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# Preface

The well-known monographs by G.S. Ladde, V. Lakshmikantham and B.G. Zhang [248], I. Győri and G. Ladas [192], L.H. Erbe, Q. Kong and B.G. Zhang [154], R.P. Agarwal, M. Bohner and W.-T. Li [3], R.P. Agarwal, S.R. Grace and D. O'Regan [8] and D.D. Bainov and D.P. Mishev [34] are devoted to the oscillation theory of functional differential equations. Each of these monographs contains nonoscillation tests, but their main objective was to present methods and results concerning oscillation of all solutions for the functional differential equations under consideration.

The main purpose of the present monograph is to consider nonoscillation and existence of positive solutions for functional differential equations and to describe their applications to maximum principles, boundary value problems and the stability of these equations.

In view of this objective, we consider a wide class of equations:

1. scalar equations and systems of different types: linear and nonlinear first-order functional differential equations, second-order equations with or without damping terms, high-order equations, systems of functional differential equations;
2. equations with variable types of delays: delay differential equations, integrodifferential equations, equations with a distributed delay, neutral equations;
3. equations with variable deviations of the argument: advanced and mixed (including both delayed and advanced terms) differential equations;
4. both continuous and impulsive equations: first- and second-order linear and first-order nonlinear impulsive differential equations;
5. specific classes of linear and nonlinear equations, as well as linear differential equations with abstract Volterra (causal) operators;
6. both initial and boundary value problems are considered for functional differential equations.

Note that we do not use methods specific only to equations with continuous parameters since we consider models with measurable coefficients and delays.

Nonoscillation results are applied

- to nonlinear nonautonomous equations of mathematical biology with both concentrated and distributed delays;

- to stability problems; and
- to boundary value problems.

Chapter 1 is a brief survey of introductory notions and ideas in nonoscillation theory: autonomous equations, characteristic equations, solution representations, differential and integral equations, and inequalities. Though elementary in its presentation (we believe it can easily be understood by senior undergraduate students), this chapter incorporates many basic ideas that will be employed later: equivalence of nonoscillation and existence of a nonnegative solution of the generalized characteristic inequality and the application of solution representation, linearization and the approach to impulsive equations. The main population dynamics equations (Hutchinson's, Lasota-Wazewska, Mackey-Glass, Nicholson's blowflies) are also introduced in Chap. 1.

Chapter 2 presents basic results for first-order linear delay equations with positive coefficients: nonoscillation criteria, comparison theorems, explicit nonoscillation and oscillation results, sufficient conditions for positivity of solutions with given initial conditions and slowly nonoscillating solutions. In Chap. 3, some of these results are generalized to equations with positive and negative coefficients; it is also illustrated that some of the results cannot be extended. Chapter 4 is concerned with a general linear equation with a distributed delay that is nonautonomous and can include integral and concentrated delay terms. The case of positive kernels of integrals and coefficients is considered, as well as terms of different signs.

In Chap. 5, nonoscillation of linear equations of advanced and mixed types is studied. The main results of this chapter are based on various fixed-point theorems. Chapter 6 is concerned with linear neutral equations of the first order that include the derivative of the unknown function both with and without delays.

In Chaps. 7 and 8, we consider linear second-order delay equations without damping and with damping, respectively. Chapter 9 deals with linear systems of delay differential equations and also higher-order differential equations. In addition to the problems considered in the previous chapters, Chap. 9 includes an extensive section on stability of nonoscillatory systems.

Chapters 10 and 11 are devoted to nonlinear equations. In Chap. 10, the linearization method is applied to various nonautonomous models of population dynamics (in particular, logistic, Lasota-Wazewska and Nicholson's blowflies equations), and all equations are considered with a distributed delay. In Chap. 11, some equations that cannot be handled with the linearization approach are studied (mostly different variations of the logistic model).

Chapters 12–14 are concerned with impulsive equations. Chapter 12 presents nonoscillation results for first-order linear impulsive differential equations with both concentrated and distributed delays. It is also demonstrated that nonoscillation of an impulsive equation can be reduced to nonoscillation of a specially constructed equation without impulses but with discontinuous coefficients. Chapter 13 deals with second-order differential equations, and generally in the models considered any linear jumps of both the solution and the first derivative can occur. In Chap. 14, linearization methods are applied to first-order nonlinear impulsive equations.

The study of many classical questions in the qualitative theory of linear $n$-th-order ordinary differential equations, such as existence and uniqueness of solutions of the interpolation boundary value problems, positivity, or a corresponding regular behavior of their Green's functions, maximum principles and stability, was connected with and even based on the notion of nonoscillation intervals of corresponding linear ordinary differential equations. In Chaps. 15–17 we create a concept of nonoscillation intervals for functional differential equations that can actually be considered as an analogue of nonoscillation theory for ordinary differential equations. Various relations between the noted properties are obtained for functional differential equations on the basis of nonoscillation. Linear and nonlinear equations with Volterra (causal) operators were previously studied in the monographs [29, 98, 239, 251]. In Chaps. 15–17, we consider equations with Volterra operators. It should be noted that it is not only a generalization but also an important instrument for studying the behavior of a corresponding component $x_r$ of a solution vector. We construct an equation for this component in Chap. 16. Even in the case of systems of ordinary differential equations, this differential equation for $x_r$ is of quite a general form that includes Volterra operators. In these chapters, we also study such questions as maximum principles, existence and uniqueness of solutions to boundary value problems, regular behavior of their Green's functions, and applications to study stability that are not considered in previous chapters.

All chapters conclude with a discussion, some open problems, and topics for possible future research.

Finally, Appendices A and B include some reference material. Appendix A contains all auxiliary notions and functional analysis results used in the monograph: definitions of functional spaces, measures and Volterra operators, compactness conditions for sets and linear operators, and fixed-point theorems in Banach spaces with or without order. These results are applied in the study of a variety of types of equations: with several concentrated and distributed delays, with general Volterra and non-Volterra equations and systems, linear and nonlinear, and continuous and impulsive. Appendix B presents existence and uniqueness conditions for all functional differential equations considered in this monograph; in addition, solution representations are given for linear equations.

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First of all, we would like to acknowledge the illuminating influence of Nikolai Viktorovich Azbelev (1922–2006) on the research leading to the present monograph. As a research advisor of L.B., E.B., and A.D., he supported and guided the authors for many years in the beginning of their research careers, and many of his ideas are reflected in the book. L.B., E.B, and A.D. are also grateful to Michael Efimovich Drakhlin (1936–2007) for his supportive and interested discussions of our research. We dedicate our monograph to the memory of our colleagues N.V. Azbelev and M.E. Drakhlin.

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# Contents

# Chapter 1
# Introduction to Oscillation Theory

## 1.1 Introduction

In this chapter, we introduce some known results for autonomous delay differential equations. We will also use these equations to illustrate some of the main ideas of this monograph for linear and nonlinear equations.

The history of delay equations, especially in the form of integrodifferential equations, goes back to the beginning of the 20th century; for example, to the works of Vito Volterra. These models are based on the idea that the derivative at a certain moment of time depends not only on the present state but on some of the previous states. However, the systematic study of delay differential equations started only in the first half of the 1950s. The best-studied models are autonomous equations where the initial point can be shifted without any influence on the solution if the initial function is shifted accordingly.

Generally, most qualitative properties of such equations are derived from explicitly constructed algebraic equations that include exponential functions called characteristic equations. In this monograph, we obtain new oscillation results for autonomous equations as corollaries of the results for general equations that are established without application of characteristic equations.

In this chapter, we introduce autonomous analogues of all equations that will later be considered in the monograph. Section 1.2 considers linear equations with several delays. Section 1.3 describes linearization techniques applied to nonlinear equations of mathematical biology. Section 1.4 involves simple linear impulsive models, while Sect. 1.5 contains an overview of models that were not explicitly discussed in the previous sections.

## 1.2 Nonoscillation of Autonomous Delay Equations with Positive Coefficients

Consider the scalar autonomous equation with several delays

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$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t - \tau_k) = f(t), \ \ \tau_k > 0, \ \ t \geq t_0. \tag{1.2.1}$$

Let us introduce

$$t_{-1} = t_0 - \max_k \tau_k$$

and consider (1.2.1) with the initial condition

$$x(t) = \varphi(t), \ \ t \in [t_{-1}, t_0]. \tag{1.2.2}$$

The existence and uniqueness result for the solution of (1.2.1), (1.2.2) is established by the so-called method of steps. If

$$\tau_{\min} = \min_k \tau_k,$$

then the solution of (1.2.1), (1.2.2) on the interval $[t_0, t_0 + \tau_{\min}]$ is

$$x(t) = \varphi(t_0) - \int_{t_0}^{t} a_k \varphi(s - \tau_k)\, ds + \int_{t_0}^{t} f(s)\, ds, \ \ t \in [t_0, t_0 + \tau_{\min}].$$

Further, we consider (1.2.1) for $t \geq t_0 + \tau_{\min}$ rather than $t \geq t_0$ with the known initial function on $[t_{-1} + \tau_{\min}, t_0 + \tau_{\min}]$. Repeating this process, we obtain existence of the unique solution of (1.2.1), (1.2.2) for any $t \geq t_0$.

So far we have not discussed properties of the initial function. In most publications, $\varphi(t)$ is assumed to be continuous and the solution continuously differentiable. However, any Lebesgue measurable function can be considered as an initial function, not necessarily continuous, if the solution can be absolutely continuous (continuous differentiability is not required). In most results of the present monograph, we consider absolutely continuous solutions.

The method of steps allows us to construct a solution for each $t \geq t_0$. However, a more useful solution representation applies the notion of *the fundamental function* $X(t, s)$, which is a solution of the homogeneous equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t - \tau_k) = 0 \tag{1.2.3}$$

for $t \geq s$ satisfying the initial conditions

$$x(t) = 0, \ \ t < s, \ \ x(s) = 1. \tag{1.2.4}$$

Direct computation implies that the unique solution of (1.2.1), (1.2.2) has the representation

$$x(t) = X(t, t_0)x(t_0) + \int_{t_0}^{t} X(t, s) f(s)\, ds - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s) a_k \varphi(s - \tau_k)\, ds, \tag{1.2.5}$$

where we assume $\varphi(s) = 0$, if $s > t_0$.

**Definition 1.1** We will say that (1.2.3) is *nonoscillatory* if there exists an initial function $\varphi$ such that the solution of initial value problem (1.2.1), (1.2.2) is eventually positive or eventually negative. Otherwise, (1.2.3) is *oscillatory*.

Together with (1.2.3), we will consider the differential inequality

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t - \tau_k) \le 0. \tag{1.2.6}$$

Theorem 1.1 includes nonoscillation criteria for (1.2.3) and will be generalized to various classes of equations with deviating arguments in the following chapters.

**Theorem 1.1** *Suppose $a_k > 0$, $\tau_k \ge 0$, $k = 1, \cdots, m$. Then the following hypotheses are equivalent*:

1) *Inequality* (1.2.6) *has an eventually positive solution.*
2) *There exists $t_1 \ge t_0$ such that the inequality*

$$u(t) \ge \sum_{k=1}^{m} a_k \exp\left\{\int_{t-\tau_k}^{t} u(s)\, ds\right\}, \tag{1.2.7}$$

   *where $u(t) = 0$ for $t < t_1$, has a solution $u(t)$ locally integrable and nonnegative for $t \ge t_1$.*
3) *There exists $t_2 \ge t_0$ such that $X(t, s) > 0,\ t \ge s \ge t_2$.*
4) *Equation* (1.2.3) *is nonoscillatory.*

The proof of a more general result will be presented in Chap. 2.
Assuming in (1.2.7)

$$u(t) = e \sum_{k=1}^{m} a_k,$$

we immediately obtain that (1.2.7) is satisfied with $u$ as above if

$$\sum_{k=1}^{m} a_k \max_k \tau_k \le \frac{1}{e}. \tag{1.2.8}$$

For autonomous equations, inequality (1.2.7) has an eventually positive solution if and only if it has a constant solution $\lambda > 0$. Then $x(t) = e^{-\lambda t}$ is a nonoscillatory solution of (1.2.3). Moreover, (1.2.3) has a nonoscillatory solution if and only if the equality corresponding to (1.2.7) has a positive constant solution (see, for example, [192]).

**Theorem 1.2** *Suppose $a_k > 0$, $\tau_k \ge 0$, $k = 1, \cdots, m$. Then the following hypotheses are equivalent*:

1) *Equation* (1.2.3) *is nonoscillatory.*

2) *The characteristic equation*

$$\lambda + \sum_{k=1}^{m} a_k e^{-\tau_k \lambda} = 0 \tag{1.2.9}$$

*has a negative root.*

The proof of the fact that oscillation of (1.2.3) is equivalent to nonexistence of real roots for characteristic equation (1.2.9) uses the Laplace transform of (1.2.3); see [192, Theorem 2.1.1] for details. If $\lambda < 0$ is a solution of (1.2.9), then $x(t) = e^{\lambda t}$ is a positive solution of (1.2.3).

*Remark 1.1* The statement of the theorem is valid for a general autonomous equation (1.2.3) with coefficients $a_k$ of arbitrary sign if we change existence of a negative root of the characteristic equation to existence of a real root of (1.2.9); see [192] for the proof. However, for $a_k > 0$, by Theorem 1.1, existence of a negative root of the characteristic equation is equivalent to the positivity of the fundamental function, which is not true for coefficients of arbitrary sign, as the following example demonstrates.

*Example 1.1* Consider the equation with a positive and a negative coefficient

$$\dot{x}(t) + 2x(t-3) - 1.9x(t-5) = 0. \tag{1.2.10}$$

Its characteristic equation

$$\lambda + 2e^{-3\lambda} - 1.9e^{-5\lambda} = 0 \tag{1.2.11}$$

has a negative root, $\lambda \approx -0.2075$. Since the fundamental function $X(t,0)$ satisfies (1.2.10) with $x(0) = 1$, $x(t) = 0$, $t < 0$,

$$X(t,0) = 1,\ t \in [0,3],\ X(t,0) = 1 - 2(t-3) = 7 - 2t,\ t \in [3,5],$$

and $X(4,0) = -1 < 0$. Moreover, $X(s+4,4) = -1$ for any $s$, and thus the fundamental function is not positive for any $t_1$ and $t > s \ge t_1$.

Positivity of the fundamental function is one of the important properties for any linear functional differential equation. In particular, by the results of Chap. 9, positivity of the fundamental function implies exponential stability of the equation (under some natural additional assumptions). For (1.2.3), we have the following result.

**Theorem 1.3** *Suppose* $a_k > 0$, $k = 1, \cdots, m$ *and there exists a negative root of* (1.2.9). *Then* (1.2.3) *is exponentially stable*; *i.e.*, *there exist positive constants* $M$ *and* $\alpha$ *such that*

$$\left|X(t,s)\right| \le M e^{-\alpha(t-s)}. \tag{1.2.12}$$

*Proof* Since $u(t) = -\lambda > 0$ is a positive solution of (1.2.7), by Theorem 1.1 the fundamental function of (1.2.3) is positive. Positivity of all the coefficients implies

$$\dot{X}(t,s) = -\sum_{k=1}^{m} a_k X(t-\tau_k, s) \le 0,$$

so the fundamental function is nonincreasing in $t$. Thus $X(t,s) \le 1$, $t \in [s, s+\tau_{\max}]$, where

$$\tau_{\max} = \max_k \tau_k,$$

and for $t \in [s+\tau_{\max}, s+2\tau_{\max}]$ we have

$$\dot{X}(t,s) = -\sum_{k=1}^{m} a_k X(t-\tau_k, s) < -\sum_{k=1}^{m} a_k X(t,s),$$

and hence

$$X(t,s) \le \exp\left\{ -\sum_{k=1}^{m} a_k (t - \tau_{\max} - s) \right\}.$$

We can continue by induction, deducing that

$$\begin{aligned} \dot{X}(t,s) &= -\sum_{k=1}^{m} a_k X(t-\tau_k, s) \\ &< -\sum_{k=1}^{m} a_k X(t,s), \;\; t \in \big[s + n\tau_{\max}, s + (n+1)\tau_{\max}\big]. \end{aligned}$$

Thus (1.2.12) is satisfied with

$$\alpha = \sum_{k=1}^{m} a_k, \;\; M = \exp\left\{ \sum_{k=1}^{m} a_k \tau_{\max} \right\},$$

which completes the proof. □

For the autonomous equation with one delay,

$$\dot{x}(t) + a x(t-\tau) = 0, \tag{1.2.13}$$

inequality (1.2.8) is necessary and sufficient for nonoscillation, which implies the following result.

**Corollary 1.1** *Let $a > 0$, $\tau > 0$. Then the following hypotheses are equivalent*:

1) *Equation (1.2.13) is nonoscillatory.*
2) *The characteristic equation*

$$\lambda + a e^{-\tau\lambda} = 0 \tag{1.2.14}$$

*has a real root.*
3) $a\tau \le 1/e$.

Theorem 1.1 can be applied to compare oscillation properties of two different autonomous equations as well as two different solutions of the same equation. Consider together with (1.2.3) the equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k x(t - \sigma_k) = 0. \tag{1.2.15}$$

**Theorem 1.4** *Suppose $a_k > 0, a_k \geq b_k, 0 \leq \sigma_k \leq \tau_k, k = 1, \cdots, m$ and there exists a negative root $\lambda_0$ of (1.2.9). Then (1.2.15) has a positive solution, and its fundamental function is positive.*

*Proof* First, we refer only to positive coefficients in (1.2.15). Let $b_k > 0$, $k = 1, \cdots, l$, $l \leq m$, and $b_k < 0$, $k = l+1, \cdots, m$ (this includes the case $m = l$ when all coefficients are positive). Consider the continuous function

$$f(\lambda) = \lambda + \sum_{k=1}^{l} b_k e^{-\sigma_k \lambda}.$$

Then $f(0) > 0$ and for $\lambda_0 < 0$ the following inequality is valid:

$$f(\lambda_0) = \lambda_0 + \sum_{k=1}^{l} b_k e^{-\sigma_k \lambda_0} \leq \lambda_0 + \sum_{k=1}^{m} a_k e^{-\tau_k \lambda_0} = 0.$$

Thus there exists $\lambda_1$, $\lambda_0 \leq \lambda_1 < 0$ that is a solution of the characteristic equation $f(\lambda) = 0$. By Theorem 1.2, the fundamental function $Z(t, s)$ of the equation

$$\dot{z}(t) + \sum_{k=1}^{l} b_k z(t - \sigma_k) = 0 \tag{1.2.16}$$

is positive, and (1.2.16) has an eventually positive solution. If $l = m$, the theorem is proven.

Now suppose $l < m$. Since (1.2.15) can be rewritten as the equation with the right-hand side

$$\dot{x}(t) + \sum_{k=1}^{l} b_k x(t - \sigma_k) = \sum_{k=l+1}^{m} (-b_k) x(t - \sigma_k),\ b_k < 0,\ k = l+1, \cdots, m,$$

the solution representation formula (1.2.5) yields that the fundamental function $x(t) = X(t, s)$ of (1.2.15) satisfies

$$x(t) = Z(t, s) - \sum_{k=l+1}^{m} b_k \int_s^t Z(t, \zeta) x(\zeta - \sigma_k)\, d\zeta,\ x(s) = 1,\ x(\zeta) = 0,\ \zeta < s,$$

where $b_k < 0$, $k = l+1, \cdots, m$. Assuming that there exist points where $x(t) \leq 0$ and recalling $x(s) = 1$, we choose a point $t^*$ such that $x(t) > 0$ for $s \leq t < t^*$ and $x(t^*) = 0$.

Then $x(t) \geq 0$ for $t \leq t^*$ and

$$x\left(t^*\right) = Z(t,s) + \sum_{k=l+1}^{m} (-b_k) \int_s^t Z(t,\zeta)x(\zeta - \sigma_k)\,d\zeta > 0$$

as a sum of two positive terms, which contradicts the assumption $x(t^*) = 0$. Thus $X(t,s) > 0$, which completes the proof. □

**Corollary 1.2** *Suppose* $a_k > 0$, $k = 1, \cdots, l$, $a_k < 0$, $k = l+1, \cdots, m$ *and any of the following conditions holds*:

1. *Equation*

$$\dot{x}(t) + \sum_{k=1}^{l} a_k x(t - \tau_k) = 0$$

*has a positive solution*.

2. *Inequality*

$$\dot{x}(t) + \sum_{k=1}^{l} a_k x(t - \tau_k) \leq 0$$

*has a positive solution*.

3. *The characteristic equation*

$$\lambda + \sum_{k=1}^{l} a_k e^{-\tau_k \lambda} = 0 \tag{1.2.17}$$

*has a negative root*.

4. *The inequality*

$$\sum_{k=1}^{l} a_k \max_{k=1,\cdots,l} \tau_k \leq \frac{1}{e}$$

*is fulfilled*.

*Then* (1.2.3) *has a positive solution*, *and its fundamental function is positive*.

Corollary 1.2 is applicable to equations with positive and negative coefficients, but the requirement that the characteristic equation (1.2.17) have a negative root is rather restrictive. For the equation with a positive and a negative coefficient

$$\dot{x}(t) + ax(t - \tau) - bx(t - \sigma) = 0,\ a, b > 0,\ \tau, \sigma \geq 0, \tag{1.2.18}$$

the following result is valid; see the argument above and Chap. 3.

**Theorem 1.5** *Equation* (1.2.18) *has a nonoscillatory solution if and only if its characteristic equation*

$$\lambda + ae^{-\tau\lambda} - be^{-\sigma\lambda} = 0 \tag{1.2.19}$$

*has a real root*.

*If $a > b$ and $\tau > \sigma$, then the following hypotheses are equivalent*:

1. *Equation* (1.2.18) *has a positive decreasing solution.*
2. *There exists a positive root $u$ of the inequality*

$$u \geq ae^{\tau u} - be^{\sigma u}. \tag{1.2.20}$$

3. *There exists a negative root of the characteristic equation* (1.2.19).
4. *The fundamental function of* (1.2.18) *is positive.*
5. *Inequality*

$$\dot{x}(t) + ax(t-\tau) - bx(t-\sigma) \leq 0$$

*has a positive decreasing solution.*

Let us now compare two solutions of different autonomous delay equations with the same delays. To this end, consider together with problem (1.2.1), (1.2.2) the nonhomogeneous equation

$$\dot{y}(t) + \sum_{k=1}^{m} b_k y(t-\tau_k) = g(t), \; t \geq t_0, \tag{1.2.21}$$

with the initial conditions

$$y(t) = \psi(t), \; t < t_0, \; y(t_0) = y_0. \tag{1.2.22}$$

**Theorem 1.6** *If $a_k > 0$, $k = 1, \cdots, m$, there exists a negative solution $\lambda_0$ of* (1.2.9),

$$a_k \geq b_k \geq 0, \; g(t) \geq f(t), \; \varphi(t) \geq \psi(t), \; t < t_0, \; y_0 \geq x_0, \tag{1.2.23}$$

*and the solution of* (1.2.1), (1.2.2) *is positive* ($x(t) > 0$), *then* $y(t) \geq x(t) > 0$.

*Proof* By Theorem 1.4, the fundamental functions $X(t,s)$ of (1.2.1) and $Y(t,s)$ of (1.2.21) are both positive. If we rewrite (1.2.1) in the form

$$\dot{x}(t) + \sum_{k=1}^{m} b_k x(t-\tau_k) = f(t) + \sum_{k=1}^{m} [b_k - a_k] x(t-\tau_k),$$

then by the solution representation formula, (1.2.23) and positivity of $Y(t,s)$ and $x(t)$ we have

$$\begin{aligned} x(t) &= Y(t,t_0)x_0 - \sum_{k=1}^{m} b_k \int_{t_0}^{t} Y(t,s)\varphi(s-\tau_k)\,ds \\ &\quad + \int_{t_0}^{t} Y(t,s)f(s)\,ds - \sum_{k=1}^{m}(a_k - b_k)\int_{t_0}^{t} Y(t,s)x(s-\tau_k)\,ds \\ &\leq Y(t,t_0)y_0 - \sum_{k=1}^{m} b_k \int_{t_0}^{t} Y(t,s)\psi(s-\tau_k)\,ds + \int_{t_0}^{t} Y(t,s)g(s)\,ds = y(t), \end{aligned}$$

and thus $y(t) \geq x(t) > 0$, which completes the proof. □

Similar methods are applicable to vector delay differential equations and systems of differential equations; see Chap. 9.

## 1.3 Nonlinear Equations of Mathematical Biology

### 1.3.1 Linearization of Nonlinear Delay Equations

In this section, we consider autonomous equations of mathematical biology (mainly population ecology). For these equations, initial conditions are usually assumed to be nonnegative with a positive initial value. Typically, solutions of these equations are positive, and oscillation about a unique positive equilibrium is considered rather than oscillation about zero.

Many autonomous equations of mathematical biology after some transformations (the positive equilibrium is shifted to zero) can be written in the form

$$\dot{x}(t) + \sum_{k=1}^{m} a_k f_k\big(x(t-\tau_k)\big) = 0, \tag{1.3.1}$$

where

$$a_k > 0, \tau_k \geq 0, f_k : \mathbb{R} \to \mathbb{R} \text{ are continuous}, \; u f_k(u) > 0, \; u \neq 0, \; k = 1, \cdots, m \tag{1.3.2}$$

and

$$\lim_{u \to 0} \frac{f_k(u)}{u} = 1, \; k = 1, \cdots, m. \tag{1.3.3}$$

The linearization method deduces nonoscillation and oscillation properties of the nonlinear equation (1.3.1) from the relevant properties of the linear equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t-\tau_k) = 0 \tag{1.3.4}$$

or of the linear equations with smaller proportional coefficients.

The following results are valid.

**Theorem 1.7** *If* (1.3.2) *is satisfied, then any nonoscillatory solution of* (1.3.1) *tends to zero.*

**Theorem 1.8** *Assume that conditions* (1.3.2) *and* (1.3.3) *hold.*

a) *If for some* $\varepsilon > 0$ *all solutions of the equation*

$$\dot{x}(t) + (1-\varepsilon) \sum_{k=1}^{m} a_k x(t-\tau_k) = 0 \tag{1.3.5}$$

*are oscillatory, then all solutions of* (1.3.1) *are also oscillatory.*

b) *If either* $0 < f_k(u) \leq u$ *for* $u > 0$ *or* $0 > f_k(x) \geq x$ *for* $x < 0$ *and there exists a nonoscillatory solution of* (1.3.4), *then* (1.3.1) *also has a nonoscillatory solution.*

For the proof of a more general result, see Chap. 10.

Some sharper oscillation results can be obtained specifically for autonomous equations; see [192, Sect. 4.1].

### 1.3.2 Hutchinson's Equation

The logistic equation

$$\dot{N}(t) = rN(t)\left[1 - \frac{N(t)}{K}\right] \tag{1.3.6}$$

is one of the most common models of population dynamics. Here $N > 0$ is the population size (or density, or a biomass), $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity of the environment. The model assumes that the per capita growth rate $\frac{1}{N(t)}\frac{dN}{dt}$ is a linear decreasing function. All positive solutions of (1.3.6) are monotone and converge to the positive equilibrium $K$, which is in contrast to ecological observations: oscillations frequently occur in nature. To incorporate oscillations in the model, Hutchinson [209] proposed the delay modification

$$\frac{dN}{dt} = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right], \tag{1.3.7}$$

where the per capita growth rate depends on the population size $\tau$ units of time ago. Hutchinson's equation (1.3.7) was studied in [213, 214, 334]; see also [192].

After the substitution

$$N(t) = Ke^{x(t)}, \tag{1.3.8}$$

the delay logistic equation (1.3.7) becomes

$$\dot{x}(t) + r\left[e^{x(t-\tau)} - 1\right] = 0, \tag{1.3.9}$$

where (1.3.9) oscillates if and only if (1.3.7) oscillates about $K$. Since $f(x) = e^x - 1$ is a continuous function satisfying the conditions in (1.3.2) and (1.3.3) and $f(x) \geq x$ for $x < 0$, Theorem 1.8 implies the following nonoscillation criterion for (1.3.7).

**Theorem 1.9** *The following hypotheses are equivalent*:

1. *Equation* (1.3.7) *has a solution nonoscillatory about* $K$.
2. *The characteristic equation of the linearized equation*

$$\lambda + re^{-\lambda\tau} = 0 \tag{1.3.10}$$

   *has a real root.*
3. *The following condition is fulfilled*:

$$r\tau \leq \frac{1}{e}. \tag{1.3.11}$$

There are numerous modifications of Hutchinson's equation (1.3.7).

1. First, the per capita growth rate may negatively depend on the population size not only at a certain moment in the past but on several past moments,

$$\frac{dN}{dt} = N(t)\left[\alpha - \sum_{k=1}^{m} \beta_k N(t-\tau_k)\right], \ \alpha, \beta_k > 0, \ \tau_k \geq 0, \ k = 1, \cdots, m; \tag{1.3.12}$$

see [192]. Then the linearized version of (1.3.12) is

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t - \tau_k) = 0, \tag{1.3.13}$$

where the positive equilibrium $N^*$ and coefficients $a_k$ are defined as

$$N^* = \frac{\alpha}{\sum_{k=1}^{m} \beta_k}, \ a_k = \beta_k N^*. \tag{1.3.14}$$

Equation (1.3.12) has solutions nonoscillatory about $N^*$ if and only if the characteristic equation of (1.3.13),

$$\lambda + \sum_{k=1}^{m} a_k e^{-\tau_k \lambda} = 0,$$

has a real root.

2. The logistic equation with several delays (1.3.12) also has the multiplicative version

$$\dot{N}(t) = rN(t) \prod_{k=1}^{m} \left(1 - \frac{N(t-\tau_k)}{K}\right) \left|1 - \frac{N(t-\tau_k)}{K}\right|^{\alpha_k - 1}, \tag{1.3.15}$$

where

$$r > 0, K > 0, \ \alpha_k > 0, \ k = 1, \cdots, m, \ \sum_{k=1}^{m} \alpha_k = 1,$$

and the right-hand side of (1.3.15) is equal to zero if $N(t - \tau_k) = K$. After the substitution

$$x(t) = \frac{N(t)}{K} - 1, \tag{1.3.16}$$

(1.3.15) becomes

$$\dot{x}(t) = -r\big(1 + x(t)\big) \prod_{k=1}^{m} x(t - \tau_k) \big|x(t - \tau_k)\big|^{\alpha_k - 1}. \tag{1.3.17}$$

An equation more general than (1.3.17) will be studied in Chap. 11.

3. The logistic equation (1.3.6) is often criticized since a linear per capita growth function does not match real ecological phenomena. It was suggested that a function more general than $1 - \frac{N}{K}$ should be chosen for this purpose. For example, the "food-limited" equation

$$\dot{N}(t) = rN(t) \frac{K - N(t)}{K + crN(t)} \tag{1.3.18}$$

was introduced in [319]. Similar to (1.3.6), all positive solutions of (1.3.18) converge monotonically to the equilibrium $K$. In contrast to (1.3.18), solutions of the time-delayed "food-limited" equation

$$\dot{N}(t) = rN(t) \frac{K - N(t-\tau)}{K + crN(t-\tau)} \tag{1.3.19}$$

may oscillate about the positive equilibrium $K$.

After the substitution (1.3.8), equation (1.3.19) has the form

$$\dot{x}(t)(t) + r\frac{e^{x(t-\tau)} - 1}{1 + cre^{x(t-\tau)}} = 0, \tag{1.3.20}$$

which is nonoscillatory if and only if its linearized equation

$$\dot{x}(t)(t) + \frac{r}{1+cr}x(t-\tau) = 0 \tag{1.3.21}$$

is nonoscillatory. Thus, (1.3.19) has a solution that does not oscillate about its positive equilibrium $K$ if and only if

$$\frac{r\tau}{1+cr} \le \frac{1}{e}; \tag{1.3.22}$$

see [168].

4. The generalization of the logistic model

$$\dot{N}(t) = rN(t)\left(1 - \frac{N(t)}{K}\right)^{\alpha} \tag{1.3.23}$$

has the delay analogue

$$\dot{N}(t) = rN(t)\sum_{k=1}^{m}\left(1 - \frac{N(t-\tau_k)}{K}\right)\left|1 - \frac{N(t-\tau_k)}{K}\right|^{\alpha_k - 1}, \tag{1.3.24}$$

where

$$r > 0,\ K > 0,\ \alpha_k > 0,\ k = 1, \cdots, m,$$

and the right-hand side of (1.3.24) is equal to zero if $N(t-\tau_k) = K$.

After the substitution (1.3.16), equation (1.3.24) becomes

$$\dot{x}(t) = -r\big(1 + x(t)\big)\sum_{k=1}^{m} x(t-\tau_k)\big|x(t-\tau_k)\big|^{\alpha_k - 1}. \tag{1.3.25}$$

An equation more general than (1.3.25) will be studied in Chap. 11.

### *1.3.3 Lasota-Wazewska Equation*

The delay differential equation

$$\dot{N}(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)},\ t \ge 0 \tag{1.3.26}$$

was introduced by Lasota and Wazewska-Czyzewska [331] to describe the survival of red blood cells in an animal. Here $N(t)$ is the number of red blood cells at time $t$, $\mu$ is the per capita death rate (the probability of death for blood cells that currently circulate) and $p$ and $\gamma$ define red blood cell production functions: $p$ can be described as the production limit when the number of cells tends to zero, the decay of cell production for large cell number becomes faster with the growth of $\gamma$ and delay $\tau$ is the time required to produce a red blood cell.

Equation (1.3.26) is considered with a nonnegative initial function and a positive initial value

$$N(t) = \varphi(t),\ \varphi(t) \ge 0,\ -\tau \le t < 0,\ N(0) = N_0 > 0. \tag{1.3.27}$$

Evidently (1.3.26) with initial condition (1.3.27) has a unique solution that is positive for any $t \ge 0$; moreover, it exceeds

$$z(t) = N_0 e^{-\mu t},$$

which is the solution of (1.3.26) with $p = 0$.

Equation (1.3.26) has a positive equilibrium $N^*$ that satisfies the equation

$$N^* = \frac{p}{\mu} e^{-\gamma N^*}. \tag{1.3.28}$$

After the change of variables

$$N(t) = N^* + \frac{1}{\gamma} x(t),$$

(1.3.26) becomes

$$\dot{x}(t) + \mu x(t) + \mu\gamma N^* \left[1 - e^{-x(t-\tau)}\right] = 0. \tag{1.3.29}$$

Then the following nonoscillation result for (1.3.26) is valid [192, 238].

**Theorem 1.10** *The following hypotheses are equivalent*:

1. *Equation* (1.3.26) *has a solution nonoscillatory about* $N^*$.
2. *The characteristic equation of the linearized equation*

$$\lambda + \mu + \mu\gamma N^* e^{-\lambda\tau} = 0 \tag{1.3.30}$$

*has a real root*.

3. *The inequality*

$$\mu\tau\gamma N^* e^{\mu\tau} \le \frac{1}{e} \tag{1.3.31}$$

*is fulfilled*.

We will not present any justification for this theorem since a proof of a more general result is included in Chap. 10.

## *1.3.4 Nicholson's Blowflies Equation*

Nicholson's blowflies equation

$$\dot{N}(t) = -\delta N(t) + pN(t-\tau)e^{-aN(t-\tau)} \tag{1.3.32}$$

was used in [177] to describe periodic oscillations in Nicholson's classic experiments [291] with the Australian sheep blowfly, *Lucila cuprina*. Here $N(t)$ is the size of the blowfly population at time $t$, $p$ is the maximum per capita daily egg production rate, $\frac{1}{a}$ is the population size at which this maximum reproduction rate is attained, $\delta$ is the per capita daily adult death rate and $\tau$ is the maturation time from an egg to a blowfly.

Equation (1.3.32) is considered with a nonnegative initial function and a positive initial value

$$N(t) = \varphi(t),\ \varphi(t) \ge 0,\ -\tau \le t < 0,\ N(0) = N_0 > 0. \tag{1.3.33}$$

Evidently (1.3.32) with initial condition (1.3.33) has a unique solution that is positive for any $t \ge 0$; moreover, it exceeds

$$z(t) = N_0 e^{-\delta t},$$

which is the solution of (1.3.32) with $p = 0$.

If $p \le \delta$, then all solutions of (1.3.32) tend to zero, which is the only equilibrium.

If $p > \delta$, then there is a positive equilibrium

$$N^* = \frac{1}{a} \ln\left(\frac{p}{\delta}\right). \tag{1.3.34}$$

After the transformation

$$N = N^* + \frac{1}{a} x, \tag{1.3.35}$$

(1.3.32) becomes

$$\dot{x}(t) + \delta x(t) - \delta x(t-\tau) e^{-x(t-\tau)} + \delta \ln\left(\frac{p}{\delta}\right)\left[1 - e^{-x(t-\tau)}\right] = 0, \tag{1.3.36}$$

where solution $N(t)$ of (1.3.32) oscillates about $N^*$ if and only if $x$ oscillates about zero.

For $\delta < p \le \delta e$, all solutions such that the initial function does not oscillate about $N^*$ are also nonoscillatory. This means that oscillation and nonoscillation properties of the solution depend on the initial function only. With appropriate initial functions, there are an infinite number of oscillatory solutions of (1.3.32); see [195] for the original result and also [86] for some generalizations.

In the case $p > \delta e$, the following oscillation and nonoscillation result for (1.3.32) is valid [192, 238].

**Theorem 1.11** *Let $p > \delta e$.*

1. *If*

$$\delta \tau e^{\delta \tau} \left[\ln\left(\frac{p}{\delta}\right) - 1\right] > \frac{1}{e}, \tag{1.3.37}$$

*then any solution of* (1.3.32) *with initial condition* (1.3.33) *oscillates about $N^*$.*

2. *If in addition $p > \delta e^2$, then any solution of* (1.3.32), (1.3.33) *oscillates about $N^*$ if and only if* (1.3.37) *is satisfied.*

A proof of a more general result is included in Chap. 10.

### 1.3.5 Mackey-Glass Equations

Let us demonstrate that the linearization technique presented above does not work for all equations of population ecology.

In [272, 279, 280], the delay equation (the Mackey-Glass, or the hematopoiesis, equation)

$$\dot{N} = \frac{aN(t-\tau)}{1+N(t-\tau)^{\gamma}} - bN(t), \tag{1.3.38}$$

where

$$a > b > 0, \ \gamma > 0, \ \tau \ge 0,$$

was applied to model white blood cell production. Here $N(t)$ is the density of mature cells in blood circulation, the function $\frac{rN_\tau}{1+N_\tau^\gamma}$ models blood cell reproduction, the time lag $\tau$ describes the maturational phase before blood cells are released into circulation and the mortality rate $bN$ was assumed to be proportional to the circulation. Equation (1.3.38) was introduced to explain the oscillations in the number of neutrophils observed in some cases of chronic myelogenous leukemia [272, 279]. Let us note that another model introduced in [279] to describe red blood cell production is described by the equation

$$\dot{N} = \frac{a}{c+N(t-\tau)^{\gamma}} - bN(t). \tag{1.3.39}$$

Equation (1.3.38) has the unique positive equilibrium

$$N^* = \left(\frac{a}{b} - 1\right)^{1/\gamma}. \tag{1.3.40}$$

After the transformation $y = N - N^*$, (1.3.38) has the form

$$\dot{y}(t) = \frac{a[y(g(t)) + N^*]}{1+(y(g(t)) + N^*)^{\gamma}} - by(t) - bN^*,$$

which can be rewritten as

$$\dot{y}(t) + by(t) - b\left[\frac{a[y(g(t)) + N^*]}{b[1+(y(g(t)) + N^*)^{\gamma}]} - N^*\right] = 0. \tag{1.3.41}$$

It is possible to check that the standard linearization techniques cannot be applied to (1.3.41). Some oscillation results can be found in [60]. For linearization methods, (1.3.2) and (1.3.3) are usually fulfilled for the appropriate choice of functions $f_k$, which allows us to obtain sufficient oscillation conditions for nonlinear models. However, the condition

$$\text{either } 0 < f_k(u) \le u, \ u > 0, \text{ or } u \le f_k(u) < 0, \ u < 0, \tag{1.3.42}$$

is more restrictive. Nicholson's blowflies equation for $P/\delta < e^2$ and the Mackey-Glass equation are examples of the models for which (1.3.42) is not satisfied. Some alternative methods were developed to study nonoscillation of nonlinear delay equations. For example, the fixed-point theory is widely used to establish existence of positive solutions, see Chap. 11.

## 1.4 Impulsive Equations

The theory of impulsive equations goes back to 1960, when it was first introduced by Milman and Myshkis [287]. Since then, impulsive equations have been intensively studied, especially in the last three decades. To illustrate the oscillation theory for impulsive differential equations, consider the autonomous equation with one delay and constant impulsive perturbations applied at time distances equal to the delay

$$\dot{x}(t) + ax(t-\tau) = 0, \ t \geq 0, \tag{1.4.1}$$

$$\Delta x(n\tau) := x\big((n\tau)^+\big) - x(n\tau) = bx(n\tau), \ n = 0, 1, \cdots, \tag{1.4.2}$$

where

$$a > 0, \ \tau \geq 0, \ b > -1. \tag{1.4.3}$$

A solution of impulsive equation (1.4.1), (1.4.2) is a function that is absolutely continuous on $(n\tau, (n+1)\tau)$, $n = 0, 1, \cdots$, left continuous at $t = n\tau$ and satisfies (1.4.2),

$$x\big((n\tau)^+\big) = (1+b)x(n\tau).$$

Let us introduce the *continuous* function

$$y(t) = (1+b)^{-([t/\tau]+1)}x(t), \ t \geq 0, \tag{1.4.4}$$

where $x(t)$ is a solution of (1.4.1), (1.4.2) and $[s]$ is the integer part of $s$.

Hence

$$x(t) = (1+b)^{[t/\tau]+1}y(t),$$
$$\dot{x}(t) = (1+b)^{[t/\tau]+1}\dot{y}(t) = -ax(t-\tau) = -a(1+b)^{[t/\tau]}y(t-\tau),$$

which implies

$$\dot{y}(t) + \frac{a}{1+b}y(t-\tau) = 0. \tag{1.4.5}$$

By (1.4.3) and (1.4.4), $y(t)$ is nonoscillatory if and only if $x(t)$ does not oscillate. This means that oscillation properties of impulsive equations (1.4.1), (1.4.2) are equivalent to the oscillation properties of (1.4.5) without impulses, which yields the following result.

**Theorem 1.12** *Let $b > -1$. Impulsive equation* (1.4.1), (1.4.2) *has a nonoscillatory solution if and only if* (1.4.5) *is nonoscillatory.*

Applying Corollary 1.1, we immediately obtain the following result.

**Theorem 1.13** *Impulsive equation* (1.4.1), (1.4.2) *has a nonoscillatory solution if and only if*

$$\frac{a\tau}{1+b} \leq \frac{1}{e}. \tag{1.4.6}$$

Consider the case where the distance between impulses is $k$ times smaller than the delay:

$$\Delta x\left(n\frac{\tau}{k}\right) = bx\left(n\frac{\tau}{k}\right), \; n = 0, 1, \cdots, \; k \in \mathbb{N}. \tag{1.4.7}$$

Then the substitution

$$y(t) = (1+b)^{-([tk/\tau]+1)}x(t), \; t \ge 0, \tag{1.4.8}$$

leads to the nonimpulsive equation for $y(t)$

$$\dot{y}(t) + a(1+b)^{-k}y(t-\tau) = 0, \tag{1.4.9}$$

which implies the following nonoscillation criterion.

**Theorem 1.14** *Impulsive equation* (1.4.1), (1.4.7) *has a nonoscillatory solution if and only if*

$$\frac{a\tau}{(1+b)^k} \le \frac{1}{e}. \tag{1.4.10}$$

If we assume that the distance between impulses is $k$ times greater than the delay

$$\Delta x(nk\tau) = bx(nk\tau), \; n = 0, 1, \cdots, \; k \in \mathbb{N}, \tag{1.4.11}$$

then $y(t)$ defined as

$$y(t) = (1+b)^{-([t/(\tau k)]+1)}x(t), \; t \ge 0, \tag{1.4.12}$$

satisfies

$$\dot{y}(t) + \frac{a}{1+b}y(t-\tau) = 0 \text{ if } t \in \big(nk\tau, (nk+1)\tau\big], \; n = 0, 1, \cdots, \tag{1.4.13}$$

and

$$\dot{y}(t) + ay(t-\tau) = 0 \text{ if } t \notin \big(nk\tau, (nk+1)\tau\big], \; n = 0, 1, \cdots. \tag{1.4.14}$$

Overall, if we define

$$c(t) = \begin{cases} a/(1+b) & \text{if } t \in (nk\tau, (nk+1)\tau], \\ a & \text{if } t \in ((nk+1)\tau, (n+1)k\tau], \end{cases} \quad n = 0, 1, \cdots, \tag{1.4.15}$$

then $y(t)$ satisfies the equation

$$\dot{y}(t) + c(t)y(t-\tau) = 0 \tag{1.4.16}$$

with a variable discontinuous coefficient. The sufficient nonoscillation condition for this equation,

$$a\tau \max\left\{1, \frac{1}{1+b}\right\} \le \frac{1}{e}, \tag{1.4.17}$$

can be deduced from the comparison result similar to Theorem 1.4 (see Chap. 2).

This model partially justifies our interest in equations with generally discontinuous coefficients. For most equations considered in this monograph, coefficients are not assumed to be continuous but just Lebesgue measurable, locally bounded functions.

Chapters 12–14 deal with impulsive equations: Chap. 12 with first-order linear equations, Chap. 13 with second-order linear equations and Chap. 14 with nonlinear equations. For second-order equations, the first derivative can be subject to impulsive conditions or the solution, or both.

The main method of these chapters is based on the reduction of impulsive delay differential equations to delay differential equations without impulses that have an explicit form, as was illustrated above for simple autonomous impulsive equations.

## 1.5 Some Other Classes of Equations

In order to illustrate models and ideas of the present monograph, let us introduce autonomous analogues of other types of equations that will be considered later:

a) the integrodifferential equation with an infinite delay

$$\dot{x}(t) + \int_{-\infty}^{t} K(t-s)x(s)\,ds = 0; \tag{1.5.1}$$

b) the integrodifferential equation with bounded delays

$$\dot{x}(t) + \sum_{j=1}^{m} \int_{t-\tau_j}^{t} K_j(t-s)x(s)\,ds = 0; \tag{1.5.2}$$

c) the equation with an infinite number of delays

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k x(t-\tau_k) = 0; \tag{1.5.3}$$

d) the equation with a distributed infinite delay

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_s R(t-s) = 0; \tag{1.5.4}$$

e) the equation with distributed bounded delays

$$\dot{x}(t) + \sum_{k=1}^{m} \int_{t-\tau_k}^{t} x(s)\,d_s R_k(t-s) = 0; \tag{1.5.5}$$

f) the neutral differential equation

$$\dot{x}(t) - b\dot{x}(t-\sigma) + \sum_{k=1}^{\infty} a_k x(t-\tau_k) = 0. \tag{1.5.6}$$

For (1.5.1)–(1.5.6), oscillation results similar to Theorems 1.1–1.6 can be obtained. For example, the analogue of Theorem 1.1 for (1.5.1) can be formulated as follows.

**Theorem 1.15** *Suppose $K(u) \geq 0$ is integrable on $[0, \infty)$. Then the following hypotheses are equivalent*:

1) *The inequality*

$$\dot{y}(t) + \int_{-\infty}^{t} K(t-s)y(s)\,ds \leq 0$$

*has an eventually positive solution.*

2) *There exists $t_1 \geq 0$ such that the inequality*

$$u(t) \geq \int_{-\infty}^{t} K(t-s)e^{\int_s^t u(\tau)\,d\tau} ds, \; t \geq t_1$$

*has a locally integrable nonnegative solution $u(t)$, where we assume $u(t) = 0$ for $t < t_1$.*

3) *There exists $t_1 \geq 0$ such that the fundamental function of* (1.5.1) *is positive*: $X(t,s) > 0$ *for* $t \geq s \geq t_1$.

4) *Equation* (1.5.1) *has a nonoscillatory solution.*

More general results for nonautonomous equations (1.5.1)–(1.5.6) are presented in Chaps. 4 and 6.

In Chap. 5, we consider linear equations of advanced and mixed types. The former type involves an advanced $x(t+\sigma)$, $\sigma > 0$, rather than a delayed argument, while the latter model includes both delayed and advanced terms and in the autonomous case has the form

$$\dot{x}(t) + ax(t-\tau) + bx(t+\sigma) = 0, \; \tau > 0, \; \sigma > 0. \tag{1.5.7}$$

As a corollary of the results of Chap. 5, we have the following theorem.

**Theorem 1.16**

1. *Suppose $a > 0$, $b > 0$, $a\tau e^{b\tau} \leq 1/e$. Then* (1.5.7) *has an eventually positive nonincreasing solution.*
2. *Suppose $a < 0$, $b < 0$, $b\sigma e^{a\sigma} \leq 1/e$. Then* (1.5.7) *has an eventually positive nondecreasing solution.*

In Chaps. 7 and 8, we consider linear second-order delay equations without and with damping. Second-order delay differential equations with damping have many applications; for example, they were applied by Minorski [288] in 1962 to the problem of stabilizing the rolling of a ship by the "activated tank method".

Chapter 9 deals with linear systems of delay differential equations; as corollaries, we obtain nonoscillation conditions for equations of higher order.

All the linear equations considered above, except the advanced and the mixed equations, can be written in the abstract form

$$\dot{x}(t) + (Hx)(t) = 0, \tag{1.5.8}$$

where $H$ is a linear bounded Volterra operator that acts from the space of locally absolutely continuous functions to the space of locally integrable functions. We say that the linear operator $H$ is a Volterra (or causal) operator if for any $b > 0$ equality $x(t) = 0$ for $t \in [0, b]$ implies $(Hx)(t) = 0$ for $t \in [0, b]$.

Linear and nonlinear equations with Volterra (causal) operators were studied in the monographs [29, 98, 239, 251] and the papers [97, 162]. In the present monograph, equations with linear Volterra operators are studied in Chaps. 15–17. In these chapters, we also consider some questions that are not considered in the previous chapters; for example, boundary value problems and positivity of the Green's functions for such problems.

## 1.6 Discussion and Open Problems

In this chapter, we presented an overview of the main nonoscillation results and methods for scalar delay autonomous equations and introduced most types of equations that will be considered later in the monograph: linear equations, nonlinear equations of mathematical biology, impulsive equations and equations with a distributed delay.

Finally, we outline some open problems and topics for research and discussion. The study of autonomous equations is not the main purpose of the present monograph, so we omit here the references to the recent results in this area, leaving the review of the literature to the reader.

1. The inequalities

$$\sum_{k=1}^{m} a_k \max_k \tau_k < \frac{1}{e} \text{ and } \sum_{k=1}^{m} a_k \min_k \tau_k > \frac{1}{e} \tag{1.6.1}$$

provide sufficient nonoscillation and oscillation conditions, respectively, for autonomous equation (1.2.3). Consider the case where both inequalities in (1.6.1) are not satisfied, and deduce explicit necessary and/or sufficient nonoscillation and oscillation conditions.

2. Is it possible to develop a general approach to deduce nonoscillation of nonlinear autonomous equations when the linearization technique cannot be applied; for example, for the Mackey-Glass equation?

   In certain cases, using linearization, nonoscillation of nonlinear equations can be established as existence of nonoscillatory solutions exceeding the equilibrium or smaller than the equilibrium if only one of the conditions $0 < f_k(u) \le u$ for $u > 0$ and $0 > f_k(x) \ge x$ for $x < 0$ holds. For example, for Nicholson's blowflies equation, only the existence of a solution $0 < x(t) < K$ can be established. Consider existence of nonoscillatory solutions in the intervals where the linearization technique is not applicable.

3. Deduce explicit nonoscillation and oscillation conditions for autonomous equations with a distributed delay. The results for some types of kernels with finite and infinite delays are presented in Chap. 4.
4. Extend linearization theory to study nonoscillation of (1.3.1) with positive and negative coefficients.

5. Extend linearization theory to study nonoscillation of the equation with several delays

$$\dot{x}(t) + f\big(x(t-\tau_1), \cdots, x(t-\tau_m)\big) = 0.$$

6. Suppose that the conditions of Theorem 1.8, Part b, hold. Prove or disprove that any solution (not necessarily positive) of this equation tends to zero.
7. Suppose that the conditions of Theorem 1.8, Part b, hold. Prove or disprove that the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k f_k\big(x(t-\tau_k)\big) - \sum_{k=1}^{l} b_k g_k\big(x(t-\delta_k)\big) = 0$$

has a nonoscillatory solution, where $b_k > 0$, $x g_k(x) > 0$, $x \neq 0$.
8. Compare nonoscillation and oscillation properties of (1.3.1) and the equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k g_k\big(x(t-\delta_k)\big) = 0.$$

9. Compare positive solutions of (1.3.1) and the corresponding differential inequality.
10. Suppose that conditions of Theorem 1.8, Part b, hold. Prove or disprove that if $\varphi(t) < x(t_0)$ and $x(t_0) > 0$, then the solution of the initial value problem for (1.3.1) is positive for $t \geq t_0$.

# Chapter 2
# Scalar Delay Differential Equations on Semiaxes

## 2.1 Introduction

This chapter deals with nonoscillation properties of scalar differential equations with a finite number of delays. There are a lot of papers devoted to oscillation conditions for this class of equations. In comparison with oscillation, there are not so many results on nonoscillation of these equations, especially in monographs on oscillation theory. One of the aims of this chapter is to consider nonoscillation together with relevant problems: differential inequalities, comparison results, solution estimations, stability and so on. The second purpose is to derive some nonoscillation methods that will be used for other classes of functional differential equations. In particular, we apply a solution representation formula, so the most important nonoscillation property is positivity of the fundamental function of the considered equation.

The chapter is organized as follows. Section 2.2 contains relevant definitions and the solution representation formula. In Sect. 2.3, we prove that the following four assertions are equivalent: nonoscillation of the equation and the corresponding differential inequality, positivity of the fundamental function and existence of a nonnegative solution for a certain nonlinear integral inequality that is constructed explicitly from the differential equation. Section 2.4 involves comparison theorems that compare oscillation properties of various equations and also solutions of these equations. Next, in Sects. 2.5 and 2.6, explicit nonoscillation conditions for several classes of equations are considered. Section 2.7 includes several oscillation conditions that will be used in the following chapters. In Sect. 2.8, we obtain estimations for solutions and for the fundamental function of nonoscillatory equations. Section 2.9 presents conditions on initial functions and initial values that imply positivity of solutions. Section 2.10 considers slowly oscillating solutions. In Sect. 2.11, connection between nonoscillation and stability is established. Finally, Sect. 2.12 involves some discussion and open problems.

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## 2.2 Preliminaries

We consider the scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0, \ t \geq 0, \tag{2.2.1}$$

under the following conditions:

(a1) $a_k$, $k = 1, \cdots, m$, are Lebesgue measurable functions essentially bounded on each finite interval $[0, b]$.
(a2) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k = 1, \cdots, m$.

Together with (2.2.1), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = f(t), \ t \geq t_0, \tag{2.2.2}$$

$$x(t) = \varphi(t), \ t < t_0, \ x(t_0) = x_0. \tag{2.2.3}$$

We also assume that the following hypothesis holds:

(a3) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable function essentially bounded in each finite interval $[t_0, b]$, and $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

**Definition 2.1** A function $x : \mathbb{R} \to \mathbb{R}$ absolutely continuous on each interval $[t_0, b]$ is called *a solution* of problem (2.2.2), (2.2.3) if it satisfies (2.2.2) for almost all $t \in [t_0, \infty)$ and equalities (2.2.3) for $t \leq t_0$.

**Definition 2.2** For each $s \geq 0$, the solution $X(t, s)$ of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0, \ x(t) = 0, \ t < s, \ x(s) = 1, \tag{2.2.4}$$

is called *a fundamental function* of (2.2.1).

We assume $X(t, s) = 0$, $0 \leq t < s$.

Theorem B.1 implies the following result.

**Lemma 2.1** *Let* (a1)–(a3) *hold. Then there exists a unique solution of problem* (2.2.2), (2.2.3) *that has the form*

$$x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)f(s)ds - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)a_k(s)\varphi\big(h_k(s)\big)ds, \tag{2.2.5}$$

*where* $\varphi(h_k(s)) = 0$, *if* $h_k(s) > t_0$.

## 2.3 Nonoscillation Criteria

**Definition 2.3** We will say that (2.2.1) *has a positive solution* for $t_0 \geq 0$ if there exist an initial function $\varphi$ and a number $x_0$ such that the solution of initial value problem (2.2.2), (2.2.3) ($f \equiv 0$) is positive.

Consider together with (2.2.1) the delay differential inequality

$$\dot{y}(t) + \sum_{k=1}^{m} a_k(t) y\big(h_k(t)\big) \leq 0. \tag{2.3.1}$$

The following theorem establishes nonoscillation criteria.

**Theorem 2.1** *Suppose $a_k(t) \geq 0$, $k = 1, \cdots, m$. Then the following hypotheses are equivalent*:

1) *There exists $t_0 \geq 0$ such that* (2.3.1) *has a positive solution for $t_0 \geq 0$.*
2) *There exist a point $t_1 \geq 0$ and a locally essentially bounded function $u(t)$ nonnegative for $t \geq t_1$ and satisfying the inequality*

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}, \; t \geq t_1, \tag{2.3.2}$$

*where we assume $u(t) = 0$, $t < t_1$.*

3) *There exists $t_1 \geq 0$ such that $X(t,s) > 0$, $t \geq s \geq t_1$.*
4) *There exists $t_1 \geq 0$ such that* (2.2.1) *has a positive solution for $t \geq t_1$.*

*Proof* 1) $\Rightarrow$ 2) Let $y(t)$ be a positive solution of (2.3.1) for $t \geq t_0$. Without loss of generality, we can assume that $y(h_k(t)) > 0$, $t \geq t_0$. By (a2), there exists a point $t_1$ such that $h_k(t) \geq t_0$ if $t \geq t_1$, $k = 1, \cdots, m$.

Denote

$$u_1(t) = -\frac{d}{dt} \ln \frac{y(t)}{y(t_1)}, \; t \geq t_0.$$

Then

$$\begin{aligned} y(t) &= y(t_1) \exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\}, \\ y\big(h_k(t)\big) &= y(t_1) \exp\left\{-\int_{t_1}^{h_k(t)} u_1(s)ds\right\}, \\ \dot{y}(t) &= -u_1(t) y(t_1) \exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\}, \; t \geq t_1. \end{aligned} \tag{2.3.3}$$

We substitute (2.3.3) into (2.3.1) and obtain

$$-u_1(t) y(t_1) \exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\} + \sum_{k=1}^{m} y(t_1) a_k(t) \exp\left\{-\int_{t_1}^{h_k(t)} u_1(s)ds\right\} \leq 0.$$

Hence

$$-\exp\left\{-\int_{t_1}^{t} u_1(s)ds\right\} y(t_1)\left[u_1(t)-\sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u_1(s)ds\right\}\right] \le 0. \quad (2.3.4)$$

Since $y(t) > 0$ for $t \ge t_0$ and $a_k(t) \ge 0$, we have $y(t_1) > 0$ and

$$u_1(t) \ge \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u_1(s)ds\right\}, \; t \ge t_1. \quad (2.3.5)$$

After denoting

$$u(t) = \begin{cases} u_1(t), & t \ge t_1 \\ 0, & t < t_1, \end{cases}$$

(2.3.5) implies (2.3.2).

2) $\Rightarrow$ 3) **Step 1**. Consider the initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(h_k(t)\big) = f(t), \; t \ge t_1, \; x(t) = 0, \; t \le t_1. \quad (2.3.6)$$

Denote

$$z(t) = \dot{x}(t) + u(t)x(t), \quad (2.3.7)$$

where $x$ is the solution of (2.3.6) and $u$ is a nonnegative solution of (2.3.2). The assumption $x(t) = 0$, $t \le t_1$ implies $z(t) = 0$ for $t \le t_1$.

The solution $x(t)$ of (2.3.7) satisfies

$$x(t) = \int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds, \quad (2.3.8)$$

$$x\big(h_k(t)\big) = \int_{t_1}^{h_k(t)} \exp\left\{-\int_{s}^{h_k(t)} u(\tau)d\tau\right\} z(s)ds, \quad (2.3.9)$$

$$\dot{x}(t) = z(t) - u(t)\int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds. \quad (2.3.10)$$

After substituting (2.3.9) and (2.3.10) into the left-hand side of (2.3.6), we have

$$\begin{aligned} & z(t) - u(t)\int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds \\ & \quad + \sum_{k=1}^{m} \int_{t_1}^{h_k(t)} \exp\left\{-\int_{s}^{h_k(t)} u(\tau)d\tau\right\} z(s)ds \\ & = z(t) - \int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds \left[u(t) - \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right]. \end{aligned}$$

Hence (2.3.6) can be rewritten in the form

$$z - Hz = f, \quad (2.3.11)$$

where

$$(Hz)(t) = \int_{t_1}^{t} \exp\left\{-\int_s^t u(\tau)d\tau\right\} z(s)ds \left[u(t) - \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right].$$

Inequality (2.3.2) yields that if $z(t) \geq 0$ then $(Hz)(t) \geq 0$ (i.e., operator $H$ is positive). Besides, the operator $H : L_\infty[t_1, b] \to L_\infty[t_1, b]$ is an integral Volterra operator with the kernel essentially bounded on $[t_1, b] \times [t_1, b]$. By Theorem A.4, operator $H$ is weakly compact in the space $L_\infty[t_1, b]$; Theorem A.7 implies that the spectral radius is $r(H) = 0 < 1$.

Thus, if in (2.3.11) $f(t) \geq 0$, then

$$z(t) = f(t) + (Hf)(t) + \left(H^2 f\right)(t) + \cdots \geq 0.$$

The solution of (2.3.6) has the form (2.3.8), with $z$ being a solution of (2.3.11). Hence, if in (2.3.6) $f(t) \geq 0$, then for the solution of this equation we have $x(t) \geq 0$. On the other hand, the solution of (2.3.6) can be presented in the form (2.2.5)

$$x(t) = \int_{t_1}^{t} X(t,s) f(s)ds.$$

As was shown above, $f(t) \geq 0$ implies $x(t) \geq 0$, and consequently the kernel of the integral operator is nonnegative; i.e., $X(t, s) \geq 0$ for $t \geq s > t_1$.

**Step 2**. Let us prove that in fact the strict inequality $X(t, s) > 0$ holds. Denote

$$x(t) = X(t, t_1) - \exp\left\{-\int_{t_1}^{t} u(s)ds\right\}, \; x(t) = 0, \; t < t_1.$$

The function $X(t, t_1)$ is a solution of homogeneous equation (2.3.6). After substituting $x(t)$ into the left-hand side of (2.3.6), we have

$$u(t)\exp\left\{-\int_{t_1}^{t} u(s)ds\right\} - \sum_{k=1}^{m} a_k(t)\exp\left\{-\int_{t_1}^{h_k(t)} u(s)ds\right\}$$
$$= \exp\left\{-\int_{t_1}^{t} u(s)ds\right\}\left[u(t) - \sum_{k=1}^{m} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right] \geq 0.$$

Hence $x(t)$ is a solution of (2.3.6) with $f(t) \geq 0$; as demonstrated above, $x(t) \geq 0$. Consequently,

$$X(t, t_1) \geq \exp\left\{-\int_{t_1}^{t} u(s)ds\right\} > 0.$$

For $s > t_1$, the inequality $X(t, s) > 0$ can be justified similarly.

3) ⇒ 4) A function $x(t) = X(t, t_1)$ is a positive solution of (2.2.1) for $t \geq t_1$.

Implication 4) ⇒ 1) is evident. □

*Remark 2.1* If there exists a nonnegative solution of inequality (2.3.2) for $t \geq t_1$, then assertions 1), 3) and 4) of Theorem 2.1 are also fulfilled for $t \geq t_1$.

We will end this section with the result on the asymptotic behavior of nonoscillatory solutions.

**Theorem 2.2** *Suppose* $a_k(t) \ge 0$, $k = 1, \cdots, m$, $\int_{t_0}^{\infty} \sum_{k=1}^{m} a_k(s)ds = \infty$. *Then, for any nonoscillatory solution of* (2.2.1), *we have* $\lim_{t\to\infty} x(t) = 0$.

*Proof* Suppose $x$ is an eventually positive solution of (2.2.1). Then $x$ is an eventually monotonically decreasing function, and hence there exists a nonnegative limit $\lim_{t\to\infty} x(t) = d < \infty$. If $d > 0$, then for some $t_1$ we have $x(t) > d - \varepsilon > 0$, $t \ge t_1$. Hence

$$x(t) = x(t_1) - \int_{t_1}^{t} \sum_{k=1}^{m} a_k(s)x\big(h_k(s)\big)ds \le x(t_0) - (d-\varepsilon)\int_{t_1}^{t} \sum_{k=1}^{m} a_k(s)ds.$$

Thus $\lim_{t\to\infty} x(t) = -\infty$, and we have a contradiction, so $d = 0$, which completes the proof. □

## 2.4 Comparison Theorems

Theorem 2.1 can be employed to obtain comparison results in oscillation theory. To this end, consider together with (2.2.1) the equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(h_k(t)\big) = 0, \; t \ge 0. \tag{2.4.1}$$

Suppose (a1) and (a2) hold for (2.4.1). Denote by $Y(t,s)$ the fundamental function of (2.4.1).

**Theorem 2.3** *Suppose* $a_k(t) \ge 0$, $a_k(t) \ge b_k(t)$, $t \ge t_0$, *and condition* 2) *of Theorem* 2.1 *holds for* (2.2.1). *Then* (2.4.1) *has a positive solution for* $t \ge t_1$ *and* $Y(t,s) > 0$ *for* $t \ge s \ge t_1$.

*Proof* By Theorem 2.1 and Remark 2.1, the fundamental function $X(t,s)$ of (2.2.1) is positive for $t \ge t_1$.

Consider the equation with the zero initial conditions

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(h_k(t)\big) = f(t), \; t \ge t_1, \; x(t) = 0, \; t \le t_1. \tag{2.4.2}$$

We will show that if $f(t) \ge 0$, then the solution of (2.4.2) is nonnegative. To this end, let us rewrite (2.4.2) in the form

$$\begin{aligned}\dot{x}(t) &+ \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) + \sum_{k=1}^{m}\big[b_k(t) - a_k(t)\big]x\big(h_k(t)\big)\\ &= f(t), \; t \ge t_1, \; x(t) = 0, t \le t_1.\end{aligned}$$

Denote

$$z(t) = \dot{x}(t) + \sum_{k=1}^{m} a_k(t) x\bigl(h_k(t)\bigr).$$

By solution representation formula (2.2.5),

$$x(t) = \int_{t_1}^{t} X(t,s) z(s) ds, \; x\bigl(h_k(t)\bigr) = \chi_{[t_1,\infty)}\bigl(h_k(t)\bigr) \int_{t_1}^{h_k(t)} X\bigl(h_k(t), s\bigr) z(s) ds,$$

where $\chi_I$ is the characteristic set of the interval $I$,

$$\chi_{[t_1,\infty)}(t) = \begin{cases} 1, \; t \geq t_1, \\ 0, \; t < t_1. \end{cases}$$

Thus (2.4.2) is equivalent to the equation

$$z - Tz = f, \tag{2.4.3}$$

where

$$(Tz)(t) = \sum_{k=1}^{m} \bigl[a_k(t) - b_k(t)\bigr] \chi_{[t_1,\infty)}\bigl(h_k(t)\bigr) \int_{t_1}^{h_k(t)} X\bigl(h_k(t), s\bigr) z(s) ds.$$

By Corollary B.1, we have the estimate

$$\bigl|X(t,s)\bigr| \leq \exp \sum_{k=1}^{m} \int_{t_1}^{b} \bigl|a_k(\tau)\bigr| d\tau, \; t_1 \leq s \leq t \leq b,$$

so the kernel of the integral operator $T$ is essentially bounded on $[t_1, b] \times [t_1, b]$. By Theorem A.4, operator $T$ is a weakly compact operator in the space $L_\infty[t_1, b]$. Theorem A.7 implies that the spectral radius $r(T) = 0 < 1$.

Theorem 2.1 implies $X(t,s) > 0$, $t \geq s \geq t_1$, and hence operator $T$ is positive. Therefore, for the solution of (2.4.3), we have

$$z(t) = f(t) + (Tf)(t) + \bigl(T^2 f\bigr)(t) + \cdots \geq 0 \text{ if } f(t) \geq 0.$$

Then, as in the proof of Theorem 2.1, we conclude that $Y(t,s) > 0$, $t \geq s \geq t_1$, and therefore $x(t) = Y(t, t_1)$ is a positive solution of (2.4.1).

Positivity of $Y(t,s)$ for an arbitrary $s > t_1$ is demonstrated similarly. □

**Corollary 2.1** *Suppose that $a_k(t) \geq 0$, $a_k(t) \geq b_k(t)$ for $t \geq t_0$ and (2.2.1) has a positive solution for $t \geq t_0$. Then there exists $t_1 \geq t_0$ such that (2.4.1) has a positive solution for $t \geq t_1$.*

Denote

$$a^+ = \max\{a, 0\}.$$

**Corollary 2.2**

1) *If the inequality*

$$\dot{x}(t) + \sum_{k=1}^{m} a_k^+(t) x\big(h_k(t)\big) \le 0 \tag{2.4.4}$$

*has an eventually positive solution, then* (2.2.1) *also has an eventually positive solution.*

2) *If condition* 2) *of Theorem* 2.1 *holds for* (2.2.1), *where inequality* (2.3.2) *is replaced by*

$$u(t) \ge \sum_{k=1}^{m} a_k^+(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)ds \right\},\ t \ge t_1, \tag{2.4.5}$$

*then* (2.2.1) *has a positive solution for $t \ge t_1$ and $X(t,s) > 0$ for $t \ge s \ge t_1$.*

*Proof* Consider the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k^+(t) x\big(h_k(t)\big) = 0.$$

Either of the two assumptions of the corollary imply that all hypotheses of Theorem 2.1 hold. Since $a_k(t) \le a_k^+(t)$ and $a_k^+(t) \ge 0$, Theorem 2.3 implies this corollary. □

Inequality (2.4.5) can be employed to obtain a comparison result that improves the statement of Theorem 2.3.

Consider the equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t) x\big(g_k(t)\big) = 0, \tag{2.4.6}$$

and suppose that the hypotheses (a1) and (a2) hold for (2.4.6); denote by $Y(t,s)$ the fundamental function of this equation.

**Theorem 2.4** *Suppose that $a_k(t) \ge 0$ and there exists $t_0 \ge 0$ such that for* (2.2.1) *anyone of assertions* 1)–4) *of Theorem* 2.1 *holds for $t \ge t_0$. If*

$$b_k(t) \le a_k(t),\ h_k(t) \le g_k(t),\ k = 1, \cdots, m, \tag{2.4.7}$$

*then there exists $t_1 \ge t_0$ such that* (2.4.6) *has a positive solution for $t \ge t_1$ and $Y(t,s) > 0$, $t \ge s \ge t_1$.*

*Proof* Theorem 2.1 implies that for some $t_1 \ge t_0$ there exists a nonnegative solution $u$ of inequality (2.3.2) for $t \ge t_1$. Inequalities (2.4.7) yield that this function is also a solution of the inequality

$$u(t) \ge \sum_{k=1}^{m} b_k^+(t) \exp\left\{ \int_{g_k(t)}^{t} u(s)ds \right\},\ t \ge t_1.$$

Hence, by Corollary 2.2, (2.4.6) has a positive solution for $t \ge t_1$ and the fundamental function of (2.4.6) is positive, which completes the proof. □

The inequality $X(t,s) > 0$ can be employed to compare solutions of two distinct differential equations. To this end, consider together with (2.2.2), (2.2.3) the initial value problem with the same delays:

$$\dot{y}(t) + \sum_{k=1}^{m} b_k(t) y\big(h_k(t)\big) = g(t), \ t \ge t_1, \tag{2.4.8}$$

$$y(t) = \psi(t), \ t < t_1, \ y(t_1) = y_0. \tag{2.4.9}$$

Suppose (a1)–(a3) hold for (2.4.8), (2.4.9). Denote by $x(t)$, $X(t,s)$ the solution and the fundamental function of problem (2.2.2), (2.2.3), where the initial point $t_0$ is replaced by $t_1$ and by $y(t)$, $Y(t,s)$ the solution and the fundamental function of problem (2.4.8), (2.4.9).

**Theorem 2.5** *Suppose that condition* 2) *of Theorem* 2.1 *holds for* (2.2.1), $x(t) > 0$ *and*

$$a_k(t) \ge b_k(t) \ge 0, \ g(t) \ge f(t), \ \varphi(t) \ge \psi(t), \ t < t_1, \ y_0 \ge x_0.$$

*Then* $y(t) \ge x(t) > 0$.

*Proof* Denote by $u(t)$ a nonnegative solution of (2.3.2). Inequality $a_k(t) \ge b_k(t)$ yields that the function $u(t)$ is also a solution of the inequality corresponding to (2.3.2) for (2.4.8). Hence, by Theorem 2.1 we have $X(t,s) > 0$ and $Y(t,s) > 0$ for $t_0 \le s < t$.

Equation (2.2.2) can be rewritten in the form

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t) x\big(h_k(t)\big) = \sum_{k=1}^{m} \big[b_k(t) - a_k(t)\big] x\big(h_k(t)\big) + f(t), \ t \ge t_1,$$

which implies

$$\begin{aligned} x(t) &= Y(t,t_1)x_0 - \sum_{k=1}^{m} \int_{t_1}^{t} Y(t,s) b_k(s) \varphi\big(h_k(s)\big) ds \\ &\quad + \int_{t_1}^{t} Y(t,s) f(s) ds - \sum_{k=1}^{m} \int_{t_1}^{t} Y(t,s)\big[a_k(s) - b_k(s)\big] x\big(h_k(s)\big) ds \\ &\le Y(t,t_1) y_0 - \sum_{k=1}^{m} \int_{t_1}^{t} Y(t,s) b_k(s) \psi\big(h_k(s)\big) ds + \int_{t_1}^{t} Y(t,s) g(s) ds = y(t), \end{aligned}$$

where $\varphi(h_k(s)) = \psi(h_k(s)) = 0$ if $h_k(s) \ge t_1$ and $x(h_k(s)) = 0$ if $h_k(s) < t_1$. Therefore $y(t) \ge x(t) > 0$. □

**Corollary 2.3** *Suppose that* $a_k(t) \ge 0$, *condition* 2) *of Theorem* 2.1 *holds for* (2.2.1) *and* $x$ *and* $y$ *are positive solutions of* (2.2.1) *and* (2.3.1) *for* $t \ge t_1$, *respectively. If* $x(t) \le y(t)$ *for* $t < t_1$ *and* $x(t_1) = y(t_1)$, *then* $x(t) \ge y(t)$ *for* $t \ge t_1$.

Since the fundamental function of any ordinary differential equation $\dot{x}(t) + a(t)x(t) = 0$ is positive, we immediately obtain the following result.

**Corollary 2.4** *If $a_k(t) \geq 0$, $k = 1, \cdots, m$, then the fundamental function $X(t,s)$ of the equation*

$$\dot{x}(t) + a(t)x(t) - \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0, \ t \geq 0,$$

*is positive for $0 \leq s \leq t$. In addition, for the solutions $y$ and $z$ of the inequalities*

$$\dot{y}(t) + a(t)y(t) - \sum_{k=1}^{m} a_k(t)y\big(h_k(t)\big) \leq 0, \ t \geq 0,$$

$$\dot{z}(t) + a(t)z(t) - \sum_{k=1}^{m} a_k(t)z\big(h_k(t)\big) \geq 0, \ t \geq 0,$$

*satisfying for any $t_0$ the equality $x(t) = y(t) = z(t)$, $t \leq t_0$, we have $y(t) \leq x(t) \leq z(t)$ for $t > t_0$.*

## 2.5 Nonoscillation Conditions, Part 1

Inequality (2.4.5) can be applied to obtain explicit nonoscillation conditions. Corollary 2.2, Part 2, immediately implies the following result if we assume $u(t) \equiv \lambda$.

**Theorem 2.6** *Suppose that there exist a point $t_1 \geq 0$ and a constant $\lambda > 0$ such that*

$$\sum_{j=1}^{m} a_j^+(t)e^{\lambda(t-h_k(t))} \leq \lambda, \ t \geq t_1.$$

*Then the fundamental function $X(t,s)$ of (2.2.1) is positive for $t \geq s \geq t_1$.*

**Theorem 2.7** *Suppose that there exists a point $t_1 \geq 0$ such that*

$$\int_{\min_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^+(s)ds \leq \frac{1}{e}, \ t \geq t_1. \tag{2.5.1}$$

*Then the fundamental function $X(t,s)$ of (2.2.1) is positive for $t \geq s \geq t_1$.*

*Proof* Let us demonstrate that the function

$$u(t) = e\sum_{k=1}^{m} a_k^+(t)$$

is a nonnegative solution of (2.4.5). By (2.5.1), we have for $t \geq t_1$

$$\begin{aligned}
&\sum_{k=1}^{m} a_k^+(t)\exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\\
&\quad= \sum_{k=1}^{m} a_k^+(t)\exp\left\{\int_{h_k(t)}^{t} e\sum_{i=1}^{m} a_i^+(s)ds\right\}\\
&\quad\le \sum_{k=1}^{m} a_k^+(t)\exp\left\{e\int_{\min_k h_k(t)}^{t} \sum_{i=1}^{m} a_i^+(s)ds\right\}\\
&\quad\le \sum_{k=1}^{m} a_k^+(t)e = u(t),
\end{aligned}$$

so $u(t)$ satisfies (2.4.5). By Corollary 2.2, the fundamental function of (2.2.1) is positive for $t \ge t_1$. □

Let us note that the constant $1/e$ is the best possible since the equation

$$\dot{x}(t) + x(t-\tau) = 0$$

is oscillatory for $\tau > 1/e$.

**Corollary 2.5** *Suppose*

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^+(s)ds < \frac{1}{e}. \tag{2.5.2}$$

*Then there exists an eventually positive solution of* (2.2.1).

**Corollary 2.6** *Suppose that there exists* $\tau > 0$ *such that* $t - h_k(t) \le \tau$, $k = 1, \cdots, m$ *and*

$$\int_{t_0}^{\infty} \sum_{k=1}^{m} a_k^+(s)ds < \infty.$$

*Then there exists an eventually positive solution of* (2.2.1).

In the monograph [192], the authors construct a counterexample that demonstrates that condition (2.5.2) is not necessary for nonoscillation of (2.2.1).

By [192, Theorem 3.4.3], the inequality

$$\limsup_{t\to\infty} \int_{\max_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j(s)ds \le 1 \tag{2.5.3}$$

is necessary for nonoscillation of (2.2.1) with nonnegative coefficients $a_k(t) \ge 0$ and monotonically nondecreasing deviations of arguments $h_k(t)$.

Let us find sufficient nonoscillation conditions when the number

$$\limsup_{t\to\infty} \int_{\max_k\{h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^+(s)ds$$

is between $1/e$ and 1.

First, consider (2.2.1) with constant delays $\tau_k(t) = \text{const}$:

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(t-\tau_k) = 0,\ \tau_k > 0,\ k = 1, \cdots, m. \tag{2.5.4}$$

**Theorem 2.8** *Suppose that there exist a number $n_0 \geq 0$ and a sequence $\{\lambda_n\}_{n=n_0}^{\infty}$, where all $\lambda_n > 1$, such that*

$$\sum_{k=1}^{m} a_k^+(t) \leq \lambda_n e^{-(\lambda_{n-1}(n\tau - t)+\lambda_n(t-(n-1)\tau))},\ (n-1)\tau \leq t \leq n\tau,\ n \geq n_0, \tag{2.5.5}$$

*where $\tau = \max_k \tau_k$.*

*Then (2.5.4) has a positive fundamental function $X(t,s)$ for $t \geq s \geq t_0 = n_0\tau$.*

*Proof* Let us demonstrate that the function

$$u(t) = \lambda_n,\ (n-1)\tau \leq t \leq n\tau,\ n \geq n_0,$$

is a solution of (2.4.5) for $t \geq n_0$.

In the interval $(n-1)\tau \leq t \leq n\tau$, we have

$$\begin{aligned}
&\sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{t-\tau_k}^{t} u(s)ds\right\} \\
&\quad \leq \sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{t-\tau}^{t} u(s)ds\right\} \\
&\quad = \sum_{k=1}^{m} a_k^+(t) \exp\left\{\int_{t-\tau}^{(n-1)\tau} \lambda_{n-1}ds + \int_{(n-1)\tau}^{t} \lambda_n ds\right\} \\
&\quad = \sum_{k=1}^{m} a_k^+(t) \exp\left\{\lambda_{n-1}(n\tau - t) + \lambda_n\big(t-(n-1)\tau\big)\right\} \leq \lambda_n = u(t).
\end{aligned}$$

Hence (2.4.5) is equivalent to (2.5.5), which completes the proof. □

By Theorem 2.4, we obtain a more general result.

**Corollary 2.7** *Suppose there exists $\tau_k > 0$ such that $t - h_k(t) \leq \tau_k$. If all the conditions of Theorem 2.8 hold, then (2.2.1) has a positive fundamental function $X(t,s)$ for $t \geq s \geq t_0$.*

*Example 2.1* Consider the equation

$$\dot{x}(t) + a(t)x(t-\tau) = 0, \tag{2.5.6}$$

where $\tau = 1$ and

$$a(t) = \begin{cases} e^{-(2n-t+1)}, & 2n-1 \leq t < 2n,\ n \geq 1, \\ 2e^{-(t-2n+1)}, & 2n \leq t < 2n+1,\ n \geq 0. \end{cases}$$

Denote

$$\lambda_{2n} = 1, \ \lambda_{2n-1} = 2.$$

Then all the conditions of Theorem 2.8 hold, and thus (2.5.6) has a positive fundamental function.

In addition, we have

$$\limsup_{t\to\infty} \int_{t-1}^{t} a(s)ds = 2\left(e^{-1} - e^{-2}\right) > \frac{1}{e}.$$

Hence (2.5.2) does not hold for (2.5.6). Thus (2.5.2) is not necessary for nonoscillation of (2.2.1).

We apply the idea of Example 2.1 to prove the following theorem.

**Theorem 2.9** *For any* $\alpha \in (1/e, 1)$, *there exists nonoscillatory equation* (2.5.6) *with* $a(t) \geq 0$ *such that*

$$\sup_{t\geq\tau} \int_{t-\tau}^{t} a(s)ds = \alpha. \tag{2.5.7}$$

*Proof* It is sufficient to prove the theorem for $\tau = 1$. Suppose $\lambda > 0$, $a > 1$. Consider (2.5.6), where $\tau = 1$ and

$$a(t) = \begin{cases} \lambda e^{-(2\lambda(a-1)n - \lambda(a-1)t + \lambda)}, & 2n-1 \leq t < 2n, \ n \geq 1, \\ \lambda a e^{-(\lambda(a-1)t - 2\lambda(a-1)n + \lambda)}, & 2n \leq t < 2n+1, \ n \geq 0. \end{cases}$$

Denote

$$\lambda_{2n} = \lambda, \ \lambda_{2n-1} = \lambda a.$$

Then all the conditions of Theorem 2.8 hold, and hence (2.5.6) has a positive fundamental function.

We have

$$\sup_{t\geq 1} \int_{t-1}^{t} a(s)ds = \frac{a}{a-1}\left(e^{-\lambda} - e^{-\lambda a}\right).$$

The function

$$f(\lambda) = \frac{a}{a-1}\left(e^{-\lambda} - e^{-\lambda a}\right)$$

has the maximum $\max f(\lambda) = f(\lambda_0) = e^{-\lambda_0}$ at the point $\lambda_0 = \frac{\ln a}{a-1}$. Let us note that

$$\lim_{a\to 1} \frac{\ln a}{a-1} = 1, \ \lim_{a\to\infty} \frac{\ln a}{a-1} = 0,$$

and take $\lambda = \frac{\ln a}{a-1}$ in the definition of function $a(t)$. Then

$$\sup_{t\geq 1} \int_{t-1}^{t} a(s)ds = e^{-\ln a/(a-1)}.$$

Since

$$\lim_{a\to 1}\sup_{t\ge 1}\int_{t-1}^{t} a(s)ds = 1/e, \quad \lim_{a\to\infty}\sup_{t\ge 1}\int_{t-1}^{t} a(s)ds = 1,$$

the continuous function $\sup_{t\ge 1}\int_{t-1}^{t} a(s)ds$ of $a$ takes all the values from the interval $(1/e, 1)$, which completes the proof. □

Now we proceed to an integral nonoscillation condition similar to Theorem 2.8.

**Theorem 2.10** *Suppose that there exist* $n_0 \ge 0$ *and a sequence* $\{\lambda_n\}_{n=n_0}^{\infty}$, *where all* $\lambda_n > 1$, *such that*

$$\lambda_{n-1}\int_{t-\tau}^{(n-1)\tau}\sum_{k=1}^{m} a_k^{+}(s)ds + \lambda_n\int_{(n-1)\tau}^{t}\sum_{k=1}^{m} a_k^{+}(s)ds \le \ln\lambda_n, \ (n-1)\tau \le t \le n\tau, \tag{2.5.8}$$

$n \ge n_0$, *where* $\tau = \max_k \tau_k$. *Then* (2.5.4) *has a positive fundamental function for* $t \ge s \ge t_0 = n_0\tau$.

*Proof* The proof is similar to the proof of the previous theorem if we put

$$u(t) = \lambda_n a(t), \ (n-1)\tau \le t \le n\tau, \ n \ge n_0,$$

where $a(t) = \sum_{k=1}^{m} a_k^{+}(s)ds$. □

**Corollary 2.8** *Suppose there exists* $\tau_k > 0$ *such that* $t - h_k(t) \le \tau_k$. *If all the conditions of Theorem* 2.10 *hold, then* (2.2.1) *has a positive fundamental function for* $t \ge s \ge t_0$.

Let us note that if in (2.5.6) we substitute the maximum delay by the minimum delay

$$\limsup_{t\to\infty}\int_{\min_k h_k(t)}^{t}\sum_{j=1}^{m} a_j(s)ds \le 1, \tag{2.5.9}$$

this condition is not necessary for nonoscillation.

*Example 2.2* The equation

$$x'(t) + 0.01x(t-10) + 0.3x(t) = 0 \tag{2.5.10}$$

is nonoscillatory since the characteristic equation $\lambda + 0.01e^{-10\lambda} + 0.3 = 0$ has two real roots, $\lambda_1 \approx -0.3261$ and $\lambda_1 \approx -0.5536$. However, (2.5.9) is not satisfied since $10(0.01 + 0.3) = 3.01 > 1$.

## 2.6 Nonoscillation Conditions, Part 2

The explicit nonoscillation condition in (2.5.1) is easily checked but contains only "the worst delay". To give a sharper result, where all delays are included, denote

$$A_{ij} = \sup_{t \geq t_1} \int_{h_i(t)}^{t} a_j^+(s)ds, \ 1 \leq i, j \leq m. \tag{2.6.1}$$

**Theorem 2.11** *Suppose there exist a point $t_1 \geq 0$ and positive numbers $x_i, i = 1, \cdots, m$ such that $A_{ij} < \infty$, $t \geq t_1$ and*

$$\ln x_i \geq \sum_{j=1}^{m} A_{ij} x_j, \ i = 1, \cdots, m. \tag{2.6.2}$$

*Then (2.2.1) has a positive fundamental function $X(t, s)$ for $t \geq s \geq t_1$.*

*Proof* Inequality (2.6.2) implies that for any $t \geq t_1$

$$x_k \geq \exp\left\{ \sum_{j=1}^{m} \int_{h_k(t)}^{t} x_j a_j^+(s)ds \right\}.$$

After introducing the function

$$u(t) = \sum_{j=1}^{m} x_j a_j^+(t), \ t \geq t_1, \ u(t) = 0, \ t \leq t_1,$$

we obtain

$$\begin{aligned}
&\sum_{k=1}^{m} a_k^+(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)ds \right\} \\
&\quad = \sum_{k=1}^{m} a_k^+(t) \exp\left\{ \int_{h_k(t)}^{t} \sum_{j=1}^{m} x_j a_j^+(s)ds \right\} \\
&\quad \leq \sum_{k=1}^{m} a_k^+(t) x_k = u(t).
\end{aligned}$$

Then all the conditions of Part 2 of Corollary 2.2 are satisfied. Hence (2.2.1) has a positive fundamental function for $t \geq t_1$. □

Theorem 2.11 contains only implicit nonoscillation conditions. To derive explicit conditions from this theorem, we consider first the equation with two delays

$$\dot{x}(t) + a(t)x\big(h(t)\big) + b(t)x\big(g(t)\big) = 0, \tag{2.6.3}$$

where

$$a(t) \geq 0, \ b(t) \geq 0, \ h(t) \leq t, \ g(t) \leq t.$$

Similar to (2.6.1), we denote (and assume $A, B, C, D$ are finite)

$$\begin{aligned} A &= \sup_{t\ge t_1}\int_{h(t)}^t a^+(s)ds,\ B = \sup_{t\ge t_1}\int_{h(t)}^t b^+(s)ds,\\ C &= \sup_{t\ge t_1}\int_{g(t)}^t a^+(s)ds,\ D = \sup_{t\ge t_1}\int_{g(t)}^t b^+(s)ds. \end{aligned} \tag{2.6.4}$$

By Theorem 2.11, the existence of positive solutions of the system

$$\ln x_1 \ge Ax_1 + Bx_2,\ \ln x_2 \ge Cx_1 + Dx_2, \tag{2.6.5}$$

implies nonoscillation of (2.6.3).

**Theorem 2.12** *Let*

$$\limsup_{t\to\infty}\int_{g(t)}^t a^+(s)ds = \limsup_{t\to\infty}\int_{g(t)}^t b^+(s)ds = 0 \tag{2.6.6}$$

*and*

$$Ae^B < \frac{1}{e}. \tag{2.6.7}$$

*Then* (2.6.3) *has an eventually positive fundamental function.*

*Proof* It is sufficient to prove the existence of a positive solution $(x_1, x_2)$, $x_1 > 0$, $x_2 > 0$ for the system

$$\ln x_1 > A_0x_1 + B_0x_2,\ \ln x_2 > 0, \tag{2.6.8}$$

where

$$A_0 = \limsup_{t\to\infty}\int_{h(t)}^t a^+(s)ds,\ \ B_0 = \limsup_{t\to\infty}\int_{h(t)}^t b^+(s)ds.$$

Assume first $A_0 > 0$, $B_0 > 0$, and put $x_1 = \frac{1}{A_0}$. Then (2.6.8) takes the form

$$B_0 < B_0x_2 < -1 - \ln A_0.$$

By (2.6.7), there exists $C > 0$ such that $B_0 < C < -1 - \ln A_0$. Therefore the pair

$$(x_1, x_2) = \left(\frac{1}{A_0}, \frac{C}{B_0}\right)$$

will be a solution of the system (2.6.8).

If $A_0 > 0$, $B_0 = 0$, then the pair $(x_1, x_2)$, where $x_1 = e$, $x_2 > 1$, is a solution of (2.6.8).

The case $A_0 = 0$, $B_0 > 0$ is treated similarly.

Existence of a positive solution of (2.6.8) in the case $A_0 = B_0 = 0$ is obvious. □

*Example 2.3* Consider the equation

$$\dot{x}(t) + ax(t-1) + bx\big(g(t)\big) = 0, \tag{2.6.9}$$

where $a, b$ are positive numbers, $\lim_{t\to\infty}(t - g(t)) = 0$. We have $A = a$, $B = b$, $C = D = 0$. Then the condition

$$ae^b < \frac{1}{e}$$

implies nonoscillation of (2.6.9).

*Example 2.4* Consider the equation

$$\dot{x}(t) + \frac{a}{t}x\left(\frac{t}{\mu}\right) + \frac{b}{t}x(t-\tau) = 0, \ t \geq t_0 > 0, \tag{2.6.10}$$

where $a > 0$, $b > 0$, $\mu > 1$, $\tau > 0$. We have $A = a\ln\mu$, $B = b\ln\mu$, $C = D = 0$. Hence, if the condition

$$a\mu^b < \frac{1}{e\ln\mu}$$

holds, then (2.6.10) has a nonoscillatory solution.

*Example 2.5* Consider the equation

$$\dot{x}(t) + \frac{a}{t\ln t}x\left(t^\alpha\right) + \frac{b}{t}x(t-\tau) = 0, \ t \geq t_0 > 1, \tag{2.6.11}$$

where $a > 0$, $b > 0$, $1 > \alpha > 0$, $\tau > 0$. We have $A = a\ln\frac{1}{\alpha}$, $B = b\ln\frac{1}{\alpha}$, $C = D = 0$. Hence, if the condition

$$a\alpha^{-b} < \frac{1}{e\ln\frac{1}{\alpha}}$$

holds, then (2.6.11) has a nonoscillatory solution.

**Theorem 2.13** *Suppose that for some $t_1 \geq 0$ at least one of the following conditions holds*:

1) $0 < A \leq \frac{1}{e}$, $B > 0$, *and there exists a number $y_0 > 0$ such that*

$$y_0 \leq -\frac{1+\ln A}{B}, \ \frac{C}{A} + Dy_0 \leq \ln y_0;$$

2) $C > 0$, $0 < D \leq \frac{1}{e}$, *and there exists a number $x_0 > 0$ such that*

$$x_0 \leq -\frac{1+\ln D}{C}, \ \frac{B}{D} + Ax_0 \leq \ln x_0.$$

*Then the fundamental function $X(t,s)$ of* (2.6.3) *is positive for $t \geq s \geq t_1$.*

*Proof* Suppose the inequalities in 1) hold. The function $y = (\ln x - Ax)/B$ has the unique maximum $y_{\max} = -\frac{1+\ln A}{B}$ at the point $x_{\max} = \frac{1}{A}$. The inequality

$$-(1+\ln A) \geq By_0 > 0$$

implies $y_{\max} > 0$, while $y_0 \leq -\frac{1+\ln A}{B}$ yields that the point $(x_{\max}, y_0)$ satisfies the first inequality in (2.6.5) in the case $y_0 < y_{\max}$. Since $\frac{C}{A} + Dy_0 < \ln y_0$, this point also satisfies the second inequality in (2.6.5). If $y_0 = y_{\max}$, then there exists $y_1 < y_0$ for which the inequality $\frac{C}{A} + Dy_1 < \ln y_1$ still holds. Then $(x_{\max}, y_1)$ is a solution of (2.6.5). If 2) holds, the proof is similar. □

**Corollary 2.9** *Suppose that there exists a point $t_1 \geq 0$ such that at least one of the following conditions holds*:

$$0 < A \leq \frac{1}{e},\ B > 0,\ \frac{C}{A} - \frac{D(1+\ln A)}{B} \leq \ln\left(-\frac{1+\ln A}{B}\right), \tag{2.6.12}$$

$$C > 0,\ 0 < D \leq \frac{1}{e},\ \frac{B}{D} - \frac{A(1+\ln D)}{C} \leq \ln\left(-\frac{1+\ln D}{C}\right). \tag{2.6.13}$$

*Then the fundamental function $X(t,s)$ of* (2.6.3) *is positive for $t \geq s \geq t_1$.*

*Proof* If (2.6.12) holds, then there exists $\varepsilon > 0$ such that for $y_0 = -\frac{1+\ln A}{B} - \varepsilon$ the first condition of Theorem 2.13 is satisfied. Similarly, (2.6.13) implies the second condition. □

*Remark 2.2* In Theorem 2.13, it is assumed that either $A > 0$, $B > 0$ or $C > 0$, $D > 0$. Including the cases where these conditions are not satisfied, by analyzing (2.6.5) we immediately obtain the following sufficient nonoscillation conditions:

1. $B = 0$, $D > 0$, $A < 1/e$, $1 + \ln D + C/e < 0$;
2. $C = 0$, $A > 0$, $D < 1/e$, $1 + \ln A + B/e < 0$;
3. $A = 0$, $D > 0$, $Ce^{B/D} + 1 + \ln D < 0$;
4. $D = 0$, $A > 0$, $be^{C/A} + \ln A + 1 < 0$;
5. $B = 0$, $C = 0$, $A < 1/e$, $D < 1/e$;
6. $A = 0$, $C = 0$, $D < 1/e$;
7. $B = 0$, $D = 0$, $A < 1/e$.

For $A = D = 0$, the situation is a little bit more complicated in that there exists an eventually positive solution if the following condition is satisfied:

8. $A = D = 0$, and there exists either $x > 0$ such that $\ln x > Be^{Cx}$ or $y > 0$ such that $\ln y > Ce^{By}$.

*Example 2.6* Consider the equation

$$\dot{x}(t) + \frac{0.2}{\pi}\sin^2 t x(t-\pi) + \frac{0.2}{\pi}\cos^2 t x(t-2\pi) = 0. \tag{2.6.14}$$

By simple calculations, we have $A = B = 0.1$, $C = D = 0.2$. Condition (2.6.12) in Corollary 2.9 is not satisfied, but inequality (2.6.13) holds. Hence (2.6.14) has an eventually positive solution.

Figure 2.1 illustrates the domain for $(x, y)$ where the inequalities of type (2.6.5) hold:

$$\ln x \geq 0.1x + 0.1y,\ \ln y \geq 0.2x + 0.2y. \tag{2.6.15}$$

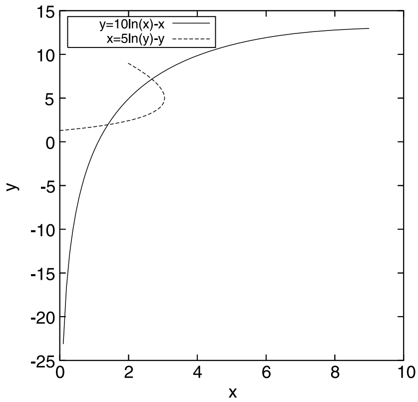


**Fig. 2.1** In the *domain between the curves*, the system of inequalities (2.6.15) has a positive solution, so (2.6.14) has an eventually positive solution. Here $A = B = 0.1$, $C = D = 0.2$

We observe that the maximum of $f(x) = 10\ln(x) - x$ is not in the domain between the curves (thus, (2.6.12) is not satisfied), while the maximum of the function $g(y) = 5\ln(y) - y$ is in the intersection domain, so (2.6.13) holds. It should be noted that Theorem 2.7 fails for this equation.

Let us present different sufficient conditions for the existence of positive solutions.

**Theorem 2.14** *Suppose that there exists a point* $t_1 \ge 0$ *such that at least one of the following conditions holds*:

1) *There exists* $y_0 > 0$ *such that* $y_0 < (1 - Ae)/B$, $Ce + Dy_0 < \ln y_0$.
2) *There exists* $x_0 > 0$ *such that* $x_0 < (1 - De)/C$, $Ax_0 + Be < \ln x_0$.

*Then the fundamental function of* (2.6.3) *is positive*.

*Proof* Suppose 1) holds. Then $Ae < 1$ and $(e, y_0)$ is a solution of the system of inequalities (2.6.5). Similarly, if 2) holds, then $(x_0, e)$ is a solution of (2.6.5). □

*Remark 2.3* In Theorem 2.14, the value $x = e$ was chosen to minimize the coefficient of $x$ in the first inequality of the system

$$\left(A - \frac{\ln x}{x}\right)x + By < 0, \ \ Cx + \left(D - \frac{\ln y}{y}\right)y < 0,$$

which is equivalent to (2.6.5), and $y = e$ minimizes the coefficient of $y$ in the second inequality.

**Corollary 2.10** *Suppose that there exists a point* $t_1 \ge 0$ *such that at least one of the following inequalities holds*:

$$Ce + \frac{D}{B}(1 - Ae) \le \ln\left(\frac{1 - Ae}{B}\right), \tag{2.6.16}$$

$$Be + \frac{A}{C}(1 - De) \le \ln\left(\frac{1 - De}{C}\right). \tag{2.6.17}$$

*Then the fundamental function of* (2.6.3) *is positive.*

Let us modify Example 2.6 to demonstrate that there are cases where either Theorem 2.13 or Theorem 2.14 can be applied while the other one fails.

*Example 2.7* Consider the following modified version of (2.6.14):

$$\dot{x}(t) + \frac{0.5}{\pi}\sin^2 t x(t-\pi) + \frac{0.08}{\pi}\cos^2 t x(t-2\pi) = 0. \tag{2.6.18}$$

Then $A = 0.25$, $B = 0.04$, $C = 0.5$, $D = 0.08$ and (2.6.16) becomes

$$0.5e + 2(1 - 0.25e) = 2 < 2.08 \approx \ln\left(\frac{1 - 0.25e}{0.04}\right);$$

i.e., (2.6.16) is satisfied and there exists an eventually positive solution of (2.6.18). Theorem 2.7 fails for (2.6.18) since $0.5 + 0.08 > 1/e$. Simple computations demonstrate that (2.6.12), (2.6.13) and (2.6.17) also fail for (2.6.18).

On the other hand, for the equation

$$\dot{x}(t) + \frac{0.2}{\pi}\sin^2 t x(t-\pi) + \frac{0.25}{\pi}\cos^2 t x(t-2\pi) = 0 \tag{2.6.19}$$

with $A = 0.1$, $B = 0.125$, $C = 0.2$, $D = 0.25$, inequality (2.6.13) is satisfied. This implies existence of an eventually positive solution for (2.6.19), while Theorem 2.7, (2.6.16), (2.6.17) and (2.6.12) fail.

Next, consider (2.2.1) with several delays.

**Theorem 2.15** *Suppose that there exists a point $t_1 \ge 0$ and an index $k$, $1 \le k \le m$, such that*

$$B_i := \sum_{j \ne k} A_{ij} \le \frac{1}{e}, \ i = 1, 2, \cdots, m, \tag{2.6.20}$$

*where $A_{ij}$ are defined in* (2.6.1), *and there exists $z > 0$ satisfying the inequalities*

$$z \le \min_{i \ne k} \frac{1 - B_i e}{A_{ik}}, \ \sum_{j \ne k} A_{kj} e + A_{kk} z \le \ln z. \tag{2.6.21}$$

*Then the fundamental function of* (2.2.1) *is positive.*

*Proof* Suppose that such $k$ exists. Let $x_i = e, i \ne k; x_i = z, i = k$. Then the first inequality in (2.6.21) implies all inequalities in (2.6.2) but the $k$-th one, which is a corollary of the latter inequality in (2.6.21). Thus (2.6.2) has a positive solution, so (2.2.1) has an eventually positive solution, which completes the proof. □

**Corollary 2.11** *Suppose there exist a point* $t_1 \geq 0$ *and an index* $k$, $1 \leq k \leq m$, *such that*

$$e \sum_{j \neq k} A_{kj} + A_{kk} B \leq \ln B, \tag{2.6.22}$$

*where* $B = \min_{i \neq k} \frac{1 - B_i e}{A_{ik}}$ *and* $A_{kj}$ *are denoted by* (2.6.1). *Then the fundamental function of* (2.2.1) *is positive.*

*Proof* Due to the continuity of the function $\ln x - A_{kk} x$, there exists $\varepsilon > 0$ such that if we substitute $z = B - \varepsilon$ instead of $B$, the inequality (2.6.22) is still valid; i.e., the second inequality in (2.6.21) is satisfied. Then $z \leq \frac{1 - B_i e}{A_{ik}}$ for any $i \neq k$, where $B_i$ are defined in (2.6.20), so the first inequality in (2.6.21) is also satisfied. By Theorem 2.15, (2.2.1) has an eventually positive solution. □

Using Theorem 2.4, we can also apply Theorem 2.13 to general equations with several delays.

**Theorem 2.16** *Suppose* $a_k(t) \geq 0$, $k = 1, \cdots, m$, *and let* $I_1 \subset I = \{1, \cdots, m\}$, $I_2 = I \backslash I_1$. *Denote*

$$a(t) = \sum_{k \in I_1} a_k(t), \;\; b(t) = \sum_{k \in I_2} a_k(t), \;\; h(t) = \min_{k \in I_1} h_k(t), \;\; g(t) = \min_{k \in I_2} h_k(t).$$

*Here we assume* $h(t) \equiv t$ *or* $g(t) \equiv t$ *if* $I_1 = \emptyset$ *or* $I_2 = \emptyset$, *respectively. Suppose that there exists a point* $t_1 \geq 0$ *such that the hypotheses of Theorem* 2.13 *or Remark* 2.2 *are satisfied, where* $A$, $B$, $C$, $D$ *are defined in* (2.6.4). *Then the fundamental function of* (2.2.1) *is positive.*

*Proof* Nonoscillation of (2.6.3) and Theorem 2.4 imply nonoscillation of (2.2.1). □

*Remark 2.4* Theorem 2.16 contains $2^m$ different nonoscillation conditions. In particular, if $I_1 = I$, $I_2 = \emptyset$, then Remark 2.2 implies Theorem 2.7. Indeed, in this case we have $a(t) = \sum_{k=1}^{m} a_k(t)$, $b(t) \equiv 0$, $h(t) = \min_{k \in I} h_k(t)$, $g(t) \equiv t$. We have

$$A = \sup_{t \geq t_1} \int_{h(t)}^{t} \sum_{k=1}^{m} a_k(s) ds, \;\; B = C = D = 0.$$

If we take $x_1 = e$, $x_2 > 1$, then inequalities (2.6.5) have the form $A \leq \frac{1}{e}$, $\ln x_2 > 0$, which is equivalent to (2.5.1).

## 2.7 Oscillation Conditions

There are many explicit oscillation conditions for equations with one delay and only a few for equations with several delays (2.2.1). We present here some explicit oscillation tests. First, let us mention two known oscillation conditions.

**Lemma 2.2** [192] *Suppose $a_k(t) \geq 0$ and at least one of the following conditions holds*:

1)

$$\liminf_{t\to\infty} \int_{\max_k h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds > \frac{1}{e},$$

2) *$h_k$ are nondecreasing and*

$$\limsup_{t\to\infty} \int_{\max_k h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds > 1.$$

*Then all solutions of* (2.2.1) *are oscillatory.*

**Lemma 2.3** [192] *Suppose $a_k(t) \geq 0$, $k = 1, \cdots, m$ and*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t)\big(t - h_k(t)\big) > \frac{1}{e}.$$

*Then all solutions of* (2.2.1) *are oscillatory.*

The conditions of Lemma 2.2 are given in the integral form but contain only the worst delay function. The inequality of Lemma 2.3 contains all the delays but is presented in the pointwise form. The following result contains all the delays and has the integral form.

**Theorem 2.17** *Suppose $a_k(t) \geq 0$, $k = 1, \cdots, m$ and there exists a set of indices $J \subset \{1, \cdots, m\}$ such that $\sum_{k\in J} a_k(t) \neq 0$ almost everywhere, $\int_{t_0}^{\infty} \sum_{i=1}^{m} a_i(s)ds = \infty$ and*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i\in J} a_i(t)} \int_{h_k(t)}^{t} \sum_{i\in J} a_i(s)ds > \frac{1}{e}. \tag{2.7.1}$$

*Then all solutions of* (2.2.1) *are oscillatory.*

*Proof* After the substitution

$$s = \int_{t_0}^{t} \sum_{k\in J} a_k(\tau)d\tau, \; y(s) = x(t), \; l_k(s) = \int_{t_0}^{h_k(t)} \sum_{k\in J} a_k(\tau)d\tau,$$

(2.2.1) has the form

$$\dot{y}(s) + \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i\in J} a_i(t)} y\big(l_k(s)\big) = 0. \tag{2.7.2}$$

Evidently oscillation of (2.2.1) is equivalent to oscillation of (2.7.2).

Since $s - l_k(s) = \int_{h_k(t)}^{t} \sum_{i\in J} a_i(s)ds$, Lemma 2.3 and condition (2.7.1) imply this theorem. □

*Remark 2.5* The first part of Lemma 2.2 can be obtained as a corollary of Theorem 2.17 for $J = \{1, \cdots, m\}$.

Consider now (2.2.1) with two delays:

$$\dot{x}(t) + a(t)x\big(h(t)\big) + b(t)x\big(g(t)\big) = 0. \tag{2.7.3}$$

**Corollary 2.12** *Suppose $a(t) \geq 0$, $b(t) \geq 0$ and at least one of the following conditions holds*:

1. $a(t) \neq 0$ *almost everywhere* (*a.e.*) *and*
$$\liminf_{t\to\infty}\left(\int_{h(t)}^{t} a(s)ds + \frac{b(t)}{a(t)}\int_{g(t)}^{t} a(s)ds\right) > \frac{1}{e};$$
2. $b(t) \neq 0$ *a.e. and*
$$\liminf_{t\to\infty}\left(\frac{a(t)}{b(t)}\int_{h(t)}^{t} b(s)ds + \int_{g(t)}^{t} b(s)ds\right) > \frac{1}{e};$$
3. $a(t) + b(t) \neq 0$ *a.e. and*
$$\liminf_{t\to\infty}\left(\frac{a(t)}{a(t)+b(t)}\int_{h(t)}^{t}\big[a(s)+b(s)\big]ds + \frac{b(t)}{a(t)+b(t)}\int_{g(t)}^{t}\big[a(s)+b(s)\big]ds\right) > \frac{1}{e}.$$

*Then all solutions of* (2.7.3) *are oscillatory.*

*Proof* We fix the sets of indices $J = \{1\}$, $J = \{2\}$ and $J = \{1, 2\}$, respectively. □

Consider (2.7.3) with a nondelay term,

$$\dot{x}(t) + a(t)x(t) + b(t)x\big(g(t)\big) = 0. \tag{2.7.4}$$

**Corollary 2.13** *Suppose $a(t) \geq 0$, $b(t) \geq 0$, $a(t) \neq 0$ a.e. and*

$$\liminf_{t\to\infty}\frac{b(t)}{a(t)}\int_{g(t)}^{t} a(s)ds > \frac{1}{e}.$$

*Then all solutions of* (2.7.3) *are oscillatory.*

*Example 2.8* By Part 3 of Corollary 2.12, the equation

$$\dot{x}(t) + \big[1+\sin(2\pi t)\big]x\big(h(t)\big) + \gamma\big[1+\sin(2\pi t)\big]x(t-1) = 0, \tag{2.7.5}$$

where $h(t) \leq t$, $\lim_{t\to\infty} h(t) = t$, is oscillatory if

$$\gamma > \frac{1}{e} \tag{2.7.6}$$

since

$$\int_{t-1}^{t}\big[1+\sin(2\pi s)\big]ds = 1 \text{ for any } t, \ \frac{b(t)}{a(t)} = \gamma.$$

However, Lemma 2.2 cannot be applied to establish oscillation since $\max\{t, t-1\} = t$, and the condition of Lemma 2.3 is not satisfied since $\liminf_{t\to\infty} \gamma[1 + \sin(2\pi t)] = 0$ for any $\gamma$.

## 2.8 Estimations of Solutions

First let us obtain a lower estimation of the fundamental function.

**Theorem 2.18** *Suppose conditions of Theorem 2.7 hold. Then*

$$X(t,s) \geq \exp\left\{-e\int_s^t \sum_{k=1}^m a_k^+(s)ds\right\}, \; t \geq s \geq t_1.$$

*Proof* Suppose first that $a_k(t) \geq 0, t \geq t_0$ and conditions of Theorem 2.1, Part 2, hold. In the proof of Theorem 2.1, it was shown that

$$X(t,t_1) \geq \exp\left\{-\int_{t_1}^t u(s)ds\right\}, \; t \geq t_1,$$

where the function $u(t)$ was denoted in Part 2 of Theorem 2.1.

The same calculations lead to the estimate

$$X(t,s) \geq \exp\left\{-\int_s^t u(s)ds\right\}, \; t \geq s \geq t_1.$$

By the proof of Theorem 2.7, the function $u(t) = e\sum_{k=1}^m a_k^+(t)$ satisfies all the conditions of Part 2 of Theorem 2.1. Hence the theorem is true for the case $a_k(t) \geq 0$.

Consider now the general case and denote by $X^+(t,s)$ the fundamental function of the equation

$$\dot{x}(t) + \sum_{k=1}^m a_k^+(t)x\big(h_k(t)\big) = 0.$$

As was proven before, $X^+(t,s) \geq \exp\{-e\int_s^t \sum_{k=1}^m a_k^+(s)ds\}, t \geq s \geq t_1$.

The fundamental function $X(t,s)$ of (2.2.1) is the solution of the initial value problem

$$\dot{x}(t) + \sum_{k=1}^m a_k^+(t)x\big(h_k(t)\big) - \sum_{k=1}^m a_k^-(t)x\big(h_k(t)\big) = 0, \; t \geq s,$$
$$x(t) = 0, \; t < s, \; x(s) = 1.$$

Hence, by solution representation formula (2.2.5) for $t \geq t_1$,

$$X(t,s) = X^+(t,s) + \int_s^t X^+(t,\tau)\sum_{k=1}^m a_k^-(\tau)X\big(h_k(\tau),s\big)d\tau.$$

By Theorem 2.7, we have $X(t,s) > 0$, $t \ge s \ge t_1$. Then

$$X(t,s) \ge X^+(t,s) \ge \exp\left\{-e\int_s^t \sum_{k=1}^m a_k^+(s)ds\right\}. \qquad \square$$

Now let us proceed to upper estimates of the fundamental function.

**Theorem 2.19** *In* (2.2.1), *let*

$$a_k(t) \ge 0,\ X(t,s) > 0,\ t \ge s \ge t_0,\ t - h_k(t) \le H,\ t \ge t_0.$$

*Then, for* $t > s \ge t_0$,

$$0 < X(t,s) \le Y(t,s) := \begin{cases} \exp\{-\int_{s+H}^t \sum_{k=1}^m a_k(\tau)d\tau\}, & t \ge s+H, \\ 1, & s \le t \le s+H. \end{cases}$$

*Proof* It is sufficient to prove the theorem for $s = t_0$ since the general case is considered similarly. Denote

$$x(t) = X(t,t_0),\ y(t) = Y(t,t_0),\ t > t_0 + H,\ y(t) = X(t,t_0),\ t \le t_0 + H.$$

Then, $x(t) = y(t)$ for $t \le t_0 + H$, and for $t \ge t_0 + H$ we have

$$\begin{aligned} &\dot{y}(t) + \sum_{k=1}^m a_k(t)y\big(h_k(t)\big) \\ &\quad = -\sum_{k=1}^m a_k(t)\exp\left\{-\int_{t_0+H}^t \sum_{k=1}^m a_k(\tau)d\tau\right\} + \sum_{k=1}^m a_k(t)r_k(t) \\ &\quad = \sum_{k=1}^m a_k(t)\left[r_k(t) - \exp\left\{-\int_{t_0+H}^t \sum_{k=1}^m a_k(\tau)d\tau\right\}\right] \ge 0, \end{aligned}$$

where

$$r_k(t) = \begin{cases} \exp\{-\int_{t_0+H}^{h_k(t)} \sum_{k=1}^m a_k(\tau)d\tau\}, & h_k(t) \ge t_0 + H, \\ X(t,t_0), & t_0 \le h_k(t) \le t_0 + H. \end{cases}$$

Theorem 2.5 implies $y(t) \ge x(t)$. Hence $Y(t,t_0) \ge X(t,t_0)$, $t \ge t_0 + H$.

Inequality $X(t,t_0) \le 1$ is valid since $X(t_0,t_0) = 1$ and $X'_t(t,t_0) \le 0$. Hence $1 = Y(t,t_0) \ge X(t,t_0)$ for $t_0 \le t \le t_0 + H$. $\square$

**Corollary 2.14** *Let*

$$a_k(t) \ge 0,\ \sum_{k=1}^m a_k(t) \ge a > 0,\ X(t,s) > 0,\ t \ge s \ge t_0,\ t - h_k(t) \le H,\ t \ge t_0.$$

*Then, for* $t > s \ge t_0$,

$$0 < X(t,s) \le Y(t,s) := \begin{cases} \exp\{-a(t-s-H)\}, & t \ge s+H, \\ 1, & s \le t \le s+H. \end{cases}$$

**Theorem 2.20** *Suppose* $a_k(t) \ge 0$, $\sum_{k=1}^m a_k(t) \ge a > 0$, $X(t,s) > 0$, $t \ge s \ge t_0$, $t - h_k(t) \le H$, $t \ge t_0$. *Then, for the solution of problem* (2.2.2), (2.2.3), *we have the estimates*

$$\left|x(t)\right| \le \left|x(t_0)\right| + \left(\|f\| + \sum_{k=1}^m \|a_k\|\|\varphi\|\right)(t - t_0), \ t_0 \le t < t_0 + H,$$

$$\left|x(t)\right| \le \left(\left|x(t_0)\right| + \frac{1}{a}\left(e^{aH} - 1\right)\sum_{k=1}^m \|a_k\|\|\varphi\|\right)e^{-a(t-t_0-H)} + \frac{\|f\|}{a}e^{aH},$$
$$t \ge t_0 + H,$$

*where*

$$\|\varphi\| = \sup_{t_0 - H \le t \le t_0}\left|\varphi(t)\right|, \ \|f\| = \sup_{t \ge t_0}\left|f(t)\right|, \ \|a_k\| = \sup_{t \ge t_0}\left|a_k(t)\right|.$$

*Proof* By solution representation formula (2.2.5), we have

$$\left|x(t)\right| \le X(t,t_0)\left|x(t_0)\right| + \int_{t_0}^t X(t,s)\left(\sum_{k=1}^m a_k(s)\left|\varphi\left(h_k(s)\right)\right| + \left|f(s)\right|\right)ds,$$

where $\varphi(t) = 0$, $t \ge t_0$.

Suppose first $t_0 \le t \le t_0 + H$. Then

$$\begin{aligned}\left|x(t)\right| &\le \left|x(t_0)\right| + \int_{t_0}^t \left(\sum_{k=1}^m a_k(s)\left|\varphi\left(h_k(s)\right)\right| + \left|f(s)\right|\right)ds \\ &\le \left|x(t_0)\right| + \left(\|f\| + \sum_{k=1}^m \|a_k\|\|\varphi\|\right)(t - t_0).\end{aligned}$$

Next, let $t \ge t_0 + H$. Then

$$\begin{aligned}\left|x(t)\right| &\le \left|x(t_0)\right|e^{-a(t-t_0-H)} + \int_{t_0}^{t_0+H} e^{-a(t-s-H)}\sum_{k=1}^m a_k(s)\left|\varphi\left(h_k(s)\right)\right|ds \\ &\quad + \int_{t_0}^t e^{-a(t-s-H)}\left|f(s)\right|ds \\ &\le \left|x(t_0)\right|e^{-a(t-t_0-H)} + \sum_{k=1}^m \|a_k\|\|\varphi\|\frac{1}{a}\left(e^{aH} - 1\right)e^{-a(t-t_0-H)} + \frac{\|f\|}{a}e^{aH}.\end{aligned}$$

□

Another integral estimation of the fundamental function can be obtained using the following result.

**Theorem 2.21** *Suppose* $a_k(t) \ge 0$ *and the fundamental function of* (2.2.1) *is positive*: $X(t,s) > 0$, $t \ge s \ge t_0$. *Then there exists* $t_1 \ge t_0$ *such that*

$$0 \le \int_{t_1}^t X(t,s)\sum_{k=1}^m a_k(s)ds \le 1. \tag{2.8.1}$$

*Proof* The function $x(t) \equiv 1$, $t > t_0$, is a solution of the problem

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = \sum_{k=1}^{m} a_k(t)\chi_{(t_0,\infty)}\big(h_k(t)\big), \ x(t) = 0, \ t \le t_0,$$

where $\chi_{(t_0,\infty)}(t)$ is the characteristic function of the interval $(t_0, \infty)$.

Hence, by (2.2.5) we have

$$1 = \int_{t_0}^{t} X(t,s) \sum_{k=1}^{m} a_k(s)\chi_{(t_0,\infty)}\big(h_k(s)\big)ds.$$

There exists $t_1 \ge t_0$ such that all $h_k(t) \ge t_0$ for $t \ge t_1$, and thus for any $t \ge t_1$

$$\int_{t_0}^{t_1} X(t,s) \sum_{k=1}^{m} a_k(s)\chi_{(t_0,\infty)}\big(h_k(s)\big)ds + \int_{t_1}^{t} X(t,s) \sum_{k=1}^{m} a_k(s)ds = 1,$$

which implies inequality (2.8.1). □

*Remark 2.6* If $t - h_k(t) \le H$, then we can take $t_1 = t_0 + H$.

## 2.9 Positivity of Solutions

Now we proceed to the analysis of positivity for solutions of problem (2.2.2), (2.2.3). We will show that if the inequality (2.3.2) has a nonnegative solution and the condition

$$0 \le \varphi(t) \le x(t_0), \ t \le t_0, \ x(t_0) > 0, \tag{2.9.1}$$

holds, then the solution of the initial value problem (2.2.1), (2.2.3) is positive. This result supplements some statements in [192].

**Theorem 2.22** *Suppose $a_k(t) \ge 0$, $f(t) \ge 0$ and there exists a nonnegative solution of the inequality*

$$u(t) \ge \sum_{k=1}^{m} a_k(t) \int_{\max\{t_0, h_k(t)\}}^{t} u(s)ds, \ t \ge t_0, \tag{2.9.2}$$

*for a certain $t_0 \ge 0$ and conditions (2.9.1) hold. Then the solution of problem (2.2.2), (2.2.3) is positive for $t \ge t_0$.*

*Proof* Let $u(t) \ge 0$, $t \ge t_0$ be a solution of (2.9.2). Denote $u(t) = 0$, $t < t_0$. Then

$$u(t) \ge \sum_{k=1}^{m} a_k(t) \int_{h_k(t)}^{t} u(s)ds, \ t \ge t_0.$$

Hence all conditions of Theorem 2.1 are satisfied, and thus the fundamental function $X(t,s)$ is positive: $X(t,s) > 0$ for $t \ge s \ge t_0$.

First assume $f \equiv 0$. Consider the auxiliary problem

$$\dot{z}(t) + \sum_{k=1}^{m} a_k(t) z\big(h_k(t)\big) = 0,\ t \geq t_0,\ z(t) = x_0,\ t \leq t_0. \tag{2.9.3}$$

Denote

$$v(t) = \begin{cases} x_0 \exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ x_0, & t < t_0, \end{cases}$$

and for a fixed $t \geq t_0$ define the sets

$$N_1(t) = \big\{k : h_k(t) \geq t_0\big\},\ N_2(t) = \big\{k : h_k(t) < t_0\big\}.$$

We obtain

$$\begin{aligned}
&\dot{v}(t) + \sum_{k=1}^{m} a_k(t) v\big(h_k(t)\big) \\
&\quad = -x_0 u(t) \exp\left\{-\int_{t_0}^{t} u(s)ds\right\} \\
&\qquad + x_0 \sum_{k \in N_1(t)} a_k(t) \exp\left\{-\int_{t_0}^{h_k(t)} u(s)ds\right\} + x_0 \sum_{k \in N_2(t)} a_k(t) \\
&\quad = -x_0 \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\left[u(t) - \sum_{k \in N_1(t)} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\right. \\
&\qquad \left. - \sum_{k \in N_2(t)} a_k(t) \exp\left\{\int_{t_0}^{t} u(s)ds\right\}\right] \\
&\quad = -x_0 \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\left[u(t) - \sum_{k \in N_1(t)} a_k(t) \exp\left\{\int_{\max\{t_0, h_k(t)\}}^{t} u(s)ds\right\}\right. \\
&\qquad \left. - \sum_{k \in N_2(t)} a_k(t) \exp\left\{\int_{\max\{t_0, h_k(t)\}}^{t} u(s)ds\right\}\right] \\
&\quad = -x_0 \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\left[u(t) - \sum_{k=1}^{m} a_k(t) \exp\left\{\int_{\max\{t_0, h_k(t)\}}^{t} u(s)ds\right\}\right] \leq 0.
\end{aligned}$$

Hence $v(t)$ is a solution of the problem

$$\dot{v}(t) + \sum_{k=1}^{m} a_k(t) v\big(h_k(t)\big) = g(t),\ t \geq t_0,\ v(t) = x_0,\ t \leq t_0,$$

with $g(t) \leq 0$. Theorem 2.5 implies that $z(t) \geq v(t) > 0$.

Conditions (2.9.1) and Corollary 2.3 imply $x(t) \geq z(t) > 0$, $t \geq t_0$. For the case $f \equiv 0$, the theorem is proven. The general case also follows from Theorem 2.5 since $f(t) \geq 0$. □

**Corollary 2.15** *Suppose* $a_k(t) \ge 0,\ f(t) \ge 0$ *and*

$$\int_{\max\{t_0, \min_k h_k(t)\}}^{t} \sum_{j=1}^{m} a_j^{+}(s)ds \le \frac{1}{e},\ t \ge t_0,$$

*for a certain* $t_0 \ge 0$ *and conditions* (2.9.1) *hold. Then the solution of the problem* (2.2.2), (2.2.3) *is positive for* $t \ge t_0$.

*Proof* As demonstrated in the proof of Theorem 2.7, the function $u(t) = e\sum_{k=1}^{m} a_k(t)$ is a solution of inequality (2.9.2). Application of Theorem 2.22 completes the proof. □

## 2.10 Slowly Oscillating Solutions for Delay Differential Equations

**Definition 2.4** A solution $x$ of (2.2.1) is said to be *slowly oscillating* if for every $t_0 \ge 0$ there exist $t_1 > t_0$, $t_2 > t_1$ such that $h_k(t) \ge t_1$ for $t \ge t_2$, $x(t_1) = x(t_2) = 0$, $x(t) > 0$, $t \in (t_1, t_2)$.

In particular, if $h_k(t) = t - \tau_k$, $\tau_k > 0$ and for every $t_0 \ge 0$ there exist $t_1 > t_0$, $t_2 > t_1$ such that $x(t_1) = x(t_2) = 0$, $x(t) > 0$, $t \in (t_1, t_2)$, $t_2 - t_1 \ge \max_k \tau_k$, then $x(t)$ is slowly oscillating.

**Theorem 2.23** *Let* $a_k(t) \ge 0$. *If there exists a slowly oscillating solution of* (2.2.1) (*inequality* (2.3.1)), *then all solutions of this equation* (*inequality*) *are oscillatory.*

*Proof* Denote by $x$ a slowly oscillating solution of (2.2.1). Suppose that this equation has a nonoscillatory solution. Then, by Theorem 2.1, for a certain $t_0 \ge 0$ the fundamental function satisfies $X(t,s) > 0$ if $t \ge s > t_0$.

There exist $t_1 > t_0$, $t_2 > t_1$ such that

$$h_k(t) \ge t_1 \text{ for } t \ge t_2,\ x(t_1) = x(t_2) = 0,\ x(t) > 0,\ t \in (t_1, t_2). \tag{2.10.1}$$

Due to solution representation formula (2.2.5), for $t \ge t_2$, solution $x(t)$ has the form

$$x(t) = -\int_{t_2}^{t} X(t,s) \sum_{k=1}^{m} a_k(s) x\big(h_k(s)\big) ds, \tag{2.10.2}$$

where $x(h_k(s)) = 0$ if $h_k(s) > t_2$. The inequality $h_k(t) \ge t_1$ for $t \ge t_2$ yields that the expression under the integral in (2.10.2) can differ from zero only if $t_1 < h_k(s) < t_2$. Therefore (2.10.1) yields that in (2.10.2) we have $x(h_k(s)) > 0$. Consequently, (2.10.2) implies $x(t) \le 0$ for each $t \ge t_2$. This contradicts the assumption that $x$ is an oscillatory solution. □

**Corollary 2.16** *Suppose* $a_k(t) \ge 0$ *and there exists a nonoscillatory solution of* (2.2.1). *Then* (2.2.1) *has no slowly oscillating solutions.*

## 2.11 Stability and Nonoscillation

In this section, we present a corollary of Theorem 9.18 that will later be obtained for systems of linear delay differential equations.

**Theorem 2.24** *Suppose that* $a_k(t) \geq 0$, $\sum_{k=1}^m a_k(t) \geq a_0 > 0$, $t - h_k(t) \leq h_0$, $k = 1, \cdots, m$, *and there exists an eventually positive solution of* (2.2.1). *Then* (2.2.1) *is exponentially stable.*

**Corollary 2.17** *Suppose that* $a_k > 0$, $k = 1, \cdots, m$, *and there exists a positive solution* $\lambda$ *of the equation*

$$\lambda = \sum_{k=1}^{m} a_k e^{\lambda \tau_k}.$$

*Then the autonomous equation*

$$\dot{x}(t) + \sum_{k=1}^{m} a_k x(t - \tau_k) = 0$$

*is exponentially stable.*

**Corollary 2.18** *Suppose that* $a_k(t) \geq 0$, $\sum_{k=1}^m a_k(t) \geq a_0 > 0$, $t - h_k(t) \leq h_0$, $k = 1, \cdots, m$, *and the conditions of anyone of Theorems* 2.7, 2.8, 2.10, 2.11, 2.15 *and* 2.16 *hold. Then* (2.2.1) *is exponentially stable.*

**Corollary 2.19** *Suppose* $a(t) \geq 0$, $b(t) \geq 0$, $a(t) + b(t) \geq a_0 > 0$, $t - h(t) \leq h_0$, $t - g(t) \leq g_0$ *and conditions of one of either Theorems* 2.12 *or* 2.14 *or Corollaries* 2.9 *or* 2.10 *hold. Then* (2.6.3) *is exponentially stable.*

In particular, all equations in Examples 2.1, 2.6 and 2.7 are exponentially stable.

## 2.12 Discussion and Open Problems

This chapter deals with some properties of a scalar delay differential equation that are equivalent to nonoscillation. For most classes of autonomous functional differential equations, nonoscillation is equivalent to existence of a real root of the characteristic equation [192].

As was demonstrated in [192], for a nonautonomous scalar linear delay differential equation, nonoscillation is equivalent to existence of a nonnegative solution for a certain nonlinear integral equation that was called “the generalized characteristic equation”. In [80], for a neutral scalar differential equation, an integral nonlinear inequality was constructed that has a nonnegative solution if and only if the fundamental function of this equation is positive.

For a scalar differential equation with one delay, the equivalence of nonoscillation and the existence of a nonnegative solution of the same inequality as in [80] was justified in [154, 232]. Unlike [154, 192, 232], it is assumed in this monograph that coefficients of the equation, delays and the initial function are not necessarily continuous but Lebesgue measurable. Such weak constraints on equation parameters are sufficient if a solution is an absolutely continuous function. Besides, a solution is not assumed to be a continuous extension of the initial function, which is a natural assumption for impulsive differential equations, considered further in Chaps. 12–14.

The main result of this chapter is Theorem 2.1, where it is demonstrated that nonoscillation is equivalent to the three other properties of (2.2.1). Such theorems are very popular for delay equations (see, for example, [192, Theorem 3.1.1]). However, in contrast to [192], we also show the equivalence of nonoscillation and positivity of the fundamental function. This property of the fundamental function is very important in stability theory, boundary value problems, control theory and generally in the qualitative theory of differential equations; we apply here positivity of the fundamental function, in particular, to prove comparison theorems.

Comparison theorems appear to be an efficient tool in oscillation theory [154, 167, 192, 228, 289]. In paper [193], a rather general comparison result was presented for a nonlinear delay differential equation. In Sect. 2.3, a similar result is obtained for a linear equation using a different technique based on the equivalence of nonoscillation and positivity of the fundamental function. Here we follow the paper [41], where some results of [193] were improved and extended to a more general class of equations.

Explicit nonoscillation conditions were obtained in Sects. 2.4 and 2.5. Theorems 2.8 and 2.10 outline the fact that for the equation

$$\dot{x}(t) + a(t)x\big(h(t)\big) = 0, \tag{2.12.1}$$

where $a(t) \geq 0$, the condition

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < \frac{1}{e} \tag{2.12.2}$$

is not necessary for nonoscillation.

Some nonoscillation conditions of Sect. 2.5 were taken from the paper [41], while Sect. 2.6 is based on [64].

Consider the equation

$$\dot{x}(t) + a(t)x(t-\tau) = 0, \tag{2.12.3}$$

where $a(t) \geq 0$, $\tau \geq 0$ and $a(t)$ is a continuous function.

For (2.12.3), the situation where

$$\liminf_{t\to\infty} \left[ a(t) - \frac{1}{\tau e} \right] = 0$$

is called *the critical case* (see, for example, [104]) because a small perturbation can change oscillation properties of (2.12.3). In [104, 109], nonoscillation and oscillation results were obtained for (2.12.3) in the critical case.

**Theorem 2.25** [104] *Let us assume that for some number $k \in \mathbb{N}$ and large $t$ we have $a(t) \le a_k(t)$, where*

$$a_k(t) := \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \frac{\tau}{8e(t\ln t\ln_2 t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2},$$

$\ln_k t = \ln\ln\cdots\ln t$.

*Then there exists a positive solution of* (2.12.3) *such that*

$$x(t) < e^{-t/\tau}\sqrt{t\ln t\ln_2 t\cdots\ln_k t}.$$

*If*

$$a(t) > a_{k-2} + \frac{\theta\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}$$

*for some $\theta > 1$, then all solutions of* (2.12.3) *are oscillatory.*

*In* [109], *the authors extend this result for* (2.12.3) *with continuous delay $\tau(t)$ and obtain the following result.*

**Theorem 2.26** [109] *Suppose in* (2.12.3) *that $\tau = \tau(t)$ is a nonnegative continuous function. If for large $t$*

$$a(t) \le \frac{1}{\tau(t)}\exp\left\{-\int_{t-\tau(t)}^{t}\frac{ds}{\tau(s)}\right\},$$

*then* (2.12.3) *has a positive solution such that*

$$x(t) < \exp\left\{-\int_{t_0-\tau(t_0)}^{t}\frac{ds}{\tau(s)}\right\}$$

*for some $t_0 \ge 0$.*

Some other nonoscillation and oscillation results in the critical case can be found in the papers [35, 105, 144, 317, 324].

For the noncritical case, a summary of some other nonoscillation results is presented in the following theorem.

**Theorem 2.27** *Suppose at least one of the following conditions holds*:

1. [302, 336] *For sufficiently large $T$ and for some $\lambda > 0$,*

$$-\lambda + \sup_{t\ge T}\sum_{i=1}^{m} a_i^+(t)\exp\bigl\{\lambda\bigl(t-h_i(t)\bigr)\bigr\} \le 0.$$

2. [303] $t - h_i(t) = \tau_i > 0$, *and there exist $\lambda > 0$ and a sufficiently large $T$ such that*

$$-\lambda + \sup_{t\ge T}\max_{j=1,\cdots,m}\sum_{k=1}^{n} p_{jk}(t)e^{\lambda\tau_k} \le 0,$$

*where $p_{jk}(t) = \frac{1}{\tau_j}\int_{t-\tau_j}^{t} a_k^+(s)ds$.*

*Then there exists a nonoscillatory solution of* (2.2.1).

There are a lot of papers devoted to explicit oscillation conditions for (2.2.1). A review of these results for equations with one delay is presented in the paper [320] and for equations with several delays in [170]. In [323, 352], a connection between oscillation properties of a linear differential equation with several constant delays and an explicitly constructed linear second-order ordinary differential equation was established.

In the following theorem, some explicit oscillation conditions for (2.2.1) with several delays are outlined.

**Theorem 2.28** *Let $a_k(t) \geq 0$, $k = 1, \cdots, m$. All solutions of* (2.2.1) *are oscillatory if any of the following conditions hold*:

1. [18] *Let $h_i(t) := t - \tau_i$, $\tau_i > 0$,*

$$\liminf_{t \to \infty} \int_t^{t+\tau_i} a_i(s)ds > 0,$$

*and at least one of the following three inequalities holds*:

a. $p_{ij}^* := \liminf_{t\to\infty} \int_{t-\tau_i}^t a_j(s)ds > 1/e$ *for some* $i, j$;

b. $[\prod_{i=1}^n \sum_{j=1}^n p_{ij}^*]^{1/n} > 1/e$;

c. $\sum_{i=1}^m p_{ij}^* + 2\sum_{i<j}^m (p_{ij}^* p_{ji}^*)^{1/2} > m/2$ *for some* $j$.

2. [208] *Coefficients and delays satisfy $a_k(t) > 0$, $0 < t - h_k(t) < \sigma$, $k = 1, \cdots, m$, and*

$$\liminf_{t \to \infty} \sum_{k=1}^m a_k(t)\big(t - h_k(t)\big) > \frac{1}{e}.$$

3. [180] *There exist indices $i_l \in \{1, \cdots, m\}$ such that*

$$\liminf_{t \to \infty} \big(t - h_{i_l}(t)\big) > 0, \ \liminf_{t \to \infty} \sum_{i=1}^m a_{i_l}(t) > 0$$

*and at least one of the following inequalities holds*:

a.

$$\liminf_{t \to \infty} \left[ \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \sum_{i=1}^m a_i(t) \exp\{\lambda\big(t - h_i(t)\big)\} \right\} \right] > 1,$$

b.

$$\liminf_{t \to \infty} \left\{ \left[ \prod_{i=1}^m a_i(t) \right]^{1/m} \left[ \sum_{i=1}^m \big(t - h_i(t)\big) \right] \right\} > \frac{1}{e}.$$

4. [302] *There exist a nonempty set $I \subset \{1, \cdots, m\}$ and constants $\tau_0, \tau_1, \tau_0 > \tau_1 > 0$, such that*

$$t - h_i(t) \geq \tau_0, \ i \in I, \ \liminf_{t \to \infty} \int_t^{t+\tau_1} \sum_{t \in I} a_i(s)ds > 0,$$

$$\limsup_{t\to\infty}\left\{\max_k \int_{h_k(t)}^t \sum_{k=1}^m a_k(s)ds\right\} < \infty,$$

*and at least one of the following inequalities holds*:

a. *For all* $\lambda > 0$ *and some* $T > 0$,

$$-\lambda + \inf_{t\ge T} \frac{\sum_{k=1}^m a_k(t)\exp\{\lambda \int_{h_k(t)}^t \sum_{i=1}^m a_i(s)ds\}}{\sum_{k=1}^m a_k(t)} > 0.$$

b.

$$\liminf_{t\to\infty} \frac{\sum_{k=1}^m a_k(t)\int_{h_k(t)}^t \sum_{i=1}^m a_i(s)ds}{\sum_{k=1}^m a_k(t)} > \frac{1}{e}.$$

5. [303] *Let* $t - h_i(t) = \tau_i > 0$, *and at least one of the following conditions holds*:

   a. *For every* $\lambda > 0$ *and sufficiently large* $T$,

$$-\lambda + \inf_{t\ge T} \min_{j=1,\cdots,m} \sum_{k=1}^m p_{jk}(t)e^{\lambda\tau_k} > 0,$$

   *where*

$$p_{jk}(t) = \frac{1}{\tau_j}\int_{t-\tau_j}^t a_k(s)ds.$$

   b.

$$\liminf_{t\to\infty} \min_{j=1,\cdots,m} \sum_{k=1}^m p_{jk}(t) > \frac{1}{e}.$$

6. [153] *There exist indices* $i_l \in \{1, \cdots, m\}$ *such that*

$$\liminf_{t\to\infty}\big(t - h_{i_l}(t)\big) > 0,\ \liminf_{t\to\infty}\sum_{i=1}^m a_{i_l}(t) > 0,$$

   *and at least one of the following inequalities holds*:

   a. *For every* $\lambda > 0$ *and* $i = 1, \cdots, m$,

$$\liminf_{t\to\infty} \frac{1}{\lambda\tau_i(t)} \sum_{k=1}^m \int_t^{t+\tau_k(t)} a_k(s)e^{\lambda\tau_k(s)}ds > 1.$$

   b. *For every* $i = 1, \cdots, m$,

$$\liminf_{t\to\infty} \frac{1}{\tau_i(t)} \sum_{k=1}^m \int_t^{t+\tau_k(t)} a_k(s)\tau_k(s)ds > \frac{1}{e}.$$

Some other oscillation results were obtained in the papers [229, 257, 258, 290].

In Sect. 2.8, we obtain lower and upper estimates of the fundamental function for a nonoscillatory equation. Applying these bounds, we can estimate a solution of the initial value problem for such equations. Moreover, we obtain here an estimation of the integral of the fundamental function; such estimations are very useful in stability theory. The results of this section were partially published in [62, 63].

In [193], several sufficient conditions on equation parameters and initial functions were established that yield that the solution of the initial value problem is positive. We supplement the results of [193] in Sect. 2.9. Namely, as is demonstrated in Sect. 2.7, if the nonlinear integral inequality has a nonnegative solution, then under certain conditions on the initial function (the same as in [193]) the solution of the initial value problem is positive. We used here the results of paper [41].

For ordinary linear differential equations of the second order, the following oscillation criterion is known: if the equation has an oscillatory solution, then all its solutions oscillate. As is well known, for delay differential equations this is not true. Y. Domshlak [140] revised the result above for differential equation (2.2.1) with monotone delays. He demonstrated that if an *associated* equation has a *slowly oscillating solution*, then every solution of (2.2.1) is oscillating. In [78, 142, 144], several new explicit sufficient conditions of oscillation were obtained by explicit construction of such slowly oscillating solutions.

In particular, the following theorem was obtained in [78].

**Theorem 2.29** [78] *Let* $A + D > 0$ *and the system*

$$\begin{cases} (AD - BC)x_1x_2 - Ax_1 - Dx_2 + 1 = 0, \\ \ln x_1 - Ax_1 - Bx_2 < 0, \\ \ln x_2 - Cx_1 - Dx_2 < 0, \end{cases} \tag{2.12.4}$$

*have a positive solution* $\{x_1, x_2\}$, *where* $A$, $B$, $C$, $D$ *are defined by* (2.6.4). *Then all solutions of* (2.6.3) *are oscillatory.*

Application of Theorem 2.29 to (2.6.9) gives the sufficient condition $ae^b > \frac{1}{e}$ for oscillation of all solutions. Note that in Example 2.3 (by application of Theorem 2.12) the inequality $ae^b < \frac{1}{e}$ implies nonoscillation of (2.6.9). Thus Theorems 2.12 and 2.29 give sharp nonoscillation and oscillation conditions for the equation with two delays and nonnegative coefficients.

Similarly, if $\mu^b < \frac{1}{e \ln \mu}$, then (2.6.10) has a nonoscillatory solution; if $a\mu^b > \frac{1}{e \ln \mu}$, then all solutions of (2.6.10) are oscillatory. If $a\alpha^{-b} < \frac{1}{e \ln(\alpha^{-1})}$, then (2.6.11) has a nonoscillatory solution. If $a\alpha^{-b} > \frac{1}{e \ln(\alpha^{-1})}$, then all solutions of (2.6.10) are oscillatory.

In Sect. 2.10, we present an oscillation criterion similar to Domshlak's result, where the existence of a slowly oscillating solution is assumed for (2.2.1) and not for the associated equation; moreover, the delays are not necessarily monotone. We prove the following result: if an equation has a nonoscillatory solution, then it has no slowly oscillating solutions; this result was also first obtained in [41].

Some other nonoscillation results for scalar delay differential equations can be found in the papers [66, 74, 106, 107, 152, 156, 284, 317, 352]. For example, in the papers [317, 352], oscillation properties of first-order delay differential equations are compared with second-order ordinary differential equations. In particular, the following theorems were obtained.

**Theorem 2.30** [317] *Assume that*

$$t - \tau(t) \geq \frac{1}{e}, \ \limsup_{t\to\infty}\left(t - \tau(t) - \frac{1}{e}\right)e^{e(t-\tau(t))} < a < 1,$$

*and the second-order ordinary differential equation*

$$\ddot{x}(t) + \frac{2}{1-a}e^{2+e(t-\tau(t))}\left(t - \tau(t) - \frac{1}{e}\right)x(t) = 0$$

*has an eventually positive solution. Then the equation* $\dot{x}(t) + x(\tau(t)) = 0$ *also has an eventually positive solution.*

**Theorem 2.31** [352] *Suppose* $p_i \in C([t_0, \infty), \mathbb{R}^+)$, $\tau_i > 0$. *Then the equation*

$$\dot{x}(t) + \sum_{i=1}^{m} p_i(t)x(t - \tau_i) = 0$$

*has a nonoscillatory solution if and only if the equation*

$$\ddot{x}(t) + \frac{2em}{\sum_{i=1}^{m}\tau_i}\sum_{i=1}^{m}\left[p_i(t) - \frac{1}{em\tau_i}\right]x(t) = 0$$

*has a nonoscillatory solution.*

Finally, let us formulate some open problems and topics for research and discussion.

1. Prove or disprove that inequality (2.12.2) implies nonoscillation of (2.12.1) without the assumption that $a(t) \geq 0$.
2. Find necessary and/or sufficient nonoscillation conditions for some partial cases of (2.2.1): equations with periodic coefficients and delays $t - h_k(t)$ and equations with delays of the form $h_k(t) = \lambda_k t^{\beta_k}$, $t \geq 1$, $0 < \lambda_k < 1$, $0 < \beta_k \leq 1$.
3. Find lower and upper bounds of the fundamental function of nonoscillatory equation (2.2.1) without the assumption that $a_k(t) \geq 0$.
4. Is it possible to extend Theorems 2.22 and 2.23 to equations with oscillatory coefficients?
5. Can Lemmas 2.2 and 2.3 be generalized to equations with positive and negative coefficients?

# Chapter 3
# Scalar Delay Differential Equations on Semiaxis with Positive and Negative Coefficients

## 3.1 Introduction

In the present chapter, we deal with the equation

$$\dot{x}(t) + a(t)x\big(h(t)\big) - b(t)x\big(g(t)\big) = 0,\ t \geq 0, \tag{3.1.1}$$

where $a(t) \geq 0$ and $b(t) \geq 0$, and with the general equation including several positive and negative terms

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) - \sum_{k=1}^{n} b_k(t)x\big(g_k(t)\big) = 0,\ t \geq 0, \tag{3.1.2}$$

where $a_k(t) \geq 0$, $k = 1, \cdots, m$, and $b_k(t) \geq 0$, $k = 1, \cdots, n$.

Such equations are more challenging for investigation, and there are only a few known nonoscillation results. The behavior of solutions is also more complicated than for equations with positive coefficients. In particular, an eventually positive solution is not in general an eventually monotone function. Existence of positive solutions does not imply positivity of the fundamental function (see Example 3.1).

The main result as in Chap. 2 is the nonoscillation criteria given in Theorem 3.2. This theorem and some comparison tests are considered in Sect. 3.2. Explicit nonoscillation conditions are presented in Sects. 3.3 and 3.4. In Sect. 3.2, we consider an equation with one delay and an oscillating coefficient. The last section includes a discussion and some open problems.

## 3.2 Nonoscillation Criteria

As a corollary of Theorem 2.3, we obtain a simple nonoscillation condition for (3.1.2).

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**Theorem 3.1** *If the delay equation*

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0 \tag{3.2.1}$$

*has a nonoscillatory solution, then* (3.1.2) *also has a nonoscillatory solution, and the fundamental function of this equation is positive.*

The aim of this chapter is to obtain conditions when the fundamental function of (3.1.1) and (3.1.2) is positive without a similar assumption on (3.2.1). We remark here that nonoscillation properties of (3.1.1) are more complicated than those of (3.2.1). In particular, existence of a nonoscillatory solution of (3.1.1) does not imply positivity of the fundamental function for this equation, as the following example demonstrates.

*Example 3.1* Consider the delay differential equation

$$\dot{x}(t) + x\big(h(t)\big) - 2x\big(g(t)\big) = 0, \tag{3.2.2}$$

where

$$h(t) = [t],\ 2n \le t < 2n+1,\ h(t) = [t-1],\ 2n+1 \le t < 2n+2,\ n \in \mathbb{N},$$

$$g(t) = [t-1],\ 2n \le t < 2n+1,\ g(t) = [t],\ 2n+1 \le t < 2n+2,\ n \in \mathbb{N}.$$

Here $[t]$ is the integer part of $t$ (the maximal integer not exceeding $t$), and $a(t) - b(t) = -1 \not\geq 0$. Integration of (3.2.2) from $2n$ to $2n+1$ leads to

$$x(2n+1) - x(2n) = -x(2n) + 2x(2n-1)$$

or $x(2n+1) = 2x(2n-1)$.

Integration of (3.2.2) from $2n+1$ to $2n+2$ gives

$$x(2n+2) - x(2n+1) = 2x(2n+1) - x(2n),$$

so $x(2n+2) = 3x(2n+1) - x(2n) = 6x(2n-1) - x(2n)$. Between integer points $t = k$ and $t = k+1$, any solution is a linear function. Denoting $Y(n) = (x(2n-1), x(2n))^T$, we get a system of difference equations in $\mathbb{R}^2$,

$$Y(n+1) = AY(n) = \begin{bmatrix} 2 & 0 \\ 6 & -1 \end{bmatrix} Y(n).$$

The eigenvalues of $A$ are 2 and $-1$, with eigenvectors $(1,2)^T$ and $(0,1)^T$, respectively. The eigenvalue $\lambda = -1$ corresponds to the fundamental function $X(t,s)$, where $s$ is an even integer $s = 2n$: $X(2n-1, 2n) = 0$, $X(2n, 2n) = 1$, $X(2(n+k)-1, 2n) = 0$,

$$X\big(2(n+k), 2n\big) = (-1)^k,\ k \in \mathbb{N}.$$

Thus the fundamental function *is not eventually positive*, while *there is a positive solution* for $x(-1) = 1$, $x(0) = 2$ corresponding to the former eigenvalue:

$$x(2n) = x(2n+1) = 2^{n+1}.$$

We consider scalar delay differential equation (3.1.1) under the following conditions:

(b1) $a(t) \ge 0$, $b(t) \ge 0$ are Lebesgue measurable functions essentially bounded on the halfline $[0, \infty)$;

(b2) $h(t),\ g(t) : [0, \infty) \to R$ are Lebesgue measurable functions, $h(t) \le t$, $g(t) \le t$, $\lim_{t\to\infty} h(t) = \lim_{t\to\infty} g(t) = \infty$.

We will assume that (b1) and (b2) are satisfied for all equations considered in this chapter, and the same assumptions hold for coefficients and delays of (3.1.2).

Together with (3.1.1), we consider the differential inequality

$$\dot{y}(t) + a(t)y\big(h(t)\big) - b(t)y\big(g(t)\big) \le 0. \tag{3.2.3}$$

The following theorem is the main result of the chapter.

**Theorem 3.2** *Suppose $a(t) \ge b(t)$, $h(t) \le g(t)$. Then the following hypotheses are equivalent*:

1) *Inequality* (3.2.3) *has an eventually positive solution with an eventually nonpositive derivative.*
2) *There exist a point $t_1 \ge 0$ and a locally essentially bounded function $u(t) \ge 0$ such that*

$$u(t) \ge a(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} - b(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\},\ t \ge t_1, \tag{3.2.4}$$

*where we assume $u(t) = 0$, $t < t_1$.*

3) *There exists $t_1 \ge 0$ such that $X(t,s) > 0$ for $t \ge s \ge t_1$, and* (3.1.1) *has an eventually positive solution with an eventually nonpositive derivative.*

*Proof* 1) $\Rightarrow$ 2). Let $y(t)$ be a positive solution with a nonpositive derivative of inequality (3.2.3) for $t \ge t_1$.

Denote

$$u(t) = -\frac{d}{dt}\ln\frac{y(t)}{y(t_1)},\ t \ge t_1,\ u(t) = 0,\ t < t_1.$$

Then $u(t) \ge 0$, $t \ge t_1$, and

$$y(t) = y(t_1)\exp\left\{-\int_{t_1}^{t} u(s)ds\right\},\ t \ge t_1. \tag{3.2.5}$$

We substitute (3.2.5) into (3.2.3) and obtain by carrying the exponent out of the brackets

$$-\exp\left\{-\int_{t_1}^{t} u(s)ds\right\}y(t_1)\left[u(t) - a(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\}\right.$$
$$\left.+ b(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\}\right] \le 0,\ t \ge t_1.$$

Thus inequality (3.2.4) holds.

2) $\Rightarrow$ 3). **Step 1**. Let us first prove the positivity of the fundamental function. Consider the initial value problem

$$\dot{x}(t)+a(t)x\big(h(t)\big)-b(t)x\big(g(t)\big)=f(t),\ t\geq t_1,\ x(t)=0,\ t\leq t_1. \tag{3.2.6}$$

Denote

$$z(t)=\dot{x}(t)+u(t)x(t),\ z(t)=0,\ t\leq t_1, \tag{3.2.7}$$

where $x$ is the solution of (3.2.6) and $u$ is a nonnegative solution of (3.2.4). Equality (3.2.7) implies

$$x(t)=\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}z(s)ds,\ t\geq t_1, \tag{3.2.8}$$

$$\dot{x}(t)=z(t)-u(t)\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}z(s)ds,\ t\geq t_1. \tag{3.2.9}$$

After substituting (3.2.8) and (3.2.9) into (3.2.6), equation (3.2.6) can be rewritten in the form

$$\begin{aligned}
&z(t)-u(t)\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}z(s)ds\\
&+a(t)\int_{t_1}^{h(t)}\exp\left\{-\int_{s}^{h(t)}u(\tau)d\tau\right\}z(s)ds\\
&-b(t)\int_{t_1}^{g(t)}\exp\left\{-\int_{s}^{g(t)}u(\tau)d\tau\right\}z(s)ds=f(t).
\end{aligned}$$

Hence we obtain the equation in $z$

$$\begin{aligned}
&z(t)-u(t)\int_{t_1}^{h(t)}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}z(s)ds\\
&+a(t)\int_{t_1}^{h(t)}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}\exp\left\{\int_{h(t)}^{t}u(\tau)d\tau\right\}z(s)ds\\
&-b(t)\int_{t_1}^{h(t)}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}\exp\left\{\int_{g(t)}^{t}u(\tau)d\tau\right\}z(s)ds\\
&-u(t)\int_{h(t)}^{t}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}z(s)ds\\
&-b(t)\int_{h(t)}^{g(t)}\exp\left\{-\int_{s}^{g(t)}u(\tau)d\tau\right\}z(s)=f(t),
\end{aligned}$$

which has the form

$$z-Tz=f, \tag{3.2.10}$$

with

$$\begin{aligned}(Tz)(t) = &\int_{t_1}^{h(t)} \exp\left\{-\int_s^t u(\tau)d\tau\right\} z(s)ds \\ &\times \left[u(t) - a(t)\exp\left\{\int_{h(t)}^t u(s)ds\right\} + b(t)\exp\left\{\int_{g(t)}^t u(s)ds\right\}\right] \\ &+ u(t)\int_{h(t)}^t \exp\left\{-\int_s^t u(\tau)d\tau\right\} z(s)ds \\ &+ b(t)\int_{h(t)}^{g(t)} \exp\left\{-\int_s^{g(t)} u(\tau)d\tau\right\} z(s)ds.\end{aligned}$$

Inequality (3.2.4) yields that if $z(t) \geq 0$, then $(Tz)(t) \geq 0$. Besides, for an arbitrary $c \geq t_1$, the operator $T : L_\infty[t_1, c] \to L_\infty[t_1, c]$ is a sum of linear integral Volterra operators with kernels essentially bounded on $[t_1, c] \times [t_1, c]$. Hence $T$ is a linear integral Volterra operator with a kernel essentially bounded on $[t_1, c] \times [t_1, c]$. By Theorem A.4, operator $T$ is a weakly compact operator on the space $L_\infty[t_1, c]$. Theorem A.7 implies that the spectral radius $r(T) = 0 < 1$.

Thus, if in (3.2.10) we have $f(t) \geq 0$, then

$$z(t) = f(t) + (Tf)(t) + \left(T^2 f\right)(t) + \cdots \geq 0.$$

The solution of (3.2.6) has the form (3.2.8), with $z$ being a solution of (3.2.10). Hence, if $f(t) \geq 0$ in (3.2.6), then for the solution of this equation we have $x(t) \geq 0$. On the other hand, the solution of (3.2.6) can be presented in the form

$$x(t) = \int_{t_1}^t X(t, s) f(s)ds.$$

As was demonstrated above, $f(t) \geq 0$ implies $x(t) \geq 0$, and consequently the kernel of the integral operator is nonnegative (i.e., $X(t, s) \geq 0$ for $t \geq s > t_1$).

**Step 2**. Let us prove that in fact the strict inequality $X(t, s) > 0$ holds. Denote

$$x(t) = X(t, t_1) - \exp\left\{-\int_{t_1}^t u(s)ds\right\},\ x(t) = 0,\ t \leq t_1.$$

Then

$$\begin{aligned}&\dot{x}(t) + a(t)x\big(h(t)\big) - b(t)x\big(g(t)\big) \\ &= u(t)\exp\left\{-\int_{t_1}^t u(s)ds\right\} - a(t)\exp\left\{-\int_{t_1}^{h(t)} u(s)ds\right\} \\ &\quad + b(t)\exp\left\{-\int_{t_1}^{g(t)} u(s)ds\right\} \\ &= \exp\left\{-\int_{t_1}^t u(s)ds\right\}\left[u(t) - a(t)\exp\left\{\int_{h(t)}^t u(s)ds\right\}\right. \\ &\quad \left. + b(t)\exp\left\{\int_{g(t)}^t u(s)ds\right\}\right] \geq 0.\end{aligned}$$

We conclude that $x(t)$ is a solution of (3.2.6) with $f(t) \geq 0$. Hence, as was proven above, $x(t) \geq 0$, and consequently

$$X(t, t_1) \geq \exp\left\{-\int_{t_1}^{t} u(s)ds\right\} > 0.$$

For $s > t_1$, the inequality $X(t, s) > 0$ can be justified similarly.

Next, let us prove the existence of an eventually positive solution with a nonpositive derivative.

Consider the following sequence of functions $u_k$, $k = 0, 1, 2, \cdots$, where $u_0(t)$, $t \geq t_1$ is a nonnegative solution of inequality (3.2.4):

$$u_{n+1} = a(t)\exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\} - b(t)\exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\}.$$

Since $a(t) \geq b(t) \geq 0$ and $g(t) \geq h(t)$, we have $u_1(t) \geq 0$, $t \geq t_1$. Inequality (3.2.4) implies $u_1(t) \leq u_0(t)$.

Further, let us prove $0 \leq u_{n+1}(t) \leq u_n(t)$ by induction. Assume that $0 \leq u_n(t) \leq u_{n-1}(t)$, and let us demonstrate that $0 \leq u_{n+1}(t) \leq u_n(t)$.

Inequality $0 \leq u_{n+1}(t)$ is evident since $u_n(t) \geq 0$. By definition,

$$u_n = a(t)\exp\left\{\int_{h(t)}^{t} u_{n-1}(s)ds\right\} - b(t)\exp\left\{\int_{g(t)}^{t} u_{n-1}(s)ds\right\}, \tag{3.2.11}$$

and we have to show that

$$\begin{aligned} & a(t)\exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\} - b(t)\exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\} \\ & \quad \leq a(t)\exp\left\{\int_{h(t)}^{t} u_{n-1}(s)ds\right\} - b(t)\exp\left\{\int_{g(t)}^{t} u_{n-1}(s)ds\right\}, \end{aligned}$$

which is equivalent to the inequality

$$\begin{aligned} & a(t)\left[\exp\left\{\int_{h(t)}^{t} u_{n-1}(s)ds\right\} - \exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\}\right] \\ & \quad \geq b(t)\left[\exp\left\{\int_{g(t)}^{t} u_{n-1}(s)ds\right\} - \exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\}\right]. \end{aligned}$$

We have

$$\begin{aligned} & \exp\left\{\int_{h(t)}^{t} u_{n-1}(s)ds\right\} - \exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\} \\ & = \exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\}\left[\exp\left\{\int_{h(t)}^{t} \big(u_{n-1}(s) - u_n(s)\big)ds\right\} - 1\right] \\ & \geq \exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\}\left[\exp\left\{\int_{g(t)}^{t} \big(u_{n-1}(s) - u_n(s)\big)ds\right\} - 1\right] \\ & = \exp\left\{\int_{g(t)}^{t} u_{n-1}(s)ds\right\} - \exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\}. \end{aligned}$$

Thus $0 \leq u_{n+1}(t) \leq u_n(t)$.

The monotone positive sequence $\{u_n(t)\}$ has a pointwise limit $u(t) \geq 0$, $t \geq t_1$. By the Lebesgue monotone convergence theorem (Theorem A.1) applied to equality (3.2.11), we have

$$u(t) = a(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} - b(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\}.$$

Then the function

$$x(t) = \exp\left\{-\int_{t_1}^{t} u(s)ds\right\}$$

is a positive solution of (3.1.1) with a nonnegative derivative.

Implication 3) ⇒ 1) is evident. □

Without the assumption that $a(t) \geq b(t)$, we have the following theorem.

**Theorem 3.3** *Suppose $h(t) \leq g(t)$, and consider the following hypotheses*:

1) *Inequality* (3.2.3) *has an eventually positive solution with an eventually nonpositive derivative.*
2) *Inequality* (3.2.4) *has a nonnegative locally integrable solution $u(t) \geq 0$, $t \geq t_1$ with $u(t) = 0$, $t < t_1$.*
3) *There exists $t_1 \geq 0$ such that $X(t,s) > 0$, $t \geq s \geq t_1$.*
4) *There exists a nonoscillatory solution of* (3.1.1).

*Then* 1) ⇒ 2) ⇒ 3) ⇒ 4).

*Proof* The implications in the statement of the theorem are justified similarly to the proof of Theorem 3.2. □

As a corollary, we obtain the following interesting result.

**Corollary 3.1** *Suppose $b(t) \geq a(t)$, $h(t) \leq g(t)$, $t \geq t_1$. Then, for the fundamental function of* (3.1.1), *we have $X(t,s) > 0$, $t \geq s \geq t_1$.*

*Proof* The function $u(t) \equiv 0$ is a solution of inequality (3.2.4) for $t \geq t_1$. □

Theorems 3.2 and 3.3 can be employed to obtain comparison results in oscillation theory. To this end, consider together with (3.1.1) the equation

$$\dot{x}(t) + a_1(t)x\bigl(h_1(t)\bigr) - b_1(t)x\bigl(g_1(t)\bigr) = 0. \qquad (3.2.12)$$

**Corollary 3.2** *If*

$$a_1(t) \leq a(t), \;\; b_1(t) \geq b(t), \;\; h(t) \leq h_1(t) \leq g_1(t) \leq g(t)$$

*and inequality* (3.2.4) *has a nonnegative locally integrable solution for $t \geq t_1$, then the fundamental function of* (3.2.12) *is positive for $t \geq t_1$.*

*If $a_1(t) \geq b_1(t)$, then* (3.2.12) *has an eventually positive solution with an eventually nonpositive derivative.*

*Proof* Let $u(t) \geq 0$ be a solution of inequality (3.2.4). Hence $u(t)$ is also a solution of (3.2.4) with parameters $a_1$, $b_1$, $h_1$, $g_1$. By Theorem 3.3, (3.2.12) has a positive fundamental function. The second part of the theorem follows from Theorem 3.2. □

Consider the autonomous equation

$$\dot{x}(t) + ax(t-\delta) - bx(t-\sigma) = 0, \tag{3.2.13}$$

where $a > 0$, $b > 0$, $\delta \geq \sigma \geq 0$.

**Corollary 3.3** *If*

$$a(t) \leq a, \;\; b(t) \geq b, \;\; t-\delta \leq h(t) \leq g(t) \leq t-\sigma, \;\; t \geq t_1,$$

*and the characteristic inequality*

$$\lambda \geq ae^{\lambda\delta} - be^{\lambda\sigma} \tag{3.2.14}$$

*has a positive solution* $\lambda > 0$, *then the fundamental function of* (3.1.1) *is positive for* $t \geq s \geq t_1$.

*If* $a(t) \geq b(t)$, *then* (3.1.1) *has an eventually positive solution with an eventually nonpositive derivative.*

In the following theorem, we deduce asymptotic properties of nonoscillatory solutions.

**Theorem 3.4** *Suppose conditions of Theorem* 3.2 *hold and* $a(t) - b(t) \geq \alpha > 0$. *Then, for any eventually positive solution* $x$ *of* (3.1.1) *with a nonpositive derivative, we have* $\lim_{t\to\infty} x(t) = 0$.

*Proof* Suppose $x(t) > 0$ and $\dot{x}(t) \leq 0$ for $t \geq t_1$ and $h(t) > t_1$ for $t > t_2$. Since $x(h(t)) \geq x(g(t))$, for $t \geq t_2$ we have

$$\dot{x}(t) \leq -a(t)x\big(h(t)\big) + b(t)x\big(h(t)\big) \leq -\alpha x\big(h(t)\big). \tag{3.2.15}$$

Hence

$$x(t) \leq x(t_2) - \alpha \int_{t_2}^{t} x\big(h(s)\big)ds, \;\; t \geq t_2.$$

A nonincreasing function $x(t)$ bounded below has a limit as $t \to \infty$. Let $\lim_{t\to\infty} x(t) > 0$. Then the limit of the right-hand side of (3.2.15) equals $-\infty$, so $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the positivity of $x(t)$. □

## 3.3 Nonoscillation Conditions, Part 1

Now we proceed to nonoscillation conditions.

**Theorem 3.5** *Suppose* $h(t) \le g(t)$. *There exists* $\lambda$, $0 < \lambda < 1$ *such that* $a(t) \ge \lambda b(t)$ *for* $t \ge t_1$ *and*

$$\sup_{t \ge t_1} \int_{h(t)}^{g(t)} \big[a(s) - \lambda b(s)\big] ds \le \frac{1}{e} \ln \frac{1}{\lambda}, \tag{3.3.1}$$

$$\sup_{t \ge t_1} \int_{h(t)}^{t} \big[a(s) - \lambda b(s)\big] ds \le \frac{1}{e}. \tag{3.3.2}$$

*Then the fundamental function of* (3.1.1) *is positive for* $t \ge s \ge t_1$. *If* $a(t) \ge b(t)$, *then there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* By inequality (3.3.2), for $t \ge t_1$, the function

$$u(t) = e\big[a(t) - \lambda b(t)\big] \tag{3.3.3}$$

is a solution of the inequality

$$u(t) \ge \big[a(t) - \lambda b(t)\big] \exp\left\{\int_{h(t)}^{t} u(s) ds\right\}, \; t \ge t_1,$$

which can be rewritten in the form

$$u(t) \ge a(t) \exp\left\{\int_{h(t)}^{t} u(s) ds\right\} - b(t) \exp\left\{\int_{g(t)}^{t} u(s) ds\right\} + b(t)\left[\exp\left\{\int_{g(t)}^{t} u(s) ds\right\} - \lambda \exp\left\{\int_{h(t)}^{t} u(s) ds\right\}\right]. \tag{3.3.4}$$

Inequality (3.3.1) implies for $u$ defined by (3.3.3)

$$\int_{h(t)}^{g(t)} u(s) ds \le \ln \frac{1}{\lambda}.$$

Thus

$$\exp\left\{\int_{g(t)}^{t} u(s) ds\right\} - \lambda \exp\left\{\int_{h(t)}^{t} u(s) ds\right\} \ge 0,$$

and so (3.3.4) yields

$$u(t) \ge a(t) \exp\left\{\int_{h(t)}^{t} u(s) ds\right\} - b(t) \exp\left\{\int_{g(t)}^{t} u(s) ds\right\}.$$

Hence $u(t)$ is a nonnegative solution of inequality (3.2.4). Theorem 3.3 implies that (3.1.1) has a positive fundamental function. The second part follows from Theorem 3.2. □

**Corollary 3.4** *Suppose* $h(t) \le g(t)$, $b(t) \le e a(t)$ *for* $t \ge t_1$ *and*

$$\sup_{t \ge t_1} \int_{h(t)}^{t} \left[a(s) - \frac{1}{e} b(s)\right] ds \le \frac{1}{e}.$$

*Then the fundamental function of* (3.1.1) *is positive for* $t \ge s \ge t_1$. *If* $a(t) \ge b(t)$, *then there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* The corollary is obtained by setting $\lambda = \frac{1}{e}$ in Theorem 3.5. □

*Remark 3.1* The coefficient $\frac{1}{e}$ of $b(s)$ in Corollary 3.4 is the best possible. Indeed, consider the equation

$$\dot{x}(t) + ax(t-\tau) - bx(t) = 0. \tag{3.3.5}$$

After the substitution $x(t) = y(t)e^{bt}$, this equation has the form

$$\dot{y}(t) + ae^{-b\tau}y(t-\tau) = 0. \tag{3.3.6}$$

Hence the inequality

$$a \le \frac{e^{b\tau}}{\tau e} \tag{3.3.7}$$

is necessary and sufficient for nonoscillation of (3.3.6) and therefore of (3.3.5).

Corollary 3.4 yields that if

$$a \le \frac{1}{e}b + \frac{1}{\tau e}, \tag{3.3.8}$$

then (3.3.5) has a nonoscillatory solution. Taking into account the equality $e^{b\tau} = 1 + b\tau + \cdots$ and comparing conditions (3.3.7) and (3.3.8), we can see that the constant $\frac{1}{e}$ is the best possible.

**Corollary 3.5** *Suppose* $a > 0$, $b > 0$, $\delta > \sigma$ *and* $0 \le (a - \frac{1}{e}b)\delta \le \frac{1}{e}$. *Then the fundamental function of* (3.2.13) *is positive for* $t \ge t_1$. *If* $a \ge b$, *then there exists an eventually positive solution with an eventually nonpositive derivative.*

Denote

$$\lambda_0 = \sup_{t \ge t_1} \frac{\int_{h(t)}^t a(s)ds - \frac{1}{e}}{\int_{h(t)}^t b(s)ds}.$$

**Corollary 3.6** *Suppose* $\lambda_0 \in (0, 1)$,

$$h(t) \le g(t),\ a(t) \ge \lambda_0 b(t),\ \int_{h(t)}^t a(s)ds \ge \frac{1}{e},\ t \ge t_1$$

*and*

$$\sup_{t \ge t_1} \int_{h(t)}^{g(t)} \big[a(s) - \lambda_0 b(s)\big]ds \le \frac{1}{e}\ln\left(\frac{1}{\lambda_0}\right),\ t \ge t_1.$$

*Then the fundamental function of* (3.1.1) *is positive for* $t \ge s \ge t_1$. *If* $a(t) \ge b(t)$, *then there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* We have

$$\lambda_0 \geq \frac{\int_{h(t)}^t a(s)ds - \frac{1}{e}}{\int_{h(t)}^t b(s)ds}, \; t \geq t_1.$$

Hence inequality (3.3.2) is satisfied for $\lambda = \lambda_0$. Inequality (3.3.1) holds by the corollary assumption. □

**Corollary 3.7** *Suppose*

$$\frac{1}{e} \leq a\delta \leq \frac{1}{e} + b\delta, \; 0 < \delta - \sigma < \delta \ln \frac{b}{a - \frac{1}{\delta e}}. \tag{3.3.9}$$

*Then the fundamental function of* (3.2.13) *is positive for* $t \geq s \geq \delta$. *If* $a \geq b$ *then there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* All the conditions of Corollary 3.6 hold for $\lambda_0 = (a\delta - \frac{1}{e})/(b\delta)$. □

*Example 3.2* Consider the equation

$$\dot{x}(t) + \frac{2}{e}x(t-1) - \frac{1.8}{e}x(t-0.9) = 0. \tag{3.3.10}$$

By Corollary 3.7, (3.3.10) has a positive fundamental function. Corollary 3.5 fails for this equation.

Now let us obtain some additional explicit nonoscillation conditions for (3.1.1) and (3.2.13).

**Lemma 3.1** *Suppose* $a > 0$, $b > 0$, $\tau > 0$, $\sigma > 0$,

$$a\delta < 1 + b\sigma, \; a\delta \leq (1 + b\sigma)\exp\left\{\frac{b\delta}{1 + b\sigma} - 1\right\}. \tag{3.3.11}$$

*Then there exists a positive solution of inequality* (3.2.14).

*Proof* Since $e^u > 1 + u$ for any positive $u$, it is sufficient to prove that there exists a positive solution of the inequality

$$\lambda \geq ae^{\lambda\delta} - b(1 + \lambda\sigma),$$

which is equivalent to the relation

$$\lambda \geq \frac{a}{1 + b\sigma}e^{\lambda\delta} - \frac{b}{1 + b\sigma}. \tag{3.3.12}$$

Inequalities (3.3.11) imply that

$$\lambda_0 = -\frac{1}{\delta} \ln \frac{a\delta}{1 + b\sigma}$$

is a positive solution of (3.3.12) and thus of inequality (3.2.14). □

**Corollary 3.8** *Let the conditions of Lemma 3.1 hold and $\delta > \sigma$. Then the autonomous equation (3.2.13) is nonoscillatory and for its fundamental function we have $X(t,s) > 0$, $t \ge s \ge \delta$. If $a \ge b$, then there exists an eventually positive solution with an eventually nonpositive derivative.*

*Example 3.3* Consider the equation

$$\dot{x}(t) + ax(t-1) - 0.5x(t-0.5) = 0.$$

By Corollary 3.8, if $0.5/e \le a < 0.686$, then the fundamental function of this equation is positive.

By Corollary 3.5, the sufficient condition of nonoscillation is $0.184 \approx 0.5/e < a < 1.5/e \approx 0.5518$. Corollary 3.7 gives the condition $\frac{1}{e} \le a \le 0.4$.

**Theorem 3.6** *Suppose that $a(t) \ge b(t) \ge 0$, $g(t) \ge h(t)$, $a(t) \ne b(t)$ almost everywhere, and there exists $t_1 \ge 0$ such that*

$$A\delta < B\sigma + 1, \ A\delta \le (1 + B\sigma)\exp\left\{\frac{B\delta}{1+B\sigma} - 1\right\}, \tag{3.3.13}$$

*where positive numbers $A$, $B$, $\tau$, $\sigma$ satisfy the inequalities (almost everywhere)*

$$\frac{a(t)}{a(t)-b(t)} \le A, \ \frac{b(t)}{a(t)-b(t)} \ge B, \ t \ge t_1, \tag{3.3.14}$$

$$\int_{h(t)}^{t} \big[a(s) - b(s)\big]ds \le \delta, \ \int_{g(t)}^{t} \big[a(s) - b(s)\big]ds \ge \sigma, \ t \ge t_1. \tag{3.3.15}$$

*Then the fundamental function of (3.1.1) is positive for $t \ge s \ge t_1$ and there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* Let us demonstrate that the function $u(t) = \lambda[a(t) - b(t)]$ for $t \ge t_1$ and $u(t) = 0$ for $t < t_1$ is a solution of inequality (3.2.4) for some $\lambda > 0$ or, equivalently, that $\lambda$ is a solution of the inequality

$$\lambda \ge \frac{a(t)}{a(t)-b(t)} \exp\left\{\lambda \int_{h(t)}^{t} \big[a(s)-b(s)\big]ds\right\} - \frac{b(t)}{a(t)-b(t)} \exp\left\{\lambda \int_{g(t)}^{t} \big[a(s)-b(s)\big]ds\right\}. \tag{3.3.16}$$

By Lemma 3.1, the inequality $\lambda \ge Ae^{\lambda\tau} - Be^{\lambda\sigma}$, where $A$, $B$, $\tau$, $\sigma$ are defined in (3.3.14), (3.3.15), has a positive solution $\lambda_0$. This number is also a positive solution of inequality (3.3.16). Theorem 3.3 implies that the fundamental function of (3.1.1) is positive, which completes the proof. □

*Example 3.4* Consider the equation

$$\dot{x}(t) + a|\sin t|x(t-2\pi) - 0.1|\sin t|x(t-\pi) = 0, \tag{3.3.17}$$

where $a > 0.1$. Assume $A = \frac{a}{a-0.1}$, $B = \frac{0.1}{a-0.1}$,

$$\delta = (a - 0.1)\int_0^{2\pi} |\sin t| dt = 4(a - 0.1),$$

$$\sigma = (a - 0.1)\int_0^{\pi} |\sin t| dt = 2(a - 0.1).$$

Inequalities (3.3.13) for (3.3.17) have the form

$$4a < 1.2, \ \ 4a \le 1.2 \exp\left(\frac{0.2}{1.2} - 1\right) \approx 0.52.$$

Hence, if $0.1 < a < 0.13$, then the fundamental function of (3.3.17) is positive. We remark that the upper bound is rather sharp (numerical simulations demonstrated oscillation for $a = 0.158$).

Let us compare conditions of Corollary 3.4 and Theorem 3.6.

*Example 3.5* Consider the equation

$$\dot{x}(t) + a|\sin t| x(t - 2\pi) - 0.1|\sin t| x\big(g(t)\big) = 0,$$

where $g(t) \ge t - 2\pi$. Then the inequality in Corollary 3.4 for this equation has the form

$$\left(a - \frac{0.1}{e}\right)\int_0^{2\pi} |\sin s| ds < \frac{1}{e}.$$

Hence, if $a \le \frac{1.4}{4e} \approx 0.129$, then the fundamental function of the equation is positive.

For $g(t) = t - \pi$, Theorem 3.6 gives a better estimate, $a < 0.13$ (see Example 3.4), of the parameter $a$ than Corollary 3.4, but the corollary gives a unique estimation for all delays $g(t)$.

## 3.4 Nonoscillation Conditions, Part 2

Now we will obtain different nonoscillation conditions for (3.1.1).

**Theorem 3.7** *Suppose $a(t) \ge b(t)$, $g(t) \ge h(t)$ and there exist $t_1 \ge 0$ and numbers $x_1 > 0$, $x_2 > 0$ such that*

$$\ln x_1 \ge x_1 \int_{h(t)}^t a(s) ds - x_2 \int_{h(t)}^t b(s) ds, \ \ t \ge t_1,$$

$$\ln x_1 \le x_1 \int_{g(t)}^t a(s) ds - x_2 \int_{g(t)}^t b(s) ds, \ \ t \ge t_1.$$

*Then the fundamental function of (3.1.1) is positive for $t \ge s \ge t_1$ and there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* Since

$$x_1 \geq \exp\left\{x_1 \int_{h(t)}^{t} a(s)ds - x_2 \int_{h(t)}^{t} b(s)ds\right\},$$
$$x_2 \leq \exp\left\{x_1 \int_{g(t)}^{t} a(s)ds - x_2 \int_{g(t)}^{t} b(s)ds\right\},$$

we have for $t \geq t_1$

$$x_1 a(t) - x_2 b(t) \geq a(t) \exp\left\{x_1 \int_{h(t)}^{t} a(s)ds - x_2 \int_{h(t)}^{t} b(s)ds\right\} - b(t) \exp\left\{x_1 \int_{g(t)}^{t} a(s)ds - x_2 \int_{g(t)}^{t} b(s)ds\right\}.$$

Then $u(t) = x_1 a(t) - x_2 b(t)$ for $t \geq t_1$, $u(t) = 0$ for $t < t_1$ is a nonnegative solution of inequality (3.2.4). By Theorem 3.2, the fundamental function of (3.1.1) is positive for $t \geq s \geq t_1$ and there exists an eventually positive solution with an eventually nonpositive derivative. □

Denote

$$B_{11} = \sup_{t \geq t_1} \int_{h(t)}^{t} a(s)ds, \;\; B_{12} = \inf_{t \geq t_1} \int_{h(t)}^{t} b(s)ds,$$
$$B_{21} = \sup_{t \geq t_1} \int_{g(t)}^{t} a(s)ds, \;\; B_{22} = \inf_{t \geq t_1} \int_{g(t)}^{t} b(s)ds.$$

**Corollary 3.9** *Suppose $a(t) \geq b(t)$, $g(t) \geq h(t)$ and there exist $t_1 \geq 0$, $x_1 > 0$, $x_2 > 0$ such that the system*

$$\begin{cases} \ln x_1 \geq x_1 B_{11} - x_2 B_{12}, \\ \ln x_2 \leq x_1 B_{21} - x_2 B_{22}, \end{cases} \tag{3.4.1}$$

*has a positive solution $(x_1, x_2)$, where $x_1 a(t) - x_2 b(t) \geq 0$ for $t \geq t_1$.*

*Then the fundamental function of (3.1.1) is positive for $t \geq s \geq t_1$ and there exists an eventually positive solution with an eventually nonpositive derivative.*

**Corollary 3.10** *Suppose $a(t) \geq b(t)$, $g(t) \geq h(t)$ and there exist $t_1 \geq 0$ and $C > 0$ such that $B_{11} \neq 0$, $B_{12} \neq 0$, $B_{21} = B_{22} = 0$,*

$$\ln(eB_{11}) \leq C \leq B_{12}, \tag{3.4.2}$$

*and for $t \geq t_1$*

$$\frac{a(t)}{B_{11}} \geq \frac{Cb(t)}{B_{12}}. \tag{3.4.3}$$

*Then the fundamental function of (3.1.1) is positive for $t \geq s \geq t_1$ and there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* Under the conditions of the corollary, system (3.4.1) takes the form

$$\begin{cases} \ln x_1 \geq x_1 B_{11} - x_2 B_{12}, \\ \ln x_2 \leq 0. \end{cases} \tag{3.4.4}$$

Inequalities (3.4.2) yield that $(x_1, x_2) = (\frac{1}{B_{11}}, \frac{C}{B_{12}})$ is a solution of (3.4.4). Inequality (3.4.3) implies $x_1 a_1(t) \geq x_2 b_1(t)$. Hence, for (3.1.1) the statement of the corollary holds. □

For (3.2.13), we can now obtain new explicit nonoscillation conditions under which there exists an eventually positive solution with an eventually nonnegative derivative.

**Corollary 3.11** *Suppose $a \geq b$, $\delta \geq \sigma$,*

$$(a\delta)^{\sigma/\delta} \ln(a\delta e) \leq b\delta \leq (a\delta)^{\sigma/\delta}. \tag{3.4.5}$$

*Then the fundamental function of* (3.2.13) *is positive for $t \geq \delta$ and there exists an eventually positive solution with an eventually nonpositive derivative.*

*Proof* For (3.2.13), system (3.4.1) has the form

$$\ln x_1 \geq a\delta x_1 - b\delta x_2, \ \ln x_2 \leq a\sigma x_1 - b\sigma x_2,$$

so it is sufficient to prove that the following system of two equations has a positive solution:

$$\begin{cases} \ln x_1 = a\delta x_1 - b\delta x_2, \\ \ln x_2 = a\sigma x_1 - b\sigma x_2. \end{cases} \tag{3.4.6}$$

We have $\frac{\ln x_1}{\ln x_2} = \frac{\delta}{\sigma}$, and hence $x_2 = x_1^{\sigma/\delta}$. After the substitution of $x_2$ in the first equation of system (3.4.6), we have

$$x_1 = \frac{1}{a\delta} \ln x_1 + \frac{b}{a} x_1^{\sigma/\delta}. \tag{3.4.7}$$

Denote

$$f(x) = x - \frac{1}{a\delta} \ln x, \ g(x) = \frac{b}{a} x^{\sigma/\delta}.$$

Equation (3.4.7) has a solution if for some $x_1 > 0$ we have $f(x_1) = g(x_1)$. Since

$$f(x) > 0, \ 0 < x < 1, \ \lim_{x\to 0} f(x) = \lim_{x\to\infty} f(x) = \infty, \ g(0) = 0, \ \lim_{x\to\infty} g(x) = \infty,$$

the function $f(x)$ has a minimum point $x_0$. We will prove that $f(x_0) \leq g(x_0)$. It will imply that there exists $x_1 \in (0, x_0]$ such that $f(x_1) = g(x_1)$ and hence (3.4.7) has a positive solution. Since

$$f'(x) = 1 - \frac{1}{a\delta x} = 0, \ x_0 = \frac{1}{a\delta}, \ f_{\min}(x_0) = \frac{1}{a\delta} \ln(a\delta e), \ g(x_0) = \frac{b}{a(a\delta)^{\sigma/\delta}},$$

inequality $f_{\min}(x_0) < g(x_0)$ has the form

$$\ln(a\delta e) < \frac{b\delta}{(a\delta)^{\sigma/\delta}},$$

which follows from the left inequality in (3.4.5). Hence the system (3.4.6) has a positive solution,

$$(x_1, x_2), \text{ where } x_1 \leq \frac{1}{a\delta}, \; x_2 = x_1^{\frac{\sigma}{\delta}} \leq \frac{1}{(a\delta)^{\frac{\sigma}{\delta}}}.$$

We have only to check the inequality $a(t)x_1 \geq b(t)x_2$ in Corollary 3.9. For (3.2.12), this inequality has the form $\frac{1}{\delta} \geq \frac{b}{(a\delta)^{\sigma/\delta}}$, or $b\delta \leq (a\delta)^{\sigma/\delta}$, which follows from the right inequality in (3.4.5). □

*Example 3.6* Consider the equation

$$\dot{x}(t) + 0.8x(t-1) - 0.7x(t-0.5) = 0.$$

By Corollary 3.11, this equation has a positive fundamental function. Corollaries 3.5 and 3.7 fail for this equation.

**Corollary 3.12** *Suppose that*

$$b \leq b(t) \leq a(t) \leq a, \; t - \delta \leq h(t) \leq g(t) \leq t - \sigma$$

*and at least one of the conditions of Corollaries 3.5, 3.7 and 3.11 hold. Then the fundamental function of (3.1.1) is eventually positive (for $t \geq s \geq t_1$) and there exists an eventually positive solution with an eventually nonpositive derivative.*

Let us illustrate the results above with several examples.

*Example 3.7* Consider the equation

$$\dot{x}(t) + ax(t-\delta) - bx\big(g(t)\big) = 0, \; t \geq t_0, \tag{3.4.8}$$

where $a \geq b$, $\delta > 0$, $g(t) \leq t$, $\lim_{t\to\infty}(t - g(t)) = 0$.

For nonoscillation conditions, we apply Corollary 3.10. We have $B_{11} = a\delta$, $B_{12} = b\delta$, $B_{21} = B_{22} = 0$. Inequality (3.4.3) becomes $C < 1$, and (3.4.2) is satisfied whenever

$$\ln(a\delta e) \leq b\delta, \; a\delta \leq 1. \tag{3.4.9}$$

Thus, if (3.4.9) holds, then (3.4.8) has a positive fundamental function and an eventually positive solution with an eventually nonpositive derivative.

If $\ln(a\delta e) > b\delta$, then [78] all solutions of (3.4.8) are oscillatory.

*Example 3.8* Consider the equation

$$\dot{x}(t) + \frac{a}{\sqrt{t}}x(t - \sqrt{t}) - \frac{b}{\sqrt{t}}x(t-\tau) = 0, \; t \geq t_0 > 0, \tag{3.4.10}$$

where $a \geq b$, $\tau > 0$. For $B_{ij}$, we have $B_{11} = a$, $B_{12} = b$, $B_{21} = B_{22} = 0$.

By the same calculations as in Example 3.7, we have that if

$$ae^{-b} \le \frac{1}{e}, \ a \le 1, \tag{3.4.11}$$

then (3.4.10) has a positive fundamental function and an eventually positive solution with an eventually nonpositive derivative. If

$$ae^{-b} > \frac{1}{e}, \tag{3.4.12}$$

then [78] all solutions of (3.4.10) are oscillatory.

*Example 3.9* Consider the equation

$$\dot{x}(t) + \frac{a}{t}x\left(\frac{t}{\mu}\right) - \frac{b}{t}x(t-\tau) = 0, \ t \ge t_0 > 0, \tag{3.4.13}$$

where $a \ge b$, $\mu > 1$, $\tau > 0$.

For $B_{ij}$, we have $B_{11} = a \ln \mu$, $B_{12} = b \ln \mu$, $B_{21} = B_{22} = 0$. If

$$a\mu^{-b} \le \frac{1}{e \ln \mu}, \tag{3.4.14}$$

then (3.4.13) has an eventually positive solution with an eventually nonpositive derivative.

If

$$a\mu^{-b} > \frac{1}{e \ln \mu}, \tag{3.4.15}$$

then [78] all solutions of (3.4.13) are oscillatory.

*Example 3.10* Consider the equation

$$\dot{x}(t) + \frac{a}{t}x\left(\frac{t}{\mu}\right) - \frac{b}{t}x\left(\frac{t}{\nu}\right) = 0, \ t \ge t_0, \tag{3.4.16}$$

where $a \ge b$, $\mu > \nu > 1$. Corollary 3.4 implies that if

$$\left(a - \frac{b}{e}\right) \ln \mu \le \frac{1}{e},$$

then (3.4.16) has an eventually positive solution with an eventually nonpositive derivative.

Let us note that if

$$(a-b)\left(1 + a \ln \frac{\mu}{\nu}\right) \ln \nu > \frac{1}{e},$$

then [78] all solutions of (3.4.16) are oscillatory.

To extend the results obtained in this chapter for equations with an arbitrary number of positive and negative coefficients, we consider the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) - \sum_{i=1}^{n} b_i(t)x\big(g_i(t)\big) = 0, \ t \ge t_0, \tag{3.4.17}$$

where for all the parameters of this equation conditions (b1) and (b2) hold. It is not difficult to generalize the results obtained for (3.1.1) to this equation.

Denote

$$a(t) = \sum_{k=1}^{m} a_k(t), \; b(t) = \sum_{i=1}^{n} a_i(t),$$

$$h(t) = \min_k \{h_k(t)\}, \; g(t) = \min_i \{g_i(t)\}, H(t) = \max_k \{h_k(t)\}, \; G(t) = \max_i \{g_i(t)\}.$$

In the future, we will need the following lemma.

**Lemma 3.2** *Suppose $c_k(t) \geq 0$, $r_k(t)$, $k = 1, \cdots, n$ are measurable functions and $x(t)$ is a continuous function. Then there exists a measurable function $r(t)$, $\min_k r_k(t) \leq r(t) \leq \max_k r_k(t)$, such that $\sum_{k=1}^n c_k(t)x(r_k(t)) = (\sum_{k=1}^n c_k(t))x(r(t))$.*

*Proof* For a fixed $t$, denote $A = \min_k r_k(t)$, $B = \max_k r_k(t)$. Then

$$\sum_{k=1}^{n} c_k(t)x\big(r_k(t)\big) \leq \sum_{k=1}^{n} c_k(t) \max_k x\big(r_k(t)\big) \leq \sum_{k=1}^{n} c_k(t) \max_{A \leq t \leq B} x(t),$$

and similarly

$$\sum_{k=1}^{n} c_k(t)x\big(r_k(t)\big) \geq \sum_{k=1}^{n} c_k(t) \min_{A \leq t \leq B} x(t).$$

By the Intermediate Value Theorem,

$$\frac{\sum_{k=1}^n c_k(t)x(r_k(t))}{\sum_{k=1}^n c_k(t)} = x(D),$$

where $A \leq D \leq B$, which completes the proof of the lemma if we denote $r(t) := D$. ☐

**Theorem 3.8** *Suppose*

$$a_k(t) \geq 0, \; b_i(t) \geq 0, \; H(t) \leq g(t), \; \sum_{k=1}^{m} a_k(t) \geq \sum_{i=1}^{n} b_i(t), \; t \geq t_1,$$

*and the inequality*

$$u(t) \geq a(t) \exp\left\{ \int_{h(t)}^{t} u(s)ds \right\} - b(t) \exp\left\{ \int_{G(t)}^{t} u(s)ds \right\}, \; t \geq t_1, \tag{3.4.18}$$

*has a nonnegative locally integrable solution satisfying $u(t) \geq 0$ for $t \geq t_1$ and $u(t) = 0$ for $t < t_1$. Then, for the fundamental function of (3.4.17), we have $X(t, s) > 0$, $t \geq s \geq t_1$ and (3.4.17) has an eventually positive solution with an eventually nonpositive derivative.*

*Proof* Let $x(t) = X(t, t_1)$. By Lemma 3.2, $x(t)$ is a solution of the equation

$$\dot{x}(t) + a(t)x\big(\tilde{h}(t)\big) - b(t)x\big(\tilde{g}(t)\big) = 0, \ t \ge t_1, \tag{3.4.19}$$

for some $h(t) \le \tilde{h}(t) \le H(t)$, $g(t) \le \tilde{g}(t) \le G(t)$. If $u(t) \ge 0$ is a solution of inequality (3.4.18), then this function is also a nonnegative solution of the inequality

$$u(t) \ge a(t)\exp\left\{\int_{\tilde{h}(t)}^{t} u(s)ds\right\} - b(t)\exp\left\{\int_{\tilde{g}(t)}^{t} u(s)ds\right\}, \ t \ge t_1. \tag{3.4.20}$$

Theorem 3.2 implies that the fundamental function $Y(t, s)$ of (3.4.19) is positive. But $Y(t, t_1) = X(t, t_1)$, so also $X(t, t_1) > 0$. Inequality $X(t, s) > 0$ for any $t \ge s \ge t_1$ is proven similarly. □

Theorems 3.5 and 3.8 immediately imply the following result.

**Corollary 3.13** *Suppose*

$$a_k(t) \ge 0, \ b_k(t) \ge 0, \ H(t) \le g(t), \ \sum_{k=1}^{m} a_k(t) \ge \sum_{i=1}^{n} b_i(t), \ t \ge t_1.$$

*If there exists* $\lambda \in (0, 1)$ *such that*

$$\limsup_{t\to\infty} \int_{h(t)}^{G(t)} \big[a(s) - \lambda b(s)\big]ds < \frac{1}{e}\ln\frac{1}{\lambda},$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \big[a(s) - \lambda b(s)\big]ds < \frac{1}{e},$$

*then, for the fundamental function of* (3.4.17), *we have* $X(t, s) > 0$, $t \ge s \ge t_1$ *and* (3.4.17) *has an eventually positive solution with an eventually nonnegative derivative.*

## 3.5 Equations with an Oscillatory Coefficient

Consider the equation with a variable delay and a variable oscillating coefficient

$$\dot{x}(t) + a(t)x\big(h(t)\big) = 0, \tag{3.5.1}$$

where $h(t) \le t$.

First, let us discuss the connection of stability and nonoscillation for (3.5.1), where the coefficient $a(t)$ can be oscillatory but in some sense its positive part "prevails". Can we claim that under the assumption $\int_0^\infty a(s)ds = \infty$ all positive solutions tend to zero? The following example demonstrates that there is no easy answer to this question.

*Example 3.11* Consider (3.5.1) with

$$a(t) = \begin{cases} \alpha + \beta, & t \in (2n, 2n+1], \\ -\alpha, & t \in (2n+1, 2n+2], \end{cases}$$

$$t - h(t) = \begin{cases} \{t\} + 1, & t \in (2n, 2n+1], \\ 0, & t \in (2n+1, 2n+2], \end{cases}$$

where $\alpha > 0$, $\beta > 0$ and $\{t\}$ is the fractional part of $t$. Then (3.5.1) becomes

$$x'(t) = \begin{cases} -(\alpha+\beta)x(2n-1), & t \in (2n, 2n+1], \\ \alpha x(t), & t \in (2n+1, 2n+2]. \end{cases} \tag{3.5.2}$$

Consider (3.5.2) for $t \geq 0$ with $x(s) = s + 1$ for $s \leq 0$. Then $x(-1) = 0$, $x(0) = 1 = x(1)$, $x(2) = e^{\alpha}$, $x(3) = e^{\alpha} - \alpha - \beta, \cdots, x(2n+1) = x(2n-1)[e^{\alpha} - \alpha - \beta]$ and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds = \int_{t-2}^{t} a(s)ds = \beta, \quad \int_{0}^{\infty} a(s)ds = \infty.$$

However, for any $\beta > 0$, there exists $\alpha > 0$ large enough that $e^{\alpha} - \alpha - \beta > 1$, which means that $x(2n+1) > x(2n-1)$, so the solution is unstable and nonoscillatory. Thus there is no constant $A$ such that for equations with oscillating $a(t)$ the inequality

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < A$$

implies stability if two additional conditions hold:

$$\liminf_{t\to\infty} \int_{t}^{t+H} a(s)ds > 0 \text{ for some } H > 0, \quad \int_{0}^{\infty} a(s)ds = \infty.$$

Example 3.11 implies that for any $\alpha > 0$ there exists a nonoscillatory equation (3.5.1) with oscillating $a(t)$ such that

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds = \alpha.$$

Some new sufficient stability results for (3.5.1) with oscillatory coefficients can be found in the recent paper [67].

Next, let us proceed to necessary nonoscillation conditions. In private discussions, Y. Domshlak and I. Stavroulakis proposed the following hypothesis.

**Conjecture** *Consider* (3.5.1) *with an oscillatory coefficient* $a(t)$. *The inequality*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < \frac{1}{e} \tag{3.5.3}$$

*implies nonoscillation of* (3.5.1), *and the condition*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} a(s)ds > \frac{1}{e} \tag{3.5.4}$$

*implies oscillation of* (3.5.1).

The following example demonstrates that the second part is incorrect; moreover, there is no $A > 0$ such that the inequality

$$\liminf_{t\to\infty} \int_{h(t)}^{t} a(s)ds > A \tag{3.5.5}$$

implies oscillation of all solutions. Thus the inequality

$$\liminf_{t\to\infty} \int_{h(t)}^{t} a(s)ds \leq \frac{1}{e}$$

is no longer necessary for nonoscillation if the coefficient of (3.5.1) may become negative.

*Example 3.12* Consider (3.5.1) with

$$t - h(t) = \begin{cases} 2, & t \in (2n, 2n+1], \\ \{t\} + 3, & t \in (2n+1, 2n+2], \end{cases} \quad a(t) = \begin{cases} -\alpha, & t \in (2n, 2n+1], \\ e^{\alpha} - 1, & t \in (2n+1, 2n+2], \end{cases}$$

where $\alpha > 0$. Then (3.5.1) becomes

$$x'(t) = \begin{cases} \alpha x(t-2), & t \in (2n, 2n+1], \\ (e^{\alpha} - 1)x(2n-2), & t \in (2n+1, 2n+2]. \end{cases} \tag{3.5.6}$$

Thus, the solution of (3.5.6) with the initial function

$$x(t) = \begin{cases} e^{\alpha(t+2)}, & t \in [-2, -1], \\ e^{\alpha} - (e^{\alpha} - 1)(t+1), & t \in [-1, 0], \end{cases}$$

is two-periodic. In fact,

$$x'(t) = \alpha e^{\alpha t},\ t \in [0,1],\ x(0) = 1 \Rightarrow x(t) = e^{\alpha t},\ t \in [0,1],$$
$$x'(t) = -\big(e^{\alpha} - 1\big)x(-2) = 1 - e^{\alpha},\ t \in [1,2]$$
$$\Rightarrow x(t) = e^{\alpha} - \big(e^{\alpha} - 1\big)(t-1),\ t \in [1,2],$$

so $x(2) = 1$. This solution is two-periodic and nonoscillatory, while

$$\liminf_{t\to\infty} \int_{h(t)}^{t} a(s)ds = \int_{2n-3}^{2n} a(s)ds = e^{\alpha} - 2\alpha - 1 > \frac{1}{e}$$

for $\alpha \geq \alpha_0 \approx 1.4522$. Here $\alpha_0$ is the only positive solution of the equation $e^x - 2x - 1 = 1/e$ (the function on the left-hand side is increasing). Moreover, lim inf exceeds $1/e$ for $\alpha > 1.4522$. Thus (3.5.6) has a nonoscillatory solution, while $A < e^{\alpha} - 2\alpha - 1$ for some $\alpha$ in (3.5.5) can be any positive number.

## 3.6 Discussion and Open Problems

For nonautonomous delay differential equations (DDEs) with positive and negative coefficients, the first oscillation result [246] was obtained only in 1984, much later than the first oscillation results for delay differential equations with positive coefficients appeared. Chuanxi and Ladas [94] obtained for the equation

$$\dot{x}(t) + a(t)x(t-\tau) - b(t)x(t-\sigma) = 0, \ t \geq t_0, \tag{3.6.1}$$

$a(t) \geq 0$, $b(t) \geq 0$, $\tau > 0$, $\sigma > 0$, the following well-known result.

**Theorem 3.9** [94] *Suppose $a(t)$ and $b(t)$ are continuous functions, $\tau > \sigma$,*

$$\int_{t-\tau+\sigma}^{t} b(s)ds \leq 1, \ a(t) \geq b(t-\tau+\sigma), \tag{3.6.2}$$

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \big[a(s) - b(s-\tau+\sigma)\big]ds > \frac{1}{e}. \tag{3.6.3}$$

*Then all the solutions of (3.6.1) are oscillatory.*

Similar results were obtained in [155, 235]. Later many publications appeared that improved the results of [94, 155, 235] and extended them to various classes of equations, including the second-order and higher-order equations and neutral type equations. Here we refer the reader to the papers [92, 143, 179, 259–261, 342, 346, 357] and references therein.

In [261], the following result was obtained.

**Theorem 3.10** [261] *Suppose $a(t)$ and $b(t)$ are continuous functions, $\tau > \sigma$, condition (3.6.2) holds and*

$$\liminf_{t\to\infty} \left( \int_{t-\tau}^{t} \big[a(s) - b(s-\tau+\sigma)\big]ds + \frac{1}{e}\int_{t-\tau+\sigma}^{t} b(s-\tau)ds \right) > \frac{1}{e}.$$

*Then all solutions of (3.6.1) are oscillatory.*

However, all these publications except [143] consider equations with constant delays only, condition (3.6.2) remains unchanged and only (3.6.3) is improved. In the present chapter, we deal with a more general case: unlike the publications above, we consider instead of (3.6.1) equation (3.1.1) with arbitrary delays and coefficients; moreover, $a(t) \geq 0$ and $b(t) \geq 0$ are not assumed to be continuous. Besides, instead of the second inequality in (3.6.2), we consider the condition $a(t) \geq b(t)$, which seems to be more natural. The method of investigation is based on the properties of linear operators in the corresponding spaces. The basic result is that the existence of a nonnegative solution of a nonlinear integral inequality, which is explicitly constructed by (3.1.1), implies positivity of the fundamental function for this equation.

Theorems of this kind are well known and widely applied for delay differential equations with positive coefficients (see Chap. 2). For (3.1.1), this result was published in [81]. As an immediate corollary of the main proposition, we obtained the comparison theorem for (3.1.1) and the result that all nonoscillatory solutions tend to zero at infinity, which is well known for equations with positive coefficients. Here we presented various explicit conditions for the existence of a nonoscillatory solution of (3.1.1). It should be noted that, unlike oscillation, nonoscillation even of (3.6.1) has not been extensively studied. We mention here [92] and [154, Theorem 2.5.2]. Examples illustrate sharpness of nonoscillation conditions obtained in the present chapter and their applications.

Finally, let us formulate some open problems as well as topics for research and discussion.

1. For (3.1.1) with coefficients of arbitrary signs, find sufficient conditions on the parameters of the equation where existence of a positive solution implies positivity of the fundamental function. In particular, these conditions include (but we believe are not reduced to) the case of nonnegative coefficients.
2. Prove or disprove:
   For any given $a(t) > b(t) \geq 0$ and $h(t) \leq t$, there exists $g(t) \leq t$ such that (3.1.1) has a nonoscillatory solution.
3. Prove or disprove:
   If $\sum_{k=1}^{m} a_k(t) \leq \sum_{k=1}^{n} b_k(t)$, then (3.4.17) has a nonoscillatory solution.
4. Prove or disprove:
   Suppose that the conditions of Theorem 3.2 hold. If $x(t_1) > 0$ and $\sup_{t \leq t_1} \varphi(t) < x(t_1)$, then the solution of the initial value problem is positive.
5. Find nonoscillation conditions for (3.1.1) without the assumption that $h(t) \leq g(t)$.
6. Prove or disprove:
   If the assumptions of Theorem 3.4 hold, then for any solution $x$ of (3.1.1) we have $\lim_{t\to\infty} x(t) = 0$.
7. Find sufficient conditions under which any nonoscillatory solution of (3.5.1) with an oscillating coefficient tends to zero.
8. Prove or disprove:
   The inequality
   $$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < \frac{1}{e} \tag{3.6.4}$$
   implies nonoscillation of (3.5.1) with an oscillatory coefficient.

# Chapter 4
# Oscillation of Equations with Distributed Delays

## 4.1 Introduction

In this chapter, we consider for any point $t_1$ the scalar equation with a distributed delay

$$\dot{x}(t) + \int\limits_{-\infty}^{t} x(s)\, d_s R(t,s) = f(t),\ t > t_1, \tag{4.1.1}$$

where $R(t,\cdot)$ is a function of a bounded variation and $f$ is locally essentially bounded, with the initial function

$$x(t) = \varphi(t),\ t < t_1. \tag{4.1.2}$$

As particular cases, homogeneous equation (4.1.1) includes the following models:

1) the differential equation with several variable delays

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(h_k(t)\big) = 0, \tag{4.1.3}$$

if we assume that $R = R_1$, where

$$R_1(t,s) = \sum_{k=1}^{m} a_k(t) \chi_{(h_k(t),\infty)}(s), \tag{4.1.4}$$

where $\chi_I$ is the characteristic function of interval $I$;
2) the integrodifferential equation

$$\dot{x}(t) + \int_{-\infty}^{t} K(t,s) x(s)\, ds = 0, \tag{4.1.5}$$

if we assume that $R = R_2$, where

$$R_2(t,s) = \int_{-\infty}^{s} K(t,\zeta)\, d\zeta; \tag{4.1.6}$$

R.P. Agarwal et al., *Nonoscillation Theory of Functional Differential Equations with Applications*, DOI 10.1007/978-1-4614-3455-9_4,

3) the mixed equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) + \int_{-\infty}^{t} K(t,s)x(s)\,ds = 0, \tag{4.1.7}$$

where $R(t,s) = R_3(t,s) = R_1(t,s) + R_2(t,s)$ and $R_1$ and $R_2$ are defined by (4.1.4) and (4.1.6), respectively;

4) the mixed equation with an infinite number of delays,

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x\big(h_k(t)\big) + \int_{-\infty}^{t} K(t,s)x(s)\,ds = 0, \tag{4.1.8}$$

which is obtained if $R(t,s) = R_4(t,s)$, where

$$R_4(t,s) = \sum_{k=1}^{\infty} a_k(t)\chi_{(h_k(t),\infty)}(s) + \int_{-\infty}^{s} K(t,\zeta)\,d\zeta. \tag{4.1.9}$$

Equation (4.1.8) is the most general among (4.1.3), (4.1.5), (4.1.7), (4.1.8). However, it should be noted that not any function of bounded variation can be presented in the form (4.1.9) as a sum of an infinite number of step functions and an absolutely continuous function. Once oscillation of (4.1.1) has been studied, the relevant results for (4.1.3), (4.1.5), (4.1.7), (4.1.8) can be deduced.

The chapter is organized as follows. Section 4.2 contains relevant definitions and known results. In Sect. 4.3, we prove that the following four assertions are equivalent: nonoscillation of the equation and the corresponding differential inequality, positivity of the fundamental function and existence of a nonnegative solution of a certain nonlinear integral inequality that is constructed explicitly from the differential equation. Section 4.4 involves comparison theorems that in particular allow us to compare oscillation properties of equations with concentrated and distributed delays. Next, in Sect. 4.5, sharp nonoscillation conditions for several classes of autonomous integrodifferential equations are considered. Further, using comparison theorems, we obtain efficient nonoscillation conditions for various classes of nonautonomous delay equations. Section 4.6 contains efficient nonoscillation and oscillation conditions. Section 4.7 considers slowly oscillating solutions, while Sect. 4.8 presents nonoscillation conditions for equations with positive and negative coefficients. Finally, Sect. 4.9 involves some discussion and open problems.

## 4.2 Preliminaries

We study initial value problem (4.1.1), (4.1.2) for the equation with a distributed delay under the following assumptions:

(a1) $R(t,\cdot)$ is a left continuous scalar function of bounded variation, and for each $s$ its variation on the segment $[t_1, s]$

$$P(t,s) = \mathrm{Var}_{\tau\in[t_1,s]}\, R(t,\tau) \tag{4.2.1}$$

is a locally integrable function in $t$ for any $s > t_1$ and $R(t,s) = R(t,t^+) = \lim_{s\to t^+} R(t,s)$ for $t < s$.

(a2) There exist $M > 0$ and $\lambda > 0$ such that for any $s$ and $t$ the inequality

$$\int_s^t \left| d_\tau R(t,\tau) \right| \le M e^{-\lambda(t-s)} \tag{4.2.2}$$

holds.

(a3) $f : [t_1, \infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi : (-\infty, t_1) \to \mathbb{R}$ is a Lebesgue measurable locally bounded function such that the Lebesgue-Stieltjes integral in (4.1.1) exists; in particular, if $\varphi$ is a continuous function then (4.1.1) is well defined for any $R(t,s)$. We assume also that for some $M_1 > 0$ and $\mu \in [0, \lambda)$

$$\left|\varphi(t)\right| \le M_1 e^{-\mu(t-t_1)}, \ t \le t_1. \tag{4.2.3}$$

**Definition 4.1** As in the previous chapters, a function $x : \mathbb{R} \to \mathbb{R}$ is called *a solution* of (4.1.1), (4.1.2) if it satisfies (4.1.1) for almost every $t \in [t_1, \infty)$, and (4.1.2) holds for $t < t_1$.

All the functions of bounded variation considered in this chapter are left continuous, so all the integrals are understood as

$$\int_{-\infty}^{t} x(s)\, d_s R(t,s) = \int_{-\infty}^{t^+} x(s)\, d_s R(t,s) = \lim_{\tau\to t^+} \int_{-\infty}^{\tau} x(s)\, d_s R(t,s);$$

in particular, for $R(t,s) = a(t)\chi_{(t,\infty)}(s)$,

$$\int_{-\infty}^{t} x(s)\, d_s R(t,s) = a(t)x(t),$$

which corresponds to a nondelay term.

**Definition 4.2** For each $s \ge t_1$, denote by $X(t,s)$ the solution of the problem

$$\dot{x}(t) + \int_{-\infty}^{t} x(\tau)\, d_\tau R(t,\tau) = 0, \ t \ge s, \ x(t) = 0, \ t < s, \ x(s) = 1. \tag{4.2.4}$$

$X(t,s)$ is called *the fundamental function* of (4.1.1).

By Theorem B.3, there exists one and only one solution of problem (4.1.1), (4.1.2), and it can be presented in the form

$$x(t) = X(t,t_1)x_0 + \int_{t_1}^{t} X(t,s) f(s) ds - \int_{t_1}^{t} X(t,s)\, ds \int_{-\infty}^{s} \varphi(\tau)\, d_\tau R(s,\tau), \tag{4.2.5}$$

where $\varphi(\tau) = 0$, if $\tau > t_1$.

We will also study (4.1.1), (4.1.2) with a bounded aftereffect; i.e., when the following hypothesis holds:

(a4) $R(t,s)=\sum_{k=1}^{\infty} R_k(t,s)$, $r(t):=\sum_{k=1}^{\infty}\int_{-\infty}^{t} d_s R_k(t,s)$ is a locally essentially bounded function and for each $t_1$ and $k$ there exists $s_k = s_k(t_1)\le t_1$ such that $R_k(t,s)=0$ for $s<s_k$, $t>t_1$, where $\inf_k(s_k(t))>-\infty$, $\lim_{t\to\infty}\inf_k s_k(t)=\infty$.

If (a4) holds, then we can introduce the function

$$h_k(t)=\inf_{s\le t}\{s\,|\,R_k(t,s)\ne 0\} \tag{4.2.6}$$

such that $\lim_{t\to\infty} h_k(t)=\infty$ and rewrite homogeneous equation (4.1.1) in the form

$$\dot{x}(t)+\sum_{k=1}^{\infty}\int_{h_k(t)}^{t} x(s)\,d_s R_k(t,s)=0,\ \ t\ge t_1. \tag{4.2.7}$$

Together with (4.1.1), let us consider the homogeneous equation

$$\dot{x}(t)+\int_{-\infty}^{t} x(s)\,d_s R(t,s)=0,\ \ t\ge t_1, \tag{4.2.8}$$

and the differential inequality

$$\dot{y}(t)+\int_{-\infty}^{t} y(s)\,d_s R(t,s)\le 0,\ \ t\ge t_1. \tag{4.2.9}$$

Let us study the existence of an eventually positive solution of (4.2.8), (4.1.2).

**Definition 4.3** Equation (4.2.8) *has a positive solution* for $t\ge t_1$ if there exists an initial function $\varphi$ such that a solution of (4.2.8), (4.1.2) is positive for $t\ge t_1$.

## 4.3 Existence of a Positive Solution—General Results

**Theorem 4.1** *Suppose* (a1)–(a3) *hold and* $R(t,\cdot)$ *is a nondecreasing function. Then the following hypotheses are equivalent*:

1) *There exists a point* $t_1$ *and an initial function* $\varphi(t)\ge 0$ *such that problem* (4.2.9), (4.1.2) *has a positive solution for* $t\ge t_1$.
2) *There exists a point* $t_1$ *such that the inequality*

$$u(t)\ge\int_{t_1}^{t}\exp\left\{\int_{s}^{t} u(\tau)\,d\tau\right\} d_s R(t,s) \tag{4.3.1}$$

*has a nonnegative locally integrable solution* $u(t)$ *for* $t\ge t_1$, *where* $u(t)=0$, $t<t_1$.
3) *There exists a point* $t_1$ *such that* $X(t,s)>0$ *for* $t>s\ge t_1$.
4) *There exists a point* $t_1$ *such that* (4.2.8) *has a positive solution for* $t\ge t_1$, *with* $x(t)=0$, $t<t_1$.

*Proof* Let us prove the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1).

1) $\Rightarrow$ 2) Let $y$ be a positive solution of (4.2.9), (4.1.2), where $\varphi(t)\ge 0$.

Since $\dot{y}(t) \le 0$ for $t \ge t_1$, the solution $y(t)$ is nonincreasing, and $u(t)$ defined for $t \ge t_1$ as

$$u(t) = -\frac{d}{dt} \ln \frac{y(t)}{y(t_1)}$$

satisfies $u(t) \ge 0$. We have

$$y(t) = y(t_1) \exp\left\{ -\int_{t_1}^{t} u(s)\, ds \right\}.$$

Inequality (4.2.9) is equivalent to

$$\dot{y}(t) + \int_{-\infty}^{t_1} \varphi(s)\, d_s R(t,s) + \int_{t_1}^{t} y(s)\, d_s R(t,s) \le 0.$$

By substituting $y$ into this inequality, we obtain

$$-y(t_1) \exp\left\{ -\int_{t_1}^{t} u(s)\, ds \right\} u(t) + \int_{-\infty}^{t_1} \varphi(s)\, d_s R(t,s)$$
$$+ y(t_1) \int_{t_1}^{t} \exp\left\{ -\int_{t_1}^{s} u(\tau)\, d\tau \right\} d_s R(t,s) \le 0,$$

which implies

$$y(t_1) \exp\left\{ -\int_{t_1}^{t} u(s)\, ds \right\} \left[ -u(t) + \int_{t_1}^{t} \exp\left\{ \int_{s}^{t} u(\tau)\, d\tau \right\} d_s R(t,s) \right] \le 0. \quad (4.3.2)$$

The first factor is positive since $y(t_1) > 0$, so the expression in the brackets is nonpositive and

$$u_1(t) \ge \int_{t_1}^{t} \exp\left\{ \int_{s}^{t} u(\tau)\, d\tau \right\} d_s R(t,s), \ t \ge t_1. \quad (4.3.3)$$

2) $\Rightarrow$ 3) As the first step, let us prove that the fundamental solution is nonnegative for $t \ge s \ge t_1$.

Consider the initial value problem

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s R(t,s) = f(t), \ t > t_1, \ x(t) = 0, \ t \le t_1. \quad (4.3.4)$$

This problem is equivalent to

$$\dot{x}(t) + \int_{t_1}^{t} x(s)\, d_s R(t,s) = f(t), \ t > t_1, \ x(t_1) = 0.$$

Denote by $z$ the function

$$z(t) = \dot{x}(t) + u(t)x(t),$$

where $u$ is a nonnegative solution of (4.3.1) and $x$ is the solution of (4.3.4). Thus

$$x(t) = \int_{t_1}^{t} \exp\left\{ -\int_{s}^{t} u(\tau)\, d\tau \right\} z(s)\, ds, \ t \ge t_1. \quad (4.3.5)$$

After substituting $x$ into (4.3.4), we obtain

$$z(t)-u(t)\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)\,d\tau\right\}z(s)\,ds$$
$$+\int_{t_1}^{t}\left(\int_{t_1}^{s}\exp\left\{-\int_{\theta}^{s}u(\tau)\,d\tau\right\}z(\theta)\,d\theta\right)d_sR(t,s)=f(t).$$

After changing the order of integration in the second integral, we have

$$z(t)-u(t)\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)\,d\tau\right\}z(s)\,ds$$
$$+\int_{t_1}^{t}z(s)\,ds\int_{s}^{t}\exp\left\{-\int_{s}^{\theta}u(\tau)\,d\tau\right\}d_\theta R(t,\theta)=f(t).$$

Thus the left-hand side is equal to

$$\begin{aligned}
&z(t)-u(t)\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)\,d\tau\right\}z(s)\,ds\\
&\quad+\int_{t_1}^{t}z(s)\,ds\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)d\tau\right\}\exp\left\{\int_{\theta}^{t}u(\tau)\,d\tau\right\}d_\theta R(t,\theta)\\
&\quad-\int_{t_1}^{t}z(s)\,ds\int_{t_1}^{s}\exp\left\{-\int_{s}^{\theta}u(\tau)\,d\tau\right\}d_\theta R(t,\theta)\\
&=z(t)-\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)\,d\tau\right\}z(s)\,ds\Bigg[u(t)\\
&\quad-\int_{t_1}^{t}\exp\left\{\int_{\theta}^{t}u(\tau)\,d\tau\right\}d_\theta R(t,\theta)\Bigg]\\
&\quad-\int_{t_1}^{t}z(s)\,ds\int_{t_1}^{s}\exp\left\{-\int_{s}^{\theta}u(\tau)\,d\tau\right\}d_\theta R(t,\theta)=f(t).
\end{aligned}$$

Consequently we obtain the operator equation

$$z-Hz=f, \tag{4.3.6}$$

which is equivalent to (4.3.4), where

$$\begin{aligned}
(Hz)(t)=&\int_{t_1}^{t}\exp\left\{-\int_{s}^{t}u(\tau)\,d\tau\right\}z(s)\,ds\\
&\times\left[u(t)-\int_{t_1}^{t}\exp\left\{\int_{\theta}^{t}u(\tau)\,d\tau\right\}d_\theta R(t,\theta)\right]\\
&+\int_{t_1}^{t}z(s)\,ds\int_{t_1}^{s}\exp\left\{-\int_{s}^{\theta}u(\tau)\,d\tau\right\}d_\theta R(t,\theta).
\end{aligned}$$

Inequality (4.3.1) yields that if $z(t)\geq 0$, then $(Hz)(t)\geq 0$; i.e., $H$ is a positive operator ($t>t_1$). Besides, in each finite interval $[t_1,b]$, the operator $H$ is a sum of integral Volterra operators, which are weakly compact in the space $L[t_1,b]$ by Theorem A.5.

Theorem A.8 implies that the spectral radius is $r(H)=0<1$, and, consequently, if in (4.3.6) the right-hand side $f$ is nonnegative, then

$$z(t)=f(t)+(Hf)(t)+\big(H^2f\big)(t)+\big(H^3f\big)(t)+\cdots\geq 0.$$

We recall that the solution of (4.3.4) has the form (4.3.5), with $z$ being a solution of (4.3.6). Thus, if in (4.3.4) we have $f \geq 0$, then $x(t) \geq 0$. On the other hand, the solution of (4.3.4) has the representation

$$x(t) = \int_{t_1}^{t} X(t,s) f(s)\, ds,$$

where $f(t) \geq 0$ implies $x(t) \geq 0$. Hence the kernel of the integral operator is nonnegative (i.e., $X(t,s) \geq 0$ for $t \geq s > t_1$).

As the second step, let us prove that $X(t,s)$ is strictly positive: $X(t,s) > 0$. To this end, consider

$$x(t) = X(t,t_1) - \exp\left\{-\int_{t_1}^{t} u(s)\, ds\right\},\ x(t) = 0,\ t < t_1,$$

and substitute $x(t)$ into the left-hand side of (4.3.4),

$$\begin{aligned}
&X_t'(t,t_1) + u(t)\exp\left\{-\int_{t_1}^{t} u(s)\, ds\right\} + \int_{t_1}^{t} X(s,t_1)\, d_s R(t,s) \\
&\quad - \int_{t_1}^{t} \exp\left\{-\int_{t_1}^{s} u(\tau)\, d\tau\right\} d_s R(t,s) \\
&= 0 + \exp\left\{-\int_{t_1}^{t} u(s)\, ds\right\}\left[u(t) - \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(s)\, ds\right\} d_s R(t,s)\right] \geq 0.
\end{aligned}$$

Hence $x(t)$ is a solution of (4.3.4) with a nonnegative right-hand side. As was demonstrated above, $x(t) \geq 0$, and consequently

$$X(t,t_1) \geq \exp\left\{-\int_{t_1}^{t} u(s)\, ds\right\} > 0.$$

For any $s > t_1$, inequality $X(t,s) > 0$ is verified in a similar way.

3) ⇒ 4) Function $x(t) = X(t,t_1)$ is a positive solution of (4.2.8) for $t \geq t_1$.

Implication 4) ⇒ 1) is obvious. □

**Corollary 4.1** *Suppose* (a1) *and* (a2) *hold*, $R(t,\cdot)$ *is a nondecreasing function for any* $t$ *and at least one of the following hypotheses holds*:

1) *Inequality* (4.2.9) *has a positive solution for any* $t \in \mathbb{R}$.
2) *The inequality*

$$u(t) \geq \int_{-\infty}^{t} \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} d_s R(t,s) \tag{4.3.7}$$

*has a nonnegative locally integrable solution* $u(t)$ *for any* $t \in \mathbb{R}$.

*Then*:

a) $X(t,s) > 0$ *for any* $t > s$.
b) *For any* $t_1$, (4.2.8) *has a positive solution for* $t \geq t_1$.

*Proof* 1) Suppose $x$ is a solution of (4.2.9) for any $t \in \mathbb{R}$. After fixing a point $t_1$ and denoting $y(t) = x(t)$ for $t \geq t_1$, $\varphi(t) = x(t)$ for $t < t_1$, we observe that $y$ is a positive solution of problem (4.2.9), (4.1.2) with a nonnegative initial function $\varphi(t)$.

2) If $u$ is a nonnegative solution of (4.3.7), then for any $t_1$ it also solves the inequality

$$u(t) \geq \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s R(t,s),$$

which completes the proof. □

Consider together with (4.1.7) the relevant inequality

$$\dot{y}(t) + \sum_{k=1}^{m} a_k(t) y\big(h_k(t)\big) + \int_{-\infty}^{t} K(t,s) y(s)\,ds \leq 0. \tag{4.3.8}$$

**Corollary 4.2** *Suppose that the following conditions are satisfied*:

(b1) $a_k \geq 0$, $k = 1, \cdots, m$ *are Lebesgue measurable locally essentially bounded functions*, $h_k : [t_0, \infty) \to R$ *are Lebesgue measurable functions satisfying* $h_k(t) \leq t$, $\limsup_{t\to\infty} h_k(t) = \infty$, $k = 1, \cdots, m$.

(b2) $K(t,s) \geq 0$ *is Lebesgue integrable over each finite square* $[t_0, b] \times [t_0, b]$, *and there exist* $M > 0$ *and* $\lambda > 0$ *such that*

$$K(t,s) \leq M e^{-\lambda(t-s)}.$$

*Then the following hypotheses are equivalent*:

1) *There exists a point* $t_1 \geq t_0$ *and an initial function* $\varphi(t) \geq 0$ *such that problem* (4.3.8), (4.1.2) *has a positive solution for* $t \geq t_1$.
2) *There exists* $t_1 \geq t_0$ *such that the inequality*

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)\,ds\right\} + \int_{t_1}^{t} K(t,s) \exp\left\{\int_s^t u(\tau)\,d\tau\right\} ds \tag{4.3.9}$$

*has a nonnegative locally integrable solution* $u$ *for* $t \geq t_1$ (*in* (4.3.9) *it is assumed that* $u(t) = 0$ *for* $t < t_1$).
3) *There exists* $t_1 \geq t_0$ *such that the fundamental function of* (4.1.7) *is positive for* $t > s \geq t_1$.
4) *There exists* $t_1 \geq 0$ *such that* (4.1.7) *has a positive solution for* $t \geq t_1$, *where* $x(t) = 0$ *for* $t < t_1$.

**Corollary 4.3** *Suppose that* $R_k(t,\cdot)$ *are nondecreasing functions for each* $t$ *and* $k \in \mathbb{N}$, (a1), (a2) *and* (a4) *hold. Then the following hypotheses are equivalent*:

1) *Inequality*

$$\dot{y}(t) + \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} y(t)\,d_s R_k(t,s) \leq 0$$

*has an eventually positive solution.*

2) *There exists $t_1$ such that the inequality*

$$u(t) \geq \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s R_k(t,s) \tag{4.3.10}$$

*has a nonnegative locally integrable solution $u$ for $t \geq t_1$ (in (4.3.10) we assume $u(t) = 0$ for $t < t_1$).*

3) *There exists $t_1$ such that $X(t,s) > 0,\ t > s \geq t_1$.*

4) *Equation (4.2.7) has an eventually positive solution.*

**Corollary 4.4** *Suppose that* (b1) *with $m = \infty$ and* (b2) *hold, as well as the following condition*:

(c1) $a(t) = \sum_{k=1}^{\infty} a_k(t)\chi_{[h_k(t),\infty)}(t)$ *is a locally essentially bounded function.*

*Then the following hypotheses are equivalent*:

1) *There exist a point $t_1$ and an initial function $\varphi(t) \geq 0$ such that the problem*

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x\big(h_k(t)\big) + \int_{-\infty}^{t} K(t,s)x(s)\,ds \leq 0, \tag{4.3.11}$$

(4.1.2) *has a positive solution for $t \geq t_1$.*

2) *There exists $t_1$ such that the inequality*

$$u(t) \geq \sum_{k=1}^{\infty} a_k(t)\exp\left\{\int_{h_k(t)}^{t} u(s)\,ds\right\} + \int_{t_1}^{t} K(t,s)\exp\left\{\int_s^t u(\tau)\,d\tau\right\} ds \tag{4.3.12}$$

*has a nonnegative locally integrable solution $u$ for $t \geq t_1$, where $u(t) = 0$ for $t < t_1$.*

3) *There exists $t_1$ such that the fundamental function of* (4.1.8) *is positive for $t \geq s \geq t_1$.*

4) *There exists $t_1$ such that* (4.1.8), *with $f \equiv 0$, has a positive solution for $t \geq t_1$, where $x(t) = 0$ for $t < t_1$.*

**Corollary 4.5** *Suppose* (b1), (b2) *and* (c1) *hold, as well as the following condition*:

(c2) *There exists a function $h_0(t) \leq t$, $\lim_{t\to\infty} h_0(t) = \infty$ such that $K(t,s) = 0$ for $s < h_0(t)$.*

*Then the following hypotheses are equivalent*:

1) *The inequality*

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x\big(h_k(t)\big) + \int_{h_0(t)}^{t} K(t,s)x(s)\,ds \leq 0 \tag{4.3.13}$$

*has an eventually positive solution.*

2) *There exists* $t_1$ *such that the inequality*

$$u(t) \geq \sum_{k=1}^{\infty} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)\, ds\right\} + \int_{h_0(t)}^{t} K(t,s) \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} ds \tag{4.3.14}$$

*has a nonnegative locally integrable solution* $u$ *for* $t \geq t_1$, *where* $u(t) = 0$ *for* $t < t_1$.

3) *There exists* $t_1$ *such that the fundamental function of* (4.1.8) *is positive for* $t > s \geq t_1$.
4) *Equation* (4.1.8) *has an eventually positive solution.*

## 4.4 Comparison Theorems

Consider together with (4.2.8) the equation with a different distributed delay

$$\dot{x}(t) + \int_{-\infty}^{t} x(s) d_s T(t,s) = 0, \;\; t \geq t_1. \tag{4.4.1}$$

We compare the properties of (4.2.8) and (4.4.1) related to the existence of a nonnegative solution. Theorem 4.1 immediately implies the following result.

**Theorem 4.2** *Suppose* $R$, $T$ *satisfy* (a1) *and* (a2) *and functions* $R(t,\cdot)$, $T(t,\cdot)$ *and the difference* $R(t,\cdot) - T(t,\cdot)$ *are nondecreasing for each* $t \geq t_1$. *If inequality* (4.3.1) *has a nonnegative solution for* $t \geq t_1$ (*with* $u(t) = 0$, $t < t_1$), *then* (4.4.1) *has a positive solution for* $t \geq t_1$ *and its fundamental solution* $Y(t,s)$ *is positive for* $t \geq s \geq t_1$.

*Proof* Let $u(t)$ be a solution of (4.3.1) nonnegative for $t \geq t_1$. Since

$$\begin{aligned} u(t) &\geq \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} d_s R(t,s) \\ &= \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} d_s T(t,s) \\ &\quad + \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} d_s \big[R(t,s) - T(t,s)\big] \\ &\geq \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} d_s T(t,s), \end{aligned}$$

then (4.4.1) has a nonnegative solution for $t \geq t_1$, with $u(t) = 0$ for $t \leq t_1$, and thus by Theorem 4.1 its fundamental solution $Y(t,s)$ is positive for $t \geq s \geq t_1$. □

**Corollary 4.6** *Let* $R$, $T$ *satisfy* (a1) *and* (a2).

1. *If the functions* $R(t,\cdot)$, $T(t,\cdot)$ *and the difference* $R(t,\cdot)-T(t,\cdot)$ *are nondecreasing for each* $t$ *and* (4.2.8) *has a nonoscillatory solution, then* (4.4.1) *also has a nonoscillatory solution.*
2. *If the functions* $R(t,\cdot)$, $T(t,\cdot)$ *and the difference* $T(t,\cdot)-R(t,\cdot)$ *are nondecreasing for each* $t$ *and all the solutions of* (4.4.1) *are oscillatory, then all the solutions of* (4.2.8) *are oscillatory.*

In the future, we will need a more advanced comparison result.

**Theorem 4.3** *Suppose that* (a1) *and* (a2) *hold for* $R$, $T$, *and*

$$\lim_{s\to-\infty} R(t,s) = \lim_{s\to-\infty} T(t,s) = 0, \quad R(t,s) \ge T(t,s) \ge 0, \tag{4.4.2}$$

*where* $R(t,s)$ *and* $T(t,s)$ *are nondecreasing in* $s$ *for each* $t$. *If inequality* (4.3.7) *has a nonnegative solution, then for any* $t_1$ (4.4.1) *has a positive solution for* $t \ge t_1$ *and its fundamental solution* $Y(t,s)$ *is positive for* $t \ge s \ge t_1$.

*Proof* Since (4.3.7) holds, we have

$$\begin{aligned}
u(t) &\ge \int_{-\infty}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s R(t,s) \\
&= \exp\left\{\int_s^t u(\tau)\,d\tau\right\} R(t,s)\Big|_{s\to-\infty}^{s=t} + \int_{-\infty}^{t} R(t,s)u(s)\exp\left\{\int_s^t u(\tau)\,d\tau\right\} ds \\
&= R(t,t) - 0 + \int_{-\infty}^{t} R(t,s)u(s)\exp\left\{\int_s^t u(\tau)\,d\tau\right\} ds \\
&\ge T(t,t) + \int_{-\infty}^{t} T(t,s)u(s)\exp\left\{\int_s^t u(\tau)\,d\tau\right\} ds \\
&= \int_{-\infty}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s T(t,s).
\end{aligned}$$

Application of Corollary 4.1 completes the proof. □

Let us compare (4.1.7) with the equation

$$\dot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) + \int_{-\infty}^{t} M(t,s)x(s)\,ds = 0. \tag{4.4.3}$$

**Theorem 4.4** *Suppose* $a_k$ *and* $b_k$, $h_k$ *and* $g_k$, *and* $K(t,s)$ *and* $M(t,s)$ *satisfy conditions* (b1) *and* (b2) *of Corollary* 4.2, $a_k(t) \ge b_k(t) \ge 0$, $h_k(t) \le g_k(t) \le t$ *and* $K(t,s) \ge M(t,s) \ge 0$ *for any* $t \ge t_1$. *If inequality* (4.3.9) *has a nonnegative solution for* $t \ge t_1$, *then* (4.4.3) *has a positive solution for* $t \ge t_1$ *and its fundamental solution* $Y(t,s)$ *is positive for* $t \ge s \ge t_1$.

*Proof* We have

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)\,ds\right\} + \int_{t_1}^{t} K(t,s) \exp\left\{\int_{s}^{t} u(\tau)\,d\tau\right\} ds$$
$$\geq \sum_{k=1}^{m} b_k(t) \exp\left\{\int_{g_k(t)}^{t} u(s)\,ds\right\} + \int_{t_1}^{t} M(t,s) \exp\left\{\int_{s}^{t} u(\tau)\,d\tau\right\} ds.$$

Corollary 4.2 implies the statement of this theorem. □

By Corollary 4.4, we obtain the following result.

**Theorem 4.5** *Let conditions* (b1), (b2) *with* $m = \infty$ *and* (c1) *of Corollary* 4.4 *hold for* (4.1.8) *and for*

$$\dot{x}(t) + \sum_{k=1}^{\infty} b_k(t) x\big(g_k(t)\big) + \int_{-\infty}^{t} M(t,s) x(s)\,ds = 0, \tag{4.4.4}$$

$K(t,s) \geq M(t,s) \geq 0$, $a_k(t) \geq b_k(t) \geq 0$ *and* $h_k(t) \leq g_k(t) \leq t$.

1. *If* (4.1.8) *has a nonoscillatory solution, then* (4.4.4) *also has a nonoscillatory solution.*
2. *If all solutions of* (4.1.8) *are oscillatory, then all solutions of* (4.4.4) *are also oscillatory.*

Any function of bounded variation can be presented as a difference of two nondecreasing functions (in $s$ for each $t$):

$$R(t,s) = P(t,s) - Q(t,s), \text{ where } \lim_{s \to -\infty} P(t,s) = \lim_{s \to -\infty} Q(t,s) = 0. \tag{4.4.5}$$

**Corollary 4.7** *Suppose that* (a1) *and* (a2) *hold.*

a) *If the inequality*

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_s P(t,s) \leq 0 \tag{4.4.6}$$

*has an eventually positive solution, then* (4.2.8) *has a nonoscillatory solution.*

b) *If the inequality*

$$u(t) \geq \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(\tau)\,d\tau\right\} d_s P(t,s) \tag{4.4.7}$$

*has a nonnegative locally integrable solution* $u$ *for all* $t \geq t_1$, *where* $u(t) = 0$ *for* $t < t_1$, *then* (4.2.8) *has a positive solution for* $t \geq t_1$ *and its fundamental solution* $X(t,s) > 0$, $t \geq s \geq t_1$.

*Proof* Let us compare the solutions of (4.2.8) and the equation

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_s P(t,s) = 0, \ t \geq t_1.$$

Theorem 4.2 yields that there exists a positive solution of (4.2.8) since $Q = P - R$ is nondecreasing in $s$ for each $t$ and all the hypotheses of Theorem 4.1 are satisfied for (4.2.8) as well. □

Theorem 4.3 generalizes comparison Theorem 2.4. It compares oscillation properties of two equations with generally different types of delays.

Now let us compare solutions of the following two problems:

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_s R(t,s) = f(t),\ x(t) = \varphi(t),\ t \le t_1,\ x(t_1) = x_0, \tag{4.4.8}$$

$$\dot{y}(t) + \int_{-\infty}^{t} y(s)\,d_s T(t,s) = g(t),\ y(t) = \psi(t),\ t \le t_1,\ y(t_1) = y_0. \tag{4.4.9}$$

Denote by $Y(t,s)$ the fundamental function of (4.4.9); we recall that $X(t,s)$ is the fundamental function of (4.4.8).

**Theorem 4.6** *Let the parameters of* (4.4.8), (4.4.9) *satisfy* (a1)–(a3). *Suppose that there exists a nonnegative solution of inequality* (4.3.1) *for* $t \ge t_1$, *functions* $R(t,\cdot)$, $T(t,\cdot)$ *and* $R(t,\cdot) - T(t,\cdot)$ *are nondecreasing for each* $t \ge t_1$ *and*

$$g(t) \ge f(t),\ \varphi(t) \ge \psi(t) \ge 0,\ t \le t_1,\ y_0 \ge x_0. \tag{4.4.10}$$

*If* $x(t) > 0$, *then* $y(t) \ge x(t) > 0$, *where* $x(t)$ *and* $y(t)$ *are solutions of* (4.4.8) *and* (4.4.9), *respectively.*

*Proof* Since $R(t,\cdot) - T(t,\cdot)$ is nondecreasing in $s$ for each $t$, then

$$\begin{aligned}
u(t) &\ge \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s R(t,s) \\
&= \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s\big(R(t,s) - T(t,s) + T(t,s)\big) \\
&= \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s\big(R(t,s) - T(t,s)\big) \\
&\quad + \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s T(t,s) \\
&\ge \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s T(t,s).
\end{aligned} \tag{4.4.11}$$

Thus, by Theorem 4.1 we have $X(t,s) > 0$, $Y(t,s) > 0$.

Equation (4.4.8) can be rewritten as

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_s T(t,s) - \int_{-\infty}^{t} x(s)\,d_s\big(T(t,s) - R(t,s)\big) = f(t),\ t \ge t_1,$$

and consequently

$$x(t) = Y(t,t_1)x_0 - \int_{t_1}^{t} Y(t,s)\,ds \int_{-\infty}^{s} \varphi(\tau)\,d_\tau T(s,\tau) + \int_{t_1}^{t} Y(t,s) f(s)\,ds$$
$$- \int_{t_1}^{t} Y(t,s)\,ds \int_{-\infty}^{s} x(\tau)\,d_\tau \big(R(s,\tau) - T(s,\tau)\big),\ t \ge t_1. \qquad (4.4.12)$$

If equality (4.4.12) is compared with

$$y(t) = Y(t,t_1)y_0 - \int_{t_1}^{t} Y(t,s)\,ds \int_{-\infty}^{s} \psi(\tau)\,d_\tau T(s,\tau) + \int_{t_1}^{t} Y(t,s) g(s)\,ds,$$

one can observe $y(t) \ge x(t) \ge 0$ since (4.4.10) holds. □

Let us present a result on the positivity of solutions for equations with a distributed delay.

**Theorem 4.7** *Suppose that $R(t,s)$ is nondecreasing in $s$ for any $t$, there exists $u \ge 0$ such that (4.3.1) has a nonnegative solution for $t \ge t_1$, $f(t) \ge 0$ and*

$$0 \le \varphi(x) \le x_0,\ x_0 > 0. \qquad (4.4.13)$$

*Then the solution of problem (4.4.8) is positive.*

*Proof* Let $u(t)$, $t \ge t_1$, be a solution of (4.3.1), where $u(t) = 0$ for $t < t_1$. By Theorem 4.1, the fundamental function of the equation in problem (4.4.8) is positive: $X(t,s) > 0$ for $t \ge s \ge t_1$. First assume $f \equiv 0$. Consider the auxiliary problem

$$\dot z(t) + \int_{-\infty}^{t} z(s)\,d_s R(t,s) = 0,\ t \ge t_1,\ x(t) = x_0,\ t \le t_1. \qquad (4.4.14)$$

Let us define the positive function $v(t) = x_0 \exp\{-\int_{t_1}^{t} u(s)ds\}$ for any $t$.

Due to inequality (4.3.1), we have

$$\begin{aligned}
&\dot v(t) + \int_{-\infty}^{t} v(s)\,d_s R(t,s)\\
&= -x_0 u(t) \exp\left\{-\int_{t_1}^{t} u(s)ds\right\} + x_0 \int_{t_1}^{t} \exp\left\{-\int_{t_1}^{s} u(\tau)\,d\tau\right\} d_s R(t,s)\\
&= -x_0 \exp\left\{-\int_{t_1}^{t} u(s)ds\right\}\left[u(t) - \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(\tau)\,d\tau\right\} d_s R(t,s)\right] \le 0.
\end{aligned}$$

Hence $v(t) > 0$ is a solution of the problem

$$\dot v(t) + \int_{-\infty}^{t} v(s)\,d_s R(t,s) = g(t),\ t \ge t_0,\ v(t) = x_0,\ t \le t_0,$$

with $g(t) \le 0$. Theorem 4.6 implies $z(t) \ge v(t) > 0$. In the general case of an arbitrary positive $f$, the application of Theorem 4.6 completes the proof. □

## 4.5 Nonoscillation Criteria for Some Autonomous Integrodifferential Equations

Consider nonoscillation conditions for the autonomous equation

$$\dot{x}(t) + \int_{-\infty}^{t} K(t,s)x(s)\,ds = 0, \tag{4.5.1}$$

where $K(t,s) = G(t-s) > 0$.

*Example 4.1* Let $K(t,s) = A\chi_{[t-h,t]}(s)$, $A > 0$, $h > 0$. We have the equation

$$\dot{x}(t) + \int_{t-h}^{t} Ax(s)\,ds = 0. \tag{4.5.2}$$

By Theorem 4.1, it has a nonoscillatory solution as far as the inequality

$$\lambda \geq \int_{t-h}^{t} Ae^{\lambda(t-s)}ds = \frac{A}{\lambda}\left(e^{\lambda h} - 1\right)$$

has a positive solution $\lambda$. Thus, if there exists $x > 0$ such that

$$f(x) = x^2 - A\left(e^{hx} - 1\right) \geq 0,$$

then (4.5.2) has a nonoscillatory solution. For this $x$, we have

$$A \leq \frac{x^2}{e^{hx} - 1}.$$

Thus, (4.5.2) is nonoscillatory if and only if

$$A \leq B_1(h) := \sup_{x>0}\left(\frac{x^2}{e^{hx} - 1}\right). \tag{4.5.3}$$

Obviously $B_1(h)$ is decreasing in $h$. For $h \approx 0.8047$, we have $A = B_1(h) = 1$, which is attained at $x \approx 1.98$. For $h \approx 0.569$, we have $A = B_1(h) = 2$ attained at $x \approx 2.8$, which illustrates the difference between equations with distributed and concentrated delays. For equations with concentrated delay $\dot{x} + ax(t-h) = 0$, the sharp nonoscillation condition $ah \leq 1/e$ implies that the delay boundary for $h$ decays twice as $a$ is doubled.

The graph of $A(h) = B_1(h)$ is presented in Fig. 4.1, left. Let us note that $hA(h) = hB_1(h)$ is unbounded in contrast to the case for the constant concentrated delay (see Fig. 4.1, right).

*Example 4.2* Let $A > 0$, $h > 0$ and

$$K(t,s) = \begin{cases} A(s+h-t), & t-s \leq h, \\ 0, & \text{otherwise}. \end{cases}$$

Then we have

$$\dot{x}(t) + \int_{t-h}^{t} A(s-t+h)x(s)\,ds = 0. \tag{4.5.4}$$

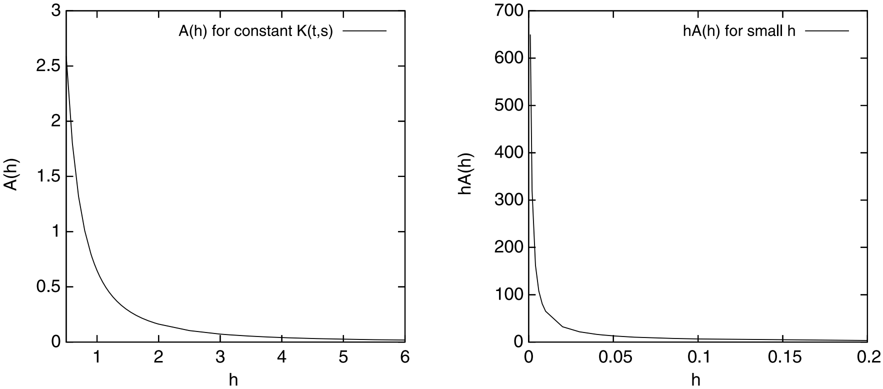



**Fig. 4.1** Function $A = A(h) = B_1(h)$ (*left*) described in (4.5.3) and the graph of $Ah = hB_1(h)$ (*right*), which tends to infinity as $h \to 0$

There exists a positive solution if for some positive $\lambda$

$$A\int_{t-h}^{t}(s-t+h)e^{\lambda(t-s)}ds = A\left(-\frac{1}{\lambda}(s-t+h)e^{\lambda(t-s)} - \frac{1}{\lambda^2}e^{\lambda(t-s)}\right)\Bigg|_{s=t-h}^{s=t}$$
$$= \frac{A}{\lambda^2}\left(e^{\lambda h} - 1 - \lambda h\right) \le \lambda.$$

Thus, if

$$A \le B_2(h) := \sup_{x>0}\left[\frac{x^3}{e^{hx} - hx - 1}\right], \tag{4.5.5}$$

then (4.5.4) is nonoscillatory. For the graph of $A = A(h) = B_2(h)$, see Fig. 4.2.

Let us also note that the supremum in (4.5.5) does not exceed

$$\sup_{x>0}\left[\frac{x^3}{h^3x^3/6 + h^2x^2/2}\right] \le \frac{6}{h^3}.$$

*Example 4.3* Let $K(t,s) = Ae^{\nu(s-t)}$, $A > 0$, $\nu > 0$. Then the equation is

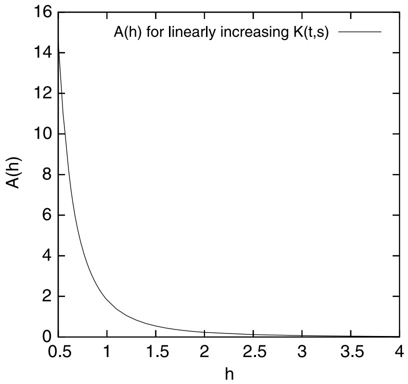
$$\dot{x}(t) + \int_{-\infty}^{t} Ae^{\nu(s-t)}x(s)\,ds = 0. \tag{4.5.6}$$

Equation (4.5.6) has a nonoscillatory solution if for some $\lambda > 0$ and $t_1 \ge 0$ we have

$$A\int_{t_1}^{t} e^{\nu(s-t)+\lambda(t-s)}ds \le \lambda.$$

The equation is autonomous, so the supremum of the left-hand side in $t$ is the integral with the lower bound $-\infty$, which diverges for $\nu \le \lambda$, so we can consider $\nu > \lambda$

**Fig. 4.2** Function $A = A(h) = B_2(h)$ described in (4.5.5)




only,

$$A\int_{-\infty}^{t} e^{\nu(s-t)+\lambda(t-s)}ds = A\int_{-\infty}^{t} e^{(\nu-\lambda)(s-t)}ds$$
$$= \frac{A}{\nu-\lambda}e^{(\nu-\lambda)(s-t)}\Big|_{s\to-\infty}^{s=t} = \frac{A}{\nu-\lambda} \le \lambda,$$

which means that the quadratic inequality

$$x^2 - \nu x + A \le 0$$

has a positive root (which is valid if and only if the discriminant is nonnegative). Thus, (4.5.6) is nonoscillatory if and only if

$$\nu^2 \ge 4A. \tag{4.5.7}$$

*Example 4.4* Further, consider the truncated Gaussian kernel $K(t,s) = Ae^{-\nu(s-t)^2}$, $A > 0$, $\nu > 0$. Then the equation is

$$\dot{x}(t) + \int_{-\infty}^{t} Ae^{-\nu(s-t)^2}x(s)\,ds = 0. \tag{4.5.8}$$

Then we obtain (here we make a substitution $\eta = s - t$) the nonoscillation condition

$$A\int_{-\infty}^{t} e^{-\nu(s-t)^2+\lambda(t-s)}ds = A\int_{-\infty}^{0} e^{-\nu\eta^2-\lambda\eta}\,d\eta$$
$$\le A\int_{-\infty}^{\infty} e^{-\nu\eta^2-\lambda\eta}\,d\eta = Ae^{\lambda^2/(4\nu)}\sqrt{\frac{\pi}{\nu}} \le \lambda,$$

and for $A \le \sqrt{\frac{\nu}{\pi}}\sup_{x>0}[xe^{-x^2/(4\nu)}]$ equation (4.5.8) has a nonoscillatory solution. The supremum on the right-hand side is attained for $x = \sqrt{2\nu}$ and equals $\sqrt{\frac{2\nu}{e}}$, so the sufficient nonoscillation condition is

$$A \le \nu\sqrt{\frac{2}{\pi e}}. \tag{4.5.9}$$

*Remark 4.1* In Example 4.4, we have obtained a simple but just sufficient nonoscillation condition. Since

$$A\int_{-\infty}^{0} e^{-\nu\eta^2-\lambda\eta}\,d\eta = \frac{A}{2}\sqrt{\frac{\pi}{\nu}}e^{\lambda^2/(4\nu)}\left[1+\operatorname{erf}\left(\frac{\lambda}{2\sqrt{\nu}}\right)\right] \geq \lambda,$$

where $\operatorname{erf}(x)$ is the Gaussian error function [1, Chap. 7]

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt,$$

this leads to the sharp nonoscillation condition

$$A \leq \sup_{x>0} \frac{2\sqrt{\nu}xe^{-x^2/(4\nu)}}{\sqrt{\pi}[1+\operatorname{erf}(x/(2\sqrt{\nu}))]}. \tag{4.5.10}$$

Using Theorem 4.3, we can obtain several efficient nonoscillation conditions.

**Theorem 4.8** *Let* (b1) *and* (b2) *hold.*

*If at least one of the hypotheses*

1) $K(t,s)=0$ *for* $t-s>h$ *and* $K(t,s)\leq B_1(h)$ *for* $t-s\leq h$, *where* $B_1(h)$ *is defined in* (4.5.3),
2) $K(t,s)=0$ *for* $t-s>h$ *and* $K(t,s)\leq B_2(h)(s-t+h)$ *for* $t-s\leq h$, *where* $B_2(h)$ *is defined in* (4.5.5),
3) *there exists* $\nu>0$ *such that* $K(t,s)\leq \nu^2 e^{-\nu(t-s)}/4$, *or*
4) *there exists* $\nu>0$ *such that* $K(t,s)\leq \nu\sqrt{\frac{2}{\pi e}}e^{-\nu(s-t)^2}$

*holds for* $t\geq s\geq t_1$ *for some* $t_1\geq 0$, *then* (4.1.5) *has a nonoscillatory solution.*

*If at least one of the hypotheses*

5) $t-h_k(t)\leq h,\ k=1,\cdots,m$ *and* $\sum_{k=1}^m a_k(t)\chi_{(h_k(t),\infty)}(s)\leq B_1(h)(s-t+h)$, *where* $B_1(h)$ *is defined in* (4.5.3),
6) $t-h_k(t)\leq h, k=1,\cdots,m$ *and* $\sum_{k=1}^m a_k(t)\chi_{(h_k(t),\infty)}(s)\leq B_2(h)(t-s+h)^2/2$, *where* $B_2(h)$ *is defined in* (4.5.5), *or*
7) *the delays are ordered* $h_1(t)\leq h_2(t)\leq\cdots\leq h_m(t)$,

$$a_1(t)\leq \frac{A}{\nu}e^{-\nu(t-h_1(t))} \text{ and } a_k(t)\leq \frac{A}{\nu}e^{-\nu t}\left[e^{\nu h_k(t)}-e^{-\nu h_{k-1}(t)}\right],\ k=2,\cdots,m,$$

*holds, then* (4.1.3) *has a nonoscillatory solution.*

*Proof* Comparing (4.1.5) to nonoscillatory equations (4.5.2), (4.5.4), (4.5.6) and (4.5.8), and applying Theorem 4.4 together with nonoscillation conditions (4.5.3), (4.5.5), (4.5.7) and (4.5.9), respectively, we immediately obtain 1)–4).

In order to deduce 5) and 6), we apply Theorem 4.3 and note that

$$T(t,s)=\sum_{k=1}^m a_k(t)\chi_{(h_k(t),\infty)}(s)$$

should be compared with $R(t,s)=\int_{-\infty}^{s} K(t,\zeta)\,d\zeta$, which equals

$$A\int_{t-h}^{s} d\zeta = A(s-t+h) \text{ and } A\int_{t-h}^{s} (\zeta-t+h)\,d\zeta = \frac{(s-t+h)^2}{2}$$

for (4.5.2) and (4.5.4), respectively. This implies 5) and 6).

To justify 7), let us first note that since $T(t,s)$ is a nondecreasing step function and $R(t,s)$ is nondecreasing in $s$ for any $t$, it is enough to check the inequality $T(t,h_k(t)^+)\le R(t,h_k(t)^+)$ (where $R(t,h_k(t)^+)=\lim_{s\to 0^+} R(t,h_k(t)+s)$) only and prove $T(t,h_1(t)^+)\le R(t,h_1(t)^+)$, $T(t,h_k(t)^+)-T(t,h_{k-1}(t)^+)\le R(t,h_k(t)^+)-R(t,h_{k-1}(t)^+)$, $k=2,\cdots,m$. Since

$$R\big(t,h_1(t)\big)=A\int_{-\infty}^{h_1(t)} e^{\nu(\zeta-t)}\,d\zeta=\frac{A}{\nu}e^{-\nu(t-h_1(t))},$$

$$R\big(t,h_k(t)^+\big)-R\big(t,h_{k-1}(t)^+\big) = A\int_{h_{k-1}(t)}^{h_k(t)} e^{\nu(\zeta-t)}\,d\zeta=\frac{A}{\nu}\big[e^{-\nu(t-h_k)}-e^{-\nu(t-h_{k-1})}\big]$$

and $T(t,h_1(t)^+)=a_1(t)$, $T(t,h_k(t)^+)-T(t,h_{k-1}(t)^+)=a_k(t)$, this implies the statement of 7), which completes the proof. □

*Example 4.5* Consider differential equation (4.1.3) with coefficients compared to (4.5.2), where $A=2$, $h\approx 0.569$. Then the delay equation

$$\dot{x}(t)+\sum_{k=1}^{55} a_k x(t-h_k)=\dot{x}(t)+\sum_{k=1}^{55} 0.02x(t-0.01k)=0 \qquad (4.5.11)$$

is nonoscillatory by Theorem 4.8, Part 5, while

$$\sum_{k=1}^{55} a_k h_k=\sum_{k=1}^{55} 0.02\;0.01k=0.308.$$

Let us also remark that the number above describes the total area of rectangles inscribed in the triangle with the legs of 0.568 ($x$-axis) and 2 ($y$-axis). Whatever partition we choose satisfying Part 5 of Theorem 4.8, the total area will be less than $1/e\approx 0.36788$ since

$$\sum_{k=1}^{m} a_k h_k > \frac{1}{e}$$

is a sufficient oscillation condition for autonomous equations [192]. Nevertheless, the area of the triangle exceeds $0.5>1/e$. Let us also note that $\sum_{k=1}^{55} a_k \max_k h_k = 0.605 > 1/e$.

## 4.6 Explicit Nonoscillation and Oscillation Conditions

Now let us proceed to explicit nonoscillation conditions.

We recall that any function of bounded variation (in $s$ for any $t$) can be represented as a difference of two nondecreasing functions of bounded variation in $s$ for any $t$

$$R(t,s)=P(t,s)-Q(t,s). \tag{4.6.1}$$

This representation of $R(t,s)$ will be further applied, with an additional assumption

$$R(t,s)=P(t,s)=Q(t,s)=0 \text{ for } s\le h(t), \tag{4.6.2}$$

where

$$h_k(t)=\inf_{s\le t}\{s \,|\, R(t,s)\ne 0\}.$$

Then

$$\mathrm{Var}_{s\in[h(\tau),\tau]}\, P(t,s)=P(t,\tau)-P\big(t,h(\tau)\big).$$

**Theorem 4.9** *Suppose that $P(t,s)$ defined by (4.6.1) is nondecreasing in $s$ for any $t$ and conditions* (a1) *and* (a2) *and one of the following two conditions holds*:

1) (a4) *is satisfied together with the inequality*

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\mathrm{Var}_{s\in[h(\tau),\tau]}\, P(t,s)\,d\tau \\ =\limsup_{t\to\infty}\int_{h(t)}^{t}\big[P(t,\tau)-P\big(t,h(\tau)\big)\big]\,d\tau<\frac{1}{e}. \tag{4.6.3}$$

2) *For some $t_1\ge t_0$, the following inequality is satisfied*:

$$\limsup_{t\to\infty}\int_{t_1}^{t}\mathrm{Var}_{s\in[h(\tau),\tau]}\, P(t,s)\,d\tau=\limsup_{t\to\infty}\int_{t_1}^{t}\big[P(t,\tau)-P\big(t,h(\tau)\big)\big]\,d\tau<\frac{1}{e}. \tag{4.6.4}$$

*Then* (4.2.8) *has a nonoscillatory solution.*

*Proof* 1) Let $t_1>t_0$ be such that

$$\int_{h(t)}^{t}\mathrm{Var}_{s\in[h(\tau),\tau]}\, P(t,s)\,d\tau<\frac{1}{e},\ t>t_1.$$

Let us choose

$$u(t)=\begin{cases} e\,\mathrm{Var}_{s\in[h(t),t]}\, P(t,s), & t>t_1,\\ 0, & t\le t_1.\end{cases} \tag{4.6.5}$$

Let $t_2>t_1$ be such that $P(t,s)=0$ if $s\le t_1$, $t>t_2$. Then, for $t>t_2$,

$$\int_{t_2}^{t}\exp\left\{\int_{s}^{t}u(\tau)\,d\tau\right\}d_sP(t,s) \\ =\int_{h(t)}^{t}\exp\left\{e\int_{s}^{t}\mathrm{Var}_{s\in[h(\tau),\tau]}\, P(\tau,s)d\tau\right\}d_sP(t,s)$$

$$\leq \int_{h(t)}^{t} \exp\left\{e\frac{1}{e}\right\} d_s P(t,s)$$
$$\leq e \operatorname{Var}_{s\in[h(t),t]} P(t,s) = u(t).$$

By Corollary 4.7, (4.2.8) has an eventually positive solution.

2) Let us choose

$$u(t) = \begin{cases} e \operatorname{Var}_{s\in[t_1,t]} P(t,s), & t > t_1, \\ 0, & t \leq t_1. \end{cases} \tag{4.6.6}$$

Then the inequalities

$$\int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)\,d\tau\right\} d_s P(t,s)$$
$$= \int_{t_1}^{t} \exp\left\{e\int_s^t \operatorname{Var}_{s\in[t_1,\tau]} P(\tau,s)d\tau\right\} d_s P(t,s)$$
$$\leq \int_{t_1}^{t} \exp\left\{e\frac{1}{e}\right\} d_s P(t,s)$$
$$\leq e \operatorname{Var}_{s\in[t_1,t]} P(t,s) = u(t)$$

and the reference to Corollary 4.7 complete the proof of the theorem. □

Denote

$$f^+(t) = \max\{f(t), 0\}$$

and

$$\tilde{h}(t) = \inf_{k=0,1,\cdots} h_k(t),$$

where $h_0$ was defined in condition (c2) of Corollary 4.5.

Let us note that for the equation

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x\big(h_k(t)\big) + \int_{h_0(t)}^{t} K(t,s)x(s)\,ds = 0 \tag{4.6.7}$$

we have

$$R(t,s) = \sum_{k=1}^{m} a_k(t)\chi_{(h_k(t),\infty)}(s) + \int_{h_0(s)}^{s} K(t,\zeta)\,d\zeta,$$

$$P(t,s) = \sum_{k=1}^{m} a_k^+(t)\chi_{(h_k(t),\infty)}(s) + \int_{h_0(s)}^{s} K^+(t,\zeta)\,d\zeta, \tag{4.6.8}$$

where function $P(t,s)$ nondecreasing in $s$ for any $t$ was defined in (4.6.1).

**Corollary 4.8** *If* (b1) *and* (b2) *with* $m = \infty$ *and* (c1) *and* (c2) *hold, and*

$$\limsup_{t\to\infty} \int_{\tilde{h}(t)}^{t} \left[\sum_{k=1}^{\infty} a_k^+(\tau) + \int_{h_0(\tau)}^{\tau} K^+(\tau,s)\,ds\right] d\tau < \frac{1}{e}, \tag{4.6.9}$$

*then* (4.6.7) *has a nonoscillatory solution.*

**Lemma 4.1** *Suppose $R(t,s)$ is nondecreasing in $s$ for each $t$ and satisfies* (a1) *and* (a2), *$h(t) \le t$, and $x(t)$ is a continuous function for $t \ge t_1$. Then there exists a measurable function $r(t)$ satisfying $h(t) \le r(t) \le t$ such that*

$$\int_{h(t)}^{t} x(s) d_s R(t,s) = a(t) x\big(r(t)\big),$$

*where $a(t) = \int_{h(t)}^{t} d_s R(t,s) > 0$ almost everywhere.*

*Proof* For any fixed $t > t_1$, we have

$$a(t) \min_{h(t) \le \zeta \le t} x(\zeta) \le \int_{h(t)}^{t} x(s) d_s R(t,s) \le a(t) \max_{h(t) \le \zeta \le t} x(\zeta), \tag{4.6.10}$$

and hence for the continuous function $x(t)$ on $[h(t), t]$

$$\frac{1}{a(t)} \int_{h(t)}^{t} x(s) d_s R(t,s) \in \Big[ \min_{h(t) \le \zeta \le t} x(\zeta), \max_{h(t) \le \zeta \le t} x(\zeta) \Big].$$

By the Intermediate Value Theorem, there exists $c \in [h(t), t]$ such that

$$\frac{1}{a(t)} \int_{h(t)}^{t} x(s) d_s R(t,s) = x(c).$$

Now, we can denote $r(t) = c$ and notice that $r$ is measurable since the medium function in (4.6.10) and $a(t)$ are measurable. □

*Remark 4.2* Since $R(t,\cdot)$ is nondecreasing, then

$$a(t) = \int_{h(t)}^{t} d_s R(t,s) = R(t,t) - R\big(t, h(t)\big) = R(t,t) - 0 = R(t,t).$$

As an application of Lemma 4.1, consider the equation with two distributed delays,

$$\dot{x}(t) + \int_{h(t)}^{t} x(s) d_s H(t,s) + \int_{g(t)}^{t} x(s) d_s G(t,s) = 0, \tag{4.6.11}$$

where for functions $H(t,s)$, $G(t,s)$, $h(t)$, $g(t)$ conditions of Lemma 4.1 hold.

Denote

$$a(t) = \int_{h(t)}^{t} d_s H(t,s), \ \ b(t) = \int_{g(t)}^{t} d_s G(t,s),$$

$$A = \limsup_{t \to \infty} \int_{h(t)}^{t} a(s) ds, \ \ B = \limsup_{t \to \infty} \int_{h(t)}^{t} a(s) b(s) ds.$$

**Theorem 4.10** *Suppose* (a1) *and* (a2) *hold for $H$, $G$, $h$ and $g$, $H(t,\cdot)$ and $G(t,\cdot)$ are nondecreasing,*

$$\limsup_{t \to \infty} \int_{g(t)}^{t} a(s) ds = \limsup_{t \to \infty} \int_{g(t)}^{t} b(s) ds = 0,$$

*and $Ae^B < 1/e$. Then* (4.6.11) *has an eventually positive fundamental function.*

*Proof* Theorem 2.12 implies that the equation

$$\dot{x}(t) + a(t)x\big(h(t)\big) + b(t)x\big(g(t)\big) = 0 \tag{4.6.12}$$

has a nonoscillatory solution. By Theorem 2.1, there exist a point $t_1 \geq 0$ and a locally essentially bounded function $u(t)$ nonnegative for $t \geq t_1$ and satisfying the inequality

$$u(t) \geq a(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\}, \; t \geq t_1, \tag{4.6.13}$$

where we assume $u(t) = 0$, $t < t_1$.

Suppose $X(t,s)$ is the fundamental function of (4.6.11) and $x(t) = X(t,t_1)$. By Lemma 4.1, there exist $r(t)$ and $p(t)$, $h(t) \leq r(t) \leq t$, $g(t) \leq p(t) \leq t$ such that $x$ is a solution of (4.6.11),

$$\dot{x}(t) + a(t)x\big(r(t)\big) + b(t)x\big(p(t)\big) = 0, \tag{4.6.14}$$

where $x(t) = 0$, $t < t_1$, $x(t_1) = 1$. If $u$ is a nonnegative solution of inequality (4.6.13), then $u$ is also a nonnegative solution of the inequality

$$u(t) \geq a(t)\exp\left\{\int_{r(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{p(t)}^{t} u(s)ds\right\}, \; t \geq t_1. \tag{4.6.15}$$

Theorem 2.1 implies that $Y(t,s) > 0$, $t \geq s \geq t_1$, where $Y(t,s)$ is the fundamental function of (4.6.14). Since $x(t) = Y(t,t_1) > 0$, (4.6.11) has a nonoscillatory solution. □

**Theorem 4.11** *If* (a1), (a2) *and* (a4) *hold*, $R_k(t,s)$, $k \in \mathbb{N}$ *are nondecreasing in* $s$ *for each* $t$ *and*

$$\liminf_{t\to\infty} \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} \mathrm{Var}_{s\in[h_k(t),\tau]}\, R_k(t,s)\, d\tau = \liminf_{t\to\infty} \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} R_k(t,\tau)\, d\tau > \frac{1}{e}, \tag{4.6.16}$$

*then all solutions of* (4.2.7) *are oscillatory.*

*Proof* Suppose there exists a nonoscillatory solution of (4.2.7). By Theorem 4.1, there exist $t_1$ and a positive locally integrable function $u$ that satisfies

$$u(t) \geq \sum_{k=1}^{\infty} \int_{t_1}^{t} \exp\left\{\int_{s}^{t} u(\tau)\, d\tau\right\} d_s R_k(t,s), \; t > t_1. \tag{4.6.17}$$

Since $e^t \geq et$ for positive $t$ and the functions $R_k(t,s)$ are nondecreasing in $s$, for $t$ such that $h_k(t) \geq t_1$ we have

$$\begin{aligned} u(t) &\geq \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} e\left(\int_{s}^{t} u(\tau)\, d\tau\right) d_s R_k(t,s) \\ &= e\sum_{k=1}^{\infty} \int_{h_k(t)}^{t} u(\tau)\, d\tau \left(\int_{h_k(t)}^{\tau} d_s R_k(t,s)\right) \end{aligned}$$

$$= e \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} u(\tau)\, \mathrm{Var}_{s\in[h_k(t),\tau]}\, R_k(t,s)\, d\tau$$

$$\geq \inf_{t\geq t_1} u(t) e \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} \mathrm{Var}_{s\in[h_k(t),\tau]}\, R_k(t,s)\, d\tau.$$

Hence

$$\inf_{t\geq t_1} u(t) \geq \inf_{t\geq t_1} \big\{u(t)\big\} e \inf_{t\geq t_1} \left\{ \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} \mathrm{Var}_{s\in[h_k(t),\tau]}\, R_k(t,s)\, d\tau \right\},$$

which implies

$$\inf_{t\geq t_1} \left\{ \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} \mathrm{Var}_{s\in[h_k(t),\tau]}\, R_k(t,s)\, d\tau \right\} \leq \frac{1}{e}.$$

The same inequality can be obtained for any $t_2 > t_1$, and consequently

$$\liminf_{t\to\infty} \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} \mathrm{Var}_{s\in[h_k(t),\tau]}\, R_k(t,s)\, d\tau \leq \frac{1}{e}.$$

We have a contradiction with (4.6.16), which completes the proof. □

**Corollary 4.9** *If* (b1) *and* (b2) *with* $m = \infty$ *and* (c1) *and* (c2) *hold*, *and*

$$\liminf_{t\to\infty} \left[ \sum_{k=1}^{\infty} a_k(t)\big(t - h_k(t)\big) + \int_{h_0(t)}^{t} K(t,s)(t-s)\, ds \right] > \frac{1}{e}, \tag{4.6.18}$$

*then all solutions of* (4.6.7) *are oscillatory.*

*Proof* Without loss of generality, we can assume that the expression in the brackets in (4.6.18) is greater than $1/e$ for $t \geq t_1$. Then, for $t \geq t_1$, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} \int_{h_k(t)}^{t} \mathrm{Var}_{s\in[h_k(t),\tau]}\, R_k(t,s)\, d\tau \\
&\quad = \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} \mathrm{Var}_{s\in[h(t),\tau]} \big(a_k(t)\chi_{(h_k(t),\infty)}(s)\big)\, d\tau \\
&\qquad + \int_{h_0(t)}^{t} \mathrm{Var}_{s\in[h_0(t),\tau]} \int_{h_0(s)}^{s} K(t,\zeta)\, d\zeta\, d\tau \\
&\quad = \sum_{k=1}^{\infty} \int_{h_k(t)}^{t} a_k(t)\, d\tau + \int_{h_k(t)}^{t} \int_{h_0(t)}^{\tau} K(t,\zeta)\, d\zeta\, d\tau \\
&\quad = \sum_{k=1}^{\infty} a_k(t)\big(t - h_k(t)\big) + \int_{h_0(t)}^{t} \int_{h_0(t)}^{\tau} K(t,\zeta)\, d\zeta\, d\tau
\end{aligned}$$

$$= \sum_{k=1}^{\infty} a_k(t)\big(t - h_k(t)\big) + \int_{h_0(t)}^{t} K(t,\zeta)\,d\zeta \int_{\zeta}^{t} d\tau$$
$$= \sum_{k=1}^{\infty} a_k(t)\big(t - h_k(t)\big) + \int_{h_0(t)}^{t} K(t,s)(t-s)\,ds > \frac{1}{e}.$$

By Theorem 4.11, all solutions of (4.6.7) are oscillatory. □

The results of this section generalize Theorems 2.9.1 and 2.9.2 in [248] and conclusions of Chap. 2.

## 4.7 Slowly Oscillating Solutions

For ordinary linear differential equations of the second order, it is known that if an equation has an oscillatory solution, then all its solutions are oscillating. As is well known, for delay differential equations this is not true. Y. Domshlak demonstrated that if an *associated* equation has a *slowly oscillatory solution*, then every solution of (4.1.3) is oscillating. In [142, 144], several new explicit sufficient conditions of oscillation were obtained by an explicit construction of such slowly oscillating solutions.

We present here a similar oscillation result for (4.2.8), only the existence of a slowly oscillating solution is assumed for (4.2.8) and not for the associated equation. This result extends Theorem 2.23 of Chap. 2 to equations with a distributed delay.

For (4.1.1) with finite memory (i.e., satisfying (a4)), the following definition can be introduced.

**Definition 4.4** A solution $x$ of (4.2.8) is called *a slowly oscillating solution* if for each $t_1 \ge t_0$ there exist $t_3 > t_2 > t_1$ such that $R(t,s) = 0$ if $t > t_3$, $s < t_2$, $x(t_2) = x(t_3) = 0$ and $x(t) > 0$, $t \in (t_2, t_3)$.

For (4.1.8), this means that for every $t_1 \ge t_0$ there exist $t_2$, $t_3$ such that $t_3 > t_2 > t_1$, $K(t,s) = 0$ for $t > t_3$, $s < t_2$, $h_k(t) \ge t_2$ for $t > t_3$ and $x(t_2) = x(t_3) = 0$, $x(t) > 0$, $t \in (t_2, t_3)$.

The following theorem is a more general case of the results obtained in Theorem 2.23 for equations of type (4.1.3).

**Theorem 4.12** *Suppose $R(t,\cdot)$ is nondecreasing for each $t$. If there exists a slowly oscillating solution of* (4.2.8) (*of inequality* (4.2.9)), *then all solutions of* (4.2.8) (*of inequality* (4.2.9)) *are oscillatory.*

*Proof* Let $x$ be a slowly oscillating solution of (4.2.8). Suppose that there exists a nonoscillatory solution of the same equation. Then, by Theorem 4.1, $t_1$ can be found such that $X(t,s) > 0$ for $t \ge s > t_1$.

By the definition of a slowly oscillating solution, there exist $t_2, t_3$ exceeding $t_1$ and satisfying

$$R(t,s) = 0 \text{ if } t > t_3,\ s < t_2,\ x(t_2) = x(t_3) = 0,\ x(t) > 0,\ t \in (t_2, t_3). \quad (4.7.1)$$

By (4.2.5), the solution $x$ can be presented in the form (with $t_3$ as the initial point)

$$x(t) = -\int_{t_3}^{t} X(t,s)\,ds \int_{-\infty}^{s} x(\tau)\,d_\tau R(s,\tau), \quad (4.7.2)$$

where $x(\tau)$ under the second integral is assumed to be zero if $\tau > t_3$.

Since in addition $R(t,s) = 0$ for $t > t_3$, $s < t_2$, the expression under the integral on the right-hand side of (4.7.2) can differ from zero only for $s \in (t_2, t_3)$. By (4.7.1), the inequality $x(t) \le 0$ is satisfied for each $t \ge t_3$ since the right-hand side in (4.7.2) is negative for $t > t_3$. Thus $x$ is an eventually nonoscillatory solution, which contradicts the assumption that it is slowly oscillating and thus oscillatory.

The proof in the case of inequality (4.2.9) is similar. □

*Remark 4.3* The statement of the theorem implies that if (4.2.8) with nondecreasing $R(t,\cdot)$ for each $t$ has a nonoscillatory solution, then (4.2.8) has no slowly oscillating solutions.

**Corollary 4.10** *If* (b1), (b2) *and* (c1) *hold,* $K(t,s)$ *and* $a_k(t)$ *are nonnegative functions and there exists a slowly oscillating solution of* (4.1.8), *then all solutions of* (4.1.8) *are oscillatory.*

## 4.8 Equations with Positive and Negative Coefficients

We consider the equation with two distributed delays

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\,d_s R(t,s) - \int_{-\infty}^{t} x(s)\,d_s T(t,s) = 0,\ t \ge t_0, \quad (4.8.1)$$

where both $R(t,s)$ and $T(t,s)$ are nondecreasing in $s$ for each $t$.

Equation (4.8.1) includes the following special cases:

1. the delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{n} a_k(t)x\big(h_k(t)\big) - \sum_{l=1}^{m} b_l(t)x\big(g_l(t)\big) = 0, \quad (4.8.2)$$

   where $a_k(t) \ge 0$, $b_l(t) \ge 0$, if we assume

$$R(t,s) = \sum_{k=1}^{n} a_k(t)\chi_{(h_k(t),\infty)}(s),\ T(t,s) = \sum_{l=1}^{m} b_l(t)\chi_{(g_l(t),\infty)}(s); \quad (4.8.3)$$

2. the integrodifferential equation

$$\dot{x}(t) + \int_{-\infty}^{t} K_1(t,s)x(s)\,ds - \int_{-\infty}^{t} K_2(t,s)\,ds = 0, \quad (4.8.4)$$

where $K_i(t,s) \ge 0$,

$$R(t,s) = \int_{-\infty}^{s} K_1(t,\zeta)\,d\zeta, \;\; T(t,s) = \int_{-\infty}^{s} K_2(t,\zeta)\,d\zeta; \tag{4.8.5}$$

3. some types of mixed equations, two of which we will consider:

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t) x\big(h_k(t)\big) - \int_{-\infty}^{t} K(t,s) x(s)\,ds = 0, \tag{4.8.6}$$

$$\begin{aligned} \dot{x}(t) &+ \sum_{k=1}^{n} a_k(t) x\big(h_k(t)\big) - \sum_{l=1}^{m} b_l(t) x\big(g_k(t)\big) \\ &+ \int_{-\infty}^{t} K_1(t,s) x(s)\,ds - \int_{-\infty}^{t} K_2(t,s) x(s)\,ds = 0, \end{aligned} \tag{4.8.7}$$

where $R(t,s)$ and $T(t,s)$ are defined similarly.

We consider scalar delay differential equation (4.8.1) under the following assumptions:

(A1) $R(t,\cdot)$ and $T(t,\cdot)$ are left continuous functions of bounded variation, and for each $s$ their variations on the segment $[t_0,s]$

$$P_R(t,s) = \operatorname{Var}_{\tau \in [t_0,s]} R(t,\tau), \;\; P_T(t,s) = \operatorname{Var}_{\tau \in [t_0,s]} T(t,\tau) \tag{4.8.8}$$

are locally integrable functions in $t$, $R(t,s) = R(t,t^+)$, $T(t,s) = T(t,t^+)$, $t < s$.

(A2) $R(t,\cdot)$, $T(t,\cdot)$ are nondecreasing functions for each $t$, $R(t,s) \ge T(t,s)$ for each $t,s$.

(A3) For each $t_1$, there exist $s_1 = s(t_1) \le t_1$ and $r_1 = r(t_1) \le t_1$ such that $R(t,s) = 0$ for $s < s_1$, $t > t_1$ and $T(t,s) = 0$ for $s < r_1$, $t > t_1$; in addition, functions $s(t)$, $r(t)$ satisfy

$$\lim_{t\to\infty} s(t) = \infty, \;\; \lim_{t\to\infty} r(t) = \infty.$$

If (A3) holds, then we can introduce the functions

$$h(t) = \inf\big\{s \mid R(t,s) \ne 0\big\}, \;\; g(t) = \inf\big\{s \mid T(t,s) \ne 0\big\}, \tag{4.8.9}$$

such that $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g(t) = \infty$, and (4.8.1) can be rewritten as

$$\dot{y}(t) + \int_{h(t)}^{t} y(s)\,d_s R(t,s) - \int_{g(t)}^{t} y(s)\,d_s T(t,s) = 0, \;\; t \ge t_0. \tag{4.8.10}$$

If (A2) and (A3) hold, then obviously $h(t) \le g(t)$.

We will apply the results of the previous sections and will also need the following trivial result.

**Lemma 4.2** *Let $\mu$ be a nonnegative function of bounded variation and $f$ be an absolutely continuous function on $[a,b]$. Suppose in addition that $f$ is nonincreasing and $\mu(a) = 0$. Then $\int_a^b f(t)\,d\mu(t) \ge 0$.*

*Proof* Evidently,

$$\int_a^b f(t)\,d\mu(t) = f(b)\mu(b) - f(a)\mu(a) - \int_a^b \mu(t)\,df(t)$$
$$= f(b)\mu(b) - \int_a^b \mu(t)\,df(t) \geq 0,$$

which completes the proof. □

Together with (4.8.10), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{y}(t) + \int_{h(t)}^t y(s)\,d_sR(t,s) - \int_{g(t)}^t y(s)\,d_sT(t,s) = f(t),\ t \geq t_0, \tag{4.8.11}$$
$$x(t) = \varphi(t),\ t < t_0,\ x(t_0) = x_0. \tag{4.8.12}$$

We also assume that the following hypothesis holds:

(A4) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi$ is such that all the integrals exist; in particular, we can consider continuous $\varphi : (-\infty, t_0) \to \mathbb{R}$.

Consider together with (4.8.10) the delay differential inequality

$$\dot{y}(t) + \int_{h(t)}^t y(s)\,d_sR(t,s) - \int_{g(t)}^t y(s)\,d_sT(t,s) \leq 0,\ t \geq t_0. \tag{4.8.13}$$

The next theorem establishes sufficient nonoscillation conditions.

**Theorem 4.13** *Suppose* (A1)–(A3) *hold. Consider the following hypotheses*:

1) *There exists* $t_1 \geq t_0$ *such that the inequality*

$$u(t) \geq \int_{h(t)}^t \exp\left\{\int_s^t u(\tau)d\tau\right\} d_sR(t,s) - \int_{g(t)}^t \exp\left\{\int_s^t u(\tau)d\tau\right\} d_sT(t,s) \tag{4.8.14}$$

*has a nonnegative locally integrable solution for* $t \geq t_1$, *where we assume* $u(t) = 0$ *for* $t < t_1$.

2) *There exists* $t_2 \geq t_0$ *such that the fundamental function* $X(t,s)$ *of* (4.8.10) *satisfies* $X(t,s) > 0$ *for* $t \geq s \geq t_2$.
3) *Equation* (4.8.10) *has a nonoscillatory solution.*
4) *Inequality* (4.8.13) *has an eventually positive solution.*

*Then the implications* 1) ⇒ 2) ⇒ 3) ⇒ 4) *are valid.*

*Proof* 1) ⇒ 2) **Step 1**. Let us prove that the fundamental solution is nonnegative for $t \geq s \geq t_1$, so in fact $t_2 = t_1$. To this end, consider the initial value problem

$$\dot{x}(t) + \int_{h(t)}^t x(s)\,d_sR(t,s) - \int_{g(t)}^t x(s)\,d_sT(t,s) = f(t),\ t \geq t_1,\ x(t) = 0,\ t \leq t_1. \tag{4.8.15}$$

Denote

$$z(t) = \dot{x}(t) + u(t)x(t), \; z(t) = 0, \; t \le t_1, \tag{4.8.16}$$

where $x$ is a solution of (4.8.15) and $u$ is a nonnegative solution of (4.8.14).

Equality (4.8.16) implies

$$x(t) = \int_{t_1}^{t} \exp\left\{-\int_s^t u(\tau)d\tau\right\} z(s)ds, \; t \ge t_1. \tag{4.8.17}$$

After substituting (4.8.17) into (4.8.15), we have

$$\begin{aligned} & z(t) - u(t)\int_{t_1}^{t} \exp\left\{-\int_s^t u(\tau)\,d\tau\right\} z(s)\,ds \\ & \quad + \int_{h(t)}^{t} \left(\int_{t_1}^{s} \exp\left\{-\int_\theta^s u(\tau)\,d\tau\right\} z(\theta)\,d\theta\right) d_s R(t,s) \\ & \quad - \int_{g(t)}^{t} \left(\int_{t_1}^{s} \exp\left\{-\int_\theta^s u(\tau)\,d\tau\right\} z(\theta)\,d\theta\right) d_s T(t,s) = f(t). \end{aligned}$$

In the second and the third integrals (in $s$), the integrand vanishes for $s < t_1$. After changing the order of integration in the second and third integrals, we have

$$\begin{aligned} & z(t) - u(t)\int_{t_1}^{t} \exp\left\{-\int_s^t u(\tau)\,d\tau\right\} z(s)\,ds \\ & \quad + \int_{h(t)}^{t} z(s)\,ds \int_s^t \exp\left\{-\int_s^\theta u(\tau)\,d\tau\right\} d_\theta R(t,\theta) \\ & \quad - \int_{g(t)}^{t} z(s)\,ds \int_s^t \exp\left\{-\int_s^\theta u(\tau)\,d\tau\right\} d_\theta T(t,\theta) = f(t). \end{aligned}$$

Thus the left-hand side equals

$$\begin{aligned} & z(t) - u(t)\int_{t_1}^{h(t)} \exp\left\{-\int_s^t u(\tau)\,d\tau\right\} z(s)\,ds \\ & \quad + \int_{t_1}^{h(t)} z(s)\,ds \int_s^t \exp\left\{-\int_s^t u(\tau)d\tau\right\} \exp\left\{\int_\theta^t u(\tau)\,d\tau\right\} d_\theta R(t,\theta) \\ & \quad - \int_{t_1}^{h(t)} z(s)\,ds \int_s^t \exp\left\{-\int_s^t u(\tau)d\tau\right\} \exp\left\{\int_\theta^t u(\tau)\,d\tau\right\} d_\theta T(t,\theta) \\ & \quad - u(t)\int_{h(t)}^{t} \exp\left\{-\int_s^t u(\tau)\,d\tau\right\} z(s)\,ds \\ & \quad + \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{t} \exp\left\{-\int_s^t u(\tau)d\tau\right\} \exp\left\{\int_\theta^t u(\tau)\,d\tau\right\} d_\theta R(t,\theta) \\ & \quad - \int_{h(t)}^{t} z(s)\,ds \int_{g(t)}^{t} \exp\left\{-\int_s^t u(\tau)d\tau\right\} \exp\left\{\int_\theta^t u(\tau)\,d\tau\right\} d_\theta T(t,\theta) \\ & \quad - \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{s} \exp\left\{-\int_s^\theta u(\tau)\,d\tau\right\} d_\theta \big[R(t,\theta) - T(t,\theta)\big] \end{aligned}$$

$$-\int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{g(t)} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta T(t,\theta),$$

which can be rewritten as

$$\begin{aligned}
& z(t) - u(t)\int_{t_1}^{h(t)} \exp\left\{-\int_{s}^{t} u(\tau)\,d\tau\right\} z(s)\,ds \\
& \quad + \int_{t_1}^{h(t)} z(s)\,ds \int_{s}^{t} \exp\left\{-\int_{h(t)}^{t} u(\tau)d\tau\right\} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta R(t,\theta) \\
& \quad - \int_{t_1}^{h(t)} z(s)\,ds \int_{s}^{t} \exp\left\{-\int_{g(t)}^{t} u(\tau)d\tau\right\} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta T(t,\theta) \\
& \quad - u(t)\int_{h(t)}^{t} \exp\left\{-\int_{s}^{t} u(\tau)\,d\tau\right\} z(s)\,ds \\
& \quad + \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta R(t,\theta) \\
& \quad - \int_{h(t)}^{t} z(s)\,ds \int_{g(t)}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta T(t,\theta) \\
& \quad - \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{s} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta\big[R(t,\theta) - T(t,\theta)\big] \\
& \quad - \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{g(t)} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta T(t,\theta).
\end{aligned}$$

This in turn is equal to

$$\begin{aligned}
& z(t) - \int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)d\tau\right\} z(s)\,ds \left[u(t) - \int_{h(t)}^{t} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta R(t,\theta)\right. \\
& \quad \left. + \int_{g(t)}^{t} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta T(t,\theta)\right] - u(t)\int_{t_1}^{h(t)} \exp\left\{-\int_{s}^{t} u(\tau)\,d\tau\right\} z(s)\,ds \\
& \quad - \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{s} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta\big[R(t,\theta) - T(t,\theta)\big] \\
& \quad - \int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{g(t)} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta T(t,\theta)
\end{aligned}$$

since $R(t,s) = 0$ for $s < h(t)$ and $T(t,s) = 0$ for $s < g(t)$. Consequently, we obtain the operator equation

$$z - Hz = f, \tag{4.8.18}$$

which is equivalent to (4.8.15), where

$$\begin{aligned}
(Hz)(t) = & \int_{t_1}^{t} \exp\left\{-\int_{s}^{t} u(\tau)\,d\tau\right\} z(s)\,ds \left[u(t) - \int_{h(t)}^{t} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta R(t,\theta)\right. \\
& \left. + \int_{g(t)}^{t} \exp\left\{\int_{\theta}^{t} u(\tau)\,d\tau\right\} d_\theta T(t,\theta)\right]
\end{aligned}$$

$$+\int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{g(t)} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta T(t,\theta)$$

$$+u(t)\int_{t_1}^{h(t)} \exp\left\{-\int_{s}^{t} u(\tau)\,d\tau\right\} z(s)\,ds$$

$$+\int_{h(t)}^{t} z(s)\,ds \int_{h(t)}^{s} \exp\left\{-\int_{s}^{\theta} u(\tau)\,d\tau\right\} d_\theta \big[R(t,\theta)-T(t,\theta)\big].$$

Let $z(t) \geq 0$. Then, by (4.8.14), the first term is positive, the last term is nonnegative due to Lemma 4.2 ($R(t,\theta)-T(t,\theta)$ is nonnegative and $\exp\{-\int_s^\theta u(\tau)\,d\tau\}$ is nonincreasing in $\theta$); i.e., operator $H$ is positive.

Besides, in each finite interval $[t_1,b]$, operator $H$ is a sum of integral Volterra operators, which by Theorem A.5 are weakly compact operators in the space $L[t_1,b]$. Theorem A.8 implies that its spectral radius $r(H)=0<1$ and consequently if in (4.8.18) the right-hand side $f$ is nonnegative, then

$$z(t)=f(t)+(Hf)(t)+\big(H^2 f\big)(t)+\big(H^3 f\big)(t)+\cdots \geq 0.$$

We recall that the solution of (4.8.15) has form (4.8.17), with $z$ being a solution of (4.8.18). Thus if in (4.8.15) we have $f(t)\geq 0$, then $x(t)\geq 0$. On the other hand, the solution of (4.8.15) has the representation

$$x(t)=\int_{t_1}^{t} X(t,s)f(s)\,ds.$$

As was demonstrated above, $f(t)\geq 0$ implies $x(t)\geq 0$. Hence the kernel of the integral operator is nonnegative; i.e., $X(t,s)\geq 0$ for $t\geq s>t_1$.

**Step 2**. Let us prove that in fact the strict inequality $X(t,s)>0$ holds. After denoting

$$x(t)=X(t,t_1)-\exp\left\{-\int_{t_1}^{t} u(s)ds\right\},\ x(t)=0,\ t\leq t_1,$$

and substituting $x(t)$ into the left-hand side of (4.8.15), we obtain

$$X'_t(t,t_1)+u(t)\exp\left\{-\int_{t_1}^{t} u(s)\,ds\right\}+\int_{h(t)}^{t} X(s,t_1)\,d_s R(t,s)$$

$$-\int_{g(t)}^{t} X(s,t_1)\,d_s T(t,s)-\int_{h(t)}^{t} \exp\left\{-\int_{t_1}^{s} u(\tau)\,d\tau\right\} d_s R(t,s)$$

$$+\int_{g(t)}^{t} \exp\left\{-\int_{t_1}^{s} u(\tau)\,d\tau\right\} d_s T(t,s)$$

$$=0+\exp\left\{-\int_{t_1}^{t} u(s)\,ds\right\}\left[u(t)-\int_{h(t)}^{t} \exp\left\{\int_{s}^{t} u(s)\,ds\right\} d_s R(t,s)\right.$$

$$\left.+\int_{g(t)}^{t} \exp\left\{\int_{s}^{t} u(s)\,ds\right\} d_s T(t,s)\right]\geq 0.$$

Therefore, $x(t)$ is a solution of (4.8.15) with a nonnegative right-hand side. Hence, as was demonstrated above, $x(t) \ge 0$. Consequently,

$$X(t,t_1) \ge \exp\left\{-\int_{t_1}^{t} u(s)ds\right\} > 0.$$

For $s > t_1$, inequality $X(t,s) > 0$ can be proven similarly.

Implication 2) $\Rightarrow$ 3) is valid since the function $x(t) = X(t,t_1)$ is a positive solution of (4.8.10) for $t \ge t_1$.

Implication 3) $\Rightarrow$ 4) is evident, which completes the proof. □

Let us compare oscillation properties of (4.8.10) and the equation with two different distributed delays

$$\dot{y}(t) + \int_{h_1(t)}^{t} y(s)\, d_s R_1(t,s) - \int_{g_1(t)}^{t} y(s)\, d_s T_1(t,s) = 0,\ \ t \ge t_0. \tag{4.8.19}$$

**Theorem 4.14** *Suppose that* (A1)–(A3) *hold for parameters of* (4.8.10) *and* (4.8.19), $h_1(t) \le h(t)$, $g_1(t) \ge g(t)$, *functions* $R(t,s) - R_1(t,s)$ *and* $T_1(t,s) - T(t,s)$ *are nondecreasing in* $s$ *for* $t \ge t_1$, *and inequality* (4.8.14) *has a nonnegative locally integrable solution for* $t \ge t_1$ (*vanishing for* $t < t_1$). *Then* (4.8.19) *has a nonoscillatory solution.*

*Proof* Let $u(t)$ be a nonnegative solution of (4.8.14). Then

$$\begin{aligned}
u(t) &\ge \int_{h(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s R(t,s) - \int_{g(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s T(t,s)\\
&\ge \int_{h_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s R(t,s) - \int_{g_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s T(t,s)\\
&= \int_{h_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s R_1(t,s) - \int_{g_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s T_1(t,s)\\
&\quad + \int_{h_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s\big[R(t,s) - R_1(t,s)\big]\\
&\quad + \int_{g_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s\big[T_1(t,s) - T(t,s)\big]\\
&\ge \int_{h_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s R_1(t,s) - \int_{g_1(t)}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} d_s T_1(t,s).
\end{aligned}$$

By Theorem 4.13, (4.8.19) has a nonoscillatory solution, which completes the proof. □

Consider the modification of (4.8.10) with several terms of each sign

$$\dot{y}(t) + \sum_{k=1}^{n} \int_{h_k(t)}^{t} y(s)\, d_s R_k(t,s) - \sum_{l=1}^{m} \int_{g_l(t)}^{t} y(s)\, d_s T_l(t,s) = 0,\ \ t \ge t_0, \tag{4.8.20}$$

where $R_k$, $T_l$, $h_k$, $g_l$ satisfy the following conditions:

(A1$^\star$) $R_k(t,\cdot)$, $T_l(t,\cdot)$ are left continuous functions of bounded variation and for each $s$ their variations $P_{R_k}(t,s)$, $P_{T_l}(t,s)$ on the segment $[t_0,s]$ are locally integrable functions in $t$, $R_k(t,s)=R_k(t,t^+)$, $T_l(t,s)=T_l(t,t^+)$, $t<s$.

(A2$^\star$) $R_k(t,\cdot)$, $T_l(t,\cdot)$ are nondecreasing functions for each $t$, and

$$\sum_{k=1}^{n} R_k(t,s) \ge \sum_{l=1}^{m} T_l(t,s) \text{ for each } t,s.$$

(A3$^\star$) $\lim_{t\to\infty} h_k(t)=\infty$, $k=1,\cdots,n$ and $\lim_{t\to\infty} g_l(t)=\infty$, $l=1,\cdots,m$.

In order to reformulate Theorem 4.13 for (4.8.20), consider the corresponding inequality

$$\dot{y}(t)+\sum_{k=1}^{n}\int_{h_k(t)}^{t} y(s)\,d_sR_k(t,s)-\sum_{l=1}^{m}\int_{g_l(t)}^{t} y(s)\,d_sT_l(t,s)\le 0,\ t\ge t_0. \tag{4.8.21}$$

**Theorem 4.15** *Suppose* (A1$^\star$)–(A3$^\star$) *hold. Consider the following hypotheses*:

1) *There exists $t_1\ge t_0$ such that the inequality*

$$u(t)\ge\sum_{k=1}^{n}\int_{h_k(t)}^{t}\exp\left\{\int_s^t u(\tau)d\tau\right\}d_sR_k(t,s) -\sum_{l=1}^{m}\int_{g_l(t)}^{t}\exp\left\{\int_s^t u(\tau)d\tau\right\}d_sT_l(t,s) \tag{4.8.22}$$

*has a nonnegative locally integrable solution for $t\ge t_1$, where $u(t)=0$ for $t<t_1$.*

2) *There exists $t_1\ge t_0$ such that $X(t,s)>0$ for $t\ge s\ge t_1$.*
3) *Equation* (4.8.20) *has a nonoscillatory solution.*
4) *Inequality* (4.8.21) *has an eventually positive solution.*

*Then the implications* 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) *are valid.*

Let us proceed to explicit nonoscillation conditions. Assuming that (A1) and (A3) hold, we will use the following condition:

(A5) $R(t,s)-T(t,s-h(t)+g(t))$ is a positive nondecreasing function in $s$.

Let us note that for the particular case of the equation with two concentrated delays (3.1.1), condition (A5) is satisfied if $a(t)\ge b(t)$ and $h(t)\le g(t)$.

**Theorem 4.16** *Suppose* (A1)–(A3) *hold for* (4.8.10) *and there exists $\lambda\in(0,1)$ such that $R(t,s)-\lambda T(t,s)$ is nonnegative and nondecreasing in $s$ for any $t$ large enough and the following two inequalities hold*:

$$\limsup_{t\to\infty}\int_{h(t)}^{g(t)}\left[R\left(s,s^+\right)-\lambda T\left(s,s^+\right)\right]ds<\frac{1}{e}\ln\frac{1}{\lambda}, \tag{4.8.23}$$

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left[R\left(s,s^+\right)-\lambda T\left(s,s^+\right)\right]ds<\frac{1}{e}. \tag{4.8.24}$$

*Then* (4.8.10) *has a nonoscillatory solution.*

*Proof* By (4.8.24), there exists $t_1 \ge t_0$ such that for $t \ge t_1$ the function

$$u(t) = e\big[R\big(t,t^+\big) - \lambda T\big(t,t^+\big)\big] \tag{4.8.25}$$

is a nonnegative solution of the inequality

$$u(t) \ge \exp\left\{\int_{h(t)}^{t} u(\tau)d\tau\right\}\big[R\big(t,t^+\big) - \lambda T\big(t,t^+\big)\big],$$

which can be rewritten in the form

$$\begin{aligned}
u(t) &\ge \exp\left\{\int_{h(t)}^{t} u(\tau)d\tau\right\}\left[\int_{h(t)}^{t} d_s R(t,s) - \lambda \int_{g(t)}^{t} d_s T(t,s)\right] \\
&= \exp\left\{\int_{h(t)}^{t} u(\tau)d\tau\right\}\int_{h(t)}^{t} d_s\big[R(t,s) - \lambda T\big(t,s-h(t)+g(t)\big)\big] \\
&= \int_{h(t)}^{t} \exp\left\{\int_{h(t)}^{t} u(\tau)d\tau\right\} d_s\big[R(t,s) - \lambda T\big(t,s-h(t)+g(t)\big)\big].
\end{aligned}$$

The function

$$\begin{aligned}
&R(t,s) - \lambda T\big(t,s-h(t)+g(t)\big) \\
&\quad = \big(R(t,s) - T\big(t,s-h(t)+g(t)\big)\big) + (1-\lambda)T\big(t,s-h(t)+g(t)\big)
\end{aligned}$$

is nondecreasing as a sum of two nondecreasing functions. Consequently, the integral becomes smaller if the function under the integral is changed by a smaller one:

$$\begin{aligned}
u(t) &\ge \int_{h(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s\big[R(t,s) - \lambda T\big(t,s-h(t)+g(t)\big)\big] \\
&= \int_{h(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s R(t,s) - \int_{g(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s T(t,s) \\
&\quad + \left[\int_{g(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s T(t,s)\right. \\
&\quad \left. - \lambda \int_{h(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s T\big(t,s-h(t)+g(t)\big)\right].
\end{aligned} \tag{4.8.26}$$

Let us demonstrate that the expression in the brackets is nonnegative. Inequality (4.8.23) implies that for $u$ defined by (4.8.25), $t$ large enough and any $s \ge g(t)$,

$$\int_{s+h(t)-g(t)}^{s} u(\tau)d\tau \le \ln\frac{1}{\lambda}, \tag{4.8.27}$$

which yields

$$\begin{aligned}
&\int_{g(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s T(t,s) - \lambda \int_{h(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s T\big(t,s-h(t)+g(t)\big) \\
&\quad = \int_{g(t)}^{t} \left(\exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} - \lambda \exp\left\{\int_{s+h(t)-g(t)}^{t} u(\tau)d\tau\right\}\right) d_s T(t,s) \\
&\quad = \int_{g(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\}\left(1 - \lambda \exp\left\{\int_{s+h(t)-g(t)}^{s} u(\tau)d\tau\right\}\right) d_s T(t,s) \ge 0.
\end{aligned}$$

By inequalities (4.8.26) and (4.8.27), we have

$$u(t) \geq \int_{h(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s R(t,s) - \int_{g(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} d_s T(t,s),$$

and hence $u$ is a nonnegative solution of (4.8.14). By Theorem 4.13, (4.8.10) has a nonoscillatory solution. □

**Corollary 4.11** *Suppose* (A1)–(A3) *and* (A5) *hold for* (4.8.10) *and*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \left[R\big(s,s^+\big) - \frac{1}{e}T\big(s,s^+\big)\right] ds < \frac{1}{e}. \tag{4.8.28}$$

*Then* (4.8.10) *has a nonoscillatory solution.*

This corollary is obtained by setting $\lambda = 1/e$ in Theorem 4.16. Similarly to Theorem 4.16, the following result can be obtained.

**Theorem 4.17** *Suppose* $n = m$, *conditions* (A1$^\star$)–(A3$^\star$) *are satisfied*, $\sum_{k=1}^{n}[R(t,s) - T(t,s-h(t)+g(t))]$ *is a positive nondecreasing function in* $s$ *for* $t$ *large enough and the inequality*

$$\limsup_{t\to\infty} \sum_{k=1}^{n} \int_{\min_k h_k(t)}^{t} \left[R_k\big(s,s^+\big) - \frac{1}{e}T_k\big(s,s^+\big)\right] ds < \frac{1}{e} \tag{4.8.29}$$

*holds. Then* (4.8.20) *has a nonoscillatory solution.*

*Remark 4.4* The coefficient $1/e$ of $b(s)$ is unimprovable. Indeed, for the equation

$$\dot{x}(t) + ax(t-\tau) - bx(t) = 0, \tag{4.8.30}$$

the inequality

$$a \leq \frac{e^{b\tau}}{\tau e} \tag{4.8.31}$$

is necessary and sufficient for nonoscillation.

**Theorem 4.18** *Suppose* (A1)–(A3) *and* (A5) *hold for* (4.8.10) *and there exist nondecreasing functions* $L(t,s)$, $D(t,s)$ *for each* $t$ *such that*

$$D(t,s) \leq T(t,s) \leq R(t,s) \leq L(t,s)$$

*and there exist finite limits*

$$\begin{aligned} B_{11} &= \lim_{t\to\infty} \int_{h(t)}^{t} L\big(s,s^+\big)ds, \quad B_{12} = \lim_{t\to\infty} \int_{h(t)}^{t} D\big(s,s^+\big)ds, \\ B_{21} &= \lim_{t\to\infty} \int_{g(t)}^{t} L\big(s,s^+\big)ds, \quad B_{22} = \lim_{t\to\infty} \int_{g(t)}^{t} D\big(s,s^+\big)ds. \end{aligned} \tag{4.8.32}$$

*Suppose in addition that the system*

$$\begin{aligned} \ln x_1 &> x_1 B_{11} - x_2 B_{12}, \\ \ln x_2 &< x_1 B_{21} - x_2 B_{22}, \end{aligned} \tag{4.8.33}$$

*has a positive solution* $(x_1, x_2)$ *such that for* $t$ *large enough*

$$x_1 L\big(t, t^+\big) \geq x_2 D\big(t, t^+\big).$$

*Then* (4.8.10) *has a nonoscillatory solution.*

*Proof* Consider the function $u(t) = x_1 L(t, t^+) - x_2 D(t, t^+)$, which is eventually nonnegative. System (4.8.33) can be rewritten as

$$x_1 > \exp\{x_1 B_{11} - x_2 B_{12}\},\ x_2 < \exp\{x_1 B_{21} - x_2 B_{22}\}.$$

By definitions (4.8.32), there exists $t_1 \geq t_0$ such that for $t \geq t_1$

$$x_1 \geq \exp\left\{x_1 \int_{h(t)}^t L(s, s+) ds - x_2 \int_{h(t)}^t D(s, s+) ds\right\} = \exp\left\{\int_{h(t)}^t u(s) ds\right\}$$

and

$$-x_2 \geq -\exp\left\{x_1 \int_{g(t)}^t L(s, s+) ds - x_2 \int_{g(t)}^t D(s, s+) ds\right\} = -\exp\left\{\int_{g(t)}^t u(s) ds\right\}.$$

Similar to the definition of $h$, $g$ in (4.8.9), let us define functions $H(t)$ and $G(t)$ for $L(t, s)$ and $D(t, s)$. Then $H(t) \leq h(t)$, $G(t) \geq g(t)$. Since $L(t, \cdot)$, $D(t, \cdot)$ are nondecreasing for each $t$, for any $t \geq t_1$ we have

$$\begin{aligned} x_1 L\big(t, t^+\big) &= \int_{H(t)}^t x_1 d_s L(t, s) \geq \int_{h(t)}^t \exp\left\{\int_{h(s)}^s u(\tau) d\tau\right\} d_s L(t, s), \\ -x_2 D\big(t, t^+\big) &= -\int_{G(t)}^t x_2 d_s D(t, s) \geq -\int_{g(t)}^t \exp\left\{\int_{g(s)}^s u(\tau) d\tau\right\} d_s D(t, s). \end{aligned}$$

The summation of the two equalities gives

$$u(t) \geq \int_{h(t)}^t \exp\left\{\int_{h(s)}^s u(\tau) d\tau\right\} d_s L(t, s) - \int_{g(t)}^t \exp\left\{\int_{g(s)}^s u(\tau) d\tau\right\} d_s D(t, s).$$

By Theorem 4.13, (4.8.10), where $R$ and $T$ are changed by $L$ and $D$, respectively, has a nonoscillatory solution. □

## 4.9 Discussion and Open Problems

It is usually believed that equations with a distributed delay provide a more realistic description for models of population dynamics and mathematical biology in general. For example, if a maturation delay is involved in the equation, then the maturation time is generally not constant but is distributed around its expectancy value.

Historically, equations with a distributed delay had been studied even before relevant models with concentrated delays appeared. For example, in 1926 Volterra [329] considered the logistic equation with a distributed delay

$$\dot{N}(t) = rN(t)\int_0^\infty k(\tau)\left[1 - \frac{N(t-\tau)}{K}\right]d\tau, \tag{4.9.1}$$

before Hutchinson's equation (the logistic equation with a concentrated delay)

$$\dot{N}(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right] \tag{4.9.2}$$

was introduced in 1948 [209].

To the best of our knowledge, the first systematic study of equations with a distributed delay can be found in the monograph of Myshkis [289], and the results obtained by 1993 are summarized in the book of Kuang [236]. Presently equations with distributed delays are being intensively studied. For various models of mathematical biology with distributed and concentrated delays, see the monographs [85, 167, 231, 236]. In most publications, integrodifferential equations are studied; however, sometimes applied models, as in the present chapter, incorporate both integral terms and concentrated delays.

Oscillation properties of an equation that is equivalent to (4.1.1) were investigated in [248] (see also references therein). The results presented in this chapter are more general than in [248] from the following points of view:

1. In [248], an additional integral condition on $R(t,s)$ is imposed, which leads to continuous coefficients $a_k$ and kernel $K$ in (4.1.3)–(4.1.8). We deal with measurable essentially bounded (locally integrable) functions. Many applied problems lead to equations with discontinuous coefficients. In addition, this is important for mathematical applications. For example, in [193] it was demonstrated that oscillation properties of difference equations can be derived from oscillation properties of some delay differential equations with discontinuous delays. As was demonstrated in [40, 44, 54] and will be discussed in Chaps. 12 and 13, we can study oscillation of nonimpulsive delay equations with discontinuous coefficients rather than the oscillation of an impulsive delay equation.
2. Explicit oscillation (nonoscillation) conditions in [248] are obtained in the case where $R(t,s)$ is nondecreasing in $s$ for each $t$ (which corresponds to positive coefficients in (4.1.3)–(4.1.8)). In some of the results, we succeed in avoiding such a constraint and consider both positive and negative terms.

Among numerous publications on the oscillation of equations with deviating arguments, we mention here [54, 178, 181, 322], which are concerned with a distributed delay. In particular, in [322] sufficient oscillation conditions were obtained for integrodifferential equations.

The main results of this chapter were published in [48], [58], [59], [68].

The results of the second part of Theorem 4.8 can be viewed in the following way. If the integrodifferential equation is approximated by the equation with several concentrated delays in such a way that the coefficient of $x(h_j(t))$ does not exceed either the integral $\int_{-\infty}^{h_j(t)} K(t,s)\,ds$ or $\int_{h_j(t)}^{h_{j-1}(t)} K(t,s)\,ds$ and the integral equation is nonoscillatory, then so is the approximating equation with several concentrated delays. This means that nonoscillation of the integrodifferential equation implies nonoscillation of its approximation by an equation with concentrated delays under certain monotonicity conditions.

By Theorem 4.3, nonoscillation of the equation

$$\dot{x}(t) + a(t)x\big(h(t)\big) = 0,\ a(t) \ge 0,\ h(t) \le t, \tag{4.9.3}$$

yields nonoscillation of the integrodifferential equation

$$\dot{x}(t) + \int_{h(t)}^{t} K(t,s)x(s)\,ds = 0,\ \text{where } \int_{h(t)}^{t} K(t,s)\,ds = a(t),\ K(t,s) \ge 0. \tag{4.9.4}$$

In fact, (4.9.3) and (4.9.4) can be written as

$$\dot{x}(t) + \int_{h(t)}^{t} x(s)\,d_s R(t,s) = 0,$$

$$\dot{x}(t) + \int_{h(t)}^{t} x(s)\,d_s T(t,s) = 0,$$

respectively.

We assume that $K(t,s) = 0$ for $s \le h(t)$. Thus, for $s \le h(t)$ we have $R(t,s) = T(t,s) = 0$ and for any $s \in (h(t), t)$

$$R(t,s) = a(t) = \int_{h(t)}^{t} K(t,s)x(s)\,ds \ge \int_{h(t)}^{s} K(t,\zeta)\,d\zeta = T(t,s).$$

For $s > t$, we have $R(t,s) = R(t,t^+) = a(t) = T(t,t^+)$. Since both $R(t,s)$ and $T(t,s)$ are nondecreasing in $s$ for any $t$, by Theorem 4.3, the inequality $R(t,s) \ge T(t,s)$ and nonoscillation of (4.9.3) imply nonoscillation of (4.9.4).

This is usually described in the following form: nonoscillation and stability properties of an equation with a distributed delay are better than for an equation with concentrated delays. We have demonstrated above that we can deduce nonoscillation of equations with concentrated delays from nonoscillation of an integrodifferential equation.

Oscillation of integrodifferential and mixed equations was studied in [71, 322]. For (4.5.2) and its modifications with a variable delay $h(t)$ and also the upper bound $\tau(t)$, where $h(t) \le \tau(t) \le t$, sharp oscillation results were recently obtained in [281, 312].

We can mention also the paper [293], where oscillation of the first-order linear retarded differential equation

$$x'(t) + \int_{0}^{h(t)} x(t-s)d_s R(t,s) = 0$$

is investigated.

Finally, let us formulate some open problems.

1. If we have the same "total weight" $a(t)$ and the maximal delay equals $h(t)$, then nonoscillation of (4.9.3) implies nonoscillation of (4.9.4). If we assume the contrary, that $h(t)$ is the minimal delay, then by Theorem 4.3 nonoscillation of the equation

$$\dot{x}(t) + \int_{-\infty}^{h(t)} K(t,s)x(s)\,ds = 0 \text{ with } \int_{-\infty}^{h(t)} K(t,s)\,ds = a(t), \tag{4.9.5}$$

where $K(t,s) \geq 0$, implies nonoscillation of (4.9.3).

Prove or disprove:

The equation with a distributed delay is nonoscillating if the equation with the same total weight $a(t)$ and the single delay concentrated at the "center of mass" (expectation) of the delay is nonoscillatory; i.e., nonoscillation of (4.9.3) implies nonoscillation of the equation

$$\dot{x}(t) + \int_{g(t)}^{t} K(t,s)x(s)\,ds = 0, \tag{4.9.6}$$

where

$$\int_{h(t)}^{t} K(t,s)(t-s)\,ds = a(t)\big(t - g(t)\big), \quad \int_{h(t)}^{t} K(t,s)\,ds = a(t). \tag{4.9.7}$$

2. For the general integrodifferential equation, there is a gap between nonoscillation and oscillation conditions. If

$$\limsup_{t\to\infty} \int_{h(t)}^{t} K(t,s)\,ds > \frac{1}{e}$$

but

$$\liminf_{t\to\infty} \int_{h(t)}^{t} K(t,s)(t-s)\,ds < \frac{1}{e},$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} K(t,s)(t-s)\,ds < 1,$$

then [71, 322] the known tests do not allow us to establish oscillation properties of the equation. Develop sharp nonoscillation conditions for integrodifferential equations, at least in the case where lim sup and lim inf coincide.

3. Most efficient nonoscillation tests for nonautonomous equations were obtained under the assumption that the delays are finite and tend to infinity as $t \to \infty$. Deduce explicit nonoscillation and oscillation conditions for equations with a distributed delay in the case where (a4) is not satisfied and the relevant kernels do not have an exponential estimate and are not bounded by a Gaussian function (which would allow us to apply the comparison with Examples 4.3 and 4.4, respectively).
4. If $R(t,s)$ is nondecreasing, obtain sufficient conditions under which any nonoscillatory solution of (4.2.8) is asymptotically and exponentially stable.

# Chapter 5
# Scalar Advanced and Mixed Differential Equations on Semiaxes

## 5.1 Introduction

This chapter deals with nonoscillation properties of scalar advanced differential equations and mixed (including delay and advanced terms) differential equations.

Equations considered here differ from equations studied in the previous chapters. It is not clear how to formulate an initial value problem. To study oscillation, we need to assume that there exists a solution of such an equation on the halfline. For equations with both advanced and delayed terms (mixed equations), there is no solution representation formula, so we cannot apply the techniques of the previous chapters. Instead, we use fixed-point theorems in Banach functional spaces.

We start with advanced equations in Sect. 5.2. Advanced differential equations appear, for example, in dynamic economic models. In Sects. 5.3–5.6, all kinds of mixed differential equations are considered, including equations with positive coefficients, with negative coefficients, and with coefficients of different signs. Section 5.7 includes some comments, and several open problems are presented.

Everywhere in this chapter, we assume that there exists a solution of considered advanced and mixed equations. However, in most of our results (for example, Theorems 5.1, 5.3, 5.4, 5.6, 5.7, 5.8, etc.), existence of a solution is justified. This solution is positive and either can be obtained using a limit of successive approximations or its existence is deduced using the Schauder Fixed-Point Theorem.

## 5.2 Advanced Equations

Consider the equation

$$\dot{x}(t) - \sum_{k=1}^{m} a_k(t)x\big(h_k(t)\big) = 0 \tag{5.2.1}$$

under the following conditions:

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(a1) $a_k(t) \geq 0, k = 1, \cdots, m$, are Lebesgue measurable functions locally essentially bounded for $t \geq 0$.
(a2) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \geq t, k = 1, \cdots, m$.

**Definition 5.1** A locally absolutely continuous function $x : [t_0, \infty) \to R$ is called *a solution* of (5.2.1) if it satisfies (5.2.1) for almost all $t \in [t_0, \infty)$.

The same definition will be used for all further advanced equations.

**Theorem 5.1** *Suppose that the inequality*

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u(s)ds\right\}, \ t \geq t_0, \tag{5.2.2}$$

*has a nonnegative solution that is integrable on each interval* $[t_0, b]$. *Then* (5.2.1) *has a positive solution for* $t \geq t_0$.

*Proof* Let $u_0(t)$ be a nonnegative solution of inequality (5.2.2). Denote

$$u_{n+1}(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\}, \ n = 0, 1, \cdots.$$

Then

$$u_1(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\} \leq u_0(t).$$

By induction, we have $0 \leq u_{n+1}(t) \leq u_n(t) \leq u_0(t)$. Hence there exists a pointwise limit $u(t) = \lim_{n\to\infty} u_n(t)$. By the Lebesgue convergence theorem (Theorem A.1), we have

$$u(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u(s)ds\right\}.$$

Then the function

$$x(t) = x(t_0) \exp\left\{\int_{t_0}^{t} u(s)ds\right\} \text{ for any } x(t_0) > 0$$

is a positive solution of (5.2.1). □

**Corollary 5.1** *If*

$$\int_t^{\max_k h_k(t)} \sum_{i=1}^{m} a_i(s)ds \leq \frac{1}{e}, \ t \geq t_0, \tag{5.2.3}$$

*then* (5.2.1) *has a positive solution for* $t \geq t_0$.

*Proof* Let $u_0(t) = e\sum_{k=1}^m a_k(t)$. Then the function $u_0$ satisfies inequality (5.2.2) at any point $t$ where $\sum_{k=1}^m a_k(t) = 0$. In the case where $\sum_{k=1}^m a_k(t) \neq 0$, inequality (5.2.3) implies

$$\begin{aligned}
&\frac{u_0(t)}{\sum_{k=1}^m a_k(t)\exp\{\int_t^{h_k(t)} u_0(s)ds\}} \\
&\quad\geq \frac{u_0(t)}{\sum_{k=1}^m a_k(t)\exp\{\int_t^{\max_k h_k(t)} u_0(s)ds\}} \\
&\quad= \frac{e\sum_{k=1}^m a_k(t)}{\sum_{k=1}^m a_k(t)\exp\{e\int_t^{\max_k h_k(t)} \sum_{i=1}^m a_i(s)ds\}} \\
&\quad\geq \frac{e\sum_{k=1}^m a_k(t)}{\sum_{k=1}^m a_k(t)e} = 1.
\end{aligned}$$

Hence $u_0(t)$ is a positive solution of inequality (5.2.2). By Theorem 5.1, (5.2.1) has a positive solution for $t \geq t_0$. □

**Corollary 5.2** *If there exists $\sigma > 0$ such that $h_k(t) - t \leq \sigma$ and*

$$\int_0^\infty \sum_{k=1}^m a_k(s)ds < \infty,$$

*then (5.2.1) has an eventually positive solution.*

**Corollary 5.3** *If there exists $\sigma > 0$ such that $h_k(t) - t \leq \sigma$ and $\lim_{t\to\infty} a_k(t) = 0$, then (5.2.1) has an eventually positive solution.*

*Proof* Under the conditions of either Corollary 5.2 or Corollary 5.3, obviously there exists $t_0 \geq 0$ such that (5.2.3) is satisfied. □

**Theorem 5.2** *Let $\int^\infty \sum_{k=1}^m a_k(s)ds = \infty$ and $x$ be an eventually positive solution of (5.2.1). Then $\lim_{t\to\infty} x(t) = \infty$.*

*Proof* Suppose $x(t) > 0$ for $t \geq t_1$. Then $\dot{x}(t) \geq 0$ for $t \geq t_1$ and

$$\dot{x}(t) \geq \sum_{k=1}^m a_k(t)x(t_1),\ t \geq t_1,$$

which implies

$$x(t) \geq x(t_1)\int_{t_1}^t \sum_{k=1}^m a_k(s)ds.$$

Thus $\lim_{t\to\infty} x(t) = \infty$. □

Consider together with (5.2.1) the equation

$$\dot{x}(t) - \sum_{k=1}^m b_k(t)x\big(g_k(t)\big) = 0 \tag{5.2.4}$$

for $t \geq t_0$. We assume that for (5.2.4) conditions (a1) and (a2) also hold.

**Theorem 5.3** *Suppose that* $t \le g_k(t) \le h_k(t)$, $0 \le b_k(t) \le a_k(t)$, $t \ge t_0$ *and the conditions of Theorem 5.1 hold. Then (5.2.4) has a positive solution for* $t \ge t_0$.

*Proof* Let $u_0(t) \ge 0$ be a solution of inequality (5.2.2) for $t \ge t_0$. Then this function is also a solution of this inequality if $a_k(t)$ and $h_k(t)$ are replaced by $b_k(t)$ and $g_k(t)$, respectively. The reference to Theorem 5.1 completes the proof. □

**Corollary 5.4** *Suppose that there exist* $\bar{a}_k > 0$ *and* $\sigma_k > 0$ *such that* $0 \le a_k(t) \le \bar{a}_k$, $t \le h_k(t) \le t + \sigma_k$, $t \ge t_0$ *and the inequality*

$$\lambda \ge \sum_{k=1}^{m} \bar{a}_k e^{\lambda \sigma_k}$$

*has a solution* $\lambda \ge 0$. *Then (5.2.1) has a positive solution for* $t \ge t_0$.

*Proof* Consider the equation with constant parameters

$$\dot{x}(t) - \sum_{k=1}^{m} \bar{a}_k x(t + \sigma_k) = 0. \tag{5.2.5}$$

Since the function $u(t) \equiv \lambda$ is a solution of inequality (5.2.2) corresponding to (5.2.5), by Theorem 5.1, (5.2.5) has a positive solution. Theorem 5.3 implies this corollary. □

**Corollary 5.5** *Suppose that* $0 \le a_k(t) \le \bar{a}_k$, $t \le h_k(t) \le t + \sigma$ *for* $t \ge t_0$ *and*

$$\sum_{k=1}^{m} \bar{a}_k \le \frac{1}{e\sigma}.$$

*Then (5.2.1) has a positive solution for* $t \ge t_0$.

*Proof* Since $\sum_{k=1}^{m} \bar{a}_k \le \frac{1}{e\sigma}$, the number $\lambda = \frac{1}{\sigma}$ is a positive solution of the inequality

$$\lambda \ge \left( \sum_{k=1}^{m} \bar{a}_k \right) e^{\lambda \sigma},$$

which completes the proof. □

Consider now the equation with positive coefficients

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(h_k(t)\big) = 0. \tag{5.2.6}$$

**Theorem 5.4** *Suppose that* $a_k(t) \ge 0$ *are continuous functions that are bounded on* $[t_0, \infty)$ *and* $h_k$ *are equicontinuous functions on* $[t_0, \infty)$ *satisfying* $0 \le h_k(t) - t \le \delta$. *Then (5.2.6) has a nonoscillatory solution.*

*Proof* In the space $C[t_0, \infty)$ of continuous functions on $[t_0, \infty)$, consider the set

$$M = \left\{ u \,\middle|\, 0 \le u \le \sum_{k=1}^{m} a_k(t) \right\}$$

and the operator

$$(Hu)(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{ -\int_t^{h_k(t)} u(s)ds \right\}.$$

If $u \in M$, then $Hu \in M$.

For the integral operator

$$(Tu)(t) := \int_t^{h_k(t)} u(s)ds,$$

we will demonstrate that $TM$ is a compact set in the space $C[t_0, \infty)$. If $u \in M$, then

$$\left\| (Tu)(t) \right\|_{C[t_0,\infty)} \le \sup_{t \ge t_0} \int_t^{t+\delta} \left| u(s) \right| ds \le \sup_{t \ge t_0} \sum_{k=1}^{m} a_k(t)\delta < \infty,$$

and hence the functions in the set $TM$ are uniformly bounded in the space $C[t_0, \infty)$.

Functions $h_k$ are equicontinuous on $[t_0, \infty)$, so for any $\varepsilon > 0$ there exists a $\sigma_0 > 0$ such that for $|t - s| < \sigma_0$ the inequality

$$\left| h_k(t) - h_k(s) \right| < \frac{\varepsilon}{2} \left( \sup_{t \ge t_0} \sum_{k=1}^{m} a_k(t) \right)^{-1}, \ k = 1, \cdots, m,$$

holds. From the relation

$$\int_{t_0}^{h_k(t_0)} - \int_t^{h_k(t)} = \int_{t_0}^{t} + \int_t^{h_k(t_0)} - \int_t^{h_k(t_0)} - \int_{h_k(t_0)}^{h_k(t)} = \int_{t_0}^{t} - \int_{h_k(t_0)}^{h_k(t)},$$

we have for

$$|t - t_0| < \min\left\{ \sigma_0, \ \frac{\varepsilon}{2 \sup_{t \ge t_0} \sum_{k=1}^{m} a_k(t)} \right\}$$

and $u \in M$ the estimate

$$\begin{aligned}
\left| (Tu)(t) - (Tu)(t_0) \right| &= \left| \int_t^{h_k(t)} u(s) - \int_{t_0}^{h_k(t_0)} u(s)ds \right| \\
&\le \int_{t_0}^{t} \left| u(s) \right| ds + \int_{h_k(t_0)}^{h_k(t)} \left| u(s) \right| ds \\
&\le |t - t_0| \sup_{t \ge t_0} \sum_{k=1}^{m} a_k(t) + \left| h_k(t) - h_k(t_0) \right| \sup_{t \ge t_0} \sum_{k=1}^{m} a_k(t) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence the set $TM$ contains functions that are uniformly bounded and equicontinuous on $[t_0, \infty)$, so by Theorem A.2 it is compact in the space $C[t_0, \infty)$; thus the set $HM$ is also compact in $C[t_0, \infty)$.

By the Schauder Fixed-Point Theorem (Theorem A.15), there exists a continuous function $u \in M$ such that $u = Hu$, and then the function

$$x(t) = \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}$$

is a bounded positive solution of (5.2.6). Moreover, since $u$ is nonnegative, this solution is nonincreasing on $[t_0, \infty)$. □

**Theorem 5.5** *Suppose that the conditions of Theorem 5.4 hold,*

$$\int_{t_0}^{\infty} \sum_{k=1}^{m} a_k(s)ds = \infty,$$

*and $x$ is a nonoscillatory solution of* (5.2.6). *Then* $\lim_{t\to\infty} x(t) = 0$.

*Proof* Let $x(t) > 0$ for $t \geq t_0$. Then $\dot{x}(t) \leq 0$ for $t \geq t_0$. Hence $x(t)$ is nonincreasing and thus has a finite limit. If $\lim_{t\to\infty} x(t) = d > 0$, then $x(t) > d$ for any $t$ and thus $\dot{x}(t) \leq -d\sum_{k=1}^{m} a_k(t)$, which implies $\lim_{t\to\infty} x(t) = -\infty$. By the assumption, $x(t)$ is positive; the contradiction proves that $\lim_{t\to\infty} x(t) = 0$. □

Let us note that we cannot guarantee any (exponential or polynomial) rate of convergence to zero even for constant coefficients $a_k$, as the following example demonstrates.

*Example 5.1* Consider the equation $\dot{x}(t) + x(h(t)) = 0$, where $h(t) = t^{t \ln t}$, $t \geq 3$, $x(3) = 1/\ln 3$. Then $x(t) = 1/(\ln t)$ is the solution, which tends to zero slower than $t^{-r}$ for any $r > 0$.

Consider now the advanced equation with positive and negative coefficients

$$\dot{x}(t) - \sum_{k=1}^{m}\left[a_k(t)x\big(h_k(t)\big) - b_k(t)x\big(g_k(t)\big)\right] = 0, \ t \geq 0. \tag{5.2.7}$$

**Theorem 5.6** *Suppose that $a_k(t)$ and $b_k(t)$ are Lebesgue measurable locally essentially bounded functions, $a_k(t) \geq b_k(t) \geq 0$, $h_k(t)$ and $g_k(t)$ are Lebesgue measurable functions, $h_k(t) \geq g_k(t) \geq t$ and inequality* (5.2.2) *has a nonnegative solution. Then* (5.2.7) *has a nonoscillatory solution; moreover, it has a positive nondecreasing and a negative nonincreasing solution.*

*Proof* Let $u_0$ be a nonnegative solution of (5.2.2), and denote

$$u_{n+1}(t) = \sum_{k=1}^{m}\left[a_k(t)\exp\left\{\int_{t}^{h_k(t)} u_n(s)ds\right\} - b_k(t)\exp\left\{\int_{t}^{g_k(t)} u_n(s)ds\right\}\right],$$

$t \ge t_0$, $n \ge 0$. We have $u_0 \ge 0$, and by (5.2.2)

$$u_0 \ge \sum_{k=1}^m a_k(t) \exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\}$$
$$\ge \sum_{k=1}^m \left[a_k(t) \exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\} - b_k(t) \exp\left\{\int_t^{g_k(t)} u_0(s)ds\right\}\right] = u_1(t).$$

Since $a_k(t) \ge b_k(t) \ge 0$ and $t \le g_k(t) \le h_k(t)$, then $u_1(t) \ge 0$.

Next, let us assume that $0 \le u_n(t) \le u_{n-1}(t)$. The assumptions of the theorem imply $u_{n+1} \ge 0$. Let us demonstrate that $u_{n+1}(t) \le u_n(t)$. This inequality has the form

$$\sum_{k=1}^m \left[a_k(t) \exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\} - b_k(t) \exp\left\{\int_t^{g_k(t)} u_n(s)ds\right\}\right]$$
$$\le \sum_{k=1}^m \left[a_k(t) \exp\left\{\int_t^{h_k(t)} u_{n-1}(s)ds\right\} - b_k(t) \exp\left\{\int_t^{g_k(t)} u_{n-1}(s)ds\right\}\right],$$

which is equivalent to

$$\sum_{k=1}^m \exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\}\left[a_k(t) - b_k(t) \exp\left\{-\int_{g_k(t)}^{h_k(t)} u_n(s)ds\right\}\right]$$
$$\le \sum_{k=1}^m \exp\left\{\int_t^{h_k(t)} u_{n-1}(s)ds\right\}\left[a_k(t) - b_k(t) \exp\left\{-\int_{g_k(t)}^{h_k(t)} u_{n-1}(s)ds\right\}\right].$$

This inequality is evident for any $0 \le u_n(t) \le u_{n-1}(t)$, $a_k(t) \ge 0$ and $b_k \ge 0$, and thus we have $u_{n+1}(t) \le u_n(t)$.

By the Lebesgue convergence theorem (Theorem A.1), there is a pointwise limit $u(t) = \lim_{n\to\infty} u_n(t)$ satisfying

$$u(t) = \sum_{k=1}^m \left[a_k(t) \exp\left\{\int_t^{h_k(t)} u(s)ds\right\} - b_k(t) \exp\left\{\int_t^{g_k(t)} u(s)ds\right\}\right], \ t \ge t_0,$$

$u(t) \ge 0$, $t \ge t_0$. Then the function

$$x(t) = x(t_0) \exp\left\{\int_{t_0}^t u(s)ds\right\}, \ t \ge t_0,$$

is a positive nondecreasing solution of (5.2.7) for any $x(t_0) > 0$ and is a negative nonincreasing solution of (5.2.7) for any $x(t_0) < 0$. □

**Corollary 5.6** *Let $a_k(t)$ and $b_k(t)$ be Lebesgue measurable locally essentially bounded functions satisfying $a_k(t) \ge b_k(t) \ge 0$, $h_k(t)$ and $g_k(t)$ be Lebesgue measurable functions, where $h_k(t) \ge g_k(t) \ge t$. Assume in addition that inequality (5.2.3) holds. Then (5.2.7) has a nonoscillatory solution.*

Consider now the equation with constant deviations of advanced arguments

$$\dot{x}(t) - \sum_{k=1}^{m} \big[a_k(t)x(t+\tau_k) - b_k(t)x(t+\sigma_k)\big] = 0, \tag{5.2.8}$$

where $a_k, b_k$ are continuous functions, $\tau_k \geq 0, \sigma_k \geq 0$.

Denote $A_k = \sup_{t \geq t_0} a_k(t)$, $\bar{a}_k = \inf_{t \geq t_0} a_k(t)$, $B_k = \sup_{t \geq t_0} b_k(t)$, $\bar{b}_k = \inf_{t \geq t_0} b_k(t)$.

**Theorem 5.7** *Suppose that* $\bar{a}_k \geq 0, \bar{b}_k \geq 0, A_k < \infty, B_k < \infty$.

*If there exists a number* $\lambda_0 < 0$ *such that*

$$\sum_{k=1}^{m} \big(\bar{a}_k e^{\lambda_0 \tau_k} - B_k\big) \geq \lambda_0, \tag{5.2.9}$$

$$\sum_{k=1}^{m} \big(A_k - \bar{b}_k e^{\lambda_0 \sigma_k}\big) \leq 0, \tag{5.2.10}$$

*then* (5.2.8) *has a nonoscillatory solution*; *moreover, it has a positive nonincreasing and a negative nondecreasing solution.*

*Proof* In the space $C[t_0, \infty)$, consider the set $M = \{u | \lambda_0 \leq u \leq 0\}$ and the operator

$$(Hu)(t) = \sum_{k=1}^{m} \left[ a_k(t) \exp\left\{ \int_t^{t+\tau_k} u(s)ds \right\} - b_k(t) \exp\left\{ \int_t^{t+\sigma_k} u(s)ds \right\} \right].$$

For $u \in M$, we have from (5.2.9) and (5.2.10)

$$(Hu)(t) \leq \sum_{k=1}^{m} \big(A_k - \bar{b}_k e^{\lambda_0 \sigma_k}\big) \leq 0,$$

$$(Hu)(t) \geq \sum_{k=1}^{m} \big(\bar{a}_k e^{\lambda_0 \tau_k} - B_k\big) \geq \lambda_0.$$

Hence $HM \subset M$.

Consider the integral operator

$$(Tu)(t) := \int_t^{t+\delta} u(s)ds, \ \delta > 0.$$

We will show that $TM$ is a compact set in the space $C[t_0, \infty)$. For $u \in M$, we have

$$\big\|(Tu)(\cdot)\big\|_{C[t_0,\infty)} \leq \sup_{t \geq t_0} \int_t^{t+\delta} \big|u(s)\big| ds \leq |\lambda_0| \delta,$$

and hence the functions in the set $TM$ are uniformly bounded in the space $C[t_0, \infty)$.

The equality

$$\int_{t_0}^{t_0+\delta} - \int_t^{t+\delta} = \int_{t_0}^{t} + \int_t^{t_0+\delta} - \int_t^{t_0+\delta} - \int_{t_0+\delta}^{t+\delta} = \int_{t_0}^{t} - \int_{t_0+\delta}^{t+\delta}$$

implies

$$\begin{aligned}\big|(Tu)(t)-(Tu)(t_0)\big| &= \left|\int_t^{t+\delta} u(s) - \int_{t_0}^{t_0+\delta} u(s)ds\right| \\ &\le \int_{t_0}^{t} \big|u(s)\big|ds + \int_{t_0+\delta}^{t+\delta} \big|u(s)\big|ds \le 2|\lambda_0||t-t_0|.\end{aligned}$$

Hence, by Theorem A.2, the set $TM$ and so the set $HM$ are compact in the space $C[t_0,\infty)$.

By the Schauder Fixed-Point Theorem (Theorem A.15), there exists a continuous function $u$ satisfying $\lambda_0 \le u \le 0$ such that $u = Hu$, and thus the function

$$x(t) = x(t_0)\exp\left\{\int_{t_0}^{t} u(s)ds\right\},\ t \ge t_0,$$

is a positive nonincreasing solution of (5.2.8) for any $x(t_0) > 0$ and is a negative nondecreasing solution of (5.2.7) for any $x(t_0) < 0$. □

Let us remark that (5.2.10) for any $\lambda_0 < 0$ implies $\sum_{k=1}^{m}(A_k - b_k) < 0$.

**Corollary 5.7** *Let* $\sum_{k=1}^{m}(A_k - b_k) < 0$, $\sum_{k=1}^{m} A_k > 0$, *and for*

$$\lambda_0 = \ln\left(\frac{\sum_{k=1}^{m} A_k}{\sum_{k=1}^{m} b_k}\right) \Big/ \max_k \sigma_k \tag{5.2.11}$$

*the inequality*

$$\sum_{k=1}^{m}\left(a_k e^{\lambda_0 \tau_k} - B_k\right) \ge \lambda_0 \tag{5.2.12}$$

*hold. Then* (5.2.8) *has a bounded positive solution.*

*Proof* The negative number $\lambda_0$ defined in (5.2.11) is a solution of both (5.2.9) and (5.2.10). By definition, it satisfies (5.2.10), and (5.2.12) implies (5.2.9). □

*Example 5.2* Consider the equation with constant advances and coefficients

$$\dot{x}(t) - ax(t+r) + bx(t+d) = 0, \tag{5.2.13}$$

where $0 < a < b$, $d > 0$, $r \ge 0$. Then $\lambda_0 = \frac{1}{d}\ln(\frac{a}{b})$ is the minimal value of $\lambda$ for which inequality (5.2.10) holds; for (5.2.13) it has the form $a - be^{\lambda d} \le 0$.

Inequality (5.2.9) for (5.2.13) can be rewritten as

$$f(\lambda) = ae^{\lambda r} - b - \lambda \ge 0,$$

where the function $f(x)$ decreases on $(-\infty, -\ln(ar)/r]$ if $\tau > 0$ and for any negative $x$ if $r = 0$; besides, $f(0) < 0$. Thus, if $f(\lambda_1) < 0$ for some $\lambda_1 < 0$, then $f(\lambda) < 0$ for any $\lambda \in [\lambda_1, 0)$. Hence the inequality

$$f(\lambda_0) = a\left(\frac{a}{b}\right)^{r/d} - b - \frac{1}{d}\ln\left(\frac{a}{b}\right) \ge 0 \tag{5.2.14}$$

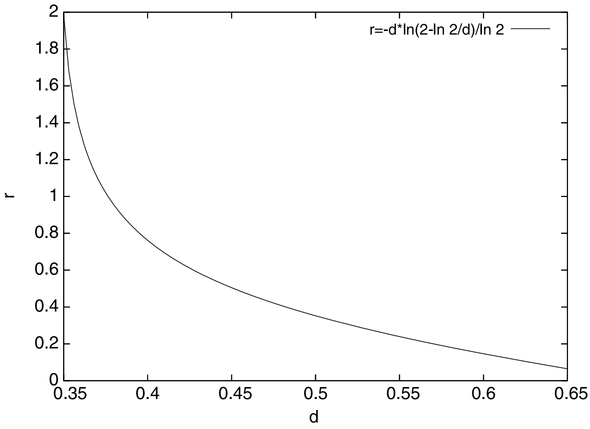


**Fig. 5.1** The domain of values $(d, r)$ satisfying inequality (5.2.15). If the values of advances $d$ and $r$ are under the curve, then (5.2.13) has a positive solution

is necessary and sufficient for the conditions of Theorem 5.7 to be satisfied for (5.2.13).

Figure 5.1 demonstrates possible values of advances $d$ and $r$ such that Corollary 5.7 implies existence of a positive solution in the case $1 = a < b = 2$. Then (5.2.14) has the form $0.5^{r/d} \geq 2 - (\ln 2)/d$, which is possible only for $d > 0.5 \ln 2 \approx 0.347$ and for these values is equivalent to

$$r \leq \frac{-d \ln(2 - \ln 2/d)}{\ln 2}. \tag{5.2.15}$$

## 5.3 Mixed Equations with Positive Coefficients

In this section, we consider the equation

$$\dot{x}(t) + a(t)x\big(g(t)\big) + b(t)x\big(h(t)\big) = 0, \ t \geq t_0, \tag{5.3.1}$$

with nonnegative bounded coefficients $a(t)$, $b(t)$, one delayed argument $g(t) \leq t$ and one advanced argument $h(t) \geq t$.

**Theorem 5.8** *Suppose $a(t)$ and $b(t)$ are continuous bounded and $g(t)$ and $h(t)$ are uniformly continuous on the interval $[t_0, \infty)$,*

$$\limsup_{t\to\infty}\big[t - g(t)\big] = r < \infty, \ \limsup_{t\to\infty}\big[h(t) - t\big] = p < \infty, \tag{5.3.2}$$

*and there exists a nonoscillatory solution of the delay differential equation*

$$\dot{y}(t) + a(t)y\big(g(t)\big) + b(t)y(t) = 0. \tag{5.3.3}$$

*Then there exists a nonoscillatory solution of* (5.3.1).

*Proof* Theorem 2.1 applied to (5.3.3) implies the existence of a function $u_0(t) \geq 0$ and $t_0 \geq 0$ such that

$$u_0(t) \geq a(t) \exp\left\{\int_{g(t)}^{t} u_0(s)ds\right\} + b(t), \ t \geq t_0. \tag{5.3.4}$$

Consider the space $\mathbf{C}[t_0, \infty)$ of all continuous and bounded functions on $[t_0, \infty)$ with supremum norm $\|\cdot\|$, and consider the operator

$$(Au)(t) := a(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{-\int_{t}^{h(t)} u(s)ds\right\}.$$

Let $S = \{u \in \mathbf{C}[t_0, \infty) \mid 0 \le u(t) \le u_0(t)\}$. Inequality (5.3.4) implies $0 \le (Au)(t) \le u_0(t)$. For $u \in S$, denote the integral operators

$$(Hu)(t) := \int_{g(t)}^{t} u(s)ds, \ (Ru)(t) := \int_{t}^{h(t)} u(s)ds.$$

Conditions (5.3.2) imply

$$\left|(Hu)(t)\right| \le r\|u_0\|, \ \left|(Ru)(t)\right| \le p\|u_0\|.$$

Hence the sets $HS$ and $RS$ are bounded in the space $\mathbf{C}[t_0, \infty)$. For $u \in S$, we obtain

$$\begin{aligned}\left|(Hu)(t_2) - (Hu)(t_1)\right| &\le \left|\int_{g(t_1)}^{g(t_2)} u(s)ds\right| + \left|\int_{t_1}^{t_2} u(s)ds\right| \\ &\le \|u_0\|\left(\left|g(t_2) - g(t_1)\right| + |t_2 - t_1|\right)\end{aligned}$$

and

$$\left|(Ru)(t_2) - (Ru)(t_1)\right| \le \|u_0\|\left(\left|h(t_2) - h(t_1)\right| + |t_2 - t_1|\right).$$

Hence the sets of functions $HS$ and $RS$ are equicontinuous. Then, by Theorem A.2, the sets $HS$ and $RS$ are compact. Therefore, the set $AS$ is also compact.

The Schauder Fixed-Point Theorem (Theorem A.15) implies that there exists a solution $u \in S$ of the equation $u = Au$. Therefore, the function

$$x(t) = x(t_1)\exp\left\{-\int_{t_1}^{t} u(s)ds\right\}, \ t \ge t_1,$$

is a positive solution of (5.3.1). □

**Corollary 5.8** *Suppose $a(t)$ and $b(t)$ are continuous bounded and $g(t)$ and $h(t)$ are continuous on* $[t_0, \infty)$, (5.3.2) *holds and*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} a(s)\exp\left\{\int_{g(s)}^{s} b(\tau)d\tau\right\}ds < \frac{1}{e}. \tag{5.3.5}$$

*Then* (5.3.1) *has a nonoscillatory solution.*

*Proof* Substituting $y(t) = z(t)\exp\{-\int_{t_0}^{t} b(s)ds\}$ in (5.3.3), we obtain

$$\dot{z}(t) + a(t)\exp\left\{\int_{g(t)}^{t} b(s)ds\right\}z\big(g(t)\big) = 0.$$

Theorem 2.7 and (5.3.5) imply that this equation and therefore (5.3.1) has a nonoscillatory solution. □

**Corollary 5.9** *Suppose* $a(t)$ *and* $b(t)$ *are continuous bounded and* $g(t)$ *and* $h(t)$ *are continuous on* $[t_0, \infty)$, $b(t)$ *is bounded on* $[t_0, \infty)$, (5.3.2) *holds and*

$$\int_{t_0}^{\infty} a(s)ds < \infty. \tag{5.3.6}$$

*Then* (5.3.1) *has a nonoscillatory solution.*

*Proof* Condition (5.3.2) implies

$$\begin{aligned}&\limsup_{t\to\infty} \int_{g(t)}^{t} a(s)\exp\left\{\int_{g(s)}^{s} b(\tau)d\tau\right\}ds \\ &\quad\le e^{r\|b\|}\limsup_{t\to\infty}\int_{g(t)}^{t} a(s)ds = 0 < \frac{1}{e}.\end{aligned}$$

By Corollary 5.8, (5.3.1) has a nonoscillatory solution. □

**Theorem 5.9** *Suppose*

$$\int_{t_0}^{\infty} a(s)ds = \infty$$

*and* $x$ *is a nonoscillatory solution of* (5.3.1). *Then* $\lim_{t\to\infty} x(t) = 0$.

*Proof* Suppose $x(t) > 0$, $t \ge t_1$ and $g(t) \ge t_1$, $t \ge t_2$. Then $\dot{x}(t) \le 0$, $t \ge t_2$. Denote $u(t) = -\frac{\dot{x}(t)}{x(t)}$, $t \ge t_2$. Then $u(t) \ge 0$, $t \ge t_2$. After substituting

$$x(t) = x(t_2)\exp\left\{-\int_{t_2}^{t} u(s)ds\right\},\ t \ge t_2, \tag{5.3.7}$$

into (5.3.1), we have

$$u(t) = a(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{-\int_{t}^{h(t)} u(s)ds\right\},\ t \ge t_2. \tag{5.3.8}$$

Hence $u(t) \ge a(t)$ and therefore $\int_{t_0}^{\infty} u(s)ds = \infty$. Equality (5.3.7) implies $\lim_{t\to\infty} x(t) = 0$, which completes the proof. □

## 5.4 Mixed Equation with Negative Coefficients

Consider now the mixed differential equation

$$\dot{x}(t) - a(t)x\big(g(t)\big) - b(t)x\big(h(t)\big) = 0,\ t \ge t_0, \tag{5.4.1}$$

where $a(t) \ge 0$, $b(t) \ge 0$, $g(t) \le t$, $h(t) \ge t$.

**Theorem 5.10** *Suppose* $a(t)$ *and* $b(t)$ *are continuous bounded*, $g(t)$ *and* $h(t)$ *are uniformly continuous on* $[t_0, \infty)$, (5.3.2) *holds and the equation*

$$\dot{y}(t) - a(t)y(t) - b(t)y\big(h(t)\big) = 0,\ t \ge t_0, \tag{5.4.2}$$

*has a nonoscillatory solution.*

*Then* (5.4.1) *has a nonoscillatory solution.*

*Proof* In the space $\mathbf{C}[t_0, \infty)$, consider the operator

$$(Bu)(t) := a(t)\exp\left\{-\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{t}^{h(t)} u(s)ds\right\}.$$

Theorem 5.1 implies that there exists a nonnegative solution $u_0(t)$ of the equality

$$u(t) = a(t) + b(t)\exp\left\{\int_{t}^{h(t)} u(s)ds\right\}, \ t \geq t_0.$$

Let $S = \{u | 0 \leq u(t) \leq u_0(t)\}$. As in the proof of Theorem 5.8, we obtain $BS \subset S$, and the set $BS$ is compact. Therefore, the equation $u = Bu$ has a nonnegative solution $u$. Hence the function

$$x(t) = x(t_0)\exp\left\{\int_{t_0}^{t} u(s)ds\right\}, \ t \geq t_0,$$

is a positive solution of (5.4.1). □

**Corollary 5.10** *Suppose $a(t)$ and $b(t)$ are continuous bounded and $g(t)$ and $h(t)$ are uniformly continuous on $[t_0, \infty)$, (5.3.2) holds and*

$$\limsup_{t\to\infty} \int_{t}^{g(t)} b(s)\exp\left\{\int_{s}^{g(s)} a(\tau)d\tau\right\}ds < \frac{1}{e}.$$

*Then (5.4.1) has a nonoscillatory solution.*

The proof of this corollary is similar to the proof of Corollary 5.8.

**Corollary 5.11** *Suppose $a(t)$ and $b(t)$ are continuous bounded and $g$ and $h$ are uniformly continuous on $[t_0, \infty)$, (5.3.2) holds and $\int_{t_0}^{\infty} b(s)ds < \infty$. Then (5.4.1) has a nonoscillatory solution.*

**Theorem 5.11** *Suppose $\int_{t_0}^{\infty} b(s)ds = \infty$ and $x$ is a nonoscillatory solution of (5.4.1). Then $\lim_{t\to\infty} x(t) = \infty$.*

The proof of this theorem is similar to the proof of Theorem 5.9.

## 5.5 Positive Delay Term, Negative Advanced Term

In this section, the equation

$$\dot{x}(t) + a(t)x\big(g(t)\big) - b(t)x\big(h(t)\big) = 0, \ t \geq t_0, \tag{5.5.1}$$

with a positive delay term and a negative advanced term is considered. In this section and the next, we will assume that the following conditions hold:

(b1) $a(t) \geq 0$, $b(t) \geq 0$ are Lebesgue measurable locally essentially bounded functions $[t_0, \infty) \to \mathbb{R}$.

(b2) $g : [t_0, \infty) \to \mathbb{R}$, $h : [t_0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $g(t) \le t$, $h(t) \ge t$, $\lim_{t\to\infty} g(t) = \infty$.

**Theorem 5.12** *Suppose* (b1) *and* (b2) *hold and* $a(t) \ge b(t)$. *Then the following conditions are equivalent*:

1. *The differential inequality*

$$\dot{x}(t) + a(t)x\big(g(t)\big) - b(t)x\big(h(t)\big) \le 0, \ t \ge t_0, \tag{5.5.2}$$

*has an eventually nonincreasing positive solution.*

2. *The integral inequality*

$$u(t) \ge a(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} - b(t)\exp\left\{-\int_{t}^{h(t)} u(s)ds\right\}, \ t \ge t_1, \tag{5.5.3}$$

*has a nonnegative locally integrable solution for some* $t_1 \ge t_0$, *where we assume* $u(t) = 0$ *for* $t < t_1$.

3. *Differential equation* (5.5.1) *has an eventually positive nonincreasing solution.*

*Proof* 1) ⇒ 2). Let $x$ be a solution of (5.5.2) such that $x(t) > 0$, $\dot{x}(t) \le 0$, $t \ge t_0$. For some $t_1 \ge t_0$, we have $g(t) \ge t_0$ for $t \ge t_1$. Denote $u(t) = -\dot{x}(t)/x(t)$, $t \ge t_1$, $u(t) = 0$, $t < t_1$. Then

$$x(t) = x(t_1)\exp\left\{-\int_{t_1}^{t} u(s)ds\right\}, \ t \ge t_1. \tag{5.5.4}$$

After substituting (5.5.4) into (5.5.2) and carrying the exponent out of the brackets, we obtain

$$\begin{aligned} -\exp\left\{-\int_{t_1}^{t} u(s)ds\right\}x(t_1)\bigg[u(t) - a(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} \\ + b(t)\exp\left\{-\int_{t}^{h(t)} u(s)ds\right\}\bigg] \le 0, \end{aligned}$$

which implies (5.5.3).

2) ⇒ 3). Suppose $u_0(t) \ge 0$ is a solution of inequality (5.5.3) for $t \ge t_1$. Consider the sequence

$$u_{n+1}(t) = a(t)\exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\} - b(t)\exp\left\{-\int_{t}^{h(t)} u_n(s)ds\right\}, \ n \ge 0. \tag{5.5.5}$$

Since $u_n(t) \ge a(t) - b(t) \ge 0$ and $u_0 \ge u_1$, then by induction

$$0 \le u_{n+1}(t) \le u_n(t) \le \cdots \le u_0(t).$$

Hence there exists a pointwise limit

$$u(t) = \lim_{n\to\infty} u_n(t).$$

The Lebesgue convergence theorem (Theorem A.1) and (5.5.5) imply

$$u(t) = a(t)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} - b(t)\exp\left\{-\int_{t}^{h(t)} u(s)ds\right\}. \tag{5.5.6}$$

Then $x(t)$ denoted by (5.5.4) is a nonnegative nonincreasing solution of (5.5.1).

Implication 3) $\Rightarrow$ 1) is evident. □

For comparison, consider now the mixed differential equation

$$\dot{x}(t) + a_1(t)x\big(g_1(t)\big) - b_1(t)x\big(h_1(t)\big) = 0,\ t \ge t_0. \tag{5.5.7}$$

**Corollary 5.12** *Suppose* (b1) *and* (b2) *hold for* $a, b, h, g, a_1, b_1, h_1, g_1$ *and*

$$b_1(t) \le b(t) \le a(t) \le a_1(t),\ g(t) \ge g_1(t),\ h(t) \le h_1(t). \tag{5.5.8}$$

*If* (5.5.7) *has an eventually positive solution with an eventually nonpositive derivative, then the same is valid for* (5.5.1).

*Proof* Suppose (5.5.7) has an eventually positive solution with an eventually nonpositive derivative. By Theorem 5.12, the integral inequality

$$u(t) \ge a_1(t)\exp\left\{\int_{g_1(t)}^{t} u(s)ds\right\} - b_1(t)\exp\left\{-\int_{t}^{h_1(t)} u(s)ds\right\},\ t \ge t_1,$$

has a nonnegative locally integrable solution $u(t)$ for some $t_1$. Inequalities (5.5.8) imply that $u(t)$ also satisfies (5.5.3). Thus, by Theorem 5.12, (5.5.1) has an eventually positive solution with an eventually nonpositive derivative. □

**Corollary 5.13** *Suppose* (b1) *and* (b2) *hold for* $t$ *sufficiently large*, $a(t) \ge b(t)$ *and*

$$b(t) \ge a(t)\left[\exp\left\{\int_{g(t)}^{t} a(s)ds\right\} - 1\right]\exp\left\{\int_{t}^{h(t)} a(s)ds\right\}.$$

*Then there exists an eventually positive solution with an eventually nonpositive derivative of* (5.5.1).

*Proof* It is easy to see that $u(t) = a(t)$ is a nonnegative solution of inequality (5.5.3). □

**Corollary 5.14** *Suppose* (b1) *and* (b2) *hold and there exist* $a > 0$, $b > 0$, $\tau > 0$, $\sigma > 0$ *such that*

$$b \le b(t) \le a(t) \le a,\ g(t) \ge t - \tau,\ h(t) \le t + \sigma.$$

*If there exists a solution* $\lambda > 0$ *of the algebraic equation*

$$-\lambda + ae^{\lambda\tau} - be^{-\lambda\sigma} = 0, \tag{5.5.9}$$

*then* (5.5.1) *has an eventually positive solution with an eventually nonpositive derivative.*

*Proof* The function $x(t) = e^{-\lambda t}$ is a positive solution of the autonomous equation

$$\dot{x}(t) + ax(t-\tau) - bx(t+\sigma) = 0, t \geq t_0. \tag{5.5.10}$$

By Corollary 5.12, (5.5.1) has a nonoscillatory solution. □

**Corollary 5.15** *Suppose* (b1) *and* (b2) *hold*, $a(t) \geq b(t)$ *and there exists a nonoscillatory solution of the delay equation*

$$\dot{x}(t) + a(t)x\bigl(g(t)\bigr) = 0. \tag{5.5.11}$$

*Then there exists an eventually positive solution with an eventually nonpositive derivative of* (5.5.1).

*Proof* Theorem 2.1 implies that there exists a nonnegative solution $u(t)$ of the inequality

$$u(t) \geq a(t) \exp\left\{\int_{g(t)}^{t} u(s) ds\right\}.$$

Hence $u(t)$ is also a nonnegative solution of inequality (5.5.3). Then, by Theorem 5.12, (5.5.1) has a nonoscillatory solution. □

*Remark 5.1* Let us recall that (5.5.11) has a nonoscillatory solution if for $t$ sufficiently large

$$\int_{g(t)}^{t} a(s)\, ds \leq \frac{1}{e}.$$

**Corollary 5.16** *Suppose* (b1) *and* (b2) *hold*, $a(t) \geq b(t)$, *integral inequality* (5.5.3) *has a nonnegative solution for* $t \geq t_1$ *and* $\int_0^{\infty}[a(s) - b(s)]ds = \infty$. *Then there exists an eventually positive solution* $x(t)$ *with an eventually nonpositive derivative of* (5.5.1) *such that* $\lim_{t\to\infty} x(t) = 0$.

*Proof* By the assumption of the corollary, there exists a nonnegative solution $u(t)$ of inequality (5.5.3) and also (by the proof of Theorem 5.12) of the relevant equation (5.5.6), and the inequality $u(t) \geq a(t) - b(t)$ is obviously satisfied. Then the function $x(t)$ defined by (5.5.4) is a solution of (5.5.1). For this solution, we have

$$0 < x(t) \leq x(t_1) \exp\left\{-\int_0^t \bigl[a(s) - b(s)\bigr] ds\right\},$$

which immediately implies $\lim_{t\to\infty} x(t) = 0$. □

Corollaries 5.15 and 5.16 imply the following statement.

**Corollary 5.17** *Suppose* (b1) *and* (b2) *hold*, $a(t) \geq b(t)$, $\int_{g(t)}^{t} a(s)\, ds \leq \frac{1}{e}$ *for* $t$ *sufficiently large and*

$$\int_0^{\infty} \bigl[a(s) - b(s)\bigr]\, ds = \infty.$$

*Then the equation*

$$\dot{x}(t) + a(t)x\big(g(t)\big) - b(t)x\big(h(t)\big) = 0$$

*has an eventually positive solution* $x(t)$ *with an eventually nonpositive derivative such that* $\lim_{t\to\infty} x(t) = 0$.

*Example 5.3* Consider the equation

$$\dot{x}(t) + 1.4x(t-0.3) - 1.3x(t+0.3) = 0. \tag{5.5.12}$$

Then $u(t) \equiv 1$ is a solution of the relevant inequality (5.5.3) since

$$1.4e^{0.3} - 1.3e^{-0.3} \approx 0.9267 < 1.$$

We remark that since $1.3 \cdot 0.3 = 0.39 > 1/e \approx 0.368$, then all solutions of both equations

$$\dot{x}(t) + 1.4x(t-0.3) = 0 \tag{5.5.13}$$

and

$$\dot{x}(t) - 1.3x(t+0.3) = 0 \tag{5.5.14}$$

are oscillatory [248]. The characteristic equation

$$-\lambda + 1.4e^{0.3\lambda} - 1.3e^{-0.3\lambda} = 0 \tag{5.5.15}$$

has three real roots: $\lambda_1 \approx -4.2282$, $\lambda_2 \approx 0.5436$ and $\lambda_3 \approx 3.3541$. Thus $e^{-\lambda_1 t}$, $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$ are three nonoscillatory solutions of (5.5.12): the first one is unbounded on $[t_0, \infty)$ and the two others are bounded and have a negative derivative.

*Example 5.4* Consider the equation

$$\dot{x}(t) + (1.375 + 0.025\sin t)x(t-0.3) - (1.325 + 0.025\cos t)x(t+0.3) = 0. \tag{5.5.16}$$

Since $1.3 \le 1.325 + 0.025\cos t \le 1.35 \le 1.375 + 0.025\sin t \le 1.4$, applying Corollary 5.14 and the results of Example 5.3, we obtain that (5.5.16) has an eventually positive solution $x(t)$ with an eventually nonpositive derivative. Moreover, since the integral of a continuous nonnegative periodic function $\int_0^\infty (0.05 + 0.025\sin t - 0.025\cos t)\,dt$ diverges, by Corollary 5.16 this solution satisfies $\lim_{t\to\infty} x(t) = 0$.

**Theorem 5.13** *Suppose* (b1) *and* (b2) *hold and* $b(t) \ge a(t)$. *Then the following conditions are equivalent*:

1. *The differential inequality*

$$\dot{x}(t) + a(t)x\big(g(t)\big) - b(t)x\big(h(t)\big) \ge 0, \ t \ge t_0, \tag{5.5.17}$$

*has an eventually positive solution with an eventually nonnegative derivative.*

2. *The integral inequality*

$$u(t) \ge b(t)\exp\left\{\int_t^{h(t)} u(s)ds\right\} - a(t)\exp\left\{-\int_{g(t)}^t u(s)ds\right\}, \ t \ge t_1, \tag{5.5.18}$$

$u(t) = 0$, $t < t_1$, *has a nonnegative locally integrable solution for some* $t_1 \ge t_0$.

3. *Differential equation* (5.5.1) *has an eventually positive solution with an eventually nonnegative derivative.*

*Proof* 1) $\Rightarrow$ 2). Let $x$ be a solution of (5.5.17) such that $x(t) > 0$, $\dot{x}(t) \ge 0$, $t \ge t_0$. For some $t_1 \ge t_0$, we have $g(t) \ge t_0$ for $t \ge t_1$. Denote $u(t) = \dot{x}(t)/x(t)$, $t \ge t_1$. Then

$$x(t) = x(t_1)\exp\left\{\int_{t_1}^{t} u(s)ds\right\},\ t \ge t_1. \tag{5.5.19}$$

After substituting (5.5.19) into (5.5.17) and carrying the exponent out of the brackets, we obtain

$$\exp\left\{\int_{t_1}^{t} u(s)ds\right\} x(t_1)\left[u(t) - b(t)\exp\left\{\int_{t}^{h(t)} u(s)ds\right\}\right.$$
$$\left. + a(t)\exp\left\{-\int_{g(t)}^{t} u(s)ds\right\}\right] \ge 0.$$

Hence (5.5.18) holds.

2) $\Rightarrow$ 3). Let $u_0(t) \ge 0$ be a solution of inequality (5.5.18). Consider the sequence

$$u_{n+1}(t) = b(t)\exp\left\{\int_{t}^{h(t)} u_n(s)ds\right\} - a(t)\exp\left\{-\int_{g(t)}^{t} u_n(s)ds\right\},\ n \ge 0. \tag{5.5.20}$$

Inequalities $u_n(t) \ge b(t) - a(t) \ge 0$ and $u_0 \ge u_1$ imply

$$0 \le u_{n+1}(t) \le u_n(t) \le \cdots \le u_0(t),$$

so there exists a pointwise limit

$$u(t) = \lim_{n\to\infty} u_n(t).$$

By the Lebesgue convergence theorem (Theorem A.1) and (5.5.20), we have

$$u(t) = b(t)\exp\left\{\int_{t}^{h(t)} u(s)ds\right\} - a(t)\exp\left\{-\int_{g(t)}^{t} u(s)ds\right\}. \tag{5.5.21}$$

Then $x(t)$ defined by (5.5.19) is a positive solution of (5.5.1) with a nonnegative derivative.

Implication 3) $\Rightarrow$ 1) is evident. □

The proofs of the following results are similar to those of the corollaries of Theorem 5.12 and thus will be omitted.

**Corollary 5.18** *Suppose* (b1) *and* (b2) *hold for* (5.5.1) *and* (5.5.7),

$$a_1(t) \le a(t) \le b(t) \le b_1(t),\ \ g(t) \ge g_1(t),\ \ h(t) \le h_1(t).$$

*If* (5.5.7) *has an eventually positive solution with an eventually nonnegative derivative, then so does* (5.5.1).

**Corollary 5.19** *Suppose* (b1) *and* (b2) *hold, for $t$ sufficiently large $b(t) \geq a(t)$, and*

$$a(t) \geq b(t)\left[\exp\left\{\int_t^{h(t)} b(s)ds\right\} - 1\right]\exp\left\{\int_{g(t)}^t b(s)ds\right\}.$$

*Then there exists an eventually positive solution of* (5.5.1) *with an eventually nonnegative derivative.*

*Proof* It is easy to see that $u(t) = b(t)$ is a nonnegative solution of inequality (5.5.18). □

**Corollary 5.20** *Suppose* (b1) *and* (b2) *hold and there exist $a > 0$, $b > 0$, $\tau > 0$ and $\sigma > 0$ such that*

$$a \leq a(t) \leq b(t) \leq b, \; g(t) \geq t - \tau, \; h(t) \leq t + \sigma.$$

*If there is a positive solution $\lambda > 0$ of the algebraic equation*

$$\lambda + ae^{-\lambda\tau} - be^{\lambda\sigma} = 0, \tag{5.5.22}$$

*then there exists an eventually positive solution of* (5.5.1) *with an eventually nonnegative derivative.*

**Corollary 5.21** *Suppose* (b1) *and* (b2) *hold, $b(t) \geq a(t)$ and there exists a nonoscillatory solution of the advanced equation*

$$\dot{x}(t) - b(t)x\big(h(t)\big) = 0. \tag{5.5.23}$$

*Then there exists an eventually positive solution of* (5.5.1) *with an eventually nonnegative derivative.*

*Remark 5.2* If for $t$ sufficiently large

$$\int_t^{h(t)} b(s)\,ds \leq \frac{1}{e},$$

then by Corollary 5.1 there exists a nonoscillatory solution of (5.5.23).

**Corollary 5.22** *Suppose* (b1) *and* (b2) *hold, $b(t) \geq a(t)$, the integral inequality* (5.5.3) *has a nonnegative solution for $t \geq t_1$ and $\int_0^\infty [b(s) - a(s)]\,ds = \infty$. Then there exists an eventually positive solution $x(t)$ of* (5.5.1) *with an eventually nonnegative derivative such that* $\lim_{t\to\infty} x(t) = \infty$.

## 5.6 Negative Delay Term, Positive Advanced Term

In this section, we consider the scalar mixed differential equation

$$\dot{x}(t) - a(t)x\big(g(t)\big) + b(t)x\big(h(t)\big) = 0, \; t \geq t_0, \tag{5.6.1}$$

for which assumptions (b1) and (b2) are satisfied.

**Theorem 5.14** *Suppose that $a(t)$ and $b(t)$ are continuous functions that are bounded on $[t_0, \infty)$, functions $g(t)$ and $h(t)$ are uniformly continuous on $[t_0, \infty)$, there exist positive numbers $a_1, a_2, b_1, b_2, \tau, \sigma, t_1$ such that*

$$a_1 \le a(t) \le a_2, \ b_1 \le b(t) \le b_2, \ t - g(t) \le \tau, \ h(t) - t \le \sigma, \ t \ge t_1, \tag{5.6.2}$$

*and the algebraic system*

$$\begin{cases} a_2 e^{y\tau} - b_1 e^{-y\sigma} \le x, \\ b_2 e^{x\sigma} - a_1 e^{-x\tau} \le y, \end{cases} \tag{5.6.3}$$

*has a solution $x = d_1 > 0$, $y = d_2 > 0$.*

*Then (5.6.1) has a nonoscillatory solution.*

*Proof* Define the operator

$$(Au)(t) = a(t) \exp\left\{ - \int_{g(t)}^{t} u(s)ds \right\} - b(t) \exp\left\{ \int_{t}^{h(t)} u(s)ds \right\}, \ t \ge t_1,$$

in the space $\mathbf{C}[t_1, \infty)$ of all functions bounded continuous on $[t_1, \infty)$ with the usual sup-norm, where we assume that $u(t) = 0$, $t \le t_1$. Let $x = d_1$, $y = d_2$ be a positive solution of system (5.6.3). Then the inequality $-d_2 \le u(t) \le d_1$ implies $-d_2 \le (Au)(t) \le d_1$. This means that $AS \subset S$, where

$$S = \left\{ u | -d_2 \le u(t) \le d_1 \right\}.$$

Now we will prove that $AS$ is a compact set in the space $\mathbf{C}[t_1, \infty)$. Denote the integral operators

$$(Hu)(t) := \int_{g(t)}^{t} u(s)ds, \ (Ru)(t) := \int_{t}^{h(t)} u(s)ds.$$

We have for $u \in S$

$$\left|(Hu)(t)\right| \le \max\{d_1, d_2\}\tau, \ \left|(Ru)(t)\right| \le \max\{d_1, d_2\}\sigma.$$

Hence the sets $HS$ and $RS$ are bounded in the space $\mathbf{C}[t_1, \infty)$.

Let $u \in S$. Then

$$\begin{aligned} \left|(Hu)(\tau_2) - (Hu)(\tau_1)\right| &\le \left| \int_{g(\tau_1)}^{g(\tau_2)} \left|u(s)\right| ds \right| + \left| \int_{\tau_1}^{\tau_2} \left|u(s)\right| ds \right| \\ &\le \max\{d_1, d_2\}\left( \left|g(\tau_2) - g(\tau_1)\right| + |\tau_2 - \tau_1| \right) \end{aligned}$$

and, similarly,

$$\left|(Ru)(\tau_2) - (Ru)(\tau_1)\right| \le \max\{d_1, d_2\}\left( \left|h(\tau_2) - h(\tau_1)\right| + |\tau_2 - \tau_1| \right).$$

Since $g$ and $h$ are uniformly continuous in $[t_1, \infty)$, functions in $HS$ and $RS$ are equicontinuous. Then, by Theorem A.2, the sets $HS$ and $RS$ are compact, and consequently $AS$ is also a compact set.

Schauder's Fixed-Point Theorem (Theorem A.15) implies that there exists a solution $u \in S$ of operator equation $u = Au$. Therefore $x(t) = x(t_1) \exp\{\int_{t_1}^{t} u(s)ds\}$, $t \ge t_1$, is a positive solution of (5.6.1). □

**Corollary 5.23** *Suppose that $a(t)$, $b(t)$ are functions which are continuous and bounded on $[t_0, \infty)$, $g \le t$ and $h \ge$ are uniformly continuous on $[t_0, \infty)$ and there exist positive numbers $a_1$, $a_2$, $b_1$, $b_2$, $\tau$, $\sigma$ such that (5.6.2) is satisfied and at least one of the following conditions holds*:

1) $b_2 < a_1, 0 < a_2 - b_1 < \frac{1}{\tau+\sigma} \ln \frac{a_1}{b_2}$,
2) $a_2 < b_1, 0 < b_2 - a_1 < \frac{1}{\tau+\sigma} \ln \frac{b_1}{a_2}$.

*Then (5.6.1) has a nonoscillatory solution.*

*Proof* It is enough to prove that system (5.6.3) has a positive solution. Suppose condition 1) holds. For the second condition, the proof is similar.

Let us define the functions $F(x) = b_2 e^{x\sigma} - a_1 e^{-x\tau}$ and $G(y) = a_2 e^{y\tau} - b_1 e^{-y\sigma}$, which are both monotonically increasing. We have $F(0) < 0$, $F(x_1) = 0$, where

$$x_1 = \frac{1}{\tau + \sigma} \ln \frac{a_1}{b_2}.$$

Since $G(y)$ is a monotone function, there exists the monotonically increasing inverse function $G^{-1}(x)$, for which we have $G^{-1}(x_2) = 0$, where $x_2 = a_2 - b_1 > 0$.

Denote $H(x) = G^{-1}(x) - F(x)$. Condition 1) implies $x_2 < x_1$, then $F(x_2) < 0$ and thus $H(x_2) > 0$. From the equality $G(y) = a_2 e^{y\tau} - b_1 e^{-y\sigma} = x$, we have $a_2 e^{y\tau} \le x + b_1$ for $y \ge 0$, and thus

$$G^{-1}(x) \le \frac{1}{\tau} \ln \frac{x + b_1}{a_2} \text{ and } H(x) \le \frac{1}{\tau} \ln \frac{x + b_1}{a_2} - b_2 e^{x\sigma} + a_1$$

for $x$ large enough. Hence $\lim_{x\to\infty} H(x) = -\infty$.

Since $H$ is continuous, there exists $x_0 > x_2 > 0$ such that $H(x_0) = 0$, i.e. $F(x_0) = G^{-1}(x_0)$. Therefore, $x_0$, $y_0 = F(x_0)$ is a solution of system (5.6.3), and (5.6.1) has a nonoscillatory solution. □

If the conditions of Corollary 5.23 do not hold, we can apply numerical methods to prove that system (5.6.3) has a positive solution.

*Example 5.5* Consider the equation

$$\begin{aligned} &\dot{x}(t) - (1.3 + 0.1 \sin t) x(t - 0.1 - 0.1 \cos t) \\ &\quad + (1.7 + 0.1 \cos t) x(t + 0.2 + 0.1 \sin t) = 0, \ t \ge 0. \end{aligned} \tag{5.6.4}$$

Then $a_1 = 1.2$, $a_2 = 1.4$, $b_1 = 1.6$, $b_2 = 1.8$, $\tau = 0.2$, $\sigma = 0.3$ and (5.6.3) has a positive solution $x = 2$, $y = 3$, since

$$a_2 e^{y\tau} - b_1 e^{-y\sigma} = 1.4 e^{0.6} - 1.6 e^{-0.9} \approx 1.9 < x = 2,$$

$$b_2 e^{x\sigma} - a_1 e^{-x\tau} \le y = 1.8 e^{0.6} - 1.2 e^{-0.4} \approx 2.48 < y = 3.$$

Hence (5.6.4) has a nonoscillatory solution.

Figure 5.2 illustrates the domain of values $(x, y)$ satisfying the system of inequalities (5.6.3) for (5.6.4), which is between the two curves.

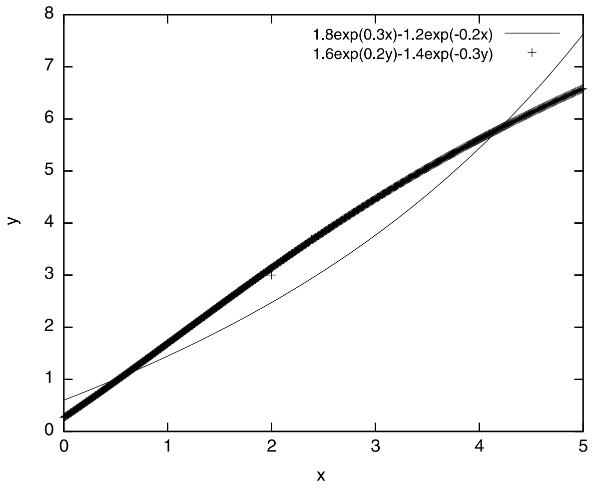


**Fig. 5.2** The *domain* of values $(x, y)$ satisfying the system of inequalities (5.6.3) for (5.6.4) is *between the curves*. The chosen value $x = 2$, $y = 3$ inside the *domain* is also marked on the graph

*Example 5.6* Consider the equation with constant coefficients and variable advance and delay

$$\dot{x}(t) - ax\big(g(t)\big) + bx\big(h(t)\big) = 0, \ t \geq t_0, \tag{5.6.5}$$

where $t \geq g(t) \geq t - 0.2$, $t \leq h(t) \leq t + 0.3$. Thus in (5.6.3) we have $\tau = 0.2$, $\delta = 0.3$. All values below the curve in Fig. 5.3 are such that the system of inequalities (5.6.3) has a positive solution and hence (5.6.5) has a nonoscillatory solution.

For comparison, we also included the line

$$0.2a + 0.3b = \frac{1}{e}. \tag{5.6.6}$$

*Remark 5.3* The autonomous equation

$$\dot{x}(t) - ax(t-\tau) + bx(t+\sigma) = 0, \ a > 0, \ b > 0, \ \tau > 0, \ \sigma > 0 \tag{5.6.7}$$

always has a positive solution $e^{\lambda t}$, where $\lambda$ is a solution of the characteristic equation

$$f(\lambda) = \lambda - ae^{-\tau\lambda} + be^{\sigma\lambda} = 0. \tag{5.6.8}$$

Since $\lim_{\lambda\to\pm\infty} f(\lambda) = \pm\infty$, there is always a real $\lambda$ satisfying (5.6.8). There is a positive solution satisfying $\lim_{t\to\infty} x(t) = \infty$ if $b < a$ and a positive solution satisfying $\lim_{t\to\infty} x(t) = 0$ if $b > a$.

## 5.7 Discussion and Open Problems

Recently results on oscillation of delay differential equations (DDEs) have taken the shape of a developed theory presented in monographs [3, 7, 154, 167, 192, 248]. Most oscillation criteria for DDEs can be extended to equations of advanced type (ADEs) (see [229, 248] and also the recent papers [10, 268, 286]). In comparison

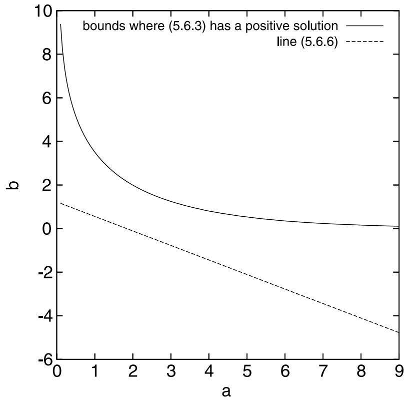


**Fig. 5.3** The *domain* of values $a, b$ such that the system of inequalities (5.6.3) for (5.6.5) has a positive solution is *under the curve*

with these papers, Sect. 5.2 contains the following new results: criteria for existence of a positive solution and a comparison theorem for advanced equations with negative coefficients (Theorems 5.1 and 5.3) and nonoscillation conditions for advanced equations with positive coefficients and with positive and negative coefficients (Theorems 5.4–5.6). We note that these two classes of advanced equations have not been considered before.

For mixed differential equations (MDEs), equations with delay and advanced arguments, there are only a few publications dealing with oscillation problems.

In this chapter, we considered a mixed differential equation,

$$\dot{x}(t) + \delta_1 a(t)x\big(g(t)\big) + \delta_2 b(t)x\big(h(t)\big) = 0,\ t \geq t_0, \tag{5.7.1}$$

with variable coefficients $a(t) \geq 0$, $b(t) \geq 0$, and one delayed $(g(t) \leq t)$ and one advanced $(h(t) \geq t)$ argument. To the best of our knowledge, oscillation of such equations has not been studied before except partial cases of autonomous equations [171, 247], equations of the second or higher order [151, 240] and equations with constant delays [337]. In [166], nonoscillation only of (5.7.1) and higher-order equations was considered, where $\delta_1$ and $\delta_2$ have the same sign. In [290], the author considers a differential equation with a deviating argument without the assumption that it is either a delay or an advanced equation, so the results of [290] can be applied to MDE (5.7.1). The results presented in this chapter and in [290] are independent.

Functional differential equations

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(r_k(t)\big) = 0$$

without the assumption that either $r_k(t) \leq t$ or $r_k(t) \geq t$ were considered in [199].

We presented results for equations with variable arguments and coefficients, one delay and one advanced term in the case where coefficients have any of four possible sign combinations.

The results of Sect. 5.2 were published in [69]. The results of Sects. 5.3 and 5.4 were published in [79], and the results of Sects. 5.5 and 5.6 were extracted from [76].

If the delayed term is positive, we not only claim the existence of a positive solution but present sufficient conditions under which its asymptotics can be deduced (i.e., a nonincreasing solution that tends to zero or a nondecreasing solution that tends to infinity).

Below we present some open problems and topics for research and discussion.

1. Prove or disprove the following conjecture:

   If (5.2.1), with $a_k(t) \geq 0$, has a nonoscillatory solution, then (5.2.7) with positive and negative coefficients also has a nonoscillatory solution.

   As the first step in this direction, prove or disprove:

   If $h(t) \geq t$ and the equation

   $$\dot{x}(t) - a^+(t)x\big(h(t)\big) = 0$$

   has a nonoscillatory solution, then the equation

   $$\dot{x}(t) - a(t)x\big(h(t)\big) = 0$$

   also has a nonoscillatory solution, where $a^+(t) = \max\{a(t), 0\}$.

   If these conjectures are valid, obtain comparison results for advanced equations.
2. Deduce nonoscillation conditions for (5.2.1) with oscillatory coefficients. Oscillation results for an equation with a constant advance and an oscillatory coefficient were recently obtained in [264].
3. Consider advanced equations with positive and negative coefficients when the numbers of positive and negative terms do not coincide.
4. Study existence and/or uniqueness problems for the initial value problem or boundary value problems for advanced differential equations. Generally speaking, it is not clear how to set up such problems.
5. Consider mixed equations of the form

   $$\dot{x}(t) + \sum_{k=1}^{m} \alpha_k a_k(t)x\big(h_k(t)\big) + \sum_{i=1}^{n} \beta_i b_i(t)x\big(g_i(t)\big) = 0, \ \alpha_k, \beta_i = \pm 1,$$

   and obtain sufficient nonoscillation conditions.
6. Prove or disprove:

   Theorem 5.14 remains true without the assumption that the system (5.6.3) has a positive solution.
7. Prove or disprove:

   If $a(t) \geq b(t) \geq 0$ and (5.5.1) has an eventually positive solution with an eventually nonpositive derivative that tends to zero, then all solutions of the ordinary differential equation

   $$\dot{x}(t) + \big[a(t) - b(t)\big]x(t) = 0 \tag{5.7.2}$$

   tend to zero as $t \to \infty$.

   If $b(t) \geq a(t) \geq 0$ and (5.5.1) has an eventually positive solution with an eventually nonnegative derivative that tends to infinity as $t \to \infty$, then all solutions of (5.7.2) tend to infinity.

8. Find sufficient conditions when (5.6.1) has a positive nonincreasing solution $x(t)$ such that $\lim_{t\to\infty} x(t) = 0$ or find sufficient conditions when (5.6.1) has a positive solution $x(t)$ with a nonnegative derivative such that $\lim_{t\to\infty} x(t) = \infty$.
9. Obtain sufficient conditions when the equation with one term that can be both advanced and delayed and an oscillating coefficient

$$\dot{x}(t) + a(t)x\big(g(t)\big) = 0 \tag{5.7.3}$$

has a positive solution. For instance, if $a(t)[t - g(t)]$ is either positive or negative for any $t$, then (5.7.3) can be rewritten in the form (5.5.1) or (5.6.1), and thus some conditions can be deduced from the results of the present chapter.
10. Prove or disprove:

Suppose $a(t) \geq 0$, $b(t) \geq 0$, $\int_0^\infty [a(t) + b(t)]dt = \infty$.

If (5.3.1) has a positive solution, then this equation is asymptotically stable.

If (5.4.1) has a positive solution, then the absolute value of any nontrivial solution tends to infinity.

# Chapter 6
# Neutral Differential Equations

## 6.1 Introduction and Preliminaries

In this chapter, we consider oscillation and nonoscillation properties of the scalar neutral differential equation

$$\dot{x}(t) - a(t)\dot{x}\big(g(t)\big) + b(t)x\big(h(t)\big) = 0,\ t \geq t_0. \tag{6.1.1}$$

The chapter is organized as follows. Section 6.1 contains relevant definitions, notations and auxiliary lemmas. Section 6.2 includes the main results of the chapter on the equivalence of nonoscillation of (6.1.1), the existence of a positive solution for a differential inequality and the existence of a nonnegative solution of some nonlinear integral inequality explicitly constructed by (6.1.1). Sections 6.2 and 6.3 include a comparison theorem and nonoscillation results for (6.1.1). Section 6.4 presents conditions when all solutions of (6.1.1) are oscillatory. These results are obtained by applying nonoscillation criteria and comparison with a differential equation containing an infinite number of delays. It is to be noted that in the cases where the neutral equation turns into a delay equation (either $a(t) \equiv 0$ or $g(t) \equiv t$), the oscillation results for (6.1.1) coincide with these known for delay equations.

Section 6.5 contains conditions on initial functions and initial values that imply positivity of the solution of the initial value problem. In Sect. 6.6, we discuss the relation between nonoscillation and existence of a slowly oscillating solution. In Sect. 6.7, we consider equations with positive and negative coefficients. Section 6.8 contains discussion and open problems.

We consider (6.1.1) under the following conditions:

(a1) $a(t)$, $b(t)$, $g(t)$ and $h(t)$ are Lebesgue measurable locally essentially bounded functions.

(a2) $a(t) \geq 0$, $\sup_{t \geq t_0} a(t) < 1$.

(a3) $g(t) \leq t$, $\lim_{t\to\infty} g(t) = \infty$ and the condition $\operatorname{mes} E = 0$ implies $\operatorname{mes} g^{-1}(E) = 0$, where $\operatorname{mes} E$ is the Lebesgue measure of the set $E$.

(a4) $h(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$.

R.P. Agarwal et al., *Nonoscillation Theory of Functional Differential Equations with Applications*, DOI 10.1007/978-1-4614-3455-9_6,

By Theorem A.10, see also papers [146, 147], the operator $S : L_\infty[t_0, b] \to L_\infty[t_0, b]$ defined by the equality

$$(Sy)(t) = \begin{cases} a(t)y(g(t)), & g(t) \geq t_0, \\ 0, & g(t) < t_0, \end{cases} \tag{6.1.2}$$

is bounded for any $b > t_0$, and its spectral radius is $r(S) < 1$.

As a corollary of this result, we have the following lemma.

**Lemma 6.1** *Suppose $a$, $g$ are Lebesgue measurable locally essentially bounded functions and conditions* (a2), (a3) *hold. Then, for every $b > t_0$ the inverse to the $I - S$ operator can be presented as $(I - S)^{-1} = I + S + S^2 + \cdots$, where $I$ is the identity operator. Operator $(I - S)^{-1}$ is positive if $a(t) \geq 0$.*

Consider now the differential equation with an infinite number of delays

$$\dot{x}(t) + \sum_{k=0}^{\infty} b_k(t) x\big(h_k(t)\big) = 0, \quad t \geq t_0, \tag{6.1.3}$$

where

$$b_0(t) = b(t), \quad b_{k+1}(t) = (Sb_k)(t), \quad h_0(t) = h(t), \quad h_{k+1}(t) = h_k\big(g(t)\big). \tag{6.1.4}$$

By induction it is easy to see that

$$\sup_{t \in [t_0, b]} \big|b_k(t)\big| \leq \sup_{t \in [t_0, b]} \big|a(t)\big|^k \sup_{t \in [t_0, b]} \big|b(t)\big|.$$

Then

$$B(t) = \sum_{k=0}^{\infty} b_k(t)$$

is an essentially locally bounded function. Equation (6.1.3) with this condition coincides with (4.1.8), where $K(t, s) \equiv 0$, so all the results for (4.1.8) obtained in Chap. 4 can be applied to (6.1.3).

Together with (6.1.1), we consider the initial value problem

$$\dot{x}(t) - a(t)\dot{x}\big(g(t)\big) + b(t)x\big(h(t)\big) = f(t), \quad t \geq t_0, \tag{6.1.5}$$

$$x(t) = \varphi(t), \dot{x}(t) = \psi(t), \quad t < t_0, \quad x(t_0) = x_0. \tag{6.1.6}$$

We also assume that the following hypothesis holds:

(a5) $f : [t_0, \infty) \to R$ is a Lebesgue measurable locally essentially bounded function and $\varphi, \psi : (-\infty, t_0) \to \mathbb{R}$ are Borel measurable bounded functions.

**Definition 6.1** An absolutely continuous function $x : \mathbb{R} \to \mathbb{R}$ on each interval $[t_0, b]$ is called *a solution* of problem (6.1.5), (6.1.6) if it satisfies (6.1.5) for almost all $t \in [t_0, \infty)$ and also satisfies (6.1.6).

**Definition 6.2** For each $s \geq t_0$, the solution $X(t,s)$ of the problem

$$\dot{x}(t) - a(t)\dot{x}\big(g(t)\big) + b(t)x\big(h(t)\big) = 0,\ x(t) = 0,\ \dot{x}(t) = 0,\ t < s,\ x(s) = 1 \tag{6.1.7}$$

is called *the fundamental function* of (6.1.1).

We assume $X(t,s) = 0, 0 \leq t < s$. Note that the fundamental functions of (6.1.1) and (6.1.3) coincide. Indeed, the condition $x(t) = 0$, $\dot{x}(t) = 0$, $t < s$ implies that (6.1.1) can be rewritten in the form $(I - S)\dot{x}(t) + b(t)x(h(t)) = 0$, $t \geq s$, which is equivalent to (6.1.3). By Theorem B.4, we have the following lemma.

**Lemma 6.2** *There exists one and only one solution of problem* (6.1.5), (6.1.6); *this solution can be presented in the form*

$$x(t) = X(t,t_0)x_0 + \int_{t_0}^{t} X(t,s)\big[(I-S)^{-1}f\big](s)ds + \int_{t_1}^{t} X(t,s)\big[(I-S)^{-1}F\big](s)ds, \tag{6.1.8}$$

*where* $F(t) = a(t)\psi(g(t)) - b(t)\varphi(h(t))$ *and* $\psi(g(t)) = 0$ *for* $g(t) \geq t_0$ *and* $\varphi(h(t)) = 0$ *for* $h(t) \geq t_0$.

**Definition 6.3** We will say that (6.1.1) *has a nonoscillatory solution* if there exists a solution of (6.1.5), (6.1.6) with $f \equiv 0$ that is eventually positive or eventually negative; otherwise, all solutions of (6.1.1) are *oscillatory*.

The same definition will be used for (6.1.3). We will need some properties of (6.1.3). Consider together with (6.1.3) the equation

$$\dot{x}(t) + \sum_{k=0}^{\infty} c_k(t)x\big(p_k(t)\big) = 0, \tag{6.1.9}$$

where functions $c_k$ are essentially locally bounded and for $p_k$ conditions (a4) hold.

The following lemma follows from Theorems 4.1 and 4.4.

**Lemma 6.3**

1) *Suppose* $b_k(t) \geq 0$ *and* (6.1.3) *has a nonoscillatory solution. Then there exists* $t_0 \geq 0$ *such that the fundamental function of* (6.1.3) *is positive for* $t \geq s \geq t_0$.
2) *Suppose all solutions of* (6.1.3) *are oscillatory and* $c_k(t) \geq b_k(t) \geq 0$, $p_k(t) \leq h_k(t)$. *Then all solutions of* (6.1.9) *are oscillatory.*

## 6.2 Nonoscillation Criteria

The following theorem establishes nonoscillation criteria.

**Theorem 6.1** *Suppose $b(t) \geq 0$. Then the following hypotheses are equivalent*:

1) *For some $t_1 \geq 0$, the differential inequality*

$$\dot{y}(t) - a(t)\dot{y}\big(g(t)\big) + b(t)y\big(h(t)\big) \leq 0 \tag{6.2.1}$$

*has a positive solution $y(t) > 0$ for $t > t_1$ satisfying $y(t) = \dot{y}(t) = 0$ for $t < t_1$.*

2) *For some $t_1 \geq 0$, the integral inequality*

$$u(t) \geq a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} \tag{6.2.2}$$

*has a nonnegative locally integrable solution $u(t)$ for $t \geq t_1$, where $u(t) = 0$ for $t < t_1$.*

3) *For some $t_1 \geq 0$, (6.1.1) has a positive solution $x(t) > 0$, $t > t_1$ with $x(t) = \dot{x}(t) = 0$ for $t < t_1$.*

4) *There exists $t_1 \geq 0$ such that the fundamental function is positive: $X(t,s) > 0$, $t \geq s \geq t_1$.*

*Proof* 1) ⇒ 2) Let $y(t)$ be a positive solution of inequality (6.2.1); i.e., $y(t) > 0$ for $t \geq t_1$ and $y(t) = \dot{y}(t) = 0$ for $t < t_1$. Then $y$ is also a solution of the inequality

$$\dot{y}(t) + (I - S)^{-1}\big[b(t)y\big(h(t)\big)\big] \leq 0, \ t \geq t_1;$$

i.e., $\dot{y}(t) \leq -(I - S)^{-1}[b(t)y(h(t))] \leq 0$ for each $t \geq t_1$ since by Lemma 6.1 the operator $(I - S)^{-1}$ is positive. Hence $y$ is nonincreasing, the function $u(t)$ defined as

$$u(t) = -\frac{d}{dt}\ln\frac{y(t)}{y(t_1)}$$

is nonnegative for $t \geq t_1$ and

$$y(t) = y(t_1)\exp\left\{-\int_{t_1}^{t} u(s)ds\right\}, \ t \geq t_1. \tag{6.2.3}$$

After substituting (6.2.3) into (6.2.1) and carrying the exponent out of the brackets, we obtain

$$-\exp\left\{-\int_{t_1}^{t} u(s)ds\right\}y(t_0)\bigg[u(t) - a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\}$$
$$- b(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\}\bigg] \leq 0,$$

which implies (6.2.2).

2) ⇒ 3) Suppose $u_0(t)$, $t \geq t_1$, is a nonnegative solution of (6.2.2). Denote

$$u_{n+1}(t) = a(t)u_n\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\}. \tag{6.2.4}$$

Since $a, b$ are nonnegative and (6.2.2) holds for $u = u_0$, we have

$$0 \leq u_{n+1}(t) \leq u_n(t) \leq \cdots \leq u_0(t).$$

Hence there exists a pointwise limit $u(t) = \lim_{n\to\infty} u_n(t)$. The Lebesgue convergence theorem (Theorem A.1) and (6.2.4) imply

$$u(t) = a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\}.$$

Obviously

$$x(t) = \exp\left\{-\int_{t_1}^{t} u(s)ds\right\},\ t \geq t_1,\ x(t) = \dot{x}(t) = 0,\ t < t_1,$$

is a nonoscillatory solution of (6.1.1).

3) ⇒ 4) Suppose $x(t) > 0$, $t \geq t_0$, $x(t) = \dot{x}(t) = 0$, $t < t_1$ is a solution of (6.1.1). Then it is also a positive solution of (6.1.3). Corollary 4.5 implies that the fundamental function of (6.1.3) is positive: $X(t,s) > 0$, $t \geq s \geq t_1$. Since (6.1.1) and (6.1.3) have the same fundamental functions, the fundamental function of (6.1.1) is also positive for $t \geq s \geq t_1$.

The implication 4) ⇒ 1) is obvious. □

*Remark 6.1* The equivalence of oscillation properties for (6.1.1) and the corresponding differential inequality was demonstrated in [91, 252].

**Corollary 6.1** *Equation* (6.1.3) *has a nonoscillatory solution if and only if* (6.1.1) *has a nonoscillatory solution.*

*Proof* Equations (6.1.1) and (6.1.3) have the same fundamental functions, and the reference to Theorem 6.1 completes the proof. □

As the next corollary of Theorem 6.1, let us obtain a comparison result. Consider the neutral differential equation

$$\dot{x}(t) - a_1(t)\dot{x}\big(g(t)\big) + b_1(t)x\big(h_1(t)\big) = 0, \tag{6.2.5}$$

where for parameters of (6.2.5) hypotheses (a1)–(a4) hold.

**Theorem 6.2**

1) *Suppose that*

$$0 \leq a_1(t) \leq a(t),\ 0 \leq b_1(t) \leq b(t),\ h(t) \leq h_1(t),$$

(6.1.1) *has a nonoscillatory solution, and its fundamental function is positive for* $t \geq t_1$. *Then* (6.2.5) *also has a nonoscillatory solution, and its fundamental function is positive for* $t \geq t_1$.

2) *Suppose that*

$$0 \leq a(t) \leq a_1(t),\ 0 \leq b(t) \leq b_1(t),\ h_1(t) \leq h(t),$$

*and all solutions of* (6.1.1) *are oscillatory. Then all solutions of* (6.2.5) *are also oscillatory.*

*Proof* 1) Theorem 6.1 yields that there exists a nonnegative solution $u$ of inequality (6.2.2). Then $u$ is also a solution of this inequality, where $a, b, g, h$ are replaced by $a_1, b_1, g_1, h_1$. By Theorem 6.1, (6.2.5) has a nonoscillatory solution.

Statement 1) immediately implies 2). □

*Remark 6.2* Another comparison theorem for (6.1.1) was obtained in [182].

**Corollary 6.2** *Let* $0 < a < 1, b > 0, \sigma \geq 0, \tau \geq 0$. *Suppose that* $0 \leq a(t) \leq a$, $0 \leq b(t) \leq b$, $g(t) = t - \sigma$, $h(t) \geq t - \tau$ *and the equation*

$$\dot{x}(t) - a\dot{x}(t - \sigma) + bx(t - \tau) = 0 \tag{6.2.6}$$

*has a nonoscillatory solution. Then* (6.1.1) *also has a nonoscillatory solution.*

*If* $a(t) \geq a \geq 0$, $b(t) \geq b \geq 0$, $g(t) = t - \sigma$, $h(t) \leq t - \tau$ *and all solutions of* (6.2.6) *are oscillatory, then all solutions of* (6.2.6) *are oscillatory.*

In Theorem 6.2, we compared nonoscillation and oscillation properties of (6.1.1) and (6.2.5) under the condition $b(t) \geq 0$, $b_1(t) \geq 0$. In the next theorem, $b_1(t)$ is not assumed to be nonnegative.

**Theorem 6.3** *Suppose* $b(t) \geq 0$, $0 \leq a_1(t) \leq a(t)$, $b_1(t) \leq b(t)$, $h(t) \leq h_1(t)$, (6.1.1) *has a nonoscillatory solution, and its fundamental function is positive for* $t \geq t_1$. *Then* (6.2.5) *also has a nonoscillatory solution, and its fundamental function is positive for* $t \geq t_1$.

*Proof* Since $0 \leq b_1^+(t) \leq b(t)$, by Theorem 6.2 the fundamental function $X^+(t, s)$ of the equation

$$\dot{x}(t) - a_1(t)\dot{x}\big(g(t)\big) + b_1^+(t)x\big(h_1(t)\big) = 0 \tag{6.2.7}$$

is positive for $t \geq t_1$.

Consider the following problem for $f \geq 0$, where $f$ is a locally essentially bounded function on $[t_1, \infty)$:

$$\dot{x}(t) - a_1(t)\dot{x}\big(g(t)\big) + b_1(t)x\big(h_1(t)\big) = f(t),\ t \geq t_1,\ x(t) = \dot{x}(t) = 0,\ t \leq t_1. \tag{6.2.8}$$

Then

$$\dot{x}(t) - a_1(t)\dot{x}\big(g(t)\big) + b_1^+(t)x\big(h_1(t)\big) = b_1^-(t)x\big(h_1(t)\big) + f(t),\ t \geq t_1,$$

$x(t) = \dot{x}(t) = 0$ for $t \leq t_1$. Hence

$$x(t) = \int_{t_1}^{t} X^+(t, s)(I - S)^{-1}\big[b_1^-(s)x\big(h_1(s)\big)\big]ds + v(t),$$

where $v(t) = \int_{t_1}^{t} X^+(t, s)(I - S)^{-1} f(s)ds$. By Corollary B.3, the function $X^+(t, s)$ is essentially bounded on $[t_1, b] \times [t_0, b]$ for any $b > t_0$.

Then $v(t) \geq 0$ and $v \in L_\infty[t_1, b]$ for any $b > t_1$.

Denote

$$(Tx)(t) = \int_{t_1}^{t} X^{+}(t,s)(I-S)^{-1}\big[b_1^{-}(s)x\big(h_1(s)\big)\big]ds.$$

Then $T = KH$, where

$$(Hx)(t) = (I-S)^{-1}\big[b_1^{-}(t)x\big(h_1(t)\big)\big],\ (Ky)(t) = \int_{t_1}^{t} X^{+}(t,s)y(s)ds.$$

Operator $H$ is a Volterra linear bounded operator (for the definition of Volterra operators, see Appendix A) in the space $L_\infty[t_1,b]$. Theorem A.4 implies that $K$ is a Volterra integral weakly compact operator in the space $L_\infty[t_1,b]$. Then $T = KH$ is a weakly compact Volterra linear operator in the space $L_\infty[t_1,b]$ as a composition of a continuous and a weakly compact operator. By Theorem A.7, its spectral radius is $r(T) = 0$.

Hence, for the solution of problem (6.2.8) we have $x(t) = (I-T)^{-1}v(t) \ge 0$ for any $v(t) \ge 0$. But we have another representation for this solution, $x(t) = \int_{t_1}^{t} X(t,s)v(s)ds \ge 0$ for $v(t) \ge 0$. Hence $X(t,s) \ge 0$ for $t \ge s \ge t_1$. The strict inequality $X(t,s) > 0$, $t \ge s \ge t_1$ can be obtained as in the proof of Theorem 2.1. □

**Corollary 6.3** *Suppose that the integral inequality*

$$u(t) \ge a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b^{+}(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} \tag{6.2.9}$$

*has a nonnegative locally integrable solution for $t \ge t_1$. Then (6.1.1) has a nonoscillatory solution, and its fundamental function is positive for $t \ge t_1$.*

Let us now compare solutions of two neutral equations. To this end, consider together with (6.1.5) and (6.1.6) the problem

$$\dot y(t) - a_1(t)\dot y\big(g(t)\big) + b_1(t)y\big(h(t)\big) = f_1(t),\ t \ge t_0, \tag{6.2.10}$$

$$y(t) = \varphi_1(t),\ \dot y(t) = \psi_1(t),\ t < t_0,\ y(t_0) = y_0. \tag{6.2.11}$$

Suppose that for parameters of (6.2.10) and (6.2.11) conditions (a1)–(a5) hold. Denote by $Y(t,s)$ the fundamental function of (6.2.10).

**Theorem 6.4** *Suppose*

$$a(t) \ge a_1(t) \ge 0,\ b(t) \ge b_1(t) \ge 0,\ f_1(t) \ge f(t),$$

$$\varphi(t) \ge \varphi_1(t),\ \psi_1(t) \ge \psi(t),\ y_0 \ge x_0.$$

*Suppose in addition that the solution $x$ of (6.1.5) and (6.1.6) is positive and the derivative $\dot x$ is nonpositive for $t > t_0$. Then $y(t) \ge x(t) > 0$, where $y$ is the solution of (6.2.10) and (6.2.11).*

*Proof* Let us rewrite (6.1.5) in the form

$$\dot{x}(t) - a_1(t)\dot{x}\big(g(t)\big) + b_1(t)x\big(h(t)\big)$$
$$= \big[a(t) - a_1(t)\big]\dot{x}\big(g(t)\big) - \big[b(t) - b_1(t)\big]x\big(h(t)\big) + f(t).$$

Then solution representation (6.1.8) for (6.1.5) and (6.1.6) has the form

$$x(t) = Y(t,t_0)x_0 + \int_{t_0}^{t} Y(t,s)\big[(I-S)^{-1}f\big](s)ds$$
$$+ \int_{t_0}^{t} Y(t,s)(I-S)^{-1}\big[\big(a(s)-a_1(s)\big)\dot{x}\big(g(s)\big) - \big(b(s)-b_1(s)\big)x\big(h(s)\big)\big]ds$$
$$+ \int_{t_0}^{t} Y(t,s)(I-S)^{-1}\big[a(s)\psi\big(g(s)\big) - b(s)\varphi\big(h(s)\big)\big]ds,$$

where $(I-S)^{-1}$ is a positive operator. For the solution $y$ of (6.2.10) and (6.2.11), we have

$$y(t) = Y(t,t_0)y_0 + \int_{t_0}^{t} Y(t,s)\big[(I-S)^{-1}f_1\big](s)ds$$
$$+ \int_{t_0}^{t} Y(t,s)(I-S)^{-1}\big[a_1(s)\psi_1\big(g(s)\big) - b_1(s)\varphi_1\big(h(s)\big)\big]ds.$$

Hence $y(t) \geq x(t) > 0$. □

## 6.3 Efficient Nonoscillation Conditions

Using Theorem 6.1 and Corollary 6.3, we will now obtain explicit nonoscillation conditions.

**Theorem 6.5** *Suppose $b^+(t) > 0$ almost everywhere and at least one of the following conditions holds*:

1) $$0 < \lambda < \liminf_{t\to\infty}\left[\frac{1}{e} - \frac{a(t)b^+(g(t))}{b^+(t)}\right]\exp\left\{\frac{1}{\lambda}\int_{g(t)}^{h(t)} b^+(s)ds\right\}, \tag{6.3.1}$$

*where* $\lambda = \limsup_{t\to\infty}\int_{g(t)}^{t} b(s)ds$;

2) $$0 < \lambda < \liminf_{t\to\infty}\left[\frac{1}{e} - \frac{a(t)b^+(g(t))}{b^+(t)}\exp\left\{\frac{1}{\lambda}\int_{h(t)}^{g(t)} b^+(s)ds\right\}\right],$$

*where* $\lambda = \limsup_{t\to\infty}\int_{h(t)}^{t} b^+(s)ds$;

3) $$0 < \lambda < \liminf_{t\to\infty}\frac{1}{e}\left[1 - \frac{a(t)b^+(g(t))}{b(t)}\exp\left\{\frac{1}{\lambda}\int_{g(t)}^{t} b^+(s)ds\right\}\right], \tag{6.3.2}$$

*where* $\lambda = \limsup_{t\to\infty}\int_{h(t)}^{t} b^+(s)ds$.

*Then* (6.1.1) *has a nonoscillatory solution, and the fundamental function of this equation is eventually positive.*

*Proof* It is sufficient to prove the theorem for $b(t) \geq 0$. In the general case, Theorem 6.3 can be applied.

1) We will show that $u(t) = \frac{b(t)}{\lambda}$ is a solution of inequality (6.2.2).

The definition of $\lambda$ and inequality (6.3.1) yield that for some $t_1 > t_0$ and $\varepsilon > 0$

$$\lambda \leq \left( \frac{1}{e^{1+\varepsilon}} - \frac{a(t)b(g(t))}{b(t)} \right) \exp\left\{ \frac{1}{\lambda} \int_{g(t)}^{h(t)} b(s)ds \right\}, \ t \geq t_1, \tag{6.3.3}$$

$$\frac{1}{\lambda} \int_{g(t)}^{t} b(s)ds \leq 1 + \varepsilon, \ t \geq t_1. \tag{6.3.4}$$

Inequality (6.3.3) implies

$$\frac{b(t)}{\lambda e^{1+\varepsilon}} \geq \frac{1}{\lambda} a(t) b\big(g(t)\big) + b(t) \exp\left\{ \frac{1}{\lambda} \int_{h(t)}^{g(t)} b(s)ds \right\}.$$

Since by (6.3.4) we have $\frac{1}{e^{1+\varepsilon}} \leq \exp\{-\frac{1}{\lambda} \int_{g(t)}^{t} b(s)ds\}$,

$$\frac{b(t)}{\lambda} \exp\left\{ -\frac{1}{\lambda} \int_{g(t)}^{t} b(s)ds \right\} \geq \frac{1}{\lambda} a(t) b\big(g(t)\big) + b(t) \exp\left\{ \frac{1}{\lambda} \int_{h(t)}^{g(t)} b(s)ds \right\}$$

for $t \geq t_1$. Thus

$$\frac{b(t)}{\lambda} \geq \frac{1}{\lambda} a(t) b\big(g(t)\big) \exp\left\{ \frac{1}{\lambda} \int_{g(t)}^{t} b(s)ds \right\} + b(t) \exp\left\{ \frac{1}{\lambda} \int_{h(t)}^{t} b(s)ds \right\}, \tag{6.3.5}$$

which implies that $u(t) = \frac{b(t)}{\lambda}$ is a nonnegative solution of inequality (6.2.2); consequently, (6.1.1) has a nonoscillatory solution.

The proof if condition 2) holds is similar, with the same solution $u(t) = \frac{b(t)}{\lambda}$ of inequality (6.2.2).

3) We will show that $u(t) = \frac{b(t)}{\lambda}$ is a solution of inequality (6.2.2).

The definition of $\lambda$ and inequality (6.3.2) imply for some $t_1 > t_0$ and $\varepsilon > 0$

$$\frac{1}{\lambda} \int_{g(t)}^{t} b(s)ds \leq 1 + \varepsilon, \ t \geq t_1,$$

$$\lambda b(t) e^{1+\varepsilon} \leq b(t) - a(t) b\big(g(t)\big) \exp\left\{ \frac{1}{\lambda} \int_{g(t)}^{t} b(s)ds \right\}, \ t \geq t_1.$$

Then

$$\lambda b(t) \exp\left\{ \frac{1}{\lambda} \int_{h(t)}^{t} b(s)ds \right\} \leq b(t) - a(t) b\big(g(t)\big) \exp\left\{ \frac{1}{\lambda} \int_{g(t)}^{t} b(s)ds \right\}$$

for $t \geq t_1$, which is equivalent to (6.3.5). Therefore, by Theorem 6.1, (6.1.1) has a nonoscillatory solution. □

**Corollary 6.4** *Suppose $b^+(t) > 0$ almost everywhere and at least one of the following conditions is satisfied*:

1) $$g(t) \leq h(t), \ 0 < \limsup_{t \to \infty} \int_{g(t)}^{t} b^+(s)ds < \liminf_{t \to \infty} \left[ \frac{1}{e} - \frac{a(t) b^+(g(t))}{b^+(t)} \right].$$

2) $$h(t) \le g(t),\ 0 < \limsup_{t\to\infty} \int_{h(t)}^{t} b^+(s)ds < \liminf_{t\to\infty}\left[\frac{1}{e} - \frac{a(t)b^+(g(t))}{b^+(t)}\right].$$

*Then* (6.1.1) *has a nonoscillatory solution, and the fundamental function of this equation is eventually positive.*

*Proof* Since $e^x > 1$ for $x > 0$, the statement of the corollary follows from conditions 1) and 2) of Theorem 6.5. □

**Corollary 6.5** *Suppose* $0 < a < 1$, $b > 0$, $\tau > 0$, $\sigma > 0$ *and at least one of the following conditions holds*:

1) $b\sigma < (\frac{1}{e} - a)e^{1-\frac{\tau}{\sigma}}$;
2) $b\tau < \frac{1}{e} - ae^{1-\frac{\sigma}{\tau}}$;
3) $b\tau < \frac{1}{e}(1 - ae^{\frac{\sigma}{\tau}})$.

*Then* (6.2.6) *has a nonoscillatory solution, and the fundamental function of this equation is eventually positive.*

*Remark 6.3*

1. The same results as in Corollary 6.5 by a different method were obtained in [167].
2. Corollaries 6.1 and 6.5 can be employed to obtain explicit nonoscillation conditions for (6.1.1).

Another set of explicit nonoscillation conditions for (6.1.1) can be obtained by applying the following result.

**Theorem 6.6** *Suppose* $b^+(t) > 0$ *almost everywhere and at least one of the following conditions holds*:

1) $$0 < \lambda < \liminf_{t\to\infty}\left[\frac{1}{e(1-a(t))} - \frac{a(t)b^+(g(t))}{b^+(t)(1-a(g(t)))}\right]\exp\left\{\frac{1}{\lambda}\int_{g(t)}^{h(t)} \frac{b^+(s)}{1-a(s)}ds\right\},$$

*where*

$$\lambda = \limsup_{t\to\infty} \int_{g(t)}^{t} \frac{b^+(s)}{1-a(s)}ds;$$

2) $$0 < \lambda < \liminf_{t\to\infty}\left[\frac{1}{e(1-a(t))} - \frac{a(t)b^+(g(t))}{b^+(t)(1-a(g(t)))}\exp\left\{\frac{1}{\lambda}\int_{h(t)}^{g(t)} \frac{b^+(s)}{1-a(s)}ds\right\}\right],$$

*where*

$$\lambda = \limsup_{t\to\infty} \int_{h(t)}^{t} \frac{b^+(s)}{1-a(s)}ds;$$

3) $$0 < \lambda < \liminf_{t\to\infty}\frac{1}{e}\left[\frac{1}{1-a(t)} - \frac{a(t)b^+(g(t))}{b^+(t)(1-a(g(t)))}\exp\left\{\frac{1}{\lambda}\int_{g(t)}^{t} \frac{b^+(s)}{1-a(s)}ds\right\}\right],$$

*where*

$$\lambda = \limsup_{t\to\infty} \int_{h(t)}^{t} \frac{b^+(s)}{1-a(s)} ds.$$

*Then* (6.1.1) *has a nonoscillatory solution, and the fundamental function of this equation is eventually positive.*

*Proof* The argument is similar to the proof of Theorem 6.5 if we assume

$$u(t) = \frac{b(t)}{\lambda(1-a(t))}.$$ □

**Corollary 6.6** *Suppose $b^+(t) > 0$ almost everywhere and at least one of the following conditions is satisfied:*

1) $g(t) \le h(t)$ *and*

$$0 < \limsup_{t\to\infty} \int_{g(t)}^{t} \frac{b^+(s)}{1-a(s)} ds < \liminf_{t\to\infty} \left( \frac{1}{e(1-a(t))} - \frac{a(t)b^+(g(t))}{b^+(t)(1-a(g(t)))} \right);$$

2) $h(t) \le g(t)$ *and*

$$0 < \limsup_{t\to\infty} \int_{h(t)}^{t} \frac{b^+(s)}{1-a(s)} ds < \liminf_{t\to\infty} \left( \frac{1}{e(1-a(t))} - \frac{a(t)b^+(g(t))}{b^+(t)(1-a(g(t)))} \right).$$

*Then* (6.1.1) *has a nonoscillatory solution, and the fundamental function of this equation is eventually positive.*

*Remark 6.4* If in (6.1.1) we assume $g(t) \equiv t$, then Theorem 6.6 implies the best possible nonoscillation condition for this delay equation:

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \frac{b^+(s)}{1-a(s)} ds < \frac{1}{e}.$$

The following theorem is a generalization of the well-known nonoscillation condition for delay differential equations.

**Theorem 6.7** *Suppose $b^+(t) >$ almost everywhere,*

$$\int_{t_1}^{\infty} b^+(s)ds < \infty, \ \limsup_{t\to\infty} \left[ \frac{1}{e} - \frac{a(t)b^+(g(t))}{b^+(t)} \right] > 0, \tag{6.3.6}$$

*and $h(t) - g(t)$ is a nonoscillatory function.*

*Then* (6.1.1) *has a nonoscillatory solution.*

*Proof* We have

$$\limsup_{t\to\infty} \int_{h(t)}^{t} b^+(s)ds = \limsup_{t\to\infty} \int_{g(t)}^{t} b^+(s)ds = 0.$$

Corollary 6.4 implies that (6.1.1) has a nonoscillatory solution. □

**Corollary 6.7** *Suppose*

$$\int_{t_1}^{\infty} b^+(s)ds < \infty,\ \limsup_{t\to\infty} a(t) < \frac{1}{e},\ \limsup_{t\to\infty} \frac{b^+(g(t))}{b^+(t)} \le 1,$$

*and $h(t) - g(t)$ is a nonoscillatory function.*

*Then* (6.1.1) *has a nonoscillatory solution.*

*Proof* For some $t_1 \ge t_1$, $\epsilon > 0$, we have

$$a(t) \le \frac{1}{e} - \epsilon,\ \frac{b^+(g(t))}{b^+(t)} \le 1 + \epsilon,\ t \ge t_1.$$

Hence

$$\frac{a(t)b^+(g(t))}{b^+(t)} \le \frac{1}{e} - \epsilon\left(1 - \frac{1}{e}\right) - \epsilon^2,\ t \ge t_1.$$

Then

$$\limsup_{t\to\infty}\left[\frac{1}{e} - \frac{a(t)b^+(g(t))}{b^+(t)}\right] > 0.$$

□

*Example 6.1* Consider the equation

$$\dot{x}(t) - a(t)\dot{x}(t-\sigma) + \frac{b}{t^{\alpha}} x\big(h(t)\big) = 0,\ t \ge t_1 > 0, \tag{6.3.7}$$

where $0 \le \limsup_{t\to\infty} a(t) < \frac{1}{e}$, $b > 0$, $\alpha > 1$, $\sigma > 0$, $\lim_{t\to\infty} h(t) = \infty$, $h(t) \le t$. By Corollary 6.7, (6.3.7) has a nonoscillatory solution.

**Theorem 6.8** *Let $b(t) \ge 0$, $\int_{t_1}^{\infty} b(s)ds = \infty$. Then, for every nonoscillatory solution of* (6.1.1), *we have $\lim_{t\to\infty} x(t) = 0$.*

*Proof* If $x(t) > 0$ for $t \ge t_1$, then for some $t_2 \ge t_1$ the function $u(t) = -\frac{\dot{x}(t)}{x(t)}$ is a nonnegative solution of (6.2.2) for $t \ge t_2$ (see the proof of Theorem 6.1). Inequality (6.2.2) implies $u(t) \ge b(t)$; hence $\int_{t_1}^{\infty} u(s)ds = \infty$. For the solution $x$ of (6.1.1), we have $x(t) = x(t_2)\exp\{-\int_{t_2}^{t} u(s)ds\}$ for $t \ge t_2$. Then $\lim_{t\to\infty} x(t) = 0$, which completes the proof. □

## 6.4 Explicit Oscillation Conditions

In this section, we assume that $b(t) \ge 0$.

Denote $p(t) = \max\{g(t), h(t)\}$.

**Theorem 6.9** *Suppose*

$$\liminf_{t\to\infty} \int_{p(t)}^{t} \left[a(s)b\big(g(s)\big)\exp\left\{\int_{g(s)}^{p(s)} b(\tau)d\tau\right\} + b(s)\exp\left\{\int_{h(s)}^{p(s)} b(\tau)d\tau\right\}\right]ds > \frac{1}{e}. \tag{6.4.1}$$

*Then all solutions of* (6.1.1) *are oscillatory.*

*Proof* Suppose there exists a nonoscillatory solution of (6.1.1). Then there exists a nonnegative solution $u$ of inequality (6.2.2) for $t \geq t_1 \geq t_1$. Let us rewrite inequality (6.2.2) in the form

$$u(t)\exp\left\{-\int_{p(t)}^{t} u(s)ds\right\} \geq a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{p(t)} u(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{p(t)} u(s)ds\right\},\ t \geq t_1.$$

Then

$$\int_{p(t)}^{t} u(s)\exp\left\{-\int_{p(s)}^{s} u(\tau)d\tau\right\}ds \geq \int_{p(t)}^{t}\left[a(s)u\big(g(s)\big)\exp\left\{\int_{g(s)}^{p(s)} u(\tau)d\tau\right\} + b(s)\exp\left\{\int_{h(s)}^{p(s)} u(\tau)d\tau\right\}\right]ds$$

for $t \geq t_1$. Inequality (6.2.2) implies $u(t) \geq b(t)$, and therefore

$$\int_{p(t)}^{t} u(s)\exp\left\{-\int_{p(s)}^{s} u(\tau)d\tau\right\}ds \geq \int_{p(t)}^{t}\left[a(s)b\big(g(s)\big)\exp\left\{\int_{g(s)}^{p(s)} b(\tau)d\tau\right\} + b(s)\exp\left\{\int_{h(s)}^{p(s)} b(\tau)d\tau\right\}\right]ds$$

for $t \geq t_1$. From

$$\int_{p(t)}^{t} u(s)\exp\left\{-\int_{p(s)}^{s} u(\tau)d\tau\right\}ds \leq \int_{p(t)}^{t} u(s)\exp\left\{-\inf_{t\geq t_1}\int_{p(t)}^{t} u(\tau)d\tau\right\}ds = \exp\left\{-\inf_{t\geq t_1}\int_{p(t)}^{t} u(\tau)d\tau\right\}\int_{p(t)}^{t} u(s)ds,$$

we conclude that

$$\exp\left\{-\inf_{t\geq t_1}\int_{p(t)}^{t} u(\tau)d\tau\right\}\inf_{t\geq t_1}\int_{p(t)}^{t} u(s)ds \geq \inf_{t\geq t_1}\int_{p(t)}^{t}\left[a(s)b\big(g(s)\big)\exp\left\{\int_{g(s)}^{p(s)} b(\tau)d\tau\right\} + b(s)\exp\left\{\int_{h(s)}^{p(s)} b(\tau)d\tau\right\}\right]ds.$$

The equality $\sup_{t\geq 0} te^{-t} = 1/e$ implies

$$\liminf_{t\to\infty}\int_{p(t)}^{t}\left[a(s)b\big(g(s)\big)\exp\left\{\int_{g(s)}^{p(s)} b(\tau)d\tau\right\} + b(s)\exp\left\{\int_{h(s)}^{p(s)} b(\tau)d\tau\right\}\right]ds \leq \frac{1}{e}.$$

This is a contradiction with (6.4.1), which proves the theorem. □

**Corollary 6.8** *Suppose* $0 < a < 1,\ b > 0,\ \tau \geq 0,\ \sigma \geq 0$ *and at least one of the following conditions holds*:

1) $\sigma \le \tau, \sigma b(a + e^{\tau-\sigma}) > 1/e$;
2) $\sigma \ge \tau, \tau b(1 + ae^{\sigma-\tau}) > 1/e$.

*Then all solutions of* (6.2.6) *are oscillatory.*

Corollary 6.1 yields that oscillation properties of (6.1.1) and (6.1.3) are equivalent. As a corollary of this statement, we can obtain additional explicit oscillation conditions.

**Theorem 6.10** *Suppose all solutions of the delay differential equation*

$$\dot{x}(t) + b(t)x\big(h(t)\big) = 0 \tag{6.4.2}$$

*are oscillatory. Then all solutions of* (6.1.1) *are oscillatory.*

*Proof* Suppose (6.1.1) has a nonoscillatory solution. Corollary 6.1 yields that (6.1.3) has a nonoscillatory solution. Hence, for some $t_1 \ge t_1$, solution $x$ of (6.1.3) with $x(t) = 0$ for $t \le t_1$ and $x(t_1) = 1$ is positive. Then

$$\dot{x}(t) + b(t)x\big(h(t)\big) \le 0, \; t \ge t_1.$$

By Theorem 2.1, (6.4.2) has a nonoscillatory solution. We have a contradiction with the assumption of the theorem, which completes the proof. □

**Corollary 6.9** *Suppose*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} b(s)ds > \frac{1}{e}.$$

*Then all solutions of* (6.1.1) *are oscillatory.*

**Corollary 6.10** *Suppose* $h(t) \equiv t$, *and*

$$\liminf_{t\to\infty} \int_{g(t)}^{t} a(s)b\big(g(s)\big) \exp\left\{\int_{g(s)}^{s} b(\tau)d\tau\right\} ds > \frac{1}{e}. \tag{6.4.3}$$

*Then all solutions of* (6.1.1) *are oscillatory.*

*Proof* If $h(t) \equiv t$, then (6.1.3) has the form

$$\dot{x}(t) + b(t)x(t) + a(t)b\big(g(t)\big)x\big(g(t)\big) + \cdots = 0. \tag{6.4.4}$$

After substituting $x(t) = y(t)\exp\{-\int_{t_1}^{t} b(s)ds\}$ in (6.4.4) and multiplying both sides by $\exp\{\int_{t_1}^{t} b(s)ds\}$, we have

$$\dot{y}(t) + a(t)b\big(g(t)\big) \exp\left\{\int_{g(t)}^{t} b(s)ds\right\} y\big(g(t)\big) + \cdots = 0.$$

Condition (6.4.3) and the proof of Theorem 6.10 imply that all solutions of this equation and therefore all solutions of (6.1.1) are oscillatory. □

**Theorem 6.11** *Suppose $h(t)$ is a nondecreasing function and all solutions of the equation*

$$\dot{x}(t) + \big((I-S)^{-1}b\big)(t)x\big(h(t)\big) = 0 \tag{6.4.5}$$

*are oscillatory. Then all solutions of (6.1.1) are oscillatory, where the operator $S$ is defined by (6.1.2) and acts in the space $L_\infty[t_1, c]$ for any $c > t_1$.*

*Proof* Equation (6.4.5) can be rewritten in the form

$$\dot{x}(t) + \sum_{k=0}^{\infty} b_k(t)x\big(h(t)\big) = 0, \tag{6.4.6}$$

where $b_k(t)$ are defined by (6.1.4). We have $h(g(t)) \le h(t)$ and hence $h_k(t) \le h(t)$, with $h_k(t)$ also defined in (6.1.4). Lemma 6.3 implies that all solutions of (6.1.3) are oscillatory. By Corollary 6.1, all solutions of (6.1.1) are also oscillatory. □

**Corollary 6.11** *Suppose $h(t)$ is a nondecreasing function and*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \big((I-S)^{-1}b\big)(s)ds > \frac{1}{e}.$$

*Then all solutions of (6.1.1) are oscillatory.*

**Corollary 6.12** *Suppose $h(t)$ is a nondecreasing function and for some $n \ge 0$ all solutions of the equation*

$$\dot{x}(t) + \sum_{k=0}^{n} b_k(t)x\big(h(t)\big) = 0$$

*are oscillatory, where the coefficients $b_k$ are defined in (6.1.4). Then all solutions of (6.1.1) are also oscillatory.*

**Corollary 6.13** *Let $0 < a < 1$, $b > 0$, $\tau \ge 0$, $\sigma \ge 0$, $b\tau e > 1 - a$. Then all solutions of (6.2.6) are oscillatory.*

*Proof* We have $((I-S)^{-1}b)(t) = b + ab\chi_{[\tau,\infty)}(t) + a^2 b\chi_{[2\tau,\infty)}(t) + \cdots$, where $\chi_I$ is the characteristic function of the interval $I$.

Hence $t \in [n\tau, (n+1)\tau)$ implies

$$\int_{t-\tau}^{t} \big((I-S)^{-1}b\big)(s)ds = b\tau + ab\tau + \cdots + a^n b\tau.$$

Then, by the conditions of the corollary,

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \big((I-S)^{-1}b\big)(s)ds = b\tau + ab\tau + \cdots + a^n b\tau + \cdots = \frac{b\tau}{1-a} > \frac{1}{e}. \quad \square$$

## 6.5 Positivity of Solutions

In this section, explicit conditions on initial data are obtained that imply positivity of the solution of initial value problem (6.1.5), (6.1.6).

**Theorem 6.12** *Suppose $b(t) \geq 0$ and for $t \geq t_1$ there exists a nonnegative solution of the inequality*

$$u(t) \geq a(t)u\big(g(t)\big)\exp\left\{\int_{\max\{t_1,g(t)\}}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{\max\{t_1,h(t)\}}^{t} u(s)ds\right\}, \tag{6.5.1}$$

*where $u(t) = 0$, $t \geq t_1$. If*

$$x(t_1) = x_0 > 0,\ \varphi(t) \leq x(t_1),\ \psi(t) \geq 0,\ f(t) \geq 0, \tag{6.5.2}$$

*then for the solution of* (6.1.5), (6.1.6) *we have $x(t) > 0$, $t \geq t_1$.*

*Proof* Let $u$ be a nonnegative solution of (6.5.1). Denote

$$v(t) = \begin{cases} x_0 \exp\{-\int_{t_1}^t u(s)ds\}, & t \geq t_1, \\ x_0, & t \leq t_1. \end{cases}$$

We have for $t \geq t_1$

$$\begin{aligned}
&\dot{v}(t) - a(t)\dot{v}\big(g(t)\big) + b(t)v\big(h(t)\big) \\
&= -x_0 u(t)\exp\left\{-\int_{t_1}^{t} u(s)ds\right\} + x_0 a(t)u\big(g(t)\big)\exp\left\{-\int_{t_1}^{g(t)} u(s)ds\right\} \\
&\quad + x_0 b(t)\exp\left\{-\int_{t_1}^{h(t)} u(s)ds\right\} \\
&= -x_0\exp\left\{-\int_{t_1}^{t} u(s)ds\right\}\left[u(t) - a(t)u\big(g(t)\big)\exp\left\{\int_{\max\{t_1,g(t)\}}^{t} u(s)ds\right\}\right. \\
&\quad \left. - b(t)\exp\left\{\int_{\max\{t_1,h(t)\}}^{t} u(s)ds\right\}\right] \leq 0.
\end{aligned}$$

Hence $v(t)$ is a solution of the problem

$$\dot{v}(t) - a(t)\dot{v}\big(g(t)\big) + b(t)v\big(h(t)\big) = r(t),$$

$$v(t) = x_0,\ t \leq t_1,\ \dot{v}(t) = 0,\ t < t_1,$$

where $r(t) \leq 0$.

The assumptions of this theorem and Theorem 6.4 imply that the solution $x$ of (6.1.5), (6.1.6) satisfies $x(t) \geq v(t) > 0$, which completes the proof. □

## 6.6 Slowly Oscillating Solutions

As was mentioned in the previous sections, if an ordinary linear differential equation of the second order has an oscillatory solution, then all its solutions are oscillatory. For delay differential equations this is not true, but under certain conditions the existence of a *slowly oscillating solution* for either the associated or the original equation implies oscillation of all solutions.

In [142, 145], several new explicit sufficient conditions of oscillation for neutral equations are obtained by an explicit construction of such slowly oscillating solutions for the associated equation.

We present here a similar oscillation criterion for a neutral equation. Unlike the results of [142, 145] and similar to the previous chapters, the existence of a slowly oscillatory solution is assumed for (6.1.1) and not for the associated equation. Moreover, the delays are not necessarily monotone. Let us start with the definition of slowly oscillating solutions for a neutral equation.

**Definition 6.4** A solution $x$ of (6.1.1) is said to be *slowly oscillating* if for every $t^0 \ge t_0$ there exist $t_1 > t^0$ and $t_2 > t^0$ such that

$$\begin{aligned} &g(t) \ge t_1, \ h(t) \ge t_1 \text{ for } t \ge t_2; \\ &x(t_2) = 0; \ x(t) > 0, \ \dot{x}(t) \le 0, \ t \in [t_1, t_2). \end{aligned} \tag{6.6.1}$$

*Remark 6.5* In the case of constant delays $g(t) = t - \tau$, $h(t) = t - \sigma$, the solution $x$ is slowly oscillating if there exists a sequence of intervals $[t_{1n}, t_{2n})$ such that $x(t_{1n}) = x(t_{2n}) = 0$, the solution is positive and its derivative is nonpositive on these intervals, while the lengths of the intervals are greater than $\max\{\tau, \sigma\}$.

**Theorem 6.13** *Suppose $b(t) \ge 0$. If there exists a slowly oscillating solution of* (6.1.1), *then all solutions of this equation are oscillatory.*

*Proof* Denote by $x$ a slowly oscillating solution of (6.1.1). Suppose that this equation also has a nonoscillatory solution. Then, by Theorem 6.1, for a certain $t^0 \ge t_0$ we have $X(t, s) > 0$ if $t \ge s > t^0$.

There exist numbers $t_1 > t^0$ and $t_2 > t^0$ such that condition (6.6.1) holds.

Due to (6.1.8), for $t \ge t_2$, the solution $x$ can be presented as

$$x(t) = \int_{t_2}^{t} X(t, s)\big[a(t)\dot{x}\big(g(s)\big) - b(t)x\big(h(s)\big)\big]ds, \tag{6.6.2}$$

where $x(g(s)) = 0$ if $g(s) > t_2$ and $x(h(s)) = 0$ if $h(s) > t_2$. The inequalities $g(t) \ge t_1$, $h(t) \ge t_1$ for $t \ge t_2$ yield that the expression under the integral in (6.6.2) can differ from zero only if $t_1 < g(t), h(s) < t_2$. Therefore, by (6.6.1) in the equality (6.6.2), we have $x(h(s)) > 0$, $\dot{x}(g(t)) \le 0$. Consequently, (6.6.2) implies $x(t) \le 0$ for each $t \ge t_2$. This contradicts the assumption that $x$ is an oscillatory solution, which completes the proof. □

**Corollary 6.14** *Suppose $b(t) \ge 0$ and there exists a nonoscillatory solution of* (6.1.1). *Then* (6.1.1) *has no slowly oscillating solutions.*

## 6.7 Neutral Equations with Positive and Negative Coefficients

In this section, we consider the equation

$$\dot{x}(t) - a(t)\dot{x}\big(g(t)\big) + b(t)x\big(h(t)\big) - c(t)x\big(r(t)\big) = 0, \ t \ge t_0, \tag{6.7.1}$$

where conditions (a1)–(a3) hold and $b(t) \ge c(t) \ge 0$, $h(t) \le r(t) \le t$.

**Theorem 6.14** *Suppose that the integral inequality*

$$u(t) \ge a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} - c(t)\exp\left\{\int_{r(t)}^{t} u(s)ds\right\}, \tag{6.7.2}$$

*where $u(t) = 0$, $t < t_1$, has a nonnegative locally integrable solution for $t \ge t_1$. Then the fundamental function of* (6.7.1) *is positive for $t \ge s \ge t_1$.*

*Proof* Suppose that $u_0(t)$, $t \ge t_1$ is a nonnegative solution of (6.7.2). Denote

$$u_{n+1}(t) = a(t)u_n\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\} - c(t)\exp\left\{\int_{r(t)}^{t} u_n(s)ds\right\}. \tag{6.7.3}$$

We have

$$u_{n+1}(t) = a(t)u_n\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u_n(s)ds\right\} + \big(b(t) - c(t)\big)\exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\} + c(t)\left[\exp\left\{\int_{h(t)}^{t} u_n(s)ds\right\} - \exp\left\{\int_{r(t)}^{t} u_n(s)ds\right\}\right].$$

By (6.7.2) and the theorem assumptions, we have $u_0(t) \ge u_1(t) \ge 0$, and hence by induction

$$0 \le u_{n+1}(t) \le u_n(t) \le \cdots \le u_0(t).$$

The monotone sequence $u_n(t)$ has a pointwise limit $u(t) = \lim_{n\to\infty} u_n(t)$. The Lebesgue convergence theorem (Theorem A.1) and (6.7.3) imply

$$u(t) = a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} - c(t)\exp\left\{\int_{r(t)}^{t} u(s)ds\right\},$$

and thus the function

$$x(t) = \exp\left\{-\int_{t_0}^{t} u(s)ds\right\},\ t \ge t_0,\ x(t) = \dot{x}(t) = 0,\ t < t_1,$$

is a nonoscillatory solution of (6.7.1), which completes the proof. □

For comparison, consider the equation

$$\dot{x}(t) - a_1(t)\dot{x}\big(g(t)\big) + b_1(t)x\big(h_1(t)\big) - c_1(t)x\big(r_1(t)\big) = 0,\ t \ge t_0. \tag{6.7.4}$$

We assume that for (6.7.4) conditions (a1)–(a3) hold and $b_1(t) \ge c_1(t) \ge 0$, $h_1(t) \le r_1(t) \le t$.

**Corollary 6.15** *Suppose*

$$a_1(t) \le a(t),\ c(t) \le c_1(t) \le b_1(t) \le b(t),$$

$$h(t) \le h_1(t) \le r_1(t) \le r(t).$$

*If inequality* (6.7.2) *has a nonnegative locally integrable solution for $t \ge t_1$, then the fundamental function of* (6.7.4) *is positive for $t \ge s \ge t_1$.*

*Proof* If $u(t)$, $t \ge t_0$, is a nonnegative solution of inequality (6.7.2), then this function is also a solution of the same inequality, where $a$, $b$, $c$, $h$, $r$ are replaced by $a_1$, $b_1$, $c_1$, $h_1$, $r_1$. □

**Corollary 6.16** *Let* $0 < a < 1$, $0 \le c \le c(t) \le b(t) \le b$, $\sigma > 0$, $\tau > 0$, $\delta > 0$, $\delta \le \tau$,

$$a(t) \le a,\ c \le c(t) \le b(t) \le b,\ g(t) = t - \sigma,\ t - \tau \le h(t) \le r(t) \le t - \delta,$$

*and suppose that the inequality*

$$\lambda\big(1 - ae^{\lambda\sigma}\big) \ge be^{\lambda\tau} - ce^{\lambda\delta} \tag{6.7.5}$$

*has a positive solution. Then* (6.7.1) *has a nonoscillatory solution.*

*Proof* The equation

$$\dot{x}(t) - a\dot{x}(t-\sigma) + bx(t-\tau) - cx(t-\delta) = 0 \tag{6.7.6}$$

has a nonoscillatory solution [192] if and only if the inequality (6.7.5) has a positive solution, which implies the result of the corollary. □

The following theorem gives explicit nonoscillation conditions based on Theorem 6.14.

**Theorem 6.15** *Suppose for $t \ge t_1$*

$$\int_{\min\{g(t),h(t)\}}^{t} \big[eb(s) - c(s)\big]ds + \ln\left[\frac{a(t)(eb(g(t)) - c(g(t)))}{b(t)} + 1\right] \le 1. \tag{6.7.7}$$

*Then* (6.7.1) *has a positive fundamental function for $t \ge s \ge t_1$.*

*Proof* Consider

$$\dot{x}(t) - a(t)\dot{x}\big(g(t)\big) + b(t)x\big(h(t)\big) - c(t)x(t) = 0 \tag{6.7.8}$$

as a comparison equation. Inequality (6.7.2) for this equation has the form

$$u(t) \ge a(t)u\big(g(t)\big)\exp\left\{\int_{g(t)}^{t} u(s)ds\right\} + b(t)\exp\left\{\int_{h(t)}^{t} u(s)ds\right\} - c(t). \tag{6.7.9}$$

If inequality (6.7.9) has a nonnegative solution, then (6.7.8) has a positive fundamental function, and hence (6.7.1) also has a positive fundamental function.

We will show that the function $u_0(t) = eb(t) - c(t)$ is a solution of inequality (6.7.9). After substituting $u(t) = u_0(t)$ in (6.7.9), we have

$$\begin{aligned} eb(t) \ge a(t)\big[eb\big(g(t)\big) - c\big(g(t)\big)\big]\exp\left\{\int_{g(t)}^{t}\big(eb(s) - c(s)\big)ds\right\} \\ + b(t)\exp\left\{\int_{h(t)}^{t}\big(eb(s) - c(s)\big)ds\right\}. \end{aligned} \tag{6.7.10}$$

Inequality (6.7.10) holds if the condition

$$eb(t) \ge \big[a(t)\big(eb\big(g(t)\big) - c\big(g(t)\big)\big) + b(t)\big]\exp\left\{\int_{\min\{g(t),h(t)\}}^{t}\big(eb(s) - c(s)\big)ds\right\}$$

is satisfied, which is equivalent to (6.7.7). The proof of the theorem is complete. □

**Corollary 6.17** *If* $b(t) \ge 0$ *and*

$$\int_{\min\{g(t),h(t)\}}^{t} b(s)ds + \frac{1}{e}\ln\left[\frac{ea(t)b(g(t))}{b(t)} + 1\right] \le \frac{1}{e},\ t \ge t_1,$$

*then* (6.1.1) *has a positive fundamental function for* $t \ge s \ge t_1$.

**Corollary 6.18** *If* $0 < a < 1$, $0 < c < b$, $\sigma \ge 0$, $0 \le \delta \le \tau$ *and*

$$(eb - c)\max\{\tau, \sigma\} + \ln\left(\frac{a(eb - c)}{b} + 1\right) \le 1,$$

*then* (6.7.6) *has a positive fundamental function.*

**Corollary 6.19** *If* $0 < a < 1$, $b > 0$, $\tau \ge 0$, $\sigma \ge 0$ *and*

$$b\max\{\tau, \sigma\} + \frac{1}{e}\ln(ea + 1) \le \frac{1}{e},$$

*then* (6.2.6) *has a positive fundamental function.*

## 6.8 Discussion and Open Problems

This chapter deals with nonoscillation and oscillation of scalar neutral differential equations.

A linear neutral type equation can be written in the two forms

$$\big(x(t)-a(t)x\big(g(t)\big)\big)'+\sum_{k=1}^{m}b_k(t)x\big(h_k(t)\big)=0, \tag{6.8.1}$$

$$\dot{x}(t)-a(t)\dot{x}\big(g(t)\big)+\sum_{k=1}^{m}b_k(t)x\big(h_k(t)\big)=0, \tag{6.8.2}$$

where $g(t)\le t$, $h_k(t)\le t$. Equations (6.8.1) and (6.8.2) are similar; however, there are differences between them. For example, unlike (6.8.2), solution $x$ of (6.8.1) is an arbitrary continuous function such that $x(t)-a(t)x(g(t))$ is differentiable. Thus (6.8.1) in general cannot be rewritten in the form (6.8.2) and vice versa. Concerning the connection of (6.8.1) with (6.8.2), we mention here the paper [145], where the oscillation of (6.8.1) was studied by applying an adjoint equation that has the form (6.8.2). For the autonomous case in the neutral part when $a(t)\equiv a$ and $g(t)\equiv t-\sigma$, (6.8.1) and (6.8.2) are the same once we consider only differentiable solutions $x(t)$. In this case, the results of this chapter coincide with the known ones.

It is to be emphasized that (6.8.1) is much better studied than (6.8.2). Extensive literature on (6.8.1) is concerned with existence and uniqueness theorems and especially stability and oscillation theories; see monographs [154, 167, 192, 201] and references therein.

Equation (6.8.2) is a natural representative of neutral type equations. There exist applied problems that can be written in the form (6.8.2); see [226]. The monograph [29] contains solvability and uniqueness results, the solution representation for (6.8.2) and elements of stability theory. The recent monograph [239] involves stability results for (6.8.2). We also mention here papers [22–24], where a new method based on the Bohl-Perron theorem was applied to investigate the stability of (6.8.2).

Though there exists a developed stability theory for (6.8.2), surprisingly there are only a few publications on its oscillation. We mention here the paper [182], where comparison results for (6.8.2) were obtained and two papers [80, 178] where positivity of the fundamental function of (6.8.2) was studied.

Theorem 6.14 and Theorem 6.15 are new. All other results of this chapter were published in [51, 52, 56]. More results on nonoscillation of neutral type equations can be found in [88, 205, 253, 270, 273, 274, 277, 304, 316, 321, 356, 358], see also [196, 212, 245, 305, 344, 345, 355].

Finally, let us state some open problems in the theory of neutral equations, possible generalizations, topics for research and discussion.

1. Consider nonoscillation and oscillation properties of (6.1.1) in the case $-1<a(t)<0$.
2. Consider nonoscillation and oscillation properties of (6.1.1) in the case $|a(t)|<1$, where $a(t)$ is an oscillatory function; for example,

$$a(t)=\alpha\sin t,\ \ |\alpha|<1.$$

3. Consider oscillation properties of (6.1.1) in the case where $b(t)$ is an oscillatory function.

4. Consider nonoscillation and oscillation properties of the following neutral equations:

   - integrodifferential neutral equation

   $$\dot{x}(t) - \int_{-\infty}^{t} K(t,s)\dot{x}(s)ds + \int_{-\infty}^{t} L(t,s)x(s)ds = 0,$$

   - neutral equation with distributed delays

   $$\dot{x}(t) - \int_{-\infty}^{t} \dot{x}(s)d_s R(t,s) + \int_{-\infty}^{t} x(s)d_s P(t,s) = 0,$$

   and mixed equations containing both delay and integral terms where for the kernels of the integral operators the following conditions hold:

   $$\int_{-\infty}^{t} \left|K(t,s)\right| ds \le \lambda < 1, \quad \int_{-\infty}^{t} \left|d_s R(t,s)\right| \le \lambda < 1.$$

5. Suppose $b(t) \ge b_0 > 0$ and (6.1.1) has a nonoscillatory solution. Prove or disprove:
   Every nonoscillatory solution of (6.1.1) tends to zero.
   Is it is true for any solution of (6.1.1)?

# Chapter 7
# Second-Order Delay Differential Equations

## 7.1 Introduction

In this chapter, we consider oscillation of second-order delay differential equations. The methods previously used for the study of first-order equations are applied here to equations of the second order.

The chapter is organized as follows. Section 7.2 contains relevant definitions and notation. In Sect. 7.3, the equivalence of four properties is established: nonoscillation of solutions of a delay differential equation and the corresponding differential inequality, positivity of the fundamental function and the existence of a nonnegative solution of a generalized Riccati inequality. In Sect. 7.4, comparison results are presented. Section 7.5 includes some explicit nonoscillation and oscillation tests. Section 7.7 contains conditions under which the solution of the initial value problem is positive. Section 7.8 discusses the results, and some open problems are outlined.

## 7.2 Preliminaries

We consider the scalar delay differential equation of the second order

$$\ddot{x}(t)+\sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big)=0 \tag{7.2.1}$$

under the following conditions:

(a1) $a_k$, $k=1,\cdots,m$, are Lebesgue measurable and locally essentially bounded functions on $[0,\infty)$.
(a2) $g_k:[0,\infty)\to\mathbb{R}$ are Lebesgue measurable functions, $g_k(t)\le t$, $t\ge 0$, $\lim_{t\to\infty} g_k(t)=\infty$, $k=1,\cdots,m$.

Together with (7.2.1), consider for each $t_0\ge 0$ the initial value problem

$$\ddot{x}(t)+\sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big)=f(t),\ t\ge t_0, \tag{7.2.2}$$

R.P. Agarwal et al., *Nonoscillation Theory of Functional Differential Equations with Applications*, DOI 10.1007/978-1-4614-3455-9_7,

$$x(t) = \varphi(t),\ t < t_0,\ x(t_0) = x_0,\ \dot{x}(t_0) = x_0'. \tag{7.2.3}$$

We also assume that the following hypothesis holds:

(a3) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

**Definition 7.1** A function $x : \mathbb{R} \to \mathbb{R}$ with derivative $\dot{x}$ locally absolutely continuous on $[t_0, \infty)$ is called *a solution* of problem (7.2.2), (7.2.3) if it satisfies (7.2.2) for almost every $t \in [t_0, \infty)$ and equalities (7.2.3) for $t \le t_0$.

**Definition 7.2** For each $s \ge 0$, the solution $X(t, s)$ of the problem

$$\begin{aligned} &\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(g_k(t)\big) = 0, t \ge s, \\ &x(t) = 0,\ t \le s,\ \dot{x}(s) = 1, \end{aligned} \tag{7.2.4}$$

is called *the fundamental function* of (7.2.1).

We assume $X(t, s) = 0$, $0 \le t < s$. Let functions $x_1$ and $x_2$ be the solutions of the problems

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(g_k(t)\big) = 0,\ t \ge t_0,\ x(t) = 0,\ t < t_0,$$

with the initial values $x_1(t_0) = 1, \dot{x}_1(t_0) = 0$ and $x_2(t_0) = 0, \dot{x}_2(t_0) = 1$, respectively. By definition, $x_2(t) = X(t, t_0)$.

Theorem B.5 implies the following result.

**Lemma 7.1** *Let* (a1)–(a3) *hold. Then there exists one and only one solution of problem* (7.2.2), (7.2.3) *that can be presented in the form*

$$\begin{aligned} x(t) = x_1(t)x_0 + X(t, t_0)x_0' + \int_{t_0}^{t} X(t, s) f(s) ds \\ - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s) a_k(s) \varphi\big(g_k(s)\big) ds, \end{aligned} \tag{7.2.5}$$

*where* $\varphi(g_k(s)) = 0$ *if* $g_k(s) > t_0$.

## 7.3 Nonoscillation Criteria

**Definition 7.3** We will say that (7.2.1) *has a positive solution* for $t > t_0$ if there exist an initial function $\varphi$ and numbers $x_0$ and $x_0'$ such that the solution of initial value problem (7.2.2), (7.2.3) ($f \equiv 0$) is positive for $t > t_0$.

Together with (7.2.1), consider the second-order delay differential inequality

$$\ddot{y}(t) + \sum_{k=1}^{m} a_k(t) y\big(g_k(t)\big) \le 0. \tag{7.3.1}$$

The following theorem establishes nonoscillation criteria.

**Theorem 7.1** *Suppose $a_k(t) \ge 0$ for $t \ge 0$, $k = 1, \cdots, m$. Then the following statements are equivalent*:

1) *Inequality* (7.3.1) *has an eventually positive solution.*
2) *There exists $t_1 \ge 0$ such that the inequality*

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t) \exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\} \le 0, \tag{7.3.2}$$

*where $u(t) = 0$ for $t < t_1$, has a nonnegative locally absolutely continuous solution for $t \ge t_1$.*

3) *There exists $t_2 \ge 0$ such that $X(t,s) > 0$, $t > s \ge t_2$.*
4) *Equation* (7.2.1) *has an eventually positive solution.*

*Proof* Let us justify the implications 1) ⇒ 2) ⇒ 3) ⇒ 4) ⇒ 1).

1) ⇒ 2) Let $y(t)$ be a positive solution of inequality (7.3.1) for $t > t_0$. Then there exists a point $t_1$ such that $g_k(t) \ge t_0$ if $t \ge t_1$. We can assume without loss of generality that $y(t_1) = 1$. Since $y(t) > 0$ and $\ddot{y}(t) \le 0$, $t \ge t_1$, we have $\dot{y}(t) \ge 0$ for $t \ge t_1$.

Let us assume the contrary, that $d = \dot{x}(t_2) < 0$ for some $t_2 > t_1$. Then $\ddot{x}(t) \le 0$ for $t \ge s$ implies $\dot{x}(t) \le \dot{x}(t_2) < 0$ for $t > t_2$, and thus $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the assumption $x(t) \ge 0$, $t \ge t_1$.

Denote $u(t) = \frac{\dot{y}(t)}{y(t)}$ if $t \ge t_1$ and $u(t) = 0$ if $t < t_1$. Then $u$ is a nonnegative locally absolutely continuous function. Equalities $\dot{y}(t) - u(t)y(t) = 0$, $y(t_1) = 1$ imply

$$\begin{aligned} &y(t) = \exp\left\{\int_{t_1}^{t} u(s)ds\right\}, \quad \dot{y}(t) = u(t)\exp\left\{\int_{t_1}^{t} u(s)ds\right\}, \\ &\ddot{y}(t) = \dot{u}(t)\exp\left\{\int_{t_1}^{t} u(s)ds\right\} + u^2(t)\exp\left\{\int_{t_1}^{t} u(s)ds\right\}, \quad t \ge t_1. \end{aligned} \tag{7.3.3}$$

We substitute (7.3.3) into (7.3.1) and obtain after carrying the exponent out of the brackets the inequality

$$\exp\left\{\int_{t_1}^{t} u(s)ds\right\}\left[\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t)\exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\}\right] \le 0. \tag{7.3.4}$$

Since $a_k(t) \ge 0$, inequality (7.3.4) implies (7.3.2).

2) ⇒ 3) Consider the initial value problem

$$\begin{aligned} &\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(g_k(t)\big) = f(t), \quad t \ge t_1, \\ &x(t) = 0, \quad t < t_1, \quad x(t_1) = \dot{x}(t_1) = 0. \end{aligned} \tag{7.3.5}$$

Denote

$$z(t) = \dot{x}(t) - u(t)x(t), \tag{7.3.6}$$

where $x$ is the solution of (7.3.5) and $u$ is a nonnegative solution of (7.3.2). From (7.3.6), we obtain

$$\begin{aligned}
x(t) &= \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds, \\
\dot{x}(t) &= z(t) + u(t)\int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds, \\
\ddot{x}(t) &= \dot{z}(t) + \dot{u}(t)\int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds \\
&\quad + u(t)\left[z(t) + u(t)\int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds\right] \\
&= \dot{z}(t) + u(t)z(t) + \left(\dot{u}(t) + u^2(t)\right)\int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds
\end{aligned} \tag{7.3.7}$$

for $t \geq t_1$.

Substituting $x$ and $\ddot{x}$ into (7.3.5), we have

$$\begin{aligned}
&\dot{z}(t) + u(t)z(t) + \left(\dot{u}(t) + u^2(t)\right)\int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds \\
&\quad + \sum_{k=1}^{m} a_k(t)\int_{t_1}^{g_k(t)} \exp\left\{\int_s^{g_k(t)} u(\tau)d\tau\right\} z(s)ds = f(t).
\end{aligned} \tag{7.3.8}$$

Equalities (7.3.5) and (7.3.6) imply $z(t_1) = 0$. Hence we can rewrite (7.3.8) in the form

$$\begin{aligned}
&\dot{z}(t) + u(t)z(t) \\
&= -\left[\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t)\exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\}\right] \\
&\quad \times \int_{t_1}^{t} \exp\left\{\int_s^t u(\tau)d\tau\right\} z(s)ds + \sum_{k=1}^{m} a_k(t)\int_{g_k(t)}^{t} \exp\left\{\int_s^{g_k(t)} u(\tau)d\tau\right\} z(s)ds \\
&\quad + f(t),\ z(t_1) = 0.
\end{aligned} \tag{7.3.9}$$

Then (7.3.9) is equivalent to the equation

$$z = Hz + p, \tag{7.3.10}$$

where

$$\begin{aligned}(Hz)(t) = \int_{t_1}^{t} \exp\left\{-\int_s^t u(\tau)d\tau\right\}\Bigg[-\Bigg(\dot{u}(s)+u^2(s)\\
+\sum_{k=1}^m a_k(s)\exp\left\{-\int_{g_k(s)}^s u(\tau)d\tau\right\}\Bigg)\int_{t_1}^s \exp\left\{\int_\tau^s u(\xi)d\xi\right\}z(\tau)d\tau\\
+\sum_{k=1}^m a_k(s)\int_{g_k(s)}^s \exp\left\{\int_\tau^{g_k(s)} u(\xi)d\xi\right\}z(\tau)d\tau\Bigg]ds,\end{aligned} \tag{7.3.11}$$

$$p(t) = \int_{t_1}^t \exp\left\{-\int_s^t u(\tau)d\tau\right\}f(s)ds.$$

Inequality (7.3.2) yields that if $z(t) \geq 0$ for $t \geq t_1$, then $(Hz)(t) \geq 0$ for $t \geq t_1$ (i.e., operator $H$ is positive).

Denote

$$c(t) = \dot{u}(t) + u^2(t) + \sum_{k=1}^m a_k(t)\exp\left\{-\int_{g_k(t)}^t u(s)ds\right\}.$$

Since $u(t)$ is absolutely continuous on each finite interval, the function $c \in L[t_1, b]$ for every $b > t_1$.

We have for $t \in [t_1, b]$

$$\begin{aligned}|(Hz)(t)| &\leq \exp\left\{\int_{t_1}^b u(\tau)d\tau\right\}\int_{t_1}^t\left(|c(s)|+\sum_{k=1}^m|a_k(s)|\right)\int_{t_1}^s|z(\tau)|d\tau ds\\
&= \exp\left\{\int_{t_1}^b u(\tau)d\tau\right\}\int_{t_1}^t\left(\int_\tau^t\left(|c(s)|+\sum_{k=1}^m|a_k(s)|\right)ds\right)|z(\tau)|d\tau.\end{aligned}$$

The kernel of the Volterra integral operator $H$ is bounded in each square $[t_1, b] \times [t_1, b]$. By Theorem A.4, operator $H$ is a weakly compact operator in the space $L_\infty[t_1, b]$; Theorem A.7 implies that the spectral radius $r(H) = 0 < 1$.

Thus, if in (7.3.10) we have $p(t) \geq 0$ for $t \geq t_1$, then

$$z(t) = p(t) + (Hp)(t) + \left(H^2 p\right)(t) + \cdots \geq 0 \text{ for } t \geq t_1.$$

If $f(t) \geq 0$ for $t \geq t_1$, then by (7.3.12) we have $p(t) \geq 0$ for $t \geq t_1$. Hence, for (7.3.8) the following statement is valid: if $f(t) \geq 0$ for $t \geq t_1$, then $z(t) \geq 0$ for $t \geq t_1$. Therefore (7.3.7) implies that the solution of (7.3.5) is nonnegative for any nonnegative right-hand side. The solution of this initial value problem can be presented in the form (7.2.5), which is

$$x(t) = \int_{t_1}^t X(t,s)f(s)ds. \tag{7.3.12}$$

As was shown above, $f(t) \geq 0$, $t \geq t_1$, implies $x(t) \geq 0$, $t \geq t_1$. Consequently, the kernel of the integral operator (7.3.12) is nonnegative. Therefore $X(t,s) \geq 0$ for $t \geq s \geq t_1$. The function $x(t) = X(t,s)$ is a nonnegative solution of (7.2.4) for $t \geq s$, and hence $x(t) \geq 0$ and $\ddot{x}(t) \leq 0$ for $t \geq s$, which implies $\dot{x}(t) \geq 0$, $t \geq s$.

Since $\dot{x}(s) = 1$ implies $\dot{x}(t) > 0$ on some interval $[s, s + \sigma]$, the strict inequality $x(t) = X(t, s) > 0$ holds for $t > s \geq t_1$.

3) $\Rightarrow$ 4) The function $x(t) = X(t, t_2)$ is a positive solution of (7.2.1).

The implication 4) $\Rightarrow$ 1) is evident. □

**Corollary 7.1** *Equation* (7.2.1) *is nonoscillatory if and only if inequality* (7.3.1) *has a positive solution.*

*Remark 7.1* If there exists a nonnegative solution of inequality (7.3.2) for $t \geq t_0$, then statements 1), 3) and 4) of the theorem are also valid for $t \geq t_0$.

## 7.4 Comparison Theorems

Theorem 7.1 can be employed for comparison of oscillation properties. To this end, together with (7.2.1), consider the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = 0, \ t \geq 0. \tag{7.4.1}$$

We assume that (a1) and (a2) hold for (7.4.1) and denote by $Y(t, s)$ the fundamental function of this equation.

**Theorem 7.2** *Suppose $a_k(t) \geq 0$, $a_k(t) \geq b_k(t)$ for $t \geq t_0$ and inequality* (7.3.2) *has a nonnegative solution for $t \geq t_0$. Then* (7.4.1) *has a positive solution for $t \geq t_0$, and $Y(t, s) > 0$, $t > s \geq t_0$.*

*Proof* Consider the problem

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = f(t), \ t \geq t_0, \ x(t) = 0, \ t < t_0, \ x(t_0) = \dot{x}(t_0) = 0. \tag{7.4.2}$$

We will demonstrate that if $f(t) \geq 0$ for $t \geq t_0$, then the solution of (7.4.2) is positive.

Problem (7.4.2) can be rewritten in the form

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big) + \sum_{k=1}^{m}\big[b_k(t) - a_k(t)\big]x\big(g_k(t)\big) = f(t), \ t \geq t_0, \tag{7.4.3}$$

$$x(t) = 0, \ t < t_0, \ x(t_0) = \dot{x}(t_0) = 0. \tag{7.4.4}$$

Let $x$ be the solution of the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big) = z(t)$$

satisfying initial conditions (7.4.4). Then

$$x(t) = \int_{t_0}^{t} X(t,s)z(s)ds, \tag{7.4.5}$$

where $X(t,s)$ is the fundamental function of (7.2.1), and

$$x\big(g_k(t)\big) = \int_{t_0}^{g_k(t)} X\big(g_k(t),s\big)z(s)ds = \int_{t_0}^{t} X\big(g_k(t),s\big)z(s)ds.$$

Substituting the equality above into (7.4.2), we obtain that (7.4.2) is equivalent to the equation

$$z - Tz = f \tag{7.4.6}$$

with

$$(Tz)(t) = \int_{t_0}^{t} \sum_{k=1}^{m} X\big(g_k(t),s\big)\big[a_k(t) - b_k(t)\big]z(s)ds,\ t \ge t_0.$$

By Theorem B.7, the fundamental function $X(t,s)$ is bounded on the square $[t_0,b]\times[t_0,b]$. Hence, by Theorem A.4, the Volterra integral operator $T$ is a weakly compact operator acting on the space $L_\infty[t_0,b]$ for every $b > t_0$. Theorem A.7 implies that the spectral radius of this operator $r(T) = 0 < 1$. By Theorem 7.1, $X(t,s) > 0$ for $t > s \ge t_0$, so operator $T$ is positive. Therefore, for the solution of (7.4.6), we have

$$z(t) = f(t) + (Tf)(t) + \big(T^2 f\big)(t) + \cdots \ge 0 \text{ if } f(t) \ge 0 \text{ for } t \ge t_0.$$

Then, similar to the proof of Theorem 7.1, we conclude that $Y(t,s) \ge 0$, $t > s \ge t_0$, and we only need to prove that the strict inequality $Y(t,s) > 0$, $t > s \ge t_0$ holds.

After denoting $y(t) = Y(t,s)$, $t \ge s$, we notice that $y(t)$ is a solution of the problem

$$\ddot{y}(t) + \sum_{k=1}^{m} b_k(t)y\big(g_k(t)\big) = 0,\ t \ge s, \tag{7.4.7}$$

$$y(t) = 0,\ t \le s,\ \dot{y}(s) = 1. \tag{7.4.8}$$

After rewriting (7.4.7) in the form

$$\ddot{y}(t) + \sum_{k=1}^{m} a_k(t)y\big(g_k(t)\big) = \sum_{k=1}^{m}\big[a_k(t) - b_k(t)\big]y\big(g_k(t)\big),$$

we see that for the solution of problem (7.4.7), (7.4.8) solution representation (7.2.5) implies

$$y(t) = Y(t,s) = X(t,s) + \int_{s}^{t} X(t,\tau)\sum_{k=1}^{m}\big[a_k(\tau) - b_k(\tau)\big]Y\big(g_k(\tau),s\big)d\tau.$$

Thus $Y(t,s) \ge X(t,s) > 0$, which completes the proof. □

**Corollary 7.2** *Suppose* $a_k(t) \geq 0$, $a_k(t) \geq b_k(t)$, $k = 1, \cdots, m$ *for* $t \geq t_0$ *and* (7.2.1) *has a positive solution for* $t > t_0$. *Then there exists* $t_1 \geq t_0$ *such that* (7.4.1) *has a positive solution for* $t > t_1$.

Denote $a^+ = \max\{a, 0\}$.

**Corollary 7.3**

1) *If the inequality*

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k^+(t) x\big(g_k(t)\big) \leq 0 \tag{7.4.9}$$

*has a positive solution for* $t > t_0$, *then there exists* $t_1 \geq t_0$ *such that* (7.2.1) *has a positive solution for* $t > t_1$.

2) *If the inequality*

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k^+(t) \exp\left\{ - \int_{g_k(t)}^{t} u(s)ds \right\} \leq 0, \ t \geq t_0, \tag{7.4.10}$$

*has a nonnegative absolutely continuous solution, where the sum contains only those terms for which* $g_k(t) \geq t_0$, *then* (7.2.1) *has a positive solution for* $t > t_0$ *and* $X(t,s) > 0$, $t > s \geq t_0$.

*Proof* Consider the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k^+(t) x\big(g_k(t)\big) = 0. \tag{7.4.11}$$

Theorem 7.1 implies that for (7.4.11) all assertions of this theorem hold. Since $a_k(t) \leq a_k^+(t)$, the reference to Theorem 7.2 completes the proof. □

Corollary 7.3 can be employed to obtain a comparison result that improves the statement of Theorem 7.2.

Consider the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t) x\big(h_k(t)\big) = 0, \ t \geq 0, \tag{7.4.12}$$

and suppose that (a1) and (a2) hold for (7.4.12) and denote by $Y(t,s)$ the fundamental function of this equation.

**Theorem 7.3** *Suppose* $a_k(t) \geq 0$ *for* $t \geq t_0$ *and there exists* $t_0 \geq 0$ *such that for* (7.2.1) *any of assertions* 1)–4) *of Theorem* 7.1 *hold. If*

$$b_k(t) \leq a_k(t), \ h_k(t) \leq g_k(t) \text{ for } t \geq t_0, \tag{7.4.13}$$

*then there exists* $t_1 \geq t_0$ *such that* (7.4.12) *has a positive solution for* $t > t_1$ *and* $Y(t,s) > 0$, $t > s \geq t_1$.

*Proof* Theorem 7.1 implies that for some $t_1 \geq t_0$ there exists a nonnegative solution $u(t)$ of inequality (7.3.2) for $t \geq t_1$. Conditions (7.4.13) yield that this function is also a solution of the inequality

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} b_k^+(t) \exp\left\{-\int_{h_k(t)}^{t} u(s)ds\right\} \leq 0, \ t \geq t_0,$$

where the sum contains only those terms for which $h_k(t) \geq t_0$. By Corollary 7.3, (7.4.12) has a positive solution for $t > t_1$ and the fundamental function of this equation is positive, which completes the proof. □

**Corollary 7.4** *Suppose $a_k(t) \geq 0$. If the ordinary differential equation*

$$\ddot{x}(t) + \left(\sum_{k=1}^{m} a_k(t)\right) x(t) = 0 \tag{7.4.14}$$

*has a positive solution for $t \geq t_0$, then (7.2.1) has a positive solution for $t > t_0$ and its fundamental function satisfies $X(t,s) > 0$ for $t > s \geq t_0$.*

*If all solutions of (7.2.1) are oscillatory, then all solutions of (7.4.14) are also oscillatory.*

Now let us compare solutions of (7.2.2), (7.2.3) and of the initial value problem

$$\ddot{y}(t) + \sum_{k=1}^{m} b_k(t) y\big(g_k(t)\big) = r(t), \ t \geq t_0, \tag{7.4.15}$$

$$y(t) = \psi(t), \ t < t_0, \ y(t_0) = y_0, \ \dot{y}(t_0) = y_0'. \tag{7.4.16}$$

Denote by $x(t)$ and $y(t)$ the solutions of (7.2.2), (7.2.3) and (7.4.15), (7.4.16), respectively, and let $Y(t,s)$ be the fundamental function of (7.4.15).

**Theorem 7.4** *Suppose that there exists a positive solution $x(t)$ of (7.3.2) for $t \geq t_0$ and*

$$a_k(t) \geq b_k(t) \geq 0, \ r(t) \geq f(t) \text{ for } t \geq t_0, \varphi(t) \geq \psi(t) \text{ for } t < t_0,$$
$$y_0 = x_0, \ y_0' \geq x_0'.$$

*Then $y(t) \geq x(t)$ for $t \geq t_0$.*

*Proof* Denote by $u(t)$ a nonnegative solution of (7.3.2). The inequality $a_k(t) \geq b_k(t)$, $t \geq t_0$, yields that the function $u(t)$ is also a solution of the inequality corresponding to (7.3.2) for (7.4.15). Hence, by Theorem 7.1 we have $Y(t,s) > 0$ for $t > s \geq t_0$.

After rewriting (7.2.2) in the form

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t) x\big(g_k(t)\big) = -\sum_{k=1}^{m} \big[a_k(t) - b_k(t)\big] x\big(g_k(t)\big) + f(t)$$

and applying solution representation (7.2.5), for the solutions of (7.2.2), (7.2.3) and (7.4.15), (7.4.16) we have

$$x(t) = y_1(t)x_0 + Y(t,t_0)x_0' - \sum_{k=1}^{m} \int_{t_0}^{t} Y(t,s)\big[a_k(s) - b_k(s)\big]x\big(g_k(s)\big)ds$$
$$- \sum_{k=1}^{m} \int_{t_0}^{t} Y(t,s)b_k(s)\varphi\big(g_k(s)\big)ds + \int_{t_0}^{t} Y(t,s)f(s)ds,$$
$$y(t) = y_1(t)y_0 + Y(t,t_0)y_0' - \sum_{k=1}^{m} \int_{t_0}^{t} Y(t,s)b_k(s)\psi\big(g_k(s)\big)ds + \int_{t_0}^{t} Y(t,s)r(s)ds,$$

where $y_1$ is the solution of (7.4.15), (7.4.16) with $r \equiv 0$, $\psi \equiv 0$, $y_0 = 1$, $y_0' = 0$ and $\varphi(g_k(s)) = \psi(g_k(s)) = 0$ if $g_k(s) > t_0$, $x(g_k(s)) = 0$ if $g_k(s) < t_0$.

Therefore $y(t) \ge x(t) > 0$, $t \ge t_0$, which completes the proof. □

In the next statement, the equality $x_0 = y_0$ is replaced by $x_0 \ge y_0$.

**Theorem 7.5** *Suppose that there exists a positive solution $x(t)$ of (7.3.2) for $t \ge t_0$,*

$$a_k(t) \ge b_k(t) \ge 0,\ \ r(t) \ge f(t) \text{ for } t \ge t_0, \varphi(t) \ge \psi(t) \text{ for } t < t_0.$$

*If $x_0 \ge y_0 > 0$ and $y_0' + \alpha y_0 \ge y_0 \ge x_0' + \alpha x_0$ for some $\alpha > 0$, then*

$$y(t) - x(t) \ge e^{-\alpha(t-t_0)}(y_0 - x_0) \text{ for } t \ge t_0.$$

*In particular, if $x_0 \ge y_0 > 0$ and $y_0' > x_0'$, then*

$$\liminf_{t\to\infty}\big[y(t) - x(t)\big] \ge 0.$$

*Proof* First assume that $f \equiv r \equiv 0$. As in the proof of Theorem 7.4, we obtain that $Y(t,s) > 0$ for $t \ge s$. The inequalities $x(t) > 0$, $\ddot{x}(t) \le 0$ for $t \ge t_0$ imply $\dot{x}(t) \ge 0$, $t \ge t_0$. Then $x(t) \ge x_0$ and hence $x(t) > x_0 e^{-\alpha(t-t_0)}$ for $t \ge t_0$ for any $\alpha > 0$.

Denote

$$u(t) = \begin{cases} x(t) - x_0 e^{-\alpha(t-t_0)}, & t \ge t_0, \\ \varphi(t), & t < t_0, \end{cases}$$
$$v(t) = \begin{cases} y(t) - y_0 e^{-\alpha(t-t_0)}, & t \ge t_0, \\ \psi(t), & t < t_0. \end{cases}$$

Then $u$ and $v$ are the solutions of the problems

$$\ddot{u}(t) + \sum_{k=1}^{m} a_k(t)u\big(g_k(t)\big) = -x_0\left[\alpha^2 e^{-\alpha(t-t_0)} + \sum_{k=1}^{m} a_k(t)e^{-\alpha(g_k(t)-t_0)}\right], \quad (7.4.17)$$

$$u(t) = \varphi(t),\ \ t < t_0,\ \ u(t_0) = 0,\ \ \dot{u}(t_0) = x_0' + \alpha x_0, \quad (7.4.18)$$

and

$$\ddot{v}(t)+\sum_{k=1}^{m} b_k(t)v\big(g_k(t)\big)=-y_0\left[\alpha^2 e^{-\alpha(t-t_0)}+\sum_{k=1}^{m} b_k(t)e^{-\alpha(g_k(t)-t_0)}\right], \quad (7.4.19)$$

$$v(t)=\psi(t),\ t<t_0,\ v(t_0)=0,\ \dot{v}(t_0)=y_0'+\alpha y_0, \quad (7.4.20)$$

respectively. The assumptions of this theorem and Theorem 7.4 imply $v(t)\ge u(t)$ for $t\ge t_0$ (i.e., $y(t)-y_0e^{-\alpha(t-t_0)}\ge x(t)-x_0e^{-\alpha(t-t_0)}$ for $t\ge t_0$), which can be rewritten in the form

$$y(t)-x(t)\ge e^{-\alpha(t-t_0)}(y_0-x_0),\ t\ge t_0.$$

If $y_0'>x_0'$, then, for a sufficiently small $\alpha>0$, we have $y_0'+\alpha y_0>x_0'+\alpha x_0$. Therefore $y(t)-x(t)\ge e^{-\alpha(t-t_0)}(y_0-x_0)$ for $t\ge t_0$, which completes the proof in the case $f\equiv r\equiv 0$.

In the general case, denote by $x_1$ and $y_1$ the solutions of problems (7.2.2), (7.2.3) and (7.4.15), (7.4.16), respectively, with $f\equiv r\equiv 0$, and by $x_2$ and $y_2$ the solutions of these problems with $x_0=x_0'=y_0=y_0'=0$. By Theorem 7.4, $y_2(t)\ge x_2(t)$ for $t\ge t_0$. Clearly, $y(t)-x(t)=[y_1(t)-x_1(t)]+[y_2(t)-x_2(t)]$, and hence

$$y(t)-x(t)\ge e^{-\alpha(t-t_0)}(y_0-x_0)\text{ for } t\ge t_0,$$

which concludes the proof. □

We obtain the most complete result if we compare two solutions $x$ and $y$ of the same equation (7.2.2). In this case, we will not assume that $a_k(t)\ge 0$ and the solutions $x$ and $y$ are positive.

**Theorem 7.6** *Assume that inequality* (7.4.10) *has a nonnegative solution for* $t\ge t_0\ge 0$, $x$ *and* $y$ *are two solutions of* (7.2.2), (7.2.3) *with right-hand sides* $f$ *and* $r$ *and initial functions* $\varphi$ *and* $\psi$, *respectively. Moreover,*

$$a_k(s)\varphi\big(g_k(s)\big)\ge a_k(s)\psi\big(g_k(s)\big)\ \textit{for } s \textit{ such that } g_k(s)<t_0;\ r(t)\ge f(t)\ \textit{for } t\ge t_0.$$

1) *If* $x_0=y_0$, $y_0'\ge x_0'$, *then* $y(t)\ge x(t)$, $t\ge t_0$.
2) *If* $x_0\ge y_0>0$ *and* $y_0'+\alpha y_0\ge x_0'+\alpha x_0$ *for some* $\alpha>0$, *then*

$$y(t)-x(t)\ge e^{-\alpha(t-t_0)}(y_0-x_0)\ \textit{for } t\ge t_0.$$

3) *If* $x_0\ge y_0>0$ *and* $y_0'>x_0'$, *then*

$$\liminf_{t\to\infty}\big[y(t)-x(t)\big]\ge 0.$$

*Proof* 1) By Corollary 7.3, the inequality $X(t,s)>0$, $t>s\ge t_0$ holds for the fundamental function of (7.2.1).

For the solutions $x$ and $y$, we have

$$x(t)=x_1(t)x_0+X(t,t_0)x_0' \\ -\sum_{k=1}^{m}\int_{t_0}^{t} X(t,s)a_k(s)\varphi\big(g_k(s)\big)ds+\int_{t_0}^{t} X(t,s)f(s)ds,$$

$$y(t) = x_1(t)y_0 + X(t,t_0)y_0'$$
$$- \sum_{k=1}^{m} \int_{t_0}^{t} X(t,s)a_k(s)\psi\big(g_k(s)\big)ds + \int_{t_0}^{t} X(t,s)r(s)ds,$$

where $\varphi(g_k(s)) = \psi(g_k(s)) = 0$ if $g_k(s) \geq t_0$. The theorem assumptions yield that $y(t) \geq x(t)$ for $t \geq t_0$.

The proof of 2) and 3) is similar to the proof of the previous theorem. □

*Remark 7.2* Explicit constructions of solutions for inequality (7.4.10) will be presented in the next section.

## 7.5 Explicit Nonoscillation and Oscillation Conditions

We will employ Corollary 7.2 to obtain explicit sufficient nonoscillation conditions.

**Theorem 7.7** *For some $t \geq t_0$, let*

$$\sup_{t \geq t_0} \sum_{k=1}^{m} \frac{a_k^+(t)\sqrt{t^3 g_k(t)}\ln g_k(t)}{\ln t} \leq \frac{1}{4}, \; t \geq t_0,$$

*and $g_k(t) > 1$ for $t \geq t_0$, $k = 1, \cdots, m$. Then there exists $t_1 \geq t_0$ such that (7.2.1) has a positive solution for $t > t_1$.*

*Proof* The statement of the theorem yields that $x(t) = \sqrt{t}\ln t$ is a positive solution of inequality (7.4.9) for $t > t_0$. In fact,

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k^+(t)x\big(g_k(t)\big) = -\frac{\ln t}{4t\sqrt{t}} + \sum_{k=1}^{m} \sqrt{g_k(t)}\ln g_k(t)$$
$$= \frac{\ln t}{t\sqrt{t}}\left[-\frac{1}{4} + \sum_{k=1}^{m} \frac{a_k^+(t)\sqrt{t^3 g_k(t)}\ln g_k(t)}{\ln t}\right] \leq 0.$$

Part 1) of Corollary 7.3 implies the statement of the theorem. □

Now we proceed to the oscillation problem and start with the following oscillation criterion.

**Theorem 7.8** *Suppose $a_k(t) \geq 0$ and there exists $\delta > 0$ such that $t - g_k(t) \leq \delta$. Then all solutions of (7.2.1) are oscillatory if and only if all solutions of ordinary differential equation (7.4.14) are oscillatory.*

*Proof* By Corollary 7.4, oscillation of (7.2.1) yields that (7.4.14) is also oscillatory.

Suppose now that all solutions of (7.4.14) are oscillatory but (7.2.1) has a positive solution $x(t) > 0, t \geq t_0$. By Theorem 7.3, the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(t-\delta) = 0 \tag{7.5.1}$$

has a positive solution for $t > t_0$, and its fundamental function $Y(t,s)$ is positive for $t > t_0$.

Denote $x(t) = Y(t,t_0) > 0$, $t \geq t_0$. This function is a solution of (7.5.1), where $x(t_0) = 0$, $\dot{x}(t_0) = 1$. Let us rewrite (7.5.1) in the form

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(t) + \sum_{k=1}^{m} a_k(t)\big[x(t-\delta) - x(t)\big] = 0. \tag{7.5.2}$$

Inequality $\ddot{x}(t) \leq 0, t > t_0$ implies that $\dot{x}(t), t > t_0$ is nonincreasing. Then $\dot{x}(t-\delta) \geq \dot{x}(t)$. After integrating this inequality from $t_0 + \delta$ to $t$, we obtain $x(t-\delta) - x(t_0) \geq x(t) - x(t_0+\delta)$, where $x(t_0) = 0$. Thus $x(t) - x(t-\delta) \leq x(t_0+\delta)$, and therefore (7.5.2) implies the inequality

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)\big[x(t) - x(t_0+\delta)\big] \leq 0.$$

Hence the function $z(t) = x(t) - x(t_0+\delta)$ is a positive solution (for $t > t_0 + \delta$) of the inequality

$$\ddot{z}(t) + \sum_{k=1}^{m} a_k(t)z(t) \leq 0.$$

Theorem 7.1 implies that (7.4.14) has a positive solution, which contradicts the assumption that all solutions of (7.4.14) are oscillatory. □

Consider the equation

$$\ddot{x}(t) + a(t)x\big(g(t)\big) = 0, \;\; ct \leq g(t) \leq t \text{ for } t \geq 0, 0 < c < 1, a(t) \geq 0 \text{ for } t \geq 0, \tag{7.5.3}$$

with continuous functions $a$ and $g$. In [283], the following result was obtained. If the ordinary differential equation

$$\ddot{x}(t) + ca(t)x(t) = 0$$

is oscillatory, then (7.5.3) is also oscillatory. We generalize this statement to (7.2.1) with several delays.

**Theorem 7.9** *Suppose $a_k(t) \geq 0$, there exist constants $c_k$ such that $0 < c_k < 1$, $g_k(t) \geq c_k t$ for $t \geq 0$, $k = 1, \cdots, m$ and the ordinary differential equation*

$$\ddot{x}(t) + \sum_{k=1}^{m} c_k a_k(t)x(t) = 0 \tag{7.5.4}$$

*is oscillatory. Then (7.2.1) is also oscillatory.*

*Proof* Suppose that delay equation (7.2.1) is nonoscillatory. Then, by Theorem 7.3, the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x(c_k t) = 0 \tag{7.5.5}$$

is also nonoscillatory. Theorem 7.1 implies that there exists $t_0 \geq 0$ such that $Y(t,s) > 0$, $t > s \geq t_0$, where $Y(t,s)$ is the fundamental function of (7.5.5). Hence $y(t) = Y(t,t_0)$ is a nonnegative solution of the problem

$$\begin{aligned} &\ddot{y}(t) + \sum_{k=1}^{m} a_k(t)y(c_k t) = 0, \ t \geq t_0, \\ &y(t) = 0, \ t \leq t_0, \ \dot{y}(t_0) = 1. \end{aligned} \tag{7.5.6}$$

Let us rewrite (7.5.5) in the form

$$\ddot{y}(t) + \sum_{k=1}^{m} c_k a_k(t)y(t) + \sum_{k=1}^{m} a_k(t)\big[y(c_k t) - c_k y(t)\big] = 0. \tag{7.5.7}$$

Since $\dot{y}$ is a nonincreasing function, $\dot{y}(c_k t) \geq \dot{y}(t)$. By integrating this inequality from $t_0$ to $t$, we obtain

$$\frac{1}{c_k} y(c_k t) - \frac{1}{c_k} y(c_k t_0) \geq y(t) - y(t_0),$$

where $y(t_0) = y(c_k t_0) = 0$. Thus, $y(c_k t) \geq c_k y(t)$, and (7.5.6) implies

$$\ddot{y}(t) + \sum_{k=1}^{m} c_k a_k(t)y(t) \leq 0 \text{ for } t \geq t_0.$$

Theorem 7.1 yields that (7.5.4) is nonoscillatory, which contradicts the assumption of the theorem. □

*Remark 7.3* As a consequence of Theorem 7.9, we obtain ($m = 1$) that the condition $\liminf_{t\to\infty} a(t)t^2 > \frac{1}{4c}$ is sufficient for oscillation. Domshlak [140, 141] replaced the constant $\frac{1}{4c}$ with a certain constant $B_c$ and showed that this constant is strict.

The following statement is well known (see, for example, [84]): if, for a certain $k$, $0 < k < 1$, the ordinary differential equation

$$\ddot{x}(t) + k\frac{g(t)}{t}a(t)x(t) = 0$$

is oscillatory, then the delay equation

$$\ddot{x}(t) + a(t)x\big(g(t)\big) = 0$$

is also oscillatory.

The corollary of the following theorem generalizes this statement.

**Theorem 7.10** *Suppose $a_k(t) \ge 0$ for $t \ge 0$ and for every $c > 0$ the ordinary differential equation*

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) \frac{g_k(t) - c}{t - c} x(t) = 0, \ t > c,$$

*is oscillatory. Then (7.2.1) is also oscillatory.*

*Proof* Suppose that (7.2.1) is nonoscillatory. Then, similar to the proof of the previous theorem, there exists $t_0 \ge 0$ such that the solution $y$ of the problem

$$\begin{aligned} &\ddot{y}(t) + \sum_{k=1}^{m} a_k(t) y\big(g_k(t)\big) = 0, \ t \ge t_0, \\ &y(t) = 0, \ t \le t_0, \ \dot{y}(t_0) = 1, \end{aligned} \tag{7.5.8}$$

is positive for $t > t_0$. For $y(t)$, we have $\ddot{y} \le 0$, so $\dot{y}(t)$ is nonincreasing. The inequality

$$y(t) - y(t_0) \ge \dot{y}(t)(t - t_0), \ t \ge t_0,$$

implies $y(t) - \dot{y}(t)(t - t_0) \ge 0$. Then the function

$$f(t) = \frac{y(t)}{t - t_0}, \ t > t_0,$$

is nonincreasing and therefore

$$\frac{y(t)}{t - t_0} \le \frac{y(g_k(t))}{g_k(t) - t_0},$$

which yields

$$y\big(g_k(t)\big) \ge \frac{g_k(t) - t_0}{t - t_0} y(t), \ t > t_0. \tag{7.5.9}$$

Then,

$$\ddot{y}(t) + \sum_{k=1}^{m} a_k(t) \frac{g_k(t) - t_0}{t - t_0} y(t) \le 0, \ t > t_0,$$

and the corresponding ordinary differential equation

$$\ddot{y}(t) + \sum_{k=1}^{m} a_k(t) \frac{g_k(t) - t_0}{t - t_0} y(t) = 0, \ t > t_0, \tag{7.5.10}$$

is nonoscillatory, which contradicts the assumption of the theorem. □

**Corollary 7.5** *Suppose $a_i(t) \ge 0$ and for some $k_i$, $0 < k_i < 1$, $i = 1, 2, \cdots, m$, the ordinary differential equation*

$$\ddot{x}(t) + \sum_{i=1}^{m} k_i \frac{g_i(t)}{t} a_i(t) x(t) = 0$$

*is oscillatory. Then (7.2.1) is also oscillatory.*

*Proof* Suppose $c > 0$. The inequality

$$k_i \frac{g_i(t)}{t} \le \frac{g_i(t) - c}{t - c} \tag{7.5.11}$$

is equivalent to the relation

$$ct - ck_i g_i(t) \le (1 - k_i) t g_i(t). \tag{7.5.12}$$

Since $\lim_{t\to\infty} g_i(t) = \infty$, there exists $t_i \ge 0$ such that (7.5.11) holds for $t \ge t_i$. Hence, for $t \ge t_0 = \max_i\{t_i\}$, inequalities (7.5.9) hold for all $i$, $i = 1, 2, \cdots, m$. Application of Theorems 7.3 and 7.9 completes the proof. □

To obtain explicit oscillation conditions for delay equations, we can apply any known oscillation test for ordinary differential equations; for example, we can use the following result.

**Lemma 7.2** [206] *Suppose $p(t) \ge 0$ is a locally integrable function and at least one of the following conditions holds*:

1) $$\limsup_{t\to\infty} t \int_t^\infty p(s)ds > 1$$

*or*

2) $$\liminf_{t\to\infty} t \int_t^\infty p(s)ds > \frac{1}{4}.$$

(*It is assumed that the conditions are satisfied if the integral diverges*.)

*Then all solutions of the equation*

$$\ddot{x}(t) + p(t)x(t) = 0$$

*are oscillatory.*

**Corollary 7.6** *Suppose $a_i(t) \ge 0$ and at least one of the following conditions holds*:

$$\limsup_{t\to\infty} t \int_t^\infty \sum_{i=1}^m \frac{g_i(s)}{s} a_i(s)ds > 1 \tag{7.5.13}$$

*or*

$$\liminf_{t\to\infty} t \int_t^\infty \sum_{i=1}^m \frac{g_i(s)}{s} a_i(s)ds > \frac{1}{4}. \tag{7.5.14}$$

*Then* (7.2.1) *is oscillatory.*

*Proof* Suppose (7.5.13) holds. Then there exists $q > 1$ such that

$$\limsup_{t\to\infty} t \int_t^\infty \sum_{i=1}^m \frac{g_i(s)}{s} a_i(s)ds > q;$$

i.e., for $k_i = \frac{1}{q} < 1$ we have

$$\limsup_{t\to\infty} t \int_t^\infty \sum_{i=1}^m k_i \frac{g_i(s)}{s} a_i(s)ds > 1.$$

The reference to Lemma 7.2 and Corollary 7.5 completes the proof. The case where (7.5.14) holds is considered similarly. □

## 7.6 Slowly Oscillating Solutions

For linear ordinary differential equations of the second order, the following oscillation criterion is well known: if an equation has an oscillatory solution, then all its solutions are oscillatory. It is known that for delay differential equations this statement is not true.

We will show that if (7.2.1) has a *slowly oscillating* solution, then all solutions of this equation are oscillatory. A similar result for delay differential equations of the first order was obtained in Chapter 2.

**Definition 7.4** A solution $x$ of (7.2.1) is *slowly oscillating* if for every $t_0 \geq 0$ there exist $t_2 > t_1 > t_0$ such that

$$g_k(t) \geq t_1 \text{ for } t \geq t_2,\ x(t_1) = x(t_2) = 0,\ x(t) > 0,\ t \in (t_1, t_2),$$

and at the point $t_2$ the function $x(t)$ has a sign change.

Let us remark that for ordinary differential equations any oscillatory solution is slowly oscillating. Thus the following theorem implies the well-known result on oscillation of all solutions of an ordinary differential equation once a solution oscillates.

**Theorem 7.11** *Suppose* $a_k(t) \geq 0$ *for* $t \geq 0$. *If there exists a slowly oscillating solution of* (7.2.1), *then all solutions of this equation are oscillatory.*

*Proof* Denote by $x$ a slowly oscillating solution of (7.2.1). Suppose that this equation has a nonoscillatory solution. Then, by Theorem 7.1, for a certain $t_0 \geq 0$ the fundamental function is positive, $X(t,s) > 0$ for $t > s \geq t_0$.

There exist $t_1 > t_0$, $t_2 > t_0$, $\sigma > 0$ such that

$$g(t) \geq t_1 \text{ for } t \geq t_2,$$
$$x(t_1) = x(t_2) = 0,\ x(t) > 0,\ t \in (t_1, t_2),\ x(t) < 0,\ t \in (t_2, t_2 + \sigma]. \tag{7.6.1}$$

Due to solution representation (7.2.5), this solution for $t \geq t_2$ can be presented in the form

$$x(t) = X(t, t_2)\dot{x}(t_2) - \sum_{k=1}^m \int_{t_2}^t X(t,s)a_k(s)x\big(g_k(s)\big)ds, \tag{7.6.2}$$

where $x(g_k(s)) = 0$ if $g_k(s) > t_2$.

Inequality $g_k(t) \geq t_1$ for $t \geq t_2$ yields that the expression under the integral in (7.6.2) may differ from zero only if $t_1 < g_k(s) < t_2$, and therefore $x(g_k(s)) > 0$ in (7.6.2).

Besides, $x(t)$ has a sign change at the point $t_2$; thus $\dot{x}(t_2) \leq 0$. Hence (7.6.2) implies $x(t) \leq 0$ for each $t \geq t_2$. This contradicts the assumption that $x$ is an oscillatory solution. □

**Corollary 7.7** *Suppose that* $a_k(t) \geq 0$ *for* $t \geq 0$ *and* (7.2.1) *has a positive solution for* $t > t_0 \geq 0$. *Then* (7.2.1) *has no slowly oscillating solutions.*

*Remark 7.4* Yu. Domshlak [140, 141] demonstrated that if $g_k$ are monotonically increasing functions and an *associated* equation has a slowly oscillating solution, then every solution of (7.2.1) is oscillatory. He obtained several new explicit sufficient oscillation conditions by explicit construction of such slowly oscillating solutions.

## 7.7 Existence of a Positive Solution

In this section, we assume that (7.2.1) is nonoscillatory and present conditions on the initial function and the initial values that imply positivity of the solution of initial value problem (7.2.2), (7.2.3).

For delay differential equations of the first order, a few papers contain such results. In paper [193], the most general result is presented. The method applied in this section uses the same ideas as in Chap. 2 for equations of the first order.

**Theorem 7.12** *Suppose that* $a_k(t) \geq 0$, $f(t) \geq 0$ *for* $t \geq 0$, $x(t)$ *is a solution of problem* (7.2.2), (7.2.3) *and* $u(t)$ *is a nonnegative solution of the inequality*

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t) \exp\left\{-\int_{\max\{t_0, g_k(t)\}}^{t} u(s)ds\right\} \leq 0, \quad t \geq t_0. \tag{7.7.1}$$

*If*

$$x_0 > 0, \quad \varphi(t) \leq x_0, \quad t \leq t_0, \quad \text{and } x_0' \geq u(t_0)x_0,$$

*then* $x(t) > 0$ *for* $t \geq t_0$.

*Proof* First assume that $f \equiv 0$. Consider the auxiliary problem

$$\ddot{z}(t) + \sum_{k=1}^{m} a_k(t) z\big(g_k(t)\big) = 0, \quad t \geq t_0,$$

$$z(t) = x_0, \quad t \leq t_0, \quad \dot{z}(t_0) = u(t_0)x_0.$$

Denote

$$v(t) = \begin{cases} x_0 \exp\{\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ x_0, & t < t_0, \end{cases}$$

and for a fixed $t \ge t_0$ define the sets

$$N_1(t) = \{k \mid g_k(t) \ge t_0\}, \ N_2(t) = \{k \mid g_k(t) < t_0\}.$$

We have

$$\begin{aligned}
&\ddot{v}(t) + \sum_{k=1}^{m} a_k(t) v\big(g_k(t)\big) \\
&= x_0 \exp\left\{\int_{t_0}^{t} u(s)ds\right\}\big(\dot{u}(t) + u^2(t)\big) \\
&\quad + x_0 \sum_{k \in N_1(t)} a_k(t) \exp\left\{\int_{t_0}^{g_k(t)} u(s)ds\right\} + x_0 \sum_{k \in N_2(t)} a_k(t) \\
&= x_0 \exp\left\{\int_{t_0}^{t} u(s)ds\right\}\left[\dot{u}(t) + u^2(t) + \sum_{k \in N_1(t)} a_k(t) \exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\}\right. \\
&\quad \left. + \sum_{k \in N_2(t)} a_k(t) \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right] \\
&= x_0 \exp\left\{\int_{t_0}^{t} u(s)ds\right\}\left[\dot{u}(t) + u^2(t) + \sum_{k \in N_1(t)} a_k(t) \exp\left\{-\int_{\max\{t_0, g_k(t)\}}^{t} u(s)ds\right\}\right. \\
&\quad \left. + \sum_{k \in N_2(t)} a_k(t) \exp\left\{-\int_{\max\{t_0, g_k(t)\}}^{t} u(s)ds\right\}\right] \\
&= x_0 \exp\left\{\int_{t_0}^{t} u(s)ds\right\}\left[\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} a_k(t) \exp\left\{-\int_{\max\{t_0, g_k(t)\}}^{t} u(s)ds\right\}\right] \\
&\le 0.
\end{aligned}$$

Therefore

$$\ddot{v}(t) + \sum_{k=1}^{m} a_k(t) v\big(g_k(t)\big) = r(t),$$

where $r(t) \le 0$, $t \ge t_0$. Inequality (7.7.1) implies (7.3.2). Since $z(t) = v(t) = x_0$, $t \le t_0$, $z_0' = v_0' = u(t_0)x_0$, Theorem 7.6 yields that $z(t) \ge v(t) > 0$ for $t \ge t_0$. Hence the hypotheses of this theorem and Theorem 7.6 imply $x(t) \ge z(t) > 0$ for $t \ge t_0$.

In the case $f \equiv 0$, the proof is complete. The general case is also a consequence of Theorem 7.6 since $f(t) \ge 0$ for $t \ge 0$. □

**Corollary 7.8** *Suppose*

$$a_k(t) \ge 0, \ f(t) \ge 0 \textit{ for } t \ge t_0 > 0; \ \varphi(t) \le x_0 \textit{ for } t \le t_0; \ x_0 > 0, \ x_0' \ge \frac{1}{2t_0} x_0$$

*and*

$$\sup_{t \ge t_0} \sum_{k=1}^{m} a_k(t) \sqrt{t^3 \max\{t_0, g_k(t)\}} \le \frac{1}{4}. \tag{7.7.2}$$

*Then the solution of problem* (7.2.2), (7.2.3) *is positive.*

*Proof* Let us demonstrate that the function $u(t)=\frac{1}{2t}$ is a solution of inequality (7.7.1). Substituting $u(t)$ into (7.7.1), we obtain

$$\begin{aligned}
&\dot{u}(t)+u^2(t)+\sum_{k=1}^m a_k(t)\exp\left\{-\int_{\max\{t_0,g_k(t)\}}^t u(s)ds\right\}\\
&=-\frac{1}{2t^2}+\frac{1}{4t^2}+\sum_{k=1}^m a_k(t)\exp\left\{-\int_{\max\{t_0,g_k(t)\}}^t \frac{1}{2s}ds\right\}\\
&=-\frac{1}{4t^2}+\sum_{k=1}^m a_k(t)\exp\left\{\ln\left(\frac{\max\{t_0,g_k(t)\}}{t}\right)^{1/2}\right\}\\
&=-\frac{1}{4t^2}+\sum_{k=1}^m a_k(t)\left(\frac{\max\{t_0,g_k(t)\}}{t}\right)^{1/2}\le 0
\end{aligned}$$

by (7.7.2), and thus (7.7.1) is satisfied for $u(t)=\frac{1}{2t}$, which completes the proof. □

## 7.8 Discussion and Open Problems

This chapter deals with nonoscillation properties of a scalar linear delay differential equation of the second order. Such equations attract the attention of many mathematicians due to their significance in applications. We mention here the monographs of A.D. Myshkis [289], S.B. Norkin [292], G.S. Ladde, V. Lakshmikantham and B.G. Zhang [248], I. Győri and G. Ladas [192], L.N. Erbe, Q. Kong and B.G. Zhang [154] and references therein.

The main result of this chapter (Theorem 7.1) states that under some natural assumptions for a delay differential equation the following four assertions are equivalent: nonoscillation of solutions of this equation and the corresponding differential inequality, positivity of the fundamental function and the existence of a nonnegative solution for the generalized Riccati inequality. The connection between hypotheses 1) and 3) of Theorem 7.1 was first established in [20]. The generalized Riccati equation appeared for the first time in [80].

The equivalence of oscillation properties of the differential equation and the corresponding differential inequality has been applied to obtain new explicit conditions for nonoscillation and oscillation and also to prove some well-known results in a different way. We employ the generalized Riccati inequality to compare oscillation properties of two equations without comparing their solutions. These results can be regarded as a natural generalization of the well-known Sturm comparison theorem for second-order ordinary differential equations.

By applying positivity of the fundamental function, we compare positive solutions of two nonoscillatory equations. There are a lot of results of this kind for delay differential equations of the first order and only a few for the second-order equations. A. D. Myshkis [289] obtained one of the first comparison theorems for second-order equations. The result presented in this chapter is more general and is proven in a different way.

The chapter also contains conditions on the initial function and the initial values that imply that the corresponding solution is positive. Such conditions are well known for first-order delay differential equations but not for second-order equations.

Some of the results presented in this chapter extend known nonoscillation tests to the case of discontinuous parameters (we only need the minimal requirement of measurability). A new technique based on a generalized Riccati inequality made it possible to deal with discontinuous parameters and to obtain explicit oscillation criteria.

In this chapter, we obtain oscillation conditions for (7.2.1) by comparing oscillation properties of this equation and a certain ordinary differential equation of the second order. The results of this chapter are published in the paper [43].

Explicit oscillation conditions different from these obtained in this chapter are presented in [84, 300, 301, 306], see also the papers [163, 174, 217, 221, 263, 305, 327, 335]. In the very interesting paper [227], many new oscillation conditions for (7.2.1) in the case $m=1$,

$$\ddot{x}(t)+a(t)x\big(h(t)\big)=0, \tag{7.8.1}$$

were obtained, where $a(t)\geq 0$.

In particular, paper [227] contains the following two results.

**Theorem 7.13** [227] *Suppose $a(t)\geq 0$ and either $h$ is nondecreasing and*

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left[h(s)+\int_0^{h(s)}\tau h(\tau)a(\tau)d\tau\right]a(s)ds>1$$

*or*

$$\liminf_{t\to\infty}\int_{h(t)}^{t}\left[h(s)+\int_0^{h(s)}\tau h(\tau)a(\tau)d\tau\right]a(s)ds>\frac{1}{e}.$$

*Then all solutions of* (7.8.1) *are oscillatory.*

**Theorem 7.14** [227] *Suppose $a(t)\geq 0$, $h(t)$ is a nondecreasing function and*

$$\limsup_{t\to\infty}\left[\int_{h(t)}^{t}a(s)h(s)ds+h(t)\int_t^{\infty}a(s)ds\right]>1.$$

*Then all solutions of* (7.8.1) *are oscillatory.*

Some positivity conditions for the fundamental function of second-order delay differential equations were established by M. I. Gil'. In particular, in [159] he suggested positivity conditions for autonomous equations and then applied these conditions to the Aizerman-Myshkis problem in the absolute stability theory of nonlinear delay differential equations. In [160], using positivity conditions from [159], under certain assumptions, positivity and boundedness of solutions for nonlinear delay differential equations were established. The results of [160] were slightly improved in [161]; see also nonoscillation results for delay differential equations of the second order in [297].

Finally, let us present some open problems.

1. Obtain oscillation and nonoscillation conditions for (7.2.1) in the case of oscillating coefficients $a_k(t)$.
2. Establish oscillation and nonoscillation properties of the equation with positive and negative coefficients

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big) - \sum_{k=1}^{n} b_k(t)x\big(h_k(t)\big) = 0,$$

where $a_k(t) \geq 0$, $b_k(t) \geq 0$.

In particular, consider two cases: $a_k(t) \geq b_k(t)$ and $a_k(t) < b_k(t)$.

3. If $a_k(t) \geq 0$, then any positive solution of (7.2.1) is a nondecreasing function.

Find sufficient conditions when this solution has a finite limit, and give an asymptotic representation of this limit.

# Chapter 8
# Second-Order Delay Differential Equations with Damping Terms

## 8.1 Introduction

In this chapter, we consider differential equations of the second order with damping, which are in some sense similar to the equations without damping described in the previous chapter. In particular, we study the existence of positive solutions with a nonnegative derivative.

The chapter is organized as follows. Section 8.2 contains relevant definitions and notation. In Sect. 8.3, the main result of this chapter is obtained. Section 8.4 deals with comparison results. Section 8.5 includes some explicit nonoscillation conditions. Section 8.6 involves discussion and states open problems.

## 8.2 Preliminaries

We consider the scalar delay differential equation of the second order

$$\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = 0, \ t \geq 0, \tag{8.2.1}$$

under the following assumptions:

(a1) $a_k$, $b_k$ are Lebesgue measurable functions locally essentially bounded on $[0, \infty)$.

(a2) $h_k$, $g_k$ are Lebesgue measurable functions,

$$h_k(t) \leq t, \ \lim_{t\to\infty} h_k(t) = \infty, \ g_k(t) \leq t, \ \lim_{t\to\infty} g_k(t) = \infty.$$

In the theory of linear second-order ordinary differential equations, the term that includes the first derivative is called the damping term, so we further refer to (8.2.1) as the equation with damping terms.

R.P. Agarwal et al., *Nonoscillation Theory of Functional Differential Equations with Applications*, DOI 10.1007/978-1-4614-3455-9_8,

Together with (8.2.1), consider for each $t_0 \geq 0$ the initial value problem with a nonzero right-hand side

$$\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = f(t),\ t \geq t_0, \tag{8.2.2}$$

$$x(t) = \varphi(t),\ \dot{x}(t) = \psi(t),\ t < t_0;\ x(t_0) = x_0,\ \dot{x}(t_0) = x_0'. \tag{8.2.3}$$

We also assume that the following hypothesis holds:

(a3) $f : [t_0, \infty) \to R$ is a Lebesgue measurable locally essentially bounded function and $\varphi, \psi : (-\infty, t_0) \to R$ are Borel measurable bounded functions.

**Definition 8.1** Suppose that a function $x : [t_0, \infty) \to R$ is differentiable, $\dot{x}$ is a locally absolutely continuous function and the functions $x$ and $\dot{x}$ are extended for $t \leq t_0$ by equalities (8.2.3). We say that the extended function $x$ is *a solution* of problem (8.2.2), (8.2.3) if it satisfies (8.2.2) for almost every $t \in [t_0, \infty)$.

**Definition 8.2** For each $s \geq 0$, the solution $X(t, s)$ of the problem

$$\begin{gathered} \ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = 0,\ t \geq s, \\ x(t) = 0,\ \dot{x}(t) = 0,\ t < s,\ x(s) = 0,\ \dot{x}(s) = 1, \end{gathered} \tag{8.2.4}$$

is called *the fundamental function* of (8.2.1).

We assume $X(t, s) = 0$, $0 \leq t < s$. Let functions $x_1$ and $x_2$ be the solutions of the problem

$$\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = 0,\ t \geq t_0,$$
$$x(t) = 0,\ \dot{x}(t) = 0,\ t < t_0,$$

with initial values $x(t_0) = 1$, $\dot{x}(t_0) = 0$ for $x_1$ and $x(t_0) = 0$, $\dot{x}(t_0) = 1$ for $x_2$, respectively. By definition, $x_2(t) = X(t, t_0)$.

Theorem B.5 implies the following result.

**Lemma 8.1** *Let* (a1)–(a3) *hold. Then there exists one and only one solution of problem* (8.2.2), (8.2.3), *and it can be presented in the form*

$$\begin{aligned} x(t) = x_1(t)x_0 + x_2(t)x_0' - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s)b_k(s)\varphi\big(g_k(s)\big)ds \\ - \sum_{k=1}^{r} \int_{t_0}^{t} X(t, s)a_k(s)\psi\big(h_k(s)\big)ds + \int_{t_0}^{t} X(t, s)f(s)ds. \end{aligned} \tag{8.2.5}$$

*The functions $\varphi(t)$ and $\psi(t)$ that describe "the prehistory" of the process are not defined for $t \geq t_0$; in* (8.2.5) *we assume $\varphi(s) = 0$ and $\psi(s) = 0$ if $s > t_0$.*

## 8.3 Nonoscillation Criteria

Denote

$$a^+ = \max\{a, 0\}, \ a^- = \max\{-a, 0\}.$$

The following theorem establishes a sufficient condition for the existence of a nonoscillatory solution.

**Theorem 8.1** *Suppose that there exist $t_0 \geq 0$ and a nonnegative locally absolutely continuous function $u$, $u(t) = 0$ for $t \leq t_0$ such that $u$ satisfies the inequality*

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{r} a_k^+(t) u\big(h_k(t)\big) \exp\left\{-\int_{h_k(t)}^{t} u(s)ds\right\} + \sum_{k=1}^{m} b_k^+(t) \exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\} \leq 0 \tag{8.3.1}$$

*and the equation*

$$\dot{z}(t) + u(t)z(t) + \sum_{k=1}^{r} a_k(t) z\big(h_k(t)\big) = 0 \tag{8.3.2}$$

*has a positive fundamental function $Z(t, s) > 0$ for $t > s > t_0$.*

*Then*:

1) *The fundamental function of (8.2.1) is positive and its derivative $X_t'$ in $t$ is nonnegative for $t > s > t_0$,*

$$X(t, s) > 0, \ X_t'(t, s) \geq 0, \ t > s > t_0.$$

2) *There exists a solution $x(t)$ of (8.2.1) such that $x(t) > 0$ and $\dot{x}(t) \geq 0$ for $t > t_0$.*

*Proof* 1) Consider the initial value problem

$$\begin{aligned} &\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{m} b_k(t) x\big(g_k(t)\big) = f(t), \ t \geq t_0, \\ &x(t) = \dot{x}(t) = 0, \ t \leq t_0. \end{aligned} \tag{8.3.3}$$

Denote

$$z(t) = \dot{x}(t) - u(t)x(t), \tag{8.3.4}$$

where $x$ is the solution of (8.3.3) and $u$ is a nonnegative solution of (8.3.1). Equality (8.3.4) implies

$$x(t) = \int_{t_0}^{t} \exp\left\{\int_{s}^{t} u(\tau)d\tau\right\} z(s)ds \tag{8.3.5}$$

and

$$\dot{x} = z + ux, \ \ddot{x} = \dot{z} + \dot{u}x + uz + u^2x = \dot{z} + uz + \big(\dot{u} + u^2\big)x.$$

Substituting $\dot{x}$ and $\ddot{x}$ into (8.3.3), we obtain

$$\dot{z}(t)+u(t)z(t)+\sum_{k=1}^{r}a_k(t)z\big(h_k(t)\big)$$
$$=-\big(\dot{u}(t)+u^2(t)\big)x(t)-\sum_{k=1}^{r}a_k(t)u\big(h_k(t)\big)x\big(h_k(t)\big)$$
$$-\sum_{k=1}^{m}b_k(t)x\big(g_k(t)\big)+f(t). \tag{8.3.6}$$

Equalities (8.3.3) and (8.3.4) imply $z(t_0)=0$. Using (8.3.5), we can rewrite (8.3.6) in the form

$$\dot{z}(t)+u(t)z(t)+\sum_{k=1}^{r}a_k(t)z\big(h_k(t)\big)$$
$$=-\left(\dot{u}(t)+u^2(t)+\sum_{k=1}^{r}a_k^+(t)u\big(h_k(t)\big)\exp\left\{-\int_{h_k(t)}^{t}u(s)ds\right\}\right.$$
$$\left.+\sum_{k=1}^{m}b_k^+(t)\exp\left\{-\int_{g_k(t)}^{t}u(s)ds\right\}\right)\int_{t_0}^{t}\exp\left\{\int_s^t u(\tau)d\tau\right\}z(s)ds$$
$$+\sum_{k=1}^{r}a_k^+(t)u\big(h_k(t)\big)\int_{h_k(t)}^{t}\exp\left\{\int_s^{h_k(t)}u(\tau)d\tau\right\}z(s)ds$$
$$+\sum_{k=1}^{m}b_k^+(t)\int_{g_k(t)}^{t}\exp\left\{\int_s^{g_k(t)}u(\tau)d\tau\right\}z(s)ds$$
$$+\sum_{k=1}^{r}a_k^-(t)u\big(h_k(t)\big)\int_{t_0}^{h_k(t)}\exp\left\{\int_s^{h_k(t)}u(\tau)d\tau\right\}z(s)ds$$
$$+\sum_{k=1}^{m}b_k^-(t)\int_{t_0}^{g_k(t)}\exp\left\{\int_s^{g_k(t)}u(\tau)d\tau\right\}z(s)ds+f(t),\ \ z(t_0)=0. \tag{8.3.7}$$

Denote by $Z(t,s)$ the fundamental function of (8.3.2) and by $Fz+f$ the right-hand side of (8.3.7). Then (8.3.7) is equivalent to the equation

$$z=Hz+p, \tag{8.3.8}$$

where

$$(Hz)(t)=\int_{t_0}^{t}Z(t,s)(Fz)(s)ds,\ \ p(t)=\int_{t_0}^{t}Z(t,s)f(s)ds. \tag{8.3.9}$$

Inequalities (8.3.1) and $Z(t,s)>0$ yield that $z(t)\geq 0$ implies $(Hz)(t)\geq 0$ (i.e., the operator $H$ is positive).

Denote

$$c(t) = \dot{u}(t) + u^2(t) + \sum_{k=1}^{r} a_k^+(t) u\big(h_k(t)\big) \exp\left\{ - \int_{h_k(t)}^{t} u(s)ds \right\} + \sum_{k=1}^{m} b_k^+(t) \exp\left\{ - \int_{g_k(t)}^{t} u(s)ds \right\}.$$

Since $u$ is locally absolutely continuous, the function $c \in L[t_0, b]$ for every $b > t_0$, where $L_{[a,b]}$ is the space of all Lebesgue integrable functions on $[a, b]$ with the usual $L_1$ norm.

By Corollary B.1, the function $Z(t, s)$ is bounded in any square $[t_0, b] \times [t_0, b]$, and hence for a certain $K > 0$ the estimate $|Z(t, s)| \leq K$ holds for $b \geq t \geq s \geq t_0$. Then we have for $t \in [t_0, b]$

$$\begin{aligned} \big|(Hz)(t)\big| &\leq K \exp\left\{ \int_{t_0}^{b} u(\tau)d\tau \right\} \int_{t_0}^{t} \Bigg( \big|c(s)\big| \\ &\quad + \sum_{k=1}^{r} \big|a_k(s)\big| \big|u\big(h_k(s)\big)\big| + \sum_{k=1}^{m} \big|b_k(s)\big| \Bigg) \int_{t_0}^{s} \big|z(\tau)\big| d\tau ds \\ &= K \exp\left\{ \int_{t_0}^{b} u(\tau)d\tau \right\} \int_{t_0}^{t} \Bigg( \int_{\tau}^{t} \Bigg[ \big|c(s)\big| + \sum_{k=1}^{r} \big|a_k(s)\big| \big|u\big(h_k(s)\big)\big| \\ &\quad + \sum_{k=1}^{m} \big|b_k(s)\big| \Bigg] ds \Bigg) \big|z(\tau)\big| d\tau. \end{aligned}$$

The kernel of the Volterra integral operator $H$ is bounded in each square $[t_0, b] \times [t_0, b]$. Hence, by Theorem A.4, the operator $H : L_\infty[t_0, b] \to L_\infty[t_0, b]$ is a weakly compact operator. By Theorem A.7, the spectral radius of a compact Volterra integral operator in the space $L_\infty[a, b]$ is equal to zero.

Thus, if in (8.3.8) we have $p(t) \geq 0$, then

$$z(t) = p(t) + (Hp)(t) + \big(H^2 p\big)(t) + \cdots \geq 0.$$

If $f(t) \geq 0$, then by (8.3.9) $p(t) \geq 0$. Hence, for (8.3.6) we have that if $f(t) \geq 0$, then the solution of this equation satisfies $z(t) \geq 0$.

Therefore equality (8.3.5) implies that the solution of (8.3.3) and its derivative are nonnegative for any nonnegative right-hand side.

The solution of (8.3.3) can be presented in the form of (8.2.5) with $\varphi(s) = \psi(s) = 0$ for $s \leq t_0$ and consequently

$$x(t) = \int_{t_0}^{t} X(t, s) f(s)ds, \; \dot{x}(t) = \int_{t_0}^{t} X_t'(t, s) f(s)ds. \tag{8.3.10}$$

As was shown above, $f(t) \geq 0$ implies $x(t) \geq 0$ and $\dot{x}(t) \geq 0$, so the kernels of the integral operators (8.3.9) are nonnegative and therefore $X(t, s) \geq 0$ and $X_t'(t, s) \geq 0$.

Since $X_t'(s, s) = 1$ implies $X_t'(t, s) > 0$ on some interval $[s, s + \sigma]$ for a certain $\sigma > 0$, the strict inequality $X(t, s) > 0$ holds for $t > s \geq t_0$.

2) The function $x(t) = X(t, t_0)$ is a positive solution of (8.2.1) with a nonnegative derivative. □

**Corollary 8.1** *Suppose $a_k(t) \le 0$ and there exist $t_0 \ge 0$ and a nonnegative locally absolutely continuous function $u$ such that the inequality*

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} b_k^+(t) \exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\} \le 0$$

*holds for $t \ge t_0$.*

*Then*:

1) $X(t, s) > 0,\ X_t'(t, s) \ge 0,\ t > s > t_0$.
2) *There exists a solution $x(t)$ of* (8.2.1) *such that $x(t) > 0,\ \dot{x}(t) \ge 0,\ t > t_0$.*

*Proof* We have to prove only positivity of the fundamental function of (8.3.2). The ordinary differential equation

$$\dot{z}(t) + u(t)z(t) = 0$$

has a positive fundamental function. Then inequality $a_k(t) \le 0$ and Theorem 2.3 imply that the fundamental function of (8.3.2) is positive, which completes the proof. □

We will demonstrate that condition 1) in Theorem 8.1 is necessary for nonoscillation of (8.2.1) with nonnegative coefficients. To this end, consider the delay differential inequality

$$\ddot{y}(t) + \sum_{k=1}^{r} a_k(t)\dot{y}\big(h_k(t)\big) + \sum_{k=1}^{m} b_k(t)y\big(g_k(t)\big) \le 0,\ \ t \ge t_0. \tag{8.3.11}$$

**Theorem 8.2** *Suppose $a_k(t) \ge 0$, $b_k(t) \ge 0$. If there exists $t_0 \ge 0$ such that inequality* (8.3.11) *has a positive solution with a nonnegative derivative for $t > t_0$, then there exists $t_1 \ge t_0$ such that inequality* (8.3.1) *has a nonnegative solution for $t \ge t_1$.*

*Proof* Let $y(t)$ be a positive solution of inequality (8.3.11) for $t > t_0$ with a nonnegative derivative. Then there exists a point $t_1$ such that $h_k(t) \ge t_0$ and $g_k(t) \ge t_0$ for $t \ge t_1$. We can assume without loss of generality that $y(t_1) = 1$.

Denote $u(t) = \frac{\dot{y}(t)}{y(t)}$ if $t \ge t_1$ and $u(t) = 0$ if $t < t_1$. Then $u$ is a nonnegative function locally absolutely continuous on $[t_1, \infty)$. The equalities $\dot{y}(t) - u(t)y(t) = 0$ and $y(t_1) = 1$ imply

$$\begin{aligned}
y(t) &= \exp\left\{\int_{t_1}^{t} u(s)ds\right\},\\
\dot{y}(t) &= u(t)\exp\left\{\int_{t_1}^{t} u(s)ds\right\},\\
\ddot{y}(t) &= \dot{u}(t)\exp\left\{\int_{t_1}^{t} u(s)ds\right\} + u^2(t)\exp\left\{\int_{t_1}^{t} u(s)ds\right\}.
\end{aligned} \tag{8.3.12}$$

Substituting (8.3.12) into (8.3.11), we obtain

$$\exp\left\{\int_{t_1}^{t} u(s)ds\right\}\left[\dot{u}(t)+u^2(t)+\sum_{k=1}^{r} a_k(t)u\big(h_k(t)\big)\exp\left\{-\int_{h_k(t)}^{t} u(s)ds\right\}\right.$$
$$\left.+\sum_{k=1}^{m} b_k(t)\exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\}\right]\le 0. \tag{8.3.13}$$

For $t \ge t_0$, we have $y(t)\ge 0$, $\dot{y}(t)\ge 0$, $a_k(t)\ge 0$, $b_k(t)\ge 0$. Consequently, the last two terms in (8.3.13) are positive. Therefore (8.3.13) implies inequality (8.3.1), which completes the proof. □

In the case $h_k(t)\equiv t$, necessary and sufficient nonoscillation conditions can be obtained as a corollary of Theorems 8.1 and 8.2. To this end, consider the equation

$$\ddot{x}(t)+a(t)\dot{x}(t)+\sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big)=0,\ t\ge 0 \tag{8.3.14}$$

and the delay differential inequality

$$\ddot{y}(t)+a(t)\dot{y}(t)+\sum_{k=1}^{m} b_k(t)y\big(g_k(t)\big)\le 0,\ t\ge 0. \tag{8.3.15}$$

For (8.3.14), condition 2) of Theorem 8.1 always holds.

**Corollary 8.2** *Suppose $a(t)\ge 0$, $b_k(t)\ge 0$, $k=1,\cdots,m$. Then, for (8.3.14), the following statements are equivalent:*

1) *There exists $t_1\ge 0$ such that inequality (8.3.15) has a positive solution with a nonnegative derivative for $t>t_1$.*
2) *There exists $t_2\ge 0$ such that the inequality*

$$\dot{u}(t)+u^2(t)+a(t)u(t)+\sum_{k=1}^{m} b_k(t)\exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\}\le 0 \tag{8.3.16}$$

*has a nonnegative solution locally absolutely continuous on $[t_2,\infty)$.*

3) *There exists $t_3\ge 0$ such that $X(t,s)>0$ and $X'_t(t,s)\ge 0$ for $t>s\ge t_3$.*
4) *There exists $t_4\ge 0$ such that (8.3.14) has a positive solution with a nonnegative derivative for $t>t_4$.*

*Remark 8.1* For equations without the damping term ($a(t)\equiv 0$), this result was obtained in the previous chapter.

## 8.4 Comparison Theorems

Theorem 8.1 can be employed for comparison of oscillation properties. To this end, consider together with (8.2.1) the equation with the same delays in the damping term

$$\ddot{x}(t)+\sum_{k=1}^{r} c_k(t)\dot{x}\big(h_k(t)\big)+\sum_{k=1}^{m} d_k(t)x\big(p_k(t)\big)=0,\ t\geq 0. \tag{8.4.1}$$

Suppose (a1) and (a2) hold for (8.4.1), and denote by $Y(t,s)$ the fundamental function of this equation.

**Theorem 8.3** *Suppose* $a_k(t)\geq 0$, $b_k(t)\geq 0$ *and the conditions of Theorem* 8.1 *hold for some* $t_0\geq 0$. *If*

$$c_k(t)\leq a_k(t),\ d_k(t)\leq b_k(t),\ p_k(t)\leq g_k(t),\ t\geq t_0, \tag{8.4.2}$$

*then* (8.4.1) *has a positive solution with a nonnegative derivative for* $t>t_0$ *and* $Y(t,s)>0$, $Y_t'(t,s)\geq 0$, $t>s>t_0$.

*Proof* By Theorem 8.1, there exists a nonnegative solution $u$ of the inequality (8.3.1) for $t\geq t_0$. Inequalities (8.4.2) yield that for $t\geq t_0$ the function $u$ is also a solution of the inequality

$$\dot{u}(t)+u^2(t)+\sum_{k=1}^{r} c_k^+(t)u\big(h_k(t)\big)\exp\left\{-\int_{h_k(t)}^{t} u(s)ds\right\}$$
$$+\sum_{k=1}^{m} d_k^+(t)\exp\left\{-\int_{p_k(t)}^{t} u(s)ds\right\}\leq 0.$$

By comparison, when Theorem 2.3 is applied to (8.3.2) and the equation

$$\dot{z}(t)+u(t)z(t)+\sum_{k=1}^{r} c_k(t)z\big(h_k(t)\big)=0, \tag{8.4.3}$$

the fundamental function of (8.4.3) is positive. Thus, Theorem 8.1 implies all the statements of this theorem. □

**Corollary 8.3** *If* $a(t)\geq 0$, $b_k(t)\geq 0$ *and the ordinary differential equation*

$$\ddot{y}(t)+a(t)\dot{y}(t)+\sum_{k=1}^{m} b_k(t)y(t)=0 \tag{8.4.4}$$

*has a positive solution with a nonnegative derivative, then* (8.3.14) *also has a positive solution with a nonnegative derivative for any* $g_k(t)\leq t$.

Let us now compare the solutions of (8.2.2), (8.2.3) and the problem

$$\ddot{y}(t)+\sum_{k=1}^{r} c_k(t)\dot{y}\big(h_k(t)\big)+\sum_{k=1}^{m} d_k(t)y\big(g_k(t)\big)=r(t),\ t\geq t_0, \tag{8.4.5}$$

$$y(t) = \bar{\varphi}(t),\ \dot{y}(t) = \bar{\psi}(t),\ t < t_0,\ y(t_0) = y_0,\ \dot{y}(t_0) = y_0'. \tag{8.4.6}$$

Denote by $x(t)$ and $X(t,s)$ the solution and the fundamental function of (8.2.2), (8.2.3) and by $y(t)$ and $Y(t,s)$ the solution and the fundamental function of (8.4.5), (8.4.6).

**Theorem 8.4** *Suppose that all the conditions of Theorem 8.1 hold, $x(t) > 0$, $\dot{x}(t) \ge 0$ for $t > t_0$ and $a_k(t) \ge c_k(t) \ge 0$, $b_k(t) \ge d_k(t) \ge 0$, $r(t) \ge f(t)$; $\varphi(t) \ge \bar{\varphi}(t)$, $\psi(t) \ge \bar{\psi}(t)$, $t < t_0$; $y_0 = x_0$, $y_0' \ge x_0'$.*

*Then $y(t) \ge x(t)$ for $t \ge t_0$ and $Y(t,s) \ge X(t,s) > 0$ for $t > s > t_0$.*

*Proof* Denote by $u$ a nonnegative solution of inequality (8.3.1). The inequalities $a_k(t) \ge c_k(t), b_k(t) \ge d_k(t)$ yield that the function $u$ is also a solution of the inequality corresponding to (8.3.1) for (8.4.5). By Theorem 2.3, the fundamental function of the equation corresponding to (8.3.2) is positive. Hence, by Theorem 8.1 the fundamental function $Y(t,s)$ of (8.4.5) is positive: $Y(t,s) > 0$, $t > s > t_0$.

Equation (8.2.2) can be rewritten in the form

$$\begin{aligned}\ddot{x}(t) &+ \sum_{k=1}^{r} c_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{m} d_k(t)x\big(g_k(t)\big)\\ &= -\sum_{k=1}^{r}\big[a_k(t) - c_k(t)\big]\dot{x}\big(h_k(t)\big) - \sum_{k=1}^{m}\big[b_k(t) - d_k(t)\big]x\big(g_k(t)\big) + f(t).\end{aligned}$$

Hence (see (8.2.5)), for the solutions of (8.2.2), (8.2.3) and (8.4.5), (8.4.6), we have

$$\begin{aligned}x(t) = y_1(t)x_0 + Y(t,t_0)x_0' &- \sum_{k=1}^{r}\int_{t_0}^{t} Y(t,s)\big[a_k(s) - c_k(s)\big]\dot{x}\big(h_k(s)\big)ds\\ &- \sum_{k=1}^{m}\int_{t_0}^{t} Y(t,s)\big[b_k(s) - d_k(s)\big]x\big(g_k(s)\big)ds\\ &- \sum_{k=1}^{r}\int_{t_0}^{t} Y(t,s)c_k(s)\psi\big(h_k(s)\big)ds\\ &- \sum_{k=1}^{m}\int_{t_0}^{t} Y(t,s)d_k(s)\varphi\big(g_k(s)\big)ds + \int_{t_0}^{t} Y(t,s)f(s)ds\end{aligned}$$

and

$$\begin{aligned}y(t) = y_1(t)y_0 + Y(t,t_0)y_0' &- \sum_{k=1}^{r}\int_{t_0}^{t} Y(t,s)c_k(s)\bar{\psi}\big(h_k(s)\big)ds\\ &- \sum_{k=1}^{m}\int_{t_0}^{t} Y(t,s)d_k(s)\bar{\varphi}\big(g_k(s)\big)ds + \int_{t_0}^{t} Y(t,s)r(s)ds,\end{aligned}$$

where $\varphi(g_k(s)) = \psi(h_k(s)) = \bar{\varphi}(g_k(s)) = \bar{\psi}(h_k(s)) = 0$ if $h_k(s) > t_0, g_k(s) > t_0$ and $x(g_k(s)) = 0$, $\dot{x}(h_k(s)) = 0$ if $g_k(t) < t_0$ or $h_k(s) < t_0$. Therefore $y(t) \geq x(t) > 0, t > t_0$.

Comparing the solutions of problem (8.2.4) and the corresponding problem for $Y(t,s)$, we see that $Y(t,s) \geq X(t,s)$ for $t > s > t_0$, which completes the proof. □

If in (8.2.2), (8.2.3) we have $b_k(t) \leq 0$ and there is no delay in the damping terms (i.e., $h_k(t) \equiv t$), then we can obtain a stronger comparison result. To this end, consider the two initial value problems

$$\begin{gathered}\ddot{x}(t) + a(t)\dot{x}(t) - \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = f(t),\ t \geq t_0,\\ x(t) = \varphi(t),\ t < t_0,\ x(t_0) = x_0,\ \dot{x}(t_0) = x'_0,\end{gathered} \tag{8.4.7}$$

and

$$\begin{gathered}\ddot{y}(t) + c(t)\dot{y}(t) - \sum_{k=1}^{m} d_k(t)y\big(g_k(t)\big) = r(t),\ t \geq t_0,\\ y(t) = \bar{\varphi}(t),\ t < t_0,\ y(t_0) = y_0,\ \dot{y}(t_0) = y'_0.\end{gathered} \tag{8.4.8}$$

Denote by $x(t)$, $y(t)$ and $X(t,s)$, $Y(t,s)$ solutions and fundamental functions of (8.4.7) and (8.4.8), respectively.

**Theorem 8.5** *Let*

$$\begin{gathered}a(t) \geq c(t), d_k(t) \geq b_k(t) \geq 0,\\ \bar{\varphi}(t) \geq \varphi(t) \geq 0,\ r(t) \geq f(t) \geq 0,\ y_0 \geq x_0 > 0,\ y'_0 \geq x'_0 \geq 0.\end{gathered}$$

*Then* $y(t) \geq x(t) > 0$ *for* $t \geq t_0$ *and* $Y(t,s) \geq X(t,s) > 0$ *for* $t > s > t_0$.

*Proof* Inequality (8.3.1) for (8.4.7) has the form

$$\dot{u}(t) + u^2(t) + a^+(t)u(t) \leq 0. \tag{8.4.9}$$

Straightforward calculations imply that the function

$$u(t) = \frac{\exp\{-\int_{t_0}^{t} a^+(s)ds\}}{\int_{t_0}^{t}\exp\{-\int_{t_0}^{s} a^+(\tau)d\tau\}ds}$$

is a nonnegative solution of the equation corresponding to inequality (8.4.9).

Then, for the equation in (8.4.7) and similarly for the equation in (8.4.8), we have $X(t,s) > 0$, $Y(t,s) > 0$ for $t > s > t_0$.

Next, let us compare solutions $x$ of (8.4.7) and solution $z$ of the problem

$$\ddot{z}(t) + a(t)\dot{z}(t) = f(t),\ z(t_0) = x_0,\ \dot{z}(t_0) = x'_0. \tag{8.4.10}$$

Since $z(t)$ can be represented as

$$z(t) = x_0 + \int_{t_0}^{t}\left[\exp\left\{-\int_{t_0}^{s} a(\tau)d\tau\right\}x'_0 + \int_{t_0}^{s}\exp\left\{-\int_{\zeta}^{s} a(\tau)d\tau\right\}f(\zeta)d\zeta\right]ds,$$

we have $z(t) > 0$ for $t \geq t_0$.

After rewriting (8.4.10) in the form

$$\ddot{z}(t) + a(t)\dot{z}(t) - \sum_{k=1}^{m} b_k(t) z\big(g_k(t)\big) = -\sum_{k=1}^{m} b_k(t) z\big(g_k(t)\big) + f(t),$$

for the solutions of problems (8.4.7) and (8.4.10) we have

$$x(t) = x_1(t)x_0 + X(t,t_0)x_0' + \sum_{k=1}^{m} \int_{t_0}^{t} X(t,s) b_k(s) \varphi\big(g_k(s)\big) ds + \int_{t_0}^{t} X(t,s) f(s) ds,$$

$$z(t) = x_1(t)x_0 + X(t,t_0)x_0' + \sum_{k=1}^{m} \int_{t_0}^{t} X(t,s) b_k(s) \varphi\big(g_k(s)\big) ds - \sum_{k=1}^{m} \int_{t_0}^{t} X(t,s) b_k(s) z\big(g_k(s)\big) ds + \int_{t_0}^{t} X(t,s) f(s) ds.$$

Hence $x(t) \ge z(t) > 0$ for $t > t_0$ and as a consequence $x_1(t) > 0$. Similarly, $y(t) > 0$, $y_1(t) > 0$.

Finally, the same computation as in the proof of the previous theorem implies the statement of the theorem. □

**Corollary 8.4** *Suppose that $b_k(t) \le 0$ and $x$ and $y$ are solutions of (8.3.14) and inequality (8.3.15), respectively, such that $x(t) = y(t)$ for $t \le t_0$ and $\dot{x}(t_0) = \dot{y}(t_0)$. Then $x(t) \ge y(t)$ for $t \ge t_0$.*

The proof is based on solution representation (8.2.5) and the inequality $X(t,s) > 0$ satisfied for $t > s > 0$.

## 8.5 Explicit Nonoscillation Conditions

We will apply Theorem 8.1 to obtain explicit sufficient nonoscillation conditions.

**Theorem 8.6** *Suppose the following conditions hold: $h_k(t) > 0$ and $g_k(t) \ge 0$ for $t \ge t_0$,*

$$\frac{1}{2} \sum_{k=1}^{r} a_k^+(t) \sqrt{\frac{t^3}{h_k(t)}} + \sum_{k=1}^{m} b_k^+(t) \sqrt{t^3 g_k(t)} \le \frac{1}{4}, \quad t \ge t_0, \tag{8.5.1}$$

$$\sum_{k=1}^{m} \int_{h(t)}^{t} a_k^+(\tau) \sqrt{\frac{\tau}{h_k(\tau)}} d\tau \le \frac{1}{e}, \quad t \ge t_0, \tag{8.5.2}$$

*where $h(t) = \min_k \{h_k(t)\}$.*

*Then (8.2.1) has a positive solution with a nonnegative derivative for $t > t_0$.*

*Proof* Let $u = \frac{1}{2t}$. Then inequality (8.3.1) takes the form

$$-\frac{1}{4t^2} + \frac{1}{2}\sum_{k=1}^{r} \frac{a_k^+(t)}{h_k(t)}\sqrt{\frac{h_k(t)}{t}} + \sum_{k=1}^{m} b_k^+(t)\sqrt{\frac{g_k(t)}{t}} \le 0,$$

which is equivalent to inequality (8.5.1).

Equation (8.3.2) with $u = \frac{1}{2t}$ becomes

$$\dot{z}(t) + \frac{1}{2t}z(t) + \sum_{k=1}^{r} a_k^+(t)z\big(h_k(t)\big) = 0. \tag{8.5.3}$$

Substituting $z(t) = \frac{v(t)}{\sqrt{t}}$ in (8.5.3), we obtain

$$\dot{v}(t) + \sum_{k=1}^{r} a_k(t)\sqrt{\frac{t}{h_k(t)}}v\big(h_k(t)\big) = 0. \tag{8.5.4}$$

Condition (8.5.2) and Theorem 2.7 yield that the fundamental function of (8.5.4), and therefore of (8.5.3), is positive. Theorem 8.1 implies the statement of the theorem. □

**Corollary 8.5** *If the inequality*

$$\frac{1}{2}a^+(t)t + \sum_{k=1}^{m} b_k^+(t)\sqrt{t^3 g_k(t)} \le \frac{1}{4}, \ t \ge t_0$$

*holds, then* (8.3.14) *has a positive solution with a nonnegative derivative for* $t > t_0$.

## 8.6 Discussion and Open Problems

This chapter deals with nonoscillation problems for a scalar linear delay differential equation of the second order. Such equations attract attention of many mathematicians due to their significance in applications. We mention here the monographs of A.D. Myshkis [289], S.B. Norkin [292], G.S. Ladde, V. Lakshmikantham and B.G. Zhang [248], I. Győri and G. Ladas [192], L.N. Erbe, Q. Kong and B.G. Zhang [154] and references therein. The monographs contain examples of physical models leading to equations of the type

$$\ddot{x}(t) + \sum_{k=1}^{r} a_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = f(t), \ h_k(t) \le t, \ g_k(t) \le t.$$

We can also mention here the paper [164], where the author considered positivity conditions for the fundamental function of the equation

$$\ddot{x}(t) + 2c_0(t)\dot{x}(t) + c_1(t)\dot{x}(t-h) + d_0(t)x(t) + d_1(t)x(t-h) + d_2(t)u(t-2h) = 0,$$

where all the coefficients are continuous functions.

The term with the first derivative is usually called "a damping term". Most publications deal with equations that do not contain the term with the first derivative. For these equations, positivity of the coefficients and a solution on the semiaxis implies that its derivative is nonnegative. This fact is very important; it is employed in most investigations on second-order delay differential equations. If the first derivative is included in the equation explicitly (i.e., the equation contains the damping term), then a sign of a solution does not uniquely define the sign of its derivative. Therefore the study of oscillation properties of the equations with the damping term is more complicated. This is the reason why such equations are much less studied than the equations without the damping term. The particular cases were considered where the damping term is not delayed (see, for example, papers [218, 230]) and where the delay is constant [172, 173].

In this chapter, we considered the general class of equations containing the damping term with deviating argument and studied properties of these equations concerned with nonoscillation. The main result is that if a generalized Riccati inequality (8.3.1) has a nonnegative solution for $t \geq t_0$, then the differential equation for $t \geq t_0$ has a positive solution with a nonnegative derivative and the fundamental function of this equation is positive. If the damping term is not delayed, this immediately yields that the following four properties are equivalent: nonoscillation of solutions of this equation and the corresponding differential inequality, positivity of the fundamental function and existence of a nonnegative solution of a generalized Riccati inequality.

We employed a generalized Riccati inequality to compare oscillation properties of two equations without comparing their solutions. One can treat these results as a natural generalization of the well-known Sturm comparison theorem for a second-order ordinary differential equation.

By applying positivity of the fundamental function, we compare positive solutions of two nonoscillatory equations.

The chapter also contains explicit nonoscillation conditions that are obtained by substituting specific solutions of the generalized Riccati inequality. The results of the chapter were published in the papers [45, 46].

Another approach that is based on comparing the oscillation properties of the delay differential equation of second order

$$\ddot{x}(t) + a(t)\dot{x}(t) + b(t)x\big(h(t)\big) = 0 \tag{8.6.1}$$

and the integrodifferential equation

$$\dot{y}(t) + \int_{t_0}^{t} e^{-\int_s^t a(\xi)d\xi} b(s)y\big(h(s)\big)ds = 0$$

is applied in [77]. In particular, the following result was obtained.

**Theorem 8.7** [77] *Let $t - h(t) \leq \delta$, $\delta > 0$, $t \geq t_0$, and let there exist positive numbers $\lambda$, $a_0$ such that at least one of the following conditions holds*:

1) $a(t) - \lambda \geq a_0$, $\lambda \geq e^{\lambda\delta} \| \frac{b^+(t)}{a(t)-\lambda} \|$;

2) $\lambda < 1$, $a(t) \geq a_0/(1-\lambda)$, $\lambda(1-\lambda)a(t) \geq e^{\lambda \|a(t)\| \delta} \| \frac{b^+(t)}{a(t)} \|$.

*Then the fundamental function and the fundamental system of* (8.6.1) *are positive for* $t > s \geq t_0$.

A connection between nonoscillation and stability for an ordinary differential equation of the second order with damping was studied in [72]. Stability results for (8.2.1), which are based on the positivity of the fundamental functions of some auxiliary delay differential equations of the first order, were obtained in [73].

Below we present some open problems and topics for research and discussion.

1. Derive oscillation properties for (8.2.1).
2. Obtain oscillation and nonoscillation properties for the following equations with damping that have not been studied in this chapter:

   the integrodifferential equation

$$\ddot{x}(t) + \int_{-\infty}^{t} K(t,s)\dot{x}(s)ds + \int_{-\infty}^{t} L(t,s)x(s)ds = 0;$$

   the differential equation with a distributed delay

$$\ddot{x}(t) + \int_{-\infty}^{t} \dot{x}(s)d_s R(t,s) + \int_{-\infty}^{t} x(s)d_s T(t,s) = 0;$$

   various mixed differential equations that contain both concentrated delay and integral terms.

3. Is it possible to obtain nonoscillation conditions for equations with a damping term and a combination of positive and negative coefficients (or an oscillating coefficient) when (8.3.1) is not satisfied? In other words, is it possible to obtain an analogue of (8.3.1) that includes both positive and negative terms?
4. Study differential equations with damping that have positive solutions with nonpositive derivatives. What is the asymptotic behavior of such solutions?

# Chapter 9
# Vector Delay Differential Equations

## 9.1 Introduction

This chapter deals with nonoscillation of systems of delay differential equations. There are several different nonoscillation definitions for such systems. In [192], a system is called nonoscillatory if there exists a solution for which at least *one component* is eventually positive. In [167], and in some results in [192], a nonoscillatory system by definition has a solution for which *all components* are eventually positive.

Due to their applications in stability and boundary value problems, we will be interested in nonnegativity of the fundamental matrix for systems of linear delay differential equations. This means that all entries of this matrix are nonnegative functions, which will imply the existence of a solution with positive components. For ordinary differential equations, nonnegativity of the fundamental matrix is equivalent to the following classical result of Wazewski.

*For solutions of the vector differential equation*

$$X'(t) + A(t)X(t) = 0$$

*and the vector differential inequality*

$$Y'(t) + A(t)Y(t) \le 0,$$

*where $X(t_0) = Y(t_0)$, the inequality $Y(t) \le X(t)$ holds if and only if $a_{ij} \le 0$, $i \ne j$,* where $a_{ij}$, $i, j = 1, \cdots, n$ are the entries of the matrix $A$.

We extend the sufficient part of this result to vector delay differential equations. We also consider the related problems of comparison of nonoscillation properties and comparison of solutions, existence of positive solutions, estimations of the fundamental matrix and the connection between nonoscillation and asymptotic stability. For scalar delay differential equations, these topics are well studied. Some of them have also been investigated for systems of delay equations: positivity of the fundamental matrix and comparison results in [139], estimation of solutions in [224] and the connection between nonoscillation and stability in [31, 36]. In this chapter, we present results on all problems mentioned above. In particular, positivity of

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the fundamental matrix implies exponential stability of the vector delay differential equation under some quite natural restrictions.

Section 9.2 contains relevant definitions and a solution representation formula. Section 9.3 deals with nonnegativity of the fundamental matrix for the vector delay equation. Section 9.4 includes some comparison results: nonoscillation properties of two delay equations and solutions of these equations are compared. In Sect. 9.5, previous results are applied to a higher-order scalar delay differential equation. Section 9.6 establishes conditions on the initial function and the initial value that imply positivity of the solution for the initial value problem. An upper estimate for entries of the fundamental matrix is also presented in Sect. 9.6. Section 9.7 demonstrates that under some natural conditions an equation with a nonnegative fundamental matrix is exponentially stable. We also give an instability condition based on comparison results. In Sect. 9.8, equations with distributed delays are considered. Finally, in Sect. 9.9, the results of the chapter are discussed and some open problems are stated.

## 9.2 Preliminaries

We consider for $t \geq 0$ systems of linear delay differential equations in two equivalent forms: the system of scalar equations

$$\dot{x}_i(t) + \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}^k(t) x_j\big(h_{ij}^k(t)\big) = 0, \; i = 1, \cdots, n \tag{9.2.1}$$

and the vector equation

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = 0, \tag{9.2.2}$$

where $A_k(t)$ are $n \times n$ matrices with entries $a_{ij}^k$, $i, j = 1, \cdots, n$, $k = 1, \cdots, m$.

System (9.2.1) can be rewritten in the form (9.2.2), but due to applications we will formulate conditions for vector equation (9.2.2) and scalar system (9.2.1) separately.

We consider vector delay differential equation (9.2.2) and scalar system (9.2.1) under the following conditions:

(a1) Coefficients $a_{ij}^k$ are Lebesgue measurable locally essentially bounded functions.

(a2) Delays $h_k, h_{ij}^k : [0,\infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $h_{ij}^k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $\lim_{t\to\infty} h_{ij}^k(t) = \infty$, $k = 1, \cdots, m$, $i, j = 1, \cdots, n$.

Together with (9.2.2), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = F(t), \; t \geq t_0, \tag{9.2.3}$$

$$X(t)=\Phi(t),\ t<t_0,\ X(t_0)=X_0, \tag{9.2.4}$$

where $\Phi(t)=[\varphi_1(t),\cdots,\varphi_n(t)]^T$ and $F$ satisfies the following hypothesis:

(a3) $F:[t_0,\infty)\to\mathbb{R}^n$, where $F(t)=[f_1(t),\cdots,f_n(t)]^T$ is a Lebesgue measurable locally essentially bounded function and $\Phi:(-\infty,t_0)\to\mathbb{R}^n$ is a Borel measurable bounded function. Here $A^T$ is the transposed matrix.

**Definition 9.1** An absolutely continuous function $X:\mathbb{R}\to\mathbb{R}^n$ on each interval $[t_0,b]$ is called *a solution* of problem (9.2.3), (9.2.4) if it satisfies (9.2.3) for almost all $t\in[t_0,\infty)$ and equalities (9.2.4) for $t\le t_0$.

In addition to problem (9.2.3), (9.2.4) where $X$, $F$ and $\Phi$ are column vector functions, we will consider this problem where $F$, $\Phi$ and $X$ are $n\times n$ matrix functions.

**Definition 9.2** For each $s\ge 0$, the solution $C(t,s)$ of the matrix problem

$$\dot{X}(t)+\sum_{k=1}^{m}A_k(t)X\big(h_k(t)\big)=0,\ X(t)=0,\ t<s,\ X(s)=I, \tag{9.2.5}$$

is called *the fundamental matrix* (*or the Cauchy matrix*) of (9.2.2), where $C(t,s)$ is an $n\times n$ matrix function and $I$ is the identity matrix.

By 0 we will also denote the zero column vector and the zero matrix. We assume $C(t,s)=0$, $0\le t<s$. By Theorem B.1, we have the following result.

**Lemma 9.1** *Let* (a1)–(a3) *hold. Then there exists one and only one solution of problem* (9.2.3), (9.2.4), *and it can be presented in the form*

$$X(t)=C(t,t_0)X_0+\int_{t_0}^{t}C(t,s)F(s)ds-\sum_{k=1}^{m}\int_{t_0}^{t}C(t,s)A_k(s)\Phi\big(h_k(s)\big)ds, \tag{9.2.6}$$

*where* $\Phi(h_k(s))=0$, *if* $h_k(s)>t_0$.

We will write $X\ge 0$, $A\ge 0$ if all entries of vector $X$ or matrix $A$ are nonnegative. Let us proceed to system (9.2.1). Together with (9.2.1), we consider for each $t_0\ge 0$ the initial value problem

$$\dot{x}_i(t)+\sum_{k=1}^{m}\sum_{j=1}^{n}a_{ij}^k(t)x_j\big(h_{ij}^k(t)\big)=f_i(t),\ i=1,\cdots,n,\ t\ge t_0, \tag{9.2.7}$$

$$x_i(t)=\varphi_i(t),\ t<t_0,\ x_i(t_0)=x_i^0, \tag{9.2.8}$$

where $\varphi_i(t)$ and $f_i$ for each $i=1,\cdots,n$ satisfy the following hypothesis:

$f_i:[t_0,\infty)\to\mathbb{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi_i:(-\infty,t_0)\to\mathbb{R}$ is a Borel measurable bounded function.

**Definition 9.3** A set of functions $x_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \cdots, n$, absolutely continuous on each interval $[t_0, b]$ is called *a solution* of problem (9.2.7), (9.2.8) if $x_i$, $i = 1, \cdots, n$ satisfy (9.2.7) for almost all $t \in [t_0, \infty)$ and equalities (9.2.8) for $t \le t_0$.

Denote $X(t) = [x_1(t), \cdots, x_n(t)]^T$, and let $A_{ij}^k$ be an $n \times n$ matrix with the only nonzero entry $a_{ij}^k$. Then (9.2.1) can be rewritten in the vector form

$$\dot{X}(t) + \sum_{k=1}^{m} \sum_{i,j=1}^{n} A_{ij}^k(t) X\big(h_{ij}^k(t)\big) = 0. \tag{9.2.9}$$

The fundamental matrix $C(t, s)$ of (9.2.9) will be called the fundamental matrix of system (9.2.1). Denote $X_0 = [x_1^0, \cdots, x_n^0]^T$, $\Phi(t) = [\varphi_1(t), \cdots, \varphi_n(t)]^T$, $F(t) = [f_1(t), \cdots, f_n(t)]^T$. By Lemma 9.1, the solution of (9.2.7), (9.2.8) can be presented as

$$X(t) = C(t, t_0) X_0 + \int_{t_0}^{t} C(t, s) F(s) ds - \sum_{k=1}^{m} \sum_{i,j=1}^{n} \int_{t_0}^{t} C(t, s) A_{ij}^k(s) \Phi\big(h_{ij}^k(s)\big) ds, \tag{9.2.10}$$

where $\Phi(h_{ij}^k(s)) = 0$, if $h_{ij}^k(s) > t_0$.

## 9.3 Main Results

**Theorem 9.1** *Suppose $a_{ij}^k(t) \le 0$, $i \ne j$, $k = 1, \cdots, m$, $t \ge t_0$ and the fundamental functions $X_i(t, s)$ of the scalar equations*

$$\dot{x}(t) + \sum_{k=1}^{m} \big[a_{ii}^k(t)\big]^+ x\big(h_k(t)\big) = 0,\ \ i = 1, \cdots, n, \tag{9.3.1}$$

*are positive for $t \ge s \ge t_0$. Then, for the fundamental matrix $C(t, s)$ of the system* (9.2.2), *we have $C(t, s) \ge 0$, $t \ge s \ge t_0$.*

*Proof* Consider first the case $a_{ii}^k \ge 0$. After introducing diagonal matrices $B_k(t) = \operatorname{diag}\{a_{11}^k, \cdots, a_{nn}^k\}$ and defining $D_k(t) = A_k(t) - B_k(t)$, $k = 1, \cdots, m$, it is evident that for the entries $d_{ij}^k$ of $D_k(t)$ we have $d_{ij}^k \le 0$, $i \ne j$, $d_{ii}^k = 0$. Denote by $Y(t, s)$ the fundamental matrix of the system

$$\dot{Y}(t) + \sum_{k=1}^{m} B_k(t) Y\big(h_k(t)\big) = 0.$$

$B_k(t)$ are diagonal matrices, and thus $Y(t, s) = \operatorname{diag}\{X_1(t, s), \cdots, X_n(t, s)\}$, where $X_i(t, s)$ are the fundamental functions of scalar equations (9.3.1), $i = 1, \cdots, n$ and $X_i$ are positive for $t \ge s \ge t_0$. Then $Y(t, s) \ge 0$ for $t \ge s \ge t_0$.

Consider the problem

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t)X\big(h_k(t)\big) = F(t),\ t \geq t_0,\ X(t) = 0,\ t \leq t_0, \tag{9.3.2}$$

which can be rewritten in the form

$$\dot{X}(t) + \sum_{k=1}^{m} B_k(t)X\big(h_k(t)\big) + \sum_{k=1}^{m} D_k(t)X\big(h_k(t)\big) = F(t),\ t \geq t_0,$$
$$X(t) = 0,\ t \leq t_0.$$

Hence, by (9.2.6), for the solution of (9.3.2), we have

$$X(t) = \int_{t_0}^{t} Y(t,s)F(s)ds - \int_{t_0}^{t} Y(t,s)\sum_{k=1}^{m} D_k(s)X\big(h_k(s)\big)ds. \tag{9.3.3}$$

If we introduce operator $H$ as

$$(HX)(t) = -\int_{t_0}^{t} Y(t,s)\sum_{k=1}^{m} D_k(s)X\big(h_k(s)\big)ds,\ \text{where } X\big(h_k(s)\big) = 0,\ h_k(s) \leq t_0,$$

then system (9.3.3) has the form

$$X - HX = G, \tag{9.3.4}$$

where $G(t) = \int_{t_0}^{t} Y(t,s)F(s)ds \geq 0$ if $F(t) \geq 0$. Let $L_\infty[t_0, b]$ be the space of functions essentially bounded on $[t_0, b]$ with the essential supremum norm for every $b > t_0$. Then (Theorem A.4) $H : L_\infty[t_0, b] \to L_\infty[t_0, b]$ is a sum of weakly compact integral Volterra operators, and hence by Theorem A.7 its spectral radius equals 0: $r(H) = 0 < 1$.

If $X \geq 0$, then $HX \geq 0$ (i.e., $H$ is positive). Thus, for a solution of (9.3.4), we have $X = (I - H)^{-1}G \geq 0$ for $G \geq 0$ since $(I - H)^{-1} = I + H + H^2 + \cdots$. Hence, for any $F \geq 0$ for the solution of (9.3.2), we have $X(t) \geq 0$. The solution representation $X(t) = \int_{t_0}^{t} C(t,s)F(s)ds$ implies $C(t,s) \geq 0$, $t \geq s \geq 0$.

In the general case, we can write $B_k(t) = B_k^+(t) - B_k^-(t)$, where

$$B_k^+(t) = \operatorname{diag}\big\{\big(a_{11}^k\big)^+, \cdots, \big(a_{nn}^k\big)^+\big\},\ B_k^-(t) = B_k^+(t) - B_k(t),$$

and prove the statement similarly. □

*Remark 9.1* Theorem 9.1 was first proven in [139] with a different method.

**Corollary 9.1** *Suppose* $a_{ij}^k(t) \leq 0$, $i \neq j$, $t \geq t_0$ *and*

$$\sup_{t \geq t_0} \int_{\max\{t_0, \min_k h_k(t)\}}^{t} \sum_{k=1}^{m} \big[a_{ii}^k(s)\big]^+ ds \leq \frac{1}{e},\ i = 1, \cdots, n. \tag{9.3.5}$$

*Then the fundamental matrix of system* (9.2.2) *satisfies* $C(t,s) \geq 0$, $t \geq s \geq t_0$.

**Theorem 9.2** *Suppose $a_{ij}^k(t) \le 0$, $i \ne j$, $k = 1, \cdots, m$, $t \ge t_0$, and the fundamental functions $X_i(t,s)$ of the scalar equations*

$$\dot{x}(t) + \sum_{k=1}^{m} \left[a_{ii}^k(t)\right]^+ x\left(h_{ii}^k(t)\right) = 0, \; i = 1, \cdots, n, \tag{9.3.6}$$

*are positive for $t \ge s \ge t_0$. Then, for the fundamental matrix of the system (9.2.1), we have $C(t,s) \ge 0$, $t \ge s \ge t_0$.*

*Proof* The proof follows from Theorem 9.1 since for system (9.2.9) equations (9.3.1) have the form (9.3.6). □

**Corollary 9.2** *Suppose $a_{ij}(t) \le 0$, $i \ne j$, $t \ge t_0$ and*

$$\sup_{t \ge t_0} \int_{\max\{t_0, \min_k h_{ii}^k(t)\}}^{t} \sum_{k=1}^{m} \left[a_{ii}^k(s)\right]^+ ds \le \frac{1}{e}, \; i = 1, \cdots, n. \tag{9.3.7}$$

*Then the fundamental matrix of the system (9.2.1) satisfies $C(t,s) \ge 0$, $t \ge s \ge t_0$.*

Unlike systems of ordinary differential equations, the condition $a_{ij}^k(t) \le 0$, $i \ne j$, $t \ge t_0$ generally is not necessary for positivity of the fundamental matrix of a system of linear delay equations, as the following example demonstrates.

*Example 9.1* Consider the system

$$\begin{aligned} \dot{x}(t) + x(t) &= 0, \\ \dot{y}(t) - 3x(t) + y(t) &= 0, \\ \dot{z}(t) + e^{-3}x(t-3) - 3y(t) + z(t) &= 0. \end{aligned} \tag{9.3.8}$$

Denote $X(t,s)$, $t \ge s \ge t_0$ the fundamental matrix of this system. By simple calculations, we have the following structure of $C(t,s)$. The first row of $C(t,s)$ is $(e^{-(t-s)}, 0, 0)$. The second row is $(3(t-s)e^{-(t-s)}, e^{-(t-s)}, 0)$. The third row is $(4.5(t-s)^2 e^{-(t-s)}, 3(t-s)e^{-(t-s)}, e^{-(t-s)})$ for $t \in [s, s+3]$ and $([4.5(t-s)^2 - (t-s) + 3]e^{-(t-s)}, 3(t-s)e^{-(t-s)}, e^{-(t-s)})$ for $t \in [s+3, \infty)$. So $C(t,s) \ge 0$, while one of the nondiagonal coefficients is positive.

However, there are at least two types of delay systems for which the condition $a_{ij}^k(t) \le 0$, $i \ne j$, $t \ge t_0$ is necessary. For the first type, all nondiagonal terms have no delay; such systems can be rewritten in the form

$$\dot{X}(t) + A_0(t)X(t) + \sum_{k=1}^{m} A_k(t)X\left(h_k(t)\right) = 0, \tag{9.3.9}$$

where $a_{ii}^0(t) = 0$, $a_{ij}^k(t) = 0$, $i \ne j$, $k = 1, \cdots, m$.

**Theorem 9.3** *Suppose that the fundamental functions of the scalar equations*

$$\dot{x}(t) + \sum_{k=1}^{m} \left[a_{ii}^{k}(t)\right]^{+} x\big(h_k(t)\big) = 0, \; i = 1, \cdots, n, \tag{9.3.10}$$

*are positive for* $t \geq s \geq t_0$. *If there exists a nonnegative coefficient* $a_{ij}^{0}(t) \geq 0$, $i \neq j$ *such that* $a_{ij}^{0}(t) \geq a_0 > 0$ *in some interval, then the fundamental matrix* $C(t,s)$ *of system* (9.3.9) *is not nonnegative.*

*Proof* The assumptions of the theorem imply that the fundamental functions $X_i(t,s)$ of the equations

$$\dot{x}(t) + a_{ii}^{0}(t)x(t) + \sum_{k=1}^{m} a_{ii}^{k}(t)x\big(h_k(t)\big) = 0, \; i = 1, \cdots, n,$$

are also positive for $t \geq s \geq t_0$. Let us assume the contrary to the statement of the theorem: $C(t,s) \geq 0$ for $t \geq s \geq t_0$, and there exist a pair of indices $(i,j)$ and a number $t_1 > t_0$ such that $a_{ij}^{0}(t) \geq a_0 > 0$ for $t \in [t_0, t_1]$; without loss of generality, we can assume $j = 1$. Coefficients $a_{ij}^{0}$ are locally bounded, so for some $\alpha > 0$ we have $|a_{ij}^{0}(t)| \leq \alpha$, $(i,j) \neq (n,1)$, $t_0 \leq t \leq t_1$.

Consider now the solution $X(t)$ of (9.3.9) with initial conditions $X(t) = 0$, $t < t_0$, $X(t_0) = B := [1, 0, \cdots, 0]^T$, which is the first column of the fundamental matrix $C(t, t_0)$. For the solution of this problem $X = [x_1, \cdots, x_n]^T$, we have $X(t) = C(t,t_0)B \geq 0$. Let us choose $\delta > 0$ satisfying $\delta < \frac{a_0}{a_0 + \alpha(n-2)}$. Then there exists $t_2 \in (t_0, t_1)$ such that for $t \in [t_0, t_2]$ we have $x_1(t) > 1 - \delta$, $x_j(t) < \delta$, $j \neq 1$, which implies

$$\sum_{j \neq n} a_{nj}^{0}(t)x_j(t) \geq a_0(1-\delta) - \alpha\delta(n-2) > 0.$$

The last equation in system (9.3.9) has the form

$$\dot{x}_n(t) + \sum_{k=1}^{m} a_{nn}^{k}(t)x_n\big(h_k(t)\big) = -\sum_{j \neq n} a_{nj}^{0}(t)x_j(t).$$

Hence

$$x_n(t) = -\int_{t_0}^{t} X_n(t,s) \sum_{j \neq n} a_{nj}^{0}(s)x_j(s)\,ds < 0, \; t_0 \leq t \leq t_2,$$

which contradicts nonnegativity of the $n$-th component $x_n$ of the first column of the fundamental matrix for $t_0 \leq t \leq t_2$. □

The second case where the condition $a_{ij}^{k}(t) \leq 0$, $i \neq j$, $t \geq t_0$ becomes necessary is the delay system of two equations with constant delays of nondiagonal terms. For simplicity, consider the system

$$\begin{cases} \dot{x_1}(t) = -a_{11}(t)x_1\big(h_{11}(t)\big) - a_{12}(t)x_2(t - \tau_{12}), \\ \dot{x_2}(t) = -a_{21}(t)x_1(t - \tau_{21}) - a_{22}(t)x_2\big(h_{22}(t)\big), \end{cases} \tag{9.3.11}$$

where $\tau_{ij} \geq 0$.

**Theorem 9.4** *Suppose the fundamental functions of the scalar equations*

$$\dot{x}(t) + [a_{ii}(t)]^+ x(h_{ii}(t)) = 0, \; i = 1, 2 \tag{9.3.12}$$

*are positive for* $t \ge s \ge t_0$. *If there exists a nonnegative nondiagonal coefficient* $a_{ij}(t) \ge 0$ $(i \ne j)$ *such that* $a_{ij}(t) \ge a_0 > 0$ *in some interval* $[t_0, t_1]$, $t_1 - t_0 > \max\{\tau_{ij}\}$, *then the fundamental matrix* $C(t, s)$ *of the system* (9.3.11) *is not a nonnegative function.*

*Proof* Denote by $X_i(t, s)$ the fundamental functions of the equations

$$\dot{x}(t) + a_{ii}(t)x(h_{ii}(t)) = 0, \; i = 1, 2. \tag{9.3.13}$$

Then $X_i(t, s) > 0$, $t \ge s \ge t_0$.

Let us assume the contrary to the statement of the theorem: $C(t, s) \ge 0$ for any $t \ge s \ge t_0$, while for some $t_1 > t_0$ we have $a_{ij}(t) \ge a_0 > 0$ for $t \in [t_0, t_1]$, by the assumption of the theorem; without loss of generality, we can assume $(i, j) = (2, 1)$.

Consider now the solution $X(t)$ of (9.3.11) with the initial conditions $X(t) = 0$, $t < t_0$, $X(t_0) = B := [1, 0]^T$, which is the first column of the fundamental matrix $C(t, t_0)$. For the solution of this problem $X = [x_1, x_2]^T$, we have $X(t) = C(t, t_0)B \ge 0$. There exist an interval $[t_0, t_2]$ and a number $\delta > 0$ such that $x_1(t) \ge 1 - \delta > 0$, $t_0 \le t \le t_2 \le t_0 + \tau_{21}$. The second equation in system (9.3.11) has the form

$$\dot{x}_2(t) + a_{22}(t)x_2(h_{22}(t)) = -a_{21}(t)x_1(t - \tau_{21}),$$

where $x_i(t) = 0$, $t < t_0$, $x_2(t_0) = 0$. Then

$$\begin{aligned} x_2(t_1) &= -\int_{t_0}^{t_1} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds \\ &= -\int_{t_0}^{t_0+\tau_{21}} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds \\ &\quad - \int_{t_0+\tau_{21}}^{t_2+\tau_{21}} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds \\ &\quad - \int_{t_2+\tau_{21}}^{t_1} X_2(t, s)a_{21}(s)x_1(s - \tau_{21})ds =: I_1 + I_2 + I_3. \end{aligned}$$

It is evident that $I_1 = 0$, $I_3 \le 0$. In the second integral, $t_0 \le s - \tau_{21} \le t_2$. Hence, in this integral $x_1(s - \tau_{21}) \ge 1 - \delta > 0$. Then $I_2 < 0$, and so $x_2(t_1) < 0$, which contradicts our assumption. □

## 9.4 Comparison Results

Now we can compare two solutions of system (9.2.3) of differential equations.

**Theorem 9.5** *Suppose the conditions of Theorem 9.1 hold, $X(t)$ is a solution of problem (9.2.3), (9.2.4) and $Y(t)$ is a solution of the same problem, where the function $F(t)$ is replaced by $G(t)$. If $G(t) \le F(t)$ for $t \ge t_0$, then $Y(t) \le X(t)$ for $t \ge t_0$.*

The proof follows from solution representation (9.2.6) and inequality $C(t,s) \ge 0$ for $t \ge s \ge t_0$.

**Theorem 9.6** *Suppose the conditions of Theorem 9.1 hold, $X(t)$ is a solution of (9.2.2), $Y(t)$ is a solution of the differential inequality*

$$\dot{Y}(t) + \sum_{k=1}^{m} A_k(t) Y\big(h_k(t)\big) \le 0, \ \ t \ge t_0, \tag{9.4.1}$$

*and $X(t) = Y(t)$, $t \le t_0$. Then $Y(t) \le X(t)$ for $t \ge t_0$.*

Now we proceed to comparison results for system (9.2.1).

**Theorem 9.7** *Suppose the conditions of Theorem 9.2 hold, $x_i(t)$ is a solution of problem (9.2.7), (9.2.8) and $y_i(t)$ is a solution of the same problem, where function $f_i(t)$ is replaced by $g_i(t)$. If $g_i(t) \le f_i(t)$ for $t \ge t_0$, then $y_i(t) \le x_i(t)$ for $t \ge t_0$.*

The proof follows from solution representation (9.2.10) and inequality $C(t,s) \ge 0$ for $t \ge s \ge t_0$.

**Theorem 9.8** *Suppose the conditions of Theorem 9.2 hold, $x_i(t)$ is a solution of (9.2.1), $y_i(t)$ is a solution of the differential inequality*

$$\dot{y}_i(t) + \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ij}^k(t) y_j\big(h_{ij}^k(t)\big) \le 0, \ \ i = 1, \cdots, n, \ \ t \ge t_0 \tag{9.4.2}$$

*and $x_i(t) = y_i(t)$ for $t \le t_0$. Then $y_i(t) \le x_i(t)$ for $t \ge t_0$.*

Consider together with (9.2.2) the following system, for which conditions (a1) and (a2) hold:

$$\dot{X}(t) + \sum_{k=1}^{m} B_k(t) X\big(g_k(t)\big) = 0, \ \ t \ge 0. \tag{9.4.3}$$

Denote by $Y(t,s)$ the fundamental matrix of system (9.4.3).

**Theorem 9.9** *Suppose that $a_{ij}^k(t) \le 0$, $b_{ij}^k(t) \le 0$ for $i \ne j$, the fundamental functions of scalar equations (9.3.1) are positive for $t \ge s \ge t_0$, $(a_{ii}^k)^+(t) \ge (b_{ii}^k)^+(t)$ and $g^k(t) \ge h^k(t)$ for $t \ge t_0$. Then $Y(t,s) \ge 0$ for $t \ge s \ge t_0$.*

The proof is similar to the proof of Theorem 2.4.

Consider together with (9.2.1) the system

$$\dot{x}_i(t) + \sum_{k=1}^{m}\sum_{j=1}^{n} b_{ij}^k(t)x_j\big(g_{ij}^k(t)\big) = 0, \ i = 1, \cdots, n, \ t \geq 0. \tag{9.4.4}$$

Suppose that (a1) and (a2) hold for system (9.4.4). Denote by $D(t,s)$ the fundamental matrix of system (9.4.4).

**Theorem 9.10** *Suppose that $a_{ij}^k(t) \leq 0$, $b_{ij}^k(t) \leq 0$ for $i \neq j$, the fundamental functions of scalar equations (9.3.6) are positive for $t \geq s \geq t_0$, $a_{ii}^k(t) \geq b_{ii}^k(t)$ and $g_{ii}^k(t) \geq h_{ii}^k(t)$ for $t \geq t_0$. Then $D(t,s) \geq 0$ for $t \geq s \geq t_0$.*

Let us compare solutions of differential equations with different matrices and right-hand sides.

To this end, consider together with (9.2.3), (9.2.4) the initial value problem

$$\dot{Y}(t) + \sum_{k=1}^{m} B_k(t)Y\big(h_k(t)\big) = G(t), \ t \geq t_0, \tag{9.4.5}$$

$$Y(t) = \Phi(t), \ t < t_0, \ Y(t_0) = Y_0. \tag{9.4.6}$$

Suppose that (a1)–(a3) hold for (9.4.5), (9.4.6). Denote by $X(t)$, $C(t,s)$ the solution and the fundamental matrix of problem (9.2.3), (9.2.4) and by $Y(t)$, $D(t,s)$ the solution and the fundamental matrix of problem (9.4.5), (9.4.6), respectively.

**Theorem 9.11** *If $a_{ij}^k(t) \leq 0$ for $i \neq j$, the fundamental functions of scalar equations (9.3.1) are positive for $t \geq s \geq t_0$, $X(t) \geq 0$ and*

$$A_k(t) \geq B_k(t), \ G(t) \geq F(t), \ Y_0 \geq X_0,$$

*then $Y(t) \geq X(t) \geq 0$, $t \geq t_0$.*

*Proof* By Theorem 9.10, the fundamental matrix $D(t,s)$ of vector equation (9.4.5) is positive for $t \geq s \geq t_0$. System (9.2.3) can be rewritten as

$$\dot{X}(t) + \sum_{k=1}^{m} B_k(t)X\big(h_k(t)\big) = \sum_{k=1}^{m}\big[B_k(t) - A_k(t)\big]X\big(h_k(t)\big) + F(t).$$

Hence, by the solution representation formula,

$$\begin{aligned} X(t) &= D(t,t_0)X_0 - \sum_{k=1}^{m}\int_{t_0}^{t} D(t,s)B_k(s)\Phi\big(h_k(s)\big)\,ds \\ &\quad + \int_{t_0}^{t} D(t,s)F(s)\,ds - \sum_{k=1}^{m}\int_{t_0}^{t} D(t,s)\big[A_k(s) - B_k(s)\big]X\big(h_k(s)\big)\,ds \\ &\leq D(t,t_0)Y_0 - \sum_{k=1}^{m}\int_{t_0}^{t} D(t,s)B_k(s)\Phi\big(h_k(s)\big)\,ds \\ &\quad + \int_{t_0}^{t} D(t,s)G(s)\,ds = Y(t), \end{aligned}$$

where $\Phi(h_k(s)) = 0$ if $h_k(s) \geq t_0$ and $X(h_k(s)) = 0$ if $h_k(s) < t_0$. Thus $X(t) \leq Y(t)$, which completes the proof. □

**Corollary 9.3** *If $a_{ij}^k(t) \leq 0$ for $i \neq j$, the fundamental functions of scalar equations (9.3.1) are positive for $t \geq s \geq t_0$ and $A_k(t) \geq B_k(t)$, then $D(t,s) \geq C(t,s) \geq 0$ for $t \geq s \geq t_0$.*

## 9.5 Higher-Order Scalar Delay Differential Equations

In this section, we consider the linear scalar delay differential equation of the $n$-th order

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}\big(h_{n-1}(t)\big) + \cdots + a_1(t)y'\big(h_1(t)\big) + a_0(t)y\big(h_0(t)\big) = 0 \tag{9.5.1}$$

for $t \geq t_0$, where for parameters of (9.5.1) and other higher-order equations it is assumed that coefficients $a_k$ are Lebesgue measurable locally essentially bounded functions, and delays $h_k(t) \leq t$ satisfy $\lim_{t\to\infty} h_k(t) = \infty$, $k = 0, \cdots, n-1$.

Together with (9.5.1), consider the initial value problem

$$\begin{aligned} y^{(n)}(t) &+ a_{n-1}(t)y^{(n-1)}\big(h_{n-1}(t)\big) + \cdots \\ &+ a_1(t)y'\big(h_1(t)\big) + a_0(t)y\big(h_0(t)\big) = f(t),\ t \geq t_0, \end{aligned} \tag{9.5.2}$$

$$y^{(k)}(t) = \varphi_k(t),\ t < t_0,\ y^{(k)}(t_0) = y_k,\ k = 0, \cdots, n-1. \tag{9.5.3}$$

**Definition 9.4** A function $y : \mathbb{R} \to \mathbb{R}$ with an $(n-1)$-th derivative $y^{(n-1)}$ absolutely continuous on each finite interval is called *a solution* of problem (9.5.2), (9.5.3) if it satisfies (9.5.2) for almost all $t \in [t_0, \infty)$ and equalities (9.5.3) for $t \leq t_0$.

**Definition 9.5** For each $s \geq 0$, the solution $Y(t,s)$ of the problem

$$\begin{aligned} y^{(n)}(t) &+ a_{n-1}(t)y^{(n-1)}\big(h_{n-1}(t)\big) + \cdots + a_1(t)y'\big(h_1(t)\big) \\ &+ a_0(t)y\big(h_0(t)\big) = 0,\ t \geq s, \end{aligned} \tag{9.5.4}$$

$$\begin{aligned} &y^{(k)}(t) = 0,\ t < s,\ k = 0, \cdots, n-1, \\ &y^{(k)}(s) = 0,\ k = 0, \cdots, n-2,\ y^{(n-1)}(s) = 1, \end{aligned} \tag{9.5.5}$$

is called *the fundamental function* of (9.5.1).

Further, we will denote by $Y_k(t,s)$ a solution of (9.5.4) with the initial conditions

$$\begin{aligned} &y^{(j)}(t) = 0,\ t < s,\ j = 0, \cdots, n-1,\ y^{(j)}(s) = 0,\ j \neq k, \\ &y^{(k)}(s) = 1,\ k = 0, \cdots, n-1, \end{aligned}$$

instead of (9.5.5). We assume $Y(t,s) = 0$ for $0 \leq t < s$ and $Y_k(t,s) = 0$ for $0 \leq t < s$, $k = 0, \cdots, n-1$; evidently $Y_{n-1}(t,s) = Y(t,s)$.

By Theorem B.5, we have the following result.

**Lemma 9.2** *There exists a unique solution of problem* (9.5.2), (9.5.3), *and it can be presented in the form*

$$y(t)=\sum_{k=0}^{n-1}Y_k(t,t_0)y_k+\int_{t_0}^{t}Y(t,s)f(s)ds-\int_{t_0}^{t}Y(t,s)\sum_{k=0}^{n-1}a_k(s)\varphi\big(h_k(s)\big)ds, \tag{9.5.6}$$

*where* $\varphi(h_k(s))=0$, *if* $h_k(s)\geq t_0$.

Denote

$$x_1(t)=y(t),\ x_2(t)=y'(t),\ \cdots,\ x_n(t)=y^{(n-1)}(t).$$

Then

$$x_1'(t)=x_2(t),\ x_2'(t)=x_3(t),\ \cdots,\ x_{n-1}'(t)=x_n(t),$$
$$x_n'(t)=-\sum_{k=1}^{n-1}a_{k-1}(t)x_k\big(h_{k-1}(t)\big).$$

Denote the column vector $X(t)=[x_1(t),x_2(t),\cdots,x_n(t)]^T$ and

$$A_0(t)=\begin{pmatrix}0&-1&0&\cdots&0\\0&0&-1&\cdots&0\\\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdots&-1\\0&0&0&0&0\end{pmatrix},\ A_k(t)=\begin{pmatrix}0&0&0&\cdots&0\\\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdots&0\\\cdot&\cdot&\cdot&\cdot&\cdot\\0&\cdots&a_{k-1}(t)&\cdots&0\end{pmatrix},\ k=1,\cdots,n,$$

where $A_k(t)$ are $n\times n$ matrices.

Hence (9.5.1) can be rewritten as the system

$$\dot{X}(t)+A_0(t)X(t)+\sum_{k=1}^{n}A_k(t)X\big(h_{k-1}(t)\big)=0, \tag{9.5.7}$$

and for its fundamental matrix $C(t,s)$ we can deduce $C_{1n}(t,s)=Y(t,s)$, where $Y(t,s)$ is the fundamental function of (9.5.1).

As a corollary of Theorem 9.1, we can obtain the following result.

**Theorem 9.12** *Suppose that* $a_k(t)\leq 0$, $k=0,\cdots,n-2$ *and the fundamental function* $X(t,s)$ *of the scalar equation*

$$\dot{x}(t)+\big[a_{n-1}(t)\big]^{+}x\big(h_{n-1}(t)\big)=0 \tag{9.5.8}$$

*is positive for* $t\geq s\geq t_0$. *Then, for the fundamental function of* (9.5.1), *we have* $Y(t,s)\geq 0$, $t\geq s\geq t_0$.

*Proof* First let us check that all the conditions of Theorem 9.1 hold for system (9.5.7). Evidently all the nondiagonal entries of matrices $A_k(t)$, $k=0,\cdots,n$ are nonpositive.

Scalar equations (9.3.1) have the form

$$\dot{x}_k(t) = 0,\ k = 1, \cdots, n-1,\ \dot{x}_n(t) + \left[a_{n-1}(t)\right]^+ x_n\left(h_{n-1}(t)\right) = 0.$$

Fundamental functions of all these equations are positive. Theorem 9.1 implies that the fundamental matrix $C(t,s)$ of system (9.5.7) is nonnegative for $t \ge s \ge t_0$. Since $C_{1n}(t,s) = Y(t,s)$, the fundamental function of (9.5.1) is nonnegative. □

**Corollary 9.4** *Suppose that* $a_k(t) \le 0$, $k = 0, \cdots, n-2$, $t \ge t_0$ *and*

$$\sup_{t \ge t_0} \int_{\max\{t_0, h_{n-1}(t)\}}^{t} \left[a_{n-1}(s)\right]^+ ds \le \frac{1}{e}. \tag{9.5.9}$$

*Then the fundamental function* $Y(t,s)$ *of* (9.5.1) *is nonnegative for* $t \ge s \ge t_0$.

**Corollary 9.5** *Suppose that* $a_k(t) \le 0$, $k = 1, \cdots, n$, $t \ge t_0$. *Then the fundamental function* $Y(t,s)$ *of* (9.5.1) *is nonnegative for* $t \ge s \ge t_0$.

**Corollary 9.6** *Suppose that the conditions of Theorem* 9.12 *hold*, $u(t)$ *is a solution of problem* (9.5.2), (9.5.3) *and* $v(t)$ *is a solution of the problem, where* $f$ *is replaced by* $g$. *If* $g(t) \le f(t)$, *then* $v(t) \le u(t)$, $t \ge t_0$.

**Corollary 9.7** *Suppose that the conditions of Theorem* 9.12 *hold*, $u(t)$ *is a solution of* (9.5.1) *and* $v(t)$ *is a solution of the differential inequality*

$$y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}\left(h_{n-1}(t)\right) + \cdots + a_0(t) y\left(h_0(t)\right) \le 0.$$

*If* $v(t) = u(t)$, $t \le t_0$, *then* $v(t) \le u(t)$, $t \ge t_0$.

*Remark 9.2* Corollary 9.7 extends the famous result obtained by Chaplygin [89] for an ordinary differential equation of the second order,

$$y''(t) + a(t)y'(t) + b(t)y(t) = 0.$$

Together with (9.5.1), consider the equation

$$z^{(n)}(t) + b_{n-1}(t) z^{(n-1)}\left(g_{n-1}(t)\right) + \cdots + b_0(t) z\left(g_0(t)\right) = 0,\ t \ge t_0. \tag{9.5.10}$$

Denote by $Z(t,s)$ the fundamental function of (9.5.10).

**Theorem 9.13** *If*

$$a_{n-1}(t) \ge b_{n-1}(t) \ge 0,\ g_{n-1}(t) \ge h_{n-1}(t) a_k(t) \le 0,$$
$$b_k(t) \le 0,\ k = 0, \cdots, n-2,\ t \ge t_0,$$

*and the fundamental function of* (9.5.1) *is nonnegative, then* $Z(t,s) \ge 0$ *for* $t \ge s \ge t_0$.

Together with (9.5.2), (9.5.3), consider the initial value problem

$$z^{(n)}(t) + b_{n-1}(t) z^{(n-1)}\left(h_{n-1}(t)\right) + \cdots + b_0(t) z\left(h_0(t)\right) = g(t),\ t \ge t_0, \tag{9.5.11}$$

$$z^{(k)}(t) = \varphi_k(t),\ t < t_0,\ z^{(k)}(t_0) = z_k,\ k = 0, \cdots, n-1. \tag{9.5.12}$$

Further, let $y(t), Y(t,s)$ be the solution and the fundamental function of problem (9.5.2), (9.5.3) and $z(t),\ Z(t,s)$ the solution and the fundamental function of problem (9.5.11), (9.5.12), respectively.

In the following theorem, we compare positive solutions and fundamental functions of two nonoscillatory equations.

**Theorem 9.14** *Suppose that* $a_{n-1}(t) \ge b_{n-1}(t) \ge 0$ *for* $t \ge t_0$, $y(t) \ge 0, t \ge t_0$,

$$a_k(t) \ge b_k(t), k = 0, \cdots, n-2,\ g(t) \ge f(t),\ y_k \le z_k,\ k = 0, \cdots, n-1,$$

*and the fundamental function of* (9.5.1) *is positive for* $t \ge s \ge t_0$*. Then* $Z(t,s) \ge Y(t,s) \ge 0$ *for* $t \ge s \ge t_0$*, and* $z(t) \ge y(t) \ge 0, t \ge t_0$.

## 9.6 Positivity and Solution Estimates

We begin this section with an analogue of Corollary 2.15 on the existence of a positive solution.

**Theorem 9.15** *Suppose that*

$$a_{ii}^k(t) \ge 0,\ a_{ij}^k(t) \le 0,\ i \ne j,\ k = 1, \cdots, m,\ F(t) \ge 0,\ t \ge t_0,$$

$0 \le \Phi(t) \le X_0$, $X_0 > 0$ *and inequality* (9.3.5) *holds. Then, for the solution* $X(t)$ *of problem* (9.2.3), (9.2.4), *we have* $X(t) \ge 0, t \ge t_0$.

*Proof* First, let $F(t) = 0$. We recall that

$$\Phi(t) = \left[\varphi_1(t), \cdots, \varphi_n(t)\right]^T,\ X_0 = \left[x_1^0, \cdots, x_n^0\right]^T,$$

where $\varphi_i(t) \le x_i^0$, $x_i^0 > 0$. Hence, for solutions $y_i(t)$ of initial value problems

$$\dot{y}(t) + \sum_{k=1}^{m} a_{ii}^k(t) y\big(h_k(t)\big) = 0,\ i = 1, \cdots, n,$$

$$y(t) = \varphi_i(t),\ t < t_0,\ y(t_0) = x_i^0,$$

Corollary 2.15 implies $y_i(t) > 0, t \ge t_0, i = 1, \cdots, n$.

Denote $Y(t) = [y_1(t), \cdots, y_n(t)]^T$. Then

$$\dot{Y}(t) + \sum_{k=1}^{m} A_k(t) Y\big(h_k(t)\big) \le 0,$$

$$Y(t) = X(t),\ t \le t_0.$$

The solution representation implies $0 \le Y(t) \le X(t)$.

For the case $F(t) \geq 0$, the proof also follows from solution representation (9.2.6). □

**Corollary 9.8** *If* $a_k(t) \leq 0$, $k = 0, \cdots, n-2$, $a_{n-1}(t) \geq 0$, $t \geq t_0$, (9.5.9) *holds and*

$$0 \leq \varphi_k(t) \leq y_k,\ \ y_k > 0,\ \ k = 1, \cdots, n,\ \ f(t) \geq 0,$$

*then for the solution* $y(t)$ *of initial value problem* (9.5.2), (9.5.3) *we have* $y(t) \geq 0$, $t \geq t_0$.

Let $\|X\|$ denote a vector norm for $X \in \mathbb{R}^n$. By $\|\cdot\|$, we will also denote the associated matrix norm

$$\|A\| = \sup_{\|x\|=1, x \in \mathbb{R}^n} \|Ax\|.$$

**Definition 9.6** Let $A(t)$ be a variable matrix. $\mu(A)$ is *the matrix measure* defined as

$$\mu(A) = \lim_{\theta \to 0^+} \frac{\|I + \theta A\| - 1}{\theta}, \tag{9.6.1}$$

where $I$ is the identity matrix.

The well-known Coppel inequality (see [103] and also [351] for the outline of the relevant facts for the matrix measure) states that solutions of the system

$$\dot{X}(t) = A(t)X(t) \tag{9.6.2}$$

satisfy the inequality

$$\big\|X(t)\big\| \leq \big\|X(t_0)\big\| \exp\left\{\int_{t_0}^{t} \mu\big(A(\tau)\big)\,d\tau\right\}. \tag{9.6.3}$$

For the fundamental matrix of system (9.6.2), the estimate

$$\big\|C(t,s)\big\| \leq \exp\left\{\int_{s}^{t} \mu\big(A(\tau)\big)\,d\tau\right\} \tag{9.6.4}$$

holds.

Let us obtain an upper estimate for the fundamental matrix of system (9.2.2).

**Theorem 9.16** *Suppose that the conditions of Theorem* 9.1 *hold and* $a_{ii}^k(t) \geq 0$, $0 \leq t - h_k(t) \leq H$ *for* $i = 1, \cdots, n$, $k = 1, \cdots, m$.

*Then*

$$\big\|C(t,s)\big\| \leq \big\|M(t,s)\big\| + \int_s^t \left[\big\|M(t,\tau)\big\|\big\|S(\tau)\big\| \exp\left\{\int_s^{\tau} \mu\big(S(\zeta)\big)\,d\zeta\right\}\right] d\tau, \tag{9.6.5}$$

*where* $M = \mathrm{diag}\{M_{11}, \cdots, M_{nn}\}$,

$$M_{ii} = \exp\left\{-\int_{s+H}^{t} \sum_{k=1}^{m} a_{ii}^k(\tau)\,d\tau\right\}, \tag{9.6.6}$$

$$S(t)=\sum_{k=1}^{m}\big[B_k(t)-A_k(t)\big],\ B_k(t)=\operatorname{diag}\big\{a_{11}^k,\cdots,a_{nn}^k\big\},\ k=1,\cdots,m. \quad (9.6.7)$$

*Proof* By Theorem 9.1, the fundamental matrix $C(t,s)$ of system (9.2.2) is nonnegative: $C(t,s)\ge 0$. On the other hand, it does not exceed the solution of the problem

$$\dot X(t)=\sum_{k=1}^{m}\big[B_k(t)-A_k(t)\big]X\big(h_k(t)\big),\ X(s)=I,\ X(t)=0,\ t<s.$$

Since the elements of the matrix $B_k(t)-A_k(t)$ are nonnegative, the entries of $X(t)$ are increasing functions. Hence

$$\dot X(t)=\sum_{k=1}^{m}\big[B_k(t)-A_k(t)\big]X\big(h_k(t)\big)\le\sum_{k=1}^{m}\big[B_k(t)-A_k(t)\big]X(t)=S(t)X(t),$$
$$X(s)=I,$$

and thus

$$\big\|C(t,s)\big\|\le\exp\bigg\{\int_s^t\mu\big(S(\zeta)\big)\,d\zeta\bigg\}.$$

Since $S(t)\ge 0$, we have $\mu(S(t))\ge 0$, and

$$\big\|C\big(h_k(t),s\big)\big\|\le\exp\bigg\{\int_s^{h_k(t)}\mu\big(S(\zeta)\big)\,d\zeta\bigg\}\le\exp\bigg\{\int_s^t\mu\big(S(\zeta)\big)\,d\zeta\bigg\} \quad (9.6.8)$$

for any $t$ and $k=1,\cdots,m$. Denote by $Y(t,s)$ the fundamental matrix of the system

$$\dot Y(t)+\sum_{k=1}^{m}B_k(t)Y\big(h_k(t)\big)=0.$$

Since $B_k(t)$ are diagonal matrices,

$$Y(t,s)=\operatorname{diag}\big\{X_1(t,s),\cdots,X_n(t,s)\big\},$$

where $X_k(t,s)$ is the fundamental function of scalar equation (9.3.1), which is positive for $t\ge s\ge t_0$. Thus $Y(t,s)\ge 0,\ t\ge s\ge t_0$.

By Lemma 2.19, we have $X_i(t,s)\le M_{ii}(t,s)$, where $M_{ii}$ are defined in (9.6.6), so

$$Y(t,s)\le M(t,s).$$

Let us fix $s$ and denote $X(t)=C(t,s)$. Since

$$\dot X(t)+\sum_{k=1}^{m}B_k(t)X\big(h_k(t,s)\big)=\sum_{k=1}^{m}\big[B_k(t)-A_k(t)\big]X\big(h_k(t)\big),$$

by solution representation formula (9.2.6) and (9.6.8) we have

$$\big\|X(t)\big\|=\bigg\|Y(t,s)+\int_s^t Y(t,\tau)\sum_{k=1}^{m}\big[B_k(\tau)-A_k(\tau)\big]X\big(h_k(\tau)\big)\,d\tau\bigg\|$$
$$\le\big\|M(t,s)\big\|$$

$$+\int_s^t \left( \|M(t,\tau)\| \left\| \sum_{k=1}^m [B_k(\tau) - A_k(\tau)] \right\| \exp\left\{ \int_s^\tau \mu\big(S(\zeta)\big)\, d\zeta \right\} \right) d\tau$$

$$= \|M(t,s)\| + \int_s^t \left[ \|M(t,\tau)\| \, \|S(\tau)\| \exp\left\{ \int_s^\tau \mu\big(S(\zeta)\big)\, d\zeta \right\} \right] d\tau,$$

which completes the proof. □

Consider now system (9.2.1).

**Theorem 9.17** *If*

$$a_{ii}^k(t) \geq 0,\ a_{ij}^k(t) \leq 0,\ i \neq j,\ k = 1, \cdots, m,$$
$$f_i(t) \geq 0,\ t \geq t_0,\ 0 \leq \varphi_i(t) \leq x_i^0,\ x_i^0 > 0$$

*and inequality* (9.3.7) *holds, then for the solution* $x_i(t)$ *of initial value problem* (9.2.7), (9.2.8) *we have* $x_i(t) \geq 0$ *for* $t \geq t_0$.

## 9.7 Positive Solutions and Stability

In this section, we establish a connection between nonoscillation properties and stability for delay differential systems. We need some definitions and results.

**Definition 9.7** Matrix $A$ is called an *$M$-matrix* if $a_{ij} \leq 0$, $i \neq j$ and there exists a nonnegative inverse matrix $A^{-1} \geq 0$.

We refer to [36, 83] for many equivalent forms of this definition.

**Definition 9.8** Equation (9.2.2) is *asymptotically stable* if for any $s \geq 0$ the fundamental matrix of this equation satisfies $\lim_{t\to\infty} C(t,s) = 0$. Equation (9.2.2) is *exponentially stable* if there exist $M > 0$ and $\alpha > 0$ such that

$$\|C(t,s)\| \leq Me^{-\alpha(t-s)},\ t \geq s \geq t_0. \tag{9.7.1}$$

The following result is presented in Appendix B as Theorem B.21.

**Lemma 9.3** *Suppose for* (9.2.2) *for some* $t_0 \geq 0$ *and* $H > 0$ *functions* $a_{ij}^k$ *are essentially bounded on* $[t_0, \infty)$, $t - h_k(t) \leq H$ *and for any function* $F$ *essentially bounded on* $[t_0, \infty)$ *the solution of the initial value problem*

$$\dot{X}(t) + \sum_{k=1}^m A_k(t) X\big(h_k(t)\big) = F(t),\ t > t_0,\ X(t) = 0,\ t \leq t_0$$

*is bounded on* $[t_0, \infty)$. *Then* (9.2.2) *is exponentially stable.*

**Corollary 9.9** *Suppose that functions $a_{ij}^k$ are essentially bounded on* $[t_0,\infty)$ *and* $t-h_k(t)\le H$. *If there exists $t_1>t_0$ such that, for any function $F$ that is essentially bounded on* $[t_0,\infty)$ *and satisfies $F(t)=0$ for $t\in[t_0,t_1]$, the solution of the initial value problem* (9.2.9) *is bounded on* $[t_0,\infty)$, *then* (9.2.2) *is exponentially stable.*

*Proof* Since the solution of (9.2.2) with the zero initial conditions and the zero right-hand side on $[t_0,t_1]$ vanishes on $[t_0,t_1]$, we can obtain by Lemma 9.3 that (9.2.2) with the initial point $t_1$ instead of $t_0$ is exponentially stable: $\|C(t,s)\|\le M_1e^{-\alpha(t-s)}$ for $t\ge s\ge t_1$.

The fundamental matrix of a vector equation with coefficients bounded on $[t_0,t_1]$ is also bounded on the square $(t,s)\in[t_0,t_1]\times[t_0,t_1]$: $|C(t,s)|\le M_2$, $t_0\le s\le t\le t_1$. Consider the matrix initial value problem with the initial point at $t_1$ and the initial matrix $\Phi(t)$, which coincides with $C(t,s)$ for $t\le t_1$ and vanishes for $t>t_1$, which implies $\Phi(h_k(t))=0$ for $t>t_1+H$. Then solution representation (9.2.6) implies for $s\in[t_0,t_1]$ (we note that $\Phi(t)=0$ for $t>t_1+H$)

$$\begin{aligned}\|C(t,s)\| &= \left\|C(t,t_1)C(t_1,s)-\sum_{k=1}^m\int_{t_1}^t C(t,\tau)A_k(\tau)\Phi\big(h_k(\tau)\big)\,d\tau\right\|\\ &= \left\|C(t,t_1)C(t_1,s)-\sum_{k=1}^m\int_{t_1}^{t_1+H} C(t,\tau)A_k(\tau)\Phi\big(h_k(\tau)\big)\,d\tau\right\|\\ &\le \|C(t,t_1)\|\,\|C(t_1,s)\|+\sum_{k=1}^m\int_{t_1}^{t_1+H}\|A_k(\tau)\|\,\|C\big(h_k(\tau),s\big)\|\,d\tau\\ &\le M_1M_2e^{-\alpha(t-t_1)}+H\sup_{t\in[t_1,t_1+H]}\sum_{k=1}^m\|A_k(\tau)\|M_1M_2e^{-\alpha(t-t_1-H)}\\ &\le Me^{-\alpha(t-s)}.\end{aligned}$$

Thus, for any $t\ge s\ge t_0$ we have $\|C(t,s)\|\le Me^{-\alpha(t-s)}$, where

$$M=\max\left\{1,\,M_1M_2e^{\alpha(t_1-t_0)}+H\left(\sup_{\tau\in[t_1,t_1+H]}\sum_{k=1}^m\|A_k(\tau)\|\right)M_1M_2e^{\alpha(t_1+H-t_0)}\right\},$$

which completes the proof. □

For a function essentially bounded on $[t_0,\infty)$, define the norm

$$\|f\|_{L_\infty}=\operatorname*{ess\,sup}_{t\ge t_0}\|f(t)\|.$$

**Theorem 9.18** *Suppose that for system* (9.2.2) *the conditions of Theorem* 9.1 *hold, for some $t_0\ge 0$ and $H>0$ the functions $a_{ij}^k$ are essentially bounded on* $[t_0,\infty)$, $t-h_k(t)\le H$, $a_{ii}^k\ge 0$, $i=1,\cdots,n$, $k=1,\cdots,m$,

$$\inf_{t\ge t_0}\sum_{k=1}^m a_{ii}^k(t)\ge a_i>0,$$

*and the matrix $B = \{b_{ij}\}$ defined as*

$$b_{ij} = \begin{cases} a_i, & i = j, \\ -\sum_{k=1}^{m} \|a_{ij}^k\|_{L_\infty}, & i \neq j, \end{cases} \tag{9.7.2}$$

*is an $M$-matrix. Then system (9.2.2) is exponentially stable.*

*Proof* We apply Lemma 9.3 and Corollary 9.9. Consider the initial value problem

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = F(t), \ X(t) = 0, \ t \leq t_0, \tag{9.7.3}$$

where $F(t) = 0$ for $t_0 \leq t \leq t_0 + H$. Since $F(t) = 0$, $t \leq t_0 + H$, we have $X(t) = 0$, $t \leq t_0 + H$.

Equation (9.7.3) can be rewritten as the system of scalar equations

$$\dot{x}_i(t) + \sum_{k=1}^{m} a_{ii}^k(t) x_i\big(h_k(t)\big) = -\sum_{j \neq i} \sum_{k=1}^{m} a_{ij}^k(t) x_j\big(h_k(t)\big) + f_i(t), \ i = 1, \cdots, n.$$

Denote by $X_i(t,s)$ the fundamental function of (9.3.1). By the assumption of the theorem $X_i(t,s) \geq 0$, $t \geq s \geq t_0$. Hence we can also rewrite (9.7.3) in the form

$$x_i(t) = -\int_{t_0}^{t} X_i(t,s) \sum_{j \neq i} \sum_{k=1}^{m} a_{ij}^k(s) x_j\big(h_k(s)\big) ds + g_i(t), \ i = 1, \cdots, n,$$

where

$$g_i(t) = \int_{t_0}^{t} X_i(t,s) f_i(s) ds.$$

By Theorem 2.21, we have

$$\big|g_i(t)\big| \leq \int_{t_0}^{t} X_i(t,s) \sum_{k=1}^{m} a_{ii}^k(s) ds \frac{\|f_i\|_{L_\infty}}{a_i} \leq \frac{\|f_i\|_{L_\infty}}{a_i},$$

which implies $\operatorname{ess\,sup}_{t \geq t_0} |g_i(t)| < \infty$.

From the inequality

$$\begin{aligned} \big|x_i(t)\big| &\leq \int_{t_0}^{t} X_i(t,s) \sum_{k=1}^{m} a_{ii}^k(s) ds \sum_{j \neq i} \sum_{k=1}^{m} \frac{\|a_{ij}^k\|_{L_\infty}}{a_i} \sup_{0 \leq s \leq t} \big|x_j(s)\big| + \|g_i\|_{L_\infty} \\ &\leq \sum_{j \neq i} \sum_{k=1}^{m} \frac{\|a_{ij}^k\|_{L_\infty}}{a_i} \sup_{0 \leq s \leq t} \big|x_j(s)\big| + \|g_i\|_{L_\infty}, \end{aligned}$$

we obtain

$$\sup_{0 \leq s \leq t} \big|x_i(s)\big| \leq \sum_{j \neq i} \sum_{k=1}^{m} \frac{\|a_{ij}^k\|_{L_\infty}}{a_i} \sup_{0 \leq s \leq t} \big|x_j(s)\big| + \|g_i\|_{L_\infty}.$$

Denote

$$y_i(t)=\sup_{0\le s\le t}\big|x_i(s)\big|,\ \ Y(t)=\big[y_1(t),\cdots,y_n(t)\big]^T,$$

$$G(t)=\big[\|g_1\|_{L_\infty},\cdots,\|g_n\|_{L_\infty}\big]^T,$$

$$c_{ij}=\begin{cases}1, & i=j,\\ -\frac{1}{a_i}\sum_{k=1}^m\|a_{ij}^k\|_{L_\infty}, & i\ne j,\end{cases}\quad C=[c_{ij}].$$

We have $CY(t)\le G$ for any $t>t_0$. Evidently, if $B$ is an $M$-matrix, then $C$ is also an $M$-matrix. Thus $C^{-1}\ge 0$ and $Y(t)\le C^{-1}G$. Hence

$$\sup_{t\ge t_0}\big|X(t)\big|=\sup_{t\ge t_0}\big|Y(t)\big|\le\big\|C^{-1}G\big\|<\infty.$$

By Lemma 9.3, (9.2.2) is exponentially stable. □

**Corollary 9.10** *Suppose that for the scalar equation*

$$\dot x(t)+\sum_{k=1}^m a_k(t)x\big(h_k(t)\big)=0 \tag{9.7.4}$$

*for some* $t_0\ge 0$ *and* $H>0$, *functions* $a_k$ *are essentially bounded on* $[t_0,\infty)$, $t-h_k(t)\le H$, $a_k\ge 0$, $k=1,\cdots,m$,

$$\inf_{t\ge t_0}\sum_{k=1}^m a_k(t)\ge a_0>0,$$

*and the fundamental function of the equation is positive.*

*Then* (9.7.4) *is exponentially stable.*

*Proof* Since the matrix $B$ in Theorem 9.1 is equal to the positive number $a_0$, then all the conditions of Theorem 9.1 hold. □

**Corollary 9.11** *Suppose that for system* (9.2.2) *the conditions of Theorem* 9.1 *hold, for some* $t_0\ge 0$ *and* $H>0$ *the functions* $a_{ij}^k$ *are essentially bounded on* $[t_0,\infty)$ *and* $t-h_k(t)\le H$. *Let matrix* $B=\{b_{ij}\}$ *be defined as in* (9.7.2).

1) *If the condition*

   (A1) *there exist two vectors* $z=[z_1,\cdots,z_n]^T$ *and* $\varepsilon=[\varepsilon_1,\cdots,\varepsilon_n]^T$ *such that* $z_i>0$, $\varepsilon_i\ge 1$ *for* $i=1,\cdots,n$, *and*

$$Bz\ge\varepsilon, \tag{9.7.5}$$

   *holds, then* (9.2.2) *is exponentially stable and*

$$\lim_{t\to\infty}\int_0^t\sum_{j=1}^n C_{ij}(t,s)ds\le z_i,\ \ i=1,\cdots,n. \tag{9.7.6}$$

2) *If there exists a matrix* $Y = \{y_{ij}\} \geq 0$ *such that* $y_{ii} > 0$ *and the inequality*

$$BY \geq I \tag{9.7.7}$$

*holds, then* (9.2.2) *is exponentially stable, and the entries of the Cauchy matrix satisfy the inequalities*

$$\lim_{t\to\infty} \int_0^t C_{ij}(t,s)ds \leq y_{ij},\ i,j = 1,\cdots,n. \tag{9.7.8}$$

*Proof* Let us note that condition (A1) is equivalent to the hypothesis that $B$ is an $M$-matrix [83] as well as to the existence of matrix $Y$ such that (9.7.7) holds. For the estimates (9.7.6) and (9.7.8), see [31, 139]. □

*Remark 9.3* It should be emphasized that the requirement that $B$ be an $M$-matrix in Theorem 9.18 is essential and becomes necessary in the case of constant coefficients $a_{ij}^k$. Note in this connection the following results obtained in [139].

**Theorem 9.19** [139] *Suppose that all coefficients* $a_{ij}^k$ *are constants,* $t - h_k(t) \leq H$ *for some* $t_0 \geq 0$ *and* $H > 0$, *and for system* (9.2.2) *the conditions of Theorem* 9.1 *hold. Then system* (9.2.2) *is exponentially stable if and only if condition* (A1) *is satisfied.*

**Theorem 9.20** [139] *Let the conditions of Theorem* 9.19 *be fulfilled and* $B$ *be an* $M$*-matrix. Then*

$$\lim_{t\to\infty} \int_0^t \sum_{j=1}^n C_{ij}(t,s)\,ds = z_i,\ i = 1,\cdots,n,$$

*and*

$$\lim_{t\to\infty} \int_0^t C_{ij}(t,s)ds = y_{ij},\ i,j = 1,\cdots,n,$$

*where* $z_i$, $i = 1,\cdots,n$ *and* $y_{ij}$, $i,j = 1,\cdots,n$ *are defined in Corollary* 9.11.

In the case of variable coefficients, the following result can be obtained.

**Theorem 9.21** *Suppose for system* (9.2.2) *the conditions of Theorem* 9.1 *hold,* $t - h_k(t) \leq H$ *and*

$$\sum_{k=1}^m \sum_{j=1}^n a_{ij}^k(t) \leq 0,\ t \geq t_0,\ i = 1,\cdots,n.$$

*Then* (9.2.2) *is not asymptotically stable.*

*Proof* Denote $Y(t) = [1,\cdots,1]^T$. Then, for $t \geq t_0 + H$, vector function $Y(t)$ is a solution of inequality (9.4.1). By Theorem 9.6, we have $X(t) \geq Y(t)$, where $X(t) = Y(t)$, $t \leq t_1$. Hence (9.2.2) is not asymptotically stable. □

For system (9.2.1), stability results similar to Theorems 9.18–9.21 can be obtained. Below we consider only an analogue of Theorem 9.18 and some corollaries of this result for $n = 2$.

**Theorem 9.22** *Suppose that for system (9.2.1) for some $t_0 \ge 0$ and $H > 0$ functions $a_{ij}^k$ are essentially bounded on $[t_0, \infty)$, $t - h_{ij}^k(t) \le H$, $i, j = 1, \cdots, n$, $k = 1, \cdots, m$,*

$$\inf_{t \ge t_0} \sum_{k=1}^{m} a_{ii}^k(t) \ge a_i > 0, \quad \int_{\min_k h_{ii}^k(t)}^{t} \sum_{k=1}^{m} a_{ii}^k(s)ds \le \frac{1}{e},$$

*and the matrix $B = \{b_{ij}\}$ defined as*

$$b_{ij} = \begin{cases} a_i, & i = j, \\ -\sum_{k=1}^{m} \|a_{ij}^k\|_{L_\infty}, & i \ne j, \end{cases} \tag{9.7.9}$$

*is an $M$-matrix. Then system (9.2.1) is exponentially stable.*

**Corollary 9.12** *Suppose in system (9.2.1) that $n = 2$, $k = 1, \cdots, m$, for some $t_0 \ge 0$ and $H > 0$ functions $a_{ij}^k$ are essentially bounded on $[t_0, \infty)$, $t - h_{ij}^k(t) \le H$, $i, j = 1, 2$,*

$$\inf_{t \ge t_0} \sum_{k=1}^{m} a_{ii}^k(t) \ge a_i > 0, \quad \int_{\min_k h_{ii}^k(t)}^{t} \sum_{k=1}^{m} a_{ii}^k(s)ds \le \frac{1}{e}, \ i = 1, 2,$$

*and*

$$\Delta = a_1 a_2 - \sum_{k=1}^{m} \left\|a_{12}^k\right\|_{L_\infty} \sum_{k=1}^{m} \left\|a_{21}^k\right\|_{L_\infty} > 0. \tag{9.7.10}$$

*Then system (9.2.1) is exponentially stable.*

*Proof* The matrix $B$ for $n = 2$ has the form

$$B = \begin{pmatrix} a_1 & -\sum_{k=1}^{m} \|a_{12}^k\|_{L_\infty} \\ -\sum_{k=1}^{m} \|a_{21}^k\|_{L_\infty} & a_2 \end{pmatrix}.$$

Hence

$$B^{-1} = \frac{1}{\Delta} \begin{pmatrix} a_2 & \sum_{k=1}^{m} \|a_{12}^k\|_{L_\infty} \\ \sum_{k=1}^{m} \|a_{21}^k\|_{L_\infty} & a_1 \end{pmatrix} \ge 0.$$

This means that the matrix $B$ is an $M$-matrix. □

Consider the following system with some nondelay terms:

$$\begin{cases} \dot{x}_1(t) + a_{11}(t)x_1(t) + a_{12}(t)x_2\big(h(t)\big) = 0, \\ \dot{x}_2(t) + a_{21}(t)x_1\big(g(t)\big) + a_{22}(t)x_2(t) = 0. \end{cases} \tag{9.7.11}$$

**Corollary 9.13** *Suppose in system (9.7.11) for some $t_0 \ge 0$ and $H > 0$ functions $a_{ij}$ are essentially bounded on $[t_0, \infty)$, $t - h(t) \le H$, $t - g(t) \le H$, $\inf_{t \ge t_0} a_{ii}(t) \ge a_i > 0$ and $\Delta > 0$, where $\Delta$ is denoted by (9.7.10) for $m = 1$. Then system (9.7.11) is exponentially stable.*

## 9.8 Systems of Differential Equations with a Distributed Delay

In the previous sections, we considered systems of equations with concentrated delays. Results obtained for these systems can be extended to systems with distributed delays, including integrodifferential systems and mixed type systems.

In this section, we will present most results of this type without proofs since all proofs are similar to the proofs for equations with concentrated delays.

### 9.8.1 Nonnegativity of Fundamental Matrices

Consider the vector equation with distributed delays

$$\dot{X}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} \big[d_s R_k(t,s)\big] X(s) = 0, \tag{9.8.1}$$

where $R_k$ are $n \times n$ matrix functions, the integral in (9.8.1) is the Lebesgue-Stieltjes integral and the lower bound satisfies $-\infty < h_k(t) \le t$.

We assume that for the parameters of (9.8.1) the following conditions hold:

(a1) all entries $r_{ij}^k$ of $n \times n$ matrices $R_k(t,\cdot)$ are left continuous scalar functions of bounded variation and for each $s$ the variation on the segment $[t_0, s]$

$$p_{ij}^k(t,s) = \mathrm{Var}_{\tau \in [t_0,s]}\, r_{ij}^k(t,\tau),\ i,j = 1,\cdots,n,\ k = 1,\cdots,m, \tag{9.8.2}$$

is a locally essentially bounded function in $t$.

(a2) $R_k(t,s) = R_k(t,t^+),\ t < s,\ R_k(t,s) = 0,\ s \le h_k(t)$, where $R_k(t,t^+) = \lim_{s\to t^+} R_k(t,s)$ and the integrals for left continuous functions $R_k(t,\cdot)$ are understood as

$$\int_a^t \big[d_s R_k(t,s)\big] x(s) = \lim_{\varepsilon \to 0^+} \int_a^{t+\varepsilon} \big[d_s R_k(t,s)\big] x(s)$$

for any $-\infty \le a \le t$, where $t > -\infty$.

Now we can define the fundamental matrix $C(t,s)$ for system (9.8.1) as was done for system (9.2.1) and obtain a solution representation formula for the initial value problem (see Theorem B.3).

**Theorem 9.23** *Let the nondiagonal entries $r_{ij}^k(t,s)$, $i \ne j$ of $R_k(t,s)$ be nonincreasing in $s$ for any $t \ge t_0$, the diagonal entries $r_{ii}^k(t,s)$ be nondecreasing in $s$ for any $t \ge t_0$ and the fundamental functions $X_i(t,s)$ of the scalar equations*

$$\dot{x_i}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x_i(s) d_s r_{ii}^k(t,s) = 0,\ i = 1,\cdots,n, \tag{9.8.3}$$

*be positive for $t \ge s \ge t_0$. Then, for the fundamental matrix $C(t,s)$ of system (9.8.1), we have $C(t,s) \ge 0$ for $t \ge s \ge t_0$.*

Any function of bounded variation can be presented as a difference of two non-decreasing functions (see the Hahn decomposition [150, p. 129]), so

$$r_{ii}^k(t,s) = u_{ii}^k(t,s) - v_{ii}^k(t,s),\ i = 1,\cdots,n,\ k = 1,\cdots,m, \tag{9.8.4}$$

where $u_{ii}^k$, $v_{ii}^k$ are nondecreasing functions in $s$ for any $t$.

**Theorem 9.24** *Suppose that the nondiagonal entries $r_{ij}^k(t,s)$, $i \neq j$ of $R_k(t,s)$ are nonincreasing in $s$ for any $t \geq t_0$ and the fundamental functions $X_i(t,s)$ of the scalar equations*

$$\dot{x}_i(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x_i(s) d_s u_{ii}^k(t,s) = 0,\ i = 1,\cdots,n, \tag{9.8.5}$$

*where $u_{ii}^k$, $v_{ii}^k$ are nondecreasing functions in $s$ for any $t$ defined in (9.8.4), are positive for $t \geq s \geq t_0$. Then, for the fundamental matrix $C(t,s)$ of system (9.8.1), we have $C(t,s) \geq 0$, $t \geq s \geq t_0$.*

**Corollary 9.14** *Suppose $r_{ij}^k(t,s)$, $i \neq j$, $t \geq t_0$, are nonincreasing in $s$ for any $t$, $r_{ii}^k(t,s) = u_{ii}^k(t,s) - v_{ii}^k(t,s)$, where $u_{ii}$ and $v_{ii}$ are nondecreasing in $s$ for any $t$ and*

$$\int_{\max\{t_0,\, \min_i h_i(t)\}}^{t} \sum_{k=1}^{m} u_{ii}^k\left(\tau, \tau^+\right) d\tau \leq \frac{1}{e},\ i = 1,\cdots,n.$$

*Then the fundamental matrix of the system (9.2.7) satisfies $C(t,s) \geq 0$ for $t \geq s \geq t_0$.*

Consider now the integrodifferential system

$$\dot{X}(t) + \sum_{l=1}^{m} \int_{h_l(t)}^{t} K_l(t,s) X(s) ds = 0. \tag{9.8.6}$$

By $k_{ij}^l$ we denote the entries of the matrix kernels $K_l(t,s)$. Then (a1) and (a2) are equivalent to the assumption that $k_{ij}^l(t,s)$ are locally essentially bounded functions, $l = 1,\cdots,m$, $i, j = 1,\cdots,n$.

**Theorem 9.25** *Let $k_{ij}^l(t,s) \leq 0$ for $i \neq j$, $t \geq t_0$, $k_{ii}^l(t,s) \geq 0$ and the fundamental functions of the scalar equations*

$$\dot{x}(t) + \sum_{l=1}^{m} \int_{h_l(t)}^{t} k_{ii}^l(t,s) x(s) ds = 0,\ i = 1,\cdots,n, \tag{9.8.7}$$

*be positive for $t \geq s \geq t_0$. Then the fundamental matrix $C(t,s)$ of system (9.8.6) satisfies $C(t,s) \geq 0$ for $t \geq s \geq t_0$.*

Since for integral equation (9.8.6) functions $u_{ii}^l(t,s)$ and $v_{ii}^k(t,s)$ defined in (9.8.4) correspond to $(k_{ii}^l)^+$ and $(k_{ii}^l)^-$, respectively, where $a^+ = \max\{a,0\}$, $a^- = -\min\{a,0\}$, the following corollary is a particular case of Corollary 9.14.

**Corollary 9.15** *Let* $k_{ij}(t,s) \le 0$, $i \ne j$, $t \ge t_0$ *and*

$$\sup_{t \ge t_0} \int_{\max\{t_0, \min_j h_j(t)\}}^t \left( \sum_{l=1}^m \int_{h_l(\tau)}^\tau \left(k_{ii}^l\right)^+(t,s)\,ds \right) d\tau < \frac{1}{e}, \; i = 1, \cdots, n. \tag{9.8.8}$$

*Then the fundamental matrix of system* (9.8.6) *satisfies* $C(t,s) \ge 0$ *for* $t \ge s \ge t_0$.

Next, let us study mixed type systems containing concentrated delay and integral terms. For simplicity, we consider such a system for $n = 2$:

$$\begin{cases} \dot{x}_1(t) + a_1(t)x_1\big(h_{11}(t)\big) + \int_{h_{12}(t)}^t K_1(t,s)x_2(s)ds = 0, \\ \dot{x}_2(t) + a_2(t)x_1\big(h_{21}(t)\big) + \int_{h_{22}(t)}^t K_2(t,s)x_2(s)ds = 0. \end{cases} \tag{9.8.9}$$

This system can be written as the vector equation

$$\dot{X}(t) + \sum_{i=1}^2 \sum_{j=1}^2 \int_{h_{ij}(t)}^t \big[d_s R_{ij}(t,s)\big] X(s) = 0, \tag{9.8.10}$$

which is a particular case of (9.8.1), if we assume

$$R_{11}(t,s) = \begin{bmatrix} a_1(t)\chi_{(h_{11}(t),\infty)}(s) & 0 \\ 0 & 0 \end{bmatrix}, \; R_{12}(t,s) = \begin{bmatrix} 0 & \int_{h_{12}(t)}^s K_1(t,\tau)\,d\tau \\ 0 & 0 \end{bmatrix}, \tag{9.8.11}$$

where $\chi_I$ is the characteristic function of the interval $I$, and $R_{21}(t,s)$, $R_{22}(t,s)$ are defined similarly.

**Theorem 9.26** *Suppose that* $a_2(t) \le 0$, $K_1(t,s) \le 0$, $t \ge t_0$, *and the fundamental functions of the scalar equations*

$$\dot{x}(t) + a_1^+(t)x\big(h_{11}(t)\big) = 0$$

*and*

$$\dot{x}(t) + \int_{h_{22}(t)}^t K_2^+(t,s)x(s)ds = 0$$

*are positive for* $t \ge s \ge t_0$. *Then the fundamental matrix* $C(t,s)$ *of system* (9.8.9) *satisfies* $C(t,s) \ge 0$ *for* $t \ge s \ge t_0$.

**Corollary 9.16** *Suppose* $a_2(t) \le 0$, $K_1(t,s) \le 0$, $t \ge t_0$ *and*

$$\sup_{t \ge t_0} \int_{h_{11}(t)}^t a_1^+(s)ds < \frac{1}{e}, \; \sup_{t \ge t_0} \int_{\max\{t_0, h_{22}(t)\}}^t \int_{h_{22}(\tau)}^\tau K_2^+(t,s)\,ds\,d\tau < \frac{1}{e}.$$

*Then the fundamental matrix* $C(t,s)$ *of system* (9.8.9) *satisfies* $C(t,s) \ge 0$, $t \ge s \ge t_0$.

Let us remark that in population dynamics the situation where the growth of a certain component of the solution is influenced negatively by the size of the same component and positively by the other components is quite typical. If various components correspond to different developmental stages, this means that species at the same stage compete for resources, while other stages contribute to the growth (which can be either maturation of juveniles or reproduction of adults). If we have a patch structure, then species within the same patch again compete for resources, while overpopulation of other patches leads to immigration and thus population growth.

### *9.8.2 Comparison Results and Positivity of Solutions*

In this section, we compare two solutions of the same system and solutions of different systems. First, let us compare two solutions of system (9.2.7) of differential equations.

**Theorem 9.27** *Suppose that the conditions of Theorem 9.23 hold, $X(t)$ is a solution of problem (9.2.7), (9.2.8) and $Y(t)$ is a solution of the same problem, where function $F(t)$ is replaced by $G(t)$. If $G(t) \le F(t)$ for $t \ge t_0$, then $Y(t) \le X(t)$ for $t \ge t_0$.*

**Corollary 9.17** *Suppose that conditions of Theorem 9.23 hold, $X(t)$ is a solution of (9.2.7), $Y(t)$ is a solution of the differential inequality*

$$\dot{Y}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} \big[d_s R_k(t,s)\big] Y(s) \le 0, \;\; t \ge t_0, \tag{9.8.12}$$

*and $X(t) = Y(t)$ for $t \le t_0$. Then $Y(t) \le X(t)$ for $t \ge t_0$.*

Let us compare solutions of systems with different delay distributions and right-hand sides. To this end, consider together with (9.2.7), (9.2.8) the initial value problem

$$\dot{Y}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} \big[d_s B_k(t,s)\big] Y(s) = G(t), \;\; t \ge t_0, \tag{9.8.13}$$

$$Y(t) = \Phi(t), \;\; t < t_0, \;\; Y(t_0) = Y_0. \tag{9.8.14}$$

Denote by $X(t)$, $C(t,s)$ the solution and the fundamental matrix of problem (9.2.7), (9.2.8) and by $Y(t)$, $D(t,s)$ the solution and the fundamental matrix of problem (9.8.13), (9.8.14), respectively. Let $b_{ij}^k$ be the entries of $B_k$.

**Theorem 9.28** *Let the entries $r_{ij}^k(t,s)$ of $R_k(t,s)$ be nonincreasing in $s$ for any $t \ge t_0$ and $i \ne j$, $r_{ii}^k(t,s)$, $b_{ii}^k(t,s)$, $i = 1, \cdots, n$ and $r_{ij}^k(t,s) - b_{ij}^k(t,s)$, $i, j = 1, \cdots, n$, be nondecreasing in $s$ for any $t$, the fundamental functions of scalar equations (9.8.3) be positive for $t \ge s \ge t_0$, $X(t) \ge 0$ for $t \ge t_0$ and the inequalities*

$$G(t) \ge F(t), \;\; t \ge t_0, \;\; Y_0 \ge X_0, \;\; R_k(t,s) \ge B_k(t,s) \ge 0, \;\; t, s \ge t_0$$

*hold. Then $Y(t) \ge X(t) \ge 0$, $t \ge t_0$.*

**Corollary 9.18** *If $r_{ij}^k(t,s)$ are nonincreasing in $s$ for any $t \geq t_0$ and $i \neq j$, $r_{ij}^k(t,s) - b_{ij}^k(t,s)$, $i,j = 1,\cdots,n$ are nondecreasing in $s$ for any $t$, and the fundamental functions of scalar equations (9.8.3) are positive for $t \geq s \geq t_0$, then we have $D(t,s) \geq C(t,s) \geq 0$, $t \geq s \geq t_0$ for the fundamental functions $D(t,s)$ of (9.8.13) and $C(t,s)$ of (9.2.7).*

As was mentioned in the introduction, the condition that the nondiagonal entries are nonpositive (nonincreasing for equations with a distributed delay), which is necessary for equations without delay, is no longer necessary for delay equations. However, if nondiagonal entries are nondelayed and the fundamental matrices of scalar equations corresponding to the diagonal entries are positive, then this condition becomes in a certain sense necessary.

Consider the system

$$\dot{X}(t) + A_0(t)X(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} \left[d_s R_k(t,s)\right] x(s) = 0. \tag{9.8.15}$$

Let $a_{ij}$ be the entries of $A_0$.

**Theorem 9.29** *If $r_{ij}^k \equiv 0$, $i \neq j$, the fundamental functions of the scalar equations*

$$\dot{x_i}(t) + a_{ii}(t)x_i(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} x_i(s) d_s r_{ii}^k(t,s) = 0, \; i = 1,\cdots,n,$$

*are positive and for some pair $(i,j)$ with $i \neq j$, we have $a_{ij}(t) \geq \alpha_0 > 0$ on some interval $[c,d]$, $d > c$, then the fundamental matrix of (9.8.15) is not positive.*

Finally, let us obtain an analogue of Lemma 4.7 on the existence of a positive solution for systems.

**Theorem 9.30** *Suppose that the nondiagonal entries $r_{ij}^k(t,s)$, $i \neq j$ of $R_k(t,s)$ are nonincreasing in $s$ for any $t \geq t_0$, the diagonal entries $r_{ii}^k(t,s)$ are nondecreasing in $s$ for any $t \geq t_0$, $F(t) \geq 0$ for $t \geq t_0$, $0 \leq \Phi(t) \leq X_0$, $X_0 > 0$ and the inequalities*

$$\int_{\max\{t_0, \min_i h_i(t)\}}^{t} \sum_{k=1}^{m} r_{ii}^k\left(\tau, \tau^+\right) d\tau \leq \frac{1}{e}, \; i = 1,\cdots,n \tag{9.8.16}$$

*hold. Then, for the solution $X(t)$ of initial value problem (9.2.7), (9.2.8), we have*

$$X(t) \geq 0, \; t \geq t_0.$$

### 9.8.3 Solution Estimates

The following auxiliary results will be used later.

Consider the scalar equation

$$\dot{x}(t)+\sum_{k=1}^{m}\int_{h_k(t)}^{t}x(s)d_sR_k(t,s)=0,\ t\geq t_0. \tag{9.8.17}$$

**Lemma 9.4** *Suppose that $R_k(t,s)$ are scalar functions nondecreasing in $s$ for any $t$ and the fundamental function $X(t,s)$ of scalar equation (9.8.17) is positive for $t\geq s\geq t_0$. Then, for any $t_1\geq t_0$ such that $h_k(t)>t_0$ for $t>t_1$, the following inequality holds*:

$$0\leq\sum_{k=1}^{m}\int_{t_1}^{t}X(t,s)R_k\big(s,s^+\big)\,ds\leq 1. \tag{9.8.18}$$

*Proof* The function $x(t)=\chi_{(t_0,\infty)}(t)$, where $\chi_I$ is the characteristic function of the interval $I$, satisfies

$$\dot{x}(t)+\sum_{k=1}^{m}\int_{h_k(t)}^{t}x(s)\,d_sR_k(t,s)=\sum_{k=1}^{m}\big[R_k\big(t,t^+\big)-R_k\big(t,t_0^+\big)\big],\ t\geq t_0,$$

$x(t)=0$, $t<t_0$, $x(t_0)=0$. By solution representation (B.1.22), we have

$$1=\int_{t_0}^{t}X(t,s)\sum_{k=1}^{m}\big[R_k\big(s,s^+\big)-R_k\big(s,t_0^+\big)\big]\,ds,\ t\geq t_0.$$

First, since $R_k(t,s)$ are nondecreasing in $s$ for any $t$ and $s\geq t_0$, the difference $R_k(s,s^+)-R_k(s,t_0)$ is nonnegative for any $s\geq t_0$. Next, by (a4) there exists $t_1\geq t_0$ such that $h_k(t)\geq t_0$ for $t\geq t_1$. By the definition of $h_k(t)$, we have $R_k(t,s)=0$ for $s\leq h_k(t)$, which implies $R(s,t_0^+)=0$, $s>t_1$. Since

$$\begin{aligned}
&\int_{t_0}^{t}X(t,s)\sum_{k=1}^{m}\big[R_k\big(s,s^+\big)-R_k\big(s,t_0^+\big)\big]\,ds\\
&\quad=\int_{t_0}^{t_1}X(t,s)\sum_{k=1}^{m}\big[R_k\big(s,s^+\big)-R_k\big(s,t_0^+\big)\big]\,ds\\
&\qquad+\int_{t_1}^{t}X(t,s)\sum_{k=1}^{m}\big[R_k\big(s,s^+\big)-R_k\big(s,t_0^+\big)\big]\,ds\\
&\quad=\int_{t_0}^{t_1}X(t,s)\sum_{k=1}^{m}\big[R_k\big(s,s^+\big)-R_k\big(s,t_0^+\big)\big]\,ds\\
&\qquad+\int_{t_1}^{t}X(t,s)\sum_{k=1}^{m}R_k\big(s,s^+\big)\,ds=1,
\end{aligned}$$

where the first term is nonnegative and $R_k(s,t_0)=0$ for any $s>t_1$, we obtain (9.8.18). □

We recall that by $\|X\|$ we denote the vector norm of $X \in \mathbb{R}^n$ and also the associated matrix norm

$$\|A\| = \sup_{\|x\|=1, x\in\mathbb{R}^n} \|Ax\|.$$

Let us obtain an upper estimate for the fundamental matrix of system (9.8.1). First, consider scalar equation (9.8.17).

**Lemma 9.5** *Assume that $R_k(t,s)$ are nondecreasing in $s$ for any $t \ge t_0$ and there exists $H > 0$ such that $t - H \le t - h_k(t) \le t$ for $t \ge t_0$, $k = 1, \cdots, m$. If the fundamental function $X(t,s)$ of (9.8.17) is positive for $t \ge s \ge t_0$, then it satisfies the estimate*

$$\left|X(t,s)\right| \le \exp\left\{-\sum_{k=1}^{m}\int_{s+H}^{t} R_k\left(\tau,\tau^+\right)d\tau\right\}. \tag{9.8.19}$$

*Proof* Since $X(t,s) > 0$ and $R_k(t,s)$ are nondecreasing in $s$ for any $t \ge s \ge t_0$, the fundamental function $X(t,s)$ of (9.8.17) is nonincreasing in $t$. Thus $X(t,s) \le X(\tau,s)$ for $h_k(t) \le \tau \le t$.

Denoting $x(t) = X(t,s)$, we obtain

$$\begin{aligned}\dot{x}(t) &= -\sum_{k=1}^{m}\int_{h_k(t)}^{t} x(s)d_sR_k(t,s)\\ &\le \begin{cases} 0, & \min_k h_k(t) < s,\\ -[\sum_{k=1}^{m}\int_s^t d_sR_k(t,s)]x(t), & \min_k h_k(t) \ge s,\end{cases}\end{aligned}$$

and consequently $\dot{x}(t) \le b(t)x(t)$, $x(s) = 1$, where

$$b(t) = \begin{cases} 0, & s \le t < s+H,\\ -[\sum_{k=1}^{m} R_k(t,t^+)], & t \ge s+H.\end{cases}$$

By the Gronwall-Bellman inequality (Lemma A.5),

$$\begin{aligned}x(t) &\le \exp\left\{\int_s^t b(\tau)\,d\tau\right\}\\ &\le \exp\left\{-\sum_{k=1}^{m}\int_{s+H}^{t} R_k\left(\tau,\tau^+\right)d\tau\right\},\end{aligned}$$

which completes the proof. □

**Theorem 9.31** *Suppose that the fundamental matrix $C(t,s)$ of system (9.8.1) satisfies $C(t,s) \ge 0$, $t \ge s \ge t_0$, $r_{ii}^k(t,s)$ are nondecreasing and $r_{ij}^k(t,s)$ are nonincreasing in $s$ for any $t$, $j \ne i$, $0 \le t - h_k(t) \le H$, $i, j = 1, \cdots, n$, $k = 1, \cdots, m$. Then, for the fundamental matrix $C(t,s)$ of system (9.8.1), the estimate*

$$\begin{aligned}\left\|C(t,s)\right\| \le \left\|M(t,s)\right\| + \int_s^t \bigg[\left\|M(t,\tau)\right\|\left\|S(\tau)\right\|\\ \times \exp\left\{\int_s^\tau \mu\left(S(\zeta)\right)d\zeta\right\}\bigg]d\tau\end{aligned} \tag{9.8.20}$$

*is valid*, *where* $M = \mathrm{diag}\{M_{11}, \cdots, M_{nn}\}$,

$$M_{ii} = \exp\left\{-\int_{s+H}^{t} \sum_{k=1}^{m} r_{ii}^{k}\big(\tau, \tau^{+}\big)\, d\tau\right\}, \tag{9.8.21}$$

$$S(t) = \sum_{k=1}^{m}\big[B_k\big(t, t^{+}\big) - R_k\big(t, t^{+}\big)\big], \tag{9.8.22}$$

$B_k(t, s) = \mathrm{diag}\{r_{11}^{k}(t, s), \cdots, r_{nn}^{k}(t, s)\}, k = 1, \cdots, m.$

Let us apply the result of Theorem 9.31 to the mixed system

$$\dot{X}(t) + \sum_{l=1}^{m} A_l(t) X\big(h_l(t)\big) + \int_{h_0(t)}^{t} K(t, s) X(s)\, ds = 0, \tag{9.8.23}$$

where $a_{ij}^{k}(t)$ are the entries of the matrices $A_k(t)$ and $k_{ij}(t, s)$ are the entries of $K(t, s)$.

**Corollary 9.19** *Suppose that* $a_{ii}^{l}(t) \ge 0, a_{ij}^{l}(t) \le 0, i \ne j, k_{ii}(t) \ge 0, k_{ij}(t) \le 0, i \ne j, i, j = 1, \cdots, n, 0 \le t - h_l(t) \le H, l = 0, 1, \cdots, m$, *and the fundamental matrices of the scalar equations*

$$\dot{x}_i(t) + \sum_{l=1}^{m} a_{ii}^{l}(t) x_i\big(h_l(t)\big) + \int_{h_0(t)}^{t} k_{ii}(t, s) x(s)\, ds = 0$$

*are positive*. *Then the fundamental matrix of* (9.8.23) *satisfies estimate* (9.8.20), *where* $M = \mathrm{diag}\{M_{11}, \cdots, M_{nn}\}$,

$$M_{ii} = \exp\left\{-\int_{s+H}^{t}\left[\sum_{l=1}^{m} a_{ii}^{l}(\tau) + \int_{h_0(\tau)}^{\tau} k_{ii}(\tau, \zeta)\, d\zeta\right] d\tau\right\},$$

$$S(t) = B(t) - \sum_{l=1}^{m} A_l(t) - \int_{h_0(t)}^{t} K(t, s)\, ds,$$

$$B(t) = \mathrm{diag}\left\{\sum_{l=1}^{m} a_{11}^{l}(t) + \int_{h_0(t)}^{t} k_{11}(t, s)\, ds, \cdots, \sum_{l=1}^{m} a_{nn}^{l}(t) + \int_{h_0(t)}^{t} k_{nn}(t, s)\, ds\right\}.$$

### *9.8.4 Nonoscillation and Stability*

In this section, we establish a connection between nonoscillation and stability for differential systems with a distributed delay. The proofs are based on Theorem B.23, Lemma 9.4 and the following auxiliary result.

**Lemma 9.6** *In (9.8.1), let the matrix functions*

$$\int_{h_k(t)}^{t} d_s R_k(t,s)$$

*be essentially bounded on* $[t_0,\infty)$ *and* $t-h_k(t)\le H$. *If there exists* $t_1>t_0$ *such that for any function* $F$ *that is essentially bounded on* $[t_0,\infty)$ *and satisfies* $F(t)=0$ *for* $t\in[t_0,t_1]$ *the solution of the initial value problem (9.2.7) with the zero initial conditions is bounded on* $[t_0,\infty)$, *then (9.8.1) is exponentially stable.*

**Theorem 9.32** *Suppose that* $r_{ij}^k(t,s)$, $i\neq j$ *are nonincreasing and* $r_{ii}^k(t,s)$ *are nondecreasing in* $s$ *for any* $t$, $k=1,\cdots,m$, *the fundamental functions* $X_i(t,s)$ *of scalar equations (9.8.3) are positive,* $t\ge s\ge t_0$, $r_{ij}^k(t,t^+)$ *are essentially bounded on* $[t_0,\infty)$, *there exist* $H>0$, $a_i>0$, $i,j=1,\cdots,n$ *such that*

$$t-h_k(t)\le H,\ k=1,\cdots,m,\ \inf_{t\ge t_0}\sum_{k=1}^{m} r_{ii}^k\big(t,t^+\big)\ge a_i,$$

*and the matrix* $B=[b_{ij}]$ *with*

$$b_{ij}=\begin{cases} a_i, & i=j,\\ -\sum_{k=1}^m \|r_{ij}^k(\cdot,\cdot^+)\|_{L_\infty}, & i\neq j,\end{cases}$$

*is an* $M$*-matrix.*

*Then system (9.8.1) is exponentially stable.*

Consider the scalar equation

$$\dot{x}(t)+\sum_{l=1}^{m} a_l(t)x\big(h_l(t)\big)+\int_{h_0(t)}^{t} K(t,s)x(s)ds=0. \tag{9.8.24}$$

**Corollary 9.20** *Suppose* $K(t,s)\ge 0$, $a_l(t)\ge 0$ *for* $t\ge t_0$, $t-h(t)\le H$, $t-h_l(t)\le H$, $a_l$ *and* $\int_{h_0(t)}^t K(s,s^+)\,ds$ *are essentially bounded on* $[t_0,\infty)$, $l=1,\cdots,m$ *and*

$$\sup_{t\ge t_0}\int_{\tilde{h}(t)}^{t}\left[\int_{h(\tau)}^{\tau} K(t,s)\,ds+\sum_{l=1}^{m} a_l(\tau)\right]d\tau<\frac{1}{e},$$

$$\int_{h(t)}^{t} K(t,\tau)\,d\tau+\sum_{l=1}^{m} a_l(t)\ge a_0>0,$$

*where*

$$\tilde{h}(t)=\max\Big\{t_0,\min_{k=0,\cdots,m} h_k(t)\Big\}.$$

*Then (9.8.24) is exponentially stable.*

## 9.9 Discussion and Open Problems

The main result of this chapter is the generalization to delay equations of the well-known Wazewski's result [332] for the ordinary vector differential equation

$$\dot{X}(t) + A(t)X(t) = 0. \tag{9.9.1}$$

By this result, (9.9.1) has a nonnegative fundamental matrix if and only if $a_{ij} \le 0$, $i \ne j$; for the proof of this theorem, see [36].

In contrast with Wazewski's classical theorem, the condition $a_{ij}^k \le 0$, $i \ne j$ is not necessary for nonnegativity of all entries of the fundamental (Cauchy) matrix $C(t,s)$ for equations with several delays, as was demonstrated in the present chapter. Theorems 9.1 and 9.13 were proven in [139] by a different method. Paper [113] deals with nonnegativity of certain entries of the fundamental matrix. In [5], the authors considered nonoscillation problems for a general vector Volterra equation on a bounded interval. Most of the results of the present chapter are contained in [70, 75].

To the best of our knowledge, [31] is the first paper where a connection between nonnegativity of the fundamental matrix and exponential stability was established, see also papers [186, 189, 190]. The relation between nonoscillation and asymptotic properties of solutions for linear and nonlinear functional differential equations was studied in the papers [183–191, 207].

In [162, 164], positivity of the fundamental function for functional differential equations of higher order, including integrodifferential equations and equations with causal operators, was considered.

Nonoscillation results for systems of autonomous equations with distributed delays and for neutral equations with distributed delays were obtained in [254].

Finally, let us formulate some open problems.

1. Suppose for (9.2.2) condition $a_{ij} \le 0$, $i \ne j$ holds and the fundamental matrix of this equation is nonnegative. Are the fundamental functions of scalar equations (9.3.1) necessarily positive?
2. Obtain explicit lower and upper estimates
$$me^{-\alpha(t-s)} \le \left\| C(t,s) \right\| \le Me^{-\beta(t-s)},\ 0 < \beta < \alpha,$$
for the fundamental matrix (in the case where it is nonnegative) of system (9.2.2) and for the fundamental function (again when it is nonnegative) of higher-order equation (9.5.1) when the system/equation is exponentially stable.
3. Is there a connection between nonoscillation and asymptotic stability of (9.5.1)?
4. Establish the relation of nonoscillation and asymptotic stability for (9.2.2) without the assumption $a_{ii}^k \ge 0$.
5. Prove or disprove:
*Suppose that $a_{ii}^k(t) \ge 0$, $a_{ij}^k(t) \le 0$, $i \ne j$, in vector equation* (9.2.2). *Then* (9.2.2) *has a nonnegative solution if and only if its fundamental matrix is nonnegative.*
6. Generalize the results of this chapter on nonoscillation of integrodifferential equations to the case of infinite delays ($h_k(t) = -\infty$).

7. Prove or disprove the following conjecture:

   *Suppose* $a_{ij} \leq 0, i \neq j,\ 0 < a_{ii} \leq 1/(e\tau),\ i, j = 1, \cdots, n$. *Then the autonomous system*

   $$\dot{X}(t) + AX(t-\tau) = 0$$

   *is exponentially stable if and only if* $A$ *is an* $M$*-matrix.*

   Can this result be extended to autonomous systems with a distributed delay?

# Chapter 10
# Linearization Methods for Nonlinear Equations with a Distributed Delay

## 10.1 Introduction

Linearization is quite a common tool for studying oscillation of nonlinear equations (see the papers [55, 192, 225, 238, 242, 243, 307, 318], where linearization results for delay differential equations with concentrated delays were obtained). For a different approach to study oscillation see, for example, [311].

The chapter is organized as follows. After preliminaries in Sect. 10.2, we present in Sect. 10.3 our main linearization theorems, which are applied in Sect. 10.4 to various population ecology models, in particular, to logistic, Lasota-Wazewska and Nicholson blowflies equations. In Sect. 10.5, we establish "the Mean Value Theorem", which claims that under certain conditions a solution of a nonlinear equation with a distributed delay also satisfies a linear equation with a single variable concentrated delay. Finally, Sect. 10.6 presents discussion and open problems.

## 10.2 Preliminaries

We consider a nonlinear differential equation with a distributed delay

$$\dot{x}(t)+\sum_{k=1}^{m} r_k(t)\int_{-\infty}^{t} f_k\big(x(s)\big)d_s R_k(t,s)=0 \tag{10.2.1}$$

as well as the equation with a nondelay term

$$\dot{x}(t)+b(t)x(t)+\sum_{k=1}^{m} r_k(t)\int_{-\infty}^{t} f_k\big(x(s)\big)d_s R_k(t,s)=0 \tag{10.2.2}$$

for $t>t_0\ge 0$. In most cases, it is assumed that for each $t$ the memory is finite:

for each $t_1$ there exists $s_1=s(t_1)\le t_1$ such that $R_k(t,s)=0$ for $s<s_1$, $t>t_1$, $k=1,\cdots,m$, and $\lim_{t\to\infty} s(t)=\infty$.

If the condition above is satisfied, then we can introduce the functions

$$h_k(t)=\inf\big\{s\le t\,|\,R_k(t,s)\ne 0\big\} \tag{10.2.3}$$

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and rewrite (10.2.1), (10.2.2) in the form

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k\big(x(s)\big) d_s R_k(t, s) = 0, \tag{10.2.4}$$

$$\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k\big(x(s)\big) d_s R_k(t, s) = 0, \tag{10.2.5}$$

for $t > t_0$. Together with (10.2.4), (10.2.5) we assume, for each $t_0 \ge 0$, that the initial condition

$$x(t) = \varphi(t), \ \ t \le t_0, \tag{10.2.6}$$

is satisfied. We consider (10.2.4) and (10.2.5) under the following assumptions:

(a1) $r_k(t) \ge 0$, $k = 1, \cdots, m$, $b(t) \ge 0$ are Lebesgue measurable functions bounded on the halfline: $r_k(t) < r_k$, $b(t) < b$, $t \ge 0$;
(a2) $h_k : [0, \infty) \to \mathbb{R}$, $k = 1, \cdots, m$ are Lebesgue measurable functions, $h_k(t) \le t$, $\lim_{t\to\infty} h_k(t) = \infty$;
(a3) $R_k(t, \cdot)$ are left continuous nondecreasing functions for any $t$, $R_k(\cdot, s)$ are locally integrable for any $s$, $R_k(t, h_k(t)) = 0$, $R_k(t, t^+) = 1$.

In (a3), the condition $R_k(t, h_k(t)) = 0$ means that the delay is finite, while $R_k(t, t^+) = 1$ corresponds to any delay equation, which is "normalized" with the coefficient $r_k(t)$; $R_k$ can be treated as probabilities that the delay at point $t$ exceeds $t - s$.

Now let us proceed to the initial function $\varphi$. This function should satisfy such conditions that the integral on the left-hand side of (10.2.4) exists almost everywhere. In particular, if $R_k(t, \cdot)$, $k = 1, \cdots, m$ is absolutely continuous for any $t$ (which allows us to write (10.2.4) as an integrodifferential equation), then $\varphi$ can be chosen as a Lebesgue measurable essentially bounded function. If $R_k(t, \cdot)$, $k = 1, \cdots, m$ is a combination of step functions (which corresponds to an equation with concentrated delays), then $\varphi$ should be a Borel measurable bounded function. For any choice of $R$, the integral exists if $\varphi$ is bounded and continuous. Thus, we assume that

(a4) $\varphi : (-\infty, 0] \to \mathbb{R}$ is a bounded continuous function

and the following hypothesis for $f_k$ is satisfied:

(a5) $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \cdots, m$ are continuous differentiable functions and $f_k'$ are locally essentially bounded functions.

*Remark 10.1* For existence and uniqueness results, in (a5) we can assume that the functions $f_k$ are locally Lipschitz rather than differentiable: for each $[a, b]$ there is an $M_k > 0$ (generally depending on $[a, b]$) such that $|f_k(x) - f_k(y)| < M_k|x - y|$ for any $x, y \in [a, b]$; for details see Theorem B.12.

Let us note that the equation with several concentrated delays

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f_k\big(x\big(h_k(t)\big)\big) = 0, \tag{10.2.7}$$

the integrodifferential equation

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} M_k(t,s) f_k\big(x(s)\big)\, ds = 0 \tag{10.2.8}$$

and the mixed equation

$$\dot{x}(t) + \sum_{k=1}^{l} r_k(t) f_k\big(x\big(h_k(t)\big)\big) + \sum_{k=l+1}^{m} r_k(t) \int_{h_k(t)}^{t} M_k(t,s) f_k\big(x(s)\big)\, ds = 0 \tag{10.2.9}$$

are partial cases of (10.2.4). Here we assume that for the initial value problem conditions (a1), (a2), (a4) and (a5) hold; besides, the parameters of (10.2.8) and (10.2.9) satisfy the following condition:

(a6) $M_k(t,s)$ are locally integrable functions, $\int_{h_k(t)}^{t} M_k(t,s)\, ds = 1$ for any $t$ and $k = 1, \cdots, m$.

We will also consider the linear equation

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s)\, d_s R_k(t,s) = 0 \tag{10.2.10}$$

corresponding to nonlinear model (10.2.4).

Denote by $h(t)$ and $H(t)$ the maximal and minimal argument functions

$$h(t) = \min_{k=1,\cdots,m} h_k(t), \;\; H(t) = \max_{k=1,\cdots,m} h_k(t). \tag{10.2.11}$$

Consider the equation with a nondelay term

$$\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s)\, d_s R_k(t,s) = 0. \tag{10.2.12}$$

Substituting

$$z = x \exp\left\{ \int_{t_0}^{t} b(\zeta)\, d\zeta \right\}, \tag{10.2.13}$$

we obtain the equation

$$\dot{z}(t) + \exp\left\{ \int_{t_0}^{t} b(\zeta)\, d\zeta \right\} \sum_{k=1}^{m} r_k(t) \times \int_{h_k(t)}^{t} z(s) \exp\left\{ -\int_{t_0}^{s} b(\zeta)\, d\zeta \right\} d_s R_k(t,s) = 0, \tag{10.2.14}$$

which can be rewritten as

$$\dot{z}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} z(s) \exp\left\{\int_s^t b(\zeta)\, d\zeta\right\} d_s R_k(t,s) = 0. \tag{10.2.15}$$

Applying Theorems 4.9 and 4.11 to this equation, we immediately obtain the following result.

**Lemma 10.1** *Suppose* (a1)–(a4) *hold and* $b(t)$ *is a measurable locally essentially bounded function.*

*If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[\int_{h_k(\tau)}^{\tau} \exp\left\{\int_s^\tau b(\zeta)\, d\zeta\right\} d_s R_k(\tau,s)\right] d\tau < \frac{1}{e}, \tag{10.2.16}$$

*then* (10.2.14) *has a nonoscillatory solution. If*

$$\liminf_{t\to\infty} \int_{H(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[\int_{h_k(t)}^{\tau} \exp\left\{\int_s^\tau b(\zeta)\, d\zeta\right\} d_s R(\tau,s)\right] d\tau > \frac{1}{e}, \tag{10.2.17}$$

*then all solutions of* (10.2.14) *are oscillatory. Here* $h(t)$, $H(t)$ *are defined in* (10.2.11).

In the following sections, we will also consider the particular cases of (10.2.10)

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) x\big(h_k(t)\big) = 0, \tag{10.2.18}$$

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} M_k(t,s) x(s)\, ds = 0 \tag{10.2.19}$$

and

$$\dot{x}(t) + \sum_{k=1}^{l} r_k(t) f_k\big(x\big(h_k(t)\big)\big) + \sum_{k=l+1}^{m} r_k(t) \int_{h_k(t)}^{t} M_k(t,s) x(s)\, ds = 0, \tag{10.2.20}$$

corresponding to (10.2.7), (10.2.8) and (10.2.9), respectively.

## 10.3 Linearized Oscillation

In this section, we assume the existence of a global solution for $t \ge 0$.

**Theorem 10.1** *Suppose* (a1)–(a5) *hold and*

$$\int_0^\infty \sum_{k=1}^{m} r_k(t) = \infty, \ \ x f_k(x) > 0, \ \ x \neq 0. \tag{10.3.1}$$

*Then, for any nonoscillatory solution $x(t)$ of* (10.2.4), *we have*

$$\lim_{t\to\infty} x(t) = 0. \tag{10.3.2}$$

*Proof* The first equality in (10.3.1) implies that at least one of the integrals of $r_k$ diverges. Let it be $r_j$. We also recall that all $r_k$ are nonnegative.

Let us assume $x(t) > 0$, $t \ge t_0$ (the case of negative $x(t)$ is treated similarly). Then by (a2) there exists $t_1 \ge t_0$ such that for $s > t_1$ we have $h(s) > t_0$. So $\dot{x}(t) < 0$ or $x(t)$ is decreasing for $t > t_1$ and $x(t)$ is bounded since $x(t) > 0$, $t \ge t_1$. Consequently, there exists a limit $d = \lim_{t\to\infty} x(t) \ge 0$. Let $d > 0$. By (10.3.1), we also have $f_k(d) = d_k > 0$. Since $f_k$ are continuous, there exists $t_2 > 0$ such that $f_k(x(t)) \ge d_k/2$, $t > t_2$ and $t_3 \ge t_2$ such that $h(t) > t_2$, $t > t_3$. Integrating from $t_3$ to infinity, we obtain

$$\int_{t_3}^{\infty} \dot{x}(\tau)\,d\tau = d - x(t_3) = -\int_{t_3}^{\infty} \sum_{k=1}^{m} r_k(\tau)\,d\tau \int_{h_k(\tau)}^{\tau} f_k\big(x(s)\big) d_s R_k(\tau, s)$$

$$\le -\int_{t_3}^{\infty} \sum_{k=1}^{m} \frac{d_k}{2} r_k(\tau)\,d\tau \le -\frac{d_j}{2} \int_{t_3}^{\infty} r_j(\tau)\,d\tau = -\infty.$$

Since $d - x(t_3)$ is finite, we obtain a contradiction, which completes the proof. □

*Remark 10.2* The example of an ordinary differential equation $x' = x(x-1)^2$ (all solutions of this equation with $x(0) \ge 1$ converge to the equilibrium $x = 1$) illustrates that the condition $x f_k(x) \ge 0$ (the nonstrict inequality for $x \ne 0$) is not enough for convergence to the zero equilibrium. Let us also comment that the inequalities $f_k(x) > 0$, $k = 1, \cdots, m$ for $x > 0$ imply convergence to zero for positive solutions, while $f_k(x) < 0$, $x < 0$ for negative solutions.

**Theorem 10.2** *Suppose* (a1)–(a5) *and* (10.3.1) *hold and*

$$\lim_{x\to 0} \frac{f_k(x)}{x} = 1, \; k = 1, \cdots, m. \tag{10.3.3}$$

*If for some $\varepsilon > 0$ all solutions of the equation*

$$\dot{x}(t) + (1-\varepsilon) \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} x(s) d_s R_k(t, s) = 0 \tag{10.3.4}$$

*are oscillatory, then all solutions of* (10.2.4) *are also oscillatory.*

*Proof* Let $x(t)$ be an eventually positive solution of (10.2.4). Then $\lim_{t\to 0} x(t) = 0$ by Theorem 10.1. By (10.3.3), for any $\varepsilon > 0$ there exists $t_1$ such that

$$f_k\big(x(t)\big) \ge (1-\varepsilon)x(t), \; t \ge t_1, \; k = 1, \cdots, m.$$

Thus

$$\dot{x}(t)+(1-\varepsilon)\sum_{k=1}^{m} r_k(t)\int_{h_k(t)}^{t} x(s)d_s R_k(t,s)$$
$$\leq \dot{x}(t)+\sum_{k=1}^{m} r_k(t)\int_{h_k(t)}^{t} f_k\big(x(t)\big)d_s R_k(t,s)=0.$$

By Theorem 4.9, (10.3.4) has a nonoscillatory solution and we have a contradiction.

In the case $x<0$, for any $\varepsilon>0$ there exists $t_1$ such that $f_k(x(t))\leq(1-\varepsilon)x(t)$, $t\geq t_1$, $k=1,\cdots,m$. Similar to the previous case, (10.3.4) has a nonoscillatory solution, which completes the proof. □

**Corollary 10.1** *Suppose* (a1), (a2), (a4), (a5), (10.3.1) *and* (10.3.3) *hold. If for some* $\varepsilon>0$ *all solutions of the equation*

$$\dot{x}(t)+(1-\varepsilon)\sum_{k=1}^{m} r_k(t)x\big(h_k(t)\big)=0$$

*are oscillatory, then all solutions of* (10.2.7) *are also oscillatory.*

**Corollary 10.2** *Suppose* (a1), (a2), (a4)–(a6), (10.3.1) *and* (10.3.3) *hold. If for some* $\varepsilon>0$ *all solutions of the equation*

$$\dot{x}(t)+(1-\varepsilon)\sum_{k=1}^{m} r_k(t)\int_{h_k(t)}^{t} M_k(t,s)x(s)\,ds=0$$

*are oscillatory, then all solutions of* (10.2.8) *are also oscillatory.*

**Corollary 10.3** *Suppose* (a1), (a2), (a4)–(a6), (10.3.1) *and* (10.3.3) *hold. If for some* $\varepsilon>0$ *all solutions of the equation*

$$\dot{x}(t)+(1-\varepsilon)\left[\sum_{k=1}^{l} r_k(t)x\big(h_k(t)\big)+\sum_{k=l+1}^{m} r_k(t)\int_{h_k(t)}^{t} M_k(t,s)x(s)\,ds\right]=0$$

*are oscillatory, then all solutions of* (10.2.9) *are oscillatory.*

Now let us proceed to nonoscillation.

**Theorem 10.3** *Suppose* (a1)–(a5) *hold and for all* $k=1,\cdots,m$ *either*

$$0<f_k(x)\leq x,\ \ x>0, \tag{10.3.5}$$

*or*

$$0>f_k(x)\geq x,\ \ x<0, \tag{10.3.6}$$

*and there exists a nonoscillatory solution of* (10.2.10). *Then there exists a nonoscillatory* (*positive or negative*, *respectively*) *solution of* (10.2.4).

*Proof* First suppose that (10.3.5) holds and there exists a nonoscillatory solution of (10.2.10). Then by Theorem 4.1 there exists $w_0(t) \ge 0$, which is a solution of the following inequality for $t \ge t_1$:

$$w_0(t) \ge \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} \exp\left\{\int_s^t w_0(\tau)\,d\tau\right\} d_s R_k(t,s). \tag{10.3.7}$$

Let us fix $b \ge t_1$ and define the operator

$$(Tu)(t) = \exp\left\{\int_{t_1}^{t} u(\tau)\,d\tau\right\} \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k\left(\exp\left\{-\int_{t_1}^{s} u(\tau)\,d\tau\right\}\right) d_s R_k(t,s)$$

for $t_1 \le t \le b$ (we assume $u(s) = 0$, $s < t_1$). For any $u$ from the interval $0 \le u \le w_0$ we have by (10.3.5)

$$0 \le (Tu)(t) \le \exp\left\{\int_{t_1}^{t} u(\tau)\,d\tau\right\} \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} \exp\left\{-\int_{t_1}^{s} u(\tau)\,d\tau\right\} d_s R_k(t,s)$$
$$= \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} \exp\left\{\int_{s}^{t} u(\tau)\,d\tau\right\} d_s R_k(t,s) \le w_0(t),$$

so $0 \le Tu \le w_0$. Thus $T$ maps a closed segment in $\mathbf{L}_\infty[t_1, b]$ onto itself.

Now let us prove that for any $b > t_1$ the operator $T$ is compact in $L_\infty[t_1, b]$. Let us fix $k$ and omit this index (the sum of $m$ compact operators is compact). Denote

$$(T_1 u)(t) = \int_{t_1}^{t} u(\tau)\,d\tau, \;\; (T_2 u)(t) = r(t) e^{u(t)} \int_{h(t)}^{t} e^{-u(s)} d_s R(t,s).$$

For any $u$ in the unit ball

$$B_1 = \left\{u \in L_\infty[t_1, b] \mid 0 \le u(t) \le 1,\; t \in [t_1, b]\right\},$$

the function

$$y(s) = \int_{t_1}^{s} u(\tau)\,d\tau$$

is continuous; moreover, all such functions are bounded ($|y(s)| \le b - t_1$) and equicontinuous:

$$\left|y(t) - y(s)\right| = \left|\int_t^s u(\tau)\,d\tau\right| \le \operatorname*{ess\,sup}_{\tau} \left|u(\tau)\right| |t - s| \le |t - s|.$$

Thus the image of the unit ball is compact by Theorem A.3.

Then operator $T_1$ is a compact operator in the space $\mathbf{L}_\infty[t_1, b]$. Moreover, it is compact as an operator $T_1 : L_\infty[t_1, b] \to C[t_1, b]$. Evidently the operator $T_2 : \mathbf{C}[t_1, b] \to L_\infty[t_1, b]$ is continuous. Then the composition $T = T_2 T_1$ is a compact operator in the space $L_\infty[t_1, b]$.

Thus, by the Schauder Fixed-Point Theorem (Theorem A.15), there exists a nonnegative solution of the equation $u = Tu$. Then the function

$$x(t) = \begin{cases} \exp\{-\int_{t_1}^{t} u(\tau)\, d\tau\}, & t \geq t_1, \\ 1, & t < t_1, \end{cases}$$

is an eventually positive solution of (10.2.4).

If (10.3.6) holds (i.e., $f(x) \geq x$ for $x < 0$), then in this case we consider the segment $-w_0(t) \leq u(t) \leq 0$ and the operator

$$(Tu)(t) = \exp\left\{-\int_{t_1}^{t} u(\tau)\, d\tau\right\} \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k\left(-\exp\left\{\int_{t_1}^{s} u(\tau)\, d\tau\right\}\right) d_s R_k(t,s)$$

in the segment, which satisfies $-w_0(t) \leq (Tu)(t) \leq 0$, as far as $-w_0(t) \leq u(t) \leq 0$. Similarly, we demonstrate that the function

$$x(t) = \begin{cases} -\exp\{\int_{t_1}^{t} u(\tau)\, d\tau\}, & t \geq t_1, \\ -1, & t < t_1, \end{cases}$$

is an eventually negative solution of (10.2.4). □

**Corollary 10.4** *Suppose* (a1), (a2), (a4) *and* (a5) *hold, for each* $k = 1, \cdots, m$ *either* (10.3.5) *or* (10.3.6) *is satisfied and there exists a nonoscillatory solution of* (10.2.18). *Then there exists a nonoscillatory* (*positive or negative*, *respectively*) *solution of nonlinear equation* (10.2.7).

**Corollary 10.5** *Suppose* (a1), (a2) *and* (a4)–(a6) *hold, for each* $k = 1, \cdots, m$ *either* (10.3.5) *or* (10.3.6) *is satisfied and there exists a nonoscillatory solution of* (10.2.19). *Then there exists a nonoscillatory* (*positive or negative*, *respectively*) *solution of nonlinear integrodifferential equation* (10.2.8).

**Corollary 10.6** *Suppose* (a1), (a2) *and* (a4)–(a6) *hold, for each* $k = 1, \cdots, m$ *either* (10.3.5) *or* (10.3.6) *is satisfied and there exists a nonoscillatory solution of* (10.2.20). *Then there exists a nonoscillatory* (*positive or negative*, *respectively*) *solution of mixed equation* (10.2.9).

## 10.4 Applications

For equations of population ecology considered in this section, existence of a global positive solution is justified in Appendix B (see Theorems B.12–B.17 and their corollaries) if we assume positive initial conditions:

$$\varphi(t) \geq 0,\ t \leq t_0,\ \varphi(t_0) > 0. \tag{10.4.1}$$

### 10.4.1 Logistic Equation

Consider the logistic equation with a distributed delay

$$\dot{N}(t) = N(t) \sum_{k=1}^{m} r_k(t) \left( 1 - \frac{1}{K} \int_{h_k(t)}^{t} N(s)\, d_s R_k(t,s) \right) \tag{10.4.2}$$

with the initial conditions

$$N(t) = \varphi(t), \ t \le t_0, \tag{10.4.3}$$

where (a1)–(a4) and (10.4.1) are satisfied and $K > 0$. The existence of a global solution is due to Theorem B.14, and this solution is positive if (10.4.1) holds.

After the substitution $N(t) = Ke^{x(t)}$, (10.4.2) has the form

$$\dot{x}(t) = -\sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f\bigl(x(s)\bigr) d_s R_k(t,s), \tag{10.4.4}$$

where the function $f(x) = e^x - 1$ satisfies both (10.3.3) and (10.3.6).

The results of Sect. 10.3 and Lemma 10.1 imply the following result.

**Theorem 10.4** *Suppose* (a1)–(a4) *and* (10.4.1) *hold. If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[ \int_{h_k(\tau)}^{\tau} d_s R_k(t,s) \right] d\tau < \frac{1}{e}, \tag{10.4.5}$$

*then* (10.4.2) *has a nonoscillatory solution about $K$. If*

$$\liminf_{t\to\infty} \int_{H(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \left[ \int_{h_k(t)}^{\tau} d_s R_k(t,s) \right] d\tau > \frac{1}{e}, \tag{10.4.6}$$

*then all solutions of* (10.4.2) *are oscillatory about $K$. Here $h(t)$, $H(t)$ are defined in* (10.2.11).

The result of Theorem 10.4 was obtained in [53, Theorem 5] using a different method. As corollaries, oscillation and nonoscillation results for the logistic equation with concentrated delays

$$\dot{N}(t) = N(t) \sum_{k=1}^{m} r_k(t) \left( 1 - \frac{N(h_k(t))}{K} \right) \tag{10.4.7}$$

can be obtained (see [192]).

**Corollary 10.7** *Let $r_k \ge 0$ be Lebesgue measurable locally essentially bounded functions, the delays $h_k(t) \le t$ satisfy $\lim_{t\to\infty} h_k(t) = \infty$ and* (10.4.1) *hold. If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{k=1}^{m} r_k(\tau)\, d\tau < \frac{1}{e}, \tag{10.4.8}$$

*then* (10.4.7) *has a solution nonoscillatory about* $K$. *If*

$$\liminf_{t\to\infty} \int_{H(t)}^{t} \sum_{k=1}^{m} r_k(\tau)\, d\tau > \frac{1}{e}, \tag{10.4.9}$$

*then all solutions of* (10.4.7) *are oscillatory about* $K$. *Here* $h(t)$, $H(t)$ *are defined in* (10.2.11).

It is also possible to deduce a result for the integrodifferential logistic equation

$$\dot{N}(t) = N(t) \sum_{k=1}^{m} r_k(t) \left( 1 - \frac{1}{K} \int_{h_k(t)}^{t} M_k(t,s) N(s)\, ds \right). \tag{10.4.10}$$

**Corollary 10.8** *Let* $r_k \geq 0$ *be Lebesgue measurable locally essentially bounded functions,* $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $M_k(t,s)$ *be nonnegative Lebesgue measurable locally integrable functions satisfying*

$$\int_{h_k(t)}^{t} M_k(t,s)\, ds = 1, \; k = 1, \cdots, m, \tag{10.4.11}$$

*and let condition* (10.4.1) *hold.*

*If*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{k=1}^{m} r_k(\tau) \int_{h_k(\tau)}^{\tau} M_k(\tau,s)\, ds\, d\tau < \frac{1}{e}, \tag{10.4.12}$$

*then* (10.4.10) *has a nonoscillatory solution about* $K$. *If*

$$\liminf_{t\to\infty} \int_{H(t)}^{t} \sum_{k=1}^{m} r_k(\tau)(t-\tau) M_k(t,\tau)\, d\tau > \frac{1}{e}, \tag{10.4.13}$$

*then all solutions of* (10.4.10) *are oscillatory about* $K$. *Here* $h(t)$, $H(t)$ *are defined in* (10.2.11).

### *10.4.2 Lasota-Wazewska Equation*

Consider the generalized Lasota-Wazewska equation [331] for the survival of red blood cells with a distributed delay

$$\dot{N}(t) = -\mu N(t) + p \int_{h(t)}^{t} e^{-\gamma N(s)}\, d_s R(t,s) \tag{10.4.14}$$

with initial conditions (10.4.3), where $\mu > 0$, $p > 0$, $\gamma > 0$, (a2) and (a3) and (10.4.1) are satisfied. The global solution of (10.4.14), (10.4.3) exists and is unique by Corollary B.4; moreover, this solution is positive if (10.4.1) holds.

The unique positive equilibrium of (10.4.14) is a solution of the equation

$$N^* = \frac{p}{\mu} e^{-\gamma N^*}. \tag{10.4.15}$$

After the change of variables

$$N(t) = N^* + \frac{1}{\gamma}x(t),$$

(10.4.14) takes the form

$$\dot{x}(t) + \mu x(t) + \mu\gamma N^* \int_{h(t)}^{t} f\big(x(s)\big) d_s R(t,s) = 0, \tag{10.4.16}$$

where the function

$$f(x) = 1 - e^{-x} \tag{10.4.17}$$

satisfies conditions (10.3.3) and (10.3.5).

Thus the results of Sect. 10.3 imply the following theorems.

**Theorem 10.5** *Suppose that* (a2), (a3) *and* (10.4.1) *hold, where* $m = 1$ *and* $R_1(t,s) = R(t,s)$, $h_1(t) = h(t)$. *If in addition there exists* $\varepsilon > 0$ *such that all solutions of the linear equation*

$$\dot{x}(t) + (1-\varepsilon)\mu x(t) + (1-\varepsilon)\mu\gamma N^* \int_{h(t)}^{t} x(s) d_s R(t,s) = 0 \tag{10.4.18}$$

*are oscillatory, then all solutions of* (10.4.14) *oscillate about* $N^*$.

**Theorem 10.6** *Suppose that* (a2), (a3) *and* (10.4.1) *hold, where* $m = 1$ *and* $R_1(t,s) = R(t,s)$, $h_1(t) = h(t)$, *and there exists a nonoscillatory solution of the linear equation*

$$\dot{x}(t) + \mu x(t) + \mu\gamma N^* \int_{h(t)}^{t} x(s) d_s R(t,s) = 0. \tag{10.4.19}$$

*Then there exists a solution of* (10.4.14) *that is nonoscillatory about* $N^*$.

For the particular case of (10.4.14) with a variable concentrated delay

$$\dot{N}(t) = -\mu N(t) + pe^{-\gamma N(h(t))}, \tag{10.4.20}$$

we obtain the following corollary that was earlier deduced in [55].

**Corollary 10.9** *Let* $\lim_{t\to\infty} \sup(t - h(t)) < \infty$. *If*

$$\liminf_{t\to\infty} \left[ \mu\gamma N^* \int_{h(t)}^{t} \exp\big\{\mu\big(s - h(s)\big)\big\}\, ds \right] > \frac{1}{e},$$

*then all solutions of* (10.4.20) *are oscillatory about* $N^*$. *If*

$$\limsup_{t\to\infty} \left[ \mu\gamma N^* \int_{h(t)}^{t} \exp\big\{\mu\big(s - h(s)\big)\big\}\, ds \right] < \frac{1}{e},$$

*then there exists a solution of* (10.4.20) *nonoscillatory about* $N^*$.

For the integrodifferential equation

$$\dot{N}(t) = -\mu N(t) + p \int_{h(t)}^{t} M(t,s) e^{-\gamma N(s)}\, ds, \tag{10.4.21}$$

Theorem 10.5 implies the following result.

**Corollary 10.10** *Let $M(t,s)$ be a Lebesgue measurable locally essentially bounded function, $M(t,s) \ge 0$, $\mu > 0$, $p > 0$, $\gamma > 0$, $\int_{h(t)}^{t} M(t,s)\, ds = 1$ for any $t > 0$. If in addition*

$$\liminf_{t\to\infty} \left[ \mu\gamma N^* \int_{h(t)}^{t} M(t,\tau)(t-\tau) \exp\{\mu(\tau - h(\tau))\}\, d\tau \right] > \frac{1}{e},$$

*then all solutions of (10.4.21) are oscillatory about $N^*$.*

*If*

$$\limsup_{t\to\infty} \left[ \mu\gamma N^* \int_{h(t)}^{t} d\tau \int_{h(\tau)}^{\tau} M(\tau,s) \exp\{\mu(s - h(s))\}\, ds \right] < \frac{1}{e},$$

*then there exists a solution of (10.4.21) nonoscillatory about $N^*$.*

To illustrate the application of Theorem 10.5 to different models, consider the mixed equation

$$\dot{N}(t) = -\mu N(t) + p \left[ \int_{h(t)}^{t} M(t,s) e^{-\gamma N(s)}\, ds + \alpha(t) e^{-\gamma N(g(t))} \right], \tag{10.4.22}$$

where $M(t,s) \ge 0$, $\mu > 0$, $p > 0$, $\gamma > 0$, $\alpha(t) \ge 0$.

**Corollary 10.11** *Let $M(t,s)$ be a Lebesgue measurable locally essentially bounded function and (a2) hold for $h(t)$ and $g(t)$,*

$$M(t,s) \ge 0,\ \mu > 0,\ p > 0,\ \gamma > 0,\ \int_{h(t)}^{t} M(t,s)\, ds + \alpha(t) = 1,\ t > 0.$$

*If*

$$\liminf_{t\to\infty} \mu\gamma N^* \int_{\max\{h(t),\, g(t)\}}^{t} \big[ M(t,\tau)(t-\tau) e^{\mu(\tau - h(\tau))} + \alpha(\tau) e^{\mu(\tau - g(\tau))} \big]\, d\tau > \frac{1}{e},$$

*then all solutions of (10.4.22) are oscillatory about $N^*$.*

*If*

$$\limsup_{t\to\infty} \mu\gamma N^* \int_{\min\{h(t), g(t)\}}^{t} \left[ \int_{h(\tau)}^{\tau} M(\tau,s) e^{\mu(s-h(s))}\, ds + \alpha(\tau)\, e^{\mu(\tau - g(\tau))} \right] d\tau < \frac{1}{e},$$

*then there exists a solution of (10.4.22) nonoscillatory about $N^*$.*

Similar results can be obtained for equations with several concentrated delays and several integral terms:

$$\dot{N}(t) = -\mu N(t) + p\left[\sum_{k=1}^{l}\int_{h_k(t)}^{t} M_k(t,s)e^{-\gamma N(s)}\,ds + \sum_{k=l+1}^{m}\alpha_k(t)e^{-\gamma N(h_k(t))}\right]. \tag{10.4.23}$$

**Corollary 10.12** *Let $M_k(t,s)$ be Lebesgue measurable locally essentially bounded functions, $(a2)$ hold for $h_k(t)$, $M_k(t,s) \ge 0$, $k = 1,\cdots,l$, $\mu > 0$, $p > 0$, $\gamma > 0$, $\alpha_k(t) \ge 0$, $k = l+1,\cdots,m$,*

$$\sum_{k=1}^{l}\int_{h_k(t)}^{t} M_k(t,s)\,ds + \sum_{k=l+1}^{m}\alpha_k(t) = 1 \textit{ for any } t > 0.$$

*If*

$$\liminf_{t\to\infty} \mu\gamma N^* \int_{\max_k\{h_k(t)\}}^{t}\left[\sum_{k=1}^{l} M_k(t,\tau)(t-\tau)e^{\mu(\tau-h_k(\tau))} + \sum_{k=l+1}^{m}\alpha_k(\tau)e^{\mu(\tau-h_k(\tau))}\right]d\tau > \frac{1}{e},$$

*then all solutions of* (10.4.22) *are oscillatory about $N^*$ .*

*If*

$$\limsup_{t\to\infty} \mu\gamma N^* \int_{\min_k\{h_k(t)\}}^{t}\left[\sum_{k=1}^{l}\int_{h_k(\tau)}^{\tau} M_k(\tau,s)e^{\mu(s-h_k(s))}\,ds + \sum_{k=l+1}^{m}\alpha_k(\tau)e^{\mu(\tau-h_k(\tau))}\right]d\tau < \frac{1}{e},$$

*then there exists a solution of* (10.4.22) *nonoscillatory about $N^*$.*

### *10.4.3 Nicholson's Blowflies Equation*

Now let us apply the results above to Nicholson's blowflies equation with a distributed delay

$$\dot{N}(t) - p\int_{h(t)}^{t} N(s)e^{-aN(s)}d_sR(t,s) + \delta N(t) = 0,\ t > t_0, \tag{10.4.24}$$

with initial conditions (10.4.3), where $p > \delta > 0$, $a > 0$ and (a2)–(a4) are satisfied.

Equation (10.4.24) has the unique positive equilibrium

$$N^* = \frac{1}{a} \ln \frac{p}{\delta}. \tag{10.4.25}$$

The global solution of (10.4.24), (10.4.3) exists and is unique by Corollary B.5 (see also [86]); moreover, this solution is positive for $t \geq t_0$ as far as the initial conditions are positive.

We can apply the linearization method after the transformation

$$N = N^* + \frac{1}{a} x, \tag{10.4.26}$$

where $N^*$ is defined in (10.4.25). Then (10.4.24) takes the form

$$\dot{x}(t) + \delta x(t) - \delta \int_{h(t)}^{t} x(s) e^{-x(s)} d_s R_k(t,s)$$
$$+ \delta \int_{h(t)}^{t} \ln\left(\frac{p}{\delta}\right)\left[1 - e^{-x(s)}\right] d_s R_k(t,s) = 0,$$

which can be rewritten as

$$\dot{x}(t) + \delta x(t) + \delta \int_{h(t)}^{t} \left[\ln\left(\frac{p}{\delta}\right)\left(1 - e^{-x(s)}\right) - x(s) e^{-x(s)}\right] d_s R(t,s) = 0. \tag{10.4.27}$$

Consider the function

$$f(x) = \frac{1}{\ln(\frac{p}{\delta}) - 1}\left[\ln\left(\frac{p}{\delta}\right)\left(1 - e^{-x}\right) - x e^{-x}\right]. \tag{10.4.28}$$

Then (10.4.27) has the form

$$\dot{x}(t) + \delta x(t) + \delta\left[\ln\left(\frac{p}{\delta}\right) - 1\right] \int_{h(t)}^{t} f\big(x(s)\big) d_s R(t,s) = 0. \tag{10.4.29}$$

**Lemma 10.2** *Let $f(x)$ be defined in* (10.4.28) *and $p > \delta > 0$.*

1) *Condition* (10.3.3) *holds.*
2) *If $p > \delta e$, $x \neq 0$, $x > 1 - \ln(p/\delta)$, then $x f(x) > 0$.*
3) *If $p > \delta e^2$, then* (10.3.5) *is satisfied.*

*Proof* Since $\lim_{x \to 0} \frac{1 - e^{-x}}{x} = \lim_{x \to 0} e^{-x} = 1$, condition (10.3.3) holds for $p > \delta$.

The function $f(x)$ defined in (10.4.28) vanishes at zero. For $p > \delta e$, its derivative satisfies

$$f'(x) = e^{-x} + \frac{x e^{-x}}{\ln(\frac{p}{\delta}) - 1} > 0 \text{ for } x > 1 - \ln\left(\frac{p}{\delta}\right),$$

and $f'(x)$ is negative otherwise, so

$$f(x) > 0 \text{ for } x > 0 \text{ and } f(x) < 0 \text{ for } 1 - \ln(p/\delta) < x < 0.$$

Further, consider $g(x) = f(x) - x$. Then

$$g'(x) = e^{-x} - 1 + \frac{xe^{-x}}{\ln(\frac{p}{\delta}) - 1}, \quad g''(x) = \frac{2 - \ln(\frac{p}{\delta}) - x}{\ln(\frac{p}{\delta}) - 1} e^{-x}.$$

Thus $g(0) = 0$, $g'(0) = 0$ and $\ln(\frac{p}{\delta}) > 2$ imply $g''(x) < 0$ for $x > 0$. Consequently, for $p > \delta e^2$ the first derivative is negative for $x > 0$ and $g(x) < g(0) = 0$, or $f(x) < x$, $x > 0$. Since also $f(x) > 0$ for $x > 0$, then (10.3.5) holds, which completes the proof. □

**Lemma 10.3** *Suppose $p > \delta e$ and a solution $N(t)$ of (10.4.24) is below the equilibrium $N(t) < N^*$ for any $t > t_1 \geq 0$. Then there exists $t^*$ such that*

$$N(t) > \frac{1}{a}, \quad t > t^*. \tag{10.4.30}$$

*Proof* Denote

$$g(x) = \frac{p}{\delta} x e^{-ax}. \tag{10.4.31}$$

According to (a2), there exists $t_2 \geq t_1$ such that $h(t) > t_1$ for $t > t_2$. Since the solution $N(t)$ is positive and continuous, there exists

$$N_1 = \min_{t \in [t_1, t_2]} N(t) < N^*. \tag{10.4.32}$$

(1) First, let us demonstrate that $N_1 > 1/a$ implies $N(t) > 1/a$ for any $t > t_1$. Assume the contrary. Denote

$$\bar{t} = \inf\left\{ t > t_2 \,\middle|\, N(t) < \frac{1}{a} \right\}.$$

By definition, $N(\bar{t}) = 1/a$ and $N^* > N(t) > 1/a$ for $t \in [t_1, \bar{t})$. Thus $g(N(t)) > N^*$ for $t \in [t_1, \bar{t})$ (see Fig. 10.1). Consequently, from (10.4.24) we have

$$\dot{N}(t) \geq \delta \Big[ \inf_{s \in [t_1, t]} g\big(N(s)\big) - N(t) \Big] > \delta\big(N^* - N^*\big) = 0, \quad t \in [t_2, \bar{t})$$

almost everywhere, and the nonnegative derivative in the segment $[t_2, \bar{t})$ contradicts the assumption $N(\bar{t}) = 1/a < N(t_2)$.

(2) Next, let us assume $m < N^*$ and prove that once $N(t) > m$, $t \in [t_1, t_2]$, then first $N(t) > m$ for any $t \geq t_1$ and second, if $c \leq g(m) < N^*$ and there is $t_3$ such that $N(t_3) = c$, then $N(t) \geq c$, $t \geq t_3$.

As in part (1) of the proof, first assume that there are points where $N(t)$ does not exceed $m$ and denote $\bar{t} = \inf\{t > t_2 \mid N(t) < m\}$. By definition, $N(\bar{t}) = m$ and $N^* > N(t) > m$ for $t \in [t_1, \bar{t})$. Since $N(t)$ is continuous and $g(m) > m$, then there exists $\varepsilon > 0$ such that $N(t) < g(m)$ for $t \in [\bar{t} - \varepsilon, \bar{t}]$. Besides, $N(t) < N^*$ for any $t$. Let us notice that

$$\min_{x \in [m, N^*]} g(x) = \min\big\{g(m), N^*\big\} \text{ and } g\big(N(t)\big) > \min\big\{g(m), N^*\big\} \text{ for } t \in [t_1, \bar{t}].$$

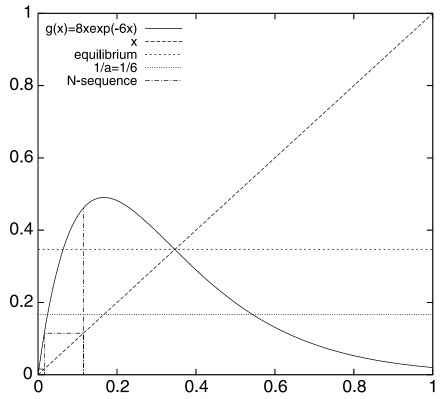


**Fig. 10.1** The function $g(x) = 8x\exp(-6x)$, the equilibrium $N^*$, $1/a = 1/6$ and $g(N_1)$, $g(g(N_1))$ in the sequence. In the example presented, $g(g(N_1))$ already exceeds $1/a$. $N_2$ in part (3) of the proof exceeds $g(N_1)$, $N_3 = N^*$

Hence, for $t \in [\bar{t} - \varepsilon, \bar{t})$ we have $N(t) < g(m)$ and

$$\dot{N}(t) \geq \delta\Big[\inf_{s\in[t_1,\bar{t}]} g\big(N(s)\big) - N(t)\Big] > \delta\big(\min\{g(m), N^*\} - \min\{g(m), N^*\}\big) = 0 \tag{10.4.33}$$

almost everywhere, which contradicts the assumption $N(\bar{t}) = m < N(\bar{t} - \varepsilon)$.

If $g(m) < N^*$, $N(t_3) = c \leq g(m)$ and there is $t_4 > t_3$, where $N(t_4) < c$. According to the previous part, $N(t) > m$ and $g(N(t)) > g(m)$ for any $t > t_1$. Then, as in (10.4.33), $\dot{N}(t) \geq 0$ in $[t_4 - \varepsilon_1, t_4]$, which contradicts $N(t_4) < N(t_4 - \varepsilon_1)$.

(3) Finally, assuming $N_1 < 1/a$, we build a sequence of $N_k$ that eventually exceeds $1/a$ and a sequence of increasing points $s_k$ such that $t \geq s_k$ implies $N(t) \geq N_k$, $t > s_k$. Let $N_1 \leq 1/a$. Consider $N_2 = \min\{0.5(N^* + 1/a), g(N_1)\}$, $N_2 \geq N_1$ (see Fig. 10.1). According to part (2) of the proof, there may be two possibilities: for some $s_2 = t_3 > t_2$ we have $N(t_3) = N_2$ and also $N(t) \geq N_2$ for $t \geq s_2$, or $N$ is increasing (see (10.4.33)) and is less than $N_2$ for any $t > t_1$. The latter is impossible. In fact, assuming $N < N_2$ implies $\dot{N} > \delta(g(N_1) - N_1) > 0$. Thus $N(t) \to \infty$ as $t \to \infty$, which contradicts $N(t) < N^*$. Hence $N(t_3) = N_2$ for some $t_3$. Similarly, we define $N_k = \min\{0.5(N^* + 1/a), g(N_{k-1})\}$. By induction, we prove that for some $s_k > s_{k-1}$ we have $N(t) \geq N_k$ for $t > s_k$. The sequence $\{N_k\}$ is nondecreasing (i.e., each element is less than $N^*$ and eventually exceeds $1/a$). Let $N_k > 1/a$. Then $s_k = t^*$, where the existence of $t^*$ is claimed in the statement of the lemma, which completes the proof. □

*Remark 10.3* Continuing the proof of Lemma 10.3, we could obtain that any solution of (10.4.24) that is less than the equilibrium converges to $N^*$.

Let us also note that according to Theorem 10.1 any nonoscillatory solution tends to zero. Thus, applying Theorems 10.2, 10.3, 4.9 and 4.11, we obtain the following results.

**Theorem 10.7** *Suppose $h(t)$, $R(t,s)$ and the initial conditions satisfy* (a2)–(a4), *$p > \delta e$. If*

$$\delta\left[\ln\left(\frac{p}{\delta}\right)-1\right]\liminf_{t\to\infty}\int_{h(t)}^{t}d\tau\int_{h(t)}^{\tau}e^{\delta(\tau-s)}d_sR(\tau,s)>\frac{1}{e}, \tag{10.4.34}$$

*then all solutions of* (10.4.24) *are oscillatory about $N^*$.*

*If $p > \delta e^2$ and*

$$\delta\left[\ln\left(\frac{p}{\delta}\right)-1\right]\limsup_{t\to\infty}\int_{h(t)}^{t}d\tau\int_{h(\tau)}^{\tau}e^{\delta(\tau-s)}d_sR(\tau,s)<\frac{1}{e}, \tag{10.4.35}$$

*then* (10.4.24) *has a solution nonoscillatory about $N^*$. For nonoscillatory solutions, we have $\lim_{t\to\infty} N(t) = N^*$.*

*Proof* Suppose that (10.4.34) holds. It is sufficient to prove that all solutions of (10.4.29) are oscillatory, where the function $f$ is denoted by (10.4.28). Let us check that all the conditions of Theorem 10.2 hold.

By Lemma 10.2, condition (10.3.3) is satisfied. After transformation of (10.4.26), the condition $x(t) > 1 - \ln(p/\delta)$ is equivalent to the inequality

$$N(t)=N^*+\frac{1}{a}x>\frac{1}{a}\left[\ln\left(\frac{p}{\delta}\right)+1-\ln\left(\frac{p}{\delta}\right)\right]=\frac{1}{a}.$$

We recall that $N^* > 1/a$ for $p > \delta e$. Hence, by Lemma 10.3, for any solution $x$ there exists $t_1$ such that $x(t) > 1 - \ln(p/\delta)$ for $t > t_1$. By Lemma 10.2, we have $xf(x) > 0$.

Condition (10.4.34) and Theorem 4.11 imply that for some $\varepsilon > 0$ all solutions of linear equation (10.3.4) are oscillatory. By Theorem 10.2, all solutions of (10.4.29) are also oscillatory.

The second part is based on Theorems 10.3 and 4.9 and is proven similarly. □

To deduce some corollaries, let us consider the following particular cases of (10.4.24): the equations with several concentrated delays

$$\dot N(t)-\frac{p}{m}\sum_{k=1}^{m}N\big(h_k(t)\big)e^{-aN(h_k(t))}+\delta N(t)=0, \tag{10.4.36}$$

the autonomous equation with a constant delay

$$\dot N(t)-pN(t-\tau)e^{-aN(t-\tau)}+\delta N(t)=0, \tag{10.4.37}$$

and the integrodifferential equation

$$\dot N(t)-p\int_{h(t)}^{t}M(t,s)N(s)e^{-aN(s)}\,ds+\delta N(t)=0. \tag{10.4.38}$$

**Corollary 10.13** *If $p > \delta e$ and*

$$\liminf_{t\to\infty}\frac{\delta}{m}\left[\ln\left(\frac{p}{\delta}\right)-1\right]\sum_{k=1}^{m}\int_{\max_k h_k(t)}^{t}e^{\delta(\tau-h_k(\tau))}\,d\tau>\frac{1}{e}, \tag{10.4.39}$$

*then all solutions of* (10.4.36) *are oscillatory about $N^*$.*

*If* $p > \delta e^2$ *and*

$$\limsup_{t\to\infty} \frac{\delta}{m}\left[\ln\left(\frac{p}{\delta}\right) - 1\right] \sum_{k=1}^{m} \int_{\min_k h_k(t)}^{t} e^{\delta(\tau - h_k(\tau))}\, d\tau < \frac{1}{e}, \tag{10.4.40}$$

*then* (10.4.36) *has a solution nonoscillatory about* $N^*$. *For nonoscillatory solutions, we have* $\lim_{t\to\infty} N(t) = N^*$.

**Corollary 10.14** [192, 238] *If* $p > \delta e$ *and*

$$\delta\left[\ln\left(\frac{p}{\delta}\right) - 1\right]\tau e^{\delta\tau} > \frac{1}{e}, \tag{10.4.41}$$

*then all solutions of* (10.4.37) *are oscillatory about* $N^*$. *If* $p > \delta e^2$ *and*

$$\delta\left[\ln\left(\frac{p}{\delta}\right) - 1\right]\tau e^{\delta\tau} < \frac{1}{e}, \tag{10.4.42}$$

*then* (10.4.37) *has a solution nonoscillatory about* $N^*$. *For nonoscillatory solutions, we have* $\lim_{t\to\infty} N(t) = N^*$.

**Corollary 10.15** *Let* $p > \delta e$, $M(t, s)$ *be a Lebesgue measurable locally essentially bounded function,* $M(t, s) \geq 0$ *and* $\int_{h(t)}^{t} M(t, s)\, ds = 1$ *for any* $t > 0$. *If*

$$\liminf_{t\to\infty} \delta\left[\ln\left(\frac{p}{\delta}\right) - 1\right] \int_{h(t)}^{t} e^{\delta(t-\tau)} M(t, \tau)(t - \tau)\, d\tau > \frac{1}{e}, \tag{10.4.43}$$

*then all solutions of* (10.4.38) *are oscillatory about* $N^*$. *If* $p > \delta e^2$ *and*

$$\limsup_{t\to\infty} \delta\left[\ln\left(\frac{p}{\delta}\right) - 1\right] \int_{h(t)}^{t} d\tau \int_{h(\tau)}^{\tau} e^{\delta(\tau-s)} M(\tau, s)\, ds < \frac{1}{e}, \tag{10.4.44}$$

*then* (10.4.38) *has a solution nonoscillatory about* $N^*$. *For nonoscillatory solutions, we have* $\lim_{t\to\infty} N(t) = N^*$.

Let us note that oscillation properties for Nicholson's blowflies equation with $\delta < p < \delta e$ are essentially different; see [195] for the constant concentrated delay and [86] for the distributed delay.

## 10.5 "Mean Value Theorem" for Equations with a Distributed Delay

Let us demonstrate that from a certain point of view an equation with a distributed delay (10.2.4) can be reduced to a linear equation with a single concentrated delay.

Denote

$$h_k(t) = \sup\left\{s \in \mathbb{R} \mid R_k(t, s) = 0\right\},\ \ h(t) = \min_k h_k(t), \tag{10.5.1}$$

$$G_k(t) = \inf\{s \le t \mid R_k(t,s) = 1\}, \ G(t) = \max_k G_k(t). \tag{10.5.2}$$

Let us note that for (10.2.7) with concentrated delays

$$h(t) = \min_k h_k(t), \ G(t) = \max_k h_k(t),$$

for integrodifferential equation (10.2.8) we have $h(t) = \min_k h_k(t)$ and, generally, $G(t) = t$; however, if for some $F_k(t) < G_k(t)$

$$M_k(t,s) = 0, \ G_k(t) < s \le t, \ M_k(t,s) > 0, \ F_k(t) < s < G_k(t),$$

then $G(t) = \max_k G_k(t)$. Further, for mixed equation (10.2.9)

$$h(t) = \min_k h_k(t), \ G(t) = \max\{h_1(t), \cdots, h_l(t), G_{l+1}(t), \cdots, G_m\},$$

where $G_k$ are defined as above.

**Theorem 10.8** *Suppose that* (a1)–(a5) *hold*, $f_k(0) = 0$, $k = 1, \cdots, m$, *and* $f_k$ *are nondecreasing functions. Then, for any* $\varphi(t)$ *there exist functions* $\xi_k(t)$ *and* $g(t)$, *where* $h(t) \le g(t) \le G(t)$ ($h(t), G(t)$ *are defined in* (10.5.1) *and* (10.5.2), *respectively*) *such that the solution of problem* (10.2.4), (10.2.6) *also satisfies the linear equation with a single concentrated delay*

$$\dot{x}(t) + \left(\sum_{k=1}^{m} r_k(t) f_k'\big(\xi_k(t)\big)\right) x\big(g(t)\big) = 0. \tag{10.5.3}$$

*Proof* Since $f_k(x(\cdot))$ is continuous for any $k$ and continuous function $x$, including the initial function satisfying (a4), then by the Mean Value Theorem for any $k$ and any $t \ge t_0$ there exists $g_k(t)$ such that $h_k(t) \le g_k(t) \le G_k(t)$ and

$$\int_{h_k(t)}^{t} f_k\big(x(s)\big)\, d_s R_k(t,s) = f_k\Big(x\big(g_k(t)\big)\Big) \int_{h_k(t)}^{t} d_s R_k(t,s) = f_k\Big(x\big(g_k(t)\big)\Big),$$

i.e., $x(t)$ is a solution of the equation

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f_k\Big(x\big(g_k(t)\big)\Big) = 0 \tag{10.5.4}$$

as well as

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \Big[f_k\Big(x\big(g_k(t)\big)\Big) - f_k(0)\Big] = 0. \tag{10.5.5}$$

By the Mean Value Theorem, the expression in the brackets equals $f_k'(\xi_k(t)) \times x(g_k(t))$, where $\xi_k(t)$ is between zero and $x(g_k(t))$, so $x(t)$ is a solution of the linear equation with several delays and nonnegative coefficients

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f_k'\big(\xi_k(t)\big) x\big(g_k(t)\big) = 0. \tag{10.5.6}$$

Let $x(t)$ be a solution of (10.5.6). For any fixed $t \geq 0$ and any $k = 1, \cdots, m$, we have

$$\min_{s\in[h(t),G(t)]} x(s) \leq x\big(g_k(t)\big) \leq \max_{s\in[h(t),G(t)]} x(s).$$

Since all the derivatives satisfy $f_k'(x) \geq 0$, we have

$$\min_{s\in[h(t),G(t)]} x(s) \sum_{k=1}^{m} r_k(t) f_k'\big(\xi_k(t)\big) \leq \sum_{k=1}^{m} r_k(t) f_k'\big(\xi_k(t)\big) x\big(g_k(t)\big)$$
$$\leq \max_{s\in[h(t),G(t)]} x(s) \sum_{k=1}^{m} r_k(t) f_k'\big(\xi_k(t)\big).$$

Solution $x(s)$ is continuous so, for any $t$, by the Intermediate Value Theorem there is $s^*(t) \in [h(t), G(t)]$ such that

$$x\big(s^*(t)\big) = \left[\sum_{k=1}^{m} r_k(t) f_k'\big(\xi_k(t)\big) x\big(g_k(t)\big)\right] \Big/ \left[\sum_{k=1}^{m} r_k(t) f_k'\big(\xi_k(t)\big)\right].$$

Choosing $g(t) = s^*(t)$ for each $t$, we obtain that the function $x$ is also a solution of (10.5.3) with one delay, which completes the proof. □

Theorem 10.8 can also be applied to linear equations.

**Corollary 10.16** *Suppose* (a1)–(a4) *hold. Then there exists* $g(t)$, *where* $h(t) \leq g(t) \leq H(t)$, *and* $h(t)$ *and* $G(t)$ *are defined in* (10.5.1) *and* (10.5.2), *respectively, such that the solution of* (10.2.10) *and* (10.2.6) *also satisfies the linear equation with a single concentrated delay*

$$\dot{x}(t) + \left(\sum_{k=1}^{m} r_k(t)\right) x\big(g(t)\big) = 0. \tag{10.5.7}$$

The following result is also an immediate corollary of Theorem 10.8.

**Corollary 10.17** *Suppose* (a1)–(a5) *hold,* $f_k(0) = 0$, $f_k$ *are nondecreasing functions,* $k = 1, \cdots, m$ *and the linear equation*

$$\dot{x}(t) + \left(\sum_{k=1}^{m} A_k r_k(t)\right) x\big(g(t)\big) = 0 \tag{10.5.8}$$

*for any* $g(t)$ *such that* $h(t) \leq g(t) \leq G(t)$ *and any* $A_k$ *satisfying*

$$0 \leq \inf_{t\in\mathbb{R}} f_k'(t) \leq A_k \leq \sup_{t\in\mathbb{R}} f_k'(t) \tag{10.5.9}$$

*has one of the following properties*:

- *all solutions of* (10.5.8) *are oscillatory*;
- *there exists a nonoscillatory solution of* (10.5.8);
- *the zero solution of* (10.5.8) *is stable* (*globally asymptotically stable*);

- *all solutions of* (10.5.8) *with nonnegative initial conditions and a positive initial value are positive* (*permanent*; *i.e., satisfy* $0 < a < x(t) < b < \infty$ *for any* $t$).

*Then* (10.2.4) *has the same property.*

For another approach to study asymptotic properties of nonlinear equations with a distributed delay see [271].

## 10.6 Discussion and Open Problems

Let us note that in this chapter we considered a general form of delays and coefficients in the following sense.

1. The distributed delay allows us, for an appropriate choice of the distribution, to consider integrodifferential equations, equations with several variable concentrated delays, and equations with both delayed and integral terms. All parameters are generally time dependent.
2. Solutions are absolutely continuous, not necessarily continuously differentiable functions. This corresponds to measurable locally essentially bounded (not necessarily continuous) kernels of integrals and coefficients.

Let us note that the chapter mainly follows the paper [65]. Some oscillation and nonoscillation results close to the results of this chapter were obtained in [275].

Finally, let us state some open problems.

1. Can the linearization scheme be applied to (10.2.4) and (10.2.5), where coefficients may be positive or negative? As an easy exercise, consider (10.2.5) with a nonnegative $r_k(t)$ and a nonpositive (or oscillatory) $b(t)$.
2. Is some linearization scheme applicable to the Mackey-Glass equations with a distributed delay, such as the equation
$$\frac{dN}{dt} = \int_{-\infty}^{t} \frac{r(s)N(r)}{1+(N(s))^{\gamma}}\, d_s R(t,s) - b(t)N(t)$$
describing the production of white blood cells and
$$\frac{dN}{dt} = \int_{-\infty}^{t} \frac{r(s)}{(K(t))^{\gamma}+(N(s))^{\gamma}}\, d_s R(t,s) - b(t)N(t)$$
modeling red blood cell production, where $r(s) \geq 0$, $b(t) \geq 0$, $K(s) > 0$, $\gamma > 0$?
3. So far, we have considered equations with a distributed delay of type (10.2.4). Consider nonoscillation and oscillation of the equation with a distributed delay
$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f_k\left(\int_{h_k(t)}^{t} x(s) d_s R_k(t,s)\right) = 0, \tag{10.6.1}$$
where (a1)–(a3) are satisfied.
4. Obtain nonoscillation results for Nicholson's blowflies equation with a distributed delay in the case $\delta e < p < \delta e^2$.

5. All results of this chapter are obtained under the assumption that all the delays are finite. Deduce nonoscillation conditions for (10.2.1) with generally an infinite delay.

# Chapter 11
# Nonlinear Models—Modifications of Delay Logistic Equations

## 11.1 Introduction

In this chapter, we study several nonlinear delay differential equations for which the linearization method of the previous chapter cannot be applied.

The delay logistic equation

$$\dot{y}(t) = r(t)y(t)\left(1 - \frac{y(h(t))}{K}\right), \ h(t) \le t, \tag{11.1.1}$$

is known as Hutchinson's equation if $r$ and $K$ are positive constants and $h(t) = t - \tau$ for a positive constant $\tau$. Hutchinson's equation was investigated by several authors, see, for example, [213, 214, 285, 334]. Delay logistic equation (11.1.1) was studied by Gopalsamy and Zhang [167, 354], who gave sufficient conditions for oscillation and nonoscillation of (11.1.1).

Publications [12, 154, 175, 176, 192, 237, 244, 255, 296, 330, 340] are devoted to various generalizations of logistic equation (11.1.1). For example, in [12, 154, 237] the authors considered the equation

$$\dot{y}(t) = r(t)y(t)\left(1 - \frac{y(h(t))}{K}\right)\left|1 - \frac{y(h(t))}{K}\right|^{\alpha-1}, \tag{11.1.2}$$

where $\alpha < 1$ (sublinear case) or $\alpha > 1$ (superlinear case).

In this chapter, we study (11.1.2) with several delays,

$$\dot{y}(t) = \sum_{k=1}^{m} r_k(t)y(t)\left(1 - \frac{y(h_k(t))}{K}\right)\left|1 - \frac{y(h_k(t))}{K}\right|^{\alpha_k-1}. \tag{11.1.3}$$

We consider the cases $\alpha_k \le 1,\ k = 1, \cdots, m,\ \alpha_k \ge 1,\ k = 1, \cdots, m$ and also the three mixed cases for $m = 2$. The case $\alpha_k = 1,\ k = 1, \cdots, m$ was considered in the previous chapter.

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Delay logistic equation (11.1.1) assumes both additive and multiplicative generalizations. For example, in [175] the authors considered the multiplicative logistic equation

$$\dot{y}(t) = r(t)y(t)\left(1 - \prod_{k=1}^{m} \frac{y(h_k(t))}{K}\right).$$

In [90, 336], oscillation properties of the nonlinear delay equation

$$\dot{y}(t) + a(t)y(t) + b(t)\prod_{k=1}^{m}\left(\frac{y(h_k(t))}{K}\right)^{\alpha_k} = 0, \ \sum_{k=1}^{m}\alpha_k = 1,$$

were studied. In [265], the authors considered the nonlinear neutral multiplicative differential equation

$$\frac{d}{dt}\big[r(t) - c(t)x(t-r)\big] + p(t)\prod_{k=1}^{m}[x(t-r_k)]^{\alpha_k} = 0,$$

where also $\sum_{k=1}^{m}\alpha_k = 1$.

In this chapter, we consider the generalized logistic equation where the sum in (11.1.3) is replaced by a product:

$$\dot{y}(t) = \prod_{k=1}^{m} r(t)y(t)\left(1 - \frac{y(h_k(t))}{K}\right)\left|1 - \frac{y(h_k(t))}{K}\right|^{\alpha_k - 1}, \ \sum_{k=1}^{m}\alpha_k = 1. \qquad (11.1.4)$$

It is interesting to discuss here some methods that are applied to obtain nonoscillation and oscillation results for the delay logistic equations and their generalizations.

Usually a differential equation is transformed into an operator equation with the following property: if the operator equation has a nonnegative solution, then the differential equation has a nonoscillatory solution. For the operator equation, either the Schauder Fixed-Point Theorem is applied, convergence of monotone approximations to a solution is demonstrated or the connection of oscillation properties of the nonlinear logistic equation and a linear delay differential equation is employed.

Here we use all three methods mentioned above. We do not assume that the parameters of (11.1.3) are continuous functions. Hence, unlike most known results, we have to apply the Schauder Fixed-Point Theorem in the space $L_\infty$ of Lebesgue measurable and locally essentially bounded functions. In the most difficult superlinear case, we transform the differential equation into an operator equation,

$$u = AuBu,$$

where the operator $A$ is monotonically increasing and $B$ is monotonically decreasing. We prove that there exist two functions $v$, $w$, where $0 \le v(t) \le w(t)$, such that

$$v(t) \le (Av)(t)(Bw)(t), \ w(t) \ge (Aw)(t)(Bv)(t).$$

Then the operator $Tu = AuBu$ maps the interval $v(t) \le u(t) \le w(t)$ into itself, and therefore we can use the Schauder Fixed-Point Theorem.

The chapter is organized as follows. In Sect. 11.2, we study the generalized logistic equation with several delays (11.1.3). For this equation, we consider the cases of sublinear and superlinear equations and also mixed equations that contain both sublinear and superlinear terms. Section 11.3 includes investigation of nonoscillation properties for the multiplicative delay logistic equation (11.1.4), which contains a product of logistic terms. Section 11.4 provides discussion and states some open problems.

## 11.2 Generalized Logistic Equation with Several Delays

### *11.2.1 Preliminaries*

Consider the scalar delay differential equation

$$\dot{x}(t) = -\big(1+x(t)\big)\sum_{k=1}^{m} r_k(t)x\big(h_k(t)\big)\big|x\big(h_k(t)\big)\big|^{\alpha_k-1},\ t \ge 0, \tag{11.2.1}$$

under the following assumptions:

(a1) $r_k$, $k=1,\cdots,m$, are Lebesgue measurable locally essentially bounded functions, $r_k(t) \ge 0$.

(a2) $h_k : [0,\infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \le t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k=1,\cdots,m$, and $\alpha_k > 0$, $k=1,2,\cdots,m$ are real numbers.

Together with (11.2.1), we consider for each $t_0 \ge 0$ the initial value problem

$$\dot{x}(t) = -\big(1+x(t)\big)\sum_{k=1}^{m} r_k(t)x\big(h_k(t)\big)\big|x\big(h_k(t)\big)\big|^{\alpha_k-1},\ t \ge t_0, \tag{11.2.2}$$

$$x(t) = \varphi(t),\ t < t_0,\ x(t_0) = x_0. \tag{11.2.3}$$

We also assume that the following hypothesis holds:

(a3) $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

**Definition 11.1** A function $x : \mathbb{R} \to \mathbb{R}$ absolutely continuous in each interval $[t_0, b]$ is called *a solution* of problem (11.2.2), (11.2.3) if it satisfies (11.2.2) for almost all $t \in [t_0,\infty)$ and equalities (11.2.3) for $t \le t_0$.

Equation (11.2.1) is obtained from the generalized logistic equation

$$\dot{y}(t) = \sum_{k=1}^{m} r_k(t)y(t)\left(1-\frac{y(h_k(t))}{K}\right)\left|1-\frac{y(h_k(t))}{K}\right|^{\alpha_k-1} \tag{11.2.4}$$

using the substitution

$$y = K(1+x).$$

Let us note that all solutions of (11.2.4) with a positive initial value and a nonnegative initial function are positive by Theorem B.17. Since for the logistic equation $y(t) > 0$ we will consider only such solutions of (11.2.1) for which the inequality

$$x(t) > -1$$

holds.

### 11.2.2 Sublinear Case $\alpha_k < 1$, $k = 1, \cdots, m$

Together with (11.2.1), consider the differential inequalities

$$\dot{x}(t) \le -\big(1 + x(t)\big) \sum_{k=1}^{m} r_k(t) x\big(h_k(t)\big) \big|x\big(h_k(t)\big)\big|^{\alpha_k - 1}, \ t \ge 0, \tag{11.2.5}$$

$$\dot{x}(t) \ge -\big(1 + x(t)\big) \sum_{k=1}^{m} r_k(t) x\big(h_k(t)\big) \big|x\big(h_k(t)\big)\big|^{\alpha_k - 1}, \ t \ge 0. \tag{11.2.6}$$

As was mentioned before, only such solutions of (11.2.1), (11.2.5) and (11.2.6) for which the condition

$$1 + x(t) > 0 \tag{11.2.7}$$

holds are considered.

**Theorem 11.1** *The following statements are equivalent*:

1. *Either inequality* (11.2.5) *has an eventually positive solution or inequality* (11.2.6) *has an eventually negative solution satisfying* $-1 < x(t) < 0$.
2. *There exist a point* $t_0 \ge 0$, *number* $c$ *and function* $\varphi : (-\infty, t_0) \to \mathbb{R}$ *with either* $\varphi(t) \ge 0$, $c > 0$ *or* $-1 < \varphi(t) \le 0$, $-1 < c < 0$, *such that the inequality*

$$u(t) \ge \left(1 + c \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right) \sum_{k=1}^{m} (F_k u)(t) \tag{11.2.8}$$

   *has a nonnegative solution locally integrable on* $[t_0, \infty)$, *where operators* $F_k$ *are defined as*

$$(F_k u)(t) = \begin{cases} |c|^{\alpha_k - 1} r_k(t) \exp\{\int_{h_k(t)}^{t} u(s)ds\} \exp\{(1-\alpha_k) \int_{t_0}^{h_k(t)} u(s)ds\}, & h_k(t) \ge t_0, \\ \frac{r_k(t)}{|c|} \exp\{\int_{t_0}^{t} u(s)ds\} |\varphi(h_k(t))|^{\alpha_k}, & h_k(t) < t_0. \end{cases}$$

3. *Equation* (11.2.1) *has a nonoscillatory solution.*

*Proof* 1) ⇒ 2) Let $x$ be a solution of (11.2.5) and $x(t) > 0$ for $t \ge t_1$. Then there exists $t_0 \ge t_1$ such that $h_k(t) \ge t_1$ for $t \ge t_0$, $k = 1, \cdots, m$. Denote $\varphi(t) = x(t)$, $t < t_0$ and $c = x(t_0)$. Then $\varphi(t) > 0$, $c > 0$.

Let $u(t) = -\frac{\dot{x}(t)}{x(t)}$, $t \geq t_0$. For the solution $x(t)$ of (11.2.5), we have $\dot{x}(t) \leq 0$, $t \geq t_0$ and consequently $u(t) \geq 0$. We can now rewrite $x$ in the form

$$x(t) = \begin{cases} c\exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ \varphi(t), & t < t_0. \end{cases} \tag{11.2.9}$$

By substituting $x$ in (11.2.5), we obtain inequality (11.2.8). Similarly (11.2.8) can be obtained if $-1 < x(t) < 0$ is a solution of (11.2.6).

2) $\Rightarrow$ 3) Let $u_0$ be a nonnegative solution of inequality (11.2.8) with $-1 < \varphi(t) \leq 0$, $-1 < c < 0$. For the sequence $\{u_n\}$ denoted by

$$u_n(t) = \left(1 + c\exp\left\{-\int_{t_0}^{t} u_{n-1}(s)ds\right\}\right)\sum_{k=1}^{m}(F_k u_{n-1})(t), \tag{11.2.10}$$

inequality (11.2.8) implies $u_1(t) \leq u_0(t)$. By induction, we can prove

$$0 \leq u_n(t) \leq u_{n-1}(t) \leq u_0(t).$$

There exists a pointwise limit of the nonincreasing nonnegative sequence $u_n(t)$. Let $u(t) = \lim_{n\to\infty} u_n(t)$. Then, by the Lebesgue convergence theorem, $u(t)$ is locally integrable and

$$\lim_{n\to\infty}(F_k u_n)(t) = (F_k u)(t),\ k = 1, \cdots, m.$$

Thus (11.2.10) implies

$$u(t) = \left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\sum_{k=1}^{m}(F_k u)(t).$$

Hence the function $x$ defined by equality (11.2.9) is an eventually negative solution of (11.2.1).

Further, let $u_0$ be a nonnegative solution of inequality (11.2.8) with $\varphi(t) \geq 0$ and $c > 0$. If inequality (11.2.8) holds for some $\varphi$, then it holds also for $\varphi \equiv 0$, so we can assume that $\varphi \equiv 0$.

If $0 < c < 1$, then denoting $c_0 = -c$ we obtain that $u_0$ is also a solution of (11.2.8) with $c_0$, $\varphi \equiv 0$ instead of $c$, $\varphi(t)$. From the previous case, it follows that there exists an eventually negative solution of (11.2.1).

Next, suppose that $c \geq 1$. From (11.2.8), we have

$$\begin{aligned} u(t) \geq \exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\sum_{k=1}^{m} c^{\alpha_k} r_k(t) \\ \times \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\}\exp\left\{(1-\alpha_k)\int_{t_0}^{h_k(t)} u(s)ds\right\}, \end{aligned}$$

where the sum includes only the terms for which $h_k(t) \geq t_0$.

There exists $0 < d \leq 1$ such that $c^{\alpha_k} > d^{\alpha_k - 1}$, $k = 1, \cdots, m$ if $c > 1$ and $d = c = 1$ if $c = 1$.

Hence $u(t) \geq \sum_{k=1}^{m}(F_k u)(t)$, where $c$ in the definition of $F_k$ is replaced by $d$ and $\varphi \equiv 0$. Denote $c_0 = -d$. Then inequality (11.2.8) holds for $-1 < c_0 < 0$ and $\varphi \equiv 0$. As was proven above, there exists a nonnegative function $u(t)$ such that

$$x(t) = c_0 \exp\left\{-\int_{t_0}^{t} u_0(s)ds\right\},\ t \geq 0,\ x(t) = 0,\ t < t_0,$$

is an eventually negative solution of (11.2.1).

Implication 3) $\Rightarrow$ 1) is evident. □

**Corollary 11.1** *Suppose that there exist $t_0$ and $A > 1$ such that the inequality*

$$u(t) \geq A \sum_{k=1}^{m} r_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \exp\left\{(1-\alpha_k)\int_{t_0}^{h_k(t)} u(s)ds\right\},\ t \geq t_0, \tag{11.2.11}$$

*has a nonnegative locally integrable solution, where the sum contains only such terms for which $h_k(t) \geq t_0$. Then* (11.2.1) *has a nonoscillatory solution.*

*Proof* The corollary follows from statement 2) of Theorem 11.1 if we assume $\varphi \equiv 0$ and $-1 < c < 0$ such that $|c|^{\alpha_k - 1} < A$, $k = 1, \cdots, m$. □

**Corollary 11.2** *If there exists an eventually positive solution of* (11.2.1), *then there exists an eventually negative solution of* (11.2.1).

*Proof* This result follows from the proof of Theorem 11.1. □

*Remark 11.1* Theorem 11.1 and its corollaries remain valid if for some or all indices $k$ we have $\alpha_k = 1$.

**Theorem 11.2** *There exists a nonoscillatory solution of* (11.2.1) *if and only if*

$$\int_0^{\infty} r_k(t)dt < \infty,\ k = 1, \cdots, m.$$

*Proof* Let

$$\int_0^{\infty} r_k(t)dt < \infty,\ k = 1, \cdots, m.$$

Then there exists $t_0$ and $A > 1$ such that $A \exp\{2\int_{t_0}^{\infty} \sum_{k=1}^{m} r_k(s)ds\} < 2$. For any nonnegative $u$, $u(t) = 0$, for $t < t_0$ we have

$$\begin{aligned} &A \sum_{k=1}^{m} r_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \exp\left\{(1-\alpha_k)\int_{t_0}^{h_k(t)} u(s)ds\right\} \\ &\quad \leq A \sum_{k=1}^{m} r_k(t) \exp\left\{\int_{t_0}^{t} u(s)ds\right\}. \end{aligned}$$

Let $u(t) = 2\sum_{k=1}^{m} r_k(t)$, $t \ge t_0$. Then $u$ is a solution of inequality (11.2.11). Corollary 11.1 implies that (11.2.1) has a nonoscillatory solution.

Suppose now that for some $i$, $1 \le i \le m$, we have $\int_0^\infty r_i(t)dt = \infty$. Let $x$ be a positive or negative solution of (11.2.1) for $t \ge t_1$. There exists $t_0 \ge t_1$ such that $h_k(t) \ge t_1$, $t \ge t_0$, $k = 1, \cdots, m$. Let $u(t) = -\frac{\dot{x}(t)}{x(t)}$, $t \ge t_0$. Then $u(t) \ge 0$. We can now rewrite $x$ in the form (11.2.10), where $c = x(t_0)$, $\varphi(t) = x(t)$, $t < t_0$. Since we consider only solutions $x(t) > -1$, we have $c > -1$.

By substituting $x$ in (11.2.1), we obtain for $t \ge t_0$ the equality

$$u(t) = \left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\sum_{k=1}^{m}(F_k u)(t),$$

which corresponds to inequality (11.2.8).

There exists $t_2 > t_0$ such that $h_k(t) \ge t_0$ for $t \ge t_2$. Then, for $t \ge t_2$ we have

$$u(t) \ge \min\{1, 1+c\}|c|^{\alpha_i - 1} r_i(t)\exp\left\{(1-\alpha_i)\int_{t_1}^{t} u(s)ds\right\}.$$

Hence

$$r_i(t) \le \frac{|c|^{1-\alpha_i}}{\min\{1, 1+c\}} u(t)\exp\left\{-(1-\alpha_i)\int_{t_1}^{t} u(s)ds\right\}, \quad t \ge t_2,$$

which implies

$$\begin{aligned}\int_{t_1}^{t} r_i(s)ds &\le \frac{|c|^{1-\alpha_i}}{\min\{1, 1+c\}}\int_{t_1}^{t} u(s)\exp\left\{-(1-\alpha_i)\int_{t_1}^{s} u(\tau)d\tau\right\}ds \\ &= \frac{|c|^{1-\alpha_i}}{\min\{1, 1+c\}(1-\alpha_i)}\left(1 - \exp\left\{-(1-\alpha_i)\int_{t_1}^{t} u(s)ds\right\}\right) \\ &\le \frac{|c|^{1-\alpha_i}}{\min\{1, 1+c\}(1-\alpha_i)}.\end{aligned}$$

Thus $\int_{t_1}^{\infty} a_i(s)ds < \infty$, which gives a contradiction. Hence all the solutions of (11.2.1) are oscillatory. □

*Remark 11.2* The sufficient part ("if") of Theorem 11.2 remains true if some or all $\alpha_k$ are equal to one.

### *11.2.3 Superlinear Case $\alpha_k > 1$, $k = 1, \cdots, m$*

**Theorem 11.3** *Suppose that for some $\varepsilon > 0$ there exists a nonoscillatory solution of the linear delay differential equation*

$$\dot{x}(t) = -\varepsilon\sum_{k=1}^{m} r_k(t)x\big(h_k(t)\big). \tag{11.2.12}$$

*Then there exists a nonoscillatory solution of* (11.2.1).

*Proof* Let $t_0 \ge 0$, $c$ and $\varphi : (-\infty, t_0) \to \mathbb{R}$ be such that $-1 < c < 0$, $\varphi(t) \le 0$, $|\varphi(t)| < |c| < \varepsilon^{1/(\alpha_k - 1)}$, $k = 1, \cdots, m$, and hence $c \le \varphi(t) \le 0$. Theorem 2.22 implies that initial value problem (11.2.12), (11.2.3) with initial function $\varphi$ and initial value $x_0 = c$ has a negative solution $x_0(t) < 0$. Denote

$$w_0(t) = -\frac{\dot{x}_0(t)}{x_0(t)}.$$

Then $w_0(t) \ge 0$ and $x_0(t) = c\exp\{-\int_{t_0}^t w_0(s)ds\}$, $t \ge t_0$.

After substituting $x_0$ into (11.2.12), we have

$$w_0(t) = \varepsilon \sum_{k=1}^m r_k(t) \times \begin{cases} \exp\{\int_{h_k(t)}^t w_0(s)ds\}, & h_k(t) \ge t_0, \\ \exp\{\int_{t_0}^t w_0(s)ds\}\frac{\varphi(h_k(t))}{c}, & h_k(t) < t_0. \end{cases}$$

Consider now the two sequences

$$w_n(t) = \left(1 + c\exp\left\{-\int_{t_0}^t w_{n-1}(s)ds\right\}\right)\sum_{k=1}^m r_k(t)$$
$$\times \begin{cases} |c|^{\alpha_k - 1}\exp\{\int_{h_k(t)}^t w_{n-1}(s)ds\}\exp\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} v_{n-1}(s)ds\}, \\ \quad h_k(t) \ge t_0, \\ \exp\{\int_{t_0}^t w_{n-1}(s)ds\}\frac{|\varphi(h_k(t))|^{\alpha_k}}{|c|}, \ h_k(t) < t_0, \end{cases}$$
$$v_n(t) = \left(1 + c\exp\left\{-\int_{t_0}^t v_{n-1}(s)ds\right\}\right)\sum_{k=1}^m r_k(t)$$
$$\times \begin{cases} |c|^{\alpha_k - 1}\exp\{\int_{h_k(t)}^t v_{n-1}(s)ds\}\exp\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} w_{n-1}(s)ds\}, \\ \quad h_k(t) \ge t_0, \\ \exp\{\int_{t_0}^t v_{n-1}(s)ds\}\frac{|\varphi(h_k(t))|^{\alpha_k}}{|c|}, \ h_k(t) < t_0, \end{cases}$$

where $w_0$ was defined above and $v_0 = 0$.

We have $|\varphi(h_k(t))|^{\alpha_k - 1} < |c|^{\alpha_k - 1} < \varepsilon$ and $1 + c\exp\{-\int_{t_0}^t w_0(s)\,ds\} < 1$. Then $w_1(t) \le w_0(t)$, $v_1(t) \ge v_0(t) = 0$ and $w_0(t) \ge v_0(t)$. Hence, by induction,

$$0 \le w_n(t) \le w_{n-1}(t) \le \cdots \le w_0(t), \ v_n(t) \ge v_{n-1}(t) \ge \cdots \ge v_0(t) = 0,$$

and $v_n(t) \le w_n(t) \le w_0(t)$.

There exist pointwise limits of the nonincreasing nonnegative sequence $w_n(t)$ and of the bounded nondecreasing sequence $v_n(t)$. If we denote $w(t) = \lim_{n\to\infty} w_n(t)$ and $v(t) = \lim_{n\to\infty} v_n(t)$, then by the Lebesgue monotone convergence theorem (see Theorem A.1) we conclude that

$$w(t) = \left(1 + c\exp\left\{-\int_{t_0}^t w(s)ds\right\}\right)\sum_{k=1}^m r_k(t)$$
$$\times \begin{cases} |c|^{\alpha_k - 1}\exp\{\int_{h_k(t)}^t w(s)ds\}\exp\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} v(s)ds\}, \\ \quad h_k(t) \ge t_0, \\ \exp\{\int_{t_0}^t w(s)ds\}\frac{|\varphi(h_k(t))|^{\alpha_k}}{|c|}, \ h_k(t) < t_0, \end{cases} \tag{11.2.13}$$

and

$$v(t) = \left(1 + c\exp\left\{-\int_{t_0}^{t} v(s)ds\right\}\right) \sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |c|^{\alpha_k - 1} \exp\{\int_{h_k(t)}^{t} v(s)ds\} \exp\{-(\alpha_k - 1) \int_{t_0}^{h_k(t)} w(s)ds\}, \\ \quad h_k(t) \geq t_0, \\ \exp\{\int_{t_0}^{t} v(s)ds\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|c|}, \; h_k(t) < t_0. \end{cases} \tag{11.2.14}$$

Denote the operator $T : L_\infty[a, b] \to: L_\infty[a, b]$ by the equality

$$(Tu)(t) = \left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right) \sum_{k=1}^{m} r_k(t)$$
$$\times \begin{cases} |c|^{\alpha_k - 1} \exp\{\int_{h_k(t)}^{t} u(s)ds\} \exp\{-(\alpha_k - 1) \int_{t_0}^{h_k(t)} u(s)ds\}, \\ \quad h_k(t) \geq t_0, \\ \exp\{\int_{t_0}^{t} u(s)ds\} \frac{|\varphi(h_k(t))|^{\alpha_k}}{|c|}, \; h_k(t) < t_0. \end{cases}$$

Equalities (11.2.13), (11.2.14) imply that for every function $u$ from the interval $v \leq u \leq w$ we have $v \leq Tu \leq w$. Theorem A.6 implies that operator $T$ is a compact operator in the space $L_\infty[t_0, b]$ for every $b > t_0$. Then, by the Schauder Fixed-Point Theorem (see Theorem A.15) there exists a nonnegative solution of equation $u = Tu$.

Thus $x(t)$ defined as

$$x(t) = \begin{cases} c\exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ \varphi(t), & t < t_0. \end{cases}$$

is a negative solution of (11.2.1), which completes the proof. □

**Corollary 11.3** *If*

$$\limsup_{t \to \infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{i=1}^{m} r_i(s)ds < \infty, \tag{11.2.15}$$

*then* (11.2.1) *has a nonoscillatory solution.*

*Proof* Inequality (11.2.15) yields that for some $\varepsilon > 0$

$$\limsup_{t \to \infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{k=1}^{m} \varepsilon r_k(s)ds < \frac{1}{e}.$$

Theorems 2.7 and 11.3 imply that (11.2.1) has a nonoscillatory solution. □

Next, consider the case where some $\alpha_k$ can equal one.

**Theorem 11.4** *Let* $\{1, \cdots, m\} = I \cup J$, *where* $\alpha_k > 1$, $k \in I$, $\alpha_k = 1$, $k \in J$. *Suppose that for some* $\varepsilon > 0$ *there exists a nonoscillatory solution of the linear delay differential equation*

$$\dot{x}(t) = -\varepsilon \sum_{k\in I} r_k(t)x\big(h_k(t)\big) - \sum_{k\in J} r_k(t)x\big(h_k(t)\big). \tag{11.2.16}$$

*Then there exists a nonoscillatory solution of* (11.2.1).

**Corollary 11.4** *If*

$$\limsup_{t\to\infty} \int_{\min_{k\in I}\{h_k(t)\}}^{t} \sum_{i\in I} r_i(s)ds < \infty,$$

$$\limsup_{t\to\infty} \int_{\min_{k\in J}\{h_k(t)\}}^{t} \sum_{i\in J} r_i(s)ds < \frac{1}{e},$$

*then* (11.2.1) *has a nonoscillatory solution.*

The proof of Theorem 11.4 and its corollary is similar to the proof of the previous theorem and its corollary.

### *11.2.4 Mixed Cases*

For simplicity, we consider in this section model (11.2.1) with $m = 2$; i.e., the equation

$$\dot{x}(t) = -\big(1 + x(t)\big)\big[r_1(t)x\big(h_1(t)\big)\big|x\big(h_1(t)\big)\big|^{\alpha_1-1} + r_2(t)x\big(h_2(t)\big)\big|x\big(h_2(t)\big)\big|^{\alpha_2-1}\big]. \tag{11.2.17}$$

General equation (11.2.1) can be studied in a similar way.

**I. Case** $\alpha_1 < 1 = \alpha_2$.

**Theorem 11.5** *Let* $\alpha_1 < 1 = \alpha_2$ *and all solutions of at least one of the equations*

$$\dot{x}(t) = -\big(1 + x(t)\big)r_1(t)x\big(h_1(t)\big)\big|x\big(h_1(t)\big)\big|^{\alpha_1-1}, \tag{11.2.18}$$

$$\dot{x}(t) = -\big(1 + x(t)\big)r_2(t)x\big(h_2(t)\big), \tag{11.2.19}$$

*be oscillatory. Then all solutions of* (11.2.17) *are also oscillatory.*

*Proof* Suppose there exists a nonoscillatory solution $x$ of (11.2.17). If $x$ is an eventually positive solution of (11.2.17), then there exists $t_0 \ge 0$ such that

$$\dot{x}(t) \le -\big(1 + x(t)\big)r_1(t)x\big(h_1(t)\big)\big|x\big(h_1(t)\big)\big|^{\alpha_1-1},\ t \ge t_0,$$

and

$$\dot{x}(t) \le -\big(1 + x(t)\big) r_2(t) x\big(h_2(t)\big), \ t \ge t_0.$$

Theorem 11.1 and Remark 11.1 imply that (11.2.18) and (11.2.19) have nonoscillatory solutions, which gives a contradiction.

The case where (11.2.1) has an eventually negative solution is similar. □

**Corollary 11.5** *Let $m = 2$, $\alpha_1 < 1 = \alpha_2$. If*

$$\textit{either } \int_0^\infty r_1(s)ds = \infty \textit{ or } \liminf_{t\to\infty} \int_{h_2(t)}^t r_2(s)ds > \frac{1}{e},$$

*then all the solutions of* (11.2.17) *are oscillatory.*

*Proof* The result follows from Theorems 11.2 and 10.4. □

**Theorem 11.6** *Let $\alpha_1 < 1 = \alpha_2$. If in addition*

$$\int_0^\infty r_k(s)ds < \infty, \ k = 1, 2,$$

*then there exists a nonoscillatory solution of* (11.2.17).

*Proof* The result follows from Remark 11.2 and Theorem 11.2. □

**II. Case** $\alpha_1 > 1 = \alpha_2$.

**Theorem 11.7** *Let $\alpha_1 > 1 = \alpha_2$. Suppose that for some $\varepsilon > 0$ there exists a nonoscillatory solution of the linear equation*

$$\dot{x}(t) = -\varepsilon r_1(t) x\big(h_1(t)\big) - r_2(t) x\big(h_2(t)\big). \tag{11.2.20}$$

*Then there exists a nonoscillatory solution of* (11.2.17).

*Proof* The proof is similar to the proof of Theorem 11.4. □

**Corollary 11.6** *Let $\alpha_1 > 1 = \alpha_2$. If*

$$\limsup_{t\to\infty} \int_{\min\{h_1(t),h_2(t)\}}^t r_1(s)ds < \infty \textit{ and } \limsup_{t\to\infty} \int_{\min\{h_1(t),h_2(t)\}}^t r_2(s)ds < \frac{1}{e},$$

*then there exists a nonoscillatory solution of* (11.2.17).

*Proof* The proof follows from Corollary 11.4 for the case $m = 2$. □

**III. Case** $\alpha_1 > 1 > \alpha_2$.

**Theorem 11.8** *Let $\alpha_1 > 1 > \alpha_2$. Suppose there exists a nonoscillatory solution of* (11.2.17) *with $\alpha_1 = 1 > \alpha_2$. Then there exists a nonoscillatory solution of* (11.2.17) *with $\alpha_1 > 1 > \alpha_2$.*

*Proof* As in the proof of Theorem 11.1 (see also Remark 11.1), there exist $t_0$, $c$ and $\varphi$ satisfying $t_0 \ge 0$, $-1 < c < 0$, $-1 < \varphi(t) \le 0$ such that (11.2.1), (11.2.3) with $m = 2$, $\alpha_1 = 1 > \alpha_2$ has a negative solution $x_0(t)$. Denote $w_0(t) = -\frac{\dot{x}_0(t)}{x_0(t)}$. Then $w_0(t) \ge 0$ and $x_0(t) = c\exp\{-\int_{t_0}^t w_0(s)ds\}$, $t \ge t_0$, where $c = x_0(t_0)$.

By substituting $x_0$ into (11.2.1) with $\alpha_1 = 1 > \alpha_2$, we have

$$w_0(t) = \left(1 + c\exp\left\{-\int_{t_0}^t w_0(s)ds\right\}\right)\sum_{k=1}^2 r_k(t) \times \begin{cases} |c|^{\alpha_k - 1}\exp\{\int_{h_k(t)}^t w_0(s)ds\}\exp\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} w_0(s)ds\}, \\ \quad h_k(t) \ge t_0, \\ \exp\{\int_{t_0}^t w_0(s)ds\}\frac{|\varphi(h_k(t))|^{\alpha_k}}{|c|}, \ h_k(t) < t_0, \end{cases}$$

where $\alpha_1 = 1$. Then, for $\alpha_1 > 1 > \alpha_2$, we have

$$w_0(t) \ge \left(1 + c\exp\left\{-\int_{t_0}^t w_0(s)ds\right\}\right)\sum_{k=1}^2 r_k(t) \times |c|^{\alpha_k - 1}\exp\left\{\int_{h_k(t)}^t w_0(s)ds\right\}\exp\left\{-(\alpha_k - 1)\int_{t_0}^{h_k(t)} w_0(s)ds\right\},$$

where the sum contains only such terms for which $h_k(t) \ge t_0$. Let $v_0(t) \equiv 0$ and consider the two sequences

$$\begin{aligned} w_n(t) &= \left(1 + c\exp\left\{-\int_{t_0}^t w_{n-1}(s)ds\right\}\right) \\ &\times \left(r_1(t)|c|^{\alpha_1 - 1}\exp\left\{\int_{h_1(t)}^t w_{n-1}(s)ds\right\}\exp\left\{-(\alpha_1 - 1)\int_{t_0}^{h_1(t)} v_{n-1}(s)ds\right\}\right. \\ &\left. + r_2(t)|c|^{\alpha_2 - 1}\exp\left\{\int_{h_2(t)}^t w_{n-1}(s)ds\right\}\exp\left\{-(\alpha_2 - 1)\int_{t_0}^{h_2(t)} w_{n-1}(s)ds\right\}\right) \end{aligned}$$

and

$$\begin{aligned} v_n(t) &= \left(1 + c\exp\left\{-\int_{t_0}^t v_{n-1}(s)ds\right\}\right) \\ &\times \left(r_1(t)|c|^{\alpha_1 - 1}\exp\left\{\int_{h_1(t)}^t v_{n-1}(s)ds\right\}\exp\left\{-(\alpha_1 - 1)\int_{t_0}^{h_1(t)} w_{n-1}(s)ds\right\}\right. \\ &\left. + r_2(t)|c|^{\alpha_2 - 1}\exp\left\{\int_{h_2(t)}^t v_{n-1}(s)ds\right\}\exp\left\{-(\alpha_2 - 1)\int_{t_0}^{h_2(t)} v_{n-1}(s)ds\right\}\right). \end{aligned}$$

Then, as in the proof of Theorem 11.5, we obtain

$$0 \le w_n(t) \le w_{n-1}(t) \le \cdots \le w_0(t), \ \ v_n(t) \ge v_{n-1}(t) \ge \cdots \ge v_0(t) = 0,$$

and $w_n(t) \geq v_n(t)$. Then there exist $w(t) = \lim_{n\to\infty} w_n(t)$ and $v(t) = \lim_{n\to\infty} v_n(t)$. Hence by the Schauder Fixed-Point Theorem there is $u$ satisfying $v \leq u \leq w$, which is a solution of the equation

$$u(t) = \left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)$$
$$\times \left(r_1(t)|c|^{\alpha_1-1}\exp\left\{\int_{h_1(t)}^{t} u(s)ds\right\}\exp\left\{-(\alpha_1-1)\int_{t_0}^{h_1(t)} u(s)ds\right\}\right.$$
$$\left.+ r_2(t)|c|^{\alpha_2-1}\exp\left\{\int_{h_2(t)}^{t} u(s)ds\right\}\exp\left\{-(\alpha_2-1)\int_{t_0}^{h_2(t)} u(s)ds\right\}\right).$$

Then

$$x(t) = \begin{cases} c\exp\{-\int_{t_0}^{t} u(s)ds\}, & t \geq t_0, \\ 0, & t < t_0, \end{cases}$$

is a negative solution of (11.2.17), (11.2.3) with $x(t_0) = c$, $\varphi \equiv 0$, which completes the proof. □

**Corollary 11.7** *Let $\alpha_1 > 1 > \alpha_2$. If*

$$\int_0^\infty r_k(s)ds < \infty, \ k = 1, 2,$$

*then there exists a nonoscillatory solution of* (11.2.17).

## *11.2.5 Generalized Logistic Equation—Main Results*

The main purpose of this section is to study oscillation of the generalized logistic equation about the unique positive equilibrium.

Consider

$$\dot{y}(t) = \sum_{k=1}^{m} r_k(t)y(t)\left(1 - \frac{y(h_k(t))}{K}\right)\left|1 - \frac{y(h_k(t))}{K}\right|^{\alpha_k-1}, \quad (11.2.21)$$

where $r_k$, $h_k$, $\alpha_k$ satisfy conditions (a1) and (a2), $K > 0$ and the initial function $\psi$ satisfies (a3) with the initial condition

$$y(t) = \psi(t), \ t < t_0, \ y(t_0) = y_0. \quad (11.2.22)$$

In this section we assume that the initial conditions satisfy

(a4) $y_0 > 0$, $\psi(t) \geq 0$, $t < t_0$.

By Theorem B.17, there exists a unique positive solution of problem (11.2.21), (11.2.22).

**Definition 11.2** A positive solution $y$ of (11.2.21) is said to be *oscillatory about* $K$ if there exists a sequence $t_n$, $t_n \to \infty$ such that $y(t_n) - K = 0$, $n = 1, 2, \cdots$; $y$ is *nonoscillatory about* $K$ if there exists $T \geq t_0$ such that $|y(t) - K| > 0$ for $t \geq T$. A solution $y$ is *eventually positive* (*eventually negative*) about $K$ if $y - K$ is eventually positive (eventually negative).

Suppose $y$ is a positive solution of (11.2.21), and define $x$ as $x = \frac{y}{K} - 1$. Then $x$ is a solution of (11.2.1) such that $1 + x > 0$.

Hence oscillation (or nonoscillation) of $y$ about $K$ is equivalent to oscillation (nonoscillation) of $x$.

By applying Theorems 11.1–11.8, we obtain the following results for (11.2.21).

**Theorem 11.9** *Let $\alpha_k < 1$, $k = 1, \cdots, m$. There exists a solution of (11.2.21) nonoscillatory about $K$ if and only if*

$$\int_0^\infty r_k(t)dt < \infty, \ k = 1, \cdots, m.$$

**Theorem 11.10** *Let $\alpha_k > 1$, $k = 1, \cdots, m$.*

*If*

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^t \sum_{i=1}^m r_i(s)ds < \infty,$$

*then (11.2.21) has a solution nonoscillatory about $K$.*

*Remark 11.3* If $m = 1$ and the parameters of (11.2.21) are continuous functions, then the results of Theorems 11.9 and 11.10 were first obtained in [12, 237].

**Theorem 11.11** *Let $m = 2$, $\alpha_1 < 1 = \alpha_2$. If either*

$$\int_0^\infty r_1(s)ds = \infty$$

*or*

$$\liminf_{t\to\infty} \int_{h_2(t)}^t r_2(s)ds > \frac{1}{e},$$

*then all solutions of (11.2.21) are oscillatory about $K$.*

*If*

$$\int_0^\infty r_k(s)ds < \infty, \ k = 1, 2,$$

*then there exists a solution of (11.2.21) nonoscillatory about $K$.*

**Theorem 11.12** *Let $m = 2$, $\alpha_1 > 1 = \alpha_2$.*

*If*

$$\limsup_{t\to\infty} \int_{\min\{h_1(t),h_2(t)\}}^t r_1(s)ds < \infty, \ \limsup_{t\to\infty} \int_{\min\{h_1(t),h_2(t)\}}^t r_2(s)ds < \frac{1}{e},$$

*then there exists a solution of (11.2.21) nonoscillatory about $K$.*

**Theorem 11.13** *Let $m = 2$, $\alpha_1 > 1 > \alpha_2$.*
*If*

$$\int_0^\infty r_k(s)ds < \infty, \ k = 1, 2,$$

*then there exists a solution of* (11.2.21) *nonoscillatory about* $K$.

## 11.3 Multiplicative Delay Logistic Equation

### *11.3.1 Preliminaries*

In this section, the scalar multiplicative delay differential equation

$$\dot{x}(t) = -r(t)\big(1 + x(t)\big) \prod_{k=1}^{m} x\big(h_k(t)\big)\big|x\big(h_k(t)\big)\big|^{\alpha_k - 1}, \ t \geq 0, \tag{11.3.1}$$

is considered under the following assumptions:

(b1) $r$ is a Lebesgue measurable locally essentially bounded function, $r(t) \geq 0$;
(b2) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k = 1, \cdots, m$, $\alpha_k > 0$, $k = 1, 2, \cdots, m$, are real numbers and $\sum_{k=1}^{m} \alpha_k = 1$.

Equation (11.3.1) is obtained from the multiplicative delay logistic equation

$$\dot{y}(t) = r(t)y(t) \prod_{k=1}^{m} \left(1 - \frac{y(h_k(t))}{K}\right) \left|1 - \frac{y(h_k(t))}{K}\right|^{\alpha_k - 1} \tag{11.3.2}$$

by the substitution $y(t) = 1 + x(t)$. Let us note that all solutions of (11.3.2) with positive initial conditions are positive by Theorem B.17, so $x(t) > -1$ as far as the initial values satisfy the inequalities

$$x(t) \geq -1, \ t \leq t_0, \ x(t_0) > -1.$$

Together with (11.3.1), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{x}(t) = -r(t)\big(1 + x(t)\big) \prod_{k=1}^{m} x\big(h_k(t)\big)\big|x\big(h_k(t)\big)\big|^{\alpha_k - 1}, \ t \geq t_0, \tag{11.3.3}$$

$$x(t) = \varphi(t), \ t < t_0, \ x(t_0) = x_0. \tag{11.3.4}$$

We also assume that the following hypothesis holds:

(b3) $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

**Definition 11.3** A function $x : \mathbb{R} \to \mathbb{R}$ absolutely continuous in each interval $[t_0, b]$ is called *a solution* of problem (11.3.3), (11.3.4) if it satisfies (11.3.3) for almost all $t \in [t_0, \infty)$ and equalities (11.3.4) for $t \leq t_0$.

### *11.3.2 Nonoscillation Criteria*

Together with (11.3.1), consider the differential inequalities

$$\dot{x}(t) \le -r(t)\big(1+x(t)\big)\prod_{k=1}^{m} x\big(h_k(t)\big)\big|x\big(h_k(t)\big)\big|^{\alpha_k-1},\ t\ge 0, \tag{11.3.5}$$

$$\dot{x}(t) \ge -r(t)\big(1+x(t)\big)\prod_{k=1}^{m} x\big(h_k(t)\big)\big|x\big(h_k(t)\big)\big|^{\alpha_k-1},\ t\ge 0. \tag{11.3.6}$$

In this section, we assume that (b1)–(b3) hold and consider only such solutions of (11.3.1), (11.3.5) and (11.3.6) for which the following condition holds:

$$1+x(t)>0. \tag{11.3.7}$$

**Theorem 11.14** *The following statements are equivalent*:

1. *Either inequality* (11.3.5) *has an eventually positive solution or inequality* (11.3.6) *has an eventually negative solution.*
2. *There exist* $t_0\ge 0$, $-\infty<c<\infty$, $\varphi:(-\infty,t_0)\to\mathbb{R}$, *either* $\varphi(t)\ge 0$, $c>0$, *or* $\varphi(t)\le 0$, $-1<c<0$, *such that the inequality*

$$u(t)\ge r(t)\left(1+c\exp\left\{-\int_{t_0}^{t}u(s)ds\right\}\right)\prod_{k=1}^{m}(F_k u)(t) \tag{11.3.8}$$

*has a nonnegative locally integrable solution on* $[t_0,\infty)$, *where*

$$(F_k u)(t)=\begin{cases}\exp\{\alpha_k\int_{h_k(t)}^{t}u(s)ds\}, & h_k(t)\ge t_0,\\ \frac{1}{|c|^{\alpha_k}}\exp\{\alpha_k\int_{t_0}^{t}u(s)ds\}|\varphi(h_k(t))|^{\alpha_k}, & h_k(t)<t_0,\end{cases}$$

3. *Equation* (11.3.1) *has a nonoscillatory solution.*

*Proof* 1) $\Rightarrow$ 2) Let $x$ be a solution of (11.3.5) and $x(t)>0$ for $t\ge t_1$. Then there exists $t_0\ge t_1$ such that $h_k(t)\ge t_1$ for $t\ge t_0$, $k=1,\cdots,m$. Denote $\varphi(t)=x(t)$, $t<t_0$ and $c=x(t_0)$, $c>0$.

Let $u(t)=-\frac{\dot{x}(t)}{x(t)}$, $t\ge t_0$. For the solution $x(t)$ of (11.3.5), we have $\dot{x}(t)\le 0$, $t\ge t_0$, and consequently $u(t)\ge 0$. The solution $x(t)$ can be rewritten in the form

$$x(t)=\begin{cases}c\exp\{-\int_{t_0}^{t}u(s)ds\}, & t\ge t_0,\\ \varphi(t), & t<t_0.\end{cases} \tag{11.3.9}$$

After substituting $x$ defined by (11.3.9) into inequality (11.3.5), we obtain (11.3.8).

Similarly, (11.3.8) can be obtained if $x(t)<0$ is a solution of (11.3.6).

2) $\Rightarrow$ 3) Let $u_0$ be a nonnegative solution of inequality (11.3.8) with $\varphi(t)\le 0$, $-1<c<0$. For the sequence $\{u_n\}$ denoted by

$$u_n(t)=r(t)\left(1+c\exp\left\{-\int_{t_0}^{t}u_{n-1}(s)ds\right\}\right)\prod_{k=1}^{m}(F_k u_{n-1})(t), \tag{11.3.10}$$

inequality (11.3.8) implies $0 \le u_1(t) \le u_0(t)$. Since the right-hand side in (11.3.8) is a nondecreasing operator in $u$, the inequalities

$$0 \le u_n(t) \le u_{n-1}(t) \le u_0(t)$$

can be easily obtained by induction.

Thus there exists a pointwise limit of the nonincreasing nonnegative sequence $u_n(t)$. Let $u(t) = \lim_{n\to\infty} u_n(t)$. By the Lebesgue monotone convergence theorem, $u(t)$ is locally integrable and

$$\lim_{n\to\infty} (F_k u_n)(t) = (F_k u)(t), \; k = 1, \cdots, m.$$

Thus (11.3.10) implies

$$u(t) = r(t)\left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\prod_{k=1}^{m}(F_k u)(t),$$

and hence the function $x$ defined by equality (11.3.9) is an eventually negative solution of (11.3.1).

The case where $u_0$ is a nonnegative solution of inequality (11.3.8) with $\varphi(t) \ge 0$, $c > 0$, is considered similarly to the proof of Theorem 11.1.

Implication 3) ⇒ 1) is evident. □

The proof of Theorem 11.14 yields the following statement.

**Corollary 11.8** *If there exists an eventually positive solution of* (11.3.1), *then there exists an eventually negative solution of* (11.3.1).

**Theorem 11.15** *If there exists a nonoscillatory solution of the linear equation*

$$\dot{x}(t) + r(t)\sum_{k=1}^{m} x\big(h_k(t)\big) = 0, \tag{11.3.11}$$

*then there is a nonoscillatory solution of* (11.3.1).

*Proof* Suppose that linear equation (11.3.11) has a nonoscillatory solution. Theorem 2.1 implies that for some $t_0 \ge 0$ there exists a solution $u(t)$, nonnegative locally integrable on $t \ge t_0$, of the inequality

$$u(t) \ge r(t)\exp\left\{\sum_{k=1}^{m}\int_{h_k(t)}^{t} u(s)ds\right\}, \; t \ge t_0,$$

and hence

$$u(t) \ge r(t)\exp\left\{\sum_{k=1}^{m}\alpha_k\int_{h_k(t)}^{t} u(s)ds\right\}, \; t \ge t_0, \tag{11.3.12}$$

where the sum contains only terms for which $h_k(t) \ge t_0$.

Then $u$ is also a solution of the inequality

$$u(t) \geq r(t)\left(1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\}\right)\prod_{k=1}^{m}\exp\left\{\alpha_k\int_{h_k(t)}^{t} u(s)ds\right\}$$

for any $-1 < c < 0$. The implication 2) $\Rightarrow$ 3) in Theorem 11.14 implies the statement of the theorem if we assume $\varphi \equiv 0$. □

**Corollary 11.9** *If*

$$\limsup_{t\to\infty}\int_{\min_k h_k(t)}^{t} r(s)ds < \frac{1}{e},$$

*then* (11.3.1) *has a nonoscillatory solution.*

*Proof* The proof follows from Theorems 11.15 and 2.7. □

**Theorem 11.16** *If for some $\varepsilon > 0$ all solutions of the linear equation*

$$\dot{x}(t) + (1-\varepsilon)r(t)\sum_{k=1}^{m}\alpha_k x\big(h_k(t)\big) = 0 \tag{11.3.13}$$

*are oscillatory, then all solutions of* (11.3.1) *are also oscillatory.*

*Proof* If there exists a nonoscillatory solution of (11.3.1), then for some $t_0$ by Theorem 11.14 there is a nonnegative solution $u$ of inequality (11.3.8) for $c > -1$. We can assume that $\varphi \equiv 0$. There exists $t_1 \geq t_0$ such that $h_k(t) \geq t_0, t \geq t_1, k = 1, \cdots, m$.

We have $u(t) \geq \min\{1, 1+c\}r(t), t \geq t_1$. Let $\int_0^\infty r(s)ds < \infty$. Then, by Corollary 2.6, (11.3.1) has a nonoscillatory solution, which contradicts the theorem assumption.

Thus $\int_0^\infty r(s)ds = \infty$, and so $\int_0^\infty u(s)ds = \infty$. Hence there exist $t_2 \geq t_0$ and $\varepsilon > 0$ such that

$$1 + c\exp\left\{-\int_{t_0}^{t} u(s)ds\right\} > 1-\varepsilon,\ t \geq t_2,$$

which implies

$$u(t) \geq (1-\varepsilon)r(t)\sum_{k=1}^{m}\exp\left\{\alpha_k\int_{h_k(t)}^{t} u(s)ds\right\},\ t \geq t_2.$$

Theorem 2.1 yields that (11.3.13) has a nonoscillatory solution, which leads to a contradiction. □

**Corollary 11.10** *If*

$$\liminf_{t\to\infty}\int_{\max_k h_k(t)}^{t} r(s)ds > \frac{1}{e},$$

*then all solutions of* (11.3.1) *are oscillatory.*

### 11.3.3 Multiplicative Logistic Equation—Main Results

In this section, we consider the equation

$$\dot{y}(t) = r(t)y(t)\prod_{k=1}^{m}\left(1 - \frac{y(h_k(t))}{K}\right)\left|1 - \frac{y(h_k(t))}{K}\right|^{\alpha_k - 1} \tag{11.3.14}$$

with the unique positive equilibrium $K$ and also the differential inequalities

$$\dot{y}(t) \geq r(t)y(t)\prod_{k=1}^{m}\left(1 - \frac{y(h_k(t))}{K}\right)\left|1 - \frac{y(h_k(t))}{K}\right|^{\alpha_k - 1}, \tag{11.3.15}$$

$$\dot{y}(t) \leq r(t)y(t)\prod_{k=1}^{m}\left(1 - \frac{y(h_k(t))}{K}\right)\left|1 - \frac{y(h_k(t))}{K}\right|^{\alpha_k - 1}, \tag{11.3.16}$$

where $r_k, h_k, \alpha_k$ satisfy conditions (b1) and (b2), $K > 0$. Equation (11.3.14) will be considered with the initial condition

$$y(t) = \psi(t),\ t < t_0,\ y(t_0) = y_0, \tag{11.3.17}$$

where the initial function $\psi$ satisfies (b3).

In this section, we assume that an additional condition,

(b4) $y_0 > 0$, $\psi(t) \geq 0$, $t < t_0$,

holds. By Theorem B.17, there exists a unique positive solution of (11.3.14), (11.3.17).

**Definition 11.4** A positive solution $y$ of (11.3.14), (11.3.17) is *oscillatory about* $K$ if there exists a sequence $t_n$, $t_n \to \infty$, such that $y(t_n) - K = 0$, $n = 1, 2, \cdots$; $y$ is *nonoscillatory about* $K$ if there exists $T \geq t_0$ such that $|y(t) - K| > 0$ for $t \geq T$.

Let $y$ be a positive solution of (11.3.14) and define $x$ as $x = \frac{y}{K} - 1$. Then $x$ is a solution of (11.3.1) such that $1 + x > 0$. Hence oscillation (or nonoscillation) of $y$ about $K$ is equivalent to oscillation (nonoscillation) of $x$.

The same equivalence connects the pairs of differential inequalities (11.3.5), (11.3.15) and (11.3.6), (11.3.16), respectively.

By applying Theorems 11.14–11.16, we obtain the following results for (11.3.14).

**Theorem 11.17** *The following statements are equivalent*:

1. *Either inequality* (11.3.15) *has a solution eventually greater than* $K$ *or inequality* (11.3.16) *has a solution eventually less than* $K$.
2. *There exist* $t_0 \geq 0$, $\varphi : (-\infty, t_0) \to \mathbb{R}$, *either* $\varphi(t) \geq 0$, $c > 0$ *or* $\varphi(t) \leq 0$, $1 < c < 0$, *such that inequality* (11.3.8) *has a nonnegative solution locally integrable on* $[t_0, \infty)$.
3. *Equation* (11.3.14) *has a solution nonoscillatory about* $K$.

**Corollary 11.11** *Suppose that for functions $p(t)$ and $g_k(t)$ conditions* (b1) *and* (b2) *hold.*

*If $p(t) \le r(t)$, $g_k(t) \ge h_k(t)$ and* (11.3.14) *has a solution nonoscillatory about $K$, then the equation*

$$\dot{y}(t) = p(t)y(t)\prod_{k=1}^{m}\left(1 - \frac{y(g_k(t))}{K}\right)\left|1 - \frac{y(g_k(t))}{K}\right|^{\alpha_k - 1} \tag{11.3.18}$$

*has a solution nonoscillatory about $K$.*

*If $p(t) \ge r(t)$, $g_k(t) \le h_k(t)$ and all solutions of* (11.3.14) *are oscillatory about $K$, then all the solutions of* (11.3.18) *are also oscillatory about $K$.*

*Proof* Let $p(t) \le r(t)$, $g_k(t) \ge h_k(t)$, and suppose that (11.3.14) has a solution nonoscillatory about $K$. Theorem 11.14 implies that inequality (11.3.8) has a nonnegative solution $u$, and $u$ is also a nonnegative solution of the inequality (11.3.8), where $r$ and $h_k$ are replaced by $p$ and $g_k$, respectively. Hence, again by Theorem 11.14, (11.3.18) has a nonoscillatory solution.

The second assertion of the theorem is a consequence of the first one. □

**Theorem 11.18** *If there exists a nonoscillatory solution of linear equation* (11.3.11), *then there exists a solution of* (11.3.14) *nonoscillatory about $K$.*

**Corollary 11.12** *If*

$$\limsup_{t\to\infty} \int_{\min_k h_k(t)}^{t} r(s)ds < \frac{1}{e},$$

*then* (11.3.14) *has a solution nonoscillatory about $K$.*

**Theorem 11.19** *If for some $\varepsilon > 0$ all solutions of linear equation* (11.3.13) *are oscillatory, then all solutions of* (11.3.14) *are oscillatory about $K$.*

**Corollary 11.13** *If*

$$\liminf_{t\to\infty} \int_{\max_k h_k(t)}^{t} r(s)ds > \frac{1}{e},$$

*then all the solutions of* (11.3.14) *are oscillatory about $K$.*

## 11.4 Discussion and Open Problems

All the results of this chapter are obtained under the assumption that coefficients, delays and initial functions are arbitrarily measurable, and not necessarily continuous, functions. It is important to avoid a usual constraint that the parameters are continuous since in many interesting applications they are not continuous. In addition, in [193] the authors proved that oscillation properties of a difference equation can be derived from oscillation properties of some delay differential equation with discontinuous delays. In Chap. 12, it will be shown that we can study oscillation of nonimpulsive delay equations with discontinuous coefficients rather than oscillation of an impulsive delay equation. We also note here the paper [175], where the parameters of the logistic equation were not assumed to be continuous functions.

Some other nonoscillation results for nonlinear delay differential equations can be found in [294]. The results of this chapter were published in the papers [47, 49].

Finally, let us present some open problems and topics for research and discussion.

1. Find necessary and/or sufficient oscillation conditions for (11.1.3) in the case where all or some of $\alpha_k > 1$.
2. Prove or disprove the inverse statement to Corollary 11.2: existence of an eventually negative solution implies existence of an eventually positive solution.
3. Consider the following generalizations of the logistic equation with concentrated delays:

   - the integrodifferential equations

$$\dot{y}(t) = y(t) \sum_{k=1}^{m} \int_{h_k(t)}^{t} M_k(t,s)\left(1 - \frac{y(s)}{K}\right)\left|1 - \frac{y(s)}{K}\right|^{\alpha_k - 1} ds;$$

   - equations of a mixed type containing terms with a concentrated delay and also integral terms;
   - the equations with a distributed delay

$$\dot{y}(t) = y(t) \sum_{k=1}^{m} \int_{-\infty}^{t} \left(1 - \frac{y(s)}{K}\right)\left|1 - \frac{y(s)}{K}\right|^{\alpha_k - 1} d_s R_k(t,s).$$

4. Consider the following multiplicative delay logistic equations:

   - the integrodifferential equations

$$\dot{y}(t) = y(t) \int_{-\infty}^{t} \prod_{k=1}^{m} L_k(t,s)\left(1 - \frac{y(s)}{K}\right)\left|1 - \frac{y(s)}{K}\right|^{\alpha_k - 1} ds,$$

$$\dot{y}(t) = y(t) \prod_{k=1}^{m} \int_{h_k(t)}^{t} L_k(t,s)\left(1 - \frac{y(s)}{K}\right)\left|1 - \frac{y(s)}{K}\right|^{\alpha_k - 1} ds;$$

   - equations of a mixed type containing concentrated delay terms and integral terms;
   - the equations with a distributed delay

$$\dot{y}(t) = y(t) \int_{-\infty}^{t} \prod_{k=1}^{m} \left(1 - \frac{y(s)}{K}\right)\left|1 - \frac{y(s)}{K}\right|^{\alpha_k - 1} d_s R_k(t,s),$$

$$\dot{y}(t) = y(t) \prod_{k=1}^{m} \int_{h_k(t)}^{t} \left(1 - \frac{y(s)}{K}\right)\left|1 - \frac{y(s)}{K}\right|^{\alpha_k - 1} d_s R_k(t,s).$$

5. Prove or disprove:
   If $r_k(t) \geq r_0 > 0$, then any solution of (11.2.21) nonoscillatory about $K$ tends to $K$.
6. Prove or disprove:
   If $r_k(t) \geq r_0 > 0$, then any solution of (11.3.14) nonoscillatory about $K$ tends to $K$.
7. Consider oscillation and nonoscillation properties of (11.2.21) with positive and negative coefficients $r_k(t)$ and with oscillatory coefficients.
8. Consider oscillation and nonoscillation properties of (11.3.14) without the assumption that $\sum_{k=1}^{m} \alpha_k = 1$.

# Chapter 12
# First-Order Linear Delay Impulsive Differential Equations

## 12.1 Introduction

The theory of impulsive differential equations goes back to the paper of Milman and Myshkis [287], which appeared in 1960. In 30 years, the results on oscillation of delay differential equations took the shape of a developed theory; see, for example, [154, 167, 249]. At the same time, it is an intensively developing field that is an object of numerous publications. However, for impulsive differential equations there were relatively few publications dealing with oscillation problems that appeared before 2000 [40, 126, 169, 353]; for a review of recent results on oscillation and nonoscillation of impulsive delay differential equations, see [11].

The main result of this chapter is that oscillation (nonoscillation) of the impulsive delay differential equation is equivalent to oscillation (nonoscillation) of a certain differential equation without impulses that can be constructed explicitly from an impulsive equation. Thus, oscillation problems (in particular, oscillation and nonoscillation criteria) for an impulsive equation can be reduced to the similar problem for a certain nonimpulsive equation.

The chapter contains the following results. Theorems 12.2 and 12.3 are concerned with the equivalence of nonoscillation, positivity of a fundamental function and existence of a positive solution for a certain inequality. They lead to explicit nonoscillation results (Theorem 12.4). Theorem 12.5 compares nonoscillation conditions for two different impulsive delay differential equations. Theorem 12.6 contains the main result of the chapter connecting oscillation of an impulsive and a nonimpulsive equation. As a corollary (Theorem 12.7), we obtain explicit oscillation conditions for an impulsive delay differential equation. Theorems 12.8–12.10 establish the relation between nonoscillation of an impulsive equation with a distributed delay and a specially constructed nonimpulsive equation.

The chapter is organized as follows. Section 12.2 contains some definitions and the solution representation formula. Section 12.3 includes results on the relation between nonoscillation, positivity of the fundamental function and existence of a positive solution for an integral inequality. Section 12.4 involves comparison results and explicit nonoscillation conditions. Section 12.5 presents the main result of this chapter, which establishes the relation of nonoscillation properties for a linear impulsive

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equation and some specially constructed nonimpulsive equation. Section 12.6 considers oscillation of a linear impulsive equation with a distributed delay. Finally, Sect. 12.7 presents a discussion and states some open problems.

## 12.2 Preliminaries

We consider the scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x\big(h_k(t)\big) = 0,\ t \ge 0 \tag{12.2.1}$$

with the linear impulsive conditions

$$x\big(\tau_j^+\big) = B_j x(\tau_j),\ j = 1, 2, \cdots, \tag{12.2.2}$$

where $x(\tau_j^+) = \lim_{t\to\tau_j^+} x(t)$, under the following assumptions:

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ are fixed points, $\lim_{j\to\infty} \tau_j = \infty$.
(a2) $A_k$, $k = 1, \cdots, m$ are Lebesgue measurable functions essentially bounded in each finite interval $[0, b]$, $B_j \in \mathbb{R}$, $j \in \mathbb{N}$.
(a3) $h_k : [0,\infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \le t$, $\lim_{t\to\infty} h_k(t) = \infty$.

Together with (12.2.1), (12.2.2), we will consider for each $t_0 \ge 0$ the initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x\big(h_k(t)\big) = f(t),\ t \ge t_0, \tag{12.2.3}$$

$$x\big(\tau_j^+\big) = B_j x(\tau_j) + \alpha_j,\ \tau_j > t_0, \tag{12.2.4}$$

$$x(t_0) = x_0,\ x(\xi) = \varphi(\xi),\ \xi < t_0. \tag{12.2.5}$$

We assume that for the initial function $\varphi$ the following hypothesis holds:

(a4) $f$ are Lebesgue measurable functions essentially bounded in each finite interval $[0, b]$, $\alpha_j \in \mathbb{R}$ and $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

**Definition 12.1** A function $x : [t_0,\infty) \to \mathbb{R}$ that is absolutely continuous on each interval $(\tau_j, \tau_{j+1}]$, $j = 0, 1, 2, \cdots$ is *a solution* of impulsive problem (12.2.3)–(12.2.5) if (12.2.3) is satisfied for almost all $t \in [0,\infty)$ and condition (12.2.5) for $t \le t_0$ and the equalities (12.2.4) hold.

**Definition 12.2** For each $s \ge 0$, the solution $X(t, s)$ of the problem

$$\begin{aligned} &\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x\big(h_k(t)\big) = 0,\ t \ge s;\ x(\xi) = 0,\ \xi < s, \\ &x\big(\tau_j^+\big) = B_j x(\tau_j),\ \tau_j > s,\ x(s) = 1, \end{aligned} \tag{12.2.6}$$

is *the fundamental function* of equation (12.2.1), (12.2.2). We assume that $X(t,s)=0$ for $0\le t<s$.

The following theorem is the scalar version of Theorem B.18.

**Theorem 12.1** *Let* (a1)–(a4) *hold. Then there exists one and only one solution of problem* (12.2.3)–(12.2.5), *and it can be presented in the form*

$$x(t)=X(t,t_0)x(t_0)+\int_{t_0}^{t}X(t,s)f(s)ds - \sum_{k=1}^{m}\int_{t_0}^{t}X(t,s)A_k(s)\varphi\big(h_k(s)\big)ds+\sum_{t_0<\tau_j\le t}X(t,\tau_j)\alpha_j, \quad (12.2.7)$$

*where* $\varphi(h_k(s))=0$, *if* $h_k(s)>t_0$.

## 12.3 Nonoscillation Criteria for Impulsive Equations

**Definition 12.3** Impulsive equation (12.2.1), (12.2.2) *has a nonoscillatory solution* if there exist $t_0>0$ and $\varphi(t)$ such that the solution of (12.2.3)–(12.2.5) with $f\equiv 0, \alpha_j=0$ is positive for $t\ge t_0$. Otherwise, all solutions of (12.2.1), (12.2.2) are *oscillatory*.

Denote for any $s$

$$A_k^s(t)=\begin{cases}A_k(t) & \text{if } t\ge s,\\ 0 & \text{if } t<s,\end{cases} \qquad h_k^s(t)=\begin{cases}h_k(t) & \text{if } t\ge s,\\ s & \text{if } t<s.\end{cases} \quad (12.3.1)$$

Everywhere we assume that a product equals one if the number of factors is equal to zero.

The following theorem establishes nonoscillation criteria.

**Theorem 12.2** *Suppose* (a1)–(a4) *hold*, $A_k(t)\ge 0$, $k=1,\cdots,m$ *and* $B_j>0$, $j\in\mathbb{N}$. *Then the following hypotheses are equivalent*:

1) *Equation* (12.2.1), (12.2.2) *has a nonoscillatory solution.*
2) *There exist* $t_1\ge 0$ *and a nonnegative locally essentially bounded solution* $u$ *of the inequality*

$$u(t)\ge\sum_{k=1}^{m}A_k^{t_1}(t)\exp\left\{\int_{h_k^{t_1}(t)}^{t}u(s)ds\right\}\prod_{h_k^{t_1}(t)\le\tau_j<t}B_j^{-1},\ t\ge t_1. \quad (12.3.2)$$

3) *There exists* $t_1\ge 0$ *such that* $X(t,s)>0$, $t_1\le s<t<\infty$.

*Proof* The scheme of the proof is 1) ⇒ 2) ⇒ 3) ⇒ 1).

1) ⇒ 2) Let $x(t)$ be a positive solution of (12.2.3)–(12.2.5), where $f \equiv 0$, $\alpha_j = 0$. By (a3), for a certain $t_1 \geq t_0$, we have $h_k(t) > t_0$, $t \geq t_1$, $k = 1, \cdots, m$.

Let us demonstrate that

$$u(t) = -\frac{d}{dt} \ln \left\{ \frac{x(t)}{x(t_1)} \prod_{t_1 \leq \tau_j < t} B_j^{-1} \right\},\ t \geq t_1$$

is a solution of (12.3.2). To this end, we integrate the equality

$$x(t) = x(t_1) \exp\left\{ -\int_{t_1}^{t} u(s)ds \right\} \prod_{t_1 \leq \tau_j < t} B_j,\ t \geq t_1 \tag{12.3.3}$$

and set $\varphi(t) = x(t)$ for $t < t_1$. Then $x(t)$, $t \geq t_1$ is a solution of (12.2.3)–(12.2.5) with the initial point $t = t_1$ and the initial function $\varphi(t) > 0$. We substitute (12.3.3) in (12.2.3) with $f \equiv 0$, $\alpha_j = 0$,

$$\begin{aligned} &-u(t) \exp\left\{ -\int_{t_1}^{t} u(s)ds \right\} \prod_{t_1 \leq \tau_j < t} B_j \\ &+ \sum_{k \in N_1} A_k(t) \exp\left\{ -\int_{t_1}^{h_k(t)} u(s)ds \right\} \prod_{t_1 \leq \tau_j < h_k(t)} B_j \\ &+ \sum_{k \in N_2} A_k(t) \varphi\big(h_k(t)\big) = 0,\ t \geq t_1, \end{aligned} \tag{12.3.4}$$

where $N_1 = \{k \mid h_k(t) \geq t_1\}$, $N_2 = \{k \mid h_k(t) < t_1\}$.

Using notation in (12.3.1), the equality (12.3.4) can be rewritten in the form

$$\begin{aligned} &-u(t) \exp\left\{ -\int_{t_1}^{t} u(s)ds \right\} \prod_{t_1 \leq \tau_j < t} B_j \\ &+ \sum_{k=1}^{m} A_k^{t_1}(t) \exp\left\{ -\int_{t_1}^{h_k^{t_1}(t)} u(s)ds \right\} \prod_{t_1 \leq \tau_j < h_k^{t_1}(t)} B_j \\ &+ \sum_{k \in N_2} A_k(t) \varphi\big(h_k(t)\big) = 0,\ t \geq t_1. \end{aligned}$$

Consequently,

$$\begin{aligned} &\left( u(t) - \sum_{k=1}^{m} A_k^{t_1}(t) \exp\left\{ \int_{h_k^{t_1}(t)}^{t} u(s)ds \right\} \prod_{h_k^{t_1}(t) \leq \tau_j < t} B_j^{-1} \right) \\ &\times \exp\left\{ -\int_{t_1}^{t} u(s)ds \right\} \prod_{t_1 \leq \tau_j < t} B_j = \sum_{k \in N_2} A_k(t) \varphi\big(h_k(t)\big) \geq 0 \end{aligned}$$

since $\varphi(t)$ is nonnegative according to our choice of the point $t_1$, which implies 3).

2) ⇒ 3) Consider (12.2.3)–(12.2.5) with the initial point $t_1$, the initial function $\varphi \equiv 0$, $\alpha_j = 0$ and initial value $x(t_1) = 0$ on the segment $[t_1, b]$:

$$\begin{aligned} &\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x\bigl(h_k(t)\bigr) = f(t),\ t \in [t_1, b],\ x(\xi) = 0,\ \xi < t_1, \\ &x(t_1) = 0,\ x\bigl(\tau_j^+\bigr) = B_j x(\tau_j), \tau_j \ge t_1. \end{aligned} \tag{12.3.5}$$

Besides, we consider the ordinary impulsive differential equation including the solution $u(t) \ge 0$ of (12.3.2) as a coefficient:

$$\dot{x}(t) + u(t)x(t) = z(t),\ t \in [t_1, b],\ x\bigl(\tau_j^+\bigr) = B_j x(\tau_j),\ x(t_1) = 0. \tag{12.3.6}$$

Direct calculations yield that the solution of (12.3.6) has the form

$$x(t) = \int_{t_1}^{t} \exp\left\{-\int_s^t u(\xi)d\xi\right\} \prod_{s \le \tau_j < t} B_j z(s)ds. \tag{12.3.7}$$

We seek the solution of problem (12.3.5) in the form (12.3.7). By substituting $x$ and $\dot{x}$ from (12.3.7) and (12.3.6) into (12.3.5), we obtain

$$\begin{aligned} &z(t) - u(t) \int_{t_1}^{t} \exp\left\{-\int_s^t u(\xi)d\xi\right\} \prod_{s \le \tau_j < t} B_j z(s)ds \\ &\quad + \sum_{k=1}^{m} A_k^{t_1}(t) \int_{t_1}^{h_k^{t_1}(t)} \exp\left\{-\int_s^{h_k^{t_1}(t)} u(\xi)d\xi\right\} \prod_{s \le \tau_j < h_k^{t_1}(t)} B_j z(s)ds = f(t). \end{aligned} \tag{12.3.8}$$

Equation (12.3.8) is of the type

$$z - Hz = f, \tag{12.3.9}$$

where

$$\begin{aligned} (Hz)(t) &= u(t) \int_{t_1}^{t} \exp\left\{-\int_s^t u(\xi)d\xi\right\} \prod_{s \le \tau_j < t} B_j z(s)ds \\ &\quad - \sum_{k=1}^{m} A_k^{t_1}(t) \int_{t_1}^{h_k^{t_1}(t)} \exp\left\{-\int_s^{h_k^{t_1}(t)} u(\xi)d\xi\right\} \prod_{s \le \tau_j < h_k^{t_1}(t)} B_j z(s)ds \\ &= \int_{t_1}^{t} \Biggl[ u(t) \exp\left\{-\int_s^t u(\xi)d\xi\right\} \prod_{s \le \tau_j < t} B_j - \sum_{k=1}^{m} \Biggl( A_k^{t_1}(t) \\ &\quad \times \exp\left\{-\int_s^{h_k^{t_1}(t)} u(\xi)d\xi\right\} \prod_{s \le \tau_j < h_k^{t_1}(t)} B_j \Biggr) \chi_{[t_1, h_k^{t_1}(t)]}(s) \Biggr] z(s)\,ds. \end{aligned}$$

For the kernel $K(t, s)$ of the operator $H$ defined above, we have

$$\bigl|K(t, s)\bigr| \le \sup_{s, t \in [t_1, b]} \prod_{s \le \tau_j < t} \max\{B_j, 1\} \left( u(t) + \sum_{k=1}^{m} \bigl|A_k(t)\bigr| \right).$$

Hence the kernel of $H$ is bounded on the set $[t_1, b] \times [t_1, b]$. By Theorem A.4, the operator $H : L_\infty[t_1, b] \to L_\infty[t_1, b]$ is a weakly compact Volterra integral operator. Therefore, by Theorem A.7, its spectral radius is equal to zero. Consequently, (12.3.9) for any $f \in L_\infty[t_1, b]$ has one and only one solution,

$$z = (I - H)^{-1} f, \tag{12.3.10}$$

where $I$ is the identity operator.

Let us demonstrate that $H$ is a positive operator. The operator $H$ can be rewritten as a sum $H = H_1 + H_2$, where

$$(H_1 z)(t) = \left( u(t) - \sum_{k=1}^{m} A_k^{t_1}(t) \exp\left\{ \int_{h_k^{t_1}(t)}^{t} u(s)ds \right\} \prod_{h_k^{t_1}(t) \le \tau_j < t} B_j^{-1} \right)$$
$$\times \int_{t_1}^{t} \exp\left\{ -\int_{s}^{t} u(\xi)d\xi \right\} \prod_{s \le \tau_j < t} B_j z(s)ds,$$
$$(H_2 z)(t) = \sum_{k=1}^{m} A_k^{t_1}(t) \int_{h_k^{t_1}(t)}^{t} \exp\left\{ -\int_{s}^{h_k^{t_1}(t)} u(\xi)d\xi \right\} \prod_{s \le \tau_j < h_k^{t_1}(t)} B_j z(s)ds,$$

where $H_2$ is obviously positive, and inequality (12.3.2) implies $H_1 \ge 0$, so $H = H_1 + H_2 \ge 0$. Since the spectral radius of $H$ is equal to zero, we have $(I - H)^{-1} = I + H + H^2 + \cdots \ge 0$. Thus, if $f \ge 0$, then the solution $z$ of (12.3.9) is nonnegative: $z \ge 0$. The solution of (12.3.5) has the form (12.3.7), where $z$ is the solution of (12.3.9). Consequently, $f \ge 0$ in (12.3.5) implies $f \ge 0$ for the solution of (12.3.5). On the other hand, the solution of problem (12.3.5) can be presented in the form (12.2.7)

$$x(t) = \int_{t_1}^{t} X(t, s) f(s)ds. \tag{12.3.11}$$

As was shown above, $f \ge 0$ implies $x \ge 0$, and therefore the kernel of the integral operator is nonnegative (i.e., $X(t, s) \ge 0$ for $t_1 \le s \le t < b$). Since $b > t_1$ is chosen arbitrarily, $X(t, s) \ge 0$ for $t_1 \le s < t < \infty$.

Let us prove that in fact the strict inequality $X(t, s) > 0$ holds. Denote

$$x(t) = X(t, t_1) - \exp\left\{ -\int_{t_1}^{t} u(s)ds \right\} \prod_{t_1 \le \tau_j < t} B_j.$$

Our purpose is to demonstrate that $x(t)$ is nonnegative. The function $x(t)$ is a solution of (12.2.3)–(12.2.5) with $x(t_1) = 0$, $\varphi \equiv 0$ and

$$f(t) = u(t) \exp\left\{ -\int_{t_1}^{t} u(s)ds \right\} \prod_{t_1 \le \tau_j < t} B_j$$
$$- \sum_{k=1}^{m} A_k^{t_1}(t) \exp\left\{ -\int_{t_1}^{h_k^{t_1}(t)} u(s)ds \right\} \prod_{t_1 \le \tau_j < h_k^{t_1}(t)} B_j$$

$$= \exp\left\{-\int_{t_1}^{t} u(s)ds\right\} \prod_{t_1 \le \tau_j < t} B_j$$
$$\times \left( u(t) - \sum_{k=1}^{m} A_k^{t_1}(t) \exp\left\{\int_{h_k^{t_1}(t)}^{t} u(s)ds\right\} \prod_{h_k^{t_1}(t) \le \tau_j < t} B_j^{-1} \right).$$

Thus (12.3.2) implies $f(t) \ge 0$, and therefore in view of (12.2.7) we have

$$x(t) = \int_{t_1}^{t} X(t,s) f(s)ds \ge 0$$

and consequently

$$X(t,t_1) \ge \exp\left\{-\int_{t_1}^{t} u(s)ds\right\} \prod_{t_1 \le \tau_j < t} B_j > 0.$$

For $s > t_1$, the inequality $X(t,s) > 0$ can be proven similarly.

3) ⇒ 1) Denote $x(t) = X(t,t_1)$. Then $x(t)$ is a positive solution of (12.2.3)–(12.2.5) with $f \equiv 0$, $\alpha_j = 0$ and the initial function $\varphi \equiv 0$, which completes the proof. □

Let us consider (12.2.1), (12.2.2) with coefficients of an arbitrary sign. Denote

$$a^+ = \max\{a, 0\},\ a^- = \max\{-a, 0\}.$$

**Theorem 12.3** *Suppose* (a1)–(a4) *hold and* $B_j > 0$.

*Consider the following three hypotheses*:

1) *The initial value problem* (12.2.3)–(12.2.5) *with the initial point* $t_0 > 0$ ($f \equiv 0$, $\alpha_j = 0$) *has a positive solution.*
2) *There exists a solution nonnegative integrable on each interval* $[t_0, b]$, *of the inequality*

$$u(t) \ge \sum_{k=1}^{m} \left(A_k^{t_0}(t)\right)^+ \exp\left\{\int_{h_k^{t_0}(t)}^{t} u(s)ds\right\} \prod_{h_k^{t_0}(t) \le \tau_j < t} B_j^{-1},\ t \ge t_0. \quad (12.3.12)$$

3) $X(t,s) > 0$, $t_0 \le s < t < \infty$.

*Then the implications* 2) ⇒ 3) *and* 2) ⇒ 1) *are valid.*

*Proof* The proof of 2) ⇒ 3) coincides with the proof of 2) ⇒ 3) in Theorem 12.2 up to the place where the operator $H$ is presented as a sum of two terms. Here we present it as a sum of three terms $H = H_1 + H_2 + H_3$, where

$$(H_1 z)(t) = \left( u(t) - \sum_{k=1}^{m} \left(A_k^{t_0}(t)\right)^+ \exp\left\{\int_{h_k^{t_0}(t)}^{t} u(s)ds\right\} \prod_{h_k^{t_0}(t) \le \tau_j < t} B_j^{-1} \right)$$
$$\times \int_{t_0}^{t} \exp\left\{-\int_{s}^{t} u(\xi)d\xi\right\} \prod_{s \le \tau_j < t} B_j z(s)ds,$$

$$(H_2z)(t)=\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+\int_{h_k^{t_0}(t)}^{t}\exp\left\{-\int_s^{h_k^{t_0}(t)}u(\xi)d\xi\right\}\prod_{s\le\tau_j<h_k^{t_0}(t)}B_jz(s)ds,$$

$$(H_3z)(t)=\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^-\int_{t_0}^{h_k^{t_0}(t)}\exp\left\{-\int_s^{h_k^{t_0}(t)}u(\xi)d\xi\right\}\prod_{s\le\tau_j<h_k^{t_0}(t)}B_jz(s)ds.$$

Again, as in Theorem 12.2, $H_1\ge 0$, $H_2\ge 0$, $H_3\ge 0$, which implies $H=H_1+H_2+H_3\ge 0$. The end of the proof coincides with that of Theorem 12.2.

Implication 2) $\Rightarrow$ 1) is evident. □

## 12.4 Explicit Nonoscillation Tests and Comparison Theorems

Now we proceed to explicit nonoscillation results. Denote

$$\underline{h}^{t_0}(t)=\min_k h_k^{t_0}(t),$$

where $h_k^s(t)$ are defined by (12.3.1).

**Theorem 12.4** *Suppose* (a1)–(a4) *hold*, $B_j>0$ *and at least one of the following three hypotheses holds*:

1) $$A_k(t)\le 0,\ t\ge t_0.$$

2) $$\sum_{k=1}^{m}\int_{\underline{h}^{t_0}(t)}^{t}\left(A_k^{t_0}(s)\right)^+\prod_{h_k^{t_0}(s)\le\tau_j<s}B_j^{-1}z(s)ds\le\frac{1}{e},\ t\ge t_0. \tag{12.4.1}$$

3) $$\sum_{k=1}^{m}\int_{\underline{h}^{t_0}(t)}^{t}\left(A_k^{t_0}(s)\right)^+ds\le 1/e\left(1+\sum_{\underline{h}^{t_0}(t)\le\tau_j<t,\ B_j<1}\ln B_j\right),\ t\ge t_0. \tag{12.4.2}$$

*Then the fundamental matrix* $X(t,s)$ *is positive for* $t_0\le s<t$ *and there exists a positive solution of* (12.2.3)–(12.2.5) *with* $f\equiv 0$, $\alpha_j=0$.

*Proof* Obviously 1) is a special case of 2). Let us prove the theorem assuming that (12.4) holds. To this end, we will demonstrate that the function

$$u(t)=e\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+\prod_{h_k^{t_0}(t)\le\tau_j<t}B_j^{-1}$$

is a nonnegative solution of inequality (12.3.12).

After substituting $u(t)$ into (12.3.12), we obtain

$$e\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+ \prod_{h_k^{t_0}(t)\le\tau_j<t} B_j^{-1}$$
$$\ge \sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+ \exp\left\{e\int_{h_k^{t_0}(t)}^{t}\sum_{i=1}^{m}\left(A_i^{t_0}(s)\right)^+ \prod_{h_i^{t_0}(s)\le\tau_j<s} B_j^{-1}\,ds\right\}\prod_{h_k^{t_0}(t)\le\tau_j<t} B_j^{-1},$$

which can be deduced from the inequality

$$e\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+ \prod_{h_k^{t_0}(t)\le\tau_j<t} B_j^{-1}$$
$$\ge \exp\left\{e\int_{\underline{h}^{t_0}(t)}^{t}\sum_{k=1}^{m}\left(A_k^{t_0}(s)\right)^+ \prod_{h_k^{t_0}(s)\le\tau_j<s} B_j^{-1}\,ds\right\}\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+ \prod_{h_k^{t_0}(t)\le\tau_j<t} B_j^{-1}.$$

After dividing this inequality by its left-hand side and taking the logarithm of both sides, we obtain

$$\sum_{k=1}^{m}\int_{\underline{h}^{t_0}(t)}^{t}\left(A_k^{t_0}(s)\right)^+ \prod_{h_k^{t_0}(s)\le\tau_j<s} B_j^{-1} z(s)ds \le \frac{1}{e},$$

which obviously follows from (12.4).

Further, let 3) hold. We will prove that $u(t) = e\sum_{k=1}^{m}(A_k^{t_0}(t))^+$ is a solution of (12.3.12), which after substituting $u$ takes the form

$$e\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+$$
$$\ge \sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+ \exp\left\{e\int_{h_k^{t_0}(t)}^{t}\sum_{i=1}^{m}\left(A_i^{t_0}(s)\right)^+ ds\right\}\prod_{h_k^{t_0}(t)\le\tau_j<t} B_j^{-1}.$$

This inequality can be deduced from

$$e\sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+$$
$$\ge \sum_{k=1}^{m}\left(A_k^{t_0}(t)\right)^+ \exp\left\{e\int_{\underline{h}^{t_0}(t)}^{t}\sum_{k=1}^{m}\left(A_k^{t_0}(s)\right)^+ ds\right\}\prod_{\underline{h}^{t_0}(t)\le\tau_j<t} B_j^{-1},$$

where the product contains only the factors for which $B_j < 1$. The latter inequality, after dividing by the left-hand side and taking the logarithm of both sides, coincides with (12.4.2), which completes the proof of the theorem. □

Let us compare oscillation properties of (12.2.1), (12.2.2) and the equation

$$\begin{aligned} &\dot{x}(t) + \sum_{k=1}^{m} \tilde{A}_k(t) x\big(\tilde{h}_k(t)\big) = 0,\ t \in [0,\infty), \\ &x\big(\tau_j^+\big) = \tilde{B}_j x(\tau_j). \end{aligned} \tag{12.4.3}$$

**Theorem 12.5** *Let the hypotheses* (a1)–(a4) *hold for* (12.2.1), (12.2.2) *and* (12.4.3). *Suppose that any* (*therefore all*) *of the hypotheses* 1)–3) *of Theorem* 12.2 *holds for* (12.2.1), (12.2.2).

*If* $A_k(t) \ge \tilde{A}_k(t) \ge 0$, $\tilde{B}_j \ge B_j > 0$ *and at least one of the hypotheses*

1) $h_k(t) \le \tilde{h}_k(t)$, $\tilde{B}_j \le 1$, $j = 1, 2, \cdots$,
2) $h_k(t) = \tilde{h}_k(t)$,

*holds*, *then for* (12.4.3) *assertions* 1)–3) *of Theorem* 12.2 *are valid.*

*Proof* By the hypothesis of this theorem, there exists a nonnegative function $u(t)$ satisfying (12.3.2). Besides, for any nonnegative function $u$, under this hypotheses of this theorem the inequality

$$\begin{aligned} &\sum_{k=1}^{m} A_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)ds\right\} \prod_{h_k(t)\le\tau_j<t} B_j^{-1} \\ &\quad \ge \sum_{k=1}^{m} \tilde{A}_k(t) \exp\left\{\int_{\tilde{h}_k(t)}^{t} u(s)ds\right\} \prod_{\tilde{h}_k(t)\le\tau_j<t} \tilde{B}_j^{-1} \end{aligned}$$

holds. Consequently, if $u$ is a solution of the inequality (12.3.2), then $u$ is also a solution of this inequality, where $A_k$, $h_k$, $B_j$ are changed by $\tilde{A}_k$, $\tilde{h}_k$, $\tilde{B}_j$. Thus, by Theorem 12.2, the other assertions of this theorem also hold. □

**Corollary 12.1** *Suppose that* (a1)–(a4) *hold for* (12.2.1), (12.2.2) *and* $B_j > 0$. *Besides, let* $0 \le A_k(t) \le A_k$, $t - h_k(t) \le h_k$, $B_j \le 1$.

*If there exists a nonoscillatory solution of the impulsive equation*

$$\dot{x}(t) + \sum_{k=1}^{m} A_k x(t - h_k) = 0,\ t \in \big[0,\infty),\ x\big(\tau_j^+\big) = B_j x(\tau_j),$$

*then* (12.2.1), (12.2.2) *also has a nonoscillatory solution.*

**Corollary 12.2** *Let* (a1)–(a4) *hold and* $A_k(t) \ge 0$. *If there exists a nonoscillatory solution of* (12.2.1) *without impulses and* $B_j \ge 1$, *then there exists a nonoscillatory solution of impulsive equation* (12.2.1), (12.2.2).

## 12.5 Reduction to Equations Without Impulses

In this section, we present a fundamental result that enables us to reduce the oscillation problem for (12.2.1), (12.2.2) to oscillation of an equation without impulses.

Consider the auxiliary equation

$$\dot{x}(t)+\sum_{k=1}^{m}A_k(t)\prod_{h_k^0(t)\le\tau_j<t}B_j^{-1}x\big(h_k(t)\big)=0,\ t\in[0,\infty),\ h_k^0(t)=\begin{cases}h_k(t) & \text{if } t\ge 0,\\ 0 & \text{if } t<0.\end{cases} \tag{12.5.1}$$

Denote by $Y(t,s)$ the fundamental function of (12.5.1).

**Theorem 12.6** *Suppose* (a1)–(a4) *hold and* $A_k(t)\ge 0$, $B_j>0$. *Then*:

1) *There exists* $t_0>0$ *such that* $X(t,s)>0$, $t_0\le s<t<\infty$ *if and only if there exists* $t_1>0$ *such that* $Y(t,s)>0$, $t_1\le s<t<\infty$.
2) *All solutions of* (12.2.1), (12.2.2) *are oscillatory if and only if all solutions of* (12.5.1) *are oscillatory.*
3) *There exists a nonoscillatory solution of* (12.2.1), (12.2.2) *if and only if* (12.5.1) *has a nonoscillatory solution.*

*Proof* 1) Let $X(t,s)>0$, $t_0\le s<t<\infty$. Then, by Theorem 12.2, there exists a solution of inequality (12.3.1) for $t\ge t_1$. This inequality coincides with (2.3.2) if we assume

$$a_k(t)=A_k(t)\prod_{h_k^0(t)\le\tau_j<t}B_j^{-1}.$$

Therefore, by Theorem 2.1, we have $Y(t,s)>0$, $t_1\le s<t<\infty$. The inverse result can be proven similarly.

2) Suppose all solutions of (12.2.1), (12.2.2) are oscillatory and (12.5.1) has a positive solution for $t\ge t_0$ for a certain $t_0$. By Theorem 2.1, the fundamental function is positive: $Y(t,s)>0$, $t_1\le s<t<\infty$. Then, as was proven in part 1), $X(t,s)>0$ for $t_2\le s<t<\infty$. Consequently, by Theorem 12.2, (12.2.1), (12.2.2) has a nonoscillatory solution, which contradicts the assumption. The inverse claim is proven similarly.

Besides, 2) implies 3), which completes the proof. □

Application of Theorem 12.6 and known oscillation (nonoscillation) results for equations without impulses leads to oscillation results for impulsive equations. As an example, we present the following statement. Denote

$$\underline{h}(t)=\min_k h_k(t),\ \bar{h}(t)=\max_k h_k(t).$$

**Theorem 12.7** *Let* (a1)–(a4) *hold for* (12.2.1), (12.2.2), $A_k(t)\ge 0$ *and* $B_j>0$. *If at least one of the following inequalities holds,* 1)

$$\liminf_{t\to\infty}\int_{\underline{h}(t)}^{t}\sum_{k=1}^{m}A_k(s)\prod_{h_k(s)\le\tau_j<s}B_j^{-1}ds>\frac{1}{e},$$

2) *$h_k$ are nondecreasing functions and*

$$\limsup_{t\to\infty}\int_{\bar{h}(t)}^{t}\sum_{k=1}^{m}A_k(s)\prod_{h_k(s)\le\tau_j<s}B_j^{-1}ds>1,$$

*then all the solutions of* (12.2.1), (12.2.2) *are oscillatory.*

This statement is obtained by applying Theorem 12.6 and Lemma 2.2.

*Remark 12.1* In Theorem 12.6, we can omit the condition $A_k(t)\ge 0$. In fact, if $x(t)$ is a solution of (12.2.1), (12.2.2), then $y(t)=x(t)\prod_{0\le\tau_j<t}B_j^{-1}$ is a solution of (12.5.1) with a suitable initial function. We omit the details of the proof of this statement.

## 12.6 Impulsive Equations with a Distributed Delay

Consider the linear delay impulsive equation

$$\dot{y}(t)+\int_{-\infty}^{t}y(s)\,d_sR(t,s)=0,\ \ t>t_0 \tag{12.6.1}$$

with the initial function

$$y(t)=\varphi(t),\ \ t<t_0 \tag{12.6.2}$$

and the impulsive conditions

$$y\big(\tau_j^+\big)=B_jy(\tau_j),\ \ j\in\mathbb{N} \tag{12.6.3}$$

under some of the following assumptions:

(b1) $R(t,\cdot)$ is a left continuous function of bounded variation, and for each $s$ its variation on the segment $[t_0,s]$

$$P(t,s)=\mathrm{Var}_{\tau\in[t_0,s]}R(t,\tau) \tag{12.6.4}$$

is a locally integrable function in $t$.

(b2) $R(t,s)=R(t,t^+)$, $t\le s$, and there exist $M>0$, $\lambda>0$ such that

$$\int_s^t\big|d_\tau R(t,\tau)\big|\le Me^{-\lambda(t-s)}.$$

(b3) For each $t_1$, there exists $s_1=s(t_1)\le t_1$ such that $R(t,s)=0$ for $s<s_1$, $t>t_1$ and $\lim_{t\to\infty}s(t)=\infty$.

(b4) $\varphi:(-\infty,t_0)\to R$ is a continuous bounded function.

(b5) $t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots$ satisfy $\lim_{j\to\infty} \tau_j = \infty$.
(b6) $B_j > 0$, $j = 1, 2, \cdots$.

**Definition 12.4** A function absolutely continuous on every interval $(\tau_j, \tau_{j+1}]$ is *a solution* of (12.6.1)–(12.6.3) if it satisfies (12.6.1) almost everywhere for $t > t_0$, initial condition (12.6.2) for $t < t_0$ and also impulsive conditions (12.6.3).

We assume that the solution is a left continuous function. Equation (12.6.1) includes equations with nonconstant delays, integrodifferential equations and equations with both integral and concentrated delay terms as special cases; for a detailed discussion, see Chap. 4.

Together with impulsive equation (12.6.1), (12.6.3), consider the equation without impulses

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s T(t,s) = 0, \ \ t > t_0, \tag{12.6.5}$$

where

$$T(t,s) = \prod_{s \le \tau_j < t} B_j^{-1} R(t,s). \tag{12.6.6}$$

**Theorem 12.8** *Suppose* (b1)–(b6) *hold. Then impulsive equation* (12.6.1), (12.6.3) *is oscillatory* (*nonoscillatory*) *if and only if nonimpulsive equation* (12.6.5) *is oscillatory* (*nonoscillatory*).

*Proof* Let $y$ be a solution of (12.6.1)–(12.6.3). Then $x(t) = \prod_{t_0 \le \tau_j < t} B_j^{-1} y(t)$ is continuous and $y(t) = \prod_{t_0 \le \tau_j < t} B_j x(t)$. After substituting $y$ and $\dot{y} = \prod_{t_0 \le \tau_j < t} B_j \dot{x}$ into (12.6.1), we have

$$\prod_{t_0 \le \tau_j < t} B_j \dot{x}(t) + \int_{t_0}^{t} x(s) \prod_{t_0 \le \tau_j < s} B_j \, d_s R(t,s) = 0, \tag{12.6.7}$$

which after multiplying by $\prod_{t_0 \le \tau_j < t} B_j^{-1}$ turns into the equation

$$\dot{x}(t) + \int_{t_0}^{t} x(s) \prod_{s \le \tau_j < t} B_j^{-1} d_s R(t,s) = \dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s T(t,s) = 0. \tag{12.6.8}$$

Conversely, if $x(t)$ is a solution of (12.6.5), (12.6.2) without impulses, where $T$ is defined by (12.6.6), then

$$y(t) = \prod_{t_0 \le \tau_j < t} B_j x(t)$$

is a solution of (12.6.1), (12.6.2), (12.6.3). Since $B_j > 0$, $x$ and $y$ are oscillatory (nonoscillatory) at the same time, which completes the proof. □

Consider now the inequalities corresponding to (12.6.1), (12.6.5):

$$\dot{y}(t) + \int_{-\infty}^{t} y(s)\, d_s R(t,s) \le 0, \ \ t > t_0, \tag{12.6.9}$$

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s T(t,s) \le 0, \ t > t_0, \tag{12.6.10}$$

$$\dot{y}(t) + \int_{-\infty}^{t} y(s)\, d_s R(t,s) \ge 0, \ t > t_0, \tag{12.6.11}$$

$$\dot{x}(t) + \int_{-\infty}^{t} x(s)\, d_s T(t,s) \ge 0, \ t > t_0. \tag{12.6.12}$$

Similar to Theorem 12.8, the following result can be obtained.

**Theorem 12.9** *Suppose* (b1)–(b6) *hold. Then inequality* (12.6.9), (12.6.3) (*or inequality* (12.6.11), (12.6.3)) *is oscillatory* (*nonoscillatory*) *if and only if* (12.6.10) (*or* (12.6.12)) *is oscillatory* (*nonoscillatory*).

**Corollary 12.3** *Suppose that the following condition is satisfied*:

(c1) $A_k(t) \ge 0$ *are locally essentially bounded functions*, *and* $h_k(t)$ *are Lebesgue measurable functions*, $h_k(t) \le t$, $k = 1, 2, \cdots$, $\lim_{t\to\infty} \inf_k h_k(t) = \infty$.

*Then the equation*

$$\dot{y}(t) + \sum_{k=1}^{m} A_k(t) y\big(h_k(t)\big) = 0 \tag{12.6.13}$$

*with impulsive conditions* (12.6.3) *is oscillatory* (*nonoscillatory*) *if and only if the equation*

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t) \prod_{h_k(t)\le\tau_j<t} B_j^{-1} x\big(h_k(t)\big) = 0 \tag{12.6.14}$$

*without impulses is oscillatory* (*nonoscillatory*). *The same claim is valid for the corresponding inequalities.*

Note that this corollary partially coincides with Theorem 12.6.

*Remark 12.2* Theorem 1 in [338] is a special case of the corollary above when $m = 2$, $h_1(t) = t$, $h_2(t) = t - \tau$.

**Corollary 12.4** *The equation*

$$\dot{y}(t) + \sum_{k=1}^{m} A_k(t) y\big(h_k(t)\big) + \int_{-\infty}^{t} K(t,s) y(s)\, ds = 0$$

*with impulsive conditions* (12.6.3) *is oscillatory* (*nonoscillatory*) *if and only if*

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t) \prod_{h_k(t)\le\tau_j<t} B_j^{-1} x\big(h_k(t)\big) + \int_{-\infty}^{t} K(t,s) \prod_{s\le\tau_j<t} B_j^{-1} x(s)\, ds = 0$$

*is oscillatory* (*nonoscillatory*). *The same result is valid for the corresponding inequalities.*

It is to be emphasized that the scheme above is not specific for the oscillation property. An arbitrary (Q)-property of solutions of linear impulsive and nonimpulsive equations can be considered in a similar way; for example, stability, exponential stability and boundedness of solutions.

**Theorem 12.10** *Let* (b1)–(b5) *hold and* $B_j, \tau_j$ *be such that for any continuous* $x(t)$ *two functions* $x(t)$ *and* $y(t) = \prod_{t_0 \le \tau_j < t} B_j x(t)$ *enjoy the* $(Q)$*-property simultaneously. Then all solutions of* (12.6.1), (12.6.2), (12.6.3) *have the* $(Q)$*-property if and only if all solutions of* (12.6.5) *have the* $(Q)$*-property.*

**Corollary 12.5** *Suppose that there exist positive* $m, M$ *such that*

$$m \le \prod_{t_0 \le \tau_j < t} |B_j| \le M$$

*for any* $t$*. Then all solutions of* (12.6.1), (12.6.2), (12.6.3) *are bounded* (*tend to zero*) *if and only if all solutions of* (12.6.5) *are bounded* (*tend to zero*).

Many similar results can be obtained as a corollary of Theorem 12.10. The popularity of this approach in the study of oscillation of impulsive equations can be explained by the simplicity of the solution correspondence condition.

## 12.7 Discussion and Open Problems

The method proposed in the present chapter, similar to the previous chapters, is based on the solution representation formula. Such formulas are widely used in stability investigations of nonimpulsive [33, 347, 359] and impulsive equations [32, 43, 341, 353]. We demonstrate that the existence of a nonoscillatory solution is equivalent to the positivity of the fundamental function. At the same time, it is equivalent to the solvability of a certain nonlinear inequality that is similar to "the generalized characteristic equation" in the monograph [249].

Oscillation properties of impulsive delay differential equations is a field of intensive research. Many results in this area were justified using the following scheme. First the equivalence of oscillation of the impulsive equation (inequality) and of some specially constructed nonimpulsive equation (inequality) is established. Further, on the basis of well-known results for the nonimpulsive case, oscillation of the impulsive equation is analyzed. To the best of our knowledge, for delay impulsive equations this method was first applied in [40] and then was employed for various classes of delay impulsive equations (see, for example, [44, 338, 339]).

In this chapter, the "oscillation equivalence" result was justified for equations with several delays and for a linear impulsive equation with a distributed delay. This approach can be extended to other properties of impulsive equations (for example, stability and asymptotic behavior).

The results of Sects. 12.3, 12.4 and 12.5 were published in [40], while the theorems of Sect. 12.6 are a part of [54].

Finally, let us state some open problems and topics for research and discussion.

1. Consider linear delay equations with the linear impulsive conditions of a more general type,
$$x\left(\tau_j^+\right) = \int_{t_0}^{\tau_j} b_j(s)x(s)\,ds + \sum_{t_0 \le \tau_k < \tau_j} \lambda_{jk} x(\tau_k), \tag{12.7.1}$$
see [13] for solution representations and some stability results for delay equations with impulsive conditions (12.7.1). Deduce explicit oscillation and nonoscillation conditions for (12.2.1) and impulsive conditions (12.7.1).
2. Once the previous problem is solved, consider $\dot{x}(t) = 0$ with impulsive conditions (12.7.1), where $b_j(s) \equiv 0$. Then oscillation (nonoscillation) of $\dot{x}(t) = 0$ with impulsive conditions (12.7.1) is equivalent to oscillation (nonoscillation) of the difference equation
$$x(n) = \sum_{k=n_0}^{n} A_{nk} x(k). \tag{12.7.2}$$
Deduce nonoscillation results for (12.7.2) and compare them to known nonoscillation conditions for Volterra difference equations.
3. Prove or disprove:
If $\sum_{k=1}^{m} A_k(t) \ge a_0 \ge 0$ and there are numbers $m > 0$ and $M > 0$ such that for any $n$
$$m \le \prod_{j=0}^{n} B_j \le M < \infty,$$
then any solution of (12.2.1), (12.2.2) with $f \equiv 0$ tends to zero at infinity.
4. Deduce sufficient nonoscillation and oscillation conditions for equations with a distributed delay and impulsive conditions (12.7.1). Stability of impulsive equations with a distributed delay was studied in [14] (with impulsive conditions that are more general than (12.2.2) but a particular case of (12.7.1)). However, there are still very few publications on oscillation of impulsive equations with a distributed delay; see [54], where there are no explicit nonoscillation and oscillation conditions.
5. Obtain an analogue of results of this chapter for the case where the impulses $B_j$ are functions of the solution $x(t)$ and for the case where impulse moments $\tau_k$ are also solution-dependent.

# Chapter 13
# Second-Order Linear Delay Impulsive Differential Equations

## 13.1 Introduction

In this chapter, we obtain explicit oscillation and nonoscillation conditions for a sufficiently general class of scalar linear second-order impulsive delay differential equations. For equations without impulses, these results coincide with known ones. We present several examples illustrating these conditions. In the first example, the impulsive differential equation is nonoscillatory while the corresponding nonimpulsive equation is oscillatory. In the second example, the impulsive differential equation is oscillatory and the corresponding nonimpulsive equation is nonoscillatory. In both examples, the sequence of values of impulses tends to one. Thus we can "improve" the oscillation nature of an equation by using a sequence of "disappearing" impulses. Such an example for a second-order impulsive *ordinary* differential equation was constructed in [341]. As follows from the previous chapter, this phenomenon is not possible for first-order differential equations.

The chapter is organized as follows. Section 13.2 contains relevant definitions and notation. In Sect. 13.3, the equivalence of the four properties for second-order impulsive delay equations is established: nonoscillation of the differential equation and the corresponding differential inequality, positivity of the fundamental function and the existence of a solution of a generalized Riccati inequality. Section 13.4 contains comparison theorems. Section 13.5 includes explicit nonoscillation and oscillation conditions. In the particular case where the values of impulses for the solution and its derivative are equal, a special nonimpulsive delay differential equation is constructed such that oscillation of an impulsive equation is equivalent to oscillation of the constructed nonimpulsive equation. As a consequence of this theorem, sharper nonoscillation results for this case of impulsive conditions are obtained. Section 13.6 compares oscillation properties of a second-order impulsive equation and some specially constructed nonimpulsive equation. Section 13.7 contains discussion and open problems.

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## 13.2 Preliminaries

We consider the scalar impulsive delay differential equation of the second order

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big) = 0, \ t \geq 0, \tag{13.2.1}$$

$$x\big(\tau_j^+\big) = A_j x(\tau_j), \ \dot{x}\big(\tau_j^+\big) = B_j \dot{x}(\tau_j), \ j = 1, 2, \cdots, \tag{13.2.2}$$

under the following conditions:

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ are fixed points, $\lim_{j\to\infty} \tau_j = \infty$.
(a2) $a_k$, $k = 1, \cdots, m$, are Lebesgue measurable locally essentially bounded functions on $[0, \infty)$, $A_j, B_j \in \mathbb{R}$, $j = 1, 2, \cdots$.
(a3) $g_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $g_k(t) \leq t$, $\lim_{t\to\infty} g_k(t) = \infty$, $k = 1, \cdots, m$.

Together with (13.2.1), (13.2.2), consider for each $t_0 \geq 0$ the initial value problem

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big)$$
$$= f(t), \ t \geq t_0, \ x(t) = \varphi(t), \ t < t_0, \ x(t_0) = \alpha_0, \ \dot{x}(t_0) = \beta_0, \tag{13.2.3}$$

$$x\big(\tau_j^+\big) = A_j x(\tau_j) + \alpha_j, \dot{x}\big(\tau_j^+\big) = B_j \dot{x}(\tau_j) + \beta_j, \ \tau_j \geq t_0. \tag{13.2.4}$$

We also assume that the following hypothesis holds:

(a4) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi : (-\infty, t_0) \to \mathbb{R}$ is a Borel measurable bounded function.

**Definition 13.1** A function $x : \mathbb{R} \to \mathbb{R}$ with a derivative absolutely continuous on each interval $(\tau_j, \tau_{j+1}]$ is called *a solution* of problem (13.2.3), (13.2.4) if it satisfies (13.2.3) for almost every $t \in [t_0, \infty)$ and equalities (13.2.4) hold.

**Definition 13.2** For each $s \geq 0$, denote by $X_0(t, s)$ and $X(t, s)$ the solutions of the problem

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big) = 0, \ t \geq s, \ x(t) = 0, \ t < s, \tag{13.2.5}$$

$$x\big(\tau_j^+\big) = A_j x(\tau_j), \dot{x}\big(\tau_j^+\big) = B_j \dot{x}(\tau_j), \ \tau_j \geq s, \tag{13.2.6}$$

with initial conditions $x(s) = 1$, $\dot{x}(s) = 0$ for $X_0(t, s)$ and $x(s) = 0$, $\dot{x}(s) = 1$ for $X(t, s)$, respectively. $X(t, s)$ is called *the fundamental function* of (13.2.1), (13.2.2). We assume $X(t, s) = 0$, $0 \leq t < s$.

Further, the solution representation formula for second-order impulsive equations will be applied, which is Theorem B.19 in the scalar case.

**Lemma 13.1** *Let* (a1)–(a4) *hold. Then there exists one and only one solution of problem* (13.2.3), (13.2.4), *and it can be presented in the form*

$$x(t) = X_0(t,t_0)\alpha_0 + X(t,t_0)\beta_0 + \int_{t_0}^{t} X(t,s)f(s)ds - \sum_{k=1}^{m}\int_{t_0}^{t} X(t,s)a_k(s)\varphi\big(g_k(s)\big)ds + \sum_{t_0\le\tau_j<t} X_0(t,\tau_j)\alpha_j + \sum_{t_0\le\tau_j<t} X(t,\tau_j)\beta_j, \tag{13.2.7}$$

*where* $\varphi(g_k(s)) = 0$, *if* $g_k(s) > t_0$.

## 13.3 Nonoscillation Criteria

As usual, we will say that (13.2.1), (13.2.2) has a positive solution for $t > t_0$ if there exist an initial function $\varphi$ and numbers $\alpha_0$ and $\beta_0$ such that the solution of initial value problem (13.2.3), (13.2.4), with $f \equiv 0$, $\alpha_j = \beta_j = 0$, $j = 1, 2, \cdots$, is positive for $t > t_0$. In this case, (13.2.1), (13.2.2) is nonoscillatory. Otherwise it is called oscillatory.

Together with (13.2.1), (13.2.2), consider the second-order delay differential inequality

$$\ddot{y}(t) + \sum_{k=1}^{m} a_k(t)y\big(g_k(t)\big) \le 0,\ \ t \ge 0, \tag{13.3.1}$$

$$y\big(\tau_j^+\big) = A_j y(\tau_j),\ \ \dot{y}\big(\tau_j^+\big) = B_j\dot{y}(\tau_j),\ \ j = 1, 2, \cdots. \tag{13.3.2}$$

The following theorem establishes nonoscillation criteria.

**Theorem 13.1** *Suppose* $a_k(t) \ge 0$, $k = 1, \cdots, m$, $A_j > 0$, $B_j > 0$, $j = 1, 2, \cdots$. *Then the following statements are equivalent*:

1) *There exists* $t_1$ *such that inequality* (13.3.1), (13.3.2) *has a positive solution with a locally essentially bounded second derivative for* $t > t_1$.
2) *There exists* $t_2 \ge 0$ *such that the inequality*

$$\dot{u}(t) + \prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u^2(t) + \sum_{k=1}^{m}\prod_{t_2\le\tau_j<t}\frac{A_j}{B_j} \times \prod_{g_k(t)\le\tau_j<t} A_j^{-1}a_k(t)\exp\left\{-\int_{g_k(t)}^{t}\prod_{t_2\le\tau_j<s}\frac{B_j}{A_j}u(s)ds\right\} \le 0 \tag{13.3.3}$$

   *has a locally absolutely continuous solution with a locally essentially bounded derivative*.
3) *There exists* $t_3 \ge 0$ *such that* $X(t,s) > 0$, $t > s \ge t_3$.
4) *There exists* $t_4 \ge 0$ *such that* (13.2.1), (13.2.2) *has a positive solution for* $t > t_4$.

*Proof* Let us prove the implications 1) ⇒ 2) ⇒ 3) ⇒ 4) ⇒ 1).

1) ⇒ 2) Let $y(t)$ be a positive solution of impulsive inequality (13.3.1), (13.3.2) for $t > t_1$. Then there exists a point $t_2$ such that $g_k(t) \geq t_1$ if $t \geq t_2$. We can assume without loss of generality that $y(t_2) = 1$.

Denote

$$u(t) = \prod_{t_2 \leq \tau_j < t} \frac{A_j}{B_j} \frac{\dot{y}(t)}{y(t)} \text{ if } t \geq t_2 \text{ and } u(t) = 0 \text{ if } t < t_2.$$

Then $u$ is a locally absolutely continuous function with a locally essentially bounded derivative. Equalities $\dot{y}(t) - \prod_{t_2 \leq \tau_j < t} B_j/A_j u(t) y(t) = 0,\ y(t_2) = 1$ and (13.3.2) imply that for $t \geq t_2$

$$\begin{aligned} y(t) &= \prod_{t_2 \leq \tau_j < t} A_j \exp\left\{\int_{t_2}^{t} \prod_{t_2 \leq \tau_j < s} \frac{B_j}{A_j} u(s)ds\right\}, \\ \dot{y}(t) &= \prod_{t_2 \leq \tau_j < t} B_j u(t) \exp\left\{\int_{t_2}^{t} \prod_{t_2 \leq \tau_j < s} \frac{B_j}{A_j} u(s)ds\right\}, \\ \ddot{y}(t) &= \prod_{t_2 \leq \tau_j < t} B_j \dot{u}(t) \exp\left\{\int_{t_2}^{t} \prod_{t_2 \leq \tau_j < s} \frac{B_j}{A_j} u(s)ds\right\} \\ &\quad + \prod_{t_2 \leq \tau_j < t} \frac{B_j^2}{A_j} u^2(t) \exp\left\{\int_{t_2}^{t} \prod_{t_2 \leq \tau_j < s} \frac{B_j}{A_j} u(s)ds\right\}. \end{aligned} \tag{13.3.4}$$

After substituting (13.3.4) into (13.3.1), we have

$$\begin{aligned} &\prod_{t_2 \leq \tau_j < t} B_j \exp\left\{\int_{t_2}^{t} \prod_{t_2 \leq \tau_j < s} \frac{B_j}{A_j} u(s)ds\right\} \Bigg[\dot{u}(t) + \prod_{t_2 \leq \tau_j < t} \frac{B_j}{A_j} u^2(t) + \sum_{k=1}^{m} \prod_{t_2 \leq \tau_j < t} \frac{A_j}{B_j} \\ &\times \prod_{g_k \leq \tau_j < t} A_j^{-1} a_k(t) \exp\left\{-\int_{g_k(t)}^{t} \prod_{t_2 \leq \tau_j < s} \frac{B_j}{A_j} u(s)ds\right\}\Bigg] \leq 0. \end{aligned} \tag{13.3.5}$$

Inequality (13.3.5) implies (13.3.3).

2) ⇒ 3) Consider the initial value problem

$$\begin{aligned} &\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(g_k(t)\big) = f(t),\ t \geq t_2, \\ &x(t) = 0,\ t < t_2,\ x(t_2) = \dot{x}(t_2) = 0, \end{aligned} \tag{13.3.6}$$

with impulsive conditions (13.2.2). Denote

$$z(t) = \dot{x}(t) - \prod_{t_2 \leq \tau_j < t} \frac{B_j}{A_j} u(t) x(t), \tag{13.3.7}$$

where $x$ is a solution of (13.3.6), (13.2.2) and $u$ is a solution of (13.3.3). From (13.3.7), (13.2.2), we obtain

$$x(t)=\int_{t_2}^{t}\exp\left\{\int_s^t\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<t}A_jz(s)ds,$$

$$\dot{x}(t)=z(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u(t)\int_{t_2}^{t}\exp\left\{\int_s^t\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<t}A_jz(s)ds,$$

$$\ddot{x}(t)=\dot{z}(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u(t)z(t)+\left(\dot{u}(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u^2(t)\right)$$
$$\times\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}\int_{t_2}^{t}\exp\left\{\int_s^t\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<t}A_jz(s)ds.$$

Substituting $x$, $\ddot{x}$ into (13.3.6), we obtain

$$\begin{aligned}&\dot{z}(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u(t)z(t)+\left(\dot{u}(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u^2(t)\right)\\&\quad\times\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}\int_{t_2}^{t}\exp\left\{\int_s^t\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<t}A_jz(s)ds\\&\quad+\sum_{k=1}^{m}a_k(t)\int_{t_2}^{g_k(t)}\exp\left\{\int_s^{g_k(t)}\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<g_k(t)}A_jz(s)ds\\&=f(t).\end{aligned}\tag{13.3.8}$$

Equality (13.3.7) implies $z(t_2)=0$ and

$$\begin{aligned}z\left(\tau_i^+\right)&=\dot{x}(\tau_i)-\prod_{t_2\le\tau_j\le\tau_i}\frac{B_j}{A_j}u(\tau_i)x(\tau_i)=B_i\dot{x}(\tau_i)-\frac{B_i}{A_i}\prod_{t_2\le\tau_j<\tau_i}\frac{B_j}{A_j}u(\tau_i)A_ix(\tau_i)\\&=B_iz(\tau_i).\end{aligned}$$

Hence we can rewrite (13.3.8) in the form

$$\begin{aligned}&\dot{z}(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u(t)z(t)\\&=-\Bigg[\dot{u}(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u^2(t)+\sum_{k=1}^{m}\prod_{t_2\le\tau_j<t}\frac{A_j}{B_j}\\&\quad\times\prod_{g_k(t)\le\tau_j<t}A_j^{-1}a_k(t)\exp\left\{-\int_{g_k(t)}^{t}\prod_{t_2\le\tau_j<s}\frac{B_j}{A_j}u(s)ds\right\}\Bigg]\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}\\&\quad\times\int_{t_2}^{t}\exp\left\{\int_s^t\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<t}A_jz(s)ds+\sum_{k=1}^{m}a_k(t)\\&\quad\times\int_{g_k(t)}^{t}\exp\left\{\int_s^{g_k(t)}\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<g_k(t)}A_jz(s)ds+f(t),\end{aligned}\tag{13.3.9}$$

$$z(t_2)=0,\ z\left(\tau_j^+\right)=B_j z(\tau_j),\ \tau_j>t_2. \tag{13.3.10}$$

Then (13.3.9), (13.3.10) is equivalent to the equation

$$z=Hz+p, \tag{13.3.11}$$

where

$$\begin{aligned}
(Hz)(t)=&\int_{t_2}^{t}\exp\left\{-\int_s^t\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\\
&\times\prod_{s\le\tau_j<t}B_j\Bigg[-\Bigg(\dot u(s)+\prod_{t_2\le\tau_j<s}\frac{B_j}{A_j}u^2(s)\\
&+\sum_{k=1}^{m}\prod_{t_2\le\tau_j<t}\frac{A_j}{B_j}\prod_{g_k(s)\le\tau_j<s}A_j^{-1}a_k(s)\\
&\times\exp\left\{-\int_{g_k(s)}^{s}\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\Bigg)\\
&\times\prod_{t_2\le\tau_j<s}\frac{B_j}{A_j}\int_{t_2}^{s}\exp\left\{\int_\tau^s\prod_{t_2\le\tau_j<\xi}\frac{B_j}{A_j}u(\xi)d\xi\right\}\prod_{\tau\le\tau_j<s}A_jz(\tau)d\tau\\
&+\sum_{k=1}^{m}a_k(s)\int_{g_k(s)}^{s}\exp\left\{\int_\tau^{g_k(s)}\prod_{t_2\le\tau_j<\xi}\frac{B_j}{A_j}u(\xi)d\xi\right\}\\
&\times\prod_{\tau\le\tau_j<g_k(s)}A_jz(\tau)d\tau\Bigg]ds,
\end{aligned}$$

$$p(t)=\int_{t_2}^{t}\exp\left\{-\int_s^t\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\prod_{s\le\tau_j<t}B_jf(s)ds. \tag{13.3.12}$$

Inequality (13.3.3) yields that $z(t)\ge 0$ for $t\ge t_2$ implies $(Hz)(t)\ge 0$ for $t\ge t_2$ (i.e., operator $H$ is positive). Denote

$$\begin{aligned}
c(t)=&\,\dot u(t)+\prod_{t_2\le\tau_j<t}\frac{B_j}{A_j}u^2(t)\\
&+\sum_{k=1}^{m}\prod_{t_2\le\tau_j<t}\frac{A_j}{B_j}\prod_{g_k(t)\le\tau_j<t}A_j^{-1}a_k(t)\exp\left\{-\int_{g_k(t)}^{t}\prod_{t_2\le\tau_j<s}\frac{B_j}{A_j}u(s)ds\right\}.
\end{aligned}$$

Since $u$ is absolutely continuous in each finite interval, we have $c\in L_\infty[t_2,b]$ for every $b>t_2$. For $t\in[t_2,b]$, we have

$$\begin{aligned}
\left|(Hz)(t)\right|\le&\exp\left\{\int_{t_2}^{b}\prod_{t_2\le\tau_j<\tau}\frac{B_j}{A_j}u(\tau)d\tau\right\}\int_{t_2}^{t}\prod_{s\le\tau_j<t}B_j\Bigg(\prod_{t_2\le\tau_j<s}\frac{B_j}{A_j}\left|c(s)\right|\\
&+\sum_{k=1}^{m}\left|a_k(s)\right|\Bigg)\int_{t_2}^{s}\prod_{\tau\le\tau_j<s}A_j\left|z(\tau)\right|d\tau ds
\end{aligned}$$

$$= \exp\left\{\int_{t_2}^{b} \prod_{t_2 \le \tau_j < \tau} \frac{B_j}{A_j} u(\tau)d\tau\right\} \int_{t_2}^{t}\left[\int_{\tau}^{t} \prod_{\tau \le \tau_j < t} B_j\left(\prod_{t_2 \le \tau_j < s} \frac{B_j}{A_j}\left|c(s)\right| + \sum_{k=1}^{m}\left|a_k(s)\right|\right) \prod_{\tau \le \tau_j < s} A_j ds\right]\left|z(\tau)\right|d\tau.$$

The kernel of the Volterra integral operator $H$ is bounded in each square $[t_2, b] \times [t_2, b]$. Hence, by Theorem A.4, we deduce that $H : L_\infty[t_2, b] \to L_\infty[t_2, b]$ is a weakly compact operator. Theorem A.7 implies that the spectral radius of integral Volterra operator $r(H) = 0$. Thus, if in (13.3.11) we have $p(t) \ge 0$ for $t \ge t_2$, then

$$z(t) = p(t) + (Hp)(t) + \left(H^2 p\right)(t) + \cdots \ge 0 \text{ for } t \ge t_2.$$

If $f(t) \ge 0$ for $t \ge t_2$, then $p(t) \ge 0$ for $t \ge t_2$ by (13.3.12). Hence, for (13.3.8) we have the following: if $f(t) \ge 0$ for $t \ge t_2$, then $z(t) \ge 0$ for $t \ge t_2$.

Therefore (13.3.7) implies that the solution of (13.3.6), (13.2.2) is nonnegative for any nonnegative right-hand side $f$.

The solution of this equation can be written in the form (13.2.7), which is

$$x(t) = \int_{t_2}^{t} X(t, s) f(s)ds. \tag{13.3.13}$$

As was shown above, $f(t) \ge 0$, $t \ge t_2$, implies $x(t) \ge 0$, $t \ge t_2$. Consequently, the kernel of the integral operator (13.3.13) is nonnegative (i.e., $X(t, s) \ge 0$ for $t \ge s \ge t_2$). The function $x(t) = X(t, s)$ is a nonnegative solution of (13.2.5) for $t \ge s$. Suppose that for certain $t_3 > s$ we have $x(t_3) = 0$ and $x(t) > 0$ for $s < t < t_3$. Then $\dot{x}(t_3) < 0$. By (13.2.7), for $t > t_3$ the solution can be presented as

$$x(t) = X(t, t_3)\dot{x}(t_3) - \sum_{k=1}^{m} \int_{t_3}^{t} X(t, s)a_k(s)\varphi\big(g_k(s)\big)ds,$$

where $\varphi(t) = x(t)$, $t < t_3$. Therefore $x(t) < 0$ for $t > t_3$, and we get a contradiction. Hence the strict inequality $x(t) = X(t, s) > 0$, $t > s \ge t_2$ holds.

3) $\Rightarrow$ 4) The function $x(t) = X(t, t_3)$ is a positive solution of (13.2.1), (13.2.2).

Implication 4) $\Rightarrow$ 1) is evident, which completes the proof. □

**Corollary 13.1** *Equation* (13.2.1), (13.2.2) *is nonoscillatory if and only if inequality* (13.3.1), (13.3.2) *is nonoscillatory.*

*Remark 13.1*

1) If there exists a nonnegative solution of inequality (13.3.3) for $t \ge t_0$, then statements 1), 3) and 4) of the theorem are also valid for $t \ge t_0$.
2) A generalized Riccati equation for a delay differential equation without impulses appeared for the first time in [43].
3) If inequality (13.3.3) has a nonnegative solution, then (13.2.1), (13.2.2) has a positive solution with a nonnegative derivative.

## 13.4 Comparison Theorems

In this section, we assume everywhere that $A_j > 0$, $B_j > 0$.

Theorem 13.1 can be employed for comparison of oscillation properties. To this end, together with (13.2.1), (13.2.2), consider the equation

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)x\big(g_k(t)\big) = 0, \ t \geq 0, \tag{13.4.1}$$

with impulsive conditions (13.2.2).

Suppose (a2) and (a3) hold for (13.4.1), and denote by $Y(t,s)$ the fundamental function of this equation.

A proof of Theorem 13.2 below and its corollary is similar to the proof of Theorem 7.2 for equations without impulses, which was first obtained in [43, Theorem 3].

**Theorem 13.2** *Suppose $a_k(t) \geq 0$, $a_k(t) \geq b_k(t)$ for $t \geq t_0$ and inequality (13.3.3) has a solution for $t \geq t_0$. Then (13.4.1), (13.2.2) has a positive solution for $t > t_0$ and $Y(t,s) > 0$, $t > s \geq t_0$.*

We recall that $a^+ = \max\{a, 0\}$.

**Corollary 13.2**

1) *If the inequality*

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k^+(t)x\big(g_k(t)\big) \leq 0 \tag{13.4.2}$$

*with impulsive conditions (13.2.2) is nonoscillatory, then (13.2.1), (13.2.2) is also nonoscillatory.*

2) *If the inequality*

$$\dot{u}(t) + \prod_{t_0 \leq \tau_j < t} B_j/A_j u^2(t) + \sum_{k=1}^{m} \prod_{t_0 \leq \tau_j < t} \frac{A_j}{B_j} \times \prod_{g_k(t) \leq \tau_j < t} A_j^{-1} a_k^+(t) \exp\left\{-\int_{g_k(t)}^{t} \prod_{t_0 \leq \tau_j < s} \frac{B_j}{A_j} u(s)ds\right\} \leq 0 \tag{13.4.3}$$

*has an absolutely continuous solution for $t \geq t_0$, then (13.2.1), (13.2.2) has a positive solution for $t > t_0$ and $X(t,s) > 0$ for $t > s \geq t_0$.*

Now let us compare solutions of (13.2.3), (13.2.4) and the problem

$$\ddot{y}(t) + \sum_{k=1}^{m} b_k(t)y\big(g_k(t)\big) = r(t), \ t \geq t_0, \ y(t) = \psi(t), \ t < t_0, \ x(t_0) = \alpha_0, \ \dot{x}(t_0) = \beta_0 \tag{13.4.4}$$

with impulsive conditions (13.2.4). Denote by $x(t)$ and $y(t)$ the solutions of (13.2.3), (13.2.4) and (13.4.4), (13.2.4), respectively.

**Theorem 13.3** *Suppose there exists a solution of* (13.3.3) *for* $t \ge t_0$, $x(t) > 0$ *for* $t \ge t_0$ *and*

$$a_k(t) \ge b_k(t) \ge 0,\ r(t) \ge f(t) \text{ for } t \ge t_0,\ \varphi(t) \ge \psi(t) \text{ for } t < t_0.$$

*Then* $y(t) \ge x(t)$ *for* $t \ge t_0$, *where* $x$ *and* $y$ *are solutions of* (13.2.3), (13.2.4) *and* (13.4.4), (13.2.4), *respectively.*

The proof is similar to the proof of Theorem 7.4; see also [43, Theorem 5].

## 13.5 Explicit Nonoscillation and Oscillation Conditions

We will employ Corollary 13.2 to obtain explicit sufficient nonoscillation conditions. Everywhere in this section, we assume that $A_j > 0$, $B_j > 0$.

**Theorem 13.4** *Suppose for some* $t_0 > 0$, $0 < q < 1$, $r > -1$, $\mu > 0$, $M > 0$, *either*

$$\sup_{t \ge t_0} \left| \prod_{t_0 \le \tau_j < t} \frac{B_j}{A_j} - 1 \right| \le q,$$
$$\sup_{t \ge t_0} \left\{ t^2 \sum_{k=1}^{m} \prod_{g_k(t) \le \tau_j < t} A_j^{-1} a_k^+(t) \left[ \frac{g_k(t)}{t} \right]^{(1-q)/2} \right\} \le \frac{(1-q)^2}{4}, \tag{13.5.1}$$

*or*

$$\mu t^r \le \sup_{t \ge t_0} \prod_{t_0 \le \tau_j < t} B_j / A_j \le M t^r,$$
$$\sup_{t \ge t_0} \left[ t^2 \sum_{k=1}^{m} \prod_{g_k(t) \le \tau_j < t} A_j^{-1} a_k^+(t) \right] \le \frac{\mu (1+r)^2}{4M}. \tag{13.5.2}$$

*Then* (13.2.1), (13.2.2) *has a positive solution with a nonnegative derivative for* $t \ge t_0$.

*Proof* Suppose (13.5.1) holds. We will show that the function $u(t) = \frac{1}{2t}$ is a solution of inequality (13.4.3). To this end, substitute this function into the left-hand side of the inequality and consider the function

$$h(t) = -\frac{1}{2t^2} + \prod_{t_0 \le \tau_j < t} \frac{B_j}{A_j} \frac{1}{4t^2} + \sum_{k=1}^{m} \prod_{t_0 \le \tau_j < t} \frac{A_j}{B_j}$$
$$\times \prod_{g_k(t) \le \tau_j < t} A_j^{-1} a_k^+(t) \exp\left\{ - \int_{g_k(t)}^{t} \prod_{t_0 \le \tau_j < s} \frac{B_j}{A_j} \frac{1}{2s} ds \right\}.$$

Denote

$$\sup_{t \ge t_0} \prod_{t_0 \le \tau_j < t} \frac{B_j}{A_j} = \alpha.$$

Then $1-q \le \alpha \le 1+q$. Hence

$$\begin{aligned}
h(t) &\le -\frac{1}{2t^2} + (1+q)\frac{1}{4t^2} \\
&\quad + \frac{1}{1-q}\sum_{k=1}^{m} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} a_k^+(t) \exp\left\{-\int_{g_k(t)}^{t} (1-q)\frac{1}{2s}ds\right\} \\
&= -\frac{1}{4t^2}(1-q) + \frac{1}{1-q}\sum_{k=1}^{m} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} a_k^+(t)\left[\frac{g_k(t)}{t}\right]^{(1-q)/2} \\
&= -\frac{1}{t^2(1-q)}\left[\frac{(1-q)^2}{4} - t^2\sum_{k=1}^{m} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} a_k^+(t)\left(\frac{g_k(t)}{t}\right)^{(1-q)/2}\right] \le 0.
\end{aligned}$$

Then $\frac{1}{2t}$ is a nonnegative solution of inequality (13.4.3), and therefore (13.2.1), (13.2.2) has a positive solution with a nonnegative derivative.

If (13.5.2) holds, then $u = \frac{1+r}{2Mt^{1+r}}$ is a nonnegative solution of (13.4.3). After substituting this function into the left-hand side of (13.4.3) and denoting by $h(t)$ the expression obtained, we have

$$\begin{aligned}
h(t) &\le -\frac{(1+r)^2}{2Mt^{2+r}} + \frac{(1+r)^2}{4M^2 t^{2+2r}} \prod_{t_0\le\tau_j<t} \frac{B_j}{A_j} + \sum_{k=1}^{m} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} \prod_{t_0\le\tau_j<t} \frac{A_j}{B_j} a_k^+(t) \\
&\le -\frac{(1+r)^2}{2Mt^{2+r}} + \frac{(1+r)^2}{4Mt^{2+r}} + \sum_{k=1}^{m} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} a_k^+(t)\frac{1}{\mu t^r} \\
&= -\frac{1}{\mu t^{2+r}}\left(\frac{\mu(1+r)^2}{4M} - t^2\sum_{k=1}^{m} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} a_k^+(t)\right) \le 0,
\end{aligned}$$

which completes the proof. □

**Corollary 13.3** *Suppose that for $t \ge t_0 \ge 0$ we have*

$$a_k(t) \le 0,\ mt^r \le \sup_{t\ge t_0} \prod_{t_0\le\tau_j<t} \frac{B_j}{A_j} \le Mt^r,\ r > -1,\ m > 0,\ M > 0.$$

*Then* (13.2.1), (13.2.2) *has a positive solution with a nonnegative derivative for $t \ge t_0$.*

**Corollary 13.4** *Suppose for some $t_0 \ge 0, 0 < q < 1, r > -1, m > 0, M > 0$ at least one of the following conditions holds*:

1) $$\sup_{t\ge t_0}\left|\prod_{t_0\le\tau_j<t} \frac{B_j}{A_j} - 1\right| \le q,\ \sup_{t\ge t_0}\left[t^2 a^+(t)\right] \le \frac{(1-q)^2}{4}.$$

2) $$mt^r \le \sup_{t\ge t_0} \prod_{t_0\le\tau_j<t} \frac{B_j}{A_j} \le Mt^r,\ \sup_{t\ge t_0}\left[t^2 a^+(t)\right] \le \frac{m(1+r)^2}{4M}.$$

*Then the ordinary differential equation*

$$\ddot{x}(t) + a(t)x(t) = 0 \tag{13.5.3}$$

*with impulsive conditions* (13.2.2) *has a positive solution with a nonnegative derivative for* $t \ge t_0$.

*Example 13.1* Consider the delay differential equation

$$\ddot{x}(t) + \frac{1}{2t^2}x(t-\delta) = 0 \tag{13.5.4}$$

with the impulsive conditions

$$x(j^+) = \frac{j}{j+1}x(j), \ \dot{x}(j) = \dot{x}(j), \ j = 1, 2, \cdots, \tag{13.5.5}$$

or

$$x(j^+) = x(j), \ \dot{x}(j^+) = \frac{j+1}{j}\dot{x}(j). \tag{13.5.6}$$

Then (13.5.4), (13.5.5) and (13.5.4), (13.5.6) are nonoscillatory.

In fact, for (13.5.4), (13.5.5), the inequality

$$t \le \prod_{t_0 \le \tau_j < t} B_j/A_j \le t + 1 \le t\left(1 + \frac{1}{t_0}\right)$$

is satisfied. Thus the first inequality in (13.5.2) holds with $r = 1$, $\mu = 1$, $M = 1 + \frac{1}{t_0}$.

Consider the second inequality of (13.5.2). While its left-hand side is less than $0.5(1 + \frac{1}{t_0 - \delta})^{[\delta]+1}$ and tends to 0.5 as $t_0 \to \infty$, its right-hand side is $\frac{t_0}{1+t_0}$ and tends to 1. Then, for sufficiently large $t_0$, inequalities (13.5.2) hold. Hence, (13.5.4), (13.5.5) is nonoscillatory. Similarly, (13.5.4), (13.5.6) is also nonoscillatory.

*Remark 13.2* Let us note that all solutions of (13.5.4) without impulses are oscillatory [84].

Next, we will obtain some additional nonoscillation conditions. Denote

$$b(t) = \sum_{k=1}^{m} \prod_{t_0 \le \tau_j < t} \frac{A_j}{B_j} \prod_{g_k(t) \le \tau_j < t} A_j^{-1} a_k^+(t).$$

**Theorem 13.5** *Suppose* $\prod_{t_0 \le \tau_j < t} B_j/A_j \le 1$. *If for* $t > t_0$ *there exists a positive solution of the nonimpulsive ordinary differential equation* $\ddot{x}(t) + b(t)x(t) = 0$, *then for* $t > t_0$ *there exists a positive solution of* (13.2.1), (13.2.2).

*Proof* Suppose $u$ is a solution of the Riccati inequality

$$\dot{u}(t) + u^2(t) + b(t) \le 0, \ t \ge t_0.$$

Then $u$ is also a solution of inequality (13.4.3). Therefore (13.2.1), (13.2.2) has a positive solution for $t > t_0$. □

**Corollary 13.5** *Suppose for some* $t_0 \geq 0$

$$\limsup_{t\to\infty} \prod_{t_0\leq\tau_j<t} \frac{B_j}{A_j} \leq 1, \ \limsup_{t\to\infty} A_j \geq 1, \tag{13.5.7}$$

*and*

$$\limsup_{t\to\infty} \left[ t^2 \sum_{k=1}^{m} \prod_{t_0\leq\tau_j<t} \frac{A_j}{B_j} a_k^+(t) \right] \leq \frac{1}{4}. \tag{13.5.8}$$

*Then* (13.2.1), (13.2.2) *is nonoscillatory.*

*Example 13.2* Consider the delay differential equation

$$\ddot{x}(t) + \frac{\alpha}{t^\beta} x\big(g(t)\big) = 0 \tag{13.5.9}$$

with the impulsive conditions

$$x\big(j^+\big) = \frac{(j+1)^k}{j^k} x(j), \ \dot{x}(j) = \dot{x}(j), \ j = 1, 2, \cdots; \ k > 0, \tag{13.5.10}$$

or

$$x\big(j^+\big) = x(j), \ \dot{x}\big(j^+\big) = \frac{j^k}{(j+1)^k} \dot{x}(j). \tag{13.5.11}$$

Here conditions (13.5.7) hold and $\prod_{t_0\leq\tau_j<t} A_j/B_j \leq Mt^k$, where $M = (1+\frac{1}{t_0})^k \to 1$ as $t_0 \to \infty$. Hence the left-hand side of (13.5.8) is less than or equal to $\alpha M \frac{t^{2+k}}{t^\beta}$. Therefore, if $\beta = 2+k$ and $\alpha < \frac{1}{4}$ or $\beta > 2+k$, then (13.5.9), (13.5.10) and (13.5.9), (13.5.11) are nonoscillatory.

Now we proceed to oscillation.

**Theorem 13.6** *Suppose* $a_k(t) \geq 0$ *and there exist* $M > 0$, $\delta > 0$ *and* $t_1 \geq 0$ *such that*

$$\sup_{t\geq t_1} \prod_{t_1\leq\tau_j\leq t} \frac{B_j}{A_j} \leq M, \ t - g_k(t) \leq \delta. \tag{13.5.12}$$

*If for some* $k$, $k = 1, 2, \cdots, m$ *we have*

$$\int^\infty \prod_{t_1\leq\tau_j\leq t} \frac{A_j}{B_j} \prod_{g_k(t)\leq\tau_j<t} A_j^{-1} a_k(t)dt = \infty \ \text{and} \int^\infty \prod_{t_1\leq\tau_j\leq t} \frac{B_j}{A_j} dt = \infty, \tag{13.5.13}$$

*then all solutions of* (13.2.1), (13.2.2) *are oscillatory.*

*Proof* Suppose (13.2.1), (13.2.2) has a positive solution. Then, for some $t_0 \geq 0$, inequality (13.3.3) has a solution $u(t)$. This function is nonincreasing and therefore $u(t) \leq u(t_0)$ and either there exists a finite limit of $u(t)$ as $t \to \infty$ or we have $\lim_{t\to\infty} u(t) = -\infty$. We will see that the latter case is impossible. Inequality (13.3.3) implies $\dot{u}(t) + \prod_{t_0\le\tau_j<t} B_j/A_j u^2(t) \le 0$. Then

$$-\frac{1}{u(t)} + \frac{1}{u(t_0)} + \int_{t_0}^{t} \prod_{t_0\le\tau_j<s} \frac{B_j}{A_j} ds \le 0,$$

and therefore $\lim_{t\to\infty} u(t)$ is a finite number.

Further, from inequality (13.3.3) we have

$$\dot{u}(t) + \prod_{t_0\le\tau_j<t} \frac{A_j}{B_j} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} a_k(t) \exp\{-\delta M u(t_0)\} \le 0.$$

Hence

$$u(t) - u(t_0) + \exp\{-\delta M u(t_0)\} \int_{t_0}^{t} \prod_{t_0\le\tau_j\le s} \frac{A_j}{B_j} \prod_{g_k(t)\le\tau_j<s} A_j^{-1} a_k(s) ds \le 0,$$

which contradicts the first condition in (13.5.13). □

**Corollary 13.6** *Suppose that $a_k(t) \ge 0$, condition (13.5.12) holds and there exist $A > 0$ and $\tau > 0$ such that $A_j \le A$, $\tau_{j+1} - \tau_j \ge \tau$. If for some $k$, $k = 1, 2, \cdots, m$ we have*

$$\int^{\infty} \prod_{\tau_j\le t} \frac{A_j}{B_j} a_k(t) dt = \infty \text{ and } \int^{\infty} \prod_{\tau_j\le t} \frac{B_j}{A_j} dt = \infty,$$

*then all solutions of (13.2.1), (13.2.2) are oscillatory.*

*Proof* The inequality

$$\int^{\infty} \prod_{\tau_j\le t} \frac{A_j}{B_j} \prod_{g_k(t)\le\tau_j<t} A_j^{-1} a_k(t) dt \ge A^{-([\frac{\delta}{\tau}]+1)} \int^{\infty} \prod_{\tau_j\le t} \frac{A_j}{B_j} a_k(t) dt = \infty$$

implies the statement of the corollary. □

**Corollary 13.7** *Suppose that $a(t) \ge 0$,*

$$\sup_{t\ge 0} \prod_{\tau_j\le t} \frac{B_j}{A_j} \le M, \int^{\infty} \prod_{\tau_j\le t} \frac{A_j}{B_j} a(t) dt = \infty \text{ and } \int^{\infty} \prod_{\tau_j\le t} \frac{B_j}{A_j} dt = \infty.$$

*Then all solutions of ordinary differential equation (13.5.3), (13.2.2) are oscillatory.*

*Example 13.3* Consider the delay differential equation

$$\ddot{x}(t) + \frac{1}{4t^2} x(t-\delta) = 0 \tag{13.5.14}$$

with the impulsive conditions

$$x(j^+) = \frac{j+1}{j} x(j), \ \dot{x}(j^+) = \dot{x}(j), \ j = 1, 2, \cdots \tag{13.5.15}$$

or

$$x(j^+) = x(j),\ \dot{x}(j^+) = \frac{j}{j+1}\dot{x}(j). \tag{13.5.16}$$

Then all solutions of (13.5.14), (13.5.15) and (13.5.14), (13.5.16) are oscillatory.

In fact,

$$\int_{t_0}^{\infty} \prod_{t_0 \le \tau_j < t} \frac{A_j}{B_j} \prod_{t-\delta \le \tau_j < t} A_j^{-1} a(t)dt \ge \left(1 - \frac{1}{t_0+1}\right)^{[\delta]+1} \int_{t_0}^{\infty} t\frac{1}{4t^2}dt = \infty$$

and

$$\int_{t_0}^{\infty} \prod_{t_0 \le \tau_j < t} \frac{B_j}{A_j} dt \ge \int_{t_0}^{\infty} \frac{1}{t+1} dt = \infty.$$

*Remark 13.3* For an ordinary differential equation (the case $\delta = 0$) with impulsive conditions (13.5.16), this result was obtained in [341] in a different way. It can also be obtained using [32, Theorem 2]. Equation (13.5.14) without impulses is nonoscillatory for all $\delta \ge 0$ [43, 341].

Simpler and more interesting results will be obtained under the assumption $A_j = B_j$, which means that the impulsive conditions are

$$x(\tau_j^+) = A_j x(\tau_j),\ \dot{x}(\tau_j^+) = A_j \dot{x}(\tau_j), \tag{13.5.17}$$

and the jumps of the function and the derivative are matched. In this case, the Riccati inequality (13.3.3) is

$$\dot{u}(t) + u^2(t) + \sum_{k=1}^{m} \prod_{g_k(t) \le \tau_j < t} A_j^{-1} a_k(t) \exp\left\{-\int_{g_k(t)}^{t} u(s)ds\right\} \le 0. \tag{13.5.18}$$

Consider the delay differential equation without impulses

$$\ddot{x}(t) + \sum_{k=1}^{m} \prod_{g_k(t) \le \tau_j < t} A_j^{-1} a_k(t) x(g_k(t)) = 0. \tag{13.5.19}$$

**Theorem 13.7** *Suppose* $a_k(t) \ge 0$. *Then* (13.2.1), (13.5.17) *is oscillatory* (*nonoscillatory*) *if and only if* (13.5.19) *is oscillatory* (*nonoscillatory*).

*Proof* Theorem 13.1 implies that nonoscillation of (13.2.1), (13.5.17) is equivalent to the existence of a nonnegative solution of inequality (13.5.18), which is equivalent to nonoscillation of nonimpulsive equation (13.5.19) by Theorem 7.1. □

Theorem 13.7 and oscillation results obtained in Chap. 7 for nonimpulsive equations imply the following theorems.

**Theorem 13.8** *If for some* $t_1 \geq 0$

$$\sup_{t \geq t_1} \sum_{k=1}^{m} \prod_{g_k(t) \leq \tau_j < t} A_j^{-1} a_k^+(t) \frac{\sqrt{t^3 g_k(t)} \ln g_k(t)}{\ln t} \leq 1/4, \; t \geq t_1,$$

*then* (13.2.1), (13.5.17) *is nonoscillatory.*

**Theorem 13.9** *Suppose that* $a_k(t) \geq 0$ *and there exists* $\delta > 0$ *such that* $t - g_k(t) < \delta$. *Then* (13.2.1), (13.5.17) *is oscillatory* (*nonoscillatory*) *if and only if the nonimpulsive ordinary differential equation*

$$\ddot{x}(t) + \sum_{k=1}^{m} \prod_{g_k(t) \leq \tau_j < t} A_j^{-1} a_k(t) x(t) = 0$$

*is oscillatory* (*nonoscillatory*).

**Theorem 13.10** *Suppose that* $a_k(t) \geq 0$ *and for some* $c_k$, $0 < c_k < 1$, *the nonimpulsive ordinary differential equation*

$$\ddot{x}(t) + \sum_{k=1}^{m} \prod_{g_k(t) \leq \tau_j < t} A_j^{-1} a_k(t) c_k \frac{g_k(t)}{t} x(t) = 0$$

*is oscillatory. Then* (13.2.1), (13.5.17) *is also oscillatory.*

## 13.6 Impulsive Equations with Damping Terms

Now let us proceed to the delay equation of the second order with concentrated delays and the damping terms

$$\ddot{y}(t) + \sum_{k=1}^{m} p_k(t) \dot{y}\big(h_k(t)\big) + \sum_{k=1}^{l} a_k(t) y\big(g_k(t)\big) = 0, \; t \geq t_0, \tag{13.6.1}$$

$$x(t) = \varphi(t), \; t < t_0, \; \dot{x}(t) = \xi(t), \; t < t_0, \tag{13.6.2}$$

and the impulsive conditions

$$\dot{y}\big(\tau_j^+\big) = A_j \dot{y}(\tau_j), \; y\big(\tau_j^+\big) = B_j(\tau_j), \tag{13.6.3}$$

under the following assumptions:

(b1) $p_k, a_k$ are Lebesgue measurable and locally essentially bounded functions on $[t_0, \infty)$.
(b2) $h_k, g_k : [t_0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $g_k(t) \leq t$, $h_j(t) \leq t$, $\lim_{t\to\infty} g_k(t) = \infty$, $\lim_{t\to\infty} h_j(t) = \infty$, $k = 1, \cdots, m$, $j = 1, \cdots, l$.
(b3) $\varphi : (-\infty, t_0) \to \mathbb{R}$, $\xi : (-\infty, t_0) \to \mathbb{R}$ are Borel measurable bounded functions.

**Lemma 13.2** *Suppose $A_j \neq 0$, $B_j \neq 0$ and $x$ is a solution of the nonimpulsive equation*

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{l} \int_{t_0}^{t} x(s) d_s R_k(t,s) = 0, \tag{13.6.4}$$

*where*

$$b_k(t) = p_k(t) \prod_{h_k(t) \le \tau_j < t} A_j^{-1},$$

$$R_k(t,s) = a_k(t)\Bigg[\prod_{t_0 \le \tau_j < g_k(t)} B_j \prod_{t_0 \le \tau_j < t} A_j^{-1} x(t_0) + \prod_{g_k(t) \le \tau_j < t} A_j^{-1} \\ \times \sum_{t_0 \le \tau_r < g_k(t)} \prod_{j=r}^{\max\{i|\tau_i < g_k(t)\}} \frac{B_j}{A_j}\big(\chi_{(\tau_{r+1},\infty)}(s) - \chi_{(\tau_r,\infty)}(s)\big)\Bigg] \tag{13.6.5}$$

*and $\chi_{(a,b)}$ is the characteristic function of the open segment $(a,b)$. Then*

$$y(t) = \prod_{t_0 \le \tau_j < t} B_j \left( x(t_0) + \int_{t_0}^{t} \prod_{t_0 \le \tau_j < s} \frac{A_j}{B_j} \dot{x}(s) ds \right) \tag{13.6.6}$$

*is a solution of impulsive equation* (13.6.1), (13.6.3).

*Proof* It is easy to see that the function $y$ defined in (13.6.6) satisfies impulsive conditions (13.6.3) and also

$$\dot{y}(t) = \prod_{t_0 \le \tau_j < t} A_j \dot{x}(t), \ \ \ddot{y}(t) = \prod_{t_0 \le \tau_j < t} A_j \ddot{x}(t). \tag{13.6.7}$$

After substituting (13.6.6), (13.6.7) into (13.6.1), we have

$$\prod_{t_0 \le \tau_j < t} A_j \ddot{x}(t) + \sum_{k=1}^{m} p_k(t) \prod_{t_0 \le \tau_j < h_k(t)} A_j \dot{x}\big(h_k(t)\big) \\ + \sum_{k=1}^{l} a_k(t) \prod_{t_0 \le \tau_j < g_k(t)} B_j \left( x(t_0) + \int_{t_0}^{g_k(t)} \prod_{t_0 \le \tau_j < s} \frac{A_j}{B_j} \dot{x}(s) ds \right) = 0.$$

Multiplication of the equality above by $\prod_{t_0 \le \tau_j < t} A_j^{-1}$ gives

$$\ddot{x}(t) + \sum_{k=1}^{m} p_k(t) \prod_{h_k(t) \le \tau_j < t} A_j^{-1} \dot{x}\big(h_k(t)\big) + \sum_{k=1}^{l} a_k(t) \Bigg( \prod_{t_0 \le \tau_j < g_k(t)} B_j \prod_{t_0 \le \tau_j < t} A_j^{-1} x(t_0) \\ + \int_{t_0}^{g_k(t)} \prod_{s \le \tau_j < t} A_j^{-1} \prod_{s \le \tau_j < g_k(t)} B_j \dot{x}(s) ds \Bigg) = 0.$$

We have

$$\int_{t_0}^{g_k(t)} \prod_{s\le\tau_j<t} A_j^{-1} \prod_{s\le\tau_j<g_k(t)} B_j \dot{x}(s)ds$$
$$= \prod_{g_k(t)\le\tau_j<t} A_j^{-1} \int_{t_0}^{g_k(t)} \prod_{s\le\tau_j\le g_k(t)} \frac{B_j}{A_j}\dot{x}(s)ds.$$

This integral on the right-hand side can be evaluated as

$$\int_{t_0}^{g_k(t)} \prod_{s\le\tau_j\le g_k(t)} \frac{B_j}{A_j}\dot{x}(s)ds = \sum_{t_0\le\tau_r<g_k(t)} \prod_{j=r}^{\max\{i|\tau_i<g_k(t)\}} \frac{B_j}{A_j}\big(x(\tau_{r+1})-x(\tau_r)\big).$$

Thus $y$ defined by (13.6.6) is a solution of (13.6.1), (13.6.3). □

**Theorem 13.11** *Suppose* (b1)–(b3) *are satisfied*, $0 < A_j \le B_j$ *and the nonimpulsive equation* (13.6.4), (13.6.2) *has a nonoscillatory solution*. *Then* (13.6.1), (13.6.2), (13.6.3) *also has a nonoscillatory solution*.

*Proof* Let $x(t)$ be a positive solution of (13.6.4) (the negative case is treated similarly). Then, in (13.6.6) with $t_0 = \tau_0$, $\tau_{n_k}$ being the greatest impulse point preceding $t$, we have

$$\prod_{t_0\le\tau_j<t} B_j^{-1} y(t)$$
$$= x(t_0) + \frac{A_0}{B_0}\big(x(\tau_1)-x(\tau_0)\big) + \frac{A_0A_1}{B_0B_1}\big(x(\tau_2)-x(\tau_1)\big)$$
$$+ \cdots + \prod_{t_0\le\tau_j<t}\frac{A_j}{B_j}\big(x(t)-x(\tau_{n_k})\big)$$
$$= x(\tau_0)\left(1-\frac{A_0}{B_0}\right) + \frac{A_0}{B_0}x(\tau_1)\left(1-\frac{A_1}{B_1}\right) + \frac{A_0A_1}{B_0B_1}x(\tau_2)\left(1-\frac{A_2}{B_2}\right)$$
$$+ \cdots + \prod_{\tau_1\le\tau_{j+1}<t} A_jB_jx(\tau_{n_k})\left(1-\frac{A_{n_k}}{B_{n_k}}\right) + \prod_{\tau_0\le\tau_j<t}\frac{A_j}{B_j}x(t) > 0,$$

if $x(t)$ is positive for $t \ge t_0$. So $y(t) > 0$ for $t \ge t_0$. □

*Remark 13.4* If $A_j = B_j$, then the relation (13.6.6) becomes

$$y(t) = \prod_{t_0\le\tau_j<t} B_jx(t), \quad R_k(t,s) = a_k(s) \prod_{g_k(t)\le\tau_j<t} B_j^{-1}\chi_{(g_k(t),\infty)}(s),$$

i.e., (13.6.4) has the form

$$\ddot{x}(t) + \sum_{k=1}^{m} b_k(t)\dot{x}\big(h_k(t)\big) + \sum_{k=1}^{l} r_k(t)x\big(g_k(t)\big) = 0,$$

where $b_k$ are defined by (13.6.5),

$$r_k(t) = a_k(t) \prod_{g_k(t) \le \tau_j < t} B_j^{-1} = a_k(t) \prod_{g_k(t) \le \tau_j < t} A_j^{-1}.$$

## 13.7 Discussion and Open Problems

The first paper on oscillation of impulsive delay differential equations [169] was published in 1989, and its results were included in monographs [167, 249]. In recent years, impulsive delay differential equations have attracted attention of many mathematicians, and numerous papers have been published on this class of equations. Most of the publications are devoted to oscillation of first-order differential equations; see, for example, [33, 40, 126, 347, 353, 359]. There are only a few papers on higher-order impulsive differential equations. We mention here publications on oscillation of second-order impulsive *ordinary* [32, 341] and delay differential equations [326]. Let us note that the main results of this chapter were published in [44, 54].

Finally, let us present some open problems and topics for research and discussion.

1. Establish explicit nonoscillation conditions for the second-order impulsive differential equation with a distributed delay (13.6.4), (13.6.2). Apply the results obtained to deduce nonoscillation conditions for (13.6.1), (13.6.2), (13.6.3).
2. Applying Remark 13.4, obtain explicit nonoscillation conditions for initial value problem (13.6.1), (13.6.2), (13.6.3) when $A_j = B_j$.
3. Consider (13.2.1) with impulsive conditions more general than (13.2.2),

$$x\left(\tau_j^+\right) = \sum_{l=1}^{j} A_{jl} x(\tau_l) + \int_{t_0}^{t} K_j(s) x(s) ds, \quad j = 1, 2, \cdots, \tag{13.7.1}$$

$$\dot{x}\left(\tau_j^+\right) = \sum_{l=1}^{j} B_{jl} x(\tau_l) + \int_{t_0}^{t} M_j(s) x(s) ds, \quad j = 1, 2, \cdots. \tag{13.7.2}$$

4. Deduce sufficient conditions under which any nonoscillatory solution of (13.2.1), (13.2.2) tends to zero.
5. Consider (13.2.1), (13.2.2) with positive and negative coefficients $a_k(t)$.

# Chapter 14
# Linearized Oscillation Theory for Nonlinear Delay Impulsive Equations

## 14.1 Introduction

Nonlinear delay differential equations arise as models of population dynamics, economics, mechanics and technology; see examples in the monographs [154, 167, 192], where the evolution of a system depends not only on its present state but also on its history. Impulses provide an adequate description of sharp system changes when the time of the change is negligible when compared to the process dynamics. In equations of population dynamics and immunology, impulses can describe short-time harvesting, hunting, vaccination [295] or removal of infected species [157].

We consider the nonlinear differential equation

$$\dot{x}(t)+\sum_{k=1}^{m} r_k(t) f_k\big(x\big(h_k(t)\big)\big)=0,\ t\neq\tau_j, \tag{14.1.1}$$

with the nonlinear impulsive conditions

$$x\big(\tau_j^+\big)=I_j\big(x(\tau_j)\big),\ j=1,2,\cdots. \tag{14.1.2}$$

For this equation, the linearized oscillation theory is developed in Sect. 14.3. Sections 14.4 and 14.4.3 contain applications. Using linearized results, explicit oscillation and nonoscillation conditions are obtained for impulsive models of population dynamics. In Sect. 14.4, results on nonoscillation and oscillation and numerical simulations for impulsive logistic equations are presented. Section 14.4.3 investigates oscillation properties of a generalized Lasota-Wazewska equation. Finally, Sect. 14.5 involves some discussion and open problems.

## 14.2 Preliminaries

Let us assume that the parameters of (14.1.1), (14.1.2) satisfy the following conditions:

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(a1) $r_k(t) \geq 0$, $k = 1, \cdots, m$ are Lebesgue measurable locally essentially bounded functions.
(a2) $h_k : [0, \infty) \to \mathbb{R}$, $k = 1, \cdots, m$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$.
(a3) $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \cdots, m$ are continuous functions, $xf_k(x) > 0$, $x \neq 0$.
(a4) $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ are fixed points, $\lim_{j\to\infty} \tau_j = \infty$.
(a5) $I_j$ are continuous functions satisfying $xI_j(x) > 0$, $x \neq 0$, $j \in \mathbb{N}$.

Together with (14.1.1), (14.1.2), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f_k\big(x\big(h_k(t)\big)\big) = 0,\ t \geq t_0,\ t \neq \tau_j, \tag{14.2.1}$$

$$x\big(\tau_j^+\big) = I_j\big(x(\tau_j)\big),\ \tau_j > t_0,\ j = 1, 2, \cdots, \tag{14.2.2}$$

$$x(t) = \varphi(t),\ t < t_0,\ x(t_0) = x_0. \tag{14.2.3}$$

We also assume that the following hypothesis holds:

(a6) $\varphi : (-\infty, t_0) \to R$ is a Borel measurable bounded function.

**Definition 14.1** A function $x : \mathbb{R} \to \mathbb{R}$ piecewise absolutely continuous in each interval $[t_0, b]$ is called *a solution* of problem (14.2.1), (14.2.2), (14.2.3) if it satisfies (14.2.1) for almost all $t \in [t_0, \infty)$, $t \neq \tau_j$, equalities (14.2.2) at $t = \tau_j$ and conditions (14.2.3) for $t \leq t_0$.

Equation (14.1.1), (14.1.2) has *a nonoscillatory solution* if it has either an eventually positive or an eventually negative solution. Otherwise all solutions of (14.1.1), (14.1.2) are *oscillatory*.

We will also consider the linear equation

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) x\big(h_k(t)\big) = 0,\ t \neq \tau_j, \tag{14.2.4}$$

$$x\big(\tau_j^+\big) = b_j x(\tau_j),\ j = 1, 2, \cdots. \tag{14.2.5}$$

Theorems 12.2 and 12.6 can be reformulated in the following way.

**Lemma 14.1** *Suppose* (a1), (a2) *and* (a4) *hold and* $b_j > 0$, $j \in \mathbb{N}$. *Then the following hypotheses are equivalent*:

1) *Equation* (14.2.4), (14.2.5) *has a nonoscillatory solution.*
2) *There exists* $t_1 \geq 0$ *such that the inequality*

$$u(t) \geq \sum_{k=1}^{m} r_k(t) \exp\left\{\int_{h_k(t)}^{t} u(s)\, ds\right\} \prod_{h_k(t) \leq \tau_j < t} b_j^{-1},\ t \geq t_1, \tag{14.2.6}$$

*has a nonnegative locally integrable solution* $u$. *Here the product is assumed to be equal to one if the number of factors is equal to zero.*

3) *The nonimpulsive equation*

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \prod_{h_k(t) \le \tau_j < t} b_j^{-1} x\big(h_k(t)\big) = 0 \tag{14.2.7}$$

*has a nonoscillatory solution.*

## 14.3 Oscillation and Nonoscillation

It is well known that it is easier to establish oscillation properties of differential inequalities than of the corresponding differential equations. In view of this, we need the following generalization of Theorem 12.2, which involves a differential inequality and also an inequality for impulses.

**Lemma 14.2** *Suppose* (a1), (a2) *and* (a4) *hold and* $b_j > 0,\ j = 1, 2, \cdots$. *Then the following hypotheses are equivalent*:

1) *The impulsive differential inequality*

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) x\big(h_k(t)\big) \le 0,\ t \ne \tau_j, \tag{14.3.1}$$

$$x\big(\tau_j^+\big) \le b_j x(\tau_j),\ \ j = 1, 2, \cdots \tag{14.3.2}$$

*has an eventually positive solution with a locally essentially bounded derivative.*

2) *There exists* $t_1 \ge 0$ *such that inequality* (14.2.6) *has a nonnegative locally integrable solution* $u$.
3) *Equation* (14.2.4), (14.2.5) *has a nonoscillatory solution.*
4) *Nonimpulsive equation* (14.2.7) *has a nonoscillatory solution.*

*Proof* Since the implication 3) ⇒ 1) is obvious, in view of Lemma 14.1 it is enough to prove that 1) implies 2). Let $x(t)$ be a positive solution of inequalities (14.3.1), (14.3.2) for $t \ge t_2$. Then there exist positive constants $c_j$, $0 < c_j \le b_j$, such that $x(\tau_j^+) = c_j x(\tau_j)$ and $t_1 \ge t_2$ such that $h_k(t) \ge t_2$ for $t \ge t_1$. Let us define

$$u(t) = -\frac{d}{dt} \ln\left\{\frac{x(t)}{x(t_1)} \prod_{t_1 < \tau_j \le t} c_j^{-1}\right\}; \tag{14.3.3}$$

i.e.,

$$x(t) = x(t_1) \exp\left\{-\int_{t_1}^{t} u(s)\, ds\right\} \prod_{t_1 \le \tau_j < t} c_j,\ \ t \ge t_2. \tag{14.3.4}$$

After substituting (14.3.4) into (14.3.1), we have

$$-u(t)\exp\left\{-\int_{t_1}^{t}u(s)\,ds\right\}\prod_{t_1\le\tau_j<t}c_j$$
$$+\sum_{k=1}^{m}r_k(t)\exp\left\{-\int_{t_1}^{h_k(t)}u(s)\,ds\right\}\prod_{t_1\le\tau_j<h_k(t)}c_j\le 0,$$

and hence

$$\exp\left\{-\int_{t_1}^{t}u(s)\,ds\right\}\prod_{t_1\le\tau_j<t}c_j\Bigg[u(t)$$
$$-\sum_{k=1}^{m}r_k(t)\exp\left\{\int_{h_k(t)}^{t}u(s)\,ds\right\}\prod_{h_k(t)\le\tau_j<t}c_j^{-1}\Bigg]\ge 0.$$

Since the first factor is positive and $0<c_j\le b_j$, the inequality

$$u(t)\ge\sum_{k=1}^{m}r_k(t)\exp\left\{\int_{h_k(t)}^{t}u(s)\,ds\right\}\prod_{h_k(t)\le\tau_j<t}c_j^{-1}$$
$$\ge\sum_{k=1}^{m}r_k(t)\exp\left\{\int_{h_k(t)}^{t}u(s)\,ds\right\}\prod_{h_k(t)\le\tau_j<t}b_j^{-1}$$

is valid, which completes the proof. □

Lemma 14.1 immediately implies the following result, which claims the equivalence of the oscillation properties for various combinations of equations and inequalities in the differential equation and the impulsive conditions.

**Lemma 14.3** *Suppose* (a1), (a2) *and* (a4) *hold and* $b_j>0$, $j\in\mathbb{N}$. *Then the following pairs of equations* (*inequalities*) *have* (*or have no*) *eventually positive solutions at the same time*: (14.2.4) *and* (14.3.2), (14.2.4) *and* (14.2.5), (14.3.1) *and* (14.2.5), *and* (14.3.1) *and* (14.3.2).

Many results in oscillation theory employ the fact that for certain differential equations any nonoscillatory solution tends to zero. Below we prove a similar property for nonlinear impulsive equations under some additional restrictions.

**Lemma 14.4** *Suppose that there exists $k$ such that*

$$\int_{t_0}^{\infty}r_k(t)=\infty,\ \liminf_{x\to\infty}f_k(x)>0, \tag{14.3.5}$$

*for $x$ sufficiently large*,

$$\left|I_j(x)\right|\le c_j|x|,\ c_j\ge 1,\ \sum_{j=1}^{\infty}(c_j-1)<\infty, \tag{14.3.6}$$

*and* (a1)–(a5) *hold. Let* $x$ *be an eventually positive solution of* (14.1.1), (14.1.2). *Then* $\lim_{t\to\infty} x(t) = 0$.

*Proof* Let $x$ be an eventually positive solution of (14.1.1), (14.1.2); i.e., $x(h_k(t)) > 0$ for $t > t_1$. Therefore $x$ is decreasing between impulses.

By (14.3.6), solution $x$ is bounded as

$$0 < x(t) \le x(t_0)\prod_{j=1}^{\infty} c_j = Mx(t_0), \text{ where } M = \prod_{j=1}^{\infty} c_j < \infty,$$

for any $t \ge t_0 \ge t_1$, since $\prod_{j=1}^{\infty} c_j$ converges whenever

$$\lim_{n\to\infty} \ln\left(\prod_{j=1}^{n} c_j\right) = \sum_{j=1}^{n} \ln c_j \le \sum_{j=1}^{n} (c_j - 1) < \infty.$$

Suppose now that for some sequence $t_k \to \infty$ we have $\lim_{k\to\infty} x(t_k) = 0$. Since $x$ is positive and decreases between impulses, for some subsequence $\{\tau_k\} \subset \{\tau_j\}$ we have $\lim_{k\to\infty} x(\tau_k) = 0$. Let us fix a small number $\varepsilon > 0$. Then there exists $k_0$ such that $x(\tau_{k_0}) < \frac{\varepsilon}{M}$. Thus, for any $t > \tau_{k_0}$, we have

$$0 < x(t) < \prod_{j=1}^{\infty} c_j x(\tau_{k_0}) < M\frac{\varepsilon}{M} = \varepsilon.$$

Hence $\lim_{t\to\infty} x(t) = 0$.

This means that either $\lim_{t\to\infty} x(t) = 0$ or $x(t) \ge m > 0$ (in the latter case, the limit, unlike the nonimpulsive equation, does not necessarily exist). The second equality in (14.3.5) implies that for some $k$ there exists $L > 0$ such that $f_k(x) \ge L$ for $x \ge m$. Let us assume $x(t) \ge m > 0$. After integrating both sides of (14.1.1), we have

$$\int_{t_1}^{\infty} \dot{x}(t)\,dt + \sum_{k=1}^{m} \int_{t_1}^{\infty} r_k(t) f_k\big(x\big(h_k(t)\big)\big)\,dt = 0. \tag{14.3.7}$$

Let $\tau_k$ be the first impulsive point not less than $t_1$; the first term in (14.3.7) satisfies the inequality (below we use the fact that $x(\tau_{j+1}) \le x(\tau_j^+)$ for any $j$)

$$\begin{aligned}
\left|\int_{t_1}^{\infty} \dot{x}(t)\,dt\right| &\le \int_{t_1}^{\infty} \big|\dot{x}(t)\big|\,dt = x(t_1) - x(\tau_k) + \sum_{i=k}^{\infty} \big[x\big(\tau_i^+\big) - x(\tau_{i+1})\big] \\
&\le x(t_1) - x(\tau_k) + \sum_{i=k}^{\infty} \big[c_i x(\tau_i) - x(\tau_{i+1})\big] \\
&\le x(t_1) + \sum_{i=k}^{\infty} (c_i - 1)x(\tau_i) + \sup_{t>t_1} x(t) \\
&\le x(t_1) + Mx(t_1)\sum_{i=k}^{\infty} (c_i - 1) + Mx(t_1) < \infty.
\end{aligned}$$

Thus the first term in (14.3.7) is finite, while the second term is infinite since $f(x(h_k(t))) \geq L > 0$ for $t > t_2$, where $t_2$ is a number such that all $h_k(t) > t_1$ for $t > t_2$ and (14.3.5) is satisfied. The contradiction proves this lemma. □

*Remark 14.1* If we have $\limsup_{x\to-\infty} f_k(x) < 0$ in (14.3.5) rather than $\liminf_{x\to\infty} f_k(x) > 0$, then any eventually negative solution of (14.1.1), (14.1.2) converges to zero.

*Example 14.1* If in Lemma 14.4 we omit the last condition in (14.3.6), then the solution, generally speaking, does not tend to zero. For instance, the equation

$$\dot{x}(t) + x(t) = 0, \ t \neq n,$$
$$x\big(n^+\big) = ex(n), \ n = 1, 2, \cdots,$$

has a nonoscillatory solution $x(t) = e^{-\{t\}}$, where $\{t\}$ is the fractional part of $t$, which does not tend to zero.

*Remark 14.2* A special case of Lemma 14.4 was obtained in [347] under the following conditions: $|I_j(x)| > 2 - c_j, \ f_k(x) \equiv x$ and $r_k, h_k$ are continuous.

Let us proceed to linearization results. The following theorem reduces the study of the oscillation properties of a nonlinear impulsive equation to the investigation of an associated linear equation.

**Theorem 14.1** *Let* (a1)–(a5), (14.3.5) *and* (14.3.6) *hold and there exist* $\delta > 0$, $a_k > 0$, $d_j > 0$, $k = 1, \cdots, m$, $j \in \mathbb{N}$ *such that*

$$\lim_{x\to 0} \frac{f_k(x)}{x} = a_k \tag{14.3.8}$$

*and* $|I_j(x)| \leq d_j|x|$ *for* $|x| < \delta$. *If for some* $\varepsilon$, $0 < \varepsilon < a_k$, *all solutions of the impulsive equation*

$$\dot{x}(t) + \sum_{k=1}^{m} (a_k - \varepsilon) r_k(t) x\big(h_k(t)\big) = 0, \ t \neq \tau_j, \tag{14.3.9}$$

$$x\big(\tau_j^+\big) = d_j x(\tau_j), \ j = 1, 2, \cdots, \tag{14.3.10}$$

*are oscillatory, then all solutions of* (14.1.1), (14.1.2) *are also oscillatory.*

*Proof* Let us assume that there exists an eventually positive solution of (14.1.1), (14.1.2), which is $x(t) > 0$, $t \geq t_1$. Then there exists $t_2 \geq t_1$ such that $h_k(t) > t_1$, $t \geq t_2$. By Lemma 14.4, we have $\lim_{t\to\infty} x(t) = 0$. Therefore, by (14.3.8), for any $\varepsilon > 0$ there exists $t_3$ such that $f(x(h_k(t))) > (a_k - \varepsilon)x(h_k(t))$ for all $t \geq t_3$.

Since $\lim_{t\to\infty} x(t) = 0$ and $|I_j(x)| \le d_j|x|$ for $|x| < \delta$, there exists $t_4 \ge t_3$ such that $|x(t)| < \delta$ for $t > t_4$ and $x(\tau_j^+) = I_j(x(\tau_j)) \le d_j x(\tau_j)$, $\tau_j > t_4$. Thus $x$ is a solution of the impulsive inequality

$$\dot{x}(t) + \sum_{k=1}^{m} (a_k - \varepsilon_k) r_k(t) x\big(h_k(t)\big) \le 0, \ t \ne \tau_j,$$

$$x\big(\tau_j^+\big) \le d_j x(\tau_j), \ j = 1, 2, \cdots.$$

Consequently, by Lemma 14.2 impulsive equation (14.3.9), (14.3.10) has a nonoscillatory solution, which leads to a contradiction.

If $x(t) < 0$ for $t \ge t_1$, then we denote $y(t) = -x(t)$, $g_k(y) = -f_k(-y)$, $\psi_j(y) = -I_j(-y)$. Then $y(t)$ is an eventually positive solution of the equation

$$\dot{y}(t) + \sum_{k=1}^{m} a_k r_k(t) g_k\big(y\big(h_k(t)\big)\big) = 0, \ t \ne \tau_j, \tag{14.3.11}$$

$$y\big(\tau_j^+\big) = \psi\big(y(\tau_j)\big), \ j = 1, 2, \cdots, \tag{14.3.12}$$

where all the parameters of (14.3.11), (14.3.12) satisfy all the assumptions of the theorem. Then, similar to the case $x(t) > 0$, we obtain that (14.3.9), (14.3.10) (where $x$ is changed by $y$) has a nonoscillatory solution, which completes the proof. □

**Corollary 14.1** *If* (a1)–(a5), (14.3.5) *and* (14.3.8) *hold*, $|I_j(x)| \le |x|$ *and for some* $\varepsilon$, $0 < \varepsilon < a_k$, *all solutions of nonimpulsive equation* (14.3.9) *are oscillatory, then all solutions of* (14.1.1), (14.1.2) *are also oscillatory.*

**Corollary 14.2** *If* (a1)–(a5), (14.3.5) *and* (14.3.8) *hold*,

$$\lim_{x\to 0} \left( \frac{|I_j(x)|}{|x|} - d_j \right) = 0 \text{ uniformly in } j \tag{14.3.13}$$

*and for some* $\varepsilon > 0$, $\delta > 0$, $\varepsilon < a_k$, *all solutions of* (14.3.9) *with the impulsive conditions*

$$x\big(\tau_j^+\big) = (d_j + \delta) x(\tau_j) \tag{14.3.14}$$

*are oscillatory, then all solutions of* (14.1.1), (14.1.2) *are also oscillatory.*

Generally, it is not necessary to assume the existence of the limit $\lim_{x\to 0} \frac{f_k(x)}{x}$. In the following theorems, we suppose only that the lower bound of this ratio is positive or the upper bound is finite.

**Theorem 14.2** *Let* (a1)–(a5) *hold and there exist* $a_k > 0$ *and* $d_j > 0$ *such that*

$$\big|f_k(x)\big| \ge a_k|x|, \ \big|I_j(x)\big| \le d_j|x|.$$

*If all solutions of the impulsive equation*

$$\dot{x}(t) + \sum_{k=1}^{m} a_k r_k(t) x\big(h_k(t)\big) = 0, \tag{14.3.15}$$

$$x\left(\tau_j^+\right) = d_j x(\tau_j), \ \ j = 1, 2, \cdots, \tag{14.3.16}$$

*are oscillatory, then all solutions of* (14.1.1), (14.1.2) *are also oscillatory.*

*Proof* Let $x(t) > 0$, $t \geq t_1$, $h_k(t) \geq t_2$, $t \geq t_2$. Then $f_k(x(h_k(t)^+)) \geq a_k x(h_k(t))$, $I_j(x(\tau_j^+)) \leq d_j x(\tau_j)$. Thus $x$ is a positive solution of the inequalities

$$\dot{x}(t) + \sum_{k=1}^{m} a_k r_k(t) x\big(h_k(t)\big) \leq 0, \ \ x\left(\tau_j^+\right) \leq d_j x(\tau_j), \ \ j = 1, 2, \cdots.$$

By Lemma 14.2, there exists a nonoscillatory solution of (14.3.15), (14.3.16). The contradiction proves the theorem in the case $x(t) > 0$. The case $x(t) < 0$, $t \geq t_1$, is treated as in the proof of Theorem 14.1. □

**Corollary 14.3** *Let* (a1)–(a5) *hold. If* $|f_k(x)| \geq a_k|x|$, $|I_j(x)| \leq |x|$ *and all solutions of* (14.3.15) *without impulses are oscillatory, then all solutions of* (14.1.1), (14.1.2) *are also oscillatory.*

**Theorem 14.3** *Suppose that* (a1)–(a5) *hold and there exist* $M_k > 0$, $d_j > 0$, $k = 1, \cdots, m$, $j \in \mathbb{N}$ *such that* $f_k(x) \leq M_k x$, $I_j(x) \geq d_j x$ *for any* $x > 0$. *If in addition there exists a nonoscillatory solution of the linear delay impulsive equation*

$$\dot{x}(t) + \sum_{k=1}^{m} M_k r_k(t) x\big(h_k(t)\big) = 0, \ \ t \neq \tau_j, \tag{14.3.17}$$

$$x\left(\tau_j^+\right) = d_j x(\tau_j), \ \ j = 1, 2, \cdots, \tag{14.3.18}$$

*then there exists a nonoscillatory* (*eventually positive*) *solution of* (14.1.1), (14.1.2).

*Proof* By Lemma 14.1, there exist $t_1$ and $w_0(t) \geq 0$, $t \geq t_1$, $w_0(t) = 0$, $t < t_1$, such that

$$w_0(t) \geq \sum_{k=1}^{m} r_k(t) M_k \exp\left\{\int_{h_k(t)}^{t} w_0(s)\, ds\right\} \prod_{h_k(t) \leq \tau_j < t} d_j^{-1}, \ \ t \geq t_1. \tag{14.3.19}$$

Let us define for any locally essentially bounded function $u(t) \geq 0$ and any integer $l > j$ (where $\tau_j$ is the first impulse point satisfying $\tau_j \geq t_1$) the function

$$\begin{aligned} x_u(t) = I_l\Bigg(\cdots I_{j+1}\Bigg(I_j\Bigg(\exp\left\{-\int_{t_1}^{\tau_j} u(s)\, ds\right\}\Bigg)\exp\left\{-\int_{\tau_j}^{\tau_{j+1}} u(s)\, ds\right\}\Bigg)\cdots\Bigg) \\ \times \exp\left\{-\int_{\tau_l}^{t} u(s)\, ds\right\}, \end{aligned} \tag{14.3.20}$$

where $\tau_{j-1} < t_1 \leq \tau_j < \tau_{j+1} < \cdots < \tau_l < t \leq \tau_{l+1}$, which is the solution of the impulsive problem

$$\begin{aligned} &\dot{x}_u(t) + u(t) x_u(t) = 0, \ \ t \geq t_1, \\ &x(t_1) = 1, x_u\left(\tau_j^+\right) = I_j\big(x_u(\tau_j)\big). \end{aligned} \tag{14.3.21}$$

We have

$$x_u\big(h_k(t)\big) = I_{i_k}\bigg(\cdots I_{j+1}\bigg(I_j\bigg(\exp\bigg\{-\int_{t_1}^{\tau_j} u(s)\,ds\bigg\}\bigg)\exp\bigg\{-\int_{\tau_j}^{\tau_{j+1}} u(s)\,ds\bigg\}\bigg)\cdots\bigg)$$
$$\times \exp\bigg\{-\int_{\tau_{i_k}}^{h_k(t)} u(s)\,ds\bigg\},$$

where $\tau_{j-1} < t_1 \le \tau_j < \tau_{j+1} < \cdots < \tau_{i_k} \le h_k(t) < \tau_{i_k+1}$. Then

$$\begin{aligned}
x_u(t) \ge{}& d_l \cdots d_{i_k+1} \exp\bigg\{-\int_{\tau_{l-1}}^{\tau_l} u(s)\,ds\bigg\}\cdots\exp\bigg\{-\int_{\tau_{i_k}}^{\tau_{i_k+1}} u(s)\,ds\bigg\}\\
&\times I_{i_k}\bigg(\cdots I_{j+1}\bigg(I_j\bigg(\exp\bigg\{-\int_{t_1}^{\tau_j} u(s)\,ds\bigg\}\bigg)\exp\bigg\{-\int_{\tau_j}^{\tau_{j+1}} u(s)\,ds\bigg\}\bigg)\cdots\bigg)\\
&\times \exp\bigg\{-\int_{\tau_l}^{t} u(s)\,ds\bigg\}\\
={}& d_l \cdots d_{i_k+1} \exp\bigg\{-\int_{\tau_{i_k}}^{t} u(s)\,ds\bigg\}\\
&\times I_{i_k}\bigg(\cdots I_{j+1}\bigg(I_j\bigg(\exp\bigg\{-\int_{t_1}^{\tau_j} u(s)\,ds\bigg\}\bigg)\exp\bigg\{-\int_{\tau_j}^{\tau_{j+1}} u(s)\,ds\bigg\}\bigg)\cdots\bigg)\\
\ge{}& \exp\bigg\{-\int_{h_k(t)}^{t} u(s)\,ds\bigg\} x_u\big(h_k(t)\big) \prod_{h_k(t)\le\tau_j<t} d_j.
\end{aligned}$$

Let us fix $b > t_1$ and define the operator $T : L_\infty[t_1, b] \to L_\infty[t_1, b]$ in the space of all functions essentially bounded on $[t_1, b]$:

$$(Tu)(t) = \sum_{k=1}^{m} r_k(t) f_k\Big(x_u\big(h_k(t)\big)\Big)\frac{1}{x_u(t)}. \tag{14.3.22}$$

For any function $u \in L_\infty[t_1, b]$ satisfying $0 \le u \le w_0$, we have

$$0 \le (Tu)(t) \le \sum_{k=1}^{m} r_k(t) M_k \frac{x_u(h_k(t))}{x_u(t)}.$$

The estimate above of $x_u(t)$, (14.3.19) and (14.3.22) yield for each $u$, $0 \le u \le w_0$,

$$\begin{aligned}
0 \le (Tu)(t) &\le \sum_{k=1}^{m} r_k(t) M_k \frac{x_u(h_k(t))}{x_u(t)}\\
&\le \sum_{k=1}^{m} r_k(t) M_k \exp\bigg\{\int_{h_k(t)}^{t} u(s)\,ds\bigg\} \prod_{h_k(t)\le\tau_j<t} d_j^{-1}\\
&\le \sum_{k=1}^{m} r_k(t) M_k \exp\bigg\{\int_{h_k(t)}^{t} w_0(s)\,ds\bigg\} \prod_{h_k(t)\le\tau_j<t} d_j^{-1} \le w_0;
\end{aligned}$$

i.e., $0 \le Tu \le w_0$.

We will also demonstrate that $T : L_\infty[t_1,b] \to L_\infty[t_1,b]$ is a compact operator. Denote

$$(H_1u)(t) = x_u(t),\ (H_2u)(t) = \frac{1}{x_u(t)},$$

where $x_u$ is the solution of impulsive problem (14.3.21). By Theorem A.6, operator $H_1$ is a compact operator in space $L_\infty[t_1,b]$. We will now prove that $H_2$ is a bounded continuous operator in this space. It is sufficient to consider the case with one impulsive point $t_1 < \tau < b$ since the general case is considered similarly by induction. By the assumption of the theorem, $I_1(x) > d_1 x$ for $x > 0$.

We have

$$\begin{aligned} x_u(t) &= \exp\left\{-\int_{t_1}^t u(s)\,ds\right\}\chi_{[t_1,\tau_1]}(t) \\ &\quad + I\left(\exp\left\{-\int_{t_1}^{\tau_1} u(s)\,ds\right\}\right)\exp\left\{-\int_{\tau_1}^t u(s)\,ds\right\}\chi_{[\tau_1,b]}(t) \\ &\ge \exp\left\{-\int_{t_1}^t u(s)\,ds\right\} + d_1\exp\left\{-\int_{t_1}^{\tau} u(s)\,ds\right\}\exp\left\{-\int_{\tau}^t u(s)\,ds\right\} \\ &\ge \max\{1,d_1\}\exp\left\{-\int_{t_1}^t u(s)\,ds\right\}, \end{aligned}$$

where $\chi_I$ is the characteristic function of the interval $I$.

Denote

$$S_{u_0} = \left\{u \in L_\infty[t_1,b] \mid \|u\|_{L_\infty} \le u_0\right\}.$$

Then, for any $u \in S_{u_0}$, we have

$$\left\|\frac{1}{x_u}\right\| \le \frac{1}{\max\{1,d_1\}} e^{u_0(b-t_1)}.$$

Hence the operator $H_2$ is a bounded operator in the space $L_\infty[t_1,b]$.

Suppose now that $\|u_n - u\|_{L_\infty} \to 0$ as $n \to \infty$, $\|u_n\|_{L_\infty} \le u_0$. Then

$$\begin{aligned} \|H_2u_n - H_2u\|_{L_\infty} &= \left\|\frac{1}{x_{u_n}} - \frac{1}{x_u}\right\|_{L_\infty} \\ &= \frac{\|x_{u_n} - x_u\|}{\|x_{u_n}\|\|x_u\|} \le \frac{1}{\max\{1,d\}} e^{2u_0(b-t_1)}\|x_{u_n} - x_u\|. \end{aligned}$$

The operator $H_1$ is compact and hence is continuous in the space $L_\infty[t_1,b]$. Then $\|x_{u_n} - x_u\| \to 0$ and therefore $\|H_2u_n - H_2u\| \to 0$. This means that operator $H_2$ is a continuous operator. By Lemma A.3, operator $T : L_\infty[t_1,b] \to L_\infty[t_1,b]$ is a compact operator.

Schauder's Fixed-Point Theorem (Theorem A.15) implies that there exists $u \in L_\infty[t_1,b]$ satisfying $0 \le u \le w_0$ such that $u = Tu$. Since $u = Tu$ implies (see (14.3.21) and (14.3.22))

$$ux_u = -\dot{x}_u = \sum_{k=1}^m r_k(t) f_k\big(x_u\big(h_k(t)\big)\big),$$

the function $x_u(t)$ that is defined by (14.3.20) for $t \geq t_1$ and vanishes for $t < t_1$ is an eventually positive solution of (14.1.1), (14.1.2), which completes the proof. □

**Corollary 14.4** *Suppose* (a1)–(a5) *hold and* $f_k(x) \leq x$, $I_j(x) \geq x$ *for any* $x > 0$. *If in addition there exists a nonoscillatory solution of* (14.2.4) *without impulses, then there exists a nonoscillatory* (*eventually positive*) *solution of* (14.1.1), (14.1.2).

**Theorem 14.4** *Suppose that* (a1)–(a5) *hold, there exist* $M_k, d_j > 0$, $k = 1, \cdots, m$, $j \in \mathbb{N}$ *such that* $f_k(x) \geq M_k x$ *and* $I_j(x) \leq d_j x$ *for any* $x < 0$. *If in addition there exists a nonoscillatory solution of the linear delay impulsive equation* (14.3.17), (14.3.18), *then there exists a nonoscillatory* (*eventually negative*) *solution of* (14.1.1), (14.1.2).

**Corollary 14.5** *Suppose that* (a1)–(a5) *hold,* $f_k(x) \geq x$ *and* $I_j(x) \leq x$ *for any* $x < 0$. *If in addition there exists a nonoscillatory solution of* (14.2.4), *then there exists a nonoscillatory* (*eventually negative*) *solution of* (14.1.1), (14.1.2).

In the case where there are no impulses ($I_j(x) = x$), Theorems 14.1–14.4 yield the results of Chapter 10 for equations with concentrated delays.

## 14.4 Applications to Equations of Mathematical Biology

As an application, impulsive models for some equations of mathematical biology can be studied on the basis of the linearized theory that was developed in the previous section. Impulsive models provide an adequate description of sharp changes in the system, such as the removal of infected species, short-time harvesting and vaccination.

### *14.4.1 Logistic Equation: Theoretical Results*

The results of Sect. 14.3 can be applied to the logistic equation

$$\dot{N}(t) = N(t) \sum_{k=1}^{m} r_k(t) \left(1 - \frac{N(h_k(t))}{K}\right) \tag{14.4.1}$$

with certain impulsive conditions, where $r_k$, $h_k$ satisfy conditions (a1) and (a2), $K > 0$. There exists a unique solution of (14.4.1) with the initial condition

$$N(t) = \psi(t) \geq 0, \; t < t_0, \; N(t_0) = y_0 > 0. \tag{14.4.2}$$

In [50], impulses of the type

$$N\left(\tau_j^+\right) - K = b_j\left(N(\tau_j) - K\right), \tag{14.4.3}$$

were considered, where the initial function $\psi$ satisfies (a6).

Similar to the nonimpulsive case, the solution of (14.4.1)–(14.4.3) is positive if all $b_j > 0$.

**Definition 14.2** A positive solution $N$ of (14.4.1)–(14.4.3) is said to be *oscillatory about* $K$ if there exists a sequence $t_n$, $t_n \to \infty$ such that $N(t_n) - K = 0$, $n = 1, 2, \cdots$; $N$ is said to be *nonoscillatory about* $K$ if there exists $t_0 \geq 0$ such that $|N(t) - K| > 0$ for $t \geq t_0$. A solution $N$ is said to be *eventually positive (eventually negative) about* $K$ if $N - K$ is eventually positive (eventually negative).

**Theorem 14.5** *Suppose that the hypotheses* (a1), (a2), (a4) *and the first equality in* (14.3.5) *hold*, $0 < b_j \leq 1$ *and for some sufficiently small* $\varepsilon > 0$ *and* $\delta > 0$ *all solutions of the equation*

$$\dot{x}(t) + (1-\varepsilon)\sum_{k=1}^{m} r_k(t)x\big(h_k(t)\big) = 0, \ t \geq t_0 \tag{14.4.4}$$

*with the impulsive conditions*

$$x\big(\tau_j^+\big) = (b_j + \delta)x(\tau_j) \tag{14.4.5}$$

*are oscillatory. Then all solutions of* (14.4.1), (14.4.3) *are oscillatory about* $K$.

*Proof* After the substitution

$$N(t) = Ke^{x(t)}, \tag{14.4.6}$$

impulsive equation (14.4.1), (14.4.3) is transformed into impulsive equation (14.1.1), (14.1.2) with

$$f_k(x) = f(x) = e^x - 1, \ I_j(x) = \ln\big(1 - b_j + b_j e^x\big).$$

Note that oscillation (or nonoscillation) of $N$ about $K$ is equivalent to oscillation (nonoscillation) of $x$.

We will now apply Theorem 14.1. Let us note that (14.3.6) holds with $c_k = 1$, and (14.3.8) is satisfied with $a_k = 1$. We have

$$\lim_{x\to 0}\frac{I_j(x)}{x} = b_j,$$

so there exists $\delta > 0$ such that for $|x| < \varepsilon$, where $\varepsilon$ is small enough,

$$\left|\frac{I_j(x)}{x}\right| \leq b_j + \delta.$$

All the conditions of Theorem 14.1, with $a_k = 1, d_j = b_j + \delta$, are satisfied. Then all solutions of (14.4.4), (14.4.5) are oscillatory about zero. Hence all solutions of (14.4.1), (14.4.3) are oscillatory about $K$. □

**Corollary 14.6** *Suppose that the hypotheses* (a1), (a2), (a4) *and the first equality in* (14.3.5) *hold and*

$$\liminf_{t\to\infty}\sum_{k=1}^{m} r_k(t)\prod_{h_k(t)\leq\tau_j\leq t} b_j^{-1}\big(t - h_k(t)\big) > \frac{1}{e}. \tag{14.4.7}$$

*Then all solutions of* (14.4.1), (14.4.3) *are oscillatory about* $K$.

*Proof* Inequality (14.4.7) implies that for some $\varepsilon > 0, \delta > 0$ we have

$$\liminf_{t\to\infty} \sum_{k=1}^{m} (1-\varepsilon) r_k(t) \prod_{h_k(t)\le\tau_j\le t} (b_j+\delta)^{-1}\big(t-h_k(t)\big) > \frac{1}{e}.$$

Lemma 2.3 yields that all solutions of the equation

$$\dot{x}(t) + \sum_{k=1}^{m} (1-\varepsilon) r_k(t) \prod_{h_k(t)\le\tau_j\le t} (b_j+\delta)^{-1} x\big(h_k(t)\big) = 0 \tag{14.4.8}$$

are oscillatory. Then, by Corollary 12.3 and Theorem 14.5, all solutions of (14.4.1), (14.4.3) are oscillatory about $K$, which completes the proof. □

In principle, the same scheme as in Theorem 14.5 is applicable to the case $b_j \ge 1$, where the solutions $x < 0$ are considered. However, unlike equations without impulses for certain $r_k \ge 0$, $b_j \ge 1$ and initial conditions, there exists $\tau_i$ such that $N(\tau_i) > 0$, while $N(\tau_i^+) < 0$ (i.e., the solution obtained becomes negative). For the logistic equation, this means extinction of the population, while impulsive perturbations can be treated as short-time harvesting. To avoid the possibility of extinction in a finite time, we impose an additional constraint:

$$\prod_{j=1}^{\infty} b_j \le M < \infty. \tag{14.4.9}$$

Condition (14.4.9) does not provide that any solution of (14.4.1), (14.4.3) satisfy $N(t) > 0$; however, under (14.4.9) there exists such a solution. For example, any solution with a nonpositive initial function and an initial value satisfying $(1-1/M)K < N(0) \le K$ (assuming $M > 1$) is positive. In the following, we will consider only such solutions.

**Theorem 14.6** *Suppose that* (a1), (a2), (a4) *and* (14.4.9) *hold*, $b_j \ge 1$ *and there exists a nonoscillatory solution of the linear delay impulsive equation*

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) x\big(h_k(t)\big) = 0, \tag{14.4.10}$$

$$x\big(\tau_j^+\big) = b_j x(\tau_j). \tag{14.4.11}$$

*Then there exists a solution of impulsive equation* (14.4.1), (14.4.3) *nonoscillatory about* $K$.

*Proof* Again, after the substitution as in (14.4.6), equation (14.4.1), (14.4.3) is transformed into impulsive equation (14.1.1), (14.1.2), with

$$f_k(x) = f(x) = e^x - 1, \;\; I_j(x) = \ln\big(1 - b_j + b_j e^x\big).$$

Let us apply Theorem 14.4. We have $f(x) \ge x$ for $x < 0$. We also notice that for $x < 0, b_j \ge 1$,

$$I_j(x) = \ln\big(1 - b_j + b_j e^x\big) \le b_j x. \tag{14.4.12}$$

In order to justify (14.4.12), it is enough to demonstrate that

$$b_j\left(e^x-1\right)\le e^{b_jx}-1,$$

or $u(x)=b_je^x-b_j-e^{b_jx}-1\le 0$ for $x<0$. Obviously $u(0)=0$ and

$$u'(x)=b_je^x-b_je^{b_jx}=b_je^x\left(1-e^{(b_j-1)x}\right)>0$$

for $x<0, b_j\ge 1$. Hence $u(x)\le 0$ for $x<0$ and therefore (14.4.12) is satisfied.

All the conditions of Theorem 14.4 with $M_k=1$, $d_j=b_j$ hold. Then (14.1.1), (14.1.2) has a nonoscillatory solution. Hence (14.4.1), (14.4.3) has a solution nonoscillatory about $K$, which completes the proof. □

**Corollary 14.7** *Suppose* (a1), (a2), (a4) *and* (14.4.9) *hold*, $b_j\ge 1$ *and there exists a nonoscillatory solution of* (14.4.10) *without impulses. Then there exists a solution of* (14.4.1), (14.4.3) *nonoscillatory about* $K$.

**Corollary 14.8** *Suppose* (a1), (a2), (a4) *and* (14.4.9) *hold*, $b_j\ge 1$ *and there exists* $\mu>0$, $t_0\ge 0$ *such that*

$$\sum_{k=1}^{m}r_k(t)\prod_{h_k(t)\le\tau_j\le t}b_j^{-1}e^{\mu[t-h_k(t)]}\le\mu,\ t\ge t_0.$$

*Then there exists a solution of* (14.4.1), (14.4.3) *nonoscillatory about* $K$.

*Proof* The statement of the corollary follows from Theorems 14.1 and 2.6. □

If instead of impulsive conditions (14.4.3) impulses of the exponential type

$$N\left(\tau_j^+\right)=K\left(\frac{N(\tau_j)}{K}\right)^{b_j},\ b_j>0, \tag{14.4.13}$$

are imposed on (14.4.1), then after the substitution (14.4.6) we obtain linear impulses (14.2.5).

Thus Theorems 14.1, 14.3 and 14.4 imply the following results.

**Theorem 14.7** *Suppose that the hypotheses* (a1), (a2) *and* (a4) *hold.*

1) *If the equality* (14.3.5) *holds*, $b_j\le 1$ *and for every sufficiently small* $\varepsilon>0$, $\delta>0$, *all solutions of* (14.4.4), (14.4.5) *are oscillatory. Then all solutions of* (14.4.1), (14.4.13) *are oscillatory about* $K$.
2) *If* $b_j\ge 1$ *and there exists a nonoscillatory solution of linear delay impulsive equation* (14.4.10), (14.4.11), *then there exists a solution of impulsive logistic equation* (14.4.1), (14.4.13) *nonoscillatory about* $K$.

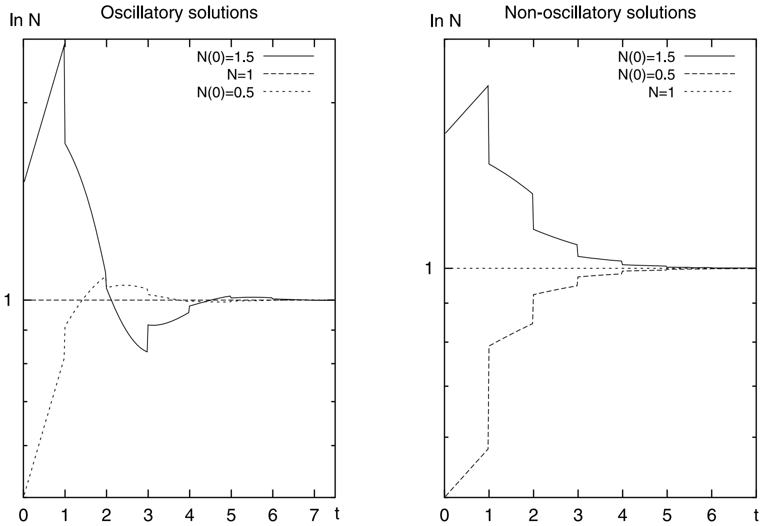


**Fig. 14.1** The solutions of (14.4.14), (14.4.15), with the zero initial function $h = K = 1$, $b = 0.5$, $a = 0.5$ (the *left graph*) and $a = 0.15$ (the *right graph*). The initial conditions are $N(0) = 0.5$ and $N(0) = 1.5$, respectively. We use the logarithmic scale in $N$

## *14.4.2 Logistic Equation: Numerical Simulations*

Here we present some numerical simulations to illustrate the results of the previous subsection. Let impulsive conditions be of type (14.4.3). Consider the logistic equation with one constant delay

$$\dot{N}(t) = aN(t)\left(1 - \frac{N(t-h)}{K}\right) \tag{14.4.14}$$

and the impulsive conditions

$$N\big(j^+\big) - 1 = b\big(N(j) - 1\big),\ \ j = 1, 2, \cdots. \tag{14.4.15}$$

Figure 14.1 presents the solutions of (14.4.14), (14.4.15) with the delay $h = 1$, the equilibrium state is $K = 1$, $b = 0.5$, and the distance between the impulses is also equal to one, with the zero initial function. Corollary 14.6 gives explicit sufficient conditions for all solutions of (14.4.14), (14.4.15) to oscillate, where the impulsive coefficient is constant $b_j = b$ and the impulses are equispaced, here we assume $\tau_{j+1} - \tau_j = 1$.

The sufficient condition is

$$ahb^{-[h]} > \frac{1}{e}, \tag{14.4.16}$$

where $[x]$ is the greatest integer not exceeding $x$.

The left graph in Fig. 14.1 presents two solutions oscillatory about $K = 1$ with $a = 0.5$ and the initial conditions $N(0) = 0.5$ and $N(0) = 1.5$, respectively. The right graph is for $a = 0.15$.

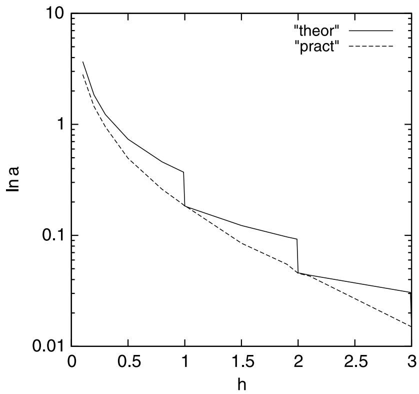


**Fig. 14.2** The critical values of $h$, $a$ for all solutions of (14.4.14), (14.4.15) to be oscillatory (theoretical curve) and the numerically computed value of $a$ for which the solution of (14.4.14) with the zero initial function and $N(0)=0.5$ begins to oscillate about $K=1$. Here $K=1$, $b=0.5$. We use the logarithmic scale in $N$

Figure 14.2 presents the critical values of $h$, $a$ for all solutions to be oscillatory (theoretical curve) and the numerically computed value of $a$ for which the solution of the equation with the zero initial function and $N(0)=0.5$ begins to oscillate about $K=1$.

Now consider the case $b>1$, $N(0)<1$. Impulsive conditions (14.4.15) cannot satisfy constraint (14.4.9). Let us study numerically the values of $a>0$, $b>1$ for which the introduction of impulsive conditions will not lead to negative solutions with $N<0$, which correspond to the extinction of the population in a finite time. In numerical simulations, we set $N(0)=0.5$ with the zero initial function. The results are presented in Fig. 14.3.

### *14.4.3 Generalized Lasota-Wazewska Equation*

Consider now an impulsive model for the generalized Lasota-Wazewska equation that describes the survival of red blood cells,

$$\dot{N}(t) = -\mu N(t) + p e^{-\gamma N(h(t))}, \; t \geq 0, \tag{14.4.17}$$

where $\mu, p, \gamma > 0$ and for $h(t)$ condition (a2) holds; for details see [192] and Chap. 10.

We consider only those solutions of (14.4.17) that correspond to the initial conditions satisfying (14.4.2). Then (14.4.17), (14.4.2) has a unique solution that is positive for all $t \geq t_0$.

The equilibrium $N^*$ of (14.4.17) is positive and satisfies the equation

$$N^* = \frac{p}{\mu} e^{-\gamma N^*}.$$

Consider the impulsive conditions

$$N\left(\tau_j^+\right) - N^* = b_j\left(N(\tau_j) - N^*\right). \tag{14.4.18}$$

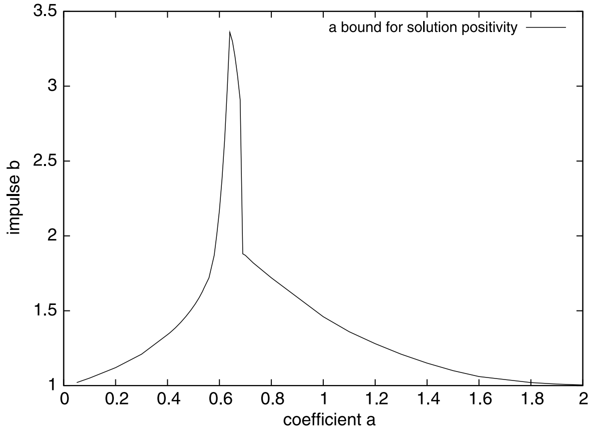


**Fig. 14.3** The values of $a, b > 1$ such that the solution is nonnegative ($N > 0$) are *under the curve*. Here $h = 1$

After the change of variables

$$N(t) = N^* + \frac{1}{\gamma}x(t),$$

impulsive equation (14.4.17), (14.4.18) takes the form

$$\dot{x}(t) + \mu x(t) + \mu\gamma N^*\bigl[1 - e^{-x(h(t))}\bigr] = 0, \tag{14.4.19}$$

$$x\bigl(\tau_j^+\bigr) = b_j x(\tau_j). \tag{14.4.20}$$

Equation (14.4.19) has the form (14.1.1), where

$$n = 2,\ r_1(t) = \mu,\ r_2(t) = \mu\gamma N^*,\ h_1(t) = t, h_2(t) = h(t),$$
$$f_1(x) = x,\ f_2(x) = 1 - e^{-x},$$

and impulsive conditions (14.4.20) are linear. All solutions of (14.4.17) are oscillatory about $N^*$ if and only if all solutions of (14.4.19) are oscillatory about zero.

Functions $f_1$ and $f_2$ satisfy conditions (a3), (14.3.8), $a_1 = a_2 = 1$ and $f_i(x) \le x$, $x \ge 0$, $f_i(x) \ge x$, $x < 0$, $i = 1, 2$. As a corollary of Theorem 14.1 and either Theorem 14.3 or 14.4, we obtain the following results.

**Theorem 14.8** *Suppose* $0 < b_j \le 1$, *and there exists* $\varepsilon > 0$ *such that all solutions of the linear equation*

$$\dot{x}(t) + (1 - \varepsilon)\mu x(t) + (1 - \varepsilon)\mu\gamma N^* x\bigl(h(t)\bigr) = 0 \tag{14.4.21}$$

*with impulsive conditions* (14.4.20) *are oscillatory. Then all solutions of* (14.4.17), (14.4.18) *are oscillatory about* $N^*$.

**Corollary 14.9** *Suppose* $0 < b_j \le 1$,

$$\limsup_{t\to\infty}\bigl(t - h(t)\bigr) < \infty,$$
$$\liminf_{t\to\infty} \mu\gamma N^* \int_{h(t)}^{t} \exp\bigl\{\mu\bigl(s - h(s)\bigr)\bigr\} \prod_{h(s)\le\tau_j<s} b_j^{-1}\,ds > \frac{1}{e}. \tag{14.4.22}$$

*Then all solutions of* (14.4.17), (14.4.18) *are oscillatory about* $N^*$.

*Proof* After the substitution $x(t)=y(t)e^{-(1-\varepsilon)\mu t}$, (14.4.21) has the form

$$\dot{y}(t)+(1-\varepsilon)\mu\gamma N^* \exp\{(1-\varepsilon)\mu\big(t-h(t)\big)\}y\big(h(t)\big)=0. \tag{14.4.23}$$

Inequalities (14.4.22) imply that for some $\varepsilon>0$

$$\liminf_{t\to\infty}(1-\varepsilon)\mu\gamma N^* \int_{h(t)}^{t} \exp\{\mu(1-\varepsilon)\big(s-h(s)\big)\} \prod_{h(s)\le\tau_j<s} b_j^{-1}\,ds > \frac{1}{e}.$$

Lemma 2.2 yields that all solutions of (14.4.23), (14.4.20) and therefore of (14.4.21), (14.4.20) are oscillatory. □

**Theorem 14.9** *Suppose* $b_j\ge 1$, *and there exists a nonoscillatory solution of the linear equation*

$$\dot{x}(t)+\mu x(t)+\mu\gamma N^* x\big(h(t)\big)=0.$$

*Then there exists a solution of* (14.4.17), (14.4.18) *nonoscillatory about* $N^*$.

**Corollary 14.10** *Suppose* $b_j\ge 1$, *and*

$$\limsup_{t\to\infty}\mu\gamma N^\star \int_{h(t)}^{t} \exp\{\mu\big(s-h(s)\big)\}ds < \frac{1}{e}.$$

*Then there exists a solution of* (14.4.17), (14.4.18) *nonoscillatory about* $N^*$.

Proofs of Theorem 14.9 and Corollary 14.10 are similar to the proofs of Theorem 14.8 and Corollary 14.9 and apply Theorem 2.7.

## 14.5 Discussion and Open Problems

Usually investigation of nonlinear delay differential equations is more complicated than for linear equations. However, in certain cases it is possible to deduce properties of a nonlinear equation from an associated linear equation. The purpose of the linearized oscillation theory is to study oscillation of the associated linear equation rather than the original nonlinear equation. Such a theory is very well developed for autonomous nonlinear delay differential equations and some of their generalizations (see monographs [154, 192] and references therein). For both nonlinear nonautonomous impulsive and nonimpulsive delay differential equations, most qualitative results were obtained without reducing them to linear equations [17, 237, 248, 262, 269, 276, 315, 343].

Only a few works deal with the linearized theory of nonimpulsive equations (see [55, 192, 225, 243, 318]), and to the best of our knowledge there are only a few recent publications [149] on the linearized theory of impulsive delay differential equations. Compared to [149] and most other publications, this chapter involves more general or additional results in the following sense:

1. Coefficients $r_k$ and delays are not assumed to be continuous.
2. In most of our results, we do not suppose that the impulses satisfy $\prod_{1\le k\le\infty} b_k < \infty$ or some equivalent condition.
3. We apply the linearization results to impulsive equations of mathematical biology such as the delay logistic equation and the generalized Lasota-Wazewska equation, which describes the survival of red blood cells.
4. Usually differential inequalities are applied for comparison of oscillation properties, and impulses are assumed to be the same. In the present chapter, we also consider inequalities for the linear impulsive conditions.

The main results of this chapter were published in [57]. The paper [11] contains a survey on oscillation and nonoscillation for linear and nonlinear impulsive delay differential equations.

Finally, let us outline some open problems and topics for research and discussion.

1. Deduce nonoscillation conditions for (14.1.1) with nonlocal impulsive conditions
$$x\big(\tau_j^+\big) = I_j\big(x(\cdot)\big), \tag{14.5.1}$$
where $x(\cdot)$ is considered for $t \in [t_0, \tau_j]$. In particular, the impulsive conditions
$$x\big(\tau_j^+\big) = x(\tau_j) + \int_{t_0}^{\tau_j} I_j\big(x(s)\big)\,dr_j(s),\ \ j = 1, 2, \cdots, \tag{14.5.2}$$
can be imposed, where $r_j$ are functions of bounded variation and $I_j$ satisfy (a5).
2. Can linearized results be applied to other impulsive equations, such as generalized and multiplicative delay logistic impulsive equations, the Nicholson blowflies equation with variable (most generally distributed) delay and impulses, the Mackey-Glass impulsive equation and others?
3. If impulsive points are prescribed, can we impose impulses satisfying (a5) such that
   - all solutions are oscillatory;
   - there exists a nonoscillatory solution; and
   - all solutions with the zero initial function and a positive initial value are nonoscillatory.
4. For impulsive nonlinear equations, deduce sufficient conditions when all positive solutions nonoscillatory about the unique positive equilibrium tend to the equilibrium.
5. Consider the linearized oscillation theory for nonlinear impulsive differential equations with a distributed delay, integrodifferential equations, mixed type equations and equations of neutral type.
6. Apply the results of this chapter to equations of mathematical biology with nonlinear impulsive conditions.

# Chapter 15
# Maximum Principles and Nonoscillation Intervals

## 15.1 Introduction

In the previous chapters, as well as in monographs on nonoscillation of functional differential equations, nonoscillation was interpreted as existence of eventually positive solutions of functional differential equations on a semiaxis. In this chapter, we try to understand the essence of nonoscillation in a more general context. Actually nonoscillation plays a very important role in the theory of linear $n$-th-order ordinary differential equations. The study of many classical questions on the qualitative theory of these equations, such as existence and uniqueness of solutions of the interpolation boundary value problems, positivity or a corresponding regular behavior of their Green's functions, maximum principles, variation problems and stability, was connected with and even essentially based on the notion of nonoscillation intervals of corresponding linear ordinary differential equations. Below (see, for example, (15.2.21)) we explain why the notion of a nonoscillation interval, defined as an interval where nontrivial solutions of a homogeneous equation do not have $n$ zeros, did not exist for a long time for delay differential equations.

In this and the next two chapters we try to create a concept of nonoscillation for functional differential equations that will be an analogue of nonoscillation theory for ordinary differential equations. We start with a definition of homogeneous equations of first order in such a form that they preserve one-dimensional fundamental systems. This leads us to an important conclusion: every first-order homogeneous functional differential equation on a nonoscillation interval is equivalent to a corresponding first-order ordinary differential equation. This explains why properties of nonoscillation functional differential equations are analogous to those of ordinary differential equations. On this basis, we discover many correlations between different properties of functional differential equations known for ordinary differential equations.

In this chapter, we present maximum principles and study nonoscillation intervals for first-order Volterra functional differential equations. The concept of nonoscillation developed in this chapter consists of two parts. The first part is to discover a nonoscillation interval for functional differential equations. It can be noted that although there is a difference between the definition of nonoscillation in the previous

R.P. Agarwal et al., *Nonoscillation Theory of Functional Differential Equations with Applications*, DOI 10.1007/978-1-4614-3455-9_15,

chapters (existence of an eventually positive solution on a semiaxis) and in this one (nontrivial solutions do not have zeros on a finite or infinite interval), almost all tests of nonoscillation coincide. The second part is to find various corollaries and applications of nonoscillation results in maximum principles, boundary value problems and stability. In several cases, we prove equivalence of nonoscillation and corresponding maximum principles. This second part is the main one in this chapter.

## 15.2 Preliminaries

Let us start with several examples demonstrating the difference between properties of first-order ordinary and delay differential equations.

Consider the first-order linear ordinary differential equation

$$x'(t) + p(t)x(t) = f(t), \ t \in [0, \omega], \tag{15.2.1}$$

with integrable coefficients $p$ and $f$. Its general solution can be represented by the classical formula

$$x(t) = \int_0^t e^{-\int_s^t p(\xi)d\xi} f(s)ds + ce^{-\int_0^t p(\xi)d\xi}. \tag{15.2.2}$$

*Example 15.1* It is clear from formula (15.2.2) that every solution of the homogeneous ordinary differential equation

$$x'(t) + p(t)x(t) = 0, \ t \in [0, \omega], \tag{15.2.3}$$

is of the form

$$x(t) = x(0)e^{-\int_0^t p(s)ds}, \ t \in [0, \omega], \tag{15.2.4}$$

and it is positive if $x(0) > 0$ and is negative if $x(0) < 0$. For the delay equation

$$x'(t) + x(0) = 0, \ t \in [0, \omega], \tag{15.2.5}$$

the solution

$$x(t) = x(0)(1 - t), \ t \in [0, \omega], \tag{15.2.6}$$

changes its sign at the point $t = 1$.

*Example 15.2* The one-point problem (15.2.1), (15.2.7), where

$$x(\omega) = 0, \tag{15.2.7}$$

has a unique solution for each positive real number $\omega$. Actually, substituting equality (15.2.7) into (15.2.2), we obtain

$$0 = x(\omega) = \int_0^\omega e^{-\int_s^\omega p(\xi)d\xi} f(s)ds + ce^{-\int_0^\omega p(\xi)d\xi} \tag{15.2.8}$$

and

$$c = -e^{\int_0^\omega p(\xi)d\xi} \int_0^\omega e^{-\int_s^\omega p(\xi)d\xi} f(s)ds. \tag{15.2.9}$$

For delay equations, a one-point problem does not necessarily have a unique solution. Actually, consider the equation

$$x'(t) + x(0) = f(t), \ t \in [0, \omega]. \tag{15.2.10}$$

Integrating it, we obtain the formula

$$x(t) = \int_0^t f(s)ds - x(0)t + x(0), \tag{15.2.11}$$

representing a general solution of (15.2.10). Setting $t = \omega$, we get

$$x(\omega) = \int_0^\omega f(s)ds - x(0)\omega + x(0), \tag{15.2.12}$$

and using the condition (15.2.7),

$$0 = \int_0^\omega f(s)ds - x(0)\omega + x(0). \tag{15.2.13}$$

If $\omega = 1$, then problem (15.2.10), (15.2.7) has a solution if and only if $\int_0^1 f(s)ds = 0$, and this solution is not unique.

*Example 15.3* Consider the equation

$$x'(t) + p(t)x\big([t]\big) = 0, \ t \in [0, \infty), \tag{15.2.14}$$

where $[t]$ is the integer part of $t$. If $p(t) \geq 0$ and $\int_t^{t+1} p(s)ds > 1$, then all nontrivial solutions of this equation oscillate in contrast with solutions of ordinary differential equation (15.2.3). The amplitudes of oscillating solutions tend to infinity if $p(t) \geq 0$ and $\int_t^{t+1} p(s)ds > 2$ in contrast with the solutions of (15.2.3), which all tend to zero.

*Example 15.4* The general solution of (15.2.1) can be written in the form

$$x(t) = \int_0^t e^{-\int_s^t p(\xi)d\xi} f(s)ds + x(0)e^{-\int_0^t p(\xi)d\xi}. \tag{15.2.15}$$

Formula (15.2.15) allows us to compare solutions (applicability of Chaplygin's theorem on the differential inequality): if the absolutely continuous function $v$ satisfies the inequalities

$$v'(t) + p(t)v(t) \geq f(t), \ t \in [0, \omega], \ v(0) \geq x(0), \tag{15.2.16}$$

then $v(t) \geq x(t)$ for $t \in [0, \omega]$. For a delay equation, this is not true. For example, solution (15.2.6) of (15.2.5) for positive $x(0)$ becomes negative for $t > 1$ and is consequently less than the trivial solution of (15.2.5).

Note that for $\omega < 1$ nontrivial solutions of homogeneous equation (15.2.5) do not have zeros, the solution of problem (15.2.5), (15.2.3) has the representation

$$x(t) = \int_0^\omega \left(\frac{t-1}{1-\omega} + \chi(t, s)\right) f(s)ds, \tag{15.2.17}$$

where

$$\chi(t,s)=\begin{cases}1, & s\le t,\\ 0, & t<s,\end{cases} \tag{15.2.18}$$

and consequently this problem has a unique solution for each integrable $f$ and Chaplygin's theorem about differential inequality is applicable.

These examples lead us to the idea that intervals $[0,\omega]$, where nontrivial solutions of homogeneous equations do not have zeros, may play an important role in the qualitative theory of linear functional differential equations since on these intervals delay differential equations preserve the basic properties of ordinary differential equations. We call such $[0,\omega]$ *nonoscillation intervals*.

The theory of delay differential equations began with the equation

$$x'(t)+\sum_{i=1}^{n}p_i(t)x\big(t-\tau_i(t)\big)=0,\ \ t\in[0,\omega], \tag{15.2.19}$$

where

$$x(s)=\varphi(s)\ \text{for}\ s<0, \tag{15.2.20}$$

and $\varphi$ is a corresponding continuous function, which is called an initial function. Note that we have to add the equality (15.2.20) to (15.2.19) in order to define what must be put instead of $x(t-\tau_i(t))$ when $t-\tau_i(t)<0$. This equation was considered in [289] as a homogeneous equation. If we want to study (15.2.19) for all possible continuous initial functions $\varphi$, the space of solutions becomes infinite-dimensional and there is no nonoscillation interval in this case. Actually, let us consider, for example, the equation

$$x'(t)+x(t-1)=0,\ \ t\in[0,1], \tag{15.2.21}$$

with all possible continuous initial functions in (15.2.20). It is clear that the solution $x$ is determined by the initial function $\varphi$ and a corresponding choice of $\varphi$ allows us to get solutions $x$ having more than any fixed number of zeros on $[0,1]$. For example, the function $x(t)=\sin\frac{\pi n}{2}t$ is a solution of (15.2.21) if we choose $\varphi(t)=\frac{\pi n}{2}\cos(\frac{\pi n}{2}(t+1))$. In the case of odd $n$, the solution is a continuously extended initial function $\varphi$.

Various examples of this sort led researchers to the opinion that the nonoscillation interval for delay equations does not exist. That is why in the classical monographs [3, 7, 9, 154, 192, 248, 289] on delay differential equations, the notion of nonoscillation is defined as existence of an eventually positive solution (i.e., existence of an initial function $\varphi$ such that the solution obtained as its continuous extension is eventually positive) on the semiaxis $[0,\infty)$ and not in the sense of the definition that we use in this chapter.

In the paper [20], the tradition of considering a solution of delay equation (15.2.19) as a continuously extended initial function $\varphi(t)$ was avoided and a homogeneous object was defined as (15.2.19) with the initial function

$$x(\xi)=0\ \text{for}\ \xi<0. \tag{15.2.22}$$

Equation (15.2.19), (15.2.22) exactly corresponds to a homogeneous equation in the theory of ordinary differential equations: the space of its solutions becomes one-dimensional, and the formula for representation of the general solution of the equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x\big(t - \tau_i(t)\big) = f(t),\ t \in [0, \omega], \tag{15.2.23}$$

with the initial function (15.2.22) is

$$x(t) = \int_0^t C(t,s)f(s)ds + C(t,0)x(0), \tag{15.2.24}$$

where $C(t,s)$ as a function of $t$ for each fixed $s$ is a solution of the equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x\big(t - \tau_i(t)\big) = 0,\ t \in [s, \omega], \tag{15.2.25}$$

$$x(\xi) = 0 \text{ for } \xi < s, \tag{15.2.26}$$

satisfying the condition $C(s,s) = 1$. $C(t,s)$ is called the Cauchy function of (15.2.23). Solution representation formula (15.2.24) generalizes the formula of the general solution

$$x(t) = \int_0^t e^{-\int_s^t p(\xi)d\xi} f(s)ds + x(0)e^{-\int_0^t p(\xi)d\xi} \tag{15.2.27}$$

of the ordinary differential equation

$$x'(t) + p(t)x(t) = f(t),\ t \in [0, \omega]. \tag{15.2.28}$$

It is clear that in the case of ordinary differential equation (15.2.28) we have $C(t,s) = e^{-\int_s^t p(\xi)d\xi}$.

Azbelev's definition of the homogeneous equation allows us to study maximum principles and to construct a theory of boundary value problems for delay differential equations and functional differential equations.

In this chapter, we consider the boundary value problem

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t),\ t \in [0, \omega], \tag{15.2.29}$$

$$lx = c, \tag{15.2.30}$$

where $B : C[0,\omega] \to L_1[0,\omega]$ or $B : C[0,\omega] \to L_\infty[0,\omega]$ is a linear continuous Volterra operator, $C[0,\omega]$ is the space of continuous functions, $L_1[0,\omega]$ is the space of integrable functions and $L_\infty[0,\omega]$ is the space of essentially bounded functions defined on $[0,\omega]$, and $l : D[0,\omega] \to \mathbb{R}$ is a linear bounded functional defined on the space of absolutely continuous functions $D[0,\omega]$.

Note that the operator $B$ can be, for example, of the forms

$$(Bx)(t) = \sum_{i=1}^{n} p_i(t)x\big(t - \tau_i(t)\big),\ t \in [0, \omega], \tag{15.2.31}$$

where

$$x(\xi) = 0, \ t \notin [0, \omega], \tag{15.2.32}$$

or

$$(Bx)(t) = \int_0^{\omega} k(t, s)x(s)ds, \ t \in [0, \omega]. \tag{15.2.33}$$

All linear combinations and superpositions of these operators are also allowed.

Delay and integrodifferential equations are important particular cases of (15.2.29). Equations with integral operators were studied, for example, in [96–98]. These equations are used in viscoelasticity [148, 165]. Note that equations in this operator form become a very important instrument in the study of systems of ordinary or functional differential equations [5, 125], when a functional differential equation for one component of the solution vector is constructed. This equation will be of the form (15.2.29). Note also that an analysis of neutral equations can be reduced to one of (15.2.29) (see [110, 138]) which is another important application of (15.2.29) in the operator form. This approach is based on the study of the inner superposition operator [146, 147].

In almost all assertions, we assume that $B$ is a Volterra operator. We define Volterra operators according to Tikhonov's definition.

**Definition 15.1** An operator $B$ is called *Volterra* if any two functions $x_1$ and $x_2$ coinciding on an interval $[0, a]$ have equal images on $[0, a]$; i.e., $(Bx_1)(t) = (Bx_2)(t)$ for $t \in [0, a]$ and for each $0 < a \le \omega$.

In the case of Volterra operator $B$ defined by (15.2.31), we assume that $\tau_i(t) \ge 0$, and in the case of the integral Volterra operator (15.2.33), we set $t$ instead of $\omega$ in the upper limit of the integral.

Let us now discuss the functional $l : D[0, \omega] \to \mathbb{R}$ in boundary condition (15.2.30). It seems to be important to consider a boundary condition in such a general form. Let us consider a model that is described by a one-point value problem, where $x(t_1) = c$ is a result of taking measurements. A possible difficulty that occurs in some real systems is noise in the system, that does not allow us to rely on the result of only one measurement taken at a moment $t = t_1$. Usually in such cases several measurements $x(t_1), x(t_2), \cdots, x(t_m)$ are taken, and then their averaged value

$$\alpha_1 x(t_1) + \alpha_2 x(t_2) + \cdots + \alpha_m x(t_m) = c$$

is used. The conditions with integrals can describe, for example, the law of conservation of energy. The general form of the functional $l : D[0, \omega] \to \mathbb{R}$ is known,

$$lx \equiv \theta x(0) + \int_0^{\omega} \phi(s)x'(s)ds, \tag{15.2.34}$$

where $\theta \in \mathbb{R}$, $\phi \in L_\infty[0, \omega]$.

Maximum principles present one of the classical parts of the qualitative theory of ordinary and partial differential equations [309]. This chapter is devoted to maximum principles for first-order functional differential equations. In the mathematical literature, there are several definitions of maximum principles. We mean results of the following three types.

1. **Maximum inequalities principle:** *The inequalities*

$$(My)(t) \geq (Mx)(t),\ t \in [0, \omega],\ ly \geq lx \tag{15.2.35}$$

*imply* $y(t) \geq x(t)$ *for* $t \in [0, \omega]$.

More generally, we can formulate this principle as follows: solutions of inequalities are greater than or less than the solution of the equation. In the case where the homogeneous problem $Mx = 0$, $lx = 0$, has only the trivial solution, problem (15.2.29), (15.2.30) has a unique solution, which can be written in the form

$$x(t) = \int_0^{\omega} G(t, s) f(s) ds + X(t), \tag{15.2.36}$$

where $X(t)$ is a solution of the homogeneous equation $(Mx)(t) = 0$, $t \in [0, \omega]$ satisfying the boundary condition $lx = c$. The kernel $G(t, s)$ is called the Green's function. On the basis of this representation, the comparison results were first formulated in the form of positivity or negativity of the corresponding Green's functions in [20].

Note also that the maximum inequality principle allows us to make conclusions of the following form: there is no negative minimum of the solutions of the nonhomogeneous boundary value problem $Mx(t) = f(t)$, $t \in [0, \omega]$, $lx = 0$, in the case of nonnegative $f$.

2. **Maximum principle as boundedness of solutions:** *There exists a positive constant* $N$ *such that* $|x| \leq N(\|f\| + |c|)$*, where* $\|f\|$ *is the norm in the spaces* $L_\infty[0, \omega]$ *or* $L_1[0, \omega]$*, respectively.*

This is actually a problem of continuous dependence of solutions on the right-hand side $f$ and the boundary condition $c$. The formula of the integral representation of the solution reduces the maximum boundedness principle to the fact that any boundary value problem has a unique solution.

If we consider (15.2.29) on the semiaxis $[0, \infty)$ (i.e., $\omega = \infty$), then the problem of exponential stability on the basis of the Bohl-Perron theorem can be reduced to the maximum boundedness principle for the Cauchy problem [29]

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t),\ t \in [0, \infty),\ x(0) = c. \tag{15.2.37}$$

The analogues of the classical Bohl-Perron theorem claim that for a wide class of linear functional differential equations (for example, for equations with a bounded memory) boundedness of solutions for every $|f| \leq 1$ implies exponential stability (i.e., solutions of the homogeneous equation $(Mx)(t) = 0$, $t \in [0, \infty)$ tend to zero as exponents $e^{-\alpha t}$ when $t \to \infty$, where $\alpha$ is a corresponding positive number). Application of the maximum boundedness principle to studying exponential stability was proposed in [22–24, 29]. The idea to use left and right (for example, the so-called Azbelev $W$-transform) regularizations with the Green's operators of the corresponding model equations, satisfying the maximum inequalities principle, in order to obtain the maximum boundedness principle, was implemented for stability investigation in the papers [31, 61–63, 121, 139, 178].

3. **Maximum boundaries principle:** *For the solutions of the homogeneous equation*

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = 0, \ t \in [0, \omega], \tag{15.2.38}$$

*at least one of the inequalities* $x(0) \le x(t) \le x(\omega)$ *or* $x(\omega) \le x(t) \le x(0)$ *is fulfilled.*

This means that the maximum and the minimum values of the solution can only be attained at the points 0 or $\omega$.

If the operator $B$ is positive (negative) and the solution $x$ is positive, then this solution decreases (increases) and the maximum boundaries principle is true. This demonstrates that in many cases nonoscillation defined in such a way implies the maximum boundaries principle. In order to estimate the nonoscillation interval, we can use the maximum inequalities principle: if we know that this principle is true and find out the absolutely continuous positive function $z$ such that $(Mz)(t) \le 0$ for $t \in [0, \omega]$, then every nontrivial solution $x$ of the homogeneous equation (15.2.38) is either positive or negative for $t \in [0, \omega]$. The maximum principle in the form of theorems on inequalities, which are based on the definition of nonoscillation introduced in this chapter, plays the central role in this area.

## 15.3 Maximum Principles in the Case of Positive Volterra Operator $(-B)$

Let us consider boundary value problems described by the scalar equation

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t), \ t \in [0, \omega], \tag{15.3.1}$$

and one of the following boundary conditions:

$$x(0) = c, \tag{15.3.2}$$

$$x(\omega) = c, \tag{15.3.3}$$

$$lx \equiv \theta x(0) + \int_0^\omega \phi(s) x'(s) ds = c. \tag{15.3.4}$$

Let us assume below that the operator $B : C[0, \omega] \to L_1[0, \omega]$ admits the representation $B = B^+ - B^-$, where $B^+ : C[0, \omega] \to L_1[0, \omega]$ and $B^- : C[0, \omega] \to L_1[0, \omega]$ are positive operators.

It is known that every $u$-bounded operator $B : C[0, \omega] \to L_1[0, \omega]$ (see Definition A.6) can be presented in the form (see Theorem A.11)

$$(Bx)(t) = \int_0^t x(s) d_s b(t, s), \ t \in [0, \omega], \tag{15.3.5}$$

and, consequently, the equation

$$(Mx)(t) \equiv x'(t) + \int_0^t x(s) d_s b(t, s) = f(t), \ t \in [0, \omega], \tag{15.3.6}$$

is a form of the representation of (15.3.1). Here the function $b(\cdot, s) : [0, \omega] \to \mathbb{R}$ is measurable for $s \in [0, \omega]$, $b(t, \cdot) : [0, \omega] \to \mathbb{R}$ has a bounded variation for almost all $t \in [0, \omega]$ and $\mathrm{Var}_{s\in[0,t]}\, b(t, s)$ is integrable. It is clear that in this case the operator $B : C[0, \omega] \to L_1[0, \omega]$ admits the representation $B = B^+ - B^-$.

The general solution of (15.3.1) with Volterra operator $B$ can be represented in the form

$$x(t) = \int_0^t C(t, s) f(s) ds + C(t, 0)x(0), \tag{15.3.7}$$

where $C(t, s)$ is called the Cauchy function of (15.3.1). Note that for (15.3.6) $C(t, s)$ as a function of $t$ for each $s$ is a solution of the equation

$$(M_s x)(t) \equiv x'(t) + \int_s^t x(s) d_s b(t, s) = 0,\ \ t \in [s, \omega], \tag{15.3.8}$$

satisfying the condition $C(s, s) = 1$.

**Theorem 15.1** *If* $(-B) : C[0, \omega] \to L_1[0, \omega]$ *is a positive Volterra operator, then* $C(t, s) > 0$ *for* $0 \le s \le t \le \omega$ *and solutions of the homogeneous equation* $Mx = 0$ *do not decrease.*

*Proof* The proof follows from the inequality

$$C'_t(t, s) = -\int_s^t C(t, \xi) d_\xi b(t, \xi) \ge 0,\ \ t \in [s, \omega]. \tag{15.3.9}$$

□

The proof of the following corollary is based on the Fredholm alternative for functional differential equations.

**Lemma 15.1** [29] *Boundary value problem* (15.3.1), (15.3.10) *has a unique solution for each* $f \in L_1[0, \omega]$, $c \in \mathbb{R}$ *if and only if the homogeneous problem* $Mx = 0$, $lx = 0$, *has only the trivial solution.*

**Corollary 15.1** *If* $(-B) : C[0, \omega] \to L_1[0, \omega]$ *is a positive Volterra operator and* $l : C[0, \omega] \to \mathbb{R}$ *is a positive nonzero functional, then boundary value problem* (15.3.1) *with the condition*

$$lx = c \tag{15.3.10}$$

*has a unique solution for each* $f \in L_1[0, \omega]$, $c \in \mathbb{R}$.

*Proof* If we suppose that a nontrivial solution $x$ of this homogeneous problem exists, then the assumption on positivity of the functional $l : C[0, \omega] \to \mathbb{R}$ leads us to the inequality $lx > 0$. This contradicts the existence of the nontrivial solution $x$ of the homogeneous problem $Mx = 0$, $lx = 0$. □

Note that it is not assumed in the following assertions that the interval $[0, \omega]$ is short enough.

Let us write boundary condition (15.3.10) in the form $lx = x(\omega) - mx$, where $m : C[0, \omega] \to \mathbb{R}$ is a linear bounded functional, and consider the boundary value problem

$$(Mx)(t) = f(t),\ t \in [0, \omega],\ x(\omega) - mx = c. \tag{15.3.11}$$

**Corollary 15.2** *If $(-B) : C[0, \omega] \to L_1[0, \omega]$ is a positive Volterra operator and $\|m\| < 1$, then boundary value problem (15.3.11) has a unique solution for each $f \in L_1[0, \omega]$, $c \in \mathbb{R}$.*

*Proof* The statement of the corollary follows from the fact that solutions of the homogeneous equation $(Mx)(t) = 0$ do not decrease. □

Consider the problem

$$(Mx)(t) = f(t),\ t \in [0, \omega],\ \sum_{j-1}^{2k} \alpha_j x(t_j) = c,\ 0 \le t_1 < t_2 < \cdots < t_{2k} \le \omega. \tag{15.3.12}$$

**Corollary 15.3** *If $(-B) : C[0, \omega] \to L_1[0, \omega]$ is a positive Volterra operator, $0 \le -\alpha_{2j-1} \le \alpha_{2j}$, $j = 1, \cdots, k$, and there exists $i$ such that $-\alpha_{2i-1} < \alpha_{2i}$, then boundary value problem (15.3.12) has a unique solution for each $f \in L_1[0, \omega]$, $c \in \mathbb{R}$.*

*Proof* The statement of the corollary follows from the fact that solutions of the homogeneous equation $(Mx)(t) = 0$ do not decrease and $t_1 < t_2 < \cdots < t_{2k}$. □

*Remark 15.1* The periodic boundary value problem

$$x'(t) = f(t),\ t \in [0, \omega],\ x(0) - x(\omega) = c, \tag{15.3.13}$$

demonstrates that the condition $\|m\| < 1$ in Corollary 15.2 and the condition on existence of such $i$ that $-\alpha_{2i-1} < \alpha_{2i}$ cannot be improved.

More advanced results can be obtained on the basis of upper and lower functions estimating the solutions of the initial value problem

$$x'(t) - p(t)x\big(h(t)\big) = 0,\ t \in [0, \omega], \tag{15.3.14}$$

$$x(\xi) = 0 \text{ for } \xi < 0, \tag{15.3.15}$$

$$x(0) = 1. \tag{15.3.16}$$

**Lemma 15.2** *Let $p \ge 0$ and the number $k$ satisfy the inequality*

$$\int_{h(t)}^{t} p(s)\chi\big(h(s)\big)ds \le \frac{e}{k}(1 - \ln k), \tag{15.3.17}$$

*where*

$$\chi(t) = \begin{cases} 1, & t \ge 0, \\ 0, & t < 0. \end{cases} \tag{15.3.18}$$

*Then the solution of* (15.3.14)–(15.3.16) *satisfies the inequality*

$$v(t) \le x(t) \le u(t), \tag{15.3.19}$$

*where*

$$v(t) = \exp\left\{\frac{k}{e}\int_0^t p(s)\chi\big(h(s)\big)ds\right\}, \tag{15.3.20}$$

$$u(t) = \exp\left\{\int_0^t p(s)\chi\big(h(s)\big)ds\right\}. \tag{15.3.21}$$

*Proof* The proof follows from the positivity of the Cauchy function $C(t,s)$ for $0 \le s \le t \le \omega$ and the inequalities $Mv \le 0$ and $Mu \ge 0$. For example, substituting $v$ defined in (15.3.20) into the differential operation

$$(Mx)(t) \equiv x'(t) - p(t)x\big(h(t)\big),$$

we obtain

$$\begin{aligned} &\exp\left\{\frac{k}{e}\int_0^t p(s)\chi\big(h(s)\big)ds\right\}\frac{k}{e}p(t)\chi\big(h(t)\big) \\ &\quad - p(t)\chi\big(h(t)\big)\exp\left\{\frac{k}{e}\int_0^{h(t)} p(s)\chi\big(h(s)\big)ds\right\} \\ &= p(t)\chi\big(h(t)\big)\exp\left\{\frac{k}{e}\int_0^t p(s)\chi\big(h(s)\big)ds\right\} \\ &\quad \times \left[\frac{k}{e} - \exp\left\{-\frac{k}{e}\int_{h(t)}^t p(s)\chi\big(h(s)\big)ds\right\}\right]. \end{aligned}$$

The inequality $(Mv)(t) \le 0$ is equivalent to

$$\frac{k}{e} \le \exp\left\{-\frac{k}{e}\int_{h(t)}^t p(s)\chi\big(h(s)\big)ds\right\},$$

or

$$\ln\frac{k}{e} \le -\frac{k}{e}\int_{h(t)}^t p(s)\chi\big(h(s)\big)ds,$$

which can be rewritten as (15.3.17). □

Let us write the functional $l : C[0,\omega] \to \mathbb{R}$ in the form $l = l^+ - l^-$, where $l^+ : C[0,\omega] \to \mathbb{R}$, $l^- : C[0,\omega] \to \mathbb{R}$ are positive linear functionals.

**Theorem 15.2** *If $p(t) \ge 0$, $h(t) \le t$ and $l^+v > l^-u$, where $v$ and $u$ are defined by the formulas* (15.3.20) *and* (15.3.21), *respectively, then boundary value problem* (15.3.14), (15.3.10) *has a unique solution for each $f \in L_1[0,\omega]$, $c \in \mathbb{R}$.*

*Proof* Existence of a nontrivial solution $x$ of the homogeneous problem $Mx = 0$, $lx = 0$, contradicts the fact that $lx = l^+x - l^-x \ge l^+v - l^-u > 0$. □

## 15.4 Nonoscillation and Positivity of Green's Functions for Positive Volterra Operator *B*

Let us consider boundary value problems described by the scalar equation

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t), \ t \in [0, \omega], \tag{15.4.1}$$

and one of the following boundary conditions:

$$x(0) = c, \tag{15.4.2}$$

$$x(\omega) = c, \tag{15.4.3}$$

$$lx \equiv \theta x(0) + \int_0^{\omega} \phi(s)x'(s)ds = c. \tag{15.4.4}$$

Let us obtain the Green's function and the solution representation for boundary value problem (15.4.1), (15.4.3). When we set $t = \omega$ in the formula (15.3.7), we obtain

$$x(\omega) = \int_0^{\omega} C(\omega, s)f(s)ds + C(\omega, 0)x(0),$$

and then, using the boundary condition (15.4.3), in the case $C(\omega, 0) \neq 0$ we can express $x(0) = (c - \int_0^{\omega} C(\omega, s)f(s)ds)/C(\omega, 0)$ and

$$x(\omega) = \int_0^{\omega} \left\{C(t, s) - \frac{C(\omega, s)}{C(\omega, 0)}C(t, 0)\right\} f(s)ds + \frac{c}{C(\omega, 0)}C(t, 0), \tag{15.4.5}$$

where $C(t, s) = 0$ if $t < s$.

Using Lemma 15.1, the fact that the space of solutions of the first-order equation $Mx = 0$ is one-dimensional and that $x(t) = C(t, 0)$ is its solution, we obtain the following assertion.

**Lemma 15.3** *If a nontrivial solution $x$ of the homogeneous equation $Mx = 0$ does not have zero at the point $t = \omega$, then the boundary value problem* (15.4.1), (15.4.3) *has a unique solution, which has the representation* (15.4.5), *and its Green's function is of the form*

$$G(t, s) = C(t, s) - \frac{C(\omega, s)}{C(\omega, 0)}C(t, 0), \tag{15.4.6}$$

*where $C(t, s) = 0$ if $t < s$.*

Let us define the operator $N : C[0, \omega] \to C[0, \omega]$ by the formula

$$(Nx)(t) = \int_t^{\omega} (Bx)(s)ds \tag{15.4.7}$$

and assume that the operator $B : C[0, \omega] \to L_1[0, \omega]$ admits representation (15.3.5). Introduce the operator $B_s : C[s, \omega] \to \mathbb{R}$ as

$$(B_s y)(t) = \int_s^t y(\xi)d_{\xi}b(t, \xi).$$

We will denote

$$Be^{\int_s^t u(\xi)d\xi} = \int_0^t e^{\int_s^t u(\xi)d\xi} d_s b(t,s).$$

Let us formulate the known result of G.G. Islamov [210, 211] (see Theorem A.12) in a convenient form.

**Lemma 15.4** *Suppose that there exists a nonnegative continuous function $v$ such that $\psi(t) \equiv v(t) - \int_t^\omega (Bv)(s)ds$, where $B$ is a Volterra operator and $\psi(t) > 0$ for $t \in [0,\omega)$. Then the spectral radius $\rho(N)$ of the operator $N : C[0,\omega] \to C[0,\omega]$ is less than one.*

*Proof* To prove the statement of the lemma, we have only to note that the operator $B : C[0,\omega] \to L_1[0,\omega]$ is a Volterra one and to apply Theorem A.12. □

**Definition 15.2** Let us say that problem (15.4.1), (15.4.4) *satisfies the condition $\Theta$* if

$$\frac{\int_s^\omega \phi(\xi)C'_\xi(\xi,s)d\xi + \phi(s)}{\theta + \int_0^\omega \phi(s)C'_s(s,0)ds} < 0. \tag{15.4.8}$$

*Remark 15.2* It will be demonstrated in Sects. 15.7 and 15.9 that for a wide class of generalized periodic boundary value problems the condition $\Theta$ is fulfilled. Here let us discuss only problems with a general form of boundary condition. Let us assume that $\theta > 0$ and $\phi(s) < -\varepsilon < 0$. Then it will be demonstrated that on nonoscillation interval $C(t,s) > 0$ and consequently in the case of the positive operator $B$ the derivative satisfies the inequality $C'_t(t,s) \le 0$ for $0 \le s \le t \le \omega$. It is obvious that the denominator is positive. The numerator will be negative if the interval $[0,\omega]$ is small enough.

For (15.4.1), we propose the following statement on eight equivalences.

**Theorem 15.3** *Let $B : C[0,\omega] \to L_1[0,\omega]$ be a positive Volterra operator. Then the following seven hypotheses are equivalent*:

1) *There exists a nonnegative absolutely continuous function $v$ such that*

$$v(\omega) - \int_t^\omega (Mv)(s)ds > 0,\ t \in [0,\omega). \tag{15.4.9}$$

2) *The spectral radius of the operator $N : C[0,\omega] \to C[0,\omega]$ is less than one.*
3) *Problem* (15.4.1), (15.4.3) *has a unique solution, and its Green's function $G(t,s)$ is negative for $0 \le t < s \le \omega$ and nonpositive for $0 \le s \le t \le \omega$.*
4) *A nontrivial solution of the homogeneous equation $(Mx)(t) = 0$, $t \in [0,\omega]$ has no zeros on $[0,\omega]$.*
5) *The Cauchy function $C(t,s)$ of* (15.4.1) *is positive for $0 \le s \le t \le \omega$.*
6) *There exists a positive continuous function $v$ such that $v(t) > Nv(t)$, $t \in [0,\omega)$.*

7) *There exists a positive essentially bounded function* $u$ *such that*

$$Be^{\int_s^t u(\xi)d\xi}(t) \leq u(t), \; t \in [0, \omega]. \tag{15.4.10}$$

*If in addition the condition* $\Theta$ *is fulfilled, then the following assertion is included in the list of equivalences*:

8) *Problem* (15.4.1), (15.4.4) *has a unique solution, and its Green's function* $P(t,s)$ *is positive for* $t, s \in [0, \omega]$.

*Proof* 1) $\Rightarrow$ 6) The function $v$ satisfies the equation $v = Nv + \psi$, where $\psi(t) = v(\omega) - \int_t^\omega (Mv)(s)ds$, $t \in [0,\omega]$, and inequality (15.4.9) implies that $\psi(t) > 0$ for $t \in [0, \omega)$.

The implication 6) $\Rightarrow$ 2) follows from Lemma A.2.

2) $\Rightarrow$ 3) The equation $x = Nx + g$, where $g = -\int_t^\omega f(s)ds$, is equivalent to problem (15.4.1), (15.4.3) for $t \in [0, \omega]$. The condition that the spectral radius of the operator $N$ be less than one implies that this problem has a unique solution. Its solution can be represented in the form

$$x(t) = g(t) + \int_0^\omega \big\{G(t,s) - G_0(t,s)\big\} f(s)ds, \tag{15.4.11}$$

where $G_0(t,s)$ is the Green's function of the boundary value problem

$$x'(t) = f(t), \; t \in [0,\omega], \; x(\omega) = 0. \tag{15.4.12}$$

We can obtain that $G_0(t,s) = -1$ for $0 \leq t < s \leq \omega$ and $G_0(t,s) = 0$ for $0 \leq s \leq t \leq \omega$. If $f \leq 0$, then $0 \leq g(t) \leq x(t)$ for $t \in [0,\omega]$. Now it follows from equality (15.4.11) that $G(t,s) \leq G_0(t,s)$ for $t, s \in [0, \omega]$.

3) $\Rightarrow$ 1) The function $v(t) = -\int_0^\omega G(t,s)ds$ satisfies condition 1).

5) $\Rightarrow$ 4) This assertion follows from the fact that $C(t,0)$ is a nontrivial solution of equation $Mx = 0$.

In order to prove 4) $\Rightarrow$ 1), we substitute $v(t) = C(t,0)$ into condition 1).

1) $\Rightarrow$ 5) The function $v(t)$ satisfies the inequalities

$$v(\omega) - \int_t^\omega \left[ v'(s_1) + \int_s^{s_1} x(\xi) d_\xi b(s_1, \xi) \right] ds_1 > 0 \tag{15.4.13}$$

for each $s$. This means that the spectral radius of the operators

$$(N_s x)(t) = \int_t^\omega (Bx)(\xi)d\xi, \; t \in [s, \omega], \tag{15.4.14}$$

is less than one for each $s \in [0, \omega)$ according to implications 1) $\Rightarrow$ 6) $\Rightarrow$ 2). Each function $C(\cdot, s)$ for a fixed $s$ is a solution of (15.3.8). The assumption $C(t, s_0) = 0$ contradicts the fact that the spectral radius of $N_{s_0}$ is less than one.

In order to prove the assertion 4) $\Rightarrow$ 7), we set $u(t) = -\frac{d}{dt} \ln C(t, 0)$.

7) $\Rightarrow$ 4) The operator $(Fy)(t) = (Be^{\int_\cdot^t y(s)ds})(t)$ is monotone, and

$$0 \leq F0 \leq Fu \leq u.$$

There exists a solution $y$ of the equation $y = Fy$ such that $0 \leq y \leq u$. Thus

$$C(t,0) = e^{-\int_0^t y(s)ds}$$

is a positive solution of the homogeneous equation $Mx = 0$.

Let us prove 5) ⇒ 8). It can be demonstrated that in the case

$$\theta + \int_0^{\omega} \phi(s) C_s'(s,0) ds \neq 0$$

the solution of boundary value problem (15.4.1), (15.4.4) can be presented in the form

$$x(t) = \int_0^{\omega} P(t,s) f(s) ds + \frac{c}{\theta + \int_0^{\omega} \phi(s) C_s'(s,0) ds}. \tag{15.4.15}$$

Here the Green's function of problem (15.4.1), (15.4.4) has the representation

$$P(t,s) = C(t,s) - \frac{\int_s^{\omega} \phi(\xi) C_{\xi}'(\xi,s) d\xi + \phi(s)}{\theta + \int_0^{\omega} \phi(s) C_s'(s,0) ds} C(t,0), \tag{15.4.16}$$

where $C(t,s) = 0$ if $t < s$. If $C(t,s)$ is positive for $0 \leq s \leq t \leq \omega$ and the condition $\Theta$ is fulfilled, then $P(t,s) > 0$ for $t, s \in [0,\omega]$.

8) ⇒ 4) If $P(t,s) > 0$ for $t, s \in [0,\omega]$, then the solution $x(t) = C(t,0)$ should be positive for $t \in [0,\omega]$. If not, $C(t_0,0) = 0$ for a corresponding $t_0$, and then $P(t_0,s) = 0$ for $s > t_0$ and we get a contradiction, which completes the proof. □

If we substitute $v = 1$ in assertion 6), then the following corollary is obtained.

**Corollary 15.4** *Let $B : C[0,\omega] \to L_1[0,\omega]$ be a positive Volterra operator and*

$$\int_0^{\omega} (B1)(s) ds < 1, \ t \in [0,\omega]. \tag{15.4.17}$$

*Then assertions* 1)–7) *are fulfilled. If in addition the condition $\Theta$ is satisfied, then assertion* 8) *is also fulfilled.*

*Example 15.5* The equation

$$x'(t) + x(0) = 0, \ t \in [0,1], \tag{15.4.18}$$

has a solution $x(t) = 1 - t$ vanishing at the point $\omega = 1$, which demonstrates that the inequality in Corollary 15.4 is sharp.

*Remark 15.3* Positivity of the operator $B$ is essential for nonpositivity of Green's function $G(t,s)$ of problem (15.4.1), (15.4.3), as the following example [178] demonstrates.

Consider the equation

$$x'(t) - x(0) = f(t), \ t \in [0,\omega], \tag{15.4.19}$$

with the boundary condition (15.4.3). All nontrivial solutions of the homogeneous equation

$$x'(t) - x(0) = 0$$

are proportional to $x(t) = 1 + t$; i.e., they are either positive or negative. According to formula (15.4.6), the Green's function of the problem (15.4.19), (15.4.3) can be constructed as

$$G(t,s) = \begin{cases} -\frac{1+t}{1+\omega}, & t < s, \\ 1 - \frac{1+t}{1+\omega}, & t > s > 0, \\ 0, & t > s = 0. \end{cases} \tag{15.4.20}$$

We see that the Green's function $G(t,s)$ changes its sign in each rectangle $(t,s) \in [0,\omega] \times [0,\omega]$. This is impossible for ordinary differential equations, where $G(t,s) < 0$ for $t < s$ and $G(t,s) = 0$ for $t > s$.

We can say even more, using for (15.3.6) the so-called generalized semigroup equality (see [328, p. 11])

$$C(\omega,0) = C(\omega,s)C(s,0) - \int_s^\omega C(\omega,\xi)\left[\int_0^s C(\eta,0)d_\eta b(\xi,\eta)\right]d\xi. \tag{15.4.21}$$

Consider now the formula of Green's function (15.4.6) for problem (15.3.6), (15.4.3). It is clear that $G(\cdot,s)$ is right continuous at the point $t = s$. If we set $t = s$ in (15.4.6), the following equality is obtained:

$$G(s,s) = 1 - \frac{C(\omega,s)}{C(\omega,0)}C(s,0) = \frac{C(\omega,0) - C(s,0)C(\omega,s)}{C(\omega,0)}. \tag{15.4.22}$$

If $b(t,\cdot)$ does not increase for $t \in [0,\omega]$, then according to (15.4.21) we get $C(\omega,0) \ge C(\omega,s)C(s,0)$ and, according to equality (15.4.22), $G(s,s) \ge 0$ for $s \in (0,\omega)$; moreover, $G(s,s) > 0$ if $\int_s^\omega [\int_0^s d_\eta b(\xi,\eta)]d\xi < 0$.

Thus, we have proven the following result.

**Theorem 15.4** *If* $p(t) \le 0$ *and* $\operatorname{mes}\{t \in [0,\omega] \mid p(t) < 0, 0 \le h(t) < t\} > 0$, *then the Green's function of the problem*

$$x'(t) + p(t)x\big(h(t)\big) = f(t), \ t \in [0,\omega], \ x(\omega) = 0, \tag{15.4.23}$$

*changes its sign in the square* $(t,s) \in (0,\omega) \times (0,\omega)$.

*Remark 15.4* Let us consider the following hypothesis:

5*) The Cauchy function $C(t,s)$ of (15.4.1) is positive for $0 < s \le t \le \omega$.

Assertions 5) and 5*) seem almost the same. The difference is only in the non-strong inequality $0 \le s \le t \le \omega$ in assertion 5). However, Theorem 15.3 will not be true if instead of assertion 5) we set 5*) since 5*) does not imply assertion 4) on nonoscillation of a nontrivial solution on the interval $[0,\omega]$, as the following example demonstrates.

*Example 15.6* Consider the equation

$$x'(t) + x(0) = f(t), \ t \in [0,\omega]. \tag{15.4.24}$$

Its Cauchy function is $C(t,s)=1>0$ for $0<s\le t\le\omega$. The nontrivial solution of the homogeneous equation

$$x'(t)+x(0)=0,\ t\in[0,\omega], \tag{15.4.25}$$

is $x(t)=C(t,0)=1-t$, which changes its sign at the point $t=1$.

*Remark 15.5* It is essential that the operator $B$ be a Volterra operator. For equations with non-Volterra operators, assertion 5) does not follow from hypothesis 4). Consider the initial value problem for the equation

$$x'(t)+x(\omega)=f(t),\ t\in[0,\omega].$$

The function $x(t)=\omega+1-t$ is the positive solution of the homogeneous equation

$$x'(t)+x(\omega)=0,\ t\in[0,\omega].$$

We can construct the Cauchy function $C(t,s)$ as

$$C(t,s)=-\frac{1}{1+\omega}+\gamma(t,s),$$

where

$$\gamma(t,s)=\begin{cases}1, & 0\le s\le t\le\omega,\\ 0, & 0\le t<s\le\omega.\end{cases}$$

The function $C(t,s)$ changes its sign in the domain $(t,s)\in[0,\omega]\times[0,\omega]$ for each $\omega$.

**Theorem 15.5** *Let $B$ be a positive operator and $[0,\omega]$ be a nonoscillation interval of the equation*

$$x'(t)+(Bx)(t)=0,\ t\in[0,\omega]. \tag{15.4.26}$$

*Then its solution $x$ and the Cauchy function $C(t,s)$ satisfy the inequalities*

$$-|x(0)|\exp\left\{\int_0^t -(B1)(s)ds\right\}\le x(t)\le |x(0)|\exp\left\{-\int_0^t (B1)(s)ds\right\},\ 0\le t\le\omega, \tag{15.4.27}$$

$$\exp\left\{-e\int_s^t (B_s1)(\xi)d\xi\right\}\le C(t,s)\le \exp\left\{-\int_s^t (B_s1)(\xi)d\xi\right\},\ 0\le s\le t\le\omega. \tag{15.4.28}$$

*Proof* The proof follows from the equivalence of nonoscillation and positivity of the Cauchy function $C(t,s)$ and the inequalities $M_sv_s(t)\le 0$, $(Mu)(t)\ge 0$, $M_su_s(t)\ge 0$, where

$$\begin{aligned} v_s(t)&=\exp\left\{-e\int_s^t (B_s1)(s)ds\right\},\\ u(t)&=\exp\left\{-\int_0^t (B1)(s)ds\right\}, \qquad (15.4.29)\\ u_s(t)&=\exp\left\{-\int_s^t (B_s1)(s)ds\right\}. \end{aligned}$$

□

## 15.5 Nonoscillation on the Semiaxis

The most interesting case is when the behavior of solutions of functional differential equations is similar to the behavior of first-order ordinary differential equations on the semiaxis. It can be demonstrated that in this case various results on the maximum boundedness principle on the semiaxis and the exponential stability are obtained.

Consider the equation

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t),\ t \in [0, \infty), \tag{15.5.1}$$

where $B : C[0, \infty) \to L_\infty[0, \infty)$ is a Volterra operator, $C[0, \infty)$ is the space of continuous functions and $L_\infty[0, \infty)$ is the space of essentially bounded functions defined on $[0, \infty)$, $f \in L_\infty[0, \infty)$. Assume that the operator $B$ admits the representation in the form of the Stieltjes integral

$$(Bx)(t) = \int_0^t x(s) d_s b(t, s),\ t \in [0, \infty), \tag{15.5.2}$$

and, consequently, the equation

$$(Mx)(t) \equiv x'(t) + \int_0^t x(s) d_s b(t, s) = f(t),\ t \in [0, \infty), \tag{15.5.3}$$

is a form of the representation of (15.5.1). Here the function $b(\cdot, s) : [0, \omega] \to \mathbb{R}$ is measurable for $s \in [0, \omega]$, $b(t, \cdot) : [0, \omega] \to \mathbb{R}$ has a bounded variation for almost all $t \in [0, \omega]$ and $\mathrm{Var}_{s\in[0,t]}\, b(t, s)$ is essentially bounded for every $\omega > 0$.

Theorem 15.3 can be extended to this equation on the semiaxis.

**Theorem 15.6** *Let $B : C[0, \infty) \to L_\infty[0, \infty)$ be a positive Volterra operator. Then the following hypotheses are equivalent*:

1) *There exists a function $v$ positive absolutely continuous on each interval $[0, \omega]$ with essentially bounded derivative $v'$ such that*

$$(Mv)(t) \le 0,\ t \in [0, \infty). \tag{15.5.4}$$

2) *For each $\omega \in (0, \infty)$, the spectral radius of the operator $N$ defined by the formula*

$$(Nx)(t) = \int_t^\omega \left\{ \int_0^s x(\xi) d_\xi b(s, \xi) \right\} ds \tag{15.5.5}$$

*is less than one*.

3) *For each $\omega \in (0, \infty)$, the problem*

$$x'(t) + \int_0^t x(s) d_s b(t, s) = f(t),\ t \in [0, \omega], \tag{15.5.6}$$

$$x(\omega) = 0, \tag{15.5.7}$$

*has a unique solution, and its Green's function $G(t, s)$ is negative for $0 \le t < s \le \omega$ and nonpositive for $0 \le s \le t \le \omega$.*

4) *A nontrivial solution of the homogeneous equation* $(Mx)(t)=0$, $t\in[0,\infty)$, *has no zeros on* $[0,\infty)$.
5) *The Cauchy function* $C(t,s)$ *of* (15.5.1) *is positive for* $0\le s\le t<\infty$.
6) *There exists a positive continuous function* $v$ *such that* $v(t)>Nv(t)$, $t\in[0,\infty)$.
7) *There exists a positive essentially bounded function* $u$ *such that*

$$Be^{\int_s^t u(\xi)d\xi}(t)\le u(t),\ t\in[0,\infty). \tag{15.5.8}$$

*If in addition the condition* $\Theta$ *is fulfilled for each* $\omega\in(0,\infty)$, *then the following assertion is included in the list of equivalences*:

8) *For each* $\omega\in(0,\infty)$, *the problem* (15.4.1), (15.4.4) *has a unique solution and its Green's function* $P(t,s)$ *is positive for* $t,s\in[0,\omega]$.

*Proof* We repeat the proof of Theorem 15.3 according to the scheme 1) ⇒ 6) ⇒ 2) ⇒ 3), 5) ⇒ 4) ⇒ 1) ⇒ 5). Only one implication, 3) ⇒ 1), cannot be used. Instead we can prove the implication 3) ⇒ 4). Actually, according to Lemma 15.1, it follows from the fact that the problem (15.5.6), (15.5.7) has a unique solution for every $\omega>0$ if the homogeneous problem (15.5.9), (15.5.7), where

$$x'(t)+\int_0^t x(s)d_sb(t,s)=0,\ t\in[0,\omega], \tag{15.5.9}$$

has only the trivial solution for every $\omega$. This means that nontrivial solutions of homogeneous equation (15.5.9) do not have zeros on $[0,\infty)$, which completes the proof. □

*Remark 15.6* Note that for the delay differential equation

$$x'(t)+p(t)x\big(t-\tau(t)\big)=0,\ t\in[0,\infty), \tag{15.5.10}$$

the inequality in assertion 7) is of the form

$$p(t)e^{\int_{t-\tau(t)}^t u(s)ds}(t)\le u(t),\ t\in[0,\infty), \tag{15.5.11}$$

and the equivalence of existence of nonoscillatory solution and inequality (15.5.4) is the well-known result (see [154, p. 29]).

## 15.6 Positivity Tests for Green's Functions Through Choice of $v(t)$

Corollary 15.4 actually claims the general principle that if the interval $[0,\omega]$ is small enough, then the behavior of solutions is similar to the behavior of solutions of ordinary differential equations; i.e., nontrivial solutions do not have zeros, and the Cauchy function $C(t,s)$ and the Green's function of the problem with boundary condition at $\omega$ preserve their signs. Note that Theorem 15.3 gives several possibilities (hypotheses 1), 6) and 7)) to obtain explicit conditions for the properties of Cauchy and Green's functions mentioned in this theorem.

In Corollary 15.4, assertion 6) was used. Choosing various functions $v$ and $u$ in assertions 1) and 7), respectively, we obtain results of another type, and their principle can be formulated as follows: if the delay $\tau$ is small enough, then the properties of solutions of the delay equation

$$x'(t) + p(t)x\big(t - \tau(t)\big) = f(t), \ t \in [0, \infty), \tag{15.6.1}$$

$$x(\xi) = 0 \text{ for } \xi < 0, \tag{15.6.2}$$

are similar to the properties of the ordinary differential equation

$$x'(t) + p(t)x(t) = f(t), \ t \in [0, \infty). \tag{15.6.3}$$

Consider the equation

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t), \ t \in [0, \infty), \tag{15.6.4}$$

where $B : C[0, \infty) \to L_\infty[0, \infty)$ is a positive Volterra operator.

**Definition 15.3** Let us determine *the function* $H : [0, \infty) \to [0, \infty)$ as the maximal possible value for which the equality $y_1(s) = y_2(s)$ for $s \in [H(t), \infty)$ implies the equality $(By_1)(s) = (By_2)(s)$ for $s \in [t, \infty)$ for each of the two continuous functions $y_1$ and $y_2 : [0, \infty) \to (-\infty, \infty)$.

It is clear that the function $H$ describes the "size" of the memory of the operator $B$. If we set $v = \exp\{-e \int_0^t (B1)(s)ds\}$ in assertion 1) of Theorem 15.6, then the following result is obtained.

**Theorem 15.7** *Let* $B : C[0, \infty) \to L_\infty[0, \infty)$ *be a positive linear Volterra operator and*

$$\int_{H(t)}^t (B1)(s)ds \le \frac{1}{e}, \ t \in (0, \infty). \tag{15.6.5}$$

*Then nontrivial solutions of the homogeneous equation* $Mx = 0$ *have no zeros for* $t \in [0, \infty)$, $C(t, s) > 0$ *for* $0 \le s \le t < \infty$, *and if in addition the condition* $\Theta$ *is fulfilled, then problem* (15.4.1), (15.4.4) *has a unique solution and its Green's function* $P(t, s)$ *is positive for* $t, s \in [0, \omega]$ *and any* $\omega \in (0, \infty)$.

Consider the particular case of (15.6.4)

$$(Mx)(t) \equiv x'(t) + \sum_{k=1}^m p_k(t)x\big(h_k(t)\big) = f(t), \ t \in [0, \infty), \tag{15.6.6}$$

where $x(s) = 0$ for $s < 0$. Assume that $h_k(t) \le t, \ p_k(t) \ge 0$ for $t \in [0, \infty)$.

For (15.6.6), the function $H(t)$ in Definition 15.3 can be determined as follows: $H(t) = \min_{1 \le k \le m} h_k(t)$, and inequality (15.6.5) has the form

$$\int_{H(t)}^t \sum_{k=1}^m p_k(s)ds \le \frac{1}{e}, \ t \in (0, \infty). \tag{15.6.7}$$

*Remark 15.7* For the equation

$$x'(t) + px(t-\tau) = 0, \ t \in [0, \infty),$$

where $p$ and $\tau$ are positive constants, an opposite inequality $p\tau > \frac{1}{e}$ implies oscillation of all solutions, see, for example, [154]. This demonstrates that inequality (15.6.7) is sharp.

Consider the equation

$$(Mx)(t) \equiv x'(t) + \sum_{k=1}^{\infty} p_k(t)x(t-k\tau) = f(t), \ t \in [0, \infty), \tag{15.6.8}$$

where $x(s) = 0$ if $s < 0$.

**Theorem 15.8** *Suppose that there exists $\beta$ such that*

$$\beta < \frac{1}{e}, \ p_1 \geq \beta p_2, \ p_1 \geq \beta^2 p_3, \cdots, \ p_1 \geq \beta^{k-1} p_k, \cdots, \tag{15.6.9}$$

*and*

$$p_1\tau + \beta \leq \frac{1}{e} \tag{15.6.10}$$

*is satisfied for* (15.6.8). *Then nontrivial solutions of the homogeneous equation $Mx = 0$ have no zeros for $t \in [0, \infty)$, $C(t, s) > 0$ for $0 \leq s \leq t < \infty$, and if in addition the condition $\Theta$ is fulfilled, then the problem* (15.6.8), (15.4.4) *has a unique solution and its Green's function $P(t, s)$ is positive for $t, s \in [0, \omega]$ and each $\omega \in (0, \infty)$.*

*Proof* Let us substitute $v(t) = e^{-\alpha t}$ into hypothesis 1) of Theorem 15.3. We obtain the inequality

$$(Mv)(t) \equiv e^{-\alpha t}\left\{-\alpha + \sum_{k=1}^{\infty} p_k(t)e^{\alpha k\tau}\right\} \leq 0, \ t \in [0, \infty).$$

Using the condition (15.6.9), we can see that this inequality is satisfied if

$$\frac{p_1}{1 - \beta e^{\alpha\tau}} \leq \alpha e^{-\alpha\tau}. \tag{15.6.11}$$

The function $g(\alpha) = \alpha e^{-\alpha\tau}$ on the right-hand side of this inequality is maximal at $\alpha = \frac{1}{\tau}$. If we substitute this value of $\alpha$ into inequality (15.6.11), the inequality

$$\frac{p_1\tau}{1 - \beta e} \leq \frac{1}{e} \tag{15.6.12}$$

is obtained and (15.6.10) is satisfied. □

Let us consider the equation

$$(Mx)(t) \equiv x'(t) + \int_0^{t-\tau} K(t,s)x(s) = f(t),\ t \in [0,\infty), \tag{15.6.13}$$

where $K(t,s)$ is a positive continuous function satisfying the inequality

$$K(t,s) \le be^{-\gamma(t-s)},\ 0 \le s \le t < \infty,\ \gamma,\ b > 0. \tag{15.6.14}$$

**Theorem 15.9** *Let inequalities* (15.6.14) *and*

$$b \le \frac{\gamma^2}{4} e^{\frac{\gamma}{2}\tau} \tag{15.6.15}$$

*be satisfied. Then nontrivial solutions of the homogeneous equation $Mx = 0$ have no zeros for $t \in [0,\infty)$, $C(t,s) > 0$ for $0 \le s \le t < \infty$, and if in addition the condition $\Theta$ is fulfilled, then problem* (15.6.13), (15.4.4) *has a unique solution and its Green's function $P(t,s)$ is positive for $t, s \in [0,\omega]$.*

*Proof* Substituting $v(t) = e^{-\alpha t}$, where $\gamma > \alpha$, into hypothesis 1) of Theorem 15.6, we obtain

$$\left(Me^{-\alpha t}\right)(t) \le -\alpha + \frac{b}{\gamma - \alpha} e^{-(\gamma-\alpha)\tau},\ 0 \le t < \infty. \tag{15.6.16}$$

The right-hand side of (15.6.16) is nonpositive if

$$be^{-(\gamma-\alpha)\tau} \le \alpha(\gamma - \alpha). \tag{15.6.17}$$

Choosing $\alpha = \gamma/2$, we obtain that inequality (15.6.15) implies nonpositivity of $(Me^{-\alpha t})(t)$ for $0 \le t < \infty$. □

*Remark 15.8* If $K(t,s) = be^{-\gamma(t-s)}$ for $0 \le s \le t < \infty$, $\gamma,\ b > 0$ and $\tau = 0$, then the inequality (15.6.15) becomes

$$b \le \frac{\gamma^2}{4}, \tag{15.6.18}$$

which is a necessary and sufficient condition for nonoscillation of the solutions of (15.6.13) [132].

## 15.7 The Generalized Periodic Problem for Positive Volterra Operator *B*

Consider the generalized periodic problem

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t),\ t \in [0,\omega], \tag{15.7.1}$$

$$x(0) - \beta x(\omega) = c. \tag{15.7.2}$$

From the formula (15.4.8) representing the Green's function in the case of the general functional $l : D[0,\omega] \to \mathbb{R}$ in the boundary condition we get the following representation for the Green's function of problem (15.7.1), (15.7.2):

$$P(t,s) = C(t,s) + \frac{\beta C(\omega,s)}{1-\beta C(\omega,0)} C(t,0). \tag{15.7.3}$$

The nontrivial solution $x(t) = C(t,0)$ of the equation

$$x'(t) + (Bx)(t) = 0, \ t \in [0,\omega], \tag{15.7.4}$$

does not increase on a nonoscillation interval in the case of the positive operator $B : C[0,\omega] \to L_1[0,\omega]$. It is clear that the condition $\Theta$ (see Definition 15.2) on a nonoscillation interval is fulfilled if $\beta < 1$, and in the case $\beta = 1$ (the periodic problem) this condition is fulfilled if $B$ is a nonzero operator.

*Remark 15.9* It is essential that the operator $B$ be a Volterra operator. For equations with non-Volterra operators, hypothesis 8) of Theorem 15.3 on positivity of the Green's function of the periodic problem (noted in the case $\beta = 1$) does not follow from assertion 4). Consider the periodic problem for the equation

$$x'(t) + x(\omega) = f(t), \ t \in [0,\omega].$$

The function $x(t) = \omega + 1 - t$ is the positive solution of the homogeneous equation

$$x'(t) + x(\omega) = 0, \ t \in [0,\omega].$$

We can construct the Green's function of the periodic problem as

$$P(t,s) = \frac{1-t}{\omega} + \gamma(t,s),$$

where

$$\gamma(t,s) = \begin{cases} 1, \ 0 \le s \le t \le \omega, \\ 0, \ 0 \le t < s \le \omega. \end{cases}$$

The Green's function $P(t,s)$ is positive in the domain $(t,s) \in [0,\omega] \times [0,\omega]$ if and only if $\omega < 1$.

Now, using Theorem 15.5, we can check when the condition $\Theta$ is fulfilled in the case $\beta > 1$.

**Theorem 15.10** *Let $B : C[0,\omega] \to L_1[0,\omega]$ be a positive Volterra operator,* $[0,\omega]$ *be a nonoscillation interval of* (15.7.4), *and*

$$\beta < \exp\left\{\int_0^\omega (B1)(s)ds\right\}. \tag{15.7.5}$$

*Then the generalized periodic problem* (15.7.1), (15.7.2) *has a unique solution and its Green's function $P(t,s)$ is positive for $t,s \in [0,\omega]$.*

*Remark 15.10* Inequality (15.7.5) cannot be improved, as the following example demonstrates. Consider the generalized periodic boundary value problem (15.7.2) for the ordinary differential equation

$$x'(t) + p(t)x(t) = f(t), \ t \in [0, \omega]. \tag{15.7.6}$$

It is clear that for this equation

$$C(t, s) = \exp\left\{-\int_s^t p(\xi)d\xi\right\} \geq 0 \text{ and } C(\omega, 0) = \exp\left\{-\int_0^\omega p(\xi)d\xi\right\}.$$

According to formula (15.7.3), the inequality $1-\beta C(\omega, 0) > 0$ is necessary and sufficient for positivity of the Green's function $P(t, s)$ of the problem (15.7.6), (15.7.2). In this case, the condition $1 - \beta C(\omega, 0) = 1 - \beta \exp\{-\int_0^\omega p(\xi)d\xi\} > 0$ is satisfied if and only if

$$\beta < \exp\left\{\int_0^\omega p(s)ds\right\}. \tag{15.7.7}$$

Note that inequality (15.7.7) is a particular case of the inequality (15.7.5) for $(Bx)(t) = p(t)x(t)$.

## 15.8 Regular Behavior of the Green's Function to a One-Point Boundary Value Problem

Consider the one-point problem

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t), \ t \in [0, \omega], \tag{15.8.1}$$

$$x(a) = c, \tag{15.8.2}$$

where $a$ is a fixed point such that $0 < a < \omega$.

**Definition 15.4** We say that the Green's function of problem (15.8.1), (15.8.2) *behaves regularly* if

$$G(t, s) \leq 0 \text{ for } 0 < t < a, \ 0 < s < \omega, \text{ and } G(t, s) \geq 0 \text{ for } a < t < \omega, \ 0 < s < \omega. \tag{15.8.3}$$

**Theorem 15.11** *Let $B : C[0, \omega] \to L_1[0, \omega]$ be a positive Volterra operator and $[0, \omega]$ be a nonoscillation interval of the equation $Mx = 0$. Then the Green's function of problem* (15.8.1), (15.8.2) *behaves regularly.*

Theorem 15.11 follows from Theorem 17.11, proven further in Chap. 17.

**Theorem 15.12** *Let $B : C_{[0,\omega]} \to L_1[0, \omega]$ be a positive Volterra operator and at least one of the inequalities*

$$\int_0^\omega (B1)(s)ds < 1, \ t \in [0, \omega], \tag{15.8.4}$$

*or*

$$\int_{H(t)}^{t} (B1)(s)ds \le \frac{1}{e}, \ t \in (0, \omega], \tag{15.8.5}$$

*be satisfied, where the function $H$ is defined in Definition 15.3. Then the Green's function of problem (15.8.1), (15.8.2) behaves regularly.*

*Proof* The proof follows from Theorems 15.3, 15.7 and 15.11 and Corollary 15.4. □

Consider the equation

$$(Mx)(t) \equiv x'(t) + \sum_{k=1}^{m} p_k(t)x\big(h_k(t)\big) = f(t), \ t \in [0, \omega], \tag{15.8.6}$$

where $x(s) = 0$ for $s < 0$ and $h_k(t) \le t$ for $t \in [0, \omega]$, $k = 1, \cdots, m$.

**Corollary 15.5** *Let $p_k \ge 0$ for $k = 1, \cdots, m$, and at least one of the inequalities*

$$\int_{0}^{\omega} \sum_{k=1}^{m} p_k(t)dt < 1, \ t \in [0, \omega], \tag{15.8.7}$$

*or*

$$\int_{h(t)}^{t} \sum_{k=1}^{m} p_k(s)ds \le \frac{1}{e}, \ t \in (0, \omega], \tag{15.8.8}$$

*be satisfied, where the function $h(t) = \min\{h_k(t), k = 1, \cdots, m\}$. Then the Green's function of problem (15.8.6), (15.8.2) behaves regularly.*

*Proof* The proof follows from Theorem 15.12. □

## 15.9 Positivity of Green's Functions for Equations Including Difference of Positive Operators

In this section, we consider the equation

$$(Mx)(t) \equiv x'(t) - (Ax)(t) + (Bx)(t) = f(t), \ t \in [0, \infty), \tag{15.9.1}$$

where $B : C[0, \infty) \to L_\infty[0, \infty)$ and $A : C[0, \infty) \to L_\infty[0, \infty)$ are positive linear continuous Volterra operators, with the general boundary condition

$$lx \equiv \theta x(0) + \int_{0}^{\omega} \phi(s)x'(s)ds = c. \tag{15.9.2}$$

The equation

$$x'(t) + \sum_{k=1}^{m} p_k(t)x\big(h_k(t)\big) = f(t), \ t \in [0, \infty), \tag{15.9.3}$$

with $x(s) = 0$ for $s < 0$ and $h_k(t) \le t$ for $t \in [0, \infty)$, where the coefficients $p_k(t)$ can change their signs, is a particular case of (15.9.1).

**Theorem 15.13** *If $B$ and $A$ are positive Volterra operators and the Cauchy function $C^+(t,s)$ of the equation*

$$x'(t) + (Bx)(t) = f(t),\ t \in [0, \infty), \tag{15.9.4}$$

*is positive for $0 \le s \le t < \infty$, then:*

1) *The Cauchy function $C(t,s)$ of (15.9.1) satisfies the inequality $C(t,s) \ge C^+(t,s) > 0$ for $0 \le s \le t < \infty$.*
2) *If in addition the inequality*

$$\frac{\int_s^\omega \phi(\xi)\{C^+(\xi,s)\}'_\xi d\xi + \phi(s)}{\theta + \int_0^\omega \phi(s)\{C^+(s,0)\}'_s ds} < 0 \tag{15.9.5}$$

*and at least one of the conditions*
 a) $(B1)(t) > (A1)(t)$, $c \ge 0$,
 *or*
 b) $B1)(t) \ge (A1)(t)$, $c > 0$,
 *is fulfilled, then the Green's function $G(t,s)$ of the boundary value problem (15.9.1), (15.9.2) is positive for $t, s \in (0,\omega)$ and satisfies the inequality $G(t,s) \ge G^+(t,s)$, where $G^+(t,s)$ is the Green's function of problem (15.9.4), (15.9.2).*

*Proof* The Cauchy problem $(Mx)(t) = f(t),\ x(0) = 0,\ t \in [0,\omega]$, is equivalent to the integral equation

$$x(t) = \int_0^t C^+(t,s)(Ax)(s)ds + \varphi(t),\ t \in [0,\omega],$$

where $\varphi(t) = \int_0^t C^+(t,s) f(s)ds$. The spectral radius of the operator $K : C[0,\omega] \to C[0,\omega]$ defined by the equality

$$(Kx)(t) = \int_0^t C^+(t,s)(Ax)(s)ds,\ t \in [0,\omega],$$

equals zero. It implies that $x(t) = (I-K)^{-1}\varphi(t) = ((I + K + K^2 + \cdots)\varphi)(t)$. If $f \ge 0$, then $\varphi \ge 0$ and $x \ge 0$. Moreover, $x(t) = \int_0^t C(t,s) f(s)ds \ge \int_0^t C^+(t,s) f(s)ds$ for each nonnegative $f$. It implies that $C(t,s) \ge C^+(t,s) > 0$ for $0 \le s \le t \le \omega$. From the fact that $A$ and $B$ are Volterra operators, it follows that $C(t,s) \ge C^+(t,s) > 0$ for $0 \le s \le t < \infty$.

Let us prove assertion 2). If condition (15.9.5) is fulfilled, then equivalence of the positivity of $C^+(t,s)$ and $G^+(t,s)$ was proven in Theorem 15.3. It is clear now that the operator $N : C[0,\omega] \to C[0,\omega]$ defined by the formula

$$(Nx)(t) = \int_0^\omega G^+(t,s)(Ax)(s)ds,\ t \in [0,\omega], \tag{15.9.6}$$

is positive. The solution $x$ of problem (15.9.1), (15.9.2) satisfies the inequality

$$x(t) = (Nx)(t) + \int_0^\omega G^+(t,s) f(s)ds + \frac{c}{\theta + \int_0^\omega \phi(s)\{C^+(s,0)\}'_s ds},\ t \in [0,\omega]. \tag{15.9.7}$$

If $f \geq 0$, then each of the conditions a) or b) implies by Lemma A.2 that the spectral radius of the operator $N : C[0,\omega] \to C[0,\omega]$ is less than one. Let us define $\varphi$ as

$$\varphi(t) = \int_0^\omega G^+(t,s)f(s)ds + \frac{c}{\theta + \int_0^\omega \phi(s)\{C^+(s,0)\}'_s ds}.$$

The solution $x$ can be represented as $x(t) = (I - N)^{-1}\varphi(t) = \varphi(t) + (N\varphi)(t) + (N^2\varphi)(t) + \cdots$. It is clear that $x(t) \geq \varphi(t)$ and $G(t,s) \geq G^+(t,s)$. This completes the proof of assertion 2). □

*Remark 15.11* Inequality (15.9.5) seems very difficult to check. We demonstrated in Remark 15.2 that in the case $\theta > 0$ and $\phi(s) < -\varepsilon < 0$ this inequality is fulfilled if the interval $[0,\omega]$ is small enough.

**Theorem 15.14** *Let $A, B : C[0,\infty) \to L_\infty[0,\infty)$ be positive linear Volterra operators and*

$$\int_{H(t)}^t (B1)(s)ds \leq \frac{1}{e},\ t \in (0,\infty), \tag{15.9.8}$$

*where the function $H(t)$ is defined in Definition 15.3. Then conditions 1) and 2) of Theorem 15.13 are fulfilled.*

Consider the particular case of (15.9.1)

$$(Mx)(t) \equiv x'(t) + \sum_{k=1}^m p_k(t)x\big(h_k(t)\big) = f(t),\ t \in [0,\infty), \tag{15.9.9}$$

where $x(s) = 0$ for $s < 0$. Assume that $h_k(t) \leq t$ for $t \in [0,\infty)$.

Let us denote $p_k^+(t) = \max\{p_k(t), 0\}$, $p_k^-(t) = \max\{-p_k(t), 0\}$. It is clear that $p_k(t) = p_k^+(t) - p_k^-(t)$. Equation (15.9.4) in this case can be written in the form

$$x'(t) + \sum_{k=1}^m p_k^+(t)x\big(h_k(t)\big) = f(t),\ t \in [0,\infty). \tag{15.9.10}$$

Conditions a) $(B1)(t) > (A1)(t)$, $c \geq 0$ or b) $(B1)(t) \geq (A1)(t)$, $c > 0$, in this case get the following forms:

$$\begin{aligned} &\text{a)} \quad \sum_{k=1}^m p_k(t)\chi\big(h_k(t)\big) > 0, c \geq 0,\ t \in (0,\infty), \\ &\text{b)} \quad \sum_{k=1}^m p_k(t)\chi\big(h_k(t)\big) \geq 0,\ c > 0,\ t \in (0,\infty). \end{aligned} \tag{15.9.11}$$

The function $H(t)$ in Definition 15.3 can be determined as follows: $H(t) = \min_{1 \leq k \leq m} h_k(t)$, and inequality (15.9.8) for (15.9.9) has the form

$$\int_{H(t)}^t \sum_{k=1}^m p_k^+(s)ds \leq \frac{1}{e},\ t \in (0,\infty). \tag{15.9.12}$$

**Corollary 15.6** *Let inequality* (15.9.12) *be satisfied. Then*:

1) *The Cauchy function* $C(t,s)$ *of* (15.9.9) *satisfies the inequality* $C(t,s) \geq C^+(t,s) > 0$ *for* $0 \leq s \leq t < \infty$.
2) *If in addition inequality* (15.9.5) *and at least one of the conditions* a) *or* b) *of* (15.9.11) *are fulfilled, then the Green's function* $G(t,s)$ *of the boundary value problem* (15.9.9), (15.9.2) *is positive for* $t, s \in (0,\omega)$ *and satisfies the inequality* $G(t,s) \geq G^+(t,s)$, *where* $G^+(t,s)$ *is the Green's function of problem* (15.9.10), (15.9.2).

Consider the delay equation

$$(Mx)(t) \equiv x'(t) + \sum_{k=1}^{\infty} p_k(t)x(t-k\tau) = f(t),\ t \in [0,\infty), \tag{15.9.13}$$

where $x(s) = 0$ if $s < 0$.

The restriction that the delay $t - H(t)$ be small can be replaced by the conditions on the rate of decrease for the coefficients.

**Theorem 15.15** *Suppose that there exists* $\beta$ *such that*

$$\beta < \frac{1}{e},\ p_1^+ \geq \beta p_2^+,\ p_1^+ \geq \beta^2 p_3^+, \cdots,\ p_1^+ \geq \beta^{k-1} p_k^+, \cdots \tag{15.9.14}$$

*and*

$$p_1^+\tau + \beta \leq \frac{1}{e} \tag{15.9.15}$$

*is satisfied for* (15.9.13). *Then*:

1) $C(t,s) > 0$ *for* $0 \leq s \leq t < \infty$.
2) *If in addition inequality* (15.9.5) *and at least one of the conditions*

$$\begin{aligned} &\text{a)} \quad \sum_{k=1}^{\infty} p_k(t)\chi\big(h_k(t)\big) > 0,\ c \geq 0,\ t \in (0,\infty),\\ &\text{b)} \quad \sum_{k=1}^{\infty} p_k(t)\chi\big(h_k(t)\big) \geq 0,\ c > 0,\ t \in (0,\infty), \end{aligned} \tag{15.9.16}$$

*is fulfilled, then the Green's function* $G(t,s)$ *of the boundary value problem* (15.9.13), (15.9.2) *is positive for* $t, s \in (0,\omega)$ *and satisfies the inequality* $G(t,s) \geq G^+(t,s)$, *where* $G^+(t,s)$ *is the Green's function of problem* (15.9.17), (15.9.2), *where*

$$x'(t) + \sum_{k=1}^{\infty} p_k^+(t)x(t-k\tau) = f(t),\ t \in [0,\infty). \tag{15.9.17}$$

*Proof* The proof follows from Theorems 15.8 and 15.14. □

Let us consider the integrodifferential equation

$$(Mx)(t) \equiv x'(t) + \int_0^{t-\tau} K(t,s)x(s)ds = f(t),\ t \in [0,\infty), \tag{15.9.18}$$

where $K(t,s)$ is an essentially bounded function. Let us write $K(t,s) = K^+(t,s) - K^-(t,s)$, where $K^+(t,s)$ and $K^-(t,s)$ are nonnegative functions.

**Theorem 15.16** *Let the inequalities*

$$K^+(t,s) \le be^{-\gamma(t-s)},\ 0 \le s \le t < \infty,\ \gamma,\ b > 0, \tag{15.9.19}$$

*and*

$$b \le \frac{\gamma^2}{4} e^{\frac{\gamma}{2}\tau} \tag{15.9.20}$$

*be satisfied. Then*:

1) $C(t,s) > 0$ *for* $0 \le s \le t < \infty$.
2) *If in addition inequality* (15.9.5) *and*

$$\int_0^{t-\tau} K(t,s)ds \ge 0,\ c > 0,\ t \in (0,\infty), \tag{15.9.21}$$

*are fulfilled, then the Green's function* $G(t,s)$ *of boundary value problem* (15.9.18), (15.9.2) *is positive for* $t,s \in (0,\omega)$ *and satisfies the inequality* $G(t,s) \ge G^+(t,s)$. *Here* $G^+(t,s)$ *is the Green's function of problem* (15.9.22), (15.9.2), *where*

$$x'(t) + \int_0^{t-\tau} K^+(t,s)x(s)ds = f(t),\ t \in [0,\infty). \tag{15.9.22}$$

*Remark 15.12* Note that inequality (15.9.20) cannot be improved (see Remark 15.8).

## 15.10 Positivity of the Cauchy and Green's Functions

Let us write positive linear continuous Volterra operators $A : C[0,\infty) \to L_\infty[0,\infty)$ and $B : C[0,\infty) \to L_\infty[0,\infty)$ in (15.9.1) in the form of the Stieltjes integrals $(Ax)(t) = \int_0^t x(s)d_s a(t,s)$ and $(Bx)(t) = \int_0^t x(s)d_s b(t,s)$, respectively, and consider the class of auxiliary equations

$$(M_s x)(t) \equiv x'(t) - (A_s x)(t) + (B_s x)(t) = f(t),\ t \in [s,\infty), \tag{15.10.1}$$

where the operators $B_s : C[s,\infty) \to L_\infty[s,\infty)$ and $A_s : C[s,\infty) \to L_\infty[s,\infty)$ are defined as

$$(A_s x)(t) = \int_s^t x(\xi)d_\xi a(t,\xi) \text{ and } (B_s x)(t) = \int_s^t x(\xi)d_\xi b(t,\xi),\ t \in [0,\infty). \tag{15.10.2}$$

**Theorem 15.17** *The following three hypotheses are equivalent*:

1) *For each* $s \in [0, \infty)$, *there exists a positive function* $v_s \in D[s, \omega]$ *for every* $\omega \in (0, \infty)$, $v_s' \in L_\infty[0, \infty)$ *such that* $(M_s v_s)(t) \le 0$ *for* $t \in [s, \infty)$.
2) *The Cauchy function* $C(t, s)$ *of* (15.9.1) *satisfies the inequality* $C(t, s) > 0$ *for* $0 \le s \le t < \infty$.
3) *There exists a function* $u_s \in L_\infty[s, \infty)$ *such that*

$$u_s(t) \ge -A_s \exp \int_\xi^t u_s(\eta) d\eta + B_s \exp \int_\xi^t u_s(\eta) d\eta, \ t \in [s, \infty), \tag{15.10.3}$$

*where*

$$A_s \exp \int_\xi^t u_s(\eta) d\eta = \int_s^t e^{\int_\xi^t u_s(\eta) d\eta} d_\xi a(t, \xi) \tag{15.10.4}$$

*and*

$$B_s \exp \int_\xi^t u_s(\eta) d\eta = \int_s^t e^{\int_\xi^t u_s(\eta) d\eta} d_\xi b(t, \xi).$$

*Proof* 1) $\Rightarrow$ 2) Let us suppose that $C(t_0, s_0) = 0$, while $C(t, s) > 0$ for $s < s_0$, $0 \le s < t < t_0$.

Let us set $V_s(t) = \frac{v_s(t)}{v_s(s)}$. Obviously $V_s(s) = 1$, $V_s(t) > 0$ for $t \ge s$, and $(M_s V_s)(t) \le 0$ for $t \in [s, \infty)$. Using the solution representation formula for (15.10.1), we obtain

$$V_{s_0}(t_0) = \int_{s_0}^{t_0} C(t, \xi) \psi(\xi) d\xi + C(t_0, s_0), \tag{15.10.5}$$

where $\psi(t) \equiv (M_{s_0} V_{s_0})(t) \le 0$ for $t \ge s_0$. Equality (15.10.5) implies $V_{s_0}(t_0) \le C(t_0, s_0)$. From the positivity of $V_{s_0}(t_0)$, we get the contradiction to our assumption that $C(t_0, s_0) = 0$.

We set $v_s(t) = C(t, s)$ and $v_s(t) = \exp(-\int_s^t u_s(\xi) d\xi)$ in order to prove 2) $\Rightarrow$ 1) and 3) $\Rightarrow$ 1), respectively.

In order to prove 2) $\Rightarrow$ 3), we set $u_s(t) = -\frac{d}{dt} \ln C(t, s)$. In this case, $C(t, s) = \exp(-\int_s^t u_s(\xi) d\xi)$. Substituting $x(t) = C(t, s)$ into (15.10.1), we obtain by carrying the exponent out of the brackets

$$-\exp\left\{-\int_s^t u_s(\eta) d\eta\right\}\left[u_s(t) + A_s \exp \int_\xi^t u_s(\eta) d\eta - B_s \exp \int_\xi^t u_s(\eta) d\eta\right] = 0, \tag{15.10.6}$$

$t \in [s, \infty)$. This proves inequality (15.10.3) and completes the proof. □

Consider now the delay differential equation

$$x'(t) - a(t) x\big(g(t)\big) + b(t) x\big(h(t)\big) = f(t), \ t \in [0, \infty), \tag{15.10.7}$$

where

$$x(\xi) = 0 \text{ for } \xi < 0, \tag{15.10.8}$$

and $a, b, f \in L_\infty[0, \infty)$ as a particular case of (15.9.1).

For (15.10.7), Theorem 15.17 takes the following form.

**Theorem 15.18** *Let $a(t) \geq 0$, $b(t) \geq 0$. Then the following three hypotheses are equivalent*:

1) *For each $s \in [0, \infty)$, there exists a positive function $v_s \in D[s, \omega]$ for every $\omega \in (0, \infty)$, $v_s' \in L_\infty[0, \infty)$ such that*

$$Mv_s(t) \equiv v_s'(t) - a(t)\chi_s\big(g(t)\big)v_s\big(g(t)\big) + b(t)\chi_s\big(h(t)\big)v_s\big(h(t)\big) \leq 0, \quad t \in [s, \infty), \tag{15.10.9}$$

*where the function $\chi(t)$ is defined by the formula*

$$\chi_s(t) = \begin{cases} 1, & t \geq s, \\ 0, & t < s. \end{cases} \tag{15.10.10}$$

2) *The Cauchy function $C(t, s)$ of (15.10.7) satisfies the inequality $C(t, s) > 0$ for $0 \leq s \leq t < \infty$.*
3) *For each $s \in [0, \infty)$, there exists a function $u_s \in L_\infty[s, \infty)$ such that*

$$u_s(t) + a(t)\chi_s\big(g(t)\big)\exp\left\{\int_{g(t)}^t u_s(\eta)d\eta\right\} - b(t)\chi_s\big(h(t)\big)\exp\left\{\int_{h(t)}^t u_s(\eta)d\eta\right\} \geq 0, \tag{15.10.11}$$

*$t \in [s, \infty)$.*

When we set $u_s = 0$ for every $s$ in assertion 3) of Theorem 15.18, the following result is obtained.

**Corollary 15.7** *Let $h(t) \leq g(t)$ and $0 \leq b(t) \leq a(t)$ for $t \in [0, \infty)$. Then the Cauchy function $C(t, s)$ of (15.10.7) satisfies the inequality $C(t, s) > 0$ for $0 \leq s \leq t < \infty$.*

*Remark 15.13* Note that in hypothesis 3) nonnegativity of the function $u_s$ is not assumed.

*Remark 15.14* The possibility should be noted that the Cauchy function $C(t, s)$ of (15.10.7) can be positive although the Cauchy function $C^+(t, s)$ of the equation

$$x'(t) + b(t)x\big(h(t)\big) = f(t), \quad t \in [0, \infty), \qquad x(\xi) = 0 \text{ for } \xi < 0, \tag{15.10.12}$$

can change its sign.

*Example 15.7* Although solutions of the equation

$$x'(t) + 10x(t - 2) = 0, \quad t \in [0, \infty), \qquad x(\xi) = 0 \text{ for } \xi < 0, \tag{15.10.13}$$

oscillate and consequently, according to Theorem 15.3, its Cauchy function $C^+(t,s)$ changes sign, all nontrivial solutions of the equation

$$x'(t) + 10x(t-2) - 11x(t-1) = 0,\ t \in [0,\infty),\qquad x(\xi) = 0 \text{ for } \xi < 0, \tag{15.10.14}$$

nonoscillate and its Cauchy function $C(t,s)$ is positive.

**Theorem 15.19** *Let $h(t) \le g(t)$ and $0 \le b(t) \le a(t)$ for $t \in [0,\infty)$. Then*:

1) *For each $s \in [0,\infty)$, the derivative $C_t'(t,s)$ of the Cauchy function $C(t,s)$ of (15.10.7) satisfies the inequality $C_t'(t,s) \ge 0$ for $0 \le s \le t < \infty$.*
2) *The boundary value problem*
$$x'(t) - a(t)x\big(g(t)\big) + b(t)x\big(h(t)\big) = f(t),\ t \in [0,\omega],\ lx = c, \tag{15.10.15}$$
*where $l : C[0,\omega] \to \mathbb{R}$ is a positive nonzero functional, has a unique solution for each $f \in L_\infty[0,\omega]$ $c \in \mathbb{R}$.*
3) *The boundary value problem*
$$x'(t) - a(t)x\big(g(t)\big) + b(t)x\big(h(t)\big) = f(t),\ t \in [0,\omega],\ lx \equiv x(\omega) - mx = c, \tag{15.10.16}$$
*where the norm of the functional $m : C[0,\omega] \to \mathbb{R}$ is less than one, has a unique solution for each $f \in L_\infty[0,\omega]$, $c \in \mathbb{R}$.*
4) *The boundary value problem*
$$x'(t) - a(t)x\big(g(t)\big) + b(t)x\big(h(t)\big) = f(t),\ t \in [0,\omega],$$
$$\sum_{j-1}^{2k} \alpha_j x(t_j) = c,\ 0 \le t_1 < t_2 < \cdots < t_{2k} \le \omega,$$
*where $0 \le -\alpha_{2j-1} \le \alpha_{2j}$, $j = 1,\cdots,k$, and there exists $i$ such that $-\alpha_{2i-1} < \alpha_{2i}$, has a unique solution for each $f \in L_\infty[0,\omega]$ $c \in \mathbb{R}$.*

*Proof* For each fixed $s \in [0,\infty)$, the Cauchy function $C(t,s)$ as a function of $t$ is a solution of the equation

$$x'(t) - a(t)x\big(g(t)\big) + b(t)x\big(h(t)\big) = 0,\ t \in [s,\infty),\qquad x(\xi) = 0 \text{ for } \xi < s, \tag{15.10.17}$$

satisfying the initial condition $C(s,s) = 1$. The conditions $h(t) \le g(t)$ and $0 \le b(t) \le a(t)$ for $t \in [0,\infty)$ imply

$$C_t'(t,s) = a(t)C\big(g(t),s\big) - b(t)C\big(h(t),s\big) \ge 0,\ t \in [s,\infty). \tag{15.10.18}$$

Proofs of assertions 2)–4) follow from the fact that every nontrivial solution is proportional to $C(t,0)$. □

Let us define the sets

$$I_s^1 = \big\{t \mid g(t) \ge s,\ h(t) \ge s\big\},\ I_s^2 = \big\{t \mid g(t) \ge s,\ h(t) < s\big\}, \tag{15.10.19}$$

$$I_s^3 = \{t \mid g(t) < s,\ h(t) \geq s\},\ I_s^4 = \{t \mid g(t) < s,\ h(t) < s\}, \tag{15.10.20}$$

where $t \in [s, \infty)$. It is clear that $I_s^1 \cup I_s^2 \cup I_s^3 \cup I_s^4 = [s, \infty)$.

**Theorem 15.20** *Let $h(t) \leq g(t)$ and $a(t) \geq 0$, $b(t) \geq 0$ for $t \in [0, \infty)$. Then hypotheses* 1) *and* 2) *below are equivalent, and each of them implies assertions* 3) *and* 4). *If we assume in addition that $a(t) \leq b(t)$ and the functions $h$ and $g$ are nondecreasing for $t \in [0, \infty)$, then assertions* 1)–4) *are equivalent.*

1) *There exists a positive function $v \in D[0, \omega]$ for every $\omega \in (0, \infty)$, $v' \in L_\infty[0, \infty)$ such that $v'(t) \leq 0$ for $t \in [0, \infty)$ and*
$$v'(t) - a(t)v\big(g(t)\big) + b(t)v\big(h(t)\big) \leq 0,\ \ t \in I_0^1. \tag{15.10.21}$$
2) *There exists a nonnegative function $u \in L_\infty[0, \infty)$ such that*
$$u(t) + a(t)\exp\left\{\int_{g(t)}^t u(\eta)d\eta\right\} - b(t)\exp\left\{\int_{h(t)}^t u(\eta)d\eta\right\} \geq 0,\ \ t \in I_0^1. \tag{15.10.22}$$
3) *The Cauchy function $C(t, s)$ of* (15.10.7) *satisfies the inequality $C(t, s) > 0$ for $0 \leq s \leq t < \infty$.*
4) *The homogeneous equation* (15.10.7), (15.10.8) ($f \equiv 0$) *has a positive solution $x(t) > 0$ for $t \in [0, \infty)$.*

*Proof* In order to prove 1) $\Rightarrow$ 2) we set $u(t) = -\frac{d}{dt}\ln\frac{v(t)}{v(0)}$.

In this case, $v(t) = \exp\{-\int_0^t u(\xi)d\xi\}$. Substituting this $v$ into (15.10.21), we obtain by carrying the exponent out of the brackets

$$-\exp\left\{-\int_0^t u(\eta)d\eta\right\}\left\{u(t) + a(t)\exp\int_{g(t)}^t u(\eta)d\eta - b(t)\exp\int_{h(t)}^t u(\eta)d\eta\right\} \leq 0, \tag{15.10.23}$$

where $t \in I_0^1$. This proves inequality (15.10.22).

Setting $v(t) = \exp\{-\int_0^t u(\xi)d\xi\}$, we prove 2) $\Rightarrow$ 1).

Let us prove the implication 1) $\Rightarrow$ 3). It is sufficient to demonstrate that assertion 1) of the theorem implies hypothesis 1) of Theorem 15.18, and then the equivalence obtained in Theorem 15.18 will complete the proof.

From the condition $h(t) \leq g(t)$, we get $I_s^1 \cup I_s^2 \cup I_s^4 = [s, \infty)$ for each $s \in [0, \infty)$. Let us set $v_s(t) = v(t)$ for $t \geq s$ and $v_s(t) = 0$ for $t < s$, where $0 \leq s < \infty$. The condition $v'(t) \leq 0$ for $t \in [0, \infty)$ implies the inequality

$$v_s'(t) - a(t)\chi_s\big(g(t)\big)v_s\big(g(t)\big) + b(t)\chi_s\big(h(t)\big)v_s\big(h(t)\big) \leq 0 \tag{15.10.24}$$

for $t \in I_s^4$. The inequalities $v'(t) \leq 0$, $a(t) \geq 0$ and $h(t) \leq g(t)$ for $t \in [0, \infty)$ imply the inequality (15.10.24) for $t \in I_s^2$. Inequality (15.10.21) implies the inequality (15.10.24) for $t \in I_s^1$. Thus

$$v_s'(t) - a(t)\chi_s\big(g(t)\big)v_s\big(g(t)\big) + b(t)\chi_s\big(h(t)\big)v_s\big(h(t)\big) \leq 0 \tag{15.10.25}$$

for $t \in [s, \infty)$, $0 \leq s < \infty$.

The reference to Theorem 15.18 completes the proof of the implication 1) $\Rightarrow$ 3).

The implication 3) $\Rightarrow$ 4) is evident since the function $C(t,0)$ is a solution of (15.10.7), (15.10.8) (for $f \equiv 0$).

In order to prove the implication 4) $\Rightarrow$ 1), we note that in this case every set $I_s^j$ can be only a finite or infinite interval for $j = 1, \cdots, 4$ and set

$$v(t) = \begin{cases} \alpha, & t \in I_0^2 \cup I_0^4, \\ C(t,0), & t \in I_0^1, \end{cases} \tag{15.10.26}$$

where the constant $\alpha$ is chosen such that the function $v : [0,\infty) \to [0,\infty)$ is continuous. The inequality $v'(t) \le 0$ can be proven by repeating the same trick as in the proof of Theorem 3.2 in Chap. 3, which completes the proof. □

*Remark 15.15* The condition $h(t) \le g(t)$ is essential, as the following example demonstrates.

*Example 15.8* Consider the equation

$$\begin{gathered} x'(t) + x(t-1) - x(t-3) = 0, \ t \in [0,\infty), \\ x(\xi) = 0 \text{ for } \xi < 0. \end{gathered} \tag{15.10.27}$$

The function $v(t) \equiv 1$ for $t \in [0,\infty)$ satisfies condition 1) of Theorem 15.20 but the Cauchy function $C(t,s) < 0$ for $t \in (s+1, s+2)$.

Note that the results of Chap. 3 on eventually nonoscillating solutions and eventually positive Cauchy functions could not be directly applied for the maximum inequalities principle. Theorem 15.20 allows us to adopt methods of Chap. 3 and to obtain Corollaries 15.8 and 15.9 on nonoscillation and positivity of the Cauchy function and the Green's function.

**Corollary 15.8** *Assume that $h(t) \le g(t)$, $a(t) \ge 0$, $b(t) \ge 0$ for $t \in [0,\infty)$ and there exists $\lambda$, $0 < \lambda < 1$, such that the inequalities*

$$b(t) \ge \lambda a(t), \ t \in [0,\infty), \tag{15.10.28}$$

$$\sup_{t\in[0,\infty)} \int_{h(t)}^{g(t)} \big[b(\xi) - \lambda a(\xi)\big] d\xi < \frac{1}{e} \ln \frac{1}{\lambda}, \tag{15.10.29}$$

$$\sup_{t\in[0,\infty)} \int_{h(t)}^{t} \big[b(\xi) - \lambda a(\xi)\big] d\xi < \frac{1}{e}, \tag{15.10.30}$$

*are satisfied. Then $C(t,s) > 0$ for $0 \le s \le t < \infty$, and the nontrivial solution of homogeneous equation* (15.10.7), (15.10.8) (*for* $f \equiv 0$) *is positive for* $0 \le t < \infty$. *If in addition $\beta C(\omega,0) < 1$, then the Green's function $G(t,s)$ of the generalized periodic problem* (15.10.7), (15.7.2) *is positive for* $t, s \in [0,\omega]$.

*Proof* Actually the justification of positivity of the Cauchy function repeats the proof of Theorem 3.5 in Chap. 3.

Let us prove first that the positive function $u = e\{b - \lambda a\}$ satisfies the inequality

$$u(t) \geq \big[b(t) - \lambda a(t)\big] \exp\left\{\int_{h(t)}^{t} u(s)ds\right\}, \; t \in I_0^1. \tag{15.10.31}$$

When we substitute this function $u$ into the inequality, we get

$$e\big[b(t) - \lambda a(t)\big] \geq \big[b(t) - \lambda a(t)\big] \exp\left\{\int_{h(t)}^{t} e\big\{b(\xi) - \lambda a(\xi)\big\}d\xi\right\}$$

and

$$1 \geq \left\{\int_{h(t)}^{t} e\big\{b(\xi) - \lambda a(\xi)\big\}d\xi\right\}.$$

Inequality (15.10.30) implies that this function $u$ satisfies inequality (15.10.31).

Inequality (15.10.31) can be rewritten in the form

$$\begin{aligned} u(t) \geq b(t) \exp\left\{\int_{h(t)}^{t} u(\xi)d\xi\right\} - a(t) \exp\left\{\int_{g(t)}^{t} u(\xi)d\xi\right\} \\ + a(t)\left[\exp\left\{\int_{g(t)}^{t} u(\xi)d\xi\right\} - \lambda \exp\left\{\int_{h(t)}^{t} u(\xi)d\xi\right\}\right], \; t \in I_0^1. \end{aligned} \tag{15.10.32}$$

Inequality (15.10.29) implies that the function $u = e\{b - \lambda a\}$ satisfies the inequality

$$\int_{h(t)}^{g(t)} u(\xi)d\xi < \ln \frac{1}{\lambda}. \tag{15.10.33}$$

Using inequality (15.10.33), we can estimate the second term on the right-hand side of inequality (15.10.32) as

$$\begin{aligned} &\exp\left\{\int_{g(t)}^{t} u(\xi)d\xi\right\} - \lambda \exp\left\{\int_{h(t)}^{t} u(\xi)d\xi\right\} \\ &= \exp\left\{\int_{g(t)}^{t} u(\xi)d\xi\right\}\left[1 - \lambda \exp\left\{\int_{h(t)}^{g(t)} u(\xi)d\xi\right\}\right] \\ &\geq \exp\left\{\int_{g(t)}^{t} u(\xi)d\xi\right\}\left[1 - \lambda \frac{1}{\lambda}\right] = 0. \end{aligned}$$

Thus

$$\exp\left\{\int_{g(t)}^{t} u(\xi)d\xi\right\} - \lambda \exp\left\{\int_{h(t)}^{t} u(\xi)d\xi\right\} \geq 0, \; t \in I_0^1.$$

Next, inequality (15.10.32) can be rewritten in the form

$$u(t) \geq b(t) \exp\left\{\int_{h(t)}^{t} u(\xi)d\xi\right\} - a(t) \exp\left\{\int_{g(t)}^{t} u(\xi)d\xi\right\}, \; t \in I_0^1.$$

The implication 2) $\Rightarrow$ 3) of Theorem (15.20) implies the positivity of the Cauchy function $C(t,s)$.

Positivity of the Green's function of the generalized periodic problem (15.10.7), (15.7.2) follows from the representation of the Green's function (15.7.3). □

**Corollary 15.9** *Assume that $h(t) \le g(t)$, $a(t) \ge 0$, $b(t) \ge 0$ for $t \in [0, \infty)$, and the inequalities*

$$b(t) \ge \frac{1}{e} a(t), \; t \in [0, \infty), \tag{15.10.34}$$

*and*

$$\sup_{t \in [0,\infty)} \int_{h(t)}^{t} \left[ b(\xi) - \frac{1}{e} a(\xi) \right] d\xi < \frac{1}{e}, \tag{15.10.35}$$

*are fulfilled. Then $C(t, s) > 0$ for $0 \le s \le t < \infty$ and the nontrivial solution of the homogeneous equation (15.10.7) (for $f \equiv 0$), (15.10.8) is positive for $0 \le t < \infty$. If in addition $\beta C(\omega, 0) < 1$, then the Green's function $G(t, s)$ of the generalized periodic problem (15.10.7), (15.7.2) is positive for $t, s \in [0, \omega]$.*

*Proof* To prove Corollary 15.9, it is enough to set $\lambda = \frac{1}{e}$ in Corollary 15.8. □

*Remark 15.16* It was demonstrated in [81] that the coefficient $\frac{1}{e}$ of $a(t)$ in (15.10.34) and (15.10.35) is sharp.

The following theorem deals with the case $h(t) \ge g(t)$, which was not considered in Chap. 3. Let us denote

$$\tau(t) = t - h(t), \; \sigma(t) = t - g(t), \; \tau_* = \inf_{t \in [0,\infty)} \tau(t), \; \sigma^* = \sup_{t \in [0,\infty)} \sigma(t).$$

**Theorem 15.21** *Let $h(t) \ge g(t)$, $b(t) \ge a(t) \ge 0$ for $t \in [0, \infty)$ and the inequalities*

$$\int_{h(t)}^{t} \big[ b(\xi) - a(\xi) \big] d\xi \le \frac{1}{e}, \; t \in [0, \infty), \tag{15.10.36}$$

$$\sup_{s \in [0,\infty)} \int_{s+\tau_*}^{s+\sigma^*} b(\xi) d\xi \le \frac{1}{e}, \tag{15.10.37}$$

*be satisfied. Then $C(t, s) > 0$ for $0 \le s \le t < \infty$, and the nontrivial solution of the homogeneous equation (15.10.7), (15.10.8) (for $f \equiv 0$) is positive for $0 \le t < \infty$. If in addition $\beta C(\omega, 0) < 1$, then the Green's function $G(t, s)$ of the generalized periodic problem (15.10.7), (15.7.2) is positive for $t, s \in [0, \omega]$.*

*Proof* Let us set

$$v_s(t) = \exp\left\{ -e \int_s^t \big[ b(\xi) \chi_s\big(h(\xi)\big) - a(\xi) \chi_s\big(g(\xi)\big) \big] d\xi \right\}, \; t \in [s, \infty), \tag{15.10.38}$$

in hypothesis 1) of Theorem 15.18. Then

$$\begin{aligned} M v_s(t) = &-e \big[ b(t) \chi_s\big(h(t)\big) - a(t) \chi_s\big(g(t)\big) \big] \\ &\times \exp\left\{ -e \int_s^t \big[ b(\xi) \chi_s\big(h(\xi)\big) - a(\xi) \chi_s\big(g(\xi)\big) \big] d\xi \right\} \end{aligned}$$

$$
\begin{aligned}
&+ b(t)\chi_s\big(h(t)\big)\exp\left\{-e\int_s^{h(t)}\big[b(\xi)\chi_s\big(h(\xi)\big)-a(\xi)\chi_s\big(g(\xi)\big)\big]d\xi\right\}\\
&- a(t)\chi_s\big(g(t)\big)\exp\left\{-e\int_s^{g(t)}\big[b(\xi)\chi_s\big(h(\xi)\big)-a(\xi)\chi_s\big(g(\xi)\big)\big]d\xi\right\}\\
=&\exp\left\{-e\int_s^{t}\big[b(\xi)\chi_s\big(h(\xi)\big)-a(\xi)\chi_s\big(g(\xi)\big)\big]d\xi\right\}\\
&\times\Bigg[-e\big[b(t)\chi_s\big(h(t)\big)-a(t)\chi_s\big(g(t)\big)\big]\\
&+ b(t)\chi_s\big(h(t)\big)\exp\left\{e\int_{h(t)}^{t}\big[b(\xi)\chi_s\big(h(\xi)\big)-a(\xi)\chi_s\big(g(\xi)\big)\big]d\xi\right\}\\
&- a(t)\chi_s\big(g(t)\big)\exp\left\{e\int_{g(t)}^{t}\big[b(\xi)\chi_s\big(h(\xi)\big)-a(\xi)\chi_s\big(g(\xi)\big)\big]d\xi\right\}\Bigg].
\end{aligned}
\tag{15.10.39}
$$

It is clear that $I_s^1 \cup I_s^3 \cup I_s^4 = [s,\infty)$. Obviously the inequality $Mv_s(t) \le 0$ is satisfied for $t \in I_s^4$. The inequality (15.10.36) implies that the inequality $Mv_s(t) \le 0$ is satisfied for $t \in I_s^1$.

In order to obtain the inequality $Mv_s(t) \le 0$ for $t \in I_s^3$ from formula (15.10.39), we have to get the inequality

$$
\begin{aligned}
&-eb(t)\chi_s\big(h(t)\big) + b(t)\chi_s\big(h(t)\big)\exp\left\{e\int_{h(t)}^{t}\big[b(\xi)\chi_s\big(h(\xi)\big)\big]d\xi\right\}\\
&\quad = b(t)\chi_s\big(h(t)\big)\left[-e+\exp\left\{e\int_{h(t)}^{t}\big[b(\xi)\chi_s\big(h(\xi)\big)\big]d\xi\right\}\right] \le 0,
\end{aligned}
\tag{15.10.40}
$$

which is fulfilled if

$$
\int_{h(t)}^{t} b(\xi)\chi_s\big(h(\xi)\big)d\xi \le \frac{1}{e},\ t \in I_s^3. \tag{15.10.41}
$$

Inequality (15.10.41) follows from the definition of $I_s^3$ that $t \le s+\sigma^*$ for $t \in I_s^3$ and from the inequality $h(\xi) \le \xi - \tau_*$ that $\chi_s(h(\xi)) = 0$ for $\xi \in [s, s+\tau_*)$. It is clear now that inequality (15.10.37) implies (15.10.41) for any $s$.

Positivity of the Green's function of generalized periodic problem (15.10.7), (15.7.2) follows from the representation of the Green's function (15.7.3). This completes the proof of the theorem. □

Consider now the autonomous delay differential equation

$$
x'(t) - ax(t-\sigma) + bx(t-\tau) = f(t),\ t \in [0,\infty) \tag{15.10.42}
$$

(i.e., we set $a(t) \equiv a$, $b(t) \equiv b$, $h(t) \equiv t-\tau$, $g(t) \equiv t-\sigma$ in (15.10.7)), where the initial function is determined by (15.10.8).

For the autonomous equation (15.10.42), we need only one function $v_0$ instead of an infinite number of functions $v_s$, $s \in [0,\infty)$ in condition 1) and only one function $u_0$ instead of $u_s$, $s \in [0,\infty)$ in condition 3) of Theorem 15.18.

**Theorem 15.22** *Let* $a > 0$, $b > 0$. *Then the following three hypotheses are equivalent*:

1) *There exists a positive function* $v \in D_\infty[0,\infty)$ *such that*
$$v'(t) - a\chi_0(t-\sigma)v(t-\sigma) + b\chi_0(t-\tau)v(t-\tau) \le 0, \ t \in [0,\infty), \quad (15.10.43)$$
*where the function* $\chi(t)$ *is defined by formula* (15.10.10).
2) *The Cauchy function* $C(t,s)$ *of* (15.10.42) *satisfies the inequality* $C(t,s) > 0$ *for* $0 \le s \le t < \infty$.
3) *There exists a function* $u \in L_\infty[0,\infty)$ *such that*
$$u(t) + a\chi_0(t-\sigma)\exp\left\{\int_{t-\sigma}^{t} u(\eta)d\eta\right\} - b\chi_0(t-\tau)\exp\left\{\int_{t-\tau}^{t} u(\eta)d\eta\right\} \ge 0 \quad (15.10.44)$$
*for* $t \in [0,\infty)$.

*If in addition* $\beta C(\omega,0) < 1$, *then the following assertion is equivalent to hypotheses* 1)–3):

4) *The Green's function* $G(t,s)$ *of the generalized periodic problem* (15.10.7), (15.7.2) *is positive for* $t,s \in [0,\omega]$.

*Proof* For autonomous equation (15.10.42), we can set $v_s(t) = v(t-s)$ in assertion 1) and $u_s(t) = u(t-s)$ in hypothesis 3) of Theorem 15.18. The reference to Theorem 15.18 completes the proof of the equivalence of assertions 1)–3). Positivity of the Green's function of the generalized periodic problem (15.10.7), (15.7.2) follows from the representation of the Green's function (15.7.3). □

The following statement for (15.10.42) follows from Corollary 15.9 and Theorem 15.21.

**Theorem 15.23** *Let* $a > 0$, $b > 0$, *and one of the conditions* a) *or* b) *be fulfilled*:

a) $\tau > \sigma$, $b \ge \frac{1}{e}a$ *and* $(b - \frac{1}{e}a)\tau < \frac{1}{e}$.
b) $\tau < \sigma$, $b \ge a$, $(b-a)\tau \le \frac{1}{e}$ *and* $b(\sigma-\tau) \le \frac{1}{e}$.

*Then the Cauchy function* $C(t,s) > 0$ *for* $0 \le s \le t < \infty$ *and the nontrivial solution of the homogeneous equation* (15.10.42), (15.10.8) (*for* $f \equiv 0$) *is positive for* $0 \le t < \infty$. *If in addition* $\beta C(\omega,0) < 1$, *then the Green's function* $G(t,s)$ *of the generalized periodic problem* (15.10.42), (15.7.2) *is positive for* $t,s \in [0,\omega]$.

## 15.11 Equations with an Oscillating Coefficient

Consider the delay equation
$$x'(t) + p(t)x\big(h(t)\big) = f(t), \ t \in [0,\infty), \quad (15.11.1)$$

$$x(\xi) = 0, \ \xi < 0, \tag{15.11.2}$$

with oscillating coefficient $p(t)$ changing its sign at the points

$$t_k \ (k = 1, 2, 3, \cdots, \ t_0 = 0).$$

**Theorem 15.24** *Let one of the following two conditions* a) *or* b) *be satisfied*:

a) *The coefficient* $p(t)$ *satisfies the condition* $p(t) \geq 0$ *for* $(t_{2k}, t_{2k+1})$ *and* $p(t) \leq 0$ *for* $(t_{2k+1}, t_{2k+2})$ *for* $k = 0, 1, 2, \cdots$, *and*

$$\int_{t_{2k}}^{t_{2k+1}} p(t)dt < 1, \ k = 0, 1, 2, \cdots, \tag{15.11.3}$$

*and the deviating argument* $h(t)$ *satisfies the condition* $t_{2k-1} \leq h(t)$ *for* $t \in [t_{2k}, t_{2k+1}]$.

b) *The coefficient* $p(t)$ *satisfies the condition* $p(t) \leq 0$ *for* $(t_{2k}, t_{2k+1})$ *and* $p(t) \geq 0$ *for* $(t_{2k+1}, t_{2k+2})$ *for* $k = 0, 1, 2, \cdots$, *and*

$$\int_{t_{2k+1}}^{t_{2k+2}} p(t)dt < 1, \ k = 0, 1, 2, \cdots, \tag{15.11.4}$$

*and the deviating argument* $h(t)$ *satisfies the condition*: $t_{2k} \leq h(t)$ *for* $t \in [t_{2k+1}, t_{2k+2}]$.

*Then* $C(t, s) > 0$ *for* $0 \leq s \leq t < \infty$.

*Proof* We use the fact that $x(t) = C(t, s)$ for each fixed $s$ as a function of $t$ is a solution of the equation

$$x'(t) + p(t)x\big(h(t)\big) = 0, \ t \in [s, \infty), \tag{15.11.5}$$

$$x(\xi) = 0, \ \xi < s. \tag{15.11.6}$$

Let us prove that condition a) implies $C(t, s) > 0$ for $0 \leq s \leq t < \infty$. Let us denote $p(t) = p^+(t) - p^-(t)$, where $p^+(t) \geq 0$, $p^-(t) \geq 0$, and consider the equation

$$x'(t) + p^+(t)x\big(h(t)\big) = 0, \ t \in [s, \infty). \tag{15.11.7}$$

It is clear that the solution $x(t) = C(t, s)$ is equal to the constants on each of the intervals $[t_{2k+1}, t_{2k+2}]$, $k = 0, 1, 2, \cdots$. The inequality (15.11.3) implies according to Corollary (15.4.1) positivity of $x(t) = C(t, s)$ for $[t_{2k}, t_{2k+1}]$, $k = 0, 1, 2, \cdots$.

Similarly it can be proven that condition b) implies $C(t, s) > 0$ for $0 \leq s \leq t < \infty$. □

Inequalities (15.11.3) and (15.11.4) are unimprovable, as the following examples demonstrate.

*Example 15.9* Consider the equation

$$x'(t) + x\big([t]\big) = 0, \ t \in [0, \infty), \tag{15.11.8}$$

where $[t]$ is the integer part of $t$. The solution of this equation is

$$x(t) = C(t,0) = \begin{cases} 1-t, & 0 \le t < 1, \\ 0, & 1 \le t. \end{cases} \tag{15.11.9}$$

*Example 15.10* Consider the equation

$$x'(t) + p(t)x\big([t]\big) = 0, \; t \in [0,\infty), \tag{15.11.10}$$

where

$$p(t) = \begin{cases} b(t), & 2k \le t \le 2k+1, \\ -a(t), & 2k+1 < t < 2k+2, \end{cases} \tag{15.11.11}$$

$a(t) \ge 0$, $b(t) > 1+\varepsilon$, $\varepsilon > 0$. The solution $x(t) = C(t,0)$ changes its sign in every interval $[2k, 2k+1]$, $k = 0, 1, 2, \cdots$.

The following example demonstrates that even positivity of the solution of a homogeneous equation with an oscillating coefficient on a corresponding interval does not imply positivity of its Cauchy function.

*Example 15.11* Consider the equation

$$x'(t) + p(t)x\big(h(t)\big) = f(t), \; t \in \left[0, \frac{7}{2}\right], \tag{15.11.12}$$

where $x(\xi) = 0$ for $\xi < 0$,

$$h(t) = \begin{cases} t-1, & 0 \le t \le 2, \\ 1, & 2 < t \le \frac{7}{2}, \end{cases}$$

$$p(t) = \begin{cases} -1, & 0 \le t \le 2, \\ 1, & 2 < t \le \frac{7}{2}. \end{cases}$$

The solution of the homogeneous equation

$$x'(t) + p(t)x\big(h(t)\big) = 0, \; t \in \left[0, \frac{7}{2}\right], \tag{15.11.13}$$

where $x(\xi) = 0$ for $\xi < 0$, is

$$x(t) = \begin{cases} 1, & t \in [0,1], \\ t, & t \in (1,2], \\ 4-t, & t \in (2, \frac{7}{2}], \end{cases} \tag{15.11.14}$$

and it is positive for $t \in [0, \frac{7}{2}]$.

The Cauchy function $C(t,s)$ for each fixed $s$ can be obtained as a solution of the equation

$$x'(t) + p(t)x\big(h(t)\big) = 0, \; t \in \left[s, \frac{7}{2}\right], \tag{15.11.15}$$

where $x(\xi) = 0$ for $\xi < s$. Considering the cases when $\frac{1}{2} < s < 1$, we get

$$C(t,s) = \begin{cases} 1, & t \in [s, s+1], \\ t - s, & t \in (s+1, 2], \\ 4 - s - t, & t \in (2, \frac{7}{2}]. \end{cases} \tag{15.11.16}$$

We see that $C(t,s)$ changes its sign for $t \in (2, \frac{7}{2}]$.

We can conclude that $C(t,s)$ changes its sign also using another approach. Consider the initial problem

$$x'(t) + p(t)x\big(h(t)\big) = f(t),\ t \in \left[0, \frac{7}{2}\right],\ x(0) = 0, \tag{15.11.17}$$

where

$$f(t) = \begin{cases} 0, \ t \in [0, \frac{1}{2}), \\ 1, \ t \in [\frac{1}{2}, 1], \\ 0, \ t \in (1, \frac{7}{2}]. \end{cases} \tag{15.11.18}$$

The solution of problem (15.11.17) in this case is

$$x(t) = \begin{cases} 0, & t \in [0, \frac{1}{2}], \\ t - \frac{1}{2}, & t \in (\frac{1}{2}, 1], \\ 1, & t \in (1, \frac{3}{2}], \\ \frac{1}{2}[1 + (t - \frac{3}{2})^2], & t \in [\frac{3}{2}, 2), \\ \frac{1}{2}(\frac{13}{4} - t), & t \in [2, \frac{7}{2}]. \end{cases} \tag{15.11.19}$$

It is clear that $x(t) < 0$ for $\frac{13}{4} < t < \frac{7}{2}$. We obtain that for the nonnegative right-hand side of (15.11.18), the solution $x$ changes its sign. This also proves that the Cauchy function $C(t,s)$ changes its sign.

The following assertion is the maximum boundaries principle for equations with oscillating coefficients.

**Theorem 15.25** *Let the condition* a) (b)) *of Theorem* 15.24 *be fulfilled. Then the absolute value of every solution $x$ of the homogeneous equation*

$$x'(t) + p(t)x\big(h(t)\big) = 0,\ t \in [0, \infty), \tag{15.11.20}$$

*where $x(\xi) = 0$ for $\xi < 0$, has its maxima only at the points $t_{2k}$ $(t_{2k+1})$ and its minima only at the points $t_{2k+1}$ $(t_{2k})$, $k = 0, 1, 2, \cdots$.*

*Proof* According to Theorem 15.24, solutions of the homogeneous equation (15.11.20) do not change their signs, which implies that $|x(t)|$ does not increase when $p(t) \geq 0$ and does not decrease when $p(t) \leq 0$. □

The maximum principle obtained in Theorem 15.25 implies various results on the existence of unique solutions of boundary value problems for (15.11.1).

**Theorem 15.26** *Let condition* a) *of Theorem* 15.24 *be satisfied. Then the following assertions are valid*:

1) *If* $l : C[0, \omega] \to \mathbb{R}$ *is a linear nonzero positive functional, then the boundary value problem* (15.11.1), (15.11.2), (15.11.21), *where*

$$lx = c, \tag{15.11.21}$$

*has a unique solution for each* $f \in L[0, \omega]$, $c \in \mathbb{R}$.

2) *Boundary value problem* (15.11.1), (15.11.2), (15.11.22), *where*

$$lx \equiv \sum_{k=1}^{n} \left\{ x(t_{2k}) - m_k x \right\} = c, \tag{15.11.22}$$

*and the norm of every linear functional* $m_k : C[t_{2k-1}, t_{2k}] \to \mathbb{R}$ *is less than one, has a unique solution for each* $f \in L_1[0, \omega]$, $c \in \mathbb{R}$.

3) *The boundary value problem* (15.11.1), (15.11.2), (15.11.23),

$$\sum_{k=0}^{n} \alpha_k x(s_{2k}) = \sum_{k=0}^{n} \beta_k x(s_{2k+1}) + c, \tag{15.11.23}$$

*where the inequalities* $t_{2k} \le s_{2k} < s_{2k+1} \le t_{2k+1}$ *and* $\alpha_k \ge \beta_k \ge 0$ *are satisfied for* $k = 0, 1, 2, \cdots, n$ *and there exists* $j$ *such that* $\alpha_j > \beta_j$, *has a unique solution for each* $f \in L[0, \omega]$, $c \in \mathbb{R}$.

4) *The boundary value problem* (15.11.1), (15.11.2), (15.11.24), *where*

$$\sum_{j=1}^{n} \left\{ \int_{t_{2j-1}}^{t_{2j}} \alpha(t) x(t) dt \right\} = c, \;\; 0 = t_0 \le t_1 < t_2 < \cdots < t_{2n-1} < t_{2n} \le \omega, \tag{15.11.24}$$

*in the case where* $\alpha(t) \ge 0$ *for* $t \in [s_j, t_{2j}]$, $\alpha(t) \le 0$ *for* $t \in [t_{2j-1}, s_j]$, $t_1 < s_1 < t_2, \cdots, t_{2n-1} < s_n < t_{2n}$, $\int_{t_{2j-1}}^{t_{2j}} \alpha(t) dt \ge 0$, $j = 1, \cdots, k$, *and there exists* $j$ *such that* $\int_{t_{2j-1}}^{t_{2j}} \alpha(t) dt > 0$, *has a unique solution for each* $f \in L[0, \omega]$, $c \in \mathbb{R}$.

*Example 15.12* The periodic problem for the equation

$$x'(t) = 0, \;\; t \in [0, \omega], \tag{15.11.25}$$

has the nontrivial solution $x(t) \equiv 1$, $t \in [0, \omega]$. This demonstrates that the condition about existence of $j$ such that $\alpha_j > \beta_j$ is essential.

The location of the points $s_k$ in assertion 3) is essential. If instead of the inequality $t_{2k} \le s_{2k} < s_{2k+1} \le t_{2k+1}$ we assume that $t_{2k} \le s_{2k} < s_{2k+1} \le t_{2k+2}$, then the result on the unique solvability is not true, as the following example demonstrates.

*Example 15.13* Consider the homogeneous problem

$$x'(t) + p(t)x(0) = 0, \;\; 2x\left(\frac{1}{2}\right) = x(1), \;\; t \in [0, 1], \tag{15.11.26}$$

where

$$p(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t \le 1. \end{cases} \tag{15.11.27}$$

This problem has the nontrivial solution

$$x(t) = \begin{cases} 1 - t, & 0 \le t < \frac{1}{2}, \\ t, & \frac{1}{2} \le t \le 1. \end{cases} \tag{15.11.28}$$

Let us obtain sufficient conditions of the exponential stability (see Definition B.17) based on the positivity of the Cauchy function.

Consider (15.11.1) with an oscillating coefficient $p(t)$, changing its sign at the points $t_k$ $(k = 1, 2, 3, \cdots)$, where $t_0 = 0$. Let us assume that there exist positive numbers $c_1$ and $c_2$ such that $c_1 < t_{k+1} - t_k < c_2$ for every $k$.

**Theorem 15.27** *Let the following conditions hold*:

a) *The coefficient* $p(t)$ *satisfies the inequalities* $p(t) \ge 0$ *for* $(t_{2k}, t_{2k+1})$, $p(t) \le 0$ *for* $(t_{2k+1}, t_{2k+2})$ *and* $\int_{t_{2k}}^{t_{2k+1}} p(t)dt < 1$ *for* $k = 0, 1, 2, \cdots$.
b) *The deviating argument* $h(t)$ *satisfies the inequalities* $t_{k-1} \le h(t)$ *for* $t \in [t_k, t_{k+1}]$, $h(t) \le t_{2k-1}$ *for* $t \in [t_{2k-1}, t_{2k}]$.
c) *There exists a number* $\gamma$ *such that*

$$\gamma_{k+1} \equiv \exp\left[-\int_{t_{2k}}^{t_{2k+1}} p(t)\chi\big(h(t), t_{2k}\big)dt\right] + \int_{t_{2k+1}}^{t_{2k+2}} \big|p(t)\big|dt \le \gamma < 1, \ \ k = 0, 1, 2, \cdots, \tag{15.11.29}$$

*where*

$$\chi(t, s) = \begin{cases} 1, & t \ge s, \\ 0, & t < s. \end{cases} \tag{15.11.30}$$

*Then* (15.11.1) *is exponentially stable.*

*Proof* According to Theorem 15.24, conditions a) and b) imply positivity of the Cauchy function $C(t, s)$ of (15.11.1), (15.11.2) for $0 \le s \le t < \infty$. The function

$$u(t) = \begin{cases} \gamma_0 \cdots \gamma_{k-1} \exp[-\int_{t_{2k}}^{t} p(t)\chi(h(s), t_{2k})ds], & t_{2k-2} \le t < t_{2k-1}, \\ \gamma_0 \cdots \gamma_{k-1} \exp[-\int_{t_{2k}}^{t_{2k+1}} p(s)\chi(h(s), t_{2k})ds] & \\ \quad + \int_{t_{2k+1}}^{t} |p(s)|ds, & t_{2k-1} \le t < t_{2k}, \end{cases} \tag{15.11.31}$$

$k = 1, 2, \cdots$, where $\gamma_0 = 1$, satisfies the inequality $(Mu)(t) \ge 0$. The positivity of $C(t, s)$ implies that $u(t) \ge x(t)$ for $t \in [0, \infty)$, where the function $x$ is a solution of the initial problem $(Mx)(t) = 0$, $t \in [0, \infty)$, $x(0) = 1$. □

*Remark 15.17* The inequality

$$\exp\left[-\int_{t_{2k}}^{t_{2k+1}} p(t)\chi\big(h(t), t_{2k}\big)dt\right] + \int_{t_{2k+1}}^{t_{2k+2}} \big|p(t)\big|dt < 1, \; k = 0, 1, 2, \cdots, \tag{15.11.32}$$

cannot be set instead of condition c), as the following example demonstrates.

*Example 15.14* Consider (15.11.20), where $h(t) \equiv t$,

$$p(t) = \begin{cases} \frac{1}{t^2}, \; 2k \le t < 2k+1, \\ 0, \;\; 2k+1 \le t < 2k+2, \end{cases} \quad k = 1, 2, \cdots. \tag{15.11.33}$$

Its nontrivial solutions tend to constants when $t \to \infty$. Note that condition c) avoids the possibility that $\lim_{k\to\infty} \int_{t_{2k}}^{t_{2k+1}} p(t)\chi(h(t), t_{2k})dt = 0$.

**Theorem 15.28** *Suppose that conditions* a) *and* b) *of Theorem* 15.27 *are fulfilled, the deviating argument* $h(t)$ *satisfies the inequality* $t - h(t) \le \tau$ *for* $t \in [0, \infty)$ *and there exists a number* $\gamma$ *such that*

$$\exp\left[-\int_{t_{2k}}^{t_{2k+1}} p(t)\chi\big(h(t), t_{2k}\big)dt\right] \\ \times \left\{1 + \exp\left[\int_{t_{2k+1}-\tau}^{t_{2k+1}} p(\xi)d\xi\right] \int_{t_{2k+1}}^{t_{2k+2}} \big|p(t)\big|dt\right\} \le \gamma < 1 \tag{15.11.34}$$

*for* $k = 0, 1, 2, \cdots$. *Then* (15.11.1) *is exponentially stable.*

*Example 15.15* Assume that $h(t) \ge 0$, $t_{2k-1} \le h(t)$ for $t \in [t_{2k}, t_{2k+1}]$, $p(t+2) = p(t)$, where

$$p(t) = \begin{cases} \ln(1+t), \; 0 \le t \le 1, \\ -\mu, \qquad\quad 1 < t < 2. \end{cases} \tag{15.11.35}$$

If $0 < \mu < \frac{4-e}{4}$, then (15.11.1) with this coefficient $p(t)$ is exponentially stable.

Consider now the generalized periodic problem

$$x'(t) + p(t)x\big(h(t)\big) = f(t), \; t \in [0, \omega], \tag{15.11.36}$$

$$x(\xi) = 0, \; \xi < 0, \tag{15.11.37}$$

$$x(0) = \beta x(\omega), \tag{15.11.38}$$

with oscillating coefficient $p(t)$ changing its sign at the points $t_k$, where $k = 1, 2, 3, \cdots, 2m-1$, of the interval $[0, \omega]$. Denote $t_0 = 0$ and $t_{2m} = \omega$.

**Theorem 15.29** *Let* $p(t) \ge 0$ *for* $t \in [0, t_1]$ *and* $p(t) \le 0$ *for* $t \in (t_1, \omega]$, $h(t) \le t_1$ *for* $t \in [t_1, \omega]$, $C(t, s) > 0$ *for* $0 \le s \le t \le \omega$, *and the inequality*

$$\left\{1 - \exp\left[\int_{t_1-\tau^*}^{t_1} p(\xi)\chi_0\big(h(\xi)\big)d\xi\right]\int_{t_1}^{\omega} p(\xi)d\xi\right\}$$
$$\times \exp\left\{-\int_0^{t_1} p(\xi)\chi_0\big(h(\xi)\big)d\xi\right\} < \frac{1}{\beta}, \tag{15.11.39}$$

*where* $\tau^* = \operatorname{ess\,sup}_{t\in[0,\omega]}\{t - h(t)\}$, *be fulfilled. Then* $C(\omega, 0) < \frac{1}{\beta}$ *and the Green's function* $G(t, s)$ *of the generalized periodic problem* (15.11.36), (15.11.38) *is positive for* $t, s \in [0, \omega]$.

*Proof* Using the conditions $C(t, s) > 0$ for $0 \le s \le t \le \omega$ and $p(t) \ge 0$ for $t \in [0, t_1]$, we obtain according to Theorem 15.5 that $C(t, 0) \le \exp\{-\int_0^t p(\xi) \times \chi_0(h(\xi))d\xi\}$ for $t \in [0, t_1]$. Using the fact that $C(t, 0)$ decreases for $t \in [0, t_1]$ and the condition $h(t) \le t_1$ for $t \in [t_1, \omega]$, we obtain

$$C(t, 0) \le \exp\left\{-\int_0^{t_1} p(\xi)\chi_0\big(h(\xi)\big)d\xi\right\}$$
$$- \exp\left[-\int_0^{t_1-\tau^*} p(\xi)\chi_0\big(h(\xi)\big)d\xi\right]\int_{t_1}^{\omega} p(\xi)d\xi. \tag{15.11.40}$$

Inequality (15.11.39) implies that $C(\omega, 0) < \frac{1}{\beta}$. Now the representation

$$G(t, s) = C(t, s) + \frac{\beta C(\omega, s)}{1 - \beta C(\omega, 0)} C(t, 0), \tag{15.11.41}$$

where $C(t, s) = 0$ for $t < s$, implies the positivity of the Green's function of the generalized periodic problem (15.11.36), (15.11.38). □

**Theorem 15.30** *Let the following conditions hold*:

a) *The coefficient* $p(t)$ *satisfies the inequalities* $p(t) \ge 0$ *for* $(t_{2k}, t_{2k+1})$, $p(t) \le 0$ *for* $(t_{2k+1}, t_{2k+2})$ *and* $\int_{t_{2k}}^{t_{2k+1}} p(t)dt < 1$ *for* $k = 0, 1, 2, \cdots, m - 1$.
b) *The deviating argument* $h(t)$ *satisfies the inequalities* $t_{k-1} \le h(t)$ *for* $t \in [t_k, t_{k+1}]$, $h(t) \le t_{2k-1}$ *for* $t \in [t_{2k-1}, t_{2k}]$.
c) *The inequality* $\gamma_1 \cdots \gamma_m < \frac{1}{\beta}$ *is satisfied, where* $\gamma_k$ *are defined by* (15.11.29).

*Then the Green's function* $G(t, s)$ *of the generalized periodic problem* (15.11.36), (15.11.38) *is positive for* $t, s \in [0, \omega]$.

*Proof* According to Theorem 15.24, conditions a) and b) imply the positivity of the Cauchy function $C(t, s)$ for $0 \le s \le t \le \omega$. Now it is clear that the function $u(t)$ defined by formula (15.11.31) satisfies the inequality $u(t) \ge C(t, 0)$ for $t \in [0, \omega]$. Condition c) implies the inequality $C(\omega, 0) < \frac{1}{\beta}$ and consequently the unique solvability of problem (15.11.36), (15.11.38). The Green's function of problem (15.11.36), (15.11.38) has representation (15.11.41), where $C(t, s) = 0$ for $t < s$. The inequalities $C(t, s) > 0$ for $0 \le s \le t \le \omega$ and $C(\omega, 0) < \frac{1}{\beta}$ imply that $G(t, s) > 0$ for $t, s \in [0, \omega]$. □

## 15.12 Positivity of the Cauchy Function and Exponential Stability

The aim of this section is to demonstrate how results on nonoscillation can be adopted to study exponential stability of functional differential equations.

Let us consider the equation

$$x'(t) - (A_1x)(t) - (A_2x)(t) + (Bx)(t) = f(t),\ t \in [0, \infty), \tag{15.12.1}$$

where the linear operators $A_1$, $A_2$, $B$ are defined by the equalities

$$(A_1x)(t) = \int_0^t x(s)d_sA_1(t,s),$$

$$(A_2x)(t) = \int_0^t x(s)d_sA_2(t,s),$$

$$(Bx)(t) = \int_0^t x(s)d_sB(t,s),$$

with the functions $A_1(t,s)$, $A_2(t,s)$ and $B(t,s)$ satisfying the conditions that $A_1(\cdot,s)$, $A_2(\cdot,s)$ and $B(\cdot,s) : [0,\omega] \to \mathbb{R}$ are measurable for $s \in [0,\omega]$, $A_1(t,\cdot)$, $A_2(t,\cdot)$ and $B(t,\cdot) : [0,\omega] \to \mathbb{R}$ have a bounded variation for almost all $t \in [0,\omega]$ and $\mathrm{Var}_{s=0}^t A_1(t,s)$, $\mathrm{Var}_{s=0}^t A_2(t,s)$ and $\mathrm{Var}_{s=0}^t B(t,s)$ are essentially bounded for every positive $\omega$. It is clear that $A_1$, $A_2$, $B$ are bounded Volterra operators on each finite interval $[0,\omega]$, and let us assume that the operators $A_1$ and $B$ are positive, $f \in L_\infty[0,\infty)$.

Let us define the operator $M : D[0,\infty) \to L_\infty[0,\infty)$, where $D[0,\infty)$ is the space of functions absolutely continuous on each interval $[0,\omega]$ such that their derivative $x' \in L_\infty[0,\infty)$, by the formula

$$(Mx)(t) \equiv x'(t) - (A_1x)(t) - (A_2x)(t) + (Bx)(t),\ t \in [0,\infty). \tag{15.12.2}$$

In order to formulate several convenient exponential stability tests for (15.12.1), we introduce the following definitions.

By Theorem A.11, the operator $A_2 : C[0,\infty) \to L_\infty[0,\infty)$ has the representation

$$(A_2x)(t) = \int_0^t x(s)d_sA_2(t,s),$$

where $A_2(t,s) = A_2^+(t,s) - A_2^-(t,s)$, and for each $t \in [0,\infty)$ the functions $A_2^+(t,\cdot)$ and $A_2^-(t,\cdot)$ are nondecreasing.

Consider the auxiliary equations

$$(M_0x)(t) \equiv x'(t) - (A_1x)(t) + (Bx)(t) = f(t),\ t \in [0,\infty), \tag{15.12.3}$$

and

$$\big(M^+x\big)(t) \equiv x'(t) - (A_1x)(t) - \big(|A_2|x\big)(t) + (Bx)(t) = f(t),\ t \in [0,\infty), \tag{15.12.4}$$

where the operator $|A_2|$ is defined by the formula

$$\big(|A_2|x\big)(t) = \int_0^t x(s)d_s\big(A_2^+(t,s) + A_2^-(t,s)\big),\ t \in [0,\infty). \tag{15.12.5}$$

In the following theorem, we use weighted spaces, see Definition A.3.

**Theorem 15.31** *Assume that the Cauchy function $C_0(t,s)$ of (15.12.3) is positive for $0 \le s \le t < \infty$, $A_1$ and $B$ are positive operators, for some $\lambda > 0$ we have $A_1, A_2, B : C^\lambda[0,\infty) \to L_\infty^\lambda[0,\infty)$ and there exist positive $\varepsilon$ and nonnegative $t_0$ such that the following inequality is satisfied:*

$$(A_1 1)(t) + \big(|A_2|1\big)(t) + \varepsilon \le (B1)(t),\ t \in [t_0,\infty). \tag{15.12.6}$$

*Then (15.12.1) is exponentially stable and its Cauchy function $C(t,s)$ satisfies the inequality*

$$\lim_{t\to\infty} \int_0^t \big|C(t,s)\big| ds \le \frac{1}{\varepsilon}. \tag{15.12.7}$$

*Proof* Without loss of generality, we consider the case $t_0 = 0$. Inequality (15.12.6) implies that the function $v(t) \equiv \frac{1}{\varepsilon}$ satisfies the inequality $f^+(t) \equiv (M^+\frac{1}{\varepsilon})(t) \ge 1$ for $t \in [0,\infty)$. Equation (15.12.4) is equivalent to the integral equation

$$x(t) = (Kx)(t) + \psi(t),\ t \in [0,\omega], \tag{15.12.8}$$

where the operator $K$ is defined by the equality

$$(Kx)(t) = \int_0^t C_0(t,s)\big(|A_2|x\big)(s)ds \tag{15.12.9}$$

and

$$\psi(t) = \int_0^t C_0(t,s)f(s)ds + C_0(t,0)\gamma, \tag{15.12.10}$$

where $\gamma = x(0)$. Positivity of the Cauchy function $C_0(t,s)$ of (15.12.3) implies that the operator $K$ is positive. Nonnegativity of the function $f^+$, positivity of the Cauchy function $C_0(t,s)$ and the equality $\gamma = v(0) = \frac{1}{\varepsilon}$ imply positivity of the function $\psi$.

By Lemma A.2, if there exists a positive continuous function $v$ such that $v - Kv > 0$, where $K$ is a positive operator, then the spectral radius $\rho(K)$ of the operator $K$ is less than one. We have demonstrated that all conditions of this theorem are fulfilled, and now we obtain that $\rho(K) < 1$. This implies the positivity of the Cauchy function $C(t,s)$ of (15.12.4), and consequently the fact that for each $|f(t)| \le 1$, where $t \in [0,\infty)$, the solution $x$ of (15.12.4) satisfies the inequality $-\frac{1}{\varepsilon} \le x(t) \le \frac{1}{\varepsilon}$ for $t \in [0,\infty)$.

According to Theorem A.13, the spectral radius of the operator $K_1 : C[0,\infty) \to C[0,\infty)$ defined by the formula

$$(K_1x)(t) = \int_0^t C_0(t,s)(A_2x)(s)ds \tag{15.12.11}$$

is less than one. Equation (15.12.1) is equivalent to the integral equation

$$x(t) = (K_1 x)(t) + \psi(t), \ t \in [0, \infty), \tag{15.12.12}$$

where the function $\psi$ is defined by the equality (15.12.10). If we set $f = f^+$, then for solutions $x$ of (15.12.12) and $y$ of (15.12.8) we get the inequality $|x(t)| \le y(t)$, $t \in [0, \infty)$. This proves inequality (15.12.7).

From solution representation formula (15.3.7), it is now clear that for each bounded $f$ a solution $x$ of (15.12.1) is bounded. Hence, by Theorem B.20, (15.12.1) is exponentially stable. □

Now we propose several explicit tests of the exponential stability obtained for the equation

$$x'(t) - a_1(t)x\big(g_1(t)\big) - a_2(t)x\big(g_2(t)\big) + b(t)x\big(h(t)\big) = f(t), \ t \in [0, \infty), \tag{15.12.13}$$

where

$$x(\xi) = 0 \text{ for } \xi < 0, \tag{15.12.14}$$

on the basis of the results of Section 15.10 on positivity of the Cauchy function $C(t, s)$.

Let us assume that $t - g_1(t)$, $t - g_2(t)$, $t - h(t)$ are bounded (in this case the operator $M$ defined by formula (15.12.2) satisfies the $\delta$-condition) and denote $\tau(t) = t - h(t)$, $\sigma_1(t) = t - g_1(t)$, $\tau_* = \inf_{t \in [0,\infty)} \tau(t)$, $\sigma^* = \sup_{t\in[0,\infty)} \sigma_1(t)$.

**Theorem 15.32** *Assume that $a_1(t) \ge 0, t - h(t) \le \delta, t - g_1(t) \le \delta, t - g_2(t) \le \delta$ for $t \in [0, \infty)$, there exists a positive $\varepsilon$ such that $b(t) \ge a_1(t) + |a_2(t)| + \varepsilon$ and at least one of the following conditions a) or b) is fulfilled*:

a) $h(t) \ge g_1(t)$ *for* $t \in [0, \infty)$,

$$\int_{h(t)}^{t} \big[b(\xi) - a_1(\xi)\big] d\xi \le \frac{1}{e}, \ t \in [0, \infty), \tag{15.12.15}$$

$$\sup_{s \in [0,\infty)} \int_{s+\tau_*}^{s+\sigma^*} b(\xi) d\xi \le \frac{1}{e}. \tag{15.12.16}$$

b) $h(t) \le g_1(t)$ *for* $t \in [0, \infty)$,

$$\sup_{t \in [0,\infty)} \int_{h(t)}^{t} \left\{ b(\xi) - \frac{1}{e} a_1(\xi) \right\} d\xi < \frac{1}{e}. \tag{15.12.17}$$

*Then* (15.12.13) *is exponentially stable*, *and its Cauchy function $C(t, s)$ satisfies the exponential estimate and the inequality*

$$\lim_{t \to \infty} \int_0^t \big|C(t, s)\big| ds \le \frac{1}{\varepsilon}. \tag{15.12.18}$$

*Proof* The proof follows from Corollary 15.9 and Theorems 15.21 and 15.31. □

Consider now the autonomous delay differential equation

$$x'(t) - a_1 x(t-\sigma_1) - a_2 x(t-\sigma_2) + bx(t-\tau) = f(t), \ t \in [0,\infty), \quad (15.12.19)$$

with the initial condition (15.12.14).

**Theorem 15.33** *Let $a_1 > 0$ and one of the following conditions a) or b) be fulfilled:*

a) $\tau < \sigma_1$, $(b-a_1)\tau \le \frac{1}{e}$ *and* $b(\sigma_1 - \tau) \le \frac{1}{e}$.
b) $\tau > \sigma_1$, $(b - \frac{1}{e}a_1)\tau < \frac{1}{e}$.

*Then:*

1) *Equation* (15.12.19) *is exponentially stable and its Cauchy function $C(t,s)$ satisfies the exponential estimate if $b - a_1 - |a_2| > 0$.*
2) *If in addition we assume that $a_2 > 0$, then* (15.12.19) *is exponentially stable and its Cauchy function $C(t,s)$ satisfies the exponential estimate if and only if $b - a_1 - a_2 > 0$.*
3) $\varepsilon = b - a_1 - |a_2| > 0$, *the Cauchy function $C(t,s)$ of* (15.12.19) *satisfies inequality* (15.12.18), *and if $a_2 > 0$, then*

$$\lim_{t\to\infty} \int_0^t \big|C(t,s)\big| ds = \frac{1}{\varepsilon}. \quad (15.12.20)$$

*Proof* Assertion 1), sufficiency in assertion 2), and inequality (15.12.18) follow from Theorems 15.23 and 15.32.

**Necessity in assertion 2).** The function $x(t) \equiv 1$ is a solution of (15.12.19), where

$$f(t) = \begin{cases} \varepsilon[a_1(1-\chi(t-\sigma_1) + a_2(1-\chi(t-\sigma_2) - b(1-\tau)\chi(t-\tau))], & 0 \le t \le t_0, \\ \varepsilon, & t_0 < t, \end{cases}$$

where $t_0 = \max\{\sigma_1, \sigma_2, \tau\}$.

By the solution representation formula, we can write

$$1 = \int_0^{t_0} C(t,s) f(s) ds + \varepsilon \int_{t_0}^t C(t,s) ds + C(t,0), \quad (15.12.21)$$

and using the conditions that the solution of the homogeneous equation ($f \equiv 0$) corresponding to (15.12.19) tends to zero when $t \to \infty$ and the Cauchy function $C(t,s)$ of (15.12.19) satisfies the exponential estimate, we obtain

$$1 = \varepsilon \lim_{t\to\infty} \int_{t_0}^t C(t,s) ds. \quad (15.12.22)$$

The Cauchy function $C(t,s)$ of (15.12.19) is positive for $0 \le s \le t < \infty$ [178]. Thus the formula (15.12.22) implies $\varepsilon > 0$.

To finish the proof of Theorem 15.33, we note that equality (15.12.20) follows from (15.12.22). □

## 15.13 General Boundary Value Problems

In this section, we will consider the cases of positive and negative Volterra operators, as well as non-Volterra operators.

Let us consider the boundary value problem

$$(Mx)(t) \equiv x'(t) + (Bx)(t) = f(t), \ t \in [0, \omega], \tag{15.13.1}$$

$$lx \equiv \theta x(0) + \int_0^\omega \phi(s) x'(s) ds = c, \tag{15.13.2}$$

where $B : C[0, \omega] \to L_\infty[0, \omega]$ is a linear continuous operator ($C[0, \omega]$ and $L_\infty[0, \omega]$ are the spaces of continuous and measurable essentially bounded functions $[0, \omega] \to \mathbb{R}$, respectively), $\theta \in \mathbb{R}$ and $\phi \in L_\infty[0, \omega]$. It should be noted that formula (15.13.2) defines the general form of a functional $l : D[0, \omega] \to \mathbb{R}$, where $D[0, \omega]$ is the space of absolutely continuous functions $x : [0, \omega] \to \mathbb{R}$. In this case, the operator $B$ can be represented in the form [216]

$$(Bx)(t) = \int_0^\omega x(s) d_s b(t, s), \ t \in [0, \omega].$$

The ideas of the proof of Theorem 15.3 can be extended to the non-Volterra equation (15.13.1). Let us formulate several results obtained due to this extension. Note that similar assertions by a similar method were also obtained in the monograph [197]. The difference is that in our approach a general form of the functional $l : D[0, \omega] \to \mathbb{R}$ is used. This allows us to consider conditions more general than the general periodic conditions (for example, conditions with integrals, which are important in the case where a boundary condition describes preservation of energy).

Let us define the operator $N : C[0, \omega] \to C[0, \omega]$ by the formula

$$(Nx)(t) = -\int_0^\omega G_0(t, s)(Bx)(s) ds, \ t \in [0, \omega], \tag{15.13.3}$$

where $G_0(t, s)$ is the Green's function of the equation

$$x'(t) = f(t), \ t \in [0, \omega], \tag{15.13.4}$$

with the boundary condition (15.13.2), and the operator $K : L_\infty[0, \omega] \to L_\infty[0, \omega]$ by the formula

$$(Kz)(t) = -(BWz)(t), \ t \in [0, \omega], \tag{15.13.5}$$

where

$$(Wz)(t) = \int_0^\omega G_0(t, s) z(s) ds, \ t \in [0, \omega]. \tag{15.13.6}$$

Using the idea of reducing boundary value problem (15.13.1), (15.13.2) to integral equations $x(t) = (Nx)(t) + \varphi(t)$, $t \in [0, \omega]$ in the space $C[0, \omega]$ and to the equation $z(t) = (Kz)(t) + \psi(t)$, $t \in [0, \omega]$ in the space $L_\infty[0, \omega]$ with the positive operators $N : C[0, \omega] \to C[0, \omega]$ and $K : L_\infty[0, \omega] \to L_\infty[0, \omega]$, respectively, we obtain the following result.

**Theorem 15.34** *Assume that* $\theta \neq 0$, $G_0(t,s)$ *preserves its sign and the operator* $N : C[0,\omega] \to C[0,\omega]$ *is positive. Then the following hypotheses are equivalent*:

1) *There exists an absolutely continuous positive function* $v$ *such that*

$$\psi(t) \equiv v(0) + \frac{1}{\theta}\int_0^\omega \phi(s)v'(s)ds + \int_0^\omega G_0(t,s)(Mv)(s)ds \geq 0, \ t \in [0,\omega], \tag{15.13.7}$$

*and the set of zeros of* $\psi$ *is not more than countable and* $\psi(s) > 0$ *if* $\text{mes}\{t \in [0,\omega] : b(t,s+) \neq b(t,s-)\} > 0$.
2) *Problem* (15.13.1), (15.13.2) *has a unique solution and its Green's function* $G(t,s)$ *satisfies the inequalities* $G(t,s) > 0$ *if* $G_0(t,s) > 0$ *for* $t,s \in (0,\omega)$ *and* $G(t,s) < 0$ *if* $G_0(t,s) < 0$ *for* $t,s \in (0,\omega)$.
3) *There exists a positive function* $u \in C[0,\omega]$ *such that* $\varphi(t) \equiv u(t) - (Nu)(t) \geq 0$ *for* $t \in [0,\omega]$ *and the set of zeros of* $\varphi$ *is not more than countable and* $\varphi(s) > 0$ *if* $\text{mes}\{t \in [0,\omega] : b(t,s+) \neq b(t,s-)\} > 0$.
4) *The spectral radius of the operator* $N : C[0,\omega] \to C[0,\omega]$ *is less than one*.
5) *The spectral radius of the operator* $K : L_\infty[0,\omega] \to L_\infty[0,\omega]$ *is less than one*.

*Remark 15.18* In order to obtain sufficient conditions of sign constancy of the Green's function $G(t,s)$, we usually assume that

$$v(0) + \frac{1}{\theta}\int_0^\omega \phi(s)v'(s)ds > 0, \ t \in [0,\omega], \tag{15.13.8}$$

and

$$\varepsilon(t) \equiv \int_0^\omega G_0(t,s)(Mv)(s)ds \geq 0, \ t \in [0,\omega]. \tag{15.13.9}$$

It is clear that $\varepsilon(t) \geq 0$ in the case $(Mv)(t) \geq 0$ and $G_0(t,s) \geq 0$ for $t,s \in [0,\omega]$ and in the case $(Mv)(t) \leq 0$ and $G_0(t,s) \leq 0$ for $t,s \in [0,\omega]$.

Various corollaries can be obtained on the basis of assertions 4) and 5). Let us formulate some of them.

**Theorem 15.35** *Let* $B$ *be a positive operator,* $\phi(s) > \theta > 0$ *for* $s \in [0,\omega]$ *and the inequality*

$$\operatorname*{ess\,sup}_{t\in[0,\omega]} B\left[-t + \frac{1}{\theta}\int_0^\omega \phi(s)ds\right](t) < 1 \tag{15.13.10}$$

*be fulfilled. Then the Green's function* $G(t,s)$ *of problem* (15.13.1), (15.13.2) *is negative for* $t,s \in (0,\omega)$, *and for each nonnegative* $f$ *the unique solution of this problem satisfies the inequalities* $x(t) \leq 0$, $t \in [0,\omega]$ *for* $c \leq 0$ *and* $x(t) < 0$, $t \in [0,\omega]$ *for* $c < 0$.

*Remark 15.19* It is clear that the inequality

$$\left(\int_0^\omega \phi(s)ds\right) \operatorname*{ess\,sup}_{t\in[0,\omega]} B1(t) < \theta, \ t \in [0,\omega], \tag{15.13.11}$$

implies inequality (15.13.10).

*Remark 15.20* Note that inequality (15.13.10) can be better than (15.13.11). If, for example, the operator $B$ is of the form $(Bx)(t) \equiv p(t)x(h(t))$, where $0 \leq h(t) \leq \omega$, then we have

$$\operatorname*{ess\ sup}_{t\in[0,\omega]} p(t)\left[\frac{1}{\theta}\int_0^{\omega}\phi(s)ds - h(t)\right] < 1. \tag{15.13.12}$$

In the particular case $(Bx)(t) \equiv p(t)x(\omega)$, we obtain the following condition for negativity of Green's function:

$$-\left[\frac{1}{\theta}\int_0^{\omega}\phi(s)ds - \omega\right] \operatorname*{ess\ sup}_{t\in[0,\omega]} p(t) < 1. \tag{15.13.13}$$

**Theorem 15.36** *Let $(-B)$ be a positive operator, $\frac{\phi(s)}{\theta} \leq 0$ for $s \in [0,\omega]$, and the inequality*

$$\operatorname*{ess\ sup}_{t\in[0,\omega]} -B\left[t - \frac{1}{\theta}\int_0^{\omega}\phi(s)ds\right](t) < 1 \tag{15.13.14}$$

*be fulfilled. Then the Green's function $G(t,s)$ of problem* (15.13.1), (15.13.2) *is positive for $t, s \in (0,\omega)$, and for each nonnegative $f$ the unique solution of this problem satisfies the inequalities $x(t) \geq 0$, $t \in [0,\omega]$ for $\frac{c}{\theta} \geq 0$ and $x(t) > 0$, $t \in [0,\omega]$ for $\frac{c}{\theta} > 0$.*

*Remark 15.21* Note that the inequality

$$\left[\omega - \frac{1}{\theta}\int_0^{\omega}\phi(s)ds\right] \operatorname*{ess\ sup}_{t\in[0,\omega]} -(B1)(t) < 1 \tag{15.13.15}$$

implies the inequality (15.13.14).

*Remark 15.22* Note that inequality (15.13.14) can be better than (15.13.15). If, for example, the operator $B$ is of the form $(Bx)(t) \equiv p(t)x(h(t))$, where $0 \leq h(t) \leq \omega$, then we get

$$\operatorname*{ess\ sup}_{t\in[0,\omega]} -p(t)\left[\frac{1}{\theta}\int_0^{\omega}\phi(s)ds - h(t)\right] < 1. \tag{15.13.16}$$

In the particular case $(Bx)(t) \equiv p(t)x(0)$, we obtain the following condition for positivity of Green's function:

$$\left|\frac{1}{\theta}\int_0^{\omega}\phi(s)ds\right| \operatorname*{ess\ sup}_{t\in[0,\omega]} -p(t) < 1. \tag{15.13.17}$$

Consider now (15.13.1) with the boundary condition

$$\varsigma x(0) + \int_0^{\omega}\eta(s)x(s)ds = c. \tag{15.13.18}$$

**Theorem 15.37** *Let the operator* $-B$ *be positive and the inequalities* $\eta(s) \leq 0$ *for* $s \in [0,\omega]$, $-\int_0^{\omega} \eta(\xi)d\xi < \varsigma$ *and*

$$\operatorname*{ess\,sup}_{t\in[0,\omega]} -B\left[t - \frac{1}{\varsigma + \int_0^{\omega}\eta(\xi)d\xi}\int_0^{\omega}\int_s^{\omega}\eta(\xi)d\xi ds\right](t) < 1 \qquad (15.13.19)$$

*be satisfied. Then the Green's function* $G(t,s)$ *of the problem* (15.13.1), (15.13.18) *is positive for* $t,s \in (0,\omega)$ *and for each nonnegative* $f$ *the unique solution of this problem satisfies the inequalities* $x(t) \geq 0$, $t \in [0,\omega]$ *for* $c \geq 0$ *and* $x(t) > 0$, $t \in [0,\omega]$ *for* $c > 0$.

*Remark 15.23* The inequality

$$\left[\omega - \frac{1}{\varsigma + \int_0^{\omega}\eta(\xi)d\xi}\int_0^{\omega}\int_s^{\omega}\eta(\xi)d\xi ds\right]\operatorname*{ess\,sup}_{t\in[0,\omega]} -(B1)(t) < 1 \qquad (15.13.20)$$

implies inequality (15.13.19).

Now consider (15.13.1) with the boundary condition

$$\lambda x(0) - \mu x(\omega) = c. \qquad (15.13.21)$$

**Theorem 15.38** *Let the operator* $-B$ *be positive and the inequalities* $0 < \mu < \lambda$ *and*

$$-\lambda\int_0^{\omega}(B1)(s)ds < \lambda - \mu \qquad (15.13.22)$$

*be satisfied. Then the Green's function* $G(t,s)$ *of problem* (15.13.1), (15.13.21) *is positive for* $t,s \in (0,\omega)$ *and for each nonnegative* $f$ *the unique solution of this problem satisfies the inequalities* $x(t) \geq 0$, $t \in [0,\omega]$ *for* $c \geq 0$ *and* $x(t) > 0$, $t \in [0,\omega]$ *for* $c > 0$.

*Remark 15.24* This assertion coincides with the corresponding statement in Remark 2.5 in the monograph [197].

**Theorem 15.39** *Let the operator* $B$ *be positive and the inequalities* $0 < \lambda < \mu$ *and*

$$\mu\int_0^{\omega}(B1)(s)ds < \mu - \lambda \qquad (15.13.23)$$

*be satisfied. Then the Green's function* $G(t,s)$ *of problem* (15.13.1), (15.13.21) *is negative for* $t,s \in (0,\omega)$ *and for each nonnegative* $f$ *the unique solution of this problem satisfies the inequalities* $x(t) \leq 0$, $t \in [0,\omega]$ *for* $c \geq 0$ *and* $x(t) < 0$, $t \in [0,\omega]$ *for* $c > 0$.

## 15.14 Discussion and Open Problems

For the first time, results on the comparison of solutions of delay differential equations and corresponding inequalities (theorems on differential inequalities) for the

Cauchy problem were obtained in the classical book by A.D. Myshkis [289] without any connection with the integral representation of solutions and their nonoscillation properties that can be explained by the definition of the homogeneous equation he used. Describing delay differential equations

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) = 0,\ \ \tau_i > 0,\ \ t \in [0,\omega], \tag{15.14.1}$$

we have to define what should be substituted instead of $x(t-\tau_i)$ when $t-\tau_i < 0$. This means that we have to add the equality

$$x(s) = \varphi(s) \text{ for } s < 0 \tag{15.14.2}$$

to (15.14.1), where $\varphi$ is a corresponding continuous function, in the description of the delay equation.

The important step in the nonoscillation study for delay differential equations was made in the paper [20], where a homogeneous object was first defined as (15.14.1) with the zero initial function

$$x(\xi) = 0 \text{ for } \xi < 0. \tag{15.14.3}$$

The delay equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x\big(t-\tau_i(t)\big) = 0,\ \ t \in [0,\omega], \tag{15.14.4}$$

where $\tau_i(t) \geq 0$ for all $t \geq 0$, $i = 1, \cdots, n$, with the initial function (15.14.3), was defined as a homogeneous object. The space of its solutions becomes one-dimensional. This allows us to define the nonoscillation interval as an interval where a nontrivial solution of (15.14.4) has no zeros. Equation (15.14.4), (15.14.3) can be handled as a homogeneous equation in the theory of ordinary differential equations. The formula for representation of the general solution of the equation

$$(Mx)(t) \equiv x'(t) + \sum_{i=1}^{n} p_i(t)x\big(t-\tau_i(t)\big) = f(t),\ \ t \in [0,\omega], \tag{15.14.5}$$

with the initial function (15.14.3), was first obtained in the early 1970s (see, for example, [20]) in the form

$$x(t) = \int_0^t C(t,s)f(s)ds + C(t,0)x(0), \tag{15.14.6}$$

where $C(t,s)$ as a function of $t$ for each fixed $s$ is a solution of the equation

$$x'(t) + \sum_{i=1}^{n} p_i(t)x\big(t-\tau_i(t)\big) = 0,\ \ t \in [s,\omega], \tag{15.14.7}$$

$$x(\xi) = 0 \text{ for } \xi < s, \tag{15.14.8}$$

satisfying the condition $C(s,s) = 1$. $C(t,s)$ is called the Cauchy function of (15.14.5). Formula (15.14.6) reduces theorems on the differential inequality for the Cauchy problem to positivity of the Cauchy function $C(t,s)$.

The idea to construct an approximate integration method for solving differential equations on the basis of comparison of solutions of equations and inequalities appeared first in the works of the famous Russian mathematician S.A. Chaplygin [89]. Publications on this problem were started by another famous Russian mathematician, N.N. Luzin [278]. Later this idea was developed by the mathematicians N.V. Azbelev, I.T. Kiguradze, V. Lakshmikantham and their groups (see, for example, [29, 223, 250]).

The first use of sign properties of Green's functions in the integral representations of solutions to boundary value problems for functional differential equations was proposed by N.V. Azbelev [20]. His definition of the homogeneous equation allows us to study maximum principles and to construct a theory of boundary value problems for delay differential equations and functional differential equations as a natural generalization of corresponding results for ordinary differential equations. For first-order functional differential equations, various assertions on positivity of Green's functions of the initial value problem, periodic problem and some other problems were obtained in the works [4, 81, 178, 197].

The connection between nonoscillation on the interval $[0,\omega]$ and differential inequalities for several boundary value problems was first proven in [111], where equivalence for (15.14.5) of the following facts in the case where $p_i \geq 0$ and $\tau_i \geq 0$ was obtained:

1) $[0,\omega]$ is the nonoscillation interval of (15.14.4), (15.14.3).
2) The Cauchy function $C(t,s)$ of (15.14.5), (15.14.3) is positive for $0 \leq s \leq t \leq \omega$.
3) Problem (15.14.5), (15.14.3), (15.14.9), where

$$x(\omega)=0, \tag{15.14.9}$$

has a unique solution, and its Green's function $G(t,s)$ is negative for $0 \leq t < s \leq \omega$ and nonpositive for $0 \leq s \leq t \leq \omega$.

4) There exists a nonnegative absolutely continuous function $v$ such that

$$v'(t) + \sum_{i=1}^{n} p_i(t) v\big(t-\tau_i(t)\big) \leq 0 \text{ for } t \in [0,\omega],\ \ v(\omega) > 0. \tag{15.14.10}$$

The implication 4) ⇒ 1) is an analogue for the first-order functional differential equations of the classical de La Vallee Poussin theorem on the differential inequality obtained in [102] for ordinary second-order equations. The assertions 1) ⇒ 2) and 1) ⇒ 3) are analogues of the corresponding results connecting nonoscillation and positivity of Green's functions for $n$-th-order ordinary differential equations [256, 333].

The theorem on equivalences was generalized on impulsive equations with impulses at fixed points in [126], with impulses at variable points in [129] and to equations with state-dependent delays in [130].

The fact that (15.14.5) is a delay equation allows us to obtain a corresponding result on equivalence of these facts on the semiaxis. For example, the choice of $v(t) = \exp\{-e\int_0^t \sum_{i=1}^n p_i(s)ds\}$ in assertion 4) leads to the inequality

$$\int_{H(t)}^{t} \sum_{k=1}^{n} p_k(s)ds \leq \frac{1}{e},\ t \in (0,\infty), \tag{15.14.11}$$

where $H(t) = \min_{1=1,\cdots,n}\{t - \tau_i(t)\}$, which guarantees nonoscillation in the sense that a nontrivial solution does not have a zero on the semiaxis $(0,\infty)$. Inequality (15.14.11) coincides with a classical nonoscillation test (in the sense of existence of an eventually positive solution) obtained in several monographs on the theory of delay differential equations (see, for example, [192]).

In the paper [178], the result on equivalences became a theorem on eight equivalences. One additional equivalent assertion for the equation

$$x'(t) + p(t)x\big(t - \tau(t)\big) = 0,\ t \in [0,\infty), \tag{15.14.12}$$

is the following:

5) There exists a positive essentially bounded function $u$ such that

$$p(t)e^{\int_{t-\tau(t)}^{t} u(s)ds}(t) \leq u(t),\ t \in [0,\infty). \tag{15.14.13}$$

Equivalence of the existence of an eventually positive solution and inequality (15.14.13) is the well-known result (see [154, p. 29]). Note that a corresponding development of this result on the basis of the ideas offered in [251] is presented in the recent paper [108].

In the paper [178], results on equivalences were extended to the equation

$$x'(t) + (Bx)(t) = f(t),\ t \in [0,\omega], \tag{15.14.14}$$

where $B : C[0,\omega] \to L_1[0,\omega]$ or $B : C[0,\omega] \to L_\infty[0,\omega]$ is a linear continuous positive Volterra operator, $C[0,\omega]$ is the space of continuous functions, $L_1[0,\omega]$ is the space of integrable functions and $L_\infty[0,\omega]$ is the space of essentially bounded functions defined on $[0,\omega]$. It should be stressed that an equation in this operator form becomes a very important instrument in the study of neutral equations [110, 138] and systems of ordinary or functional differential equations [5]. In the next chapter, we construct an equation for one of the components of the solution vector, and this equation has the form of (15.14.14).

The fact that the Cauchy function $C(t,s)$ as a function of $t$ for a fixed $s$ satisfies (15.14.7), (15.14.8) is a basis for various results on maximum principles for functional differential equations. The results about maximum boundaries principles and their corollaries for (15.14.14) are obtained in Sect. 15.3 of this chapter. They are based on the papers [121, 122]. In contrast with the well-known results of the monograph [197], a smallness of $\|B\|$ is not assumed. Results for the equation of the form

$$x'(t) - (Ax)(t) + (Bx)(t) = f(t),\ t \in [0,\omega], \tag{15.14.15}$$

where $A, B : C[0,\omega] \to L_1[0,\omega]$ or $C[0,\omega] \to L_\infty[0,\omega]$ are linear continuous positive Volterra operators, are based on results of the paper [121]. A generalization of these results for the neutral equations was presented in the paper [138]. A particular case of this equation is the delay differential equation

$$x'(t) - a(t)x\big(g(t)\big) + b(t)x\big(h(t)\big) = f(t), \ t \in [0, \infty), \tag{15.14.16}$$

where

$$x(\xi) = 0 \text{ for } \xi < 0. \tag{15.14.17}$$

Our results on nonoscillation and positivity of the corresponding Green's functions do not rely on the fact that the coefficient $b$ is small. It is even demonstrated that all solutions of the equation

$$x'(t) + b(t)x\big(h(t)\big) = 0, \ t \in [0, \infty), \tag{15.14.18}$$

can be oscillating, but all solutions of (15.14.16), where $f \equiv 0$, can nonoscillate, and the Cauchy function of (15.14.16) can be positive for $0 \le s \le t < \infty$. In the case $h(t) \le g(t)$, the main condition obtained in this chapter as well as in Chap. 3 claims that the difference $|be - a|$ has to be small enough. There were no results of this sort in the case $h(t) \ge g(t)$. In Sect. 15.10, on the basis of [121] we obtained that the smallness of $|b - a|$ and $|h - g|$ implies nonoscillation of (15.14.16) and positivity of its Cauchy function $C(t, s)$.

Let us now discuss the boundary conditions. In the well-known monograph [197], the generalized periodic condition

$$\nu x(0) + \mu x(\omega) = c \tag{15.14.19}$$

was considered. We study the boundary condition in the form

$$lx = c, \tag{15.14.20}$$

where $l : D[0, \omega] \to \mathbb{R}$ is a linear bounded functional defined on the space of absolutely continuous functions $D[0, \omega]$. The general form of this functional $l : D[0, \omega] \to \mathbb{R}$ is

$$lx \equiv \theta x(0) + \int_0^\omega \phi(s)x'(s)ds, \tag{15.14.21}$$

where $\theta \in \mathbb{R}$, $\phi \in L_\infty[0, \omega]$. Condition (15.14.19) can be obtained as a particular case of (15.14.20) if we set $\phi(t) \equiv \mu$, $\theta = \nu + \mu$ in (15.14.21). It seems to be important to consider general boundary condition (15.14.21) since conditions with integrals can describe, for example, the laws of conservation of energy.

Concerning results of Sect. 15.8 based on [26, 121], we note that there are no other results on the sign of the Green's function of the one-point boundary value problem

$$x'(t) + (Bx)(t) = f(t), \ t \in [0, \omega], \ x(\theta) = 0, \tag{15.14.22}$$

in the case $0 < \theta < \omega$.

Let us describe several possible schemes in the study of the exponential stability. First of all, we reduce differential equation (15.12.1) to equivalent integral equation (15.12.8) using the left regularization. Then we get the fact that the integral equation has a unique solution in the space $C[0, \infty)$. In order to make it, we estimate the norm or the spectral radius of the operator $K : C[0,\infty) \to C[0,\infty)$. These estimates are based on the positivity of the Cauchy function $C_0(t,s)$ of corresponding model equation (15.12.3); i.e., on the maximum inequalities principle for the model equation. As a result of the fact that $\|K\| < 1$ or $\rho(K) < 1$, we obtain the unique solvability, which together with solution representation formula (15.3.7) implies the maximum boundedness principle. On the other hand, analogues of the Bohl-Perron theorem reduce the exponential stability to the maximum boundedness principle (see [29, p. 95]), and thus we get the exponential stability of the given equation (15.12.1). The right regularization (the so-called Azbelev $W$-transform) presents another possibility for reducing differential equation (15.12.1) to the integral equation $z = \Omega z + f$, where the operator $\Omega : L_\infty[0,\infty) \to L_\infty[0,\infty)$ can be defined, for example, as $\Omega = A_2 C_0$, where $(C_0 z)(t) = \int_0^t C_0(t,s) z(s) ds$. If the Cauchy function $C_0(t,s)$ of the model equation (15.12.3) satisfies the exponential estimate and the norm of the operator $\Omega : L_\infty[0,\infty) \to L_\infty[0,\infty)$ is less than one, then the maximum boundedness principle is true for (15.12.1). The foundations of this approach to studying stability were obtained by N.V. Azbelev and his followers in the series of papers [22–24, 39] and in the book [29], and then were formulated in a complete form in the book [30]. Possibilities of this approach were demonstrated in the paper [178]. New results based on this approach were obtained in the recent papers [61–63, 121], for this approach see also the papers [19, 25, 308, 333].

It should be stressed that this approach is based on the positivity of the Cauchy functions of corresponding model equations. Problems of stability of the zero solution of a wide class of nonlinear equations can be reduced to the maximum boundedness principle and the exponential stability of its linear approximations [61]. Some applications can be found in [4, 61].

The results on the exponential stability of first-order delay equations of Sect. 15.11 are based on corresponding results of the paper [121] and present corollaries of nonoscillation and positivity of the Cauchy function. For an illustration of their possibilities in stabilization, let us consider the equation

$$x'(t) + bx(t-\tau) = 0, \ t \in [0,\infty), \tag{15.14.23}$$

where $b$ and $\tau$ are constants such that $b\tau > \frac{\pi}{2}$. It is known that all its nontrivial solutions oscillate and the amplitudes tend to infinity when $t \to \infty$. Consider now the equation

$$x'(t) - ax(t-\sigma) + bx(t-\tau) = f(t), \ t \in [0,\infty). \tag{15.14.24}$$

If $a > 0$, $\tau < \sigma$, $0 < (b-a)\tau \le \frac{1}{e}$ and $b(\sigma - \tau) \le \frac{1}{e}$, then (15.14.24) is exponentially stable. This demonstrates that stabilization of (15.14.24) can be achieved by adding the control $u$ in the form $u(t) = ax(t-\sigma)$ in the equation

$$x'(t) + bx(t-\tau) = f(t) + u(t), \ t \in [0,\infty). \tag{15.14.25}$$

This result can be extended to (15.14.16) with a variable coefficient and delays. It is essential in applications that the time to apply the control $u$ using previous states of the process be more than the delay in (15.14.18), i.e., $h(t) \ge g(t)$ (in (15.14.24) we have $\sigma > \tau$). There are no other results in this case. Thus the smallness of $|b-a|$

and $|h - g|$ and the dominance of the positive term such that a positive number $\varepsilon$ exists such that $b > a + \varepsilon$ implies stability of (15.14.16).

The maximum principles for the equation

$$x'(t) + p(t)x\big(h(t)\big) = 0,\ t \in [0, \infty), \tag{15.14.26}$$

with oscillating coefficient $p(t)$, for example, such that $p(t) \geq 0$ for $(t_{2k}, t_{2k+1})$ and $p(t) \leq 0$ for $(t_{2k+1}, t_{2k+2})$ for $k = 0, 1, 2, \cdots$, were first studied in [123]. The condition on the deviating argument $t_{2k-1} \leq h(t)$ for $t \in [t_{2k}, t_{2k+1}]$ and the inequality

$$\int_{t_{2k}}^{t_{2k+1}} p(t)dt < 1,\ k = 0, 1, 2, \cdots, \tag{15.14.27}$$

imply nonoscillation of the solution and positivity of the Cauchy function $C(t, s)$ for $0 \leq s \leq t < \infty$ for (15.14.26). This nonoscillation result is the first one that assumes that the integral of the coefficient $p(t)$ is small on each of the intervals of positivity separately and does not assume that the integral of the positive part $p^+(t)$ (defined as usual by the equalities $p(t) = p^+(t) - p^-(t),\ p^+(t) \geq 0,\ p^-(t) \geq 0$) on the semiaxis converges or is of the form

$$\int^{\infty} p^+(t)dt < 1. \tag{15.14.28}$$

Results on the exponential stability of (15.14.26) are based on its nonoscillation and positivity of the Cauchy function. Note that the approach developed in Sect. 15.12 assumes the dominance of the positive term and does not work here. There are almost no results about stability of equations with an oscillating coefficient $p(t)$. In the paper [194], various necessary and sufficient conditions of stability and asymptotic stability of the equation

$$x'(t) + p(t)x(t - \tau) = 0,\ t \in [0, \infty),\ \tau = \text{const}, \tag{15.14.29}$$

with an oscillating coefficient $p(t)$ were obtained in the case where $p(t) \to 0$ for $t \to \infty$. In the results on the exponential stability of Sect. 15.11 based on the paper [123], this case was not considered.

In Sect. 15.13, the boundary value problem

$$x'(t) + (Bx)(t) = f(t),\ t \in [0, \omega],\ lx = c, \tag{15.14.30}$$

with a non-Volterra operator $B : C[0, \omega] \to L_\infty[0, \omega]$ was considered.

Here $l : D[0, \omega] \to \mathbb{R}$ is a linear bounded functional defined on the space of absolutely continuous functions $D[0, \omega]$. The general form of this functional $l$ is defined by the formula (15.14.21). The general form of this functional $l : D[0, \omega] \to \mathbb{R}$ allows us to describe the known scheme of regularization [178, 197, 198] in a general form and to see the types of problems where this scheme can be applied.

Finally, let us formulate several open problems.

1. Theorem 15.5 claims that the Green's function of the problem

$$x'(t) + p(t)x\big(h(t)\big) = f(t),\ t \in [0, \omega],\ x(\omega) = 0, \tag{15.14.31}$$

changes its sign in the square $(t,s) \in (0,\omega)\times\ (0,\omega)$ if $p(t) \le 0$ and $\text{mes}\{t \in [0,\omega]|\ p(t) < 0,\ 0 \le h(t) \le t\} > 0$. Is it possible to obtain a result on nonpositivity of the Green's function for the problem

$$x'(t) + b(t)x\big(h(t)\big) - a(t)x\big(g(t)\big) = f(t),\ t \in [0,\omega],\ x(\omega) = 0, \quad (15.14.32)$$

in the case $b(t) \ge a(t)$ or for the problem with the more general equation

$$x'(t) + (Bx)(t) - (Ax)(t) = f(t),\ t \in [0,\omega],\ x(\omega) = 0, \quad (15.14.33)$$

in the case of a dominant operator $B$?

2. In Sect. 15.10, various coefficient tests of nonoscillation and positivity of the Cauchy function $C(t,s)$ were obtained. The problem is to obtain a corresponding coefficient test in the case where the operators $A$ and $B$ are of different forms; for example, where one of the operators $A$ and $B$ is an integral operator and the second is a deviating argument operator

$$(Ax)(t) = \int_0^t k(t,s)x(s)ds,\ (Bx)(t) = b(t)x\big(h(t)\big),\ t \in [0,\infty), \quad (15.14.34)$$

where

$$x(\xi) = 0 \text{ for } \xi < 0. \quad (15.14.35)$$

3. In Sect. 15.11, (15.14.26) with an oscillating coefficient $p(t)$ was studied. In addition, obtain an exponential estimate for the Cauchy function of this equation; i.e., deduce conditions where there exist positive constants $\alpha$ and $N$ such that $|C(t,s)| \le Ne^{-\alpha(t-s)}$ for $0 \le s \le t < \infty$.
4. Almost all results on the maximum principles were obtained in the case of Volterra operators $A$ and $B$. Can the maximum boundaries principle be extended to the case of non-Volterra operators?
5. Is it possible to obtain results on sign properties of the Green's function for the problem

$$x'(t) + b(t)x\big(h(t)\big) = f(t),\ t \in [0,\omega],\ x(a) = 0,\ 0 < a < \omega,$$

without the assumption that $h(t) \le t$ for $t \in [0,\omega]$?

# Chapter 16
# Systems of Functional Differential Equations on Finite Intervals

## 16.1 Introduction

Consider the system of functional differential equations

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij} x_j)(t) = f_i(t), \ t \in [0, \omega], \ i = 1, \cdots, n, \tag{16.1.1}$$

where $x = \mathrm{col}(x_1, \cdots, x_n)$, $B_{ij} : C[0, \omega] \to L_1[0, \omega]$ or $B_{ij} : C[0, \omega] \to L_\infty[0, \omega]$, $i, j = 1, \cdots, n$, are linear continuous operators and $C[0, \omega]$, $L_1[0, \omega]$, $L_\infty[0, \omega]$ are the spaces of continuous, integrable and essentially bounded functions $y : [0, \omega] \to \mathbb{R}$, respectively.

Let $l : C[0, \omega] \to \mathbb{R}^n$ be a linear bounded functional. If the homogeneous boundary value problem $(M_i x)(t) = 0, \ t \in [0, \omega], i = 1, \cdots, n, lx = 0$ has only the trivial solution, then the boundary value problem

$$(M_i x)(t) = f_i(t), \ t \in [0, \omega], \ i = 1, \cdots, n, \ lx = \alpha \tag{16.1.2}$$

has for each $f = \mathrm{col}(f_1, \cdots, f_n)$, where $f_i \in L_1[0, \omega]$, $i = 1, \cdots, n$, and $\alpha \in \mathbb{R}^n$, a unique solution, which has the representation [29]

$$x(t) = \int_0^\omega G(t, s) f(s) ds + X(t)\alpha, \ t \in [0, \omega], \tag{16.1.3}$$

where the $n \times n$ matrix $G(t, s)$ is called the Green's matrix of problem (16.1.2) and $X(t)$ is the $n \times n$ fundamental matrix of the system $(M_i x)(t) = 0, i = 1, \cdots, n$, such that $lX = I$ ($I$ is the identity $n \times n$ matrix). It is clear from solution representation (16.1.3) that the matrices $G(t, s)$ and $X(t)$ determine all properties of solutions.

If the conditions

$$(M_i x)(t) \geq (M_i y)(t), \ t \in [0, \omega], \ i = 1, \cdots, n, \ lx = ly, \tag{16.1.4}$$

imply

$$x_i(t) \geq y_i(t), \ t \in [0, \omega], \ i = 1, \cdots, n, \tag{16.1.5}$$

this property is a basis of the monotone methods in approximate integration [250].

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As a particular case of system (16.1.1), let us consider the delay system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j\big(h_{ij}(t)\big) = f_i(t),\ i = 1, \cdots, n,\ t \in [0, \omega], \tag{16.1.6}$$

$$x(\xi) = 0 \text{ for } \xi < 0, \tag{16.1.7}$$

where $p_{ij}$ are integrable or essentially bounded functions and $h_{ij}$ are measurable functions such that $h_{ij}(t) \le t$ for $i, j = 1, \cdots, n$, $t \in [0, \omega]$.

Wazewski's classical theorem claims [332] that the condition

$$p_{ij} \le 0 \text{ for } j \ne i,\ i, j = 1, \cdots, n \tag{16.1.8}$$

is necessary and sufficient for the property (16.1.4) ⇒ (16.1.5) for the Cauchy problem for the system of ordinary differential equations

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t) = f_i(t),\ i = 1, \cdots, n,\ t \in [0, \omega]. \tag{16.1.9}$$

Solution representation formula (16.1.3) yields that implication (16.1.4) ⇒ (16.1.5) is true if all entries of the matrices $G(t, s)$ and $X(t)$ are nonnegative.

We focus our attention on the problem of comparison for only one of the components of the solution vector. Let $k_i$ be either 1 or 2. In this chapter, we consider the property that the conditions

$$(-1)^{k_i}\big[(M_i x)(t) - (M_i y)(t)\big] \ge 0,\ t \in [0, \omega],\ lx = ly,\ i = 1, \cdots, n \tag{16.1.10}$$

imply that for a fixed component $x_r$ of the solution vector the inequality $x_r(t) \ge y_r(t)$, $t \in [0, \omega]$, is satisfied. This property is weaker than the implication (16.1.4) ⇒ (16.1.5) and, as will be obtained below, leads to essentially weaker limitations on the given system. From solution representation formula (16.1.3), it follows that this property is reduced to sign constancy of all entries in the $r$-th row of the Green's matrix only.

The main idea of our approach is to construct a corresponding scalar functional differential equation of the first order,

$$x_r'(t) + (Bx_r)(t) = f^*(t),\ t \in [0, \omega], \tag{16.1.11}$$

for the $r$-th component of a solution vector, where $B : C[0, \omega] \to L_1[0, \omega]$ is a linear continuous operator, $f^* \in L_1[0, \omega]$. It should be stressed that we get this quite complicated functional differential equation for the description of behavior of the component $x_r$ even in the case of systems of ordinary differential equations. This circumstance explains why we had to consider a scalar equation in such a form in Chap. 15. Equations of the form (16.1.11) are constructed in Sects. 16.3 and 16.4. Then the technique of analysis of the first-order scalar functional differential equations, developed in Chap. 15, is used. On this basis, in Sect. 16.3 we obtain necessary and sufficient conditions for nonpositivity or nonnegativity of entries in the $n$-th row of the Cauchy and the Green's matrices in the form of theorems on differential inequalities. Simple coefficient tests of the sign constancy for the $n$-th row of the Cauchy and the Green's matrices are proposed in

Sects. 16.3 and 16.4. In Sect. 16.3, the results of this type for the Cauchy problem (i.e., $lx \equiv \mathrm{col}(x_1(0), \cdots, x_n(0))$) and Volterra operators $B_{ij} : C[0,\omega] \to L_1[0,\omega]$ are obtained. In this case, the operator $B : C[0,\omega] \to L_1[0,\omega]$ in (16.1.11) is a Volterra operator. In Sects. 16.4 and 16.5, we consider other boundary conditions that imply that the operator $B : C[0,\omega] \to L_1[0,\omega]$ is not a Volterra one even in the case where all $B_{ij} : C[0,\omega] \to L_1[0,\omega]$, $i, j = 1, \cdots, n$, are Volterra operators. It requires a special technique to work with (16.1.11) when the operator $B : C[0,\omega] \to L_1[0,\omega]$ is not Volterra. It should be emphasized that in many of our results the interval $[0,\omega]$ is not assumed to be short.

In this chapter, we consider the boundary value problems with boundary conditions of the form

$$l_i x_i = 0, \ i = 1, \cdots, n, \tag{16.1.12}$$

where $l_i : C_{[0,b]} \to \mathbb{R}$, $i = 1, \cdots, n$ are linear bounded functionals. Note that each of the types of boundary conditions

$$x_i(0) = 0, \ i = 1, \cdots, n, \tag{16.1.13}$$

$$x_i(\omega) = 0, \ i = 1, \cdots, n, \tag{16.1.14}$$

$$x_i(0) = 0, \ x_j(\omega) = 0, \ i = 1, \cdots, k, \ \ j = k+1, \cdots, n, \tag{16.1.15}$$

$$x_i(0) = \beta x_i(\omega), \ i = 1, \cdots, n \tag{16.1.16}$$

is a particular case of condition (16.1.12).

## 16.2 Nonnegativity and Nonpositivity of Green's Matrices

In this section, we consider the system

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij} x_j)(t) = f_i(t), \ i = 1, \cdots, n, \ t \in [0,\omega], \tag{16.2.1}$$

with boundary conditions of the form

$$l_i x_i = 0, \ i = 1, \cdots, n, \tag{16.2.2}$$

where $B_{ij} : C[0,\omega] \to L_\infty[0,\omega]$ are linear continuous operators and $l_i : C[0,\omega] \to \mathbb{R}$ are linear bounded functionals, $i, j = 1, \cdots, n$.

**Theorem 16.1** *Let the following conditions be fulfilled*:

1) *The Green's functions $g_i(t,s)$ $(i = 1, \cdots, n)$ of $n$ scalar boundary value problems for the diagonal equations*

$$(m_i x)(t) \equiv x_i'(t) + (B_{ii} x_i)(t) = f_i(t), \ t \in [0,\omega], \ l_i x_i = 0, \tag{16.2.3}$$

*exist, preserve their signs for $t, s \in (0,\omega)$ and are such that*

$$\int_0^\omega \left| g_i(t,s) \right| \varphi(s) ds > 0, \ t \in [0,\omega], \tag{16.2.4}$$

*for each positive measurable essentially bounded function $\varphi$.*

2) *The nondiagonal operators* $B_{ij}$ $(i, j = 1, \cdots, n,\ j \neq i)$ *are positive or negative such that the operator* $R : C[0, \omega] \to C[0, \omega]$ *defined by the formula*

$$(Rx)(t) = \operatorname{col}\left(-\int_0^{\omega} g_i(t, s) \sum_{j=1,\, j\neq i}^{n} (B_{ij}x_j)(s)ds\right)_{i=1}^{n}, \ t \in [0, \omega], \quad (16.2.5)$$

*is positive.*

*Then the following assertions* a), b) *and* c) *are equivalent*:

a) *There exists a vector function* $v \in C[0, \omega]$ *with positive absolutely continuous components* $v_i : [0, \omega] \to [0, \infty)$ *such that* $l_i v_i = 0$ *and*

$$\int_0^{\omega} g_i(t, s)(M_i v)(s)ds > 0,\ t \in [0, \omega],\ i = 1, \cdots, n. \quad (16.2.6)$$

b) *The spectral radius of the operator* $R : C[0, \omega] \to C[0, \omega]$ *is less than one.*
c) *Boundary value problem* (16.2.1), (16.2.2) *has a unique solution for every right-hand side* $f = \operatorname{col}(f_1, \cdots, f_n)$ *such that* $f_i \in L_\infty[0, \omega]$, $i = 1, \cdots, n$ *and the entries of its Green's matrix preserve their sign and satisfy the inequalities*

$$g_i(t, s)G_{ij}(t, s) \geq 0,\ t, s \in [0, \omega],\ i, j = 1, \cdots, n, \quad (16.2.7)$$

$$\left|G_{ii}(t, s)\right| \geq \left|g_i(t, s)\right|,\ t, s \in [0, \omega],\ i = 1, \cdots, n. \quad (16.2.8)$$

*Proof* a) ⇒ b) The proof of this implication is based on Lemma A.2.

The function $v$ satisfies the boundary value problem

$$(M_i x)(t) = \varphi_i(t),\ l_i x_i = 0,\ i = 1, \cdots, n,\ t \in [0, \omega], \quad (16.2.9)$$

where $\varphi_i(t) = (M_i v)(t)$, $t \in [0, \omega]$. It is clear that the function $v$ also satisfies the integral equation $x(t) - (Rx)(t) = \psi(t)$, $t \in [0, \omega]$, where

$$\psi(t) = \operatorname{col}\left(\int_0^{\omega} g_i(t, s)\varphi_i(s)ds\right)_{i=1}^{n}, \ t \in [0, \omega]. \quad (16.2.10)$$

The condition a) implies that all components $\psi_i(t)$, $i = 1, \cdots, n$ of the vector $\psi(t)$ are positive for $t \in [0, \omega]$. The reference to Lemma A.2 completes the proof.

b) ⇒ c) If the spectral radius of the operator $R : C[0, \omega] \to C[0, \omega]$ is less than one, then the sequence $\{x^m\}$ of vectors $x^m = (x_1^m, \cdots, x_n^m)$, where $x^m = Rx^{m-1} + \psi$, $x^0 = \psi$, $\psi \in C[0, \omega]$, converges to the solution of the equation $x = Rx + \psi$, which is equivalent to boundary value problem (16.2.1), (16.2.2). This means that boundary value problem (16.2.1), (16.2.2) has a unique solution, while for $\psi$ with nonnegative components $\psi_i$, $i = 1, \cdots, n$ we obtain $x_i \geq \psi_i \geq 0$, $i = 1, \cdots, n$.

If $f_i$ preserves its sign for $i = 1, \cdots, n$ and is such that

$$\int_0^{\omega} g_i(t, s) f_i(s)ds \geq 0,\ t \in [0, \omega], \quad (16.2.11)$$

then

$$\psi_i(t) = \int_0^{\omega} g_i(t, s) f_i(s)ds \geq 0,\ t \in [0, \omega], \quad (16.2.12)$$

and consequently $x_i(t) \geq \psi_i(t)$, $i = 1, \cdots, n$. The inequality (16.2.7) has been proven.

In order to prove inequality (16.2.8), we set $f_j = 0$ for $j \neq i$, $j = 1, \cdots, n$. In this case, we obtain

$$x_i(t) - \psi_i(t) = \int_0^{\omega} \big[G_{ii}(t,s) - g_i(t,s)\big] f_i(s)ds, \; t \in [0,\omega], \; i = 1, \cdots, n. \tag{16.2.13}$$

The inequality $x_i(t) \geq \psi_i(t)$ implies inequality (16.2.8).

c) $\Rightarrow$ a) In order to prove this implication we can set $v(t) = \int_0^{\omega} G(t,s)\Xi ds$, where $\Xi = \mathrm{col}(e_1, \cdots, e_n)$ and $e_i$ equals 1 or $-1$ so that $e_i g_i(t,s) \geq 0$, $t, s \in [0,\omega]$. □

*Remark 16.1* Condition 1) of Theorem 16.1 is fulfilled for generalized periodic problem (16.2.1), (16.1.16) and is not fulfilled for problems consisting of (16.2.1) with conditions (16.1.13), (16.1.14) and (16.1.15). For these conditions, the following result is valid.

**Theorem 16.2** *Let the following conditions be fulfilled*:

1) *Green's functions $g_i(t,s)$ $(i = 1, \cdots, n)$ of $n$ scalar boundary value problems for diagonal equations*

$$(m_i x_i)(t) \equiv x_i'(t) + (B_{ii} x_i)(t) = f_i(t), \; t \in [0,\omega], \; l_i x_i = c_i, \tag{16.2.14}$$

*exist and preserve their signs.*

2) *The nondiagonal operators $B_{ij}$ $(i, j = 1, \cdots, n, \; j \neq i)$ are positive or negative such that the operator $R : C[0,\omega] \to C[0,\omega]$ defined by formula* (16.2.5) *is positive.*

*Then the assertions* a) *and* b) *are equivalent and each of them implies* c).

a) *There exists a vector function $v \in C[0,\omega]$ with positive absolutely continuous components $v_i : [0,\omega] \to [0,\infty)$ such that the solution $w_i$ of the problem*

$$(m_i w_i)(t) \equiv w_i'(t) + (B_{ii} w_i)(t) = (M_i v)(t), \; t \in [0,\omega], \; l_i w_i = l_i v_i, \tag{16.2.15}$$

*is positive for $t \in [0,\omega]$ for every $l = 1, \cdots, n$.*

b) *The spectral radius of the operator $R : C[0,\omega] \to C[0,\omega]$ is less than one.*

c) *Boundary value problem* (16.2.1), (16.2.2) *has a unique solution for every right-hand side $f = \mathrm{col}(f_1, \cdots, f_n)$ such that $f_i \in L_\infty[0,\omega]$, $i = 1, \cdots, n$, and the entries of its Green's matrix preserve sign and satisfy the inequalities*

$$g_i(t,s) G_{ij}(t,s) \geq 0, \; t, s \in [0,\omega], \; i, j = 1, \cdots, n, \tag{16.2.16}$$

$$\big|G_{ii}(t,s)\big| \geq \big|g_i(t,s)\big|, \; t, s \in [0,\omega], \; i = 1, \cdots, n. \tag{16.2.17}$$

*Proof* a) $\Rightarrow$ b) It is clear that the vector function $v$ also satisfies the integral equation $x(t) - (Rx)(t) = w(t)$, $t \in [0,\omega]$, where the vector function $w$ satisfies the boundary value problem $(m_i x)(t) = \varphi_i(t)$, $l_i x_i = l_i v_i$, $i = 1, \cdots, n$, $t \in [0,\omega]$, where $\varphi_i(t) =$

$(M_i v)(t)$, $t \in [0, \omega]$. Condition a) implies that all components $w_i(t)$, $i = 1, \cdots, n$ of the vector $w(t)$ are positive for $t \in [0, \omega]$. The reference to Lemma A.2 completes the proof.

b) $\Rightarrow$ a) Let us set $f_i(t) \equiv 1$ or $f_i(t) \equiv -1$ for $t \in [0, \omega]$, depending on the sign of $g_i(t, s)$, and $c_i$, where $i = 1, \cdots, n$, such that the solution $w$ of the problem

$$(m_i x)(t) = f_i(t),\ l_i x_i = c_i,\ i = 1, \cdots, n,\ t \in [0, \omega], \tag{16.2.18}$$

is positive for $t \in [0, \omega]$. The spectral radius of the operator $R : C[0, \omega] \to C[0, \omega]$ is less than one. Then the sequence $\{x^m\}$ of vectors $x^m = (x_1^m, \cdots, x_n^m)$, where $x^m = Rx^{m-1} + w$, $x^0 = w$, converges to the solution $v$ of the equation $x = Rx + w$, which is equivalent to boundary value problem (16.2.1), (16.2.2), where $l_i x_i = c_i$, $i = 1, \cdots, n$. This solution $v$ is positive and satisfies all conditions of a).

The proof of the implication b) $\Rightarrow$ c) is similar to the proof of the same implication in Theorem 16.1. □

**Corollary 16.1** *Let all $B_{ij}$ $(i, j = 1, \cdots, n)$ be Volterra operators, nondiagonal operators $B_{ij}$ $(i, j = 1, \cdots, n, i \neq j)$ be negative and the Cauchy functions $c_i(t, s)$ of the diagonal equations $(m_i x)(t) \equiv x_i'(t) + (B_{ii} x_i)(t) = f_i(t)$, $t \in [0, \infty)$ be positive for $0 \le s \le t < \infty$. Then $C_{ij}(t, s) \ge 0$, $C_{ii}(t, s) > 0$ for $0 \le s \le t < \infty$, $i, j = 1, \cdots, n$.*

*Proof* On every finite interval $[0, \omega]$, the spectral radius of the compact Volterra operator $R$ defined by (16.2.5), where $g_i(t, s) = c_i(t, s)$, is zero. The conditions of the corollary imply $C_{ij}(t, s) \ge 0$, $C_{ii}(t, s) > 0$ for $0 \le s \le t \le \omega$, $i, j = 1, \cdots, n$. Now, using the fact that all $B_{ij}$ $(i, j = 1, \cdots, n)$ are Volterra operators, we can extend these inequalities to the semiaxis $[0, \infty)$. □

In the following theorem, we use weighted spaces; see Definition A.3.

**Theorem 16.3** *Let $B_{ij} : C[0, \infty) \to L_\infty[0, \infty)$ be Volterra linear bounded operators, $i, j = 1, \cdots, n$, and the following conditions be fulfilled:*

1) *For some $\lambda > 0$ we have $B_{ij} : C^\lambda[0, \infty) \to L_\infty^\lambda[0, \infty)$.*
2) *The Cauchy functions $c_i(t, s)$ $(i = 1, \cdots, n)$ of $n$ scalar diagonal equations*

$$(m_i x_i)(t) \equiv x_i'(t) + (B_{ii} x_i)(t) = f_i(t),\ t \in [0, \infty), \tag{16.2.19}$$

   *are positive for $0 \le s \le t < \infty$.*
3) *The nondiagonal operators $B_{ij}$ $(i, j = 1, \cdots, n,\ j \neq i)$ are negative.*
4) *There exist a constant vector $z = \mathrm{col}(z_1, \cdots, z_n)$ and a number $\varepsilon > 0$ such that*

$$\sum_{j=1}^{n} (B_{ij} z_j)(t) \ge \varepsilon > 0,\ t \in [0, \infty). \tag{16.2.20}$$

*Then the following assertions are true:*

a) *Boundary value problems (16.2.1), (16.1.16) with $\beta_i \le 1$ $(i = 1, \cdots, n)$ have a unique solution for every right-hand side $f = \mathrm{col}(f_1, \cdots, f_n)$ such that $f_i \in$*

*$L_\infty[0,\infty)$, $i=1,\cdots,n$ and entries of their Green's matrices satisfy the inequalities $G_{ij}(t,s)\geq 0$ $(i,j=1,\cdots,n)$, while $G_{ii}(t,s)\geq g_i(t,s)>0$ for $t,s\in[0,\omega]$ and every $\omega>0$.*

b) *For every bounded right-hand side $f=\text{col}(f_1,\cdots,f_n)$, the solution $x=\text{col}(x_1,\cdots,x_n)$ is bounded on the semiaxis $[0,\infty)$.*

c) *The Cauchy matrix $C(t,s)$ satisfies the exponential estimate; i.e., there exist positive $\alpha$ and $N$ such that*

$$0\leq C_{ij}(t,s)\leq Ne^{-\alpha(t-s)},\ 0\leq s\leq t<\infty,\ i,j=1,\cdots,n. \tag{16.2.21}$$

d) *In the case where $\varepsilon\geq 1$ in (16.2.20), the inequalities*

$$0\leq\int_0^t\sum_{j=1}^n C_{ij}(t,s)ds\leq z_i,\ 0\leq t<\infty,\ i=1,\cdots,n \tag{16.2.22}$$

$$0\leq\int_0^\omega\sum_{j=1}^n G_{ij}(t,s)ds\leq z_i,\ 0\leq t<\omega,\ i=1,\cdots,n \tag{16.2.23}$$

*are valid, and in the case where the constant $n\times n$-matrix $Y(t)=\{y_{ij}\}_{i,j=1}$ satisfies*

$$\sum_{j=1}^n(B_{ij}y_{jk})(t)\geq\delta_{ik},\ t\in[0,\infty),\ i=1,\cdots,n, \tag{16.2.24}$$

*where $\delta_{ik}=1$, $i=j$, $\delta_{ik}=0$, $i\neq j$, the following inequalities hold:*

$$0\leq\int_0^t C_{ij}(t,s)ds\leq y_{ij},\ 0<t<\infty,\ i,j=1,\cdots,n, \tag{16.2.25}$$

$$0\leq\int_0^\omega G_{ij}(t,s)ds\leq y_{ij},\ 0<t\leq\omega,\ i,j=1,\cdots,n. \tag{16.2.26}$$

*Proof* Conditions 3) and 4) imply that the operators $B_{ii}$ are nonzero for $i=1,\cdots,n$. According to assertion 8 of Theorem 15.3, Green's functions $g_i(t,s)$ of the generalized periodic problems

$$(m_ix_i)(t)\equiv x_i'(t)+(B_{ii}x_i)(t)=f_i(t),\ t\in[0,\omega],\ x_i(0)=\beta_ix_i(\omega), \tag{16.2.27}$$

where $i=1,\cdots,n$, are positive for $t,s\in[0,\omega]$.

The constant vector $v=\text{col}(z_1,\cdots,z_n)$ satisfies assertion a) of Theorem 16.1 and, according to assertion c) of the same theorem, we obtain assertion a) of Theorem 16.3. Thus the entries of the Green's matrix of problem (16.2.1), (16.1.16), where $\beta_i\leq 1$ for $i=1,\cdots,n$, satisfy the inequalities $G_{ij}(t,s)\geq 0$, $G_{ii}(t,s)>g_i(t,s)>0$ for $t,s\in[0,\omega]$ for every positive $\omega$.

From the conditions 2) and 3), according to Corollary 16.1, we have $C_{ij}(t,s)\geq 0$ and $C_{ii}(t,s)>0$ for $0\leq s\leq t<\infty$, $i=1,\cdots,n$.

Vector function $z=\text{col}(z_1,\cdots,z_n)$ satisfies the system

$$(M_ix)(t)=f_i(t),\ t\in[0,\infty),\ i=1,\cdots,n,$$

where $f_i(t) = \sum_{j=1}^{n}(B_{ij}z_j)(t)$, $i = 1, \cdots, n$ and the initial condition $x_i(0) = z_i$, $i = 1, \cdots, n$. The solution representation formula

$$x(t) = \int_0^t C(t,s)f(s)ds + C(t,0)x(0),$$

where $f = \mathrm{col}(f_1, \cdots, f_n)$, in this case leads us to the equality

$$z = \int_0^t C(t,s)f(s)ds + C(t,0)z.$$

By condition 4), all components $f_i$ of the right-hand side $f = \mathrm{col}(f_1, \cdots, f_n)$ are positive. Nonnegativity of all entries $C_{ij}(t,s)$, $i, j = 1, \cdots, n$ in this case implies that

$$\int_0^t C(t,s)f(s)ds \le z,\ C(t,0)z \le z,\ t \in [0,\infty),$$

and for every vector function $\phi = \mathrm{col}(\phi_1, \cdots, \phi_n)$ such that $|\phi| \le f$, we also get boundedness of solutions on the semiaxis. Thus, for every bounded right-hand side, the solution is bounded on the semiaxis, which by the Bohl-Perron theorem (Theorem B.20) implies the exponential estimate (16.2.21). Positivity of the matrices $C(t,s)$ and $G(t,s)$ implies inequalities (16.2.22), (16.2.23), (16.2.25) and (16.2.26). □

For the delay system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j\big(h_{ij}(t)\big) = f_i(t),\ i = 1, \cdots, n,\ t \in [0,\infty), \tag{16.2.28}$$

$$x_i(\xi) = 0 \text{ for } \xi < 0,\ i = 1, \cdots, n, \tag{16.2.29}$$

which is a particular case of (16.2.1), we propose the following result that can be obtained as a corollary of Theorem 16.3 if we also use a small trick described in the proof of Theorem 16.7 below.

**Theorem 16.4** *Let $t - h_{ij}(t) \le \delta$, $p_{ij} \le 0, i \ne j,\ i, j = 1, \cdots, n$,*

$$\int_{h_{ii}(t)}^t p_{ij}(s)ds \le \frac{1}{e},\ i = 1, \cdots, n,\ t \in (0,\infty), \tag{16.2.30}$$

*and there exist a constant vector $z = \mathrm{col}(z_1, \cdots, z_n)$ and a number $\varepsilon > 0$ such that*

$$\sum_{j=1}^{n} p_{ij}(t)z_j \ge \varepsilon > 0,\ t \in [0,\infty). \tag{16.2.31}$$

*Then assertions* a), b) *and* c) *of Theorem* 16.3 *are true, and in the case of additional assumption $0 \le t - h_{ij}(t)$ assertion* d) *is also true.*

Note that results on the exponential stability of systems obtained in [87, 207] follow from this theorem.

Let us define the constant vector $z = \text{col}(z_1, \cdots, z_n)$ as

$$z = P^{-1}E, \tag{16.2.32}$$

where $P$ is the $n \times n$ matrices $P = \{p_{ij}\}_{i,j=1}^{n}$, $E = \text{col}(1, \cdots, 1)$ and $n \times n$-matrix $Y(t) = \{y_{ij}\}_{i,j=1}^{n}$ as

$$Y = P^{-1}. \tag{16.2.33}$$

For systems with constant coefficients $p_{ij}$ we propose the following result.

**Theorem 16.5** *Let* $0 \le t - h_{ij}(t) \le \delta$, $p_{ij} \le 0$, $i \ne j$, $i, j = 1, \cdots, n$, $p_{ii}(t - h_{ii}(t)) \le \frac{1}{e}$ *for* $i = 1, \cdots, n$, $t \in (0, \infty)$.

*Then the assertions* a), b) *and* c) *are equivalent and each of them implies* d):

a) *All components* $z_i$ $(i = 1, \cdots, n)$ *of the vector* $z$ *defined by equality* (16.2.32) *are positive.*
b) *For every bounded right-hand side* $f = \text{col}(f_1, \cdots, f_n)$, *the solution* $x = \text{col}(x_1, \cdots, x_n)$ *is bounded on the semiaxis* $[0, \infty)$.
c) *The Cauchy matrix* $C(t, s)$ *satisfies the exponential estimate*; *i.e.*, *there exist positive* $\alpha$ *and* $N$ *such that*

$$0 \le C_{ij}(t, s) \le N e^{-\alpha(t-s)}, \quad 0 \le s \le t < \infty, \quad i, j = 1, \cdots, n. \tag{16.2.34}$$

d) *Boundary value problems* (16.1.16) *with* $\beta_i \le 1$ $(i = 1, \cdots, n)$ *have a unique solution for every right-hand side* $f = \text{col}(f_1, \cdots, f_n)$ *such that* $f_i \in L_\infty[0, \infty)$, $i = 1, \cdots, n$, *and entries of their Green's matrices satisfy the inequalities* $G_{ij}(t, s) \ge 0$ $(i, j = 1, \cdots, n)$, *while* $G_{ii}(t, s) \ge g_i(t, s) > 0$ *for* $t, s \in [0, \omega]$ *and every* $\omega > 0$.

**Corollary 16.2** *Let all coefficients* $p_{ij}$ *be constants*, *the conditions of Theorem* 16.5 *be satisfied and all components* $z_i$ $(i = 1, \cdots, n)$ *of the vector* $z = \text{col}(z_1, \cdots, z_n)$ *defined by equality* (16.2.32) *be positive. Then*

$$\lim_{t \to \infty} \int_0^t \sum_{j=1}^{n} C_{ij}(t, s) ds = z_i, \quad 0 \le s \le t < \infty, \tag{16.2.35}$$

$$\lim_{t \to \infty} \int_0^t C_{ij}(t, s) ds = y_{ij}, \quad 0 < t < \infty, \quad i, j = 1, \cdots, n, \tag{16.2.36}$$

*and the Green's matrix of periodic problem* (16.2.28), (16.2.37), *where*

$$x_i(0) = x_i(\omega), \quad i = 1, \cdots, n, \tag{16.2.37}$$

*satisfies*

$$\int_0^\omega \sum_{j=1}^{n} G_{ij}(t, s) ds = z_i, \quad 0 \le t \le \omega, \quad i = 1, \cdots, n \tag{16.2.38}$$

$$\int_0^\omega G_{ij}(t, s) ds = y_{ij}, \quad 0 \le t \le \omega, \quad i, j = 1, \cdots, n. \tag{16.2.39}$$

## 16.3 Positivity of the *n*-th Row of the Cauchy Matrix

In this section, we consider the equation

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij} x_j)(t) = f_i(t), \ t \in [0, \omega], \ i = 1, \cdots, n, \tag{16.3.1}$$

where $B_{ij} : C[0, \omega] \to L_\infty[0, \omega]$ are linear bounded Volterra operators for $i, j = 1, \cdots, n$. Its general solution has the representation

$$x(t) = \int_0^t C(t, s) f(s) ds + C(t, 0) x(0),$$

where $C(t, s)$ is the Cauchy matrix,

$$f(t) = \mathrm{col}\big(f_1(t), \cdots, f_n(t)\big), \ x(t) = \mathrm{col}\big(x_1(t), \cdots, x_n(t)\big).$$

Together with (16.3.1), consider the auxiliary system of the order $n - 1$

$$x_i'(t) + \sum_{j=1}^{n-1} (B_{ij} x_j)(t) = f_i(t), \ t \in [0, \omega], \ i = 1, \cdots, n-1, \tag{16.3.2}$$

and denote by $K(t, s) = \{K_{ij}(t, s)\}_{i,j=1,\cdots,n-1}$ its Cauchy matrix. Denote by $G(t, s) = \{G_{ij}(t, s)\}_{i,j=1,\cdots,n}$ and $P(t, s) = \{P_{ij}(t, s)\}_{i,j=1,\cdots,n}$ the Green's matrices of the problems consisting of (16.3.1) and one of the boundary conditions

$$x_i(0) = 0, \ i = 1, \cdots, n-1, \ x_n(\omega) = 0, \tag{16.3.3}$$

or

$$x_i(0) = 0, \ i = 1, \cdots, n-1, \ x_n(0) = x_n(\omega), \tag{16.3.4}$$

respectively.

Let us start with the following statement explaining how the scalar functional differential equation for one of the components of the solution vector can be constructed.

**Lemma 16.1** *The component $x_n$ of the solution vector for the problem (16.3.1) satisfies the scalar functional differential equation*

$$(M x_n)(t) \equiv x_n'(t) + (B x_n)(t) = f^*(t), \ t \in [0, \omega], \tag{16.3.5}$$

*where*

$$(B x_n)(t) \equiv -\sum_{i=1}^{n-1} B_{ni} \left[ \int_0^t \sum_{j=1}^{n-1} K_{ij}(\xi, s)(B_{jn} x_n)(s) ds \right](t) + (B_{nn} x_n)(t), \ t \in [0, \omega], \tag{16.3.6}$$

*and*

$$f^*(t) = f_n(t) - \sum_{i=1}^{n-1} B_{ni}\left[\int_0^t \sum_{j=1}^{n-1} K_{ij}(\xi, s) f_j(s)ds\right](t)$$
$$- \sum_{i=1}^{n-1} B_{ni}\left[\sum_{j=1}^{n-1} K_{ij}(\xi, 0)x_j(0)\right](t). \tag{16.3.7}$$

*Proof* Using the Cauchy matrix $K(t,s) = \{K_{ij}(t,s)\}_{i,j=1,\cdots,n-1}$ of system (16.3.2), we obtain

$$x_i(t) = -\int_0^t \sum_{j=1}^{n-1} K_{ij}(t,s)(B_{jn}x_n)(s)ds + \int_0^t \sum_{j=1}^{n-1} K_{ij}(t,s) f_j(s)ds$$
$$+ \sum_{j=1}^{n-1} K_{ij}(t,0)x_j(0) \tag{16.3.8}$$

for each $i$. Substitution of these representations into the $n$-th equation of system (16.3.1) leads to (16.3.5), where the operator $B$ and the function $f^*$ are described by formulas (16.3.6) and (16.3.7), respectively. □

**Theorem 16.6** *Let all entries of the $(n-1)\times(n-1)$ Cauchy matrix $K(t,s)$ of system (16.3.2) be nonnegative, $B_{nn}$ be a positive operator and each of the operators $B_{jn}$ and $B_{nj}$ is the either positive or negative, while the product $-B_{nj}B_{jn}$ is the positive operator for $j = 1,\cdots,n-1$.*

*If $B_{ni}$ for $i = 1,\cdots,n-1$ are negative operators, then the following five assertions are equivalent*:

1) *There exists an absolutely continuous vector function $v$ such that $v' \in L_\infty[0,\omega]$, $v_n(t) > 0$, $v_i(0) \le 0$ for $i = 1,\cdots,n-1$, $(M_i v)(t) \le 0$ for $i = 1,\cdots,n$ and $v_n(\omega) - \int_t^\omega (M_n v)(s)ds > 0$ for $t \in [0,\omega)$.*
2) *$C_{nn}(t,s) > 0$, $C_{nj}(t,s) \ge 0$ for $j = 1,\cdots,n-1$, $0 \le s \le t \le \omega$.*
3) *The boundary value problem (16.3.1), (16.3.3) has a unique solution, and its Green's matrix satisfies the inequalities $G_{nj}(t,s) \le 0$ for $j = 1,\cdots,n$, $t,s \in [0,\omega]$ and $G_{nn}(t,s) < 0$ for $0 \le t < s \le \omega$.*
4) *If in addition $B$ defined by equality (16.3.6) is a nonzero operator, then the boundary value problem (16.3.1), (16.3.4) has a unique solution and its Green's matrix satisfies the inequalities $P_{nj}(t,s) \ge 0$ for $j = 1,\cdots,n$ and $P_{nn}(t,s) > 0$ for $t,s \in [0,\omega]$.*
5) *The $n$-th component of the solution vector $x$ of the homogeneous system $M_i x = 0$, $i = 1,\cdots,n$ such that $x_i(0) \ge 0$, $i = 1,\cdots,n-1$, $x_n(0) > 0$ is positive for $t \in [0,\omega]$.*

*If $B_{ni}$ for $i = 1,\cdots,n-1$ are positive operators, then the following five assertions are equivalent*:

1*) *There exists an absolutely continuous vector function $v$ such that $v' \in L_\infty[0,\omega]$, $v_n(t) > 0$, $v_i(0) \ge 0$, $(M_i v)(t) \ge 0$ for $i = 1,\cdots,n-1$, $(M_n v)(t) \le 0$ and $v_n(\omega) - \int_t^\omega (M_n v)(s)ds > 0$ for $t \in [0,\omega)$.*

2*) $C_{nn}(t,s) > 0$, $C_{nj}(t,s) \le 0$ *for* $j = 1, \cdots, n-1$, $0 \le s \le t \le \omega$.
3*) *The boundary value problem* (16.3.1), (16.3.3) *has a unique solution and its Green's matrix satisfies the inequalities* $G_{nj}(t,s) \ge 0$ *for* $j = 1, \cdots, n-1$, $G_{nn}(t,s) \le 0$ *for* $t, s \in [0,\omega]$ *and* $G_{nn}(t,s) < 0$ *for* $0 \le t < s \le \omega$.
4*) *If in addition* $B$ *defined by equality* (16.3.6) *is a nonzero operator, then the boundary value problem* (16.3.1), (16.3.4) *has a unique solution and its Green's matrix satisfies the inequalities* $P_{nj}(t,s) \le 0$ *for* $j = 1, \cdots, n$, $P_{nn}(t,s) > 0$ *for* $t, s \in [0,\omega]$.
5*) *The* $n$*-th component of the solution vector* $x$ *of the homogeneous system* $M_i x = 0$, $i = 1, \cdots, n$ *such that* $x_i(0) \le 0$, $i = 1, \cdots, n-1$, $x_n(0) > 0$ *is positive for* $t \in [0,\omega]$.

*Proof* Let us start with the implications 1) ⇒ 2) and 1*) ⇒ 2*). By Lemma 16.1, the component $x_n$ of the solution vector of system (16.3.1) satisfies (16.3.5). From the positivity of the operator $-B_{nj}B_{jn}$, it follows that $B$ is a positive operator. Each of the conditions 1) and 1*) implies that $(Mv_n)(t) \le 0$, where the operator $M$ is defined by (16.3.5), for $t \in [0,\omega]$. By Theorem 15.3, the Cauchy function $R(t,s)$ of the equation $Mv_n = 0$ is positive for $0 \le s \le t \le \omega$, $t \in [0,\omega]$.

From the solution representation formula and Lemma 16.1, it follows that

$$x_n(t) = \int_0^t \sum_{j=1}^n C_{nj}(t,s) f_j(s)ds = \int_0^t R(t,s) f^*(s)ds, \ t \in [0,\omega]. \tag{16.3.9}$$

If $B_{nj}$ is a negative operator for each $j = 1, \cdots, n-1$ and $f_i \ge 0$ for $i = 1, \cdots, n$, then $f^* \ge 0$. If $B_{nj}$ is a positive operator for each $j = 1, \cdots, n-1$, and $f_i \le 0$ for $i = 1, \cdots, n$, then $f^* \ge 0$. The positivity of $R(t,s)$ implies that $x_n$ is nonnegative and consequently $C_{nj}(t,s) \ge 0$ for $0 \le s \le t \le \omega$ and $j = 1, \cdots, n$.

If we set $f_j = 0$ and $x_j(0) = 0$ for $j = 1, \cdots, n-1$, then

$$x_n(t) = \int_0^t C_{nn}(t,s) f_n(s)ds = \int_0^t R(t,s) f_n(s)ds, \ t \in [0,\omega], \tag{16.3.10}$$

and it is evident that $C_{nn}(t,s) = R(t,s)$, which implies that $C_{nn}(t,s) > 0$ for $0 \le s \le t \le \omega$.

Let us prove the implication 1) ⇒ 3). By Lemma 16.1, the component $x_n$ of the solution vector for system (16.3.1) satisfies (16.3.5). From the positivity of the operator $-B_{nj}B_{jn}$, it follows that $B$ is a positive operator. Condition 1) implies that $(Mv_n)(t) \le 0$ for $t \in [0,\omega]$. By Theorem 15.3, the Green's function $G_M(t,s)$ of the scalar boundary value problem

$$(Mx_n)(t) \equiv x_n'(t) + (Bx_n)(t) = f^*(t), \ t \in [0,\omega], \ \ x_n(\omega) = 0 \tag{16.3.11}$$

exists and satisfies the inequalities $G_M(t,s) < 0$ for $0 \le t \le s \le \omega$ and $G_M(t,s) \le 0$ for $0 \le s \le t \le \omega$. Lemma 16.1, the representations of solutions of boundary value problem (16.3.1), (16.3.3) and scalar one-point problem (16.3.11) imply the equality

$$x_n(t) = \int_0^t \sum_{j=1}^n G_{nj}(t,s) f_j(s)ds = \int_0^t G_M(t,s) f^*(s)ds, \ t \in [0,\omega]. \tag{16.3.12}$$

If $B_{nj}$ is a negative operator for each $j = 1, \cdots, n-1$ and $f_i \geq 0$ for $i = 1, \cdots, n$, then $f^* \geq 0$. The nonpositivity of $G_M(t,s)$ implies that $x_n$ is nonnegative and consequently $G_{nj}(t,s) \leq 0$ for $t, s \in [0, \omega]$ and $j = 1, \cdots, n$.

If we set $f_j = 0$ and $x_j(0) = 0$ for $j = 1, \cdots, n-1$, then

$$x_n(t) = \int_0^{\omega} G_{nn}(t,s) f_n(s) ds = \int_0^{\omega} G_M(t,s) f(s) ds, \ t \in [0, \omega], \tag{16.3.13}$$

and it is obvious that $G_{nn}(t,s) = G_M(t,s)$, which implies that $G_{nn}(t,s) < 0$ for $0 \leq t < s \leq \omega$.

The proofs of the implications 1*) ⇒ 3*), 1) ⇒ 4) and 1*) ⇒ 4*) are similar.

In order to prove 3) ⇒ 1), we set $v(t) = y(t)$, where $y$ is a solution of the boundary value problem

$$(M_i x)(t) = -1, \ i = 1, \cdots, n, \ t \in [0, \omega], \tag{16.3.14}$$

$$x_i(0) = 0, \ i = 1, \cdots, n-1, \ x_n(\omega) = 0. \tag{16.3.15}$$

In this case, $v_i(0) = 0$, $i = 1, \cdots, n-1$ and

$$v_n(t) = -\int_0^{\omega} \sum_{j=1}^{n} G_{nj}(t,s) ds > 0, \ t \in [0, \omega). \tag{16.3.16}$$

In order to prove 3*) ⇒ 1*), we set $v(t) = y(t)$, where $y$ is the solution of the boundary value problem consisting of the equations

$$(M_i x)(t) = 1, \ i = 1, \cdots, n-1, \ (M_n x)(t) = -1, \ t \in [0, \omega] \tag{16.3.17}$$

and boundary conditions (16.3.15).

In order to prove the implications 2) ⇒ 1) and 2*) ⇒ 1*), we set $v_i(t) = C_{in}(t, 0)$ for $i = 1, \cdots, n$, $t \in [0, \omega]$.

4) ⇒ 2) The entry $P_{nn}(t,s)$ coincides with the Green's function $Q(t,s)$ of the periodic problem for scalar equation (16.3.5), which has the representation

$$Q(t,s) = R(t,s) + \frac{R(t,0) R(\omega, s)}{1 - R(\omega, 0)}, \tag{16.3.18}$$

where $R(t,s)$ is the Cauchy function of (16.3.5) and $R(t,s) = 0$ if $0 \leq t < s \leq \omega$. Thus the positivity of the Green's matrix of problem (16.3.4) implies the positivity of $Q(t,s)$ and by Theorem 15.3 also the positivity of $R(t,s)$ for $0 \leq s \leq t \leq \omega$. If $f_i(t) \geq 0$ for $t \in [0, \omega]$, $i = 1, \cdots, n$, then $f^*$ defined by formula (16.3.7) satisfies the inequality $f^*(t) \geq 0$ for $t \in [0, \omega]$. It follows from formula (16.3.9) that $C_{nj}(t,s) \geq 0$. If we set $f_j = 0$ and $x_j(0) = 0$ for $j = 1, \cdots, n-1$, then we get equality (16.3.10) and consequently $C_{nn}(t,s) = R(t,s) > 0$ for $0 \leq s \leq t \leq \omega$.

4*) ⇒ 2*) can be proven similarly.

5) ⇒ 1) and 5*) ⇒ 1*) are justified as follows. Consider the solution vector of the problem $M_i x = 0$, $i = 1, \cdots, n$, $x_i(0) = 0$, $i = 1, \cdots, n-1$, $x_n(0) = \gamma$, with $\gamma > 0$. By assertion 5), the component $x_n(t) > 0$ for $t \in [0, \omega]$. Now we can set $v_n(t) = x_n(t)$, $v_i(0) = 0$, $i = 1, \cdots, n-1$.

In order to prove 2) ⇒ 5) and 2*) ⇒ 5*), consider the homogeneous system $M_i x = 0$, $i = 1, \cdots, n$. By Lemma 16.1, the component $x_n$ satisfies (16.3.5), where the operator $B$ is defined by formula (16.3.6) and

$$f^*(t) = -\sum_{i=1}^{n-1} B_{ni}\left(\sum_{j=1}^{n-1} K_{ij}(\xi, 0)x_j(0)\right)(t)$$

for this homogeneous system. From the formula (16.3.10), it is clear that the Cauchy function $R(t,s)$ of the scalar first-order equation (16.3.5) coincides with the entry $C_{nn}(t,s)$ of the Cauchy matrix $C(t,s)$ of system (16.3.1). The general solution of (16.3.5) can be written as

$$x(t) = \int_0^t C_{nn}(t,s) f^*(s)ds + C_{nn}(t,0)x_n(0).$$

From the properties of the Cauchy matrix, it follows that the vector

$$\mathrm{col}\left\{C_{1n}(t,0), \cdots, C_{nn}(t,0)\right\}$$

is a solution of the initial problem $M_i x = 0$, $i = 1, \cdots, n$, $x_i(0) = 0$, $i = 1, \cdots, n-1$, $x_n(0) = 1$. The conditions $x_i(0) \geq 0$ and the negativity of the operators $B_{ni}$ for $i = 1, \cdots, n-1$ in assertion 5) (the conditions $x_i(0) \leq 0$ and positivity of the operators $B_{ni}$ for $i = 1, \cdots, n-1$, in assertion 5*)) imply that $f^*(t) \geq 0$ for $t \in [0, \omega]$. Positivity of $C_{nn}(t,s)$ now implies that $x_n(t) \geq C_{nn}(t,0)x_n(0) > 0$. □

*Remark 16.2* Assertions 1) ⇒ 5) and 1*) ⇒ 5*) are analogues for the $n$-th component of the solution vector of the $n$-th-order functional differential systems of the classical de La Vallee Poussin theorem on the differential inequality obtained in [102] for ordinary second-order differential equations. Assertions 5) ⇒ 2), 5*) ⇒ 2*), 5) ⇒ 3) and 5*) ⇒ 3*) are analogues of the corresponding assertions connecting nonoscillation and positivity of Green's functions for the $n$-th-order ordinary differential equations [256].

Let us consider the delay differential system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j\big(t - \tau_{ij}(t)\big) = f_i(t),\ i = 1, \cdots, n,\ t \in [0, \infty), \qquad (16.3.19)$$

where the delay $\tau_{ij} \geq 0$ for $i, j = 1, \cdots, n$ as a particular case of system (16.3.1).

Let us introduce the following notation:

$$p_{ij}^* = \operatorname{ess\,sup} p_{ij}(t),\ \ p_{ij*} = \operatorname{ess\,inf} p_{ij}(t),\ \ \tau_{ij}^* = \operatorname{ess\,sup} \tau_{ij}(t),$$
$$\tau_{ij*} = \operatorname{ess\,inf} \tau_{ij}(t),\ \ p_{ij}^+(t) = \max\left\{0, p_{ij}(t)\right\}.$$

**Theorem 16.7** *Let the following conditions be fulfilled*:

1) $p_{ij} \leq 0$ *for* $i \neq j,\ i, j = 1, \cdots, n-1$.
2) $p_{jn} \geq 0$, $p_{nj} \leq 0$ *for* $j = 1, \cdots, n-1$, $p_{nn} \geq 0$.

3) $\tau_{ii}^*(p_{ii}^+)^* \le \frac{1}{e}$ *for* $i = 1, \cdots, n-1$.
4) *There exists a positive* $\alpha$ *such that* $\tau_{ii}^*\alpha \le \frac{1}{e}$ *for* $i = 1, \cdots, n$ *and*

$$p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t)e^{\alpha\tau_{nj}(t)} \le \alpha$$
$$\le \min_{1\le i\le n-1}\left\{-p_{in}(t)e^{\alpha\tau_{in}(t)} + \sum_{j=1}^{n-1} p_{ij}(t)e^{\alpha\tau_{ij}(t)}\right\}, \ t \in [0, \infty). \quad (16.3.20)$$

*Then the entries of the* $n$*-th row of the Cauchy matrix of system* (16.3.19) *satisfy the inequalities* $C_{nn}(t, s) > 0$, $C_{nj}(t, s) \ge 0$ *for* $j = 1, \cdots, n-1$, $0 \le s \le t < \infty$.

*Proof* By Corollary 16.1, all the entries of the $(n-1) \times (n-1)$ Cauchy matrix of the system

$$x_i'(t) + \sum_{j=1}^{n-1} p_{ij}(t)x_j\big(t - \tau_{ij}(t)\big) = f_i(t), \ i = 1, \cdots, n-1, \ t \in [0, \infty)$$

of the order $n-1$ are nonnegative. Let us extend the coefficients $p_{ij}$ and $\tau_{ij}$ to the interval $[-\tau^*, 0)$, where $\tau^* = \max_{i,j=1,\cdots,n} \tau_{ij}^*$, as $p_{ij}(t) = 0$ for $i \ne j$ and $p_{ii} = \alpha$, $\tau_{ij} = 0$, $i, j = 1, \cdots, n$, and consider system (16.3.19) also on $[-\tau^*, \infty)$. Let us set $v_i(t) = -e^{-\alpha t}$ for $i = 1, \cdots, n-1$ and $v_n(t) = e^{-\alpha t}$ in condition 1) of Theorem 16.6. We obtain that this condition is fulfilled if $\alpha$ satisfies the following system of inequalities:

$$\alpha \le -p_{in}(t)e^{\alpha\tau_{in}(t)} + \sum_{j=1}^{n-1} p_{ij}(t)e^{\alpha\tau_{nj}(t)}, \ i = 1, \cdots, n-1, \ t \in [-\tau^*, \infty), \quad (16.3.21)$$

$$p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t)e^{\alpha\tau_{nj}(t)} \le \alpha, \ t \in [-\tau^*, \infty). \quad (16.3.22)$$

Now, by virtue of Theorem 16.6, all the entries of the $n$-th row of the Cauchy matrix satisfy the inequalities $C_{nj}(t, s) \ge 0$ for $j = 1, \cdots, n-1$ and $C_{nn}(t, s) > 0$ for $-\tau^* \le s \le t < \infty$. The Cauchy matrices of system (16.3.19) on the interval $[0, \infty)$ and on the interval $[-\tau^*, \infty)$ clearly coincide in the triangle $0 \le s \le t < \infty$, which completes the proof. □

For the ordinary differential system

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j(t) = f_i(t), \ i = 1, \cdots, n, \ t \in [0, \infty), \quad (16.3.23)$$

Theorem 16.7 implies the following result.

**Theorem 16.8** *Let the following conditions be fulfilled*:

1) $p_{ij} \le 0$ *for* $i \ne j$, $i, j = 1, \cdots, n-1$.
2) $p_{jn} \ge 0$, $p_{nj} \le 0$ *for* $j = 1, \cdots, n-1$, $p_{nn} \ge 0$.

3) *There exists a positive* $\alpha$ *such that*

$$p_{nn}(t) - \sum_{j=1}^{n-1} p_{nj}(t) \le \alpha \le \min_{1 \le i \le n-1} \left\{ -p_{in}(t) + \sum_{j=1}^{n-1} p_{ij}(t) \right\}, \ t \in [0, \infty). \tag{16.3.24}$$

*Then the entries of the n-th row of the Cauchy matrix of system* (16.3.23) *satisfy the inequalities* $C_{nn}(t,s) > 0$, $C_{nj}(t,s) \ge 0$ *for* $j = 1, \cdots, n-1$, $0 \le s \le t < \infty$.

Consider now the ordinary differential system of the second order

$$\begin{aligned} x_1'(t) + p_{11}(t)x_1(t) + p_{12}(t)x_2(t) = f_1(t), \\ x_2'(t) + p_{21}(t)x_1(t) + p_{22}(t)x_2(t) = f_2(t), \end{aligned} \quad t \in [0, \infty). \tag{16.3.25}$$

**Theorem 16.9** *Let the following conditions be fulfilled*:

1) $p_{11} \ge 0$, $p_{12} \ge 0$, $p_{21} \le 0$, $p_{22} \ge 0$.
2) *There exists a positive* $\alpha$ *such that*

$$p_{22}(t) - p_{21}(t) \le \alpha \le p_{11}(t) - p_{12}(t), \ t \in [0, \infty). \tag{16.3.26}$$

*Then the entries of the second row of the Cauchy matrix of system* (16.3.25) *satisfy the inequalities* $C_{21}(t,s) \ge 0$, $C_{22}(t,s) > 0$ *for* $0 \le s \le t < \infty$.

*Remark 16.3* If coefficients $p_{ij}$ are constants, then the second condition in Theorem 16.9 is as follows:

$$p_{22} - p_{21} \le p_{11} - p_{12}, \ p_{11} - p_{12} > 0. \tag{16.3.27}$$

*Remark 16.4* Let us demonstrate that condition (16.3.27) (and consequently inequality (16.3.26)) is the best possible in the corresponding case. It is known that for each fixed $s$ the $2 \times 2$ matrix $C(t,s)$ is the fundamental matrix $X(t)$ of system (16.3.25) satisfying the condition $C(s,s) = I$, where $I$ is the $2 \times 2$ identity matrix. Theorem 16.9 claims that entries in the second row of the fundamental matrix are nonnegative. The characteristic equation of the system

$$\begin{aligned} x_1'(t) + p_{11}x_1(t) + p_{12}x_2(t) = 0, \\ x_2'(t) + p_{21}x_1(t) + p_{22}x_2(t) = 0, \end{aligned} \quad t \in [0, \infty), \tag{16.3.28}$$

with constant coefficients is

$$\lambda^2 + (p_{11} + p_{22})\lambda + p_{11}p_{22} - p_{12}p_{21} = 0, \tag{16.3.29}$$

and its roots are real if and only if

$$(p_{11} - p_{22})^2 \ge -4p_{12}p_{21}. \tag{16.3.30}$$

Instead of inequalities (16.3.27) let us consider

$$p_{22} - p_{21} \le p_{11} - p_{12} + \varepsilon, \ p_{11} - p_{12} + \varepsilon > 0, \tag{16.3.31}$$

where $\varepsilon$ is any positive constant. We can set $p_{11} = p_{22}$, and then the first inequality has the form $p_{12} - p_{21} \le \varepsilon$. If $p_{12}p_{21} < 0$, then inequality (16.3.30) is not satisfied and consequently each entry of the fundamental and the Cauchy matrices oscillates.

Let us prove the following assertions, giving an efficient test for nonnegativity of the entries in the $n$-th row of the Cauchy matrix in the case where the coefficients $|p_{nj}|$ are small enough for $j = 1, \cdots, n$.

**Theorem 16.10** *Let the following conditions be fulfilled*:

1) $p_{ij} \le 0$ *for* $i \ne j,\ i, j = 1, \cdots, n-1$.
2) $p_{jn} \ge 0,\ p_{nj} \le 0$ *for* $j = 1, \cdots, n-1,\ p_{nn} \ge 0$.
3) $\tau_{nn} = \text{const} > 0,\ \tau_{ij} = 0$ *for others*.
4) *The following inequalities are fulfilled*:

$$p_{nn}(t)\tau_{nn} \exp\left\{\tau_{nn}\sum_{j=1}^{n-1}|p_{nj}|^*\right\} \le \frac{1}{e},\ t \in [0, \infty), \tag{16.3.32}$$

$$\frac{1}{\tau_{nn}} + \sum_{j=1}^{n-1}|p_{nj}|^* \le \min_{1\le i\le n-1}\left\{-p_{in}(t) + \sum_{j=1}^{n-1}p_{ij}(t)\right\},\ t \in [0, \infty). \tag{16.3.33}$$

*Then the entries of the n-th row of the Cauchy matrix of system* (16.3.19) *satisfy the inequalities* $C_{nn}(t, s) > 0,\ C_{nj}(t, s) \ge 0$ *for* $j = 1, \cdots, n-1,\ 0 \le s \le t < \infty$.

*Proof* Consider the left-hand side

$$p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1}p_{nj}(t)e^{\alpha\tau_{nj}(t)} \le \alpha,\ t \in [0, \infty), \tag{16.3.34}$$

of inequality (16.3.20). Using condition 3), we obtain that the inequality

$$p_{nn}(t) \le \left[\alpha - \sum_{j=1}^{n-1}|p_{nj}|^*\right] e^{-\alpha\tau_{nn}},\ [0, \infty), \tag{16.3.35}$$

implies inequality (16.3.34). The right-hand side in inequality (16.3.35) attains its maximum for $\alpha = \frac{1}{\tau_{nn}} + \sum_{j=1}^{n-1}|p_{nj}|^*$. Substituting this $\alpha$ into (16.3.35) and the right part of (16.3.20), we obtain inequalities (16.3.32) and (16.3.33). □

*Remark 16.5* It should be noted that (16.3.32) is the best possible in the following sense. If $p_{nj} = 0,\ j = 1, \cdots, n-1,\ p_{nn} = \text{const} > 0$, then (16.3.32) has the form

$$p_{nn}\tau_{nn} \le \frac{1}{e},\ t \in [0, \infty), \tag{16.3.36}$$

$C_{nn}(t, s) = c_n(t, s)$, where $c_n(t, s)$ is the Cauchy function of the diagonal equation

$$x_n'(t) + p_{nn}x(t - \tau_{nn}) = 0,\ t \in [0, \infty). \tag{16.3.37}$$

The opposite inequality $p_{nn}\tau_{nn} > \frac{1}{e}$ implies oscillation of all solutions [192] and, by Theorem 15.3, $c_n(t, s)$ changes its sign. Now it is clear that we cannot substitute

$$p_{nn}(t)\tau_{nn} \exp\left\{\tau_{nn}\sum_{j=1}^{n-1}|p_{nj}|^*\right\} \le \frac{1+\varepsilon}{e},\ t \in [0, \infty), \tag{16.3.38}$$

where $\varepsilon$ is any positive number, instead of inequality (16.3.32).

Let us consider the second-order scalar differential equation

$$(Ny)(t) \equiv y''(t) + p_{11}(t)y'\big(t-\tau_{11}(t)\big) + p_{12}(t)y\big(t-\tau_{12}(t)\big) = f_1(t), \quad (16.3.39)$$

$t \in [0,\infty)$, where $y(\theta) = y'(\theta) = 0$ for $\theta < 0$, and the corresponding differential system of the second order

$$\begin{aligned} &x_1'(t) + p_{11}(t)x_1\big(t-\tau_{11}(t)\big) + p_{12}(t)x_2\big(t-\tau_{12}(t)\big) = f_1(t), \\ &x_2'(t) - x_1(t) = 0, \end{aligned} \quad (16.3.40)$$

$t \in [0,\infty)$, where $x_1(\theta) = x_2(\theta) = 0$ for $\theta < 0$.

It should be noted that the entry $C_{21}(t,s)$ of the Cauchy matrix of system (16.3.40) coincides with the Cauchy function $W(t,s)$ of the scalar second-order equation (16.3.39) and $C_{11}(t,s) = W_t'(t,s)$. If the function $y(t)$ is a solution of the Cauchy problem $(Ny)(t) = 0$, $t \in [0,\infty)$, $y(0) = 1$, $y'(0) = 0$, then $C_{22}(t,0) = y(t)$ and $C_{12}(t,0) = y'(t)$.

**Theorem 16.11** *Assume that $p_{12} \geq 0$, $p_{11}^*\tau_{11}^* \leq \frac{1}{e}$ and there exists a positive number $\alpha$ such that $\alpha\tau_{11}^* \leq \frac{1}{e}$ and*

$$\alpha^2 + p_{12}(t)e^{\alpha\tau_{12}(t)} \leq \alpha p_{11}(t)e^{\alpha\tau_{11}(t)}, \quad t \in [0,\infty). \quad (16.3.41)$$

*Then the entries of the second row of the Cauchy matrix of system* (16.3.40) *satisfy the inequalities $C_{21}(t,s) \geq 0$, $C_{22}(t,s) > 0$ for $0 \leq s \leq t < \infty$.*

*Proof* In order to prove the theorem, we set $v_1(t) = -\alpha e^{-\alpha t}$, $v_2(t) = e^{-\alpha t}$ in assertion 1) of Theorem 16.6. □

**Theorem 16.12** *Assume that $p_{12} \geq 0$, $p_{11}^*\tau_{11}^* \leq \frac{1}{e}$, $\tau_{11} \geq \tau_{12}$ and*

$$4p_{12}(t) \leq p_{11*}^2, \quad t \in [0,\infty). \quad (16.3.42)$$

*Then the entries of the second row of the Cauchy matrix of system* (16.3.40) *satisfy the inequalities $C_{21}(t,s) \geq 0$, $C_{22}(t,s) > 0$ for $0 \leq s \leq t < \infty$.*

*Proof* In order to prove the theorem, we set $\alpha = \frac{p_{11*}}{2}$ in Theorem 16.11. □

*Remark 16.6* Inequality (16.3.42) is the best possible in the following sense. Let us consider the system with constant coefficients

$$\begin{aligned} &x_1'(t) + p_{11}x_1(t) + p_{12}x_2(t) = f_1(t), \\ &x_2'(t) - x_1(t) = 0, \end{aligned} \quad (16.3.43)$$

$t \in [0,\infty)$. The characteristic equation for this system has real roots if and only if the inequality $4p_{12} \leq p_{11}^2$ is fulfilled.

## 16.4 Positivity of the Fixed $n$-th Row of Green's Matrices

Consider the boundary value problem

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij} x_j)(t) = f_i(t),\ t \in [0, \omega],\ i = 1, \cdots, n, \tag{16.4.1}$$

$$l_i x_i = c_i,\ i = 1, \cdots, n, \tag{16.4.2}$$

where $B_{ij} : C[0, \omega] \to L_1[0, \omega]$ are linear bounded operators for $i, j = 1, \cdots, n$ and $l_i : C_{[0,\omega]} \to \mathbb{R}$, $i = 1, \cdots, n$ are linear boundary functionals.

Together with problem (16.4.1), (16.4.2), let us consider the auxiliary problem consisting of the system

$$(m_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n-1} (B_{ij} x_j)(t) = f_i(t),\ t \in [0, \omega],\ i = 1, \cdots, n-1, \tag{16.4.3}$$

of the order $n - 1$ and the boundary conditions

$$l_i x_i = c_i,\ i = 1, \cdots, n-1. \tag{16.4.4}$$

Let us assume that this auxiliary boundary value problem has a unique solution for every vector $\mathrm{col}\{f_1, \cdots, f_{n-1}\}$ with essentially bounded components $f_i$ and every constant vector $\mathrm{col}\{c_1, \cdots, c_{n-1}\}$ and denote by $K(t, s) = \{K_{ij}(t, s)\}_{i,j=1,\cdots,n-1}$ Green's matrix of problem (16.4.3), (16.4.4) and by $G(t, s) = \{G_{ij}(t, s)\}_{i,j=1,\cdots,n}$ Green's matrix of problem (16.4.1), (16.4.2).

**Lemma 16.2** *Let problem* (16.4.3), (16.4.4) *have a unique solution for each vector* $f = \mathrm{col}\{f_1, \cdots, f_n\}$ *with essentially bounded components* $f_i$ *and each constant vector* $c = \mathrm{col}\{c_1, \cdots, c_n\}$. *Then the component* $x_n$ *of the solution vector of system* (16.4.1) *satisfies the scalar functional differential equation*

$$x_n'(t) + (Bx_n)(t) = f^*(t),\ t \in [0, \omega], \tag{16.4.5}$$

*where the operator* $B : C[0, \omega] \to L_1[0, \omega]$ *and the function* $f^* \in L_1[0, \omega]$ *are defined by the equalities*

$$(Bx_n)(t) \equiv -\sum_{i=1}^{n-1} B_{ni} \left\{ \int_0^{\omega} \sum_{j=1}^{n-1} K_{ij}(\cdot, s)(B_{jn} x_n)(s) ds \right\}(t) + (B_{nn} x_n)(t),\ t \in [0, \omega], \tag{16.4.6}$$

$$f^*(t) = f_n(t) - \sum_{i=1}^{n-1} B_{ni} \left\{ \int_0^{\omega} \sum_{j=1}^{n-1} K_{ij}(\cdot, s) f_j(s) ds \right\}(t) - \sum_{i=1}^{n-1} (B_{ni} u_i)(t), \tag{16.4.7}$$

*where* $u = \mathrm{col}\{u_1, \cdots, u_{n-1}\}$ *is the solution of the system*

$$(m_i x)(t) = 0,\ t \in [0, \omega],\ i = 1, \cdots, n-1, \tag{16.4.8}$$

*satisfying condition* (16.4.4).

*Proof* Using Green's matrix $K(t,s) = \{K_{ij}(t,s)\}_{i,j=1}^{n-1}$ of problem (16.4.3), (16.4.4), we obtain

$$x_i(t) = -\int_0^{\omega} \sum_{j=1}^{n-1} K_{ij}(t,s)(B_{jn}x_n)(s)ds + \int_0^{\omega} \sum_{j=1}^{n-1} K_{ij}(t,s) f_j(s)ds + u_i(t) \tag{16.4.9}$$

for every $i \in \{1, \cdots, n-1\}$. Substitution of these representations in the $n$-th equation of the system (16.4.1) leads to (16.4.5), where the operator $B$ and the function $f^*$ are described by the formulas (16.4.6) and (16.4.7), respectively. □

Consider the boundary value problem

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n}(B_{ij}x_j)(t) = f_i(t),\ t \in [0,\omega],\ i = 1, \cdots, n, \tag{16.4.10}$$

$$l_i x_i = c_i,\ i = 1, \cdots, n-1,\ x_n(\omega) = c_n, \tag{16.4.11}$$

where $B_{ij} : C[0,\omega] \to L_1[0,\omega]$ are linear continuous operators for $i, j = 1, \cdots, n$.

Let us present an auxiliary assertion [115]. Consider scalar equation (16.4.5) and the integral operator $N : C_{[0,\omega]} \to C_{[0,\omega]}$ defined by the equality

$$(Nx)(t) = \int_t^{\omega} \left\{ \int_0^{\omega} x(\xi) d_{\xi} b(s,\xi) \right\} ds.$$

**Lemma 16.3** *Let the operator $B : C[0,\omega] \to L_1[0,\omega]$ be a positive operator admitting the representation*

$$(Bx)(t) = \int_0^{\omega} x(s) d_s b(t,s),\ t \in [0,\omega].$$

*Then the following assertions are equivalent*:

1) *There exists a nonnegative absolutely continuous function $v$ such that the set of zeros of $\psi$, where*

$$\psi(t) = v(t) - \int_t^{\omega} \int_0^{\omega} v(\xi)\, d_{\xi} b(s,\xi)\, ds,$$

*is not more than countable and $\psi(s) > 0$ if* $\mathrm{mes}\{t \in [0,\omega] : b(t,s+) \neq b(t,s-)\} > 0$.
2) *The spectral radius of the operator $N : C[0,\omega] \to C[0,\omega]$ is less than one.*
3) *The boundary value problem, which consists of* (16.4.5) *and boundary condition $x_n(\omega) = 0$, has a unique solution, and its Green's function is nonpositive for $t, s \in [0,\omega]$, while $G(t,s) < 0$ for $0 < t < s < \omega$.*

**Theorem 16.13** *Let problem* (16.4.3), (16.4.4) *have a unique solution for every vector* $\mathrm{col}\{f_1, \cdots, f_{n-1}\}$ *with essentially bounded components $f_i$ and every constant vector* $\mathrm{col}\{c_1, \cdots, c_{n-1}\}$, *all entries of its $(n-1) \times (n-1)$ Green's matrix $K(t,s)$ be nonnegative and the operators $B_{in}$, $-B_{ni}$ and $B_{nn}$ be positive operators for $i = 1, \cdots, n-1$. Then the following two assertions are equivalent*:

1) *There exists an absolutely continuous vector function* $v$ *such that* $v_n(t) > 0$, $(M_i v)(t) \le 0$, *for* $t \in [0, \omega]$, $i = 1, \cdots, n$, *and the solution of the homogeneous equation* $(m_i u)(t) = 0$ *for* $t \in [0, \omega]$, $i = 1, \cdots, n-1$ *satisfying the conditions* $l_i u_i = l_i v_i$, $i = 1, \cdots, n-1$ *is nonpositive.*
2) *Boundary value problem* (16.4.1), (16.4.2) *has a unique solution for every integrable* $f = \mathrm{col}(f_1, \cdots, f_n)$ *and* $c = \mathrm{col}(c_1, \cdots, c_n) \in \mathbb{R}^n$, *and entries of the* $n$*-th row of its Green's matrix satisfy the inequalities*: $G_{nj}(t,s) \le 0$ *for* $j = 1, \cdots, n$, $t, s \in [0, \omega]$, *while* $G_{nn}(t,s) < 0$ *for* $0 \le t < s \le \omega$.

*Proof* Let us now prove the implication 1) $\Rightarrow$ 2) of the theorem. By Lemma 16.2, the component $x_n$ of the solution vector of problem (16.4.1), (16.4.2) satisfies (16.4.5). Condition 1) by Lemma 16.3 implies that the Green's function $G_N(t,s)$ of the boundary value problem

$$x_n'(t) + (Bx_n)(t) = f^*(t), \ t \in [0, \omega], \ x(\omega) = 0 \tag{16.4.12}$$

exists and satisfies the inequalities $G_N(t,s) \le 0$ for $t, s \in [0, \omega]$ and $G_N(t,s) < 0$ for $0 \le t < s \le \omega$. Lemma 16.2, the representations of solutions of boundary value problem (16.4.1), (16.4.2) and scalar one-point problem (16.4.12) imply the equality

$$x_n(t) = \int_0^\omega \sum_{j=1}^n G_{nj}(t,s) f_j(s) ds = \int_0^\omega G_N(t,s) f^*(s) ds, \ t \in [0, \omega]. \tag{16.4.13}$$

If $B_{nj}$ is a negative operator for every $j = 1, \cdots, n-1$ and $f_i \le 0$ for $i = 1, \cdots, n$, then $f^* \le 0$. The nonpositivity of $G_N(t,s)$ implies that $x_n$ is nonnegative and consequently $G_{nj}(t,s) \le 0$ for $t, s \in [0, \omega]$ and $j = 1, \cdots, n$.

If we set $f_j = 0$ for $j = 1, \cdots, n-1$ and $l_j x_j = 0$ for $j = 1, \cdots, n$, then

$$x_n(t) = \int_0^\omega G_{nn}(t,s) f_n(s) ds = \int_0^\omega G_N(t,s) f_n(s) ds, \ t \in [0, \omega], \tag{16.4.14}$$

and it is clear that $G_{nn}(t,s) = G_N(t,s)$. It is known from Lemma 16.3 that $G_N(t,s) < 0$ for $0 \le t < s \le \omega$, which implies that $G_{nn}(t,s) < 0$ for $0 \le t < s \le \omega$.

In order to prove 2) $\Rightarrow$ 1), let us define $v_i$ $(i = 1, \cdots, n)$ as

$$v_i(t) = w_i(t), \ i = 1, \cdots, n-1, \ v_n(t) = w_n(t) + 1, \ t \in [0, \omega], \tag{16.4.15}$$

where $w_i$ $(i = 1, \cdots, n)$ is a solution to the problem

$$\begin{gathered} w_i'(t) + \sum_{j=1}^n (B_{ij} w_j)(t) = -(B_{in} 1)(t), \ i = 1, \cdots, n, \ t \in [0, \omega], \\ l_i w_i = 0, \ i = 1, \cdots, n-1, \ w_n(\omega) = 0. \end{gathered} \tag{16.4.16}$$

Evidently the functions $v_i$ $(i = 1, \cdots, n)$ satisfy the homogeneous system

$$v_i'(t) + \sum_{j=1}^n (B_{ij} v_j)(t) = 0, \ i = 1, \cdots, n, \ t \in [0, \omega], \tag{16.4.17}$$

and $v_n(t) > 0$ for $t \in [0, \omega]$. □

**Theorem 16.14** *Let problem* (16.4.3), (16.4.4) *have a unique solution, all entries of its* $(n-1) \times (n-1)$ *Green's matrix* $K(t,s)$ *be nonpositive and* $B_{nn}, -B_{in}$ *and* $-B_{ni}$ *be positive operators for* $i = 1, \cdots, n-1$. *Then the following two assertions are equivalent*:

1*) *There exists an absolutely continuous vector function* $v$ *such that* $v_n(t) > 0$, $(M_n v)(t) \le 0$, $(M_i v)(t) \ge 0$ *for* $t \in [0,\omega]$, $i = 1, \cdots, n-1$ *and the solution of the homogeneous equation* $(m_i u)(t) = 0$ *for* $t \in [0,\omega]$, $i = 1, \cdots, n-1$, *satisfying the conditions* $l_i u_i = l_i v_i$, $i = 1, \cdots, n-1$, *is nonnegative.*
2*) *The boundary value problem* (16.4.1), (16.4.2) *has a unique solution for every integrable* $f = \mathrm{col}(f_1, \cdots, f_n)$ *and* $c = \mathrm{col}(c_1, \cdots, c_n) \in \mathbb{R}^n$, *and the entries of the* $n$*-th row of its Green's matrix satisfy the inequalities* $G_{nj}(t,s) \ge 0$ *for* $j = 1, \cdots, n-1$, $G_{nn}(t,s) \le 0$ *for* $t, s \in [0,\omega]$ *and* $G_{nn}(t,s) < 0$ *for* $0 \le t < s \le \omega$.

The proof of this theorem is similar to the proof of Theorem 16.13.

## 16.5 Nonpositivity Conditions for the *n*-th Row of Green's Matrices

In this section, we present sufficient nonpositivity conditions for the entries of the $n$-th row of Green's matrices for a system of ordinary and delay differential equations.

First we consider the system of ordinary differential equations

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t) = f_i(t), \ i = 1, \cdots, n, \ t \in [0,\omega], \tag{16.5.1}$$

with the boundary conditions

$$x_i(0) = x_i(\omega) + c_i, \ i = 1, \cdots, n-1, \ x_n(\omega) = c_n. \tag{16.5.2}$$

**Theorem 16.15** *Let the following conditions be fulfilled*:

1) $p_{ij} \le 0$ *for* $i \ne j, \ i, j = 1, \cdots, n-1$.
2) $p_{jn} \ge 0$, $p_{nj} \le 0$ *for* $j = 1, \cdots, n-1$, $p_{nn} \ge 0$.
3) *There exists a positive number* $\alpha$ *such that*

$$p_{nn}(t) - \sum_{j=1}^{n-1} p_{nj}(t) \le \alpha \le \min_{1 \le i \le n-1} \left\{ -p_{in}(t) + \sum_{j=1}^{n-1} p_{ij}(t) \right\}, \ t \in [0,\omega]. \tag{16.5.3}$$

*Then problem* (16.5.1), (16.5.2) *has a unique solution for every integrable vector function* $f = \mathrm{col}(f_1, f_2, \cdots, f_n)$ *and* $c = \mathrm{col}(c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$, *and the entries of the* $n$*-th row of Green's matrix of boundary value problem* (16.5.1), (16.5.2) *satisfy the inequalities* $G_{nj}(t,s) \le 0$ *for* $j = 1, \cdots, n$ *for* $t, s \in [0,\omega]$ *and* $G_{nn}(t,s) < 0$ *for* $0 \le t < s \le \omega$.

*Proof* Condition 1) and the inequality

$$0 < \alpha \le \min_{1\le i\le n-1}\left\{-p_{in}(t) + \sum_{j=1}^{n-1} p_{ij}(t)\right\},\ t \in [0,\omega], \tag{16.5.4}$$

imply, according to Theorems 16.1 and 16.2, the nonnegativity of all entries of Green's matrix $K(t,s)$ of the boundary value problem

$$x_i'(t) + \sum_{j=1}^{n-1} p_{ij}(t)x_j(t) = f_i(t),\ i = 1,\cdots,n-1,\ t \in [0,\omega], \tag{16.5.5}$$

$$x_i(0) = x_i(\omega) + c_i,\ i = 1,\cdots,n-1. \tag{16.5.6}$$

Let us set $v_i(t) = -e^{-\alpha t}$ for $i = 1,\cdots,n-1$ and $v_n(t) = e^{-\alpha t}$ in condition 1) of Theorem 16.13. We obtain that this condition is fulfilled if $\alpha$ satisfies

$$\alpha \le -p_{in}(t) + \sum_{j=1}^{n-1} p_{ij}(t),\ i = 1,\cdots,n-1,\ t \in [0,\omega], \tag{16.5.7}$$

$$p_{nn}(t) - \sum_{j=1}^{n-1} p_{nj}(t) \le \alpha,\ t \in [0,\omega]. \tag{16.5.8}$$

Now, by Theorem 16.13, all entries of the $n$-th row of Green's matrix satisfy the inequalities $G_{nj}(t,s) \le 0$ for $j = 1,\cdots,n-1$ and, using Lemma 16.3, we can conclude that $G_{nn}(t,s) < 0$ for $0 \le t < s \le \omega$. □

Consider now the ordinary differential system of the second order

$$\begin{aligned} x_1'(t) + p_{11}(t)x_1(t) + p_{12}(t)x_2(t) = f_1(t),\\ x_2'(t) + p_{21}(t)x_1(t) + p_{22}(t)x_2(t) = f_2(t), \end{aligned}\quad t \in [0,\omega], \tag{16.5.9}$$

with the conditions

$$x_1(0) = x_1(\omega) + c_1,\ x_2(\omega) = c_2. \tag{16.5.10}$$

From Theorem 16.15, as a particular case for $n = 2$, we obtain the following result.

**Theorem 16.16** *Let the following two conditions be fulfilled*:

1) $p_{11} \ge 0,\ p_{12} \ge 0,\ p_{21} \le 0,\ p_{22} \ge 0.$
2) *There exists a positive $\alpha$ such that*

$$p_{22}(t) - p_{21}(t) \le \alpha \le p_{11}(t) - p_{12}(t),\ t \in [0,\omega]. \tag{16.5.11}$$

*Then problem* (16.5.9), (16.5.10) *has a unique solution for every integrable* $f = \operatorname{col}(f_1, f_2)$ *and* $c = \{c_1, c_2\} \in \mathbb{R}^2$, *and the entries of the second row of Green's matrix of problem* (16.5.9), (16.5.10) *satisfy the inequalities* $G_{2i}(t,s) \le 0$ *for* $i = 1, 2,\ t,s \in [0,\omega]$ *and* $G_{22}(t,s) < 0$ *for* $0 \le t < s < \omega$.

*Remark 16.7* If coefficients $p_{ij}$ are constants, the second condition in Theorem 16.16 is as follows:

$$p_{22} - p_{21} \le p_{11} - p_{12}, \ p_{11} - p_{12} > 0. \tag{16.5.12}$$

*Remark 16.8* Let us demonstrate that inequality (16.5.12) is the best possible in the corresponding case and the condition

$$p_{22} - p_{21} \le p_{11} - p_{12} + \varepsilon, \ p_{11} - p_{12} + \varepsilon > 0 \tag{16.5.13}$$

cannot be set instead of (16.5.12). The characteristic equation of the system

$$\begin{aligned} x_1'(t) + p_{11}x_1(t) + p_{12}x_2(t) = 0, \\ x_2'(t) + p_{21}x_1(t) + p_{22}x_2(t) = 0, \end{aligned} \quad t \in [0, \omega], \tag{16.5.14}$$

with constant coefficients is as follows:

$$\lambda^2 + (p_{11} + p_{22})\lambda + p_{11}p_{22} - p_{12}p_{21} = 0. \tag{16.5.15}$$

If we set $p_{11} = p_{22} = 0$, $p_{21} < 0$, $p_{12} > 0$, $p_{12} - p_{21} < \varepsilon$, then the roots are $\lambda_1 = i\sqrt{-p_{12}p_{21}}$, $\lambda_2 = -i\sqrt{-p_{12}p_{21}}$, and the problem

$$\begin{aligned} x_1'(t) + p_{12}x_2(t) = 0, \\ x_2'(t) + p_{21}x_1(t) = 0, \end{aligned} \quad t \in [0, \omega], \tag{16.5.16}$$

$$x_1(0) = x_1(\omega), \ x_2(\omega) = 0, \tag{16.5.17}$$

has a nontrivial solution for $\omega = \frac{2\pi}{\sqrt{-p_{12}p_{21}}}$.

Let us consider the system of delay differential equations

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t)x_j\big(t - \tau_{ij}(t)\big) = f_i(t), \ i = 1, \cdots, n, \ t \in [0, \omega], \tag{16.5.18}$$

$$x_i(\xi) = 0 \text{ for } \xi < 0, \ i = 1, \cdots, n \tag{16.5.19}$$

with the boundary conditions

$$x_i(0) = x_i(\omega) + c_i, \ i = 1, \cdots, n-1, \ x_n(\omega) = c_n. \tag{16.5.20}$$

**Theorem 16.17** *Let the following conditions be fulfilled*:

1) $p_{ij} \le 0$ *for* $i \ne j, \ i, j = 1, \cdots, n-1$.
2) $p_{jn} \ge 0$, $p_{nj} \le 0$ *for* $j = 1, \cdots, n-1$, $p_{nn} \ge 0$.
3) $\tau_{ii} = 0$ *for* $i = 1, \cdots, n-1$.
4) *There exists a positive number* $\alpha$ *such that for* $t \in [0, \omega]$ *we have*

$$\begin{aligned} p_{nn}(t)e^{\alpha\tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t)e^{\alpha\tau_{nj}(t)} \le \alpha \\ \le \min_{1 \le i \le n-1} \left\{ -p_{in}(t)e^{\alpha\tau_{in}(t)} + p_{ii}(t) + \sum_{j=1, i \ne j}^{n-1} p_{ij}(t)e^{\alpha\tau_{ij}(t)} \right\}. \end{aligned} \tag{16.5.21}$$

*Then problem* (16.5.18), (16.5.20) *has a unique solution for every integrable* $f = \mathrm{col}(f_1, \cdots, f_n)$ *and* $c = \{c_1, \cdots, c_n\} \in \mathbb{R}^n$ *and the entries of the* $n$*-th row of Green's matrix of problem* (16.5.18), (16.5.20) *satisfy the inequalities* $G_{nj}(t,s) \le 0$ *for* $t, s \in [0, \omega]$, $j = 1, \cdots, n$ *and* $G_{nn}(t,s) < 0$ *for* $0 \le t < s \le \omega$.

*Proof* By Theorems 16.1 and 16.2, all the entries of $(n-1) \times (n-1)$ Green's matrix $K(t,s)$ of the problem, consisting of the system

$$x_i'(t) + \sum_{j=1}^{n-1} p_{ij}(t) x_j\big(t - \tau_{ij}(t)\big) = f_i(t), \; i = 1, \cdots, n-1, \; t \in [0, \omega], \qquad (16.5.22)$$

and the boundary conditions $x_i(0) = x_i(\omega) + c_i$, $i = 1, \cdots, n-1$, are nonnegative.

Let us set $v_i(t) = -e^{-\alpha t}$ for $i = 1, \cdots, n-1$ and $v_n(t) = e^{-\alpha t}$ in condition 1) of Theorem 16.13. We obtain that this condition is fulfilled if $\alpha$ satisfies the following system of inequalities:

$$\alpha \le -p_{in}(t) e^{\alpha \tau_{in}(t)} + p_{ii}(t) + \sum_{j=1, i \ne j}^{n-1} p_{ij}(t) e^{\alpha \tau_{ij}(t)}, \; i = 1, \cdots, n-1, \; t \in [0, \omega], \qquad (16.5.23)$$

$$p_{nn}(t) e^{\alpha \tau_{nn}(t)} - \sum_{j=1}^{n-1} p_{nj}(t) e^{\alpha \tau_{nj}(t)} \le \alpha, \; t \in [0, \omega]. \qquad (16.5.24)$$

Now, by virtue of Theorem 16.13, all entries of the $n$-th row of Green's matrix of problem (16.5.18), (16.5.20) satisfy the inequalities $G_{nj}(t,s) \le 0$ for $t, s \in [0, \omega]$, $j = 1, \cdots, n$ and $G_{nn}(t,s) < 0$ for $0 \le t < s \le \omega$. □

*Remark 16.9* It was explained in Sect. 16.4 that in the case of the ordinary system ($\tau_{ij} = 0$, $i, j = 1, \cdots, n$) with constant coefficients $p_{ij}$, inequality (16.5.21) is the best possible in the corresponding case.

Let us consider the second-order scalar differential equation

$$y''(t) + p_{11}(t) y'(t) + p_{12}(t) y\big(t - \tau_{12}(t)\big) = f_1(t), \; t \in [0, \omega], \qquad (16.5.25)$$

where $y(\xi) = y'(\xi) = 0$ for $\xi < 0$, with the boundary conditions

$$y'(0) = y'(\omega) + c_1, \; y(\omega) = c_2, \qquad (16.5.26)$$

and the corresponding differential system of the second order

$$\begin{aligned} &x_1'(t) + p_{11}(t) x_1(t) + p_{12}(t) x_2\big(t - \tau_{12}(t)\big) = f_1(t), \\ &x_2'(t) - x_1(t) = 0, \end{aligned} \quad t \in [0, \omega], \qquad (16.5.27)$$

where $x_1(\xi) = x_2(\xi) = 0$ for $\xi < 0$, with the boundary conditions

$$x_1(0) = x_1(\omega) + c_1, \; x_2(\omega) = c_2. \qquad (16.5.28)$$

It should be noted that the entry $G_{21}(t,s)$ of Green's matrix of system (16.5.27), (16.5.28) coincides with Green's function $W(t,s)$ of the problem (16.5.25), (16.5.26) for the scalar second-order equation and $C_{11}(t,s) = W_t'(t,s)$.

**Theorem 16.18** *Assume that* $p_{12} \geq 0$ *and there exists* $\alpha > 0$ *such that*

$$\alpha^2 + p_{12}(t)e^{\alpha\tau_{12}(t)} \leq \alpha p_{11}(t), \ t \in [0, \omega]. \tag{16.5.29}$$

*Then problem* (16.5.27), (16.5.28) *has a unique solution for every integrable* $f = \mathrm{col}(f_1, f_2)$ *and* $c = \mathrm{col}(c_1, c_2) \in \mathbb{R}^2$, *and the entries of the second row of Green's matrix of this problem satisfy the inequalities* $G_{2j}(t,s) \leq 0, \ j = 1, 2, \ t, s \in (0, \omega)$, $G_{22}(t,s) < 0$ *for* $0 \leq t < s < \omega$.

*Proof* In order to prove the theorem, we set $v_1(t) = -\alpha e^{-\alpha t}$, $v_2(t) = e^{-\alpha t}$ in assertion 1) of Theorem 16.13. □

*Remark 16.10* Inequality (16.5.29) is the best possible in the following sense. Let us add $\varepsilon$ to the right-hand side. We obtain the inequality

$$\alpha^2 + p_{12}(t)e^{\alpha\tau_{12}(t)} \leq \alpha p_{11}(t) + \varepsilon, \ t \in [0, \omega], \tag{16.5.30}$$

and the statement of Theorem 16.18 is not true. Let us set the coefficients as constants $p_{11} = 0$ and $0 < p_{12} < \varepsilon$. It is clear that inequality (16.5.30) is fulfilled if we set $\alpha$ small enough. Consider the homogeneous boundary value problem

$$\begin{aligned} &x_1'(t) + p_{12}x_2(t) = 0, \ x_2'(t) - x_1(t) = 0, \ t \in [0, \omega], \\ &x_1(0) = x_1(\omega), \ x_2(\omega) = 0. \end{aligned} \tag{16.5.31}$$

The components $x_1, x_2$ of the solution vector are periodic, and for $\omega = \frac{2\pi}{\sqrt{p_{12}}}$ the boundary value problem (16.5.31) has a nontrivial solution.

Let us prove the following result, which gives an efficient test for nonpositivity of the entries of the $n$-th row of Green's matrix in the case where the coefficients $|p_{nj}|$ are small enough for $j = 1, \cdots, n-1$.

**Theorem 16.19** *Let the following conditions be fulfilled*:

1) $p_{ij} \leq 0$ *for* $i \neq j, i, j = 1, \cdots, n-1$.
2) $p_{jn} \geq 0, \ p_{nj} \leq 0, \ p_{nn} \geq 0$ *for* $j = 1, \cdots, n-1$.
3) $\tau_{nn} = \mathrm{const}, \ \tau_{ij} = 0$ *for other* $i, j$.
4) *The following inequalities are fulfilled*:

$$p_{nn}(t)\tau_{nn} \exp\left\{\tau_{nn}\sum_{j=1}^{n-1}|p_{nj}|^*\right\} \leq \frac{1}{e}, \ t \in [0, \omega], \tag{16.5.32}$$

$$\frac{1}{\tau_{nn}} + \sum_{j=1}^{n-1}|p_{nj}|^* \leq \min_{1 \leq i \leq n-1}\left\{-p_{in}(t) + \sum_{j=1}^{n-1}p_{ij}(t)\right\}, \ t \in [0, \omega]. \tag{16.5.33}$$

*Then problem* (16.5.18), (16.5.20) *has a unique solution for every integrable vector function* $f = \mathrm{col}(f_1, f_2, \cdots, f_n)$ *and constant* $c = (c_1, c_2, \cdots, c_n) \in \mathbb{R}^n$, *and the entries of the* $n$*-th row of its Green's matrix satisfy the inequalities* $G_{nj}(t,s) \leq 0$ *for* $j = 1, \cdots, n$ *and* $G_{nn}(t,s) < 0$ *for* $0 \leq t < s < \omega$.

*Proof* Let us set $v_i(t) = -e^{-\alpha t}$ for $i = 1, \cdots, n-1$ and $v_n(t) = e^{-\alpha t}$ in condition 1) of Theorem 16.14,

$$p_{nn}(t)e^{\alpha\tau_{nn}} - \sum_{j=1}^{n-1} p_{nj}(t) \leq \alpha \leq \min_{1\leq i\leq n-1}\left\{-p_{in}(t) + \sum_{j=1}^{n-1} p_{ij}(t)\right\},\ t \in [0,\omega]. \tag{16.5.34}$$

On the left-hand side, we have the inequality

$$p_{nn}(t)e^{\alpha\tau_{nn}} - \sum_{j=1}^{n-1} p_{nj}(t) \leq \alpha,\ t \in [0,\omega], \tag{16.5.35}$$

which is fulfilled when

$$p_{nn}(t) \leq \left[\alpha - \sum_{j=1}^{n-1} |p_{nj}|^*\right] e^{-\alpha\tau_{nn}},\ t \in [0,\omega]. \tag{16.5.36}$$

The right-hand side in inequality (16.5.36) attains its maximum for $\alpha = \frac{1}{\tau_{nn}} + \sum_{j=1}^{n-1} |p_{nj}|^*$. Substituting this $\alpha$ into (16.5.36) and the right-hand side of (16.5.34), we obtain inequalities (16.5.32) and (16.5.33). □

*Remark 16.11* It can be stressed that we do not require smallness of the interval $[0,\omega]$ in Theorems 16.17–16.19.

*Remark 16.12* It can be noted that inequality (16.5.32) is the best possible in the following sense. If $p_{nj} = 0$ for $j = 1, \cdots, n-1$, $p_{nn} = \text{const} > 0$, then system (16.5.18) and inequality (16.5.32) become of the forms

$$x_i'(t) = f_i(t),\ i = 1, \cdots, n-1,\ x_n'(t) + p_{nn}x_n(t - \tau_{nn}) = f_n(t),\ t \in [0,\omega], \tag{16.5.37}$$

$$p_{nn}\tau_{nn} \leq \frac{1}{e},\ t \in [0,\omega], \tag{16.5.38}$$

respectively. The opposite of the inequality in (16.5.38), $p_{nn}\tau_{nn} > \frac{1}{e}$, implies oscillation of all solutions of the equation

$$x_n'(t) + p_{nn}x(t - \tau_{nn}) = 0,\ \in [0,\omega]. \tag{16.5.39}$$

This implies that the homogeneous problem

$$x_i'(t) = 0,\ i = 1, \cdots, n-1,\ x_n'(t) + p_{nn}x_n(t - \tau_{nn}) = 0,\ t \in [0,\omega], \tag{16.5.40}$$

$$x_i(0) = x_i(\omega),\ i = 1, \cdots, n-1,\ x_n(\omega) = 0 \tag{16.5.41}$$

has nontrivial solutions for corresponding $\omega$. Evidently, we cannot substitute

$$p_{nn}(t)\tau_{nn}\exp\left\{\tau_{nn}\sum_{j=1}^{n-1} |p_{nj}|^*\right\} \leq \frac{1+\varepsilon}{e},\ t \in [0,\omega], \tag{16.5.42}$$

where $\varepsilon$ is any positive number, instead of inequality (16.5.32).

## 16.6 Discussion and Open Problems

In this chapter, we considered boundary value problems for the system of functional differential equations

$$(M_i x)(t) \equiv x_i'(t) + \sum_{j=1}^{n} (B_{ij} x_j)(t) = f_i(t),\ t \in [0,\omega],\ i = 1, \cdots, n,\ lx = \alpha, \tag{16.6.1}$$

where $x = \mathrm{col}(x_1, \cdots, x_n)$, $B_{ij} : C[0,\omega] \to L_1[0,\omega]$ or $B_{ij} : C[0,\omega] \to L_\infty[0,\omega]$, $i, j = 1, \cdots, n$, are linear continuous operators, $C[0,\omega]$, $L_1[0,\omega]$ and $L_\infty[0,\omega]$ are the spaces of continuous, integrable and essentially bounded functions $y : [0,\omega] \to \mathbb{R}$, respectively, and $l : C[0,\omega] \to \mathbb{R}^n$ is a linear bounded functional.

The property that the conditions

$$(M_i x)(t) \geq (M_i y)(t),\ t \in [0,\omega],\ i = 1, \cdots, n,\ lx = ly, \tag{16.6.2}$$

imply,

$$x_i(t) \geq y_i(t),\ t \in [0,\omega],\ i = 1, \cdots, n, \tag{16.6.3}$$

is a basis for the monotone technique [250].

S.A. Chaplygin was the first to note the importance of this property and proposed an approximate integration method on this basis [325]. A series of papers, starting with the paper by N.N. Luzin [278], considered various aspects of Chaplygin's approximate method. The well-known monograph by V. Lakshmikantham and S. Leela [250] was one of the most important contributions in this area. The book by M.A. Krasnosel'skii and others [233] was devoted to approximate methods for operator equations. These ideas have been developed in various publications on the monotone technique for the approximate solution of boundary value problems for systems of functional differential equations. The works by N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina [28] and I. Kiguradze and B. Puza [219, 222, 223] determine the modern level of this topic for functional differential equations.

Wazewski's classical theorem claims [332] that the condition

$$p_{ij} \leq 0 \text{ for } j \neq i,\ i, j = 1, \cdots, n \tag{16.6.4}$$

is necessary and sufficient for the property (16.6.2) ⇒ (16.6.3) for the Cauchy problem for the system of ordinary differential equations

$$x_i'(t) + \sum_{j=1}^{n} p_{ij}(t) x_j(t) = f_i(t),\ i = 1, \cdots, n,\ t \in [0,\omega]. \tag{16.6.5}$$

It would seem that the condition (16.6.4) or its analogues for various functional differential equations and boundary value problems sets a natural bound on the use of monotone methods. An attempt to overcome this limitation was made in the paper [113], where the idea to compare only one component of the solution vector was first formulated. Let $k_i$ be either 1 or 2. The paper [113] considered the conditions under which the inequalities

$$(-1)^{k_i}\big[(M_i x)(t) - (M_i y)(t)\big] \geq 0,\ t \in [0,\omega],\ lx = ly,\ i = 1,\cdots,n \qquad (16.6.6)$$

yield that for a corresponding fixed component $x_r$ of the solution vector the relation

$$x_r(t) \geq y_r(t),\ t \in [0,\omega] \qquad (16.6.7)$$

is satisfied. This property is weaker than the implication (16.6.2) ⇒ (16.6.3) and, as we see, leads to essentially weaker restrictions on the given system.

By solution representation formula (16.1.3), the property that (16.6.2) implies (16.6.3) is valid if all the entries of the matrices $G(t,s)$ and $X(t)$ are nonnegative and the implication (16.6.6) ⇒ (16.6.7) is reduced to the property that all entries of the $r$-th row only of the Green's matrix preserve their sign. The results on sign preservation for the entries in the $r$-th row of the Green's matrices were obtained in the papers [5, 113, 115].

The application of results on positivity of entries of the $r$-th row in the study of stability can be outlined as one of the most important open problems. The second one is application of these results to the study of nonlinear boundary value problems.

The main results of this chapter are based on the idea of constructing a scalar functional differential equation for one of the components of a solution vector. Quite a different idea, to reduce scalar integrodifferential equations (in a general case they can also be vector equations)

$$x'(t) + p(t)x\big(h(t)\big) + \int_0^t K(t,s)x(s)ds = 0,\ t \in [0,\infty), \qquad (16.6.8)$$

to systems of ordinary or functional differential equations and then to analyze these systems, was presented in the papers [2, 6, 131, 133, 134, 136]. Results on stability and bifurcation were obtained there on the basis of this idea. Note that such an approach allows us to understand what are autonomous and periodic equations in the case of (16.6.8) and to get results on the exponential stability of integrodifferential equations even in cases where the corresponding equation

$$x'(t) + p(t)x\big(h(t)\big) = 0,\ t \in [0,\infty)$$

is exponentially unstable. For a wide class of kernels $K(t,s)$, the system of ordinary or functional differential equations is finite-dimensional. In the case of more general kernels $K(t,s)$, this system is infinite-dimensional. One of the first systematic studies on the theory of infinite systems of ordinary differential equations

$$x_i'(t) + \sum_{j=1}^{\infty} p_{ij}(t)x(t) = f_i(t),\ i = 1,2,3,\cdots,\ t \in [0,\omega] \qquad (16.6.9)$$

can be found in the papers by K.P. Persidskii [298, 299], where existence and uniqueness results for the Cauchy problem were obtained and various examples demonstrating principal differences of finite and infinite systems were presented. Theorems on existence, uniqueness and convergence for the infinite systems were discussed in the well-known monograph by R. Bellman [38, pp. 204–216]. Important developments in the area of countable systems were presented in the book [313]. An approach to infinite-dimensional systems on the basis of the general theory of functional differential equations was started in [266, 267]. In [135, 137], the extension of results about positivity of a Green's matrix and exponential stability, obtained in Sect. 16.2 of this chapter, on infinite-dimensional systems is presented.

Finally, we outline some open problems.

1. Obtain theorems on differential inequalities for nonlinear systems based on sign constancy of the entries in the $r$-th row of Green's matrices and consequently of the property (16.6.6) $\Rightarrow$ (16.6.7).
2. Construct an analogue of the approximate integration monotone technique for nonlinear systems based on the property (16.6.6) $\Rightarrow$ (16.6.7).
3. Obtain assertions on maximum principles for a corresponding component $x_r$ of the solution vector. Based on these results, deduce existence and uniqueness tests for linear and nonlinear boundary value problems.
4. Obtain stability results for systems based on positivity of entries in a corresponding line of the Cauchy matrix.

# Chapter 17
# Nonoscillation Intervals for *n*-th-Order Equations

## 17.1 Introduction

In this chapter, we consider the $n$-th-order functional differential equation

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1} \big(B_j x^{(j)}\big)(t) = f(t),\ t \in [0, \omega], \tag{17.1.1}$$

where $B_j : C[0, \omega] \to L_\infty[0, \omega],\ j = 0, \cdots, n-1$ are linear continuous operators, $C[0, \omega]$ is the space of continuous functions and $L_\infty[0, \omega]$ is the space of measurable essentially bounded functions $x : [0, \omega] \to \mathbb{R}$. The case where $B_j : C[0, \omega] \to L_1[0, \omega],\ j = 0, \cdots, n-1$, where $L_1[0, \omega]$ is the space of integrable functions $x : [0, \omega] \to \mathbb{R}^1$, can be considered similarly. Operator $B_j$ can be of the forms

$$(B_j x)(t) = \sum_{i=1}^{m} p_i(t) x\big(h_i(t)\big),\ t \in [0, \omega], \tag{17.1.2}$$

$$x(\xi) = 0 \text{ for } \xi \notin [0, \omega], \tag{17.1.3}$$

or

$$(B_j x)(t) = \int_0^\omega K(t, s) x(s) ds,\ t \in [0, \omega], \tag{17.1.4}$$

and their linear combinations and superpositions also can be considered.

Let us start with the $n$-th-order ordinary differential equation

$$(\pounds x)(t) \equiv x^{(n)}(t) + \sum_{i=0}^{n-1} p_i(t) x^{(i)}(t) = f(t),\ t \in [0, \omega], \tag{17.1.5}$$

with essentially bounded coefficients $p_i$ and $f$. This equation will be considered with the interpolation boundary conditions

$$x^{(i)}(t_j) = c_j^i, \tag{17.1.6}$$

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where

$$0 \le t_1 < t_2 < \cdots < t_m \le \omega, \; i = 0, \cdots, k_j - 1, \; j = 1, \cdots, m, \; k_1 + \cdots + k_m = n. \tag{17.1.7}$$

Problem (17.1.5), (17.1.6) is called a de La Vallee Poussin problem. If the homogeneous problem

$$(\pounds x)(t) \equiv x^{(n)}(t) + \sum_{i=0}^{n-1} p_i(t) x^{(i)}(t) = 0, \; t \in [0, \omega], \tag{17.1.8}$$

$$x^{(i)}(t_j) = 0, \tag{17.1.9}$$

$$0 \le t_1 < t_2 < \cdots < t_m \le \omega, \; i = 0, \cdots, k_j - 1, \; j = 1, \cdots, m, \; k_1 + \cdots + k_m = n \tag{17.1.10}$$

has only the trivial solution, then the solution of (17.1.5), (17.1.6) has the representation

$$x(t) = \int_0^{\omega} G(t,s) f(s) ds + X(t), \tag{17.1.11}$$

where $X(t)$ is a solution of homogeneous equation (17.1.8) satisfying the condition (17.1.6) and $G(t,s)$ is called the Green's function of the boundary value problem (17.1.5), (17.1.6).

In order to explain the main goal of this chapter, let us present two definitions.

**Definition 17.1** $[0, \omega]$ is *a nonoscillation interval* of the equation $\pounds x = 0$ if every nontrivial solution does not have more than $n - 1$ zeros on this interval counting each zero according to its multiplicity.

**Definition 17.2** The Green's function of de La Vallee Poussin problem (17.1.5), (17.1.6) *behaves regularly* if $G(t,s)(t - t_1)^{k_1} \cdots (t - t_m)^{k_m} \ge 0$ for $t, s \in [0, \omega]$.

In this chapter, we generalize the following well-known result for ordinary differential equations to functional differential equations.

**Theorem A** [93, 256] *If* $[0, \omega]$ *is a nonoscillation interval of* (17.1.8), *then all Green's functions of de La Vallee Poussin problem* (17.1.5), (17.1.6) *behave regularly.*

## 17.2 Homogeneous Functional Differential Equations of the *n*-th Order

The modern theory of $n$-th-order differential equations with a delayed argument was started by A.D. Myshkis in the beginning of the 1950s with the equation

$$x''(t) + q(t) x'\big(t - \theta(t)\big) + p(t) x\big(t - \tau(t)\big) = f(t), \; t \in [0, \infty), \tag{17.2.1}$$

$$x(\xi) = \varphi(\xi),\ x'(\xi) = \psi(\xi) \text{ for } \xi < 0, \tag{17.2.2}$$

where $f$, $p$, $q$, $\tau$, $\theta$, $\varphi$ and $\psi$ are continuous functions, $\tau \geq 0$ and $\theta \geq 0$.

A natural question arises: which object will operate as a homogeneous equation in the case of this delay equation?

In the first publications, a homogeneous delay equation was introduced in the following form:

$$x''(t) + q(t)x'\big(t - \theta(t)\big) + p(t)x\big(t - \tau(t)\big) = 0,\ t \in [0, \infty), \tag{17.2.3}$$

$$x(\xi) = \varphi(\xi),\ x'(\xi) = \varphi'(\xi) \text{ for } \xi \leq 0. \tag{17.2.4}$$

If $\varphi$ in (17.2.4) is a fixed, twice differentiable function, only the Cauchy problem can be considered and a notion of the fundamental system cannot be introduced.

If (17.2.3), (17.2.4) is considered for all possible twice differentiable functions $\varphi$, then the space of its solutions is infinite-dimensional. As a result, a solution can have any number $n$ of zeros on a finite interval. For example,

$$x''(t) + x(t - 2\pi) = 0,\ t \in [0, \pi]$$

has the solutions $x = \sin nt$ if we choose $\varphi(\xi) = n^2 \sin n\xi$. Examples of this sort led mathematicians to the conclusion that for differential equations with a delayed argument a nonoscillation interval does not exist. That is why in the classical books on the oscillation theory of delay equations, nonoscillation was considered only as the existence of an eventually positive solution on the semiaxis [192, 248] and not on a finite interval. There were no analogues of the integral representation (17.1.11) and a notion of Wronskian for such an equation. In this sense, such an equation does not act as the homogeneous equation in the theory of ODEs.

N.V. Azbelev in [20] did not follow the tradition of considering a solution of delay equation (17.2.1) as a continuous extension of initial function $\varphi(t)$ and defined a homogeneous object as (17.2.3) with the zero initial functions

$$x(\xi) = 0,\ x'(\xi) = 0 \text{ for } \xi < 0. \tag{17.2.5}$$

As will be demonstrated below, (17.2.3), (17.2.5) is an analogue of the homogeneous equation for (17.2.1) in the theory of ODEs: the space of its solutions becomes two-dimensional, the formula for representation of the general solution of (17.2.1) is

$$x(t) = \int_0^t C(t,s) f(s)ds + x_1(t)x(0) + x_2(t)x'(0),$$

where $x_1, x_2$ are two solutions of the homogeneous equation such that $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 0$, $x_2'(0) = 1$, and $C(t,s)$ (as a function of $t$ for each fixed $s$) is a solution of the equation

$$x''(t) + q(t)x'\big(t - \theta(t)\big) + p(t)x\big(t - \tau(t)\big) = 0,\ t \in [s, \infty),$$

$$x(\xi) = 0,\ x'(\xi) = 0 \text{ for } \xi < s,$$

satisfying the conditions $C(s,s) = 0$, $C_t^{'}(s,s) = 1$. The behavior of the fundamental system $x_1$, $x_2$ of solutions of (17.2.3), (17.2.5) determines the existence and uniqueness of solutions and essentially also the stability and the sign properties of Green's functions for boundary value problems for delay differential equations and functional differential equations, which are their natural generalization.

Note that the equation

$$x''(t) + q(t)x\big(t-\theta(t)\big) + p(t)x\big(t-\tau(t)\big) = g(t),\ t \in [0,\infty),\\ x(\xi) = 0,\ x'(\xi) = 0 \text{ for } \xi < 0, \tag{17.2.6}$$

where $g(t) = f(t) - q(t)\psi(t-\theta(t))\sigma(t-\theta(t)) - p(t)\varphi(t-\tau(t))\sigma(t-\tau(t)), t \in [0,\infty), \sigma(t) = 1, t < 0, \sigma(t) = 0, 0 \le t$, is equivalent to (17.2.1), (17.2.2).

It is clear now that "traditional" homogeneous equation (17.2.3), (17.2.4) is nonhomogeneous with a special right-hand side according to Azbelev's definition of the homogeneous equation. Properties of homogeneous equation (17.2.3), (17.2.5) allow us to analyze the behavior of solutions for nonhomogeneous delay equation (17.2.1), (17.2.2).

The space of solutions of the homogeneous equation

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1}\big(B_j x^{(j)}\big)(t) = 0,\ t \in [0,\omega], \tag{17.2.7}$$

in the case of Volterra operators $B_j,\ j = 0,\cdots,n-1$, is $n$-dimensional. If homogeneous problem (17.2.7), (17.1.9) has only the trivial solution, then the solution of (17.1.1), (17.1.6) has the representation

$$x(t) = \int_0^\omega G(t,s)f(s)ds + X(t), \tag{17.2.8}$$

where $X(t)$ is a solution of the homogeneous equation (17.2.7) satisfying conditions (17.1.6), and the kernel $G(t,s)$ of this integral representation is called Green's function of problem (17.1.1), (17.1.6), see [28].

## 17.3 Wronskian of the Fundamental System

Consider the one-point problem

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1}\big(B_j x^{(j)}\big)(t)\\ = f(t),\ x^{(i-1)}(\mu) = c_i,\ i = 1,\cdots,n,\ t \in [0,\omega]. \tag{17.3.1}$$

This problem has a unique solution for each $f \in L_\infty[0,\omega]$ and $c_i \in \mathbb{R}$ ($i = 1,\cdots,n$) if and only if the homogeneous problem

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1}\big(B_j x^{(j)}\big)(t) = 0,\ x^{(i-1)}(\mu) = 0,\ i = 1,\cdots,n,\ t \in [0,\omega] \tag{17.3.2}$$

has only a trivial solution and, consequently, the Wronskian

$$W(t) = \begin{vmatrix} x_1(t) & \cdots & x_n(t) \\ & \cdots & \\ x_1^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{vmatrix} \equiv \left| x_1(t), \cdots, x_{n-1}(t), x_n(t) \right|, \tag{17.3.3}$$

of the fundamental system $x_1(t), \cdots, x_{n-1}(t), x_n(t)$ of the homogeneous equation $(Mx)(t) = 0$, $t \in [0, \omega]$ does not have a zero at the point $t = \mu$.

Let us assume that $W(0) \neq 0$ and define the ordinary differential operation

$$(\pounds x)(t) \equiv \frac{1}{W(t)} \left| x_1(t), \cdots, x_n(t), x(t) \right|, \ t \in [0, \omega], \tag{17.3.4}$$

where $W(t)$ is the Wronskian of the fundamental system $x_1(t), \cdots, x_{n-1}(t), x_n(t)$ of the equation $(Mx)(t) = 0$, $t \in [0, \omega]$. We have actually proven the following fact.

**Lemma 17.1** *The following three assertions are equivalent*:

1) *The Wronskian* $W(t) \neq 0$ *for* $t \in [0, \omega]$.
2) *The boundary value problem* (17.3.1) *has a unique solution for each* $f \in L_\infty[0, \omega]$, $c_i \in \mathbb{R}$ $(i = 1, \cdots, n)$ *and* $\mu \in [0, \omega]$.
3) *There exists an ordinary differential equation of* $n$*-th order* $(\pounds x)(t) = 0$, $t \in [0, \omega]$ *with essentially bounded coefficients*, *which is equivalent to the homogeneous equation* $(Mx)(t) = 0$, $t \in [0, \omega]$ *in the sense that every solution of one of them is also a solution of the other one*.

Note that for the second-order equation ($n = 2$) the condition $W(t) \neq 0$ is equivalent to Sturm's separation theorem (between two adjacent zeros of a given nontrivial solution there is one and only one zero of each solution that is linearly independent of the given one).

Sufficiency of $W(t) \neq 0$ can be easily proven from the contrary. Let $x_1$ and $x_2$ be two solutions such that $x_1(\alpha) = x_1(\beta) = 0$ but $x_2(t) > 0$ for $t \in [\alpha, \beta]$. Consider the function

$$y(t) = \frac{x_1(t)}{x_2(t)}.$$

It is clear that $y(\alpha) = y(\beta) = 0$. According to Rolle's theorem, there is a point $\theta$ such that $y'(\theta) = 0$, where $\alpha < \theta < \beta$. This is in contradiction with the fact that

$$y'(t) = \left\{ \frac{x_1(t)}{x_2(t)} \right\}' = \frac{x_1'(t)x_2(t) - x_1(t)x_2'(t)}{[x_2(t)]^2} = \frac{W(t)}{[x_2(t)]^2} \neq 0.$$

*Example 17.1* Consider the equation

$$x''(t) - 2x(0) = 0, \ t \in [0, \omega]. \tag{17.3.5}$$

Its fundamental system is $x_1(t) = (t-1)^2$, $x_2(t) = t$, and the Wronskian is $W(t) = 1 - t^2$, $W(1) = 0$. The problem

$$x''(t) - 2x(0) = 0, \ t \in [0, \omega], \ x(1) = 0, \ x'(1) = 1 \tag{17.3.6}$$

has no solution. Sturm's separation theorem is not valid: there exists the positive solution $x_2(t) = t$ and solutions with two zeros for each $\omega > 1$.

## 17.4 Nonoscillation of Functional Differential Equations

In this section, we obtain sufficient conditions under which $[0,\omega]$ is the nonoscillation interval of functional differential equation (17.2.7).

Let $x_1,\cdots,x_n$ be a fundamental system of homogeneous equation (17.2.7). Assume that $W(0)\neq 0$. Define a normal chain of the Wronskians according to [282],

$$W_1(t)=x_1(t),\ W_i(t)=\begin{vmatrix} x_1(t) & \cdots & x_i(t) \\ & \cdots & \\ x_1^{(i-1)}(t) & \cdots & x_i^{(i-1)}(t) \end{vmatrix},\ i=2,\cdots,n, \tag{17.4.1}$$

where the solutions $x_1,\cdots,x_n$ satisfy the initial conditions such that

$$W_n(0)=\begin{vmatrix} 0 & \cdots 0\ 1 \\ 0 & \cdots 1\ 0 \\ \cdots & \cdots\cdots \\ 1 & \cdots 0\ 0 \end{vmatrix}. \tag{17.4.2}$$

It is clear that $W_n(t)$ is the Wronskian $W(t)$ of (17.2.7). The existence of the equivalent ordinary differential equation $\pounds x=0$ (see Lemma 17.1) allows us to generalize the result of G. Mammana [282] (in the reduction of N.V. Azbelev) for functional differential equation (17.2.7).

**Theorem 17.1** $[0,\omega)$ *is a nonoscillation interval of* (17.2.7) *if and only if* $W(0)\neq 0$ *and each one of the elements of the normal chain of the Wronskians* $W_i(t)$ $(i=1,\cdots,n)$ *does not have a zero in the interval* $(0,\omega)$.

All elements of the normal chain of the Wronskians are continuous. This implies the following qualitative property of functional differential equation (17.2.7).

**Corollary 17.1** *If* $W(0)\neq 0$, *then there exists a nonoscillation interval of* (17.2.7).

The condition $W(0)\neq 0$ is essential, as the following example demonstrates.

*Example 17.2* Consider the equation $x''(t)-2x(1)=0$. Its solutions have the form $x(t)=at^2-bt+b$, where $a$ and $b$ are constants. We can choose, for example, $x_1(t)=t^2$ as one of its solutions. It is clear that $x_1(0)=x_1'(0)=0$ and there is no nonoscillation interval $[0,\omega]$ of this equation. Note that in this equation $W(0)=0$.

In the case of Volterra operators $B_j$ $(j=0,\cdots,n-1)$, we have $W(0)\neq 0$.

**Corollary 17.2** *If all* $B_j$ $(j=0,\cdots,n-1)$ *are Volterra operators*, *then there exists a nonoscillation interval of* (17.2.7).

The following Azbelev's result is known, see, for example [100, 101].

**Lemma 17.2** *If* $W(t) \neq 0$ *and the boundary value problem* (17.2.7), (17.4.3), *where*

$$x^{(i)}(0) = 0,\ x^{(j)}(\mu) = 0,\ i = 0, \cdots, k-1,\ j = 0, \cdots, n-k-1 \tag{17.4.3}$$

*for every* $1 \leq k \leq n-1$ *and* $\mu \in (0, \omega]$ *has only the trivial solution, then the Wronskians* $W_{n-k}(t)$ *do not have a zero for* $t \in (0, \omega]$, $1 \leq k \leq n-1$.

*Proof* The proof follows from the fact that the equality $W_{n-k}(\mu) = 0$ for a corresponding $\mu \in (0, \omega]$ implies that the system

$$\sum_{i=1}^{n-k} C_i x_i^{(r)}(\mu) = 0,\ r = 0, \cdots, n-k-1 \tag{17.4.4}$$

has nontrivial solutions with respect to $C_i$. In this case, $x(t) = \sum_{i=1}^{n-k} C_i x_i(t)$ is a nontrivial solution of (17.2.7) satisfying condition (17.4.3). □

Let us denote by $W_{k,n-k}(t,s)$ Green's functions of the problem (17.4.5), (17.4.3), where

$$x^{(n)}(t) = z(t),\ t \in [0, \omega]. \tag{17.4.5}$$

The solution of problem (17.4.5), (17.4.3) has the representation

$$x(t) = \int_0^\omega W_{k,n-k}(t,s) z(s) ds,\ t \in [0, \omega]. \tag{17.4.6}$$

After substituting it into the equation

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1} \left(B_j x^{(j)}\right)(t) = f(t),\ t \in [0, \omega], \tag{17.4.7}$$

we obtain the equation

$$z(t) = -\sum_{j=0}^{n-1} \left( B_j \int_0^\omega \frac{\partial^j W_{k,n-k}(\xi, s)}{\partial \xi^j} z(s) ds \right)(t) + f(t) \tag{17.4.8}$$

for the function $z \in L_\infty[0, \omega]$.

If the inequality

$$\operatorname*{ess\ sup}_{t \in [0,\omega]} \sum_{j=0}^{n-1} \|B_j\| \int_0^\omega \left| \frac{\partial^j W_{k,n-k}(t,s)}{\partial t^j} \right| ds < 1 \tag{17.4.9}$$

is fulfilled, then $W_{n-k}(t) \neq 0$ for $t \in (0, \omega]$.

The argument above leads to the following result.

**Theorem 17.2** *If the Wronskian* $W(t)$ *does not have a zero for* $t \in [0, \omega]$ *and inequality* (17.4.9) *is fulfilled for every* $k = 1, \cdots, n-1$, *then* $[0, \omega]$ *is a nonoscillation interval of the equation*

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1} \left(B_j x^{(j)}\right)(t) = 0,\ t \in [0, \omega]. \tag{17.4.10}$$

*Remark 17.1* Estimates of Green's functions $W_{k,n-k}(t,s)$ of the boundary value problems (17.4.5), (17.4.3) were obtained in the paper [37], and estimates of the derivatives $\frac{\partial^j W_{k,n-k}(t,s)}{\partial t^j}$ in the thesis [101]. Note that the same idea (known as Azbelev's $W$-transform) can be used also for the one-point boundary value problem (17.4.7), (17.4.11), where

$$x^{(j)}(\mu) = 0, \ j = 0, \cdots, n-1. \tag{17.4.11}$$

Green's function $W_{0,n}(t,s)$ of the auxiliary problem (17.4.5), (17.4.11) has the form

$$W_{0,n}(t,s) = \varkappa^1(t,s) W^1_{0n}(t,s) + \varkappa^2(t,s) W^2_{0n}(t,s), \tag{17.4.12}$$

$$\frac{\partial^i}{\partial t^i} W_{0,n}(t,s) = \varkappa^1(t,s) \frac{\partial^i}{\partial t^i} W^1_{0n}(t,s) + \varkappa^2(t,s) \frac{\partial^i}{\partial t^i} W^2_{0n}(t,s), \tag{17.4.13}$$

where

$$W^1_{0,n}(t,s) = \begin{cases} \frac{(t-s)^{n-1}}{(n-1)!}, & 0 \le s \le t \le \omega, \\ 0, & 0 \le t < s \le \omega, \end{cases} \tag{17.4.14}$$

$$\frac{\partial^i}{\partial t^i} W^1_{0,n}(t,s) = \begin{cases} \frac{(t-s)^{n-i-1}}{(n-i-1)!}, & 0 \le s \le t \le \omega, \\ 0, & 0 \le t < s \le \omega, \end{cases} \tag{17.4.15}$$

$$W^2_{0,n}(t,s) = \begin{cases} 0, & 0 \le s \le t \le \omega, \\ -\frac{(t-s)^{n-1}}{(n-1)!}, & 0 \le t < s \le \omega, \end{cases} \tag{17.4.16}$$

$$\frac{\partial^i}{\partial t^i} W^2_{0,n}(t,s) = \begin{cases} 0, & 0 \le s \le t \le \omega, \\ -\frac{(t-s)^{n-i-1}}{(n-i-1)!}, & 0 \le t < s \le \omega, \end{cases} \tag{17.4.17}$$

$$\varkappa^1(t,s) = \begin{cases} 1, \ \mu \le s \le t \le \omega, \\ 0, \text{ otherwise}, \end{cases} \tag{17.4.18}$$

$$\varkappa^2(t,s) = \begin{cases} 1, \ 0 \le s \le t \le \mu, \\ 0, \text{ otherwise}. \end{cases} \tag{17.4.19}$$

**Theorem 17.3** *If*

$$\operatorname*{ess\ sup}_{t \in [0,\omega]} \sum_{j=0}^{n-1} \|B_j\| \frac{\omega^{n-j}}{(n-j)!} < 1, \tag{17.4.20}$$

*then* $W(t) \neq 0$ *for* $t \in [0, \omega]$.

*Proof* By inequality (17.4.20), there exists a unique solution of problem (17.4.7), (17.4.11) and, according to Lemma 17.1, this inequality also implies the inequality $W(t) \neq 0$ for $t \in [0, \omega]$. □

Various necessary and sufficient conditions of nonvanishing Wronskians in the case of Volterra operators $B_i$ $(i = 1, \cdots, n)$ were obtained also in [241].

Consider, for example, the delay equation

$$x^{(n)}(t) + \sum_{j=0}^{n-1} p_i(t)x^{(j)}\big(t - \tau_j(t)\big) = 0,\ \ \tau_j(t) \ge 0,\ \ t \in [0, \infty), \tag{17.4.21}$$

$$x^{(j)}(\xi) = 0,\ \ \xi < 0,\ \ j = 0, \cdots, n-1. \tag{17.4.22}$$

Denote $\tau_i^* = \sup_{t\in[0,\infty)} \tau_i(t)$, $|p_i|^* = \sup |p_i(t)|$.

The following result was obtained in [241].

**Theorem 17.4** *Assume that there exists a positive $\alpha$ such that*

$$\alpha^n - \sum_{j=0}^{n-1} \alpha^j \exp\big\{\alpha\tau_j^*\big\}|p_j|^* \ge 0,\ \ t \in [0, \infty). \tag{17.4.23}$$

*Then the Wronskian of the fundamental system of* (17.4.21) *satisfies the inequality $W(t) \ne 0$ for $t \in [0, \infty)$.*

To prove Theorem 17.4, we set $v(t) = e^{-\alpha t}$ into condition 3) of Theorem 17.10 in the next section.

For the equation

$$x^{(n)}(t) + p_0(t)x\big(t - \tau_0(t)\big) = 0,\ \ \tau_0(t) \ge 0,\ \ t \in [0, \infty), \tag{17.4.24}$$

$$x(\xi) = 0,\ \ \xi < 0, \tag{17.4.25}$$

the special choice of $\alpha$ leads us to the following assertion, which was obtained in [241].

**Theorem 17.5** *If*

$$\tau_0^* \sqrt[n]{|p_0(t)|} \le \frac{n}{e},\ \ t \in [0, \infty), \tag{17.4.26}$$

*then the Wronskian of the fundamental system of equation* (17.4.24) *with initial function* (17.4.25) *satisfies the inequality $W(t) \ne 0$ for $t \in [0, \infty)$.*

## 17.5 Nonoscillation and Regular Behavior of Green's Functions

The results of two previous sections are based on the equivalence of the homogeneous functional differential

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1} \big(B_j x^{(j)}\big)(t) = 0,\ \ t \in [0, \omega], \tag{17.5.1}$$

and the ordinary differential equation

$$(\pounds x)(t) = 0,\ \ t \in [0, \omega], \tag{17.5.2}$$

where the differential operation $\pounds$ is defined by formula (17.3.4). In this section, we obtain the equivalence of the nonhomogeneous functional differential equation

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1} \big(B_j x^{(j)}\big)(t) = f(t),\ t \in [0, \omega], \tag{17.5.3}$$

and the ordinary differential equations

$$(\pounds x)(t) = (Pf)(t),\ t \in [0, \omega]. \tag{17.5.4}$$

In order to describe the structure of the operator $P : L_\infty[0, \omega] \to L_\infty[0, \omega]$, we will use the boundary value problems

$$(Mx)(t) = f(t),\ x^{(i-1)}(\mu) = 0,\ i = 1, \cdots, n,\ t \in [0, \omega] \tag{17.5.5}$$

for every $\mu \in (0, \omega]$ and the operators $H_\mu : L_\infty[0, \omega] \to L_\infty[0, \omega]$ defined by the equality

$$(H_\mu f)(t) \equiv -\sum_{j=0}^{n-1} \left( B_j \int_0^\omega \frac{\partial^j}{\partial \xi^j} G_\mu(\xi, s) f(s) ds \right)(t),\ t \in [0, \omega], \tag{17.5.6}$$

where $\frac{\partial^j}{\partial \xi_j} G_\mu(\xi, s)$ $(j = 0, \cdots, n-1)$ are Green's function and its derivatives in $\xi$ for the one-point problem (17.5.5). Note that the operators $B_j$ act on the functions

$$y_j(\xi) = \int_0^\omega \frac{\partial^j}{\partial \xi^j} G_\mu(\xi, s) f(s) ds,\ \xi \in [0, \omega], \tag{17.5.7}$$

as on functions of the variable $\xi$.

Results on the regular behavior of Green's functions are based on the following statement, which reduces the problem of positivity of the operator $P$ to sign constancy of Green's functions and their derivatives of one-point boundary value problems (17.5.5) for every $\mu \in (0, \omega]$.

**Theorem 17.6** *Let $W(t) \neq 0$ for $t \in [0, \omega]$. Then any solution of functional differential equation* (17.5.3) *also satisfies ordinary differential equation* (17.5.4), *where the differential operation $\pounds$ is defined by formula* (17.3.4), *and the bounded linear operator $P : L_\infty[0, \omega] \to L_\infty[0, \omega]$ is positive if the operators $H_\mu : L_\infty[0, \omega] \to L_\infty[0, \omega]$ are positive for every $\mu \in (0, \omega]$.*

*Proof* It follows from Lemma 17.1 that the condition $W(t) \neq 0$ for $t \in [0, \omega]$ implies the existence of a unique solution of boundary value problem (17.5.5) for each $f \in L_\infty[0, \omega]$ and $\mu \in (0, \omega]$. Denote its solution by $g_\mu(t)$. The general solution of (17.5.3) can be presented in the form

$$x(t) = \sum_{i=1}^{n} C_i x_i(t) + g_\mu(t),\ t \in [0, \omega]. \tag{17.5.8}$$

When we substitute this representation into the differential operation $\pounds$ defined by formula (17.3.4), we get

$$(\pounds x)(t) = \frac{1}{W(t)} \left| x_1(t), \cdots, x_n(t), \sum_{i=1}^{n} C_i x_i(t) + g_\mu(t) \right|$$
$$= \frac{1}{W(t)} \left| x_1(t), \cdots, x_n(t), g_\mu(t) \right|, \quad (17.5.9)$$

where $W(t)$ is the Wronskian of the fundamental system $x_1(t), \cdots, x_{n-1}(t), x_n(t)$ of (17.5.1). This means that the solution $x$ of functional differential equation (17.5.3) satisfies ordinary differential equation (17.5.4), where the operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$ is defined by the equality

$$(Pf)(t) = \frac{1}{W(t)} \left| x_1(t), \cdots, x_n(t), \int_0^\omega G_\mu(t,s) f(s) ds \right|, \quad (17.5.10)$$

where $G_\mu(t,s)$ is Green's function of problem (17.5.5). Using the boundary conditions in one-point boundary problem (17.5.5), we obtain for $t = \mu$ the equality

$$(Pf)(t) = \frac{\partial^n}{\partial t^n} \int_0^\omega G_\mu(t,s) f(s) ds$$
$$= f(t) - \sum_{j=0}^{n-1} \left( B_j \int_0^\omega \frac{\partial^j}{\partial \xi^j} G_\mu(\xi,s) f(s) ds \right)(t), \quad (17.5.11)$$

which completes the proof. □

Let us now formulate an obvious result that presents an extension of Theorem A of Sect. 17.1 to functional differential equations.

**Lemma 17.3** *Let the operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$ defined by formula* (17.5.11) *be positive for every $\mu \in (0,\omega]$ and $[0,\omega]$ be a nonoscillation interval of* (17.5.1). *Then all Green's functions of de La Vallee Poussin problems* (17.5.1), (17.1.6) *behave regularly.*

Let us define the operators $K_\mu : L_\infty[0,\omega] \to L_\infty[0,\omega]$ for every $\mu \in (0,\omega]$ by the equality

$$(K_\mu z)(t) = -\sum_{j=0}^{n-1} \left( B_j \int_0^\omega \frac{\partial^j}{\partial \xi^j} W_{0,n}(\xi,s) z(s) ds \right)(t), \ t \in [0,\omega], \quad (17.5.12)$$

where $\frac{\partial^j}{\partial \xi^j} W_{0,n}(\xi,s), \ j = 0, \cdots, n-1$ are the Green's function and its derivatives in $\xi$ for the one-point problem

$$x^{(n)}(t) = z(t), \ t \in [0,\omega], \quad (17.5.13)$$
$$x^{(j)}(\mu) = 0, \ j = 0, \cdots, n-1. \quad (17.5.14)$$

Note that the functions $\frac{\partial^j}{\partial \xi^j} W_{0,n}(\xi,s)$ $(j = 0, \cdots, n-1)$ are defined by formulas (17.4.12)–(17.4.19).

The following result allows us to verify the conditions of Theorem 17.6.

**Lemma 17.4** *For every $j = 0, \cdots, n-1$, let either $B_i$ be a zero operator or $(-B_j)$ be positive and the derivative $\frac{\partial^j}{\partial \xi^j} W_{0,n}(\xi, s)$ of the Green's function be nonnegative or $B_j$ be positive and the derivative $\frac{\partial^j}{\partial \xi^j} W_{0,n}(\xi, s)$ be nonpositive and the spectral radii $\rho(K_\mu)$ of the operators $K_\mu : L_\infty[0, \omega] \to L_\infty[0, \omega]$ be less than one for every $\mu \in (0, \omega]$. Then the operator $P : L_\infty[0, \omega] \to L_\infty[0, \omega]$ is positive.*

*Proof* The solution of problem (17.5.13), (17.5.14) can be presented in the form

$$x(t) = \int_0^\omega W_{0,n}(t, s) z(s) ds, \ \ t \in [0, \omega]. \tag{17.5.15}$$

When we substitute this representation into (17.5.3), we obtain

$$z(t) = (K_\mu z)(t) + f(t), \ \ t \in [0, \omega]. \tag{17.5.16}$$

From positivity of $K_\mu : L_\infty[0, \omega] \to L_\infty[0, \omega]$ and the inequality $\rho(K_\mu) < 1$ for every $\mu \in (0, \omega]$, we obtain that problem (17.5.5) has a unique solution for every $\mu$ and $z \geq 0$ for every $f \geq 0$. For the solution $x$ of problem (17.5.5) and its derivatives $x^{(j)}$, we have the representation

$$x^{(j)}(t) = \int_0^\omega \frac{\partial^j}{\partial t^j} W_{0,n}(t, s) z(s) ds, \ \ t \in [0, \omega], \tag{17.5.17}$$

and consequently

$$\text{sign}_{t \in [0,\omega]} x^{(j)}(t) \cdot \text{sign}_{(t,s) \in (0,\omega) \times (0,\omega)} \frac{\partial^j}{\partial t^j} W_{0,n}(t, s) > 0 \text{ for } t, s \in (0, \omega).$$

We mean here that $\text{sign}_{t \in [0,\omega]} x^{(j)}(t) > 0$ if there exists a point $t_0 \in (0, \omega)$ such that $x^{(j)}(t_0) > 0$ and $\text{sign}_{t \in [0,\omega]} x^{(j)}(t) < 0$ if there exists a point $t_0 \in (0, \omega)$ such that $x^{(j)}(t_0) < 0$. Similarly, we understand the inequalities

$$\text{sign}_{(t,s) \in (0,\omega) \times (0,\omega)} \frac{\partial^j}{\partial t^j} W_{0,n}(t, s) > 0$$

and

$$\text{sign}_{(t,s) \in (0,\omega) \times (0,\omega)} \frac{\partial^j}{\partial t^j} W_{0,n}(t, s) < 0$$

for $(t, s) \in (0, \omega) \times (0, \omega)$. Now we can conclude that

$$\text{sign}_{(t,s) \in (0,\omega) \times (0,\omega)} \frac{\partial^j}{\partial t^j} W_{0,n}(t, s) \cdot \text{sign}_{(t,s) \in (0,\omega) \times (0,\omega)} \frac{\partial^j}{\partial t^j} G_\mu(\xi, s) > 0$$

for $(t, s) \in (0, \omega) \times (0, \omega)$. Now it is clear that the operators $H_\mu : L_\infty[0, \omega] \to L_\infty[0, \omega]$ are positive for every $\mu \in (0, \omega]$.

The inequality $W(t) \neq 0$ for $t \in [0, \omega]$ follows from the existence of a unique solution of problem (17.5.5) for every $\mu$ and Lemma 17.1. Now we have proven that all the conditions of Theorem 17.6 are fulfilled and consequently, according to it, the operator $P$ is positive. This completes the proof. □

**Theorem 17.7** *Let either $n$ be even and the operators $(-B_j)$ be positive for every even $j$ and be zero operators for every odd $j$ or $n$ be odd and the operators $(-B_j)$ be positive for every odd $j$ and be zero operators for every even $j$ and the spectral radii $\rho(K_\mu)$ of the operators $K_\mu : L_\infty[0,\omega] \to L_\infty[0,\omega]$ be less than one for every $\mu \in (0,\omega]$. Then the operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$ is positive.*

*Proof* In order to prove this theorem, we have only to note that the case of non-negativity of $\frac{\partial^i}{\partial t^i} W^1_{0,n}(t,s)$ and $\frac{\partial^i}{\partial t^i} W^2_{0,n}(t,s)$ allows us to obtain positivity of the operators $K_\mu : L_\infty[0,\omega] \to L_\infty[0,\omega]$. The statement of the theorem now follows from Lemma 17.4. □

**Corollary 17.3** *Let either $n$ be even and the operators $(-B_j)$ be positive for every even $j$ and be zero operators for every odd $j$ or $n$ be odd and the operators $(-B_j)$ be positive for every odd $j$ and be zero operators for every even $j$ and*

$$\operatorname*{ess\,sup}_{t\in[0,\omega]} \sum_{j=0}^{n-1} \|B_j\| \frac{\omega^{n-j}}{(n-j)!} < 1. \tag{17.5.18}$$

*Then the operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$ is positive.*

*Proof* Inequality (17.5.18) implies that the norms and the spectral radii of the operators $K_\mu : L_\infty[0,\omega] \to L_\infty[0,\omega]$ for every $\mu \in (0,\omega]$ are less than one. Reference to Theorem 17.7 completes the proof. □

In the case where all $B_j$ are Volterra operators, the solution $x(t)$ of problem (17.5.5) on $[0,\mu]$ does not depend on values of $x(t)$ on the interval $(\mu,\omega]$. This allows us to consider the boundary value problems

$$(Mx)(t) \equiv x^{(n)}(t) + \sum_{j=0}^{n-1} \left(B_j^\mu x^{(j)}\right)(t) = f(t),\ t \in [0,\mu],$$

$$x^{(i-1)}(\mu) = 0,\ i = 1,\cdots,n, \tag{17.5.19}$$

instead of (17.5.5), where the operators $B_j^\mu : C[0,\mu] \to L_\infty[0,\mu]$ are such that $(B_j^\mu y)(t) = (B_j y)(t)$ for $t \in [0,\mu]$. For every $\mu \in (0,\omega]$, define the operators $H_\mu : L_\infty[0,\mu] \to L_\infty[0,\mu]$ and $K_\mu : L_\infty[0,\mu] \to L_\infty[0,\mu]$ by the equalities

$$(H_\mu z)(t) \equiv -\sum_{j=0}^{n-1} \left( B_j^\mu \int_0^\mu \frac{\partial^j}{\partial \xi^j} G_\mu(\xi,s) z(s) ds \right)(t),\ t \in [0,\mu], \tag{17.5.20}$$

and

$$(K_\mu z)(t) \equiv -\sum_{j=0}^{n-1} \left( B_j^\mu \int_0^\mu \frac{\partial^j}{\partial \xi^j} W_{0,n}(\xi,s) z(s) ds \right)(t),\ t \in [0,\mu], \tag{17.5.21}$$

respectively.

Similarly to Theorems 17.6 and 17.7, we obtain the following results.

**Theorem 17.8** *Let $B_j$ be Volterra operators and $W(t) \neq 0$ for $t \in [0, \omega]$. Then solutions of functional differential equation (17.5.3) also satisfy the ordinary differential equation (17.5.4), where the differential operation £ is defined by formula (17.3.4) and the bounded linear operator $P : L_\infty[0, \omega] \to L_\infty[0, \omega]$ is positive if the operators $H_\mu : L_\infty[0, \mu] \to L_\infty[0, \mu]$ are positive for every $\mu \in (0, \omega]$.*

**Theorem 17.9** *Let $(-1)^{n-j-1} B_j$ be positive Volterra operators and the spectral radii $\rho(K_\mu)$ of the operators $K_\mu : L_\infty[0, \mu] \to L_\infty[0, \mu]$ be less than one for every $\mu \in (0, \omega]$. Then the operator $P : L_\infty[0, \omega] \to L_\infty[0, \omega]$ is positive.*

The following result was obtained in [241].

**Theorem 17.10** *Let $B_j$ be Volterra operators and $(-1)^{n-j+1} B_j$ be positive operators for $j = 0, \cdots, n-1$. Then the following four assertions are equivalent:*

1) *The boundary value problem (17.5.19) is uniquely solvable and its Green's function satisfies the inequalities $\frac{\partial^j}{\partial t^j} G_\mu(t, s)(-1)^{n-j} \geq 0$ for $(t, s) \in [0, \mu] \times [0, \mu]$ and $\frac{\partial}{\partial t^j} G_\mu(t, s)(-1)^{n-j} > 0$ for $0 \leq t < s \leq \mu$, $j = 0, \cdots, n-1$.*
2) *$W(t) \neq 0$ for $t \in [0, \omega]$.*
3) *There exists a function $v$ with absolutely continuous derivatives $v^{(i)}$ ($i = 0, 1, \cdots, n-1$) such that $(-1)^i v^{(i)}(t) > 0$, $i = 0, \cdots, n-1$, $(-1)^n (Mv)(t) \geq 0$, $t \in [0, \omega)$, $(-1)^i v^{(i)}(\omega) \geq 0$, $i = 0, \cdots, n-1$, $\sum_{i=0}^{n-1} |v^{(i)}(\omega)| > 0$.*
4) *Let $x$ be a solution of the homogeneous equation $(Mx)(t) = 0$ such that $(-1)^j x^{(j)}(\omega) \geq 0$, $j = 0, \cdots, n-1$. Then, from the inequality $\sum_{j=0}^{n-1} |x^{(j)}(\omega)| > 0$, it follows that $(-1)^j x^{(j)}(t) > 0$, $t \in [0, \omega)$, and if $x^{(j)}(\omega) = 0$, $j = 0, \cdots, n-1$, then $x(t) \equiv 0$, $t \in [0, \omega]$.*

The following result is an analogue of Theorem A for functional differential equations.

**Theorem 17.11** *Assume that $[0, \omega]$ is a nonoscillation interval for (17.5.1) and at least one of the following three conditions 1)–3) is fulfilled:*

1) *$n$ is even and the operators $(-B_j)$ are positive for every even $j$ and are zero operators for every odd $j$.*
2) *$n$ is odd and the operators $(-B_j)$ are positive for every odd $j$ and are zero operators for every even $j$.*
3) *$(-1)^{n-j+1} B_j$ are positive Volterra operators for $j = 0, \cdots, n-1$.*

*Then Green's functions of the de La Vallee Poussin problems (17.5.3), (17.5.22), where*

$$x^{(i)}(t_j) = 0, \tag{17.5.22}$$

$$0 \leq t_1 < t_2 < \cdots < t_m \leq \omega,\ i = 0, \cdots, k_j - 1,\ j = 1, \cdots, m,\ k_1 + \cdots + k_m = n, \tag{17.5.23}$$

*behave regularly; i.e., $G(t, s)(t - t_1)^{k_1} \cdots (t - t_m)^{k_m} \geq 0$ for $t, s \in [0, \omega]$.*

*Proof* The fact that $[0, \omega]$ is a nonoscillation interval means also that the Wronskian $W(t) \neq 0$ for $t \in [0, \omega]$ since the opposite assumption $W(\mu) = 0$ for $\mu \in [0, \omega]$ leads us to the existence of a nontrivial solution with zero of multiplicity $n$ at the point $\mu$. The fact that $W(t) \neq 0$ for $t \in [0, \omega]$ and each of the conditions 1)–3) according to Theorems 17.7 and 17.9 implies the positivity of the operator $P$ and the reference to Theorem A completes the proof. □

The following statement allows us to obtain nonoscillation without estimates of Green's functions $W_{k,n-k}(t, s)$ of problems (17.4.5), (17.4.3) and its derivatives $\frac{\partial^j W_{k,n-k}(\cdot,s)}{\partial t^j}$.

**Theorem 17.12** *Let the Wronskian satisfy $W(t) \neq 0$ for $t \in [0, \omega]$, the operator $P : L_\infty[0, \omega] \to L_\infty[0, \omega]$ be positive and there exist positive functions $v_1, \cdots, v_{n-1}$ with absolutely continuous derivatives $v_j^{(i)}$ $(i = 0, 1, \cdots, n-1,\ j = 1, \cdots, n-1)$ such that the conditions 1) and 2) are fulfilled*:

1) *The system of $n-1$ Wronskians $w_k(t) = |v_k(t), \cdots, v_{n-1}(t)|$ satisfies the inequalities $w_k(t) > 0$ for $t \in [0, \omega)$, $k = 1, \cdots, n-1$.*
2) *For every $k$, the system of $n-k-1$ Wronskians*

$$w_{kl}(t) = |v_k(t), \cdots, v_{l-1}, v_{l+1}(t), \cdots, v_{n-1}(t)|$$

*satisfies the inequalities $w_{kl}(t) > 0$ for $t \in [0, \omega)$, $k < l \leq n-1$.*

*Then the differential inequalities*

$$(-1)^{n-i}(Mv_i)(t) \geq 0 \text{ for } t \in [0, \omega],\ i = 1, \cdots, n-1, \tag{17.5.24}$$

*imply nonoscillation of solutions of* (17.5.1) *on* $[0, \omega]$.

*Proof* The proof is based on the result of Theorem 4.1 in the paper [256] for ordinary differential equation (17.1.8). □

**Lemma 17.5** *Let there exist positive functions $v_1, \cdots, v_{n-1}$ with absolutely continuous derivatives $v_j^{(i)}$ $(i = 0, 1, \cdots, n-1,\ j = 1, \cdots, n-1)$ such that conditions 1) and 2) of Theorem 17.12 are fulfilled. Then the differential inequalities*

$$(-1)^{n-i}(\pounds v_i)(t) \geq 0 \text{ for } t \in [0, \omega],\ i = 1, \cdots, n-1, \tag{17.5.25}$$

*imply nonoscillation of solutions of* (17.1.8) *on* $[0, \omega]$.

The inequality $W(t) \neq 0$ for $t \in [0, \omega]$, according to Lemma 17.1, implies that there exists an ordinary differential equation of the $n$-th order $(\pounds x)(t) = 0$, $t \in [0, \omega]$, where $\pounds$ is defined by formula (17.3.4), which is equivalent to homogeneous equation (17.5.1) in the sense that every solution of one of them is also a solution of the other one.

The positivity of the operator $P : L_\infty[0, \omega] \to L_\infty[0, \omega]$ implies that differential inequalities (17.5.25) follow from differential inequalities (17.5.24). According to Lemma 17.5, the interval $[0, \omega]$ is a nonoscillation one of the equation $(\pounds x)(t) = 0$. According to Lemma 17.1, $[0, \omega]$ is a nonoscillation interval of (17.5.1).

*Remark 17.2* Note that in the case of the third-order equation (i.e., in equation (17.5.1) we have $n = 3$) conditions 1), 2) and (17.5.24) have the following form:

$$v_1 > 0,\ v_2 > 0,\ \left|v_1(t), v_2(t)\right| > 0,\ (Mv_1)(t) \geq 0,\ (Mv_2)(t) \leq 0 \text{ for } t \in [0,\omega). \tag{17.5.26}$$

*Remark 17.3* There are several standard collections of the functions $v_1, \cdots, v_{n-1}$ satisfying conditions 1) and 2) of Theorem 17.12. For example, use of the collection

$$v_i(t) = (t+\varepsilon)^i(\omega - t)^{n-i},\ i = 1, \cdots, n-1,\ \varepsilon > 0 \tag{17.5.27}$$

leads us to the following form of inequalities (17.5.24):

$$\frac{(-1)^{n-i-1}}{n!}\sum_{k=0}^{n-1}\big(B_k v_i^{(k)}\big)(t) \leq 1,\ i = 1, \cdots, n-1,\ t \in (0,\omega). \tag{17.5.28}$$

We obtain a new nonoscillation test for functional differential equations.

**Theorem 17.13** *Let the Wronskian* $W(t) \neq 0$ *for* $t \in [0,\omega]$, *the operator* $P : L^\infty_{[0,\omega]} \to L^\infty_{[0,\omega]}$ *be positive and inequalities* (17.5.28) *with functions* $v_i$ $(i = 1, \cdots, n-1)$ *defined by* (17.5.27) *be fulfilled. Then* $[0,\omega)$ *is a nonoscillation interval of* (17.5.1).

Two other possible collections of the functions $v_i$ $(i = 1, \cdots, n-1)$ are

$$v_i(t) = (t+\varepsilon)^{\beta_i},\ i = 1, \cdots, n-1,\ \beta_1 < \beta_2 < \cdots < \beta_{n-1},\ \varepsilon > 0 \tag{17.5.29}$$

and

$$v_i(t) = e^{\beta_i t},\ i = 1, \cdots, n-1,\ \beta_1 < \beta_2 < \cdots < \beta_{n-1}. \tag{17.5.30}$$

## 17.6 Tests for Differential Equations with Deviating Arguments

Let us formulate our results for the delay differential equation

$$x^{(n)}(t) + p(t)x\big(t - \tau(t)\big) = f(t),\ t \in [0,\omega], \tag{17.6.1}$$

$$x(\xi) = 0,\ \xi \notin [0,\omega]. \tag{17.6.2}$$

**Theorem 17.14** *Let* $(-1)^{n+1}p \geq 0$ *and at least one of the following three conditions* 1), 2) *or* 3) *be fulfilled*:

1) $\tau \geq 0$ *and*

$$\int_0^\omega \left|p(t)\right| dt \leq \frac{(n-1)!}{\omega^{n-1}}. \tag{17.6.3}$$

2) *n is even and*

$$ess \sup_{t\in[0,\omega]} \left|p(t)\right| < \frac{n!}{\omega^n}. \tag{17.6.4}$$

3) $\tau \geq 0$,

$$\operatorname{ess} \sup_{t\in[0,\omega]} \tau(t) \sqrt[n]{\operatorname{ess} \sup_{t\in[0,\omega]} \left|p(t)\right|} \leq \frac{n}{e} \tag{17.6.5}$$

*and*

$$\int_0^\omega \left|p(t)\right| dt \leq \frac{n^n(n-1)!}{(n-1)^{n-1}\omega^{n-1}}. \tag{17.6.6}$$

*Then problem* (17.6.1), (17.5.22) *is uniquely solvable for every* $f \in L_\infty[0,\omega]$ *and its Green's functions behave regularly*; *i.e.*, $G(t,s)(t-t_1)^{k_1}\cdots(t-t_m)^{k_m} \geq 0$ *for* $t, s \in [0,\omega]$.

*Proof* In the case of condition 1), we have to write one-point problem (17.6.1), (17.5.19) in the equivalent form

$$x(t) = -\int_0^\omega W_{0n}(t,s)p(s)x\big(s-\tau(s)\big)ds + \int_0^\omega W_{0n}(t,s)f(s)ds,\ t \in [0,\omega], \tag{17.6.7}$$

$$x(\xi) = 0,\ \xi < 0,$$

where $W_{0n}(t,s)$ is Green's function of the problem

$$x^{(n)}(t) = f(t),\ t \in [0,\omega],\ x^{(i-1)}(\mu) = 0,\ i = 1,\cdots,n, \tag{17.6.8}$$

defined by formula (17.4.12). The operator $R_\mu : C[0,\mu] \to C[0,\mu]$ defined by the equality

$$(R_\mu x)(t) = -\int_0^\mu W_{0n}(t,s)p(s)x\big(s-\tau(s)\big)ds,\ t \in [0,\mu], \tag{17.6.9}$$

$$x(\xi) = 0,\ \xi < 0,$$

is positive. It is clear from formula (17.4.16) that

$$\left|W_{0n}(t,s)\right| \leq \frac{\omega^{n-1}}{(n-1)!},\ t,s \in [0,\omega]. \tag{17.6.10}$$

Inequality (17.6.3) implies that the spectral radius of every operator $R_\mu$ is less than one for every $\mu \in (0,\omega]$. The operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$ defined in (17.5.4) is positive. Inequality (17.6.3) also implies nonoscillation of equation

$$x^{(n)}(t) + p(t)x\big(t-\tau(t)\big) = 0,\ t \in [0,\omega], \tag{17.6.11}$$

$$x(\xi) = 0,\ \xi \notin [0,\omega], \tag{17.6.12}$$

on $[0,\omega]$. The reference to Theorem 17.11 completes the proof of Theorem 17.14 in case 1).

Let us assume now that condition 2) is fulfilled. Inequality (17.6.4), according to Theorem 17.3, implies that $W(t) \neq 0$ for $t \in [0, \omega]$. Theorem 17.1 and Lemma 17.2 reduce nonoscillation of the interval $[0, \omega]$ to the unique solvability of the problems (17.6.1), (17.4.3) for every $k$ and $\mu \in (0, \omega]$, which is equivalent to the integral equation

$$x(t) = -\int_0^\omega W_{k,n-k}(t,s)p(s)x\big(s-\tau(s)\big)ds + \int_0^\omega W_{k,n-k}(t,s)f(s)ds, \ t \in [0,\omega], \tag{17.6.13}$$

$$x(\xi) = 0, \ \xi \notin [0,\omega]. \tag{17.6.14}$$

This integral equation has a unique solution if

$$\max_{t\in[0,\omega]} \int_0^\omega \big|W_{k,n-k}(t,s)p(s)\big|ds < 1, \ t \in [0,\omega], \tag{17.6.15}$$

for every $k$ and $\mu \in (0, \omega]$. Inequality (17.6.15) follows from (17.6.4) and the equality

$$\max_{t\in[0,\omega]} \int_0^\omega \big|W_{k,n-k}(t,s)\big|ds = \frac{t^k|\mu - t|^{n-k}}{n!}, \ t \in [0,\omega]. \tag{17.6.16}$$

Thus condition 2) implies positivity of the operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$ and nonoscillation of (17.6.11), (17.6.12) on the interval $[0, \omega]$. The existence of a unique solution and regular behavior of Green's functions of de La Vallee Poussin problems (17.6.1), (17.5.22) follows now from Theorem 17.11.

Let us assume now that condition 3) of the theorem is fulfilled. Inequality (17.6.5) implies that $W(t) \neq 0$ for $t \in [0, \omega]$ and together with the sign of the coefficient $p$ positivity of the operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$. Every problem (17.6.1), (17.4.3) is uniquely solvable if the spectral radii of the operators $R_\mu : C[0,\mu] \to C[0,\mu]$ defined by the equality

$$(R_\mu x)(t) = \int_0^\mu \big|W_{k,n-k}(t,s)p(s)\big|x\big(s-\tau(s)\big)ds, \ t \in [0,\omega], \tag{17.6.17}$$

$$x(\xi) = 0, \ \xi < 0, \tag{17.6.18}$$

are less than one for every $\mu \in (0, \omega]$. Using the estimate

$$\big|W_{k,n-k}(t,s)\big| \le \frac{(n-1)^{n-1}}{n^n(n-1)!}, \ t \in [0,\omega], \tag{17.6.19}$$

we obtain that inequality (17.6.6) guarantees nonoscillation of (17.6.11), (17.6.18) on the interval $[0, \omega]$. Reference to Theorem 17.11 completes the proof. □

*Remark 17.4* An essentiality of the condition $(-1)^{n+1}p \ge 0$ for nonpositivity of Green's function $G(t,s)$ of the problem

$$x'(t) + p(t)x\big(t-\tau(t)\big) = f(t), \ t \in [0,\omega], \ x(\omega) = 0, \tag{17.6.20}$$

demonstrates the example of the boundary value problem

$$x'(t) - x(0) = f(t), \ t \in [0, \omega], \ x(\omega) = 0, \tag{17.6.21}$$

for which the Green's function is

$$G(t,s) = \begin{cases} -\frac{1+t}{1+\omega}, & 0 < t < s, \\ \frac{\omega - t}{1+\omega}, & t > s > 0, \\ 0, & t > s = 0. \end{cases} \tag{17.6.22}$$

We see that Green's function $G(t, s)$ changes its sign in the rectangle $(t, s) \in [0, \omega] \times [0, \omega]$ for each positive $\omega$. Note that this case is impossible for ordinary differential equations, where $G(t, s) < 0$ for $t < s$ and $G(t, s) = 0$ for $t > s$.

*Remark 17.5* It was proven in Theorem 15.4 that the condition $(-1)^{n+1} p \geq 0$ in the case of $n = 1$ is essential for sign constancy of Green's function. Actually, if $p(t) \leq 0$ and $\text{mes}\{t \in [0, \omega] : p(t) < 0, 0 \leq h(t) \leq t\} > 0$, then the Green's function of the problem

$$x'(t) + p(t)x\big(h(t)\big) = f(t), \ t \in [0, \omega], \ x(\omega) = 0,$$

changes its sign in the square $(t, s) \in (0, \omega) \times (0, \omega)$.

The case of the opposite sign of the coefficient $p$ is considered in the following statement.

**Theorem 17.15** *Let $n$ be even, $n \geq 4$, $p \geq 0$, $t_1 = 0$, $t_m = \omega$ and the inequality*

$$\operatorname*{ess\,sup}_{t \in [0,\omega]} p(t) < \frac{8(n-2)!}{\omega^n} \tag{17.6.23}$$

*be fulfilled. Then problem (17.6.1), (17.5.22) has a unique solution for every $f \in L_\infty[0, \omega]$ and its Green's functions behave regularly; i.e., $G(t, s)(t - t_1)^{k_1} \cdots (t - t_m)^{k_m} \geq 0$ for $t, s \in [0, \omega]$.*

*Proof* Consider the auxiliary problem

$$x''(t) = \varphi(t), \ x(0) = 0, \ x(\omega) = 0. \tag{17.6.24}$$

Denote its Green's function by $g(t, s)$. The solution of (17.6.24) has the representation

$$x(t) = \int_0^\omega g(t, s)\varphi(s)ds. \tag{17.6.25}$$

If we set $\varphi(t) = x''(t)$, we obtain the equality

$$x(t) = \int_0^\omega g(t, s)x''(s)ds. \tag{17.6.26}$$

Substituting its right-hand side instead of $x$ into (17.6.1), we obtain

$$(M_0x)(t) \equiv x^{(n)}(t) + p(t)\sigma\big(t-\tau(t)\big)\int_0^\omega g\big(t-\tau(t),s\big)x''(s)ds$$
$$= f(t),\ t \in [0,\omega], \tag{17.6.27}$$

$$x(\xi) = 0,\ \xi \notin [0,\omega], \tag{17.6.28}$$

where

$$\sigma(t-\tau(t)) = \begin{cases} 1, & t-\tau(t) \in [0,\omega], \\ 0, & t-\tau(t) \notin [0,\omega]. \end{cases} \tag{17.6.29}$$

It is clear that Green's functions of de La Vallee Poussin problems (17.6.1), (17.5.22) and (17.6.27), (17.5.22) coincide. Let us check the conditions of Theorem 17.11 for (17.6.27). Condition 1) of Theorem 17.11 is fulfilled for (17.6.27). Let us obtain the Wronskian $W(t) \neq 0$ for $t \in [0,\omega]$. The operator $K_\mu : L_\infty[0,\omega] \to L_\infty[0,\omega]$ defined by equality (17.5.12) is

$$(K_\mu z)(t) = -p(t)\sigma\big(t-\tau(t)\big)\int_0^\omega g\big(t-\tau(t),s\big)\int_0^\omega \frac{\partial^2}{\partial s^2}W_{0,n}(s,\xi)z(\xi)d\xi ds, \tag{17.6.30}$$

$t \in [0,\omega]$, where $W_{0,n}(s,\xi)$ is defined by (17.4.12). Using the inequalities

$$\int_0^\omega \frac{\partial^2}{\partial s^2}W_{0,n}(t,\xi)d\xi \le \frac{\omega^{n-2}}{(n-2)!},\ t \in [0,\omega], \tag{17.6.31}$$

and

$$\sigma(t-\tau(t))\int_0^\omega \big|g(t-\tau(t),s)\big|ds \le \frac{\omega^2}{8},\ t \in [0,\omega], \tag{17.6.32}$$

we see that (17.6.23) implies that the spectral radii of the operators $K_\mu : L_\infty[0,\omega] \to L_\infty[0,\omega]$ are less than one for every $\mu \in (0,\omega]$. This means that $W(t) \neq 0$ for $t \in [0,\omega]$. According to Theorem 17.7, the operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$, introduced in equality (17.5.4), is positive. Now setting $v_k(t) = (t+\varepsilon)^k(\omega-t)^{n-k}$, $k = 1,\cdots,n-1$, we obtain, according to Theorem 17.12, nonoscillation of solutions of (17.6.11), (17.6.12) on the interval $[0,\omega]$. Reference to Theorem 17.11 completes the proof. □

**Theorem 17.16** *Let $n$ be an odd number, $n \ge 3$, $t_1 = 0$, $t_m = \omega$ and the inequality*

$$\operatorname*{ess\ sup}_{t\in[0,\omega]} \big|p(t)\big| < \frac{(n-1)!}{\omega^n} \tag{17.6.33}$$

*be fulfilled. Then problem (17.6.1), (17.5.22) is uniquely solvable for every $f \in L_\infty[0,\omega]$ and its Green's functions behave regularly*; i.e., $G(t,s)(t-t_1)^{k_1}\cdots(t-t_m)^{k_m} \ge 0$ *for* $t,s \in [0,\omega]$.

*Proof* We use the obvious equalities

$$x(t) = x(0) + \int_0^t x'(s)ds = x(\omega) - \int_t^\omega x'(s)ds. \tag{17.6.34}$$

Setting this representation into (17.6.1), we obtain the equation

$$(M_0x)(t) \equiv x^{(n)}(t) - p^-(t)\sigma\big(t-\tau(t)\big)\int_0^{t-\tau(t)} x'(s)ds$$
$$- p^+(t)\sigma\big(t-\tau(t)\big)\int_{t-\tau(t)}^{\omega} x'(s)ds = f(t),\ t \in [0,\omega], \tag{17.6.35}$$

where $\sigma(t-\tau(t))$ is defined by formula (17.6.29).

The operator $K_\mu : L^\infty_{[0,\omega]} \to L_\infty[0,\omega]$ defined by equality (17.5.12) is

$$(K_\mu z)(t) = p^-(t)\sigma\big(t-\tau(t)\big)\int_0^{t-\tau(t)}\left\{\int_0^\omega \frac{\partial}{\partial s}W_{0n}(s,\xi)z(\xi)d\xi\right\}ds$$
$$+ p^+(t)\sigma\big(t-\tau(t)\big)\int_{t-\tau(t)}^{\omega}\left\{\int_0^\omega \frac{\partial}{\partial s}W_{0n}(s,\xi)z(\xi)d\xi\right\}ds,\ t \in [0,\omega], \tag{17.6.36}$$

where $W_{0,n}(s,\xi)$ is defined by (17.4.14) and (17.4.16) and $p(t) = p^+(t) - p^-(t)$, where $p^+(t) \geq 0$, $p^-(t) \geq 0$. This operator $K_\mu : L_\infty[0,\omega] \to L_\infty[0,\omega]$ is positive for odd $n$ and every $\mu \in (0,\omega]$, and for its norm we get

$$\|K_\mu\| \leq \frac{\omega^{n-1}}{(n-1)!}\operatorname*{ess\,sup}_{t\in\{0,\omega]}\sigma\big(t-\tau(t)\big)\big\{p^-(t)\big(t-\tau(t)\big) + p^+(t)\big(\omega - t + \tau(t)\big)\big\}, \tag{17.6.37}$$

$t \in [0,\omega]$. Now it is clear that inequality (17.6.33) implies $\|K_\mu\| < 1$. The operator $P : L_\infty[0,\omega] \to L_\infty[0,\omega]$, introduced in equality (17.5.4), is positive, according to Theorem 17.7.

Setting $v_k(t) = (t+\varepsilon)^k(\omega - t)^{n-k}$ in Theorem 17.12, we can obtain nonoscillation of the equation $(M_0x)(t) = 0$ on the interval $[0,\omega]$. Thus conditions of Theorem 17.11 (in the case of condition 2)) are fulfilled for (17.6.35). Reference to Theorem 17.11 completes the proof. □

Consider the two-point boundary value problem for the equation with delay

$$x''(t) + \sum_{j=1}^n q_j(t)x'\big(g_j(t)\big) + \sum_{i=1}^m p_i(t)x\big(h_i(t)\big)$$
$$= f(t),\ t \in [0,\omega],\ x(0) = 0,\ x(\omega) = 0, \tag{17.6.38}$$
$$x^{(j)}(\xi) = 0,\ \xi < 0,\ j = 0,1.$$

**Theorem 17.17** *Let $q_j \geq 0$, $j = 1,\cdots,n$ and two inequalities*

$$\omega\sum_{j=1}^n q_j(t) + \frac{\omega^2}{4}\sum_{i=1}^m p_i^+(t) \leq 2 \tag{17.6.39}$$

*and*

$$2\omega_1\sum_{j=1}^n q_j(t) + \omega_1^2\sum_{i=1}^m p_i^-(t) \leq 2,\ \omega_1 > \omega \tag{17.6.40}$$

*be fulfilled, where* $p_i(t) = p_i^+(t) - p_i^-(t)$, $i = 1, \cdots, m$. *Then Green's function* $G(t,s)$ *of problem* (17.6.38) *is nonpositive in the square* $(t,s) \in (0,\omega) \times (0,\omega)$.

*Proof* Consider the equation

$$x''(t) + \sum_{j=1}^{n} q_j(t)x'\big(g_j(t)\big) - \sum_{i=1}^{m} p_i^-(t)x\big(h_i(t)\big) = f(t),\ t \in [0,\omega]. \tag{17.6.41}$$

If we set $v(t) = (\omega_1 - t)^2$ in assertion 3) of Theorem 17.10, then according to this theorem we obtain nonoscillation of the equation

$$x''(t) + \sum_{j=1}^{n} q_j(t)x'\big(g_j(t)\big) - \sum_{i=1}^{m} p_i^-(t)x\big(h_i(t)\big) = 0,\ t \in [0,\omega]. \tag{17.6.42}$$

Theorem 17.11 implies that the Green's function $G^-(t,s)$ of the two-point problem

$$\begin{aligned} x''(t) &+ \sum_{j=1}^{n} q_j(t)x'\big(g_j(t)\big) - \sum_{i=1}^{m} p_i^-(t)x\big(h_i(t)\big) \\ &= f(t),\ t \in [0,\omega],\ x(0) = 0,\ x(\omega) = 0 \end{aligned} \tag{17.6.43}$$

is nonpositive. It is even negative in $0 < t < s < \omega$. Boundary problem (17.6.38) is equivalent to the integral equation

$$x(t) = -\int_0^{\omega} G^-(t,s) \sum_{i=1}^{m} p_i^+(s)x\big(h_i(s)\big)ds + \int_0^{\omega} G^-(t,s)f(s)ds,\ t \in [0,\omega]. \tag{17.6.44}$$

The function $v(t) = t(\omega - t)$ satisfies the inequality

$$v''(t) + \sum_{j=1}^{n} q_j(t)v'\big(g_j(t)\big) + \sum_{i=1}^{m} p_i(t)v\big(h_i(t)\big) \le 0,\ t \in [0,\omega], \tag{17.6.45}$$

and

$$v(t) \ge -\int_0^{\omega} G^-(t,s) \sum_{i=1}^{m} p_i^+(s)v\big(h_i(s)\big)ds,\ t \in [0,\omega]. \tag{17.6.46}$$

The spectral radius of the operator

$$(Rx)(t) \equiv -\int_0^{\omega} G^-(t,s) \sum_{i=1}^{m} p_i^+(s)x\big(h_i(s)\big)ds,\ t \in [0,\omega], \tag{17.6.47}$$

is less than one according to Theorem A.12. For nonnegative $f$, we get a nonpositive solution of (17.6.44). It proves that the Green's function of (17.6.38) is nonpositive. □

*Remark 17.6* The condition $q_j(t) \ge 0$ is essential. The Green's function of the problem

$$x''(t) - x'(0) = f(t),\ t \in [0,\omega],\ x(0) = 0,\ x(\omega) = 0,$$

can be constructed, and it changes sign in each square $(t,s) \in (0,\omega) \times (0,\omega)$.

## 17.7 Discussion and Open Problems

The linear homogeneous $n$-th-order differential equation

$$(\pounds x)(t) \equiv x^{(n)}(t) + \sum_{i=0}^{n-1} p_i(t)x^{(i)}(t) = 0, \; t \in [0,\omega], \tag{17.7.1}$$

is one of the most important objects in the qualitative theory of ordinary differential equations. The results of the papers [93, 256] were devoted to the de La Vallee Poussin problem

$$(\pounds x)(t) \equiv x^{(n)}(t) + \sum_{i=0}^{n-1} p_i(t)x^{(i)}(t) = f(t), \; t \in [0,\omega], \tag{17.7.2}$$

$$x^{(i)}(t_j) = c_j^i, \tag{17.7.3}$$

where

$$0 \le t_1 < t_2 < \cdots < t_m \le \omega, \; i = 0, \cdots, k_j - 1, \; j = 1, \cdots, m, \; k_1 + \cdots + k_m = n. \tag{17.7.4}$$

Existence and uniqueness of its solution are reduced to nonoscillation of homogeneous equation (17.7.1). The solution of problem (17.7.2), (17.7.3) has the representation

$$x(t) = \int_0^\omega G(t,s)f(s)ds + X(t), \tag{17.7.5}$$

where $X(t)$ is a solution of the homogeneous equation (17.7.1) satisfying the conditions (17.7.3), $G(t,s)$ is called the Green's function of the boundary value problem (17.7.2), (17.7.3). It should be noted that the kernel $G(t,s)$ for each fixed $s$, as a function of the argument $t$ only, satisfies homogeneous equation (17.7.1) on the intervals $[0,s)$ and $(s,\omega]$. Representation (17.7.5) explains why the attention of the authors of the classical monographs and papers on the theory of ordinary differential equations [203, 220, 314] was focused on homogeneous equation (17.7.1).

The notion of a nonoscillation interval plays a fundamental role in the theory of linear ordinary differential equations of the $n$-th order. The concept and first results allowing us to estimate nonoscillation intervals of ordinary differential equations were obtained by G. Mammana [282]. It was demonstrated in the paper by A.J. Levin [256] that nonoscillation is connected with many very different properties of linear ordinary differential equations such as the theorems on differential inequalities (under corresponding conditions, solutions of differential inequalities are greater than or less than solutions of equations), exponential stability of solutions, zones of Lyapunov's stability, regular behavior of Green's functions of interpolation problems, theory of oscillatory kernels [158], the Polia-Mammana representation [308] of the operator $\pounds$ as a product of the first-order differential operators with real coefficients and many others.

Used and developed in the classical monographs [192, 248] on nonoscillation of delay differential equations, the notion of nonoscillation as existence of eventually positive or negative solutions does not suit use in this direction. This is why we started with introducing homogeneous equations in the second section and the definition of a nonoscillation interval (see Definition 17.1).

Our goal in this chapter was to propose the concept of nonoscillation intervals for functional differential equations and to obtain results concerning regular behavior of Green's functions of the de La Vallee Poussin problem. We follow the concept by N.V. Azbelev [20] in the frame of which the notions of homogeneous equations, fundamental systems and Wronskians were first formulated. Properties of the Wronskian $W(t)$ of delay differential equations of the $n$-th order were studied in [241]. The fact that $W(t) \neq 0$ for $t \geq 0$ in the case of second-order differential equation

$$x''(t) + \sum_{j=1}^{n} q_j(t)x'\big(g_j(t)\big) + \sum_{i=1}^{m} p_i(t)x\big(h_i(t)\big) = 0,\ t \in [0,\infty), \tag{17.7.6}$$

$x(\xi) = 0$ for $\xi < 0$, is equivalent to the Sturm separation theorem (between two adjacent zeros of a nontrivial solution there is one and only one zero of each other linear independent nontrivial solution). The first results of nonvanishing $W(t)$ in the case of the delay differential equation

$$x''(t) + \sum_{i=1}^{m} p_i(t)x\big(h_i(t)\big) = 0,\ p_i(t) \geq 0,\ t \in [0,\infty), \tag{17.7.7}$$

$$x(\xi) = 0 \text{ for } \xi < 0 \tag{17.7.8}$$

were obtained in [20], where delays were assumed to be small. The maximal delay size was estimated through coefficients of (17.7.7). The essence of this smallness is that there is at most one zero of nontrivial solutions on each of the intervals $[h(t), t]$, where $h(t) = \min_{i\in\{1,\cdots,m\}} h_i(t)$. Some extension of this result on neutral delay differential equations was obtained in [110]. In [241], it was proven that $W(t) \neq 0$ for $t \geq 0$ for the equation

$$x''(t) + p(t)x\big(h(t)\big) = 0,\ p(t) \geq 0,\ t \in [0,\infty), \tag{17.7.9}$$

$$x(\xi) = 0 \text{ for } \xi < 0,$$

with nondecreasing $h(t)$. In the paper [116], the Sturm separation theorem was obtained for (17.7.7), (17.7.8) with nondecreasing $h_i$ $(i = 1,\cdots,m)$ under several assumptions, smallness of the differences $|h_i - h_j|$ being the main one. It was demonstrated that in this case oscillation properties of this equation are similar to those for (17.7.9), (17.7.8). This smallness was estimated through the coefficients of (17.7.7). The essence of this smallness is that there is no zero of a nontrivial solution together with a zero of its derivative on the intervals $[h_i(t), h_j(t)]$ $(i, j = 1,\cdots,m)$. The results about growth of the Wronskian $W(t)$ of (17.7.7), (17.7.8) were obtained in [20, 110, 241]. For ordinary differential equations, a correlation between growth of the Wronskian and the existence of unbounded solutions was obtained by P. Hartman in the classical monograph [203] and in the paper by P. Hartman and A. Winter [204]. In the paper [117], the differential inequality for the Wronskian

$$W'(t) \geq \sum_{i=1}^{n} p_i(t)C\big(t, h_i(t)\big)W\big(h_i(t)\big),\ t \in [0,\infty) \tag{17.7.10}$$

was obtained. Here $C(t,s)$ is the Cauchy function of (17.7.7). On this basis, results on the growth of amplitudes of oscillating solutions were obtained. One of the results, for example, established the conditions of Lyapunov's stability of this equation [117]: every solution of (17.7.9), (17.7.8) with nondecreasing coefficients $p$ and $h$, where $p$ is positive and bounded, is bounded if and only if

$$\int^{\infty}\big(t-h(t)\big)dt<\infty. \tag{17.7.11}$$

Results of this sort have solved the known problem of A.D. Myshkis [289], who proved that there exists an unbounded solution of equation

$$x''(t)+px(t-\varepsilon)=0,\ t\in[0,\infty) \tag{17.7.12}$$

for each couple of positive constants $p$ and $\varepsilon$. The problem of solution unboundedness in the case of nonconstant coefficients was formulated in his book as one to be solved.

For estimates of the length of nonoscillation intervals for functional differential equations, results about estimates of Green's functions and their derivatives for the auxiliary boundary value problems

$$x^{(n)}(t)=f(t),\ t\in[0,\mu], \tag{17.7.13}$$

$$x^{(i)}(0)=0,\ x^{(j)}(\mu)=0,\ i=0,\cdots,k-1,\ j=0,\cdots,n-k-1, \tag{17.7.14}$$

obtained in [37, 101], can be used. Note that MATLAB allows us to get these estimates, too.

Results about the positivity of Green's function for impulsive functional differential equations were obtained in [126–128].

The approach of this chapter is based on the results of the papers [26, 27]. Presentation of the results follows the paper [124]. The main result of this chapter claims that nonoscillation implies regular behavior of Green's functions under corresponding additional conditions on the positivity of operators $B_j$ $(j=0,\cdots,n-1)$.

The basic results about Wronskians and their properties can be found in [118]. Applications to analysis of oscillation and asymptotic properties of corresponding partial functional differential equations were presented in [119]. The right regularization (known also as Azbelev's $W$-transform) reduces boundary value problems to analysis of corresponding functional operator equations in the space of essentially bounded functions. This idea works in Sect. 17.5 of this chapter. Of course, this idea can also be used in the analysis of partial functional differential equations that was discussed in [120], where results on maximum principles for functional equations in the space of three variables were obtained. A development of this idea is to use Green's function of a corresponding equation of the order $n-k$ in Azbelev's $W$-transform for functional differential equations of the $n$-th order. As a result, we get a boundary value problem for $k$-th-order functional differential equations. This idea was applied in the papers [112, 114] in the case of $n$-th-order functional differential equations with ordinary derivatives and in [82] for hyperbolic equations.

Finally, let us state some open problems.

1. Estimate nonoscillation intervals $[0, \omega]$ in the case of non-Volterra operators $B_j$, $j = 0, \cdots, n-1$.
2. Estimate nonoscillation intervals for oscillatory solutions of $n$-th-order functional differential equations.
3. Prove or disprove the analogue of Rolle's theorem in the following form:
   If $[0, \omega]$ is a nonoscillation interval, then under corresponding conditions the fact that the right-hand side $f$ changes its sign yields that solutions of problems (17.1.1), (17.1.9), (17.1.10) have at least $n + 1$ zeros on $[0, \omega]$.

# Appendix A
# Useful Theorems from Analysis

## A.1 Vector Spaces

Denote by $\mathbb{R}^n$ the space of all vectors $X = [x_1, \cdots, x_n]^T$, where $x_k \in \mathbb{R}$, $k = 1, \cdots, n$ are real numbers and $T$ is the matrix transposition operation.

By $\|\cdot\|$ we denote a norm in $\mathbb{R}^n$. In particular,

$$\|X\|_p = \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p}, \ 1 \le p < \infty,$$
$$\|X\|_\infty = \max_{1 \le k \le n} |x_k|.$$

For any $n \times n$ matrix $A = (a_{ij})$, $i, j = 1, \cdots, n$, we define a matrix norm corresponding to the vector norm by the equality

$$\|A\| = \sup_{\|X\|=1} \frac{\|AX\|}{\|X\|}.$$

For example,

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|, \ \|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

For a chosen norm in $\mathbb{R}^n$, we define the matrix measure

$$\mu(A) = \lim_{\varepsilon \to 0^+} \frac{\|I + \varepsilon A\| - 1}{\varepsilon},$$

where $I$ is the identity matrix. For example, for the norm $\|\cdot\|_\infty$, we have

$$\mu(A) = \max_{1 \le i \le n} \left\{ a_{ii} + \sum_{j \ne i} |a_{ij}| \right\}.$$

A vector or matrix $A$ is nonnegative (we will write $\ge 0$) if all the entries are nonnegative numbers. We also use the following definition.

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**Definition A.1** Matrix $A$ is *an* $M$*-matrix* if $a_{ij} \le 0$, $i \ne j$, $A$ is invertible and the inverse matrix satisfies $A^{-1} \ge 0$.

For many equivalent definitions and properties of $M$-matrices, see [83].

## A.2 Functional Spaces

For all functional spaces we fix the norm $\|\cdot\|$ in $\mathbb{R}^n$. Suppose $X : [a,b] \to \mathbb{R}^n$ or $X : [a,\infty) \to \mathbb{R}^n$ is a vector-valued function. By $C[a,b]$, $C[a,\infty)$ we denote the Banach spaces of all vector functions continuous on $[a,b]$ or $[a,\infty)$ with the norm

$$\|X\|_{C[a,b]} = \max_{a \le t \le b} \|X(t)\|,$$
$$\|X\|_{C[a,\infty)} = \sup_{t \ge a} \|X(t)\|.$$

By $L_p[a,b]$, $L_p[a,\infty)$, $1 \le p < \infty$, we denote the Banach space of all Lebesgue integrable vector functions with the norm

$$\|X\|_{L_p[a,b]} = \left(\int_a^b \|X(s)\|^p ds\right)^{\frac{1}{p}}, \quad \|X\|_{L_p[a,\infty)} = \left(\int_a^\infty \|X(s)\|^p ds\right)^{\frac{1}{p}}.$$

If $p = 1$, we will write $L_1[a,b] = L[a,b]$.

By $L_\infty[a,b]$ and $L_\infty[a,\infty)$ we denote the Banach space of all Lebesgue measurable essentially bounded vector functions on either $[a,b]$ or $[a,\infty)$ with the norm

$$\|X\|_{L_\infty[a,b]} = \operatorname{ess} \sup_{a \le t \le b} \|X(t)\|,$$
$$\|X\|_{L_\infty[a,\infty)} = \operatorname{ess} \sup_{a \le t < \infty} \|X(t)\|.$$

The following theorem is known as the Lebesgue monotone convergence theorem.

**Theorem A.1** [150, III.6.16, III.6.17] *Let* $f_1, f_2, \cdots$ *be a scalar pointwise positive nondecreasing sequence of Lebesgue measurable functions*; *i.e.*, $0 \le f_n(t) \le f_{n+1}(t)$. *Then the pointwise limit* $f(t) = \lim_{n\to\infty} f_n(t)$ *is a Lebesgue measurable function and*

$$\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt. \tag{A.2.1}$$

*If a sequence* $\{f_n\}$ *is monotone* (*increasing or decreasing*) *and there exists* $g \in L[a,b]$ *such that* $|f_n(t)| \le |g(t)|$, *then there is a pointwise limit* $f$ *almost everywhere on* $[a,b]$ *that satisfies* $f \in L[a,b]$ *and* (A.2.1) *holds.*

Together with the Lebesgue measure, we need a notion of the Borel measure.

**Definition A.2** The $\sigma$-algebra generated by all open intervals of $(a, b)$ is called *the Borel $\sigma$-algebra*. The restriction of the Lebesgue measure on the Borel $\sigma$-algebra is called *the Borel measure*.

The function $X : [a, b] \to \mathbb{R}^n$ is *absolutely continuous* if there exists $Y \in L[a, b]$ such that $X(t) = X(a) + \int_a^t Y(s)ds$. An absolutely continuous function is continuous and differentiable almost everywhere.

A scalar function $g : [a, b] \to \mathbb{R}$ is *a function of bounded variation* if

$$\mathrm{Var}_{t\in[a,b]}\, g(t) := \sup_P \sum_{i=0}^{n-1} |g(t_{i+1}) - g(t_i)| < \infty,$$

where the supremum is taken over the set of all partitions $P$ of the interval $[a, b]$.

If $g$ is differentiable and its derivative is integrable, then its variation satisfies

$$\mathrm{Var}_{t\in[a,b]}\, g(t) = \int_a^b |g'(s)|ds.$$

For the matrix-valued function $R(t) = [r_{ij}(t)]_{i,j=1}^n$, we can define its variation as

$$\mathrm{Var}_{t\in[a,b]}\, R(t) = \left[\mathrm{Var}_{t\in[a,b]}\, r_{ij}(t)\right]_{i,j=1}^n.$$

**Lemma A.1** *Suppose $X = [x_1, \cdots, x_n]^T$ is a function continuous on $[a, b]$ and all entries of the matrix function $R$ have a bounded variation on $[a, b]$. Then*

$$\left\| \int_a^b dR(s)X(s) \right\| \le \left\| \mathrm{Var}_{t\in[a,b]}\, R(t) \right\| \|X\|_{C[a,b]}.$$

*Proof* We have

$$\begin{aligned}
&\left\| \int_a^b d_s R(s)X(s) \right\| \\
&= \left\| \left[ \int_a^b \sum_{j=1}^n x_j(s) dr_{ij}(s) \right]_{i=1}^n \right\| \\
&\le \left\| \left[ \sum_{j=1}^n \|x_j\|_{C[a,b]}\, \mathrm{Var}_{t\in[a,b]}\, r_{ij}(t) \right]_{i=1}^n \right\| \\
&\le \left\| \mathrm{Var}_{t\in[a,b]}\ R(t) \right\| \|X\|_{C[a,b]}.
\end{aligned}$$

□

The scalar function $f : [a, b] \to R$ is *Borel measurable* if for any interval $(c, d) \subset [a, b]$ the set $g^{-1}(c, d)$ belongs to the Borel $\sigma$-algebra on $[a, b]$.

For a Borel measurable function $f : [a, b] \to \mathbb{R}$ and a function $g : [a, b] \to \mathbb{R}$ of bounded variation, the Lebesgue-Stieltjes integral [202]

$$\int_a^b f(s)dg(s)$$

can be defined, which coincides with the Lebesgue integral if $g(t) \equiv t$. We have

$$\int_a^b |f(s)|dg(s) \leq \sup_{t\in[a,b]} |f(t)| \operatorname{Var}_{t\in[a,b]} g(t).$$

## A.3 Sets in Functional Spaces

In this section, we follow the monograph [192].

A set $M$ of a Banach space $B$ is *convex* if for every $x, y \in M$ and for every $\lambda \in [0, 1]$

$$\lambda x + (1 - \lambda)y \in M.$$

We say that a set $M$ of a Banach space $B$ is *compact* if every sequence of $M$ contains a subsequence that converges to an element of $M$. A set $M$ is *relatively compact* if every sequence of $M$ contains a subsequence that converges to an element of $B$.

A set $M$ of functions continuous on $[a, b]$ is *equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in [a, b]$ with $|t_2 - t_1| < \delta$ and for all $f \in M$ we have $|f(t_2) - f(t_1)| < \varepsilon$. A set $M$ is called *uniformly bounded* if there exists a positive number $K$ such that $|f(t)| \leq K$ for all $t \in [a, b]$ and for all $f \in M$.

We can now formulate the Ascoli-Arzela theorem.

**Theorem A.2** [150, 215] *A set $M \subset C[a, b]$ is compact if it is equicontinuous and uniformly bounded.*

We will also apply the following result.

**Theorem A.3** [150, IV.6.26] *The set $M \subset \mathbf{L}_\infty[a, b]$ is compact if and only if for any $\varepsilon > 0$ the segment $[a, b]$ can be presented as a union of a finite number of measurable subsets $E_i \subset [a, b]$ such that for every $E_i$, $u \in M$ and any $t, s \in E_i$ we have*

$$|u(t) - u(s)| < \varepsilon.$$

**Definition A.3** If $B$ is a Banach space of vector functions, then by $B_\lambda$, $\lambda > 0$, we define the weighted space of all functions $y \in B$ such that $y_\lambda := ye^{\lambda t} \in B$. The space $B_\lambda$ is a Banach space with the norm $\|y\|_{B_\lambda} = \|y_\lambda\|_B$.

## A.4 Linear Operators in Functional Spaces

By $r(P)$ we denote the spectral radius of the bounded linear operator $P$ in the Banach space $B$ [215]:

$$r(P) = \lim_{n\to\infty} \|P^n\|^{\frac{1}{n}}.$$

Evidently $r(P) \leq \|P\|$.

**Definition A.4** *A cone* is a closed subset $K$ of a Banach space $B$ such that

a) $x_0 \in K$ implies $\lambda x_0 \in K$, $\lambda \geq 0$,
b) if $x_1 \in K, x_2 \in K$, then $x_1 + x_2 \in K$, and
c) if $x_0 \in K, -x_0 \in K$, then $x_0 = 0$.

We write $x \geq y$ if $x - y \in K$. We will say that operator $A$ is *a positive operator* if $x \geq y$ implies $Ax \geq Ay$.

In all functional spaces used in this monograph, we apply only the cone of the nonnegative functions $K = \{x | x(t) \geq 0\}$.

**Lemma A.2** [233, pp. 86, 87] *Let the operator $R : C[a,b] \to C[a,b]$ be compact and positive and a vector function $v(t) = [v_1(t), \cdots, v_n(t)]^T$ with positive components $v_i(t) > 0$ for $t \in [a,b]$, $i = 1, \cdots, n$ be such that $\psi(t) \equiv v(t) - (Rv)(t) > 0$. Then the spectral radius $r(R)$ of the operator $R : C[a,b] \to C[a,b]$ is less than one.*

In all functional spaces, we consider the integral operator

$$(V_1X)(t) = \int_a^b K(t,s)X(s)ds \tag{A.4.1}$$

and the Volterra integral operator

$$(V_2X)(t) = \int_a^t K(t,s)X(s)ds. \tag{A.4.2}$$

There are many books (for example, [150, 215, 234, 348]) where boundedness and compactness conditions for integral operators have been obtained. In particular, we will need the following result.

**Theorem A.4** *Suppose the kernel $K : [a,b] \times [a,b] \to \mathbb{R}^n \times \mathbb{R}^n$ of integral operators $V_1$, $V_2$ is a measurable function.*

1) [150, p. 519, VI.9.53] *If $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\text{ess} \sup_{a \leq t \leq b} \int_a^b \|K(t,s)\|^q ds = M < \infty,$$

*then any operator $V_1$, $V_2$ is a compact operator in the space $L_p[a,b]$, $1 < p < \infty$, and $\|V_i\| \leq M^{1/q}$, $i = 1, 2$.*

2) [29, p. 18] *Suppose for almost all $t$ that the function $K(t,\cdot)$ has at each point $s$ finite one-sided limits and there exists a scalar function $v \in L[a,b]$ such that*

$$\|K(\cdot,s)\| \leq v(\cdot)$$

*for each $s \in [a,b]$. Then any operator $V_1$, $V_2$ is a compact operator in the space $L[a,b]$.*

3) [150, p. 519, VI.9.57] *If*

$$\operatorname*{ess\ sup}_{a\le t,s\le b}\|K(t,s)\| = M < \infty,$$

*then anyone of the operators* $V_1$ *and* $V_2$ *is a weakly compact operator in the space* $L_\infty[a,b]$, $\|V_i\| \le M$, $i = 1, 2$ *and its square is compact in the space* $L_\infty[a,b]$.

In fact, [150, p. 519, VI.9.53, remark to Theorem VI.8.10] contains a more general result.

**Theorem A.5** [150, p. 519, VI.9.57] *Let* $R(t,\cdot)$ *be a function of bounded variation for any* $t$, $R(\cdot,s)$ *be Lebesgue measurable, essentially bounded and*

$$\operatorname*{ess\ sup}_{t\in[a,b]}\int_a^b |d_s R(t,s)| \le M < \infty.$$

*Then any of the operators*

$$(T_1X)(t) = \int_a^b X(s)d_sR(t,s), \tag{A.4.3}$$

$$(T_2X)(t) = \int_a^t X(s)d_sR(t,s),\ t\in[a,b], \tag{A.4.4}$$

*is a compact operator in the space* $L_p[a,b]$, $1 < p < \infty$, *and a weakly compact operator in the space* $L[a,b]$, *and its square is compact in the space* $L[a,b]$.

**Theorem A.6** *If* $h \in L_\infty[a,b]$, *then the linear integral operator*

$$(Hx)(t) = \begin{cases} \int_a^{h(t)} x(s)ds & \text{if } h(t)\in[a,b],\\ 0 & \text{if } h(t)\notin[a,b],\end{cases}$$

*is a compact operator in* $L_\infty[a,b]$.

*Proof* Let $\varepsilon > 0$ be an arbitrary number. Divide the set $h([a,b])\cap[a,b]$ into a finite number of subsets $F_i, i = 1,\cdots,n$, such that for every $\tau_1,\tau_2\in F_i$ we have $|\tau_1-\tau_2| < \varepsilon$. Denote

$$E_i = h^{-1}(F_i), i = 1,\cdots,n,\ \ E_0 = \left\{t\in[a,b] : h(t)\notin[a,b]\right\},$$
$$S = \left\{x\in L_\infty[a,b] : \|x\| = 1\right\},\ \ M = HS.$$

Let us show that $M$ is a compact set in $L_\infty[a,b]$.

For any set $E_i$, $i = 1,\cdots,n$, we have

$$\sup_{t,s\in E_i}|(Hx)(t)-(Hx)(s)| = \sup_{t,s\in E_i}\left|\int_{h(t)}^{h(s)} x(\tau)d\tau\right| \le \sup_{t,s\in E_i}|h(t)-h(s)| < \varepsilon.$$

If $i = 0$, then $\sup_{t,s\in E_i}|h(t)-h(s)| = 0$. Theorem A.3 implies that $M = HS$ is a compact set, and hence $H$ is a compact operator. □

**Theorem A.7** [348, p. 153], [349, 350] *Suppose that the linear Volterra integral operator* $V_2 : L_p[a,b] \to L_p[a,b]$, $1 \le p \le \infty$, *is a compact operator for* $1 \le p \le \infty$ *or a weakly compact operator for* $p = 1$ *or* $p = \infty$. *Then the spectral radius of this operator is equal to zero and the integral equation* $x - V_2 x = f$ *has a unique solution* $x \in L_p[a,b]$ *for any* $f \in L_p[a,b]$.

This result has an important generalization.

**Definition A.5** Let $P$ be a linear operator in the space $H$ of the functions $X : [a,b] \to \mathbb{R}^n$. We say that $P$ is *a Volterra* (*or causal*) *operator* if for any $a < c < b$ the equality $X(t) = 0$, $a \le t \le c$, implies $(PX)(t) = 0$, $a \le t \le c$.

**Theorem A.8** [97, 310] *If* $P : L_p[a,b] \to L_p[a,b]$, $1 \le p \le \infty$, *is a linear Volterra compact operator for* $1 \le p \le \infty$ *or a weakly compact operator for* $p = 1$ *or* $p = \infty$, *then* $r(P) = 0$ *and the operator equation* $x - Px = f$ *has a unique solution* $x \in L_p[a,b]$ *for any* $f \in L_p[a,b]$.

It is easy to see that if $r(P) < 1$, then there exists the inverse operator $(I - P)^{-1} = I + P + P^2 + \cdots$, where $I$ is the identity operator and $(I - P)^{-1} \ge 0$ if $P \ge 0$.

**Theorem A.9** [21] *Let* $V : L_p[a,b] \to L_p[a,b]$, $1 \le p \le \infty$, *be a linear Volterra bounded operator and* $P : L_p[a,b] \to L_p[a,b]$, $1 \le p \le \infty$, *be a linear Volterra compact operator. Then* $r(V + P) = r(V)$.

Suppose $B : [a,b] \to \mathbb{R}^n \times \mathbb{R}^n$ is a measurable essentially bounded matrix function and $g : [a,b] \to \mathbb{R}$, $g(t) \le t$, is a measurable locally bounded scalar function. We denote the inner superposition (composition) operator by the equality

$$(SX)(t) = \begin{cases} B(t)X(g(t)), & g(t) \ge a, \\ 0, & g(t) < a. \end{cases}$$

**Theorem A.10** [29] *Suppose that* $\mathrm{mes}\{e\} = 0 \Rightarrow \mathrm{mes}\{g^{-1}(e) \cap [a,b]\} = 0$, *where* mes *is the Lebesgue measure. Then the operator* $S$ *acts in the space* $L_\infty[a,b]$ *and is bounded with the norm estimation* $\|S\|_{L_\infty} \le \operatorname{ess\,sup}_{a<t<b} \|B(t)\|$.

*If in addition*

$$\sup_{e \subset [a,b], \mathrm{mes}\, e > 0} \frac{\mathrm{mes}\{g^{-1}(e)\}}{\mathrm{mes}\{e\}} < \infty, \ \ \mathrm{mes}\, e > 0,$$

*then the operator* $S$ *acts in the space* $L[a,b]$ *and is bounded.*

Operator $S$ with a delay $g(t) \le t$ is a linear Volterra operator.

**Definition A.6** [233] An operator $B$ is *u-bounded* if for every nonzero element $x$ of the cone $K$ there exists $\beta(x) \ge 0$ such that $Bx \le \beta(x)u$.

**Theorem A.11** [216] *Every u-bounded operator $B : C[a,b] \to L[a,b]$ can be presented in the form of the Stieltjes integral*

$$(Bx)(t) = \int_a^b x(s)d_s b(t,s), \tag{A.4.5}$$

*where $b(\cdot, s) : [a,b] \to R$ is measurable and the function $b(t,\cdot) : [a,b] \to \mathbb{R}$ has a bounded variation $p(t) = \mathrm{Var}_{s\in[a,b]}\, b(t,s)$, while $p$ is integrable.*

Consider the integral operator $N : C[a,b] \to C[a,b]$ defined by the equality

$$(Nx)(t) = \int_a^b G(t,s)\left[\int_a^b x(\xi)d_\xi b(s,\xi)\right] ds. \tag{A.4.6}$$

**Theorem A.12** [210, 211] *Assume that $G(\cdot, s)$ is a continuous function for almost all $s$, $G(t,\cdot)$ is a Lebesgue integrable function for all $t$, $G(t,s) \geq 0$, the function $b(t,s)$ is nondecreasing in $s$ for every $t \in [a,b]$, there exists a positive continuous function $v$ such that $\psi(t) \equiv v(t) - (Nv)(t) \geq 0$ and the set of zeros of $\psi$ is not more than countable and $\psi(s) > 0$ if $\mathrm{mes}\{t \in [a,b] : b(t,s+) \neq b(t,s-)\} > 0$.*

*Then the spectral radius $r(N)$ of the operator $N : C[a,b] \to C[a,b]$ is less than one.*

Let us formulate Theorem 5.3 of [233, p. 79] in a convenient form.

**Theorem A.13** *Let a linear operator $A : C[a,b] \to C[a,b]$ be positive and a linear operator $B : C[a,b] \to C[a,b]$ satisfy the inequality $-Ax \leq Bx \leq Ax$ for every nonnegative function $x$. Then, for the spectral radius, we have $r(B) \leq r(A)$.*

## A.5 Nonlinear Operators

Let $F : D \subset B_1 \to B_2$ be a nonlinear operator acting from the subset $D$ of the Banach space $B_1$ to the Banach space $B_2$. We say that $F$ is *a continuous operator* if for any $x_0 \in D$ and any sequence $\{x_n\} \subset D$ the condition $x_n \to x_0$ implies $Fx_n \to Fx_0$. Operator $F$ is *bounded* if for any bounded set $M \subset D$ the set $F(M)$ is bounded. Operator $F$ is *a compact operator* if for any bounded closed set $M \subset D$ the set $F(M)$ is compact.

Some well-known facts will be summarized in the following lemma.

**Lemma A.3**

1. *Suppose $f(t,x)$, $t \in [a,b]$, $x \in (-\infty, \infty)$ is a scalar continuous function. Then the operator $(Fx)(t) = f(t, x(t))$ is a continuous bounded operator in the space $L_\infty[a,b]$.*
2. *Any superposition of a compact continuous operator and a continuous operator in Banach spaces is a compact continuous operator.*

**Theorem A.14** (The Banach contraction principle) *Let operator $F$ map $D$ into $D$, where $D \subset B$ is a closed subset of the Banach space $B$. If for some $0 < \lambda < 1$ and any $x, y \in D$ the inequality*

$$\|Fx - Fy\| \le \lambda \|x - y\|$$

*holds, then the equation $x = Fx$ has a unique solution $x \in D$.*

**Theorem A.15** (Schauder Fixed-Point Theorem) *Let the compact continuous operator $F$ map $D$ into $D$, where $D \subset B$ is a nonempty convex, bounded, closed subset of the Banach space $B$.*

*Then the equation $x = Fx$ has a solution $x \in D$.*

Many nonlinear compact operators are obtained when investigating various differential and functional differential equations. We will give an example for the scalar impulsive equation

$$\dot{x}(t) + u(t)x(t) = 0, \ t \in [a, b], \tag{A.5.1}$$

$$x(a) = x_0, \ x\big(\tau_j{}^+\big) = I_j\big(x(\tau_j)\big), \ a < \tau_1 < \cdots < \tau_n = b, \tag{A.5.2}$$

where $u : [a, b] \to \mathbb{R}$ is an essentially bounded function and $I_j : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function.

**Lemma A.4** *Denote $Fu = x$, where $x$ is the solution of problem* (A.5.1), (A.5.2). *Then $F : L_\infty[a, b] \to L_\infty[a, b]$ is a compact operator.*

*Proof* Let $n = 1$; i.e., there exists only one impulse point $a < \tau < b$. Then the solution of (A.5.1), (A.5.2) can be rewritten in the form

$$x(t) := (Fx)(t) = x_0 \exp\left\{-\int_a^t u(s)ds\right\}\chi_{[a,\tau]}(t) + x_0 I\left(\exp\left\{-\int_a^\tau u(s)ds\right\}\right)\exp\left\{-\int_\tau^t u(s)ds\right\}\chi_{[\tau,b]}(t),$$

where $\chi_J$ is the characteristic function of the interval $J$.

Since the integral operators $\int_a^t u(s)ds$, $\int_\tau^t u(s)ds$ and the functional $\int_a^\tau u(s)ds$ are compact operators in the space $L_\infty[a, b]$, Lemma A.3 implies that operator $F$ is a compact operator in the space $L_\infty[a, b]$.

The general case can be proven similarly by induction. □

## A.6 Gronwall-Bellman and Coppel Inequalities

**Lemma A.5** *Suppose*

$$u(t) \le c + \int_a^t f(s)u(s)\,ds,$$

*where* $c > 0$ *and we have scalar nonnegative functions* $u, f \in L_\infty[a, b]$. *Then*

$$u(t) \le c e^{\int_a^t f(s) ds}.$$

**Lemma A.6** *For the solution of the vector ordinary differential equation*

$$\dot{x}(t) = A(t)x(t),$$

*the inequality*

$$\|x(t)\| \le \|x(t_0)\| e^{\int_{t_0}^t \mu(A(s))ds}$$

*holds, where* $A(t)$, $t \ge t_0$ *is a locally essentially bounded vector function and* $\mu(A(t))$ *is a matrix measure of the matrix* $A(t)$.

# Appendix B
# Existence and Uniqueness Theorems, Solution Representations

In this appendix, we present existence and uniqueness conditions for solutions of all functional differential equations considered in this monograph; in addition, for linear equations, solution representations are given. Many other existence and solution representation results can be found in [29, 95, 96, 201].

## B.1 Linear Functional Differential Equations

### *B.1.1 Differential Equations with Several Concentrated Delays*

We consider for $t \geq 0$ the vector equation

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = 0, \tag{B.1.1}$$

where $A_k(t)$ are $n \times n$ matrices with entries $a_{ij}^k$, $i, j = 1, \cdots, n$, $k = 1, \cdots, m$ under the following conditions:

(a1) Functions $a_{ij}^k$ are Lebesgue measurable and locally essentially bounded.
(a2) Delays $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, $k = 1, \cdots, m$.

Together with (B.1.1), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = F(t),\ t \geq t_0, \tag{B.1.2}$$

$$X(t) = \Phi(t),\ t < t_0,\ X(t_0) = X_0 \in \mathbb{R}^n, \tag{B.1.3}$$

where $\Phi(t) = [\varphi_1(t), \cdots, \varphi_n(t)]^T$ and $F = [f_1(t), \cdots, f_n(t)]^T$ satisfy the following hypothesis:

(a3) $F : [t_0, \infty) \to \mathbb{R}^n$ is a Lebesgue measurable locally essentially bounded function and $\Phi : (-\infty, t_0) \to \mathbb{R}^n$ is a Borel measurable bounded function.

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Here $A^T$ is the transposed matrix.

**Definition B.1** Function $X : \mathbb{R} \to \mathbb{R}^n$, which is locally absolutely continuous on $[t_0, \infty)$, is called *a solution* of problem (B.1.2), (B.1.3) if it satisfies (B.1.2) for almost all $t \in [t_0, \infty)$ and equalities (B.1.3) for $t \le t_0$.

In addition to problem (B.1.2), (B.1.3), where $X$, $F$ and $\Phi$ are column vector functions, we will consider the problem where $F(t)$, $\Phi(t)$ and solution $X(t)$ are $n \times n$ matrix functions.

By 0 we will also denote the zero column vector and the zero matrix.

**Definition B.2** The $n \times n$ matrix function $C(t, s)$ that satisfies the problem

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = 0, \ \ X(t) = 0, \ \ t < s, \ \ X(s) = I, \tag{B.1.4}$$

for each $s \ge 0$, is called *the fundamental matrix* (*or the Cauchy matrix*) of (B.1.1). Here $I$ is the identity matrix, and we assume that $C(t, s) = 0$, $0 \le t < s$.

**Theorem B.1** *Let* (a1)–(a3) *hold. Then there exists a unique solution of problem* (B.1.2), (B.1.3), *and it can be represented in the form*

$$X(t) = C(t, t_0) X_0 + \int_{t_0}^{t} C(t, s) F(s) ds - \sum_{k=1}^{m} \int_{t_0}^{t} C(t, s) A_k(s) \Phi\big(h_k(s)\big) ds \tag{B.1.5}$$

*for* $t \ge t_0$, *where* $\Phi(h_k(s)) = 0$, *if* $h_k(s) > t_0$.

*Proof* In order to demonstrate existence and uniqueness, it is sufficient to prove that there exists a unique solution of problem (B.1.2), (B.1.3) on the interval $[t_0, c]$ for any $c > t_0$.

Denote

$$X_h(t) = \begin{cases} X(h(t)) & \text{if } h(t) \ge t_0, \\ 0 & \text{if } h(t) < t_0, \end{cases}$$

and

$$\Phi^h(t) = \begin{cases} \Phi(h(t)) & \text{if } h(t) < t_0, \\ 0 & \text{if } h(t) \ge t_0. \end{cases}$$

Then $X(h(t)) = X_h(t) + \Phi^h(t)$, $t \ge t_0$, and the problem (B.1.2), (B.1.3) takes the form

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X_{h_k}(t) = G(t), \ \ t \ge t_0, \ \ X(t_0) = X_0, \tag{B.1.6}$$

where

$$G(t) = F(t) - \sum_{k=1}^{m} A_k(t)\Phi^{h_k}(t).$$

Denote by $\chi_{[\alpha,\beta]}(t)$ the characteristic function of the interval $[\alpha, \beta]$, and assume that $\chi_{[\alpha,\beta]}(t) \equiv 0$ if $\alpha > \beta$. Recall that we consider the initial value problem on the finite interval $[t_0, c]$.

Since

$$X(t) = X(t_0) + \int_{t_0}^{t} \dot{X}(s)ds,$$

we have

$$X_{h_k}(t) = \chi_{[t_0,c]}\big(h_k(t)\big)\left(X(t_0) + \int_{t_0}^{\max\{h_k(t),t_0\}} \dot{X}(s)ds\right). \tag{B.1.7}$$

Hence problem (B.1.6) can be rewritten as

$$\dot{X}(t) + \sum_{k=1}^{m} \chi_{[t_0,c]}\big(h_k(t)\big)A_k(t)\int_{t_0}^{\max\{h_k(t),t_0\}} \dot{X}(s)ds = R(t),$$

where

$$R(t) = G(t) - \sum_{k=1}^{m} \chi_{[t_0,c]}\big(h_k(t)\big)A_k(t)X(t_0).$$

This means that problem (B.1.6) has the form

$$Y(t) = \int_{t_0}^{t} B(t,s)Y(s)ds + R(t), \ \ t \geq t_0, \tag{B.1.8}$$

where

$$Y(t) = \dot{X}(t), \ \ B(t,s) = -\sum_{k=1}^{m} \chi_{[t_0,c]}\big(h_k(t)\big)\chi_{[t_0,\max\{h_k(t),t_0\}]}(s)A_k(t).$$

Consider the linear integral operator

$$(TY)(t) = \int_{t_0}^{t} B(t,s)Y(s)ds$$

in the space $L_\infty[t_0, c]$.

Evidently $\operatorname{ess\,sup}_{t,s\in[t_0,c]} \|B(t,s)\| < \infty$. Theorem A.4 implies that $T$ is a weakly compact integral Volterra operator. By Theorem A.7, its spectral radius is equal to zero. Thus, by this theorem the integral equation (B.1.8) has a unique solution $Y(t)$ for $t \geq t_0$.

Consequently,

$$X(t) = \begin{cases} X(t_0) + \int_{t_0}^{t} Y(s)ds, & t \geq t_0, \\ \Phi(t), & t < t_0, \end{cases}$$

is the unique solution of problem (B.1.2), (B.1.3).

Next, let us demonstrate that the function

$$X(t) = C(t, t_0)X_0 + \int_{t_0}^{t} C(t, s)G(s)ds, \tag{B.1.9}$$

where $C$ is the fundamental matrix of (B.1.1), is a solution of problem (B.1.6). For the convenience of the reader, we will write $X(h(t))$ instead of $X_h(t)$, assuming that $X(h(t)) = 0$, if $h(t) < t_0$.

Equality (B.1.9) implies

$$\begin{aligned}
X\big(h_k(t)\big) &= C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{\max\{h_k(t),t_0\}} C\big(h_k(t), s\big)G(s)ds \\
&= C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{t} C\big(h_k(t), s\big)G(s)ds \\
&\quad - \int_{\max\{h_k(t),t_0\}}^{t} C\big(h_k(t), s\big)G(s)ds \\
&= C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{t} C\big(h_k(t), s\big)G(s)ds,
\end{aligned}$$

since $C(t, s) = 0$ for $t < s$. We consider the left-hand side of (B.1.6) if $X$ is supposed to have the form (B.1.9). Using the expression above for $X(h_k(t))$ and the definition of the fundamental function $C(t, s)$, we obtain

$$\begin{aligned}
&\dot{X}(t) + \sum_{k=1}^{m} A_k(t)X\big(h_k(t)\big) \\
&= \dot{C}(t, t_0)X(t_0) + G(t) + \int_{t_0}^{t} \dot{C}(t, s)G(s)ds \\
&\quad + \sum_{k=1}^{m} A_k(t)\Big[C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{t} C\big(h_k(t), s\big)G(s)ds\Big] \\
&= \Big[\dot{C}(t, t_0) + \sum_{k=1}^{m} A_k(t)C\big(h_k(t), t_0\big)\Big]X(t_0) \\
&\quad + \int_{t_0}^{t} \Big[\dot{C}(t, s) + \sum_{k=1}^{m} A_k(t)C\big(h_k(t), s\big)\Big]G(s)ds + G(t) = G(t),
\end{aligned}$$

which completes the proof. □

## *B.1.2 Mixed Equations with an Infinite Number of Delays*

Consider the vector equation

$$\dot{X}(t) + \sum_{k=1}^{\infty} A_k(t)X\big(h_k(t)\big) + \int_{-\infty}^{t} K(t, s)X(s)\, ds = 0, \tag{B.1.10}$$

where in addition to (a1) and (a2) the following condition holds:

(a4) $a(t) = \sum_{k=1}^{\infty} \|A_k(t)\|$ is a locally essentially bounded function, where the series converges uniformly on any bounded interval $[t_0, b]$; $K(t, s)$ is a measurable locally essentially bounded function such that $\sup_{t \geq t_0} \int_{-\infty}^{t_0} \|K(t, s)\| ds < \infty$.

It is evident that (B.1.1) with an infinite number of concentrated delays

$$\dot{X}(t) + \sum_{k=1}^{\infty} A_k(t) X\big(h_k(t)\big) = 0$$

and the integrodifferential equation

$$\dot{X}(t) + \int_{-\infty}^{t} K(t, s) X(s)\, ds = 0$$

are partial cases of (B.1.10).

Together with (B.1.10), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{X}(t) + \sum_{k=1}^{\infty} A_k(t) X\big(h_k(t)\big) + \int_{-\infty}^{t} K(t, s) X(s)\, ds = F(t),\ t \geq t_0, \tag{B.1.11}$$

$$X(t) = \Phi(t),\ t < t_0,\ X(t_0) = X_0, \tag{B.1.12}$$

where for the initial function $\Phi(t)$ and the right-hand side $F(t)$ condition (a3) holds.

**Definition B.3** A function $X : \mathbb{R} \to \mathbb{R}^n$ locally absolutely continuous on $[t_0, \infty)$ is called *a solution* of problem (B.1.11), (B.1.12), if it satisfies (B.1.11) for almost all $t \in [t_0, \infty)$ and equalities (B.1.12) for $t \leq t_0$.

**Definition B.4** For each $s \geq 0$, the solution $C(t, s)$ of the problem

$$\begin{aligned} &\dot{X}(t) + \sum_{k=1}^{\infty} A_k(t) X(h_k(t)) + \int_{-\infty}^{t} K(t, s) X(s)\, ds = 0, \\ &X(t) = 0,\ t < s,\ X(s) = I, \end{aligned} \tag{B.1.13}$$

is called *the fundamental matrix* of (B.1.1).

**Theorem B.2** *Let* (a1)–(a4) *hold. Then there exists a unique solution of problem* (B.1.11), (B.1.12), *and it can be presented in the form*

$$\begin{aligned} X(t) = C(t, t_0) X_0 &+ \int_{t_0}^{t} C(t, s) F(s) ds - \sum_{k=1}^{\infty} \int_{t_0}^{t} C(t, s) A_k(s) \Phi\big(h_k(s)\big) ds \\ &- \int_{t_0}^{t} C(t, s) \int_{-\infty}^{s} K(s, \tau) \Phi(\tau) d\tau ds, \end{aligned} \tag{B.1.14}$$

*where* $\Phi(\xi) = 0$, *if* $\xi > t_0$.

*Proof* We follow the scheme of the proof of Theorem B.1 using the notation of this proof.

It is sufficient to prove that there exists a unique solution of problem (B.1.11), (B.1.12) on the interval $[t_0, c]$ for any $c > t_0$. This problem, similar to (B.1.6), has the form

$$\dot{X}(t) + \sum_{k=1}^{\infty} A_k(t) X_{h_k}(t) + \int_{t_0}^{t} K(t,s) X(s)\, ds = G(t),\ t \geq t_0,\ X(t_0) = X_0, \tag{B.1.15}$$

where

$$G(t) = F(t) - \sum_{k=1}^{\infty} A_k(t) \Phi^{h_k}(t) - \int_{-\infty}^{t_0} K(t,s) \Phi(s)\, ds.$$

By the equality $X(t) = X(t_0) + \int_{t_0}^{t} \dot{X}(s)\, ds$, we have

$$\begin{aligned}\int_{t_0}^{t} K(t,s) X(s)\, ds &= \left( \int_{t_0}^{t} K(t,s)\, ds \right) X(t_0) + \int_{t_0}^{t} K(t,s) \int_{t_0}^{s} \dot{X}(\tau)\, d\tau\, ds \\ &= \left( \int_{t_0}^{t} K(t,s)\, ds \right) X(t_0) + \int_{t_0}^{t} \left( \int_{s}^{t} K(t,\tau)\, d\tau \right) \dot{X}(s)\, ds.\end{aligned}$$

Using (B.1.7), problem (B.1.15) can be rewritten as

$$\begin{aligned}&\dot{X}(t) + \sum_{k=1}^{\infty} \chi_{[t_0,c]}\big(h_k(t)\big) A_k(t) \int_{t_0}^{\max\{h_k(t), t_0\}} \dot{X}(s) ds \\ &\quad + \int_{t_0}^{t} \int_{s}^{t} \big(K(t,\tau)\, d\tau\big) \dot{X}(s)\, ds = R(t),\end{aligned}$$

where

$$R(t) = G(t) - \sum_{k=1}^{\infty} \chi_{[t_0,c]}\big(h_k(t)\big) A_k(t) X(t_0) - \left( \int_{t_0}^{t} K(t,s)\, ds \right) X(t_0).$$

Thus, problem (B.1.15) has the form

$$Y(t) = \int_{t_0}^{t} B(t,s) Y(s) ds + R(t),\ t \geq t_0, \tag{B.1.16}$$

where

$$Y(t) = \dot{X}(t),$$

$$B(t,s) = -\sum_{k=1}^{\infty} \chi_{[t_0,c]}\big(h_k(t)\big) \chi_{[t_0, \max\{h_k(t), t_0\}]}(s) A_k(t) - \int_{s}^{t} K(t,\tau)\, d\tau.$$

Consider the linear integral operator

$$(TY)(t) = \int_{t_0}^{t} B(t,s) Y(s) ds$$

in the space of $L_\infty[t_0, c]$.

Evidently $\operatorname{ess\,sup}_{t_0 \le t, s \le c} \|B(t,s)\| < \infty$. Theorem A.4 implies that $T$ is a weakly compact integral Volterra operator in the space $L_\infty[t_0, c]$. By Theorem A.7, its spectral radius is equal to zero. Thus, by this theorem, integral equation (B.1.16) has a unique solution $Y(t)$, $t \ge t_0$.

Consequently,

$$X(t) = \begin{cases} X(t_0) + \int_{t_0}^t Y(s)ds, & t \ge t_0, \\ \Phi(t), & t < t_0, \end{cases}$$

is the unique solution of problem (B.1.11), (B.1.12).

Further, let us show that the function

$$X(t) = C(t, t_0)X(t_0) + \int_{t_0}^t C(t,s)G(s)ds, \tag{B.1.17}$$

where $C$ is the fundamental matrix of (B.1.10), is a solution of problem (B.1.15). Equality (B.1.17) implies

$$\begin{aligned} X\big(h_k(t)\big) &= C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{\max\{h_k(t), t_0\}} C\big(h_k(t), s\big)G(s)ds \\ &= C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{t} C\big(h_k(t), s\big)G(s)ds \\ &\quad - \int_{\max\{h_k(t), t_0\}}^{t} C\big(h_k(t), s\big)G(s)ds \\ &= C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{t} C\big(h_k(t), s\big)G(s)ds. \end{aligned}$$

Since $C(\tau, s) = 0$ for $\tau < s$, we have

$$\int_{t_0}^s K(t,\tau)C(\tau,s)\,d\tau = 0.$$

Hence

$$\begin{aligned} \int_{t_0}^t K(t,s)\int_{t_0}^s C(s,\tau)G(\tau)d\tau ds &= \int_{t_0}^t \left(\int_s^t K(t,\tau)C(\tau,s)d\tau\right)G(s)ds \\ &= \int_{t_0}^t \left(\int_{t_0}^t K(t,\tau)C(\tau,s)d\tau\right)G(s)ds. \end{aligned}$$

We consider the left-hand side of (B.1.15) if $X$ is supposed to have the form (B.1.17). With the help of the relations above, we have

$$\begin{aligned} &\dot{X}(t) + \sum_{k=1}^{\infty} A_k(t)X\big(h_k(t)\big) + \int_{t_0}^t K(t,s)X(s)ds \\ &= \dot{C}(t,t_0)X(t_0) + G(t) + \int_{t_0}^t \dot{C}(t,s)G(s)ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty} A_k(t)\left[C\big(h_k(t), t_0\big)X(t_0) + \int_{t_0}^{t} C\big(h_k(t), s\big)G(s)ds\right] \\
&+\int_{t_0}^{t} K(t,s)\left[C(s,t_0)X(t_0) + \int_{t_0}^{s} C(s,\tau)G(\tau)d\tau\right]ds \\
=&\left[\dot{C}(t,t_0) + \sum_{k=1}^{\infty} A_k(t)C\big(h_k(t), t_0\big)\right]X(t_0) + \int_{t_0}^{t} K(t,s)C(s,t_0)ds X(t_0) \\
&+\int_{t_0}^{t}\left[\dot{C}(t,s) + \sum_{k=1}^{\infty} A_k(t)C\big(h_k(t), s\big)\right]G(s)ds + G(t) \\
&+\int_{t_0}^{t}\left(\int_{t_0}^{t} K(t,\tau)C(\tau,s)d\tau\right)G(s)ds \\
=&\left[\dot{C}(t,t_0) + \sum_{k=1}^{\infty} A_k(t)C\big(h_k(t), t_0\big) + \int_{t_0}^{t} K(t,s)C(s,t_0)ds\right]X(t_0) \\
&+\int_{t_0}^{t}\left[\dot{C}(t,s) + \sum_{k=1}^{\infty} A_k(t)C\big(h_k(t), s\big) + K(t,\tau)C(\tau,s)d\tau\right]G(s)ds + G(t) \\
=&\,G(t),
\end{aligned}
$$

which completes the proof. □

### B.1.3 Equations with a Distributed Delay

In this section, we consider the vector equation

$$
\dot{X}(t) + \int_{-\infty}^{t} d_s R(t,s)X(s) = 0 \tag{B.1.18}
$$

for $t \geq t_0$, where the following conditions are satisfied:

(a5) Any entry $r_{ij}(t,s)$ of $R(t,s)$ is a left continuous function of bounded variation in $s$ for any $t$, and for each $s$ its variation on the segment $[t_0, s]$

$$
p_{ij}(t,s) = \mathrm{Var}_{\tau\in[t_0,s]}\, r_{ij}(t,\tau)
$$

is a locally integrable function in $t$ and

$$
\sup_{t\geq t_0} \int_{-\infty}^{t_0} p_{ij}(t,s)ds < \infty.
$$

(a6) $R(t,s) = R(t,t^+)$, $t < s$, where $R(t,t^+) = \lim_{s\to t, t<s} R(t,s)$ and the integrals for left continuous functions are understood as

$$
\int_{-\infty}^{t} d_s R(t,s)x(s) = \lim_{\varepsilon\to 0^+} \int_{-\infty}^{t+\varepsilon} d_s R(t,s)x(s).
$$

Equation (B.1.10) is a partial case of (B.1.18) if we denote

$$R(t,s)=\sum_{k=1}^{\infty}A_k(t)\chi_{(h_k(t),\infty)}(s)+\int_{-\infty}^{s}K(t,\zeta)\,d\zeta.$$

Together with (B.1.18), we consider for each $t_0\ge 0$ the initial value problem

$$\dot{X}(t)+\int_{-\infty}^{t}d_sR(t,s)X(s)=F(t),\ t\ge t_0, \tag{B.1.19}$$

$$X(t)=\Phi(t),\ t<t_0,\ X(t_0)=X_0, \tag{B.1.20}$$

where $\Phi(t)$ and $F(t)$ satisfy the following hypothesis:

(a7) The initial function $\Phi(t)$ is a continuous function, and $F(t)$ is a Lebesgue measurable locally essentially bounded function.

**Definition B.5** A function $X:\mathbb{R}\to\mathbb{R}^n$ locally absolutely continuous on $[t_0,\infty)$ is called *a solution* of problem (B.1.19), (B.1.20), if it satisfies (B.1.19) for almost all $t\in[t_0,\infty)$ and equalities (B.1.20) for $t\le t_0$.

**Definition B.6** For each $s\ge 0$, the solution $C(t,s)$ of the problem

$$\dot{X}(t)+\int_{-\infty}^{t}d_sR(t,s)X(s)=0,\ X(t)=0,\ t<s,\ X(s)=I, \tag{B.1.21}$$

is called *the fundamental matrix* of (B.1.18).

**Theorem B.3** *Let* (a5), (a6) *and* (a7) *hold. Then there exists a unique solution of problem* (B.1.19), (B.1.20), *and it can be presented in the form*

$$X(t)=C(t,t_0)X_0+\int_{t_0}^{t}C(t,s)F(s)ds-\int_{t_0}^{t}C(t,s)\,ds\int_{-\infty}^{s}d_\tau R(s,\tau)\Phi(\tau), \tag{B.1.22}$$

*where* $\Phi(\tau)=0$, *if* $\tau>t_0$.

*Proof* We follow the scheme of the proof of Theorems B.1 and B.2.

It is sufficient to prove that there exists a unique solution of problem (B.1.19), (B.1.20) on the interval $[t_0,c]$ for any $c>t_0$. This problem has the form

$$\dot{X}(t)+\int_{t_0}^{t}d_sR(t,s)X(s)\,ds=G(t),\ \ t\ge t_0,\ \ X(t_0)=X_0, \tag{B.1.23}$$

where

$$G(t)=F(t)-\int_{-\infty}^{t_0}d_sR(t,s)\Phi(s)\,ds.$$

By the equality $X(t)=X(t_0)+\int_{t_0}^{t}\dot{X}(s)\,ds$ and simple calculations, we have

$$\begin{aligned}\int_{t_0}^{t} d_s R(t,s)X(s) &= \big[R\big(t,t^+\big) - R(t,t_0)\big]X(t_0) + \int_{t_0}^{t} d_s R(t,s)\int_{t_0}^{t} \dot{X}(\tau)\,d\tau \\ &= \big[R\big(t,t^+\big) - R(t,t_0)\big]X(t_0) + \int_{t_0}^{t} \dot{X}(s)\bigg(\int_{s}^{t} d_\tau R(t,\tau)\bigg)ds \\ &= \big[R\big(t,t^+\big) - R(t,t_0)\big]X(t_0) + \int_{t_0}^{t} \big(R\big(t,t^+\big) - R(t,s)\big)\dot{X}(s)\,ds,\end{aligned}$$

so problem (B.1.23) can be rewritten as

$$\dot{X}(t) + \int_{t_0}^{t} \big[R\big(t,t^+\big) - R(t,s)\big]\dot{X}(s)\,ds = \mathcal{R}(t),$$

where

$$\mathcal{R}(t) = G(t) - \big[R\big(t,t^+\big) - R(t,t_0)\big]X(t_0).$$

Thus, problem (B.1.23) has the form

$$Y(t) = \int_{t_0}^{t} B(t,s)Y(s)ds + \mathcal{R}(t),\ \ t \ge t_0, \tag{B.1.24}$$

where

$$Y(t) = \dot{X}(t),\ \ B(t,s) = -R\big(t,t^+\big) + R(t,s).$$

Consider the linear integral operator

$$(TY)(t) = \int_{t_0}^{t} B(t,s)Y(s)ds$$

in the vector space $L_\infty[t_0,c]$. Evidently $\operatorname{ess\,sup}_{t_0 \le t,s \le c} \|B(t,s)\| < \infty$. Theorem A.4 implies that integral Volterra operator $T$ is weakly compact in the space $L_\infty[t_0,c]$. By Theorem A.7, its spectral radius is equal to zero, and hence integral equation (B.1.7) has a unique solution $Y(t)$ for $t \ge t_0$. Consequently,

$$X(t) = \begin{cases} X(t_0) + \int_{t_0}^{t} Y(s)ds, & t \ge t_0, \\ \Phi(t), & t < t_0, \end{cases}$$

is the unique solution of problem (B.1.19), (B.1.20).

Let us demonstrate that the function

$$X(t) = C(t,t_0)X_0 + \int_{t_0}^{t} C(t,s)G(s)ds, \tag{B.1.25}$$

where $C$ is the fundamental matrix of (B.1.18), is a solution of problem (B.1.23).

Since $C(\tau,s) = 0$ for $\tau < s$, we have $\int_{t_0}^{s} d_s R(t,\tau)C(\tau,s) = 0$, which implies

$$\begin{aligned}\int_{t_0}^{t} d_s R(t,s)\int_{t_0}^{s} C(s,\tau)G(\tau)d\tau &= \int_{t_0}^{t}\bigg(\int_{\tau}^{t} d_s R(t,s)C(s,\tau)G(\tau)\bigg)d\tau \\ &= \int_{t_0}^{t}\bigg(\int_{t_0}^{t} d_\tau R(t,\tau)C(\tau,s)\bigg)G(s)ds.\end{aligned}$$

Finally, consider the left-hand side of (B.1.23), where $X$ has form (B.1.25). Applying the relations above, we obtain

$$
\begin{aligned}
&\dot{X}(t) + \int_{t_0}^{t} d_s R(t,s) X(s) \\
&= \dot{C}(t,t_0)X(t_0) + G(t) + \int_{t_0}^{t} \dot{C}(t,s)G(s)ds \\
&\quad + \int_{t_0}^{t} d_s R(t,s)\left[C(s,t_0)X(t_0) + \int_{t_0}^{s} C(s,\tau)G(\tau)d\tau\right] \\
&= \left[\dot{C}(t,t_0) + \int_{t_0}^{t} d_s R(t,s)C(s,t_0)\right]X(t_0) + \int_{t_0}^{t} \dot{C}(t,s)G(s)ds \\
&\quad + G(t) + \int_{t_0}^{t}\left(\int_{t_0}^{t} d_\tau R(t,\tau)C(\tau,s)\right)G(s)ds \\
&= \left[\dot{C}(t,t_0) + \int_{t_0}^{t} d_s R(t,s)C(s,t_0)\right]X(t_0) \\
&\quad + \int_{t_0}^{t}\left[\dot{C}(t,s) + \int_{t_0}^{t} d_\tau R(t,\tau)C(\tau,s)\right]G(s)ds + G(t) = G(t),
\end{aligned}
$$

which completes the proof. □

## B.1.4 Equations of Neutral Type

In this subsection, we consider the vector neutral differential equation

$$\dot{X}(t) - A(t)\dot{X}\big(g(t)\big) + B(t)X\big(h(t)\big) = 0, \ t \geq 0 \tag{B.1.26}$$

under the following conditions:

- (a8) $A(t), B(t), g(t), h(t)$ are Lebesgue measurable locally essentially bounded functions.
- (a9) $\operatorname{ess\,sup}_{t\geq 0} \|A(t)\| \leq q < 1$.
- (a10) $g(t) \leq t$, $\operatorname{mes} E = 0 \Longrightarrow \operatorname{mes} g^{-1}(E) = 0$, where $\operatorname{mes} E$ is the Lebesgue measure of the set $E$.
- (a11) $h(t) \leq t$, $g(t) \leq t$, $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g(t) = \infty$.

As in Appendix A, denote by $S$ the composition operator

$$(Sy)(t) = \begin{cases} A(t)y(g(t)), & g(t) \geq t_0, \\ 0, & g(t) < t_0. \end{cases}$$

If $\sup_{t\geq t_0} \|A(t)\| \leq q < 1$, then by Theorem A.9 the operator $S$ maps the space $L_\infty[t_0,c]$ onto itself for any $c > t_0$, $S$ is bounded and $\|S\|_{L_\infty[t_0,c]\to L_\infty[t_0,c]} \leq q < 1$.

Together with (B.1.26), we consider for each $t_0 \geq 0$ the initial value problem

$$\dot{X}(t) - A(t)\dot{X}\big(g(t)\big) + B(t)X\big(h(t)\big) = F(t), \ t \geq t_0, \tag{B.1.27}$$

$$X(t) = \Phi(t), \ \dot{X}(t) = \Psi(t), \ t < t_0, \ X(t_0) = X_0. \tag{B.1.28}$$

We also assume that the following hypothesis holds:

(a12) $F : [t_0, \infty) \to \mathbb{R}^n$, where $F(t) = [f_1(t), \cdots, f_n(t)]^T$, is a Lebesgue measurable locally essentially bounded function and $\Phi, \Psi : (-\infty, t_0) \to \mathbb{R}^n$ are Borel measurable bounded functions.

**Definition B.7** A function $X : \mathbb{R} \to \mathbb{R}^n$ locally absolutely continuous on $[t_0, \infty)$ is called *a solution* of problem (B.1.27), (B.1.28) if it satisfies (B.1.27) for almost all $t \in [t_0, \infty)$ and also satisfies (B.1.28).

**Definition B.8** For each $s \geq t_0$, the solution $C(t, s)$ of the problem

$$\begin{aligned} &\dot{X}(t) - A(t)\dot{X}(g(t)) + B(t)X(h(t)) = 0, \\ &X(t) = 0, \ \dot{X}(t) = 0, \ t < s, \ X(s) = I, \end{aligned} \tag{B.1.29}$$

is called *the fundamental matrix* of (B.1.26).

We assume that $C(t, s) = 0$, $0 \leq t < s$.

**Theorem B.4** *Let* (a8)–(a12) *hold. Then there exists a unique solution of problem* (B.1.27), (B.1.28), *and it can be presented in the form*

$$\begin{aligned} X(t) = C(t, t_0)X_0 &+ \int_{t_0}^{t} C(t, s)\big[(I - S)^{-1}F\big](s)ds \\ &+ \int_{t_0}^{t} C(t, s)\big[(I - S)^{-1}G\big](s)ds, \end{aligned} \tag{B.1.30}$$

*where* $G(t) = A(t)\psi(g(t)) - B(t)\Phi(h(t))$ *and* $\Psi(g(t)) = 0$ *for* $g(t) \geq t_0$ *and* $\Phi(h(t)) = 0$ *for* $h(t) \geq t_0$.

*Proof* Let us rewrite problem (B.1.27), (B.1.28) in the form

$$\begin{aligned} (I - S)\dot{X}(t) + B(t)X\big(h(t)\big) &= F(t) + G(t), \ t \geq t_0, \\ X(t) = \dot{X}(t) &= 0, \ t < t_0, \ X(t_0) = X_0. \end{aligned}$$

Since in the space $L_\infty[t_0, c]$ the norm of operator $S$ is less than one and $(I - S)^{-1} = I + S + S^2 + \cdots$, problem (B.1.27), (B.1.28) is equivalent to

$$\dot{X}(t) + \sum_{k=0}^{\infty} B_k(t)X\big(h_k(t)\big) = \big[(I - S)^{-1}F\big](t) + \big[(I - S)^{-1}G\big](t), \ t \geq t_0, \tag{B.1.31}$$

$$X(t) = \dot{X}(t) = 0, \ t < 0, \ X(t_0) = X_0, \tag{B.1.32}$$

where

$$B_0(t) = B(t),\ B(t) = A(t)B\big(g(t)\big),$$
$$B_k(t) = A(t)A\big(g(t)\big)\cdots A\big(g^{k-1}(t)\big)B\big(g^k(t)\big),$$
$$h_0(t) = h(t), h_k(t) = h\big(g^k(t)\big), g^k(t) = g\big(g\big(\cdots g(t)\big)\big).$$

For the solution of (B.1.31), (B.1.33), we have by Theorem B.2 representation (B.1.30). Equations (B.1.27) and (B.1.31) have the same fundamental matrix. Hence (B.1.30) is also a solution representation for problem (B.1.27), (B.1.28). □

### *B.1.5 Higher-Order Scalar Delay Differential Equations*

Similar to equations of the first order considered in the previous subsections of this appendix, we can consider all kinds of vector equations of the $n$-th order. For simplicity, we will present existence and uniqueness results only for a scalar delay differential equation of the $n$-th order and obtain a solution representation formula for this equation.

Consider for $t \ge 0$ the linear scalar delay differential equation of the $n$-th order

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}\big(h_{n-1}(t)\big) + \cdots + a_0(t)y\big(h_0(t)\big) = 0, \tag{B.1.33}$$

where for parameters of (B.1.33) and other high-order equations it is assumed that coefficients $a_k(t)$ are Lebesgue measurable locally essentially bounded functions, and delays $h_k(t) \le t$ satisfy $\lim_{t\to\infty} h_k(t) = \infty$, $k = 0, \cdots, n-1$.

Together with (B.1.33), consider the initial value problem

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}\big(h_{n-1}(t)\big) + \cdots + a_0(t)y\big(h_0(t)\big) = f(t),\ t \ge t_0, \tag{B.1.34}$$

$$y^{(k)}(t) = \varphi_k(t),\ t < t_0,\ y^{(k)}(t_0) = y_k,\ k = 0, \cdots, n-1. \tag{B.1.35}$$

**Definition B.9** A function $y : \mathbb{R} \to \mathbb{R}$ with an $(n-1)$-th derivative $y^{(n-1)}$ absolutely continuous on each finite interval is called *a solution* of problem (B.1.34), (B.1.35) if it satisfies (B.1.34) for almost all $t \in [t_0, \infty)$ and equalities (B.1.35) for $t \le t_0$.

**Definition B.10** For each $s \ge 0$, the solution $Y(t, s)$ of the problem

$$y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}\big(h_{n-1}(t)\big) + \cdots + a_0(t)y\big(h_0(t)\big) = 0,\ t \ge s, \tag{B.1.36}$$

$$y^{(k)}(t) = 0,\ t < s,\ k = 0, \cdots, n-1, \tag{B.1.37}$$

$$y^{(k)}(s) = 0,\ k = 0, \cdots, n-2,\ y^{(n-1)}(s) = 1, \tag{B.1.38}$$

is called *the fundamental function* of (B.1.33).

Further, we will denote by $Y_k(t, s)$, $k = 0, \cdots, n-1$ a solution of (B.1.36) with the initial conditions

$$y^{(j)}(t) = 0,\ t < s,\ j = 0, \cdots, n-1,\ y^{(j)}(s) = 0,\ j \ne k,$$
$$y^{(k)}(s) = 1,\ k = 0, \cdots, n-1,$$

instead of (B.1.37), (B.1.38). It is evident that $Y_{n-1}(t, s) = Y(t, s)$.

We assume that $Y(t,s)=0$ for $0\le t<s$ and $Y_k(t,s)=0$ for $0\le t<s$, $k=0,\cdots,n-1$.

**Theorem B.5** *There exists a unique solution of problem* (B.1.34), (B.1.35), *and it can be presented in the form*

$$y(t)=\sum_{k=0}^{n-1}Y_k(t,t_0)y_k+\int_{t_0}^{t}Y(t,s)f(s)ds$$
$$-\int_{t_0}^{t}Y(t,s)\sum_{k=0}^{n-1}a_k(s)\varphi_k\big(h_k(s)\big)ds, \qquad \text{(B.1.39)}$$

*where* $\varphi_k(h_k(s))=0$, *if* $h_k(s)>t_0$.

*Proof* Let us define the vectors

$$X=[x_1,\cdots,x_n]^T,\ x_1=y,\cdots,x_n=y^{(n-1)},$$
$$F=\big[0,\cdots,0,f(t)\big]^T,\ \Phi=[\varphi_0,\cdots,\varphi_{n-1}]^T,\ WX_0=[y_0,\cdots,y_{n-1}]^T,$$

and $n\times n$ matrices

$$A_0(t)=\begin{pmatrix}0&-1&0&\cdots&0\\0&0&-1&\cdots&0\\\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdots&-1\\0&0&0&0&0\end{pmatrix},\ A_k(t)=\begin{pmatrix}0&0&0&\cdots&0\\\cdot&\cdot&\cdot&&\cdot\\0&0&0&\cdots&0\\\cdot&\cdot&\cdot&&\cdot\\0&\cdots&a_{k-1}(t)&\cdots&0\end{pmatrix},\ k=1,\cdots,n.$$

Then problem (B.1.34), (B.1.35) has the form

$$\dot X(t)+A_0(t)X(t)+\sum_{k=1}^{n}A_k(t)X\big(h_{k-1}(t)\big)=F(t),\ t\ge t_0, \qquad \text{(B.1.40)}$$
$$X(t)=\Phi(t),\ t<t_0,\ X(t_0)=X_0. \qquad \text{(B.1.41)}$$

For problem (B.1.40), (B.1.41), all the conditions of Theorem B.1 hold, so this problem has a unique solution.

Further, let us prove that for this solution representation (B.1.39) holds. Denote

$$y_{h_k}(t)=\begin{cases}y(h_k(t)) & \text{if } h_k(t)\ge t_0,\\ 0 & \text{if } h_k(t)<t_0,\ k=0,\cdots,n-1\end{cases}$$

and

$$\varphi_k^{h_k}(t)=\begin{cases}\varphi_k(h_k(t)) & \text{if } h_k(t)<t_0,\\ 0 & \text{if } h_k(t)\ge t_0.\end{cases}$$

Then $y(h_k(t))=y_{h_k}(t)+\varphi_k^{h_k}(t)$ for $t\ge t_0$ and the problem (B.1.34), (B.1.35) takes the form

$$y^{(n)}(t)+\sum_{k=0}^{n-1}a_k(t)y_{h_k}^{(k)}(t)=g(t),\ y^{(k)}(t_0)=y_k,\ k=0,\cdots,n-1, \qquad \text{(B.1.42)}$$

where

$$g(t) = f(t) - \sum_{k=0}^{n-1} a_k(t)\varphi_k^{h_k}(t).$$

Let us demonstrate that the function

$$y(t) = \sum_{k=0}^{n-1} Y_k(t,t_0)y_k + \int_{t_0}^{t} Y(t,s)g(s)ds \tag{B.1.43}$$

is a solution of problem (B.1.42), where $Y$ is the fundamental matrix of (B.1.33); i.e., the solution of problem (B.1.36)–(B.1.38). For the convenience of the reader, we will write $y(h_k(t))$ instead of $y_{h_k}(t)$, assuming that $y(h_k(t)) = 0$, if $h_k(t) < t_0$.

Equality (B.1.43) implies

$$y^{(n)}(t) = \sum_{k=0}^{n-1} Y_k^{(n)}(t,t_0)y_k + g(t) + \int_{t_0}^{t} Y^{(n)}(t,s)g(s)ds,$$

$$y^{(k-1)}(t) = \sum_{i=0}^{n-1} Y_i^{(k-1)}(t,t_0)y_i + \int_{t_0}^{t} Y^{(k-1)}(t,s)g(s)ds,\ k = 1,\cdots,n,$$

$$y^{(k-1)}\big(h_k(t)\big) = \sum_{i=0}^{n-1} Y_i^{(k-1)}\big(h_k(t),t_0\big)y_i + \int_{t_0}^{t} Y^{(k-1)}\big(h_k(t),s\big)g(s)ds.$$

Consider the left-hand side of (B.1.42), where $y$ has the form (B.1.43). Then

$$\begin{aligned}
& y^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)y^{(k)}\big(h_k(t)\big) \\
&= \sum_{k=0}^{n-1} Y_k^{(n)}(t,t_0)y_k + g(t) + \int_{t_0}^{t} Y^{(n)}(t,s)g(s)ds \\
&\quad + \sum_{k=0}^{n-1} a_k(t)\left[\sum_{i=0}^{n-1} Y_i^{(k-1)}\big(h_k(t),t_0\big)y_i + \int_{t_0}^{t} Y^{(k-1)}\big(h_k(t),s\big)g(s)ds\right] \\
&= \sum_{k=0}^{n-1}\left[Y_k^{(n)}(t,t_0) + \sum_{i=0}^{n-1} a_i(t)Y_k^{(i)}\big(h_i(t),t_0\big)\right]y_k \\
&\quad + \int_{t_0}^{t}\left[Y^{(n)}(t,s) + \sum_{k=0}^{n-1} a_k(t)Y^{(k)}\big(h_k(t),s\big)\right]g(s)ds + g(t) = g(t),
\end{aligned}$$

which completes the proof. □

## B.2 Estimations of the Fundamental Matrix

**Theorem B.6** *Let conditions* (a5) *and* (a6) *hold. Then, for the fundamental function of* (B.1.18), *the following estimation holds*:

$$\|C(t,s)\| \le \exp\left\{\int_s^t \| \operatorname{Var}_{s\le\tau\le\xi} R(\xi,\tau)\| d\xi\right\}. \tag{B.2.1}$$

*Proof* Denote $X(t) = C(t, t_0)$. We have

$$\dot{X}(t) + \int_{-\infty}^t d_s R(t,s) X(s) = 0, t \ge t_0,$$
$$X(t_0) = I,\ \ X(t) = 0, t < t_0.$$

Hence

$$X(t) = I - \int_{t_0}^t \int_{-\infty}^s d_\tau R(s,\tau) X(\tau) ds = I - \int_{t_0}^t \int_{t_0}^s d_\tau R(s,\tau) X(\tau) ds.$$

Then, by Lemma A.1,

$$\|X(t)\| \le 1 + \int_{t_0}^t \| \operatorname{Var}_{t_0\le\tau\le s} R(s,\tau)\| \max_{t_0\le\tau\le s} \|X(\tau)\| ds.$$

Denote $y(t) = \max_{t_0\le\tau\le t} \|X(\tau)\|$. Hence

$$y(t) \le 1 + \int_{t_0}^t \| \operatorname{Var}_{t_0\le\tau\le s} R(s,\tau)\| y(s) ds.$$

Thus, by the Gronwall-Bellman inequality (Lemma A.5),

$$\|C(t,t_0)\| \le y(t) \le \exp\left\{\int_{t_0}^t \| \operatorname{Var}_{t_0\le\tau\le s} R(s,\tau)\| ds\right\}.$$

The general case is proven similarly. □

**Corollary B.1** *Suppose conditions* (a1)–(a3) *hold. Then, for the fundamental function of* (B.1.1), *the following estimation holds*:

$$\|C(t,s)\| \le \exp\left\{\int_s^t \sum_{k=1}^m \|A_k(\xi)\| d\xi\right\}.$$

**Corollary B.2** *Suppose conditions* (a1)–(a5) *hold. Then, for the fundamental function of* (B.1.10), *the following estimation holds*:

$$\|X(t,s)\| \le \exp\left\{\int_s^t \left(\sum_{k=1}^\infty \|A_k(\xi)\| + \int_s^\xi \|K(\xi,\tau)\| d\tau\right) d\xi\right\}.$$

Now consider (B.1.33), where its coefficients $a_k(t)$ are Lebesgue measurable locally essentially bounded functions, and the delays $h_k(t) \le t$ satisfy $\lim_{t\to\infty} h_k(t) = \infty$, $k = 0, \cdots, n-1$.

**Lemma B.1** *The solution of the initial value problem*

$$y^{(n)}(t) = f(t),\ y^{(k)}(a) = 0,\ k = 0, \cdots, n-2,\ y^{(n-1)}(a) = 1 \tag{B.2.2}$$

*has the form*

$$y(t) = \frac{(t-a)^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau. \tag{B.2.3}$$

*Proof* Differentiating (B.2.3), we obtain

$$\dot{y}(t) = \frac{(t-a)^{n-2}}{(n-2)!} + \frac{1}{(n-2)!}\int_a^t (t-\tau)^{n-2} f(\tau)d\tau.$$

Repeating the calculations, we have

$$y^{(n-1)}(t) = 1 + \int_a^t f(\tau)d\tau.$$

Hence

$$y(a) = y'(a) = \cdots = y^{(n-2)}(a) = 0, y^{(n-1)}(a) = 1$$

and $y^{(n)}(t) = f(t)$, which means that (B.2.3) is the unique solution of (B.2.2). □

**Theorem B.7** *For the fundamental function* $Y(t,s)$ *of* (B.1.33), *the following inequality holds for* $t_0 \le s \le t \le b$:

$$|Y(t,s)| \le M \exp\left\{\int_s^t M \sum_{k=0}^{n-1} |a_k(\tau)| d\tau\right\}, \quad \max_{0\le k\le n-1} \frac{(b-t_0)^k}{k!}. \tag{B.2.4}$$

**Theorem B.8** *Suppose conditions* (a8)–(a11) *hold. For the fundamental function of* (B.1.26), *the inequality*

$$\|C(t,s)\| \le \exp\left\{\int_s^t \left[(I-\bar{S})^{-1}b\right](\tau)d\tau\right\}, \quad s \le t, \tag{B.2.5}$$

*holds, where* $a(t) = \|A(t)\|$, $b(t) = \|B(t)\|$ *and the operator*

$$(\bar{S}x)(t) = \begin{cases} a(t)x(g(t)), & g(t) \ge s, \\ 0, & g(t) < s, \end{cases}$$

*acts in the scalar space* $L_\infty[s,c]$ *for any* $c > s$.

*Proof* For $X(t) = C(t,s)$ we have

$$(I-S)\dot{X}(t) = -B(t)X\big(h(t)\big),\ X(t) = 0,\ t < s,\ X(s) = I.$$

Hence $\dot{X}(t) = -[(I-S)^{-1}Y](t)$, where $Y(t) = B(t)X(h(t))$, and thus

$$\|X(t)\| \le 1 + \int_s^t \left\|\left[(I-S)^{-1}Y\right](\tau)\right\| d\tau.$$

Since

$$\big[(I-S)^{-1}Y\big](\tau) = B(\tau)X\big(h(\tau)\big) + A(\tau)B\big(g(\tau)\big)X\big(h\big(g(\tau)\big)\big) + A(\tau)A\big(g(\tau)\big)B\big(g\big(g(\tau)\big)\big)X\big(h\big(g\big(g(\tau)\big)\big)\big) + \cdots,$$

the estimate

$$\begin{aligned}\big\|\big[(I-S)^{-1}Y\big](\tau)\big\| &\le \big[b(\tau) + a(\tau)b\big(g(\tau)\big) \\ &\quad + a(\tau)a\big(g(\tau)\big)b\big(g\big(g(\tau)\big)\big) + \cdots\big] \sup_{s\le\xi\le t}\|X(\xi)\| \\ &= \big[(I-\bar{S})^{-1}b\big](\tau) \sup_{s\le\xi\le\tau}\|X(\xi)\|\end{aligned}$$

holds. Denote $z(t) = \sup_{s\le\xi\le t}\|X(\xi)\|$. We have

$$z(t) \le 1 + \int_s^t \big[(I-\bar{S})^{-1}b\big](\tau)z(\tau)d\tau.$$

Hence, by the Gronwall-Bellman inequality (Lemma A.5),

$$z(t) \le \exp\left\{\int_s^t \big[(I-\bar{S})^{-1}b\big](\tau)d\tau\right\}, \ s \le t,$$

which implies (B.2.5). □

**Corollary B.3** *Suppose $c > t_0$ is an arbitrary number. If $t_0 \le s \le t \le c$, then*

$$\|C(t,s)\| \le \exp\left\{\int_{t_0}^c \big[(I-\bar{S})^{-1}b\big](\tau)d\tau\right\}, \tag{B.2.6}$$

$$\|C_t(t,s)\| \le \sup_{t_0\le\xi\le c}\big[(I-\bar{S})^{-1}b\big](\xi)\exp\left\{\int_{t_0}^c \big[(I-\bar{S})^{-1}b\big](\tau)d\tau\right\}. \tag{B.2.7}$$

*Proof* Inequality (B.2.6) is evident.

By the proof of Theorem B.8, we have for $X(t) = C(t,s)$

$$\|\dot{X}(t)\| \le \big[(I-\bar{S})^{-1}b\big](t) \sup_{s\le\xi\le t}\|X(\xi)\|,$$

which implies (B.2.7). □

## B.3 Nonlinear Delay Differential Equations

We consider for $t \ge 0$ the nonlinear scalar differential equation with a distributed delay

$$\dot{x}(t) + \sum_{k=1}^m r_k(t)\int_{-\infty}^t f_k\big(x(s)\big)d_sR_k(t,s) = 0, \tag{B.3.1}$$

this equation with a nondelay term

$$\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{-\infty}^{t} f_k\big(x(s)\big) d_s R_k(t,s) = 0, \tag{B.3.2}$$

and equations with finite delays

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k\big(x(s)\big) d_s R_k(t,s) = 0, \tag{B.3.3}$$

$$\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) \int_{h_k(t)}^{t} f_k\big(x(s)\big) d_s R_k(t,s) = 0, \tag{B.3.4}$$

for $t > t_0 \geq 0$.

If we assume that for each $t_1$ there exists $s_1 = s(t_1) \leq t_1$ such that $R_k(t,s) = 0$ for $s < s_1$, $t > t_1$ and $\lim_{t\to\infty} s(t) = \infty$, then we can introduce the functions

$$h_k(t) = \inf\big\{s \leq t \,|\, R_k(t,s) \neq 0\big\} \tag{B.3.5}$$

and rewrite (B.3.1), (B.3.2) in the form (B.3.3), (B.3.4).

Together with (B.3.1)–(B.3.4), we assume for each $t_0 \geq 0$ the initial condition

$$x(t) = \varphi(t), \ t \leq t_0. \tag{B.3.6}$$

We consider (B.3.1)–(B.3.4) under the following assumptions:

(a13) $r_k(t)$, $k = 1, \cdots, m$, $b(t)$ are Lebesgue measurable essentially bounded functions on the halfline: $|r_k(t)| \leq r_k$, $|b(t)| \leq b$, $t \geq 0$.
(a14) $h_k : [0,\infty) \to \mathbb{R}$, $k = 1, \cdots, m$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$.
(a15) $R_k(t,\cdot)$ are left continuous nondecreasing functions for any $t$, $R_k(\cdot,s)$ are locally integrable for any $s$, either $R_k(t,h_k(t)) = 0$ or $\lim_{s\to-\infty} R_k(t,s) = 0$ for any $t$ and $R_k(t,t^+) = \int_{-\infty}^{t} d_s R(t,s) \leq 1$.

In (a15), the condition $R_k(t,h_k(t)) = 0$ means that the delay is finite, while $R_k(t,t^+) = 1$ and $\int_{-\infty}^{t} d_s R(t,s) \leq 1$ correspond to any delay equation that is "normalized" with the coefficient not exceeding $r_k(t)$. Now let us proceed to the initial function $\varphi$. This function should satisfy conditions such that the integral on the left-hand side of (B.3.3) exists almost everywhere. In particular, if $R_k(t,\cdot)$ are absolutely continuous for any $t$ (which allows us to write (B.3.3) as an integrodifferential equation), then $\varphi$ can be chosen as a Lebesgue measurable essentially bounded function. If $R_k(t,\cdot)$ are a combination of step functions (which corresponds to an equation with concentrated delays), then $\varphi$ should be a Borel measurable bounded function. For any choice of $R_k$, the integral exists if $\varphi$ is bounded and continuous. Thus, we assume:

(a16) $\varphi : (-\infty, 0] \to \mathbb{R}$ is a bounded continuous function.
(a17) $f_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \cdots, m$ are continuously differentiable and $f_k'$ are locally essentially bounded functions.

For (B.3.1) and (B.3.2), we assume in addition that $f_k$ are bounded on any interval $(-\infty, t_0]$.

*Remark B.1* For the existence and uniqueness results, in (a17) we can assume that the functions $f_k$ are locally Lipschitz rather than differentiable: for each $[a, b]$ there is an $M_k > 0$ (generally depending on $[a, b]$) such that $|f_k(x) - f_k(y)| < \mu_k |x - y|$ for any $x, y \in [a, b]$, $k = 1, \cdots, m$.

**Definition B.11** A function $X : (-\infty, c] \to \mathbb{R}^n$ absolutely continuous on $[t_0, c]$ is called *a local solution* of problem (B.3.3), (B.3.6) if for some $c > t_0$ it satisfies (B.3.3) for almost all $t \in [t_0, c]$ and equalities (B.3.6) for $t \le t_0$.

**Definition B.12** A function $X : \mathbb{R} \to \mathbb{R}^n$ locally absolutely continuous on $[t_0, \infty)$ is called *a global solution* of problem (B.3.3), (B.3.6) if it satisfies (B.3.3) for almost all $t \in [t_0, \infty)$ and equalities (B.3.6) for $t \le t_0$.

The same definitions are true for (B.3.1), (B.3.2) and (B.3.4).

**Theorem B.9** *Suppose* (a13)–(a17) *hold. Then, for any* $K > 0$ *and* $x(t_0) = x_0$ *such that* $|x_0| < K$*, there exists a unique local solution of anyone of* (B.3.1)–(B.3.4) *with initial condition* (B.3.6) *satisfying* $|x(t)| \le K$.

*Proof* Let us prove the theorem for (B.3.3). For the other equations, the proof is similar.

Let us fix $K > 0$. Condition (a17) implies that for some $\eta > 0$, $\mu > 0$ and any $|x| \le K$, $|y| \le K$ we have $|f_k(x)| \le \eta$, $|f_k(x) - f_k(y)| \le \mu |x - y|$. Problem (B.3.3), (B.3.6) can be rewritten in the form

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) \int_{\max\{t_0, h_k(t)\}}^{t} f_k\big(x(s)\big) d_s R_k(t, s) = g(t), \tag{B.3.7}$$

where

$$g(t) = -\sum_{k=1}^{m} r_k(t) \int_{\min\{t_0, h_k(t)\}}^{t_0} f_k\big(\varphi(s)\big) d_s R_k(t, s).$$

Then (B.3.7) is equivalent to the operator equation

$$x = Hx, \tag{B.3.8}$$

where

$$\begin{aligned}(Hx)(t) = &-\int_{t_0}^{t} \sum_{k=1}^{m} r_k(s) \int_{\max\{t_0, h_k(s)\}}^{s} f_k\big(x(\tau)\big) d_\tau R_k(s, \tau) ds \\ &+ \int_{t_0}^{t} g(s) ds + x_0.\end{aligned}$$

For any $b > t_0$, denote the set $M_b = \{x \in L_\infty[t_0, b] \mid \|x\|_{L_\infty} \le K\}$. Since $r_k$ and $g$ are locally bounded, we have for any $x \in M_b$

$$|(Hx)(t)| \le \eta \int_{t_0}^{b} \sum_{k=1}^{m} |r_k(s)| ds + \int_{t_0}^{b} |g(s)| ds + |x_0|.$$

Since $|x_0| < K$, we have $|(Hx)(t)| \le K$ for some $b > t_0$. Let us fix such $b$ and denote it by $b_1$. Then $HM_{b_1} \subset M_{b_1}$. We also have for $x, y \in M_{b_1}$

$$\begin{aligned} &|(Hx)(t) - (Hy)(t)| \\ &\quad \le \int_{t_0}^{t} \sum_{k=1}^{m} |r_k(s)| \int_{\max\{t_0, h_k(s)\}}^{s} \big| f_k\big(x(\tau)\big) - f_k\big(y(\tau)\big)\big| d_\tau R_k(s,\tau) ds, \end{aligned}$$

and for some value $b_2 < b_1$, $\lambda \in (0, 1)$ and $\mu = \min_k \mu_k$

$$\|Hx - Hy\|_{L_\infty[t_0,b_2]} \le \mu \int_{t_0}^{b_2} \sum_{k=1}^{m} |r_k(s)|(s - t_0) ds \|x - y\|_{L_\infty[t_0,b_2]} \le \lambda < 1.$$

Hence, for the set $M_{b_2} \subset L_\infty[t_0, b_2]$, all the conditions of the Banach contraction principle (Theorem A.14) are satisfied. Thus problem (B.3.3), (B.3.6) has a unique local solution, which completes the proof. □

Next, consider the scalar nonlinear delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{m} r_k(t) f_k\big(x\big(h_k(t)\big)\big) = 0 \tag{B.3.9}$$

as well as this equation with a nondelay term

$$\dot{x}(t) + b(t)x(t) + \sum_{k=1}^{m} r_k(t) f_k\big(x\big(h_k(t)\big)\big) = 0. \tag{B.3.10}$$

Equations (B.3.9) and (B.3.10) can be rewritten in the form (B.3.1) and (B.3.2) if we denote $R_k(t, s) = \chi_{(h_k(t),\infty)}(s)$, where $\chi_I$ is the characteristic function of the interval $I$. This leads to the following corollary of Theorem B.9.

**Theorem B.10** *Suppose that conditions* (a13)–(a17) *hold. Then there exists a unique local solution of both* (B.3.9), (B.3.6) *and* (B.3.10), (B.3.6).

Consider now the scalar nonlinear integrodifferential equations

$$\dot{x}(t) + \sum_{j=1}^{m} r_j(t) \int_{-\infty}^{t} K_j(t, s) f_j\big(x(s)\big) ds = 0, \tag{B.3.11}$$

$$\dot{x}(t) + b(t)x(t) + \sum_{j=1}^{m} r_j(t) \int_{-\infty}^{t} K_j(t, s) f_j\big(x(s)\big) ds = 0, \tag{B.3.12}$$

$$\dot{x}(t) + \sum_{j=1}^{m} r_j(t) \int_{h_j(t)}^{t} K_j(t,s) f_j\big(x(s)\big) ds = 0, \tag{B.3.13}$$

$$\dot{x}(t) + b(t)x(t) + \sum_{j=1}^{m} r_j(t) \int_{h_j(t)}^{t} K_j(t,s) f_j\big(x(s)\big) ds = 0, \tag{B.3.14}$$

where for the kernels $K_j(t,s)$ the following condition holds:

(a18) $K_j(t,s) \geq 0$ is a Lebesgue integrable function such that $\sup_t \int_{-\infty}^{t} K_j(t,s) \leq 1$ for (B.3.11), (B.3.12) and $\int_{h_j(t)}^{t} K_j(t,s) ds \leq 1,\ j = 1, \cdots, m$ for (B.3.13), (B.3.14).

**Theorem B.11** *Suppose* (a13), (a14) *and* (a16)–(a18) *hold. Then there exists a unique local solution of either* (B.3.13) *or* (B.3.12) *with initial condition* (B.3.6).

Further, let us study the existence of global solutions for nonlinear equations.

**Theorem B.12** *Suppose that* (a13)–(a17) *hold,* $r_k(t) \leq 0$ *and*

$$f_k(x) > 0 \text{ for any } x > 0 \text{ and } \frac{f_k(x)}{x} \text{ are bounded for } x > 0. \tag{B.3.15}$$

*If the initial function* $\varphi(t)$ *satisfies*

$$\varphi(t) \geq 0,\ t \leq t_0,\ \varphi(t_0) > 0, \tag{B.3.16}$$

*then there exists a unique global solution of* (B.3.4), (B.3.6) ((B.3.2), (B.3.6)) *that is positive for* $t \geq t_0$.

*Proof* By Theorem B.9, there exists a unique local solution. This solution is either global or there exists a $t_1$ such that either

$$\liminf_{t \to t_1^-} x(t) = -\infty \tag{B.3.17}$$

or

$$\limsup_{t \to t_1^-} x(t) = \infty. \tag{B.3.18}$$

Let us demonstrate that under (B.3.15), (B.3.16) the solution of (B.3.4) is positive. In fact, by (B.3.15), as far as $x(t)$ is positive, the derivative is not less than $-b(t)x(t)$. Since by (B.3.16) $x(t_0) > 0$, the solution of the ordinary differential equation $\dot{x}(t) = -b(t)x(t)$ is positive for $t > t_0$ as well as the solution of (B.3.4), which contradicts (B.3.17).

Now let us prove that (B.3.18) is impossible. By (B.3.15) and (a13), there exist $r_k > 0$, $M_k > 0$ such that $f_k(x) \leq M_k x$, $|r_k(t)| \leq r_k$. The solution of (B.3.4) does not exceed the solution of the equation

$$\dot{y}(t) = \sum_{k=1}^{m} r_k M_k y,\ t > t_0. \tag{B.3.19}$$

Thus $x(t) \le x(t_0) + e^{\sum_{k=1}^m r_k M_k (t-t_0)}$, so there is no point $t_2$ where (B.3.18) can be valid, which completes the proof. □

Consider now the following equations, which can be rewritten in the form (B.3.4):

$$\dot{x}(t) = -b(t)x(t) + \sum_{k=1}^{m} r_k(t) f_k\big(x\big(h_k(t)\big)\big),\ t > t_0, \tag{B.3.20}$$

$$\dot{x}(t) = -b(t)x(t) + \sum_{j=1}^{m} r_j(t) \int_{h_k(t)}^{t} K_j(t,s) f_k\big(x(s)\big) ds,\ t > t_0, \tag{B.3.21}$$

$$\dot{x}(t) = -b(t)x(t) + \sum_{j=1}^{m} r_j(t) \int_{-\infty}^{t} K_j(t,s) f_k\big(x(s)\big) ds,\ t > t_0. \tag{B.3.22}$$

**Theorem B.13** *Suppose that* $r_j(t) \ge 0, K_j(t,s) \ge 0$, *conditions* (a13), (a14), (a16)–(a18) *and* (B.3.15), (B.3.16) *hold.*

*Then there exists a unique global solution of anyone of* (B.3.20)–(B.3.22) *with initial condition* (B.3.6) *that is positive for* $t \ge 0$.

As an application of Theorem B.12, consider first the Lasota-Wazewska equation

$$\dot{N}(t) = -\mu(t)N(t) + \sum_{k=1}^{m} p_k(t) \int_{h_k(t)}^{t} e^{-\gamma_k N(s)} d_s R_k(t,s),\ t \ge 0, \tag{B.3.23}$$

where $\gamma_k > 0$, $\mu(t) \ge 0$, $p_k(t) \ge 0$, for $\mu(t)$ and $p_k(t)$ condition (a13) holds, for $h_k$ condition (a14) holds and for $R_k$ condition (a15) is satisfied, $k = 1, \cdots, m$.

**Corollary B.4** *Suppose condition* (B.3.16) *holds. Then there exists a unique global solution of* (B.3.23), (B.3.6) *that is positive for* $t \ge t_0$.

*Proof* Equation (B.3.23) has the form (B.3.20) for $f_k(x) = e^{-\gamma_k x}$. For these functions, all the conditions of Theorem B.12 hold. □

Consider now Nicholson's blowflies equation with a distributed delay

$$\dot{N}(t) = \sum_{k=1}^{m} p_k(t) \int_{h_k(t)}^{t} N(s) e^{-a_k N(s)} d_s R_k(t,s) - \delta(t)N(t),\ t \ge 0, \tag{B.3.24}$$

where $\delta(t) \ge 0$, $p_k(t) \ge 0$, for $\delta(t)$ and $p_k(t)$ condition (a13) holds, for $h_k$ condition (a14) is satisfied, for $R_k$ condition (a15) is valid and $a_k > 0$, $k = 1, \cdots, m$.

**Corollary B.5** *Suppose condition* (B.3.16) *holds. Then there exists a unique global solution of* (B.3.24), (B.3.6) *that is positive for* $t \ge t_0$.

The proof is similar to the proof of the previous theorem.

Consider two partial cases of (B.3.24):

Nicholson's blowflies equation with concentrated delays

$$\dot{N}(t)=\sum_{k=1}^{m} p_k(t)N\big(h_k(t)\big)e^{-a_k N(h_k(t))}-\delta(t)N(t),\ t\geq 0, \tag{B.3.25}$$

where $\delta(t)\geq 0$, $p_k(t)\geq 0$, for $\delta(t)$ and $p_k(t)$ condition (a13) holds, for $h_k$ condition (a14) is satisfied and $a_k>0$, $k=1,\cdots,m$; and Nicholson's blowflies integrodifferential equation

$$\dot{N}(t)=\sum_{k=1}^{m} p_k(t)\int_{h_k(t)}^{t} T_k(t,s)N(s)e^{-a_k N(s)}ds-\delta(t)N(t),\ t\geq 0, \tag{B.3.26}$$

where $\delta(t)\geq 0$, $p_k(t)\geq 0$, for $\delta(t)$ and $p_k(t)$ condition (a13) holds, for $h_k$ condition (a14) is satisfied, for $T_k$ condition (a18) is valid and $a_k>0$, $k=1,\cdots,m$.

**Corollary B.6** *Suppose that condition* (B.3.16) *holds. Then there exists a unique global solution of* (B.3.25), (B.3.6) *that is positive for* $t\geq t_0$.

**Corollary B.7** *Suppose that condition* (B.3.16) *holds. Then there exists a unique global solution of* (B.3.26), (B.3.6) *that is positive for* $t\geq t_0$.

Consider now the logistic equation with distributed delays

$$\dot{x}(t)=x(t)\left[\sum_{k=1}^{m} r_k(t)\left(1-\frac{1}{K}\int_{h_k(t)}^{t} x(s)d_s R_k(t,s)\right)\right],\ t>0, \tag{B.3.27}$$

where $r_k(t)\geq 0$ and $r_k$ satisfy (a13), for $h_k$ condition (a14) holds, for $R_k$ (a15) is valid, $R_k(t,\cdot)$ is a nondecreasing function and $K>0$.

The form of (B.3.27) is different from (B.3.4), so we cannot apply Theorem B.12. Let us prove existence, uniqueness and positivity of the solution for the logistic equation.

**Theorem B.14** *Suppose* $K>0$ *and* (a13)–(a16) *and* (B.3.16) *hold. Then there exists a unique global solution of* (B.3.27), (B.3.6) *that is positive for* $t\geq t_0$.

*Proof* Existence of a local solution for (B.3.27), (B.3.6) is justified by calculations similar to the proof of Theorem B.9. Since $x(t_0)>0$, we can assume that this local solution is positive. Suppose that $[t_0,c)$, where $c$ may be $\infty$, is the maximum interval of existence of such a solution. For any $t\in[t_0,c)$, we have

$$x(t)=x(t_0)\exp\left\{\int_{t_0}^{t}\left[\sum_{k=1}^{m} r_k(\tau)\left(1-\frac{1}{K}\int_{h_k(\tau)}^{\tau} x(s)d_s R_k(\tau,s)\right)\right]d\tau\right\},$$

hence the solution $x$ is positive. Thus, for $t\in[t_0,c)$ we have $\dot{x}(t)\leq\sum_{k=1}^{m} r_k x(t)$, and the solution $x$ does not exceed the solution of the ordinary differential equation

$\dot{y}(t) = \sum_{k=1}^{m} r_k y(t)$ with the same initial value, which is bounded on any bounded interval $[t_0, t_1]$. This implies $c = \infty$, and therefore (B.3.27) has a global solution. □

The equations

$$\dot{x}(t) = x(t)\left[\sum_{j=1}^{m} r_j(t)\left(1 - \frac{1}{K}\, x\big(h_j(t)\big)\right)\right],\ t > 0, \tag{B.3.28}$$

$$\dot{x}(t) = x(t)\left[\sum_{j=1}^{m} r_j(t)\left(1 - \frac{1}{K}\int_{h_j(t)}^{t} K_j(t,s)x(s)ds\right)\right],\ t > 0, \tag{B.3.29}$$

generalize the well-known delay logistic equation and the integrodifferential logistic equation.

**Theorem B.15** *Suppose that* $K > 0$, $r_j(t) \geq 0$, $K_j(t,s) \geq 0$ *and* (a13)–(a16), (a18) *and* (B.3.16) *hold. Then there exists a unique global solution of* (B.3.28), (B.3.6) ((B.3.29), (B.3.6)) *that is positive for* $t \geq 0$.

Consider now two new classes of nonlinear equations with concentrated delays,

$$\dot{x}(t) = -b(t)x(t) + \sum_{k=1}^{m} r_k(t) f_k\big(x(t), x\big(h_1(t)\big), \cdots, x\big(h_n(t)\big)\big),\ t \geq 0, \tag{B.3.30}$$

and

$$\dot{x}(t) = x(t)\sum_{k=1}^{m} g_k\big(t, x(t), x\big(h_1(t)\big), \cdots, x\big(h_n(t)\big)\big),\ t \geq 0. \tag{B.3.31}$$

Consider also the following conditions:

(A1) $f_k : \mathbb{R}^{n+1} \to \mathbb{R}$, $k = 1, \cdots, m$, are continuous functions that satisfy the local Lipschitz condition that for any $a, b$, $0 \leq a < b$ there exist $\lambda_i$ such that for each $u_i, v_i \in [a, b]$

$$|f_k(u_0, u_1, \cdots, u_n) - f(v_0, v_1, \cdots, v_n)| \leq \sum_{i=0}^{n} \lambda_i |u_i - v_i|$$

and the inequality

$$f_k(u_0, u_1, \cdots, u_n) \geq 0 \text{ for } u_0 \geq 0, \cdots, u_n \geq 0.$$

(A2) $g_k : [0, \infty) \times \mathbb{R}^{n+1} \to \mathbb{R}$, $k = 1, \cdots, m$ are continuous functions satisfying the local Lipschitz condition that for any $a, b$, $0 \leq a < b$ there exist $\lambda_i$ such that for each $u_i, v_i \in [a, b]$

$$|g_k(t, u_0, u_1, \cdots, u_n) - g(t, v_0, v_1, \cdots, v_n)| \leq \sum_{i=0}^{n} \lambda_i |u_i - v_i|$$

and the inequality

$$g_k(t, u_0, u_1, \cdots, u_n) \leq r(t) \text{ for } t \geq 0,\ u_0 \geq 0, \cdots, u_n \geq 0$$

for some locally essentially bounded function $r(t) \geq 0$.

We have now the following existence results.

**Theorem B.16** *Let* $r_k(t) \geq 0$ *and conditions* (a13), (a14), (a16), (A1) *and* (B.3.16) *hold. Then there exists a unique global solution of* (B.3.30), (B.3.6) *that is positive for* $t \geq t_0$.

The proof is similar to the proof of Theorem B.12.

**Theorem B.17** *Let conditions* (A2) *and* (B.3.16) *hold. Then there exists a unique global solution of* (B.3.31), (B.3.6) *that is positive for* $t \geq t_0$.

The proof is similar to the proof of Theorem B.14.

For instance, Theorem B.16 can be applied to the generalized Mackey-Glass equation

$$\dot{x}(t) = \sum_{k=1}^{m} \frac{x^{\alpha_k}(h_k(t))}{1 + x^{\beta_k}(g_k(t))} - b(t)x(t), \ 0 < \alpha_k \leq \beta_k + 1, \ \beta_k > 0.$$

Theorem B.17 is applicable to the generalized logistic equation

$$\dot{x}(t) = x(t) \sum_{k=1}^{m} r_k(t)\big(1 - x\big(h_k(t)\big)\big)\big|1 - x\big(h_k(t)\big)\big|^{\alpha_k - 1}, \ \alpha_k > 0,$$

and to the multiplicative delay logistic equation

$$\dot{x}(t) = r(t)x(t) \prod_{k=1}^{m}\big(1 - x\big(h_k(t)\big)\big)\big|1 - x\big(h_k(t)\big)\big|^{\alpha_k - 1}, \ \alpha_k > 0, \sum_{k=1}^{m} \alpha_k = 1.$$

## B.4 Linear Delay Impulsive Differential Equations

### *B.4.1 First-Order Impulsive Equations*

Some results of this subsection were taken from [15, 16, 42].

We consider the vector delay differential equation

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = F(t), \ t \geq t_0, \tag{B.4.1}$$

with the linear impulsive conditions

$$X\big(\tau_j^+\big) = B_j X(\tau_j) + D_j, \ j = 1, 2, \cdots, \tag{B.4.2}$$

where

$$A_k(t) = \big(a_{il}^k(t)\big)_{i,l=1}^n, \ F(t) = \big[f_1(t), \cdots, f_n(t)\big]^T,$$
$$B_j = \big(b_{il}^j\big)_{i,l=1}^n, \ D_j = \big[d_1^j, \cdots, d_n^j\big]^T,$$

under the following assumptions:

(a19) $t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ are fixed points, $\lim_{j\to\infty} \tau_j = \infty$, $B_j$ are invertible matrices;
(a20) $a_{il}^k$, $f_i$ are locally essentially bounded functions;
(a21) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \le t$.

Together with (B.4.1), (B.4.2), we will consider for each $t_0 \ge 0$ the initial conditions

$$X(\xi) = \Phi(\xi),\ \xi < t_0,\ X(t_0) = X_0, \tag{B.4.3}$$

where $\Phi(t) = [\varphi_1(t), \cdots, \varphi_n(t)]^T$.

We assume that for the initial function $\Phi$ the following hypothesis holds:

(a22) $\Phi : (-\infty, t_0) \to \mathbb{R}^n$ is a Borel measurable bounded vector function.

**Definition B.13** A function $X : \mathbb{R}^n \to \mathbb{R}^n$ absolutely continuous on each interval $(\tau_j, \tau_{j+1}]$ is *a solution* of impulsive problem (B.4.1)–(B.4.3) if (B.4.1) is satisfied for almost all $t \in [0, \infty)$ and the equalities (B.4.2), (B.4.3) hold.

**Definition B.14** For each $s \ge 0$, the solution $C(t, s)$ of the problem

$$\dot{X}(t) + \sum_{k=1}^{m} A_k(t) X\big(h_k(t)\big) = 0,\ \text{where } t \ge s;\ X(\xi) = 0,\ \xi < s, \tag{B.4.4}$$

$$X\big(\tau_j^+\big) = B_j X(\tau_j),\ \tau_j > s,\ X(s) = I, \tag{B.4.5}$$

is called *the fundamental matrix* of problem (B.4.1), (B.4.2).

We assume that $C(t, s) = 0$ for $t < s$.

**Lemma B.2** *Let the hypotheses* (a19)–(a21) *hold. For the fundamental matrix of* (B.4.1), (B.4.2), *the following estimate is valid*:

$$\|C(t, s)\| \le \prod_{s \le \tau_j < t} \|B_j\| \exp\left\{ \int_s^t \sum_{k=1}^{m} \|A_k(\zeta)\| d\zeta \right\}.$$

*Proof* Let $\tau_{i-1} < s \le \tau_i$. Then, for $t \in [s, \tau_i]$, the solution $X$ of problem (B.4.4), (B.4.5) can be presented as

$$X(t) = I - \int_s^t \sum_{k=1}^{m} A_k(\zeta) X\big(h_k(\zeta)\big)\, d\zeta,\ X(\xi) = 0,\ \xi < s,$$

and hence

$$\|X(t)\| \le 1 + \int_s^t \sum_{k=1}^{m} \|A_k(\zeta)\| \sup_{\xi \in [s,\zeta]} \|X(\xi)\| d\zeta. \tag{B.4.6}$$

Denote $y(t) = \sup_{\zeta \in [s,t]} \|X(\zeta)\|$. Then, for the function $y(t)$, inequality (B.4.6) yields

$$y(t) \le 1 + \int_s^t \sum_{k=1}^{m} \|A_k(\zeta)\| y(\zeta) d\zeta$$

and the Gronwall-Bellman inequality (Lemma A.5) implies

$$y(t) \le \exp\left\{\int_s^t \sum_{k=1}^m \|A_k(\zeta)\| d\zeta\right\}.$$

Thus, for the solution $X$ of problem (B.4.4), (B.4.5), we have obtained the estimate

$$\|X(t)\| \le \exp\left\{\int_s^t \sum_{k=1}^m \|A_k(\zeta)\|\, d\zeta\right\} \tag{B.4.7}$$

for $\tau_{i-1} < s, t \le \tau_i$. Let $\tau_{i-1} < s < \tau_i < t \le \tau_{i+1}$. Then

$$X(t) = X\big(\tau_i^+\big) - \int_{\tau_i}^t \sum_{k=1}^m A_k(\zeta) X\big(h_k(\zeta)\big)\, d\zeta.$$

Thus inequality (B.4.7) and the impulsive condition $X(\tau_i^+) = B_i X(\tau_i)$ imply the estimate

$$\|X(t)\| \le \|B_i\| \exp\left\{\int_s^{\tau_i} \sum_{k=1}^m \|A_k(\zeta)\|\, d\zeta\right\} + \int_{\tau_i}^t \sum_{k=1}^m \|A_k(\zeta)\| \|X(h_k(\zeta))\|\, d\zeta.$$

Again, denoting $y(t) = \max_{\zeta\in[s,t]} \|X(\zeta)\|$, we obtain

$$y(t) \le \|B_i\| \exp\left\{\int_s^{\tau_i} \sum_{k=1}^m \|A_k(\zeta)\|\, d\zeta\right\} + \int_{\tau_i}^t \sum_{k=1}^m \|A_k(\zeta)\| y(\zeta) d\zeta.$$

Repeating the previous argument gives

$$\begin{aligned}\|X(t)\| &\le \|B_i\| \exp\left\{\int_s^{\tau_i} \sum_{k=1}^m \|A_k(\zeta)\|\, d\zeta\right\} \exp\left\{\int_{\tau_i}^t \sum_{k=1}^m \|A_k(\zeta)\|\, d\zeta\right\}\\ &= \|B_i\| \exp\left\{\int_s^t \sum_{k=1}^m \|A_k(\zeta)\| d\zeta\right\},\ t \in (\tau_i, \tau_{i+1}].\end{aligned}$$

Now let $\tau_{i-1} < s < \tau_i < \cdots < \tau_{j-1} < t < \tau_j$. After considering the solution $x$ of problem (B.4.4), (B.4.5), $X(s) = I$ in the intervals $(\tau_{i+1}, \tau_{i+2}], \cdots, (\tau_{j-1}, \tau_j]$ and at the point $\tau_j^+$, we obtain the required inequality for $\|C(t,s)\|$, which completes the proof. □

**Corollary B.8** *For any* $b > 0$, *the function* $C(t,s)$ *is bounded in* $[t_0, b] \times [t_0, b]$.

**Theorem B.18** *Let* (a19)–(a22) *hold. Then there exists a unique solution of problem* (B.4.1)–(B.4.3). *This solution can be presented in the form*

$$\begin{aligned} X(t) &= C(t,t_0)X(t_0) + \int_{t_0}^t C(t,s)F(s)ds\\ &\quad - \sum_{k=1}^m \int_{t_0}^t C(t,s)A_k(s)\Phi\big(h_k(s)\big)ds + \sum_{t_0\le\tau_j\le t} C(t,\tau_j)D_j, \end{aligned} \tag{B.4.8}$$

*where* $\Phi(h_k(s)) = 0$, *if* $h_k(s) > t_0$.

*Proof* Note that, by Theorem B.1, on each interval $[\tau_{j-1}, \tau_j]$ there exists a unique solution of initial value problem (B.4.1), (B.4.3) without impulses, and therefore there exists a unique solution of initial problem (B.4.1)–(B.4.3).

We first establish equality (B.4.8) for $\Phi = 0$, $X(t_0) = 0$, $D_j = 0$, $j = 0, 1, 2, \cdots$. Let us demonstrate that in this case the solution of initial value problem (B.4.1)–(B.4.3) has the form

$$X(t) = \int_{t_0}^{t} C(t,s)F(s)ds. \tag{B.4.9}$$

By differentiating the equality (B.4.9) in $t$, we obtain

$$\dot{X}(t) = F(t) + \int_{t_0}^{t} C_t'(t,s)F(s)ds \tag{B.4.10}$$

since $C(t,t) = I$. Equalities (B.4.10) and $C(t,s) = 0$ for $t < s$ imply

$$X\big(h_k(t)\big) = \int_{t_0}^{h_k(t)} C\big(h_k(t), s\big)F(s)ds = \int_{t_0}^{t} C\big(h_k(t), s\big)F(s)ds,$$

which together with equality (B.4.10) proves that $X$ is a solution of problem (B.4.1), where $X(\xi) = 0$ for $\xi < t_0$, $X(t_0) = 0$.

It remains to show that $X$ satisfies impulsive conditions (B.4.2). Let $i$ be a fixed positive integer and let $\{t_k\}_{k=1}^{\infty} \subset (\tau_i, \tau_{i+1})$ be a sequence such that $t_k$ tends to $\tau_i$ as $k \to \infty$. We will prove that the equality

$$\lim_{t_k \to \tau_i^+} \int_{t_0}^{t_k} C(t_k, s)F(s)ds = \int_{t_0}^{\tau_i} C\big(\tau_i^+, s\big)F(s)ds \tag{B.4.11}$$

holds; i.e., that the limit under the integral is possible.

Denote $G_k(s) = C(t_k, s)F(s)$, $G(s) = S(\tau_i^+, s)F(s)$. Evidently $\lim_{k\to\infty} G_k(s) = G(s)$. Besides, Lemma B.2 implies

$$\|G_k(s)\| \le \prod_{0 \le j < i} \|B_j\| \exp\left\{ \int_{t_0}^{\tau_{i+1}} \sum_{k=1}^{m} \|A_k(\zeta)\| d\zeta \right\} \|F(s)\|,$$

and therefore the functions $G_k(s)$ are uniformly bounded for $s < \tau_{i+1}$.

By the Lebesgue convergence theorem (Theorem A.1), we obtain (B.4.11). The function $C(t,s)$ satisfies the impulsive condition $C(\tau_i^+, s) = B_i C(\tau_i, s)$. Thus equality (B.4.11) implies

$$\begin{aligned} B_i X(\tau_i) &= B_i \lim_{t_k \to \tau_i^+} \int_{t_0}^{t_k} C(t_k, s)F(s)ds \\ &= \int_{t_0}^{\tau_i} B_i C(\tau_i, s)F(s)ds = \int_{t_0}^{\tau_i} C\big(\tau_i^+, s\big)F(s)ds = X\big(\tau_i^+\big). \end{aligned}$$

Hence $X(\tau_i^+) = B_i X(\tau_i)$.

Let us proceed to the case of arbitrary initial conditions and $D_j \ne 0$. We notice that the series $\sum_{j=0}^{\infty} C(t, \tau_j)D_j$ converges since for each $t > 0$ this series contains only a finite number of terms with $\tau_j \le t$. By direct calculation, it is possible to check that the solution of the problem

$$\begin{aligned} &\dot{X}(t)+\sum_{i=1}^{m} A_i(t)X(h_i(t)) = F(t) - \sum_{i=1}^{m} A_i(t)\Phi(h_i(t)), \\ &X(\xi)=0,\ \xi<t_0,\ \Phi(\zeta)=0,\ \zeta\ge t_0, \end{aligned} \tag{B.4.12}$$

coincides with the solution of problem (B.4.1), (B.4.3).

The solution $x_0(t)$ of the problem (B.4.12), (B.4.2), $X(t_0)=0$ can be presented as

$$X_0(t)=\int_{t_0}^{t} C(t,s)F(s)ds - \sum_{i=1}^{m}\int_{t_0}^{t} C(t,s)A_i(s)\Phi\big(h_i(s)\big)\,ds.$$

Since $C(t,s)$ is the fundamental matrix, we have

$$C_t'(t,\tau_j)D_j + \sum_{i=1}^{m} A_i(t)C\big(h_i(t),\tau_j\big)D_j = 0,\ \ j=0,1,2,\cdots,$$

and hence

$$\sum_{j=0}^{\infty} C_t'(t,\tau_j)D_j + \sum_{j=0}^{\infty}\sum_{i=1}^{m} A_i(t)C\big(h_i(t),\tau_j\big)D_j = 0$$

and $X = X_0 + \sum_{j=0}^{\infty} C(t,\tau_j)D_j$ satisfies (B.4.12). Let us note that the initial condition and the impulsive conditions are also satisfied, which completes the proof. □

### *B.4.2 Second-Order Impulsive Equations*

We consider the scalar equation

$$\ddot{x}(t)+\sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big)=0,\ t\ge t_0, \tag{B.4.13}$$

with the impulsive conditions

$$x\big(\tau_j^+\big)=A_jx(\tau_j),\ \dot{x}\big(\tau_j^+\big)=B_j\dot{x}(\tau_j),\ \tau_j>t_0, \tag{B.4.14}$$

under the following assumptions:

(a23) $t_0=\tau_0<\tau_1<\tau_2<\cdots$ are fixed points, $\lim \tau_j=\infty$, $A_j\ne 0$, $B_j\ne 0$;
(a24) $a_k$, $k=1,\cdots,m$, are Lebesgue measurable functions that are locally essentially bounded on $[t_0,\infty)$;
(a25) $g_k : [t_0,\infty) \to \mathbb{R}$ are Lebesgue measurable functions, $g_k(t) \le t$, $\lim_{t\to\infty} g_k(t)=\infty$, $k=1,\cdots,m$.

Together with (B.4.13) and (B.4.14), consider for each $t_0 \geq 0$ the initial value problem

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big) = f(t),\ t \geq t_0, \tag{B.4.15}$$

$$x(t) = \varphi(t),\ t < t_0,\ x(t_0) = \alpha_0,\ \dot{x}(t_0) = \beta_0, \tag{B.4.16}$$

$$x\big(\tau_j^+\big) = A_j x(\tau_j) + \alpha_j, \dot{x}\big(\tau_j^+\big) = B_j \dot{x}(\tau_j) + \beta_j,\ \tau_j > t_0. \tag{B.4.17}$$

We also assume that the following hypothesis holds:

(a26) $f : [t_0, \infty) \to \mathbb{R}$ is a Lebesgue measurable and locally essential function and $\varphi : (-\infty, t_0) \to \mathbb{R}^n$ is a Borel measurable bounded function.

**Definition B.15** A function $x : \mathbb{R} \to \mathbb{R}$ with derivative $\dot{x}$ that is absolutely continuous on each interval $[\tau_j, \tau_{j+1})$ is called *a solution* of problem (B.4.15)–(B.4.17) if it satisfies (B.4.15) for almost all $t \in [t_0, \infty)$ and equalities (B.4.17), (B.4.16) hold.

**Definition B.16** For each $s \geq t_0$, denote by $X_0(t, s)$ and $X(t, s)$ the solutions of the problem

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)x\big(g_k(t)\big) = 0, t \geq s,\ x(t) = 0, t < s, \tag{B.4.18}$$

$$x\big(\tau_j^+\big) = A_j x(\tau_j), \dot{x}\big(\tau_j^+\big) = B_j \dot{x}(\tau_j),\ \tau_j > s, \tag{B.4.19}$$

with the initial conditions $x(s) = 1$, $\dot{x}(s) = 0$ for $X_0(t, s)$ and $x(s) = 0$, $\dot{x}(s) = 1$ for $X(t, s)$.

$X(t, s)$ is called *the fundamental function* of (B.4.13), (B.4.14).

We assume $X(t, s) = 0$, $t < s$.

**Lemma B.3** *Suppose that* (a23)–(a25) *hold. Then the fundamental function* $X(t, s)$ *of impulsive equation* (B.4.13), (B.4.14) *and its derivative* $X_t'(t, s)$ *in* $t$ *are essentially bounded on any square* $[t_0, b] \times [t_0, b]$.

*Proof* Suppose for simplicity that $m = 1$, and consider the equation

$$\ddot{x}(t) + a(t)x\big(g(t)\big) = 0$$

with impulsive condition (B.4.14). Suppose $t_0 \leq s < t \leq b$, and denote $x(t) = X(t, s)$. Let $t_0 \leq \tau_{i-1} < s < t < \tau_i < b$. Then

$$\dot{x}(t) = 1 - \int_s^t a(\eta)x\big(g(\eta)\big)d\eta, \tag{B.4.20}$$

which implies

$$x(t) = t - s - \int_s^t \bigg(\int_s^\theta a(\eta)x\big(g(\eta)\big)d\eta\bigg)d\theta$$

and

$$|x(t)| \le b - t_0 + \int_{t_0}^{b} |a(\eta)| d\eta \int_{t_0}^{t} \max_{t_0 \le \xi \le \theta} |x(\xi)| d\theta.$$

Denote $y(t) = \max_{t_0 \le \xi \le t} |x(\xi)|$. Hence

$$y(t) \le b - t_0 + \int_{t_0}^{b} |a(\eta)| d\eta \int_{t_0}^{t} y(\theta) d\theta.$$

Applying the Gronwall-Bellman inequality (Lemma A.5), we obtain

$$|x(t)| \le y(t) \le (b - t_0) \exp\left\{(b - t_0) \int_{t_0}^{b} |a(\eta)| d\eta\right\} := L_1.$$

By (B.4.20) and the definition of $y(t)$, we also have

$$|\dot{x}(t)| \le 1 + \int_{t_0}^{b} |a(\eta)| L_1 d\eta := M_1.$$

Suppose now that $t_0 \le \tau_{i-1} < s \le \tau_i < t < \tau_{i+1} < b$. Then

$$\begin{aligned} \dot{x}(t) &= \dot{x}\big(\tau_i^+\big) - \int_{\tau_i}^{t} a(\eta) x\big(g(\eta)\big) d\eta \\ &= B_i \dot{x}(\tau_i) - \int_{\tau_i}^{t} a(\eta) x\big(g(\eta)\big) d\eta \end{aligned}$$

and

$$\begin{aligned} x(t) &= x\big(\tau_i^+\big) + \int_{\tau_i}^{t} \left[B_i \dot{x}(\tau_i) - \int_{\tau_i}^{\xi} a(\eta) x\big(g(\eta)\big) d\eta\right] d\xi \\ &= A_i x(\tau_i) + \int_{\tau_i}^{t} \left[B_i \dot{x}(\tau_i) - \int_{\tau_i}^{\xi} a(\eta) x\big(g(\eta)\big) d\eta\right] d\xi. \end{aligned}$$

Hence

$$|x(t)| \le A_i L_1 + B_i M_1 (b - t_0) + \int_{t_0}^{b} |a(\eta)| d\eta \int_{t_0}^{t} y(\theta) d\theta,$$

which implies by the Gronwall-Bellman inequality (Lemma A.5)

$$|x(t)| \le y(t) \le \big(A_i L_1 + B_i M_1 (b - t_0)\big) \exp\left\{(b - t_0) \int_{t_0}^{b} |a(\eta)|\, d\eta\right\} := L_2.$$

We also have

$$|\dot{x}(t)| \le B_i M_1 + \int_{t_0}^{b} |a(\eta)| L_2\, d\eta := M_2.$$

Repeating this process, we obtain that $X(t,s)$ and $X'_t(t,s)$ are essentially bounded on any square $[t_0, b] \times [t_0, b]$. □

**Lemma B.4** *Suppose that the assumptions of Lemma* B.3 *hold. Then the solution of impulsive equation* (B.4.15)–(B.4.17) *with* $\alpha_j = \beta_j = 0$, $j = 0, 1, \cdots$, *and* $\varphi = 0$ *can be presented as*

$$x(t) = \int_{t_0}^{t} X(t,s) f(s) ds. \tag{B.4.21}$$

*Proof* Differentiating (B.4.21) once leads to the equation

$$\dot{x}(t) = \int_{t_0}^{t} X'_t(t,s) f(s)\, ds, \tag{B.4.22}$$

while the second differentiation gives the relation

$$\ddot{x}(t) = f(t) + \int_{t_0}^{t} X''_{tt}(t,s) f(s)\, ds \tag{B.4.23}$$

since $X'_t(s,s) = I_n$, $X(s,s) = 0$ for each $s$. Equality (B.4.21) and $X(t,s) = 0$ for $t \le s$ imply

$$x\big(g_k(t)\big) = \int_{t_0}^{\max\{g_k(t), t_0\}} X\big(g_k(t), s\big) f(s)\, ds = \int_{t_0}^{t} X\big(g_k(t), s\big) f(s)\, ds.$$

Consequently, by the definition of the fundamental function, we have

$$\begin{aligned} \ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x\big(g_k(t)\big) &= f(t) + \int_{t_0}^{t} X''_{tt}(t,s) f(s)\, ds \\ &\quad + \int_{t_0}^{t} \sum_{k=1}^{m} a_k(t) X\big(g_k(t), s\big) f(s)\, ds = f(t), \end{aligned}$$

and therefore (B.4.21) is a solution of the equation.

Next, let us prove that (B.4.21) also satisfies the impulsive conditions. Let $i$ be a fixed positive integer and $\{t_j\}_{j=1}^{\infty} \subset (\tau_i, \tau_{i+1})$ be a sequence tending to $\tau_i^+$ as $j \to \infty$. Let us prove the relation

$$\lim_{t_j \to \tau_i^+} \int_{t_0}^{t_j} X(t_j, s) f(s)\, ds = \int_{t_0}^{\tau_i} X\big(\tau_i^+, s\big) f(s)\, ds. \tag{B.4.24}$$

By Lemma B.3, the functions under the integral are uniformly bounded, and hence the Lebesgue convergence theorem implies (B.4.24). Similarly, the equality for the derivative of $X(t,s)$ in $t$ is obtained:

$$\lim_{t_j \to \tau_i^+} \int_{t_0}^{t_j} X'_t(t_j, s) f(s)\, ds = \int_{t_0}^{\tau_i} X'_t\big(\tau_i^+, s\big) f(s)\, ds. \tag{B.4.25}$$

The fundamental function $X(t,s)$ satisfies the impulsive conditions

$$X\big(\tau_i^+, s\big) = A_i X(\tau_i, s), \;\; X'_t\big(\tau_i^+, s\big) = B_i X'_t(\tau_i, s).$$

By (B.4.24), (B.4.25) and (B.4.22), we obtain for $x(t)$ defined by (B.4.21)

$$
\begin{aligned}
x\big(\tau_i^+\big) &= \lim_{t_j \to \tau_i^+} \int_{t_0}^{t_j} X(t_j, s) f(s)\, ds = \int_{t_0}^{\tau_i} X\big(\tau_i^+, s\big) f(s)\, ds \\
&= \int_{t_0}^{\tau_i} A_i X(\tau_i, s) f(s)\, ds = A_i x(\tau_i), \\
\dot{x}\big(\tau_i^+\big) &= \lim_{t_j \to \tau_i^+} \int_{t_0}^{t_j} X_t'(t_j, s) f(s)\, ds = \int_{t_0}^{\tau_i} X_t'\big(\tau_i^+, s\big) f(s)\, ds \\
&= \int_{t_0}^{\tau_i} B_i X_t'(\tau_i, s) f(s)\, ds = B_i \dot{x}(\tau_i),
\end{aligned}
$$

and consequently $x(t)$ satisfies the impulsive conditions, which completes the proof. □

**Theorem B.19** *Let* (a23)–(a26) *hold. Then there exists a unique solution of problem* (B.4.15)–(B.4.17), *and the solution can be presented in the form*

$$
\begin{aligned}
x(t) = X_0(t, t_0)\alpha_0 + X(t, t_0)\beta_0 &+ \int_{t_0}^{t} X(t, s) f(s) ds \\
&- \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s) a_k(s) \varphi\big(g_k(s)\big) ds + \sum_{t_0 < \tau_j \le t} X_0(t, \tau_j)\alpha_j \\
&+ \sum_{t_0 < \tau_j \le t} X(t, \tau_j)\beta_j, \qquad \text{(B.4.26)}
\end{aligned}
$$

*where* $\varphi(g_k(s)) = 0$, *if* $g_k(s) > t_0$.

*Proof* By considering the solution of (B.4.15)–(B.4.17) successively on $[t_0, \tau_1)$, $[\tau_1, \tau_2), \cdots$, we obtain that there exists a unique solution of the initial value problem. We claim that it coincides with (B.4.26).

In fact, the solution of the problem

$$
\begin{aligned}
&\ddot{x}(t) + \sum_{k=1}^{m} a_k(t) x(g_k(t)) = f(t) - \sum_{k=1}^{m} a_k(t) \varphi(g_k(t)), \\
&x(\zeta) = 0, \zeta < t_0, \varphi(\zeta) = 0, \zeta \ge t_0,
\end{aligned} \qquad \text{(B.4.27)}
$$

also solves (B.4.15)–(B.4.17), with $\alpha_j = \beta_j = 0$ for every $j$. Therefore, by Lemma B.4, the solution of (B.4.27) can be presented as

$$
x_0(t) = \int_{t_0}^{t} X(t, s) f(s) ds - \sum_{k=1}^{m} \int_{t_0}^{t} X(t, s) a_k(s) \varphi\big(g_k(s)\big) ds.
$$

Let us observe that the functions $X_0(t, s)$ and $X(t, s)$ satisfy homogeneous equation (B.4.27) ($f \equiv 0$, $\varphi = 0$), and therefore their linear combination

$$
x_1(t) = X_0(t, t_0)\alpha_0 + X(t, t_0)\beta_0 + \sum_{t_0 < \tau_j \le t} X_0(t, \tau_j)\alpha_j + \sum_{t_0 < \tau_j \le t} X(t, \tau_j)\beta_j
$$

also satisfies the homogeneous equation.

Thus $x(t) = x_0(t) + x_1(t)$ also satisfies (B.4.27). It is easy to check that it also satisfies impulsive conditions (B.4.17).

Indeed, for example, at $\tau_i$

$$\begin{aligned} x_1(\tau_i^+) &= X_0(\tau_i^+, t_0)\alpha_0 + X(\tau_i^+, t_0)\beta_0 + \sum_{t_0<\tau_j\le\tau_i} X_0(\tau_i^+, \tau_j)\alpha_j \\ &\quad + \sum_{t_0<\tau_j\le\tau_i} X(\tau_i^+, \tau_j)\beta_j \\ &= A_i X_0(\tau_i - 0, t_0)\alpha_0 + A_i X(\tau_i - 0, t_0)\beta_0 \\ &\quad + A_i \sum_{t_0<\tau_j<\tau_i} X_0(\tau_i, \tau_j)\alpha_j + \alpha_i + A_i \sum_{t_0<\tau_j<\tau_i} X(\tau_i, \tau_j)\beta_j \\ &= A_i x_1(\tau_i) + \alpha_i \end{aligned}$$

since $X_0(\tau_i, \tau_i) = X(\tau_i, \tau_i) = 0$.

Therefore $x(t) = x_0(t) + x_1(t)$ satisfies both (B.4.27) and impulsive conditions (B.4.17).

Consequently, the function $x(t)$ defined by (B.4.26) is a solution of impulsive problem (B.4.15)–(B.4.17), which completes the proof. □

## B.5 Bohl-Perron Theorems

In this monograph, we apply nonoscillation results to stability investigations of some classes of linear functional differential equations. Such applications are based on Bohl-Perron theorems, which we present below following [30]; see also the papers [22–24] and monographs [99, 200, 239].

Consider the equation

$$\dot{x}(t) + (Hx)(t) = f(t),\ t \ge t_0,\ x(t_0) = 0, \tag{B.5.1}$$

where $x(t) \in \mathbb{R}^n$, $H : C[t_0, \infty) \to L_\infty[t_0, \infty)$ is a linear Volterra bounded operator, $f \in L_\infty[t_0, \infty)$, where $C[t_0, \infty)$ is the space of continuous functions bounded on $[t_0, \infty)$. The solution of (B.5.1) that is a locally absolutely continuous function satisfying (B.5.1) almost everywhere has the form (see [29, 98])

$$x(t) = \int_{t_0}^{t} X(t, s) f(s) ds,$$

where as usual $X(t, s)$ is the fundamental matrix of (B.5.1).

**Definition B.17** Equation (B.5.1) is *exponentially stable* if there exist $\alpha > 0$ and $K > 0$ such that $\|X(t, s)\| \le K e^{-\alpha(t-s)}$.

Equations (B.1.2), (B.1.10) and (B.1.18) are partial cases of (B.5.1).

We recall the definition of weighted spaces.

**Definition B.18** If $B$ is a Banach space of vector functions, then by $B^\lambda, \lambda > 0$, we define *the weighted space* of all functions $y \in B$ such that $y^\lambda := ye^{\lambda t} \in B$. The space $B^\lambda$ is a Banach space with the norm $\|y\|_{B^\lambda} = \|y^\lambda\|_B$.

For the following theorem, see [29, Theorem 5.3.5].

**Theorem B.20** *Assume that for any* $f \in L_\infty[t_0, \infty)$ *the solution of* (B.5.1) *satisfies* $x \in C[t_0, \infty)$. *Suppose also that for some* $\lambda > 0$ *we have* $H : C^\lambda[t_0, \infty) \to L^\lambda_\infty[t_0, \infty)$, *where* $H : C[t_0, \infty) \to L_\infty[t_0, \infty)$ *is a linear Volterra bounded operator. Then* (B.5.1) *is exponentially stable.*

If the operator $H : C[t_0, \infty) \to L_\infty[t_0, \infty)$ has the form

$$(Hx)(t) = \sum_{k=1}^{m} A_k(t)x\big(h_k(t)\big), \; h_k(t) \le t,$$

then condition $H : C^\lambda[t_0, \infty) \to L^\lambda_\infty[t_0, \infty)$ is satisfied for any $\lambda > 0$ as far as the elements $a_{ij}$ of the matrices $A_k$ are functions essentially bounded on $[t_0, \infty)$ and $t - h_k(t) \le \delta$ for some $\delta > 0$.

The following results are corollaries of Theorem B.20.

**Theorem B.21** *Suppose conditions* (a1) *and* (a2) *hold, for some* $t_0 \ge 0$, $\delta > 0$, *functions* $a^k_{ij}$ *are essentially bounded on* $[t_0, \infty)$, $t - h_k(t) \le \delta$ *and for any function* $f$ *essentially bounded on* $[t_0, \infty)$ *the solution of the initial value problem*

$$\dot{x}(t) + \sum_{k=1}^{m} A_k(t)x\big(h_k(t)\big) = f(t), \; t > t_0, \; x(t) = 0, \; t \le t_0,$$

*is bounded on* $[t_0, \infty)$. *Then* (B.1.2) *is exponentially stable.*

Consider now the differential equation with distributed delay (B.1.18).

**Theorem B.22** *Suppose for* (B.1.18) *conditions* (a5) *and* (a6) *hold and there exists* $\lambda > 0$ *such that*

$$\operatorname*{ess\,sup}_{t > -\infty} \left\| \int_{-\infty}^{t} e^{\lambda(t-s)} d_s R(t,s) \right\| < \infty.$$

*If for some* $t_0 \ge 0$ *and any function* $f$ *essentially bounded on* $[t_0, \infty)$ *the solution of the initial value problem*

$$\dot{x}(t) + \int_{-\infty}^{t} d_s R(t,s)x(s) = f(t), t > t_0, \; x(t) = 0, \; t \le t_0,$$

*is bounded on* $[t_0, \infty)$, *then* (B.1.18) *is exponentially stable.*

Consider now (B.1.18) with bounded delays

$$\dot{x}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} d_s R_k(t,s)x(s) = 0. \tag{B.5.2}$$

**Theorem B.23** *Suppose that for* (B.5.2) *conditions* (a2), (a5) *and* (a6) *hold*, $t - h_k(t) \le \delta$ *and for any function* $f$ *essentially bounded on* $[t_0, \infty)$ *the solution of the initial value problem*

$$\dot{x}(t) + \sum_{k=1}^{m} \int_{h_k(t)}^{t} d_s R_k(t,s) x(s) = f(t), t > t_0,\ x(t) = 0,\ t \le t_0,$$

*is bounded on* $[t_0, \infty)$. *Then* (B.5.2) *is exponentially stable.*

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