier transform.

As pointed out by [10], the ST is closely related to the MWT with the non-dimensional frequency $\nu_0 = 1$, namely,

$$(Ss)(\tau, f) = \sqrt{f} e^{2\pi i \tau f} (W_\varphi s) \left(\tau, \frac{1}{f} \right). \tag{6}$$

We can see that the scaling variable α in the MWT is exactly the reciprocal of the Fourier frequency f and the ST is the MWT represented in the time-frequency plane with an amplitude dilation and a phase modulation.

3. Statistical Measurements in Brain Functionality Studies

3.1. Power Spectral Density

Since brain activities are characterized by multiple oscillators from different frequency bands, it is crucial to identify the spectral information of brain signals in order to understand its underlying mechanisms. One fundamental measure in spectral analysis is the power spectral density function. Given a time series x, the classic Fourier-based definition of a power spectral density function can be expressed as

$$(Px)(f) = E\{|\hat{x}(f)|^2\}, \tag{7}$$

where $E\{\cdot\}$ is the expectation operator. The definition is only valid with stationary time series whose spectral characteristics do not change over time [23]. To study the dynamics of brain activities, non-stationarity must be addressed.

The extension of the power spectral density function with the wavelet transform introduced in [17] leads to a time-scale distribution

$$(WP_\psi x)(\tau, \alpha) = E\{|(W_\psi x)(\tau, \alpha)|^2\}. \tag{8}$$

Here, we propose to extend the definition of the power spectral density function directly to the time-frequency domain using the ST

$$(SPx)(\tau, f) = E\{|(Sx)(\tau, f)|^2\}. \tag{9}$$

From (6), we can easily see that these two power density functions are closely related, i.e.,

$$(SPx)(\tau, f) = |f| \cdot (WP_\varphi x)(\tau, \frac{1}{f}) \tag{10}$$

or

$$\frac{1}{|f|} \cdot (SPx)(\tau, f) = (WP_\varphi x)(\tau, \frac{1}{f}). \tag{11}$$

Theoretically, the Stockwell power density function is the Morlet wavelet power density function transformed in the time-frequency plane and multiplied by a dilation term $|f|$. The term $\frac{1}{|f|}$ in (11), however, indicates that the Morlet wavelet power density function magnifies the low frequency components and suppresses the high frequency components. At very low frequencies, $\frac{1}{|f|}$ may also cause numerical instability. These artifacts, as illustrated later in Section 4.1, may cause ambiguity in interpretation of the Morlet wavelet power density function.

In practice, the expectation operator in above definitions can be estimated either by convolving these measures with a smoothing temporal window for a single trial signal, or by averaging these quantities across trials for event-related signals. The former assumes that the signal is locally stationary and the latter assumes cross-trial stationarity in the signal.

3.2. Coherence Function

Cognitive processing depends not only on the activities of local neuronal assembly, but also on the dynamic communication between different assemblies of neurons [22]. To study the interaction of two time series, the coherence function is often used. It is classically defined through the Fourier transform, namely,

$$(Cxy)(f) = \frac{|E\{\hat{x}(f) \cdot \overline{\hat{y}(f)}\}|^2}{(Px)(f) \cdot (Py)(f)}. \tag{12}$$

The Schwartz inequality guarantees that $(Cxy)(f)$ takes values between 0 and 1. The coherence function measures the linear relationship between any two time series in the frequency domain. More specifically, when noise is absent, $(Cxy)(f) = 1$ for two linear dependent time series $x(t)$ and $y(t)$ and $(Cxy)(f) = 0$ if the time series are completely independent. The magnitude of the coherence function, representing the strength of the linear correlation between time series, is often used to describe functional connectivity between two brain areas in neuroscience [20].

The extension of the coherence function to the time-frequency plane can be easily done by substituting the frequency representation with a joint time-frequency representation. Note that the Stockwell coherence and Morlet wavelet coherence defined in that fashion lead to an equivalent quantity

$$\begin{aligned}(SCxy)(\tau, f) &= \frac{|E\{(Sx)(\tau, f) \cdot \overline{(Sy)(\tau, f)}\}|^2}{(SPx)(\tau, f) \cdot (SPy)(\tau, f)} \\ &= (WC_{\varphi}xy)(\tau, \frac{1}{f}).\end{aligned} \tag{13}$$

These time-varying coherence functions can reveal the dynamics of the interaction between different brain areas. Therefore, it is very useful to provide insight about the mechanisms of brain functional activities.

3.3. Phase-Locking Value

The coherence function measures the linear correlation that is affected by changes in both the amplitude and the phase of two time series. It has been reported recently that the instantaneous phase of brain oscillation is associated with particular neuronal firing patterns and high temporal precision of neuronal activity [26]. Therefore, new measures of interrelationships based on the phase synchronization of signals have been proposed. Among those measures, the phase-locking value

(PLV) attracts increasing interests in studying brain functional connectivity due to its intuition and easy computation [18, 19, 21].

The PLV is a statistical quantity bounded between 0 and 1, which is subject to fluctuations when randomness is introduced in the signals. It can be defined in the time-scale domain using a complex wavelet transform

$$(WPLV_\psi xy)(\tau,\alpha) = \left| E\left\{ \frac{(W_\psi y)(\tau,\alpha)\cdot\overline{(W_\psi x)(\tau,\alpha)}}{|(W_\psi y)(\tau,\alpha)|\cdot|(W_\psi x)(\tau,\alpha)|} \right\} \right|. \tag{14}$$

The term inside the expectation operator is actually the instantaneous phase difference of the wavelet representations between two time series, i.e.,

$$\frac{(W_\psi y)(\tau,\alpha)\cdot\overline{(W_\psi x)(\tau,\alpha)}}{|(W_\psi y)(\tau,\alpha)|\cdot|(W_\psi x)(\tau,\alpha)|} = e^{i[(\phi_{W_\psi y})(\tau,\alpha)-(\phi_{W_\psi x})(\tau,\alpha)]}. \tag{15}$$

where $(\phi_{W_\psi x})(\tau,\alpha)$ is the instantaneous phase of the wavelet transform of time series x. The PLV measures the inter-trial variability of this phase difference at time τ and scale α. If the phase difference does not vary much across the trials, the PLV is close to 1; it is close to zero otherwise.

The extension of the PLV using the ST can be easily done by

$$(SPLVxy)(\tau,f) = \left| E\left\{ \frac{(Sy)(\tau,f)\cdot\overline{(Sx)(\tau,f)}}{|(Sy)(\tau,f)|\cdot|(Sx)(\tau,f)|} \right\} \right|. \tag{16}$$

The phase relation between the ST and the MWT, derived from (6), can be written as

$$(\phi_{Ss})(\tau,f) = 2\pi\tau f + (\phi_{W_\varphi s})(\tau,\frac{1}{f}) \tag{17}$$

where $(\phi_{Ss})(\tau,f)$ and $(\phi_{W_\varphi s})(\tau,\frac{1}{f})$ are the phases of the Stockwell and Morlet wavelet representations at time τ and frequency f, respectively. This leads to another equivalent quantity between the Stockwell approach and Morelet wavelet approach:

$$\begin{aligned}(SPLVxy)(\tau,f) &= \left| E\left\{ e^{i[(\phi_{Sy})(\tau,f)-(\phi_{Sx})(\tau,f)]} \right\} \right| \\ &= \left| E\left\{ e^{i[(\phi_{W_\varphi y})(\tau,\frac{1}{f})-(\phi_{W_\varphi x})(\tau,\frac{1}{f})]} \right\} \right| \\ &= (WPLV_\varphi xy)(\tau,\frac{1}{f}).\end{aligned} \tag{18}$$

3.4. Remarks

The benefits of the ST over other representations are naturally carried over in defining these Stockwell-based measures in brain functionality studies. Particularly, the interpretation of statistical measures in real applications requires tests of significance. One commonly used technique for significance tests is the nonparametric bootstrap method [9], where the significance level is determined by a bootstrap procedure that involves creating re-samples of the data set by random rearrangements of the trial order independently for each recording site. In

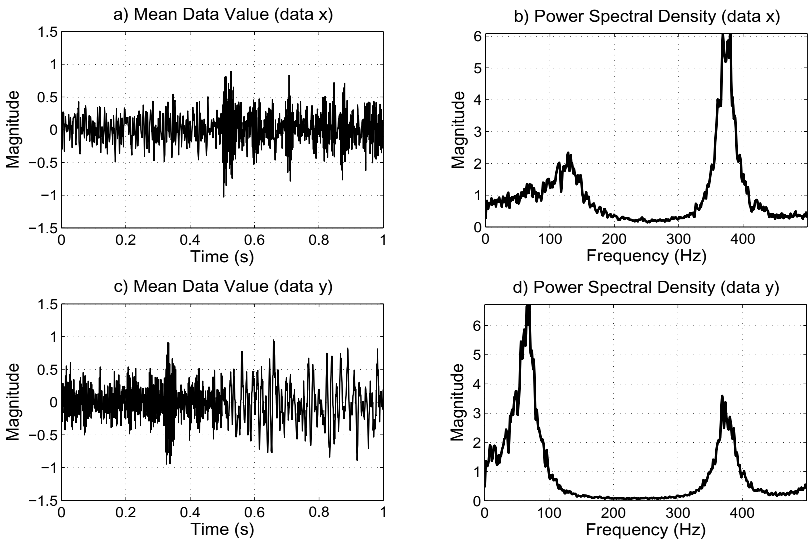

FIGURE 1. The mean value of a) time series x and c) time series y, and b) and d) their corresponding power spectral density functions, respectively.

general, a large number of bootstrap samples need to be considered in order to obtain a proper estimation of distribution. Since the statistical quantities have to be calculated for each re-sample, bootstrap significant tests for transform-based measures require intensive computation. Fortunately, a fast computational scheme is available for the ST, making the use of the Stockwell approach more practical for applications.

4. Simulations and Applications

4.1. Simulations

We construct two non-stationary time series by combining two simple linear systems with different temporal occurrence. During 0–0.5 s, System 1

$$\begin{aligned} x(t) &= 0.6x(t-1) - 0.2y(t-2) - 0.4x(t-3) + \epsilon_1(t) \\ y(t) &= -0.5y(t-1) + 0.6y(t-3) + \epsilon_2(t) \end{aligned} \tag{19}$$

occurs and during 0.5–1 s, System 2

$$\begin{aligned} x(t) &= -0.5x(t-1) - 0.3y(t-2) + 0.6x(t-3) + \epsilon_3(t) \\ y(t) &= 0.8y(t-1) + 0.6y(t-2) - 0.7y(t-3) + \epsilon_4(t) \end{aligned} \tag{20}$$

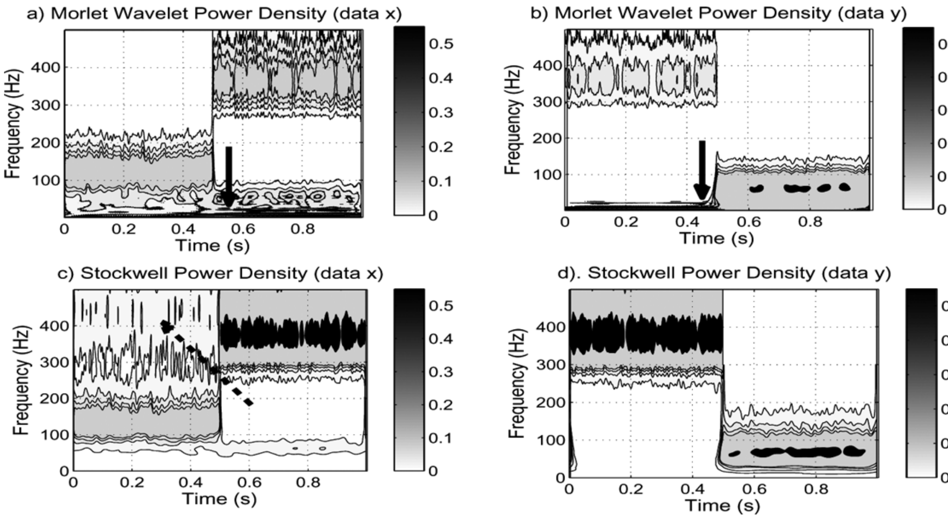

FIGURE 2. Contour plots of Morlet wavelet power density functions of a) time series x and b) time series y, and c) and d) Stockwell power density functions of time series x and y, respectively.

appears. Here the $\epsilon_i(t)$ are zero-mean uncorrelated white noise with identical variances. Both systems indicate that x is linearly dependent on y.

Fifty trials of simulated data were generated by Monte Carlo simulations with sample rate 1024 Hz and duration 1 s. Figure 1 shows the time representations of the mean value of time series x and y, and their corresponding power spectral density functions. Time series x has two significant peaks occurred within the frequency ranges 100–150 Hz and 350–400 Hz. And time series y has two significant frequency peaks: one is within the frequency range 50–100 Hz and the other within 350–400 Hz. Due to its linear dependence to y, time series x has also one subtle frequency peak appears within the frequency range 50–100 Hz. Hence significant coherence between x and y is expected to be in the frequency ranges 50–100 Hz and 350–400 Hz. As stated before, the Fourier-based approach does not provide temporal information, meaning it cannot tell when these frequency components occur.

Figure 2 displays the power density functions of x and y in the time-frequency plane based on the MWT and the ST, respectively. To make them comparable, the normalized power densities are shown in the figure. The non-stationarity characteristics of the times series are clearly shown by both approaches: x contains mostly a high frequency (350–400 Hz) component during the first half second and a low frequency (100-150 Hz) component during the second half second, while y

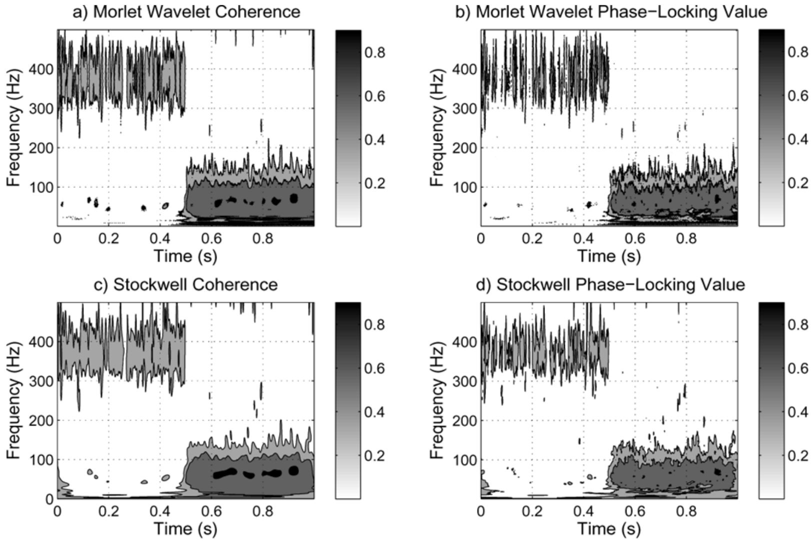

FIGURE 3. Contour plots of the interaction between simulated data x and y: a) coherence and b) phase-locking value based on the Morlet wavelet transform, and c) coherence and d) phase-locking value based on the Stockwell transform.

contains a low frequency (50–100 Hz) component for the first half second and a high frequency (350–400 Hz) component for the second half second.

As expected, because of the dependence of x on y, the Stockwell power density function of x shows a low-amplitude component of high frequency (350–400 Hz) during 0–0.5 s and also a low-amplitude component of low frequency (50–100 Hz) during 0.5–1 s. As shown in Figure 2 a), however, the subtle high frequency (350–400 Hz) component of x during 0–0.5 s (indicated by the dashed arrow in Figure 2 c)) is invisible in its Morlet wavelet power density. In addition, there exists strong artifacts at low frequencies (< 20 Hz) in the Morlet wavelet power densities for both x and y, as indicated by the solid arrows in Figures 2 a) and 2 b). These can be explained by the term $\frac{1}{|f|}$ in (11) in the Morlet wavelet power density, which may magnify the low frequency components, suppress high frequency components, and cause numerical instability due to possible division by very small numbers. The bias towards the low frequencies can be further confirmed by comparing the amplitude of the low frequency components of the Fourier power density to that of the Morlet wavelet power density.

The dynamic interaction between x and y is studied through the time-frequency coherence and PLV. As shown in Figure 3, both the Morlet wavelet approach and Stockwell approach reveal the interaction of x and y peaked at the

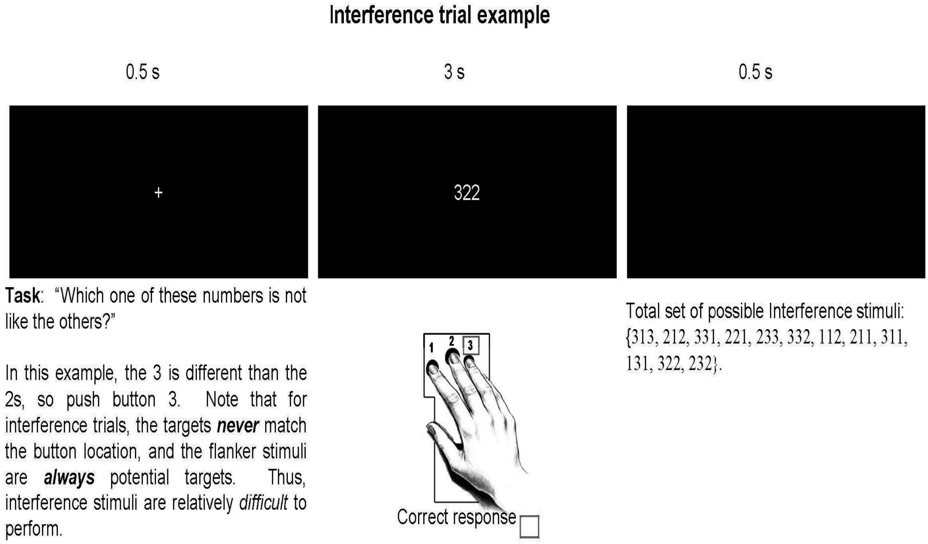

FIGURE 4. An illustration of the multi-source interference task.

frequency range 350–400 Hz during 0–0.5s and at the frequency range 50–100 Hz during 0.5–1 s. As discussed in Section 3, there is no theoretical difference between these two approaches in defining coherence and PLV. The numerical results also confirm this relation. Generally speaking, the Stockwell-based measures appear more smooth compared to the Morlet wavelet-based measures. This is because the ST is calculated directly in the time-frequency plane, but for the MWT, the scaling variable must be converted to a Fourier frequency by taking its reciprocal. Additionally, the availability of fast algorithms makes the computation of Stockwell-based measures much faster than the Morlet wavelet approach. The former approach only takes a quarter of the time to compute compared to the latter. Note that the implementations in this paper are done in MATLAB R2007a. The ST is implemented based on (5) and the continuous wavelet transform is calculated using the wavelet toolbox in MATLAB.

4.2. An Application in Magnetoencephalography

Now, we apply the Stockwell approach to study the activities of motor cortices when subjects performed the Multi-Source Interference Task (MSIT) [2] using their right hands. The MSIT combines multiple dimensions of cognitive interference in a single task, which can be used to investigate mental or behavioral diseases such as Attention Deficit Hyperactivity Disorder (ADHD) in clinical studies [3]. See Figure 4 for details of the MSIT. Fifty interference trials were recorded for two right-handed participants (SB and DM, represented by their initials). One hundred fifty-one channel whole-head MEG (sample rate = 625Hz) was recorded

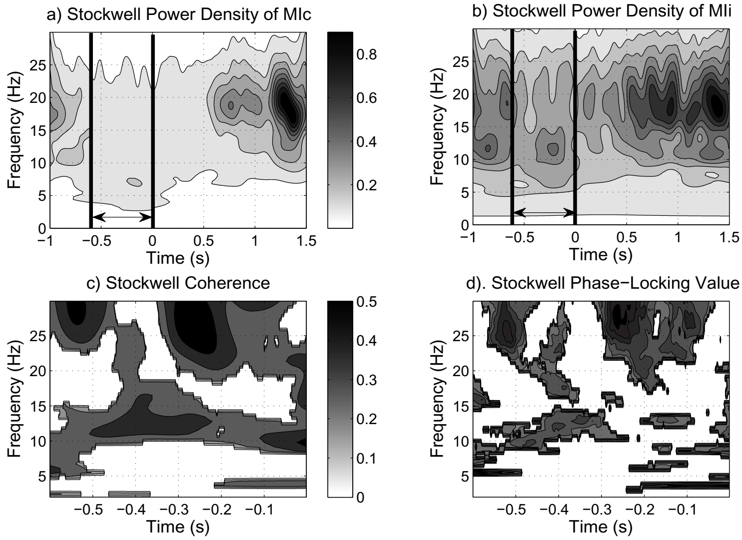

FIGURE 5. Subject SB: contour plots of the Stockwell power density of a) MIc and b) MIi, c) the Stockwell coherence and d) the Stockwell phase-locking value.

continuously for 400 seconds. Time zero is represented as a press of the button. The signals at contralateral and ipsilateral motor cortices (MIc and MIi) were extracted using the beamformer technique [5] and filtered with a low pass filter (1–30 Hz). Several preprocessing steps have been applied to the data, including temporal normalization to give the data equal weight and ensemble mean subtraction to remove first order non-stationarity [8].

For each subject, we calculate the power density function, coherence function and PLV based on the ST with the preprocessed data −1–1.5 s. We also investigate the statistical significance of coherence and PLV quantities using the bootstrap method with 500 re-samples and significance level $\alpha = 0.01$. Since the Stockwell time-frequency representation often contains artifacts at the two ends of a time series due to circular Fourier spectrum shifting in the implementation, we investigate the Stockwell coherence and PLV only during the time period −0.6–0 s. Another reason we are particularly interested in this period is that the reaction time of those two subjects is approximately 0.6 s, which suggests that subjects are processing their cognitive tasks within the time interval.

The results for subjects SB and DM are presented in Figure 5 and Figure 6, respectively. The Stockwell power density functions show the activities of these two motor cortices around 10 Hz and 20 Hz. The former is known as *mu* wave, which appears to be associated with the motor cortex, and the latter is another

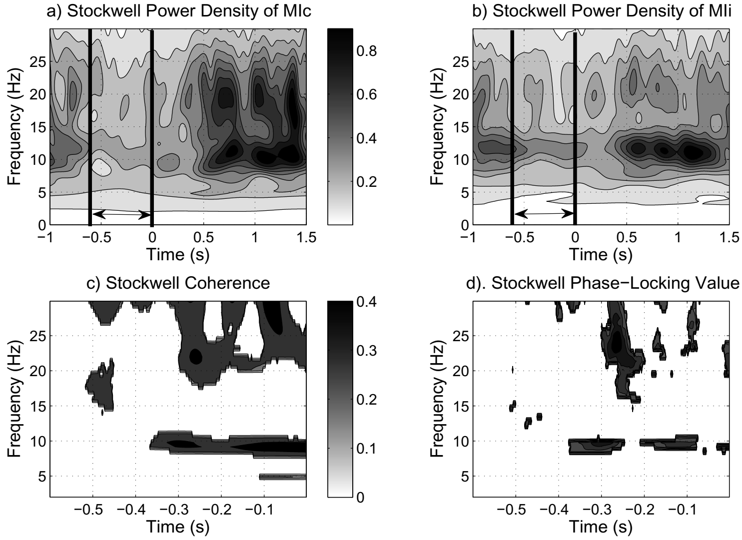

Figure 6. Subject DM: contour plots of the Stockwell power density of a) MIc and b) MIi, c) the Stockwell coherence and d) the Stockwell phase-locking value.

important brain rhythm called *beta* rhythm. It has been discovered that during movements, the motor cortex exhibits a pronounced decrease of *beta* amplitudes whereas a strong *beta* power rebound occurs when movements are stopped. Such *beta* rebounds can also be observed by our results. See [4] for details for brain rhythms.

The Stockwell coherence and PLV indicate the functional connection between the MIc and MIi under the MIST. Figure 5 and Figure 6 show that the significant connection happens mainly around frequency bands of 10–14 Hz and 25 Hz. For the 10–14 Hz frequency band, our results is consistent with the results found in [25], where activities of MIi and predominantly cortico-cortical coupling around 8–12 Hz has been observed under the unimanual auditorily paced finger-tapping task. The connection around 25 Hz in this experiment is new and needs to be further investigated. The PLV measure are generally consistent with the coherence measure. However, since the PLV measures the connection due to phase synchronization only, the PLV shows less frequency coupling than the coherence function does. This phenomenon is correctly reflected by our results in Figure 5 and Figure 6.

The common limitation of studying brain signals is the unavailability of large amounts of data. Statistical measurements with few samples may combine with artifacts. In order to improve accuracy, grand average results among more subjects need to be studied and will be further considered in the future.

5. Conclusions

In this paper, we have introduced the Stockwell-based approach to study the dynamics of brain functions, including revealing the brain activities by the power density function and investigating the functional interaction among different brain cortices by the coherence function and phase-locking value. Due to the intimate connection with the MWT, the comparison between these MWT and Stockwell approaches have been presented. The advantages of using ST can be seen from its easy interpretation, fast computation and better energy distribution. We have demonstrated the performance of the ST-based approach using a real MEG study, which shows significant functional connections between contralateral and ipsilateral motor cortices under the MSIT. In conclusion, the Stockwell approach is an intuitive, straightforward non-parametric tool to study the dynamics of non-stationary signals.

In addition, it is of interest to investigate the use of the variants of the Stockwell transforms in spectral analysis. For example, the recently developed modified Stockwell transforms [14, 15] define a family of the Stockwell transforms parametrized by s, where $1 < s < \infty$. When $s = 1$, the modified Stockwell transform is the classic Stockwell transform. By choosing a specific value for s, the modified Stockwell transforms modulate frequencies in such a way that the spectral information are highlighted at lower frequencies and compressed at higher frequencies. Hence, the modified Stockwell transforms may be useful to study the low frequency brain activities.

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Cheng Liu
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario, M3J 1P3
Canada
e-mail: `chengliu@yorku.ca`

William Gaetz
Department of Medical Imaging
University of Toronto
27 King's College Circle
Toronto, Ontario, M5S 1A1
Canada
e-mail: `bill.gaetz@gmail.com`

Hongmei Zhu
Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario, M3J 1P3
Canada
e-mail: `hmzhu@yorku.ca`