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Guang-ya Chen
Xuexiang Huang · Xiaoqi Yang

Vector Optimization

Set-Valued and
Variational Analysis

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To Our Parents

Preface

Vector optimization model has found many important applications in decision making problems such as those in economics theory, management science, and engineering design (since the introduction of the Pareto optimal solution in 1896). Typical examples of vector optimization model include maximization/minimization of the objective pairs (time, cost), (benefit, cost), and (mean, variance) etc.

Many practical equilibrium problems can be formulated as variational inequality problems, rather than optimization problems, unless further assumptions are imposed. The vector variational inequality was introduced by Giannessi (1980). Extensive research on its relations with vector optimization, the existence of a solution and duality theory has been pursued.

The fundamental idea of the Ekeland's variational principle is to assign an optimization problem a slightly perturbed one having a unique solution which is at the same time an approximate solution of the original problem. This principle has been an important tool for nonlinear analysis and optimization theory. Along with the development of vector optimization and set-valued optimization, the vector variational principle introduced by Nemeth (1980) has been an interesting topic in the last decade.

Fan Ky's minimax theorems and minimax inequalities for real-valued functions have played a key role in optimization theory, game theory and mathematical economics. An extension was proposed to vector payoffs was introduced by Blackwell (1955).

The Wardrop equilibrium principle was proposed for a transportation network. Until only recently, all these equilibrium models are based on a single cost. Vector network equilibria were introduced by Chen and Yen (1993) and are one of good examples of vector variational inequality applications.

This book studies vector optimization models, vector variational inequalities, vector variational principles, vector minimax inequalities and vector network equilibria and summarizes the recent theoretical development on these topics.

The outline of the book is as follows.

In Chapter 2, we examine vector optimization problems with a fixed domination structure, a variable domination structure and a set-valued function respectively. We will investigate optimality conditions, duality and topological properties of solutions for these problems.

In Chapter 3, we study existence, duality, gap function and characterization of a solution of vector variational inequalities. We will also explore set-valued vector variational inequalities and vector complementarity problems.

In Chapter 4, we present unified variational principles for vector-valued functions and set-valued functions respectively. We will also explore well-posedness properties of vector-valued/set-valued optimization problems.

In Chapter 5, we consider minimax inequalities for vector-valued and set-valued functions.

In Chapter 6, we consider weak vector equilibrium, vector equilibrium and continuous-time vector equilibrium principles.

One characteristic of the book is that special attention is paid to problems of set-valued and variable ordering nature. To deal with various nonconvex problems with vector objectives, the nonlinear scalarization method has been extensively used throughout the book. Most results of this book are original and should be interesting to researchers and graduates in applied mathematics and operations research. Readers can benefit from new methodologies developed in the book.

We are indebted to Franco Giannessi and Kok Lay Teo for their continuous encouragement and valuable advice and comments on the book. We are thankful to Xinmin Yang and Shengjie Li for their joint research collaboration on some parts of the book. The first draft of the book was typed by Hui Yu, whose assistance is appreciated. We acknowledge that the research of this book has been supported by the National Science Foundation of China and the Research Grants Council of Hong Kong, SAR, China.

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Introduction and Mathematical Preliminaries

In this chapter, we will present necessary mathematical concepts and results, which will be used in the later chapters. Most of the results can be found in the books: Aubin and Ekeland [5], Aubin and Frankowska [6], Rockafellar and Wets [168], Sawaragi, Nakayama and Tanino [176] and Yu [221]. Some new mathematical concepts and results on nonlinear scalarization functions will also be given.

1.1 Convex Cones and Minimal Points

Vector optimization problems (or multiobjective programming problems or multicriteria decision making problems) have close relations with orderings or preferences in objective spaces. It is known that orderings in a vector space can be defined by convex cones.

Let Y be a topological vector space, and $S \subset Y$ a nonempty subset. The topological interior, topological boundary and topological closure of S are denoted by $intS$, ∂S and clS, respectively.

A set $K \subset X$ is said to be convex if, for any $x_1, x_2 \in K, \lambda \in [0,1]$, we have $\lambda x_1 + (1-\lambda)x_2 \in K$.

A set C is called a cone if, for any $\lambda \geq 0, \lambda C \subset C$.

A set C is called a convex cone if $C + C \subset C$ and, for any $\lambda \geq 0, \lambda C \subset C$.

Let $B \subset C \backslash \{0\}$ be a subset. B is called a base of C if, for each $c \in C$, there exist $b \in B$ and $\lambda \geq 0$ such that $c = \lambda b$.

A convex cone C in Y is called pointed if

$$C \cap (-C) = \{0\}.$$

An ordering relation $\prec$ is said to be

(i) Reflexive if $x \prec x$;
(ii) Asymmetric if $x \prec y, y \prec x \Longrightarrow x = y$;

(iii) Transitive if $x \prec y, y \prec z \Longrightarrow x \prec z$.

An ordering relation is called a partial order if it satisfies reflexive, asymmetric and transitive conditions.

In principle, any nonempty subset C of Y can define an ordering relation by

$$y \leq_C z \iff z - y \in C, \quad \forall y, z \in Y.$$

However, only some particular subsets C of Y can define ordering relations with nice and useful properties. In this book, we restrict our attention to two cases: (i) C is a convex cone in Y and (ii) C is a convex subset of Y with $0 \in \partial C$. We emphasize that, throughout the book, we will discuss under case (i) unless explicitly stated otherwise.

If C is a convex cone in Y and C defines an ordering relation of Y, then C is called an ordering cone. If C is a pointed and convex cone, then the ordering relation $\leq_C$ is a partial order. If the interior $intC$ of C is nonempty, we can define a strict ordering relation "$\leq_{intC}$" in Y as follows: for any $y, z \in Y$,

$$y \leq_{intC} z \iff z - y \in intC.$$

Similarly, we can define an ordering relation "$\geq_C$" and a strict ordering relation "$\geq_{intC}$".

By (Y, C), we denote an ordered space with the ordering of Y defined by set C. Suppose that $intC \neq \emptyset$. We can define an ordering relation "$\not\leq_C$"and a strict ordering relation "$\not\leq_{intC}$" as follows: for any $y, z \in Y$

$$y \not\leq_C z \iff z - y \not\leq_C 0;$$

$$y \not\leq_{intC} z \iff z - y \notin intC.$$

Similarly, we can define an ordering relation "$\not\geq_C$" and a strict ordering relation "$\not\geq_{intC}$".

We also define the following ordering relations: for any $y, z \in Y$,

$$y \leq_{C\backslash\{0\}} z \iff z - y \in C\backslash\{0\},$$

$$y \not\leq_{C\backslash\{0\}} z \iff z - y \notin C\backslash\{0\}.$$

Given two subsets of Y, say A and B, the following ordering relationships on sets are defined:

$$\begin{aligned}
&A \leq_C B \iff \eta \leq_C \xi, \quad \forall \eta \in A, \ \xi \in B;\\
&A \leq_{intC} B \iff \eta \leq_{intC} \xi, \quad \forall \eta \in A, \ \xi \in B;\\
&A \leq_{C\backslash\{0\}} B \iff \eta \leq_{C\backslash\{0\}} \xi, \quad \forall \eta \in A, \ \xi \in B;\\
&A \not\leq_C B \iff \eta \not\leq_C \xi, \quad \forall \eta \in A, \ \xi \in B;\\
&A \not\leq_{intC} B \iff \eta \not\leq_{intC} \xi, \quad \forall \eta \in A, \ \xi \in B;\\
&A \not\leq_{C\backslash\{0\}} B \iff \eta \not\leq_{C\backslash\{0\}} \xi, \quad \forall \eta \in A, \xi \in B.
\end{aligned}$$

Let A and B be two sets. We denote by $A\backslash B$ the difference of A and B.

Lemma 1.1. *Let C be an ordering cone in Y. Then, for any $a, b, c \in Y$,*

(i) $a \geq_C b \implies a + c \geq_C b + c$;
(ii) $a \geq_{intC} b \implies a + c \geq_{intC} b + c$;
(iii) $a \geq_{C\backslash\{0\}} b \implies a + c \geq_{C\backslash\{0\}} b + c$;
(iv) $a \not\geq_C b \implies a + c \not\geq_C b + c$;
(v) $a \not\geq_{intC} b \implies a + c \not\geq_{intC} b + c$;
(vi) $a \not\geq_{C\backslash\{0\}} b \implies a + c \not\geq_{C\backslash\{0\}} b + c$.

The same is true for $\leq_C, \leq_{intC}, \leq_{C\backslash\{0\}}, \not\leq_C, \not\leq_{intC}$ and $\not\leq_{C\backslash\{0\}}$ respectively.

Lemma 1.2. *Let C be a convex ordering cone in Y. Then, for any $a, b, c \in Y$,*

(i) $a \leq_C b \leq_C c \implies a \leq_C c$;
(ii) $a \leq_C b \leq_{C\backslash\{0\}} c \implies a \leq_{C\backslash\{0\}} c$;
(iii) $a \leq_C b \leq_{intC} c \implies a \leq_{intC} c$;
(iv) $a \not\leq_{intC} b \geq_{intC} c \implies a \not\geq_{intC} c$;
(v) $a \not\leq_{intC} b \geq_C c \implies a \not\geq_{intC} c$;
(vi) $a \not\geq_{intC} b \leq_{intC} c \implies a \not\leq_{intC} c$;
(vii) $a \not\geq_{intC} b \leq_C c \implies a \not\leq_{intC} c$.

Let Y^* be the topological dual space of Y and C a convex cone of Y. Set

$$C^* = \{f \in Y^* : \langle f, x \rangle \geq 0, \forall x \in C\},$$

where $\langle f, x \rangle$ denotes the value of f at x. C^* is called the dual cone (or positive polar cone) of C. Sometimes, we also use C^+ to denote the dual cone of C.

We set

$$C^{+i} = \{f \in Y^* : \langle f, x \rangle > 0, \forall x \in C\backslash\{0\}\}.$$

Proposition 1.3. *[96] Let (Y, C) be an ordered Banach space with $C \subset Y$ being a convex cone. Consider the following properties that a convex cone $C \subset Y$ may possess:*

(i) *C is a pointed and convex cone;*
(ii) *C has a base;*
(iii) *$intC^* \neq \varnothing$.*

Then (iii) $\implies$ (ii) $\implies$ (i); if Y is some Euclidean space, and C is closed, then all three properties are equivalent.

Definition 1.4. *Let Y be a topological vector space ordered by a convex cone C in Y or a convex subset C of Y with $0 \in \partial C$. Let $A \subset Y$ be a nonempty set. A point $y^* \in A$ is called a minimal point of A if*

$$(A - y^*) \cap (-C\backslash\{0\}) = \emptyset;$$

A point $y^* \in A$ is called a maximal point of A if

$$(A - y^*) \cap (C\backslash\{0\}) = \emptyset.$$

We denote the set of all minimal points of A and the set of all maximal points of A by $\text{Min}_C A$ and $\text{Max}_C A$, respectively.

Definition 1.5. *Let Y be a topological vector space ordered by a convex cone C in Y. Let A be a nonempty subset of Y. A is said to have the lower (upper) domination property if, for each y, there is a point $y^* \in Min_C A$ (or $Max_C A$) such that $y \in y^* + C$ (or $y \in y^* - C$).*

Proposition 1.6. *Let Y be a topological vector space ordered by a closed and convex cone C in Y. If $A \subset Y$ is a nonempty compact set, then A has the lower (upper) domination property, hence $Min_C A \neq \varnothing$ ($Max_C A \neq \varnothing$).*

Thus, we obtain immediately that $A \subset \text{Min}_C A + C$ (or $A \subset \text{Max}_C A - C$).

Definition 1.7. *Let $C \subset Y$ be a convex cone or a convex subset of Y with $0 \in \partial C$ and $intC \neq \emptyset$, $A \subset Y$ be a nonempty subset. A point $y^* \in A$ is called a weakly minimal point of A if*

$$A \cap (y^* - intC) = \varnothing.$$

A point $y^* \in A$ is called a weakly maximal point of A if

$$A \cap (y^* + intC) = \varnothing.$$

We denote the set of all weakly minimal points of A and the set of all weakly maximal points of A by $\text{Min}_{intC} A$ and $\text{Max}_{intC} A$, respectively.

Definition 1.8. *Let Y be a topological vector space ordered by a convex cone C or a convex subset C with $0 \in \partial C$. Let $K \subset X$ and $f : K \to Y$ be a vector-valued function. $x^* \in K$ is said to be a minimal solution of f on K if*

$$(f(K) - f(x^*)) \cap (-C \backslash \{0\}) = \emptyset.$$

Suppose that $intC \neq \emptyset$. $x^* \in K$ is said to be a weakly minimal solution of f on K if

$$(f(K) - f(x^*)) \cap (-intC) = \emptyset.$$

We denote the set of all minimal solutions of f on K and the set of all weakly minimal solutions of f on K by $\text{Min}_C(f, K)$ and $\text{Min}_{intC}(f, K)$ respectively.

Definition 1.9. *Let Y be a topological vector space ordered by a convex cone C or a convex subset C with $0 \in \partial C$. Let $K \subset X$ and $f : K \to Y$ be a vector-valued function. $y^* \in K$ is said to be a minimal point of f on K if there is a $x^* \in Min_C(f, K)$ such that $y^* = f(x^*)$. $y^* \in K$ is said to be a weakly minimal point of f on K if there is a $x^* \in Min_{intC}(f, K)$ such that $y^* = f(x^*)$.*

We denote the set of all minimal points of f on K and the set of all weakly minimal points of f on K by $\text{Min}_C f(K)$ and $\text{Min}_{intC} f(K)$ respectively.

Definition 1.10. *Let Y be a topological vector space ordered by a convex cone C or a convex subset C with $0 \in \partial C$. Let $K \subset X$ and $f : K \longrightarrow Y$ be a vector-valued function. $x^* \in K$ is said to be a local minimal solution of f on K if there exists a neighborhood $U(x^*)$ of x^* such that*

$$(f(K \cap U(x^*)) - f(x^*)) \cap (-C \backslash \{0\}) = \emptyset.$$

Suppose that $intC \neq \emptyset$. $x^* \in K$ is said to be a local weakly minimal solution of f on K if there exists a neighborhood $U(x^*)$ of x^* such that

$$(f(K \cap U(x^*)) - f(x^*)) \cap (-intC) = \emptyset.$$

Let $C : Y \rightrightarrows Y$ be a set-valued function (i.e., for every $y \in Y$, $C(y)$ is a subset of Y) such that for each $y \in Y$, $C(y)$ is a convex cone or a convex set with $0 \in \partial C(y)$, for all $y \in Y$.

The set-valued function C or the family of sets $\{C(y) : y \in Y\}$ is called a domination structure on Y. The domination structure describes a variable ordering structure or a variable preference structure when dealing with minimal points of a set.

We define relations $\leq_{C(y)}$, $\not\leq_{C(y)}$, $\leq_{C(y)\backslash\{0\}}$, $\not\leq_{C(y)\backslash\{0\}}$, $\leq_{intC(y)}$, and $\not\leq_{intC(y)}$ with respect to the convex cone $C(y)$ as follows: for any $y_1, y_2 \in Y$,

$$y_1 \leq_{C(y)} y_2 \iff y_2 - y_1 \in C(y);$$

$$y_1 \not\leq_{C(y)} y_2 \iff y_2 - y_1 \notin C(y);$$

$$y_1 \leq_{C(y)\backslash\{0\}} y_2 \iff y_2 - y_1 \in C(y) \setminus \{0\};$$

$$y_1 \not\leq_{C(y)\backslash\{0\}} y_2 \iff y_2 - y_1 \notin C(y) \setminus \{0\};$$

$$y_1 \leq_{intC(y)} y_2 \iff y_2 - y_1 \in intC(y);$$

$$y_1 \not\leq_{intC(y)} y_2 \iff y_2 - y_1 \notin intC(y).$$

Similarly, we can define $\geq_{C(y)}$, $\not\geq_{C(y)}$, $\geq_{C(y)\backslash\{0\}}$, $\not\geq_{C(y)\backslash\{0\}}$, $\geq_{intC(y)}$, and $\not\geq_{intC(y)}$.

Yu [221] proposed the following solution concepts for vector optimization problems with a variable domination structure.

Definition 1.11. *Let $C : Y \rightrightarrows Y$ be convex cone valued. Let A be a nonempty subset of Y. A point $y^* \in A$ is called a nondominated minimal point of A if*

$$A \cap (y^* - C(y)) = \{y^*\}, \quad \forall y \in A.$$

We denote the set of all nondominated minimal points of A by $\mathrm{Min}_{C(y)}A$. It is clear that a nondominated minimal point of A is a minimal point of A with respect to $C(y)$ for every $y \in A$.

Definition 1.12. *Let A be a nonempty subset of Y. Let $C : Y \rightrightarrows Y$ be convex cone valued with $intC(y) \neq \emptyset, \forall y \in Y$. A point $y^* \in A$ is called a weakly nondominated minimal point of A if*

$$A \cap (y^* - intC(y)) = \varnothing, \quad \forall y \in A. \tag{1.1}$$

We denote the set of all weakly nondominated minimal points of A by $\text{Min}_{intC(y)}A$.

Using the ordering notation, (1.1) is equivalent to that, for any $y_1, y_2 \in A$, it follows that

$$y_1 \not\leq_{intC(y_2)} y^*.$$

In fact, Definitions 1.11 and 1.12 deal with a similar "minimal" case as in Definitions 1.4 and 1.7. By the same way, we can define a nondominated maximal point and a weakly nondominated maximal point of A similar to "maximal" in Definitions 1.4 and 1.7.

We propose the following alternative concepts of nondominated minimal points for vector optimization problems with a variable domination structure.

Definition 1.13. *Let $C : Y \rightrightarrows Y$ be convex set valued or convex cone valued, and int $C(y) \neq \emptyset, \forall y \in Y$. Let A be a nonempty subset in Y. A point $y^* \in A$ is called a nondominated-like minimal point of A, if*

$$(A - y^*) \cap (-C(y^*) \backslash \{0\}) = \emptyset.$$

A point y^* is called to be a weakly nondominated-like minimal point of A, if

$$(A - y^*) \cap (-intC(y^*)) = \varnothing.$$

We denote the set of all nondominated-like minimal points of A and the set of all weakly nondominated-like minimal points of A by $\text{LMin}_{C(y)}A$ and $\text{LMin}_{intC(y)}A$, respectively.

The following example shows that the two definitions of weakly nondominated minimal points given in Definitions 1.12 and 1.13 may be different.

Example 1.14. Let $Y = \mathbb{R}^2$ be a 2-dimensional Euclidean space, and $A = \{(y_1, y_2)^\top \in \mathbb{R}^2 : 1 \leq y_1 \leq 2, y_2 = 1\}$. Let

$$C(y) = \{(d_1, d_2)^\top \in \mathbb{R}^2 : d_2 + kd_1 \geq 0, d_1 \geq 0\},$$

where $y = (2-k, 1)^\top,\ 0 \leq k \leq 1$. It is easy to verify that only $y^1 = (1,1)^\top$ is a weakly nondominated minimal point of A. But, by definition, both $y_1 = (1,1)^\top$ and $y_2 = (2,1)^\top$ are weakly nondominated-like minimal points of A.

Let $C : X \rightrightarrows Y$ be a set-valued function such that for each $x \in Y$, $C(x)$ is a nonempty convex cone or a nonempty convex set with $0 \in \partial C(x)$, for all $x \in X$.

The set-valued function C or the family of sets $\{C(x) : x \in X\}$ is also called a domination structure on Y. The domination structure describes a variable ordering structure or a variable preference structure in vector optimization problems with an objective function.

We define relations $\leq_{C(x)}$, $\not\leq_{C(x)}$, $\leq_{C(x)\setminus\{0\}}$, $\not\leq_{C(x)\setminus\{0\}}$, $\leq_{intC(x)}$, and $\not\leq_{intC(x)}$ with respect to the convex cone $C(x)$ as follows: for any $y_1, y_2 \in Y$,

$$y_1 \leq_{C(x)} y_2 \iff y_2 - y_1 \in C(x);$$

$$y_1 \not\leq_{C(x)} y_2 \iff y_2 - y_1 \notin C(x);$$

$$y_1 \leq_{C(x)\setminus\{0\}} y_2 \iff y_2 - y_1 \in C(x) \setminus \{0\};$$

$$y_1 \not\leq_{C(x)\setminus\{0\}} y_2 \iff y_2 - y_1 \notin C(x) \setminus \{0\};$$

$$y_1 \leq_{intC(x)} y_2 \iff y_2 - y_1 \in intC(x);$$

$$y_1 \not\leq_{intC(x)} y_2 \iff y_2 - y_1 \notin intC(x).$$

Similarly, we can define $\geq_{C(x)}$, $\not\geq_{C(x)}$, $\geq_{C(x)\setminus\{0\}}$, $\not\geq_{C(x)\setminus\{0\}}$, $\geq_{intC(x)}$, and $\not\geq_{intC(x)}$.

Definition 1.15. *Let $C : X \rightrightarrows Y$ be convex set valued with $0 \in \partial C(x), \forall x \in X$ or convex cone valued. Suppose that $K \subset X$ and $f : K \to Y$ is a vector-valued function. $x^* \in K$ is said to be a nondominated-like minimal solution of f with respect to $C(x)$ if*

$$(f(K) - f(x^*)) \cap (-C(x^*)\setminus\{0\}) = \emptyset.$$

The set of all nondominated-like minimal solutions of f with respect to $C(x)$ is denoted by $LMin_{C(x))} f(K)$.

Suppose that $intC(x) \neq \emptyset, \forall x \in X$. $x^* \in K$ is said to be a weakly nondominated-like minimal solution of f with respect to $C(x)$ if

$$(f(K) - f(x^*)) \cap (-intC(x^*)) = \emptyset.$$

The set of all weakly nondominated-like minimal solutions of f with respect to $C(x)$ is denoted by $LMin_{intC(x)} f(K)$.

Definition 1.16. *Let (Y, C) be an ordered Hausdorff topological vector space and $A \subset Y$. A point $z \in A$ is called an infimum point of A if,*

(i) $y \not\leq_{C\setminus\{0\}} z$, $\forall y \in A$ and

(ii) there exists a sequence $\{z_k\} \subset A$ such that $z_k \to z$ as $k \to \infty$.

We denote by $InfA$ the set of infimum points of A.

A point $z \in A$ is called a supremum point of A if,

(i) $y \not\geq_{C\setminus\{0\}} z$, $\forall y \in A$ and

(ii) there exists a sequence $\{z_k\} \subset A$ such that $z_k \to z$ as $k \to \infty$.

We denote by $SupA$ the set of supremum points of A.

Clearly, if z is a minimal point of A, then z is an infimum point of A.

Definition 1.17 ([177]). *Let (Y, C) be an ordered vector space, and $A \subset Y$ be nonempty. $a_0 \in A$ is called an upper bound of A if $a_0 \geq_C a, \forall a \in A$. If a_0 is an upper bound of A and $a_0 \leq_C b$ for any upper bound b of A, then a_0 is unique and called the absolute supremum (least upper bound) of A. We denote $a_0 = ASup_C A$. Similarly, we can define the absolute infimum (largest lower bound) of A and denote it by $AInf_C A$.*

Definition 1.18 (Luc [142]).

(i) *The cone C is called Daniell if any decreasing sequence having a lower bound converges to its infimum;*
(ii) *A subset A of Y is said to be minorized, if there is a $y \in Y$ such that $A \subset \{y\} + C$.*

Consider the scalar optimization problem:

$$(\text{P}) \quad \min_{x \in K} \varphi(x),$$

where $K \subset X$ is a nonempty set and $\varphi : X \to \mathbb{R}$ is a real-valued function.

(i) $x^* \in K$ is called an optimal solution of (P) if

$$\varphi(x^*) \leq \varphi(x), \quad \forall x \in K.$$

(ii) $x^* \in K$ is called a local optimal solution of (P) if there exists a neighborhood $U(x^*)$ of x^* such that

$$\varphi(x^*) \leq \varphi(x), \quad \forall x \in K \cap U(x^*).$$

1.2 Elements of Set-Valued Analysis

In this section, we present necessary concepts and results in set-valued analysis. More detailed investigation of set-valued analysis can be found in Aubin and Frankowska [6] and Aubin and Ekeland [5]. Some particular concepts and results of set-valued analysis are presented in the following context.

Let X, Y be two Hausdorff topological spaces and $F : X \rightrightarrows Y$ a set-valued function.

Definition 1.19. *F is said to be closed if its graph*

$$Gr(F) = \{(x, y) : x \in X, y \in F(x)\}$$

is closed.

Definition 1.20. **(i)** *F is said to be upper semicontinuous (u.s.c. in short) at $x_0 \in X$ if, for any neighborhood $V(F(x_0))$ of the set $F(x_0)$, there exists a neighborhood $U(x_0)$ of the point x_0 such that*

$$F(x) \subset V(F(x_0)), \quad \forall x \in U(x_0).$$

(ii) *F is said to be lower semicontinuous (l.s.c., in short) at $x_0 \in X$ if, for any $y \in F(x_0)$ and any neighborhood $V(y_0)$ of y_0, there exists a neighborhood $U(x_0)$ of the point x_0 such that*

$$F(x) \cap V(y_0) \neq \varnothing, \quad \forall x \in U(x_0).$$

(iii) *F is said to be continuous at x_0 if F is both u.s.c. and l.s.c. at x_0.*
(iv) *F is said to be continuous on X if it is continuous at every $x \in X$.*

Proposition 1.21. *[5] Let X be a topological space and Y a locally convex topological vector space. Suppose that $F : X \rightrightarrows Y$ is a set-valued function which is u.s.c., nonempty and closed-valued. Then F is closed.*

Definition 1.22. *A set-valued function $F : X \rightrightarrows Y$ is said to have open lower sections if the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for every $y \in Y$.*

Proposition 1.23 (Tian [192]). *Let X be a topological space and Y a convex set of a topological vector space. Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ be set-valued functions with open lower sections. Then*

(i) *the set-valued function $M : X \rightrightarrows Y$, defined by $M(x) = co(G(x))$ for all $x \in X$, has open lower sections;*
(ii) *the set-valued function $Q : X \rightrightarrows Y$, defined by $Q(x) = G(x) \cap F(x)$ for all $x \in X$, has open lower sections.*

Definition 1.24. *Let $F : Y \rightrightarrows Y$ be a set-valued function.*

(i) The vector-valued function $e : Y \to Y$ is said to be a selection of F if $e(y) \in F(y)$, for every $y \in Y$.

(ii) $e : Y \to Y$ is said to be a continuous selection of F if e is a selection of F and e is continuous on Y.

Theorem 1.25 (Generalized Browder Selection Theorem). *Let K be a nonempty compact subset of a Hausdorff topological vector space, and let V be a subset of a topological vector space. Suppose that $H : K \rightrightarrows V$ is a set-valued function with nonempty convex values and has open lower sections. Then there exists a continuous selection $h : K \to V$ of H. Moreover, $h(K)$ is contained in the convex hull of a finite subset $M \subset V$.*

Proof. For each $v \in V$, $H^{-1}(v)$ is open, and each point $x \in K$ lies in at least one of these open subsets. Since K is compact, there exists a finite set $M = \{v_1, \cdots, v_k\} \subset V$ such that $K = \cup_{i=1}^{k} H^{-1}(v_i)$. Let $\{\beta_1, \cdots, \beta_k\}$ be a partition of unit subordinated to this covering, i.e., each β_i is a continuous function from K to $[0,1]$, which vanishes outside of $H^{-1}(v_i)$, while $\sum_{i=1}^{k} \beta_i(x) = 1$ for all x in K.

Now, we define the continuous function $h : K \to co(M)$ by $h(x) := \sum_{i=1}^{k} \beta_i(x) v_i$. Clearly, $\beta_i(x) > 0$ implies that $x \in H^{-1}(v_i)$ and therefore $v_i \in H(x)$. Thus $h(x)$ is a convex linear combination of points of $H(x)$. Since $H(x)$ is assumed to be convex for each $x \in K$, it follows that $h(x) \in H(x)$. The theorem is proved. ■

Theorem 1.26 (Browder Fixed Point Theorem). *Let K be a nonempty, compact and convex subset of a Hausdorff topological vector space. Suppose $F : K \rightrightarrows K$ is a set-valued function with nonempty convex values and open lower sections. Then F has a fixed point in K.*

Theorem 1.27 (Fan-Glicksber-Kakutani). *Let K be a nonempty compact subset of a real locally convex Hausdorff vector topological space. If $F : K \rightrightarrows K$ is upper semi-continuous and, for any $x \in K$, $F(x)$ is a nonempty, convex and closed subset, then F has a fixed point in K.*

Definition 1.28. *A nonempty topological space is said to be acyclic if all of its reduced Čech homology groups over the rational vanish.*

In particular, any contractible space is acyclic, and hence any convex or star-shaped set in a topological vector space is acyclic.

Theorem 1.29. *[159] Let K be a compact convex subset of a locally convex Hausdorff topological vector space, and $F : K \rightrightarrows K$ be an upper semicontinuous set-valued function with nonempty, closed and acyclic values. Then F has a fixed point in K.*

Now, we introduce the concepts of the contingent tangent cone of a set and the contingent derivative of a set-valued function.

Let X and Y be two topological vector spaces, and K a nonempty subset of X.

Definition 1.30. *Let $\bar{x} \in K$. The set $T(K, \bar{x}) \subset X$ is called a contingent tangent cone to K at $\bar{x}$ if*

$$T(K,\bar{x}) = \{x \in X : \exists \{x_k\} \subset X \text{ and } \{h_k\} \subset \mathbb{R}_+ \backslash \{0\}, \\ s.t.\ x_k \to x, h_k \to 0 \text{ and } \bar{x} + h_k x_k \in K, \forall k\}.$$

We know that (i) if X is a normed space, then $T(K, \bar{x})$ is closed and (ii) if K is a convex set, then $T(K, \bar{x})$ is also convex.

Obviously, if $(X, ||\cdot||)$ is a normed space, then

$$T(K,\bar{x}) = \cap_{\epsilon>0} \cap_{\alpha>0} \cap_{0<h<\alpha} (((K - \bar{x})/h) + \epsilon B),$$

where $B = \{x \in X : ||x|| = 1\}$.

Definition 1.31. *[6] Let $G : X \rightrightarrows Y$ be a set-valued function, and let $(\bar{x}, \bar{y})$ be a point of $Gr(G)$. We denote by $DG(\bar{x}, \bar{y})$ the set-valued function from X to Y whose graph is the contingent tangent cone $T(Gr(G), (\bar{x}, \bar{y})) \subset X \times Y$. $DG(\bar{x}, \bar{y})$ is called the contingent derivative of G at $(\bar{x}, \bar{y})$.*

It is useful to note that $y \in DG(\bar{x}, \bar{y})(x)$ if and only if there exist $\{h_k\} \subset \mathbb{R}_+ \setminus \{0\}$ and $\{(x_k, y_k)\} \subset X \times Y$, such that $h_k \to 0$, $(x_k, y_k) \to (x, y)$ and $\bar{y} + h_k y_k \in G(\bar{x} + h_k x_k)$ for all n.

Let X and Y be two Banach spaces. We denote by $L(X, Y)$ the set of all linear continuous operators from X to Y. The value of a linear operator $f : X \to Y$ at a point x is denoted by $\langle f, x \rangle$. For any $A \in L(X, Y)$, we introduce a norm

$$||A||_L = \sup\{||A(x)|| : ||x|| \leq 1\}.$$

Since Y is a Banach space, $L(X, Y)$ is also a Banach space with the norm $|| \cdot ||_L$ (or $\| \cdot \|$ in short).

Definition 1.32. *Let $f : K \subset X \to L(X, Y)$ be a vector-valued function. f is said to be Fréchet differentiable at $x_0 \in K$ if there exists a linear continuous operator $\Phi : X \to L(X, Y)$, such that*

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - \Phi(x - x_0)||}{||x - x_0||} = 0.$$

Φ is called the Fréchet derivative of f at x_0. If f is Fréchet differentiable at every x of K, f is said to be Fréchet differentiable on K.

Definition 1.33. *Let $f : K \subset X \to Y$ be a vector-valued function. f is said to be Gâteaux differentiable at $x_0 \in K$ if there exists a linear function $Df(x_0) : X \to Y$ such that, for any $v \in X$,*

$$\langle Df(x_0), v \rangle = \lim_{t \searrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

$Df(x_0)$ is called the Gâteaux derivative of f at x_0. If f is Gâteaux differentiable at every x of K, f is said to be Gâteaux differentiable on K.

Theorem 1.34 (Knaster, Kuratowski and Mazurkiewicz (KKM, in short) Theorem). *Let E be a subset of a topological vector space V. For each $x \in E$, let a closed and convex set $F(x)$ in V be given such that $F(x)$ is compact for at least one $x \in E$. If the convex hull of every finite subset $\{x_1, x_2, \cdots, x_k\}$ of E is contained in the corresponding union $\cup_{i=1}^{n} F(x_i)$, then $\cap_{x \in E} F(x) \neq \varnothing$.*

A set-valued $F : E \rightrightarrows E$ function is called a KKM map if we have $co\{x_1, \cdots, x_k\} \subset \cup_{i=1}^{k} F(x_i)$ for every finite subset $\{x_1, \cdots, x_k\}$ of E.

Definition 1.35. *Let T be a mapping from X into $L(X, Y)$. T is called v-hemicontinuous if, for every $x, y \in X$, the mapping $t \to \langle T(x + ty), y \rangle$ is continuous at 0^+.*

Definition 1.36. *Let X and Y be topological vector spaces, $C \subset Y$ be a nonempty convex cone with $intC \neq \varnothing$ and $C \neq \{0\}$ or Y. Let $T : X \to L(X, Y)$ be a mapping.*

(i) *T is called C-monotone, if, for every $x, y \in X$,*

$$\langle T(x) - T(y), x - y \rangle \geq_C 0;$$

(ii) *T is called strictly C-monotone, if, for every x, $y \in X$ and $x \neq y$,*

$$\langle T(x) - T(y), x - y \rangle \geq_{intC} 0.$$

Definition 1.37. *Let X and Y be topological vector spaces and $C \subset Y$ be a convex cone. The set-valued function $T : X \rightrightarrows L(X, Y)$ is said to be C-monotone if and only if*

$$\langle u_2 - u_1, y - x \rangle \geq_C 0, \quad \forall x, y \in X, u_1 \in F(x), u_2 \in F(y).$$

It is clear that any selection of a C-monotone set-valued function is also C-monotone.

Definition 1.38. *Let X and Y be Banach spaces, $C \subset Y$ be a convex cone with nonempty interior $intC$ and $intC^* \neq \varnothing$. Let K be a convex and unbounded subset of X. We say that a mapping $T : K \to L(X, Y)$ is weakly coercive on K if there exist $x_0 \in K$ and $c \in intC^*$ such that*

$$\langle c \circ T(x) - c \circ T(x_0), x - x_0 \rangle / ||x - x_0|| \to +\infty,$$

whenever $x \in K$ and $||x|| \to +\infty$.

It is easy to see that if $Y = \mathbb{R}$, then $L(X, Y) = X^*$, $intC^* = \mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$, and the weakly coercive condition reduces a standard coercive condition in "scalar" variational inequality.

1.3 Nonlinear Scalarization Functions

A useful approach for analyzing a vector optimization problem is to reduce it to a scalar optimization problem. Nonlinear scalarization functions play an important role in this reduction in the context of nonconvex vector optimization problems.

Let Y be a Hausdorff topological vector space, $C \subset Y$ a closed and convex cone of Y with nonempty interior $intC$.

Definition 1.39. *A function $\psi : Y \to I\!R$ is monotone if, for any $y_1, y_2 \in Y$,*

$$y_1 \geq_C y_2 \Longrightarrow \psi(y_1) \geq \psi(y_2).$$

A function $\psi : Y \to I\!R$ is strictly monotone if, for any $y_1, y_2 \in Y$,

$$y_1 \geq_{intC} y_2 \Longrightarrow \psi(y_1) > \psi(y_2).$$

A function $\psi : Y \to I\!R$ is strongly monotone if, for any $y_1, y_2 \in Y$,

$$y_1 \geq_{C\setminus\{0\}} y_2 \Longrightarrow \psi(y_1) > \psi(y_2).$$

The following nonlinear scalarization function is of fundamental importance to our analysis. The original version is due to Gerstewitz [77]. Its first appearance in English seems to be due to Luc [142].

Definition 1.40. *Given a fixed $e \in intC$ and $a \in Y$, the nonlinear scalarization function is defined by:*

$$\xi_{ea}(y) = \min\{t \in I\!R : y \in a + te - C\}, \quad y \in Y. \tag{1.2}$$

Proposition 1.41. *The function ξ_{ea} is well-defined, that is, the minimum in (1.2) is attained.*

Proof. For any $y \in Y$, define

$$L = \{\ \lambda \in \mathbb{R} :\ y \in \lambda e - C\}.$$

It is sufficient to show that L is bounded from below and a closed subset in $\mathbb{R}$.

Suppose that

$$\{\lambda_k\} \subset L \text{ and } \lambda_k \to \lambda^*, \qquad \text{as } k \to +\infty.$$

We have

$$\lambda_k e - y \in C, \quad \forall k.$$

By the closedness of C, we have

$$\lambda^* e - y \in C.$$

It implies that $\lambda^* \in L$. Thus, L is closed.

Assume that, for each $r \in \mathbb{R}$, there exist $\lambda_r \in \mathbb{R}$ such that $\lambda_r < r$ and $y \in \lambda_r e - C$. By Lemma 1.51 (ii), there exists $\alpha \in \mathbb{R}$ such that $y \notin \alpha e - C$. By Lemma 1.51(iii),

$$y \notin \mu e - C, \quad \forall \mu < \alpha,$$

a contradiction. Thus, L is bounded from below. ∎

If Y is the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, and $C = \mathbb{R}^\ell_+$, $e = (e_1, e_2, \cdots, e_\ell)^\top$, $a = (a_1, a_2, \cdots, a_\ell)^\top$, then the function ξ_{ea} may be rewritten as

$$\xi_{ea}(y) = \max\{(y_i - a_i)/e_i : 1 \le i \le \ell\}, \text{ for } y = (y_1, y_2, \cdots, y_\ell)^\top.$$

It can be verified that ξ_{ea} is a continuous and convex function on Y, and it is monotone and strictly monotone.

Remark 1.42. The function ξ_{ea} plays an important role in many areas of multicriteria, or vector optimization problems. Note, however, that the function ξ_{ea} is not strongly monotone. It is for this reason that the function ξ_{ea} is more useful in dealing with weakly minimal points.

Proposition 1.43. *For any fixed $e \in intC$, $y \in Y$ and $r \in \mathbb{R}$, we have*

(i) $\xi_{e0}(y) < r \iff y \in re - intC$;
(ii) $\xi_{e0}(y) \le r \iff y \in re - C$;
(iii) $\xi_{e0}(y) = r \iff y \in re - \partial C$;
(iv) $\xi_{e0}(re) = r$.

Proof. Follows directly from Definition 1.40 of ξ_{ea}. ∎

Sometimes, we denote ξ_{e0} by ξ_e.

Proposition 1.44. *Let $C = \{y \in Y : f(y) \le 0, f \in \Gamma\}$, where $\Gamma \subset Y^* \backslash \{0\}$. Assume that $intC \ne \varnothing$. Let $e \in intC$ and $a \in Y$. Then, for $y \in Y$,*

$$\xi_{ea}(y) = \sup_{f \in \Gamma} \left\{ \frac{f(y) - f(a)}{f(e)} \right\}.$$

Proof. Firstly, we prove that, for all $f \in \Gamma$, $f(e) < 0$. Assume to the contrary, i.e., there exists $f_0 \in \Gamma$ such that $f_0(e) \ge 0$. Since $f_0 \ne 0$ and f_0 is a linear functional, there exists an $y_0 \in Y$ such that $f_0(y_0) < 0$. Observe that $e \in intC$. Thus, if $\alpha > 0$ is small enough, we have $e - \alpha y_0 \in C$. It follows from the definition of C that

$$0 \ge f_0(e - \alpha y_0) = f_0(e) - \alpha f_0(y_0) > 0,$$

a contradiction.

Furthermore, since $y \in a + \xi_{ea}(y)e - C$,

$$f(y - \xi_{ea}(y)e - a) \ge 0, \quad \forall f \in \Gamma.$$

Since f is linear,

$$f(y) - \xi_{ea}(y)f(e) - f(a) \ge 0.$$

As $f(e) < 0$, we have

$$\xi_{ea}(y) \geq \frac{f(y) - f(a)}{f(e)}, \quad \forall f \in \Gamma.$$

Consequently,

$$\xi_{ea}(y) \geq \sup_{f \in \Gamma} \left\{ \frac{f(y) - f(a)}{f(e)} \right\}.$$

Conversely, let

$$t_0 = \sup_{f \in \Gamma} \left\{ \frac{f(y) - f(a)}{f(e)} \right\}.$$

Then

$$\frac{f(y) - f(a)}{f(e)} \leq t_0, \quad \forall f \in \Gamma.$$

Observing that $f(e) < 0$ and that f is linear, we have

$$f(y - a - t_0 e) \geq 0, \quad \forall f \in \Gamma,$$

which implies that $y - a - t_0 e \leq_C 0$ by the definition of C. By the definition of ξ_{ea}, we have

$$t_0 \geq \xi_{ea}(y) = \min\{t \in \mathbb{R} : y \in a + te - C\}. \qquad \blacksquare$$

Corollary 1.45. *Let* $C := \{y \in Y : f_i(y) \leq 0, f_i \in Y^*, i = 1, 2, \cdots, m\}$. *Then*

$$\xi_{ea}(y) = \max_{1 \leq i \leq m} \left\{ \frac{f_i(y) - f_i(a)}{f_i(e)} \right\}, \quad \forall y \in Y,$$

$$\xi_{e0}(y) = \max_{1 \leq i \leq m} \left\{ \frac{f_i(y)}{f_i(e)} \right\}, \quad \forall y \in Y.$$

Corollary 1.46. *Let* $Y = \mathbb{R}^\ell$ *and* $C = \mathbb{R}^\ell_+$, $e = (1, 1, \cdots, 1)^\top \in \mathbb{R}^\ell$. *Then, for any* $a \in \mathbb{R}^\ell$, $y \in \mathbb{R}^\ell$,

$$\xi_{ea}(y) = \max_{1 \leq i \leq \ell} [y_i - a_i],$$

$$\xi_{e0}(y) = \max_{1 \leq i \leq \ell} [y_i],$$

Proof. In Corollary 1.45, let $m = \ell$ and $f_i(y) = -y_i, i = 1, 2, \cdots, \ell$. Thus $C = \{y \in Y : f_i(y) \leq 0, f_i \in Y^* \backslash \{0\}, i = 1, 2, \cdots, \ell\}$. Then the conclusion follows directly from Corollary 1.45. $\blacksquare$

Proposition 1.47. *For* $e \in intC$, $a \in Y$ *and* $b \in -C$,

$$\xi_{ea}(y - b) \geq \xi_{ea}(y),$$

and the equality holds for $b \in C \cap (-C)$.

Proof. The conclusions follow directly from the monotonicity of ξ_{ea}. ■

Now we introduce a nonlinear scalarization function for a variable domination structure.

Let Y be a locally convex Hausdorff topological vector space. Let $C : Y \rightrightarrows Y$ be a set-valued function and, for any $y \in Y$, $C(y)$ be a proper, closed and convex cone with $intC(y) \neq \varnothing$ and $e : Y \to Y$ be a vector-valued function and for any $y \in X$, $e(y) \in intC(y)$. Let Y^* be the dual space of Y, equipped with weak star topology. Let $C^* : Y \rightrightarrows Y^*$ be defined by

$$C^*(y) = \{\phi \in X^* : \langle \phi, z \rangle \geq 0, \quad \forall z \in C(y)\}, \quad \forall y \in Y.$$

Thus, the set

$$B^*(y) = \{\phi \in C^*(y) : \langle \phi, e(y) \rangle = 1\}$$

is a weak star compact base of the cone $C^*(y)$.

Definition 1.48. *The nonlinear scalarization function $\xi : Y \times Y \to I\!R$ is defined by*

$$\xi(y, z) = \min \{\lambda \in I\!R : \ z \in \ \lambda e(y) - C(y)\}, \quad (y, z) \in Y \times Y.$$

Remark 1.49. **(i)** Let C be a proper, closed and convex cone in Y with $intC \neq \varnothing$, and let $e \in intC$. Recall that in Definition 1.40

$$\xi_{e0}(z) = \min\{t \in \mathbb{R} : z \in te - C\}, \quad z \in Y.$$

If, for any $y \in Y$, $C(y) = C$ and $e(y) = e$ in Definition 1.48, then $\xi(y, z)$ reduces to $\xi_{e0}(z)$.

(ii) Let $e \in int \bigcap_{y \in Y} C(y) \neq \varnothing$. A nonlinear scalarization function in [42] is defined as

$$\xi_e(y, z) = \inf\{t \in \mathbb{R} : z \in te - C(y)\}. \tag{1.3}$$

We note that if for any $y \in Y$, $e(y) = e$, the function $\xi(y, z)$ reduces to $\xi_e(y, z)$. In the new definition of $\xi(y, z)$ (Definition 1.48), the assumption $int \bigcap_{y \in Y} C(y) \neq \varnothing$ is removed.

Lemma 1.50. *[78] For each $y \in Y$,*

$$Y = \cup\{\lambda e(y) - intC(y) : \ \lambda \in I\!R^+ \backslash \{0\}\}.$$

Lemma 1.51. *For $\lambda \in I\!R$ and $y \in Y$, we set $C_\lambda(y) = \lambda e(y) - \ C(y)$.*

(i) *If $z \in C_\lambda(y)$ holds for some $\lambda \in I\!R$, and $y \in Y$, then*

$$z \in \mu e(y) - intC(y), \quad \textit{for each } \mu > \lambda;$$

moreover,

$$z \in \mu e(y) - C(y), \quad \textit{for each } \mu > \lambda.$$

(ii) *For each $y, z \in Y$, there exists a real number $\lambda \in I\!R$ such that $z \notin C_\lambda(y)$.*
(iii) *Let $z \in Y$. If $z \notin C_\lambda(y)$ for some $\lambda \in I\!R$, and $y \in Y$, then*

$$z \notin C_\mu(y), \text{ for each } \mu < \lambda.$$

Proof. (i) Let $\mu > \lambda$ and let $z \in C_\lambda(y)$ hold for some $y \in Y$. We have

$$\mu e(y) - z = (\mu - \lambda)e(y) + \lambda e(y) - z \in intC(y) + C(y) \subset intC(y).$$

Thus,

$$z \in \mu e(y) - intC(y) \subset \mu e(y) - C(y).$$

(ii) Let us assume that there exist $y_0, z_0 \in Y$ such that, for all $\lambda \in \mathbb{R}$, $z_0 \in C_\lambda(y_0)$. From (i), we have

$$z_0 \in \lambda e(y_0) - intC(y_0), \quad \text{for all } \lambda \in \mathbb{R}.$$

Thus,

$$\{\lambda e(y_0) - z_0 : \lambda \in \mathbb{R}\} \subset intC(y_0);$$

equivalently,

$$\{-\lambda e(y_0) - z_0 : \lambda \in \mathbb{R}\} \subset intC(y_0).$$

From Lemma 1.50, we have

$$Y = \{\lambda e(y_0) - intC(y_0) : \lambda \in \mathbb{R}^+ \backslash \{0\}\}.$$

Therefore, for each $y \in Y$, there exist $c \in intC(y_0)$ and $\alpha \in \mathbb{R}^+ \backslash \{0\}$ such that

$$-y = \alpha e(y_0) - c;$$

then,

$$\begin{aligned} y &= -\alpha e(y_0) + c \\ &= (-\alpha e(y_0) - z_0) + c + z_0 \\ &\in intC(y_0) + intC(y_0) + z_0 \\ &= z_0 + intC(y_0). \end{aligned}$$

Thus

$$Y \subset z_0 + intC(y_0).$$

This contradicts $C(y_0) \neq Y$.

(iii) Let

$$z \notin C_\lambda(y), \quad \text{for some } \lambda \in \mathbb{R} \text{ and } y \in Y.$$

Suppose that, for some $\mu < \lambda$, $z \in C_\mu(y)$. From (ii), we have that $z \in C_\lambda(y)$. This contradicts the assumption. ∎

Proposition 1.52. *The function $\xi : Y \times Y \to I\!R$ is well defined.*

Proof. For any $y, z \in Y$, define

$$L = \{ \lambda \in \mathbb{R} : \ z \in \lambda e(y) - C(y)\}.$$

It is sufficient to show that L is bounded from below and a closed subset in $\mathbb{R}$.

Suppose that

$$\{\lambda_k\} \subset L \text{ and } \lambda_k \to \lambda^*, \qquad \text{as } k \to +\infty.$$

We have

$$\lambda_k e(y_0) - z \in C(y), \quad \forall n.$$

By the closedness of $C(y)$, we have

$$\lambda^* e(y) - z \in C(y).$$

It implies that $\lambda^* \in L$. Thus, L is closed.

Assume that, for each $r \in \mathbb{R}$, there exist $\lambda_r \in \mathbb{R}$ such that $\lambda_r < r$ and $z \in \lambda_r e(y) - C(y)$. By Lemma 1.51 (ii), there exists $\alpha \in \mathbb{R}$ such that $z \notin \alpha e(y_0) - C(y)$. By Lemma 1.51(iii),

$$z \notin \mu e(y) - C(y), \quad \forall \mu < \alpha,$$

a contradiction. Thus, L is bounded from below. ■

Proposition 1.53. *For any $(y, z) \in Y \times Y$,*

$$\xi(y, z) = \max_{\phi \in B^*(y)} \frac{\langle \phi, \ z \rangle}{\langle \phi, e(y) \rangle},$$

where $B^*(y)$ is a base of $C^*(y)$.

Proof. We show firstly,

$$\xi(y, z) = \sup\nolimits_{\phi \in C^*(y) \setminus \{0\}} \frac{\langle \phi, \ z \rangle}{\langle \phi, e(y) \rangle}.$$

Since $\xi(y, z) = \min\{\lambda \in \mathbb{R} : z \in \lambda e(y) - C(y)\}$, $z \in \xi(y, z)e(y) - C(y)$, equivalently,

$$\xi(y, z)e(y) - z \in C(y).$$

For any $\phi \in C^*(y) \setminus \{0\} \subset C^*(y)$, we have $\langle \phi, \xi(y, z)e(y) - z \ \rangle \geq 0$, equivalently,

$$\xi(y, z)\langle \phi, \ e(y) \rangle - \langle \phi, \ z \rangle \geq 0.$$

Because $e(y) \in intC(y)$ and $\phi \in C^*(y)\backslash\{0\}$, then, we have $\langle \phi, e(y)\rangle > 0$. So $\xi(y,z) \geq \frac{\langle \phi, y\rangle}{\langle \phi, e(y)\rangle}$. That is to say,

$$\xi(y,z) \geq \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}.$$

On the other hand, let

$$\lambda_0 = \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}.$$

So, for any $\phi \in C^*\backslash\{0\}$, $\lambda_0 \geq \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}$. Since $\langle \phi, e(y)\rangle > 0$, $\lambda_0\langle \phi, e(y) - z\rangle \geq 0$. Then, $\lambda_0 e(y) - z \in C(y)$, i.e. $z \in \lambda_0 e(y) - C(y)$. From the definition of ξ, $\lambda_0 \geq \xi(y,z) = \min\{\lambda \in \mathbb{R} : z \in \lambda e(y) - C(y)\}$, i.e.,

$$\xi(y,z) \leq \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}.$$

So we have

$$\xi(y,z) = \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}.$$

Since $B^*(y)$ is the base of $C^*(y)$ for any $y \in Y$, $\phi \in C^*(y)\backslash\{0\}$, there are $\lambda > 0$, and $\varphi \in B^*(y)$ such that $\phi = \lambda\varphi$. So for any $y \in Y$,

$$\frac{\langle \phi, y\rangle}{\langle \phi, e(y)\rangle} = \frac{\langle \lambda\varphi, y\rangle}{\langle \lambda\varphi, e(y)\rangle} = \frac{\langle \varphi, y\rangle}{\langle \varphi, e(y)\rangle}.$$

So we have

$$\sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle} = \sup_{\phi \in B^*(y)} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}.$$

i.e.

$$\xi(y,z) = \sup_{\phi \in B^*(y)} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}.$$

Since $B^*(y)$ is weak star compact, $\xi(y,z) = \max_{\phi \in B^*(y)} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle}$. ■

Proposition 1.54. *For each $r \in \mathbb{R}$ and $y, z \in Y$, the following statements are true.*

(i) $\xi(y,z) < r \Longleftrightarrow z \in re(y) - intC(y)$.
(ii) $\xi(y,z) \leq r \Longleftrightarrow z \in re(y) - C(y)$.
(iii) $\xi(y,z) \geq r \Longleftrightarrow z \notin re(y) - intC(y)$.
(iv) $\xi(y,z) > r \Longleftrightarrow z \notin re(y) - C(y)$.

(v) $\xi(y,z)=r \Longleftrightarrow z \in re(y)-\partial C(y)$.

Proof. We only prove (i). The proofs for other assertions are similar and omitted. Indeed,

$$\begin{aligned}
\xi(y,z)<r &\Longleftrightarrow \max_{\phi\in B^*(y)} \frac{\langle\phi,z\rangle}{\langle\phi,e(y)\rangle} < r \\
&\Longleftrightarrow \langle\phi,z\rangle < r\langle\phi,e(y)\rangle, \forall\phi\in B^*(y) \\
&\Longleftrightarrow \langle\phi,re(y)-z\rangle > 0, \forall\phi\in B^*(y) \\
&\Longleftrightarrow \langle\phi,re(y)-z\rangle > 0, \forall\phi\in C^*(y)\backslash\{0\} \\
&\Longleftrightarrow re(y)-z\in intC(y) \\
&\Longleftrightarrow z\in re(y)-intC(y).
\end{aligned}$$

■

Proposition 1.55. *Let Y be a locally convex Hausdorff topological vector space. Then, for any given $y \in Y$,*

(i) *$\xi(y,\cdot)$ is positively homogenous;*
(ii) *$\xi(y,\cdot)$ is strictly monotone, that is, if $z_1 \geq_{intC(y)} z_2$, then*

$$\xi(y,z_2) < \xi(y,z_1).$$

Proof. (i) Let $\mu>0$. For $z\in Y$, we have

$$\begin{aligned}
\xi(y,\mu z) &= \max_{\phi\in B^*(y)} \frac{\langle\phi,\mu\, z\rangle}{\langle\phi,e(y)\rangle} \\
&= \mu \max_{\phi\in B^*(y)} \frac{\langle\phi,\ z\rangle}{\langle\phi,e(y)\rangle} \\
&= \mu\ \xi(y,z).
\end{aligned}$$

(ii) Let $z_1 \geq_{intC(y)} z_2$. Set $r=\xi(y,z_1)$. By the definition of $\xi(y,z_1)$, we have

$$z_2 \in z_1 - intC(y) \subset re(y)-C(y)-intC(y) \subset re(y)-intC(y).$$

By Proposition 1.54 (i), we have

$$\xi(y,z_2) < r = \xi(y,z_1).$$ ■

Proposition 1.56. *For any fixed $y\in Y$, and any $z_1,z_2\in Y$,*

(i) *$\xi(y,z_1+z_2)\le \xi(y,z_1)+\xi(y,z_2)$;*

(ii) *$\xi(y,z_1-z_2)\ge \xi(y,z_1)-\xi(y,z_2)$.*

Proof. (i)

$$\begin{aligned}\xi(y, z_1 + z_2) &= \max_{\phi \in B^*(y)} \frac{\langle \phi, z_1 + z_2 \rangle}{\langle \phi, e(y) \rangle} \\ &\leq \max_{\phi \in B^*(y)} \frac{\langle \phi, z_1 \rangle}{\langle \phi, e(y) \rangle} + \max_{\phi \in B^*(y)} \frac{\langle \phi, z_2 \rangle}{\langle \phi, e(y) \rangle} \\ &= \xi(y, z_1) + \xi(y, z_2).\end{aligned}$$

(ii) It follows from (i) that

$$\xi(y, z_1) = \xi(y, z_1 - z_2 + z_2) \leq \xi(y, z_1 - z_2) + \xi(y, z_2).$$

Then, $\xi(y, z_1) - \xi(y, z_2) \leq \xi(y, z_1 - z_2)$. This implies that (ii) holds. ∎

Theorem 1.57. *Let Y be a locally convex Hausdorff topological vector space, and let $C : Y \rightrightarrows Y$ be a set-valued function such that for each $y \in Y$, $C(y)$ is a proper, closed, convex cone in Y with $intC(y) \neq \varnothing$. And let $e : Y \to Y$ be a continuous selection of the set-valued function $intC(\cdot)$. Define a set-valued function $W : Y \rightrightarrows Y$ by $W(y) = Y \backslash intC(y)$, for $y \in Y$. We have*

(i) *If W is upper semi-continuous, then $\xi(\cdot, \cdot)$ is upper semi-continuous on $Y \times Y$;*
(ii) *If C is upper semi-continuous, then $\xi(\cdot, \cdot)$ is lower semi-continuous on $Y \times Y$.*

Proof. (i) In order to show that $\xi(\cdot, \cdot)$ is upper semi-continuous, we must check, for any $\lambda \in \mathbb{R}$, the set

$$A := \{(y, z) \in Y \times Y : \xi(y, z) \geq r\}$$

is closed. Let $(y_\alpha, z_\alpha) \in A$ and $(y_\alpha, z_\alpha) \to (y_0, z_0)$. We have $\xi(y_\alpha, z_\alpha) \geq r$, that is to say, by Proposition 1.54 (iii), that

$$z_\alpha \notin re(y_\alpha) - intC(y_\alpha).$$

Namely, $re(y_\alpha) - z_\alpha \in Y \backslash intC(y_\alpha) = W(y_\alpha)$. Since $e(\cdot)$ is continuous on Y, $(re(y_\alpha) - z_\alpha, y_\alpha) \to (re(y_0) - z_0, y_0)$. Since W is upper semi-continuous and closed-valued, by Proposition 1.21, W is closed. So $re(y_0) - z_0 \in W(y_0)$. Namely, $z_0 \notin re(y_0) - intC(y_0)$. By Proposition 1.54 (iii), it is equivalent to $\xi(y_0, z_0) \geq r$. So, A is closed, i.e., $\xi(\cdot, \cdot)$ is upper semi-continuous on $Y \times Y$.

(ii) In order to show $\xi(\cdot, \cdot)$ is lower semi-continuous, we must check, for any $\lambda \in \mathbb{R}$, the set

$$B := \{(y, z) \in Y \times Y : \xi(y, z) \leq r\}$$

is closed. Let $(y_\alpha, z_\alpha) \in B$ and $(y_\alpha, z_\alpha) \to (y_0, z_0)$. We have $\xi(y_\alpha, z_\alpha) \leq r$, it is to say, by Proposition 1.54 (ii),

$$z_\alpha \in re(y_\alpha) - C(y_\alpha).$$

Since $e(\cdot)$ is continuous on Y, $(re(y_\alpha) - z_\alpha, y_\alpha) \to (re(y_0) - z_0, y_0)$. Since $C(\cdot)$ is upper semi-continuous and closed-valued, by Proposition 1.21, C is closed. So $re(y_0) - z_0 \in C(y_0)$. Namely, $z_0 \in re(y_0) - C(y_0)$. By Proposition 1.54 (ii), it is equivalent to $\xi(y_0, z_0) \leq r$. So, B is closed, i.e., $\xi(\cdot, \cdot)$ is lower semi-continuous on $Y \times X$. ■

Remark 1.58. (**i**) If Y is a paracompact space, and $intC^{-1}(x) = \{y \in Y : x \in intC(y)\}$ is an open set and for each $y \in Y$, $intC(y) \neq \varnothing$ and $C(y)$ is convex, by the Browder continuous selection theorem, $intC(\cdot)$ has a continuous selection $e(\cdot)$.
(**ii**) If $e \in int \bigcap_{y \in Y} C(y)$, we could let, for any $y \in Y$, $e(y) = e$. The function e is also continuous.

The following examples are to show that if C (W, respectively) is not upper semi-continuous, then $\xi(\cdot, \cdot)$ is not lower semi-continuous (upper semi-continuous, respectively) even if all the other conditions of Theorem 1.57 are satisfied.

Example 1.59. Let $Y = \mathbb{R}^2$, the 2-dimensional Euclidean space. Let

$$A = cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, \frac{1}{2} \leq y_1 \leq \frac{3}{2}\}),$$
$$B = cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, 0 \leq y_1 \leq \frac{3}{2}\}),$$
$$C = cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, \frac{1}{2} \leq y_1 \leq 2\}).$$

The set-valued map $C : Y \rightrightarrows Y$ is defined by

$$C((y_1, y_2)^\top) = \begin{cases} A, & \text{if } y_1 = 0; \\ B, & \text{if } y_1 > 0; \\ C, & \text{if } y_1 < 0. \end{cases}$$

Thus,

$$W((y_1, y_2)^\top) = \begin{cases} Y \backslash intA, & \text{if } y_1 = 0; \\ Y \backslash intB, & \text{if } y_1 > 0; \\ Y \backslash intC, & \text{if } y_1 < 0. \end{cases}$$

Let $e = (1, 1)^\top$ and for any $y = (y_1, y_2)^\top \in Y$, $e(y) = e$.

Note that for any $y \in Y$, $intC(y) \neq \varnothing$ and $e \in intC(y)$. We also note that $W(\cdot)$ is upper semi-continuous, so $\xi(\cdot, \cdot)$ is upper semi-continuous on $Y \times Y$. But $C(\cdot)$ is not upper semi-continuous. Note that the level set of the function ξ at 0,

$$\begin{aligned} L(\xi, 0) &= \{((y_1, y_2)^\top, (z_1, z_2)^\top) \in \mathbb{R}^2 \times \mathbb{R}^2 : \xi((y_1, y_2)^\top, (z_1, z_2)^\top) \leq 0\} \\ &= (\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 = 0\} \times (-A)) \cup \\ &\quad (\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 > 0\} \times (-B)) \\ &\quad \cup(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 < 0\} \times (-C)), \end{aligned}$$

is not a closed set. That is to say, $\xi(\cdot, \cdot)$ is not lower semi-continuous.

Example 1.60. Let $Y = \mathbb{R}^2$, the 2-dimensional Euclidean space. Let

$$\begin{aligned} A &= cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, \frac{1}{2} \leq y_1 \leq \frac{3}{2}\}), \\ B &= cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, 0 \leq y_1 \leq 2\}), \end{aligned}$$

The set-valued map $C : Y \rightrightarrows Y$ is defined by

$$C((y_1, y_2)^\top) = \begin{cases} B, & \text{if } y_1 = 0; \\ A, & \text{if } y_1 \neq 0. \end{cases}$$

Then,

$$W((y_1, y_2)^\top) = \begin{cases} Y \backslash intB, & \text{if } y_1 = 0; \\ Y \backslash intA, & \text{if } y_1 \neq 0. \end{cases}$$

Let $e = (1, 1)^\top$ and for any $y = (y_1, y_2)^\top \in Y$, $e(y) = e$.

Note that for any $y \in Y$, $intC(y) \neq \varnothing$ and $e \in intC(y)$. We also note that $C(\cdot)$ is upper semi-continuous, so $\xi(\cdot, \cdot)$ is lower semi-continuous on $Y \times Y$. But $W(\cdot)$ is not upper semi-continuous. Note that the strict level set of the function ξ at 0,

$$\begin{aligned} L_s(\xi, 0) &= \{((y_1, y_2)^\top, (z_1, z_2)^\top) \in \mathbb{R}^2 \times \mathbb{R}^2 : \xi((y_1, y_2)^\top, (z_1, z_2)^\top) < 0\} \\ &= (\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 = 0\} \times (-intB)) \cup \\ &\quad (\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 \neq 0\} \times (-intA)) \end{aligned}$$

is not an open set. That is to say, $\xi(\cdot, \cdot)$ is not upper semi-continuous.

1.4 Convex and Generalized Convex Functions

In this section, we introduce some concepts of (generalized) convexity for vector-valued and set-valued functions.

Let X, Y be two topological vector spaces, $C \subset Y$ a convex cone with nonempty interior $intC$.

Definition 1.61. *(i) A set $A \subset Y$ is said to be C-bounded below if there exists b such that $A \subset b + C$.*

(ii) A set $A \subset Y$ is said to be C-bounded above if there exists b such that $A \subset b - C$.
(iii) A is said to be C order bounded if A is both C-bounded below and C-bounded above.
(iv) A set $A \subseteq Y$ is said to be strongly C-bounded below if there exists $b \in -C$ such that $A \subset b + intC$.
(v) A set $A \subset Y$ is said to be C-convex if $A + C$ is a convex set.

Let $K \subset X$ be a nonempty subset, and let $f : K \to Y$ be a vector-valued function. We denote the C-epigraph of f by

$$\text{epi}_C f = \{(x, y) \in K \times Y : x \in K, y \in f(x) + C\}.$$

Definition 1.62. *Let $K \subset X$ be a convex set and $f : K \to Y$ be a vector-valued function, K a nonempty convex subset of Y.*

(**i**) *f is C-convex on K if, for any $x_1, x_2 \in K, \lambda \in [0, 1]$,*

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq_C \lambda f(x_1) + (1 - \lambda)f(x_2),$$

i.e.,

$$f(\lambda x_1 + (1 - \lambda)x_2) \in \lambda f(x_1) + (1 - \lambda)f(x_2) - C;$$

(**ii**) *f is strictly C-convex on K if, for any $x_1, x_2 \in K, x_1 \neq x_2, \lambda \in [0, 1]$,*

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq_{intC} \lambda f(x_1) + (1 - \lambda)f(x_2),$$

i.e.,

$$f(\lambda x_1 + (1 - \lambda)x_2) \in \lambda f(x_1) + (1 - \lambda)f(x_2) - intC;$$

(**iii**) *f is C-quasiconvex on K if, for $y \in Y$, $x_1, x_2 \in K, \lambda \in [0, 1]$,*

$$f(x_1), f(x_2) \in y - C \text{ implies } f(\lambda x_1 + (1 - \lambda)x_2) \in y - C;$$

(**iv**) *f is strictly C-quasiconvex, if, for $y \in Y$, $x_1, x_2 \in K$, $x_1 \neq x_2$, $\lambda \in (0, 1)$,*

$$f(x_1), f(x_2) \in y - C \text{ implies } f(\lambda x_1 + (1 - \lambda)x_2) \in y - intC.$$

Definition 1.62 is a generalization of the convexity and quasiconvexity of real-valued functions, respectively.

$f : K \to Y$ is called a C-concave function on K if $-f$ is C-convex on K. Similarly, we can define the strict C-concavity and (strict) C-quasiconcavity of vector-valued functions.

Proposition 1.63. *Let K be a nonempty convex subset of X. Assume that f is Gâteaux differentiable on K. Let Y be a Hausdorff topological space ordered by a closed and convex cone C. If f is C-convex on K, then, for every $x, y \in K$,*

$$f(y) \geq_C f(x) + \langle Df(x), y - x\rangle,$$

where $Df(x)$ is the Gâteaux derivative of f at x.

Proof. Let f be Gâteaux differentiable on K. If f is C-convex on K, then for any $x, y \in K$, and $t \in (0, 1)$,

$$f(ty + (1-t)x) \in tf(y) + (1-t)f(x) - C,$$

$$f(x + t(y-x)) \in tf(y) + (1-t)f(x) - C,$$

i.e.,

$$f(y) \in f(x) + \frac{f(x + t(y-x)) - f(x)}{t} + C.$$

As $t \to 0+$, we have

$$f(y) \geq_C f(x) + \langle Df(x), y - x \rangle.$$ ∎

Similarly, we have

Proposition 1.64. *Let K be a nonempty convex subset of X. Assume that f is Gâteaux differentiable on K. Let Y be a Hausdorff topological space ordered by a closed and convex cone C. If f is C-concave on K, then, for every $x, y \in K$,*

$$f(y) \leq_C f(x) + \langle Df(x), y - x \rangle.$$

Definition 1.65. *Let C be a closed, convex and pointed cone in Y with the nonempty interior $intC$, K be a closed and convex subset of X. Let $x, y \in K$, and $t \in (0, 1)$. A set-valued function $F : K \rightrightarrows Y$ is said to be:*

(i) *Type I C-convex iff, $F(tx + (1-t)y) \subset tF(x) + (1-t)F(y) - C$;*
(ii) *Type II C-convex iff, $tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + C$;*
(iii) *Type I C-concave iff, $tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) - C$;*
(iv) *Type II C-concave iff, $F(tx + (1-t)y) \subset tF(x) + (1-t)F(y) + C$;*

Remark 1.66. It is not difficult to see that

(i) F is type I C-convex iff, $-F$ is type II C-concave;

(ii) F is type II C-convex iff, $-F$ is type I C-concave.

If F is a single-valued function, then both type I and type II convexity (concavity, respectively) are equivalent to the usual C-convexity (usual C-concavity, respectively).

Proposition 1.67 (Luc [142]). *f is C-convex if and only if $epi_C f$ is a convex set. Moreover, assuming that Y is separated and C is closed, f is C-convex if and only if $\varphi \circ f$ is a convex function for every $\varphi \in C^*$.*

We denote the level set of f at a point $y \in Y$ by $\mathrm{Lev}_f(y)$, i.e.,

$$\mathrm{Lev}_f(y) = \{x \in X : f(x) \in y - C\}.$$

Proposition 1.68 (Luc [142]). *The following statements hold:*

(**i**) *f is C-quasiconvex if and only if $Lev_f(y)$ is convex for each $y \in Y$;*
(**ii**) *f is C-quasiconvex if and only if $\xi_{ea} \circ f$ is quasiconvex for a fixed $e \in intC$ and every $a \in Y$, where $\xi_{ea}(y) = min\{t \in I\!R : y \in a + te - C\}, y \in Y$.*

Definition 1.69. *Let $C : X \rightrightarrows Y$ be closed convex cone valued, $K \subset X$ a nonempty convex set and $f : K \to Y$. f is said to be $C(x)$-convex if, for any $x_1, x_2 \in K$, $\lambda \in [0,1]$, there holds*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_{C(\lambda x_1+(1-\lambda)x_2)} \lambda f(x_1) + (1-\lambda)f(x_2).$$

The $C(x)$-epigraph of f is defined by

$$\mathrm{epi}_{C(x)} f = \{(x,y) \in K \times Y : x \in K, y \in f(x) + C(x)\}.$$

Definition 1.70. *Let $C : X \rightrightarrows Y$ be convex cone valued. C is said to be a convex process if,*
(i) for any $x_1, x_2 \in X$, $C(x_1) + C(x_2) \subset C(x_1 + x_2)$;
(ii) for any $\lambda > 0$, $x \in X$, $C(\lambda x) = \lambda C(x)$.

Proposition 1.71. *If the $C(x)$-epigraph of f is convex, then f is $C(x)$-convex. On the other hand, if f is $C(x)$-convex and $C : X \rightrightarrows Y$ is a convex process, then the $C(x)$-epigraph of f is convex.*

Proof. Assume that the $C(x)$-epigraph of f is convex. Let $(x_1, f(x_1))$ and $(x_2, f(x_2)) \in \mathrm{epi}_{C(x)} f$ and $0 \leq \lambda \leq 1$. Then $\lambda(x_1, f(x_1)) + (1-\lambda)(x_2, f(x_2)) \in \mathrm{epi}_{C(x)} f$, i.e., $(\lambda x_1 + (1-\lambda)x_2, \lambda f(x_1) + (1-\lambda)f(x_2)) \in \mathrm{epi}_{C(x)} f$. Thus it holds that

$$f(\lambda x_1 + (1-\lambda)x_2) \leq_{C(\lambda x_1+(1-\lambda)x_2)} \lambda f(x_1) + (1-\lambda)f(x_2).$$

So, f is $C(x)$-convex.

On the other hand, assume that f is $C(x)$-convex. Let $(x_1, y_1), (x_2, y_2) \in \mathrm{epi}_{C(x)} f$. and $0 \leq \lambda \leq 1$. Then

$$y_1 - f(x_1) \in C(x_1) \text{ and } y_2 - f(x_2) \in C(x_2).$$

Thus

$$\lambda(y - f(x_1)) + (1-\lambda)(y_2 - f(x_2)) \in \lambda C(x_1) + (1-\lambda)C(x_2).$$

Since $C(x)$ is a convex process, we have

$$\lambda C(x_1) + (1-\lambda)C(x_2) \subset C(\lambda x_1 + (1-\lambda)x_2).$$

Thus

$$\lambda y_1 + (1-\lambda)y_2 - \lambda f(x_1) - (1-\lambda)f(x_2) \in C(\lambda x_1 + (1-\lambda)x_2).$$

It follows from the $C(x)$-convexity of f that

$$\lambda y_1 + (1-\lambda)y_2 - f(\lambda x_1 + (1-\lambda)x_2) \in C(\lambda x_1 + (1-\lambda)x_2).$$

So $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in \mathrm{epi}_{C(x)} f$. ∎

Proposition 1.72. *Let Y be a Banach space. Let K be a nonempty convex subset of X. Let $C : X \rightrightarrows Y$ be closed, covex cone valued and upper semicontinuous. Assume that f is Gâteaux differentiable on K. If f is $C(x)$-convex on K, then, for every $x, y \in K$,*

$$f(y) \geq_{C(x)} f(x) + \langle Df(x), y - x \rangle.$$

Proof. By $C(x)$-convexity of f, we have

$$f(ty + (1-t)x)) \in tf(y) + (1-t)f(x) - C(ty + (1-t)x).$$

That is,

$$f(y) \in f(x) + \frac{f(x + t(y-x)) - f(x)}{t} + C(ty + (1-t)x).$$

Let $t \to 0$, it follows from the closedness of $C(x)$ and the u.s.c. of C that

$$f(y) \in f(x) + \langle Df(x), y - x \rangle + C(x). \qquad \blacksquare$$

Definition 1.73. *$f : K \to Y$ is called C-convexlike on K if, for any $x_1, x_2 \in K$ and any $\lambda \in (0,1)$, there exists $x_3 \in K$ such that*

$$\lambda f(x_1) + (1-\lambda) f(x_2) - f(x_3) \in C;$$

f is called C-subconvexlike on K if there exists a $\theta \in intC$ such that, for any $x_1, x_2 \in K$, any $\epsilon > 0$, there exists $x_3 \in K$ satisfying

$$\epsilon\theta + \lambda f(x_1) + (1-\lambda) f(x_2) - f(x_3) \in C.$$

Remark 1.74. It is not necessary for K to be convex in Definition 1.73. Suppose that K is a convex subset of a topological vector space. It is clear that C-convexity implies C-convexlikeness, and, in turn, C-convexlikeness implies C-subconvexlikeness. Moreover, for each point $c \in Y$, $f(\cdot) + c$ is C-subconvexlike if and only if f is C-subconvexlike (see Jeyakumer [119]).

Definition 1.75 (Yang [201]). *$f : K \to Y$ is said to be generalized C-subconvexlike on K if there exists a $\theta \in intC$ such that, for any $x_1, x_2 \in X$ and any $\epsilon > 0$, there exist $x_3 \in K$ and $\eta > 0$ satisfying*

$$\epsilon\theta + \lambda f(x_3) + (1-\lambda) f(x_3) - \eta f(x_3) \in C.$$

Proposition 1.76. *The following statements are equivalent:*

(i) *f is generalized C-subconvexlike on K;*
(ii) *$cone(f(K)) + intC$ is convex;*
(iii) *for any $\theta \in intC$, any $x_1, x_2 \in K$ and any $\lambda \in (0,1)$, there exist $x_3 \in K$ and $\eta > 0$ such that*

$$\theta + \lambda f(x_1) + (1-\lambda) f(x_3) - \eta f(x_3) \in intC.$$

Proof. First, we show that (i) $\Rightarrow$ (ii). Let

$$A = cone(f(K)) + intC, \quad y_1, y_2 \in A, \ \lambda \in (0,1).$$

Then, there exist $\alpha_i \geqslant 0$, $x_i \in K$, and $s_i \in intC$, $i = 1,2$, satisfying

$$y_i = \alpha_i f(x_i) + s_i, \quad i = 1,2.$$

Set

$$\bar{y} = \lambda y_1 + (1-\lambda)y_2 = \lambda\alpha_1 f(x_1) + (1-\lambda)\alpha_2 f(x_2) + s_0$$

where $s_0 = \lambda s_1 + (1-\lambda)s_2 \in intC$. Thus we can find a symmetric neighborhood U of the null element of Y such that $s_0 + U \subset intC$. If $\alpha_1 = 0$ or $\alpha_2 = 0$, then obviously $\bar{y} \in A$. Without loss of generality, we can assume that $\alpha_1 > 0$ and $\alpha_2 > 0$. Set

$$\alpha = \lambda\alpha_1 + (1-\lambda)\alpha_2, \quad \beta = (\lambda\alpha_1)/\alpha.$$

Then $\beta \in (0,1)$ and

$$\bar{y} = \alpha(\beta f(x_1) + (1-\beta)f(x_2)) + s_0. \tag{1.4}$$

By the definition of generalized C-subconvexlikeness, there exists $\theta \in intC$ such that, for the above $\beta \in (0,1)$, $x_1, x_2 \in K$, and any $\varepsilon > 0$, we can find $x_3 := x_3(\beta, x_1, x_2, \varepsilon) \in K$ and $\eta := \eta(\beta, x_1, x_2, \varepsilon) > 0$ satisfying

$$\bar{k} := \varepsilon\theta + \beta f(x_1) + (1-\beta)f(x_2) - \eta f(x_3) \in C. \tag{1.5}$$

Since U is absorbing and symmetric, we can select an $\varepsilon > 0$ small enough such that $-\varepsilon\alpha\theta \in U$, so that

$$s_0 - \varepsilon\alpha\theta \in s_0 + U \subset intC. \tag{1.6}$$

By (1.4) - (1.6), we obtain

$$\begin{aligned}\bar{y} &= \alpha\bar{k} + \alpha\eta f(x_3) + (s_0 - \varepsilon\alpha\theta)\\ &\in C + cone(f(K)) + intC\\ &\subset cone(f(K)) + intC = A.\end{aligned}$$

Therefore, A is convex.

Next, we show that (ii) $\Longrightarrow$ (iii). Let $\theta \in intC$, $x_1, x_2 \in K$, $\lambda \in (0,1)$. Obviously,

$$f(x_i) + \theta \in cone(f(K)) + intC, \quad i = 1,2.$$

Since $cone(f(K)) + intC$ is convex, we have also

$$\begin{aligned}\bar{y} &:= \theta + \lambda f(x_1) + (1-\lambda)f(x_2)\\ &= \lambda(f(x_1) + \theta) + (1-\lambda)(f(x_2) + \theta)\\ &\in cone(f(K)) + intC.\end{aligned}$$

This implies that there exist $x_3 \in K$ and $\alpha \geq 0$ such that

$$\bar{y} \in \alpha f(x_3) + intC.$$

If $\alpha \neq 0$, then, take $\eta = \alpha$, the proof for (ii) $\Longrightarrow$ (iii) is completed.

If $\alpha = 0$, then $\bar{y} \in intC$. Thus, there is a symmetric neighborhood V of the null element of Y such that

$$\bar{y} + V \subset intC.$$

Since V is absorbing and symmetric, for arbitrarily fixed $x_3' \in K$, we can find an $\eta > 0$ small enough such that $-\eta f(x_3') \in V$. Hence

$$\bar{y} - \eta f(x_3') \in \bar{y} + V \subset intC.$$

So (ii) $\Longrightarrow$ (iii).

Finally, we show that (iii) $\Longrightarrow$ (i). Suppose that the condition (iii) holds, and $\bar{\theta}$ is a given point in $intC$. Let $\theta = \varepsilon\bar{\theta}$. For any $\varepsilon > 0$, $\theta \in intC$, by the condition (iii), for any $x_1, x_2 \in K$, and $\lambda \in (0,1)$, and any $\varepsilon > 0$, there exist $x_3 \in K$ and $\eta > 0$ such that

$$\theta + \lambda f(x_1) + (1-\lambda) f(x_2) - \eta f(x_3) \in intC.$$

Thus, f is a generalized C-subconvexlike vector-valued function on K. ■

Example 1.77. Let $K = \{(1,0)^\top, (0,1)^\top\} \subset \mathbb{R}^2, C = \mathbb{R}^2_+$, and let $f = I : K \to \mathbb{R}^2$ be the identity vector-valued function. Obviously,

$$cone(f(K)) + intC = int\mathbb{R}^2_+,$$

which is a convex subset in $\mathbb{R}^2$. By Proposition 1.76, f is a generalized C-subconvexlike vector-valued function on K. But

$$f(K) + intC = K + int\mathbb{R}^2_+$$

is not convex. f is not C-subconvexlike on K.

Proposition 1.78. *f is C-subconvexlike on K if and only if the set $f(K) + intC$ is convex.*

Proof. The proof of this proposition is similar to that of Proposition 1.76 and thus omitted. ■

When $Y = \mathbb{R}^\ell$, Proposition 1.78 can be found in Li and Wang [138].

Theorem 1.79 (Gordan-Form Alternative Theorem). *Let $K \subset X$ be a nonempty subset, and let C be a closed, convex and pointed cone with nonempty interior $intC$. If f is generalized C-subconvexlike on K, then exactly one of the following statements is true:*

(i) *there exists $\bar{x} \in K$ such that $f(\bar{x}) \leq_{intC} 0$;*
(ii) *there exists $\mu \in C^* \backslash \{0\}$ such that $\langle \mu, f(x) \rangle \geq 0$, for any $x \in K$.*

Proof. Obviously, (i) and (ii) cannot hold simultaneously. Otherwise, a contradiction is introduced. In fact, we have $0 > \langle \mu, f(\bar{x}) \rangle \geq 0$.

Suppose that (i) is not true. Then,

$$0 \notin cone(f(K)) + intC.$$

In fact, assume that

$$0 \in cone(f(K)) + intC.$$

There exist $\bar{x} \in K$ and $\alpha \geq 0$ such that $0 \in \alpha f(\bar{x}) + intC$ and $\alpha > 0$, since $0 \notin intC$. Thus

$$-f(\bar{x}) \in (1/\alpha) intC \subset intC,$$

i.e., $f(\bar{x}) \leq_{intC} 0$. This is a contradiction.

By Proposition 1.76, $cone(f(K)) + intC$ is a convex set with a nonempty interior. By the separation theorem for convex sets, there exists $\mu \in Y^* \backslash \{0\}$ such that

$$\langle \mu, \alpha f(x) + s \rangle \geq 0, \quad \forall \alpha \geq 0, x \in K, s \in intC.$$

Setting $\alpha = 0$, we obtain

$$\langle \mu, s \rangle \geq 0, \quad \forall x \in intC.$$

Thus,

$$\langle \mu, s \rangle \geq 0, \quad \forall s \in C,$$

i.e., $\mu \in C^* \backslash \{0\}$. Also, setting $\alpha = 1$, we obtain

$$\langle \mu, f(x) + s \rangle \geq 0, \quad \forall x \in K, s \in intC.$$

Letting $s \to 0$, we have

$$\langle \mu, f(x) \rangle \geq 0, \quad \forall x \in K,$$

so that (ii) holds. The proof is thus completed. ■

Now, we define the generalized convexity of set-valued functions.

Definition 1.80. *Let X_0 be a nonempty convex subset of X and $F : X_0 \rightrightarrows Y$ a set-valued function.*

(**i**) *F is said to be properly quasi C-convex on X_0 if, for any $x_1, x_2 \in X_0$ and $\lambda \in [0,1]$,*

$$\textit{either } F(x_1) \subset F(\lambda x_1 + (1-\lambda)x_2) + C,$$
$$\textit{or } F(x_2) \subset F(\lambda x_1 + (1-\lambda)x_2) + C.$$

(**ii**) *F is said to be naturally quasi C-convex on X_0 if, for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$,*

$$F(\lambda x_1 + (1-\lambda)x_2) \subset co(F(x_1) \cup F(x_2)) - C,$$

where $co(A)$ denotes the convex hull of A.

Similarly, we can define the proper quasi C-concavity and natural quasi C-concavity.

Remark 1.81. Definition 1.80 is a generalization of the concepts of proper quasi C-convexity and natural quasi C-convexity for vector-valued functions in Ferro [71] and Tanaka [188]. Note that if $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then both proper quasi C-convexity and natural quasi C convexity reduce to the ordinary quasiconvexity for real-valued functions.

Proposition 1.82. *Let $F : X_0 \rightrightarrows Y$ be a set-valued function and X_0 a nonempty convex subset. If F is naturally quasi C-convex, then, for given $e \in intC$ and $a \in Y$, $\xi_{ea} \circ F$ is naturally quasi $\mathbb{R}_+$-convex. If F is properly quasi C-convex, then, for given $e \in intC$ and $a \in Y$, $\xi_{ea} \circ F$ is properly quasi $\mathbb{R}_+$-convex.*

Proof. We prove only the first conclusion of the proposition. Take any $x_1, x_2 \in X_0$, $\lambda \in [0,1]$ and $y \in F(\lambda x_1 + (1-\lambda)x_2)$. By the natural quasi C-convexity of F, there exist $y_i \in F(x_1) \cup F(x_2)$ and $\alpha_i \geq 0, i = 1, 2, \cdots, k$ and $c \in C$ such that $\sum_{i=1}^k \alpha_i = 1$ and $y = \sum_{i=1}^k \alpha_i y_i - c$. Therefore

$$\xi_{ea}(y) = \xi_{ea}(\sum_{i=1}^n \alpha_i y_i - c).$$

Since ξ_{ea} is a convex and monotone function, we have

$$\begin{aligned} \xi_{ea}(y) &\in \sum_{i=1}^k \alpha_i \xi_{ea}(y_i) - \mathbb{R}_+ \\ &\subset co\{\xi_{ea} \circ F(x_1) \cup \xi_{ea} \circ F(x_2)\} - \mathbb{R}_+. \end{aligned}$$

Thus, $\xi_{ea} \circ F$ is naturally quasi $\mathbb{R}_+$-convex. ■

Definition 1.83. *Let X be a convex subset of a topological vector space and Y be a Hausdorff topological vector space. Let $F : X \times X \rightrightarrows Y$ be a set-valued function. Let $C : X \rightrightarrows Y$ be a set-valued function. Given any finite subset $\Lambda = \{x_1, x_2, \cdots, x_k\}$ in X and any $x \in co\{x_1, x_2, \cdots, x_k\}$,*

(i) *F is said to be strongly type I C-diagonally quasi-convex (SIC-DQC, in short) in the second argument if, for some* $x_i \in \Lambda$,

$$F(x, x_i) \subset C(x);$$

(ii) *F is said to be strongly type II C-diagonally quasi-convex (SIIC-DQC, in short) in the second argument if, for some* $x_i \in \Lambda$,

$$F(x, x_i) \cap C(x) \neq \varnothing;$$

(iii) *F is said to be weakly type I C-diagonally quasi-convex (WIC-DQC, in short) in the second argument if, for some* $x_i \in \Lambda$,

$$F(x, x_i) \cap -intC(x) = \varnothing;$$

(iv) *F is said to be weakly type II C-diagonally quasi-convex (WIIC-DQC, in short) in the second argument if, for some* $x_i \in \Lambda$,

$$F(x, x_i) \not\subseteq -intC(x).$$

Remark 1.84. When $Y = \mathbb{R}$, $C(x) = \mathbb{R}_+$ for any $x \in X$, and $F : X \times X \to Y$ is a single-valued function, the above four kinds of C-diagonal quasi-convexities reduce to r-diagonal quasi-convexity for a single-valued function in Zhou and Chen [224].

It is easy to verify the following proposition.

Proposition 1.85. *If, for each* $x \in X$, $C(x)$ *is a pointed, closed and convex cone in* Y*, then the following statements hold:*

(i) *SIC-DQC implies SIIC-DQC;*
(ii) *SIC-DQC implies WIC-DQC;*
(iii) *WIC-DQC implies WIIC-DQC.*

1.5 Notations

The following notations will be used in the later chapters.

Spaces

$\mathbb{R}$: the real line
$\mathbb{R}^m$: real m-dimensional space
$\mathbb{R}^m_+$: the nonnegative orthant of $\mathbb{R}^m$
(X, D): ordered decision space
(Y, C): ordered objective space
(Z, P): ordered constraint space

(X, d): metric space

Vectors

$x^\top$: the transpose of a vector x
$x^\top y$ or xy: the standard inner product of vectors x and y
$\|x\|$: the norm of x
$x \geq_C y$, $x \geq_{C\backslash\{0\}} y$, $x \geq_{intC} y$, $x \not\geq_C y$, $x \not\geq_{intC} y$: the orderings induced by C

Functions

$\varphi : X \to \mathbb{R}$: a scalar function
$f : X \to Y$: a vector valued function with domain X and range Y
$f \circ g$ or $f(g)$: composition of functions f and g
∇f or Df: derivative of f
f^{-1}: the inverse of f
$F : X \rightrightarrows Y$: a set-valued function
DF: contingent derivative of a set-valued function F
$D_e F$: contingent epideriative of a set-valued function F
$D_g F$: generalized contingent epideriative of a set-valued function F
$d(x, D)$: distance function from vector x to set D
haus(A, B) Hausdorff distance between sets A and B
ξ_{ea}, ξ_e, ξ: nonlinear scalarization functions

Sets

$\in$, $\notin$: element membership
$\subset$, $\subseteq$, $\not\subset$: set inclusion
$\emptyset$: empty set
$\cup$, $\cap$, $\times$: union, intersection, Cartesian product
$A \backslash B$: the difference of sets A and B
A^c: the complement of set A
$intC$: the interior of a set C
∂C: the boundary of a set C
clC: the closure of a set C
B_r: the closed ball centered at 0 with radius r in a normed space
$B(x, r)$: the closed ball centered at x with radius r in a normed space
Y^*: the dual space of space Y
$L(X, Y)$: the set of all the continuous linear operators from topological vector space X to topological vector space Y
C^* or C^+: the dual cone of C
$\text{Min}_C A$: the set of minimal points of A
$\text{Max}_C A$: the set of maximal points of A
$\text{Min}_{intC} A$: the set of weakly minimal points of A

$\mathrm{Max}_{intC}A$: the set of weakly maximal points of A

$\mathrm{Min}_{C(y)}A$: the set of minimal points of A with respect to the variable domination structure $C(y)$

$\mathrm{Max}_{C(y)}A$: the set of maximal points of A with respect to the variable domination structure $C(y)$

$\mathrm{Min}_{intC(y)}A$: the set of weakly minimal points of A with respect to the variable domination structure $C(y)$

$\mathrm{Max}_{intC(y)}A$: the set of weakly maximal points of A with respect to the variable domination structure $C(y)$

$\mathrm{LMin}_{C(y)}A$: the set of nondominated-like minimal points of A with respect to the variable domination structure $C(y)$

$\mathrm{LMax}_{C(y)}A$: the set of nondominated-like maximal points of A with respect to the variable domination structure $C(y)$

$\mathrm{LMin}_{intC(y)}A$: the set of weakly nondominated-like minimal points of A with respect to the variable domination structure $C(y)$

$\mathrm{LMax}_{intC(y)}A$: the set of weakly nondominated-like maximal points of A with respect to the variable domination structure $C(y)$

$\mathrm{LMin}_{intC(x)}f(S)$: the set of weakly nondominated-like minimal points of f over S with respect to the variable domination structure $C(x)$

$\mathrm{LMax}_{intC(x)}f(S)$: the set of weakly nondominated-like maximal points of f over S with respect to the variable domination structure $C(x)$

$epi_C f$: the epigraph of function f

$Lev_f(y)$: the level set of function f

$co(A)$: the convex hull of set A

$cone(A)$: the cone generated by set A

$T(K, x)$ the contingent tangent cone of K at $x \in \mathrm{cl}K$

$\mathrm{ASup}_C A$: the absolute supremum of set A with respect to C

$\mathrm{AInf}_C A$: the absolute infimum of set A with respect to C

$\mathrm{Sup}_C A$ the set of suprema of set A with respect to C

$\mathrm{Inf}_C A$: the set of infima of set A with respect to C

$U(x)$: a neighborhood of point x

$\mathrm{Dom}(F)$: the domain of function F

$\mathrm{Gr}(F)$: the graph of function F

$\mathrm{Range}(F)$: the range of function F

D_C^{w+}: weak C-dual cone of cone D

D_C^{s+}: strong C-dual cone of cone D

$PM(K, C)$: the set of Benson's proper minimal points

$argmin(X, J)$: the set of weakly minimal solutions of the vector optimization problem (X, J)

$\partial^w f$: the set of weak subgradients of the vector-valued function f

Problems

Let X, (Y, C), (Z, P) be spaces, $K \subset X$, $f : X \to Y$, $F : X \rightrightarrows Y$.

(VOP): $\operatorname{Min}_{C} \limits_{x\in B} f(x)$, where $B = \{x \in X : g(x) \leq_P 0, x \in K\}$

(VOPV): $\operatorname{Min}_{C(x)} \limits_{x\in B} f(x)$, where $B = \{x \in X : g(x) \leq_P 0, x \in K\}$ and $C : X \rightrightarrows Y$

(VUP): $\operatorname{Min}_{C} \limits_{x\in X} f(x)$

(VOK): $\operatorname{Min}_{C} \limits_{x\in K} f(x)$

(VOKV): $\operatorname{Min}_{C(x)} \limits_{x\in K} f(x)$, where $C : X \rightrightarrows Y$

(VUP$_\mu$): $\operatorname{Min}\limits_{x\in X} \langle \mu, f(x)\rangle$

(VMP): $\operatorname{Min}_{C} \limits_{x\in Q} f(x)$, where $Q = \{x \in X : g(x) \leq_P 0\}$

(VPL): $\operatorname{Min}_{C} \limits_{x\in X} L(x,T)$, where $L(x,T) = f(x) + T(g(x))$ and $T \in L(Z,Y)$

(VDL): $\operatorname{Max}_{C} \bigcup\limits_{T\in L_+(Z,Y)} \Phi(T)$, where $\Phi(T) = PM(L(X,T),C)$

(SOK): $\operatorname{Min}_{C} \limits_{x\in K} F(x)$

(SOKV): $\operatorname{Min}_{C(x)} \limits_{x\in K} F(x)$, where $C : X \rightrightarrows Y$

(PVOP): constrained paramteric vector optimization problem

(WVVI): weak vector variational inequality problem

(VVI): vector variational inequality problem

(GVVI): generalized (general) vector variational inequality problem

(GWVVI): generalized weak vector variational inequality problem

(SWVVI): set-valued weak vector variational inequality problem

(GPVVI): generalized vector pre-variational inequality problem

(GPQVVI): generalized vector pre-quasi-variational inequality problem

(VEQ$_1$): vector equilibrium problem

(GVQVI): generalized vector quasivariational inequality problem

(GVQVI)$_w$: perturbed GVQVI problem induced by parameter w

(DGVVI): dual general vector variational inequality

(VCP): (weak) vector complementarity problem

(PVCP): positive vector complementarity problem

(SVCP): strong vector complementarity problem

(WMEP): weak (vector) minimal element problem

(VUMP): vector unilateral minimization problem

2

Vector Optimization Problems

The concept of nondominated solution with variable domination structure was introduced by Yu [221]. This is a generalization of the minimal solution in multicriteria decision making problems. Various theories of nondominated solutions with variable domination structure were established in Yu [221], Tanino and Sawaragi [190].

It is worth noting that the solution concept with variable orderings in Yu [221] is given in the sense that a candidate point is guaranteed to be optimal only if it is not dominated by any other reference point with respect to their corresponding ordering. As such not much progress has been made in this direction. In Chen [25], another kind of nondominated solutions with variable domination structure was given with respect to the ordering of the candidate point in the context of vector variational inequalities. The related concept of nondominated solutions with variable domination structure in the context of vector optimization was given in Section 1.1.

In this chapter we consider three kinds of vector optimization problems, i.e., the problem with a fixed domination structure, the problem with a variable domination structure and the problem with a set-valued function. We investigate optimality conditions, characterizations and topological properties of solutions for these problems. In particular, we investigate weak duality, strong duality and exact penalization of vector optimization problems in terms of augmented Lagrangian and nonlinear Lagrangian.

2.1 Vector Optimization (VO)

In this section, we investigate firstly the case where the domination structure is a convex set but it is not necessarily a convex cone. The need for such an extension comes from the fact that there exists a large class of problems where, if we insist that the domination structure be a convex cone, then it must be $\{0\}$. In this case each feasible solution will be nondominated, and our analysis will become useless.

Suppose that Y is a Banach space and $C \subset Y$ is nonempty and convex with $0 \in \partial C$ and $intC \neq \emptyset$.

Let $S \subset Y$. For any $c \in C \backslash \{0\}$, we set

$$S(c) = \{y + \beta c : \beta \geq 0, y \in S\}.$$

Clearly, if S is convex, then $S(c)$ is a convex set and, for any set $S \subset Y$, $S \subset S(c)$. We have the following proposition.

Proposition 2.1. *Let $S \subset Y$ be a convex set, C a nonempty open convex set satisfying $0 \in \partial C$. Then, for any point $c \in C$, we have*

$$Min_C S = Min_C S(c).$$

Proof. It is obvious that $c \neq 0$ since $0 \in \partial C$, C is open and $c \in C$. Let $y^* \in Min_C S$. Suppose that $y^* \notin Min_C S(c)$. Then, there exist $y_0 \in S$ and $\beta_0 \geq 0$ such that

$$y_0 + \beta_0 c - y^* \in -C. \tag{2.1}$$

Since $y^* \in Min_C S$, we see that

$$(S - y^*) \cap -C = \emptyset. \tag{2.2}$$

By the standard separation theorem for convex sets, we obtain $\varphi \in Y^* \backslash \{0\}$ and $r \in \mathbb{R}$ such that

$$\varphi(y - y^*) \leq r, \quad \forall y \in S \tag{2.3}$$

and

$$\varphi(c') \geq r, \quad \forall c' \in -C. \tag{2.4}$$

Taking $y = y^*$ in (2.3), we have

$$r \geq 0. \tag{2.5}$$

Setting $y = y_0$ in (2.3), we obtain

$$\varphi(y_0 - y^*) \leq r. \tag{2.6}$$

The combination of (2.1) and (2.4) yields

$$\varphi(y_0 - y^*) + \beta_0 \varphi(c) \geq r.$$

Again, using (2.4), we have

$$\varphi(y_0 - y^*) \geq r + \beta_0 \varphi(-c) \geq r + \beta_0 r. \tag{2.7}$$

(2.7) together with (2.3) gives us

$$\beta_0 r \leq 0. \tag{2.8}$$

From (2.1) and (2.2), we deduce that

$$\beta_0 > 0. \tag{2.9}$$

It follows from (2.8) and (2.9) that

$$r \leq 0. \tag{2.10}$$

By (2.5) and (2.10), we have $r = 0$. Thus, (2.3) and (2.4) become

$$\varphi(y - y^*) \leq 0, \quad \forall y \in S \tag{2.11}$$

and

$$\varphi(c') \geq 0, \quad \forall c' \in -C. \tag{2.12}$$

Now, we prove from (2.12) that

$$\varphi(c') > 0, \quad \forall c' \in -C. \tag{2.13}$$

Suppose to the contrary that there exists a $c' \in -C$ such that $\varphi(c') = 0$. As $\varphi \neq 0$, there exists $\bar{y} \in Y$ such that $\varphi(\bar{y}) > 0$. From $c' \in -C$, we see that there exists a real number $\alpha > 0$ such that $c' - \alpha\bar{y} \in -C$. By (2.12), we should have

$$\begin{aligned} 0 &\leq \varphi(c' - \alpha\bar{y}) \\ &= \varphi(c') - \alpha\varphi(\bar{y}) \\ &= -\alpha\varphi(\bar{y}) < 0, \end{aligned}$$

a contradiction. So (2.13) holds. Using (2.1) and (2.13), we have

$$\varphi(y_0 - y^*) > 0,$$

contradicting (2.11). This proves that $Min_C S \subset Min_C S(c)$. In what follows, we show that $Min_C S \supset Min_C S(c)$. Let $y^* \in Min_C S(c)$. Then,

$$(S(c) - y^*) \cap -C = \emptyset. \tag{2.14}$$

Since $S \subset S(c)$, it follows that

$$(S - y^*) \cap -C = \emptyset. \tag{2.15}$$

Assume that $y^* = y_1 + \beta_1 c$ with $y_1 \in S$ and $\beta_1 \geq 0$. If $\beta_1 = 0$, then $y^* = y_1 \in S$. This together with (2.15) implies that $y^* \in Min_C S$. Suppose that $\beta_1 > 0$. By (2.14) and the separation theorem for convex sets, we have $\varphi_1 \in Y^* \backslash \{0\}$, $r_1 \in \mathbb{R}$ such that

$$\varphi_1(y - y_1 - \beta_1 c) \leq r_1, \quad \forall y \in S(c). \tag{2.16}$$

and

$$\varphi_1(c'') \geq r_1, \quad \forall c'' \in -C. \tag{2.17}$$

From (2.16), we have

$$r_1 \geq 0.$$

Now we show that $r_1 = 0$. Otherwise, by $0 \in \partial C$, we can choose a sequence $\{c_k\} \subset -C$ such that $c_k \to 0$. Thus, from (2.17), we have

$$0 = \lim_{k \to +\infty} \varphi_1(c_k) \geq r_1 > 0,$$

a contradiction. Arguing as in the first half of the proof, we can further show by contradiction that

$$\varphi_1(c'') > 0, \quad \forall c'' \in -C. \tag{2.18}$$

On the other hand, setting $y = y_1$ in (2.16), we obtain

$$\varphi_1(-\beta_1 c) \leq 0,$$

contradicting (2.18) since $-\beta_1 c \in -C$. The proof is complete. ■

Lemma 2.2. *Let C be a nonempty, open and convex set of Y with $0 \in \partial C$ and $y \in intT(C,0)$. Then*

$$\{\alpha y : 0 < \alpha \leq 1\} \cap C \neq \varnothing.$$

Proof. If the conclusion is not true, then $\{\alpha y : 0 < \alpha \leq 1\} \cap C = \varnothing$. Set $L = \{\alpha y : 0 < \alpha \leq 1\}$. By the separation theorem for convex sets, there exist a continuous linear function $\psi \in Y^* \backslash \{0\}$, and $r \in \mathbb{R}$ such that

$$\psi(y) \geq r, \quad \text{if } y \in C,$$

$$\psi(y) \leq r, \quad \text{if } y \in L.$$

Showing as in the proof of the second part of Proposition 2.1, we can prove that $r = 0$. Thus,

$$\psi(y) \geq 0, \quad \text{if } y \in C,$$

$$\psi(y) \leq 0, \quad \text{if } y \in L.$$

Since $y \in intT(C,0)$, there exist an open ball $N(y) \subset intT(C,0)$ and $y_0 \in N(y)$, such that $\psi(y_0) < \psi(y)$. Hence, there exist a sequence $\{y_k\} \subset C$, $y_k \to 0$ and a sequence of nonnegative real numbers $\{\alpha_k\}$ with the limit $+\infty$, such that $y_0 = \lim_{k \to \infty} \alpha_k(y_k)$ and $\psi(y_0) = \lim_{k \to 0} \psi(\alpha_k y_k)$. Since $\psi(y_k) \geq 0$, for all k, we have $\psi(y_0) \geq 0$. On the other hand, it follows from $y \in L$ and $\psi(y) \leq 0$ that $\psi(y_0) < \psi(y) \leq 0$. This leads to a contradiction. Thus $L \cap C \neq \varnothing$. ■

Theorem 2.3. *Let S be a convex subset of Y, C a nonempty, open and convex set of Y with $0 \in \partial C$. Then*

$$Min_C S = Min_{intT(C,0)} S.$$

Proof. Let $y_0 \in \text{Min}_{intT(C,0)}S$. Then $(y_0 - intT(C,0) \cup \{0\}) \cap S = \{y_0\}$. Since $y_0 - C \cup \{0\} \subset y_0 - intT(C,0) \cup \{0\}$ and $y_0 \in S$, it follows that

$$(y_0 - C \cup \{0\}) \cap S = \{y_0\}.$$

That is, $y_0 \in \text{Min}_C S$. Hence $\text{Min}_{intT(C,0)}S \subset \text{Min}_C S$.

On the other hand, let $y_0 \in \text{Min}_C S$. Without loss of generality, we can suppose that $y_0 = 0$. Thus, we suppose that $(-C \cup \{0\}) \cap S = \{0\}$. We will prove that $(-intT(C,0) \cup \{0\}) \cap S = \{0\}$. Suppose that $y_1 \in (-intT(C,0) \cup \{0\}) \cap S$ and $y_1 \neq 0$. Then $y_1 \in -intT(C,0) \cap S$. Set

$$L = \{y : y = \alpha y_1, 0 < \alpha \leq 1\}.$$

By Lemma 2.2, we have $L \cap (-C) \neq \varnothing$. Let $y' \in L \cap (-C)$. Then $y' \neq 0$. Since $y_1 \in S$ and S is a convex set, $L = (0, y_1] \subset S$ and $(0, y'] \subset S$. Thus $y' \in (-C) \cap S$. This contradicts that $(-C \cup \{0\}) \cap S = \{0\}$. Thus we obtain

$$(-intT(C,0) \cup \{0\}) \cap S = \{0\}. \quad \blacksquare$$

Suppose that X, Y, and Z are Banach spaces over $\mathbb{R}$, f maps from X into Y and g maps from X into Z. Let $C \subset Y$ be a nonempty convex subset with $0 \in \partial C$, $P \subset Z$ a closed and convex cone and $K \subset X$ a convex set.

We consider the vector constrained optimization problem:

$$\text{(VOP)} \qquad \text{Min}_C\, f(x), \quad x \in B$$

where $B = \{x \in X : g(x) \leq_P 0,\ x \in K\}$.

Remark 2.4. If $T \in L(X, Z)$, $z_0 \in Z$, then the problem

$$Min_C\{f(x) : T(x) = z_0\},$$

with affine constraints, may be obtained from the problem (VOP) for $K = X$, $P = \{0\}$ and $g(x) = T(x) - z_0$, $\forall x \in X$.

Theorem 2.5. *Let C be a convex subset of Y with $0 \in \partial C$ and $intC \neq \varnothing$. For the vector optimization problem (VOP), the following results hold:*

(i) *Suppose that there exists a continuous linear functional $\varphi \in Y^*$, satisfying $\varphi(c) > 0$, for all $c \in C \backslash \{0\}$, such that $\bar{x} \in B$ is an optimal solution of the following optimization problem $P(\varphi)$:*

$$\min_{x \in B} \varphi(f(x)).$$

Then $\bar{x}$ is a nondominated minimal solution of the problem (VOP).

(ii) *Let $f(B)$ be a convex set of Y and $\bar{x}$ a nondominated minimal solution of the problem (VOP). Then there exists a continuous linear functional $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for all $c \in intC$, such that $\bar{x}$ is an optimal solution of the problem $P(\varphi)$.*

Proof. (i) If $\bar{x}$ is not a nondominated minimal solution of (VOP), then there exist $c \in C\backslash\{0\}$ and $x \in B$ such that $f(\bar{x}) - f(x) = c$. Thus, we have

$$\varphi(f(\bar{x})) = \varphi(f(x)) + \varphi(c).$$

Since $\varphi(c) > 0$ for all $c \in C\backslash\{0\}$, $\varphi(f(\bar{x})) > \varphi(f(x))$. This contradicts the fact that $\bar{x}$ is an optimal solution of $P(\varphi)$.

(ii) Let $\bar{x}$ be a nondominated minimal solution of (VOP). We have

$$(f(\bar{x}) - C \cup \{0\}) \cap f(B) = \{f(\bar{x})\}.$$

Since $0 \in \partial C$, we have

$$(f(\bar{x}) - intC) \cap f(B) = \varnothing.$$

By the separation theorem for convex sets, there exist $\varphi \in Y^*\backslash\{0\}$ and $r \in \mathbb{R}$ such that

$$\varphi(v) \leq r \text{ if } v \in f(\bar{x}) - intC,$$
$$\varphi(u) \geq r \text{ if } u \in f(B).$$

Thus,

$$\phi(f(\bar{x}) - c) \leq \phi(f(x)), \quad \forall x \in B, c \in intC.$$

As $0 \in \partial C$ and $intC \neq \emptyset$, we can choose a sequence $\{c_k\} \subset intC$ such that $c_k \to 0$ as $k \to \infty$. Consequently,

$$\phi(f(\bar{x}) - c_k) \leq \phi(f(x)), \quad \forall x \in B, k.$$

Passing to the limit as $k \to \infty$, we obtain

$$\phi(f(\bar{x})) \leq \phi(f(x)), \quad \forall x \in B.$$

That is, $\bar{x}$ is an optimal solution of $P(\phi)$.

Furthermore, since $f(\bar{x}) \in f(B)$, for every $c \in intC$ and $v = f(\bar{x}) - c$, we have

$$r \geq \varphi(f(\bar{x}) - c) = \varphi(f(\bar{x})) - \varphi(c) \geq r - \varphi(c).$$

Thus $\varphi(c) \geq 0$, for all $c \in intC$. Arguing as in the second part of the proof of Proposition 2.1, we can show that $\varphi(c) > 0$ for all $c \in intC$. The proof is complete. ■

In what follows, we need the standard Lagrange multiplier theorem for a scalar optimization problem with operatorial convex constraints, which can be found in Barbu and Procparu [9].

Assume that $P \subset Z$ is a closed and convex cone with nonempty interior $intP$. Let $h : X \to \mathbb{R}$ be a convex function and $g : X \to Z$ be P-convex and $K \subset X$ be convex. Consider the scalar convex programming problem:

$$\text{(CP)} \qquad \begin{aligned} &\min h(x) \\ &\text{s.t. } x \in K, \\ &\quad g(x) \leq_P 0. \end{aligned}$$

Theorem 2.6. *Consider the scalar convex programming problem (CP). Suppose that the Slater constraint qualification holds: there exists $x_0 \in K$ such that $g(x_0) \in -intP$. Then $x^* \in K$ is an optimal solution of (CP) if and only if there exists $u^+ \in P^+$ such that*

$$h(x^*) + u^+(g(x^*)) \leq h(x) + u^+(g(x)), \quad \forall x \in K.$$

We establish now characterizations of a solution for the problem (VOP) in terms of Lagrange multipliers . Suppose that $P \subset Z$ is a closed and convex cone with the nonempty interior $intP$, and $K \subset X$ is a convex set. Let $g : X \to Z$ be P-convex.

Theorem 2.7. *Let $f(B)$ be a convex subset of Y. Assume that there exists a point $x_0 \in B$ such that $g(x_0) \in -intP$. Let $\bar{x}$ be a nondominated minimal solution of (VOP). Then there exist $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for all $c \in intC$ and a continuous linear vector-valued function $M : Z \to Y$ such that $M(P) \subset intT(C,0) \cup \{0\}$, $M \circ g(\bar{x}) = 0$ and*

$$\varphi(f(\bar{x}) + M \circ g(\bar{x})) \leq \varphi(f(x) + M \circ g(x)), \quad \forall x \in K.$$

Proof. Suppose that $\bar{x}$ is a nondominated minimal solution of (VOP). By Theorem 2.5(ii), there exists $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for all $c \in intC$, such that

$$\varphi(f(\bar{x})) = \min_{x \in B} \varphi(f(x)).$$

Theorem 2.6 guarantees that there exists $u^+ \in P^+$ with $u^+(g(\bar{x})) = 0$ and

$$\varphi(f(\bar{x})) \leq \varphi(f(x)) + u^+(g(x)), \; \forall x \in K.$$

Choose $\bar{c} \in intC$. Let the vector-valued function $M : Z \to Y$, be defined by $M(z) = u^+(z)\bar{c}/\varphi(\bar{c})$. Since $u^+(z) \geq 0$ for all $z \in P$ and $\varphi(\bar{c}) > 0$, $M(P) \subset intT(C,0) \cup \{0\}$, $M \circ g(\bar{x}) = 0$ and M is a continuous linear vector-valued function. Since $u^+(g(\bar{x})) = 0$ and $\varphi(M \circ g(x)) = \varphi(u^+(g(x))\bar{c}/\varphi(\bar{c})) = u^+(g(x))$, we have

$$\varphi(f(\bar{x}) + M \circ g(\bar{x})) \leq \varphi(f(x) + M \circ g(x)), \quad \forall x \in K. \qquad \blacksquare$$

Corollary 2.8. *Let $C \subset Y$ be a nonempty open and convex set with $0 \in \partial C$. Let the conditions of Theorem 2.7 hold. Then there exists a continuous linear vector-valued function $M : Z \to Y$, such that $M(P) \subset intT(C,0) \cup \{0\}$, $M \circ g(\bar{x}) = 0$ and $\bar{x}$ is a nondominated minimal solution of the unconstrained vector optimization problem:*

$$\underset{x \in K}{Min_C}(f(x) + M \circ g(x)).$$

Proof. By Theorem 2.7 and in view of $C = intC$, there exists a continuous linear vector-valued function $M : Z \to Y$, such that $M(P) \subset intT(C,0) \cup \{0\}$, $M \circ g(\bar{x}) = 0$ and

$$\varphi(f(\bar{x}) + M \circ g(\bar{x})) \leq \varphi(f(x) + M \circ g(x)), \quad \forall x \in K,$$

where $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for all $c \in C$. By Theorem 2.5(i), $\bar{x}$ is a nondominated minimal solution of the following unconstrained problem:

$$\underset{x \in K}{\mathrm{Min}_C}(f(x) + M \circ g(x)). \qquad \blacksquare$$

We establish now the Kuhn-Tucker condition for the problem (VOP). Assume that $f : X \to Y$, $g : X \to Z$ are Fréchet differentiable vector-valued functions and g is P-convex.

Definition 2.9. *The generalized constraint qualification condition is said to hold at $\bar{x}$, if there exists a closed and convex cone $G \subset X$ such that $G \cap K \subset T(B, \bar{x})$, where*

$$K = \{h \in X : g'(\bar{x})(h) \in P(-P, g(\bar{x}))\} \tag{2.19}$$

and $P(-P, g(\bar{x}))$ is the closed and convex hull of $T(-P, g(\bar{x}))$.

Let

$$H = \{u^+(g'(\bar{x})) : u^+ \in P^+(-P, g(\bar{x}))\}. \tag{2.20}$$

We have the following relation between K and H.

Proposition 2.10. *Let K and H be defined by (2.19) and (2.20), respectively. Then $clH = K^+$.*

Proof. First we prove that $clH \subset K^+$. Clearly, we need only to show that $H \subset K^+$. Let $u^+ \in P^+(-P, g(\bar{x}))$ and $u^+(g'(\bar{x})) \in H$. Then, $u^+(v) \geq 0, \forall v \in P(-P, g(\bar{x}))$. Now suppose that $h \in K$. Then, $g'(\bar{x})(h) \in P(-P, g(\bar{x}))$. It follows that $u^+(g'(\bar{x})(h)) \geq 0$, or, $(u^+(g'(\bar{x}))(h) \geq 0$. That is, $u^+(g'(\bar{x})) \in K^+$. Now we prove that $K^+ \subset clH$. Let $h_1 \in K^+$. Then

$$h_1(h) \geq 0, \text{ for any } h \text{ satisfying } g'(\bar{x})(h) \in P(-P, g(\bar{x})). \tag{2.21}$$

Suppose that $h_1 \notin clH$. Since clH is a closed and convex cone, there exists $h_2 \in Z^*$ such that

$$h_2(v_1) \geq 0, \quad \forall v_1 \in H \tag{2.22}$$

and

$$h_2(h_1) < 0. \tag{2.23}$$

By (2.22), we have

$$u^+(g'(\bar{x})(h_2)) \geq 0, \quad \forall u^+ \in P^+(-P, g(\bar{x})).$$

Thus,

$$g'(\bar{x})(h_2) \in \left[P^+(-P, g(\bar{x}))\right]^+ = P(-P, g(\bar{x})).$$

By (2.21), we have $h_2(h_1) \geq 0$, contradicting (2.23). So $h_1 \in clH$. ■

Definition 2.11. *Assume that the generalized constraint qualification condition holds at $\bar{x}$. $H(G)$ is said to hold if*

(i) $K^+ + G^+$ *is closed,*
(ii) H *is closed.*

Proposition 2.12. *Let C be a convex subset of Y with $0 \in \partial C$, S be a subset of Y and $y_0 \in S$ a nondominated minimal point of S. Then*

$$T(S, y_0) \cap (-intC) = \varnothing.$$

Proof. Suppose that $y \in T(S, y_0) \cap (-intC)$. Since $0 \notin intC$, $y \neq 0$. There exist a sequence $\{y^k\} \subset S$ with limit y_0 and a sequence of nonnegative real numbers $\{\alpha_k\}$, such that $\lim_{k \to \infty} \alpha_k(y^k - y_0) = y$. Since y is an interior point of $-C$, there exist an open ball $N(y) \subset -intC$ and a positive integer number $\bar{k}$, such that if $k \geq \bar{k}$ we have $\alpha_k(y^k - y_0) \in N(y)$. Choose $k_0 \geq \bar{k}$, such that $\alpha_{k_0} \geq 1$ and $y^{k_0} \neq y_0$ (since $y \neq 0$ and $\alpha_k \to +\infty$, such a k_0 can be chosen.) Thus there exists $c_0 \in intC$ such that $\alpha_{k_0}(y^{k_0} - y_0) = -c_0$, $y_0 - y^{k_0} = c_0/\alpha_{k_0}$. Since $0 \in \partial C$ and C is a convex set, we have

$$(0, c_0] = \{c = \alpha c_0 : 0 < \alpha \leq 1\} \subset intC.$$

Thus $c_0/\alpha_{k_0} \in intC$. This contradicts the fact that y_0 is a nondominated minimal point of S. ■

The following lemma is an elementary property of tangent cones (see Borwein [19]).

Lemma 2.13. *Let $f : X \to Y$ be a Fréchet differentiable vector-valued function and $B \subset X$. Let $\bar{x} \in B$. Then*

$$f'(\bar{x})(T(B, \bar{x})) \subset T(f(B), f(\bar{x})).$$

Proof. Let $u \in T(B, \bar{x})$. We show that $f'(\bar{x})(u) \in T(f(B), f(\bar{x}))$. Indeed, there exists $\{a_k\} \subset B$ and $\{t_k\} \subset \mathbb{R}_+ \backslash \{0\}$ with $t_k \to 0$ such that $(a_k - \bar{x})/t_k \to u$. Note that $\lim_{k\to+\infty}(f(a_k) - f(\bar{x}))/t_k \in T(f(B), f(\bar{x}))$. Consequently, $\lim_{k\to+\infty}(f(a_k) - f(\bar{x}))/t_k = f'(\bar{x})(u) \in T(f(B), f(\bar{x}))$. The proof is complete. ∎

Theorem 2.14. *Let $C \subset Y$ be a convex set with $0 \in \partial C$ and a nonempty interior $intC$. Let $\bar{x}$ be a minimal solution of the problem (VOP). Assume that $f(B)$ is a convex set and g satisfies the generalized constraint qualification condition and $H(G)$ holds. Then there exist $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for $c \in intC$ and $u^+ \in P^+(P, g(\bar{x}))$, such that*

$$\varphi \circ f'(\bar{x}) - u^+ \circ g'(\bar{x}) \in G^+.$$

Proof. Set $S = f(B)$ and $y_0 = f(\bar{x})$. By Proposition 2.12, we have

$$T(f(B), f(\bar{x})) \cap (-intC) = \varnothing.$$

Since $f(B)$ is a convex set, $T(f(B), f(\bar{x}))$ is a convex cone. By the separation theorem for convex sets, there exists a continuous linear functional $\varphi \in Y^*$, such that

$$\varphi(v) \le \xi \text{ if } v \in -intC,$$
$$\varphi(u) \ge \xi \text{ if } u \in T(f(B), f(\bar{x})).$$

Since $0 \in T(f(B), f(\bar{x}))$, $\xi \le 0$. If there exists a point $c_0 \in intC$, such that $\varphi(-c_0) = r < 0$, it follows from $0 \in \partial C$ that $\lambda c_0 \in intC$ for all λ : $0 < \lambda \le 1$. Choosing $0 < \lambda_1 < \min(\xi/r, 1)$, we have $\varphi(-\lambda_1 c_0) = \lambda_1 \varphi(-c_0) > (\xi/r)\varphi(-c_0) = \xi$. Since $-\lambda_1 c_0 \in -intC$, it is a contradiction. Hence, we have $\xi \ge 0$. Thus, it is necessary that $\xi = 0$. So, $\varphi(c) \ge 0, \forall c \in intC$. Further argument as in the second part of the proof of Proposition 2.1 confirms that $\varphi(c) > 0$ for all $c \in intC$. By Lemma 2.13,

$$f'(\bar{x})(T(B, \bar{x})) \subset T(f(B), f(\bar{x})).$$

Since the generalized constraint qualification condition $G \cap K \subset T(B, \bar{x})$ holds at $\bar{x}$, we have

$$f'(x)(G \cap K) \subset T(f(B), f(\bar{x})).$$

Hence we have $\varphi(u) \ge 0$ if $u \in f'(\bar{x})(G \cap K)$, that is

$$\varphi \circ f'(\bar{x})(h) \ge 0, \quad \forall h \in G \cap K.$$

Since the hypothesis $H(G)$ holds, we have

$$\varphi \circ f'(\bar{x}) \in (K \cap G)^+ = K^+ + G^+.$$

Observing that

$$K^+ = H = \{u^+ \circ g'(\bar{x}) : u^+ \in P^+(-P, g(\bar{x}))\},$$

we obtain

$$\varphi \circ f'(\bar{x}) - u^+ \circ g'(\bar{x}) \in G^+,$$

where $\varphi \in C^+$ satisfying $\varphi(c) > 0$ for all $c \in intC$ and $u^+ \in P^+(-P, g(\bar{x}))$. ■

Proposition 2.15. *A local weakly minimal solution of a C-convex vector-valued function f over a convex set $K \subset X$ with respect to a convex cone C is a global one.*

Proof. Let x^* be a local weakly minimal solution of $f(x)$ over K. Thus, for some neighborhood V of x^*,

$$f(x) - f(x^*) \notin -intC, \quad \forall x \in V \cap K. \tag{2.24}$$

Suppose, if possible, that x^* is not a global weakly minimal solution. Then, there is some $y \in K$ for which

$$f(y) - f(x^*) \in -intC.$$

For $0 < \alpha < 1$, $x^* + \alpha(y - x^*) \in K$, since K is convex. Since f is C-convex,

$$\begin{aligned} f(x^* + \alpha(y - x^*)) - f(x^*) &\in -C + \alpha f(y) + (1 - \alpha) f(x^*) - f(x^*) \\ &= -C + \alpha(f(y) - f(x^*)) \\ &\in -C - intC \\ &\subset -intC, \end{aligned}$$

which contradicts (2.24), since $x^* + \alpha(y - x^*) \in V \cap K$ for sufficiently small positive α. ■

2.2 VO with a Variable Domination Structure

Let Y be a real normed space, and let $C : Y \rightrightarrows Y$ be a set-valued function such that, for each $y \in Y$, the set $C(y)$ is a nonempty convex set with $0 \in \partial C(y)$. Assume that $intC(y) \neq 0, \forall y \in Y$, and $\cap_{y \in S} C(y) \setminus \{0\} \neq \emptyset, S \subset Y$.

Now we establish results corresponding to those in Section 2.1.

Suppose that $S \subset Y$. For any $c \in \bigcap\limits_{y \in S} C(y) \backslash \{0\}$, let

$$S(c) = \{y + \beta c : \beta \geq 0, y \in S\}.$$

It is obvious that $S \subset S(c)$ and, if S is convex, then $S(c)$ is also convex.

Proposition 2.16. *Let $S \subset Y$ be a convex set, $C : Y \rightrightarrows Y$ be a set-valued function such that, for each $y \in S$, $C(y)$ is a nonempty open convex set with $0 \in \partial C(y)$. Then, for any point $c \in \bigcap_{y \in S} C(y)$, we have*

$$LMin_{C(y)}S = LMin_{C(y)}S(c).$$

Proof. It is obvious that $c \neq 0$ since, for any $y \in S$, $0 \in \partial C(y)$, $C(y)$ is open and $c \in C(y)$. Now we can follow the proof of Proposition 2.1 with $Min_C S$ replaced by $LMin_{C(y)}S$, $Min_C S(c)$ replaced by $LMin_{C(y)}S(c)$, C replaced by $C(y^*)$. ■

Theorem 2.17. *Let S be a convex subset of Y, $C : Y \rightrightarrows Y$ be a set-valued function such that, for each $y \in S$, $C(y)$ is a nonempty open convex set with $0 \in \partial C(y)$. Then,*

$$LMin_{C(y)}S = LMin_{intT(C(y),0)}S.$$

Proof. The proof of Theorem 2.3 works if we replace C with $C(y_0)$. ■

Now we assume that X, Y, and Z are Banach spaces, f maps from X into Y and g maps from X into Z. Let $C : X \rightrightarrows Y$ be a set-valued function such that, for each $x \in X$, $C(x)$ is a nonempty and convex set with $0 \in \partial C(x)$, $P \subset Z$ a closed and convex cone and $K \subset X$ a convex set.

We consider a constrained vector optimization problem with a variable domination structure:

$$(VOPV) \qquad \operatorname{Min}_{C(x)} f(x), \quad x \in B$$

where $B = \{x \in X : g(x) \leq_P 0,\ x \in K\}$, and $C : X \rightrightarrows Y$ is a set-valued function.

Theorem 2.18. *Let $C : X \rightrightarrows Y$ be a set-valued function such that, for each $x \in X$, $C(x)$ is a convex subset of Y with $0 \in \partial C(x)$ and $intC(x) \neq \varnothing$. For the vector optimization problem (VOPV), the following results hold:*

(i) *Let $\bar{x} \in B$. Suppose that there exists a continuous linear functional $\varphi \in Y^*$, satisfying $\varphi(c) > 0$, for all $c \in C(\bar{x})\backslash\{0\}$, such that $\bar{x} \in B$ is an optimal solution of the following optimization problem $P(\varphi)$:*

$$\min_{x \in B} \varphi(f(x)).$$

Then $\bar{x}$ is a nondominated-like minimal solution of the problem (VOPV).

(ii) *Let $f(B)$ be a convex subset of Y and $\bar{x}$ a nondominated-like minimal solution of the problem (VOPV). Then there exists a continuous linear functional $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for all $c \in intC(\bar{x})$, such that $\bar{x}$ is an optimal solution of the problem $P(\varphi)$.*

Proof. The proof is the same as that of Theorem 2.5 except that C is replaced by $C(\bar{x})$ and (VOP) is replaced by (VOPV). ■

Let g be a vector-valued function from X into Z and g a P-convex vector-valued function. Suppose that $C : X \rightrightarrows Y$ is a set-valued function such that, for each $x \in X$, $C(x)$ is a convex subset of Y with $0 \in \partial C(x)$ and nonempty interior $intC(x)$, $P \subset Z$ a closed and convex cone with nonempty interior $intP$, and $K \subset X$ a convex set.

Theorem 2.19. *Let $f(B)$ be a convex subset of Y. Assume that there exists a point $x_0 \in B$ such that $g(x_0) \in -intP$. Let $\bar{x}$ be a nondominated-like minimal solution of (VOPV). Then there exist $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for all $c \in intC(\bar{x})$ and a continuous linear vector-valued function $M : Z \to Y$ such that $M(P) \subset intT(C(\bar{x}), 0) \cup \{0\}$, $M \circ g(\bar{x}) = 0$ and*

$$\varphi(f(\bar{x}) + M \circ g(\bar{x})) \leq \varphi(f(x) + M \circ g(x)), \quad \forall x \in K.$$

Proof. The proof of Theorem 2.7 works with C replaced by $C(\bar{x})$. ■

Corollary 2.20. *Let $C : X \rightrightarrows Y$ be a set-valued function such that, for each $x \in X$, $C(x)$ is a nonempty open and convex set with $0 \in \partial C(x)$. Let the conditions of Theorem 2.19 hold. Then there exists a continuous linear vector-valued function $M : Z \to Y$, such that $M(P) \subset intT(C(\bar{x}), 0) \cup \{0\}$, $M \circ g(\bar{x}) = 0$ and $\bar{x}$ is a nondominated-like minimal solution of the unconstrained vector optimization problem with variable domination structure:*

$$\underset{x \in K}{Min_{C(x)}}(f(x) + M \circ g(x)).$$

Proof. The proof is the same as that of Corollary 2.8 except that C is replaced by $C(\bar{x})$. ■

Theorem 2.21. *Let $C : X \rightrightarrows Y$ be set-valued function such that, for each $x \in X$, $C(x)$ is a convex set with $0 \in \partial C(x)$ and nonempty interior $intC(x)$. Let $\bar{x}$ be a nondominated-like minimal solution of the problem (VOPV). Assume that $f(B)$ is a convex set and g satisfies the generalized constraint qualification condition given in Definition 2.9 and $H(G)$ in Definition 2.11 holds. Then there exist $\varphi \in Y^*$ satisfying $\varphi(c) > 0$ for $c \in intC(\bar{x})$ and $u^+ \in P^+(-P, g(\bar{x}))$, such that*

$$\varphi \circ f'(\bar{x}) - u^+ \circ g'(\bar{x}) \in G^+.$$

Proof. The proof of Theorem 2.14 works if C is replaced by $C(\bar{x})$. ■

The following result provides a characterization of a weakly nondominated-like point in terms of the nonlinear scalar function ξ given in (1.3).

Theorem 2.22. *Let Y be a normed space and $A \subset Y$ a nonempty subset. Let $C : Y \rightrightarrows Y$ be a set-valued function such that, for each $y \in Y$, $C(y)$ is a proper, closed and convex cone in Y. Let $\bar{C} = \cap_{y \in Y} C(y)$ and $e \in int\bar{C}$. Let $y^* \in A$. Then y^* is a weakly nondominated-like minimal point of A if and only if*

$$\min_{u \in A - y^*} \xi_e(y^*, u) = 0.$$

Proof. Suppose that $y^* \in A$ is a weakly nondominated-like minimal point of A with respect to the variable domination structure C. By definition, we have

$$(A - y^*) \cap (-intC(y^*)) = \varnothing,$$

equivalently, for each $a \in A$,

$$a - y^* \notin -intC(y^*).$$

By Proposition 1.54, the above inequality holds if and only if

$$\xi_e(y^*, a - y^*) \geq 0, \quad \forall a \in A.$$

Observe that $\xi_e(y^*, 0) = 0$. Obviously, the theorem holds. ■

2.3 Characterizations of Solutions for VO

In this section, we deal with characterizations of the Benson's proper minimal solution for a vector optimization problem.

Let X be a nonempty subset of some Hausdorff topological space and $C \subset Y$ and $P \subset Z$ be closed and convex cones with $intC \neq \emptyset$ and $intP \neq \emptyset$. Let (Y, C) and (Z, P) be two ordered locally convex Hausdorff topological spaces.

Definition 2.23 (Benson [12]). *Let $K \subset Y$ be a nonempty subset. $\bar{y}$ is called a Benson's proper minimal point of K with respect to C, if*

$$clcone(K + C - \bar{y}) \cap (-C) = \{0\}.$$

The set of all Benson's proper minimal points is denoted by $PM(K, C)$.

We consider an unconstrained vector optimization problem

$$(VUP) \qquad \underset{x \in X}{\mathrm{Min}_C}\ f(x),$$

where $f : X \to Y$ is a vector-valued function.

Definition 2.24. *A point $\bar{x}$ is called a Benson's proper minimal solution of (VUP), if $f(\bar{x})$ is a Benson's proper minimal point of the set $f(X)$.*

Next, we establish characterizations of Benson's proper minimal solutions, such as, scalarization, Lagrangian multipliers, saddle-point criterion, duality under a generalized cone-subconvexlikeness and the vector variational inequality, respectively.

(I) Scalarization Characterizations

Lemma 2.25 (Borwein [19]). *Let $C_1, C_2 \subset Y$ be two closed and convex cones such that $C_1 \cap C_2 = \{0\}$. If C_2 is pointed and locally compact, then*

$$(-C_1^+) \cap (C_2^{+i}) \neq \varnothing.$$

Lemma 2.26. *Let $C \subset Y$ be a closed and convex cone with the nonempty interior $intC$, and let $\bar{x} \in X$. Then*

(i) $cone(f(X) + intC - f(\bar{x})) = cone(f(X) - f(\bar{x})) + intC$;
(ii) $clcone(f(X) + intC - f(\bar{x})) = clcone(f(X) + C - f(\bar{x}))$.

Proof. (i) This is obvious, since $intC$ is a cone.

(ii) It is sufficient if the following relation holds:

$$cone(f(X) + C - f(\bar{x})) \subset clcone(f(X) + intC - f(\bar{x})).$$

Indeed, let

$$y \in cone(f(X) + C - f(\bar{x})).$$

Then, there exist $\alpha > 0, x \in X$ and $c \in C$ such that

$$y = \alpha(f(x) + c - f(\bar{x})).$$

Since C is convex, there exists a sequence $\{c_k\} \subset intC$ such that $c = \lim_{k\to\infty} c_k$. Set

$$y_k := \alpha(f(x) + c_k - f(\bar{x})) \in cone(f(X) + intC - f(\bar{x})).$$

Then,

$$\lim_{k\to\infty} y_k = \alpha(f(x) + c - f(\bar{x})) = y.$$

Hence,

$$y \in clcone(f(X) + intC - f(\bar{x})).$$

The proof is complete. ■

We consider a scalar minimization problem for the problem (VUP):

$$(VUP_\mu) \qquad \min_{x\in X} \langle \mu, f(x) \rangle,$$

where $\mu \in Y^* \backslash \{0\}$.

Theorem 2.27. *Let $C \subset Y$ be a closed, convex and pointed cone with nonempty interior $intC$, and let C be also locally compact. Let $\bar{x} \in K$, and let the vector-valued function $f(x) - f(\bar{x})$ be generalized C-subconvexlike on X. Then, $\bar{x}$ is a Benson's proper minimal solution of (VUP) if and only if there exists $\mu \in C^{+i}$ such that $\bar{x}$ is an optimal solution of (VUP_μ).*

Proof. Suppose that $\bar{x} \in X$ is a Benson's proper minimal solution of (VUP). By Definition 2.23, we have

$$clcone(f(X) + C - f(\bar{x})) \cap (-C) = \{0\}.$$

By Lemma 2.26 (ii), we have also

$$clcone(f(X) + intC - f(\bar{x})) \cap (-C) = \{0\}.$$

By Lemma 2.26 (i) and Proposition 1.76,

$$cone(f(X) + intC - f(\bar{x})) = cone(f(X) - f(\bar{x})) + intC$$

is a convex cone, since $f(x) - f(\bar{x})$ is generalized C-subconvexlike on X. Thus, by Lemma 2.25, there exists $\bar{\mu} \in C^{+i}$ such that

$$\bar{\mu} \in (clcone(f(X) + intC - f(\bar{x}))^+.$$

Thus, we obtain

$$\langle \bar{\mu}, f(x) - f(\bar{x}) \rangle \geq 0, \quad \forall x \in X, \tag{2.25}$$

and so, $\bar{x}$ is an optimal solution of $(VUP_{\bar{\mu}})$. Conversely, suppose that there exists $\bar{\mu} \in C^{+i}$ such that $\bar{x}$ is an optimal solution of $(VUP_{\bar{\mu}})$, i.e., (2.25) holds.

For any $u \in cone(f(X) + C - f(\bar{x}))$, there exist $\alpha > 0$, $x \in X$ and $c \in C$ such that

$$u = \alpha(f(x) + c - f(\bar{x})).$$

From (2.25), we have

$$\langle \bar{\mu}, u \rangle = \alpha \langle \mu, f(x) - f(\bar{x}) \rangle + \alpha \langle \bar{\mu}, c \rangle \geq 0. \tag{2.26}$$

For arbitrarily fixed $y \in clcone(f(X) + C - f(\bar{x}))$, there exists a sequence $\{u_k\}$ such that

$$u_k \in cone(f(X) + C - f(\bar{x})), \quad y = \lim_{k \to \infty} u_k.$$

By (2.26), we have

$$\langle \bar{\mu}, u_k \rangle \geq 0, \quad \forall k \in N.$$

Letting $k \to \infty$, we obtain

$$\langle \bar{\mu}, y \rangle \geq 0, \quad \forall y \in cone(f(X) + C - f(\bar{x})). \tag{2.27}$$

Now, assume that $\bar{x} \in X$ is not a Benson's proper minimal solution of (VUP). By Definition 2.23, there exists $y' \in -C \backslash \{0\}$ such that

$$y' \in clcone(f(X) + C - f(\bar{x})).$$

Since $\bar{\mu} \in C^{+i}$, we have $\langle \bar{\mu}, y' \rangle < 0$, which contradicts (2.27). The proof is complete. ∎

(II) Lagrangian Multiplier Characterizations

We consider the following vector constrained optimization problem:

$$(VMP) \qquad \text{Min}_{C} f(x), \quad x \in Q$$

where $Q = \{x \in X : g(x) \leq_P 0\}$, g is a vector valued function from X into Z, and $P \subseteq Z$ is a closed and convex cone with nonempty interior $intP$.

We say that (VMP) satisfies the generalized Slater constraint qualification condition if there exists an $x' \in K$ such that $g(x') \leq_{intP} 0$.

By $L(Z, Y)$, we denote the set of all continuous linear vector-valued functions from Z into Y. A subset $L_+(Z, Y)$ of $L(Z, Y)$ is defined as

$$L_+(Z, Y) := \{T \in L(Z, Y) : T(P) \subset C\}.$$

A vector-valued Lagrangian function for (VMP) is defined as a vector-valued function $L : X \times L_+ \to Y$,

$$L(x, T) := f(x) + T(g(x)), \quad (x, T) \in X \times L_+(Z, Y).$$

We consider an unconstrained vector minimization problem induced by (VMP):

$$(VPL) \qquad \text{Min}_{C} \ L(x, T), \quad x \in X$$

We need to introduce the concept of generalized cone-subconvexlikeness for the ordered pair (f, g).

Let $f : X \to Y$ and $g : X \to Z$ be two vector-valued functions. Let $h(x) = (f(x), g(x))$, $x \in X$. An ordered pair (f, g) is said to be generalized $C \times P$-subconvexlike on X if the vector-valued function $h : X \to Y \times Z$ is generalized $C \times P$-subconvexlike on X.

Lemma 2.28. *Let (f, g) be generalized $C \times P$-subconvexlike on X. Then*

(i) *for each $\mu \in C^+ \backslash \{0\}$, $(\langle \mu, f \rangle, g)$ is generalized $I\!R_+ \times P$-subconvexlike on X;*

(ii) *for each $T \in L_+(Z, Y)$, $L(x, T) = f(x) + T(g(x))$ is generalized C-subconvexlike on X.*

Proof. We prove only (ii), because the proof of (i) is similar to that of (ii). By Proposition 1.76, for any $(\theta_1, \theta_2) \in int(C \times P) = (intC) \times (intP)$, any x_1, $x_2 \in X$ and any $\lambda \in (0, 1)$, there exist $x_3 \in X$ and $\eta > 0$ such that

$$(\theta_1, \theta_2) + \lambda(f(x_1), g(x_2)) + (1 - \lambda)(f(x_2), g(x_2)) - \eta(f(x_3), g(x_3)) \in intC \times intP,$$

which implies that

$$\theta_1 + \lambda f(x_1) + (1-\lambda)f(x_2) - \eta f(x_3) \in intC, \tag{2.28}$$

$$\theta_1 + \lambda g(x_1) + (1-\lambda)g(x_2) - \eta g(x_3) \in intP \subset P, \tag{2.29}$$

From $T \in L_+(Z, Y)$ and (2.29), we obtain

$$T(\theta_1) + \lambda T(g(x_1)) + (1-\lambda)T(g(x_2)) - \eta T(g(x_3)) \in T(P) \subset C. \tag{2.30}$$

Set $\theta := \theta_1 + T(\theta_2)$. Then $\theta \in intC + C \subset intC$. Thus, by (2.28) and (2.30), we have

$$\theta + \lambda(f(x_1) + T(g(x_1)) + (1-\lambda)(f(x_2) + T(g(x_2)) - \eta(f(x_3) + T(g(x_3)) \\ \in intC + C \subset intC.$$

■

Theorem 2.29. *Let $C \subset Y$ be a closed, convex and pointed cone with nonempty interior $intC$, and let C be locally compact. Let $\bar{x} \in K$, let $(f(x) - f(\bar{x}), g(x))$ be generalized $C \times P$-subconvexlike on K. Let (VMP) satisfy the generalized Slater constraint qualification. Then $\bar{x}$ is a Benson's proper minimal solution of (VMP) if and only if there exists $T \in L_+(Z, Y)$ with $T(g(\bar{x})) = 0$ and $\bar{x}$ is a Benson's proper minimal solution of (VPL).*

Proof. Suppose that $\bar{x} \in K$ is a Benson's proper minimal solution of (VMP). By Theorem 2.27 (where $X = K$), there exists $\bar{\mu} \in C^{+i}$ such that $\bar{x}$ is an optimal solution of $(P_{\bar{\mu}})$, i.e.,

$$\langle \bar{\mu}, f(x) - f(\bar{x}) \rangle \geq 0, \quad \forall x \in K. \tag{2.31}$$

From (2.31), it is easy to verify that the following system is inconsistent:

$$(\langle \bar{\mu}, f(x) - f(\bar{x}) \rangle, g(x)) \in -int(\mathbb{R}_+ \times P), \quad \forall x \in X.$$

By Lemma 2.28 and $\bar{\mu} \in C^{+i} \subset C^+ \backslash \{0\}$, the ordered pair $(\langle \bar{\mu}, f(x) - f(\bar{x}) \rangle, g(x))$ is generalized $\mathbb{R}_+ \times P$-subconvexlike on X. By Theorem 1.79, there exists $w^* = (r, \lambda) \in (\mathbb{R}_+ \times P)^+ \backslash \{(0,0)\}$ such that

$$r\langle \bar{\mu}, f(x) - f(\bar{x}) \rangle + \langle \lambda, g(x) \rangle = \langle w^*, (\langle \bar{\mu}, f(x) - f(\bar{x}) \rangle, g(x)) \rangle \\ \geq 0, \quad \forall x \in X. \tag{2.32}$$

If $r = 0$, then $\lambda \neq 0$. In this case, (2.32) becomes

$$\langle \lambda, g(x) \rangle \geq 0, \quad \forall x \in X. \tag{2.33}$$

By the generalized Slater constraint qualification, there exists $x' \in K \subset X$ such that $g(x') \leq_{intP} 0$. Observing that $g(x') \in -intP$ and $\lambda \in P^+ \backslash \{0\}$, so obviously we have also $\langle \lambda, g(x') \rangle < 0$. This is a contradiction. So $r \neq 0$. Since $r \in \mathbb{R}_+$, this implies that $r > 0$. Setting $x = \bar{x}$ in (2.33), we have $\langle \lambda, g(\bar{x}) \rangle \geq 0$. We have also $\langle \lambda, g(\bar{x}) \rangle \leq 0$, since $\bar{x} \in K$, i.e. $g(\bar{x}) \leq_P 0$. Therefore,

$$\langle \lambda, g(\bar{x}) \rangle = 0. \tag{2.34}$$

Set $\sigma := r\bar{\mu}$, we obtain $\sigma \in C^{+i}$. We can select a $c \in C$ such that $\langle \sigma, c \rangle = 1$.

Define

$$T(z) := \langle \lambda, z \rangle c, \quad z \in Z.$$

Obviously, $T \in L(Z, Y)$. Notice that $\lambda \in P^+$, so that we have

$$\langle \lambda, p \rangle \geq 0, \quad \forall p \in P.$$

Hence,

$$T(p) = \langle \lambda, p \rangle c \in C, \quad \forall p \in P,$$

i.e., $T \in L_+(Z, Y)$. By (2.34), we obtain

$$T(g(\bar{x})) = \langle \lambda, g(\bar{x}) \rangle c = 0. \tag{2.35}$$

Since $r > 0$, (2.32) can be rewritten as

$$\langle \bar{\mu}, f(x) - f(\bar{x}) \rangle + (1/r)\langle \lambda, g(x) \rangle \geq 0, \quad \forall x \in X. \tag{2.36}$$

Notice that $1 = \langle \sigma, c \rangle = \langle r\bar{\mu}, c \rangle$. Thus

$$\begin{aligned} (1/r)\langle \lambda, g(x) \rangle &= \langle \bar{\mu}, c \rangle \langle \lambda, g(x) \rangle \\ &= \langle \bar{\mu}, \langle \lambda, g(x) \rangle c \rangle \\ &= \langle \bar{\mu}, T(g(x)). \end{aligned} \tag{2.37}$$

From (2.35) to (2.37), we obtain

$$\langle \bar{\mu}, L(x, T) - L(\bar{x}, T) \rangle \geq 0, \quad \forall x \in X.$$

Thus, $\bar{x}$ is an optimal solution of the following scalar minimization problem:

$$\min_{x \in X} \langle \bar{\mu}, L(x, T) \rangle.$$

From (2.35), we have also

$$L(x, T) - L(\bar{x}, T) = f(x) - f(\bar{x}) + T(g(x)).$$

Hence, by Lemma 2.28, $L(x, T) - L(\bar{x}, T)$ is generalized C-subconvexlike on X. Using Theorem 2.27, $\bar{x}$ is a Benson's proper minimal solution of (VPL).

Now, we suppose that there exists $T \in L_+(Z, Y)$ such that $T(g(\bar{x})) = 0$ and $\bar{x} \in K$ is a Benson's proper minimal solution of (VPL). Using Theorem 2.27 and Lemma 2.28, there exists $\bar{\mu} \in C^{+i}$ such that $\bar{x}$ is an optimal solution of the following scalar minimization problem:

$$\min_{x \in X} \langle \bar{\mu}, L(x, T) \rangle.$$

Then, we have

$$\langle \bar{\mu}, f(\bar{x}) + T(g(\bar{x})) \rangle \leq \langle \bar{\mu}, f(x) + T(g(x)) \rangle, \quad \forall x \in X. \tag{2.38}$$

Observe that $g(x) \leq_P 0, \forall x \in K$. From $T \in L_+(Z, Y)$, we have $T(g(x)) \in -C$, and so

$$\langle \bar{\mu}, T(g(x)) \rangle \leq 0, \quad \forall x \in K. \tag{2.39}$$

By (2.35), (2.38) and (2.39), we obtain

$$\begin{aligned}\langle \bar{\mu}, f(\bar{x}) \rangle &= \langle \bar{\mu}, f(\bar{x}) + T(g(\bar{x})) \rangle \\ &\leq \langle \bar{\mu}, f(x) \rangle + \langle \bar{\mu}, T(g(x)) \rangle \\ &\leq \langle \bar{\mu}, f(x) \rangle,\end{aligned}$$

for all $x \in X$, which shows that there exists $\bar{\mu} \in C^{+i}$ such that $\bar{x} \in K$ is an optimal solution of $(P_{\bar{\mu}})$, where $X = K$. Hence, $\bar{x}$ is a Benson's proper minimal solution of (VMP) by Theorem 2.27. The proof is complete. ■

(III) Saddle-Point Characterizations

We will characterize the Benson's proper minimal solution by using saddle points of a real-valued Lagrangian function of (VMP). To this aim, we need the following concepts.

The real-valued Lagrangian function $L_{\bar{\mu}} : X \times P^+ \to \mathbb{R}$ for (VMP) is defined as

$$L_{\bar{\mu}}(x, \lambda) := \langle \bar{\mu}, f(x) \rangle + \langle \lambda, g(x) \rangle, \quad (x, \lambda) \in X \times P^+,$$

where $\bar{\mu} \in C^{+i}$.

Definition 2.30. *(VMP) is said to satisfy the saddle-point criterion at $\bar{x} \in X$ for some $\bar{\mu} \in C^{+i}$, if there exists $\bar{\lambda} \in P^+$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of the Lagrangian function $L_{\bar{\mu}}(x, \lambda)$, that is,*

$$L_{\bar{\mu}}(\bar{x}, \lambda) \leq L_{\bar{\mu}}(\bar{x}, \bar{\lambda}) \leq L_{\bar{\mu}}(x, \bar{\lambda}), \quad \forall (x, \lambda) \in X \times P^+. \tag{2.40}$$

Theorem 2.31. *Let $C \subset Y$ be a closed, convex and pointed cone with nonempty interior int$C$, and let C be locally compact. Let $\bar{x} \in K$ and $(f(x) - f(\bar{x}), g(x))$ be generalized $C \times P$-subconvexlike on X. Let (VMP) satisfy the generalized Slater constraint qualification. Then, $\bar{x}$ is a Benson's proper minimal solution of (VMP) if and only if there exists $\bar{\mu} \in C^{+i}$ such that (VMP) satisfies the saddle-point criterion at $\bar{x}$ for $\bar{\mu}$.*

Proof. First, suppose that $\bar{x} \in X$ is a Benson proper minimal solution of (VMP). By Theorem 2.29, there exists $T \in L_+(Z, Y)$ such that $T(g(\bar{x})) = 0$ and $\bar{x}$ is a Benson's proper minimal solution of (VPL). From the proof of Theorem 2.29, we know that $L(x, T) - L(\bar{x}, T)$ is generalized C-subconvexlike on X. Therefore, by Theorem 2.27, there exists $\bar{\mu} \in C^{+i}$ such that $\bar{x}$ is an optimal solution of the following scalar minimization problem:

$$\min_{x \in X} \langle \bar{\mu}, L(x, T) \rangle.$$

Thus, we have

$$\langle \bar{\mu}, f(\bar{x}) + T(g(\bar{x})) \rangle \leq \langle \bar{\mu}, f(x) + T(g(x)) \rangle, \quad \forall x \in X. \tag{2.41}$$

Set $\bar{\lambda} := \bar{\mu} \circ T$. Then $\bar{\lambda} \in C^{+i}$. Hence, by the above inequality, we have

$$\begin{aligned} L_{\bar{\mu}}(\bar{x}, \bar{\lambda}) &= \langle \bar{\mu}, f(\bar{x}) \rangle + \langle \bar{\lambda}, g(\bar{x}) \rangle \\ &= \langle \bar{\mu}, f(\bar{x}) + T(g(\bar{x})) \rangle \\ &\leq \langle \bar{\mu}, f(x) + T(g(x)) \rangle \\ &= \langle \bar{\mu}, f(x) \rangle + \langle \bar{\lambda}, g(x) \rangle \\ &= L_{\bar{\mu}}(x, \bar{\lambda}), \quad \forall x \in X. \end{aligned} \tag{2.42}$$

Since $\bar{x} \in K$, $\langle \lambda, g(\bar{x}) \rangle \leq 0, \forall \lambda \in P^+$. Furthermore,

$$\langle \bar{\lambda}, g(\bar{x}) \rangle = \langle \bar{\mu}, T(g(\bar{x})) \rangle = 0,$$

since $T(g(\bar{x})) = 0$. Then

$$\begin{aligned} L_{\bar{\mu}}(\bar{x}, \lambda) &= \langle \bar{\mu}, f(\bar{x}) \rangle + \langle \lambda, g(\bar{x}) \rangle \\ &\leq \langle \bar{\mu}, f(\bar{x}) \rangle + \langle \bar{\lambda}, g(\bar{x}) \rangle \\ &= L_{\bar{\mu}}(\bar{x}, \bar{\lambda}), \quad \forall \lambda \in P^+. \end{aligned}$$

From (2.41) and (2.42), (VMP) satisfies the saddle-point criterion at $\bar{x}$ for $\bar{\mu}$.

Next, suppose that there exists $\bar{\mu} \in C^{+i}$ such that (VMP) satisfies the saddle-point criterion at $\bar{x}$ for $\bar{\mu}$. From Definition 2.30, there exists $\bar{\lambda} \in C^+$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of the Lagrangian function $L_{\bar{\mu}}(x, \lambda)$. By (2.17), for all $x \in K \subset X$,

$$\langle \bar{\mu}, f(\bar{x}) \rangle + \langle \bar{\lambda}, g(\bar{x}) \rangle \leq \langle \bar{\mu}, f(x) \rangle + \langle \bar{\lambda}, g(x) \rangle, \tag{2.43}$$

and, for all $\lambda \in C^+$,

$$\langle \lambda, g(\bar{x}) \rangle \leq \langle \bar{\lambda}, g(\bar{x}) \rangle. \tag{2.44}$$

Taking $\lambda = \alpha \bar{\lambda} \in C^+$, $\alpha \geq 0$, from (2.44), we get

$$(1 - \alpha) \langle \bar{\lambda}, g(\bar{x}) \rangle \geq 0, \quad \forall \alpha \geq 0,$$

which implies that $\langle \bar{\lambda}, g(\bar{x}) \rangle = 0$. Notice that $\langle \bar{\lambda}, g(x) \rangle \leq 0$, $\forall x \in K$. From (2.43), we obtain

$$\langle \bar{\mu}, f(\bar{x}) \rangle \leq \langle \bar{\mu}, f(x) \rangle, \quad \forall x \in K.$$

Hence, $\bar{x}$ is an optimal solution of the following scalar minimization problem:

$$\min_{x \in K} \langle \bar{\mu}, f(x) \rangle.$$

By Theorem 2.27, where $X := K$, $\bar{x}$ is a Benson's proper minimal solution of (VMP). The proof is complete. ■

(IV) Duality Characterizations

Now, we characterize the Benson's proper minimal solution by means of a dual problem of (VMP). Let C be a closed, convex and pointed cone with $intC \neq \emptyset$.

Set
$$L(X,T) := \{L(x,T) : x \in X\}.$$

The set-valued function

$$\Phi(T) := PM(L(X,T),C), \quad T \in L_+(Z,Y),$$

is called the proper dual function for (VMP).

The set-valued optimization problem

$$(VDL) \qquad \mathrm{Max}_C \cup_{T\in L_+(Z,Y)} \Phi(T)$$

is called the dual problem of (VMP).

A point $\bar{y} \in \cup_{T\in L_+(Z,Y)}\Phi(T)$ is called a maximal point of (VDL) if

$$\bar{y} \not\leq_{C\setminus\{0\}} y, \quad y \in \cup_{T\in L_+(Z,Y)}\Phi(T).$$

Theorem 2.32 (Weak Duality). *Let $x \in K$ be any feasible solution of (VMP), and let $y \in \cup_{T\in L_+(Z,Y)}\Phi(T)$ be any feasible point of (VDL). Then*

$$f(x) \not\leq_{C\setminus\{0\}} y.$$

Proof. Assuming that the conclusion is not true, we have $y - f(x) \in C\setminus\{0\}$. Since $x \in K$, $T(g(x)) \in -C$, $\forall T \in L_+(Z,Y)$. Hence, we obtain

$$\begin{aligned} y - L(x,T) &= y - f(x) - T(g(x)) \in C\setminus\{0\} + C \\ &\subset C\setminus\{0\}, \quad \forall x \in K, T \in L_+(Z,Y). \end{aligned} \tag{2.45}$$

On the other hand, from $y \in \cup_{T\in L_+(Z,Y)}\Phi(T)$, there exists $\bar{T} \in L_+(Z,Y)$ such that
$$y \in \Phi(\bar{T}) = PM(L(X,\bar{T}),C).$$

Hence $y \in Min_C(L(X,\bar{T}))$. By the definition of the minimal point, it follows that
$$y - L(x,\bar{T}) \notin C\setminus\{0\}, \quad \forall x \in X.$$

Consequently, we have

$$y - L(x,\bar{T}) \notin C\setminus\{0\}, \quad \forall x \in K,$$

which contradicts (2.45). ■

Lemma 2.33. *Let $C \subset Y$ be a closed, convex and pointed cone with the nonempty interior int$C$, and let C be also locally compact. Let $\bar{x} \in K$ and $f(\bar{x}) \in \cup_{T\in L_+(Z,Y)}\Phi(T)$, and let $(f(x) - f(\bar{x}), g(x))$ be generalized $C \times P$-subconvexlike on X. Then $\bar{x}$ is a Benson's proper minimal solution of (VMP) and $f(\bar{x})$ is a maximal point of (VDL).*

Proof. From

$$f(\bar{x}) \in \cup_{T\in L_+(Z,Y)}\Phi(T),$$

there exists $\bar{T} \in L_+(Z, Y)$ such that

$$f(\bar{x}) \in \Phi(\bar{T}) = PM(L(X, \bar{T}), C).$$

This implies that there exists $\hat{x} \in X$ such that $f(\bar{x}) = L(\hat{x}, \bar{T})$ and $\hat{x}$ is a Benson's proper minimal solution of (VPL), where $T = \bar{T}$. Since $(f(x) - f(\bar{x}), g(x))$ is generalized $C \times P$ subconvexlike on X, by Lemma 2.28 and $f(\bar{x}) = L(\hat{x}, \bar{T})$,

$$L(x, \bar{T}) - L(\hat{x}, \bar{T}) = f(x) - f(\bar{x}) + \bar{T}(g(x))$$

is generalized C-subconvexlike on X. By Theorem 2.27, there exists $\bar{\mu} \in C^{+i}$ such that $\hat{x}$ is an optimal solution of the problem $\min_{x\in X}\langle\bar{\mu}, L(x, \bar{T})\rangle$. Hence, we have

$$\langle\bar{\mu}, f(\bar{x})\rangle \leq \langle\bar{\mu}, f(x)\rangle + \langle\bar{\mu}, \bar{T}(g(x))\rangle, \quad \forall x \in X.$$

Since $x \in K$, $\bar{T}(g(x)) \in -C$, so $\langle\bar{\mu}, \bar{T}(g(x))\rangle \leq 0$. Therefore, we obtain

$$\langle\bar{\mu}, f(\bar{x})\rangle \leq \langle\bar{\mu}, f(x)\rangle, \quad \forall x \in K.$$

This shows that $\bar{x}$ is an optimal solution of $(VUP_{\bar{\mu}})$, where $X = K$. So, $\bar{x}$ is a Benson's proper minimal solution of (VMP) by Theorem 2.27.

By Theorem 2.32, we know that

$$y - f(\bar{x}) \notin C\backslash\{0\}, \quad \forall y \in \cup_{T\in L_+(Z,Y)}\Phi(T).$$

Since

$$f(\bar{x}) \in \cup_{T\in L_+(Z,Y)}\Phi(T),$$

$f(\bar{x})$ is a maximal point of (VDL). ■

Theorem 2.34 (Strong Duality). *Let $C \subset Y$ be a closed, convex and pointed cone with the nonempty interior int$C$, and let C be also locally compact. Let $\bar{x} \in K$, let $f(x) - f(\bar{x})$ be generalized C-subconvexlike on K, and let $(f(x) - f(\bar{x}), g(x))$ be generalized $C \times P$-subconvexlike on X. Let (VMP) satisfies the generalized Slater constraint qualification. Then $\bar{x}$ is a Benson's proper minimal solution of (VMP) if and only if $f(\bar{x})$ is a maximal point of (VDL).*

Proof. Suppose that $\bar{x} \in K$ is a Benson's proper minimal solution of (VMP). By Theorem 2.29, there exists $\bar{T} \in L_+(Z, Y)$ with $\bar{T}(g(\bar{x})) = 0$ and $\bar{x}$ satisfies

$$f(\bar{x}) \in PM(L(X, \bar{T}), C) = \Phi(\bar{T}) \subset \cup_{T \in L_+(Z,Y)} \Phi(T).$$

By Lemma 2.33, $f(\bar{x})$ is a maximal point of (VDL). Conversely, suppose that $f(\bar{x})$ is a maximal point of (VDL). Then $f(\bar{x})$ is a feasible point of (VDL), i.e.,

$$f(\bar{x}) \in \cup_{T \in L_+(Z,Y)} \Phi(T).$$

Using Lemma 2.33, it follows from $\bar{x} \in K$ that $\bar{x}$ is a Benson's proper minimal solution of (VMP). ∎

2.4 Continuity of Solutions for VO

Consider the constrained parametric vector optimization problem

$$\text{(PVOP)} \qquad \text{Min}_{\mathbb{R}^\ell_+} f(x), \quad \text{s. t. } g(x,t) \le b(t), \quad x \in \Omega, t \in T,$$

where $\Omega \subset \mathbb{R}^p$ is a nonempty and convex subset, T is a nonempty subset of a metric space, and $f : \mathbb{R}^p \to \mathbb{R}^\ell$, $g : \mathbb{R}^p \times T \to \mathbb{R}^m$, $b : T \to \mathbb{R}^m$ are continuous vector-valued functions, with $g(\cdot, t)$ convex, $\forall t \in T$. If Ω and T are fixed, then $(PVOP)$ is specified by the three elements f, g, b. Thus, problem $(PVOP)$ can be denoted as problem $q = (g, b, f)$. Denote by $C(\Omega, \mathbb{R}^\ell)$ the space of continuous functions from Ω into $\mathbb{R}^\ell$, with the metric

$$\rho_1(f_1, f_2) = \sup_{x \in \Omega} \|f_1(x) - f_2(x)\|,$$

where $\|\cdot\|$ denotes a norm in $\mathbb{R}^\ell$. Define similarly the metric ρ_2 for $C(\Omega \times T, \mathbb{R}^m)$ and ρ_3 for $C(T, \mathbb{R}^m)$. Then $(PVOP)$ may be considered as an element of the space $Z = C(\Omega, \mathbb{R}^\ell) \times C(\Omega \times T, \mathbb{R}^m) \times C(T, \mathbb{R}^m)$, with the metric $\rho = \rho_1 + \rho_2 + \rho_3$. Denote by Q a subspace of Z corresponding to problems $(PVOP)$. Denote also $Y = C(\Omega \times T, \mathbb{R}^m) \times C(T, \mathbb{R}^m)$.

Let

$$X(g, b) = \{x \in \Omega : g(x,t) \le b(t), \forall t \in T\}$$

be the feasible region for (PVOP) and $x_0 \in X(g, b)$. If, for some neighborhood $V(x_0)$ of x_0,

$$f(x) - f(x_0) \in W, \quad \forall x \in V(x_0) \cap X(g, b),$$

where $W = \mathbb{R}^\ell \backslash (-int\mathbb{R}^\ell_+)$, then x_0 is called a local weakly minimal solution of (P). Denote by $M(q)$ the set of all the local weakly minimal solutions of $(PVOP)$. If

$$f(x) - f(x_0) \in W, \quad \forall x \in X(g, b),$$

then x_0 is called a (global) weakly minimal solution of (PVOP). Denote by $M_g(q)$ the set of all the weakly minimal solutions of (PVOP).

Lemma 2.35. *Let $(g_0, b_0) \in Y$, and let T be compact. Assume that the Slater condition holds, i.e., there exists $s \in \Omega$ such that*

$$g_0(s,t) < b_0(t), \quad \forall t \in T. \tag{2.46}$$

Then, the set-valued function X is l.s.c. at (g_0, b_0). If, in addition, Ω is compact, then X is also u.s.c. at (g_0, b_0).

Proof. . Let G be open, and let $G \cap X(g_0, b_0) \neq \varnothing$. Now, $X(g_0, b_0) = \Omega \cap A$, where

$$A = \{x \in \mathbb{R}^p : g_0(x,t) \leq b_0(t), \quad \forall t \in T\}.$$

From the Slater condition, $\Omega \cap intA \neq \varnothing$. Furthermore, by the convexity of $g_0(\cdot, t)\forall t$, the Slater condition and g_0 and b_0 are continuous, as well as the compactness of T, we have

$$intA = \{x \in \mathbb{R}^n : g_0(x,t) < b_0(t), \quad \forall t \in T\}.$$

Let $x' \in G \cap X(g_0, b_0)$. Then there is a sequence $\{z_j\} \subset \Omega \cap intA$ with $\{z_j\} \to x'$; and, for some j, $z_j \in G$. Since G is open, $G \cap \Omega \cap intA \neq \varnothing$.

Let $x_0 \in G \cap \Omega \cap intA$. Suppose, if possible, that there does not exist $\epsilon > 0$ such that the ball $B(x_0, \epsilon) \subset X(g_k, b_k)$ for all sufficiently large k. Then, for each $\epsilon = \dfrac{1}{k}$, $k \in N$, there exists x_k for which $x_k \notin X(g_k, b_k)$ and $||x_k - x_0|| < \dfrac{1}{k}$, so $\{x_k\} \to x_0$. Since $x_k \notin X(g_k, b_k)$, there is a subsequence $\{z_j\}$ of $\{x_k\}$, an integer $i \in \{1, 2, \cdots, m\}$, and a sequence $\{t_j\} \subset T$, such that the component i of $g_j(z_j, t_j) - b_j(t_j)$ is positive for each j. Since T is compact, a suitable subsequence $\{t_j\} \to t_0 \in T$. Let $j \to \infty$; the component i of $g_0(x_0, t_0) - b_0(t_0) \geq 0$, contradicting $g_0(x_0, t) < b_0(t_0)$. Hence, for some $\epsilon > 0$, and all sufficiently large k, $B(x_0, \epsilon) \subset X(g_k, b_k)$; hence $G \cap X(g_k, b_k) \neq \varnothing$. Thus, X is l.s.c. at (g_0, b_0).

Now, let Ω be compact; let $(g_k, b_k) \to (g_0, b_0) \in Y$; let

$$x_k \in X(g_k, b_k), \quad \{x_k\} \to x^*, \forall k \in N.$$

Thus, uniform convergence and continuity on the compact set Ω give $x^* \in X(g_0, b_0)$; hence, X is u.s.c. at (g_0, b_0). ■

Theorem 2.36. *Let $q_0 = (g_0, b_0, f_0) \in Q$, and let T and Ω be compact. Then M is u.s.c. at q_0.*

Proof. If M is not u.s.c. at q_0, then there is an open set $G \supset M(q_0)$ and a sequence $\{q_k\} \subset Q$ such that, for each k, $M(q_k)$ is not contained in G. Thus, for each k, there is $x_k \in M(q_k) \backslash G$. Since Ω is compact, $\{x_k\}$ may be replaced by a convergent subsequence, say $\{x_k\} \to x_0$, with $x_0 \in X(g_0, b_0)$ by the continuity of g_0 and b_0. Since M is not u.s.c. at q_0, $x_0 \notin M(q_0)$. Thus, from the definition of local weakly minimal solutions, there exist $V(x_0) \subset G$ and $z_0 \in X(g_0, b_0) \cap V(x_0)$ such that

$$f_0(x_0) - f_0(z_0) \in int\mathbb{R}^\ell_+,$$

where $V(x_0)$ denotes some neighborhood of x_0. By Lemma 2.35, X is lower semicontinuous at (g_0, b_0), so there exists a sequence $\{z_k\} \to z_0$, with $z_k \in X(g_k, b_k)$, $\forall k$. Since $\{x_k\} \to x_0$ and $\{z_k\} \to z_0$,

$$f_k(x_k) - f(z_k) \in int\mathbb{R}^\ell_+,$$

for all sufficiently large k, contradicting $x_k \in M(q_k)$. ■

Lemma 2.37. *Let $(g_k, b_k) \to (g_0, b_0)$; let $C_k = X(g_k, b_k)$ and $C_0 = X(g_0, b_0)$ be convex sets; let C_0 contain two points a, b with $2d := ||a - b|| > 0$; let the set-valued function X be l.s.c. at (g_0, b_0). If $0 < \omega < d$, then*

$$C_0' \equiv \{x \in C_0 : ||x - a|| = \omega\} \neq \varnothing;$$

$$C_k' \equiv \{x \in C_k : ||x - a|| = \omega\} \neq \varnothing.$$

Proof. If $0 < \omega < d$, then $b \notin B(a, \omega)$. Since C_0 is convex, the line segment $[a, b] \subset C_0$ and $[a, b]$ intersects C_0'. So, $C_0' \neq \varnothing$. Since X is l.s.c. at (g_0, b_0) and $(g_k, b_k) \to (g_0, b_0)$, there exist u_k, $v_k \in C_k$, with $\{u_k\} \to a$ and $\{v_k\} \to b$. Now $d(u_k, v_k) > \omega$ provided that

$$||u_k - a|| < \frac{\delta - \omega}{2}, ||v_k - b|| < \frac{\delta - \omega}{2}, \text{ with } \delta = ||a - b|| > \omega,$$

for k sufficiently large. Also, we have $[u_k, v_k] \subset C_k$, since C_k is convex. Hence, $C_k' \neq \varnothing$. ■

Theorem 2.38. *Let $\Omega \subset \mathbb{R}^p$ be compact and convex; let $q_0 = (g_0, b_0, f_0) \in Q$ be a convex problem. For each $x_0 \in M(q_0)$, assume that the Slater constraint qualification (2.46) holds, and assume that the following coercivity condition holds, i.e., there are a vector $\delta \equiv \delta(x_0) \in \mathbb{R}^\ell$ and a positively increasing function τ, with $\tau(0) = 0$, depending on δ and x_0, such that*

$$\delta f_0(x) - \delta f_0(x_0) \geq 4\tau(||x - x_0||).$$

Then the set-valued function M is l.s.c. at q_0, in the domain of convex problems (PVOP).

Proof. Let

$$\{q_k\} \equiv \{(g_k, b_k, f_k)\} \to q_0 \equiv (g_0, b_0, f_0),$$

with each q_k a convex (PVOP) problem. Let $x_0 \in M(q_0)$ and $\delta = \delta(x_0)$. Let

$$N(\alpha) = \{x \in \mathbb{R}^p : \varphi(x, X(g_0, b_0)) < \alpha\},$$

where $\varphi(\cdot, \cdot)$ is the metric function in $\mathbb{R}^p$ from a point to the set. By Lemma 2.35, X is u.s.c. at (g_0, b_0). Hence, $X(g_k, b_k) \subset N(\alpha)$ for all sufficiently large

k. With d, C_0', C_k' in Lemma 2.37, noting that the feasible sets are convex, and with $0 < \omega < \min\{d, \alpha\}$, Lemma 2.37 shows that $C_0' \neq \varnothing, C_k' \neq \varnothing$. Since $\{q_k\} \to q_0$ uniformly, we deduce that $|\delta f_k(x') - \delta f_0(x')| < \tau \equiv \tau(\omega)$, for all x' with $||x' - x_0|| = \omega$ and all sufficiently large k. For such x',

$$\begin{aligned} \delta f_k(x_0) &< \delta f_0(x_0) + \tau, && \text{by the uniform convergence,} \\ &\leq \delta f_0(x') - 4\tau + \tau, && \text{by the coercivity condition,} \\ &< \delta f_k(x') + \tau - 4\tau + \tau, && \text{by the uniform convergence.} \end{aligned}$$

Thus,

$$\delta f_k(x') > \delta f_k(x_0) + 2\tau.$$

Since X is l.s.c. at (g_0, b_0) by Lemma 2.35, there exists a sequence $\{x_k\} \to x_0$, with $x_k \in X(g_k, b_k)$ for each k. Hence,

$$\delta f_k(x') > \delta f_k(x_k) + 2\tau - \tau,$$

for all sufficiently large k, by the uniform convergence. Consequently, x' is not a minimal solution of $\delta f_k(x)$ over x feasible for q_k with $\|x - x_0\| \leq \omega$. Now, such a minimal solution exists, say at $z = z_k$, since the intersection of the feasible set of q_k and the set $\{x : \|x - x_0\| \leq \omega\}$ is compact; therefore,

$$z_k \in \{x \in \mathbb{R}^n : ||x - x_0|| < \omega\}.$$

Hence, z_k is a local weakly minimal solution of $\delta f_k(x)$ over the feasible set of q_k; it is global, since q_k is convex.

Suppose, if possible, that z_k is not a weakly minimal solution of q_k. Then, for some feasible x,

$$f_k(x) - f_k(z_k) \in -int\mathbb{R}^\ell_+.$$

But $0 \neq \delta \in \mathbb{R}^\ell_+$, so that

$$\delta f_k(x) - \delta f_k(z_k) < 0,$$

contradicting the minimality of z_k. Thus, z_k is a weakly minimal solution of q_k. Choose a sequence $\{\omega_j\} \to 0+$. For each ω_j, for sufficiently large k, say all $k > k_j$, q_k has a weakly minimal solution satisfying $||z_k - x_0|| < \omega_j$. Thus M is l.s.c.. ■

2.5 Set-Valued VO with a Fixed Domination Structure

Optimizations with set-valued objective functions are closely related to problems in stochastic programming, fuzzy programming, optimal control and the duality of vector optimization problems. If the values of a given function vary in a specified region, this fact could be described using a membership function in theory of fuzzy sets or using information on distributions of the function values. In this general setting, probability distributions or membership functions are not needed because only sets are considered. Optimal control problems with differential inclusions belong to this class of set-valued optimization problems as well. Set-valued optimization seems to have the potential to become a bridge between different areas in optimization. And it is a substantial extension of standard optimization theory. Set-valued analysis is the most important tool for such an advancement in continuous optimization. And conversely, the development of set-valued analysis receives important impulses from set-valued optimization. In this section, we consider set-valued optimization problems with fixed domination structures.

Let X and Y be normed spaces, and let Y be ordered by a convex cone $C \subset Y$. Let K be a nonempty subset of X, and let $F : X \rightrightarrows Y$ be a set-valued function. The epigraph of F is defined by

$$epi_C F = \{(x, y) \in (X, Y) : y \in F(x) + C\}.$$

Let $F : K \rightrightarrows Y$ be a set-valued function. Let a pair $(\bar{x}, \bar{y})$ with $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ be given.

Definition 2.39 (Jahn and Rauh [118]). *A function $D_e F(\bar{x}, \bar{y}) : X \to Y$, whose epigraph equals the contingent cone to the epigraph of F at $(\bar{x}, \bar{y})$, i.e.,*

$$epi_C D_e F(\bar{x}, \bar{y}) = T(epi_C F, (\bar{x}, \bar{y}))$$

is called the contingent epiderivative of F at $(\bar{x}, \bar{y})$.

It is worth noting that the contingent epiderivative exists only for some specially ordered spaces.

Definition 2.40. *Let $\bar{x} \in X, \bar{y} \in Y$ and $G(x) = \{y \in Y : (x, y) \in T(epi_C F, (\bar{x}, \bar{y}))\}$. A set-valued function $D_g F(\bar{x}, \bar{y}) : K - \{\bar{x}\} \rightrightarrows Y$ is called the generalized contingent epiderivative of F at $(\bar{x}, \bar{y})$, if, for any $x \in K - \{\bar{x}\}$,*

$$D_g F(\bar{x}, \bar{y})(x) = \begin{cases} Min_C G(x), & \text{if } G(x) \neq \varnothing, \\ \varnothing, & \text{if } G(x) = \varnothing. \end{cases}$$

Note that, for some $x \in K - \{\bar{x}\}$, the set $\{y \in Y : (x, y) \in T(\mathrm{epi}_C F, (\bar{x}, \bar{y}))\}$ may be empty. In this case, we have $D_g F(\bar{x}, \bar{y})(x) = \varnothing$.

The following lemma is needed.

Lemma 2.41. *[142] Let $C \subset Y$ be a closed and convex cone. Assume that C is Daniell. Let $A \subset Y$ be nonempty, closed and minorized. Then $Min_C A$ is nonempty.*

Theorem 2.42. *Let C be a pointed, closed and convex cone, and let C be Daniell. Let, for every $x \in K$, the set $G(x) = \{y \in Y : (x, y) \in T(epi_C F, (\bar{x}, \bar{y}))\}$ be minorized. Then, for all $x \in K$, $D_g F(\bar{x}, \bar{y})(x)$ exists.*

Proof. Since the contingent cone is always closed in a normed space, for every $x \in K$, $G(x)$ is minorized and closed. By Lemma 2.41, $\text{Min}_C G(x)$ is nonempty, i.e., $D_g F(\bar{x}, \bar{y})$ is well-defined. ∎

Theorem 2.43. *Let C be a closed, pointed and convex cone in Y, and let $K = X$. Let, for all $x \in X$, $D_g F(\bar{x}, \bar{y})(x) \neq \emptyset$. Then $D_g F(\bar{x}, \bar{y})$ is strictly positive homogeneous. Moreover, if F is C-convex and the set*

$$G(x) = \{y \in Y : (x, y) \in T(epiF, (\bar{x}, \bar{y}))\}$$

fulfills the domination property for all $x \in X$, then $D_g F(\bar{x}, \bar{y})$ is subadditive.

Proof. We take any $\alpha > 0$ and $x \in X$. Then we obtain

$$\begin{aligned}
\frac{1}{\alpha} D_g F(\bar{x}, \bar{y})(\alpha x) &= \text{Min}_C\{\frac{1}{\alpha} y \in Y \ : \ (\alpha x, y) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))\} \\
&= \text{Min}_C\{u \in Y \ : \ (\alpha x, \alpha u) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))\} \\
&= \text{Min}_C\{u \in Y \ : \ (x, u) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))\} \\
&= D_g F(\bar{x}, \bar{y})(x).
\end{aligned}$$

Thus

$$D_g F(\bar{x}, \bar{y})(\alpha x) = \alpha D_g F(\bar{x}, \bar{y})(x).$$

Next, for $x_1, x_2 \in X$, $y_1 \in D_g F(\bar{x}, \bar{y})(x_1)$, $y_2 \in D_g F(\bar{x}, \bar{y})(x_2)$, we have $(x_1, y_1) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))$ and $(x_2, y_2) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))$. Since F is C-convex, $\text{epi}_C F$ is convex and $T(\text{epi}_C F, (\bar{x}, \bar{y}))$ is a convex cone, we have

$$(x_1 + x_2, y_1 + y_2) \in T(\text{epi}_C F, (\bar{x}, \bar{y})),$$

implying

$$D_g F(\bar{x}, \bar{y})(x_1) + D_g(\bar{x}, \bar{y})(x_2) \subset G(x_1 + x_2).$$

By the domination property, we have

$$G(x_1 + x_2) \subset \text{Min}_C G(x_1 + x_2) + C = D_g(F(\bar{x}, \bar{y})(x_1 + x_2) + C.$$

Thus

$$D_g F(\bar{x}, \bar{y})(x_1) + D_g F(\bar{x}, \bar{y})(x_2) \subset D_g F(\bar{x}, \bar{y})(x_1 + x_2) + C$$ ∎

Lemma 2.44. *[118] Let $F : X \rightrightarrows Y$ be a set-valued function. Let $(\bar{x}, \bar{y}) \in Gr(F)$. If the contingent epiderivative $D_e F(\bar{x}, \bar{y})$ exists, then it is unique.*

Now we consider the relation between the generalized contingent epiderivative and the contingent epiderivative, and we give an existence theorem of the contingent epiderivative in a complete vector lattice.

Theorem 2.45. *Let (Y, C) be an ordered complete vector lattice. Then, for any $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$, the contingent epiderivative $D_eF(\bar{x}, \bar{y})$ exists and*

$$D_eF(\bar{x}, \bar{y})(x) = Inf_C \ \{y \in Y : (x, y) \in T(epi_C F, (\bar{x}, \bar{y}))\}, \quad \forall x \in X.$$

Proof. We define

$$f(x) = \mathrm{Inf}_C\{y \in Y : (x, y) \in T(\mathrm{epi}_C F, (\bar{x}, \bar{y}))\}, \quad \forall x \in X.$$

Since (Y, C) is an ordered complete vector lattice, $f(x)$ is well defined for every $x \in X$ and $f(x)$ is single-valued. Now we prove that $f = D_eF(\bar{x}, \bar{y})$. By the order completeness of Y, we have

$$f(x) + C \supset G(x), \quad \forall x \in X.$$

For $(x, y) \in T(\mathrm{epi}_C F, (\bar{x}, \bar{y}))$, we have $y \in G(x) \subset f(x) + C$. Thus $(x, y) \in \mathrm{epi}_C f$. Hence

$$T(\mathrm{epi}_C F, (\bar{x}, \bar{y})) \subset \mathrm{epi}_C f.$$

Conversely, for any $x \in X$, it follows from the vector completeness that

$$(x, f(x)) \subset T(\mathrm{epi}_C F, (\bar{x}, \bar{y})).$$

Thus

$$\begin{aligned}
\mathrm{epi}_C f &\subset T(\mathrm{epi}_C F, (\bar{x}, \bar{y})) + \{0\} \times C \\
&= \mathrm{epi}_C D_eF(\bar{x}, \bar{y}) + \{0\} \times C \\
&= \mathrm{epi}_C D_eF(\bar{x}, \bar{y}) \\
&= T(\mathrm{epi}_C F, (\bar{x}, \bar{y})).
\end{aligned}$$

Hence we have

$$\mathrm{epi}_C f = T(\mathrm{epi}_C F, (\bar{x}, \bar{y})).$$

Consequently, f equals the epiderivative $D_eF(\bar{x}, \bar{y})$ which, by Lemma 2.44, is unique. ■

Theorem 2.46. *Let X, Y be real normed spaces, let $K = X$, let $C \subset X$ be a pointed and convex cone and $F : K \rightrightarrows Y$ a set-valued function. Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$ be given. If the contingent epiderivative $D_eF(\bar{x}, \bar{y})$ exists and the set $G(x) = \{y \in Y : (x, y) \in T(epi_C F, (\bar{x}, \bar{y}))\}$ fulfills the domination property for all $x \in X$, then*

$$epi_C D_eF(\bar{x}, \bar{y}) = epi_C D_gF(\bar{x}, \bar{y}).$$

Proof. By the definition of D_gF, we have

$$\begin{aligned}
\mathrm{epi}_C D_gF(\bar{x}, \bar{y}) &\subset T(\mathrm{epi}_C F, (\bar{x}, \bar{y})) + \{0\} \times C \\
&= \mathrm{epi}_C D_eF(\bar{x}, \bar{y}) + \{0\} \times C \\
&= \mathrm{epi}_C D_eF(\bar{x}, \bar{y}).
\end{aligned}$$

Thus,

$$\text{epi}_C D_g F(\bar{x}, \bar{y}) \subset \text{epi}_C D_e F(\bar{x}, \bar{y}).$$

Conversely, we suppose that

$$(x, \tilde{y}) \in epi_C D_e F(\bar{x}, \bar{y}) \text{ and } (x, \tilde{y}) \notin \text{epi} D_g F(\bar{x}, \bar{y}),$$

i.e.,

$$\tilde{y} \notin D_g F(\bar{x}, \bar{y})(x) + C,$$

or

$$\tilde{y} \notin \text{Min}_C\{y \in Y : (x, y) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))\} + C.$$

Since $(x, \tilde{y}) \in \text{epi}_C D_e F(\bar{x}, \bar{y})$, i.e., $(x, \tilde{y}) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))$, we have

$$\tilde{y} \in \{y \in Y : (x, y) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))\}.$$

By the domination property of $G(x)$,

$$\tilde{y} \in \text{Min}_C\{y \in Y : (x, y) \in T(epi_C F, (\bar{x}, \bar{y}))\} + C = D_g F(\bar{x}, \bar{y})(x) + C.$$

This is a contradiction. Hence,

$$\text{epi}_C D_e F(\bar{x}, \bar{y}) = \text{epi}_C D_g F(\bar{x}, \bar{y}).$$ ■

We consider a set-valued optimization problem:

(SOK) $$\text{Min}_{x \in K} {}_C F(x),$$

where $F : X \rightrightarrows Y$ is a set-valued function and $K \subset X$.

Definition 2.47. *Let the ordering cone C have the nonempty interior $intC$. A pair $(\bar{x}, \bar{y})$ with $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$ is called a (weakly) minimal pair of the problem (SOK) if $\bar{y}$ is a (weakly) minimal point of the set of $F(K)$, where*

$$F(K) = \cup_{x \in K} F(x).$$

We can obtain a unified necessary and sufficient optimality condition for a weakly minimal solution of (SOK).

Theorem 2.48. *Let C have nonempty interior $intC$, and let $(\bar{x}, \bar{y}) \in Gr(F)$ be a (weakly) minimal pair of (SOK). Then*

$$D_g F(\bar{x}, \bar{y})(x - \bar{x}) \subset W, \quad \forall x \in K,$$

where $W = Y \backslash (-intC)$.

Proof. We prove by contradiction. Suppose that there exists an $x \in K$ such that

$$D_gF(\bar{x},\bar{y})(x-\bar{x}) \not\subseteq W.$$

Then, there is a $z \in D_gF(\bar{x},\bar{y})(x-\bar{x})$ with $z \in -intC$. By the definition of $D_gF(\bar{x},\bar{y})$, we have

$$(x-\bar{x},z) \in T(\text{epi}_C F,(\bar{x},\bar{y})).$$

Then there are sequences $\{(x_k,y_k)\}_{k\in N}$ in $\text{epi}_C F$ and $\{\lambda_k\}_{k\in N}$ of positive real numbers with $(\bar{x},\bar{y}) = \lim_{k\to\infty}(x_k,y_k)$ and

$$(x-\bar{x},z) = \lim_{k\to\infty}\lambda_k(x_k-\bar{x},y_k-\bar{y}).$$

Thus there exists an $M \in N$ with

$$y_k \in \{\bar{y}\} - intC, \quad \forall k \geq M.$$

Since $(x_k,y_k) \in \text{epi}_C F$, there is a $\tilde{y_k} \in F(x_k)$ with $y_k \in \{\tilde{y_k}\} + C$. Thus

$$\tilde{y_k} \in \{y_k\} - C \subset \{\bar{y}\} - intC - C = \{\bar{y}\} - intC, \quad \forall n \geq M.$$

Hence $(\bar{x},\bar{y})$ is not a weakly minimal pair of (SOK). ∎

We need the following lemma for deriving a sufficient optimality condition.

Lemma 2.49. *Let K be a nonempty convex subset of X. Let C be a closed, convex and pointed cone being Danielll, let F be C-convex, and let, for every $x \in K$ and $y \in F(x)$, $G(x-\bar{x})$ be minorized. Moreover, let the set $G(x-\bar{x})$ fulfill the domination property, for all $x \in K$, where*

$$G(x) = \{y \in Y : (x,y) \in T(epi_C F,(\bar{x},\bar{y}))\}.$$

Then, for every $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$,

$$F(x) - \{\bar{y}\} \subset D_gF(\bar{x},\bar{y})(x-\bar{x}) + C, \quad \forall x \in K.$$

Proof. Take arbitrary elements $x \in K$ and $y \in F(x)$. We define a sequence $\{(x_k,y_k)\}_{k\in N}$ with

$$x_k = \bar{x} + \frac{1}{k}(x-\bar{x}), \quad \forall k \in N,$$
$$y_k = \bar{y} + \frac{1}{k}(y-\bar{y}), \quad \forall k \in N.$$

Since K is convex and F is C-convex, it follows that, for all $k \in N$,

$$x_k = (1-\frac{1}{k})\bar{x} + \frac{1}{k}x \in K,$$

and

$$y_k = (1 - \frac{1}{k})\bar{y} + \frac{1}{k}y \in F((1 - \frac{1}{k})\bar{x} + \frac{1}{k}x) + C = F(x_k) + C.$$

Hence, the elements of the sequence $(x_k, y_k)_{k \in N}$ belong to $\text{epi}_C F$ and this sequence converges to $(\bar{x}, \bar{y})$. Moreover, we obtain

$$\lim_{k \to \infty} k(x_k - \bar{x}, y_k - \bar{y}) = (x - \bar{x}, y - \bar{y}).$$

Consequently, we get $(x - \bar{x}, y - \bar{y}) \in T(\text{epi}F, (\bar{x}, \bar{y}))$, i.e.,

$$y - \bar{y} \in G(x - \bar{x}) = \{y \in Y : (x - \bar{x}, y) \in T(\text{epi}_C F, (\bar{x}, \bar{y}))\}.$$

By the definition of $D_g(\bar{x}, \bar{y})$ and the domination property, we have

$$G(x - \bar{x}) \subset D_g F(\bar{x}, \bar{y})(x - \bar{x}) + C.$$

Thus

$$F(x) - \{\bar{y}\} \subset D_g F(\bar{x}, \bar{y})(x - \bar{x}) + C.$$

■

Theorem 2.50. *Let the assumptions in Lemma 2.49 hold, and let C have nonempty interior int$C$. If, for $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$,*

$$\varnothing \neq D_g F(\bar{x}, \bar{y})(x - \bar{x}) \subset W, \quad \forall x \in K,$$

where $W = Y \backslash (-intC)$, then $(\bar{x}, \bar{y})$ is a weakly minimal pair of (SOK).

Proof. By the assumptions, we have

$$D_g F(\bar{x}, \bar{y})(x - \bar{x}) \cap (-intC) = \varnothing, \quad \forall x \in K.$$

Thus,

$$(D_g F(\bar{x}, \bar{y})(x - \bar{x}) + C) \cap (-intC) = \varnothing, \quad \forall x \in K.$$

By Lemma 2.49, we have

$$(F(x) - \{\bar{y}\}) \cap (-intC) = \varnothing, \quad \forall x \in K.$$

Thus, it means that $\bar{y}$ is a weakly minimal point of $F(K)$. ■

Subgradients for vector-valued functions were extensively considered (see Thibault [194], Zowe [227] and the references therein). In Yang [206], the existence of a weak subgradient for a convex relation (i.e., a set-valued function with a convex graph) was considered. In the sequel, we consider the existence of a weak subgradient for a general set-valued function without the restriction of the convex relation. We obtain a sufficient optimality condition for set-valued optimization problems in terms of weak subgradients of set-valued functions.

Definition 2.51. *Let* $intC \neq \emptyset$, $K \subset X$, $f : X \to Y$ *and* $\bar{x} \in K$. *A continuous linear vector-valued function* $A : X \to Y$ *is called a weak subgradient of* f *at* $\bar{x}$ *if*

$$f(x) - f(\bar{x}) - A(x - \bar{x}) \in W, \quad \forall x \in K,$$

where $W = Y \backslash (-intC)$.

We denote by $\partial_C^w f(\bar{x})$ the set of all weak subgradients of f.

Definition 2.52. *Let* $K \subset X$, $f : X \to Y$ *and* $\bar{x} \in K$. *A continuous linear vector-valued function* $A : X \to Y$ *is called a strong subgradient of* f *at* $\bar{x}$ *if*

$$f(x) - f(\bar{x}) - A(x - \bar{x}) \in C, \quad \forall x \in K.$$

Definition 2.53. *Let* C *have a nonempty interior* $intC$, *let* $F : K \rightrightarrows Y$ *be a set-valued function and* $\bar{x} \in K$. *A continuous linear vector-valued function* $A : X \to Y$ *is called a weak subgradient of* F *at* $\bar{x}$ *if*

$$F(x) - F(\bar{x}) - A(x - \bar{x}) \subset W, \quad \forall x \in K,$$

where $W = Y \backslash (-intC)$.

Lemma 2.54. *Let* K *and* C *have nonempty interiors* $intK$ *and* $intC$, *respectively, let* K *be a convex subset of* X. *Let* $F : K \rightrightarrows Y$ *be* C*-convex on* K, *let* F *be upper semicontinuous at* $\bar{x} \in intK$, *and let* $-F(\bar{x})$ *be minorized. Then* $epiF$ *is a convex subset of* $X \times Y$ *and* $int(epi_C F) \neq \varnothing$.

Proof. It is easy to verify that $epi_C F$ is convex. We prove that $int(epi_C F) \neq \varnothing$. Since $-F(x)$ is minorized, there is a $\tilde{y} \in Y$ with $F(\bar{x}) \subset \{\tilde{y}\} - intC$. Since $\bar{x} \in intK$ and F is upper semicontinuous at $\bar{x}$, there is a neighborhood U of the zero in X so that $\{\bar{x}\} + U \subset K$ and

$$F(\bar{x} + U) \subset \{\tilde{y}\} - intC.$$

For an arbitrarily chosen $\bar{y} \in \tilde{y} + intC$, there is an open neighborhood V of the zero in Y with

$$\{\bar{y}\} + V \subset \{\tilde{y}\} + intC.$$

Thus, we conclude

$$\begin{aligned}\{\tilde{y}\} + V - F(\{\bar{x}\} + U) &\subset \{\tilde{y}\} + intC - (\{\tilde{y}\} - intC) \\ &\subset intC + intC \\ &\subset C.\end{aligned}$$

Hence, we get

$$(\{\bar{x}\} + U, \{\bar{y}\} + V) \in \text{epi}F,$$

i.e., $int(\text{epi}F) \neq \varnothing$. ∎

It follows from the proof of the preceding lemma that $intM \neq \varnothing$, where

$$M = \{(x, y) \in X \times Y : x \in K, \ y \in F(x) + intC\}.$$

Theorem 2.55. *Let C have nonempty interior $intC$, K be a convex subset of X with nonempty interior $intK$, $\bar{x} \in intK$ be given. Let $F : K \rightrightarrows Y$ be C-convex, upper semicontinuous at $\bar{x}$ and $F(\bar{x}) - C$ be convex. Let $F(\bar{x})$ and $-F(\bar{x})$ be minorized, and let the set equation*

$$F(\bar{x}) \cap (F(\bar{x}) - intC) = \varnothing$$

be fulfilled. Then there exists a weak subgradient A of F at $\bar{x} \in intK$ satisfying, for every $x \in K$, the property

$$A(x - \bar{x}) \notin -intC \Leftrightarrow A(x - \bar{x}) \in C,$$

holds.

Proof. We define the set $D = C - \{x\}$ and the set-valued function $H : K \rightrightarrows Y$ with

$$H(x) = F(x + \bar{x}) - F(\bar{x}), \quad \forall x \in K.$$

Then $0 \in D$, D is convex, H is upper semicontinuous at 0, and $H(0)$ is minorized. In order to see that H is C-convex, take arbitrary $x_1, x_2 \in K$ and $\lambda \in (0, 1)$. Then

$$\begin{aligned}
&\lambda H(x_1) + (1 - \lambda x_2)H(x_2) \\
&= \lambda F(x_2 + \bar{x}) + (1 - \lambda)F(x_2 + \bar{x}) - \lambda F(\bar{x}) - (1 - \lambda)F(\bar{x}) \\
&\subset F(\lambda x_1 + (1 - \lambda)x_2 + \bar{x}) + C - F(\bar{x}) + C \\
&\subset H(\lambda x_1 + (1 - \lambda)x_2) + C.
\end{aligned}$$

Next we set

$$B = \{(x, y) \in X \times Y : x \in D, y \in H(x) + intC\}.$$

By Lemma 2.54, we obtain $intB \neq \varnothing$. Now we show that $(0, 0) \notin B$. Suppose that $(0, 0) \in B$. Then there is a $y \in H(0)$ so that $0 \in y + intC$, which implies $H(0) \cap (-intC) \neq \varnothing$, i.e.,

$$(F(\bar{x}) - F(\bar{x})) \cap (-intC) \neq \varnothing$$

which contradicts the assumption condition. By the separation theorem for convex sets, there is a nonzero $(-\rho, \sigma) \in X^* \times Y^*$ such that

$$-\rho(x) + \sigma(y) \geq 0, \quad \forall (x, y) \in B.$$

If $\sigma = 0$, then $-\rho(x) \geq 0$, $\forall x \in D$. Because of $0 \in intD$, we obtain $\rho = 0$, contradicting $(-\rho, \sigma) \neq (0, 0)$. Hence we get $\sigma \neq 0$. Moreover, observing that $\sigma \in C^*$, there is a $\bar{y} \in intC$ with $\sigma(\bar{y}) = 1$. We now define a vector-valued function $A : X \rightarrow Y$ by

$$A(x) = \rho(x)\bar{y}, \quad \forall x \in X.$$

Obviously, A is linear and continuous. Next, we assert, for this function A, that

$$y - A(x) \notin -intC, \quad \forall x \in D, y \in H(x).$$

Suppose that there exist an $x \in D$ and a $y \in H(x)$ with

$$y - A(x) \in -intC.$$

Since $\sigma \in C^*\backslash\{0\}$, we then get

$$0 > \sigma(y - A(x)) = \sigma(x) - \rho(x)\sigma(\bar{y}) = \sigma(y) - \rho(x).$$

This is a contradiction. Hence,

$$y - A(x) \notin -intC, \quad \forall x \in D, y \in H(x),$$

i.e., A is a weak subgradient of F at $\bar{x}$. Finally, for every $x \in D$, we get

$$\begin{aligned} A(x) \notin -intC &\Rightarrow \rho(x)\bar{y} \notin -intC \\ &\Rightarrow \rho(x) \geq 0 \\ &\Rightarrow A(x) \in C. \end{aligned}$$

■

Remark 2.56. (**i**) The following implication shows that the assumption condition "$F(\bar{x}) \cap (F(\bar{x}) - intC) = \varnothing$" is rather restrictive for the set $F(\bar{x})$:

$$intF(\bar{x}) \neq \varnothing \Rightarrow F(\bar{x}) \cap (F(\bar{x}) - intC) \neq \varnothing.$$

Hence the assumption condition can only be fulfilled for a set $F(\bar{x})$ with an empty interior.

Indeed, if $intF(\bar{x})$ is nonempty, then there are a $\bar{y} \in F(\bar{x})$ and a neighborhood M of $\bar{y}$ so that $M \subset F(\bar{x})$. Consequently, we obtain

$$F(\bar{x}) \cap (F(\bar{x}) - intC) \subset M \cap (M - intC) \neq \varnothing.$$

(**ii**) If, as a special case, $F : K \to Y$ is single-valued, then the assumption condition is always fulfilled.

Theorem 2.57. *Let C have a nonempty interior $intC$. If there exists a weak subgradient A of F at $\bar{x} \in K$ such that*

$$A(x - \bar{x}) \in C, \quad \forall x \in K,$$

then, every $\bar{y} \in F(\bar{x}), (\bar{x}, \bar{y})$ is a weakly minimal pair of (SOK), and we have the property

$$F(\bar{x}) \cap (F(\bar{x}) - intC) = \varnothing.$$

Proof. Since A is a weak subgradient of F at $\bar{x} \in K$, we have

$$F(x) - F(\bar{x}) - \{A(x - \bar{x})\} \subset W, \quad \forall x \in K,$$

where $W = Y \backslash (-intC)$. Thus, for every $\bar{y} \in F(\bar{x})$, we have

$$F(x) - \bar{y} \subset \{A(x - \bar{x})\} + W \subset C + W = W, \quad \forall x \in K,$$

resulting in

$$F(K) \cap (\{\bar{y}\} - intC) = \varnothing,$$

i.e., $\bar{y}$ is a weak minimal point of $F(K)$. Thus we have

$$F(\bar{x}) - F(\bar{x}) - \{A(0)\} \subset W,$$

resulting in

$$F(\bar{x}) - F(\bar{x}) \subset W.$$

Hence

$$F(\bar{x}) \cap (F(\bar{x}) - intC) = \varnothing. \quad ■$$

Remark 2.58. In the special case $K = X$, the assumption "$A(x - \bar{x}) \in C, \forall x \in K$" reads

$$A(x - \bar{x}) \in C, \quad \forall x \in X.$$

Then we can conclude

$$A(x) \in C \cap (-C) = \{0\}, \quad \forall x \in X,$$

which means that $A = 0$, or in other words, 0 is a weak subgradient of F at $\bar{x} \in X$. Hence, we obtain the standard assumption known from the theory of subgradients in convex analysis.

2.6 Set-Valued VO with a Variable Domination Structure

In this section, we consider set-valued optimization problems with a variable domination structure.

Let X and Y be real normed spaces, and $K \subset X$ be nonempty. Let $C : X \rightrightarrows Y$ be a cone-valued function, i.e., for every $x \in X$, the set $C(x)$ is a closed and convex cone with nonempty interior $intC(x)$.

Let $F : X \rightrightarrows Y$ be a set-valued function. We consider a set-valued optimization problem with variable domination structure C:

$$\text{(SOKV)} \qquad \operatorname*{Min}_{x \in K}{}_{C(x)} F(x).$$

Definition 2.59. *Let $\bar{x} \in K$ and $\bar{y} \in F(\bar{x})$.*

(i) The pair $(\bar{x}, \bar{y})$ is called a nondominated-like minimal solution pair of (SOKV) if

$$(F(K) - \bar{y}) \cap (-C(\bar{x})) = \{\bar{y}\}.$$

(ii) The pair $(\bar{x}, \bar{y})$ is called a weakly nondominated-like minimal solution pair of (SOKV) if

$$(F(K) - \bar{y}) \cap (-intC(\bar{x})) = \{\bar{y}\}.$$

Under certain conditions, we shall see that a set-valued optimization problem (SOKV) can be transformed into an equivalent vector-valued optimization problem in the sense that their solution sets of nondominated-like minimal solution pairs are identical.

Definition 2.60. *We say that a cone-valued function $C : X \rightrightarrows Y$ is pointed on $K \subset X$ if the cone $\cup_{x \in K} C(x)$ is pointed, i.e.,*

$$(\cup_{x \in K} C(x)) \cap (- \cup_{x \in K} C(x)) = \{0\}.$$

Remark 2.61. A cone valued function $C : K \rightrightarrows Y$ is pointed if and only if

$$C(x_1) \cap (-C(x_2)) = \{0\}, \quad \forall x_1, x_2 \in K.$$

Definition 2.62. *Let $f : X \to Y$ be a vector-valued function and $C : X \rightrightarrows Y$ be a cone valued function. We say that C is weakly upper f-monotone if, for any $x_1, x_2 \in X$, $x_1 \neq x_2$,*

$$f(x_1) - f(x_2) \notin C(x_2) \backslash \{0\} \Rightarrow C(x_1) \subset C(x_2).$$

Proposition 2.63. *Let $K \subset X$ and $C : X \rightrightarrows Y$ be a pointed cone-valued function on K. Let $f : X \to Y$ be a vector-valued function and $F : X \rightrightarrows Y$ be given by*

$$F(x) = f(x) + C(x), \quad x \in X.$$

(i) *Suppose that C is weakly upper f-monotone. If $\bar{x} \in K$ is a nondominated-like minimal solution of the vector optimization problem:*

$$(VOKV) \qquad \underset{x \in K}{Min_{C(x)}} \ f(x),$$

then $(\bar{x}, f(\bar{x}))$ is a nondominated-like minimal solution pair of the set-valued optimization problem:

$$(SOKV) \qquad \underset{x \in K}{Min_{C(x)}} \ F(x).$$

(ii) *If $(\bar{x}, \bar{y})$ is a nondominated-like minimal solution pair of (SOKV), then $\bar{x}$ is a nondominated-like minimal solution of (VOKV) and $\bar{y} = f(\bar{x})$.*

Proof. Suppose that $\bar{x} \in K$ is a nondominated-like minimal solution of (VOKV). Then

$$f(x) - f(\bar{x}) \notin -C(\bar{x})\backslash\{0\}, \quad \forall x \in K.$$

We claim that

$$(f(x) - f(\bar{x}) + C(x)) \cap (-C(\bar{x})\backslash\{0\}) = \varnothing, \quad \forall x \in K.$$

Indeed, if there exists $\hat{x} \in K$ such that

$$(f(\hat{x}) - f(\bar{x}) + C(\hat{x})) \cap (-C(\bar{x})\backslash\{0\}) \neq \varnothing,$$

then there exists $\bar{c} \in C(\bar{x})$ and $\bar{c} \neq 0$ such that

$$-\bar{c} \in f(\hat{x}) - f(\bar{x}) + C(\hat{x}).$$

It follows that

$$f(\hat{x}) - f(\bar{x}) \in -\bar{c} - C(\hat{x}).$$

From the pointedness and weak upper monotonicity of C, we have

$$f(\hat{x}) - f(\bar{x}) \in -\bar{c} - C(\bar{x}).$$

Since C is pointed and $\bar{c} \neq 0$, it follows that

$$f(\hat{x}) - f(\bar{x}) \in -C(\bar{x})\backslash\{0\},$$

which contradicts the fact that $\bar{x}$ is a nondominated-like minimal solution. Thus, we have

$$y - f(\bar{x}) \notin -C(\bar{x})\backslash\{0\}, \quad \forall y \in F(x), x \in K.$$

Hence $(\bar{x}, f(\bar{x}))$ is a nondominated-like minimal solution pair of (SOKV).

Now we prove that (ii) holds. To this end, let us assume that $(\bar{x}, \bar{y})$ is a nondominated-like minimal solution pair of (SOKV). Then

$$\bar{y} \in F(\bar{x}) = f(\bar{x}) + C(\bar{x})$$

and

$$y - \bar{y} \notin -C(\bar{x})\backslash\{0\}, \quad \forall y \in F(K). \tag{2.47}$$

It is clear that $\bar{y} = f(\bar{x})$. We are now ready to prove that $\bar{x}$ is a nondominated-like minimal solution of (VOKV). Suppose to the contrary that $\bar{x}$ is not a nondominated-like minimal solution of (VOKV). Then, for some point $x_0 \in K\backslash\{\bar{x}\}$,

$$f(x_0) - f(\bar{x}) \in -C(\bar{x})\backslash\{0\}.$$

But $f(\bar{x}) = \bar{y}$, and therefore

$$f(x_0) - \bar{y} \in -C(\bar{x})\backslash\{0\},$$

contradicting (2.47). Hence, $\bar{x}$ is a nondominated-like minimal solution of (VOKV). ■

Contingent cone and contingent derivative are important tools for dealing set-valued optimization problems. They are now employed to derive necessary and sufficient optimality conditions for nondominated-like minimal pairs of set-valued optimization problems with a variable domination structure.

Theorem 2.64. *Let X and Y be real normed spaces. Let $C : X \rightrightarrows Y$ be a cone-valued function such that $intC(x) \neq \emptyset$ for every $x \in X$. Let $K \subset X$ be a nonempty subset, and let $F : K \rightrightarrows Y$ be a set-valued function such that $F(x) \neq \emptyset$ for every $x \in K$. If $(\bar{x}, \bar{y})$ is a weakly nondominated-like minimal solution pair of (SOKV), then*

$$DF(\bar{x}, \bar{y})(x - \bar{x}) \subset W(\bar{x}), \quad \forall x \in K,$$

where $W(x) = Y \backslash (-intC(x))$ and $DF(\bar{x}, \bar{y})$ is the contingent derivative of F at $(\bar{x}, \bar{y})$.

Proof. We proceed by contradiction. Let $(\bar{x}, \bar{y}) \in K \times F(\bar{x})$ be a weakly nondominated-like minimal solution pair of (SOKV). Suppose that there is an $x_0 \in K$ such that

$$DF(\bar{x}, \bar{y})(x_0 - \bar{x}) \not\subset W(\bar{x}).$$

Then there is a point z such that $z \in DF(\bar{x}, \bar{y})(x_0 - \bar{x})$ and $z \in -intC(\bar{x})$. By the definition of DF, we have

$$(x_0 - x, z) \in T_{\mathrm{Gr}(F)}(\bar{x}, \bar{y}).$$

So there are sequences $\{(x_k, y_k)\}$ in $\mathrm{Gr}(F)$ and $\{\lambda_k\}$ of positive numbers for which

$$(\bar{x}, \bar{y}) = \lim_{n \to \infty} (x_k, y_k)$$

and

$$(x_0 - \bar{x}, z) = \lim_{n \to \infty} \lambda_k (x_k - \bar{x}, y_k - \bar{y}).$$

We assert that there is a positive integer M so that

$$y_k \in \bar{y} - intC(\bar{x}), \quad \forall k \geq M.$$

Since $y_k \in F(x_k) \subset \cup_{x \in K} F(x)$, it follows that $(\bar{x}, \bar{y})$ cannot be a weakly nondominated-like minimal solution pair of (SOKV). ■

We need the following lemma.

Lemma 2.65. *[6] Let X and Y be normed spaces, $f : K \to Y$ be a single-valued function, where $K \subset X$ is open, $M : X \rightrightarrows Y$ be a set-valued function and $L \subset X$. Let $F : X \rightrightarrows Y$ be a set-valued function defined by*

$$F(x) = \begin{cases} f(x) - M(x), & \textit{if } x \in L \\ \emptyset, & \textit{if } x \notin L. \end{cases}$$

If f is Fréchet differentiable at $x \in K \cap dom(F)$, then, for every $y \in F(x)$,

$$DF(x,y)(u) \subset \begin{cases} f'(x)(u) - DM(x, f(x) - y)(u), & \textit{if } u \in T(L,x) \\ \emptyset, & \textit{if } u \notin T(L,x). \end{cases}$$

If M is constant, then

$$DF(x,y)(u) = f'(x)(u) - T(M, f(x) - y).$$

Corollary 2.66. *Let the assumptions of Theorem 2.64 hold. Let C be a constant cone-valued function, i.e., $C(x) = C$, for every $x \in K$, where C is a pointed, closed and convex cone with $intC \neq \varnothing$. Let $f : K \to Y$ be a Fréchet differentiable single-valued function. If $(\bar{x}, f(\bar{x})) \in intK \times Y$ is a weakly nondominated-like minimal solution pair of (SOKV), then $\bar{x}$ satisfies the vector variational inequality*

$$f'(\bar{x})(x - \bar{x}) \in W, \quad \forall x \in K,$$

where $W = Y \backslash (-C)$.

Proof. Set $F(x) = f(x) + C$ for every $x \in K$. If f is Fréchet differentiable, then for $(x,y) \in \mathrm{Gr}F$ we have, by Lemma 2.65,

$$DF(x,y)(u) = f'(x)u - T(C, f(x) - y), \quad \forall u \in T(K,x).$$

Thus from Theorem 2.64, we have

$$f'(x)(x - \bar{x}) - T(C, f(\bar{x}) - f(\bar{x})) \subset W, \quad \forall x \in K \cap (\bar{x} + T(K,\bar{x})),$$

i.e.,

$$f'(\bar{x})(x - \bar{x}) - T(C,0) \subset W, \quad \forall x \in K \cap (\bar{x} + T(K,\bar{x})).$$

Since C is a closed convex cone and $\bar{x} \in intK$, we have $T(C,0) = C$ and $T(K,\bar{x}) = X$. Hence

$$f'(\bar{x})(x - \bar{x}) \subset W, \quad \forall x \in K.$$

■

Next, we establish a sufficient condition for (SOKV). To this end, we need some concepts.

Definition 2.67. *Let $F : X \rightrightarrows Y$ be a set-valued function. F is said to be $C(\bar{x})$ pseudo-convex at $(\bar{x}, \bar{y}) \in Gr(F)$ if and only if*

$$F(x) - \bar{y} \subset DF(\bar{x}, \bar{y})(x - \bar{x}) + C(\bar{x}), \quad \forall x \in Dom(F).$$

Definition 2.68. *Let K be a nonempty convex subset of X, and let $\bar{x} \in K$ and $\bar{y} \in F(K)$. F is said to be $C(\bar{x})$-convex at $(\bar{x}, \bar{y})$ if, for any $x', x'' \in K$ and $\lambda \in (0,1)$,*

$$\lambda F(x') + (1-\lambda)F(x'') \subset F(\lambda x' + (1-\lambda)x'') + C(\bar{x}).$$

Remark 2.69. The order pseudo-convex function is a generalization of the pseudo-convex function given in Aubin and Frankowska [6]. If C is a constant function, the C-convexity at $\bar{y}$ of F reduces to the usual C-convexity of F.

Definition 2.70 (Thibault [193]). *A set-valued function $F : X \rightrightarrows Y$ is said to be compactly approximable at $(\bar{x}, \bar{y}) \in Gr(F)$, if, for every $\bar{x} \in X$, there are a set-valued function $I\!R$ from X into the set of all nonempty compact subsets of Y, a neighborhood V of $\bar{x}$ in X, and a real function $r : (0,1] \times X \to (0, +\infty)$ satisfying*

(i) $\lim_{(t,x)\to(0_+,\bar{x})} r(t,x) = 0,$
(ii) *for every $x \in V$ and $t \in (0,1]$*

$$F(\bar{x} + tx) \subset \bar{y} + t(R(\bar{x}) + r(t,x)B_Y),$$

where B_Y is the closed unit ball centered at the origin of Y.

Lemma 2.71. *[183] Let $C \subset Y$ be a nonempty, pointed, closed and convex cone. Let $F : X \rightrightarrows Y$ be a set-valued function. Let $(x_0, y_0) \in Gr(F)$. If F is compactly approximable at (x_0, y_0), then*

$$D(F + C)(x_0, y_0)(u) = DF(x_0, y_0)(u) + C, \quad \forall u \in X.$$

Proposition 2.72. *Let the set-valued function $F : X \rightrightarrows Y$ be compactly approximable at $(\bar{x}, \bar{y}) \in Gr(F)$ and let $Dom(F)$ be a nonempty convex subset. Let F be $C(x)$-convex at $(\bar{x}, \bar{y})$. Then F is $C(\bar{x})$ pseudo-convex at $(\bar{x}, \bar{y})$.*

Proof. Fix a point $(x, y) \in \mathrm{Gr}(F)$. We define a sequence $\{(x_k, y_k)\}$ in $X \times Y$ by

$$x_k = \bar{x} + \frac{1}{k}(x - \bar{x}), \quad y_k = \bar{y} + \frac{1}{k}(y - \bar{y}), \quad \forall k \in N.$$

Since $\mathrm{dom}(F)$ is convex and F is $C(\bar{x})$-convex at $(\bar{x}, \bar{y})$, it follows that, for all $k \in N$,

$$x_k = (1 - \frac{1}{k})\bar{x} + \frac{1}{k}x \in \mathrm{Dom}(F)$$

and

$$\begin{aligned} y_k &= (1 - \frac{1}{k})\bar{y} + \frac{1}{k}y \in F((1 - \frac{1}{k})\bar{x} + \frac{1}{k}x) + C(\bar{x}) \\ &= F(x_k) + C(\bar{x}). \end{aligned}$$

Thus, $(x_k, y_k) \in \mathrm{Gr}(F + C(\bar{x}))$, for every $k \in N$, and $(x_k, y_k) \to (\bar{x}, \bar{y})$ as $k \to \infty$. Moreover, we have

$$\lim_{n\to\infty} n(x_k - \bar{x}, y_k - \bar{y}) = (x - \bar{x}, y - \bar{y}).$$

Consequently,

$$(x - \bar{x}, y - \bar{y}) \in T_{\mathrm{Gr}(F+C(\bar{x}))}(\bar{x}, \bar{y}).$$

Therefore,

$$y - \bar{y} \in D(F + C(\bar{x}))(\bar{x}, \bar{y})(x - \bar{x}), \quad \forall x \in \mathrm{Dom}(F), y \in F(x).$$

F being compactly approximable at $(\bar{x}, \bar{y})$, by Lemma 2.71, we have

$$D(F + C(\bar{x}))(\bar{x}, \bar{y})(x - \bar{x}) = DF(\bar{x}, \bar{y})(x - \bar{x}) + C(\bar{x}).$$

Thus

$$y - \bar{y} \in DF(\bar{x}, \bar{y})(x - \bar{x}) + C(\bar{x}), \quad \forall x \in \mathrm{Dom}(F), y \in F(x).$$

So F is $C(\bar{x})$ pseudo-convex at $(\bar{x}, \bar{y})$. ■

Theorem 2.73. *Let X and Y be real normed spaces. Let $C : X \rightrightarrows Y$ be a cone-valued function such that $intC(x) \neq \varnothing$ for every $x \in X$. Let $K \subset X$ be a nonempty convex subset and let $F : X \rightrightarrows Y$ be compactly approximable at $(\bar{x}, \bar{y}) \in Gr(F)$. Suppose that F is $C(\bar{x})$-convex at $(\bar{x}, \bar{y})$, $C(\bar{x})$ is a pointed, closed and convex cone, and*

$$\varnothing \neq DF(\bar{x}, \bar{y})(x - \bar{x}) \subset W(\bar{x}), \quad \forall x \in K,$$

where $W(\bar{x}) = Y \backslash (-intC(\bar{x}))$. Then $(\bar{x}, \bar{y})$ is a weakly nondominated-like minimal solution pair of (SOKV).

Proof. By assumption, we have

$$DF(\bar{x}, \bar{y})(x - \bar{x}) \cap (-intC(\bar{x})) = \varnothing, \quad \forall x \in K.$$

Since $C(\bar{x})$ is pointed and convex, we have

$$(DF(\bar{x}, \bar{y})(x - \bar{x}) + C(\bar{x})) \cap (-intC(\bar{x})) = \varnothing, \quad \forall x \in K.$$

By Proposition 2.72, F is $C(\bar{x})$ pseudo-convex at $(\bar{x}, \bar{y})$. Thus we have

$$(F(x) - \bar{y}) \cap (-intC(\bar{x})) = \varnothing, \quad \forall x \in K.$$

So $(\bar{x}, \bar{y})$ is a weakly nondominated-like solution pair of (SOKV). ■

2.7 Augmented Lagrangian Duality for VO

The conventional Lagrangian function is discussed in section 2.3 where convexity is posed to guarantee the strong duality. However, such a strong duality result may be not true if the convexity is not assumed. Let $Y = \mathbb{R}^\ell$, $\ell = 1$, $X = [0, +\infty)$, $f(x) = x$, $g(x) = x - x^2$. Consider the problem:

$$\inf_{x \in X} f(x), \quad \text{s.t. } g_1(x) \leq 0.$$

The conventional Lagrangian for this problem is

$$L(x, \lambda) = f(x) + \lambda g_1(x) = x + \lambda(x - x^2), \ \forall x \in X, \ \lambda \geq 0.$$

It is easy to check that $\inf_{x \in X} L(x, \lambda) = -\infty, \forall \lambda > 0$ and $\inf_{x \in X} L(x, 0) = 0$. Thus, $\sup_{\lambda \geq 0} \inf_{x \in X} L(x, \lambda) = 0$. However, the optimal value of the original constrained problem is 1.

In this section, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^\ell \cup \{-\infty, +\infty\}$, where $-\infty$ is an imaginary point, each of whose coordinates is $-\infty$, and the imaginary point $+\infty$ is analogously understood. Let $C = \mathbb{R}^\ell_+ \cup \{+\infty\}$ and $intC = int\mathbb{R}^\ell_+ \cup \{+\infty\}$. Without confusion, we shall not differentiate the $-\infty$, $+\infty$ in $\mathbb{R}^\ell \cup \{-\infty, +\infty\}$ and the $-\infty$ and $+\infty$ in the extended real space. The same terminology and notation such as minimal solutions and ordering relations for standard vector optimization problems will be used for (EOP). In particular, for the space $\mathbb{R}^\ell \cup \{-\infty, +\infty\}$, we use the following orderings: for any $z^1 = (z^1_1, \cdots, z^1_\ell)$, $z^2 = (z^2_1, \cdots, z^2_\ell) \in \mathbb{R}^\ell \cup \{-\infty, +\infty\}$,

$$z^1 \leq_C z^2 \iff z^1_i \leq z^2_i, \quad i = 1, \cdots, \ell;$$
$$z^1 \leq_{C \setminus \{0\}} z^2 \iff z^1_i \leq z^2_i, \quad i = 1, \cdots, \ell \text{ with at least one } i \text{ such that } z^1_i < z^2_i;$$
$$z^1 \leq_{intC} z^2 \iff z^1_i < z^2_i, \quad i = 1, \cdots, \ell.$$

Moreover, let $A \subset \mathbb{R}^\ell$ be a nonempty set. By $z^* \in \mathrm{Inf}_C A$, we mean that
(i) $z^* \in \mathbb{R}^\ell \cup \{+\infty, -\infty\}$;
(ii) $z \not\leq_{C \setminus \{0\}} z^*, \forall z \in A$;
(iii) $\exists z_k \in A$ such that $z_k \to z^*$.
The point $z^* \in \mathrm{Inf}_C A$ is called an infimum point of A.

Meanwhile, we define $z^* \in \mathrm{Sup}_C A$ if and only if $-z^* \in \mathrm{Inf}_C(-A)$.

Definition 2.74. *Let $f : \mathbb{R}^n \to \mathbb{R}^\ell \cup \{-\infty, +\infty\}$ be an extended vector-valued function. Then f is said to be proper if $f(x) > -\infty, \forall x \in \mathbb{R}^n$ and there exists some $x \in \mathbb{R}^n$ such that $f(x) < +\infty$.*

Consider the primal vector optimization problem (EOP):

$$\operatorname{Inf}_C \underset{x \in \mathbb{R}^n}{} f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^\ell \cup \{-\infty, +\infty\}$ is a proper extended vector-valued function.

Finally, the set of minimal solutions and weakly minimal solutions of (EOP) are denoted respectively by $\operatorname{Min}_C(f, \mathbb{R}^n)$ and $\operatorname{Min}_{intC}(f, \mathbb{R}^n)$, namely,

$$\operatorname{Min}_C(f, \mathbb{R}^n) = \{x^* \in \mathbb{R}^n : f(x) \not\leq_{C \setminus \{0\}} f(x^*),\ \forall x \in \mathbb{R}^n\},$$

$$\operatorname{Min}_{intC}(f, \mathbb{R}^n) = \{x^* \in \mathbb{R}^n : f(x) \not\leq_{intC} f(x^*),\ \forall x \in \mathbb{R}^n\}.$$

Let $\bar{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\ell \cup \{+\infty\}$ be a perturbed function such that $\bar{f}(x, 0) = f(x), \forall x \in \mathbb{R}^n$. Define the optimal value function by

$$p(u) = \operatorname{Inf}_C\{\bar{f}(x, u) : x \in \mathbb{R}^n\}, \quad u \in \mathbb{R}^m.$$

Obviously, $p(0)$ is the set of the infimum points of (EOP). It is also clear that $\operatorname{Min}_C f(\mathbb{R}^n) \subset p(0)$.

A function $\sigma : \mathbb{R}^m \to \mathbb{R}_+ \cup \{+\infty\}$ is called an augmenting function if it is proper, lower semicontinuous, and convex with the unique minimum value 0 at $0 \in \mathbb{R}^n$.

Definition 2.75. *Let $\sigma : \mathbb{R}^m \to \mathbb{R}_+ \cup \{+\infty\}$ be an augmenting function and $e = (1, \cdots, 1)^\top \in \mathbb{R}^\ell$. The augmented Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \times (0, +\infty) \rightrightarrows \mathbb{R}^\ell \cup \{+\infty, -\infty\}$ is a set-valued function defined by*

$$L(x, y, r) = \operatorname{Inf}_C\{\bar{f}(x, u) + r\sigma(u)e - \langle y, u\rangle e : u \in \mathbb{R}^m\},$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m, r \in (0, +\infty)$.

The augmented Lagrangian dual function is a set-valued function defined by

$$\Psi(y, r) = \operatorname{Inf}_C \cup_{x \in \mathbb{R}^n} L(x, y, r), \quad y \in \mathbb{R}^m, r \in (0, +\infty).$$

The augmented Lagrangian dual problem is a set-valued optimization problem defined by

$$\text{(DEOP)} \quad \operatorname{Sup}_C \Psi(y, r) \quad \text{subject to } (y, r) \in \mathbb{R}^m \times (0, +\infty).$$

Denote by Q the set of all the supermum points of the dual problem (DEOP).

We have the following proposition, whose proof is elementary and omitted.

Proposition 2.76. *For any $x \in \mathbb{R}^n, y \in \mathbb{R}^m, r \in (0, +\infty)$ and $z \in L(x, y, r)$, we have $z \not\geq_C f(x)$.*

Definition 2.77. *Let X be a set. Let $f : X \to \mathbb{R}^\ell \cup \{-\infty, +\infty\}$ be an extended vector-valued function.*

(i) f is said to be externally stable if, for any $x \in X$, there exists a minimal solution x^* of f on X such that $f(x^*) \leq_C f(x)$.

(ii) f is said to be Inf-externally stable if, for any $x \in X$, there exists a $z^* \in Inf_C\ f(X)$ such that $z^* \leq_C f(x)$.

(iii) f is said to be bounded below on X if there exists $z \in I\!R^\ell$ such that $f(x) \geq_C z, \forall x \in X$.

Proposition 2.78. *Let $A \subseteq I\!R^\ell \cup \{-\infty, +\infty\}$ be a nonempty set and there exists $z_0 \in I\!R^\ell$ such that $a \geq_C z_0, \forall a \in A$. Then, for any $a \in A$, there exists $a^* \in Inf_C A$ such that $a^* \leq_C a$.*

Proof. Clearly, $-\infty \notin A$. If $A = \{+\infty\}$, then the conclusion holds automatically. Now assume that $A \backslash \{+\infty\} \neq \emptyset$. Let $a \in A$. Consider the following two cases:

(i) $a \leq_{intC} +\infty$.

(ii) $a = +\infty$.

Suppose that case (i) holds. Then $A_1 = \{b \in A : b \leq_C a\} = \{b \in A : z_0 \leq_C b \leq_C a\}$ is a nonempty and bounded subset of $\mathbb{R}^\ell$. Thus, clA_1 is a nonempty and compact subset of $\mathbb{R}^\ell$. It follows that there exists $a^* \in clA_1$ such that

$$b \not\leq_{C\backslash\{0\}} a^*, \quad \forall b \in clA_1. \tag{2.48}$$

Since $a^* \in clA_1$, by the definition of A_1, we see that $z_0 \leq_C a^* \leq_C a$. So $a^* \in \mathbb{R}^\ell$ and $a^* \leq_C a$. Moreover, there exists $a_k \in A_1 \subseteq A$ such that $a_k \to a^*$. Finally, we show that $b \not\leq_{C\backslash\{0\}} a^*, \forall b \in A$. Indeed, if $b \notin A_1$, it can be easily shown by contradiction that $b \not\leq_{C\backslash\{0\}} a^*$. If $b \in A_1$, it follows from (2.48) that $b \not\leq_{C\backslash\{0\}} a^*$. Thus, we have proved $a^* \in \mathrm{Inf}_C A$.

Suppose that case (ii) holds. Then choose $a_1 \in A$ such that $a_1 \leq_{intC} +\infty$. Replacing a in the proof of case (i) by a_1, the conclusion follows. ■

Corollary 2.79. *Let X be a set. Let $f : X \to I\!R^\ell \cup \{-\infty, +\infty\}$ be an extended vector-valued function, which is bounded below on X. Then f is Inf-externally stable on X.*

Proof. The conclusion follows directly from Proposition 2.78. ■

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $r > 0$. It is obvious (by setting $u = 0$) that if $\bar{f}(x, \cdot) + r\sigma(\cdot) - \langle y, \cdot \rangle$ is Inf-externally stable on $\mathbb{R}^m$, then there exists $z \in L(x, y, r)$ such that $z \leq_C f(x)$. The following theorem can be straightforwardly proved.

Theorem 2.80. *Weak Duality. Let $x \in I\!R^n, y \in I\!R^m, r \in (0, +\infty)$. Assume that there exists $z \in L(x, y, r)$ such that $z \leq_C f(x)$. Then*

$$z^2 \not\geq_{C\backslash\{0\}} f(x), \quad \forall z^2 \in \Psi(y, r).$$

The following corollary follows immediately from Theorem 2.80.

Corollary 2.81. *Let the assumption in Theorem 2.80 hold. Then*

$$z^2 \not\geq_{intC} z^1, \quad \forall z^1 \in p(0), z^2 \in Q.$$

Definition 2.82. *(i) A function* $g : I\!R^n \to I\!R \cup \{+\infty, -\infty\}$ *is called level-bounded if, for any* $\alpha \in I\!R$, *the set* $\{x \in I\!R^n : g(x) \leq \alpha\}$ *is bounded.*

(ii) A function $h : I\!R^n \times I\!R^m \to I\!R \cup \{+\infty, -\infty\}$ *with values* $h(x, u)$ *is called level-bounded in* x *locally uniformly in* u *if for each* $\bar{u} \in I\!R^m$ *and* $\alpha \in I\!R$, *there exists a neighborhood* $V_{\bar{u}}$ *of* $\bar{u}$ *along with a bounded set* $D \subset I\!R^n$ *such that* $\{x \in I\!R^n : h(x, v) \leq \alpha\} \subset D$ *for all* $v \in V_{\bar{u}}$.

Now we introduce the function ξ defined by

$$\xi(z) = \max\{z_1, \cdots, z_\ell\}, \ \forall z = (z_1, \cdots, z_\ell) \in \mathbb{R}^\ell \cup \{-\infty, +\infty\}.$$

It is easy to check that ξ is an increasing, continuous, subadditive, positively homogenous and convex function.

It is also clear that

$$\xi(z + te) = \xi(z) + t, \ \forall z \in \mathbb{R}^\ell \cup \{+\infty\}, t \in \mathbb{R} \cup \{+\infty\}.$$

The following lemma will be frequently used in the sequel.

Lemma 2.83. *[176] Let* X *be a subset of* $I\!R^n$ *and* $f : X \to I\!R^\ell \cup \{-\infty, +\infty\}$ *be a proper vector-valued function such that each component* f_i $(i = 1, \cdots, l)$ *is lsc. Suppose that* $x_0 \in X$ *is such that* $f(x_0) \in I\!R^\ell$ *and* $X_1 = \{x \in X : f(x) \leq_C f(x_0)\}$ *is a compact set. Then there exists* $x^* \in X_1$ *such that* $f(x) \not\leq_{C\setminus\{0\}} f(x^*), \forall x \in X$.

Theorem 2.84. *Strong Duality. Assume that, for any* $x \in I\!R^n, y \in I\!R^m, r \in (0, +\infty)$, *there exists* $z^* \in L(x, y, r)$ *such that* $z^* \leq_C f(x)$. *Suppose that* $\bar{f}(x, u)$ *is a proper vector-valued function such that each of its component function* $\bar{f}_i(x, u)$ *is lsc and* $\xi(\bar{f}(x, u))$ *is level-bounded in* x *locally uniformly in* u. *Suppose further that there exist* $\bar{y} \in I\!R^m$, *and* $\bar{r} > 0$ *such that*

$$Inf_C\{\xi(\bar{f}(x, u)) + \bar{r}\sigma(u) - \langle \bar{y}, u \rangle) : x \in I\!R^n, u \in I\!R^m\} \geq_{intC} -\infty. \qquad (2.49)$$

Then $p(0) \subseteq Q$.

Proof. Let $\bar{z} \in p(0)$. Obviously, $\bar{z} \leq_{intC} +\infty$. In addition, from (2.49) we deduce that $\bar{z} \geq_{intC} -\infty$. Hence $\bar{z} \in \mathbb{R}^\ell$. By the assumption on $\xi(\bar{f}(x, u))$ and setting $u = 0$, we see that $\xi(\bar{f}(x, 0)) = \xi(f(x))$ is level-bounded. Moreover, $\bar{z} \in p(0)$ implies that there exists $\{x_k\} \subseteq \mathbb{R}^n$ such that $f(x_k) \to \bar{z}$ as $k \to +\infty$. Consequently, $\xi(f(x_k)) \to \xi(\bar{z})$. In addition, the lsc of $\bar{f}_i$ and $\bar{f}(x, 0) = f(x)$, $\forall x \in \mathbb{R}^n$ imply that each component function f_i of f is lsc and therefore, $\xi(f)$ is lsc. Thus we see that the set $\{x : \xi(f(x)) \leq \xi(\bar{z}) + 1\}$ is a nonempty and closed set, hence a nonempty and compact set because $\xi(f(x)) = \xi(\bar{f}(x, 0))$ is level-bounded. As a result, we can assume, without loss of generality, that $x_k \to x^*$. By the lsc of f_i , we have

$$f_i(x^*) \leq \liminf_{k \to +\infty} f_i(x_k) = \lim_{k \to +\infty} f_i(x_k) = \bar{z}_i, \ i = 1, \cdots, \ell,$$

where $\bar{z}_i$ is the ith coordinate of $\bar{z}$. Thus we obtain $f(x^*) \leq_C \bar{z}$. It follows that $f(x^*) = \bar{z}$ because $\bar{z} \in p(0)$. Let

$$A(\bar{y}, r) = \{(x,u) : \bar{f}(x,u) + r\sigma(u)e - \langle \bar{y}, u\rangle e \leq_C f(x^*)\}, \ r \geq \bar{r}+1.$$

Clearly, $(x^*, 0) \in A(\bar{y}, r)$. Let $(x,u) \in A(\bar{y}, r)$. Then

$$\xi(f(x^*)) \geq \xi(\bar{f}(x,u)) + \bar{r}\sigma(u) - \langle \bar{y}, u\rangle + (r-\bar{r})\sigma(u) \geq m_0 + (r-\bar{r})\sigma(u),$$

where

$$m_0 = \inf\{\xi(\bar{f}(x,u)) + \bar{r}\sigma(u) - \langle \bar{y}, u\rangle : x \in \mathbb{R}^n, u \in \mathbb{R}^m\} > -\infty.$$

So we get

$$\sigma(u) \leq \frac{\xi(f(x^*)) - m_0}{r - \bar{r}} \leq \xi(f(x^*)) - m_0, \quad r \geq \bar{r} + 1. \tag{2.50}$$

Since σ is an augmenting function, we deduce that σ is level-bounded and $U_1 = \{u : \sigma(u) \leq \xi(f(x^*)) - m_0\}$ is compact. Let

$$F = \{(x,u) \in \mathbb{R}^n \times U_1 : \xi(\bar{f}(x,u)) \leq \xi(f(x^*)) + \max_{u \in U_1}\langle \bar{y}, u\rangle\},$$

$$B(\bar{y}, r) = \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m : \xi(\bar{f}(x,u)) + r\sigma(u) - \langle \bar{y}, u\rangle \leq \xi(f(x^*))\}.$$

Since $\xi(\bar{f})$ is level-bounded in x locally uniformly in u, we deduce that $A(\bar{y}, r) \subseteq B(\bar{y}, r) \subseteq F$. Using the fact that $\xi(\bar{f}(x,u))$ is level-bounded in x locally uniformly in u and that it is also lsc, it can be shown by contradiction that F is compact. By the lsc of $\bar{f}_i$ and σ, we know that $A(\bar{y}, r)$ is closed. So $A(\bar{y}, r)$ is compact. Let $r_k \uparrow +\infty$. Assume that $r_k \geq \bar{r} + 1$ when $k \geq k_0$. Since $A(\bar{y}, r_k)$ is compact when $k \geq k_0$ and each $\bar{f}_i(x,u) + r_k\sigma(u) - \langle \bar{y}, u\rangle$ $(i = 1, \cdots, \ell)$ is lsc in (x,u), we deduce that $\exists (x_k^*, u_k^*) \in A(\bar{y}, r_k) \subseteq F$ such that, for $k \geq k_0$,

$$\begin{aligned} &\bar{f}(x_k^*, u_k^*) + r_k\sigma(u_k^*)e - \langle \bar{y}, u_k^*\rangle e \\ &\qquad \in \mathrm{Inf}_C\{\bar{f}(x,u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e : (x,u) \in \mathbb{R}^n \times \mathbb{R}^m\}. \end{aligned} \tag{2.51}$$

Let us show that

$$\begin{aligned} &\bar{f}(x_k^*, u_k^*) + r_k\sigma(u_k^*)e - \langle \bar{y}, u_k^*\rangle e \\ &\qquad \in \Psi(\bar{y}, r_k), \ k \geq k_0 \end{aligned} \tag{2.52}$$

When $k \geq k_0$, by (2.51), we get

$$\bar{f}(x,u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e \not\leq_{C\setminus\{0\}} \bar{f}(x_k^*, u_k^*) - r_k\sigma(u_k^*)e + \langle \bar{y}, u_k^*\rangle e,$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Therefore,

$$\bar{f}(x_k^*, u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e \not\leq_{C\setminus\{0\}} \bar{f}(x_k^*, u_k^*) - r_k\sigma(u_k^*)e + \langle \bar{y}, u_k^*\rangle e,$$

for all $u \in \mathbb{R}^m$, and $k \geq k_0$. That is, for $k \geq k_0$,

$$\begin{aligned}&\bar{f}(x_k^*, u_k^*) + r_k\sigma(u_k^*)e - \langle \bar{y}, u_k^*\rangle e\\ &\in \mathrm{Inf}_C\{\bar{f}(x_k^*, u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e : u \in \mathbb{R}^m\}\\ &= L(x_k^*, \bar{y}, r_k).\end{aligned}$$

We prove by contradiction that (2.52) holds. Suppose that there exist $k_1 \geq k_0$, $x' \in \mathbb{R}^n$ and $z \in L(x', \bar{y}, r_{k_1})$ such that

$$\bar{f}(x_{k_1}^*, u_{k_1}^*) + r_{k_1}\sigma(u_{k_1}^*)e - \langle \bar{y}, u_{k_1}^*\rangle e \geq_{C\setminus\{0\}} z \tag{2.53}$$

It follows that there exist $u_k \in \mathbb{R}^m$ and $z_k \in \mathbb{R}^\ell$ such that

$$z = z_k + \bar{f}(x', u_k) + r_{k_1}\sigma(u_k)e - \langle \bar{y}, u_k\rangle e \text{ and } z_k \to 0, \text{ as } k \to +\infty \tag{2.54}$$

and

$$\begin{aligned}&\bar{f}(x', u_k) + r_{k_1}\sigma(u_k)e - \langle \bar{y}, u_k\rangle e + z_k\\ &\qquad \leq_{C\setminus\{0\}} \bar{f}(x_{k_1}^*, u_{k_1}^*) + r_{k_1}\sigma(u_{k_1}^*)e - \langle \bar{y}, u_{k_1}^*\rangle e.\end{aligned} \tag{2.55}$$

Just as we have shown that $A(\bar{y}, r) \subseteq F$ is bounded when $r \geq_C \bar{r} + 1$, we can show by employing (2.55) that $\{u_k\}$ is bounded. Without loss of generality, we assume that $u_k \to u'$. Letting $k \to \infty$ in (2.54), (2.55) and applying the lsc of the function $\bar{f}$ and σ, we obtain

$$\begin{aligned}z &\leq_C \bar{f}(x', u') + r_{k_1}\sigma(u')e - \langle \bar{y}, u'\rangle e\\ &\leq_C \bar{f}(x_{k_1}^*, u_{k_1}^*) + r_{k_1}\sigma(u_{k_1}^*)e - \langle \bar{y}, u_{k_1}^*\rangle e.\end{aligned} \tag{2.56}$$

Noting that $z \in L(x', \bar{y}, r_{k_1})$, we deduce from the first inequality in (2.56) that

$$z = \bar{f}(x', u') + r_{k_1}\sigma(u')e - \langle \bar{y}, u'\rangle e.$$

This fact combined with (2.51) and (2.53) yields a contradiction.

As $\{(x_k^*, u_k^*)\} \subseteq F$ and F is compact, without loss of generality, we assume that

$$(x_k^*, u_k^*) \to (x_0, u_0) \in F.$$

It follows from (2.50) that

$$\sigma(u_k) \leq \frac{\xi(f(x^*)) - m_0}{r_k - \bar{r}}, \quad k \geq k_0.$$

Therefore,

$$\sigma(u_0) \leq \liminf_{k\to\infty}\sigma(u_k^*) \leq 0.$$

As a result, $u_0 = 0$. Moreover, we have

$$\bar{f}(x_k^*) - \langle \bar{y}, u_k^*\rangle e \leq_{C\setminus\{0\}} \bar{f}(x_k^*, u_k^*) + r_k\sigma(u_k^*)e - \langle \bar{y}, u_k^*\rangle e \leq_{C\setminus\{0\}} f(x^*). \tag{2.57}$$

Namely,

$$\bar{f}_i(x_k^*, u_k^*) - \langle \bar{y}, u_k^* \rangle \leq_{C \setminus \{0\}} f_i(x^*), \quad i = 1, \cdots, \ell.$$

Thus

$$f_i(x_0) = \bar{f}_i(x_0, 0) \leq \lim \inf_{n \to \infty} \bar{f}_i(x_k^*, u_k^*) - 0 \leq f_i(x^*), \quad i = 1, \cdots, \ell. \quad (2.58)$$

Since $f(x^*) \in p(0)$, $f(x_0) = f(x^*)$. This fact combined with (2.58) yields

$$\lim_{n \to \infty} \bar{f}_i(x_k^*, u_k^*) = f_i(x^*), \quad i = 1, \cdots, \ell. \quad (2.59)$$

(2.57) and (2.59) yield

$$\lim_{n \to \infty} \bar{f}(x_k^*, u_k^*) + r_k \sigma(u_k^*) e - \langle \bar{y}, u_k^* \rangle e = f(x^*).$$

Furthermore, by Theorem 2.80, we have

$$z \not\geq_{C \setminus \{0\}} f(x^*), \quad \forall z \in \Psi(y, r), y \in \mathbb{R}^m, r > 0.$$

Hence $z^* = f(x^*) \in Q$. ■

It is worth mentioning that if $\bar{f}(x, u)$ is not proper or lsc, and that f is not proper, then Theorem 2.84 may not be valid even for a scalar optimization problem. Let us look at the following example.

Example 2.85. Let $\ell = m = n = 1$, $\sigma(u) = |u|, \forall u \in \mathbb{R}$, $f(x) = +\infty, \forall x \in \mathbb{R}$. Let

$$\bar{f}(x, u) = \begin{cases} 0, & \text{if } u \neq 0, x \in [0,1]; \\ +\infty, & \text{else.} \end{cases}$$

Clearly, $\bar{f}(x, u)$ is level-bounded in x locally uniformly in u. $\bar{f}$ is also proper, but not lsc. Simple calculation gives us the augmented Lagrangian: for any $y \in \mathbb{R}$, $r > 0$,

$$\bar{l}(x, y, r) = \begin{cases} \inf\limits_{u \neq 0} \{-yu + r|u|\}, & \text{if } x \in [0,1]; \\ +\infty, & \text{else.} \end{cases}$$

For any $r > 0$, $\bar{\psi}(0, r) = \inf\limits_{u \neq 0} r|u| = 0 > -\infty$. However, from the expression of $\bar{l}(x, y, r)$, we deduce that $\bar{l}(x, y, r) \leq 0, \forall x \in [0,1], y \in \mathbb{R}, r > 0$. Hence

$$\sup_{y \in \mathbb{R}, r > 0} \bar{\psi}(y, r) \leq 0.$$

So

$$\sup_{y \in \mathbb{R}, r > 0} \bar{\psi}(y, r) < \inf_{x \in \mathbb{R}} f(x) = +\infty.$$

That is, Theorem 2.84 fails.

The following proposition further illustrates the relationship between the dual map Ψ and solutions of (EOP).

Proposition 2.86. *Assume that $\bar{f} : I\!R^n \times I\!R^m \to I\!R^\ell \cup \{+\infty\}$ is a proper extended vector-valued function such that each of its component function $\bar{f}_i(x, u)$ is lsc. Assume further that there exist $\bar{y}, \bar{r}$ such that*

$$\bar{f}(x, u) + \bar{r}\sigma(u)e - \langle \bar{y}, u \rangle e \geq_C m_1 e, \quad \forall x, u,$$

where m_1 is a real number. Let $r_k \to +\infty$ and each x_k a weakly minimal solution to

$$(EOP_k) \quad Inf_C \ L(x, \bar{y}, r_k) \quad \text{subject to } x \in I\!R^n.$$

If $x_k \to x^*$ and $\{\xi(f(x_k))\}$ is bounded from above, then $x^* \in Min_{intC}(f, I\!R^n)$.

Proof. Suppose to the contrary that there exist $x' \in \mathbb{R}^n$ and $\epsilon_0 > 0$ such that

$$f(x') - f(x^*) \leq_C -\epsilon_0 e.$$

Let

$$U_k = \{u \in \mathbb{R}^m : \bar{f}(x', u) + r_k\sigma(u)e - \langle \bar{y}, u \rangle e \leq_C f(x')\}.$$

As argued in the proof of Theorem 2.84, we can show that there exist $k_0 > 0$ and a compact set U_0 such that U_k is compact and $U_k \subset U_0$ when $k > k_0$. Thus, for every $k > k_0$, there exists $u_k \in U_k$ such that u_k is a minimal solution of the problem:

$$\text{Inf}_C \ \bar{f}(x', u) + r_k\sigma(u)e - \langle \bar{y}, u \rangle e, \quad \text{subject to } u \in \mathbb{R}^m.$$

Since x_k is a weakly minimal solution to (EOP_k), we deduce that $\exists z_k \in \text{Inf}\{\bar{f}(x_k, \mathbb{R}^m) + r_k\sigma(\mathbb{R}^m) - \langle \bar{y}, \mathbb{R}^m \rangle e\}$ such that

$$\bar{f}(x', u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k \rangle e \not\leq_{intC} z_k, \ k > k_0. \tag{2.60}$$

As

$$z_k \in \text{Inf}_C\{\bar{f}(x_k, u) + r_k\sigma(u) - \langle \bar{y}, u \rangle e : u \in \mathbb{R}^m\}, \tag{2.61}$$

we deduce that, for every $k > k_0$, there exists a sequence $\{u_k^j\}$ such that

$$\bar{f}(x_k, u_k^j) + r_k\sigma(u_k^j)e - \langle \bar{y}, u_k^j \rangle e \to z_k, \quad \text{as } j \to +\infty.$$

It is not difficult to prove that, for every $k > k_0$, $\{u_k^j\}$ is bounded. So we assume without loss of generality that $u_k^j \to u_k'$ as $j \to \infty$. Moreover, $\bar{f}_i$ and σ are lsc, therefore,

$$\bar{f}(x_k, u_k') + r_k\sigma(u_k')e - \langle \bar{y}, u_k' \rangle e \leq_C z_k, \quad k > k_0. \tag{2.62}$$

(2.61) and (2.62) yield

$$z_k = \bar{f}(x_k, u_k') + r_k\sigma(u_k')e - \langle \bar{y}, u_k' \rangle e, \quad k > k_0. \tag{2.63}$$

Substituting (2.63) into (2.60), we get for $k > k_0$

$$\bar{f}(x', u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k \rangle e \not\leq_{intC} \bar{f}(x_k, u_k') - r_k\sigma(u_k')e + \langle \bar{y}, u_k' \rangle e. \tag{2.64}$$

The combination of (2.61) and (2.63) yields

$$\begin{aligned} f(x_k) &= \bar{f}(x_k, 0) + r_k\sigma(0)e - \langle \bar{y}, 0 \rangle e \\ &\not\leq_{C\setminus\{0\}} -\bar{f}(x_k, u_k') - r_k\sigma(u_k')e - \langle \bar{y}, u_k' \rangle e, \quad k > k_0. \end{aligned}$$

As a result,

$$f(x_k) \not\leq_{C\setminus\{0\}} (r_k - \bar{r})\sigma(u_k')e + m_1 e, \quad k > k_0.$$

Thus

$$\xi(f(x_k)) \geq (r_k - \bar{r})\sigma(u_k') - m_1, \quad k > k_0.$$

It follows that $\{u_k'\}$ is bounded because$\{\xi(f(x_k)\}$ is bounded above. Without loss of generality, we assume that $u_k' \to u^*$. Since $\{\xi(f(x_k))\}$ is bounded above, arguing as in the proof of Theorem 2.84, we can prove that $u^* = 0$. By the lsc of $\bar{f}_i$ and σ, we obtain

$$f_i(x^*) \leq_C \liminf_{n\to\infty} f_i(x_k, u_k') + r_k\sigma(u_k') + \langle \bar{y}, u_k' \rangle, \quad i = 1, \cdots, \ell.$$

So there exists $k_1 > k_0$ such that

$$\bar{f}(x_k, u_k') + r_k\sigma(u_k')e - \langle \bar{y}, u_k' \rangle e \geq_C f(x^*) - \epsilon_0/2e, \quad k \geq k_1 > k_0 \tag{2.65}$$

Note that $u_k \in U_k$, i.e.,

$$\bar{f}(x', u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k \rangle e \leq_C f(x'), \quad k > k_0.$$

This inequality combined with (14) yields

$$\bar{f}(x', u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k \rangle e - f(x^*) \leq_C -\epsilon_0 e, \quad k > k_0. \tag{2.66}$$

(2.65) and (2.66) jointly yield, for $k \geq k_1 > k_0$,

$$\begin{aligned} &\bar{f}(x', u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k \rangle e - \bar{f}(x_k, u_k') - r_k\sigma(u_k')e + \langle \bar{y}, u_k' \rangle e \\ &\leq_C -\epsilon_0/2e, \end{aligned} \tag{2.67}$$

(2.67) contradicts (2.64). ■

2.8 Augmented Lagrangian Penalization for VO

Consider the primal vector optimization problem (EOP):

$$\operatorname{Inf}_C \underset{x \in \mathbb{R}^n}{f(x)}$$

where $f : \mathbb{R}^n \to \mathbb{R}^\ell \cup \{-\infty, +\infty\}$ is a proper extended vector-valued function.

Let $\bar{f} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^\ell \cup \{+\infty\}$ be a perturbed function such that $\bar{f}(x, 0) = f(x), \forall x \in \mathbb{R}^n$. The optimal value function is defined by

$$p(u) = \operatorname{Inf}_C\{\bar{f}(x, u) : x \in \mathbb{R}^n\}, \quad u \in \mathbb{R}^m.$$

Let $\sigma : \mathbb{R}^m \to \mathbb{R}_+ \cup \{+\infty\}$ be an augmenting function. The augmented Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^m \times (0, +\infty) \rightrightarrows \mathbb{R}^\ell \cup \{+\infty, -\infty\}$ is a set-valued function defined by

$$L(x, y, r) = \operatorname{Inf}_C\{\bar{f}(x, u) + r\sigma(u)e - \langle y, u\rangle e : u \in \mathbb{R}^m\},$$

where $y \in \mathbb{R}^m, r \in (0, +\infty)$.

The set-valued dual function is defined by

$$\Psi(y, r) = \operatorname{Inf}_C \cup_{x \in \mathbb{R}^n} L(x, y, r), \quad y \in \mathbb{R}^m, r \in (0, +\infty).$$

Definition 2.87. *$\bar{f}(x, u)$ is said to be Inf-externally stable in x when $\|u\| > 0$ is sufficiently small if there exists a neighborhood W_1 of $0 \in \mathbb{R}^m$ such that, for any fixed $u \in W_1$, $\forall x \in \mathbb{R}^n$, there exists $z_u \in p(u)$ such that $z_u \leq_C \bar{f}(x, u)$.*

Theorem 2.88. *Assume*

(a) $\bar{f}(x, u)$ is proper and each of its component function $\bar{f}_i(x, u)$ is lsc; $\bar{f}(x, u)$ is Inf-externally stable in x when $\|u\|$ is sufficiently small; $\xi(\bar{f}(x, u))$ is level-bounded in x locally uniformly in u.

(b) there exist $\bar{y} \in \mathbb{R}^m, \bar{r} > 0$ and $m_1 \in \mathbb{R}$ such that

$$\bar{f}(x, u) + \bar{r}\sigma(u)e - \langle \bar{y}, u\rangle e \geq_C m_1 e, \quad \forall x, u.$$

Then there exists $r^* > 0$ such that when $r \geq r^*$

$$p(0) \subseteq \Psi(\bar{y}, r) \tag{2.68}$$

if and only if there exist a neighborhood W of $0 \in \mathbb{R}^m$ and a scalar $r' > 0$ such that

$$z_u \not\leq_{intC} z + \langle \bar{y}, u\rangle e - r'\sigma(u)e, \quad \forall z_u \in p(u), z \in p(0), u \in W. \tag{2.69}$$

Proof. Sufficiency. Let

$$\eta(z) = \min\{z_1, \cdots, z_\ell\}, \quad \forall z = (z_1, \cdots, z_\ell) \in \mathbb{R}^\ell \cup \{-\infty, +\infty\}.$$

We assert that $\eta(p(0)) = \{\eta(z) : z \in p(0)\}$ is bounded above by some $M > 0$. Otherwise, there exists a sequence $z_k \in p(0)$ such that $z_k \to +\infty$ as $k \to +\infty$. Arbitrarily fix a $z_0 \in p(0)$. As shown in the proof of Theorem 2.84, $z_0 \in \mathbb{R}^\ell$.

Moreover, We obtain $(z_0 - z_k) \to -\infty$, which contradicts (2.69) when $u = 0$. Now we prove by contradiction that (2.69) implies (2.68). Suppose that $\exists z_k^* \in p(0)$ and $r_k \uparrow +\infty$ such that

$$z_k^* \notin \Psi(\bar{y}, r_k) = \mathrm{Inf}_C\{L(x, \bar{y}, r_k) : x \in \mathbb{R}^n\} \tag{2.70}$$

Since $z_k^* \in p(0)$, arguing as in the proof of Theorem 2.84, we conclude that, for each k,

$$z_k^* = f(x_k^*) \tag{2.71}$$

for some $x_k^* \in \mathrm{Min}_C(f, \mathbb{R}^n)$.

Consider the following two cases:

(i) $f(x_k^*) \in \mathrm{Inf}_C\{\bar{f}(x,u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e : (x,u) \in \mathbb{R}^n \times \mathbb{R}^m\}$;

(ii) $f(x_k^*) \notin \mathrm{Inf}_C\{\bar{f}(x,u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e : (x,u) \in \mathbb{R}^n \times \mathbb{R}^m\}$.

If case (i) occurs, noticing that $f(x_k^*) = \bar{f}(x_k^*, 0) + r_k\sigma(0)e - \langle \bar{y}, 0\rangle e$, we deduce that

$$\bar{f}(x,u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e \not\leq_{C\setminus\{0\}} f(x_k^*), \quad \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m.$$

Letting $x = x_k^*$, we obtain

$$\bar{f}(x_k^*, u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e \not\leq f(x_k^*), \quad u \in \mathbb{R}^m.$$

This relation shows that $f(x_k^*) \in L(x_k^*, \bar{y}, r_k)$ since $f(x_k^*) = \bar{f}(x_k^*, 0) + r_k\sigma(0)e - \langle \bar{y}, 0\rangle e$. This fact combined with (2.70) yields that there exist $x_k \in \mathbb{R}^n$ and $z_k \in L(x_k, \bar{y}, r_k)$ such that

$$z_k \leq_C z_k^*. \tag{2.72}$$

Furthermore, for each z_k, there exists a sequence $\{u_{k,j}\}$ such that

$$\bar{f}(x_k, u_{k,j}) + r_k\sigma(u_{k,j})e - \langle \bar{y}, u_{k,j}\rangle e \to z_k \text{ as } j \to \infty.$$

As a result,

$$\begin{aligned} &m_1 e + (r_k - \bar{r})\sigma(u_{k,j})e \\ &\leq_C \bar{f}(x_k, u_{k,j}) + \bar{r}\sigma(u_{k,j})e - \langle \bar{y}, u_{k,j}\rangle e + (r_k - \bar{r})\sigma(u_{k,j})e \to z_k. \end{aligned}$$

The formula above combined with (2.72) yields

$$m_1 + (r_k - \bar{r})\sigma(u_{k,j}) \leq \eta(z_k) \leq \eta(z_k^*) \leq M.$$

Thus, $\sigma(u_{k,j}) \leq M - m_1$ when $k \geq k_0$, where k_0 satisfies $r_{k_0} \geq \bar{r} + 1$. So, for each $k \geq k_0$, $\{u_{k,j}\}$ is bounded. Without loss of generality, we assume that $u_{k,j} \to u_k$. It follows from the lsc of $\bar{f}_i$ and σ that

$$\bar{f}(x_k, u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k\rangle e \leq_C z_k, \quad k \geq k_0$$

However, $z_k \in L(x_k, \bar{y}, r_k)$. Therefore,

$$z_k = \bar{f}(x_k, u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k\rangle e, \quad k \geq k_0 \tag{2.73}$$

The combination of (2.71), (2.72) and (2.73) yields

$$\bar{f}(x_k, u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k\rangle e \not\leq_{C\setminus\{0\}} f(x_k^*), \quad k \geq k_0 \tag{2.74}$$

It follows that

$$m_1 e + (r_k - \bar{r})\sigma(u_n)e \leq_C \bar{f}(x_k, u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k\rangle e \leq_C f(x_k^*) = z_k^*.$$

Therefore,

$$m_1 + (r_k - \bar{r})\sigma(u_n) \leq \eta(z_k) \leq \eta(z_k^*) \leq M.$$

Namely,

$$(r_k - \bar{r})\sigma(u_k) \leq M - m_1. \tag{2.75}$$

So $\{u_k\}$ is bounded. Without loss of generality, we assume that $u_k \to u'$. It follows from (2.75) that $\sigma(u') \leq \liminf_{k\to\infty}\sigma(u_k) \leq 0$. Hence $u' = 0$. Thus $u_k \in W_1$ when $k \geq k_1 \geq k_0$. By the external stability of $\bar{f}(x, u_k)$ in x when n is sufficiently large, we get $z_{u_k} \in p(u_k)$ such that

$$z_{u_k} \leq_C \bar{f}(x_k, u_k). \tag{2.76}$$

(2.74) and (2.76) jointly yield

$$z_{u_k} + r_k\sigma(u_k)e - \langle \bar{y}, u_k\rangle e \not\leq_{C\setminus\{0\}} f(x_k^*),$$

which contradicts (2.69) because $r_k \to +\infty$ as $k \to +\infty$.

If case (ii) occurs, noticing that $f(x_k^*) = \bar{f}(x_k^*, 0) + r_k\sigma(0)e - \langle \bar{y}, 0\rangle e$, we conclude that there exists $(x_k, u_k) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$\bar{f}(x_k, u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k\rangle e \not\leq_{C\setminus\{0\}} f(x_k^*).$$

Thus, we have returned to (2.74) in case (i). So once again, we will be led to a contradiction.

Necessity. We also prove by contradiction. Suppose that $\exists u_k \to 0, r_k \uparrow +\infty, z_k \in p(u_k)$ and $z_k^* \in p(0)$ such that

$$z_{u_k} \leq_{intC} z_k^* + \langle \bar{y}, u_k\rangle e - r_k\sigma(u_k)e.$$

Then $\exists x_k \in \mathbb{R}^n, x_k^* \in \mathrm{Min}_C(f, \mathbb{R}^n)$ such that

$$\bar{f}(x_k, u_k) \leq_{intC} f(x_k^*) + \langle \bar{y}, u_k\rangle e - r_k\sigma(u_k)e.$$

Note that $f(x_k^*) \in \Psi(\bar{y}, r_k)$ when $k \geq k_2$ is such that $r_{k_2} \geq \max\{r', \bar{r} + 1)$. By the definition of $\Psi(\bar{y}, r_k)$, we see that for any $w_k \in \mathrm{Inf}_C\{\bar{f}(x_k, u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e : u \in \mathbb{R}^m\}$, we have

$$w_k \not\leq_{C\setminus\{0\}} f(x_k^*), \quad k \geq k_2. \tag{2.77}$$

Let

$$Q_k = \{u \in \mathbb{R}^m : \bar{f}(x_k, u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e \leq_C \\ \bar{f}(x_k, u_k) + r_k\sigma(u_k)e - \langle \bar{y}, u_k\rangle e \leq_{intC} f(x_k^*)\}.$$

It is easy to check (as in the proof of sufficiency) that when $k \geq k_3 \geq k_2$, the set $\{u \in \mathbb{R}^m : \bar{f}(x_k, u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e \leq_C f(x_k^*)\}$ is compact. Thus Q_k is compact when $k \geq k_3 \geq k_2$. Consequently, we obtain $u_k' \in Q_k$ such that, for any $k \geq k_3 \geq_C k_2$,

$$\bar{f}(x_k, u_k') + r_k\sigma(u_k')e - \langle \bar{y}, u_k'\rangle e \\ \in \text{Inf}_C\{\bar{f}(x_k, u) + r_k\sigma(u)e - \langle \bar{y}, u\rangle e : u \in \mathbb{R}^m\},$$

and

$$\bar{f}(x_k, u_k') + r_k\sigma(u_k')e - \langle \bar{y}, u_k'\rangle e \leq_{intC} f(x_k^*),$$

which contradicts (2.77) when $k \geq k_3 \geq k_2$. ■

The following simple example verifies Theorem 2.88.

Example 2.89. Let $p = 1$, $\ell = 2$, $X = [-1, 1]$. Let $f(x) = (|x|, -|x|), \forall x \in \mathbb{R}$; $h(x) = x, \forall x \in \mathbb{R}$. Consider the following constrained vector optimization problem

$$(VOP_1) \qquad \begin{array}{ll} \inf_C f(x) & \\ \text{s.t. } x \in X, & \\ \quad h(x) \leq 0. & \end{array}$$

Denote by $X_0 = \{x \in X : h(x) \leq 0\}$ the feasible set of (VOP_1). Define

$$\bar{f}(x) = \begin{cases} f(x), & \text{if } x \in X_0; \\ +\infty, & \text{otherwise.} \end{cases}$$

Then (VOP_1) is equivalent to (EOP) (in the sense that the two problems have the same sets of (weakly) minimal solutions).

Let $u \in \mathbb{R}$. It is easy to see that

$$\bar{f}(x, u) = \begin{cases} g(x), \text{ if } h(x) \leq u, x \in X \\ +\infty, \text{ otherwise} \end{cases}$$

is a perturbed function of f. Let $\sigma(u) = |u|, \forall u \in \mathbb{R}$. It is routine to check that condition (a) in Theorem 2.88 holds. Let $\bar{y} = 0, \bar{r} = 1$. Then, we can verify that condition (b) in Theorem 2.88 holds. Furthermore, it can be computed that

$$p(0) = \{(x, -x) : x \in [0, 1]\}.$$

$$p(u) = \begin{cases} \{(x, -x) : x \in [0, 1]\}, & \text{if } u \geq 0 \text{ and } u \in W \\ \{(-x, x) : x \leq u\}, & \text{if } u < 0 \text{ and } u \in W, \end{cases}$$

where

$$W = \{u \in \mathbb{R} : |u| < 1\}.$$

Take $r' = 1$. It can be easily checked that (2.69) holds. By Theorem 2.88, there exists $r^* > 0$ such that when $r \geq r^*$, $p(0) \subset \Psi(\bar{y}, r)$. As a matter of fact, we can choose $r^* = 3$. Whenever $r \geq r^*$,

$$\begin{aligned}\Psi(\bar{y}, r) &= \mathrm{Inf}_C \bigcup_{x \in \mathbb{R}} L(x, \bar{y}, r) \\ &= \mathrm{Inf}_C \bigcup_{x \in \mathbb{R}} [\mathrm{Inf}_C \{\bar{f}(x, u) + r|u|e\}]_{u \in \mathbb{R}} \\ &= \mathrm{Inf}_C \bigcup_{x \in X} [\mathrm{Inf}_C \{(|x|, -|x|) + r|u|e : u \geqq x\}] \\ &= \mathrm{Inf}_{C\, x \in [-1,1]} (V_1 \cup V_2),\end{aligned}$$

where $V_1 = \{(x, -x) : x \in [0, 1]\}$ and $V_2 = \{(-(r+1)x, -(r-1)x) : x \in [-1, 0]\}$. Thus,

$$\Psi(\bar{y}, r) = V_1 = \{(x, -x) : x \in [0, 1]\}.$$

Consequently,

$$p(0) = \Psi(\bar{y}, r), \quad r \geq r^* = 3.$$

(2.68) indicates the uniformly exact penalization. That is, there exists a common $r^* > 0$ such that $z^* \in \Psi(\bar{y}, r)$ for each $z^* \in p(0)$ whenever $r \geq r^*$.

For application purpose, we may need only the following weaker version of exact penalization, which requires weaker conditions.

Theorem 2.90. *Assume (a) as in Theorem 2.88 and*
(b') there exist $\bar{y} \in \mathbb{R}^m, \bar{r} > 0$ such that

$$inf_{x,u} \xi(\bar{f}(x, u)) + \bar{r}\sigma(u) - \langle \bar{y}, u \rangle > -\infty.$$

Let $z^* \in p(0)$. Then there exists $r^* > 0$ such that $z^* \in \Psi(\bar{y}, r)$ whenever $r \geq r^*$ if and only if there exist a neighborhood W of $0 \in \mathbb{R}^m$ and $r' > 0$ such that

$$z_u \not\leq_{intC} z^* + \langle \bar{y}, u \rangle e - r'\sigma(u)e, \quad \forall z_u \in p(u), u \in W.$$

Proof. The proof of this theorem is almost the same as that of Theorem 2.88, we need only to replace z_k^* with z^* and η with ξ in the proof of Theorem 2.88. ■

2.9 Nonlinear Lagrangian Duality for VO

In this section, we discuss a nonlinear Lagrangian approach to weak and strong duality results where no convexity is required for the problem data.

Let $Y = \mathbb{R}^\ell$ be an ℓ-dimensional Euclidean space, and $C = \mathbb{R}^\ell_+$. Let $e = (1, \cdots, 1) \in intC$, and $e_i = (0, 0, \cdots, 1, 0, \cdots, 0)$ (the ith component is 1 and the other components are 0's), $i = 1, \cdots, \ell$. Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set, $f = (f_1, \ldots, f_\ell) : K \to \mathbb{R}^\ell$ be a vector-valued function such that each of its component function f_i is l.s.c., and $g_j : K \to \mathbb{R}$ be l.s.c. for any $j \in \{1, \cdots, m\}$.

Consider the following constrained vector optimization problem (VOP):

$$\operatorname{Min}_{C} f(x), \quad x \in B \tag{2.78}$$

where $B = \{x \in K : g_j(x) \leq 0, j = 1, \cdots, m\}$. It is clear that B is closed. We denote by $\operatorname{Min}_C f(B)$ and $\operatorname{Inf}_C f(B)$ the set of minimal points and the set of infimum points of (VOP) respectively.

Without loss of generality, we assume throughout this section that

$$\min_{1 \leq i \leq \ell} \inf_{x \in K} f_i(x) \geq 0.$$

If this assumption does not hold, then consider the following optimization problem (VOP$'$):

$$\begin{array}{ll} \operatorname{Min}_C & (\exp(f_1(x)) + 1, \cdots, \exp(f_\ell(x)) + 1) \\ \text{s.t.} & x \in K, g_j(x) \leq 0, \quad j = 1, \cdots, m. \end{array}$$

It is clear that the sets of minimal solutions and weakly minimal solutions of (VOP) are the same as that of (VOP$'$), respectively.

For $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$, and $\gamma \in (0, +\infty)$, let

$$\|u\|_\gamma = \left[\sum_{j=1}^m |u_j|^\gamma\right]^{1/\gamma}.$$

Let $y^1 = (y^1_1, \cdots, y^1_m), y^2 = (y^2_1, \cdots, y^2_m) \in \mathbb{R}^m$, define the notation of componentwise product for y^1 and y^2:

$$y^1 * y^2 \equiv (y^1_1 y^2_1, \cdots, y^1_m y^2_m).$$

A vector-valued function $p : \mathbb{R}^\ell_+ \times \mathbb{R}^m \to \mathbb{R}^\ell$ is called increasing if for any $(z^i, y^i) \in \mathbb{R}^\ell_+ \times \mathbb{R}^m (i = 1, 2)$ with $(z^1, y^1) - (z^2, y^2) \in C \times \mathbb{R}^m_+$, we have

$$p(z^1, y^1) \geq_C p(z^2, y^2).$$

Let p be an increasing vector-valued function defined on the domain $\mathbb{R}^\ell_+ \times \mathbb{R}^m$ such that each of its component function p_i is l.s.c. and p enjoys the following two properties:

(A) There exist positive real numbers $a_j (j = 1, \cdots, m)$ such that for any $z \in \mathbb{R}^\ell_+, y = (y_1, \cdots, y_m)$ with (z, y) belonging to the domain of p, $p(z, y) \geq_C z$ and $p(z, y) \geq_C (\max_{1 \leq j \leq m} \{a_j y_j\}) e$.

(B) $\forall z \in C, p(z, 0, \cdots, 0) = z$.

It is easy to prove the following elementary proposition.

Proposition 2.91. *Let $p(z, y) = p'(p'(z, y), y)$, where p' is an increasing function with properties (A) and (B). Then p is also an increasing function having properties (A) and (B).*

Example 2.92. Let $z = (z_1, \cdots, z_\ell), y = (y_1, \cdots, y_m)$, and $(z, y) \in C \times \mathbb{R}^m$. Some examples of the increasing function p defined on $C \times \mathbb{R}^m$ having properties (A) and (B) are as follows:

(i) $p_\infty(z, y) = \sum_{i=1}^{\ell} \max\{z_i, y_1, \cdots, y_m\} e_i$;
(ii) $p_\gamma(z, y) = \sum_{i=1}^{\ell} \left(z_i^\gamma + \sum_{j=1}^{m} y_j^{+\gamma}\right)^{1/\gamma} e_i, \quad 0 < \gamma < \infty$;
(iii) $p(z, y) = z + (\sum_{j=1}^{m} b_j y_j^+) e$, where $b_j > 0, j = 1, \cdots, m$.

Let

$$F(x, d) = (f(x), d * g(x)),$$

where $d = (d_1, \cdots, d_m) \in \mathbb{R}_+^m$ and $g(x) = (g_1(x), \cdots, g_m(x))$.

Let p be an increasing function defined on $C \times \mathbb{R}^m$ with properties (A) and (B). The nonlinear Lagrangian function corresponding to p for (VOP) is defined as

$$N(x, d) = p(F(x, d)).$$

The following problem (DVOP)

$$\sup_{d \in \mathbb{R}_+^m} q(d), \tag{2.79}$$

where $q(d) = \mathrm{Inf}_C N(K, d), \forall d \in \mathbb{R}_+^m$ is called the nonlinear Lagrangian dual problem to (VOP) corresponding to p.

Remark 2.93. If p is convex, e.g., all the p's except p_γ in the case of $\gamma \in (0, 1)$ in Example 2.92, the problem of computing $q(d)$:

$$\inf_{x \in X} p(F(x, d))$$

is a type of convex composite multiobjective optimization problems studied in [121].

It is elementary to prove the following results.

Lemma 2.94. *Let p be an increasing function with properties (A) and (B). Then $p(F(x, d)) = f(x), \forall x \in B,\ d \in \mathbb{R}_+^m$.*

Proposition 2.95. *Weak Duality. $\forall x \in B, d \in \mathbb{R}_+^m$, $(q(d) - f(x)) \cap (C \backslash \{0\}) = \emptyset$.*

Corollary 2.96. *If* $x^* \in B$ *satisfies* $f(x^*) \in Sup_C q(\mathbb{R}^m_+)$, *then*

$$x^* \in Min_{intC}(f, B).$$

Corollary 2.97. $[Sup_C q(\mathbb{R}^m_+) - Inf_C f(B)] \cap intC = \emptyset$.

Definition 2.98. *Let* $X \subset \mathbb{R}^n$ *be a set and* $f : X \to \mathbb{R}^\ell$ *be a vector-valued function. The set* $f(X)$ *is said to be externally stable if for any* $x \in X$, *there exists a minimal solution* $x^* \in X$ *of* f *on* X *such that* $f(x^*) \leq_C f(x)$.

The following lemma on external stability can be easily proved.

Lemma 2.99. *Let* $X \subset \mathbb{R}^n$ *be a compact subset. Let* $f : X \to \mathbb{R}$ *be a vector-valued function such that each of its component functions is l.s.c.. Then* $f(X)$ *is externally stable.*

It is routine to prove the next lemma.

Lemma 2.100. *Let* $s : C \times \mathbb{R}^m \to \mathbb{R}$ *be an increasing l.s.c. function. Let* $f : X \to C$ *be a vector-valued function such that each component function* f_i *is l.s.c.. Let* $g_j : X \to R$ $(j = 1, \cdots, m)$ *be l.s.c.. Then* $s(f(x), g(x))$ *is l.s.c. on* X.

Let $\xi(z) = \max_{1 \leq i \leq \ell}\{z_i\}$, $\forall z = (z_1, \cdots, z_\ell)$.

Clearly, ξ is an increasing, continuous, subadditive, positively homogeneous and convex function.

Definition 2.101. *Let* $X \subset \mathbb{R}^n$ *be an unbounded set. A vector-valued function* $f : X \to \mathbb{R}^\ell$ *is said to be coercive on* X *if*

$$\lim_{\|x\| \to +\infty, x \in X} \xi(f(x)) \to +\infty,$$

where $\|.\|$ *is a norm of* $\mathbb{R}^n$.

The following result establishes a proper relation between (VOP) and (DVOP).

Theorem 2.102. *Strong Duality. Assume that* X *is closed,* $f(x) \geq_C 0, \forall x \in X$ *and* f *is coercive on* X *if* X *is unbounded. Then*

$$Inf_C f(B) \subset Sup_C q(\mathbb{R}^m_+).$$

Proof. Let $z^* \in \text{Inf}_C f(B)$. Then $\exists x^1_k \in B$ such that $f(x^1_k) \to z^*$ as $k \to +\infty$. It follows that $\xi(f(x^1_k)) \to \xi(z^*)$ as $k \to +\infty$. Therefore, $\{x^1_k\}$ is a bounded sequence by the coercivity of f on X. Since B is closed, there exists a subsequence $\{x^1_{k_j}\}$ such that $x^1_{k_j} \to x^*$ for some $x^* \in B$. Note that $f_i(x^*) \leq \liminf_{j \to +\infty} f_i(x^1_{k_j}) = (z^*)_i, i = 1, \cdots, \ell$, where $(z^*)_i$ denotes the ith component of z^*. We have $f(x^*) \leq_C z^*$. This with $z^* \in \text{Inf}_C f(B)$ implies that $f(x^*) = z^*$.

Hence $x^* \in \mathrm{Min}_C f(B)$. Since f is coercive on X, we deduce that $\exists N > 0$ such that

$$\xi(f(x)) \geq \xi(f(x^*)) + 1, \ \forall x \in X_1 = \{x \in X : \|x\| > N\}. \tag{2.80}$$

We claim that

$$f(x) \not\leq_{C\setminus\{0\}} f(x^*), \quad \forall x \in X_1. \tag{2.81}$$

Otherwise, $\xi(f(x)) \leq \xi(f(x^*))$, contradicting (2.80).

Let $d = ke, k = 1, 2, \cdots$. Since $X_2 = \{x \in X : \|x\| \leq N\}$ is a nonempty compact set and $x^* \in X_2$, by Lemmas 2.99 and 2.100, we obtain a sequence $\{x_k^2\} \subseteq X_2$ such that each x_k^2 is a minimal solution to the problem: $\min\limits_{x \in X_2} p(f(x), kg(x))$ and

$$p(f(x_k^2), kg(x_k^2)) \leq_C p(f(x^*), kg(x^*)) = f(x^*). \tag{2.82}$$

We show that this fact combined with (2.81) yields that $p(F(x_k^2, d)) \in q(k, \cdots, k) = \mathrm{Inf}_C p(F(K, d))$.

(i) It is obvious that if $x \in X_2$, $p(F(x_k^2, d)) \not\geq_{C\setminus\{0\}} p(F(x, d))$.

(ii) Suppose that $\exists\ \bar{x} \in X_1$ such that

$$p(F(x_k^2, d)) \geq_{C\setminus\{0\}} p(F(\bar{x}, d)). \tag{2.83}$$

Note that

$$p(F(x_k^2, d)) \leq_C f(x^*)$$

and

$$f(x^*) \not\geq_{C\setminus\{0\}} f(\bar{x}).$$

Then

$$p(F(x_k^2, d)) \not\geq_{C\setminus\{0\}} f(\bar{x}). \tag{2.84}$$

By (2.83) and (2.84),

$$p(F(\bar{x}, d)) \not\geq_{C\setminus\{0\}} f(\bar{x}),$$

a contradiction with the property (A).

It follows from $\{x_k^2\} \subset X_2$ that there exists a subsequence $\{x_{k_j}^2\}$ such that $x_{k_j}^2 \to x_0 \in X_2$.

Let us show that $x_0 \in B$. If not, $d(x_0, B) \geq \delta_0$ for some $\delta_0 > 0$. It follows that $d(x_{k_j}^2, B) \geq \delta_0/2$ when j is sufficiently large.

Let $X_3 = \{x \in X_2 : d(x, B) \geq \delta_0/2\}$ and $\bar{g}(x) = \max\limits_{1 \leq j \leq m} g_j(x)$. Since $\bar{g}(x) > 0, \forall x \in X_3$, X_3 is compact and $\bar{g}$ is l.s.c, we deduce that $\min\limits_{x \in X_3} \bar{g}(x) = m_0 > 0$.

By property (A) of the function p, there exist positive numbers $a_i (i = 1, \cdots, m)$ such that

$$p(f(x_{k_j}^2), k_i g(x_{k_j}^2)) \geq_C (m_0 k_j \min_{1 \leq i \leq m} a_i)e,$$

when j is sufficiently large, which contradicts (2.82). So $x_0 \in B$.

Applying property (A) and (2.82), we have

$$f(x_{k_j}^2) \leq_C p(f(x_{k_j}^2), k_i g(x_{k_j}^2)) \leq_C f(x^*).$$

Thus,

$$f_i(x_{k_j}^2) \leq p_i(f(x_{k_j}^2), k_j g(x_{k_j}^2)) \leq f_i(x^*), \quad i = 1, \cdots, \ell. \tag{2.85}$$

Applying the lower limit to (2.85) by letting $j \to \infty$, we conclude that $f_i(x_0) \leq f_i(x^*), i = 1, \cdots, \ell$, which implies that

$$f(x_0) = f(x^*) \tag{2.86}$$

since $x^* \in \text{Min}_C(f, B)$.

(2.86) combined with (2.85) as well as $x_{k_j}^2 \to x_0$ yields that

$$p(f(x_{k_j}^2), k_j g(x_{k_j}^2)) \to f(x^*), \quad \text{as } j \to +\infty.$$

Finally, it follows directly from Proposition 2.95 that

$$(q(d) - f(x^*)) \cap (C \backslash \{0\}) = \emptyset, \quad \forall d \in \mathbb{R}_+^m.$$

The proof is complete. ∎

Example 2.103. The condition that f is coercive on X is important to guarantee the validity of Theorem 2.102. Otherwise, it may fail even if B is compact. Let $\ell = 1, K = [0, +\infty), f(x) = 1/(x+1), \forall x \in K, g_1(x) = x - 1$, if $0 \leq x \leq 1; g_1(x) = 1/\sqrt{x} - 1/x$, if $1 < x < +\infty$, $p(y_1, y_2) = \max\{y_1, y_2\}, \forall y_1, y_2 \in \mathbb{R}$.

Consider the problem:

$$\inf f(x) \text{ s.t. } x \in K, \ g_1(x) \leq 0.$$

It is easy to see that $B = [0, 1]$ (which is compact) and $\inf_{\mathbb{R}_+} f(B) = \{1/2\}$.

$$\begin{aligned} p(f(x), dg_1(x)) &= \max\{f(x), dg_1(x)\} \\ &= \max\{1/(x+1), d(1/\sqrt{x} - 1/x)\}, \quad \forall x \in X \backslash B, d \geq 0. \end{aligned}$$

Clearly, $q(d) = 0, \forall d \geq 0$. It follows that $\sup_{\mathbb{R}_+} q(\mathbb{R}_+) = \{0\}$. Hence $\inf_{\mathbb{R}_+} f(B) \subset \sup_C q(\mathbb{R}_+)$ does not hold.

We also observe that, for $z^* \in \inf_{\mathbb{R}_+} f(B)$, there does not exist $d^* \in \mathbb{R}_+$ such that $z^* \in q(d^*)$. Indeed, let $\ell = 1, X = [1/2, +\infty)$ and $f(x) = 1/x$, if $x \in [1/2, 1]$; $f(x) = 2 - x$, if $x \in [1, 2]$; $f(x) = x - 2$, if $x \in (2, +\infty)$. Let $g_1(x) = x - 1$.

Consider the problem:

$$\inf_{x \in X} f(x), \quad \text{s.t. } g_1(x) \leq 0.$$

Let $N(x, d) = \max\{f(x), dg_1(x)\}, \ d \geq 0, \ x \in X$. Then it is not difficult to derive the following fact: $q(d) = d/(1+d), \forall d \geq 0$. Clearly, $q(d) < 1 = \inf_{x \in B} f(x), \forall d > 0$.

Based on some conditions on the constraint functions, we also have the following result.

Theorem 2.104. *Let* $\bar{g}(x) = \max_{1\le j\le m} g_j(x)$. *Assume that there exist* $N > 0$ *and* $m_1 > 0$ *such that*

$$\bar{g}(x) \ge m_1, \quad \forall x \in X \text{ with } \|x\| > N. \tag{2.87}$$

Then $Inf_C f(B) \subseteq Sup_C q(I\!R^m_+)$.

Proof. It follows from (2.87) that B is a nonempty compact set. For any $z^* = f(x^*) \in \mathrm{Inf}_C f(B)$, by Proposition 2.95 we have that

$$(q(d) - f(x^*)) \cap (C\backslash\{0\}) = \emptyset, \; \forall d \in \mathrm{I\!R}^m_+.$$

Furthermore, whenever $x \in X$ with $\|x\| > N$,

$$p(f(x), kg(x)) \ge_C (km_1 \min_{1\le i\le m}\{a_i\})e \ge_{\mathrm{int}C} f(x^*) + e,$$

when k is sufficiently large. Consequently, when k is sufficiently large, the set

$$\{x \in X : p(f(x), kg(x)) \le_C f(x^*)\}(\subseteq \{x \in X : \|x\| \le N\})$$

is a nonempty compact set. Therefore, when k is sufficiently large, $\exists x_k \in X$ with $\|x_k\| \le N$ such that x_k is a minimal solution to the problem

$$\min_{x\in X} p(f(x), kg(x))$$

with

$$f(x_k) \le_C p(f(x_k), kg(x_k)) \le_C f(x^*). \tag{2.88}$$

Since $\|x_k\| \le N$ for k sufficiently large, it follows that there exists a subsequence $\{x_{k_i}\}$ converging to $x' \in X$. We can show as in the proof of Theorem 2.102 that $x' \in B$. This fact combined with (2.88) yields that $f(x') \le_C f(x^*)$. Therefore, $f(x') = f(x^*)$ since $x^* \in \mathrm{Min}_C(f, B)$. Hence, $p(f(x_{k_i}), k_i g(x_{k_i})) \to f(x^*)$. So $f(x^*) \in \mathrm{Sup}_C q(\mathrm{I\!R}^m_+)$ and the proof is complete. ∎

The following proposition further clarifies the relation between (VOP) and (DVOP).

Proposition 2.105. *Let* $d^k \in I\!R^m_+, \forall k$ *and* $d^k \to +\infty$ *as* $k \to \infty$ *(i.e.,* $d^k_i \to +\infty, \forall i$ *as* $k \to +\infty$*). Suppose that each* x^k *is a weakly minimal solution to* $Inf_C\, N(x, d^k)$ *with* $x \in K$. *Then any limiting point of* $\{x^k\}$ *is a weakly minimal solution to VOP.*

Proof. Without loss of generality, suppose that $x^k \to x^*$. We can show by contradiction that $x^* \in B$. In fact, if $d(x^*, B) \geq \delta_0$ for some $\delta_0 > 0$, then $d(x^k, B) \geq \delta_0/2$, when k is sufficiently large. Since $x^k \to x^*$, we deduce that $\|x^k - x^*\| \leq 1$ when k is sufficiently large.

Let $X_4 = \{x \in X : d(x, B) \geq \delta_0/2, \|x - x^*\| \leq 1\}$. Then $x^k \in X_4$ when k is sufficiently large. Let $\bar{g}(x) = \max_{1 \leq i \leq m} g_i(x)$. Then $\bar{g}(x^k) \geq \min_{x \in X_3} \bar{g}(x) = m_1 > 0$ when k is sufficiently large. So

$$\begin{aligned} p(f(x^k), d^k * g(x^k)) &\geq_C \bar{g}(x^k)(\min_{1 \leq i \leq m} a_i \min_{1 \leq i \leq m} d_i^k)e \\ &\geq_C (m_1 \min_{1 \leq i \leq m} a_i \min_{1 \leq i \leq m} d_i^k)e \\ &\geq_{intC} f(x_0) \end{aligned} \tag{2.89}$$

for any fixed $x_0 \in B$ and k large enough. Moreover, by Lemma 2.94,

$$f(x_0) = p(f(x_0), d^k * g(x_0)). \tag{2.90}$$

The combination of (2.89) and (2.90) contradicts the fact that x_k is a weakly minimal solution to $\operatorname{Min}_{C, x \in K} p(f(x), d^k * g(x))$. Therefore, $x^* \in B$.

Now we show that $x^* \in \operatorname{Min}_{intC}(f, B)$. Otherwise, $\exists x'' \in B$ such that $f(x'') \leq_{intC} f(x^*)$. Therefore,

$$f(x'') \leq_{intC} f(x_k), \tag{2.91}$$

when k is sufficiently large since each component function of f is l.s.c..

Note that

$$f(x'') = p(f(x''), d^k * g(x''))$$

and

$$p(f(x_k), d^k * g(x_k)) \geq_C f(x_k),$$

it follows from (2.91) that

$$p(f(x_0), d^k * g(x_0)) \leq_{intC} p(f(x_k), d^k * g(x_k)),$$

when k is sufficiently large. Namely, x_k is not a weakly minimal solution to

$$\min_{x \in X} p(f(x), d^k * g(x))$$

when k is sufficiently large, which cannot be true. The proof is complete. ■

Remark 2.106. A vector-valued function $p : \mathbb{R}_+^\ell \times \mathbb{R}_+^m \to \mathbb{R}^\ell$ is called increasing if for any $(z^i, y^i) \in \mathbb{R}_+^\ell \times \mathbb{R}_+^m (i = 1, 2)$ with $(z^1, y^1) - (z^2, y^2) \in \mathbb{R}_+^\ell \times \mathbb{R}_+^m$, we have

$$p(z^1, y^1) \geq_C p(z^2, y^2).$$

Let p be an increasing vector-valued function defined on the domain $\mathbb{R}^{\ell}_{+} \times \mathbb{R}^{m}_{+}$ such that each of its component function p_i is l.s.c. and p enjoys the following two properties:

(A') There exist positive real numbers $a_j (j = 1, \cdots, m)$ such that for any $z \in \mathbb{R}^{\ell}_{+}, y = (y_1, \cdots, y_m)$ with (z, y) belonging to the domain of p, $p(z, y) \geq_C z$ and $p(z, y) \geq_C (\max_{1 \leq j \leq m} \{a_j y_j\})e$.

(B') $\forall z \in \mathbb{R}^{\ell}_{+}, p(z, 0, \cdots, 0) = z$.

Examples of such p having properties (A') and (B') are the restrictions of p_∞, p_γ, p considered in Example 2.92 to $C \times \mathbb{R}^{m}_{+}$.

If p is defined on the domain $\mathbb{R}^{\ell}_{+} \times \mathbb{R}^{m}_{+}$ as above, then it can be shown that all the results in this section also hold for the case where

$$g^{+}(x) = (g_1^{+}(x), \cdots, g_m^{+}(x)),$$

$$F_{+}(x, d) = (f(x), d * g^{+}(x)),$$

$$N_{+}(x, d) = p(F_{+}(x, d)).$$

Next we consider the saddle point problem of the nonlinear Lagrangian.

Let p be an increasing function defined on $\mathbb{R}^{\ell}_{+} \times \mathbb{R}^{m}$ enjoying properties (A) and (B) and $F(x, d) = (f(x), d * g(x))$.

Let

$$N(x, d) = p(F(x, d)).$$

Definition 2.107. *The point $(x^*, d^*) \in K \times \mathbb{R}^{m}_{+}$ is called a saddle point of the nonlinear Lagrangian $N(x, d)$ if*

(i) $N(x, d^*) - N(x^*, d^*) \not\leq_{C \setminus \{0\}} 0, \ \forall x \in K;$
(ii) $N(x^*, d) - N(x^*, d^*) \not\geq_{C \setminus \{0\}} 0, \ \forall d \in \mathbb{R}^{m}_{+}.$

It should be noted that a saddle point may not exist even if all the conditions of Theorem 2.102 hold. The following proposition presents the relationship among a saddle point of $N(x, d)$, a minimal solution of (VOP) and a maximal solution of (DVOP).

Proposition 2.108. *The point $(x^*, d^*) \in K \times \mathbb{R}^{m}_{+}$ is a saddle point of $N(x, d)$ if and only if x^* is a minimal solution of (VOP), $f(x^*) \in q(d^*)$ and d^* is a minimal solution to (DVOP).*

In the following, we compare the conventional Lagrangian function with a special class of nonlinear Lagrangian functions.

We define a Lagrangian function as follows

$$L'(x, d) = f(x) + \sum_{j=1}^{m} d_j g_j(x) e,$$

where the dual variable $d = (d_1, \cdots, d_m) \in \mathbb{R}^m_+, x \in X$.

It is clear that the following inequality holds:

$$\left(\sum_{i=1}^{m} b_i^{\gamma}\right)^{1/\gamma} \geq \sum_{i=1}^{m} b_i, \quad \forall b_i \geq 0, \gamma \in (0,1]. \tag{2.92}$$

Let $\gamma \in (0,1]$. Consider the following class of nonlinear Lagrangian functions:

$$N_\gamma(x,d) = \sum_{i=1}^{\ell} \left[f_i^{\gamma}(x) + \sum_{j=1}^{m} d_j^{\gamma} g_j^{+\gamma}(x) \right]^{1/\gamma} e_i,$$

where $x \in X, d = (d_1, \cdots, d_m) \in \mathbb{R}^m_+$. It follows from (2.92) that

$$N_\gamma(x,d) \geq_C f(x) + \sum_{j=1}^{m} d_j g_j^+(x) e \geq_C L'(x,d), \quad \forall x \in X, \quad d \in \mathbb{R}^m_+. \tag{2.93}$$

This inequality allows us to establish the following conclusion.

Proposition 2.109. *Assume that $\gamma \in (0,1]$. Any saddle point of L' is also a saddle point of N_γ.*

Let $\bar{f}(x) = f(x), x \in B$ and $\bar{f}(x) = +\infty, x \in \mathbb{R}^n \backslash X_0$. It is obvious that (VOP) is identical to:

$$\operatorname*{Inf}_{x \in \mathbb{R}^n}{}_C\, \bar{f}(x)$$

Let $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$. Set

$$\bar{f}(x,u) = \begin{cases} f(x), & \text{if } g_j(x) \leq u_j, j = 1, \cdots, m, x \in X; \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\sigma : \mathbb{R}^m \to \mathbb{R}$ be an augmenting function. Simple reasoning gives us the augmented Lagrangian:

$$L(x,y,r) = \begin{cases} f(x) + \left[\operatorname*{inf}_{v \in \mathbb{R}^m_+}{}_C \{ r\sigma(g(x)+v) - \langle y, g(x)+v \rangle \} \right] e, \\ \qquad \text{if } x \in X, y \in \mathbb{R}^m, r > 0, \\ +\infty, \quad \text{else,} \end{cases}$$

where $g(x) = (g_1(x), \cdots, g_m(x))$.

Let the dual map be

$$\Psi(y,r) = \operatorname{Inf}_C \cup_{x \in \mathbb{R}^n} L(x,y,r), \quad y \in \mathbb{R}^m, r \in (0, +\infty).$$

The augmented Lagrangian dual problem (DVOP′) is

$$\operatorname{Sup}_C_{y\in \mathbb{R}^m, r\in(0,+\infty)} \quad \Psi(y,r).$$

It is easy to check that the augmented Lagrangian function $L(x, y, r)$ defined above has the following properties:

(I) $L(x,y,r) \geq_C f(x), \quad \forall x \in B, \forall y \in \mathbb{R}^m, r > 0$; $L(x,0,r) \geq_C f(x), \quad \forall x \in X, y \in \mathbb{R}^m, r > 0$; $L(x,0,r) = f(x), \quad \forall x \in B, \forall r > 0$.

(II) If $x \in X\setminus B$, then $L(x,y,r) \to +\infty, \quad \forall y \in \mathbb{R}^m$ as $r \to +\infty$.

In the following, we consider the relationship between the augmented Lagrangian $L(x, y, r)$ (defined above) and a special class of nonlinear Lagrangian in terms of their saddle points.

Let $\gamma \in (0,1]$. Consider the following class of nonlinear Lagrangians for (VOP):

$$N_\gamma(x,d) = \sum_{i=1}^{\ell} \left[f_i^\gamma(x) + \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x) \right]^{1/\gamma} e_i,$$

where $x \in X, d = (d_1, \cdots, d_m) \in \mathbb{R}^m_+$, $e_i = (0, \cdots 1, 0, \cdots 0)$ (the ith component being 1 and all the other components being 0's) and $g_j^+(x) = \max(g_j(x), 0)$.

It is easy to see that

$$N_\gamma(x,d) \geq_C N_1(x,d) = \bar{f}(x) + \left[\sum_{j=1}^{m} d_j g_j^+(x) \right] e,$$

where $x \in X, d \in \mathbb{R}^m_+, \gamma \in (0,1]$.

Proposition 2.110. *Let the augmented Lagrangian $L(x, y, r)$ for (VOP) be defined as above. Then (x^*, y^*, r^*) is a saddle point of $L(x, y, r)$ if and only if x^* is a minimal solution to (VOP), $g(x^*) \in \Psi(y^*, r^*)$ and $(y^*, r^*, g(x^*))$ solves the dual problem (DVOP′).*

Proof. The proof is similar to that of the usual saddle point theorem for a vector optimization problem. ∎

Proposition 2.111. *Let the augmenting function σ be finite everywhere. Let the augmented Lagrangian $L(x, y, r)$ for (VOP) be defined as above. If (x^*, y^*, r^*) is a saddle point of L, then there exists $d^* \in \mathbb{R}^m_+$ such that (x^*, d^*) is also a saddle point of $N_\gamma(x, d)$, where $\gamma \in (0,1]$.*

Proof. Since (x^*, y^*, r^*) is a saddle point of $L(x, y, r)$, by Proposition 2.108, $x^* \in B$ is a minimal solution to (VOP) and

$$\begin{aligned} L(x,y^*,r^*) &= f(x) + [\operatorname{Inf}_C\{r^*\sigma(g(x)+v) - \langle y^*, g(x)+v\rangle : v \in \mathbb{R}^m_+\}]e \\ &\not\leq L(x^*,y^*,r^*) = f(x^*), \quad \forall x \in X. \end{aligned}$$

Since σ is finite everywhere, we deduce that σ is locally Lipschitz near $0 \in \mathbb{R}^m$. Thus, there exist $\theta > 0$ and $\delta > 0$ such that when $\|u\| \leq \delta$, $\sigma(u) \leq \theta\|u\|$, where $\|u\| = \sum_{j=1}^m |u_j|$ and $u = (u_1, \cdots, u_m)$. If $x \in X \backslash B$ and $\sum_{j=1}^m g_j^+(x) \leq \delta$, then

$$\inf\{r^*\sigma(g(x)+v) - \langle y^*, g(x)+v\rangle : v \in \mathbb{R}_+^m\} \leq (\theta r^* + \|y^*\|)\sum_{j=1}^m g_j^+(x)$$

(This follows by setting $v_j = -g_j(x)$ if $g_j(x) \leq 0$ and $v_j = 0$ if $g_j(x) > 0$). Let $d^* = (d_1^*, \cdots, d_m^*) \in \mathbb{R}_+^m$ be chosen such that $\delta \min_{1\leq j\leq m} d_j^* e \geq_C g(x^*)$ and $\min_{1\leq j\leq m} d_j^* \geq \theta r^* + \|y^*\|$.

If $x \in X \backslash B$ and $\sum_{j=1}^m g_j^+(x) \leq \delta$, then

$$\begin{aligned} N_\gamma(x, d^*) - N_\gamma(x^*, d^*) &\geq_C N_1(x, d^*) - f(x^*) \\ &= f(x) + \left[\sum_{j=1}^m d_j^* g_j^+(x)\right] e - g(x^*) \\ &\geq_C f(x) + \left[(\theta r^* + \|y^*\|)\sum_{j=1}^m g_j^+(x)\right] e - g(x^*) \\ &\geq_C L(x, y^*, r^*) - f(x^*) \end{aligned}$$

Thus $N_\gamma(x, d^*) \not\leq N_\gamma(x^*, d^*)$.

If $x \in X \backslash B$ and $\sum_{j=1}^m g_j^+(x) > \delta$, then

$$f(x) \not\leq_{C\backslash\{0\}} f(x^*) + \delta \min{}_{1\leq j\leq m} d_j^* e$$

because $f(x) \geq_C 0$. In addition, we have

$$\begin{aligned} N_\gamma(x, d^*) - N_\gamma(x^*, d^*) &\geq_C N_1(x, d^*) - f(x^*) \\ &\geq_C f(x) - f(x^*) + \delta \min_{1\leq j\leq m} d_j^* e. \end{aligned}$$

Hence

$$N_\gamma(x, d^*) \not\leq_{C\backslash\{0\}} N_\gamma(x^*, d^*).$$

If $x \in B$, then $N_\gamma(x, d^*) = g(x) \not\leq N_\gamma(x^*, d^*) = g(x^*)$.

Finally, we show that

$$N_\gamma(x^*, d) \not\geq N_\gamma(x^*, d^*), \quad \forall d \in \mathbb{R}_+^m$$

This is obvious because $N_\gamma(x^*, d) = N_\gamma(x^*, d^*) = g(x^*)$. ■

2.10 Nonlinear Penalization for VO

In this section, we consider exact penalization results for a vector optimization problem via a nonlinear penalty function.

Let $X = \mathbb{R}^n, Y = \mathbb{R}^\ell$ be an ℓ-dimensional Euclidean space, and $C = \mathbb{R}^\ell_+$. Let $K \subseteq \mathbb{R}^n$ be a nonempty closed set, $f = (f_1, \dots, f_\ell) : K \to \mathbb{R}^\ell$ be a vector-valued function such that each of its component function f_i is l.s.c., and $g_j : K \to \mathbb{R}$ be l.s.c. for any $j \in \{1, \cdots, m\}$.

Consider the following constrained vector optimization problem (VOP):

$$\operatorname*{Inf}_{x \in B}{}_C f(x), \tag{2.94}$$

where $B = \{x \in K : g_j(x) \leq 0, j = 1, \cdots, m\}$. It is clear that B is closed. We denote by $\mathrm{Min}_C f(B)$ and $\mathrm{Inf}_C f(B)$ the set of minimal points and the set of infimum points of (VOP) respectively.

Consider the following nonlinear penalty function:

$$N_\gamma(x, d) = p_\gamma(f(x), d * g^+(x)) = \sum_{i=1}^{\ell} \left[f_i^\gamma(x) + \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x) \right]^{1/\gamma} e_i,$$

where $0 < \gamma < +\infty$, and $e_i = (0, 0, \cdots, 1, 0, \cdots, 0)$ (the ith component is 1 and the other components are 0's), $i = 1, \cdots, \ell$.

If $\ell = 1$, this class of nonlinear penalty functions can be considered as a composition of the following two functions

$$\left[y_0^\gamma + \sum_{j=1}^{m} y_j^\gamma \right]^{1/\gamma}, \text{ and } (f_1(x), d_1 g_1(x), \cdots, d_m g_m(x)),$$

while the first one is a so-called increasing positively homogenous function, see [172].

If $\gamma = 1$ and $\ell = 1$, then nonlinear penalty function $N_\gamma(x, d)$ reduces to the classical l_1 scalar penalty function

$$f_1(x) + \sum_{j=1}^{m} d_j g_j^+(x).$$

Let $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$. We associate (VOP) with a perturbed problem:

$$\operatorname*{Inf}_{x \in B(u)}{}_C \; f(x), \tag{2.95}$$

where $B(u) = \{x \in X : g_j(x) \leq u_j, j = 1, \cdots, m\}$. We will denote by $\mathrm{Min}_C f(B(u))$ and $\mathrm{Inf}_C f(B(u))$ the sets of minimal points and infimum points of (VOP_u) respectively.

We need the following lemma.

Lemma 2.112. *For any $x_0 \in B(u)$, there exists $z^* \in Inf_C f(B(u))$ such that $z^* \leq_C f(x_0)$.*

Proof. It follows immediately from Proposition 2.78. ■

Definition 2.113. *We say that (VOP_u) is γ-rank uniformly weakly stable if there exist $\delta > 0$ and $M > 0$ such that*

$$\left[\frac{Inf_C f(B(u)) - Inf_C f(B)}{\|u\|_\gamma^\gamma} + Me\right] \cap (-intC) = \emptyset, \tag{2.96}$$

for any $u \in I\!R_+^m$ with $0 < \|u\|_\gamma \leq \delta$.

It is not hard to show that the restriction $u \in \mathrm{I\!R}_+^m$ in the definition of the γ-rank uniform weak stability can be replaced by $u \in \mathrm{I\!R}^m$. This is also true for the γ-rank weak stability in the following Definition 2.115.

In the definition of γ-rank uniform weak stability of (VOP), the term "uniform" shows the difference from the usual stability in which $\mathrm{Inf}_C f(B)$ in (2.96) is replaced by a specific point of $\mathrm{Inf}_C f(B)$ and the fact that different points of $\mathrm{Inf}_C f(B)$ may have different M's in (2.96).

Let $0 < \gamma_1 < \gamma_2$. It is not hard to see that if (VOP) is γ_2-rank uniformly weakly stable, then it is also γ_1-rank uniformly weakly stable.

Theorem 2.114. *If (VOP) is γ-rank uniformly weakly stable, then $\exists d^* \in I\!R_+^m$ such that when $d - d^* \in I\!R_+^m$,*

$$Inf_C f(B) \subseteq q_\gamma(d), \tag{2.97}$$

where $q_\gamma(d) = Inf_C N_\gamma(K, d)$. The converse is also true.

Proof. We begin by proving the first half of this theorem.

If $\mathrm{Inf}_C f(B) = \emptyset$, then the conclusion holds automatically. Now we assume that $\mathrm{Inf}_C f(B) \neq \emptyset$.

Let $\eta(z) = \min_{1 \leq i \leq l} z_i, \forall z = (z_1, \cdots, z_\ell) \in \mathrm{I\!R}^\ell$. We show by contradiction that $\eta(\mathrm{Inf}_C f(B)) = \{\eta(z) : z \in \mathrm{Inf}_C f(B)\}$ is bounded from above by some $M' > 0$. Otherwise, there exists $z_k \in \mathrm{Inf}_C f(B)$ such that $z_k \to +\infty$. Since $\mathrm{Inf}_C f(B) \neq \emptyset$, it follows that for any $\delta > 0$, $B(u_\delta) \supset X(0) = B \neq \emptyset$, where $u_\delta = (0, 0, \cdots, 0, \delta) \in \mathrm{I\!R}_+^m$. Suppose that $x_0 \in B \subset B(u_\delta)$. Then by Lemma 2.112, there exists $z_\delta \in \mathrm{Inf}_C f(B(u_\delta))$ such that

$$z_\delta \leq_C f(x_0).$$

Hence,

$$(z_\delta - z_k)/\|u_\delta\|_\gamma^\gamma \leq_C (f(x_0) - z_k)/\|u_\delta\|_\gamma^\gamma \to -\infty \text{ as } k \to \infty,$$

which contradicts (2.96) because $\delta > 0$ can be arbitrarily small.

Suppose that there exists $d_k = (d_{k,1}, \cdots, d_{k,m}) \to +\infty$ and $z_k \in \mathrm{Inf}_C f(B)$ such that $z_k \notin \inf_{x \in X} N_\gamma(x, d_k)$.

By $z_k \in \mathrm{Inf}_C f(B)$, it follows that $\exists x_k^j$ such that $g(x_k^j) \leq 0$ and $f(x_k^j) \to z_k$ as $j \to \infty$.

It follows from $z_k \notin \inf_{x\in X} N_\gamma(x, d_k)$ that $\exists x'_k \in X$ such that

$$N_\gamma(x'_k, d_k) \leq_{C\setminus\{0\}} z_k.$$

That is,

$$\sum_{i=1}^{\ell}\left[f_i^\gamma(x'_k)+\sum_{j=1}^{m}(d_{k,j}^\gamma g_j^{+\gamma}(x'_k))\right]^{1/\gamma} e_i \leq_{C\setminus\{0\}} z_k. \tag{2.98}$$

Using (2.98), we deduce that $\max_{1\leq j\leq m} g_j(x'_k) > 0$ since $z_k \in \mathrm{Inf}_C f(B)$. (2.98) also implies that

$$\sum_{j=1}^{m} d_{k,j}^\gamma g_j^{+\gamma}(x'_k) \leq (z_k)_i^\gamma - f_i^\gamma(x'_k) \leq (z_k)_i^\gamma, \quad i = 1, \cdots, \ell, \tag{2.99}$$

where $(z_k)_i$ denotes the ith component of vector z_k.

That is, $[\sum_{j=1}^m d_{k,j}^\gamma g_j^{+\gamma}(x'_k)]^{1/\gamma} \leq \eta(z_k) \leq M'$.

It follows that $g_j^+(x'_k) \to 0$ $(j = 1, \cdots, m)$ as $k \to +\infty$.

Now let $u_{k,j} = g_j^+(x'_k)$ and $u_k = (u_{k,1}, \cdots, u_{k,m})$. Clearly, $\|u_k\|_\gamma > 0$ and $\|u_k\|_\gamma \to 0$. It follows from (2.99) that $\|u_k\|_\gamma^\gamma \min_{1\leq j\leq m} d_{k,j}^\gamma \leq (z_k)_i^\gamma - f_i^\gamma(x'_k)$. By Lemma 2.112, we deduce that $\exists v_k \in \mathrm{Inf}_C f(B(u_k))$ such that $v_k \leq_C f(x'_k)$. By the mean-value theorem, we have $(z_k)_i^\gamma - (v_k)_i^\gamma = \gamma(s_k)_i^{\gamma-1}((z_k)_i - (v_k)_i)$, where $(s_k)_i \in ((v_k)_i, (z_k)_i)$. Therefore, it follows from (2.99) that if $\gamma \leq 1$, then

$$\|u_k\|_\gamma^\gamma \min_{1\leq j\leq m} d_{k,j}^\gamma \leq \gamma(s_k)_i^{\gamma-1}((z_k)_i-(v_k)_i) \leq \gamma(v_k)_i^{\gamma-1}((z_k)_i-(v_k)_i); \tag{2.100}$$

if $\gamma < 1$, then

$$\|u_k\|_\gamma^\gamma \min_{1\leq j\leq m} d_{k,j}^\gamma \leq \gamma M'^{\gamma-1}((z_k)_i - (v_k)_i). \tag{2.101}$$

Since $\inf_{x\in X} f_i(x) > 0, \forall i$, it follows that

$$\min_{1\leq i\leq m}(v_k)_i \geq m_2 > 0. \tag{2.102}$$

Let $M'' = \max\{M'^{\gamma-1}, m_2^{\gamma-1}\}$. The combination of (2.100), (2.101) and (2.102) yields that

$$\|u_k\|_\gamma^\gamma \min_{1\leq j\leq m} d_{k,j}^\gamma \leq \gamma M''((z_k)_i - (v_k)_i),$$

i.e.,

$$\frac{(v_k)_i - (z_k)_i}{\|u_k\|_\gamma^\gamma} \leq -\frac{\min_{1\leq j\leq m} d_{k,j}^\gamma}{\gamma M''},$$

which contradicts (2.96). Thus (2.97) holds.

Now we prove the second half of the theorem by contradiction.

Suppose that $\exists u_k = (u_{k,1}, \cdots, u_{k,m}) \in \mathbb{R}^m_+$ with $u_k \to 0^+$ and $z_k \in \mathrm{Inf}_C f(B(u_k)), v_k \in \mathrm{Inf}_C f(B)$ such that

$$(z_k - v_k)/\|u_k\|^\gamma_\gamma \to -\infty, \text{ as } k \to +\infty,$$

where the virtual element $-\infty$ is such that for any $\alpha \in \mathbb{R}^1_+, -\infty \leq_{intC} -\alpha e$. Then $\exists x_k \in X$ with $g_j(x_k) \leq u_{k,j}, \forall j$ such that

$$(f(x_k) - v_k)/\|u_k\|^\gamma_\gamma \to -\infty, \text{ as } k \to +\infty. \tag{2.103}$$

By the assumption of the theorem, $\exists d^* = (d^*_1, \cdots, d^*_m) \in \mathbb{R}^m_+$ such that when $d - d^* \in \mathbb{R}^m_+$, $v_k \in \inf_{x \in X} N_\gamma(x, d)$. Therefore,

$$N_\gamma(x_k, d^*) \not\leq_{C\setminus\{0\}} v_k. \tag{2.104}$$

We assume that $i^* \in \{1, \cdots, \ell\}$ is such that

$$\left[f^\gamma_{i^*}(x_k) + \sum_{j=1}^m d^{*\gamma}_j g^{+\gamma}_j(x_k)\right]^{1/\gamma} \geq (v_k)_{i^*}.$$

Namely,

$$f^\gamma_{i^*}(x_k) - (v_k)^\gamma_{i^*} \geq -\sum_{j=1}^m d^{*\gamma}_j g^{+\gamma}_j(x_k). \tag{2.105}$$

It follows from (2.103) and (2.104) that $\max_{1\leq j\leq m} g_j(x_k) > 0$. So, from (2.105), we deduce that

$$f^\gamma_{i^*}(x_k) - (v_k)^\gamma_{i^*} \geq - \max_{1\leq j\leq m} d^{*\gamma}_j \|u_k\|^\gamma_\gamma.$$

That is,

$$[(v_k)^\gamma_{i^*} - f^k_{i^*}(x_k)]/\|u_k\|^\gamma_\gamma \leq \max_{1\leq j\leq m} d^{*\gamma}_j. \tag{2.106}$$

Since

$$(v_k)^\gamma_{i^*} - f^\gamma_{i^*}(x_k) = \gamma s^{\gamma-1}_k((v_k)_{i^*} - f_{i^*}(x_k)), \quad s_k \in (f_{i^*}(x_k), (v_k)_i),$$

it follows from the assumption on f that there exists $a > 0$ such that

$$(v_k)^\gamma_{i^*} - f^k_{i^*}(x_k) \geq \gamma a((v_k)_{i^*} - f_{i^*}(x_k)). \tag{2.107}$$

(2.106) and (2.107) yield that

$$[f_{i^*}(x_k) - (v_k)_{i^*}]/\|u_k\|^\gamma_\gamma \geq - \max_{1\leq j\leq m} d^{*\gamma}_j/(ka),$$

which contradicts (2.103). The proof is complete. ■

Definition 2.115. *(i) Let $z^* \in Inf_C f(B)$. The problem (VOP) is said to be γ-rank weakly stable at z^* if there exist positive real numbers δ_{z^*} and M_{z^*} such that*

$$\left[\frac{Inf_C f(B(u)) - z^*}{\|u\|_\gamma^\gamma} + M_{z^*}e\right] \cap (-intC) = \emptyset,$$

for any $u \in I\!R_+^m$ with $0 < \|u\|_\gamma \leq \delta_{z^*}$.

(ii) The problem (VOP) is said to be γ-rank weakly stable if it is γ-rank weakly stable at every $z^* \in Inf_C f(B)$.

The following simple example shows that (VOP) is 1-rank weakly stable but not 1-rank uniformly weakly stable. Let $n = 1$, $\ell = 2$, $K = \mathbb{R}$, $m = 1$. Let $f(x) = (\exp(-x^{1/2}),\ \exp(-x^{1/2}))$ if $x > 0$; $f(x) = (\exp(x), \exp(-x))$ if $x \leq 0$. Let $g(x) = x, \forall x \in \mathbb{R}$.

It is easy to check that

$$\text{Inf}_C f(B) = \{(\exp(x), \exp(-x)) : x \leq 0\}$$

and, for any $u > 0$,

$$\begin{aligned}&\text{Inf}_C f(B(u))\\ &= \{(\exp(-u^{1/2}),\ \exp(-u^{1/2}))\} \cup \{(\exp(x), \exp(-x)) : x < -u^{1/2}\}.\end{aligned}$$

It is elementary to prove that (VOP) is 1-rank weakly stable but not 1-rank uniformly weakly stable.

It is clear that if (VOP) is γ-rank uniformly weakly stable, then (VOP) is γ-rank weakly semi-stable.

The proof of the next theorem is similar to that of Theorem 2.114 and thus omitted.

Theorem 2.116. *Let $z^* \in Inf_C f(B)$. Then (VOP) is γ-rank weakly stable at z^* if and only if there exists a $d^* \in I\!R_+^m$ such that $z^* \in q_\gamma(d)$ whenever $d - d^* \in I\!R_+^m$.*

Corollary 2.117. *(VOP) is γ-rank weakly stable if and only if, for every z^*, there exists a $d^* \in I\!R_+^m$ such that $z^* \in q_\gamma(d)$ whenever $d - d^* \in I\!R_+^m$.*

The next theorem uses a well-known condition in the study of sensitivity of a constrained optimization problem, i.e., the compactness of the feasible set with a small perturbation. Under this condition, the set of minimal points of (VOP) and that of $N_\gamma(\cdot, d)$ are nonempty. The conclusion follows directly from Theorem 2.114.

Theorem 2.118. *Assume that there exists $u^0 = (u_1^0, \cdots, u_m^0) \in intI\!R_+^m$ with $\|u^0\| > 0$ sufficiently small such that $X(u^0) = \{x \in X : g_j(x) \leq u_j^0, \forall j\}$ is compact. If (VOP) is γ-rank uniformly weakly stable, then there exists $d^* \in I\!R_+^m$ such that when $d - d^* \in I\!R_+^m$,*

$$Min_C f(B) \subseteq \bar{q}_\gamma(d),$$

where $\bar{q}_\gamma(d)$ *is the set of minimal points of* $N_\gamma(\cdot, d)$ *over* K*. The converse is also true.*

The following theorem establishes further relationship between the solutions of (VOP) and that of the penalty problems based on N_γ.

Theorem 2.119. *Assume that* $B \neq \emptyset$ *and there exists* $d^* = (d_1^*, d_2^*, \cdots, d_m^*) \in I\!R_+^m$ *such that for all* d *satisfying* $d - d^* \in I\!R_+^m$, $x^* \in X$ *is a minimal solution of the problem* $\underset{x \in K}{Min_C} N_\gamma(x, d)$*, then* x^* *is a minimal solution of (VOP).*

Proof. Let x^* be a minimal solution of $\underset{x \in K}{\mathrm{Min}_C} N_\gamma(x, d)$ for any d satisfying $d - d^* \in \mathrm{I\!R}_+^m$. Then we have

$$N_\gamma(x, d) - N_\gamma(x^*, d) \not\leq_{C \setminus \{0\}} 0, \ \forall x \in X, d \text{ satisfying } d - d^* \in \mathrm{I\!R}_+^m.$$

For any $x_0 \in B$, we have $N_\gamma(x_0, d) = f(x_0), \forall d \in \mathrm{I\!R}_+^m$ by Lemma 2.94. Thus,

$$f(x_0) - \sum_{i=1}^{\ell} \left[f_i^\gamma(x^*) + \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x^*) \right]^{1/\gamma} e_i \not\leq_{C \setminus \{0\}} 0, \tag{2.108}$$

where $\forall x_0 \in B, d - d^* \in \mathrm{I\!R}_+^m$. We claim that $g_j^+(x^*) = 0, \forall j$ (i.e., $x^* \in B$). Otherwise, $\sum_{j=1}^m g_j^{+\gamma}(x^*) > 0$.

It follows from (2.108) that there exists $i^* \in \{1, \cdots, \ell\}$ such that

$$f_{i^*}^\gamma(x_0) - f_{i^*}^\gamma(x^*) \geq \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x^*) \geq (\min_{1 \leq j \leq m} d_j^\gamma) \sum_{j=1}^{m} g_j^{+\gamma}(x^*).$$

Hence,

$$\max_{1 \leq i \leq \ell} \{f_i^\gamma(x_0) - f_i^\gamma(x^*)\} \geq \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x^*) \geq (\min_{1 \leq j \leq m} d_j^\gamma) \sum_{j=1}^{m} g_j^{+\gamma}(x^*),$$

which is impossible if we let $d_j \to +\infty, \forall j$. Therefore, $x^* \in B$. It follows directly from Lemma 2.94 and (2.108) that $x^* \in \mathrm{Min}_C f(B)$ and the proof is complete. ■

3

Vector Variational Inequalities

The concept of a vector variational inequality was introduced by Giannessi [79] in a finite dimensional space. Chen and Yang [40] considered general vector variational inequalities and vector complementary problems in infinite dimensional spaces, and Chen [25] considered vector variational inequalities with a variable ordering structure. Yang [207] studied inverse vector variational inequalities and its relations with a vector optimization problem. Through the last twenty years of development, existence results of solutions for several kinds of vector variational inequalities have been derived and the vector variational inequality problem has found many of its applications in vector optimization, set-valued optimization, approximate analysis of vector optimization problems and vector network equilibrium problems. Because of these applications, the study of vector-variational inequalities has attracted wide attention.

In this chapter, we will study relations between vector variational inequalities (VVI) and vector optimization problems, existence of a solution of (VVI), inverse VVI, gap functions of VVI, set-valued VVI, and vector complementarity problems. We will investigate these with or without a variable ordering structure.

3.1 Vector Variational Inequalities (VVI)

Let X and Y be Hausdorff topological vector spaces. By $L(X,Y)$, we denote the set of all linear continuous vector-valued functions from X into Y. For $l \in L(X,Y)$, the value of linear vector-valued function l at x is denoted by $\langle l, x\rangle$. Let $C \subset Y$ be a nonempty convex cone. Then (Y, C) is an ordered Hausdorff topological vector space.

Definition 3.1. *A vector variational inequality* (VVI) *is a problem of finding* $x^* \in K$ *such that*

$$(VVI) \qquad \langle T(x^*), x - x^*\rangle \not\leq_{C\setminus\{0\}} 0, \quad \forall x \in K,$$

where $T : K \to L(X, Y)$ and $K \subset X$ is a nonempty subset.

Let the topological interior $intC$ of the cone C be nonempty. We have the following definition.

Definition 3.2. *A weak vector variational inequality $(WVVI)$ is a problem of finding $x^* \in K$ such that*

$$(WVVI) \qquad \langle T(x^*), x - x^* \rangle \not\leq_{intC} 0, \quad \forall x \in K,$$

where $T : K \to L(X, Y)$ and $K \subset X$ is a nonempty subset.

Consider a vector optimization problem:

$$\text{(VOK)} \qquad \operatorname*{Min}_{x \in K}{}_C\, f(x),$$

where $f : X \to Y$ is a vector-valued function. The following proposition provides a relationship between the (WVVI) and the vector optimization problem (VOK).

Proposition 3.3. *Assume f is Gâteaux differentiable with Gâteaux derivative Df. Let K be a convex subset of X and $T = Df$. If x is a weakly minimal solution of (VOK), then x solves $(WVVI)$. Assume further that f is a $C-$convex vector-valued function. If x solves $(WVVI)$, then x is a weakly minimal solution of (VOK).*

Proof. Let x be a weakly minimal solution of (VOK). For any $y \in K$, we have $x + t(y - x) \in K$, $\forall t \in (0, 1)$. Then

$$f(x + t(y - x)) - f(x) \not\leq_{intC} 0, \quad \forall t \in (0, 1),$$
$$\frac{f(x + t(y - x)) - f(x)}{t} \not\leq_{intC} 0, \quad \forall t \in (0, 1).$$

Let $t \to 0+$. Then

$$\langle Df(x), y - x \rangle \not\leq_{intC} 0, \quad \forall y \in K.$$

Noting that $T = Df$, x solves (WVVI).

Conversely, let x solve (WVVI). Then

$$\langle Df(x), y - x \rangle \not\leq_{intC} 0, \quad \forall y \in K.$$

Since f is $C-$convex, for any $y \in K$,

$$\begin{aligned} & f(y) - f(x) - \langle Df(x), y - x \rangle \in C \\ \iff & f(y) - f(x) \in \langle Df(x), y - x \rangle + C \\ \Longrightarrow & f(y) - f(x) \in W + C \subset W, \end{aligned}$$

where $W = Y\backslash(-intC)$. Thus

$$f(y) - f(x) \not\leq_{intC} 0, \quad \forall y \in K,$$

i.e., x solves (VOK). ■

The following establishes when a solution of (VVI) is also a solution of (VOK).

Proposition 3.4. *Let K be a convex subset of X and $T = Df$. Let C be a pointed and convex cone in Y. If f is $C-$convex and x solves (VVI), then x is a minimal solution of (VOK).*

Proof. Suppose that x is not a minimal solution of (VOK). Then there is a $y \in K$ such that $f(x) - f(y) \geq_{C\backslash\{0\}} 0$. Since f is $C-$convex, we have

$$f(y) - f(x) - \langle Df(x), y - x\rangle \geq_C 0.$$

Thus

$$\langle T(x), y - x\rangle = \langle Df(x), y - x\rangle \in -C - C\backslash\{0\} \subset -C\backslash\{0\},$$

a contradiction. ■

In the following example, we show that a minimal solution of (VOK) may not be a solution of (VVI).

Example 3.5. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Consider the problem

$$\text{Min}_C \ f(x), \text{ subject to } x \in [-1, 0]$$

where $f(x) = (x, x^2 + 1)^\top$. It is clear that every $x \in [-1, 0]$ is a minimal solution of the problem. Let $x = 0$. Then, for $y = -1$,

$$\nabla f(x)(y - x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in -\mathbb{R}^2_+.$$

Thus $x = 0$ is not a solution of (VVI).

However, we can show that the following holds.

Proposition 3.6. *Assume that $T = Df$ holds and f is C-concave. If x^* is a minimal solution of (VOK), then x solves also (VVI).*

Proof. Suppose that x is a minimal solution of (VOK). Then x is a weakly minimal solution of (VOK). If x^* does not solve (VVI), then there exists a $x \in K$ such that

$$\langle T(x^*), x - x^*\rangle \leq_{C\backslash\{0\}} 0.$$

By the concavity assumption,

$$f(x) - f(x^*) \leq_C \langle Df(x^*), x - x^* \rangle = \langle T(x^*), x - x^* \rangle \leq_{C\setminus\{0\}} 0.$$

Since C is convex, we have

$$f(x) \leq_{C\setminus\{0\}} f(x^*),$$

which contradicts that x^* is a minimal solution of (VOK). ■

Moreover, a minimal solution of (VOK) can be characterized by a so-called Minty VVI.

Theorem 3.7. *Giannessi [81] Let $X = I\!R^n$, $Y = I\!R^\ell$ and $C = I\!R^\ell_+$. Let*

$$f(x) := (f_1(x), \cdots, f_\ell(x))^\top, \text{ and } T(x) = \nabla f(x) := (\nabla f_1(x), \cdots, \nabla f_\ell(x))^\top$$

be the Jacobian (an $\ell \times n$ matrix) of the vector-valued function f at x. Let f be $I\!R^n_+$-convex and v-hemicontinuous on K and K be convex. Then, x^* is a minimal solution of (VOK) if and only if it is a solution of the following Minty VVI:

$$T(x)(x - x^*) \not\leq_{I\!R^\ell_+\setminus\{0\}} 0, \ \forall x \in K. \tag{3.1}$$

Proof. Suppose that x^* is a minimal solution of (VOK). If x^* is not a solution of the Minty VVI (3.1), then there exists $\bar{x} \in K$ such that

$$T(\bar{x})(\bar{x} - x^*) \leq_{I\!R^\ell_+\setminus\{0\}} 0,$$

that is,

$$T(\bar{x})(x^* - \bar{x}) \geq_{I\!R^\ell_+\setminus\{0\}} 0.$$

Since f is $I\!R^\ell_+$-convex, we have

$$f(x^*) - f(\bar{x}) \geq_{I\!R^\ell_+} T(\bar{x})(x^* - \bar{x}) \geq_{I\!R^\ell_+\setminus\{0\}} 0.$$

Then

$$f(\bar{x}) - f(x^*) \leq_{I\!R^\ell_+\setminus\{0\}} 0,$$

it is a contradiction.

Conversely, let x^* be a solution of the Minty VVI (3.1). Suppose on the contrary that there is $\bar{x} \in K$ such that

$$f(\bar{x}) - f(x^*) \leq_{I\!R^\ell_+\setminus\{0\}} 0. \tag{3.2}$$

Since K is convex, $\bar{x}(\alpha) := \alpha x^* + (1 - \alpha)\bar{x} \in K, \forall \alpha \in [0, 1]$. Because of the convexity of f and of the Lagrange Mean Value Theorem, there exists $\bar{\alpha} \in (0, 1)$, such that

$$\frac{d}{d\alpha} f(\bar{x}(\bar{\alpha})) \geq_{I\!R^\ell_+} f(x^*) - f(\bar{x}). \tag{3.3}$$

From (3.2) and (3.3), we have

$$\langle T(\hat{x}), x^* - \bar{x} \rangle \geq_{\mathbb{R}^\ell_+ \setminus \{0\}} 0,$$

where $\hat{x} = \bar{x}(\bar{\alpha})$. Multiplying both sides of this inequality by $1 - \bar{\alpha}$,

$$\langle T(\hat{x}), x^* - \hat{x} \rangle \geq_{\mathbb{R}^\ell_+ \setminus \{0\}} 0,$$

which contradicts the assumption. ■

Next we show that (VVI) is a necessary optimality condition for a Geoffrion properly minimal solution of (VOK).

Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^\ell$ and $C = \mathbb{R}^\ell_+$. Let $f(x) := (f_1(x), \cdots, f_\ell(x))^\top$. A point $x^* \in K$ is said to a Geoffrion properly minimal solution of (VOK) [76] if there exists a scalar $M > 0$ such that, for each i,

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M,$$

for some j such that $f_j(x) > f_j(x^*)$ whenever $x \in K$ and $f_i(x) < f_i(x^*)$. Every Geoffrion properly minimal solution is a minimal solution.

Let $T(x) = \nabla f(x) := (\nabla f_1(x), \cdots, \nabla f_\ell(x))^\top$ be the Jacobian (an $\ell \times n$ matrix) of the vector-valued function f at x.

Proposition 3.8. *Assume that f and K are convex. If x^* is a Geoffrion properly minimal solution for (VOK), then x^* is a solution of (VVI).*

Proof. Since x^* is a Geoffrion properly minimal solution for (VOK), it follows from [76] that there exists $\lambda \in int\mathbb{R}^\ell_+$ such that x^* solves the following problem

$$\min \lambda^\top f(x), \quad \text{subject to } x \in K.$$

Then we have

$$\nabla(\lambda^\top f)(x^*)(x - x^*) \geq 0, \quad \forall x \in K.$$

Noticing that $\nabla(\lambda^\top f) = \lambda^\top \nabla f$. Thus x^* satisfies

$$\nabla f(x^*)(x - x^*) \not\leq_{C \setminus \{0\}} 0, \quad \forall x \in K.$$

Therefore x^* is a solution of (VVI). ■

A function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be pseudolinear on K if g is both pseudoconvex and pseudoconcave on K. It is known [45] that, g is a pseudolinear function if and only if, for any pair of x and y, there exists a scalar $\eta(x, y) > 0$ such that

$$g(y) - g(x) = \eta(x, y) \nabla g(x)^\top (y - x).$$

Any linear function and any linear fractional function (ratio of linear functions) is pseudolinear.

Let $f(x) =: (f_1(x), \cdots, f_\ell(x))^\top$, each f_i is pseudolinear. Then there exists a scalar $\eta_i(x,y) > 0$ such that

$$f_i(y) - f_i(x) = \eta_i(x,y)\nabla f_i(x)^\top (y-x). \tag{3.4}$$

Consider the vector pseudolinear optimization problem

$$\mathrm{Min}_C f(x), \quad \text{subject to } x \in K. \tag{3.5}$$

Then we have

$$f(y) - f(x) = (\eta_1(x,y)\nabla f_1(x)^\top (y-x), \cdots, \eta_\ell(x,y)\nabla f_\ell(x)^\top (y-x))^\top. \tag{3.6}$$

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^{\ell \times n}$ be a matrix-valued function defined by

$$T(x) = (\eta_1(x,y)\nabla f_1(x), \cdots, \eta_\ell(x,y)\nabla f_\ell(x))^\top. \tag{3.7}$$

Consider the following vector variational inequality problem of finding $x \in K$ such that

$$T(x)(y-x) \not\leq_{C\setminus\{0\}} 0\ , \quad \forall y \in K, \tag{3.8}$$

where T is defined by (3.7).

The following shows an equivalent condition between VOK(3.5) and VVI(3.8). Let f and T be defined by (3.4) and (3.7)

Theorem 3.9. *Assume that K is convex and each $f_i (i = 1, 2, \cdots, \ell)$ is pseudolinear. The point x^* is a minimal solution of VOK(3.5) if and only if x^* solves VVI(3.8).*

Proof. Let x^* be a minimal solution of VOK(3.5). There is no $x \in K$ such that

$$f(x) - f(x^*) \leq_{C\setminus\{0\}} 0,$$

that is

$$(\eta_1(x^*,x)\nabla f_1(x^*)(x-x^*), \cdots, \eta_\ell(x^*,x)\nabla f_\ell(x^*)(x-x^*))^\top \leq_{C\setminus\{0\}} 0.$$

Thus, there is no $x \in K$ such that

$$T(x^*)(x-x^*) \leq_{C\setminus\{0\}} 0.$$

Then x^* solves VVI(3.8). ∎

Proposition 3.10. *Let X and Y be Banach spaces. Let $C : X \rightrightarrows Y$ be closed, convex cone valued and u.s.c.. Let $K \subset X$ be a nonempty convex subset. Let f be a Gâteaux differentiable function with Gâteaux derivative denoted by $T(x) = Df(x)$. If $\bar{x} \in K$ is a weakly nondominated-like minimal solution of f on K, then $\bar{x}$ is a solution of the following generalized weak vector variational inequality problem*

$$(GWVVI) \quad \langle T(\bar{x}), x - \bar{x}\rangle \not\leq_{intC(\bar{x})} 0, \quad \forall x \in K.$$

Conversely, if f is $C(x)$-convex on K and $\bar{x}$ is a solution of (GWVVI), then $\bar{x}$ is a weakly nondominated-like minimal solution of f on K.

Proof. Suppose that $\bar{x} \in K$, and $\bar{x}$ is a weakly nondominated-like minimal solution of f on K. For any $x \in K$, we have $\bar{x} + t(x - \bar{x}) \in K$, $\forall t \in (0, 1]$. By Definition 1.15,

$$f(\bar{x} + t(x - \bar{x})) - f(\bar{x}) \not\leq_{intC(\bar{x})} 0, \quad \forall t \in (0, 1],$$

then

$$\frac{f(\bar{x} + t(x - \bar{x})) - f(\bar{x})}{t} \not\leq_{intC(\bar{x})} 0, \quad \forall t \in (0, 1].$$

Let $t \to 0+$. Thus

$$\langle T(\bar{x}), x - \bar{x} \rangle \notin -intC(\bar{x}), \quad \forall x \in K.$$

So $\bar{x}$ is a solution of (WVVI). Conversely, let f be C-convex. Then, by Proposition 1.72,

$$f(y) - f(x) \in \langle T(x), y - x \rangle + C(x), \quad \forall x, y \in K.$$

Since $\bar{x}$ is a solution of (GWVVI), we have

$$\langle T(\bar{x}), y - \bar{x} \rangle \not\leq_{intC(\bar{x})} 0, \quad \forall x \in K,$$

i.e.

$$\langle T(\bar{x}), y - \bar{x} \rangle \in W(\bar{x}), \quad \forall y \in K,$$

where $W(\bar{x}) = Y \backslash (-intC(\bar{x}))$. Then

$$f(y) - f(\bar{x}) \in W(\bar{x}), \quad \forall y \in K,$$

that is,

$$f(y) - f(\bar{x}) \notin -intC(\bar{x}), \quad \forall y \in K.$$

So $\bar{x}$ is a nondominated-like minimal solution of f on K. ■

Corollary 3.11. *Let Y be a real normed space and $K \subset Y$. Let $f : Y \to Y$ be a Gâteaux differentiable vector-valued function with Gâteaux derivative denoted by $F(y) = Df(y)$. Let f be $C(y)$-convex on K. Let $C : Y \rightrightarrows Y$ be an u.s.c. set-valued function such that for each $y \in Y$, $C(y)$ is a proper, closed and convex cone in Y. Let $\bar{C} = \cap_{y \in Y} C(y)$ and $e \in int\bar{C}$. Let $\bar{y} \in K$. Then $\bar{y}$ is a weakly nondominated-like minimal solution of f on K if and only if*

$$\min_{y \in K} \xi_e(\bar{y},\ F(\bar{y})(y - \bar{y})) = 0.$$

Proof. By Proposition 3.10, $\bar{y}$ is a weakly nondominated-like minimal solution of f on K if and only if

$$F(\bar{y})(y - \bar{y}) \notin -intC(\bar{y}), \quad \forall y \in K.$$

By Proposition 1.54, the above inequality holds if and only if

$$\xi_e(\bar{y}, F(\bar{y})(y - \bar{y})) \geq 0, \quad \forall y \in K.$$

Observing that $\xi_e(\bar{y}, 0) = 0$, the theorem holds. ■

The relations between vector variational inequality problems and set-valued optimization problems are shown in Section 2.4, see Theorems 2.48, 2.50, 2.64 and 2.73.

Now we consider existence of solutions for weak vector variational inequalities (WVVI).

Lemma 3.12 (Generalized Linearization Lemma). *Let the mapping $T : X \to L(X, Y)$ be monotone and v-hemicontinuous. Then the following two problems are equivalent for each convex subset K in X:*

(i) $x \in K$, $\langle T(x), y - x \rangle \not\leq_{intC} 0, \quad \forall y \in K$;
(ii) $x \in K$, $\langle T(y), y - x \rangle \not\leq_{intC} 0, \quad \forall y \in K$.

Proof. Let x be a solution given by (i). Since T is monotone,

$$\langle T(y) - T(x), y - x \rangle \geq_C 0, \quad y \in K,$$

$$\langle T(y), y - x \rangle \geq_C \langle T(x), y - x \rangle \not\leq_{intC} 0, \quad \forall y \in K.$$

Thus we have

$$\langle T(y), y - x \rangle \not\leq_{intC} 0, \quad \forall y \in K.$$

Suppose (ii) holds. For any $y \in K$, $0 < \lambda < 1$,

$$\langle T(\lambda y + (1 - \lambda)x), \lambda y + (1 - \lambda)x - x \rangle \not\leq_{intC} 0.$$

Dividing by λ, we have

$$\langle T(x + \lambda(y - x)), y - x \rangle \not\leq_{intC} 0.$$

Let $\lambda \to 0+$, we obtain (i) since T is v-hemicontinuous. ■

Definition 3.13. *A mapping $A \in L(X, Y)$ is called completely continuous, if it maps weakly convergent sequences to strongly convergent ones.*

In next theorem, we shall use the weak topology of X and the norm topology of Y.

Theorem 3.14. *Assume that X is a reflexive Banach space and $K \subset X$ is convex. Assume that (Y, C) is an ordered Banach space with $intC \neq \varnothing$ and $intC^* \neq \varnothing$. Let the mapping $T : K \to L(X, Y)$ be monotone, v-hemicontinuous and let, for any $y \in K$, $T(y)$ be completely continuous mapping on X. If*

(**i**) *K is compact, or*
(**ii**) *K is closed, T is weakly coercive on K,*

then the vector variational inequality (WVVI) is solvable.

Proof. We set

$$F_1(y) = \{x \in K : \langle T(x), y - x\rangle \not\leq_{intC} 0\}, \quad y \in K.$$

For (i), we prove that F_1 is a KKM map on K. Let $\{x_1, \cdots, x_k\} \subset K$, $\sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0$. Suppose that $x = \sum_{i=1}^{n} \alpha_i x_i \notin \cup_{i=1}^{n} F_1(x_i)$. Then

$$\langle T(x), x_i - x\rangle \leq_{intC} 0, \quad \forall i,$$

$$\langle T(x), x\rangle = \sum_{i=1}^{n} \alpha_i \langle T(x), x_i\rangle \leq_{intC} \sum_{i=1}^{n} \alpha_i \langle T(x), x\rangle = \langle T(x), x\rangle.$$

It is impossible, so we obtain

$$co\{x_1, \cdots, x_k\} \subset \cup_{i=1}^{n} F_1(x_i),$$

i.e., F_1 is a KKM map on K.

Let

$$F_2(y) = \{x \in K : \langle T(y), y - x\rangle \not\leq_{intC} 0\}, \quad y \in K.$$

We have $F_1(y) \subset F_2(y)$ for all $y \in K$. Indeed, let $x \in F_1(y)$, so that $\langle T(x), y - x\rangle \not\leq_{intC} 0$. By the monotonicity of T, it follows that

$$\langle T(y), y - x\rangle \geq_C \langle T(x), y - x\rangle \not\leq_{intC} 0,$$

that is, $x \in F_2(y)$. Thus F_2 is also a KKM map on K. By Lemma 3.12, we have

$$\cap_{y\in K} F_1(y) = \cap_{y \in K} F_2(y).$$

We observe that for each $y \in K$, $F_2(y)$ is a (weakly) compact subset in K. Indeed, for any $y \in K$ and any $x \in clF_2(y)$, there exists a sequence $\{x_k\}_{n\in N} \subset F_2(y)$ such that x_k weakly converges to x. Since $T(y)$ is completely continuous, we have that $\langle T(y), y - x_k\rangle \in W$ strongly converges to $\langle T(y), y - x\rangle$. The strong closedness of W implies that $\langle T(y), y - x\rangle \in W$, that is, $x \in F_2(y)$. Hence, for each $y \in K$, $F_2(y)$ is weakly closed. Since we consider a weakly topology on X and K is (weakly) compact, $F_2(y)$ is compact for each $y \in K$.

By the KKM theorem (see Theorem 1.34), we have

$$\cap_{y\in K} F_1(y) = \cap_{y\in K} F_2(y) \neq \varnothing.$$

Hence, there exists an $\bar{x} \in K$ such that

$$\langle T(\bar{x}), y - \bar{x}\rangle \not\leq_{intC} 0, \quad \forall x \in K.$$

Consider the case (ii). Firstly, we prove the following conclusion: if $c \in intC^*$ and x^* is a solution of the (scalar) variational inequality (VI)$_c$:

$$(VI)_c \qquad x \in K, \quad \langle c \circ T(x), y - x \rangle \geq 0, \quad \forall x \in K,$$

then x^* is a solution of (WVVI).

Indeed, suppose that x^* is not a solution of (WVVI). Then $\langle T(x^*), y - x^* \rangle \leq_{intC} 0$ for some $y \in K$. Thus, by $c \in intC^*$

$$\langle c \circ T(x^*), y - x^* \rangle < 0,$$

i.e., x^* is not a solution of $(VI)_c$.

For (ii), it is sufficient to prove that $(VI)_c$ has a solution, where c is the one in the weak coercive condition. Let B_r denote a closed ball of center 0 and radius r in X (for the norm in X). By the Hartmann-Stampacchia theorem, there exists a solution x_r of the variational inequality problem

$$x_r \in K \cap B_r, \quad \langle c \circ T(x_r), y - x_r \rangle \geq 0, \quad \forall y \in K \cap B_r.$$

Choose $r \geq ||x^*||$ with x^* as x_0 in the weak coercive condition. Then, we have $\langle c \circ T(x_r), x^* - x_r \rangle \geq 0$. Moreover,

$$\begin{aligned}
&\langle c \circ T(x_r), x^* - x_r \rangle \\
&= -\langle c \circ T(x_r) - c \circ T(x^*), x_r - x^* \rangle + \langle -c \circ T(x^*), x_r - x^* \rangle \\
&\leq -\langle c \circ T(x_r) - c \circ T(x^*), x_r - x^* \rangle + ||c \circ T(x^*)|| ||x_r - x^*|| \\
&= -||x_r - x^*||(\langle c \circ T(x_r) - c \circ T(x^*), x_r - x^* \rangle / ||x_r - x^*|| - ||c \circ T(x^*)||).
\end{aligned}$$

Now if $||x_r||$ is unbounded, we assume without loss of generality that $||x_r|| \to \infty$. By the above inequality and the weak coercivity of T, we may choose r large enough such that $\langle c \circ T(x_r), x^* - x_r \rangle < 0$, which contradicts

$$\langle c \circ T(x_r), x^* - x_r \rangle \geq 0.$$

If $||x_r||$ is bounded, then we assume by the reflexivity of X and without loss of generality that $||x_r|| \to \bar{x} \in X$ as $r \to \infty$. For any $y \in K$, there exists $\bar{r} > 0$ such that when $r \geq \bar{r}$, $y \in K \cap B_r$. Thus

$$\langle c \circ T(x_r), y - x_r \rangle \geq 0.$$

Letting $r \to \infty$, we have

$$\langle c \circ T(\bar{x}), y - \bar{x} \rangle \geq 0.$$

Hence $\bar{x}$ solves (VI)$_c$. ■

Now we consider the existence of solutions for vector variational inequalities with variable domination structures. We assume that X and Y are two Banach spaces.

Let $K \subset X$ be a nonempty, closed and convex subset, and let $T : K \to L(X,Y)$ be a vector-valued function. Let $C : X \rightrightarrows Y$ be a set-valued function, that is, for every $x \in X$, $C(x)$ is a closed and convex cone with nonempty interior $intC(x)$.

Consider the following (WVVI):

$$x_0 \in K, \quad \langle T(x_0), x - x_0 \rangle \not\leq_{intC(x_0)} 0, \quad \forall x \in K. \tag{3.9}$$

Definition 3.15. *Let $T : X \to L(X,Y)$ be a vector-valued function. T is said to be $C(x)$-monotone on X if for any $x, y \in X$,*

$$\langle T(y) - T(x), y - x \rangle \geq_{C(x)} 0.$$

Lemma 3.16. *[Generalized Linearization Lemma] Let $T : X \to L(X,Y)$ be C-monotone and v-hemicontinuous on X. Then the following problems (I) and (II) are equivalent for any convex subset K:*

(i) $x \in K, \quad \langle T(x), y - x \rangle \not\leq_{intC(x)} 0, \quad \forall y \in K;$
(ii) $x \in K, \quad \langle T(y), y - x \rangle \not\leq_{intC(x)} 0, \quad \forall y \in K.$

Proof. Since T is $C(x)$-monotone on X, we have

$$\langle T(y) - T(x), y - x \rangle \geq_{C(x)} 0, \quad \forall y \in K.$$

Let $x \in K$ be a solution of (I). For the ordered Banach space $(Y, C(x))$ with a fixed $x \in K$, we have, by (iv) of Lemma 1.1

$$\langle T(y), y - x \rangle \not\leq_{intC(x)} 0, \quad \forall y \in Y,$$

that is (II).

Now, we suppose that (II) holds. Then, for any $y \in K$ and $\lambda \in (0,1)$, by the convexity of K, we have

$$\langle T(\lambda y + (1-\lambda)x), \lambda y + (1-\lambda)x - x \rangle \not\leq_{intC(x)} 0.$$

Dividing by λ, we have

$$\langle T(x + \lambda(y - x)), y - x \rangle \not\leq_{intC(x)} 0.$$

Thus (I) is derived by the v-hemicontinuity of T and the closedness of $W(x) = Y \backslash (-intC(x))$, for a fixed $x \in K$ as $\lambda \searrow 0$. ■

Set for every $y \in K$

$$F_1(y) = \{x \in K : \langle T(x), y - x \rangle \not\leq_{intC(x)} 0\};$$

$$F_2(y) = \{x \in K : \langle T(y), y - x \rangle \not\leq_{intC(x)} 0\}.$$

Now we equip X with a weak topology, Y with a strong topology and $L(X,Y)$ with the strong operator topology.

Lemma 3.17. *Let $K \subset X$ be weakly compact. Let $T : K \to L(X,Y)$ be a vector-valued function, and let, for every $y \in K$, $T(y)$ be a completely continuous operator. Let the set-valued function $W : K \to Y$ with $W(x) = Y \backslash (-intC(x))$ for every $x \in K$ be upper semicontinuous on K. Then $F_2(y)$ is weakly closed for every $y \in K$.*

Proof. We denote the weakly closed hull of $F_2(y)$ by $\overline{F_2(y)}$. There exists a sequence $\{x_k\}_{k \in N} \subset F_2(y)$ converges weakly to some $x \in K$. For every $k \in N$, we have

$$\langle T(y), y - x_k \rangle \not\leq_{intC(x_k)} 0,$$

that is

$$\langle T(y), y - x_k \rangle \in W(x_k), \quad \forall k \in N.$$

Since $T(y)$ is completely continuous and W is upper semicontinuous on K, we have

$$\langle T(y), y - x \rangle \in W(x),$$

that is $x \in F_2(y)$. ■

Theorem 3.18. *Let X be a reflexive Banach space, and Y a Banach space. Let $K \subset X$ be a nonempty bounded, closed and convex subset in X. Let $C : X \rightrightarrows Y$ be a set-valued function, such that, for every $x \in X$, $C(x)$ is a closed, pointed and convex cone with nonempty interior $intC(x)$. Let the set-valued function $W : K \rightrightarrows Y$ with $W(x) = Y \backslash (-intC(x))$ for every $x \in K$ be upper semicontinuous on K. Let the vector-valued function $T : K \to L(X,Y)$ be $C(x)$-monotone and v-hemicontinuous on K, and let, for every $y \in K, T(y)$ be a completely continuous operator. Then the weak vector variational inequality WVVI(3.9) is solvable.*

Proof. We first prove that F_1 is a KKM mapping on K. Suppose that $\{x_1, \cdots, x_k\} \subset K$, $\sum_{i=1}^k \alpha_i = 1, \alpha_i \geq 0, i = 1, \cdots, k$, $x = \sum_{i=1}^k \alpha_i x_i \notin \cup_{i=1}^k F_1(x_i)$. Then

$$\langle T(x), x_i - x \rangle \geq_{intC(x)} 0, \quad i = 1, \cdots, n.$$

Thus

$$\sum_{i=1}^{k} \alpha_i \langle T(x), x_i - x \rangle \geq_{intC(x)} 0,$$

that is

$$\langle T(x), x \rangle - \langle T(x), x \rangle \geq_{intC(x)} 0,$$

which is absurd. Thus we deduce that

$$conv\{x_1, \cdots, x_k\} \subset \cup_{i=1}^k F_1(x_i),$$

so that F_1 is a KKM mapping on K. Now we have $F_1(y) \subset F_2(y)$ for $y \in K$. Indeed, let $x \in F_1(y)$, so that

$$\langle T(x), y - x \rangle \not\leq_{intC(x)} 0.$$

Since T is C-monotone on K, we have

$$\langle T(y), y - x \rangle - \langle T(x), y - x \rangle \geq_{C(x)} 0.$$

By (iv) of Lemma 1.1, we find that

$$\langle T(y), y - x \rangle \not\leq_{intC(x)} 0,$$

that is, $x \in F_2(y)$. Thus, F_2 is also a KKM mapping. By Lemma 3.16 we have that

$$\cap_{y \in K} F_1(y) = \cap_{y \in K} F_2(y).$$

Besides, by Lemma 3.17 $F_2(y)$ is weakly closed for every $y \in K$.

Now we observe that K is weakly compact since X is a reflexive Banach space and K is a bounded, close and convex subset in X. Since $F_2(y) \subset K$ and weak closedness of $F_2(y)$, $F_2(y)$ is a weak compact. If we equip X with the weak topology, we can use the KKM theorem for F_2. Thus $\cap_{y \in K} F_2(y) \neq \varnothing$. so that $\cap_{y \in K} F_1(y) \neq \varnothing$, thus there exists $x_0 \in K$ such that $x_0 \in \cap_{y \in K} F_1(y)$, that is

$$\langle T(x_0), y - x_0 \rangle \not\leq_{intC(x_0)} 0, \quad \forall y \in K.$$

This completes the proof. ■

Set

$$C_+ = co\{C(x) : x \in K\}.$$

Then if, for every $x \in K$, $C(x)$ is a closed, pointed and convex cone with the nonempty interior $intC(x)$, C_+ is also a closed, and convex cone with nonempty interior $intC_+$. We assume that C_+ is also pointed.

We extend the Definition 1.38.

Definition 3.19. *Let $K \subset X$ be a convex and unbounded subset. We shall say that $T : K \to L(X, Y)$ is coercive on K if there exist $x_0 \in K$ and $s \in C_+^* \backslash \{0\}$ such that*

$$\langle s \circ T(x) - s \circ T(x_0), x - x_0 \rangle / ||x - x_0|| \to +\infty,$$

whenever $x \in K$ and $||x|| \to \infty$; here $s \circ T(x) = s(T(x))$.

Let X and Y be topological vector spaces, K a nonempty subset of X, H a nonempty subset of Y, and $M : K \to Y$ a mapping. We want to study conditions under which the system:

$$\text{(S)} \qquad \exists y \in K, \quad \text{s.t.} \quad M(y) \in H,$$

will have or will not have a solution.

Definition 3.20. *[73] Let Z be a subset of $\mathbb{R}$. The real function $w : Y \to \mathbb{R}$ is called a weak separation function if*

$$H^w = \{h \in Y : w(h) \notin Z\} \supseteq H.$$

The real function $s : Y \to \mathbb{R}$ is called a strong separation function if

$$H^s = \{h \in Y : s(h) \notin Z\} \subseteq H.$$

Lemma 3.21. *[73] Let the sets H, K, Z and the mapping M be given. Then:*

(i) *The system (S) and the system*

$$w(M(y)) \subset Z, \quad \forall y \in K,$$

are not simultaneously possible, whatever the weak separation function w may be;

(ii) *The system (S) and the system*

$$s(M(y)) \subseteq Z, \quad \forall y \in K,$$

are not simultaneously possible, whatever the strong separation function s may be.

Theorem 3.22. *Let X, Y, C, W and T satisfy the assumption conditions in Theorem 3.18. Let $K \subset X$ be a closed, convex and unbounded subset. Moreover let T be coercive on K with $s \in C_+^*$. Then the weak vector variational inequality WVVI(3.9) is solvable.*

Proof. Note that $x_0 \in K$ is a solution of (WVVI) iff the system

$$(S') \qquad \exists y \in K, \quad \text{s.t. } \langle T(x_0), y - x_0\rangle \in -intC(x_0)$$

is impossible. Observe that

$$C_+^* \backslash \{0\} \subset C^*(x) \backslash \{0\}, \quad \forall x \in K.$$

We assume that $s \in C_+^* \backslash \{0\}$ is the same one as in the coercive condition. We set in Lemma 3.21

$$Z = (-\infty, 0], \quad H = intC(x_0), \quad M(y) = \langle T(x_0), y - x_0\rangle,$$

then $w(h) = s(h), h \in Y$ is a weak separation function for the system (S'). Thus, by Lemma 3.21(i), in order to show the existence of solutions for (WVVI), it is sufficient to prove that the classical scalar variational inequality

$$(\mathrm{VI})_s \qquad x \in K, \quad \langle s \circ T(x), y - x\rangle \geq 0, \quad \forall x \in K,$$

has a solution in K.

B_r denote the closed ball in X with the center at 0 and the radius r. In the spacial case where

$$Y = \mathbb{R}, \quad C(x) = \mathbb{R}_+ \quad \forall x \in K \cap B_r.$$

Theorem 3.18 guarantees the existence of a solution x_r of the following (VI)

$$\text{(VI)} \qquad x \in K \cap B_r, \quad \langle s \circ T(x), y - x \rangle \geq 0, \quad \forall y \in K \cap B_r.$$

Choose $r > ||x_0||$, with x_0 as in the coercive condition. Then, we have

$$\langle s \circ T(x_r), x_0 - x_r \rangle \geq 0.$$

Moreover,

$$\begin{aligned}
&\langle s \circ T(x_r), x_0 - x_r \rangle \\
&= -\langle s \circ T(x_r) - s \circ T(x_0), x_r - x_0 \rangle + \langle s \circ T(x_0), x_r - x_0 \rangle \\
&\leq -\langle s \circ T(x_r) - s \circ T(x_0), x_r - x_0 \rangle + ||s \circ T(x_0)|| ||x_r - x_0|| \\
&\leq -||x_r - x_0||(\langle s \circ T(x_r) - s \circ T(x_0), x_r - x_0 \rangle / ||x_r - x_0|| + ||s \circ T(x_0)||.
\end{aligned}$$

Now, if $||x_r|| = r$, for all r, we may choose r large enough so that the above inequality and the coercivity of T imply

$$\langle s \circ T(x_r), x_0 - x_r \rangle < 0,$$

which contradicts

$$\langle s \circ T(x_r), x_0 - x_r \rangle \geq 0.$$

Hence, there exists r such that $||x_r|| < r$. Now, $\forall x \in K$, we choose $\epsilon > 0$ small enough such that $x_r + \epsilon(x - x_r) \in K \cap B_r$ and thus

$$\langle s \circ T(x_r), \epsilon(x - x_r) \rangle \geq 0, \quad \forall x \in K,$$

that is

$$\langle s \circ T(x_r), x - x_r \rangle \geq 0, \quad \forall x \in K,$$

which shows that x_r is the solution of $(\text{VI})_s$. By the above claim, x_r is a solution of (WVVI). ■

Remark 3.23. If $Y = \mathbb{R}, C(x) = \mathbb{R}_+, \forall x \in K$, the (WVVI) collapses to the usual scalar variational inequality (VI):

$$x_0 \in K, \langle T(x_0), x - x_0 \rangle \geq 0, \quad \forall x \in K.$$

Then the theorem 3.22 collapses to the following theorem 3.24.

Theorem 3.24. *Let X be a reflexive Banach space, and K be a nonempty bounded, closed and convex subset in X. Let $T : K \to X^*$ be monotone and hemicontinuous on K. Then the variational inequality (VI) is solvable.*

Obviously, the hemicontinuity of T in Theorem 3.18 is equivalent to the continuity for each one dimensional flat $L \subset X$. Thus, Theorem 3.24 is essentially the Hartmann-Stampacchia theorem for variational inequalities.

This section is concluded by a discussion on the scalarization of (VVI). Consider the following form of a (VI) of finding $x^* \in K$ such that

$$\langle \lambda, \langle T(x^*), x - x^* \rangle \rangle \geq 0, \quad \forall x \in K. \tag{3.10}$$

If $\lambda \in intC^*$ and x^* is a solution of VI(3.10), then x^* is a solution of the (VVI). If $\lambda \in C^* \setminus \{0\}$ and x^* is a solution of VI(3.10), then x^* is a solution of the (WVVI).

3.2 Inverse VVI

In this section, we study inverse VVI problems and establish their equivalences with given VVI problems. An inverse VVI is also called a dual VVI in the literature.

Let $T : X \longrightarrow L(X, Y)$ be a function, and $h : X \to Y$ is a function. The (VVI_h) problem consists in finding $x^* \in X$, such that

$$\langle T(x^*), x - x^* \rangle \not\leq_{C \setminus \{0\}} h(x^*) - h(x), \ \forall x \in X.$$

Assume that T is one-to-one (injective). Define $T' : L(X, Y) \to X$ as follows:

$$T'(l) := -T^{-1}(-l), \quad \forall l \in \text{Domain}(T') = -\text{Range}(T).$$

If T is linear, then $T' = T^{-1}$.

The inverse VVI of the (VVI_h) problem is defined as: finding $l^* \in \text{Domain}(T')$, such that

$$\langle l - l^*, T'(l^*) \rangle \not\leq_{C \setminus \{0\}} h^*_{\leq}(l^*) - h^*_{\leq}(l), \quad \forall l \in L(X, Y),$$

where $h^*_{\leq}(l) := \text{Max}_C\{\langle l, x \rangle - h(x) : x \in X\}$ is the vector conjugate function of h. This problem is denoted by (IVVI_h).

Definition 3.25. *Let $h : X \to Y$ be a C-convex function. The Fenchel conjugate of h is the set-valued function $h^*_{\leq} : L(X, Y) \rightrightarrows Y$, such that*

$$h^*_{\leq}(l) := Max_C\{\langle l, x \rangle - h(x) : x \in X\}, \quad l \in L(X, Y).$$

Thus, a generalization of Young's inequality follows immediately:

Lemma 3.26. *Let h and $h^*_{\leq}$ be as in Definition 3.25. Then*

$$h(x) + h^*_{\leq}(l) - \langle l, x \rangle \not\leq_{C \setminus \{0\}} 0, \quad \forall x \in X, l \in L(X, Y).$$

Let $h : X \to Y$ and $x^* \in X$. We define the subgradient of h at x^* by

$$\partial_{\leq} h(x^*) = \{l \in L(X,Y) : h(x) - h(x^*) \not\leq_{C\setminus\{0\}} \langle l, x - x^* \rangle, \forall x \in X\}.$$

We also define the strong subgradient of h at x^* by

$$\partial^s_{\leq} h(x^*) = \{l \in L(X,Y) : h(x) - h(x^*) \geq_C \langle l, x - x^* \rangle, \forall x \in X\}.$$

Theorem 3.27. *Let X be a Hausdorff topological vector space and (Y, C) be an ordered Hausdorff topological vector space. The function T is one-to-one and $h : X \to Y$ is continuous. Assume that $h^*_{\leq}(l) \neq \varnothing, \forall l \in L(X,Y)$.*

(i) If x^* is a solution of (VVI_h), then $l^* = -T(x^*)$ is a solution of ($IVVI_h$) and the following relation is satisfied:

$$\langle l^*, x^* \rangle \in h(x^*) + h^*_{\leq}(l^*).$$

(ii) If l^* is a solution of ($IVVI_h$), C is connected, i.e., $C \cup (-C) = Y$, and $\partial_{\leq} h(x^*) \neq \varnothing$, where $x^* = -T'(l^*)$, then x^* is a solution of (VVI_h).

Proof. (i) Let x^* be a solution of (VVI_h)

$$x^* \in X : \langle T(x^*), x - x^* \rangle \not\leq_{C\setminus\{0\}} h(x^*) - h(x), \quad \forall x \in X,$$

$$-\langle T(x^*), x^* \rangle - h(x^*) \not\leq_{C\setminus\{0\}} -\langle T(x^*), x \rangle - h(x), \quad \forall x \in X.$$

That is

$$-\langle T(x^*), x^* \rangle - h(x^*) \in h^*_{\leq}(-T(x^*)) \tag{3.11}$$

$$-\langle T(x^*), x^* \rangle - h(x^*) - h^*_{\leq}(l) \subseteq h^*_{\leq}(-T(x^*)) - h^*_{\leq}(l), \quad \forall l \in L(X,Y)$$

If $l^* = -T(x^*)$ is not a solution of (IVVI_h), then there exists $l \in L(X,Y)$, such that

$$\langle l - l^*, T'(l^*) \rangle \leq_{C\setminus\{0\}} h^*_{\leq}(l^*) - h^*_{\leq}(l).$$

It follows from (3.11) that

$$\langle l - l^*, T'(l^*) \rangle \leq_{C\setminus\{0\}} -\langle T(x^*), x^* \rangle - h(x^*) - h^*_{\leq}(l),$$

$$-\langle l, x^* \rangle \leq_{C\setminus\{0\}} -h(x^*) - h^*_{\leq}(l).$$

It is a contradiction with the definition of $h^*_{\leq}(l)$. Then l^* is a solution of (IVVI_h).

It is easy to verify that

$$\langle l^*, x^* \rangle \in h(x^*) + h^*_{\leq}(l^*).$$

(ii) Let l^* be a solution of (IVVI_h). Let $x^* = -T'(l^*)$. Then $l^* = -T(x^*)$. That is

$$\langle l - l^*, T'(l^*) \rangle \not\leq_{C\setminus\{0\}} h^*_{\leq}(l^*) - h^*_{\leq}(l), \quad \forall l \in L(X,Y)$$

$$\langle l + T(x^*), -x^* \rangle \not\leq_{C\setminus\{0\}} h^*_{\leq}(-T(x^*)) - h^*_{\leq}(l), \tag{3.12}$$

$$-\langle T(x^*), x^* \rangle - \langle l, x^* \rangle + h^*_{\leq}(l) \not\leq_{C\setminus\{0\}} h^*_{\leq}(-T(x^*)).$$

Since $\partial_{\leq} h(x^*) \neq \varnothing$, let $l \in \partial_{\leq} h(x^*)$, then

$$\langle l, x^* \rangle - h(x^*) \in h^*_{\leq}(l).$$

It follows from (3.12) that

$$-\langle T(x^*), x^* \rangle - h(x^*) \not\leq_{C\setminus\{0\}} h^*_{\leq}(-T(x^*)).$$

From the definition of $h^*_{\leq}$ and C being connected, we get

$$-\langle T(x^*), x^* \rangle - h(x^*) \in h^*_{\leq}(-T(x^*)).$$

If $x^* = -T'(l^*)$ is not a solution of the (VVI$_h$), then $\exists u \in X$, such that

$$\langle T(x^*), u - x^* \rangle \not\leq_{C\setminus\{0\}} h(x^*) - h(u),$$

$$-\langle T(x^*), x^* \rangle - h(x^*) \not\leq_{C\setminus\{0\}} -\langle T(x^*), u \rangle - h(u).$$

Then we obtain

$$-\langle T(x^*), x^* \rangle - h(x^*) \notin h^*_{\leq}(-T(x^*)),$$

a contradiction. Therefore $x^* = -T'(l^*)$ is a solution of the (VVI$_h$). This completes the proof. ■

Corollary 3.28. *Assume that $A : X \to X^*$ is one-to-one and that $h_1 : X \to I\!R$ be a lower semi-continuous convex function. x^* is a solution of the following variational inequality: find $x^* \in X$, such that*

$$\langle Ax^*, x - x^* \rangle \geq h_1(x) - h_1(x^*), \quad \forall x \in X$$

if and only if $l^* = -Ax^* \in X^*$ is a solution of the inequality which consists in finding $l^* \in Domain(A')$, such that

$$\langle A'l^*, l - l^* \rangle \geq h_1^*(l^*) - h_1^*(l), \quad \forall l \in X^*,$$

and the following identity is satisfied

$$h_1(x^*) + h_1^*(l^*) = \langle l^*, x^* \rangle,$$

where h_1^* is the Fenchel conjugate of h_1,

$$h_1^*(l) = \max\{\langle l, x \rangle - h_1(x) : x \in X\}.$$

Proof. This follows from Theorem 3.27 by letting $C = \mathbb{R}_+$. ∎

The inverse weak VVI (for short, IWVVI$_h$) of (WVVI$_h$) is defined as: find $l^* \in \text{Domain}(T')$, such that:

$$\langle l - l^*, T'(l^*) \rangle \not\leq_{intC} h^*_<(l^*) - h^*_<(l), \quad \forall l \in L(X,Y)$$

where $f^*_<(l) = \text{Max}_{intC}\{l(x) - f(x) : x \in X\}$ is the weak vector conjugate function of f.

Let $h : X \to Y$ and $x^* \in X$. We define the weak subgradient of h at x^* by

$$\partial_< h(x^*) = \{l \in L(X,Y) : h(x) - h(x^*) \not\leq_{intC} \langle l, x - x^* \rangle, \forall x \in X\}.$$

Theorem 3.29. *Let X be a Hausdorff topological vector space and (Y, C) be an ordered Hausdorff topological vector space with $intC \neq \varnothing$. The function T is one-to-one and $h : X \to Y$ is continuous. Assume that $h^*_<(l) \neq \varnothing, \forall l \in L(X,Y)$.*

(i) If x^* is a solution of (WVVI$_h$), then $l^* = -T(x^*)$ is a solution of IWVVI$_h$ and the following inclusion is satisfied:

$$\langle l^*, x^* \rangle \in h(x^*) + h^*_<(l^*).$$

(ii) If l^* is a solution of (IWVVI$_h$), C is connected, i.e., $C \cup (-C) = X$, and $\partial_< h(x^*) \neq \varnothing$, where $x^* = -T'(l^*)$, then x^* is a solution of (WVVI$_h$).

Proof. The proof follows the same lines as that of Theorem 3.27 by replacing ordering '$\not\leq_{C\setminus\{0\}}$' by $\not\leq_{intC}$ and $h^*_{\leq}(l)$ by $h^*_<(l)$. ∎

In the following some examples are given to show the application of a weak inverse VVI.

Example 3.30. (Vector approximation). Consider the vector approximation problem

$$\text{Min}_C(\|a_1 - x\|^2, \cdots, \|a_\ell - x\|^2) \quad \text{subject to} \quad x \in X,$$

where X is a Hilbert space with the inner product $[\cdot,\cdot]$, $a_i (i = 1, \cdots, \ell)$ is a fixed element of X, $\|x\|^2 = [x, x]$. If $x^* \in X$ is a weakly minimal solution, i.e.,

$$(\|a_1 - x\|^2, \cdots, \|a_\ell - x\|^2) \not\leq_{intC} (\|a_1 - x^*\|^2, \cdots, \|a_\ell - x^*\|^2), \ \forall x \in X,$$

then

$$([a_1, x - x^*], \cdots, [a_\ell, x - x^*]) \not\leq_{intC} ([a_1, x - x^*], \cdots, [a_\ell, x - x^*])$$

Hence, this is a weak VVI (WVVI$_h$) with $Y = \mathbb{R}^\ell$, $T(x) = ([x, \cdot], \cdots, [x, \cdot])$, $h(x) = -([a_1, x], \cdots, [a_\ell, x])$.

We can verify that $l^* = -T(x^*)$ satisfies

$$\langle l - l^*, -x^* \rangle \not\leq_{intC} h^*_<(l^*) - h^*_<(l), \quad \forall l \in L(X, \mathbb{R}^m).$$

Since $T'(l^*) = -x^*$, l^* is a solution of the inverse VVI

$$\langle l - l^*, T'(l^*) \rangle \not\leq_{intC} h^*_<(l^*) - h^*_<(l), \quad \forall l \in L(X, \mathbb{R}^m).$$

Consider the vector unconstrained optimization problem (for short, VUP)

$$\mathrm{Min}_C f(x), \quad \text{subject to} \quad x \in \mathbb{R}^n,$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^\ell$ is a differentiable vector-valued function.

Let $h : X \rightrightarrows Y$ be a set-valued function and $x^* \in X$. We define the weak subgradient of h at x^* by

$$\partial^s_< h(x^*) = \{l \in L(X, Y) : h(x) - h(x^*) \not\leq_{intC} \langle l, x - x^* \rangle, \forall x \in X\}.$$

(VUP) is said to be weakly stable if the set-valued mapping $W : \mathbb{R}^m \to \mathbb{R}^\ell$

$$W(u) = -\mathrm{Max}_{intC}\{-\phi(x, u) : x \in \mathbb{R}^n\}$$

has a weak subgradient at $u = 0$.

Now we construct the dual problem (for short, DVUP) of (VUP) as follows

$$\text{(DVUP)} \qquad \mathrm{Min}_C - \phi^*_<(0, \Gamma), \quad \text{subject to} \quad \Gamma \in \mathbb{R}^{n \times \ell},$$

where $\phi : \mathbb{R}^n \times \mathbb{R}^\ell \longrightarrow \mathbb{R}^\ell$ is the perturbation function satisfying

$$\phi(x, 0) = h(x), \quad \forall x \in \mathbb{R}^n.$$

Proposition 3.31. *Assume that (VUP) is weakly stable and C is connected. If x^* is a solution of (VUP), then there exists $\Gamma_0 \in \mathbb{R}^{n \times \ell}$ such that $l^* = -\nabla f(x^*)$ is a solution of the inverse vector variational inequality and Γ_0 is a solution of (DVUP) and satisfy the inclusion*

$$(l^{*\top}, \Gamma_0) \in \partial_< \phi(x^*, 0).$$

Proof. Let x^* be a weakly minimal solution of VUP. Then, x^* is a solution of the following (WVVI): find $x^* \in \mathbb{R}^n$, such that

$$\nabla f(x^*)^\top (x - x^*) \not\leq_{intC} 0, \quad \forall x \in \mathbb{R}^n.$$

Then from Theorem 3.29, $l^* = -\nabla f(x^*)$ satisfies

$$(l - l^*)^\top (-x^*) \not\leq_{intC} 0^*_<(l^*) - 0^*_<(l), \quad \forall l \in \mathbb{R}^{n \times \ell},$$

where $0^*_<(l) = \mathrm{Max}_{intC}\{l^\top x : x \in \mathbb{R}^n\}$. This is the inverse (VVI) of (WVVI) if we let $T = \nabla h$ and $-x^* = T'(l^*), f = 0$.

It is easy to verify that the weak inverse relation of (VUP) and (DVUP) holds: for any $x \in \mathbb{R}^n$, $\Gamma \in \mathbb{R}^{n \times \ell}$,

$$\phi(x,0) \notin -\phi^*_<(0,\Gamma) - intC.$$

From [176], if (VUP) is weakly stable, then there exists a solution $\Gamma_0 \in \mathbb{R}^{n\times \ell}$ of (DVUP) satisfying

$$(0,\Gamma_0) \in \partial_< \phi(x^*,0).$$

Assume that C is connected. Then, from the vector variational inequality,

$$-\nabla f(x^*)^\top x \leq_C 0, \quad \forall x \in \mathbb{R}^n.$$

Hence $(l^{*\top}, \Gamma_0) \in \partial_< \phi(x^*,0)$. ■

We now consider the inverse VVI with a variable ordering relation. Let $T : X \longrightarrow L(X,Y)$ be a function, and $h : X \to Y$ be a function. Let $C : X \rightrightarrows Y$ be a set-valued function such that, for every $x \in X$, $C(x)$ is a nonempty convex cone. (VVI^v_h) with a variable ordering relation consists in finding $x^* \in X$, such that

$$\langle T(x^*), x - x^*\rangle \not\leq_{C(x^*)\setminus\{0\}} h(x^*) - h(x), \ \forall x \in X.$$

Let $h : X \to Y$. We define the subgradient of h at x^* with a variable ordering relation by

$$\partial^v_{\leq} h(x^*) = \{l \in L(X,Y) : h(x) - h(x^*) \not\leq_{C(x^*)\setminus\{0\}} \langle l, x - x^*\rangle, \forall x \in X\}.$$

The inverse VVI of (VVI^v_h) with a variable ordering relation is defined as: finding $l^* \in \text{Domain}(T')$, such that

$$\langle l - l^*, T'(l^*)\rangle \not\leq_{C(x^*)\setminus\{0\}} h^{*v}_{\leq}(l^*) - h^{*v}_{\leq}(l), \quad \forall l \in L(X,Y),$$

where $h^{*v}_{\leq}(l) := \text{Max}_{C(x^*)}\{\langle l, x\rangle - h(x) : x \in X\}$ is the vector conjugate function of h with a variable ordering relation. This problem is denoted by (IVVI^v_h).

It is clear that if let $l \in \partial^v_{\leq} h(x^*)$, then

$$\langle l, x^*\rangle - h(x^*) \in h^{*v}_{\leq}(l).$$

Theorem 3.32. *Let X and Y be Hausdorff topological vector spaces and $C : X \rightrightarrows Y$ be a set-valued function such that, for every $x \in X$, $C(x)$ is a nonempty convex cone. The function T is one-to-one and $h : X \to Y$ is continuous. Assume that $h^{*v}_{\leq}(l) \neq \varnothing, \forall l \in L(X,Y)$.*

(i) If x^* is a solution of (VVI^v_h), then $l^* = -T(x^*)$ is a solution of (IVVI^v_h) and the following relation is satisfied:

$$\langle l^*, x^*\rangle \in h(x^*) + h^{*v}_{\leq}(l^*).$$

(ii) If l^* is a solution of (IVVI^v_h), C is connected, i.e., $C(x) \cup (-C(x)) = Y$, for every $x \in X$, and $\partial^v_{\leq} h(x^*) \neq \varnothing$, where $x^* = -T'(l^*)$, then x^* is a solution of (VVI^v_h).

Proof. (i) Let x^* be a solution of (VVI$_h^v$):

$$x^* \in X : \langle T(x^*), x - x^* \rangle \not\leq_{C(x^*)\setminus\{0\}} h(x^*) - h(x), \quad \forall x \in X,$$

$$-\langle T(x^*), x^* \rangle - h(x^*) \not\leq_{C(x^*)\setminus\{0\}} -\langle T(x^*), x \rangle - h(x), \quad \forall x \in X.$$

That is

$$-\langle T(x^*), x^* \rangle - h(x^*) \in h_{\leq}^{*v}(-T(x^*)) \tag{3.13}$$

$$-\langle T(x^*), x^* \rangle - h(x^*) - h_{\leq}^{*v}(l) \subseteq h_{\leq}^{*}(-T(x^*)) - h_{\leq}^{*}(l), \quad \forall l \in L(X, Y).$$

If $l^* = -T(x^*)$ is not a solution of (IVVI$_h^v$), then there exists $l \in L(X, Y)$, such that

$$\langle l - l^*, T'(l^*) \rangle \leq_{C(x^*)\setminus\{0\}} h_{\leq}^{*v}(l^*) - h_{\leq}^{*v}(l).$$

It follows from (3.13) that

$$\langle l - l^*, T'(l^*) \rangle \leq_{C(x^*)\setminus\{0\}} -\langle T(x^*), x^* \rangle - h(x^*) - h_{\leq}^{*v}(l),$$

$$-\langle l, x^* \rangle \leq_{C(x^*)\setminus\{0\}} -h(x^*) - h_{\leq}^{*v}(l).$$

It is a contradiction with the definition of $h_{\leq}^{*v}(l)$. Then $l^* = -T(x^*)$ is a solution of (IVVI$_h^v$).

It is easy to verify that

$$\langle l^*, x^* \rangle \in h(x^*) + h_{\leq}^{*v}(l^*).$$

(ii) Let l^* be a solution of (IVVI$_h^v$). Let $x^* = -T'(l^*)$. Then $l^* = -T(x^*)$. That is

$$\langle l - l^*, T'(l^*) \rangle \not\leq_{C(x^*)\setminus\{0\}} h_{\leq}^{*v}(l^*) - h_{\leq}^{*v}(l), \quad \forall l \in L(X, Y)$$

$$\langle l + T(x^*), -x^* \rangle \not\leq_{C(x^*)\setminus\{0\}} h_{\leq}^{*v}(-T(x^*)) - h_{\leq}^{*v}(l), \tag{3.14}$$

$$-\langle T(x^*), x^* \rangle - \langle l, x^* \rangle + h_{\leq}^{*v}(l) \not\leq_{C(x^*)\setminus\{0\}} h_{\leq}^{*v}(-T(x^*)).$$

Since $\partial_{\leq}^{v} h(x^*) \neq \varnothing$, let $l \in \partial_{\leq}^{v} h(x^*)$, then

$$\langle l, x^* \rangle - h(x^*) \in h_{\leq}^{*v}(l).$$

It follows from (3.14) that

$$-\langle T(x^*), x^* \rangle - h(x^*) \not\leq_{C(x^*)\setminus\{0\}} h_{\leq}^{*v}(-T(x^*)).$$

From the definition of $h_{\leq}^{*v}$ and C being connected, we get

$$-\langle T(x^*), x^* \rangle - h(x^*) \in h_{\leq}^{*v}(-T(x^*)).$$

If $x^* = -T'(l^*)$ is not a solution of the (VVI$_h^v$), then $\exists u \in X$, such that

$$\langle T(x^*), u - x^* \rangle \not\leq_{C(x^*)\setminus\{0\}} h(x^*) - h(u),$$

$$-\langle T(x^*), x^* \rangle - h(x^*) \not\leq_{C(x^*)\setminus\{0\}} -\langle T(x^*), u \rangle - h(u).$$

Then we obtain

$$-\langle T(x^*), x^* \rangle - h(x^*) \notin h^{*v}_{\leq}(-T(x^*)),$$

a contradiction. Therefore $x^* = -T'(l^*)$ is a solution of the (VVI^v_h). This completes the proof. ∎

Consider the inverse WVVI with a variable ordering relation. Let $T : X \longrightarrow L(X, Y)$ be a function, and $h : X \to Y$ be a function. Let $C : X \rightrightarrows Y$ be a set-valued function such that, for every $x \in X$, $C(x)$ is a convex cone with nonempty interior. The (WVVI^v_h) problem with a variable ordering relation consists in finding $x^* \in X$, such that

$$\langle T(x^*), x - x^* \rangle \not\leq_{intC(x^*)} h(x^*) - h(x), \ \forall x \in X.$$

Let $h : X \to Y$. We define the weak subgradient of h at x^* with a variable ordering relation by

$$\partial^v_{<} h(x^*) = \{l \in L(X, Y) : h(x) - h(x^*) \not\leq_{intC(x^*)} \langle l, x - x^* \rangle, \forall x \in X\}.$$

The inverse WVVI of (WVVI^v_h) with a variable ordering relation is defined as: finding $l^* \in \text{Domain}(T')$, such that

$$\langle l - l^*, T'(l^*) \rangle \not\leq_{intC(x^*)} h^{*v}_{<}(l^*) - h^{*v}_{<}(l), \quad \forall l \in L(X, Y),$$

where $h^{*v}_{<}(l) := \text{Max}_{intC(x^*)}\{\langle l, x \rangle - h(x) : x \in X\}$ is the weak vector conjugate function of h with a variable ordering relation. This problem is denoted by (IWVVI^v_h).

Theorem 3.33. *Let X and Y be Hausdorff topological vector spaces and $C : X \rightrightarrows Y$ be a set-valued function such that, for every $x \in X$, $C(x)$ is a nonempty convex cone with nonempty interior. The function T is one-to-one and $h : X \to Y$ is continuous. Assume that $h^{*v}_{<}(l) \neq \varnothing, \forall l \in L(X, Y)$.*

(i) If x^* is a solution of (WVVI^v_h), then $l^* = -T(x^*)$ is a solution of (IWVVI^v_h) and the following relation is satisfied:

$$\langle l^*, x^* \rangle \in h(x^*) + h^{*v}_{<}(l^*).$$

(ii) If l^* is a solution of (IWVVI^v_h), C is connected, i.e., $C(x) \cup (-C(x)) = Y$, for every $x \in X$, and $\partial^v_{<} h(x^*) \neq \varnothing$, where $x^* = -T'(l^*)$, then x^* is a solution of (WVVI^v_h).

3.3 Gap Functions for VVI

The concept of a gap function is well-known both in the context of convex optimization and variational inequalities. The minimization of gap functions is a viable approach for solving variational inequalities. In this section, we generalize the gap function for variational inequalities to vector variational inequalities. The convexity and differentiability of gap functions are also studied.

Let X and Y be Banach spaces, and let $C \subset Y$ be a closed and convex cone with nonempty interior $intC$.

Consider following vector variational inequality problem (VVI) of finding $y \in K$ such that

$$\langle T(y), x - y\rangle \not\leq_{C\setminus\{0\}} 0, \quad \forall x \in K,$$

where K is a closed subset of X and $T : X \to L(X,Y)$ is a function.

The weak vector variational inequality problem (WVVI) of finding $y \in K$ such that

$$\langle T(y), x - y\rangle \not\leq_{intC} 0, \quad \forall x \in K.$$

Definition 3.34. *Let C be a convex and closed cone in Y with nonempty interior and $K \subset X$ be a closed set.*

(i) *A set-valued function $\phi : K \rightrightarrows Y$ is said to be a gap function of (VVI) if*
  1. *$0 \in \phi(y)$ if and only if y solves (VVI);*
  2. *$0 \not\geq_{C\setminus\{0\}} \phi(x)$, $x \in K$.*

(ii) *A set-valued function $\phi_w : K \rightrightarrows Y$ is said to be a gap function of (WVVI) if*
  1. *$0 \in \phi_w(y)$ if and only if y solves (WVVI);*
  2. *$0 \not\geq_{intC} \phi_w(x), \forall x \in K$.*

Let

$$\langle T(x), x - K\rangle = \cup\{\langle T(x), x - z\rangle : z \in K\}.$$

Define the set-valued function $\phi : K \rightrightarrows Y$ by

$$\phi(x) := \mathrm{Max}_C\langle T(x), x - K\rangle, \quad x \in K.$$

Theorem 3.35. *Let C be a convex and pointed cone in Y. The set-valued function $\phi(x) = Max_C\langle T(x), x - K\rangle$ is a gap function for (VVI).*

Proof. We first prove that $0 \in \phi(y)$ if and only if y solves (VVI).

Suppose that y solves (VVI). Then

$$\langle T(y), y - x\rangle \not\geq_{C\setminus\{0\}} 0, \quad \forall x \in K.$$

In particular, let $x = y$, then

$$\langle T(y), y - y\rangle = 0.$$

Thus,

$$0 \in \langle T(y), y - K \rangle.$$

We assert that $0 \in \phi(y)$. Otherwise, if there exits some $z \in K$, such that $\langle T(y), y - z \rangle \geq_{C \setminus \{0\}} \langle T(y), y - y \rangle = 0$, then this contradicts the fact that y solves (VVI).

Conversely, suppose $0 \in \phi(y)$. If y does not solve (VVI), then there exists $x \in K$, such that

$$\langle T(y), x - y \rangle \leq_{C \setminus \{0\}} 0,$$

$$\langle T(y), y - x \rangle \geq_{C \setminus \{0\}} 0 = \langle T(y), y - y \rangle.$$

Thus, $0 \notin \phi(y)$, a contradiction.

Moreover, for any $x \in K$, $0 \in \langle T(x), x - x \rangle$. By the definition of $\phi(x)$, we have

$$0 \not\geq_{C \setminus \{0\}} \phi(x), \quad \forall x \in K.$$

The proof is complete. ■

Define the set-valued function $\phi_w : K \rightrightarrows Y$ by

$$\phi_w(x) := \mathrm{Max}_{intC} \langle T(x), x - K \rangle, \quad \forall x \in K.$$

Theorem 3.36. *The set-valued function $\phi_w(x)$ is a gap function for $(WVVI)$.*

Proof. The proof is similar to that for Theorem 3.35 and is omitted. ■

We now extend the definition of a gap function to a general VVI as follows.

Definition 3.37. *Let C be a closed and convex cone in Y. The general vector variational inequality problem (for short, GVVI) consists of finding $y \in X$ such that*

$$(GVVI) \qquad \langle T(y), x - y \rangle + h(x) - h(y) \not\leq_{C \setminus \{0\}} 0, \quad \forall x \in X,$$

where $T : X \to L(X, Y)$ is assumed to be injective, and $h : X \to Y$ is assumed to be a function.

Definition 3.38. *A set-valued function $\phi' : X \rightrightarrows Y$ is said to be a gap function of the $(GVVI)$ iff*

1. $0 \not\geq_{C \setminus \{0\}} \phi'(x), \quad \forall x \in X;$
2. $0 \in \phi'(y)$ *if and only if y solves $(GVVI)$.*

Define the set-valued function $\phi' : X \rightrightarrows Y$ by

$$\phi'(x) = Max_C(\langle T(x), x - X \rangle + h(x) - h(X)), \quad x \in X.$$

It follows immediately that

$$\phi'(y) = h(y) + g(-T(y)) + \langle T(y), y \rangle,$$

where g is the Fenchel conjugate of h. We now have a much simpler proof that ϕ' is a gap function for (GVVI), and the meaning of "gap" is now apparent.

Theorem 3.39. *Assume that C is a pointed, closed and convex cone. Then ϕ' is a gap function for problem (GVVI).*

Proof. (1) The fact that $\phi'(y) \not\leq_{C\setminus\{0\}} 0$ follows directly from Young's inequality of Lemma 3.26.

(2) Suppose that $0 \in \phi'(y)$. If y does not solve (GVVI), then there exists some $x \in X$, such that

$$\langle T(y), y - x\rangle - h(x) + h(y) \geq_{C\setminus\{0\}} 0 = \langle T(y), y - y\rangle - h(y) + h(y),$$

then $0 \notin \phi'(y)$, a contradiction. On the other hand, suppose that y solves (GVVI). Then

$$\langle T(y), x - y\rangle + h(x) - h(y) \not\leq_{C\setminus\{0\}} 0 \quad \forall x \in X.$$

That is,

$$\langle T(y), y - x\rangle + h(y) - h(x) \not\geq_{C\setminus\{0\}} 0 \quad \forall x \in X.$$

By the definition of $\phi'(y)$, we have that

$$\phi'(y) \not\geq_{C\setminus\{0\}} 0.$$

This together with (i) yields $0 \in \phi'(y)$ by the pointedness of C. ■

The above generalization of (VVI) can be easily extended to (WVVI).

Definition 3.40. *The general weak vector variational inequality (for short, GWVVI) consists of finding $y \in X$ such that*

$$\langle T(y), x - y\rangle - h(y) + h(x) \not\leq_{intC} 0, \quad \forall x \in X,$$

where $T : X \to L(X, Y)$ is assumed to be injective, and $h : X \to Y$ is assumed to be a function.

Definition 3.41. *A set-valued function $\phi'_w : X \rightrightarrows Y$ is said to be a gap function of the (GWVVI) iff*

1. $0 \not\geq_{intC} \phi'_w(x), \quad \forall x \in X$;
2. $0 \in \phi'_w(y)$ *if and only if y solves (GWVVI).*

Define the set-valued function $\phi'_w : X \rightrightarrows Y$ by

$$\phi'_w(x) = \mathrm{Max}_{intC}(\langle T(x), x - X\rangle + h(x) - h(X)), \quad \forall x \in X.$$

Theorem 3.42. *The set-valued function $\phi'_w : X \rightrightarrows Y$ is a gap function for (GWVVI).*

The above gap functions are of set-valued nature. Special single-valued gap functions can be constructed in terms of nonlinear scalarization functions. Let $T : X \to L(X, Y)$ and $K \subset X$ be a compact set. Consider the following weak vector variational inequality problem of finding $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle \not\leq_{intC} 0, \quad \forall x \in K. \tag{3.15}$$

Theorem 3.43. *Let $e \in intC$. Then $x^* \in K$ solves WVVI(3.15) if and only if $g(x^*) = 0$, where*

$$g(x) = \min_{x \in K} \xi_{e0}(\langle T(x), y - x \rangle), \tag{3.16}$$

is a non-positive function.

Proof. Assume that $x^* \in K$ solves the problem WVVI(3.15). Then, it follows from Proposition 1.43 that

$$\begin{aligned}
&\langle T(x^*), x - x^* \rangle \not\leq_{intC} 0, \quad \forall x \in K \\
&\Leftrightarrow \langle T(x^*), x - x^* \rangle \notin -intC, \quad \forall x \in K \\
&\Leftrightarrow \xi_{e0}(\langle T(x^*), x - x^* \rangle) \geq 0, \quad \forall x \in K \\
&\Leftrightarrow \min_{x \in K} \ \xi_{e0}(\langle T(x^*), x - x^* \rangle) \geq 0.
\end{aligned}$$

It is clear that $\xi_{e0}(\langle T(x^*), x^* - x^* \rangle) = 0$. Hence, $g(x^*) = 0$. ■

In the special case where $Y = \mathbb{R}^\ell$ and $C = \mathbb{R}^\ell_+$, the nonlinear scalar function may be expressed in the following equivalent form:

$$\xi_{ea}(y) = \max_{1 \leq i \leq \ell} \frac{y_i - a_i}{e_i}. \tag{3.17}$$

Corollary 3.44. *Let $Y = \mathbb{R}^\ell$ and $C = \mathbb{R}^\ell_+$ and let*

$$g_1(x) = \min_{y \in K} \max_{1 \leq i \leq \ell} \{\langle T_i(x), y - x \rangle\}, \quad x \in K, \tag{3.18}$$

where $T(x) = [T_1(x), \cdots, T_\ell(x)]^\top$. Then $x^* \in K$ solves WVVI problem (3.15) if and only if $g_1(x^*) = 0$.

Proof. It follows from Theorem 3.43 and (3.17) by letting $a = 0$ and $e = (1, \cdots, 1)^T \in \mathbb{R}^\ell$. ■

Consider the following vector variational inequality problem of finding $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle \not\leq_{C \setminus \{0\}} 0, \quad \forall x \in K. \tag{3.19}$$

Theorem 3.45. *Let $e \in intC$. Then $x^* \in K$ solves VVI(3.19) only if $g(x^*) = 0$, where $g(x)$ is defined by (3.16). If $Y = \mathbb{R}^\ell$ and $C = \mathbb{R}^\ell_+$, then $x \in K$ solves VVI only if $g_1(x^*) = 0$, where $g_1(x)$ is defined by (3.18).*

Proof. Note that a solution of VVI(3.19) is also a solution of WVVI(3.15). The results follow from Theorem 3.43 and Corollary 3.44. ■

The following example shows that Theorem 3.45 is only a necessary condition for VVI.

Example 3.46. Let $K = [-1, 0]$ and $C = \mathbb{R}^2_+$. Consider the VVI defined as follows:

$$(1, 2x)(y - x) \not\leq_{\mathbb{R}^2_+ \setminus \{0\}} 0, \qquad \forall y \in K.$$

Then $x = 0$ is not a solution of VVI(3.19), but $g_1(0) = 0$. Hence Theorem 3.45 is only a necessary condition for VVI.

Next we study the gap function of vector variational inequalities with a variable ordering relation.

Let $X = Y$. Consider the following vector variational inequality problem of find $y^* \in K$, such that

$$\langle T(y^*), y - y^* \rangle \notin -intC(y^*), \quad \forall y \in K, \tag{3.20}$$

where $K \subseteq Y$ is a subset, $T : Y \to L(Y, Y)$ is a function and $C : Y \rightrightarrows Y$ is a set-valued function.

By Proposition 1.54, (3.20) holds if and only if

$$\xi(y^*, \langle T(y^*), y - y^* \rangle) \geq 0, \quad \forall y \in K.$$

Note that $\xi(y^*, 0) = 0$. Then (3.20) holds if and only if

$$\min_{y \in K} \xi(y^*, \langle T(y^*), y - y^* \rangle) = 0.$$

Thus we have the following result.

Theorem 3.47. *Let Y be a real normed space and $K \subset Y$ be a nonempty subset. Let $C : Y \rightrightarrows Y$ be a set-valued map such that for each $y \in Y$, $C(y)$ is a proper, closed and convex cone in Y and C linear. Let $\bar{C} = \cap_{y \in Y} C(y)$ and $k^0 \in int\bar{C}$. Let $y^* \in K$. Then y^* is a solution of vector variational inequality (3.20), if and only if*

$$\min_{y \in K} \xi(y^*, \langle T(y^*), y - y^* \rangle) = 0. \tag{3.21}$$

Under some appropriate conditions, the gap function for both the (general) vector variational inequality and (general) weak vector variational inequality can be shown to be convex.

Definition 3.48. *Let K be an affine subset of X. The function $T : K \to L(X, Y)$ is affine if, $\forall x', x'' \in K, \forall \alpha, \beta \in \mathbb{R}$, with $\alpha + \beta = 1$ we have*

$$T(\alpha x' + \beta x'') = \alpha T(x') + \beta T(x'').$$

Lemma 3.49. *Let $T : K \to L(X, Y)$ be a function. If T is affine and monotone, then the function $\langle T(\cdot), \cdot \rangle : K \to Y$ is C-convex.*

Proof. Given $t \in (0,1)$, $x', x'' \in K$,

$$\begin{aligned}
& \langle T(tx' + (1-t)x''), tx' + (1-t)x'' \rangle - t\langle T(x'), x' \rangle - (1-t)\langle T(x''), x'' \rangle \\
= \; & t^2 \langle T(x'), x' \rangle + (1-t)^2 \langle T(x''), x'' \rangle + t(1-t)(\langle T(x'), x'' \rangle + \langle T(x''), x' \rangle) \\
& -t\langle T(x'), x' \rangle - (1-t)\langle T(x''), x'' \rangle \\
= \; & -t(1-t)\langle T(x' - x''), x' - x'' \rangle \\
\leq_C \; & 0.
\end{aligned}$$

■

It is well-known that the Fenchel conjugate of a scalar valued function is convex in the usual definition. With the above definition of convexity for set-valued functions, this notion is now affirmative for a vector-valued function.

Lemma 3.50. *Let $h : X \to Y$ be a C-convex function and let, for any $u \in L(X,Y)$, the set $\{\langle u, x \rangle - h(x) : x \in X\}$ satisfy the domination property. Then the Fenchel conjugate of h is type I C-convex.*

Proof. By definition, the Fenchel conjugate of the vector-valued function h is a set-valued function $g : L(X,Y) \rightrightarrows Y$ such that

$$g(u) = \mathrm{Max}_C\{\langle u, x \rangle - h(x) : x \in X\}.$$

We have, $\forall t \in (0,1), u', u'' \in L(X,Y)$,

$$\begin{aligned}
& g(tu' + (1-t)u'') \\
& = \mathrm{Max}_C\{\langle tu' + (1-t)u'', x \rangle - h(x) : x \in X\} \\
& \subset \{\langle tu' + (1-t)u'', x \rangle - f(x) : x \in X\} \\
& = \{t(\langle u', x \rangle - h(x)) + (1-t)(\langle u'', x \rangle - h(x)) : x \in X\} \\
& \subset t\{(\langle u', x \rangle - h(x)) : x \in X\} + (1-t)\{\langle u'', x \rangle - h(x)) : x \in X\} \\
& \subset t\mathrm{Max}_C\{(\langle u', x \rangle - h(x) : x \in X\} - C \\
& \qquad +(1-t)\mathrm{Max}_C\{(\langle u'', x \rangle - h(x)) : x \in X\} - C \\
& = tg(u') + (1-t)g(u'') - C.
\end{aligned}$$

Then g is type I C-convex. ■

Lemma 3.51. *If $g : L(X,Y) \rightrightarrows Y$ is type I C-convex, and $T : X \to L(X,Y)$ is affine, then the composite $g \circ T : X \rightrightarrows Y$ is type I C-convex.*

Proof. Given $x', x'' \in X$ and $t \in (0,1)$,

$$\begin{aligned}
g \circ T(tx' + (1-t)x'') &= g(T(tx' + (1-t)x'')) \\
&= g(tT(x') + (1-t)T(x'')) \\
&\subset tg(T(x')) + (1-t)g(T(x'')) - C \\
&= tg \circ T(x') + (1-t)g \circ T(x'') - C.
\end{aligned}$$

■

Theorem 3.52. *Let C be a closed and convex cone in Y and let for any $u \in L(X,Y)$ the set $\{\langle u,x\rangle - h(x) : x \in X\}$ satisfy the domination property. Consider the problem (GVVI). If T is affine and monotone, and $h : X \to Y$ is C-convex, then the gap function ϕ' is type I C-convex.*

Proof. By definition, the gap function $\phi'(x)$ can be rewritten as,

$$\phi'(x) = g \circ (-T)(x) + \langle T(x), x\rangle + h(x),$$

where the Fenchel conjugate g of h is type I C-convex by Lemma 3.50. Since T is affine, so is $-T$. By Lemma 3.51, $g \circ (-T)$ is type I C-convex. By Lemma 3.49 $\langle T(\cdot), \cdot\rangle$ is C-convex, and hence ϕ' is type I C-convex. ■

Recall that

$$\Phi(x) = \mathrm{Max}_C\langle T(x), x - K\rangle, \quad x \in K,$$

$$\Phi_w(x) = \mathrm{Max}_{intC}\langle T(x), x - K\rangle, \quad x \in K,$$

respectively.

It is worth noting that (VVI) is equivalent to the following set-valued optimization problem:

$$\mathrm{Min}_C \Phi(x), \text{ subject to } x \in K, \tag{3.22}$$

and (WVVI) is equivalent to the following set-valued optimization problem:

$$\mathrm{Min}_C \Phi_w(x), \text{ subject to } x \in K. \tag{3.23}$$

If T is a vector-valued function from X into X^*, then (VVI) and (WVVI) become the ordinary variational inequality problem and the gap functions Φ and Φ_w reduce to Auslender's gap function [7]. Thus, set-valued optimization problems (3.22) and (3.23) reduce to a scalar optimization problem:

$$\min \varphi(x), \text{ subject to } x \in K,$$

where $\varphi(x) = \max\langle T(x), x - K\rangle$. If φ is differentiable, then the above mathematical programming problem may be solved by a descent algorithm which possesses a global convergence property [95]. Therefore, it is very important and valuable to discuss differential properties of gap functions Φ and Φ_w of vector variational inequalities. In sequel, let X and Y be two real Banach spaces. Let θ and Θ denote the origin points of Y and $L(X,Y)$, respectively. For any $A \in L(X,Y)$, we introduce the norm:

$$||A||_L = \sup\{||A(x)||_Y : ||x|| \leq 1\}.$$

Since Y is a Banach space, $L(X,Y)$ is also a Banach space with the norm $||\cdot||_L$.

3.4 Set-valued VVI

In this section, we develop existence of a solution for a set-valued VVI using a selection of a set-valued function. We also discuss gap functions of a set-valued VVI.

Let X and Y be two Banach spaces, $K \subset X$ and $T : K \rightrightarrows L(X,Y)$. Consider the set-valued WVVI of finding $x^* \in K$ and $\bar{t} \in T(x^*)$ such that

$$\langle \bar{t}, x - x^* \rangle \not\leq_{intC} 0, \quad \forall x \in K. \tag{3.24}$$

Lemma 3.53. *Let $T_1 : K \to L(X,Y)$ be a selection of $T : K \rightrightarrows L(X,Y)$. If x^* is a solution of the following WVVI: finding $x^* \in K$ such that*

$$\langle T_1(x^*), x - x^* \rangle \not\leq_{intC} 0, \quad \forall x \in K,$$

then x^* is a solution of the set-valued WVVI.

Theorem 3.54. *Let X and Y be Banach spaces and K be a nonempty compact and convex subset of X. Let C be a proper, closed, and convex cone and $intC \neq \varnothing$, and Assume further that $T : K \rightrightarrows L(X,Y)$ is C-monotone; and there is a v-hemicontinuous selection T_1 of T on K. Let, for any $y \in K$, $T_1(y)$ be completely continuous on X. Then there exists a solution to the WVVI(3.24).*

Proof. By the assumption, there is a v-hemicontinuous selection $T_1 : K \to L(X,Y)$ such that $T_1(x) \in T(x), \forall x \in K$ and T_1 is v-hemicontinuous on K. It is clear that T_1 is also C-monotone. Then all conditions of Theorem 3.14 are satisfied. Thus, there is a solution x^* to the problem WVVI(3.26). By Lemma 3.53, x^* is a solution of WVVI(3.25). ∎

We consider the set-valued WVVI with a variable ordering relation. Let $C : K \rightrightarrows Y$ be cone-valued and $T : K \rightrightarrows L(X,Y)$. Consider the set-valued weak vector variational inequality with a variable ordering relation of finding $x^* \in K$ and $\bar{t} \in T(x^*)$ such that

$$\langle \bar{t}, x - x^* \rangle \not\leq_{intC(x)} 0, \quad \forall x \in K. \tag{3.25}$$

Let $f : X \to Y$ be a vector-valued function. Consider the vector optimization problem with variable domination structure $C(x)$:

$$(VOKV). \qquad \operatorname*{Min}_{x \in K}{}_{C(x)} f(x)$$

We set $C_0 = \cap_{x \in K} C(x)$, and assume that $intC_0 \neq \varnothing$.

Proposition 3.55. *Let $f : K \to Y$ be C_0-convex and continuous at $x^* \in K$ and $T(x) = \partial_{<} f(x)$ with respect to the cone C_0. If x^* is a weakly nondominated-like solution of the vector optimization problem (VOKV), then $(x^*, 0)$ solves (GWVVI).*

Proof. Suppose that x^* is a weakly nondominated-like solution of (VOKV). Then

$$f(x) - f(x^*) \nleq_{intC(x^*)} 0, \quad \forall x \in K.$$

This implies that

$$f(x) - f(x^*) \nleq_{intC_0} 0, \quad \forall x \in K.$$

By the definition of C_0-weak subgradient, $0 \in T(x^*) = \partial_< f(x)$. Thus (GWVVI) is satisfied with the given x^* and $\bar{t} = 0$. ■

Proposition 3.56. *Let $f : K \to Y$ be C_0-convex and continuous at $x^* \in K$ and $T(x) = \partial^s_{C_0} f(x)$. If $(x^*, \bar{t})$ solves (GWVVI), then x^* is a weakly nondominated-like solution of (VOKV).*

Proof. Suppose that $(x^*, \bar{t})$ solves (GWVVI). Let $W(x^*) = Y \setminus -intC(x^*)$. Then

$$\langle \bar{t}, x - x^* \rangle \in W(x^*), \quad \forall x \in K.$$

It follows from the definition of a C_0-strong subgradient that

$$f(x) - f(x^*) - \langle \bar{t}, x - x^* \rangle \in C_0 \subset C(x^*), \quad \forall x \in K.$$

Combining these relations, we obtain

$$f(x) - f(x^*) \in W(x^*) + C(x^*) \subset W(x^*), \quad \forall x \in K.$$

Thus x^* is a weakly nondominated-like solution of (VOKV). ■

Lemma 3.57. *Let $T_1 : K \to L(X, Y)$ be a selection of $T : K \rightrightarrows L(X, Y)$. If x^* is a solution of the following (WVVI): finding $x^* \in K$ such that*

$$\langle T_1(x^*), x - x^* \rangle \nleq_{intC(x)} 0, \quad \forall x \in K, \tag{3.26}$$

then x^* is a solution of (3.25).

Proof. Assume that $x^* \in K$ is a solution of WVVI(3.26). That is,

$$\langle T_1(x^*), x - x^* \rangle \nleq_{intC(x)} 0, \quad \forall x \in K.$$

Let $\bar{t} = T_1(x^*)$. Then $\bar{t} \in T(x^*)$ and

$$\langle \bar{t}, x - x^* \rangle \nleq_{intC(x)} 0, \quad \forall x \in K.$$

Therefore, $x^* \in K$ is a solution of WVVI(3.25). ■

We apply this lemma to derive the existence of a solution for WVVI(3.25).

Let $W : K \rightrightarrows Y$ be a set-valued mapping. The *graph* of W on K is defined by

$$\mathcal{G}(W) = \{(x, y) | x \in K, y \in W(x)\}.$$

Definition 3.58. *$T : K \rightrightarrows L(X,Y)$ is $C(x)$-pseudomonotone if, for every pair of points $x \in K, y \in K$ and for all $t' \in T(x), t'' \in T(y)$, we have*

$$\langle t', y-x\rangle \not\leq_{intC(x)} 0 \text{ implies } \langle t'', y-x\rangle \not\leq_{intC(x)} 0.$$

Definition 3.59. *$T_1 : K \to L(X,Y)$ is $C(x)$-pseudomonotone if, for every pair of points $x \in K, y \in K$, we have*

$$\langle T_1(x), y-x\rangle \not\leq_{intC(x)} 0 \text{ implies } \langle T_1(y), y-x\rangle \not\leq_{intC(x)} 0.$$

The following lemma provides a connection between the pseudomonotonicity properties of a set-valued mapping and that of its selection.

Lemma 3.60. *Let $T : K \rightrightarrows L(X,Y)$ be a set-valued mapping and T_1 be a selection of T. If T is $C(x)$-pseudomonotone, then T_1 is also $C(x)$-pseudomonotone.*

Theorem 3.61. *Let X and Y be Banach spaces and K be a nonempty weakly compact and convex subset of X. Let $C : K \rightrightarrows Y$ be a set-valued mapping such that, for each $x \in K$, $C(x)$ is a proper, closed, and convex cone with apex at the origin and $intC(x) \neq \varnothing$, and $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus (-intC(x))$ be such that the graph $\mathcal{G}(W)$ of W is weakly closed in $X \times Y$. Suppose that $T(x)$ is a nonempty set of $L(X,Y)$, for each $x \in K$. Assume further that*

(P) $T : K \rightrightarrows L(X,Y)$ is $C(x)$-pseudomonotone;
(C) there is a continuous selection T_1 of T on K.

Then there exists a solution to the WVVI(3.25).

Proof. By the assumption, there is a continuous selection $T_1 : K \to L(X,Y)$ such that $T_1(x) \in T(x), \forall x \in K$. It follows from Lemma 3.60 that T_1 is also $C(x)$-pseudomonotone. Then all conditions of Theorem 3.1 in [124] are satisfied. Thus, there is a solution x^* to the problem WVVI(3.26). By Lemma 3.57, x^* is a solution of WVVI(3.25). ■

Next we construct the gap functions for set-valued VVIs.

Let $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$ and $K \subset X$ a compact subset. Assume that $T : K \rightrightarrows L(X, \mathbb{R}^\ell)$ is a set-valued mapping with a compact set $T(x)$ for each x.

Consider the (WVVI) with the set-valued mapping T, which consists in finding $x^* \in K$, and $\bar{t} \in T(x^*)$ such that

$$\langle \bar{t}, y-x^*\rangle \not\leq_{intC} 0, \quad \forall y \in K. \tag{3.27}$$

Recall that $\phi : K \subset X \to \mathbb{R}$ is said to be a gap function of WVVI(3.27) if
(i) $\phi(x) \leq 0, \quad \forall x \in K$;
(ii) $0 = \phi(x^*)$ if and only if x^* is a solution of WVVI(3.27).
Let $x, y \in K$ and $t \in T(x)$. Denote

$$\langle t, y\rangle = ((\langle t, y\rangle)_1, \cdots, (\langle t, y\rangle)_\ell),$$

i.e., $(\langle t, y\rangle)_i$ is the i-th component of $\langle t, y\rangle$, $i = 1, \cdots, \ell$. We define two mappings $\phi_1 : K \times L(X, \mathbb{R}^\ell) \to R$ and $\phi : K \to \mathbb{R}$ as follows

$$\phi_1(x, t) = \min_{y\in K} \max_{1\le i\le \ell} (\langle t, y - x\rangle)_i \tag{3.28}$$

and

$$\phi(x) = \max\{\phi_1(x, t) | t \in T(x)\}. \tag{3.29}$$

Since K is compact, $\phi_1(x, t)$ is well-defined. If X is a Hausdorff topological vector space, then $g_1(x, t)$ is a lower semi-continuous function in x (see Corollary 22 in [5]). Since $T(x)$ is a compact set, $\phi(x)$ is well-defined.

For $x \in K$ and $t \in T(x)$, it is easy to see that

$$\phi_1(x, t) = \min_{y\in K} \max_{1\le i\le \ell} (\langle t, y - x\rangle)_i \le 0.$$

Theorem 3.62. *$\phi(x)$ defined by (3.29) is a gap function of WVVI(3.27).*

Proof. It is clear that $\phi_1(x, t) \le 0, \quad \forall x \in K, t \in T(x)$. Thus $\phi(x) \le 0, \forall x \in K$. If $0 = \phi(x^*)$, then there exists $\bar{t} \in T(x^*)$ such that $\phi(x^*, \bar{t}) = 0$. Consequently, we have

$$\min_{y\in K} \max_{1\le i\le \ell} (\langle \bar{t}, y - x^*\rangle)_i = 0,$$

if and only if, for any $y \in K$,

$$\max_{1\le i\le \ell} (\langle \bar{t}, y - x^*\rangle)_i \ge 0$$

from which it follows that, for any $y \in K$,

$$\langle \bar{t}, y - x^*\rangle \not\le_{intC} 0$$

if and only if x^* is a solution of WVVI(3.27). ■

By Theorem 3.62, the solution of WVVI(3.27) is equivalent to finding a global solution x^* to the following optimization problem

$$\max_{x\in K} \phi(x), \tag{3.30}$$

with $\phi(x^*) = 0$.

Recall that

$$\phi(x) = \max\{\phi_1(x, t) | t \in T(x)\}.$$

It is clear that the optimization problem (3.30) is equivalent to the following generalized semi-infinite programming problem

$$\begin{aligned}\max_{x,s} \quad & s \\ \text{s.t.} \quad & \phi_1(x,t) \le s, \quad \forall t \in T(x), \\ & \phi_1(x,t_1) = s, \quad \exists t_1 \in T(x), \\ & x \in K.\end{aligned}$$

Note also that $\bar{t}$ of WVVI(3.27) does not depend on the vector $y \in K$. For the case where the linear operator $\bar{t}$ depends on the vector $y \in K$, we have another type of vector variational inequalities.

Consider the generalized WVVI with the set-valued mapping T, which consists in finding $x^* \in K$, such that $\forall y \in K, \exists \bar{t}(y) \in T(\bar{x})$ satisfying

$$\langle \bar{t}(y), y - x^* \rangle \not\leq_{intC} 0. \tag{3.31}$$

Next, let us consider the gap function for the generalized WVVI(3.31). To this end, for $x \in K$, let

$$S_x = \{t | t : K \to T(x)\},$$

that is, S_x is the set of all operators t from K to $T(x)$.

Let $x \in K$ and $t \in S_x$. Then $t(y) \in T(x), \forall y \in K$. Define two mappings ϕ_1^* and ϕ^* as follows.

$$\phi_1^*(x,t) = \min_{y \in K} \max_{1 \le i \le \ell} ((\langle t(y), y - x \rangle)_i, \tag{3.32}$$

where $(\langle t(y), y \rangle)_i$ is the i-th component of $\langle t(y), y \rangle$, $i = 1, \cdots, \ell$, and

$$\phi^*(x) = \max\{\phi_1^*(x,t) | t \in S_x\} \tag{3.33}$$

We have the following result.

Theorem 3.63. $\phi^*(x)$ defined by (3.33) is a gap function of WVVI(3.31).

Proof. It is clear that $\phi_1^*(x,t) \le 0, \forall x \in K, t \in S_x$ and hence $\phi^*(x) \le 0, \quad \forall x \in K$.

Assume that x^* is a solution of WVVI(3.31). Let $y \in K$. Since x^* is a solution of WVVI(3.31), it follows that, for each $y \in K$, there is a $\bar{t}(y) \in T(x^*)$ such that

$$\langle \bar{t}(y), y - x^* \rangle \not\leq_{intC} 0$$

from which it follows that

$$\max_{1 \le i \le \ell} ((\langle \bar{t}(y), y - x^* \rangle)_i \ge 0.$$

Thus, an operator $\bar{t}$ from K into $T(x^*)$ has been defined. Then $\bar{t} \in S_{\bar{x}}$ and

$$\max_{1 \le i \le \ell} ((\langle \bar{t}(y), y - x^* \rangle)_i \ge 0, \quad \forall y \in K.$$

Hence

$$\phi_1^*(x^*, \bar{t}) = \min_{y \in K} \max_{1 \le i \le \ell} (\langle \bar{t}(y), y - x^* \rangle)_i \ge 0.$$

So $\phi_1^*(x^*, \bar{t}) = 0$. Also it is clear that, for any $t \in S_{x^*}$,

$$\max_{1 \le i \le \ell} (\langle t(y), x^* - x^* \rangle)_i = 0$$

from which it follows that $\phi_1^*(x^*, t) = 0$ and consequently $0 = \phi^*(x^*)$.

If $0 = \phi^*(x^*)$, then there exists $\bar{t} \in S_{x^*}$ such that $\phi_1^*(x^*, \bar{t}) = 0$. Thus

$$\min_{y \in K} \max_{1 \le i \le \ell} (\langle \bar{t}(y), y - x^* \rangle)_i = 0.$$

So we have, for any $y \in K$,

$$\max_{1 \le i \le \ell} (\langle \bar{t}(y), y - x^* \rangle)_i \ge 0.$$

Hence, for any $y \in K$,

$$\langle \bar{t}(y), y - x^* \rangle \not\leq_{intC} 0.$$

Therefore x^* is a solution of WVVI(3.31). ■

3.5 Stability of Generalized Set-valued Quasi-VVI

The study concerning the sensitivity and stability of variational inequalities is important because almost all variational inequalities are solved for a specified fixed set of data. Consequently, the computed solution could be considerably inaccurate or could even become infeasible when the data is subject to disturbances. In the last decade, the properties of continuity and Lipschitz continuity of the locally unique solution to parametric variational inequalities were investigated, and a global stability result was established for generalized quasivariational inequalities. For the case in which the solution set is not a singleton, the upper semicontinuity property of the solution set was obtained in Gong [92]. This section aims to establish some stability results for the solution set of a generalized vector quasivariational inequality. Under suitable conditions, we obtain that the solution set is closed and upper semicontinuous.

Let Z be a finite dimensional vector space. Let X and Y be two metric spaces. Let $C : X \rightrightarrows Z$ be a cone-valued function which induces a variable domination structure in Z. Let $K \subset X$, and let $S : K \rightrightarrows X$ and $F : K \rightrightarrows L(X, Z)$ be two set-valued functions. Further assume that

$$intC(x) \neq \emptyset, \quad x \in K.$$

The generalized vector quasivariational inequality problem is of finding $x^* \in S(x^*)$ and $z^* \in F(x^*)$ such that

$$(GVQVI) \qquad \langle z^*, y - x^* \rangle \not\leq_{intC(x^*)} 0, \quad \forall y \in S(x^*).$$

For a given generalized vector quasivariational inequality problem, we define the perturbed (GVQVI) problem as follows.

Let K and W be two nonempty, closed and convex subsets in X and Y, respectively. Let $S : K \times W \rightrightarrows K$ and $F : K \times W \rightrightarrows L(X, Z)$ be two set-valued functions. Let $C : K \times W \rightrightarrows Z$ be a family of domination structures such that $intC(x, w) \neq \varnothing$, for every $x \in K$ and $w \in W$. The perturbed (GVQVI) problem is : Find $x^* \in S(x^*, w)$ and $z^* \in F(x^*, w)$ such that

$$(GVQVI)_w \qquad \langle z^*, y - x^* \rangle \not\leq_{intC(x^*,w)} 0, \quad \text{for any } y \in S(x^*, w).$$

In order to prove the stability result for the (GVQVI) problem, we assume that the following hypothesis is satisfied.

Assumption 3.64. **(i)** *$S(x, w) \neq \varnothing$, for $x \in K$ and $w \in W$;*
(ii) *$F(x, w) \neq \varnothing$, for $x \in K$ and $w \in W$.*

For brevity, let us introduce the following notations:

(i) $G(x, w) \equiv \{(x, z) : z \in F(x, w)\}$, for $x \in K$ and $w \in W$;
(ii) $A(w) \equiv \cup_{x \in K} G(x, w)$, for every $w \in W$;
(iii) $A \equiv \cup_{w \in W} A(w)$;
(iv) for each $w \in W$,

$$I(w) = \{(x, z) \in A(w) : \langle z, y - x \rangle \not\leq_{intC(x,w)} 0 \text{ for any } y \in S(x, w)\}.$$

Note that $I(w)$ is the solution set of the $(\text{GVQVI})_w$ problem.

Lemma 3.65. *[5] Let X and Y be two metric spaces and $Q : X \rightrightarrows Y$ be a compact-valued function. Then Q is u.s.c. at x^* if and only if, for any $\epsilon > 0$, there exists an $\eta > 0$ such that*

$$Q(x) \subset B(Q(x^*), \epsilon), \text{ for each } x \in B(x, \eta),$$

where $B(x^*, \eta)$ denotes the ball with the center at x^* and the radius η and where

$$B(Q(x^*, \epsilon)) = \cup_{y \in Q(x^*)} B(y, \epsilon).$$

Lemma 3.66. *Let $H : X \rightrightarrows I\!R^\ell$ be a set-valued function. Suppose that H is l.s.c. at $x^* \in X$, that $H(x)$ is a convex set, and that $intH(x) \neq \varnothing$ for every $x \in N_\epsilon(x^*)$, where $N_\epsilon(x^*) \subset X$ is a neighborhood of x^*. Let $\{x_k\} \subset X$ and $\{y_k\} \subset I\!R^n$ be two sequences converging to x^* and y^*, respectively. If $y^* \in intH(x^*)$, then $y_k \in intH(x_k)$ except for a finite number of k's.*

Proof. Note that $y^* \in intH(x^*)$. Thus, there exists an $\epsilon > 0$ such that

$$y^* + B_\epsilon \subset H(x^*),$$

where B_ϵ denotes the closed ball in $\mathbb{R}^\ell$ with center at 0 and radius ϵ. Since $y_k \to y^*$, there exists a number $K > 0$ such that

$$||y_k - y^*|| < \epsilon/2, \quad \forall k \geq K.$$

Proceeding by contradiction, we assume that there exists an $N_0 > 0$, such that $y_k \notin intH(x_k)$ for $k \geq K_0$. Since $H(x)$ is a convex set and $intH(x) \neq \varnothing$ for all $x \in X_\epsilon(x^*)$, we may assume without loss of generality that $H(x_k)$ is a convex set and $intH(x_k) \neq \varnothing$ for $k \geq K_0$. Then, by the separation theorem for convex sets, there exists a vector $\alpha_k \in \mathbb{R}^\ell$ with $\|\alpha_k\| = 1$ such that

$$H(x_k) \subset \{y \in \mathbb{R}^\ell : \langle \alpha_k, y \rangle \leq \langle \alpha_k, y_k \rangle\}, \quad \text{for } k \geq K_0.$$

Let $K_1 = \max\{K, K_0\}$ and

$$\bar{y}_k = y^* + (\epsilon/2 + \langle \alpha_k, y_k - y^* \rangle)\alpha_k, \quad \text{for } k \geq K_1.$$

Note that

$$\begin{aligned} ||\bar{y}_k - y^*|| &= |\epsilon/2 + \langle \alpha_k, y_k - y^* \rangle| \cdot ||\alpha_k|| \\ &\leq \epsilon/2 + |\langle \alpha_k, y_k - y^* \rangle| \\ &\leq \epsilon/2 + ||\alpha_k|| \cdot ||y_k - y^*|| \\ &\leq \epsilon. \end{aligned}$$

Thus, we have

$$\bar{y}_k \in y^* + B_\epsilon, \text{ for } n \geq N_1.$$

Since

$$\langle \alpha_k, \bar{y}_k \rangle = \langle \alpha_k, y^* \rangle + (\epsilon/2) + \langle \alpha_k, y_k - y^* \rangle = \epsilon/2 + \langle \alpha_k, y_k \rangle,$$

and for each $y \in H(x_k)$

$$\begin{aligned} ||\bar{y}_k - y|| &\geq |\langle \alpha_k, \bar{y}_k - y \rangle| \\ &= |(\langle \alpha_k, \bar{y}_k \rangle - \langle \alpha_k, y_k \rangle) + (\langle \alpha_k, y_k \rangle - \langle \alpha_k, y \rangle)| \\ &> \epsilon/3, \end{aligned}$$

it follows that

$$\bar{y}_k \notin H(x_k) + B_{\epsilon/3}, \text{ for } k \geq K_1.$$

Since $y^* + B_\epsilon$ is a compact set, we can assume without loss of generality that $\bar{y}_k \to \bar{y}$. Obviously,

$$\bar{y} \in y^* + B_\epsilon \subset H(x^*).$$

Since H is l.s.c. at x^*, there exist a number $K_2 > 0$ and a sequence $\{\tilde{y}_k\}$ such that

$$\tilde{y}_k \in H(x_k), \text{ for all } k \geq K_2, \text{ and } \tilde{y}_k \to \bar{y}.$$

Then,

$$\bar{y}_k - \tilde{y}_k \to 0, \text{ as } k \to \infty.$$

However, this contradicts the fact that

$$\bar{y}_k \notin H(x_k) + B_{\epsilon/3}, \text{ for } k \geq K_1.$$

Thus, the number for which $y_k \notin intH(x_k)$ is finite. This completes the proof. ■

Theorem 3.67. *Let K be a closed set. For any given $w \in W$, suppose that the following conditions are satisfied:*

(i) *$S(\cdot, w)$ is l.s.c. on K;*
(ii) *$C(\cdot, w)$ is l.s.c. on K and $C(x, w)$ is a convex set for every $x \in K$;*
(iii) *$F(\cdot, w)$ is u.s.c. on K and $F(x, w)$ is a closed set for every $x \in K$.*

Then, the solution set $I(w)$ of the perturbed (GVQVI) problem is closed.

Proof. Without loss of generality, suppose that $w = 0$. Take any sequence $\{(x_k, z_k)\} \subset I(0)$ satisfying $(x_k, z_k) \to (x^*, z^*)$. By the closedness of K, it follows that $x^* \in K$. Then, by condition (iii), we have $z^* \in F(x^*, 0)$. Suppose that $(x^*, z^*) \notin I(0)$. Then, there exists $\bar{x} \in S(x^*, 0)$ such that

$$\langle z^*, \bar{x} - x^* \rangle \not\leq_{intC(x^*,0)} 0.$$

By the lower semicontinuity of $S(\cdot, 0)$, there exists an $\bar{x}_k \in S(x_k, \cdot)$ except for a finite number of $n's$ such that $\bar{x}_k \to \bar{x}$. Then, from the continuity of $f(x, y, z) \equiv \langle z, y - z \rangle$, we obtain

$$\langle z_k, \bar{x}_k - x_k \rangle \to \langle z^*, \bar{x} - x^* \rangle.$$

Thus, it follows from Lemma 3.66 that

$$\langle z_k, \bar{x}_k - x_k \rangle \not\leq_{intC(x_k)} 0,$$

except for a finite number of $n's$, which contradicts that $\{(x_k, y_k)\} \subset I(0)$ satisfying $(x_k, z_k) \to (x^*, z^*)$. This completes the proof. ■

Remark 3.68. Let

$$Z = \mathbb{R}, X = \mathbb{R}^n, W = \mathbb{R}^r, L(X, Z) = \mathbb{R}^n, C(x, w) = \mathbb{R}_+,$$

for $x \in K$ and $w \in W$. Then the problem $(\text{GVQVI})_w$ reduces to the classical perturbed generalized quasivariational inequality (in short, (GQVI)), which was considered in Tobin [195]. For the corresponding perturbed (GQVI), the following examples 3.69, 3.70, 3.71 show the necessity of conditions of Theorem 3.67.

Example 3.69. Let $K = [1,2], W = \mathbb{R}, C(x,w) = \mathbb{R}_+, S(x,w) = [1,2]$,

$$F(x,w) = \begin{cases} [1, 1.5), & x = 1, \\ [1, 1.2), & x \neq 1, \end{cases}$$

for each $x \in K$ and $w \in W$.

Note that, for each $w \in W$ and each $x \in K$, $F(x,w)$ is not a closed set. Clearly, for any given $w \in W$, the solution set $I(w) = \{(1,z) : z \in [1, 1.5)\}$ of the perturbed (GQVI) problem is not closed.

Example 3.70. Let $K = (-1,1), W = \mathbb{R}$ and

$$F(x,w) = 0, C(x,w) = \mathbb{R}_+, S(x,w) = (-1,1), \quad x \in K, w \in W.$$

Note that the set K is not closed as required in Theorem 3.67. Clearly, for any given $w \in W$, the solution set

$$I(w) = \{(x,0) : x \in (-1,1)\}$$

of the perturbed (GQVI) problem is not closed.

Example 3.71. Let $K = [1,2], W = \mathbb{R}, C(x,z) = \mathbb{R}_+, S(x,w) = [1,2]$,

$$F(x,w) = \begin{cases} [1, 2], & x = 1, \\ [1, 3], & x \neq 1, \end{cases}$$

for $x \in K$ and $w \in W$.

Note that, for any $w \in W$, $F(\cdot, w)$ is not u.s.c. at $x = 1$. Clearly, for any given $w \in W$, the solution set

$$I(w) = \begin{cases} \{(1,z) : z \in [1,2], x = 1\} \\ \{(x,z) : z \in [1,z], x \neq 1\}, \end{cases}$$

of the perturbed (GQVI) problem is not closed.

Theorem 3.72. *Let A be a compact set, and let K and W be closed sets. Suppose that the Assumption 3.64 holds and that the following conditions are satisfied:*

(i) *$S(\cdot,\cdot)$ is l.s.c. on $K \times W$;*
(ii) *$C(\cdot,\cdot)$ is l.s.c. on $K \times W$ and $C(x,w)$ is a convex set for each $(x,w) \in K \times W$;*
(iii) *$F(\cdot,\cdot)$ is u.s.c. on $K \times W$ and $F(x,w)$ is a closed set for each $(x,w) \in K \times W$.*

Then, the solution set $I(w)$ of the perturbed $(GVQVI)_w$ problem is u.s.c. on W.

Proof. Without loss of generality, we assume that $w = 0$. Let us establish the result by contradiction. Suppose that $I(w)$ is not u.s.c. at $0 \in W$. By the compactness of A and Theorem 3.67, $I(w)$ is a compact-valued function on W. Then, by Lemma 3.65, there exists an $\epsilon > 0$ such that, for any $1/k > 0$, $k = 1, 2, 3, \cdots$, we can find

$$\{w_k\} \subset B(0, 1/k), \text{ and } (x_k, z_k) \in I(w_k),$$

satisfying

$$\{(x_k, z_k)\} \not\subseteq B(I(0), \epsilon). \tag{3.34}$$

By the compactness of A, we can assume that $(x_k, z_k) \to (x^*, z^*)$. Then, from (3.34), we have

$$(x^*, z^*) \notin I(0). \tag{3.35}$$

Since $F(\cdot, \cdot)$ is u.s.c. on $K \times W$ and $F(x, w)$ is a closed set for each $(x, w) \in K \times W$, we obtain $z^* \in F(x^*, 0)$. Then $(x^*, z^*) \in A(0)$. Now, by (3.35), there exists a $y^* \in C(x^*, 0)$ such that

$$\langle z^*, y^* - z^* \rangle \in -intC(x^*, 0).$$

Then, for $y^* \in S(x^*, 0)$, it follows from the lower semicontinuity of $S(\cdot, \cdot)$ that there exits a sequence $\{y_k\}$ with $y_k \in S(x_k, w_k)$ such that $y_k \to y^*$, using the continuity of $f(x, y, z) = \langle z, y - x \rangle$, we obtain

$$\langle z_k, y_k - x_k \rangle \to \langle z^*, y^* - x^* \rangle.$$

By Lemma 3.66, except for a finite number of $n's$, we have

$$\langle z_k, y_k - x_k \rangle \in -intS(x_k, w_k).$$

This contradicts the fact that $(x_k, z_k) \in I(w_k)$, and Hence the proof is complete. ■

3.6 Existence of Solutions for Generalized Pre-VVI

In this section, we study the existence of solutions for two more general classes of vector variational inequalities.

Let X and Z be real locally convex Hausdorff topological vector spaces, and (Y, C) be an ordered locally convex Hausdorff topological vector space, ordered by a closed and convex cone C with nonempty interior $intC$. Let $K \subset X$ and $E \subset Z$ be nonempty subsets. Let $\eta : K \times K \to K$ be a vector-valued function. Assume that $V : K \rightrightarrows E$ and $\bar{K} : K \rightrightarrows K$ are two set-valued functions. Assume that $H : K \times E \to L(X, Y)$ is a vector-valued function.

Consider the following generalized vector pre-variational inequality problem (in short, GPVVI) of finding $\bar{x} \in K, \bar{z} \in V(\bar{x})$, s.t.

$$\langle H(\bar{x}, \bar{z}), \eta(y, \bar{x})\rangle \not\leq_{intC} 0, \quad \forall y \in K,$$

and the generalized vector pre-quasi-variational inequality problem (in short, GPQVVI) of finding $\bar{x} \in \bar{K}(\bar{x}), \bar{z} \in V(\bar{x})$, s.t.

$$\langle H(\bar{x}, \bar{z}), \eta(y, \bar{x})\rangle \not\leq_{intC} 0, \quad \forall y \in \bar{K}(\bar{x}).$$

These vector variational inequality problems arise from the optimality conditions for a vector optimization problem where the feasible set is so-called η-connected, see Yang [209]. When $\eta(x, y) = x - y$ and $H(x, z) = T(x), \forall z \in E$, (GPVVI) reduces to (WVVI). In the sequel, we will use the nonlinear scalar function ξ_e, $e \in intC$ (see Chapter 1):

$$\xi_e(y) = \min\{t \in \mathbb{R} : y \in te - C\}, \quad y \in Y.$$

The function ξ_e is continuous and strictly monotone. For a set $Q \subset Y$, let

$$\xi_e(Q) = \cup_{y \in Q} \xi_e(y).$$

It is well known that $F : X \to Y$ is a C-quasi-convex vector-valued function if and only if $\xi_e \circ F$ is a quasi-convex function on X, see Proposition 1.68.

Theorem 3.73. *Let K be a nonempty, compact and convex subset of X and E be a nonempty convex subset of Z. Let $\eta : K \times K \to K$ be a continuous set-valued function satisfying $\eta(x, x) = 0$, $\forall x \in K$, and let $V : X \rightrightarrows Z$ be a set-valued function which is upper semicontinuous, nonempty, closed and convex valued. Let $V^{-1}(z) = \{x \in X : z \in V(x)\}$ be open for each $z \in Z$. Let $H : K \times E \to L(X, Y)$ be a continuous vector-valued function. If there exists $e \in intE$ such that the function $\xi_e(\langle H(x, z), \eta(y, x)\rangle)$ is quasi-convex with respect to x, then there exists $\bar{x} \in K$, $\bar{z} \in V(\bar{x})$ such that*

$$\langle H(\bar{x}, \bar{z}), \eta(y, \bar{x})\rangle) \not\leq_{intC} 0, \quad \forall y \in K.$$

Proof. By the given conditions and the generalized Browder's selection theorem (Theorem 1.25), there exists a continuous vector-valued function $f : K \to Z$ such that $f(x) \in V(x)$, $\forall x \in K$.

Let

$$\phi(x, z, y) = \langle H(x, z), \eta(y, x)\rangle, \quad x \in K, z \in E, y \in K.$$

Then ϕ is a continuous vector-valued function from $K \times E \times K$ into Y. For simplicity, for any fixed e, we denote ξ_e by ξ.

Now, for $k = 1, 2, \cdots$, we define a set-valued function $F_k : K \rightrightarrows K$ by

$$F_k(x) = \{y \in K : (\xi \circ \phi)(x, f(x), y) - \min_{u \in K}(\xi \circ \phi)(x, f(x), u) < \frac{1}{k}\},$$

where $x \in K$. Since K is a compact subset and ξ is continuous, for any $x \in K$, $(\xi \circ \phi)(x, f(x), y)$ is a continuous function in y. It follows that $(\xi \circ \phi)(x, f(x), K)$

is a compact set in $\mathbb{R}$, and hence $F_k(x)$ is nonempty for all k.

Under the conditions of Theorem 3.73, we show that the following Lemmas hold which are parts of the proof for Theorem 3.73.

Lemma 3.74. *For every k, x, $F_k(x)$ is a convex set.*

Proof. Let $y_1, y_2 \in F_k(x)$ and $\lambda \in (0,1)$. Then $y_1, y_2 \in K$ and there exist $\alpha_i > 0, i = 1,2$ such that

$$(\xi \circ \phi)(x, f(x), y_i) = \min_{u \in K}(\xi \circ \phi)(x, f(x), u) + \frac{1}{k} - \alpha_i, \quad i = 1, 2.$$

Let $\alpha_0 = \min\{\alpha_1, \alpha_2\}$ and $t_0 = \min_{u \in K}(\xi \circ \phi)(x, f(x), u) + \frac{1}{k} - \alpha_0$. Then

$$(\xi \circ \phi)(x, f(x), y_i) \leq t_0, \quad i = 1, 2.$$

Hence,

$$y_i \in \mathrm{Lev}_{(\xi \circ \phi)(x, f(x), \,\cdot\,)}(t_0) = \{u \in K : (\xi \circ \phi)(x, f(x), u) \leq t_0\}.$$

By the assumption that $(\xi \circ \phi)(x, f(x), \cdot)$ is quasi-convex, the set $\mathrm{Lev}_{(\xi \circ \phi)}(x, f(x), \cdot)(t_0)$ is a convex set. Then $\lambda y_1 + (1-\lambda)y_2 \in \mathrm{Lev}_{(\xi \circ \phi)(x, f(x), \cdot)}(t_0)$, that is

$$(\xi \circ \phi)(x, f(x), \lambda y_1 + (1-\lambda)y_2) \leq t_0.$$

Therefore we have that

$$(\xi \circ \phi)(x, f(x), \lambda y_1 + (1-\lambda)y_2) - \min_{u \in K}(\xi \circ \phi)(x, f(x), u) < \frac{1}{k}.$$

Then $\lambda y_1 + (1-\lambda)y_2 \in F_k(x)$. ■

Let for each $y \in K$

$$\begin{aligned} F_k^{-1}(y) &= \{x \in K : y \in F_k(x)\} \\ &= \{x \in K : (\xi \circ \phi)(x, f(x), y) - \min_{u \in K}(\xi \circ \phi)(x, f(x), u) < \frac{1}{k}\}. \end{aligned}$$

Lemma 3.75. *For each $y \in K$, the set $F_k^{-1}(y)$ is open.*

Proof. Suppose that the conclusion is not true. Then there exist $x_0 \in F_k^{-1}(y)$ and a net $\{x_\alpha\}$ such that $x_\alpha \to x_0$, $x_\alpha \notin F_k^{-1}(y)$ for every α, namely, $y \notin F_k(x_\alpha)$. Therefore

$$(\xi \circ \phi)(x_\alpha, f(x_\alpha), y) - \min_{u \in K}(\xi \circ \phi)(x_\alpha, f(x_\alpha), u) \geq \frac{1}{k}.$$

Since the function $(x, y) \to (\xi \circ \phi)(x, f(x), y)$ is continuous and K is compact, by a well known result in mathematical programming, the function $M : x \to \min_{u \in K}(\xi \circ \phi)(x, f(x), u)$ is continuous in x. Therefore we have

$$(\xi \circ \phi)(x_0, f(x_0), y) - \min_{u \in K}(\xi \circ \phi)(x_0, f(x_0), u) \geq \frac{1}{k},$$

which contracts $y \in F_k(x_0)$. Hence $F_k^{-1}(y)$ is open. ■

Hence we have shown that all assumptions in Theorem 1.26 are satisfied. Then there exists $x_k \in K$ such that

$$x_k \in F_k(x_k), \quad k = 1, 2, \cdots.$$

Since K is compact and V is u.s.c., we can assume that $x_k \to \bar{x} \in K$ and $f(x_k) \to f(\bar{x}) \in V(\bar{x})$. Then by the definition of F_k we have for each $k = 1, 2, \cdots$,

$$(\xi \circ \phi)(x_k, f(x_k), x_k) < \min_{u \in K}(\xi \circ \phi)(x_k, f(x_k), u) + \frac{1}{k},$$

and as $k \to +\infty$

$$(\xi \circ \phi)(\bar{x}, f(\bar{x}), \bar{x}) \leq \min_{u \in K}(\xi \circ \phi)(\bar{x}, f(\bar{x}), u).$$

Since $\eta(\bar{x}, \bar{x}) = 0$ and $\xi(0) = 0$,

$$\min_{u \in K}(\xi \circ \phi)(\bar{x}, f(\bar{x}), u) = 0.$$

That is

$$(\xi \circ \phi)(\bar{x}, f(\bar{x}), u) \geq 0, \quad \forall u \in K$$

By the strict monotonicity of ξ, we have

$$\phi(\bar{x}, f(\bar{x}), u) \notin -intC, \quad \forall u \in K.$$

Let $\bar{z} = f(\bar{x}) \in V(\bar{x})$. Then

$$\langle H(\bar{x}, \bar{z}), \eta(y, \bar{x}) \rangle \notin -intC, \quad \forall u \in K.$$

■

As a direct result, we have:

Corollary 3.76. *Let K be a nonempty, compact and convex subset of X. Let $T : K \to L(X, Y)$ be a continuous vector-valued function. If there exists $e \in intC$ such that the function $\xi_e(\langle T(x), y - x \rangle)$ is quasi-convex with respect to y, then there exists $\bar{x} \in K$ such that*

$$\langle T(\bar{x}), y - \bar{x} \rangle \not\leq_{intC} 0, \quad \forall y \in K.$$

Proof. Assume that Z is a Hausdorff topological vector space. Define $V(x) = Z, x \in X$ and $\eta(y,x) = y - x$. Then $V^{-1}(u) = X$ is open for any $u \in Z$. Let $H(x,z) = T(x)$, if $x \in K, z \in Z$. The result follows from Theorem 3.73. ■

Now we consider the following generalized vector pre-quasi-variational inequality problem of finding $\bar{x} \in \bar{K}(\bar{x})$ and $\bar{z} \in V(\bar{x})$ such that

$$(GPQVVI) \qquad \langle H(\bar{x},\bar{z}), \eta(y,\bar{x})\rangle \not\leq_{intC} 0, \quad \forall y \in \bar{K}(\bar{x}).$$

We have the following result.

Theorem 3.77. *Let K and E be two nonempty compact and convex subsets of X and Z, respectively. Let $V : X \rightrightarrows Z$ be an upper semicontinuous, closed, convex and vector-valued function. Let $\eta : K \times K \to K$ be a continuous vector-valued function such that $\eta(x,x) = 0$, $\forall x \in K$. Let $H : K \times E \to L(X,Y)$ be a continuous vector-valued function. Let $\bar{K} : K \rightrightarrows K$ be a continuous set-valued function with a compact-valued $\bar{K}(x)$. If there exist $e \in intC$ such that the function $\xi_e(\langle H(x,z), \eta(y,x)\rangle) : K \to I\!R$ is quasi-convex with respect to y, then there exist $\bar{x} \in \bar{K}(\bar{x})$, $\bar{z} \in V(\bar{x})$ such that*

$$\langle H(\bar{x},\bar{z}), \eta(y,\bar{x})\rangle \not\leq_{intC} 0, \quad \forall y \in \bar{K}(\bar{x}).$$

Proof. For simplicity, for any fixed e, we denote still ξ_e by ξ. As in the proof of Theorem 3.73, let

$$\phi(x,z,y) = -\langle H(x,z), \eta(y,x)\rangle, \quad x \in K, z \in E, y \in K.$$

Then ϕ is a continuous vector-valued function. Define

$$M(x,z) = \mathrm{Min}_{intC}\{\phi(x,z,y) : y \in \bar{K}(x)\}, \quad x \in K, z \in K.$$

Then $M : K \times E \rightrightarrows Y$ is a set-valued function. ■

Under the conditions of Theorem 3.77, we show that the following lemmas hold which are parts of the proof for Theorem 3.77.

Lemma 3.78. *M is a closed set-valued function.*

Proof. We show that $\mathrm{Gr}(M)$ is closed. Let a net $(x_\alpha, z_\alpha, y_\alpha) \in \mathrm{Gr}(M)$ converge to (x_0, z_0, y_0). Note that $(x_\alpha, z_\alpha, y_\alpha) \in \mathrm{Gr}(M)$ is equivalent to $y_\alpha \in M(x_\alpha, z_\alpha) = \mathrm{Min}_{intC}\{\phi(x_\alpha, z_\alpha, y) : y \in \bar{K}(x_\alpha)\}$. Then there exists $u_\alpha \in \bar{K}(x_\alpha)$ such that $y_\alpha = \phi(x_\alpha, z_\alpha, u_\alpha)$. By the compactness of K, we can assume, without loss of generality, that $u_\alpha \to u_0 \in K$. Then, by the continuity of ϕ,

$$y_0 = \phi(x_0, z_0, u_0).$$

Suppose that $y_0 \notin M(x_0, z_0)$. Then, there exist $u' \in \bar{K}(x_0)$ and $p \in C$ such that $y_0 = \phi(x_0, z_0, u') + p$. Since $\bar{K}$ is l.s.c., there exist $u'_\alpha \in \bar{K}(x_\alpha), \forall\alpha$ such that $u'_\alpha \to u'$. Then

$$\phi(x_\alpha, z_\alpha, u'_\alpha) - y_\alpha = -p + (y_0 - y_\alpha) + \phi(x_\alpha, z_\alpha, u'_\alpha) - \phi(x_0, z_0, u'),$$

and for α large enough,

$$\phi(x_\alpha, z_\alpha, u'_\alpha) - y_\alpha \leq_{intC} 0,$$

which is a contradiction to the assumption $y_\alpha \in M(x_\alpha, z_\alpha)$. Thus M is a closed function. ■

Now we define a set-valued function $\nabla : K \times E \rightrightarrows K$ by

$$\nabla(x, z) := \{u \in \bar{K}(x) : \xi \circ \phi(x, z, u) = \min_{v \in \bar{K}(x)} \xi \circ \phi(x, z, v)\}.$$

It is clear that $\nabla(x, z)$ is nonempty for any $(x, z) \in K \times E$.

Lemma 3.79. *For each (x, z), $\nabla(x, z)$ is a closed function.*

Proof. Let $\{(x_\alpha, z_\alpha, u_\alpha)\}$ be a net in $\mathrm{Gr}(\nabla)$ such that

$$(x_\alpha, z_\alpha, u_\alpha) \to (x_0, z_0, u_0).$$

Then $u_\alpha \in \bar{K}(x_\alpha)$ and

$$\xi \circ \phi(x_\alpha, z_\alpha, u_\alpha) = \min_{v \in \bar{K}(x_\alpha)} \xi \circ \phi(x_\alpha, z_\alpha, v).$$

By the continuity of $\bar{K}$, $u_0 \in \bar{K}(x_0)$. By the strict monotonicity of ξ, it is easy to verify that

$$\phi(x_\alpha, z_\alpha, u_\alpha) \in \mathrm{Min}_{intC}\{\phi(x_\alpha, z_\alpha, v) : v \in \bar{K}(x_\alpha)\}.$$

Since $M(x, z)$ is a closed function,

$$\phi(x_0, z_0, u_0) \in \mathrm{Min}_{intC}\{\phi(x_0, z_0, v) : v \in \bar{K}(x_0)\}.$$

Now we show that

$$\xi \circ \phi(x_0, z_0, u_0) = \min_{v \in \bar{K}(x_0)} \xi \circ \phi(x_0, z_0, v).$$

Since $\xi \circ \phi(x, z, \bar{K}(x))$ is l.s.c., there exists $t_k \in \mathbb{R}$ such that $t_k = \xi \circ \phi(x_k, z_k, v_k)$ and

$$t_k \to \min_{v \in \bar{K}(x_0)} \xi \circ \phi(x_0, z_0, v).$$

Suppose that

$$\xi \circ \phi(x_0, z_0, u_0) \neq \min_{v \in \bar{K}(x_0)} \xi \circ \phi(x_0, z_0, v).$$

Then

$$\xi \circ \phi(x_0, z_0, u_0) > \min_{v \in \bar{K}(x_0)} \xi \circ \phi(x_0, z_0, v).$$

Therefore, for k large enough, we have that

$$\xi \circ \phi(x_k, z_k, v_k) > t_k,$$

which is a contradiction. Hence, Lemma 3.79 holds. ■

Lemma 3.80. *For each (x, z), $\nabla(x, z)$ is a convex set.*

Proof. Let $u_1, u_2 \in \nabla(x, z)$ and $\lambda \in (0, 1)$. Let

$$r_0 = \min_{v \in \bar{K}(x)} \xi \circ \phi(x, z, v).$$

Then

$$\xi \circ \phi(x, z, u_1) = \xi \circ \phi(x, z, u_2) = r_0.$$

Since $\bar{K}(x)$ is a convex set, $\lambda u_1 + (1 - \lambda)u_2 \in \bar{K}(x)$, by the assumption, the function $u \to \xi \circ \phi(x, z, u)$ is quasi-convex. Then the set

$$A = \{u \in K : \xi \circ \phi(x, z, u) \leq r_0\}$$

is convex and since $u_1, u_2 \in A$, $\lambda u_1 + (1 - \lambda)u_2 \in A$. Then

$$\xi \circ \phi(x, z, \lambda u_1 + (1 - \lambda)u_2) \leq r_0.$$

Noting that $\bar{K}(x)$ is convex, we have

$$\xi \circ \phi(x, z, \lambda u_1 + (1 - \lambda)u_2) = r_0.$$

Then $\lambda u_1 + (1 - \lambda)u_2 \in \nabla(x, z)$. ■

Now we complete the proof of Theorem 3.77. Since V is u.s.c., $V(K)$ is a compact set. Let $W : K \times E \rightrightarrows K \times E$:

$$W(x, z) = \nabla(x, z) \times V(x), \quad (x, z) \in K \times E.$$

Then W is upper semicontinuous and for each $x \in K$, $z \in E$, $W(x, z)$ is a nonempty, closed and convex subset. By Fan-Glicksberg-Kakustani theorem, there exists $(\bar{x}, \bar{z}) \in W(\bar{x}, \bar{z})$. Hence

$$\bar{x} \in \bar{K}(\bar{x}), \quad \xi \circ \phi(\bar{x}, \bar{z}, \bar{x}) = \min_{v \in \bar{K}(\bar{x})} \xi \circ \phi(\bar{x}, \bar{z}, v),$$

and $\bar{z} \in V(\bar{x})$. By the strict monotonicity of ξ, we have

$$\phi(\bar{x}, \bar{z}, \bar{x}) \in \mathrm{Min}_{intC}\{\phi(\bar{x}, \bar{z}, v) : v \in \bar{K}(\bar{x})\}.$$

Hence, for any $y \in \bar{K}(\bar{x})$,

$$\phi(\bar{x}, \bar{z}, y) - \phi(\bar{x}, \bar{z}, \bar{x}) \not\geq_{intC} 0.$$

Since $\eta(x, x) = 0$, $\phi(\bar{x}, \bar{z}, \bar{x}) = 0$, we have

$$\phi(\bar{x}, \bar{z}, y) \not\geq_{intC} 0, \quad \forall y \in \bar{K}(\bar{x}),$$

i.e.,

$$\langle H(\bar{x}, \bar{z}), \eta(y, \bar{x})\rangle \not\leq_{intC} 0, \quad \forall y \in \bar{K}(\bar{x}).$$

■

3.7 Existence of Solutions for Equilibrium Problems

The equilibrium problem is a generalization of variational inequalities. It contains many important mathematical models as special cases, for instance, optimization problems, problems of Nash equilibrium, variational inequalities, complementarity problems and fixed point problem, etc (See Blum and Oettli [18]). It is known that a vector equilibrium problem includes vector optimization, vector variational inequality, vector complementarity problems as special cases. In this section, we consider two kinds of vector equilibrium problems with a variable domination structure, and establish the existence of solutions for these problems.

Let Y be a Hausdorff topological vector space, and let X and Z be nonemtpy subsets of two Hausdorff topological vector spaces, respectively. Let $K : X \rightrightarrows X$, $T : X \rightrightarrows Z$ and $C : X \rightrightarrows Y$ be set-valued functions with nonempty values. Let $f : X \times Z \times X \rightrightarrows Y$ be a set-valued function.

Consider the following vector equilibrium problem of finding $(\bar{x}, \bar{z}) \in X \times Z$ such that $\bar{x} \in K(\bar{x}), \bar{z} \in T(\bar{x})$ and

$$(VEQ_1) \qquad f(\bar{x}, \bar{z}, u) \not\leq_{intC(\bar{x})} 0, \quad \forall u \in K(\bar{x}).$$

First we present a lemma.

Lemma 3.81. *[5] Assume that X_1 and Y_1 are Hausdorff topological spaces and Z_1 is a compact topological space. Let $f : X_1 \times Z_1 \to Y_1$ be a vector-valued function. Define a set-valued function $F : X_1 \rightrightarrows Y_1$ by*

$$F(x) = \{f(x, z) : z \in Z_1\}, \quad x \in X_1.$$

If f is continuous on $X_1 \times Z_1$, then F is u.s.c.

We have the following existence result for the problem (VEQ_1).

Theorem 3.82. *Let X and Z be nonempty compact convex sets of two locally convex Hausdorff topological vector spaces, respectively. Suppose that*

(i) $K : X \rightrightarrows X$ *is a set-valued function with nonempty closed convex values and open lower sections;*
(ii) $T : X \rightrightarrows Z$ *is upper semicontinuous with nonempty closed acyclic values;*
(iii) $C : X \rightrightarrows Y$ *is a set-valued function, which satisfies the following conditions:*
  (a) *for any* $u \in X$, *the set* $\{(x, z) \in X \times Z : f(x, z, u) \leq_{intC(x)} 0\}$ *is open in* $X \times X$;
  (b) *for any* $z \in Z$, *the set-valued function* $f(\cdot, z, \cdot)$ *is weakly type II C-diagonally quasi-convex in the third argument.*

Then there exists $(\bar{x}, \bar{z}) \in X \times Z$ *such that* $\bar{x} \in K(\bar{x})$, $\bar{z} \in T(\bar{x})$,

$$f(\bar{x}, \bar{z}, u) \not\leq_{intC(\bar{x})} 0, \quad \forall u \in K(\bar{x}).$$

Proof. Let us define a set-valued function $P : X \times Z \rightrightarrows X$ by

$$P(x, z) = \{u \in X : f(x, z, u) \leq_{intC(x)} 0\}, \quad \forall (x, z) \in X \times Z.$$

The theorem will be proven if we can show that there exists $(\bar{x}, \bar{z}) \in X \times Z$ satisfying $\bar{x} \in K(\bar{x})$, $\bar{z} \in T(\bar{x})$ and $K(\bar{x}) \cap P(\bar{x}, \bar{z}) = \varnothing$. To this end, we first show that for each $(x, z) \in X \times Z$,

$$x \notin coP(x, z). \tag{3.36}$$

If not, there would exist $(x^*, z^*) \in X \times Z$ such that $x^* \in P(x^*, z^*)$. That is, there exists a finite subset $\{x_1, x_2, \cdots, x_k\} \subset P(x^*, z^*)$ such that $x^* \in co\{x_1, x_2, \cdots, x_k\}$. Therefore, we have $f(x^*, z^*, x_i) \subset -intC(x^*)$ for $i = 1, 2, \cdots, n$, contradicting (b). Hence (3.36) holds.

Now we define another set-valued function $G : X \times Z \rightrightarrows X$ by

$$G(x, z) = K(x) \cap coP(x, z), \quad \forall (x, z) \in X \times Z.$$

For each $u \in X$, the set $\{(x, z) \in X \times Z : f(x, z, u) \subset -intC(x)\}$ is open by hypothesis (a), or equivalently the set $P^{-1}(u)$ is open for each $u \in X$, and so P has an open lower section. Since K has an open lower section by hypothesis (i), the set-valued function G has also an open lower section by Proposition 1.23.

Next let $U = \{(x, z) \in X \times Z : G(x, z) \neq \varnothing\}$. We have two cases to consider.

Case 1: $U = \varnothing$.
We note that in this case $K(x) \cap coP(x, z) = \varnothing$ for each $(x, z) \in X \times Z$. In particular, for each $(x, z) \in X \times Z$ we have

$$K(x) \cap P(x, z) = \varnothing.$$

By hypothesis (i) and the fact that X is a compact convex set, use Browder's fixed point theorem , see Theorem 1.26, to assert the existence of a fixed point $\bar{x} \in K(\bar{x})$. Also from (ii) we have $T(\bar{x}) \neq \varnothing$. So, picking $\bar{z} \in T(\bar{x})$, we have

$$K(\bar{x}) \cap P(\bar{x}, \bar{z}) = \varnothing.$$

Therefore, in this particular case the assertion of the theorem holds.

Case: 2: $U \neq \varnothing$.

The fact that $G : X \times Z \rightrightarrows X$ has open lower sections in $X \times Z$ and $U = \cup_{v \in X} G^{-1}(v)$ implies that U is open. We now define a set-valued function $H : X \times Z \rightrightarrows X$ by

$$H(x, z) = \begin{cases} G(x, z), & \text{if } (x, z) \in U \\ K(x), & \text{if } (x, z) \in X \times Y \backslash U. \end{cases}$$

Then, for each $v \in X$, we have

$$H^{-1}(v) = G^{-1}(v) \cup (K^{-1}(v) \times Z),$$

which is open, and whence H has an open lower section. It follows from the generalized Browder's selection theorem (see Theorem 1.25) that there exists a continuous selection $h : X \times Z \to X$ for H. Now consider the set-valued function $M : X \times Z \rightrightarrows X \times Z$ given by

$$M(x, z) = (h(x, z), T(x)),$$

which clearly has nonempty closed acyclic values. If we show that M is upper semicontinuous, then, by Theorem 1.29, M has a fixed point $(\bar{x}, \bar{z}) \in M(\bar{x}, \bar{z})$. Moreover, $(\bar{x}, \bar{z}) \notin U$. Suppose to the contrary that $(\bar{x}, \bar{z}) \in U$. Then

$$\bar{x} = h(\bar{x}, \bar{z}) \in G(\bar{x}, \bar{z}) = K(\bar{x}) \cap coP(\bar{x}, \bar{z}),$$

so that $\bar{x} \in coP(\bar{x}, \bar{z})$. But this is a contraction to (3.36). Hence $(\bar{x}, \bar{z}) \notin U$. Therefore,

$$(\bar{x}, \bar{z}) \in K(\bar{x}) \times T(\bar{x}) \text{ and } G(\bar{x}, \bar{z}) = \varnothing.$$

Thus $\bar{x} \in K(\bar{x})$, $\bar{z} \in T(\bar{x})$ and $K(\bar{x}) \cap coP(\bar{x}, \bar{z}) = \varnothing$. In particular, $K(\bar{x}) \cap P(\bar{x}, \bar{z}) = \varnothing$. So the assertion of the theorem also holds in this case.

It remains to prove that $M : X \times Z \rightrightarrows X \times Z$ is upper continuous. We observe that $X \times Z$ is compact, $h : X \times Z \to X$ is continuous, and $T : X \rightrightarrows Z$ is upper semi-continuous. By Lemma 3.81, M is upper semicontinuous. This completes the proof.

■

Corollary 3.83. *Let X and Z be nonempty compact convex subsets in two locally convex Hausdorff topological vector spaces, respectively. Let Y be a Hausdorff topological vector space and $C \subset Y$ be a given subset with $intC \neq \varnothing$. Assume:*

(i) *$K : X \rightrightarrows X$ is a set-valued function having nonempty closed convex values and open lower sections;*
(ii) *$T : X \rightrightarrows Z$ is an upper semicontinuous set-valued function with nonempty closed acyclic values; and*
(iii) *$f : X \times Z \times X \rightrightarrows Y$ is a set-valued function satisfying the following conditions:*
  (a) for each $u \in X$, $f(x, z, u)$ is lower semicontinuous in (x, z);
  (b) for each $z \in Z$, $f(\cdot, z, \cdot)$ is weakly type II C-diagonally quasi-convex in the third argument.

Then, there exists $(\bar{x}, \bar{z}) \in X \times Z$ such that $\bar{x} \in K(\bar{x})$, $\bar{z} \in T(\bar{x})$, and

$$f(\bar{x}, \bar{z}, u) \not\leq_{intC} 0, \quad \forall u \in K(\bar{x}).$$

Proof. We shall invoke Corollary 3.83 to prove this corollary. Let us set $C(x) = C$ for all $x \in X$. The lower semicontinuity of $f(x, z, u)$ in (x, z) implies that the set

$$\{(x, z) \in X \times Z : f(x, z, u) \leq_{intC} 0\}$$

is open in $X \times Z$ for any $u \in X$. Thus all conditions of Theorem 3.82 are satisfied and the corollary follows immediately. ■

Corollary 3.84. *Let X be a nonempty, compact and convex subset of a locally convex Hausdorff topological vector space. Assume:*

(i) *$C : X \rightrightarrows Y$ is a set-valued function with $intC(x) \neq \emptyset$ for all $x \in X$;*
(ii) *$F : X \times X \rightrightarrows Y$ is a set-valued function satisfying the following conditions:*
  (a) for each $y \in X$, the set $\{x \in X : F(x, y) \leq_{intC(x)} 0\}$ is open in X;
  (b) $F(x, y)$ is weakly type II C-diagonally quasi-convex in y.

Then there exists $\bar{x} \in X$ such that

$$F(\bar{x}, y) \not\leq_{intC(\bar{x})} 0, \quad \forall y \in X.$$

Proof. This corollary will be proven by invoking Theorem 3.82. To this end, let $Z = \{\bar{z}\}$ be a singleton. Let $K(x) = X$, $T(x) = \{\bar{z}\}$ for every $x \in X$, and let $f(x, z, y) = F(x, y)$ for all $(x, z, y) \in X \times Z \times X$. Then all the conditions of Theorem 3.82 are satisfied and the corollary follows immediately. ■

Theorem 3.85. *Assume all the hypotheses of Theorem 3.82 are satisfied except for the compactness of the sets X and Z. Suppose further that $A \subset X$ and $D \subset Z$ are two nonempty compact convex subsets and $B \subset A$ is a nonempty subset such that*

(a)′ $K(B) \subset A$;
(b)′ for each $x \in A$, the set $(K(x) \cap A) \times (T(x) \cap B)$ is nonempty;
(c)′ for each $x \in A \backslash B$, there exists $u \in K(x) \cap A$ such that

$$f(\bar{x}, z, u) \leq_{intC(x)} 0, \forall z \in T(x).$$

Then there exists $(\bar{x}, \bar{z}) \in B \times D$ *such that* $\bar{x} \in K(\bar{x})$, $\bar{z} \in T(\bar{x})$ *and*

$$f(\bar{x}, \bar{z}, u) \not\leq_{intC(\bar{x})} 0, \quad \forall u \in K(\bar{x}).$$

Proof. Define $G : A \rightrightarrows A$ by

$$G(x) = K(x) \cap A, \quad x \in A.$$

For each $x \in A$, $G(x)$ is nonempty and convex by $(b)'$ and G has open lower sections. Define a set-valued function $M : A \rightrightarrows D$ by

$$M(x) = T(x) \cap D, \quad x \in A.$$

It is easy to see that the set-valued function M is also upper semi-continuous with closed acyclic values. Note that, by $(a)'$ we have

$$G(x) = \begin{cases} K(x), & \text{if } x \in B, \\ K(x) \cap A, & \text{otherwise.} \end{cases}$$

It follows from Theorem 3.82 that there exists $(\bar{x}, \bar{z}) \in G(\bar{x}) \times M(\bar{x})$ such that

$$f(\bar{x}, \bar{z}, u) \not\leq_{intC(\bar{x})} 0, \quad \forall u \in G(\bar{x}). \tag{3.37}$$

We claim that $\bar{x} \in B$. If not, $\bar{x} \in A \backslash B$, then, by $(c)'$, there would exist a point $y \in K(\bar{x}) \cap A = G(\bar{x})$ such that

$$f(\bar{x}, z, y) \leq_{intC(\bar{x})} 0, \quad \forall z \in T(\bar{x}).$$

Thus,

$$f(\bar{x}, \bar{z}, y) \leq_{intC} 0,$$

contradicting (3.37). So $\bar{x} \in B$ such that $G(\bar{x}) = K(\bar{x})$. Since $\bar{z} \in M(\bar{x}) = T(\bar{x}) \cap D \subset T(\bar{x})$, we conclude that $\bar{x} \in K(\bar{x})$, $\bar{z} \in T(\bar{x})$ and

$$f(\bar{x}, \bar{z}, u) \not\leq_{intC(\bar{x})} 0, \quad \forall u \in K(\bar{x}).$$

The theorem is proved. ■

Theorem 3.86. *Assume that all the hypotheses of Theorem 3.82 are satisfied, except that the condition (iv) is replaced by*

$(iv)'$ $f : X \times Z \times X \rightrightarrows Y$ *satisfies:*
- (a) $\forall u \in X$, *the set* $\{(x, z) \in X \times Z : f(x, z, u) \cap -intC(x) \neq \varnothing\}$ *is open in* $X \times Z$*;*
- (b) $\forall z \in Z$, $f(\cdot, z, \cdot)$ *is weakly type I C-diagonally quasi-convex in the third argument.*

Then there exists $(\bar{x}, \bar{z}) \in X \times Z$ *such that* $\bar{x} \in K(\bar{x}), \bar{z} \in T(\bar{x})$ *and*

$$f(\bar{x}, \bar{z}, u) \not\leq_{intC(\bar{x})} 0, \quad \forall u \in K(\bar{x}).$$

Proof. We proceed as in the proof of Theorem 3.82. But we need to modify the set-valued function $P : X \times Z \rightrightarrows Z$ to be

$$P(x, z) = \{u \in X : f(x, z, u) \cap -intC(x) \neq \varnothing\}, \quad \forall (x, z) \in X \times Z.$$

Then it is easy to show that $x \notin co(P(x, z))$ is valid for all $(x, z) \in X \times Z$ due to the fact that $f(\cdot, z, \cdot)$ is weakly type I C-diagonally quasi-convex in the third argument. The rest of the proof is similar to that of Theorem 3.82. ■

Theorem 3.87. *Assume that all the hypotheses of Theorem 3.82 are satisfied, except that the condition (iv) is replaced by*

(iv)'' $f : X \times Z \times X \rightrightarrows Y$ *satisfies:*
- *(a)* $\forall u \in X,$ *the set* $\{(x, z) \in X \times Z : f(x, z, u) \cap C(x) = \varnothing\}$ *is open in* $X \times Z$*;*
- *(b)* $\forall z \in Z, f(\cdot, z, \cdot)$ *is strongly type II C-diagonally quasi-convex in the third argument.*

Then there exists $(\bar{x}, \bar{z}) \in X \times Z$ *such that* $\bar{x} \in K(\bar{x}), \bar{z} \in T(\bar{x})$ *and*

$$f(\bar{x}, \bar{z}, u) \cap C(\bar{x}) \neq \varnothing, \quad \forall u \in K(\bar{x}).$$

Theorem 3.88. *Assume that all the hypotheses of Theorem 3.82 are satisfied, except that the condition (iv) is replaced by*

(iv)''' $f : X \times Z \times X \rightrightarrows Y$ *satisfies:*
- *(a)* $\forall u \in X,$ *the set* $\{(x, z) \in X \times Z : f(x, z, u) \subset C(x) \neq \varnothing\}$ *is open in* $X \times Z$*;*
- *(b)* $\forall z \in Z, f(\cdot, z, \cdot)$ *is strongly type I C-diagonally quasi-convex in the third argument.*

Then there exists $(\bar{x}, \bar{z}) \in X \times Z$ *such that* $\bar{x} \in K(\bar{x}), \bar{z} \in T(\bar{x})$ *and*

$$f(\bar{x}, \bar{z}, u) \subset C(\bar{x}), \quad \forall u \in K(\bar{x}).$$

The proof of Theorem 3.87, as well as Theorem 3.88, is similar to that of Theorem 3.82 and is therefore omitted.

3.8 Vector Complementarity Problems (VCP)

The concept of vector complementarity problems was introduced in Chen and Yang [40]. Relations among vector complementarity problems, vector variational inequalities, vector optimization problems and minimal element problems were obtained. The existence of solutions for (positive) vector complementarity problems was also derived in [40, 208]. Further results on the existence of solutions for other type of vector complementarity problems can be found in Fu [74], Yu and Yao [222].

Let (X, D) and (Y, C) be two ordered Banach spaces. We assume that the interior $intC$ of the ordering cone C is nonempty and C is closed and convex.

The weak C-dual cone D_C^{w+} of D is defined by

$$D_C^{w+} = \{g \in L(X, Y) : \langle g, x \rangle \not\leq_{intC} 0, \quad \forall x \in D\}.$$

The strong C-dual cone D_C^{s+} of D is defined by

$$D_C^{s+} = \{g \in L(X, Y) : \langle g, x \rangle \geq_C 0, \quad \forall x \in D\}.$$

It is obvious that D_C^{w+} and D_C^{s+} are nonempty, since the null linear function in $L(X, Y)$ belongs to D_C^{w+} and D_C^{s+}. It is easy to prove that $D_C^{s+} \subset D_C^{w+}$ if C is pointed.

When $Y = \mathbb{R}$, the weak and strong C-dual cones of D reduce to the dual cone D^* of D.

We will prove that the weak and strong C-dual cones of D are algebraically closed and the strong C-dual cone of D is convex.

Definition 3.89. *Let X be a linear space and A be a subset of X.*

(i) *The algebraic interior of A, say $corA$, is defined by*

$$corA = \{\bar{x} \in A : \forall x \in X, \exists \delta > 0, \bar{x} + tx \in A, \forall t \in (0, \delta)\};$$

(ii) *A is called algebraically open if $A = corA$;*

(iii) *A is called algebraically closed if the complement A^c of A is algebraically open.*

If X is a topological vector space and C is closed and convex, then $corC = intC$.

Proposition 3.90. *Let (X, D) and (Y, C) be ordered Banach spaces, and let $intC \neq \varnothing$. Then the weak C-dual cone D_C^{w+} is algebraically closed.*

Proof. We will prove that $(D_C^{w+})^c$ is algebraically open. Let $g_0 \notin cor(D_C^{w+})^c$. Then, there exists $g \in L(X, Y)$, such that $g_0 + (1/k)g \notin (D_C^{w+})^c$, for k large enough. Hence,

$$g_0 + (1/k)g \in D_C^{w+};$$

i.e., for k large enough,

$$\langle g_0 + (1/k)g, x \rangle \not\leq_{intC} 0, \quad x \in D. \tag{3.38}$$

Assume that $g_0 \in (D_C^{w+})^c$, so that $g_0 \notin D_C^{w+}$. Then there exists $x_0 \in D$ such that $\langle g_0, x_0 \rangle \leq_{intC} 0$. Therefore, for any $y \in Y$, we have

$$-\langle g_0, x_0 \rangle + ty \geq_C 0, \text{ for } t > 0 \text{ small enough.}$$

Letting

$$y = y_1 - \langle g, x_0 \rangle, \quad y_1 \in Y, t = 1/k,$$

we see that

$$\begin{aligned} -\langle g_0, x_0 \rangle + (1/k)y &= -\langle g_0, x_0 \rangle + (1/k)(y_1 - \langle g, x_0 \rangle) \\ &= -\langle g_0 + (1/k)g, x_0 \rangle + (1/k)y_1 \geq_C 0 \end{aligned}$$

where $y_1 \in Y$ and k is large enough. Hence, we deduce that

$$-\langle g_0 + (1/k)g, x_0 \rangle \in corC;$$

i.e.,

$$\langle g_0 + (1/k)g, x_0 \rangle \leq_{intC} 0,$$

which contradicts (3.38). It follows that $g_0 \notin (D_C^{w+})^c$. Hence,

$$(D_C^{w+})^c = cor(D_C^{w+})^c.$$

Thus, $(D_C^{w+})^c$ is algebraically open. Then D_C^{w+} is algebraically closed. ■

Proposition 3.91. *Let (X, D) and (Y, C) be ordered Banach spaces, and let $intC \neq \varnothing$. Then the strong C-dual cone D_C^{s+} is algebraically closed and convex.*

Proof. It is similar to the proof of Proposition 3.90 to prove the algebraic closedness of D_C^{s+}. We show that D_C^{s+} is convex. Let $g_1, g_2 \in D_C^{s+}, 0 < \lambda < 1$. For any $x \in D$,

$$\langle \lambda g_1 + (1-\lambda) g_2, x \rangle = \lambda \langle g_1, x \rangle + (1-\lambda)\langle g_2, x \rangle \geq_C 0.$$

Note that

$$\lambda g_1 + (1-\lambda) g_2 \in D_C^{s+}.$$

Thus, D_C^{s+} is convex. ■

Remark 3.92. In general, D_C^{w+} is not necessarily a convex cone, since the weak ordering $\not\leq_{intC}$ in Y is not transitive. In fact, let $X = \mathbb{R}$, $D = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$. Then $D_C^{s+} = \mathbb{R}^2_+$ is convex, but

$$D_C^{w+} = \{x \in \mathbb{R}^2 : x = (x_1, x_2)^\top, x_1 \leq_D 0 \text{ or } x_2 \leq_D 0\}$$

is not convex.

Now we consider three types of vector complementarity problems. Let $T : X \to L(X, Y)$ be a vector-valued function and let C be a closed, pointed and convex cone with $intC \neq \varnothing$.

A (weak) vector complementarity problem (VCP) is the problem of finding $x \in D$, such that

$$\langle T(x), x\rangle \not\geq_{intC} 0, \quad T(x) \in D_C^{w+}.$$

A positive vector complementarity problem (PVCP) is the problem of finding an $x \in D$ such that

$$\langle T(x), x \not\geq_{intC} 0, \quad T(x) \in D_C^{s+}.$$

A strong vector complementarity problem (SVCP) is the problem of finding an $x \in D$ such that

$$\langle T(x), x\rangle = 0, \quad T(x) \in D_C^{s+}.$$

We denote the sets of solutions of (VCP), (PVCP) and (SVCP) by N, N_P, and N_S, respectively. There are close relations between vector complementarity problems and vector optimization problems. In this connection we consider the following vector optimization problem:

$$(VOP_w) \qquad \text{Min}_C\{\langle T(x), x\rangle : x \in D, T(x) \in D_C^{w+}\}.$$

We denote the set of all weakly minimal solutions of (VOP_w) by E_w, and we set $f(x) = \langle T(x), x\rangle$ and $H_w = f(E_w)$.

Theorem 3.93. *If $H_w \neq \varnothing$ and there exists $z \in H_w$ such that $z \not\geq_{intC} 0$, then the vector complementarity problem (VCP) is solvable.*

Proof. Let $z \in H_w$ and $z \not\geq_{intC} 0$. There exists a point $x \in D$ such that $T(x) \in D_C^{w+}$, $z = f(x) = \langle T(x), x\rangle \not\geq_{intC} 0$. So x is a solution of (VCP). ■

Theorem 3.94. *If there exists at most a finite number of solutions of the vector complementarity problem (VCP), then (VCP) is solvable if and only if $H_w \neq \varnothing$ and there exists $z \in H_w$ such that $z \not\geq_{intC} 0$.*

Proof. Let x_1 be a solution of (VCP). If $x_1 \in E_w$, we are done. If $x_1 \notin E_w$, by the definition of weak minimal solutions, there exists $x_2 \in D$ such that $T(x_2) \in D_C^{w+}$ and $\langle T(x_2), x_2\rangle \leq_{intC} \langle T(x_1), x_1\rangle \not\geq_{intC} 0$. Thus we obtain

$$\langle T(x_2), x_2\rangle \not\geq_{intC} 0,$$

hence x_2 is a solution of (VCP) and $x_1 \neq x_2$, continuing this procedure, by the finiteness of the number of solutions of (VCP), there exists $x_k \in D$ such that x_k is a solution of (VCP), $x_k \in E_w$, $z = \langle T(x_k), x_k\rangle \not\geq_{intC} 0, z \in H_w$.

Conversely, we finish the proof from Theorem 3.93. ■

We next consider the existence of solutions of the positive vector complementarity problem (PVCP). To this end, we observe the following vector optimization problem:

$$(VOP_s) \qquad \mathrm{Min}_C\{\langle T(x), x\rangle : x \in D, T(x) \in D_C^{s+}\}.$$

We denote the set of all minimal solutions of (VOP_s) by E_s and set $H_s = f(E_s)$.

Similarly, we can prove the following result.

Theorem 3.95. *If there exist at most a finite number of solutions of the (PVCP), then the (PVCP) is solvable if and only if $H_s \neq \varnothing$ and there exists $z \in H_s$, such that $z \not\geq_{intC} 0$.*

Remark 3.96. **(i)** The equivalent relation in Theorem 3.94 is a generalization of the corresponding results in Borwein [20];

(ii) The results on the nonemptiness of H_w and H_s can be found in Chen and Yang [40].

In the sequel, we will show the relationship among the vector optimization problem (VOP_{wl}), the vector complementarity problem (VCP), the weak minimal element problem (WMEP), the vector variational inequality (VVI) and the vector unilateral minimization problem (VUMP).

Define the feasible set associated to T by

$$\mathcal{F}_w = \{x \in X : x \in D, T(x) \in D_C^{w+}\}.$$

Problem VOP_{wl}: finding $x \in \mathcal{F}_w$ such that

$$\langle l, x\rangle \in \mathrm{Min}_{intC}\{\langle l, y\rangle : y \in \mathcal{F}_w\},$$

where $l \in L(X, Y)$ is given;

Problem WMEP: finding $x \in \mathcal{F}_w$ such that $x \not\geq_{intC} y, \quad \forall y \in \mathcal{F}_w$;

Problem VCP: finding $x \in \mathcal{F}_w$ such that

$$\langle T(x), x\rangle \not\geq_{intC} 0;$$

Problem WVVI: finding $x \in D$ such that

$$\langle T(x), y - x\rangle \not\leq_{intC} 0, \quad \forall y \in D;$$

Problem VUMP: finding $x \in D$ such that x is a weakly minimal solution of the vector optimization problem:

$$\mathrm{Min}_{x \in D}{}_C\, f(x),$$

where $f : X \to Y$ is given.

Definition 3.97. *Let (X, D) and (Y, C) be ordered Banach spaces. Let $l : X \to Y$ be a linear vector-valued function and $intD \neq \emptyset$, $intC \neq \emptyset$. l is called a weak positive linear operator if*

$$x \not\geq_{intD} 0 \text{ implies } \langle l, x \rangle \not\geq_{intC} 0.$$

In particular, when $Y = I\!R$, $x \not\geq_{intC} 0$ implies $\langle l, x \rangle \leq 0$, l is called a weak positive linear functional.

Jameson [114] defines l as a positive (or monotone) operator if $\langle l, D \rangle \subset C$. Generally, there is no inclusion relation between a positive linear operator and a weak positive linear operator.

Example 3.98. If $Y = X$, $intC \neq \emptyset$, then the unit operator from X to itself is a weak positive sublinear operator.

Example 3.99. Let $X = \mathrm{I\!R}^2$, $Y = \mathrm{I\!R}^3$, $D = \mathrm{I\!R}^2_+$, $C = \mathrm{I\!R}^3_+$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$. Define the operator l as follows:

$$\langle l, x \rangle = \begin{pmatrix} x_1 - x_2 \\ 2x_2 \\ x_1 \end{pmatrix} \in \mathrm{I\!R}^3.$$

Then, for any $x, y \in X$, $x \not\geq_{intD} y$ implies $\langle l, x \rangle \not>_{intC} \langle l, y \rangle$. So l is a weak positive linear operator.

Definition 3.100. *Let X, Y be Banach spaces and l be a linear operator from X to Y. If the image of any bounded subset of X is a self-sequentially compact subset in Y, then l is said to be weakly completely continuous.*

Definition 3.101. *Let (X, D) and (Y, C) be Banach spaces. The norm $||\cdot||$ in X is said to be strictly monotonically increasing on D if, for each $y \in D$,*

$$x \in (\{y\} - intD) \cap D \text{ implies } ||x|| < ||y||.$$

Theorem 3.102. *Let (X, D) and (Y, C) be ordered Banach spaces, $intD \neq \emptyset$, $intC \neq \emptyset$. Suppose that*

(i) *$T = Df$ is the Fréchet derivative of the C-convex function f from X to Y;*
(ii) *l is a weak positive linear operator;*
(iii) *There exists $x \in \mathcal{F}_w$ such that $T(x)$ is one to one and weak completely continuous;*
(iv) *X is a topological dual space of a real normed space and the norm $||\cdot||$ in X is strictly monotonically increasing on D;*

If vector variational inequality (WVVI) is solvable, then (VOP_{wl}), (WMEP), (VCP) and (VUMP) have a solution, respectively.

Since the assertions which guarantee Theorem 3.102 are in various degrees of generality, we shall treat them in a sequence of propositions, each with its own hypotheses.

Proposition 3.103. *Let $T = Df$ be the Fréchet derivative of $f : X \to Y$. Then $x \in D$ solves (VUMP) implies that x solves (WVVI); if, in addition, f is C-convex, then conversely, x solves (WVVI) implies that it solves (VUMP).*

Proof. Let x be a solution of (VUMP). Since D is a convex cone, we get

$$f(x) \not\geq_{intC} f(x + t(w - x)), \quad 0 < t < 1, w \in D,$$

i.e.,

$$(f(x + t(w - x)) - f(x))/t \not\leq_{intC} 0.$$

Let t tend to 0 from the right, we have

$$\langle Df(x), w - x \rangle \not\leq_{intC} 0, \quad \forall w \in D,$$

which is the weak vector variational inequality.

Conversely, let x solve the weak vector variational inequality:

$$\langle T(x), w - x \rangle \not\leq_{intC} 0, \quad \forall w \in D.$$

Since f is C-convex, by Proposition 1.63, we have, for any $w \in D$,

$$\begin{aligned} f(w) - f(x) &\geq_C \langle Df(x), w - x \rangle \\ &= \langle T(x), w - x \rangle \not\leq_{intC} 0, \end{aligned}$$

i.e., $f(w) \not\leq_{intC} f(x), \forall w \in D$. Thus, x is a solution of (VUMP). ■

Proposition 3.104. *x solves (WVVI) implies that x solves (VCP).*

Proof. According to the assumption,

$$\langle T(x), y - x \rangle \not\leq_{intC} 0, \quad \forall y \in D.$$

We set $y = 0$, then $\langle T(x), x \rangle \not\geq_{intC} 0$. We set $y = w + x$, $w \in D$, then $y \in D$ and $\langle T(x), w \rangle \not\leq_{intC} 0$, i.e., $T(x) \in D_C^{w+}$. Hence x solves the (VCP). ■

Remark 3.105. Generally, the converse conclusion in Proposition 3.104 does not hold. However, if T is conegative (T is called conegative if $\langle T(x), x \rangle \leq_C 0, \forall x \in D$), then the converse conclusion also holds.

Proposition 3.106. *Let l be a weak positive linear operator. Then x solves (WMEP) implies that x solves (VOP_{wl}).*

Proof. This assertion is immediate from the definition of the weak positive linear operator. ■

Lemma 3.107 (Jahn [116]). *Let A be a nonempty subset of an ordered space (Y, C) with $C \subset Y$ being a convex cone and $intC \neq \varnothing$. Let Y be the topological dual space of a real normed space $(Z, ||\cdot||)$. Suppose that there exists $y \in Y$ such that the set $(y - C) \cap A$ is weakly closed and bounded below and the norm $||\cdot||$ in Y is strictly monotonically increasing. Then the set A has at least one minimal element.*

Lemma 3.108. *If (WVVI) is solvable, then the feasible set $\mathcal{F}_w$ is nonempty.*

Proof. Let x be a solution of (WVVI), that is,

$$\langle T(x), y - x\rangle \not\leq_{intC} 0, \quad y \in D.$$

We set $y = z + x, z \in D$, then $y \in D$ since D is a convex cone and $\langle T(x), z\rangle \not\leq_{intC} 0$ for all $z \in D$. So $T(x) \in D_C^{w+}$ and $x \in \mathcal{F}_w$. ■

It is easy to fulfill the following lemma.

Lemma 3.109. *If the norm $||\cdot||$ in an ordered Banach space (X, D) is strictly monotonically increasing, then the order intervals*

$$[a, b] = \{x \in X : a \leq_D x \leq_D b\}$$

in X are bounded, where $a, b \in X$.

Proposition 3.110. *If the (WVVI) is solvable, and*

(i) *there exists x in $\mathcal{F}_w$ such that $T(x) : X \to Y$ is a one-to-one vector-valued function and it is weak completely continuous;*
(ii) *Y is the topological dual space of a normed space $(X, ||\cdot||)$ and the norm $||\cdot||$ in X is strictly monotonically increasing,*

then the (VOP_{wl}) has at least one solution.

Proof. By the assumptions and Lemma 3.108, $\mathcal{F}_w \neq \varnothing$. Let $x \in \mathcal{F}_w$ such that $T(x)$ is one to one and weakly completely continuous, and $\{y_k\} \subset (\mathcal{F}_w)_x, y_k \to y$ (weakly), where

$$(\mathcal{F}_w)_x = (\{x\} - D) \cap \mathcal{F}_w \subset (\{x\} - D) \cap D = [0, x].$$

$[0, x]$ is the order interval, i.e.,

$$[0, x] = \{y \in X : y \geq_D 0 \text{ and } y \leq_D x\}.$$

The order interval $[0, x]$ is bounded by Lemma 3.109, so $(\mathcal{F}_w)_x$ is also bounded.

We observe that $\langle T(x), (\mathcal{F}_w)_x\rangle$ is a self-sequentially compact, since $T(x)$ is weak completely continuous. Thus there exists a subsequence $\langle T(x), y_{k_i}\rangle \subset$

$\langle T(x), (\mathcal{F}_w)_x \rangle$, which converges to $z \in \langle T(x), (\mathcal{F}_w)_x \rangle$. We obtain a point $y_0 \in (\mathcal{F}_w)_x$ such that

$$\langle T(x), y_{k_i} \rangle \to \langle T(x), y_0 \rangle \quad \text{(strongly)}.$$

On the other hand, if $y_k \to y$ (weakly) and $T(x)$ is weak completely continuous, we derive

$$\langle T(x), y_k \rangle \to \langle T(x), y \rangle \quad \text{(strongly)}.$$

By the uniqueness of the convergence, we obtain

$$\langle T(x), y_0 \rangle = \langle T(x), y \rangle.$$

Since $T(x)$ is one to one, we have $y_0 = y$, i.e., $y \in (\mathcal{F}_w)_x$ and $(\mathcal{F}_w)_x$ is weakly closed. By Lemma 3.107, $\mathcal{F}_w$ has a weakly minimal element. ■

Definition 3.111. *Let (X, D) and (Y, C) be ordered Banach spaces. The operator $T : X \to L(X, Y)$ is said to be positive if*

$$\langle T(x), y \rangle \geq_C 0, \quad \forall x, y \in D.$$

The vector-valued function $F : X \to Y$ is called positive if $F(x) \in C$, for all $x \in D$.

The following corollary is elementary from the definition.

Corollary 3.112. *T is positive if and only if $T(x)$ is positive for any $x \in D$.*

We consider the positive vector complementarity problem (PVCP). The feasible set related to the (PVCP) is

$$\mathcal{F}_s = \{x \in X : x \in D, T(x) \in D_C^{s+}\}.$$

For a given $l \in L(X, Y)$, we consider following three problems: the vector optimization problem (VOP$_{s_l}$), the weak minimal element problem (WMEP$_s$) and the positive vector complementarity problem (PVCP).

Problem VOP$_{s_l}$: finding $x \in \mathcal{F}_s$ such that

$$\langle l, x \rangle \in \mathrm{Min}_{intC}\{\langle l, y \rangle : y \in \mathcal{F}_s\},$$

where $l \in L(X, Y)$;
Problem WMEP$_s$: finding $x \in \mathcal{F}_s$ such that $x \not\geq_{intD} y, \quad \forall y \in \mathcal{F}_s$;
Problem PVCP: finding $x \in \mathcal{F}_s$ such that $\langle T(x), x \rangle \not\geq_{intD} 0$.

Proposition 3.113. *Let T be strictly monotone and x be the solution of (PVCP). Then, x is a weakly minimal element of $\mathcal{F}_s$.*

Proof. It is elementary that $x \in \mathcal{F}_s \subset D$. If $x \in \partial D$, then x solves (WMEP$_s$). Otherwise, there exists $x' \in \mathcal{F}_s$ such that $x' \leq_{intD} x$, so

$$x = x - x' + x' \in intD + D \subset intD,$$

this is a contradiction. If $x \in intD$, by the strict monotonicity of T,

$$\langle T(x), x - u\rangle \geq_{intC} \langle T(u), x - u\rangle, \text{ for each } u \in \mathcal{F}_s, u \neq x.$$

If $x \geq_{intD} u$, $\langle T(u), x - u\rangle \geq_{intC} 0$, then for $c \in intC$,

$$0 \not< \langle T(x), x\rangle \geq_C \langle T(x), u\rangle + c.$$

Therefore,

$$\langle T(x), u\rangle + c \not\geq_{intC} 0.$$

Then $\langle T(x), u) \geq_C 0$ does not hold, since $c \in intC$, which is contrary to the assumption condition of $x \in \mathcal{F}_s$. So $x \geq_{intD} u$ does not hold, that is, $x \not\geq_{intD} u$. Hence, x solves (WMEP$_s$). ■

Remark 3.114. Proposition 3.113 is a generalization of the corresponding result of Riddell [163].

Proposition 3.115. *x solves (PVCP) implies that x solves (WVVI).*

Proof. By the definition of (PVCP), $x \in D$, $\langle T(x), x\rangle \not\geq_{intC} 0$, for all $x \in D$; i.e., for any $y \in D$,

$$\langle T(x), x \not\geq_{intC} 0 \leq \langle T(x), y\rangle$$

and

$$\langle T(x), y - x\rangle \not\leq_{intC} 0,$$

which is (WVVI). ■

Remark 3.116. In Proposition 3.104, we prove that x solves (WVVI) implies that x solves (VCP). In this proposition we prove an inverse relation under the condition that T is a positive operator. On the other hand, if T is a positive operator, it is elementary that x solves (WVVI) implies that x solves (PVCP). We have shown that (WVVI) and (PVCP) are equivalent if T is a positive operator.

As a summary of the above results, we have following theorem.

Theorem 3.117. *Let (X, D) and (Y, C) be ordered Banach spaces and $intD \neq \varnothing$, $intC \neq \varnothing$. Suppose that*

(i) *$T = Df$ is the Fréchet derivative of the C-convex vector-valued function $f : X \to Y$;*
(ii) *$l \in L(X, Y)$ is a weak positive linear operator;*
(iii) *T is strictly monotone.*

If the (PVCP) is solvable, then (VOP_{s_l}), ($WMEP_s$), (PVCP), (WVVI) and (VUMP) have at least one common solution.

3.9 VCP with a Variable Domination Structure

In this section, we investigate a vector complementarity problem with a variable ordering relation. We establish existence results of a solution for a vector complementarity problem under an inclusive type condition. We also obtain some equivalence results among a vector complementarity problem, a vector variational inequality problem, a vector optimization problem, a weak minimal element problem, and a vector unilateral optimization problem under some monotonicity conditions and some inclusive type conditions in ordered Banach spaces.

Let (X, D) be an ordered Banach space, Y be a Banach space, $\{C(x) : x \in X\}$ be a family of closed, pointed, and convex cones in Y with nonempty interior $intC(x)$ for all $x \in X$, and $T : X \to L(X, Y)$. Throughout this section, we assume that $C : X \rightrightarrows Y$ be upper semicontinuous.

Consider the following three kinds of vector complementarity problems with a variable ordering relation.

(Weak) Vector Complementarity Problem (VCP): finding $x \in D$, such that

$$\langle T(x), x\rangle \not\geq_{intC(x)} 0, \quad \langle T(x), y\rangle \not\leq_{intC(x)} 0, \quad \forall y \in D.$$

Positive Vector Complementarity Problem (PVCP): finding $x \in D$, such that

$$\langle T(x), x\rangle \not\geq_{intC(x)} 0, \quad \langle T(x), y\rangle \leq_{C(x)} 0, \quad \forall y \in D.$$

Strong Vector Complementarity Problem (SVCP): finding $x \in D$, such that

$$\langle T(x), x\rangle = 0, \quad \langle T(x), y\rangle \leq_{C(x)} 0, \quad \forall y \in D.$$

Remark 3.118. In here, without confusion, for example, we use the same notation (VCP) for a vector complementarity problem with a variable ordering relation as that for the problem with a fixed ordering relation.

If $C(x) = C$ for all $x \in X$, where C is a closed, pointed, and convex cone in Y with nonempty interior $intC$, then (VCP), (PVCP), and (SVCP) reduce to the problems considered in section 3.8.

Next we establish some existence results of (VCP) involving a variable ordering structure .

The feasible set of (VCP) is

$$\mathcal{F} = \{x \in X : x \in D, \langle T(x), y\rangle \not\leq_{intC(x)} 0, \quad \forall y \in D\}.$$

Let $f(x) = \langle T(x), x\rangle$ for all $x \in D$. We consider the following vector optimization problem (VOP):

$$\text{Min}_{intC(x)} f(x) \qquad \text{subject to} \quad x \in \mathcal{F}.$$

Theorem 3.119. *Assume that $LMin_{intC(x)} f(\mathcal{F}) \neq \varnothing$. If there exists $x \in LMin_{intC(x)} f(\mathcal{F})$ such that $f(x) \not\geq_{intC(x)} 0$, then the vector complementarity problem (VCP) is solvable.*

Proof. Let $x \in \mathrm{LMin}_{intC(x)} f(\mathcal{F})$ and $f(x) \not\geq_{intC(x)} 0$. Then $x \in D$ and

$$\langle T(x), x\rangle = f(x) \not\geq_{intC(x)} 0, \quad \langle T(x), y\rangle \not\leq_{intC(x)} 0, \quad \forall y \in D.$$

It follows that x is a solution of (VCP). This completes the proof. ■

Definition 3.120. *Let $f : X \to Y$. We say that $C(x)$ satisfies an f-inclusive condition if, for any $x, y \in X$,*

$$f(x) \leq_{intC(y)} f(y) \quad \textit{implies that} \quad C(x) \subset C(y).$$

This inclusive condition requires that any two of the family of closed and convex cones satisfy an inclusion relation so long as their corresponding variables satisfy certain conditions. It is easy to see that, if $C(x) = C$ for all $x \in X$, where C is a closed, pointed, and convex cone in Y, then $C(x)$ satisfies the f-inclusive condition.

Example 3.121. Let $X = Y = \mathbb{R}^2$ and $D = \mathbb{R}^2_+$. Define $f(u) = (3x+2, 3y+2)^\top$ and

$$C(u) = \begin{cases} \{(r\cos x, r\sin x) : \ r \geq 0, \ 0 \leq x \leq \pi/8\}, \\ \qquad\qquad \text{if } x \in (-\infty, \pi/8], \ y \in (-\infty, +\infty); \\ \{(r\cos x, r\sin x) : \ r \geq 0, \ 0 \leq x \leq x\}, \\ \qquad\qquad \text{if } x \in (\pi/8, \pi/2), \ y \in (-\infty, +\infty); \\ D, \\ \qquad\qquad \text{if } x \in [\pi/2, +\infty), \ y \in (-\infty, +\infty) \end{cases}$$

for all $u = (x, y) \in X$. Then it is easy to see that $C(u)$ satisfies the f-inclusive condition. In fact, for any $u = (x_1, y_1) \in X$ and $v = (x_2, y_2) \in X$, if $f(u) \leq_{intC(v)} f(v)$, then $f(v) - f(u) \in intC(v) \subset intD$ and so $x_1 < x_2$. Therefore, $C(u) \subset C(v)$ and $C(x)$ satisfies the f-inclusive condition.

Theorem 3.122. *Suppose that C satisfies the f-inclusive condition and that there exist at most a finite number of solutions for (VCP). Then (VCP) is solvable if and only if $LMin_{intC(x)} f(\mathcal{F}) \neq \varnothing$, and there exists $x \in LMin_{intC(x)} f(\mathcal{F})$ such that $f(x) \not\geq_{intC(x)} 0$.*

Proof. Let x_1 be a solution of (VCP). Then

$$\langle T(x_1), x_1\rangle \not\geq_{intC(x_1)} 0, \quad \langle T(x_1), y\rangle \not\leq_{intC(x_1)} 0, \quad \forall y \in D.$$

If $x_1 \in \mathrm{LMin}_{intC(x)} f(\mathcal{F})$, then

$$f(x_1) = \langle T(x_1), x_1\rangle \not\geq_{intC(x_1)} 0$$

and we are done. If $x_1 \notin \mathrm{LMin}_{intC(x)} f(\mathcal{F})$, by the definition of a weakly minimal-like solution, there exists $x_2 \in D$ such that

$$\langle T(x_2), y \rangle \not\leq_{intC(x_2)} 0, \quad \forall y \in D$$

and

$$f(x_2) = \langle T(x_2), x_2 \rangle \leq_{intC(x_1)} \langle T(x_1), x_1 \rangle = f(x_1) \not\geq_{intC(x_1)} 0.$$

This implies that

$$f(x_2) = \langle T(x_2), x_2 \rangle \not\geq_{intC(x_1)} 0.$$

Since $f(x_2) \leq_{intC(x_1)} f(x_1)$, and C satisfies the f-inclusive condition, it follows that $C(x_2) \subset C(x_1)$ and so

$$f(x_2) = \langle T(x_2), x_2 \rangle \not\geq_{intC(x_2)} 0.$$

Thus, x_2 is a solution of (VCP) and $x_2 \neq x_1$. Continuing this process, there exists $x_k \in D$ such that x_k is a solution of (VCP) and $x_k \in \text{LMin}_{intC(x)} f(\mathcal{F})$ since (VCP) has at most a finite number of solutions. Therefore,

$$f(x_k) = \langle T(x_k), x_k \rangle \not\geq_{intC(x_k)} 0.$$

The only if part follows from Theorem 3.119 and we complete the proof. ■

Remark 3.123. (1) If $C(x) = C$ for all $x \in X$, where C is a closed, pointed, and convex cone in Y, then C satisfies the f-inclusive condition and Theorem 3.122 is the same as Theorem 3.2 of Chen and Yang [40].

(2) If $C(x) = C$ for all $x \in X$, where C is a closed, pointed, and convex cone in $Y = (-\infty, +\infty)$, then we obtain the results in Borwein [20].

We next consider the positive vector complementarity problem (PVCP): finding $x \in D$ such that

$$\langle T(x), x \rangle \not\geq_{intC(x)} 0, \quad \langle T(x), y \rangle \geq_{C(x)} 0, \quad \forall y \in D.$$

Let

$$\mathcal{F}_0 = \{x \in X : \; x \in D, \; \langle T(x), y \rangle \geq_{C(x)} 0, \quad \forall y \in D\}.$$

Consider the following vector optimization problem $(\text{VOP})_0$

$$\text{Min}_{C(x)} f(x) \qquad \text{subject to} \quad x \in \mathcal{F}_0.$$

Similarly, we can prove the following results.

Theorem 3.124. *If $LMin_{C(x)} f(\mathcal{F}_0) \neq \emptyset$ and there exists $x \in LMin_{C(x)} f(\mathcal{F}_0)$ such that $f(x) \not\geq_{intC(x)} 0$, then (PVCP) is solvable.*

Theorem 3.125. *Suppose that C satisfies the f-inclusive condition and that there exist at most a finite number of solutions of (PVCP). Then (PVCP) is solvable if and only if $LMin_{C(x)} f(\mathcal{F}_0) \neq \emptyset$ and there exists $x \in LMin_{C(x)} f(\mathcal{F}_0)$ such that $f(x) \not\geq_{intC(x)} 0$.*

Next we consider equivalences between vector complementarity problems and weak minimal element problems.

Let (X, D) be an ordered Banach space with $intD \neq \varnothing$, Y be a Banach space, and $\{C(x) : x \in X\}$ be a family of closed, pointed, and convex cones in Y such that $intC(x) \neq \varnothing$ for all $x \in X$. Let $T : X \to L(X, Y)$ be a given map and $f : X \to Y$ be a given operator.

Recall that the feasible set of (VCP) associated with T is defined by:

$$\mathcal{F} = \{x \in X : \ x \in D, \ \langle T(x), y \rangle \not\leq_{intC(x)} 0, \quad \forall y \in D\}.$$

We now consider the following five problems.

The vector optimization problem $(\text{VOP})_l$: for a given $l \in L(X, Y)$, finding $x \in \mathcal{F}$ such that $l(x) \in \text{Min}_{intC(x)} l(\mathcal{F})$;

The weak minimal element problem (WMEP): finding $x \in \mathcal{F}$ such that $x \in \text{Min}_{intD}\mathcal{F}$;

The vector complementarity problem (VCP): finding $x \in \mathcal{F}$ such that $\langle T(x), x \rangle \not\geq_{intC(x)} 0$;

The vector variational inequality problem (WVVI): finding $x \in D$ such that

$$\langle T(x), y - x \rangle \not\leq_{intC(x)} 0, \quad \forall y \in D;$$

The vector unilateral optimization problem (VUOP): finding $x \in D$ such that $f(x) \in \text{LMin}_{intC(x)} f(D)$.

Definition 3.126. *([4]) A linear operator $l : X \to Y$ is called weakly positive with respect to the variable ordering relation $C(x)$ if, for any $x, y \in X$, $x \not\geq_{intD} y$ implies that $l(x) \not\geq_{intC(x)} l(y)$.*

Theorem 3.127. *Let (X, D) be an ordered Banach space with $intD \neq \varnothing$, Y be a Banach space, and $\{C(x) : x \in X\}$ be a family of closed, pointed, and convex cones in Y such that $intC(x) \neq \varnothing$ for all $x \in X$. Suppose*

(1) $T = Df$ is the Frechet derivative of a convex operator $f : X \to Y$;
(2) l is a weakly positive linear operator with respect to the variable ordering relation $C(x)$;
(3) there exists $x \in \mathcal{F}$ such that Tx is one to one and weak completely continuous;
(4) X is a topological dual space of a real normed space and the norm $\|\cdot\|$ in X is strictly monotonically increasing on D.

If (WVVI) is solvable, then $(VOP)_l$, (WMEP), (VCP), and (VUOP) are also solvable.

Remark 3.128. If $C(x) = C$ for all $x \in X$, where C is a closed, pointed, and convex cone in Y, then Theorem 3.127 coincides with Theorem 4.1 of Chen and Yang [40].

We need the following propositions to establish Theorem 3.127.

Proposition 3.129. *Let $T = Df$ be the Frechet derivative of an operator $f : X \to Y$. Then x solves (VUOP) implies x solves (WVVI). If, in addition, f is a $C(x)$-convex function, then conversely, x solves (WVVI) implies x solves (VUOP).*

Proof. Let x be a solution of (VUOP). Then

$$x \in D \text{ and } f(x) \in \mathrm{LMin}_{intC(x)} f(D),$$

i.e.,

$$f(x) \not\geq_{intC(x)} f(y), \ \forall y \in D.$$

Since D is a convex cone,

$$f(x) \not\geq_{intC(x)} f(x + t(w - x)), \quad 0 < t < 1, \ w \in D.$$

It follows that

$$\frac{1}{t}[f(x + t(w - x)) - f(x)] \not\leq_{intC(x)} 0.$$

Since f is Frechet differentiable on X, letting $t \to 0^+$, we get

$$\langle Df(x), w - x\rangle \not\leq_{intC(x)} 0, \quad \forall w \in D,$$

which is (WVVI).

Conversely, let x solve (WVVI). Then

$$\langle T(x), w - x\rangle \not\leq_{intC(x)} 0, \quad \forall w \in D.$$

Since f is $C(x)$-convex, by Proposition 1.72,

$$f(w) - f(x) \geq_{C(x)} \langle Df(x), w - x\rangle \not\leq_{intC(x)} 0, \quad \forall w \in D$$

and so

$$f(w) \not\leq_{intC(x)} f(x), \quad \forall w \in D,$$

which is the (VUOP). This completes the proof. ■

Definition 3.130. *A map $T : X \to L(X, Y)$ is called co-negative with respect to the variable ordering relation $C(x)$ if $\langle T(x), x\rangle \leq_{C(x)} 0$ holds for all $x \in D$.*

Proposition 3.131. *If x solves (WVVI), then x also solves (VCP). Conversely, if T is co-negative with respect to the variable ordering relation $C(x)$, then x solves (VCP) implies x solves (WVVI).*

Proof. Let x be a solution of (WVVI). Then

$$\langle T(x), y - x\rangle \not\leq_{intC(x)} 0, \quad \forall y \in D.$$

Letting $y = 0$, we get $\langle T(x), x\rangle \not\geq_{intC(x)} 0$. For $y = w + x$ with any $w \in D$, we have

$$\langle T(x), w \rangle \not\leq_{intC(x)} 0, \quad \forall w \in D.$$

Thus, x is a solution of the (VCP).

Conversely, let x solve the (VCP). Then

$$\langle T(x), x \rangle \leq_{C(x)} 0 \not\geq_{intC(x)} \langle T(x), y \rangle, \quad \forall y \in D.$$

This implies

$$\langle T(x), x \rangle \not\geq_{intC(x)} \langle T(x), y \rangle, \quad \forall y \in D$$

and so

$$\langle T(x), y - x \rangle \not\leq_{intC(x)} 0, \quad \forall y \in D.$$

This completes the proof. ■

Proposition 3.132. *Let l be a weakly positive linear operator with respect to the variable ordering relation $C(x)$. Then x solves (WMEP) implies x solves (VOP)$_l$.*

Proof. Let x be a solution of (WMEP). Then $x \in \mathcal{F}$ and

$$x \not\geq_{intD} \mathcal{F} = \{x \in X : \ x \in D, \ \langle T(x), y \rangle \not\geq_{intC(x)} 0, \quad \forall y \in D\}.$$

For any $z \in \mathcal{F}$, $x \not\geq_{intD} \mathcal{F}$ implies $x \not\geq_{intD} z$. Since l is a weakly positive linear operator with respect to the variable ordering relation $C(x)$, it follows that $l(x) \not\geq_{intC(x)} l(z)$ and so

$$l(x) \not\geq_{intC(x)} l(\mathcal{F}),$$

which is (VOP)$_l$. This completes the proof. ■

Definition 3.133. *Let (X, D) be an ordered Banach space and A a nonempty subset of X.*

(1) If, for some $x \in X$, $A_x = (\{x\} - D) \cap A \neq \varnothing$, then A_x is called a section of the set A.

(2) A is called weakly closed if $\{x_k\} \subset A$, $x \in X$, $\langle x^*, x_k \rangle \to \langle x^*, x \rangle$ for all $x^* \in X^*$, then $x \in A$.

Lemma 3.134. *If (WVVI) is solvable, then the feasible set $\mathcal{F}$ is nonempty.*

Proof. Let x be a solution of (WVVI). Then

$$\langle T(x), y - x \rangle \not\leq_{intC(x)} 0, \quad \forall y \in D.$$

Taking $y = z + x$ with any $z \in D$, we know that $y \in D$ and

$$\langle T(x), z \rangle \not\leq_{intC(x)} 0, \quad \forall z \in D.$$

Thus, $x \in \mathcal{F}$. This completes the proof. ■

Lemma 3.135. *If the norm $\|\cdot\|$ in an ordered Banach space X is strictly monotonically increasing, then the order intervals in X are bounded.*

Proposition 3.136. *Suppose (WVVI) is solvable and*

(1) there exists x in $\mathcal{F}$ such that Tx is one to one and weak completely continuous;
(2) X is the topological dual space of a real normed space $(Z, \|\cdot\|_z)$ and the norm $\|\cdot\|$ in X is strictly monotonically increasing.

Then (WMEP) has at least one solution.

Proof. By the assumption and Lemma 3.134, $\mathcal{F} \neq \varnothing$. Let $x \in \mathcal{F}$ be a point such that $T(x)$ is one to one and weak completely continuous, and $\{y_k\} \subset \mathcal{F}$ with $y_k \to y$ (weakly). Since

$$\mathcal{F}_x = (\{x\} - D) \cap \mathcal{F} \subset (\{x\} - D) \cap D = [0, x],$$

by Lemma 3.4, $[0, x]$ is bounded and so is $\mathcal{F}_x$. Since Tx is weak completely continuous, $\langle T(x), \mathcal{F}_x\rangle$ is a self-sequentially compact set and so $\langle T(x), y_k\rangle \subset \langle T(x), \mathcal{F}_x\rangle$ implies there exists a subsequence $\langle T(x), y_{k_i}\rangle$ which converges to $z \in \langle T(x), \mathcal{F}_x\rangle$. We get a point $y_0 \in \mathcal{F}_x$ such that

$$\langle T(x), y_{k_i}\rangle \to \langle Tx, y_0\rangle \quad \text{(strongly)}.$$

On the other hand, since $y_k \to y$ (weakly) and $T(x)$ is weak completely continuous,

$$\langle T(x), y_k\rangle \to \langle T(x), y\rangle \quad \text{(strongly)}.$$

By the uniqueness of the limit, we get $\langle T(x), y\rangle = \langle T(x), y_0\rangle$. Since Tx is one to one, $y = y_0$ and so $y \in \mathcal{F}_x$. Since $\mathcal{F}_x$ is weakly closed, it follows from Lemma 3.2 that $\mathcal{F}$ has a weakly minimal point a such that $a \not\geq_{intD} \mathcal{F}$. Therefore, (WMEP) has at least one solution. This completes the proof. ∎

Definition 3.137. *Let (X, D) be an ordered Banach space, Y be a Banach space, and $\{C(x) : x \in X\}$ be a family of closed, pointed, and convex cones in Y. A map $T : X \to L(X, Y)$ is called positive with respect to the variable ordering relation $C(x)$ if*

$$\langle T(x), y\rangle \geq_{C(x)} 0, \quad \textit{for any}\ x, y \in D.$$

Equivalently,

$$\langle T(x), y\rangle \in C(x), \quad \forall x, y \in D.$$

An operator $K : X \to Y$ is called positive if

$$K(x) \geq_{C(x)} 0, \quad \textit{for any}\ x \in D.$$

We now consider the positive vector complementarity problem (PVCP): Finding $x \in D$ such that

$$\langle T(x), x\rangle \not\geq_{intC(x)} 0, \quad \langle T(x), y\rangle \geq_{C(x)} 0, \quad \forall y \in D.$$

The feasible set related to (PVCP) is defined as

$$\mathcal{F}_0 = \{x \in X : \ x \in D, \ \langle T(x), y\rangle \geq_{C(x)} 0, \quad \forall y \in D\}.$$

Let us consider the following problems.

The vector optimization problem $(\text{VOP})_{l0}$: finding $x \in \mathcal{F}_0$ such that $l(x) \in \text{LMin}_{intP} l(\mathcal{F}_0)$.

The weak minimal element problem $(\text{WMEP})_0$: finding $x \in \mathcal{F}_0$ such that $x \in \text{LMin}_{intD}\mathcal{F}_0$.

The positive vector complementarity problem (PVCP): finding $x \in \mathcal{F}_0$ such that

$$\langle T(x), x\rangle \not\geq_{intC(x)} 0.$$

The vector variational inequality problem (VVIP): finding $x \in D$ such that

$$\langle T(x), y - x\rangle \not\leq_{intC(x)} 0, \quad \forall y \in D.$$

The vector unilateral optimization problem (VUOP): for a given map $f : X \to Y$, finding $x \in D$ such that $f(x) \in \text{LMin}_{intP} f(D)$.

Definition 3.138. *A map $T : X \to L(X, Y)$ is said to be strictly monotone with respect to the variable ordering relation $C(x)$ if*

$$\langle T(x) - T(y), x - y\rangle \geq_{intC(x)} 0, \quad \forall x, y \in X, x \neq y.$$

Definition 3.139. *We say that $C(x)$ satisfies an inclusive condition if, for any $x, y \in X$,*

$$x \leq_{intD} y \quad \textit{implies that} \quad C(x) \subset C(y).$$

It is easy to see that, if $C(x) = C$ for all $x \in X$, where C is a closed, pointed, and convex cone in Y, then $C(x)$ satisfies the inclusive condition.

Example 3.140. Let $X = (-\infty, +\infty)$, $D = [0, +\infty)$, $Y = R^2$, and

$$C(x) = \begin{cases} \{(r\cos x, r\sin x) : \ r \geq 0, \ 0 \leq x \leq \pi/8\}, & \text{if } x \in (-\infty, \pi/8]; \\ \{(r\cos x, r\sin x) : \ r \geq 0, \ 0 \leq x \leq x\}, & \text{if } x \in (\pi/8, \pi/2); \\ D, & \text{if } x \in [\pi/2, +\infty) \end{cases}$$

for all $x \in X$. Then it is easy to check that $C(x)$ satisfies the inclusive condition.

Proposition 3.141. *Let T be strictly monotone with respect to the variable ordering relation $C(x)$ and x a solution of (PVCP). If C satisfies the inclusive condition, then x is a weakly minimal point of $\mathcal{F}_0$ (i.e., x solves $(WMEP)_0$).*

Proof. It is easy to see that $x \in \mathcal{F}_0 \subset D$. If $x \in \text{bd}(D)$ (where $\text{bd}(D)$ denotes the boundary of D), then x solves $(\text{WMEP})_0$. Otherwise, there exists $x' \in \mathcal{F}_0$ such that $x \geq_{intD} x'$ and so

$$x = x - x' + x' \in intD + D \subset intD,$$

which is a contradiction. If $x \in intD$, by the strict monotonicity of T,

$$\langle Tx, x - u\rangle \geq_{intC(x)} \langle T(u), x - u\rangle, \quad \forall u \in \mathcal{F}_0, u \neq x.$$

Suppose $x \geq_{intD} u$. Since T is positive, $\langle T(u), x - u\rangle \geq_{C(u)} 0$ and

$$\langle T(x), x - u\rangle \geq_{intC(x)} \langle T(u), x - u\rangle \geq_{C(u)} 0.$$

By the assumption, we get $P(u) \subset C(x)$ and so

$$\begin{aligned} &\langle T(x), x - u\rangle \\ &\in \langle T(u), x - u\rangle + intC(x) \subset C(u) + intC(x) \\ &\subset C(x) + intC(x) = intC(x). \end{aligned}$$

It follows that

$$\langle T(x), x - u\rangle \geq_{intC(x)} 0$$

and thus

$$0 \not\leq_{intC(x)} \langle T(x), x\rangle \geq_{C(x)} \langle T(x), u\rangle + k$$

for some $k \in intC(x)$. This implies

$$\langle T(x), u\rangle + k \not\geq_{intC(x)} 0.$$

Since $k \in intC(x)$ and $x \in \mathcal{F}_0$,

$$\langle T(x), u\rangle + k \in C(x) + intC(x) \subset intC(x)$$

and so

$$\langle T(x), u\rangle + k \geq_{intC(x)} 0,$$

which is a contradiction. Therefore, $x \geq_{intD} u$ does not hold, that is $x \not\geq_{intD} u$. It follows that $x \not\geq_{intD} \mathcal{F}_0$ and x solves $(\text{WMEP})_0$. This completes the proof.■

Proposition 3.142. *If x solves (PVCP), then x also solves (WVVI).*

Proof. Suppose x solves (PVCP). Then $x \in D$ and

$$\langle T(x), x\rangle \not\geq_{intC(x)} 0, \quad (Tx, y) \geq_{C(x)} 0, \quad y \in D.$$

If $\langle T(x), y - x\rangle \leq_{intC(x)} 0$, then

$$\langle T(x), x\rangle = -\langle T(x), y - x\rangle + \langle T(x), y\rangle \in intC(x) + C(x) \subset intC(x)$$

and so

$$\langle T(x), x\rangle \geq_{intC(x)} 0,$$

which is a contradiction. It follows that

$$\langle T(x), y - x\rangle \not\leq_{intC(x)} 0$$

and x solves (WVVI). This completes the proof. ■

Similarly, we can get other equivalence conditions. We have the following theorem.

Theorem 3.143. *Let (X, D) be an ordered Banach space with $intD \neq \varnothing$, Y be a Banach space, and $\{C(x) : x \in X\}$ be a family of closed, pointed, and convex cones in Y such that $intC(x) \neq \varnothing$ for all $x \in X$. Suppose that C satisfies the inclusive condition and*

(1) $T = Df$ is the Frechet derivative of the convex operator $f : X \to Y$;
(2) l is a weakly positive linear operator with respect to the variable ordering relation $C(x)$;
(3) T is strictly monotone with respect to the variable ordering relation $C(x)$.

If (PVCP) is solvable, then $(VOP)_{l0}$, $(WMEP)_0$, (PVCP), (WVVI), and (VUOP) have at least a common solution.

4

Vector Variational Principles

Ekeland's variational principle is an important tool for nonlinear analysis and optimization theory. It can be applied to derive the famous Caristi-Kirk's fixed point theorem and fixed point theorem for directional contractions, and to improve the so-called Ambrosetti-Rabinowitz "Mountain Pass" theorem. It was employed to improve and generalize Morse's critical theory. It has important applications in the geometry theory of Banach spaces and in the study of nonlinear operators in Banach spaces. It has many applications in control theory. It is used to study the existence of optimal solutions, optimality conditions for mathematical programming problems, stability and well-posedness of optimization problems, approximate optimal solutions and approximate duality theory and approximate saddle point theory, development of approximate algorithms for mathematical programming. It also has important applications in convex analysis. Along with the development of vector optimization and set-valued optimization, many authors have tried to improve it, generalize it and find as many applications as possible.

In the first part of this chapter, we obtain variants of variational principles for vector-valued functions and develop variational principles for set-valued functions. We derive vector versions of "Drop theorem", "Petal theorem" and "Caristi-Kirk fixed point theorem". We establish equivalences among these theorems and vector variational principles.

The study of well-posedness of an optimization problem is to investigate the behavior of the variable when the corresponding objective function value is close to the optimal value. In scalar optimization, the notion of well-posedness originates from Tykhonov [196] in dealing with unconstrained optimization problems. Its extension to the constrained case was introduced by Levitin and Polyak [132]. Since then, various notions of well-posedness have been defined and extensively studied (see [145], [225], [226] and a recent monograph [59]). In vector-valued and set-valued optimization, there have also been quite a number of publications on the topic of well-posedness (see, e.g., [11, 144, 141, 54, 99, 100, 101] and the references therein). The last two sections of this chapter are based on the results from [99] and [100]. We shall follow the embedding technique employed in [226] to introduce the notion of extended well-posedness for vector-valued and set-valued optimization problems. Corresponding to different understandings of "approximation" of the objective values to the optimal value set, we shall introduce different notions of well-posedness in vector-valued and set-valued optimization. We shall provide various characterizations and criteria for these types of extended well-posedness, generalizing most of Zolezzi's results in [226]. We shall derive new variants of vector variational principles and apply them to present sufficient conditions for these notions of extended well-posedness.

4.1 Variational Principles for Vector-Valued Functions

In this section, we present a unified variational principle for vector-valued functions. Generally speaking, this principle includes Nemeth's, Tammer's and Isac's variational principles for vector-valued functions as its special cases. And in some sense, it also includes Dentcheva and Helbig's variational principle for vector-valued functions as its special case. We establish a generalized variational principle for vector-valued functions without a normality assumption. The main tool we used in this section is Hausdorff maximality principle and scalarization method.

In 1972, Ekeland presented his original variational principle (see [62, 63]). We state it as the following Theorem 4.1.

Theorem 4.1 (Ekeland's variational principle). *Let (X, d) be a complete metric space, $\varphi : X \to I\!R \cup \{+\infty\}$ a proper lower semicontinuous (in short, l.s.c) function which is bounded below. Let there be given $\epsilon > 0$ and $\bar{x} \in X$ such that*

$$\varphi(\bar{x}) \leq \inf_{x \in X} \varphi(x) + \epsilon.$$

Then, for any $\lambda > 0$, there exists $x_\epsilon \in X$ such that

(i) $\varphi(x_\epsilon) \leq \varphi(\bar{x})$;
(ii) $d(x_\epsilon, \bar{x}) \leq \lambda$;
(iii) $\varphi(x_\epsilon) < \varphi(x) + \epsilon/\lambda d(x, x_\epsilon), \quad \forall x \in X \backslash \{x_\epsilon\}$.

Generally speaking, there are two ways to prove this theorem: one is to apply the Hausdorff maximality principle to the epigraph of f under the order induced by Phelps' cone, and the other is to utilize a dissipative dynamic system.

Ekeland's variational principle was extended to the vector case by many authors, for instance, Nameth [151], Tammer [185], Isac [111].

First of all, we recall some basic concepts and some previous results in this field.

We say that a set A is a partially ordered set if there is a partial order defined on A, and we denote "$\prec$".

Let A be an arbitrary set and "$\prec$" be a partial order on A. We say that A is totally ordered by "$\prec$" if, for any $a, b \in A$, either $a \prec b$ or $b \prec a$ holds.

The following theorem, known as the Hausdorff maximality principle, is equivalent to the so-called axiom of choice, and Zorn's lemma, which are fundamental theorems in abstract analysis (see, e.g., [4]).

Theorem 4.2. *Let A be a set partially ordered by relation "$\prec$". Then there exists a maximal subset (with respect to the set inclusion relation), which is totally ordered by "$\prec$".*

Let Y be a Hausdorff topological vector space ordered by a nontrivial convex cone C. We introduce the following definitions.

A subset A of Y is called full if $A = (A + C) \cap (A - C)$.

We say that a cone C is normal if C is a pointed and convex cone and the zero point of Y has a neighborhood base consisting of full sets.

Let $C_0 \subset C$ be a convex cone and $\{x_\alpha\}_{\alpha \in I}$ be a net of C_0. The net $\{x_\alpha\}_{\alpha \in I}$ is said to be C_0 increasing if $x_\alpha - x_\beta \in C_0$, whenever $\alpha \geq \beta$.

A convex cone $C_0 \subset C$ is called C bound regular (sequentially C bound regular) if each C_0 increasing and C order bounded net (sequence) in C_0 converges to an element of C_0.

C_0 is said to be complete if any Cauchy net $\{c_v\}_{v \in I}$ of C_0 converges to some point $c_0 \in C_0$.

Let X be a nonempty set, Y be a Hausdorff topological vector space, and $C \subset Y$ be a nontrivial convex cone. A vector-valued function $r : X \times X \to Y$ is called a C metric function if it satisfies the following conditions: for any $x, y, z \in X$,

(i) $r(x, x) = 0$;
(ii) $r(x, y) \leq_C 0$ implies $x = y$;
(iii) $r(x, y) = r(y, x)$;
(iv) $r(x, z) \leq_C r(x, y) + r(y, z)$.

Let $B(0)$ denote a neighborhood base of the zero element of Y. Let

$$X(U, a) = \{x \in X : r(a, x) \in U\}, \qquad U \in B(0), a \in X.$$

If $\{X(U, a) : U \in B(0), a \in X\}$ form a neighborhood base for a Hausdorff topology on X, then the resulting topology is called the Hausdorff topology induced by r. We denote by (X, r) the Hausdorff space with its topology induced by r. If (X, r) is complete, we say that (X, r) is a complete C metric space.

Remark 4.3. **(i)** We have slightly modified the definition of C metric r in Nemeth [151]. If C is pointed, then our definition of C is the same as the one in Nemeth [151].

(ii) Generally speaking, a C metric r function may not induce a Hausdorff topology on X unless some special assumption is made on C.

We assume that X is a topological vector space, Y is a topological vector space ordered by a nontrivial convex set C, $c_0 \in C\backslash\{0\}$, $f : X \to Y$ is a vector-valued function.

Let U be a closed subset of X. We say that f is C lower semicontinuous (l.s.c.) on U if, for any $y \in Y$, the set $\{x \in U : f(x) \in y - clC\}$ is closed.

We say that f is C order bounded below (some times, we call it C bounded below or C lower bounded) on U if $f(x) \geq_C y_0$, $\forall x \in U$ for some $y_0 \in Y$.

f is said to be (c_0, C) lower semicontinuous on U if, for any $t \in \mathbb{R}$, the set $\{x \in U : f(x) \in tc_0 - clC\}$ is closed.

f is called submonotone (with respect to C) if, from the conditions:

(a) $\lim_\mu x_\mu = x$, where $\{x_\mu\}_{\mu \in I}$ is a net in X, $(I, \leq)$ is a totally ordered set;

(b) $f(x_v) \leq_C f(x_\mu)$, whenever $v \geq \mu$,

it follows that $f(x) \leq_C f(x_v), \forall v \in I$.

If C is a closed and convex cone, then it is not difficult to see that if f is l.s.c on X, then f is both (c_0, C) l.s.c. and submonotone (with respect to C).

Example 4.4. Let $U = [-2, 2] \subset \mathbb{R}$ and $C = \mathbb{R}^2_+$. Let $f = (f_1, f_2) : U \to \mathbb{R}^2$ be defined as follows:

$$f_1(x) = x,$$
$$f_2(x) = \begin{cases} x^2, & \text{if } x \in [-2, -1], \\ x + 3, & \text{if } x \in (-1, 1), \\ 1/2, & \text{if } x = 1, \\ x^2, & \text{if } x \in (1, 2]. \end{cases}$$

Then f is $((3/2, 2), C)$ l.s.c. on U.

Now we state several forms of Ekeland's variational principle for vector-valued functions and some of their corollaries obtained by Nemeth [151], Tammer [185] and Isac [111].

Definition 4.5. *Let M be a subset of Y, and H be a subset of C. A point $x \in M$ is called an H near to minimal point of M if*

$$(x - H - C) \cap M = \varnothing.$$

First, we introduce Tammer's variational principle for vector-valued functions.

Let K be a nonempty closed subset of X and C be a convex subset of Y.

Definition 4.6. *Let $\epsilon > 0$. A point $x_\epsilon \in K$ is called an ϵ-minimal solution of f on K if*

$$f(x) - f(x_\epsilon) + \epsilon c^0 \not\leq_{C\setminus\{0\}} 0, \quad \forall x \in U.$$

Definition 4.7. *Let $\epsilon > 0$ and $intC \neq \emptyset$. A point $x_\epsilon \in K$ is called a weakly ϵ-minimal solution of f on K if*

$$f(x) - f(x_\epsilon) + \epsilon c^0 \not\leq_{intC} 0, \quad \forall x \in K.$$

The following assumption is made.

Assumption 4.8. *C and D are proper subsets of Y and c_0 is an element of $Y \setminus \{0\}$. The following conditions hold:*

(a) C is an open convex subset of Y with $0 \in clC \setminus C$ and

$$Y = \cup\{clC + \alpha c_0 | \alpha \in I\!R\}.$$

(b) $\alpha c_0 \in D \setminus \{0\}$ for each $\alpha \in I\!R_+$, and $0 \in clD \setminus D$.
(c) $clC + (D\setminus\{0\}) \subset C$.
(d) $\partial C + \partial C \subset clC$.

Theorem 4.9 (Tammer [185]). *Let Y be a topological vector space and X be a real Banach space. Let Assumption 4.8 hold. Assume that $f : X \to Y$ is (c_0, C) l.s.c. and C bounded below on U.*

Given $\epsilon > 0$, x_0 is a weakly ϵ-minimal solution of f on K. Then there exists an $x_\epsilon \in K$ such that

(i) $f(x) - f(x_\epsilon) + \epsilon c^0 \notin -D\setminus\{0\}, \quad \forall x \in K;$
(ii) $||x_\epsilon - x_0|| \leq \sqrt{\epsilon};$
(iii) $f(x) - f(x_\epsilon) + \sqrt{\epsilon}||x - x_\epsilon||c^0 \notin -D\setminus\{0\}, \quad \forall x \in K.$

The following corollary can be obtained by replacing both C and D in Theorem 4.9 by $intC$.

Corollary 4.10 (Tammer [185]). *Assume that C is a convex cone with nonempty interior and $c_0 \in intC$. Assume that $f : X \to Y$ is (c_0, C) l.s.c. and C bounded from below on K.*

Given $\epsilon > 0$, x_0 is an ϵ-minimal solution of f on K. Then there exists an $x_\epsilon \in K$ such that

(i) *x_ϵ is a weakly ϵ-minimal solution of f on K;*
(ii) $||x_\epsilon - x_0|| \leq \sqrt{\epsilon};$
(iii) $f(x) - f(x_\epsilon) + \sqrt{\epsilon}||x - x_\epsilon||c^0 \not\leq_{intC} 0, \quad \forall x \in K.$

Second, we present Isac's vector variational principle.

Theorem 4.11 (Isac [111]). *Let (X, d) be a complete metric space, Y a locally convex Hausdorff space and $C \subset Y$ a normal cone. Let $c^0 \in C \backslash \{0\}$, and $\epsilon > 0$. Assume that $x^* \in X$ satisfies*

(i) $f(x) - f(x^*) + \epsilon c^0 \not\leq_C 0, \forall x \in X$;
(ii) $\forall x \in X, \alpha > 0$, *the set* $\{y \in Y : f(y) - f(x) + \alpha d(x, y) c^0 \leq_C 0\}$ *is closed.*

Then there exist $\lambda_1 > 0$, *and* $x' \in X$ *such that*

(iii) $f(x') \leq_C f(x^*)$,
(iv) $d(x', x^*) \leq \lambda_1$,
(v) $f(x) - f(x') + \epsilon/\lambda_1 d(x', x) c^0 \not\leq_C 0, \quad \forall x \in X \backslash \{x'\}$.

Theorem 4.13 below is a variant of Theorem 6.1 of Nemeth [151]. To prove it, we need the following lemma.

Now, let X be a topological space and Y a locally convex Hausdorff space, $C \subset Y$ a convex cone with nonempty interior $intC$.

Lemma 4.12. *Let $C_0 \subset C$ be C bound regular complete. Assume that there exists $U \in B(0)$ such that $H = C_0 \backslash U \neq \varnothing$ and that there exists $x^* \in X$ such that $(f(x^*) - C) \cap f(X)$ is C bounded below. Then, for any $\epsilon > 0$, there exists an $x'_\epsilon \in X$ such that*

(i) $f(x'_\epsilon) \leq_C f(x^*)$;
(ii) $(f(x'_\epsilon) - \epsilon H - C) \cap f(X) = \varnothing$.

Proof. We prove it by contradiction. Suppose that $\exists \epsilon > 0$ such that the conclusion of Lemma 4.12 does not hold. Then, for $x_1 = x^*$, $(f(x_1) - \epsilon H - C) \cap f(X) \neq \varnothing$, i.e., $\exists x_2 \in X$, $h_1 \in H$ such that

$$f(x_2) - f(x_1) + \epsilon h_1 \leq_C 0.$$

For x_2, $\exists x_3 \in X$, $h_2 \in H$ such that

$$f(x_3) - f(x_2) + \epsilon h_2 \leq_C 0.$$

Continuing the process, for x_k, $\exists x_{k+1} \in X$, $h_k \in H$ such that

$$f(x_{k+1}) - f(x_k) + \epsilon h_k \leq_C 0,$$

we have

$$f(x_{k+1}) - f(x_1) + \epsilon \sum_{i=1}^{k} h_i \leq_C 0, \quad \forall k \in N.$$

Since $(f(x^*) - C) \cap f(X)$ is C bounded below, it follows that $\exists y_0 \in Y$ such that $f(x_k) \geq_C y_0$, $\forall k \in N$. So $\{\sum_{i=1}^{k} h_i\}$ is C_0 increasing, C bounded. Hence $\{\sum_{i=1}^{k} h_i\}$ is convergent. Consequently, $h_k \in U$ when k is large enough, which contradicts the assumption $H = C_0 \backslash U$. ■

Theorem 4.13. *Let C be a nontrivial closed and convex cone, $C_0 \subset C$ a convex cone which is C bound regular complete and $C_0 \cap -C \subset -C_0$. Let (X, r) be a complete C_0 metric space. Let $r(\cdot, a)$ be continuous with respect to the topology of X induced by r for any $a \in X$. Let f be submonotone (with respect to C and the topology of X induced by r). Assume that there exists an $x^* \in X$ such that*

(i) $(f(x^*) - C) \cap f(X)$ *has a C lower bound.*

Then, for any $\epsilon > 0$, there exists x_ϵ such that

(ii) $f(x^*) - f(x_\epsilon) - \epsilon r(x^*, x_\epsilon) \geq_C 0,$
(iii) $f(x_\epsilon) - f(x) - \epsilon r(x_\epsilon, x) \not\geq_C 0, \quad \forall x \in X \backslash \{x_\epsilon\}.$

If there exists $U \in B(0)$ such that $H = C_0 \backslash U \neq \varnothing$, then there exists $x'_\epsilon \in X$ such that

(iv) $f(x'_\epsilon) \leq_C f(x^*),$
(v) $(f(x'_\epsilon) - \epsilon H - C) \cap f(X) = \varnothing.$

For x'_ϵ as above, there is an $x''_\epsilon \in X$ such that

(vi) $f(x''_\epsilon) - f(x) - \epsilon r(x''_\epsilon, x) \not\geq_C 0, \quad \forall x \in X \backslash \{x''_\epsilon\};$
(vii) $r(x'_\epsilon, x''_\epsilon) \in U.$

Proof. Define a relation $\prec$ on $T = \{(x, f(x)) \in X \times Y : x \in X, f(x) \leq_C f(x^*) - \epsilon r(x^*, x)\}$ by putting $(x, f(x)) \prec (y, f(y))$ iff

$$f(y) - f(x) - \epsilon r(x, y) \geq_C 0.$$

It is easy to see that $\prec$ is a partial order on T. Applying the Hausdorff maximality principle, we have a subset Z_1 of T, which is maximal with respect to the set inclusion relation, and also totally ordered with respect to $\prec$, with $(x^*, f(x^*))$ as its upper bound (with respect to the order $\prec$). We will show that Z_1 contains its infimum with respect to $\prec$.

We introduce the relation $\leq$ on $X_0 = \{x \in X : (x, f(x)) \in Z_1\}$ by putting $x \leq y$ iff $(x, f(x)) \prec (y, f(y))$. Then X_0 is totally ordered with respect to $\leq$. Now let us demonstrate that the filter of its lower section (for the concept of the lower section of a totally ordered set and the concept of the filter of its lower section, the reader may refer to [122]) is Cauchy by contradiction.

Suppose that there exists a neighborhood U' of 0 such that, for each $z \in X_0$, $\exists x, y \in X_0$ with $x \leq z, y \leq z$ such that

$$r(x, y) \notin U'.$$

Suppose that $x \leq y$. Put $v_1 = y, v_2 = x$. Then $r(v_2, v_1) \notin U'$ and

$$f(v_1) - f(v_2) - \epsilon r(v_2, v_1) \geq_C 0.$$

Starting with x instead of z, we can continue this procedure. As a result, we can obtain a decreasing sequence $\{v_k\}$ in X_0 such that

$$r(v_{2k}, v_{2k-1}) \notin U', \quad \forall k. \tag{4.1}$$

From the definition of $\leq$ on X_0, we also have

$$f(v_k) - f(v_{k+1}) - \epsilon r(v_{k+1}, v_k) \geq_C 0, \quad \forall k.$$

So we have

$$f(v_1) - f(v_{k+1}) - \epsilon \sum_{i=1}^{k} r(v_{i+1}, v_i) \geq_C 0.$$

Since

$$f(v_k) \in (f(x^*) - C) \cap f(X), \quad \forall k,$$

by (i), the sequence $\{f(v_k)\}$ has a C lower bound, say y_0, i.e.,

$$f(v_{k+1}) - y_0 \geq_C 0,$$

this relation yields

$$f(v_1) - y_0 - \epsilon \sum_{i=1}^{k} r(v_{i+1}, v_i) \geq_C 0,$$

which shows that $\sum_{i=1}^{k} r(v_{i+1}, v_i)$ are C order bounded. This simply contradicts (4.1).

The contradiction shows that the lower section of X_0 forms a Cauchy filter, which converges by the completeness of X to $x_\epsilon \in X$.

Since f is submonotone with respect to C, we have

$$f(p) - f(x_\epsilon) \geq_C 0, \text{ for every } p \in X_0. \tag{4.2}$$

Let p be an arbitrary point in X_0. For every $p \leq q$, we have

$$f(q) - f(p) - \epsilon r(p, q) \geq_C 0,$$

which, together with (4.2), yields

$$f(q) - f(x_\epsilon) - \epsilon r(p, q) \geq_C 0.$$

Letting $p \to x_\epsilon$ in this relation and taking into account the fact that C is closed and $r(\cdot, q)$ is continuous, it follows that

$$f(q) - f(x_\epsilon) - \epsilon r(x_\epsilon, q) \geq_C 0,$$

that is, $(x_\epsilon, f(x_\epsilon)) \prec (q, f(q))$, for each q with $(q, f(q)) \in Z_1$. Now that Z_1 is maximal, $(x_\epsilon, f(x_\epsilon))$ must be in Z_1 and it is the infimum of Z_1 with respect to $\prec$.

The last assertion implies also that there does not exist any v in $X\backslash\{x_\epsilon\}$ such that

$$(v, f(v)) \prec (x_\epsilon, f(x_\epsilon)).$$

Thus (ii) and (iii) have been proved.

If H is defined as in the theorem, then by Lemma 4.12, then we deduce the existence of x'_ϵ such that (iv) and (v) hold. If we proceed as above taking x'_ϵ in place of x^*, then we can get an x''_ϵ so as to have (iii) and (ii) with x''_ϵ instead of x'_ϵ, that is, to have the relation

$$f(x''_\epsilon) \leq_C f(x'_\epsilon) - \epsilon r(x''_\epsilon, x'_\epsilon). \tag{4.3}$$

Suppose that (vi) does not hold. Then, we have $r(x'_\epsilon, x''_\epsilon) \in C_0\backslash U = H$, that is,

$$f(x'_\epsilon) - \epsilon r(x'_\epsilon, x''_\epsilon) - C = f(x'_\epsilon) - \epsilon H - C,$$

this relation together with (4.3) contradicts (v). ■

Remark 4.14. **(i)** Theorem 4.13 is slightly different from Theorem 6.1 in Nemeth [151]. We do not require that C be pointed and normal.

(ii) If the relation $\prec$ is defined on $f(V)$ as that in Nemeth [151]:

$$\forall f(v_1), f(v_2) \in f(V), f(v_1) \prec f(v_2) \text{ iff } f(v_1) - f(v_2) + r(v_1, v_2) \leq_C 0.$$

This relation may not be well-defined. For example, $V = [-1, 1] \subset \mathbb{R}, E = \mathbb{R}, C = C_0 = \mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}, r(v_1, v_2) = |v_1 - v_2|, \forall v_1, v_2 \in V, f(v) = v^2, \forall v \in V$. Observe that $f(-1) = f(1) = 1 \in f(V), f(1/2) = 1/4 \in f(V)$. Thus,

$$f(1/2) - f(1) + |1/2 - 1| = 1/4 \leq_C 0, \tag{4.4}$$

hence $1/4 \prec 1$ holds.

However,

$$f(1/2) - f(-1) + |1/2 - (-1)| = 3/4 \not\leq_C 0, \tag{4.5}$$

hence $1/4 \prec 1$ does not hold. (4.4) and (4.5) lead us to confusion about the relation $\prec$.

Now we will take a unified approach to the above three vector variational principles: Nemeth's, Tammer's and Isac's vector variational principles, and we will explore the relationship between the unified principle and the three principles (or their variants).

Theorem 4.15. *Let C be a convex cone, $C_0 \subset C$ be a C bound regular complete convex cone and $C_0 \cap -C \subset -C_0$. Suppose that (X, r) is a complete C_0 metric space. Assume that one of conditions (i) and (ii) holds*

(i) *f is submonotone (with respect to C) and C is closed and, $\forall a \in X, r(\cdot, a)$ is continuous with respect to the topology of X induced by r,*

(ii) $\forall \alpha > 0, \forall x \in X, \{y \in X : f(y) + \alpha r(x,y) - f(x) \leq_C 0\}$ *is closed.*

Furthermore, assume that there exists an $x^* \in X$ *such that*

(iii) $(f(x^*) - C) \cap f(X)$ *is* C *lower bounded.*

Then, for any $\epsilon > 0$, *there exists an* $x_\epsilon \in X$ *such that*

(iv) $f(x^*) - f(x_\epsilon) - \epsilon r(x^*, x_\epsilon) \geq_C 0$; *and*
(v) $f(x_\epsilon) - f(x) - \epsilon r(x_\epsilon, x) \not\geq_C 0, \quad \forall x \in X \backslash \{x_\epsilon\}$.

If there is a $U \in B(0)$ *such that* $H = C_0 \backslash U \neq \varnothing$, *then there exists* $x'_\epsilon \in X$ *such that*

(vi) $f(x'_\epsilon) \leq_C f(x^*)$; *and*
(vii) $(f(x'_\epsilon) - \epsilon H - C) \cap f(X) = \varnothing$.

For x'_ϵ *as above, there exists an* $x''_\epsilon \in X$ *satisfying*

(viii) $f(x''_\epsilon) - f(x) - \epsilon r(x''_\epsilon, x) \not\geq_C 0, \quad \forall x \in X \backslash \{x''_\epsilon\}$; *and*
(ix) $r(x'_\epsilon, x''_\epsilon) \in U$.

Proof. By Theorem 4.13, if (i) holds, then Theorem 4.15 is true.

Now we assume that (ii) holds. We follow the proof of Theorem 4.13. We need only to show that $(x_\epsilon, f(x_\epsilon))$ is an infimum of Z_1 with respect to the relation "$\prec$" defined in the proof of Theorem 4.13. Taking $\alpha = \epsilon$ in (ii), we know that $\forall x \in X_0$, $X(x) = \{y \in X : y \leq x\} = \{y \in X : f(y) - f(x) + \epsilon r(x,y) \leq_C 0\}$ is closed. So

$$f(x_\epsilon) - f(x) + \epsilon r(x_\epsilon, x) \leq_C 0, \quad \forall x \in X_0.$$

That is, $(x_\epsilon, f(x_\epsilon)) \prec (x, f(x))$ in Z_1. Since Z_1 is maximal, $(x_\epsilon, f(x_\epsilon))$ must be in Z_1, so $(x_\epsilon, f(x_\epsilon))$ is an infimum of Z_1 with respect to $\prec$. The rest of the proof is the same as that of Theorem 4.13 with only some notation changes.■

It is easy to see that Theorem 4.15 is stronger than Theorem 4.13, hence also stronger than Nemeth's vector variational principle (Theorem 6.1 in Nemeth [151]).

Lemma 4.16. *Let* (X, d) *be a complete metric space,* C *be a convex cone and* $c^0 \in C \backslash \{0\}$ *be such that there exists a* $\lambda \in C^*$ *with* $\lambda(c^*) > 0$. *Let* $C_0 = \{\alpha c^0 : \alpha \geq 0\}$, $r(x,y) = d(x,y)c^0$, $\forall x, y \in X$. *Then*

(i) C_0 *is a* C *bound regular complete convex cone;*
(ii) (X, r) *is a complete* C_0 *metric space and* $\forall a \in X, r(\cdot, a)$ *is continuous; and*
(iii) $C_0 \cap -C \subset -C_0$.

Proof. (i) C_0 is complete. In fact, for any Cauchy net $\{\alpha_v c^0\}$, and any $U \in B(0)$, $\exists v_0$ such that $\alpha_{v_1} c^0 - \alpha_{v_2} c^0 \in U$ whenever $v_1, v_2 \geq v_0$. Given $\lambda \in C^*$ such that $\lambda(c^0) > 0, |\alpha_{v_1} - \alpha_{v_2}| = \lambda(u)/\lambda(c^0)$ for some $u \in U$. As λ is continuous, it is easy to see $\{\alpha_v\}$ is a Cauchy net, and $\{\alpha_v\}$ converges to some $\alpha_0 \geq 0$. Hence, $\{\alpha_v c^0\}$ converges to $\alpha_0 c^0$. Now we prove that C_0 is C bound regular. Given a monotonically increasing net $\{\alpha_v c^0\} \subset C_0$, i.e., $\alpha_{v_1} \geq \alpha_{v_2}$, whenever $v_1 \geq v_2$, and it is C bound, i.e., $\exists c^1 \in C$ such that $\alpha_v c^0 \leq_C c^1, \forall v$, then, given $\lambda \in C^*$ with $\lambda(c^0) > 0$, we have $\alpha_v \lambda(c^0) \leq \lambda(c^1)$, which implies that $\{\alpha_v\}$ is bounded. This combined with the monotonicity of $\{\alpha_v\}$ yields that $\{\alpha_v\}$ converges to α_0. So $\alpha_v c^0 \to \alpha_0 c^0 \in C_0$. Moreover, it is obvious that C_0 is a convex cone.

(ii) It is not hard to see that (X, r) is a C_0 metric space and the topology induced by r is equivalent to that of (X, d). So (X, r) is C_0 complete. It is easy to see that $\forall a \in X, r(\cdot, a)$ is continuous since $d(\cdot, a)$ is continuous.

(iii) If $\alpha c^0 \in -C$, then $\lambda(\alpha c^0) = \alpha\lambda(c^0) \leq 0$, which implies $\alpha \leq 0$. Hence $\alpha = 0$, i.e., $\alpha c^0 = 0 \in -C_0$. ■

Corollary 4.17. *Let (X, d) be a complete metric space, C a convex cone, $c^0 \in C\backslash\{0\}$ be such that $\exists\lambda \in C^*$ with $\lambda(c^0) > 0$. Let $f : X \to Y$ be C order lower semicontinuous. Given $\epsilon > 0$, $x^* \in X$ satisfying*

(iv) $(f(x^*) - C) \cap f(X)$ *is C lower bounded; and*
(v) $f(x) - f(x^*) + \epsilon c^0 \not\leq_C 0, \forall x \in X.$

Then for any $\lambda_1 > 0$, there exists an $x' \in X$ such that

(vi) $f(x') \leq_C f(x^*)$;
(vii) $d(x', x^*) \leq \lambda_1$; *and*
(viii) $f(x) - f(x') + \epsilon/\lambda_1 d(x', x)c^0 \not\leq_C 0, \forall x \in X\backslash\{x'\}.$

Proof. Replacing $r(x, y)$ with $d(x, y)c^0/\lambda_1$, choosing U such that $c^0 \notin U$ and $\lambda(u)/\lambda(c^0) \leq 1$ for any $u \in U$, letting $H = C_0\backslash U$ (C_0 is as defined in Lemma 4.16) and applying Lemma 4.16 and Theorem 4.15, we can establish Corollary 4.17. ■

The following Corollary 4.18 improves Theorem 4.11.

Corollary 4.18. *Let (X, d) be a complete metric space, C a convex cone, $c^0 \in C\backslash\{0\}$ be such that there exists a $\lambda \in C^*$ with $\lambda(c^0) > 0$. Given $\epsilon > 0$, there exists an $x^* \in X$ satisfies*

(i′) $f(x) - f(x^*) + \epsilon c^0 \not\leq_C 0, \quad \forall x \in X;$

and $(f(x^*) - C) \cap f(X)$ is C lower bounded. Let (ii) in Theorem 4.11 hold (or f is submonotone with respect to C and C is closed). Then for any $\lambda_1 > 0$, there exists an $x' \in X$ such that (iii),(iv) and (v) in Theorem 4.11 hold.

Since the proof of Corollary 4.18 is the same as that of the Corollary 4.17, we omit it.

To illustrate the relationship between Corollary 4.17 and Theorem 4.11, Corollary 4.18 and Theorem 4.13, we prove the following two lemmas.

Lemma 4.19. *If C is a closed and convex cone, then the following two statements are equivalent:*

(ix) $c^0 \in C\backslash\{0\}$, $\exists\lambda \in C^*$ *such that* $\lambda(c^0) > 0$;
(x) $c^0 \in C\backslash(-C)$.

Proof. Assume that (ix) hold. Then, by $\lambda(c^0) > 0$, we know by contradiction that $c^0 \in C\backslash -C$, i.e., (x) holds true.

Now we assume that (x) holds. We prove, by contradiction, that (ix) holds. Suppose that $\forall\lambda \in C^*, \lambda(c^0) = 0$. Then $c^0 \in (C^*)^* = C$. In addition, $\lambda(-c^0) = 0$, $\forall\lambda \in C^*$, which implies that $-c^0 \in (C^*)^* = C$, i.e., $c^0 \in -C$. Hence $c^0 \in C \cap (-C)$, contradicting (x). ■

Lemma 4.20. *The following statements are true.*

(xi) *If C is a nontrivial, pointed, closed and convex cone, then* $\forall c^0 \in C\backslash\{0\}$, $\exists\lambda \in C^*$ *such that* $\lambda(c^0) > 0$;
(xii) *If C is a normal cone, then* $\forall c^0 \in C\backslash\{0\}$, $\exists\lambda \in C^*$ *such that* $\lambda(c^0) > 0$; *and*
(xiii) *If C is a nontrivial convex cone with nonempty interior $intC$, then* $\forall c^0 \in intC$, $\forall\lambda \in C^*\backslash\{0\}, \lambda(c^0) > 0$.

Proof. (xi) When C is a nontrivial pointed, closed and convex cone, by Lemma 4.19, we know (xi) holds;

(xii) When C is a normal cone, $Y^* = C^* - C^*$. We show that by contradiction that (xii) holds. Otherwise, suppose $\exists c^0 \in C\backslash\{0\}$ such that $\forall\lambda \in C^*$ $\lambda(c^0) = 0$. Then $\forall\mu \in Y^*$, $\mu(c^0) = 0$, hence $c^0 = 0$, which contradicts $c^0 \neq 0$. Hence (xii) holds.

(xiii) $c^0 \in intC \neq \varnothing$. We prove that $\forall\lambda \in C^*\backslash\{0\}$, $\lambda(c^0) > 0$. Otherwise, if $\lambda_0 \in C^*\backslash\{0\}$ such that $\lambda_0(c^0) = 0$. Let $z_0 \in Y$ with

$$\lambda_0(z_0) > 0. \tag{4.6}$$

As $c^0 \in intC$, $c^0 - \delta z_0 \in C$ when $\delta > 0$ is small enough. Hence $\lambda_0(c_0 - \delta z_0) \geq 0 \Longrightarrow \lambda_0(z_0) \leq 0$, contradicting (4.6). So (xiii) holds. ■

Remark 4.21. By Lemma 4.19 and Lemma 4.20, the assumptions on the dominating cone C in Corollary 4.17 and Corollary 4.18 are much weaker than those in Theorem 4.11 and Theorem 4.13.

Remark 4.22. **(i)** If f is C lower bounded, then **(v)** in Corollary 4.17 holds automatically;
(ii) So long as we slightly strengthen the conditions of Tammer's vector variational principle (Corollary 4.10): let f be C order lower semicontinuous. We can have a stronger version of Tammer's vector variational principle (see Corollary 4.10) by setting $\lambda_1 = \sqrt{\epsilon}, c^0 \in intC$.

Remark 4.23. It is obvious that (i) in Theorem 4.11 implies (i') in Corollary 4.18. Hence, if (ii) in Theorem 4.11 holds, then Corollary 4.18 is an improvement of Isac's vector variational principle.

Now we show other generalizations of Isac's vector variational principle (see Isac [111]).

Let (X, d) be a complete metric space, and let (Y, C) be an ordered Hausdorff topological vector space in which the ordering is induced by a closed and convex cone C with nonempty interior.

Definition 4.24. *We say that a vector-valued function $\Phi : X \times X \to Y$ is a half distance if the following properties are satisfied:*

1. $\Phi(x, x) = 0, \quad \forall x \in X;$
2. $\Phi(x, y) \leq_C \Phi(x, z) + \Phi(z, y), \quad \forall x, y, z \in X.$

The family of half distance is not empty (see Isac [111]). For example, every distance measure is a half distance and for every $g : X \to Y$, the function defined by

$$\Phi(x, y) = g(y) - g(x), \quad \forall x, y \in X$$

is a half distance. Moreover, if L is an arbitrary vector space and $T : L \to Y$ is a subadditive function such that $T(0) = 0$, then, for every $h : X \to L$, the function defined by

$$\Phi(x, y) = T(h(y) - h(x)), \quad \forall x, y \in X,$$

is a half distance. Thus, the family of half distance is a rich one.

Definition 4.25. *Let $\Gamma : X \rightrightarrows X$ be a dynamic system. $x^* \in X$ is said to be a critical point of Γ if $\{x^*\} = \Gamma(x^*)$.*

The following theorem about the existence of a critical point will be used.

Theorem 4.26 (Dancs-Hegedus-Medvegyev [50]). *Let (X, d) be a complete metric space and $\Gamma : X \rightrightarrows X$ be a dynamic system. If the following conditions are satisfied:*

(i) *$\Gamma(x)$ is a closed set, $\forall x \in X$;*
(ii) $x \in \Gamma(x), \forall x \in X;$
(iii) $x_2 \in \Gamma(x_1) \Longrightarrow \Gamma(x_2) \subset \Gamma(x_1), \forall x_1, x_2 \in X;$
(iv) *for every sequence $\{x_k\}_{k \in N} \subset X$ satisfying $x_{k+1} \in \Gamma(x_k)$, we have*

$$\lim_{n \to \infty} d(x_k, x_{k+1}) = 0.$$

Then, Γ has a critical point $x^* \in X$. Moreover, for any $\hat{x} \in X$, there is a critical point of Γ in $\Gamma(\hat{x})$.

Let us recall the nonlinear scalarization function ξ_{ca} discussed in Chapter 1: For $c \in intC$, $a \in Y$, the nonlinear function is defined by

$$\xi_{ca}(y) = \min\{t \in \mathbb{R} : y \in a + tc - C\}, \quad \forall y \in Y.$$

We know that ξ_{ca} is convex, continuous, increasing and strictly increasing.

Now we set $a = 0$, denote ξ_{c0} by ξ_c for simplicity. In this case, ξ_c is subadditive, convex, continuous, increasing and strictly increasing.

Next theorem is a generalization of the Ekeland's variational principle for a vector-valued function and an improvement of Isac's vector variational principle (see Theorem 4.11).

Theorem 4.27. *Let (X, d) be a metric space and $\Phi : X \times X \to Y$ be a half distance. If, for any element $c^0 \in intC$, the following assumptions are satisfied:*

(i) *for any $x \in X$, the set $\{y \in X : \Phi(x,y) + c^0 d(x,y) \leq_C 0\}$ is closed; and*
(ii) *there exist $v_0 \in X$ and $w_0 \in Y$ such that $\Phi(v_0, x) \geq_C w_0, \forall x \in X$,*

then there exists an $x^* \in X$ such that

$$\Phi(x^*, x) + c^0 d(x^*, x) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\}.$$

Proof. Consider the dynamic system

$$\Gamma(x) = \{y \in X : \Phi(x,y) + c^0 d(x,y) \leq_C 0\}, \quad \forall x \in X.$$

The claim is proved if we show that Γ has a critical point in X. To this aim, it is sufficient to verify the assumptions of Dancs-Hegdus-Medvegyev Theorem (Theorem 4.26).

From assumption (i), we have that $\Gamma(x)$ is closed $\forall x \in X$, that is, (i) of Theorem 4.26 is satisfied. Using the properties of d and Φ, we have that $x \in \Gamma(x)$, $\forall x \in X$ which means that (ii) of Theorem 4.26 is also satisfied. To verify (iii) of Theorem 4.26, we consider two elements $x_1, x_2 \in X$ such that $x_2 \in \Gamma(x_1)$. We need to show that $\Gamma(x_2) \subset \Gamma(x_1)$. It follows from $x_2 \in \Gamma(x_1)$ that

($\mathbf{i_1}$) $\Phi(x_1, x_2) + c^0 d(x_1, x_2) \leq_C 0.$

Let $z \in \Gamma(x_2)$. Then

($\mathbf{i_2}$) $\Phi(x_2, z) + c^0 d(x_2, z) \leq_C 0.$

We will have $z \in \Gamma(x_1)$ if we show that

$$\Phi(x_1, z) + c^0 d(x_1, z) \leq_C 0.$$

From (i_1), there is an element $c_1 \in C$, such that

$$\Phi(x_1, x_2) + c^0 d(x_1, x_2) = -c_1,$$

and from (i_2), there is an element $c_2 \in C$, such that

$$\Phi(x_2, z) + c^0 d(x_2, z) = -c_2.$$

Since Φ is a half distance, there is an element $c_3 \in C$, such that

$$\Phi(x_1, z) = \Phi(x_1, x_2) + \Phi(x_2, z) - c_3.$$

Moreover, it is obvious that there exists an element $c_4 \in C$ such that

$$c^0 d(x_1, z) = c^0 d(x_1, x_2) + c^0 d(x_2, z) - c_4.$$

Thus, we have

$$\begin{aligned}
&\Phi(x_1, z) + c^0 d(x_1, z) \\
&= \Phi(x_1, x_2) + \Phi(x_2, z) - c_3 + c^0 d(x_1, z) \\
&= \Phi(x_1, x_2) + \Phi(x_2, z) - c_3 + c^0 d(x_1, x_2) + c^0 d(x_2, z) - c_4 \\
&= -c_1 - c_2 - c_3 - c_4 \leq_C 0,
\end{aligned}$$

which shows that $\Gamma(x_2) \subset \Gamma(x_1)$. Then (iii) of Theorem 4.26 is satisfied. To verify (iv) of Theorem 4.26, we consider a sequence $\{x_k\}_{k\in N} \subset X$, such that $x_{k+1} \in \Gamma(x_k)$, $\forall k \in N$ with an arbitrary x_1 in X. We have

$$\Phi(x_k, x_{k+1}) + c^0 d(x_k, x_{k+1}) \leq_C 0, \quad \forall k.$$

Thus, we have

$$\sum_{i=1}^{k} (\Phi(x_i, x_{i+1}) + c^0 d(x_i, x_{i+1}))) \leq_C 0, \quad \forall k.$$

Since ξ_{c^0} is increasing, we get

$$\xi_{c^0}(\Phi(x_k, x_{k+1})) + c^0 d(x_k, x_{k+1})) \leq 0, \quad \forall k.$$

As Φ is a half distance and ξ_{c^0} is increasing, we deduce that

$$\xi_{c^0}(\sum_{i=1}^{k} (\Phi(x_i, x_{i+1}) + c^0 d(x_i, x_{i+1}))) \leq 0, \quad \forall k.$$

Thus,

$$\xi_{c^0}(\Phi(x_1, x_{k+1})) + \sum_{i=1}^{k} d(x_i, x_{i+1}) \leq 0, \quad \forall k.$$

That is,

$$\sum_{i=1}^{k} d(x_i, x_{i+1}) \leq -\xi_{c^0}(\Phi(x_1, x_{k+1})), \quad \forall k. \tag{4.7}$$

Since $\Phi(v_0, x_{k+1}) \leq_C \Phi(v_0, x_1) + \Phi(x_1, x_{k+1})$, we have

$$-\Phi(x_1, x_{k+1}) \leq_C \Phi(v_0, x_1) - \Phi(v_0, x_{k+1}), \quad \forall k. \tag{4.8}$$

By (4.7) and (4.8), we obtain

$$\sum_{i=1}^{k} d(x_i, x_{i+1}) \leq \xi_{c^0}(\Phi(v_0, x_1) - \Phi(v_0, x_{k+1})), \quad \forall k.$$

By assumption (ii), we have

$$\sum_{i=1}^{k} d(x_i, x_{i+1}) \leq \xi_{c^0}(\Phi(v_0, x_1) - w_0), \quad \forall k.$$

We denote $s_k = \sum_{i=1}^{k} d(x_i, x_{i+1})$. Since the sequence $\{s_k\}_{k\in N}$ is monotonically increasing and bounded above, it is a convergent sequence. Thus $\sum_{i=1}^{\infty} d(x_i, x_{i+1})$ is a convergent series. Hence the sequence $\{d(x_k, x_{k+1})\}_{k\in N}$ converges to 0. Thus we have shown that assumption (iv) of Theorem 4.26 is satisfied. By Theorem 4.26, Γ has a critical point in X, and the proof is complete. ∎

Corollary 4.28. *If all the assumptions in Theorem 4.27 are satisfied, then* $\exists x^* \in \Gamma(v_0)$, *such that*

$$\Phi(x^*, x) + c^0 d(x^*, x) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\},$$

where $\Gamma(x) = \{y \in X : \Phi(x, y) + c^0 d(x, y) \leq_C 0\}$ *and the element* v_0 *is the same as the one in Theorem 4.27.*

Proof. By the second conclusion of Theorem 4.26, there exists a critical point of Γ in $\Gamma(v_0)$. Thus, the conclusion follows from Theorem 4.27. ∎

Next we will present an alternative version of vector variational principle which has a close relation to the ϵ-minimal solution of vector optimization problems.

Definition 4.29. *Let* $A \subset Y$ *and* $\epsilon > 0$. *An element* $y_\epsilon \in A$ *is said to be an* ϵ*-minimal point of* A *with respect to* $c^0 \in intC$ *if there exists no element* y *of* A *such that*

$$y_\epsilon \in y + C_{\epsilon c^0}, \tag{4.9}$$

where $C_{\epsilon c^0} = \epsilon c^0 + C\backslash\{0\}$.

We will denote the set of all ϵ-minimal points of A with respect to c^0 by $\text{Min}_{\epsilon c^0}(A)$.

Lemma 4.30. *If* $y_\epsilon \in Min_{\epsilon c^0}(A)$, *then*

$$\xi_{c^0}(y - y_\epsilon) \geq -\xi_{c^0}(\epsilon c^0) = -\epsilon, \quad \forall y \in A.$$

Proof. Observe that $y_\epsilon \in M_{\epsilon c^0}(A)$ is equivalent to

$$A \cap \{y_\epsilon - \epsilon c^0 - C\backslash\{0\}\} = \varnothing,$$

i.e.,

$$(A - y_\epsilon) \cap \{-\epsilon c^0 - C\backslash\{0\}\} = \varnothing.$$

By the properties of ξ_{c^0}, we have

$$\xi_{c^0}(y - y_\epsilon) > -\epsilon, \quad \forall y \in A, y \neq y_\epsilon - \epsilon c^0.$$

Thus

$$\xi_{c^0}(y - y_\epsilon) > -\xi_{c^0}(\epsilon c^0), \quad \forall y \in A, y \neq y_\epsilon - \epsilon c^0.$$

Then

$$\xi_{c^0}(y - y_\epsilon) \geq -\xi_{c^0}(\epsilon c^0), \quad \forall y \in A.$$

■

Lemma 4.31. *Let* (X, d) *be a metric space, let* $f : X \to Y$ *be a vector-valued function. Given* $\epsilon > 0$, *let* $x_\epsilon \in X$ *and*

$$\xi_{c^0}(f(x_\epsilon)) < \xi_{c^0}(f(x)) + \sqrt{\epsilon} d(x_\epsilon, x), \quad \forall x \in X, x \neq x_\epsilon. \tag{4.10}$$

Then $f(x_\epsilon) \in Min_C(f_{\epsilon c^0}(X))$, *where* $f_{\epsilon c^0}(x) = f(x) + \sqrt{\epsilon} d(x_\epsilon, x)c^0, \forall x \in X$.

Proof. It follows from (4.10) that

$$\begin{aligned}
&\xi_{c^0}(f(x) + \sqrt{\epsilon} d(x_\epsilon, x)c^0 - f(x_\epsilon)) \\
&= \xi_{c^0}(f(x) - f(x_\epsilon)) + \sqrt{\epsilon} d(x_\epsilon, x) \\
&\geq \xi_{c^0}(f(x)) + \sqrt{\epsilon} d(x_\epsilon, x) - \xi_{c^0}(f(x_\epsilon)) \\
&> 0.
\end{aligned}$$

Thus,

$$f(x) + \sqrt{\epsilon} d(x_\epsilon, x)c^0 - f(x_\epsilon) \not\leq_C 0, \quad \forall x \neq x_\epsilon.$$

So $f(x_\epsilon) \in \mathrm{Min}_C(f_{\epsilon c^0}(X))$. ■

Theorem 4.32. *Let* (X, d) *be a complete metric space, and let* (Y, C) *be an ordered Hausdorff topological vector space with* $intC \neq \emptyset$. *Let* $f : X \to Y$ *be a vector-valued function and* C *bounded below. Assume that, for a given* $\epsilon > 0$ *and for every* $x \in X$, *the set*

$$\{y \in X : f(y) - f(x) + \sqrt{\epsilon} d(x, y)c^0 \leq_C 0\},$$

is closed.

Then, for every point $x^0 \in X$ *satisfying* $f(x^0) \in Min_{\epsilon c^0}(f(X))$, *there exists a point* $x_\epsilon \in X$ *such that*

1. $f(x_\epsilon) \leq_C f(x^0)$;
2. $d(x^0, x) \leq \sqrt{\epsilon}$;
3. $f_{\epsilon c^0}(x_\epsilon) \in Min_C(f_{\epsilon c^0}(X))$,

where $f_{\epsilon c^0}(x) = f(x) + \sqrt{\epsilon} d(x_\epsilon, x) c^0, \forall x \in X$.

Proof. Since $f(x^0) \in \mathrm{Min}_{\epsilon c^0}(f(X))$, by Lemma 4.30, $\forall x \in X$,

$$\xi_{c^0}(f(x) - f(x^0) \geq -\xi_{c^0}(\epsilon c^0) = -\epsilon.$$

Let

$$\Phi(x, y) = \xi_{c^0}(f(y) - f(x)), \quad x, y \in X.$$

Thus, assumption (ii) of Theorem 4.27 is satisfied since ξ_{c^0} is subadditive, $\Phi(x, y)$ is a half distance and $\Phi : X \times X \to \mathbb{R}$ is a real-valued function. Observe that

$$f(y) - f(x) + \sqrt{\epsilon} d(x, y) c^0 \leq_C 0,$$

is equivalent to

$$f(y) - f(x) \leq_C -\sqrt{\epsilon} d(x, y) c^0. \tag{4.11}$$

By the properties of ξ_{c^0}, we have that (4.11) holds if and only if

$$\xi_{c^0}(f(y) - f(x)) \leq -\sqrt{\epsilon} d(x, y),$$

if and only if

$$\Phi(x, y) + \sqrt{\epsilon} d(x, y) \leq 0.$$

By the closedness of the set $\{y \in Y : f(y) - f(x) + \sqrt{\epsilon} d(x, y) c^0 \leq_C 0\}$, the set $\{y : \Phi(x, y) + \sqrt{\epsilon} d(x, y) \leq 0\}$ is closed. Hence the assumption (i) of Theorem 4.27 is satisfied. Then it follows from Corollary 4.28 that there exists $x_\epsilon \in \Gamma(x^0) = \{y \in X : \Phi(x^0, y) + \sqrt{\epsilon} d(x^0, y) \leq 0\}$, such that

$$\Phi(x_\epsilon, x) + \sqrt{\epsilon} d(x_\epsilon, x) > 0, \quad \forall x \in X, x \neq x_\epsilon. \tag{4.12}$$

Since $x_\epsilon \in \Gamma(x^0)$,

$$\xi_{c^0}(f(x_\epsilon) - f(x^0)) + \sqrt{\epsilon} d(x^0, x_\epsilon) \leq 0.$$

It follows that

$$\xi_{c^0}(f(x_\epsilon) - f(x^0)) \leq 0.$$

Therefore,

$$f(x_\epsilon) \leq_C f(x^0).$$

Moreover, by Lemma 4.30,

$$\xi_{c^0}(f(x_\epsilon) - f(x^0)) \geq -\epsilon.$$

Thus, $\sqrt{\epsilon} d(x^0, x_\epsilon) \leq \epsilon$, i.e., $d(x^0, x_\epsilon) \leq \sqrt{\epsilon}$. Finally, by (4.12), we obtain that, for any $x \in X$ and $x \neq x_\epsilon$,

$$\xi_{c^0}(f(x) - f(x_\epsilon)) + \sqrt{\epsilon} d(x_\epsilon, x^0) > 0,$$

implying

$$(f(x) + \sqrt{\epsilon} d(x_\epsilon, x) c^0) \cap (f(x_\epsilon) - C \backslash \{0\}) = \varnothing, \quad \forall x \in X, x \neq x_\epsilon.$$

Hence, $f(x_\epsilon) \in \mathrm{Min}_C(f_{\epsilon c^0}(X))$. ■

4.2 Variational Principles for Set-Valued Functions

In this section, we introduce the concept of approximate optimal solutions for set-valued functions and provide a sufficient condition for the existence of approximate optimal solutions for set-valued functions. We present a variational principle for set-valued functions.

Let X be a nonempty set, Y a locally convex Hausdorff space, $C \subset Y$ a nonempty, nontrivial, pointed, closed and convex cone with nonempty interior $intC$. Let $e \in intC$.

A set-valued function $F : X \rightrightarrows Y$ is said to be proper if $dom(F) = \{x \in X : F(x) \neq \emptyset\} \neq \emptyset$.

Definition 4.33. *Let $F : X \rightrightarrows Y$ be a set-valued function, and $\epsilon > 0$. A point $x^* \in X$ is said to be an ϵ-minimal solution of F on X, if there exists a $y^* \in F(x^*)$ such that*

(i) $(F(x^*) - y^*) \cap (-C\backslash\{0\}) = \emptyset$;
(ii) $(F(x) - y^* + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset$, *for all* $x \in X\backslash\{x^*\}$.

A set-valued function $F : X \rightrightarrows Y$ is said to be C order bounded below on X if $\exists y \in Y$ such that

$$F(x) - y \subset C, \text{ for all } x \in X.$$

If $dom(F) = \emptyset$, then F is always regarded as being bounded below.

Theorem 4.34. *If $F : X \rightrightarrows Y$ is proper, compact-valued and C order bounded below, then, for any $\epsilon > 0$, there exist an $x^* \in X$ and a $y^* \in F(x^*)$ such that*

(i) $(F(x^*) - y^*) \cap (-C\backslash\{0\}) = \emptyset$;
(ii) $(F(x) - y^* + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset, \quad \forall x \in X\backslash\{x^*\}$.

Proof. We prove the conclusion by contradiction.

Suppose that there exists a real number $\epsilon_0 > 0$ such that the conclusion of this theorem does not hold. Arbitrarily take an $x_1 \in dom(F)$ and a $y_1 \in F(x_1)$. Since $F(x_1)$ is compact, by the domination property of the compact set $F(x_1)$, there exists a $y_1' \in \mathrm{Min}_C(F(x_1))$ such that $y_1' - y \in -C$. At this time, (ii) cannot hold with y^* replaced by y_1'. So, there exist $x_2 \in X$ and $y_2 \in F(x_2)$ such that

$$y_2 - y_1' + \epsilon_0 e \leq_C 0. \tag{4.13}$$

Since $F(x_2)$ is compact, we deduce that there exists a y_2' such that $y_2' \leq_C y_2$ and $y_2' \in \mathrm{Min}_C(F(x_2))$. This combined with (4.13) yields

$$y_2' - y_1' + \epsilon_0 e \leq_C 0.$$

Once again, for y_2', (ii) does not hold. We deduce $\exists x_3 \in X$ and $y_3' \in \mathrm{Min}_C(F(x_3))$ such that

$$y_3' - y_2' + \epsilon_0 e \leq_C 0.$$

....

Hence,

$$\sum_{i=2}^{k}(y_i' - y_{i-1}' + \epsilon_0 e) = y_k' - y_1' + (k-1)\epsilon_0 e \leq_C 0, \quad \forall k \geq 2.$$

This implies

$$(y_k' - y_1')/(k-1) + \epsilon_0 e \leq_C 0, \quad k \geq 2.$$

Since F is C order bounded below, $\exists y \in Y$ such that $y_k' - y \geq_C 0$, for all $k \in N$. So we have

$$(y - y_1')/(k-1) + \epsilon_0 e \leq_C 0, \text{ for all } k \geq 2.$$

Letting $k \to \infty$, we have $\epsilon_0 e \leq_C 0$, which is impossible. ■

From Theorem 4.34 and Definition 4.33, we obtain an existence result of an ϵ-minimal solution of F on X.

Theorem 4.35. *If $F : X \rightrightarrows Y$ is proper, compact-valued and C order bounded below, then for any real number $\epsilon > 0$, the set of ϵ-minimal solutions of F on X is nonempty.*

Theorem 4.36. *Let (X, d) be a complete metric space, and Y a locally convex Hausdorff space, C a nontrivial, pointed, closed and convex cone with nonempty interior $intC$ and $e \in intC$. Let $F : X \rightrightarrows Y$ be a set-valued function satisfying*

(i) *F is proper and compact-valued on X;*
(ii) *F is u.s.c. on X and bounded below on X.*

Given a real number $\epsilon > 0$, two points $x_1 \in dom(F)$ and $y_1 \in F(x_1)$ such that

(iii) $(F(x_1) - y_1) \cap (-C \backslash \{0\}) = \varnothing$;
(iv) $(F(x) - y_1 + \epsilon e) \cap (-C \backslash \{0\}) = \varnothing$, *for all $x \in X \backslash \{x_1\}$.*

Then, for any real number $\lambda > 0$, there exist $x_2 \in dom(F)$ and $y_2 \in F(x_2)$ such that

(v) $y_2 \leq_C y_1$;
(vi) $d(x_1, x_2) \leq \lambda$;
(vii) $(F(x_2) - y_2) \cap (-C \backslash \{0\}) = \varnothing$;
(viii) $(F(x) - y_2 + \epsilon e) \cap (-C \backslash \{0\}) = \varnothing$, *for all $x \in X \backslash \{x_2\}$.*
(ix) $(F(x) - y_2 + \epsilon/\lambda d(x, x_2) e) \cap (-C) = \varnothing$, *for all $x \in X \backslash \{x_2\}$.*

Proof. We consider a set-valued function $F_1 : X \rightrightarrows Y$:

$$F_1(x) = \{y \in F(x) : y \leq_C y_1\} = F(x) \cap (y_1 - C).$$

By the assumptions of this theorem, we know that F_1 is also proper (since $y_1 \in F_1(x_1) \neq \varnothing$), compact-valued and C order bounded below.

Now we define a real function as

$$f(x) = \begin{cases} \min\{\xi_e(y - y_1) : y \in F_1(x)\}, & x \in dom(F); \\ +\infty, & x \notin dom(F). \end{cases}$$

We will show that $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, lower semicontinuous and bounded below. In fact, if $F_1(x) = \varnothing$, then $f(x) = +\infty$; if $F_1(x) \neq \varnothing$, $-\infty < f(x) < +\infty$ by (i). By (iii), $f(x_1) \geq 0$. On the other hand, $f(x_1) \leq \xi_e(y_1 - y_1) = 0$. So $f(x_1) = 0$, which implies that f is proper. Since F_1 is C order bounded below on X, we have that f is bounded below on X. In order to show that f is l.s.c. on X, we need only to show that, for all $t \in \mathbb{R}$, $A = \{x \in X : f(x) \leq t\}$ is closed. Suppose $\bar{x}_k \in A$ and $\bar{x}_k \to x^*$. By the definition of f, there exists $\bar{y}_k \in F_1(\bar{x}_k)$ (which implies $\bar{y}_k \leq_C y_1$) such that

$$\xi_e(\bar{y}_k - y_1) \leq t \tag{4.14}$$

and

$$\bar{y}_k - y_1 \leq_C 0. \tag{4.15}$$

Since F is u.s.c. at x^*, there exists a $z_k \in F(x^*)$ such that

$$z_k - \bar{y}_k \to 0. \tag{4.16}$$

By the compactness of $F(x^*)$, there exists a subsequence $\{z_{k_i}\}$ of $\{z_k\}$ and $z^* \in F(x^*)$ such that $z_{k_i} \to z^*$. This combined with (4.15) and (4.16) yields $\bar{y}_{k_i} \to z^*$ and $z^* - y_1 \in -C$. Hence $z^* \in F_1(x^*) \cap (y_1 - C) = F_1(x^*)$. By (4.14), $\xi_e(\bar{y}_{k_i} - y_1) \leq t$. Letting $k \to +\infty$, we have $\xi_e(z^* - y_1) \leq t$. So $f(x^*) \leq t$, i.e., $x^* \in A$. Therefore, the set A is closed. Besides, by (iv), we have $f(x) + \epsilon \geq 0 = f(x_1)$.

For the function f, applying Ekeland's variational principle, we obtain that, for any real number $\lambda > 0$, there exists an $x_2 \in X$ such that

$$f(x_2) \leq f(x_1) = 0; \tag{4.17}$$

$$d(x_1, x_2) \leq \lambda; \tag{4.18}$$

$$f(x) + \epsilon/\lambda d(x, x_2) > f(x_2), \forall x \in X \backslash \{x_2\}. \tag{4.19}$$

From (4.17), we deduce $F(x_2) \neq \varnothing$, and (4.18) is just (vi).

Suppose that $\xi_e(y_2' - y_1) = \min\{\xi_e(y - y_1) : y \in F_1(x_2)\}$, $y_2' \in F(x_2)$. By the compactness of $F(x_2)$ and the domination property of the set $F(x_2)$, there exists a $y_2 \in \mathrm{Min}_C(F(x_2))$ with $y_2 \leq_C y_2'$. By the monotonicity of ξ_e, we have

$$\xi_e(y_2 - y_1) = \min\{\xi_e(y - y_1) : y \in F_1(x_2)\} = \xi_e(y_2' - y_1). \tag{4.20}$$

Hence, (v) and (vii) hold. By (v), (iv) and (iii), we know (viii) holds. Finally, let us show that (ix) holds. From (4.19) and (4.20) and the definition of f, we deduce that if $x \notin dom(F_1)$ then (iv) holds automatically; and, for any $x \in dom(F_1)\backslash\{x_2\}$,

$$\min\{\xi_e(y - y_1) : y \in F_1(x)\} - \xi_e(y_2 - y_1) + \epsilon/\lambda d(x, x_2) > 0.$$

That is,

$$\xi_e(y - y_1) - \xi_e(y_2 - y_1) + \epsilon/\lambda d(x, x_2) > 0, \quad \forall x \in dom(F_1)\backslash\{x_2\}, y \in F(x).$$

Consequently, we have

$$\xi_e(y - y_2) + \epsilon/\lambda d(x, x_2) > 0, \quad \forall x \in dom(F_1)\backslash\{x_2\}, y \in F_1(x),$$

i.e.,

$$y - y_2 + \epsilon/\lambda d(x, x_2)e \not\leq_C 0, \quad \forall x \in dom(F_1)\backslash\{x_2\}, y \in F_1(x).$$

When $y \in F(x)\backslash F_1(x)$, we can show by contradiction that

$$y - y_2 + \epsilon/\lambda d(x, x_2)e \not\leq_C 0, \quad \forall x \in dom(F_1)\backslash\{x_2\}.$$

Otherwise, there exists an $x' \in dom(F_1)\backslash\{x_2\}$ and

$$y' \in F(x')\backslash F_1(x') \tag{4.21}$$

such that

$$y' - y_2 + \epsilon/\lambda d(x', x_2)e \leq_C 0.$$

This implies $y' - y_2 \in -intC$. From (v), we derive $y' \leq_C y_2 \leq_C y_1$. This means $y' \in F_1(x')$, which contradicts (4.21). So (ix) holds. The proof is complete. ■

Next, we will establish a general Ekeland's variational principle for set-valued functions in complete order metric spaces and complete metric spaces. This principle is a generalization of some results of Nemeth [151], Tammer [185], and Isac [111].

To this aim, we need some additional concepts for set-valued functions.

Let X be a Hausdorff topological vector space, and Y a locally convex vector space. Let $C \subset Y$ be a nonempty pointed and convex cone and Y be endowed with the order $\leq_C$ induced by C.

Definition 4.37. *A set-valued function $F : X \rightrightarrows Y$ is said to be submonotone with respect to C at $x_0 \in X$ if, under the following conditions:*

(i) $\lim_\alpha x_\alpha = x$, *where $\{x_\alpha\}_{\alpha \in I}$ is a net in X indexed by the totally ordered set $(I, \leq)$,*

(ii) $y_\alpha \leq_C y_\beta$ *wherever* $y_\alpha \in F(x_\alpha)$, $y_\beta \in F(x_\beta)$ *and* $\alpha \geq \beta$,

it follows that there exists a $y \in F(x)$ *such that*

$$y \leq_C y_\alpha, \quad \forall \alpha \in I.$$

Let $C_0 \subset C$ be a convex cone in Y, and let $r : X \times X \to Y$ be a C_0-metric on X. We now make the following assumptions on the set-valued function $F : X \rightrightarrows Y$.

Assumption 4.38. *Let* $\epsilon > 0$ *be given. For every* $x_0 \in X$, $y_0 \in F(x_0)$, *and for every net* $\{(x_\alpha, y_\alpha)\}_{\alpha \in I}$ *in Gr(F) with the property that* $x_\alpha \to x \in X$ *and* $y_\alpha - y_0 + \epsilon r(x_\alpha, x_0) \leq_C 0, \forall \alpha \in I$, *there exists a* $y \in F(x)$ *such that*

$$y - y_0 + \epsilon r(x, x_0) \leq_C 0.$$

Assumption 4.39. *For each* $x \in X$ *and each* $y \in Y$, $F(x) \cap (y - C)$ *is compact.*

Now, we introduce the following concept of approximate solutions for a set-valued optimization problem.

Definition 4.40. *Let* $F : X \rightrightarrows Y$ *be a set-valued function and* $H \subset C$. *We say that the pair* $(\bar{x}, \bar{y}) \in X \times Y$ *is an* H *near to the minimal solution of* F *on* X *if* $\bar{y} \in Min_C(F(\bar{x}))$ *and*

$$(F(x) - \bar{y} + H) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{\bar{x}\}.$$

Remark 4.41. When F is a real function, $C = \{t \in \mathbb{R} : t \geq 0\}$, $H = \{t \in \mathbb{R} : t \geq \epsilon\}$, this definition reduces to the definition of an ϵ-minimal solution of a real function.

Proposition 4.42. *Let* C *be closed. If* F *is nonempty compact-valued and u.s.c., then* F *is submonotone with respect to* C *at every point of* X.

Proof. Fix $x \in X$. Let $\{(x_\alpha, y_\alpha)\}_{\alpha \in I}$ be a net in $Gr(F)$ satisfying properties (i) and (ii) of Definition 4.37. Then $x_\alpha \to x$ in X and

$$y_\alpha - y_\beta \leq_C 0, \text{ whenever } \alpha \geq \beta.$$

Since F is compact-valued and u.s.c., there exists a convergent subnet of $\{y_\alpha\}_{\alpha \in I}$ (again denoted by $\{y_\alpha\}_{\alpha \in I}$) with limit $y \in F(x)$. Letting $y_\alpha \to y$ in the above relation and taking into account the fact that C is closed, we have $y - y_\alpha \leq_C 0$, $\forall \alpha \in I$, proving that F is submonotone at x. ∎

Proposition 4.43. *Let* C *be closed, and let* $r(\cdot, a)$ *be continuous for every* $a \in X$. *If* F *is compact-valued and u.s.c., then Assumption 4.38 holds.*

Proof. Let $\epsilon > 0$. Fix $(x_0, y_0) \in Gr(F)$. Consider a net $\{(x_\alpha, y_\alpha)\}_{\alpha \in I}$ in $Gr(F)$ with the property that $x_\alpha \to x$ and

$$y_\alpha - y_0 + \epsilon r(x_\alpha, x_0) \leq_C 0, \quad \forall \alpha \in I.$$

By the same argument as in the proof of Proposition 4.42, we may assume that

$$(x_\alpha, y_\alpha) \to (x, y) \in Gr(F).$$

Taking into account the fact that C is closed and $r(\cdot, x_0)$ is continuous and taking the limit in the above relation, we obtain

$$y - y_0 + \epsilon r(x, x_0) \leq_C 0.$$

Hence, Assumption 4.38 holds. ■

Proposition 4.44. *If F is submonotone with respect to C, then $F(x)$ has the domination property for every $x \in X$.*

Proof. Fix $x \in X$ and $y \in F(x)$. It is clear that $\leq_C$ is still a partial order on $D = \{z \in F(x) : z \leq_C y\}$. According to the Hausdorff maximality principle, there exists a maximal totally ordered $Z \subset D$. Let $I = \{(z, T_z) : z \in Z\}$, where T_z determined by $z \in Z$ is the lower section $\{a \in Z : a \leq_C z\}$. Setting

$$(a, T_a) \leq (b, T_b), \text{ whenever } T_b \subset T_a,$$

we see that I is totally ordered due to the fact that Z is totally ordered. Define the net $\phi : I \to Z$ as the function

$$\alpha = (a, T_a) \to z_\alpha = a.$$

Then, Z is indexed by the totally ordered set I with the property that

$$z_\alpha \leq_C z_\beta, \text{ whenever } \alpha \geq \beta.$$

By the submonotonicity of F at x (simply taking $x_\alpha \equiv x$ for all α), we can find an element $\bar{y} \in F(x)$ such that

$$\bar{y} \leq_C z_\alpha, \quad \forall \alpha \in I.$$

Clearly,

$$(\bar{y} - C) \cap F(x) = \{\bar{y}\},$$

by the maximality of Z. Therefore,

$$\bar{y} \in \mathrm{Min}_C(F(x)) \text{ and } \bar{y} \leq_C y,$$

establishing the fact that $F(x)$ has the domination property. ■

Next, we give a sufficient condition for the existence of approximate optimal solutions to set-valued functions.

Proposition 4.45. *Let $C_0 \subset C$ be a complete C-bound regular convex cone, and let $U \subset B(0)$ be such that $H = C_0 \backslash U \neq \varnothing$. Suppose that $F : X \rightrightarrows Y$ is a set-valued function such that $F(x)$ has the domination property for every $x \in X$, and F is C order lower bounded on X. Then F has an H near to the minimal solution on X.*

Proof. First, let us prove that there exist an $\bar{x} \in X$ and a $\bar{y} \in F(\bar{x})$ such that

$$(F(x) - \bar{y} + H) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{\bar{x}\}.$$

We proceed by contradiction. Arbitrarily fix $x_1 \in X$ and $y_1 \in F(x_1)$. Then, we can choose $x_{i+1} \in X \backslash \{x_i\}$ and $y_{i+1} \in F(x_{i+1})$ such that

$$(y_{i+1} - y_i + H) \cap (-C) \neq \varnothing, \text{ for } i = 1, 2, \cdots .$$

Then, we have

$$y_{i+1} - y_i + h_i \leq_C 0,$$

for some $h_i \in H$. Summing up these relations from $i = 1$ to k, we get

$$y_{k+1} - y_1 + \sum_{i=1}^{k} h_i \leq_C 0. \tag{4.22}$$

Since F is C order lower bounded on X, we find that $\{\sum_{i=1}^{k} h_i\}$ is C bounded and clearly it is C_0 increasing. It follows from the C bound regularity of C_0 that $\{\sum_{i=1}^{k} h_i\}$ is convergent, therefore, $h_k \in U$ for sufficiently large k. This contradicts the assumption that $H = C_0 \backslash U$, and thus the existence of $(\bar{x}, \bar{y})$ is established.

Second, by the domination property of $F(\bar{x})$, we can find an element $y' \in \text{Min}_C(F(\bar{x}))$ such that $y' \leq_C \bar{y}$. Thus

$$(F(x) - y' + H) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{\bar{x}\},$$

and so the pair $(\bar{x}, y')$ is an H near to the minimal solution of F on X. ■

Proposition 4.46. *The conclusion of Proposition 4.45 remains true when the condition "F is C-order lower-bounded on X" is replaced by*

(*C*) *There exist $x'_1 \in X$ and $y'_1 \in F(x'_1)$ such that F is C-order lower-bounded on the set $\{x \in X : (y'_1 - C) \cap F(x) \neq \varnothing\}$.*

Proof. Repeat the argument as in the proof of Proposition 4.45, with (x_1, y_1) replaced by (x'_1, y'_1) to generate the corresponding elements $x_i, y_i, h_i, i \geq 2$, so as to have (4.22) with y_1 replaced y'_1. Then apply condition (C) to show that $\{\sum_{i=1}^{k} h_i\}$ is C bounded. The rest of the proof now follows in a similar fashion. ■

Now we will derive a general Ekeland's variational principle for set-valued functions.

Let $\epsilon > 0$ be given, and let $r : X \times X \to Y$ be a C_0-metric on X. Suppose that $F : X \rightrightarrows Y$ is a set-valued function such that $F(x)$ has the domination property for every $x \in X$. Then $\mathrm{Min}_C(F(x)) \neq \varnothing$, for every $x \in X$, and we have a nonempty set-valued function $E : X \rightrightarrows Y$ defined by

$$E(x) := \mathrm{Min}_C(F(x)), \quad \forall x \in X.$$

Let us introduce a relation $\prec$ on $T = Gr(E)$ as follows:

$$(x_1, y_1) \prec (x_2, y_2) \text{ iff } y_1 - y_2 + \epsilon r(x_1, x_2) \leq_C 0, \text{ for } (x_1, y_1), (x_2, y_2) \in T.$$

Lemma 4.47. *The relation $\prec$ is a partial order on T.*

Proof. The relation $\prec$ is clearly well defined on T. It remains to show that it is reflexive, antisymmetric and transitive.

(i) Reflexivity. For $(x, y) \in T$, we clearly have $(x, y) \prec (x, y)$, since $y - y + \epsilon r(x, x) = 0 \leq_C 0$.

(ii) Antisymmetry. Let $(x_1, y_1), (x_2, y_2) \in T$, with $(x_1, y_1) \prec (x_2, y_2)$ and $(x_2, y_2) \prec (x_1, y_1)$. Then we have

$$y_1 - y_2 + \epsilon r(x_1, x_2) \leq_C 0, \tag{4.23}$$

$$y_2 - y_1 + \epsilon r(x_2, x_1) \leq_C 0. \tag{4.24}$$

The combination of (4.23) and (4.24) yields

$$\epsilon r(x_1, x_2) + \epsilon r(x_2, x_1) = 2\epsilon r(x_1, x_2) \leq_C 0,$$

so that $r(x_1, x_2) \leq_C 0$. Thus $r(x_1, x_2) = 0$ and hence $x_1 = x_2$. Furthermore, it follows from (4.23) and (4.24) that $\pm(y_1 - y_2) \leq_C 0$, which implies $y_1 = y_2$. Therefore, $(x_1, y_1) = (x_2, y_2)$.

(iii) Transitivity. Let $(x_i, y_i) \in T, i = 1, 2, 3$, with $(x_1, y_1) \prec (x_2, y_2)$ and $(x_2, y_2) \prec (x_3, y_3)$. Then

$$y_1 - y_2 + \epsilon r(x_1, x_2) \leq_C 0, \tag{4.25}$$

$$y_2 - y_3 + \epsilon r(x_2, x_3) \leq_C 0. \tag{4.26}$$

The combination of (4.25) and (4.26) yields

$$y_1 - y_3 + \epsilon(r(x_1, x_2) + r(x_2, x_3)) \leq_C 0.$$

Using

$$r(x_1, x_3) \leq_C r(x_1, x_2) + r(x_2, x_3),$$

we have that

$$y_1 - y_3 + \epsilon r(x_1, x_3) \leq_C 0,$$

and hence $(x_1, y_1) \prec (x_3, y_3)$. ∎

Theorem 4.48. *Let Y be a locally convex space ordered by a nonempty pointed and convex cone $C \subset Y$, and let (X, r) be a complete C_0 metric space, where $C_0 \subset C$ is a complete C bound regular convex cone. Let $\epsilon > 0$ and $F : X \rightrightarrows Y$ be a set-valued function satisfying*

(i) *F is submonotone with respect to C and C is closed,*

or

(ii) *C is closed and Assumptions 4.38 and 4.39 hold.*

Suppose that the following **(iii)** *holds:*

(iii) *there exists $(x_0, y_0) \in Gr(F)$ such that F is C order lower bounded on $X_1 = \{x \in X : (y_0 - C) \cap F(X) \neq \varnothing\}$.*

Then, for every (x_0, y_0) satisfying **(iii)**, *there exists $(x^*, y^*) \in Gr(E)$ such that*

(iv) $y_0 - y^* - \epsilon r(x_0, x^*) \geq_C 0$;
(v) $(F(x) - y^* + \epsilon r(x, x^*)) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{x^*\}$.

Moreover, let $U \in B(0)$. If $H = C_0 \backslash U \neq \varnothing$, then

(vi) *there exists $(x_1, y_1) \in Gr(E)$ with $y_1 \leq_C y_0$ such that*

$$(F(x) - y_1 + \epsilon H) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{x_1\},$$

i.e., (x_1, y_1) is an ϵH near to the minimal solution of F on X.

For every such (x_1, y_1) in **(vi)**, *there exists $(x_\epsilon, y_\epsilon) \in Gr(E)$ such that* **(iv)**, *and* **(v)** *hold with (x_0, y_0) and (x^*, y^*) replaced by (x_1, y_1) and (x_ϵ, y_ϵ), respectively. Moreover, (x_ϵ, y_ϵ) is an ϵH near to the minimal solution of F on X, satisfying*

(vii) $r(x_1, x_\epsilon) \in U$.

Proof. Let

$$E(x) = \mathrm{Min}_C F(x), \quad x \in X.$$

By Proposition 4.44 and (i) or (ii), $F(x)$ has the domination property for every $x \in X$. Thus, we see that the set-valued function $E(x) \neq \emptyset, \forall x \in X$. By (iii) and the fact that $F(x_0)$ has the domination property, there exists $y_0' \in \mathrm{Min}_C(F(x_0))$ such that $y_0' \leq_C y_0$ and F is C lower bounded on

$$X_1' = \{x \in X : (y_0' - C) \cap F(x) \neq \varnothing\},$$

since $X_1' \subset X_1$. Now, we consider the partial order $\prec$ on $T = Gr(E)$ as given in Lemma 4.47. Let

$$Z_1 = \{(x, y) \in T : (x, y) \prec (x_0, y_0')\}.$$

Applying the Hausdorff maximality principle to $(Z_1, \prec)$, we obtain a maximal totally ordered subset Z_0 of Z_1.

As in the proof of Proposition 4.44, let us write

$$Z_0 = \{(x_\alpha, y_\alpha)\}_{\alpha \in I},$$

where $(I, \leq)$ is a totally ordered index set with the property that

$$(x_\alpha, y_\alpha) \prec (x_\beta, y_\beta), \text{ whenever } \alpha \prec \beta.$$

We shall show that Z_0 contains its minimal point with respect to the relation $\prec$.

First, let us introduce a relation Δ on $X_0 = \{x_\alpha\}_{\alpha \in I}$ by defining

$$x_\alpha \Delta x_\beta, \text{ whenever } (x_\alpha, y_\alpha) \prec (x_\beta, y_\beta).$$

This relation is well defined. Indeed, we need only to show that if (x_α, y_α), $(x_\beta, y_\beta) \in Z_0$ and $x_\alpha = x_\beta$, then $y_\alpha = y_\beta$. We may assume that $(x_\alpha, y_\alpha) \prec (x_\beta, y_\beta)$, so that

$$y_\alpha - y_\beta + \epsilon r(x_\alpha, x_\beta) \leq_C 0.$$

Since $x_\alpha = x_\beta$, we have $y_\alpha - y_\beta \leq_C 0$. This and the fact that $y_\alpha, y_\beta \in \mathrm{Min}_C(F(x_\alpha))$ yield $y_\alpha = y_\beta$.

Thus, $x_\beta \Delta x_\alpha$ whenever $\alpha \geq \beta$, and so X_0 is totally ordered with respect to Δ. Moreover, the filter of its lower section is Cauchy. To verify this, let us assume the contrary: there exists a neighborhood $U' \in B(0)$ such that, for each s in X_0, there exist p, q in X_0, $p\Delta s$ and $q\Delta s$, such that $r(p,q) \notin U'$. Fix s and let p, q be as above. We can suppose that $p\Delta q$. Put $x_1 = q, x_2 = p$. Then, $r(x_2, x_1) \notin U'$ and

$$z_2 - z_1 + \epsilon r(x_2, x_1) \leq_C 0, \text{ with } (x_i, z_i) \in Z_0, i = 1, 2.$$

Starting with p instead of s, we can continue this procedure. Accordingly we can determine the decreasing sequence $\{x_k\}$ in X_0 such that, for $k = 1, 2, \cdots$,

$$r(x_{2k}, x_{2k-1}) \notin U', \tag{4.27}$$

and $(x_k, z_k) \in Z_0$. From the definition of the relation Δ, we also have that

$$z_{k+1} - z_k + \epsilon r(x_{k+1}, x_k) \leq_C 0, \text{ for all } k.$$

Summing up this relations, we get

$$z_{k+1} - z_1 + \epsilon \sum_{i=1}^{k} r(x_{i+1}, x_i) \leq_C 0.$$

Since each x_k is in X_1' and $\{z_k\}$ is C order lower bounded, the set $\{\sum_{i=1}^k r(x_{i+1}, x_i) : k = 1, 2, \cdots\}$ is C order bounded. Then $\{\sum_{i=1}^k r(x_{i+1}, x_i)\}$ is convergent and this clearly contradicts (4.27).

This contradiction shows that the lower sections of X_0 form a Cauchy filter, which converges by the completeness of X to some $x^* \in X$, or equivalently $\lim_\alpha x_\alpha = x^*$.

Now we show that $x^* \in X_0$ holds under condition (i) or (ii).

Case 1. Suppose that (i) holds. Let $\alpha \geq \beta$. Then, we have $(x_\alpha, y_\alpha) \prec (x_\beta, y_\beta)$, so that

$$y_\alpha - y_\beta + \epsilon r(x_\alpha, x_\beta) \leq_C 0,$$

and hence

$$y_\alpha \leq_C y_\beta.$$

Since $x_\alpha \to x^*$ and F is submonotone with respect to C, there exists $y \in F(x^*)$ such that

$$y - y_\alpha \leq_C 0, \quad \forall \alpha \in I. \tag{4.28}$$

Also,

$$y_\alpha - y_\beta + \epsilon r(x_\alpha, x_\beta) \leq_C 0, \text{ whenever } \alpha \geq \beta. \tag{4.29}$$

From (4.28) and (4.29), we obtain

$$y - y_\beta + \epsilon r(x_\alpha, x_\beta) \leq_C 0, \quad \forall \alpha \geq \beta. \tag{4.30}$$

Letting $x_\alpha \to x^*$ in (4.30) and taking into account the fact that C is closed and $r(\cdot, x_\beta)$ is continuous, it follows that

$$y - y_\beta + \epsilon r(x^*, x_\beta) \leq_C 0, \quad \forall \beta \in I. \tag{4.31}$$

Since $F(x^*)$ has the domination property by Proposition 4.44, we have $y^* \leq_C y$ for some $y^* \in E(x^*)$, and hence

$$(x^*, y^*) \in T.$$

This combined with (4.31) yields

$$y^* - y_\beta + \epsilon r(x^*, x_\beta) \leq_C 0, \quad \forall \beta \in I,$$

so that

$$(x^*, y^*) \prec (x_\beta, y_\beta), \quad \forall (x_\beta, y_\beta) \in Z_0,$$

and hence $(x^*, y^*) \in Z_0$ by the maximality of Z_0. Thus $x^* \in X_0$ and (x^*, y^*) is the minimal point of Z_0.

Case 2. Suppose that (ii) holds. Fix $\beta \in I$. Then

$$y_\alpha - y_\beta + \epsilon r(x_\alpha, x_\beta) \leq_C 0, \text{ whenever } \alpha \geq \beta.$$

Since $x_\alpha \to x^*$, it follows from Assumption 4.38 that there exists $\bar{y}_\beta \in F(x^*)$ such that

$$\bar{y}_\beta - y_\beta + \epsilon r(x^*, x_\beta) \leq_C 0, \quad \forall \beta \in I. \tag{4.32}$$

As $F(x^*)$ has the domination property, we can find an element $y^*_\beta \in E(x^*)$ such that $y^*_\beta \leq_C \bar{y}_\beta$, so that $(x^*, y^*_\beta) \in T$. This combined with (4.32) yields

$$y^*_\beta - y_\beta + \epsilon r(x^*, x_\beta) \leq_C 0, \quad \forall \beta \in I.$$

Hence,

$$(x^*, y^*_\beta) \prec (x_\beta, y_\beta), \quad \forall (x_\beta, y_\beta) \in Z_0.$$

For each $\beta \in I$, define

$$B_\beta = \{(x^*, z) \in \{x^*\} \times E(x^*) : (x^*, z) \prec (x_\beta, y_\beta)\}.$$

Clearly,

$$B_{\beta_1} \subset B_{\beta_2}, \quad \forall \beta_1 < \beta_2. \tag{4.33}$$

We show that each B_β is nonempty and compact.
The nonemptiness of B_β has been proved already. Now we prove that B_β is closed. Let $\{(x^*, z_\alpha)\} \subset B_\beta$ be a net, and let $z_\alpha \to z^*$. Then $(x^*, z_\alpha) \prec (x_\beta, y_\beta)$, i.e.,

$$z_\alpha - y_\beta + \epsilon r(x^*, x_\beta) \leq_C 0.$$

By the closedness of C, we deduce that

$$z^* - y_\beta + \epsilon r(x^*, x_\beta) \leq_C 0,$$

namely, $(x^*, z^*) \in B_\beta$. Hence B_β is closed. Note that

$$B_\beta \subset \{x^*\} \times (F(x^*) \cap (y_\beta - C)).$$

By Assumption 4.39, $F(x^*) \cap (y_\beta - C)$ is compact, hence, $\{x^*\} \times (F(x^*) \cap (y_\beta - C))$ is compact. This fact, combined with the closedness of B_β, yields that B_β is compact.
By the maximality of Z_0, we conclude $(x^*, y^*) \in Z_0$. Furthermore, we have that $x^* \in X_0$ by the definition of X_0 and (x^*, y^*) is the minimal point of Z_0.

So, in both cases, we have proved that $x^* \in X_0$. We also have that $(x^*, y^*) \in Z_0$ and it is the minimal point of Z_0 with respect to $\prec$.

By $(x^*, y^*) \in Z_1$, we see that $(x^*, y^*) \prec (x_0, y'_0)$, which together with $y'_0 \leq_C y_0$ implies that

$$y^* - y_0 + \epsilon r(x^*, x_0) \leq_C 0,$$

and so (iv) is established.

That (v) holds can be shown by contradiction. Indeed, suppose that

$$(F(x') - y^* + \epsilon r(x', x^*)) \cap (-C) \neq \varnothing,$$

for some $x' \in X\backslash\{x^*\}$. Then, there exists $y' \in F(x')$ such that

$$y' - y^* + \epsilon r(x', x^*) \in -C.$$

As $F(x')$ has the domination property, there is an element $\bar{y} \in E(x')$ satisfying $\bar{y} \leq_C y'$. Hence

$$\bar{y} - y^* + \epsilon r(x', x^*) \leq_C 0,$$

i.e., $(x', \bar{y}) \prec (x^*, y^*)$ and $(x', \bar{y}) \in Z_0$. It follows that $(x', \bar{y}) = (x^*, y^*)$, contradicting $x' \in X\backslash\{x^*\}$. We have proved (v).

Let $U \in B(0)$ and $H = C_0 \setminus U \neq \varnothing$. Then by applying Proposition 4.46 with ϵH replacing H, we conclude the existence x_1 in X with the property (vi).

If we proceed as above taking (x_1, y_1) in place of (x_0, y_0), we can get a pair $(x_\epsilon, y_\epsilon) \in Gr(E)$ so as to have (iv) and (v) with (x_0, y_0) and (x^*, y^*) replaced by (x_1, y_1) and (x_ϵ, y_ϵ), respectively. Moreover, the pair (x_ϵ, y_ϵ) is an ϵH near to the minimal solution of F.

Finally, we show by contradiction that (vii) holds. To this end, suppose that $r(x_1, x_\epsilon) \notin U$. Then, $x_1 \neq x_\epsilon$ and

$$r(x_1, x_\epsilon) \in H. \tag{4.34}$$

Since (iv) holds with (x_0, y_0) and (x^*, y^*) replaced by (x_1, y_1) and (x_ϵ, y_ϵ), respectively, we have

$$y_1 - y_\epsilon - \epsilon r(x_1, x_\epsilon) \geq_C 0. \tag{4.35}$$

As $x_\epsilon \neq x_1$, (vi) yields

$$(y_\epsilon - y_1 + \epsilon H) \cap (-C) = \varnothing. \tag{4.36}$$

But, from (4.34) and (4.35), we deduce that

$$y_\epsilon - y_1 + \epsilon r(x_1, x_\epsilon) \in (y_\epsilon - y_1 + \epsilon H) \cap (-C) \neq \varnothing,$$

contradicting (4.36). Hence, (vii) holds. ■

Corollary 4.49. *Let (X, d) be a complete metric space, and let Y be a locally convex Hausdorff space ordered by the nonempty, pointed and convex cone C. Let $c^0 \in C\backslash\{0\}$ be such that $\theta(c^0) > 0$ for some θ in C^*. Let $\epsilon > 0$ be given, and let $F : X \rightrightarrows Y$ be a set-valued function satisfying:*

(i) *F is submonotone with respect to C and C is closed; or*
(ii) *C is closed and Assumptions 4.38 and 4.39 hold with $r(x, y) = d(x, y)c^0/\lambda_1$ where $\lambda_1 > 0$.*

Suppose that **(iii)** *below holds:*

(iii) *there exists $(x_1, y_1) \in Gr(E)$ such that*

$$(F(x) - y_1 + \epsilon c^0) \cap (-C) = \varnothing, \quad \forall x \in X\backslash\{x_1\},$$

and $F(X) \cap (y_1 - X)$ is C lower bounded.

Then, for every $\epsilon > 0$, there exists $(x_\epsilon, y_\epsilon) \in Gr(E)$ such that :

(iv) $y_\epsilon - y_1 + (\epsilon/\lambda_1)d(x_1, x_\epsilon)c^0 \leq_C 0$;
(v) $(F(x) - y_\epsilon + (\epsilon/\lambda_1)d(x_1, x_\epsilon)c^0) \cap (-C) = \varnothing, \forall x \in X \backslash \{x_\epsilon\}$;
(vi) $d(x_1, x_\epsilon) \leq \lambda_1$.

Proof. Let us set

$$r(x, x') = d(x, x')c^0/\lambda_1$$

in Theorem 4.48 and choose

$$U = \{u \in X : \theta(u)/\theta(c^0) < 1\}.$$

Clearly, $c^0 \notin U$. Also, let

$$C_0 = \{\alpha c^0 : \alpha \geq 0\} \text{ and } H = C_0 \backslash U.$$

Now, we can apply Lemma 4.16 to show that C_0 is a complete C bound regular convex cone in C. Then, we conclude from (vi) of Theorem 4.48 that there exists $(x_\epsilon, y_\epsilon) \in Gr(E)$ such that (iv), (v) and $r(x_1, x_\epsilon) \in U$ hold. It is easy to see that $r(x_1, x_\epsilon) \in U$ is equivalent to (vi). ■

4.3 Equivalents of Variational Principles for Vector-Valued Functions

In this section, we will establish new vector variants of "Drop Theorem", "Petal Theorem" and "Caristi-Kirk Fixed Point Theorem". We will derive equivalents between those theorems and variational principles for vector-valued functions. We will obtain a fixed point theorem for directional contractions as an application of vector variational principles.

Let Y be a locally convex Hausdorff space ordered by a nontrivial convex cone $C \subset Y$ with nonempty interior $intC$. Let (X, r) be a complete C metric space, where $r : X \times X \to Y$ is C metric function.

First of all, we state an immediate consequence of Theorem 4.15.

Theorem 4.50. *Let Y be a locally convex Hausdorff space ordered by a nontrivial convex cone C, $C_0 \subset C$ a C bound regular complete convex cone, and $C_0 \cap -C \subset -C_0$. Let (X, r) be a complete C_0 metric space. Let $f : X \to Y$ be a vector-valued function such that $\exists w_0 \in Y$ such that*

$$f(x) \geq_C w_0, \quad \forall x \in X.$$

Suppose that $\forall x \in X, \alpha > 0$, the set $\{y \in X : f(y) - f(x) + \alpha r(x, y) \leq_C 0\}$ is closed (or f is submonotone (w.r.t. C), C is closed and $\forall a \in X$, $r(a, \cdot)$ is continuous.)

Then there exists an $x^* \in X$ such that

$$f(x) - f(x^*) + r(x^*, x) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\}.$$

Proof. It is easy to verify that all conditions of Theorem 4.15 hold with $\epsilon = 1$. By (v) of Theorem 4.15, it follows that this theorem holds true. ■

Now we explore the equivalents of Theorem 4.50. To this end, we need the following assumption.

Assumption 4.51. *$h : X \times X \to Y$ is a half distance function. $\forall x \in X$, $\forall \alpha > 0$, the set $\{y \in X : h(x,y) + \alpha r(x,y) \leq_C 0\}$ is closed (or $h(x,\cdot)$ is submonotone with respect to C and C is closed and $\forall \alpha \in X$, $r(a,\cdot)$ is continuous with respect to the topology of X induced by r). $\exists x_0 \in X$, $w_0 \in Y$ such that $h(x_0, x) \geq_C w_0, \forall x \in X$.*

Let

$$D_0 = \{x \in X : h(x_0, x) + r(x_0, x) \leq_C 0\}.$$

Theorem 4.52. *All the following Theorems A, B, C and D are true and equivalent to Theorem 4.50.*

Theorem A. Let Assumption 4.51 hold. Then there exists an $x^* \in D_0$ such that

$$h(x^*, x) + r(x^*, x) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\}. \tag{4.37}$$

Theorem B. Let Assumption 4.51 hold. Let the set-valued function $T : X \rightrightarrows Y$ satisfy the condition

$$\forall x' \in D_0, \exists x \in T(x') \text{ such that } h(x', x) + r(x', x) \leq_C 0. \tag{4.38}$$

Then there exists an $x^* \in D_0$ such that $x^* \in T(x^*)$.

Theorem C. Let Assumption 4.51 hold. Let $M \subset X$ satisfy

$$\forall x' \in D_0 \backslash M, \exists x \in X \text{ such that } x \neq x' \text{ and } h(x', x) + r(x', x) \leq_C 0. \tag{4.39}$$

Then there exists an $x^* \in D_0 \cap M$.

Theorem D. Let Assumption 4.51 hold. Let the following hold:

$$\forall x' \in D_0, \text{ satisfying } \exists x_1 \in X \text{ such that } h(x', x_1) \leq_{C \backslash \{0\}} 0, \text{ we have}$$
$$x_2 \in X \backslash \{x'\} \text{ such that } h(x', x_2) + r(x', x_2) \leq_C 0. \tag{4.40}$$

Then there exists an $x^* \in D_0$ such that

$$h(x^*, x) \not\leq_{C \backslash \{0\}} 0, \quad x \in X.$$

Proof. Firstly, we prove that Theorem A is equivalent to Theorem 4.50.

Theorem 4.50 $\Longrightarrow$ Theorem A:

Let $f(x) = h(x_0, x)$. Then all the conditions of Theorem 4.50 are satisfied. It follows that there is an $x^* \in D_0$ such that

$$f(x) + r(x^*, x) - f(x^*) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\}.$$

That is

$$h(x_0, x) + r(x^*, x) - h(x_0, x^*) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\}.$$

Since h is a half distance function, we have

$$h(x_0, x^*) - h(x_0, x) \leq_C h(x^*, x).$$

Hence, (4.37) holds true.

Theorem A $\Longrightarrow$ Theorem 4.50:

Let

$$h(x, y) = f(y) - f(x), \quad \forall x, y \in X.$$

Arbitrarily fixing an $x_0 \in X$, by the assumption of Theorem 4.50, we know that Assumption 4.51 holds. Applying Theorem A, we have an $x^* \in X$ such that

$$h(x^*, x) + r(x^*, x) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\},$$

hence,

$$f(x) - f(x^*) + r(x^*, x) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\}.$$

Now, we turn to proving that Theorem A, B, C and D are equivalent to each other.

Theorem A $\Longrightarrow$ Theorem B:

By Theorem A, there exists an $x^* \in D_0$ such that (4.37) holds. We claim that $x^* \in T(x^*)$. Otherwise, by (4.38), $\exists x \in T(x^*), x \neq x^*$ such that

$$h(x^*, x) + r(x^*, x) \leq_C 0,$$

which contradicts (4.37).

Theorem B $\Longrightarrow$ Theorem C:

Define $T(x') = X \backslash \{x'\}, \forall x' \in X$. Suppose that $D_0 \cap M = \varnothing$, i.e., $\forall x' \in D_0$ implies $x' \notin M$. Then by (4.39), $\exists x \in X \backslash \{x'\} = T(x')$ such that

$$h(x', x) + r(x', x) \leq_C 0,$$

i.e., (4.38) holds. By Theorem B, $\exists x^* \in D_0$ such that $x^* \in T(x^*)$, which contradicts the definition of T.

Theorem C $\Longrightarrow$ Theorem D:

Let

$$M = \{x \in X : h(x, y) \not\leq_{C\backslash\{0\}} 0, \forall y \in X\}.$$

Then $\forall x' \in D_0 \backslash M, \exists x_1 \in X$ with $h(x', x_1) \leq_{C\backslash\{0\}} 0$.

By (4.40), we have $x_2 \in X\backslash\{x'\}$ such that

$$h(x', x_2) + r(x', x_2) \leq_C 0.$$

That is (4.39) holds. By Theorem C, $\exists x^* \in D_0 \cap M$, i.e.,

$$h(x^*, y) \leq_{C\backslash\{0\}} 0, \quad \forall y \in X.$$

Theorem D $\Longrightarrow$ Theorem C:

Assume that M is as in Theorem C. $\forall x' \in D_0$, if $x' \in M$, then Theorem C is proved. Now we suppose that $\exists x' \in D_0 \backslash M$. Hence, by the assumption of M, $\exists x_1 \in X$ with $x_1 \neq x'$ such that

$$h(x', x_1) + r(x', x_1) \leq_C 0. \tag{4.41}$$

So

$$h(x', x_1) \leq_{C\backslash\{0\}} 0.$$

If $x_1 \in M$, since $x' \in D_0$,

$$h(x_0, x') + r(x_0, x') \leq_C 0.$$

By (4.41),

$$h(x', x_1) + r(x', x_1) \leq_C 0.$$

So we have

$$h(x_0, x') + h(x', x_1) + r(x_0, x') + r(x', x_1) \leq_C 0.$$

Since h and r are a half distance function and a C_0 metric function, respectively, it follows that

$$h(x_0, x_1) + r(x_0, x_1) \leq_C 0.$$

Thus $x_1 \in D_0 \cap M$. Otherwise, $x_1 \in D_0 \backslash M$. By (4.39), $\exists x_2 \neq x_1$ such that

$$h(x_1, x_2) + r(x_1, x_2) \leq_C 0. \tag{4.42}$$

By (4.41), (4.42) and the triangle inequalities of h and r, we have

$$h(x', x_2) + r(x', x_2) \leq_C 0.$$

And it is easy to see that $x' \neq x_2$, since $h(x', x_2) \leq_{C\setminus\{0\}} 0$ and $h(x_1, x_2) \leq_{C\setminus\{0\}} 0$, then $h(x', x_2) \leq_{C\setminus\{0\}} 0$, which derives that (4.40) holds. By Theorem D, $\exists x^* \in D_0$ such that

$$h(x^*, x) \not\leq_{C\setminus\{0\}} 0, \quad \forall x \in X. \tag{4.43}$$

We assert $x^* \in M$. Otherwise, by (4.39), $\exists x_3 \in X$ $x_3 \neq x^*$ and

$$h(x^*, x_3) + r(x^*, x_3) \leq_C 0.$$

This demonstrates that

$$h(x^*, x_3) + r(x^*, x_3) \leq_{C\setminus\{0\}} 0,$$

since $r(x^*, x_3) \leq_{C\setminus\{0\}} 0$, which contradicts (4.43).

Theorem C $\Longrightarrow$ Theorem A:
For any $x' \in X$, define

$$B(x') = \{x \in X\setminus\{x'\} : h(x', x) + r(x', x) \leq_C 0\}.$$

Let

$$M = \{x \in X : B(x) = \varnothing\}.$$

If $x' \in D_0\setminus M$, then by the definition of M, $\exists x \in B(x')$ such that $x' \neq x$ and

$$h(x', x) + r(x', x) \leq_C 0,$$

hence (4.39) holds. By Theorem C, $\exists x^* \in D_0 \cap M$, which implies that $B(x^*) = \varnothing$, i.e., Theorem A holds. ■

Remark 4.53. (i) Theorem 7.1 of Nemeth [151] is a special case of Theorem B and Theorem B is a new variant of Caristi-Kirk fixed point theorem.
(ii) Theorem C is the vector form of Oettli and Thera's result [153].
(iii) Theorem D corresponds to the existence result of Takahashi [184] and Tammer [186].
(iv) Let $x \in X$. If $h(x, \cdot)$ is C order lower semicontinuous, then $h(x, \cdot)$ is submonotone with respect to C.

We give a new standard assumption.

Assumption 4.54. *(X, d) is a complete metric space, Y is a locally convex Hausdorff space ordered by a nontrivial convex cone C, $c^0 \in C\setminus\{0\}$ is such that $\exists \lambda \in C^*$ such that $\lambda(c^0) > 0$, $h : X \times X \to Y$ is a half distance function. $\forall x \in X$, $\forall \alpha > 0$, the set $\{y \in X : h(x, y) + \alpha d(x, y)c^0 \leq_C 0\}$ is closed (or $h(x, \cdot)$ is submonotone and C is closed). $\exists x_0 \in X$, $w_0 \in Y$ such that $h(x_0, x) \geq_C w_0, \forall x \in X$.*

Let

$$D_0' = \{x \in X : h(x_0, x) + d(x_0, x)c^0 \leq_C 0\}.$$

The following theorem is a special form of Theorem 4.50 (with $r(x, y) = d(x, y)c^0, \forall x, y \in X$ and $C_0 = \{\alpha c^0 : \alpha \geq 0\}$).

Theorem 4.55. *Let (X, d) be a complete metric space, and Y a locally convex Hausdorff space ordered by a nontrivial convex cone C. Let $c^0 \in C\backslash\{0\}$ be such that $\exists\lambda \in C^*$ with $\lambda(c^0) > 0$. Let $f : X \to Y$ be a vector-valued function such that $\exists w_0 \in Y$ such that*

$$f(x) \geq_C w_0, \quad \forall x \in X.$$

Suppose that $\forall x \in X$, $\forall\alpha > 0$, the set $\{y \in X : f(y) - f(x) + \alpha d(x, y)c^0 \leq_C 0\}$ is closed (or f is submonotone with respect to C and C is closed.) Then $\exists x^* \in X$ such that

$$f(x) - f(x^*) + d(x^*, x)c^0 \not\leq_{C\backslash\{0\}} 0, \quad \forall x \in X\backslash\{x^*\}.$$

If we set $C_0 = \{\alpha c^0 : \alpha \geq 0\}$, $r(x, y) = d(x, y)c^0$, then the following theorem is a special form of Theorem 4.52.

Theorem 4.56. *All the following theorems are true and they are equivalent to Theorem 4.55.*

Theorem A′. Let Assumption 4.54 hold. Then $\exists x^* \in D_0'$ such that

$$h(x^*, x) + d(x^*, x)c^0 \not\leq_C 0, \quad \forall x \in X\backslash\{x^*\}. \tag{4.44}$$

Theorem B′. Let Assumption 4.54 hold, and let $T : X \rightrightarrows Y$ satisfy

$$\forall x' \in D_0', \exists x \in F(x') \text{ such that } h(x', x) + d(x', x)c^0 \leq_C 0.$$

Then $\exists x^* \in D_0'$ such that $x^* \in T(x^*)$.

Theorem C′. Let Assumption 4.54 hold, and let $M \subset X$ satisfy: $\forall x' \in D_0'\backslash M$, there exists $x \in X$ such that

$$x \neq x' \text{ and } h(x', x) + d(x', x)c^0 \leq_C 0. \tag{4.45}$$

Then $\exists x^* \in D_0' \cap M$.

Theorem D′. Let Assumption 4.54 hold, and let the following (4.46) hold:

For all $x' \in D_0'$ satisfying $\exists x_1 \in X$ with $h(x', x_1) \leq_{C\backslash\{0\}} 0$, we have $x_2 \in X\backslash\{x'\}$ such that $h(x', x_2) + d(x', x_2)c^0 \leq_C 0$. (4.46)

Then $\exists x^* \in D_0'$ such that

$$h(x^*, x) \not\leq_{C\backslash\{0\}} 0, \quad \forall x \in X.$$

Proof. By letting $C_0 = \{\alpha c^0 : \alpha \geq 0\}$, $r(x, y) = d(x, y)c^0$ and applying Theorem 4.52, the theorem is proved. ■

In sequel, we will consider some applications of the above results.

Theorem 4.57. *Let Assumption 4.54 hold. Let A be a closed set, $x_0 \in A \subset X$ and $B \subset X$ be a bounded set. Let $r_1 \geq d(b, x_0)$, for some $b \in X$, and*

$$A_{r_1} = \{x \in A : d(b,x) \leq r_1\}.$$

If $h : X \times X \to Y$ is a half distance function and

$$h(x,y) \leq_C -\epsilon_0 c^0, \quad \forall x \in A, y \in B \tag{4.47}$$

for some $\epsilon_0 > 0$, then, $\exists \alpha \geq 0$, and $x^* \in C_1$, where

$$C_1 = \{y \in X : h(x^0, y) + \alpha d(x^0, y)c^0 \leq_C 0\} \cap A_{r_1},$$

such that

$$B \subset C_2 = \{y \in X : h(x^*, y) + \alpha d(x^*, y)c^0 \leq_C 0\},$$

and $C_2 \cap A_{r_1} = \{x^*\}$.

Proof. Obviously, A_{r_1} is a closed set. So $(A_{r_1}, \alpha d)$ is complete, where $0 < \alpha < \epsilon_0/(r_1 + r)$, $r = \sup\{d(b,x) : x \in B\}$. Replace X and d in Theorem A' with A_{r_1} and αd, respectively. By Theorem A', $\exists x^* \in C_1$ such that

$$h(x^*, x) + \alpha d(x^*, x)c^0 \not\leq_C 0, \quad \forall x \in A_{r_1} \backslash \{x^*\}.$$

Hence, $\forall x \in A_{r_1}$, $x \neq x^*$, we have $x \notin C_2$. Obviously, $x^* \in C_2$ since $0 \in -C$, so $C_2 \cap A_{r_1} = C_1$. If $x \in B$, then

$$\begin{aligned} h(x^*, x) + \alpha d(x^*, x)c^0 &\leq_C -\epsilon_0 c^0 + \alpha(d(x^*, b) + d(b, x))c^0 \\ &\leq_C -\epsilon_0 c^0 + \alpha(r_1 + r)c^0 \\ &\leq_C 0. \end{aligned}$$

So $x \in C_2$. ∎

Now, we will apply Theorem 4.50 to obtain a fixed point theorem for a directional contraction function in vector form, which is a generalization of a fixed point theorem in Clarke [46].

Let Y be a locally convex Hausdorff space ordered by a nontrivial, closed and convex cone C, $C_0 \subset C$ a C bound, regular complete convex cone, $C_0 \cap -C \subset -C_0$. Let (X, d) be a complete C_0 metric space, $\forall a \in X$, $r(a, \cdot)$ be continuous.

For any $x, y \in X$, the open segment $]x, y[$ is the set of all points z (if any) in X distinct from x and y satisfying $r(x,z) + r(z,y) = r(x,y)$.

A vector-valued function $T : X \to X$ is said to be a directional contraction function provided T is continuous and $\exists \delta \in (0,1)$ such that whenever $x \in X$ with $Tx \neq x$, $\exists y \in]x, Tx[$ such that $r(Tx, Ty) \leq_C \delta r(x,y)$.

Theorem 4.58. *Let Y be a locally convex Hausdorff space ordered by a nontrivial, closed and convex cone C, $C_0 \subset C$ a C bound regular complete convex cone, $C_0 \cap -C \subset -C_0$. Let (X, r) be a complete C_0 metric space. In addition, $\forall a \in X$, $r(a, \cdot)$ is continuous. Then, every directional contraction function admits a fixed point.*

Proof. Let $T : X \to X$ be a directional contraction function.

Define $f : X \to X$ by

$$f(x) = r(x, Tx), \quad \forall x \in X.$$

We will show that f is submonotone with respect to C. In fact, for any net $\{x_\mu\} \subset X$ such that $\lim x_\mu = x$ and $f(x_v) \leq_C f(x_\mu), v \geq \mu$, we will show that $f(x) \leq_C f(x_\mu), \forall \mu$.

Suppose that this is not true. Then $\exists \mu_0$ with

$$f(x_{\mu_0}) - f(x) \not\geq_C 0.$$

Thus $\exists \lambda \in C^* \backslash \{0\}$ such that

$$\lambda(f(x_{\mu_0}) - f(x)) < -\epsilon_0,$$

for some $\epsilon_0 > 0$. That is

$$\lambda(r(x_{\mu_0}, Tx_{\mu_0})) < \lambda(r(x, Tx)) - \epsilon_0. \tag{4.48}$$

On the other hand,

$$\forall v, \; r(x, Tx) \leq_C r(x, x_v) + r(x_v, Tx_v) + r(Tx_v, Tx). \tag{4.49}$$

By the continuity of T and the definition of (X, r) being a C_0 metric space, $\forall U \in B(0)$, $\exists v_0$, whenever $v \geq v_0$,

$$r(x, x_v) + r(Tx_v, Tx) \in U.$$

Let us choose U to be such that

$$\lambda(u) < \epsilon_0/2, \quad \forall u \in U.$$

By (4.49), we have

$$\lambda(r(x, Tx)) \leq \epsilon_0/2 + \lambda(r(x_v, Tx_v)).$$

Note that $\lambda(r(x_v, Tx_v)) \leq \lambda(r(x_{\mu_0}, Tx_{\mu_0})), v \geq \mu_0$. So we have

$$\lambda(r(x, Tx)) \leq \epsilon_0/2 + \lambda(r(x_{\mu_0}, Tx_{\mu_0})). \tag{4.50}$$

But (4.50) contradicts (4.48). Thus we have proved that f is submonotone with respect to C.

By the definition of f, we have $f(x) \geq_C 0, \forall x \in X$.

By Theorem 4.50 with $\epsilon = (1 - \delta)/2$, $\exists x^* \in X$ such that

$$f(x) - f(x^*) + ((1 - \delta)/2) r(x, x^*) \not\leq_C 0, \quad \forall x \in X \backslash \{x^*\}.$$

That is,

$$r(x,Tx) - r(x^*,Tx^*) + ((1-\delta)/2)r(x,x^*) \not\leq_C 0, \quad \forall x \in X\backslash\{x^*\}. \tag{4.51}$$

If $x^* = Tx^*$, then we have proved the theorem. Otherwise, we assume that $x^* \neq Tx^*$. Since T is a directional contraction function, by definition, we have that $\exists x_1 \in]x^*, Tx^*[$ such that

$$r(x^*,Tx^*) = r(x_1,x^*) + r(x_1,Tx^*) \tag{4.52}$$

and

$$r(Tx_1,Tx^*) \leq_C \delta r(x,x^*). \tag{4.53}$$

By (4.51), we have

$$r(x_1,Tx_1) - r(x^*,Tx^*) + (1-\delta)r(x_1,x^*) \not\leq_C 0. \tag{4.54}$$

Substituting (4.52) into (4.54), we obtain

$$r(x_1,Tx_1) - r(x_1,Tx^*) - \delta/2r(x_1,x^*) \not\leq_C 0,$$

$$r(Tx_1,Tx^*) - \delta r(x_1,x^*) \not\leq_C 0,$$

which contradicts (4.53). The contradiction proves the theorem. ■

4.4 Equivalents of Variational Principles for Set-Valued Functions

In this section, we will establish set-valued variants of "Petal Theorem" and "Cristi-Kirk Point Theorem". We will also establish equivalence between these theorems and the vector variational Principe for set-valued functions (see Huang [98]).

We make the following assumption.

Assumption 4.59. *Y is a locally convex Hausdorff space ordered by a non-trivial convex cone $C \subset Y$. $C_0 \subset C$ is a C bound regular complete convex cone and $C_0 \cap -C \subset -C_0$. $r : X \times X \to C_0$ is a C_0 metric function, (X,r) is a complete C_0 metric space. $F : X \rightrightarrows Y$ is a set-valued function and $\forall x \in X$, $F(x)$ has the domination property. There exist $x_0 \in X$ and $y_0 \in F(x_0)$ such that F is C order lower bounded on $X_1 = \{x \in X : (y_0 - C) \cap F(x) \neq \varnothing\}$. Either of the following two statements holds:*

*(**I**) C is closed, $\forall a \in X$, $r(a,\cdot)$ is continuous with respect to the topology of X induced by r and F is submonotone with respect to C;*

*(**II**) $\forall x_0 \in X$, $y_0 \in F(x_0)$, a net $\{x_\alpha\} \subset X$, $x_\alpha \to \bar{x} \in X$ and $y_\alpha \in F(x_\alpha)$ such that $y_\alpha - y_0 + r(x_\alpha, x_0) \leq_C 0$, it follows that $\exists \bar{y} \in F(\bar{x})$ such that $\bar{y} - y_0 + r(\bar{x}, x_0) \leq_C 0$.*

The following theorem is a direct consequence of Theorem 4.48.

Theorem 4.60. *Let the Assumption 4.59 hold. Then there exists $x^* \in X$ and $y^* \in Min_C(F(x^*))$ such that*

$$y^* \leq_C y_0 \text{ and } (F(x) - y^* + r(x, x^*)) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{x^*\}. \quad (4.55)$$

Proof. Let $\epsilon = 1$ in Theorem 4.48. The former formula of (4.55) follow from (iv) of Theorem 4.48 and the latter formula of (4.55) is just (v) of Theorem 4.48. ■

Theorem 4.61. *All the following theorems are true and equivalent to Theorem 4.60.*

Theorem A''. Let Assumption 4.59 hold. In addition, we assume that

(a'') $\forall \bar{x} \in X, \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$, there exists $x_1 \in X_1$ such that $(F(x_1) - \bar{y}) \cap (-C \backslash \{0\}) \neq \varnothing$, it follows that $\exists x_2 \in X_1 \backslash \{\bar{x}\}$ and $y_2 \in F(x_2)$ such that $y_2 - \bar{y} + r(\bar{x}, x_2) \leq_C 0$.

Then there exist $x^* \in X_1$ and $y^* \in \mathrm{Min}_C(F(x^*))$ with $y^* \leq_C y_0$ such that

$$(F(x) - y^*) \cap (-C \backslash \{0\}) = \varnothing, \quad \forall x \in X \backslash \{x^*\}.$$

Theorem B''. Let Assumption 4.59 hold. Let $T : X \rightrightarrows X$ be a set-valued function such that (b'') holds:

(b'') $\forall \bar{x} \in X_1, \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$, there exist $x \in \bar{T}(\bar{x})$ and $y \in F(x)$ such that $y - \bar{y} + r(x, \bar{x}) \leq_C 0$.

Then there exist $x^* \in X_1$ and $y^* \in F(x^*)$ such that

$$x^* \in T(x^*), \quad y^* \leq_C y_0.$$

Theorem C''. Let Assumption 4.59 hold. Let $M \subset X$ has the property (c''):

(c'') $\forall \bar{x} \in X_1 \backslash M, \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$, there exist $x \in X$, $y \in F(x)$ such that $x \neq \bar{x}$ and $y - \bar{y} + r(x, \bar{x}) \leq_C 0$.

Then, there exist $x^* \in M \cap X_1$ and $y^* \in F(x^*)$ such that $y^* \leq_C y_0$.

Proof. Theorem 4.60 $\Longrightarrow$ Theorem A'':

It follows from Theorem 4.60 that there exist $x^* \in X_1$ and $y^* \in \mathrm{Min}_C(F(x^*))$ such that (4.55) holds. Now we show that

$$(F(x) - y^*) \cap (-C \backslash \{0\}) = \varnothing, \quad \forall x \in X \backslash \{x^*\}.$$

Otherwise, $\exists x_1 \in X \backslash \{x^*\}$ and $y_1 \in F(x_1)$ such that $y_1 - y^* \in -C\backslash\{0\}$. Thus, $x_1 \in X_1$ and $y_1 \leq_C y_0$. According to (a''), there exist $x_2 \in X_1 \backslash \{x^*\}$ and $y_2 \in F(x_2)$ such that

$$y_2 - y^* + r(x^*, x_2) \leq_C 0,$$

which contradicts (4.55).

Theorem 4.60 $\Longrightarrow$ Theorem B'':

It follows from Theorem (4.60) that there exist $x^* \in X_1$ and $y^* \in \text{Min}_C(F(x^*))$ such that (4.55) holds. Let us show that $x^* \in T(x^*)$. Otherwise, by (b''), $\exists x \in T(x^*) \backslash \{x^*\}$ and $y \in F(x)$ such that

$$y - y^* + r(x, x^*) \leq_C 0,$$

which contradicts (4.55).

Theorem 4.60 $\Longrightarrow$ Theorem C'':

Theorem 4.60 implies that there exist $x^* \in X_1$ and $y^* \in \text{Min}_C(F(x^*))$ such that (4.55) holds. Let us show that $x^* \in X_1 \cap M$. Otherwise, $x^* \in X_1 \backslash M$ and $y^* \leq_C y_0$. By (c''), $\exists x \in X \backslash \{x^*\}$ and $y \in F(x)$ such that

$$y - y^* + r(x, x^*) \leq_C 0,$$

which contradicts (4.55).

Theorem C'' $\Longrightarrow$ Theorem 4.60:

Let $M = \{\bar{x} \in X : \exists \bar{y} \in F(\bar{x})$ such that $(F(x) - \bar{y} + r(x, \bar{x})) \cap (-C) = \varnothing, \forall x \in X \backslash \{\bar{x}\}\}$. We show that the conclusion of Theorem 4.60 holds. Let $x_1 \in X \backslash M$. If $y_1 \in F(x_1)$ with $y_1 \leq_C y_0$, by the definition of M, there exists $x_2 \in X \backslash \{x_1\}$ and $y_2 \in F(x_2)$ such that

$$y_2 - y_1 + r(x_1, x_2) \leq_C 0,$$

which implies that $y_2 \leq_C y_1 \leq_C y_0$, so $x_2 \in X_1 \backslash \{x_1\}$. By Theorem C'', $\exists x^* \in M \cap X_1$ and $y^* \in F(x^*)$ with $y^* \leq_C y_0$, i.e., $\exists x^* \in X_1$ and $y^* \in F(x^*)$ with $y^* \leq_C y_0$ such that

$$y^* \leq_C y_0 \text{ and } (F(x) - y^* + r(x, x^*)) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{x^*\}. \quad (4.56)$$

By the domination property of $F(x^*)$, there exists $y' \in \text{Min}_C(F(x^*))$ such that $y' \leq_C y^*$. By (4.56),

$$y' \leq_C y_0 \text{ and } (F(x) - y' + r(x, x^*)) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{x^*\},$$

i.e., Theorem 4.60 holds.

Theorem $B'' \Longrightarrow$ Theorem 4.60:

Let $T(x) = \{w \in X\backslash\{x\} : \exists y_1 \in F(w)$ and $y_2 \in F(x)$ such that $y_1 - y_2 + r(x,w) \leq_C 0\}$. We prove by contradiction. Suppose that Theorem 4.60 fails. Then $\forall \bar{x} \in X_1, \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$, $\exists v \in X\backslash\{\bar{x}\}$ and $y_1 \in F(v)$ such that

$$y_1 - \bar{y} + r(\bar{x}, v) \leq_C 0.$$

Obviously, $v \in F(\bar{x})$. So (b'') holds. By Theorem B'', $\exists x^* \in X_1$ such that $x^* \in T(x^*)$, which contradicts the definition of T.

Theorem $A'' \Longrightarrow$ Theorem C'':

We prove by contradiction. Suppose that $X_1 \cap M = \varnothing$. Let us verify that (a'') holds. $\forall \bar{x} \in X_1$, $\forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$, $\exists x_1 \in X_1$ such that $(F(x_1) - \bar{y}) \cap (-C\backslash\{0\}) \neq \varnothing$. By (c''), $\exists x_2 \in X\backslash\{\bar{x}\}$ and $y_2 \in F(x_2)$ such that $y_2 - \bar{y} + r(\bar{x}, x_2) \leq_C 0$. Therefore, (a'') holds. By Theorem A'', $\exists x^* \in X_1$ and $y^* \in \mathrm{Min}_C(F(x^*))$ with $y^* \leq_C y_0$ such that

$$(F(x) - y^*) \cap (-C\backslash\{0\}) = \varnothing, \quad \forall x \in X\backslash\{x^*\}. \tag{4.57}$$

Once again, by (c''), $\exists x_3 \in X\backslash\{x^*\}$ and $y_3 \in F(x_3)$ such that

$$y_3 - y^* + r(x_3, x^*) \leq_C 0,$$

which contradicts (4.57). ■

Remark 4.62. Theorem A'' is a Takahashi type existence theorem for a minimal solution for a set-valued function. Theorem B'' is a fixed point theorem of Caristi-Kirk type. Theorem C'' is a set-valued version of Oettli and Thera's result (see Oettli and Thera [153]).

To present the equivalence of a relatively special variational principle for set-valued functions, we make the following standard assumption.

Assumption 4.63. *Let (X, d) be a complete metric space, Y be a locally convex Hausdorff space, $C \subset Y$ be a nonempty nontrivial convex cone, $c^0 \in C\backslash\{0\}$ is such that $\exists \lambda \in C^*$ such that $\lambda(c^0) > 0$. Let Y be ordered by C. Let $F :\rightrightarrows Y$ be a set-valued function with $F(x) \neq \varnothing$ for all $x \in X$ and for every $x \in X$, $F(x)$ has domination property. There exist $x_0 \in X$ and $y_0 \in F(x_0)$ such that F is C order lower bounded on $X_1 = \{x \in X : (y_0 - C) \cap F(x) \neq \varnothing\}$. Furthermore, either of the following two conditions holds:*

($\mathbf{I_1}$) C is closed and F is submonotone with respect to C;
($\mathbf{II_1}$) the condition (II) of Assumption 4.59 holds with $r(x_1, x_2) = d(x_1, x_2) c^0, \forall x_1, x_2 \in X$.

Applying Theorem 4.60 and Lemma 4.16, it is not difficult to prove the following Theorem 4.64, which is a special case of Theorem 4.60.

Theorem 4.64. *Let Assumption 4.63 hold. Then there exists an* $x^* \in X$ *and* $y^* \in Min_C(F(x^*))$ *such that*

$$y^* \leq_C y_0 \text{ and } (F(x) - y^* + d(x, x^*)c^0) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{x^*\}.$$

Theorem 4.65. *All the following theorems are true and equivalent to Theorem 4.64.*

Theorem A_1''. Let Assumption 4.63 hold. In addition, assume that (a_1'') holds:

(a_1'') $\forall \bar{x} \in X_1, \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$, there exists $x_1 \in X_1$ such that $(F(x_1) - \bar{y}) \cap (-C\backslash\{0\}) \neq \varnothing$, it follows that $\exists x_2 \in X_1 \backslash \{\bar{x}\}$ and $y_2 \in F(x_2)$ such that $y_2 - \bar{y} + d(\bar{x}, x_2)c^0 \leq_C 0$.

Then there exist $x^* \in X_1$ and $y^* \in \mathrm{Min}_C(F(x^*))$ with $y^* \leq_C y_0$ such that

$$(F(x) - y^*) \cap (-C\backslash\{0\}) = \varnothing, \quad \forall x \in X \backslash \{x^*\}.$$

Theorem B_1''. Let Assumption 4.63 hold. Let $T : X \rightrightarrows X$ be a set-valued function such that (b_1'') holds:

(b_1'') $\forall \bar{x} \in X, \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$ there exist $x \in T(\bar{x})$ and $y \in F(x)$ such that $y - \bar{y} + d(x, \bar{x})c^0 \leq_C 0$.

Then there exist $x^* \in X_1$ and $y^* \in F(x^*)$ such that

$$x^* \in T(x^*), \quad y^* \leq_C y_0.$$

Theorem C_1''. Let Assumption 4.63 hold. Let $M \subset X$ have the property (c_1''):

(c_1'') $\forall \bar{x} \in X_1 \backslash M$, $\forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0$, there exist $x \in X, y \in F(x)$ such that $x \neq \bar{x}$ and $y - \bar{y} + d(x, \bar{x})c^0 \leq_C 0$.

Then there exist $x^* \in M \cap X_1$ and $y^* \in F(x^*)$ such that $y^* \leq_C y_0$.

Proof. Let $C_0 = \{\alpha c^0 : \alpha \geq 0\}, r(x_1, x_2) = d(x_1, x_2)c^0, \forall x_1, x_2 \in X$. Applying Lemma 4.16 and Theorem 4.61, we can easily derive the conclusion of the theorem. ∎

Finally, we will present stronger versions of Theorem B'' and B_1'', respectively. Using the stronger results, we can obtain the existence of maximal solutions for vector optimization problems.

Let

$$D_0 = \{x \in X : (F(x) + r(x, x_0)) \cap (y_0 - C) \neq \varnothing\},$$

where x_0 and y_0 are as in Assumption 4.59. It is obvious that $D_0 \subset X_1$.

Theorem 4.66. *Let Assumption 4.59 hold. Let $T : X \rightrightarrows X$ be a set-valued function with $F(x) \neq \varnothing$ for all $x \in X$ such that $\forall \bar{x} \in D_0$, $\exists \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0 - r(\bar{x}, x_0), \forall x \in T(\bar{x})$ and $y \in F(x)$ such that $y - \bar{y} + r(x, \bar{x}) \leq_C 0$.*

Then $\exists x^* \in D_0$ and $y^* \in F(x^*)$ such that

$$T(x^*) = \{x^*\}, \quad y^* \leq_C y_0 - r(x^*, x_0).$$

Proof. Firstly, we prove that $\exists x^* \in T(x^*)$ and $y^* \in \mathrm{Min}_C(F(x^*))$ such that $y^* \leq_C y_0 - r(x_0, x^*)$ (thus $x^* \in D_0$). By (iv) and (v) of Theorem 4.48, there exist $x^* \in X_1, y^* \in \mathrm{Min}_C(F(x^*))$ with

$$\begin{aligned} &y^* \leq_C y_0 - r(x_0, x^*) \text{ and} \\ &(F(x) - y^* + r(x, x^*)) \cap (-C) = \varnothing, \quad \forall x \in X \backslash \{x^*\}, \end{aligned} \tag{4.58}$$

The first formula in (4.58) implies $x^* \in D_0$. Now we show that $x^* \in T(x^*)$. Otherwise, $x^* \notin T(x^*)$. By the assumption of this theorem, $\forall x \in T(x^*)$, $\exists y \in F(x)$ such that $y - y^* + r(x, x^*) \leq_C 0$, which contradicts the second formula in (4.58). Hence $x^* \in T(x^*)$. Finally, we show by contradiction that $T(x^*) = \{x^*\}$. Otherwise, $\exists x_1 \in T(x^*) \backslash \{x^*\}$. By the assumption of this theorem, there exists $y_1 \in F(x_1)$ such that $y_1 - y^* + r(x_1, x^*) \leq_C 0$, which contradicts (4.58). So x^* is just what we desire in the conclusion. ■

Now we let

$$D_0' = \{x \in X : F(x) + d(x, x_0)c^0 \cap (y_0 - C) \neq \varnothing\},$$

where x_0, y_0 are as in Assumption 4.63. It is obvious that $D_0' \subset X_1$.

Theorem 4.67. *Let Assumption 4.63 hold. Let $T : X \rightrightarrows X$ be a set-valued function with $F(x) \neq \varnothing$, for all $x \in X$ such that $\forall \bar{x} \in D_0', \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0 - d(\bar{x}, x_0)c^0$, $\forall x \in T(\bar{x})$ and $y \in F(x)$ such that $y - \bar{y} + d(x, \bar{x})c^0 \leq_C 0$.*

Then $\exists x^* \in D_0'$ and $y^* \in F(x^*)$ such that

$$T(x^*) = \{x^*\}, \quad y^* \leq_C y_0 - d(x^*, x_0)c^0.$$

Proof. The proof is very similar to that of Theorem 4.66. ■

As applications of the above theorems, we derive existence theorems of maximal solutions for vector optimization problems.

Theorem 4.68. *Let U be a nonempty set, Y a locally convex Hausdorff space ordered by a nontrivial convex cone $C \subset Y$. Let a convex cone $C_0 \subset C$ be C bound regular such that $C_0 \cap -C \subset -C_0$. Let $r : X \times X \to Y$ be a C_0 metric function, (X, r) a complete C_0 metric linear space. Let $C_1 \subset X$ be a nonempty nontrivial convex cone and C_1 induces an order in X. Let $f : U \to X$ be a vector-valued function. If there exist a complete subset $X_2 \subset f(U)$ and a set-valued function $F : X_2 \rightrightarrows Y$ with $F(x) \neq \varnothing$ for all $x \in X_2$ such that*

(i) *$\forall x \in X_2$, $F(x)$ has the domination property;*
(ii) *the set-valued function $\Gamma : X \rightrightarrows X$, $\Gamma(x) = f(U) \cap (C_1 + x)$ is such that $\Gamma(X_2) \subset X_2$;*
(iii) *there exist $x_0 \in X$ and $y_0 \in F(x_0)$ such that F is C order lower bounded on $X_3 = \{x \in X_2 : F(x) \cap (y - C) \neq \varnothing\}$;*
(iv) *C is closed, F is submonotone with respect to C and $\forall a \in X$, $r(a, \cdot)$ is continuous with respect to the topology of X induced by r; or $\forall x_0 \in X$ and $y_0 \in F(x_0)$, and a net $\{x_\alpha\} \subset X$, $x_\alpha \to \bar{x} \in X$ and $y_\alpha \in F(x_\alpha)$ such that $y_\alpha - y_0 + r(x_\alpha, x_0) \leq_C 0$, it follows $\exists \bar{y} \in F(\bar{x})$ such that $\bar{y} - y_0 + r(\bar{x}, x_0) \leq_C 0$.*
(v) *$\forall \bar{x} \in D_0, \forall \bar{y} \in \Gamma(\bar{x})$ with $\bar{y} \leq_C y_0 - r(\bar{x}, x_0), \forall x \in \Gamma(\bar{x})$, $\exists y \in F(x)$ such that $y - \bar{y} + r(x, \bar{x}) \in -C$, where $D_0 = \{x \in X_2 : (F(x) + r(x, x_0)) \cap (y_0 - C) \neq \varnothing\}$.*

Then, there exists $u^* \in U$, which is a maximal solution of f on U, and $y^* \in F(f(u^*))$ such that $y^* - y_0 + r(x_0, f(u^*)) \leq_C 0$.

Proof. Applying Theorem 4.66 by setting $X = X_2$, $T = \Gamma$. It follows that $\exists x^* \in D_0$ and $y^* \in F(x^*)$ with $y^* \leq_C y_0 - r(x^*, x_0)$ such that $\Gamma(x^*) = \{x^*\}$. Suppose that $x^* = f(u^*)$, $u^* \in U$, then the conclusion of this theorem follows. ■

Theorem 4.69. *Let U be a nonempty set, Y a locally convex Hausdorff space ordered by a nontrivial convex cone C. Let $c^0 \in C$ be such that $\exists \lambda \in C^*$ such that $\lambda(c^0) > 0$. Let $(X, ||\cdot||)$ be a Banach space, $C_1 \subset X$ a nonempty nontrivial convex cone and C_1 induces an order in X. Let $f : U \to X$ be a vector-valued function. If there exist a complete subset $X_2 \subset f(U)$ and a set-valued function $F : X_2 \rightrightarrows Y$ with $F(x) \neq \varnothing$ for all $x \in X_2$ such that:*

(i) *for all $x \in X_2$, $F(x)$ has the domination property;*
(ii) *the set-valued function $\Gamma : X \rightrightarrows X, \Gamma(x) = f(U) \cap (C_1 + x)$ is such that $\Gamma(X_2) \subset X_2$;*
(iii) *there exist $x_0 \in X_2$ and $y_0 \in F(x_0)$ such that F is C order lower bounded on $X_3 = \{x \in X_2 : F(x) \cap (y_0 - C) \neq \varnothing\}$;*
(iv) *C is closed, F is submonotone with respect to C, or $\forall x_0 \in X$ and $y_0 \in F(x_0)$ and a net $\{x_\alpha\} \subset X$, $x_\alpha \to \bar{x} \in X$ and $y_\alpha \in F(x_\alpha)$ such that $y_\alpha - y_0 + ||\bar{x} - x_0||c^0 \leq_C 0$, it follows that $\exists \bar{y} \in F(\bar{x})$ such that $\bar{y} - y_0 + ||\bar{x} - x_0||c^0 \leq_C 0$.*
(v) *for all $\bar{x} \in D_0, \forall \bar{y} \in F(\bar{x})$ with $\bar{y} \leq_C y_0 - ||\bar{x} - x_0||c^0, \forall x \in \Gamma(\bar{x}), \exists y \in F(x)$ such that $y - \bar{y} + ||x - \bar{x}||c^0 \leq_C 0$, where $D_0 = \{x \in X_2 : (F(x) + ||x - x_0||c^0) \cap (y_0 - C) \neq \varnothing\}$.*

Then there exists $u^* \in U$, which is a maximal solution of f on U, and $y^* \in F(f(u^*))$ such that $y^* - y_0 + ||x_0 - f(u^*)||c^0 \leq_C 0$.

Proof. The proof is very similar to that of Theorem 4.68. ■

4.5 Extended Well-Posedness in Vector-Valued Optimization

In this and next sections, we assume (unless stated otherwise) that (X, d) is a metric space, Y is a normed space ordered by a nontrivial pointed, closed and convex cone C with nonempty interior $intC$, $e \in intC$ is a fixed element and $+\infty$ is a virtual element such that $\forall \alpha > 0, \alpha e \leq_C +\infty$. We also assume that the parametric space (P, ρ) is a metric space, a point $p^* \in P$ is fixed and L is a closed ball in P with the center p^* and a positive radius.

Suppose that $A \subset Y \cup \{+\infty\}$. Denote by $\text{Inf}_{intC}A$ the set of weak infima of A. Here by $a^* \in \text{Inf}_{intC}A$, we mean

(a) $a^* = +\infty$ if $A = \{+\infty\}$

or

(b) the following conditions are satisfied:

(i) $a^* \in Y$;

(ii) $a - a^* \not\leq_{intC} 0, \quad \forall a \in A$;

and

(iii) there exists a sequence $\{a_k\} \subset A$ such that $a_k \to a^*$ as $n \to +\infty$.

In this section, we consider well-posedness of vector-valued optimization problems.

Let $J : X \to Y \cup \{+\infty\}$ and $I : X \times L \to Y \cup \{+\infty\}$ be extended vector-valued functions such that $I(x, p^*) = J(x), \forall x \in X$, where $+\infty$ is a virtual element such that $\forall \alpha > 0, \alpha e \leq_C +\infty$.

The function $I(., p)$ is said to be proper if there exists $x \in X$ such that $I(x, p) \leq_{intC} +\infty$.

The problems are set as follows.

The original problem:

$$(X, J) : \quad \text{Inf}_{intC}\{J(x) : x \in X\}.$$

The perturbed problem corresponding to parameter p:

$$(X, I(., p)) : \quad \text{Inf}_{intC}\{I(x, p) : x \in X\}.$$

Note that the original problem (X, J) is the same as problem $(X, I(., p^*))$.

Let

$$V(p) = \text{Inf}_{intC}\{I(x, p) : x \in X\}.$$

Recall that, by $y \in V(p)$, we mean

(a) $y = +\infty$ if $I(x, p) \equiv +\infty, \forall x \in X$;

or

(b) the following conditions are satisfied:

(i) $y \in Y$;

(ii) $\forall x \in X, I(x, p) - y \not\leq_{intC} 0$;

and

(iii) there exists a sequence $\{x_k\} \subset X$ such that $I(x_k, p) \to y$ as $n \to +\infty$.

Throughout the section, we always assume $V(p) \neq \emptyset$ wherever the symbol $V(p)$ appears. Denote by $argmin_{intC}(X, I(.,p))$ the set of weakly minimal solutions of $I(.,p)$ on X.

Remark 4.70. If y is a weakly minimal point of $I(X,p)$, then $y \in V(p)$.

Suppose that (Z, d_1) is a metric space, $z \in Z$ and $Z_0 \subset Z$, denote by $d(z, Z_0) = \inf \{d_1(z, z_0) : z_0 \in Z_0\}$ the distance function from point z to set Z_0. Recall that $\xi_e : Y \to \mathbb{R}$ in Chapter 1 is defined as

$$\xi_e(y) = \min \{t \in \mathbb{R} : y \in te - C\}, \quad \forall y \in Y.$$

Noting that $C = \{c \in Y : l(c) \leq 0, \forall l \in -C^*\}$, the next proposition follows immediately from Proposition 1.44.

Proposition 4.71. *For any $y \in Y$,*

$$\xi_e(y) = \sup_{\lambda \in C^* \backslash \{0\}} \frac{\lambda(y)}{\lambda(e)}.$$

By Proposition 4.71, it is clear that if there exists $\lambda \in C^* \backslash \{0\}$ such that $\lambda(f)$ is bounded below on X, then $\xi_e(f)$ is also bounded below on X, where $f : X \to Y$ is a vector-valued function.

Throughout this section, we make the following assumption.

Assumption 4.72. *For any $p \in L$, $I(.,p)$ is proper and $\xi_e(I(x,p))$ is bounded below on X.*

Now we introduce three notions of extended well-posedness for vector optimization problems.

Definition 4.73. *Problem (X, J) is called well-posed in the weakly extended sense if*

$$argmin_{intC}(X, J) \neq \emptyset; \tag{4.59}$$

$$V(p) \neq \emptyset, \forall p \in L; \tag{4.60}$$

$$\begin{aligned} &[\textit{for any sequences } p_k \to p^* \textit{ in } P \textit{ and } \{x_k\} \textit{ in } X \textit{ such that} \\ &d(I(x_k, p_k), V(p_k)) \to 0, \textit{ there exist a subsequence } \{x_{k_i}\} \textit{ of } \{x_k\} \\ &\textit{and some point } x^* \in argmin_{intC}(X, J) \textit{ such that } x_{k_i} \to x^*.] \end{aligned} \tag{4.61}$$

Definition 4.74. *Problem (X, J) is called well-posed in the extended sense if (4.59) and (4.60) hold and*

[*for any sequences* $p_k \to p^*$ *in* P *and* $\{x_k\}$ *in* X *such that*
$\exists\{\alpha_k\}, \alpha_k \geq 0, \alpha_k \to 0$ *and* $y_k \in V(p_k)$ *with*
$I(x_k, p_k) \leq_C y_k + \alpha_k e$, *there exist a subsequence* $\{x_{k_i}\}$ *of c*
$\{x_k\}$ *and some point* $x^* \in argmin_{intC}(X, J)$ *such that*
$x_{k_i} \to x^*$.] (4.62)

Definition 4.75. *Problem* (X, J) *is called well-posed in the strongly extended sense if (4.59) and (4.60) hold and*

[*for any sequences* $p_k \to p^*$ *in* P *and* $\{x_k\}$ *in* X *such that*
$\liminf_{n\to+\infty}[\inf_{y\in V(p_k)} \xi_e(y - I(x_k, p_k))] \geq 0$, *there exists a subsequence*
$\{x_{k_i}\}$ *of* $\{x_k\}$ *and some point* $x^* \in argmin_{intC}(X, J)$
such that $x_{k_i} \to x^*$.] (4.63)

The sequences $\{x_k\}$ in (4.61), (4.62), and (4.63) are called a strongly asymptotically minimizing sequence, asymptotically minimizing sequence, weakly asymptotically minimizing sequence, respectively.

Remark 4.76. (a) It is not difficult to see that a strongly asymptotically minimizing sequence is also asymptotically minimizing and an asymptotically minimizing sequence is also weakly asymptotically minimizing.

(b) Due to (a), we know that the strongly extended well-posedness implies the extended well-posedness and that the extended well-posedness implies the weakly extended well-posedness. However, the converse may not be true.

Example 4.77. Let $X = [0, +\infty)$, $Y = C[0,1] \times \mathbb{R}$, $C = C_1 \times \mathbb{R}_+$, where $C[0,1]$ stands for the set of continuous functions defined on the interval $[0,1]$ and $C_1 = \{f \in C[0,1] : f(t) \geq 0, \forall t \in [0,1]\}$. Let $e = (1,1) \in \text{int}C$, $P = \mathbb{R}_+, p^* = 0$ and

$$I(x,p) = \begin{cases} (0,0), & \text{if } x \in [0,1]; \\ (f_x(t), 1/k + x - k) + (0,p), & \text{if } x \in (k, k+1], k = 1, 2, ..., \end{cases}$$

where

$$f_x(t) = \begin{cases} -kt(1 - 1/(k+1))^{k-2} + 1/(k+1), \\ \qquad\qquad \text{if } t \in [0, 1 - 1/(k+1)), x \in (k, k+1]; \\ -kt^{k-1} + 1/(k+1), \\ \qquad\qquad \text{if } t \in [1 - 1/(k+1), 1], x \in (k, k+1]. \end{cases}$$

Now we show that problem (X, J) is well-posed in the weakly extended sense but not well-posed in the extended sense.

It is not hard to verify that $V(p) = \{(0,0)\}$, $\forall p \in P$; $argmin_{intC}(X, J) = [0,1]$; $\lambda = (0,1) \in C^*\backslash\{0\}$ is such that $\lambda(I(x,p))$ is bounded below for any

$p \in P$ and so Assumption 4.72 holds; for any sequences $x_k \in X, p_k \in P$ such that $p_k \to 0$ and $I(x_k, p_k) \to 0$, we have $x_k \in [0,1]$ when k is sufficiently large. Thus there exist a subsequence x_{k_i} and $x^* \in [0,1]$ such that $x_{k_i} \to x^*$. So (X, J) is well-posed in the weakly extended sense. However, for the sequence $x_k = k + 1/k, p_k = 1/k (k \geq 2)$ such that $I(x_k, p_k) = (f_{k+1/k}(t), 3/k) \leq_C (0,0) + (1/(k+1), 3/k)$ there exists no subsequence x_{k_i} and $x^* \in [0,1]$ such that $x_{k_i} \to x^*$.

Example 4.78. Let $X = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $P = \mathbb{R}_+$, $e = (1,1)$, $p^* = 0$ and

$$I(x,p) = \begin{cases} (0,0), & \text{if } x \in [0,1]; \\ (k, x-k) + (0,p), & \text{if } x \in (k, k+1], k = 1, 2, ... \end{cases}$$

We show that problem (X, J) is well-posed in the extended sense but not well-posed in the strongly extended sense.

It is clear that $V(p) = (0,0)$ for any $p \in P$; $argmin_{intC}(X, J) = [0,1]$; $I(.,p)$ is bounded below for any $p \in P$ and thus Assumption 4.72 holds naturally; for any sequences x_k, p_k such that $p_k \to p^*$ and $I(x_k, p_k) \leq_C (0,0) + \epsilon_k(1,1)$ for some $\{\epsilon_k\} \subset \mathbb{R}_+$ with $\epsilon_k \to 0$ we have $x_k \in [0,1]$ when k is sufficiently large. Hence there exists a subsequence x_{k_i} and $x^* \in [0,1]$ such that $x_{k_i} \to x^*$. So (X, J) is well-posed in the extended sense. However, for the sequence $x_k = k, p_k = 1/k$, it is easy to verify that x_k is weakly asymptotically minimizing corresponding to p_k and there exists no subsequence x_{k_i} and $x^* \in [0,1]$ such that $x_{k_i} \to x^*$. Hence, (X, J) is not well-posed in the strongly extended sense.

(c) When $Y = \mathbb{R}$, $C = \mathbb{R}_+$, all the three types of extended well-posedness reduce to the extended well-posedness defined in [226].

(d) Any one of three types of well-posedness implies that $argmin_{intC}(X, J)$ is compact.

For simplicity, we write $argmin_{intC}(p)$ instead of $argmin_{intC}(X, I(.,p))$; problem(p) instead of problem$(X, I(.,p))$. Thus, problem$(p^*) = (X, J)$.

Now we give some criteria and characterizations of the three notions of extended well-posedness.

Let $e \in intC$. For any $\epsilon > 0$, $p \in L$, we denote

$$\epsilon - argmin_{intC}(p) = \{x \in X : I(x,p) - I(x',p) - \epsilon e \not\geq_{intC} 0, \forall x' \in X\}.$$

Let

$$M(\epsilon, p) = \{x \in X : I(x,p) - y - \epsilon e \not\geq_{intC} 0, \forall y \in V(p)\}.$$

Clearly, $\epsilon - argmin_{intC}(p) \subset M(\epsilon, p)$. However, the reverse inclusion may not hold.

The following proposition is about the nonemptiness of the set $\epsilon - argmin_{intC}(p)$.

Proposition 4.79. *Let Assumption 4.72 hold. Then* $\epsilon - argmin_{intC}(p) \neq \emptyset, \forall \epsilon > 0, p \in L$.

Proof. Since Assumption 4.72 holds, for any $p \in L$, $\xi_e(I(x,p))$ is bounded below. So for any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that

$$\xi_e(I(x_\epsilon, p)) \leq \inf_{x \in X} \xi_e(I(x,p)) + \epsilon. \tag{4.64}$$

We assert that $x_\epsilon \in \epsilon - argmin_{intC}(p)$. Otherwise, there exists $x' \in X$ such that

$$I(x', p) - I(x_\epsilon, p) + \epsilon e \leq_{intC} 0.$$

So

$$\xi_e(I(x', p)) - \xi_e(I(x_\epsilon, p) \leq \xi_e(I(x', p) - I(x_\epsilon, p)) < -\epsilon,$$

that is,

$$\xi_e(I(x_\epsilon, p)) > \xi_e(I(x', p)) + \epsilon \geq \inf_{x \in X} \xi_e(I(x,p)) + \epsilon,$$

which contradicts (4.64). ■

M can be viewed as a set-valued function from $\text{int}\mathbb{R}_+ \times L$ into X.

Proposition 4.80. *If $M(.,.)$ is u.s.c. at $(0, p^*)$, $argmin_{intC}(p^*)$ is nonempty and compact, then problem (p^*) is well-posed in the strongly extended sense. Conversely, if problem (p^*) is well-posed in the strongly extended sense, then M is u.s.c. at $(0, p^*)$.*

Proof. Let $p_k \to p^*$, $\epsilon_k \to 0$ and $u_k \in \epsilon_k - argmin_{intC}(p_k)$. Consider

$$T_k = \{x \in X : d(x, argmin_{intC}(p^*)) < 1/k\}, \quad \forall k.$$

Then, by the u.s.c. of M at $(0, p^*)$, there exists a subsequence u_{k_l} of u_k such that $u_{k_l} \in T_{k_l}, \forall l$. By the compactness of $argmin_{intC}(p^*)$, there exists a further subsequence of u_{k_l} that converges to a point in $argmin_{intC}(p^*)$. This proves the first half of the proposition. Arguing by contradiction, it is easy to prove the second half of the proposition. ■

Definition 4.81. *We say that problem (p^*) is stable in the weakly extended sense (respectively, in the extended sense, in the strongly extended sense) if*

$argmin_{intC}(p^*) \neq \emptyset$ and for every sequence $p_k \to p^*$
and every strongly asymptotically (respectively, asymptotically,
weakly asymptotically) minimizing sequence $\{x_k\}$
corresponding to $\{p_k\}$, we have
$$d(x_k, argmin_{intC}(p^*)) \to 0. \tag{4.65}$$

Proposition 4.82. *If problem (p^*) is well-posed in the weakly extended (respectively, extended, strongly extended) sense, then (4.65) holds (respectively). Conversely, if (4.65) holds, then (p^*) is well-posed in the weakly extended (respectively, extended, strongly extended) sense provided $argmin_{intC}(p^*)$ is compact.*

Proof. We only prove the case of well-posedness in the weakly extended sense. The other two cases can be similarly proved. Assume that (p^*) is well-posed in the weakly extended case. Suppose that (4.65) fails. Then, for suitable sequences $p_k \to p^*$, x_k strongly asymptotically minimizing corresponding to p_k, and for some $\epsilon > 0$, we have

$$\epsilon \leq d(x_k, argmin_{intC}(p^*)), \quad \forall n.$$

Weakly extended well-posedness of (p^*) implies the existence of a subsequence $\{x_{k_i}\}$ such that $x_{k_i} \to u \in argmin_{intC}(p^*)$; hence,

$$0 < \epsilon \leq \limsup_{i \to +\infty} d(x_{k_i}, argmin_{intC}(p^*)) \leq d(x_{k_i}, u) = 0,$$

a contradiction. This proves the first half of the proposition. Conversely, assume (4.65). If $p_k \to p^*$ and if x_k is strongly asymptotically minimizing corresponding to p_k, for each k, we find a point $u_k \in argmin_{intC}(p^*)$ such that $d(x_k, u_k) \to 0$; by compactness, we get weakly extended well-posedness. ∎

Given nonempty subsets A, B of X, consider the excess of A to B defined by

$$e(A, B) = \sup \{d(a, B) : a \in A\}.$$

The Hausdorff distance between A and B is defined as

$$\text{haus}(A, B) = \max \{e(A, B), e(B, A)\}.$$

For a bounded metric space, a sequence of subsets A_k and a subset B thereof, $e(A_k, B) \to 0$ iff $d(a_k, B) \to 0$ for every selection $a_k \in A_k$. Therefore, a reformulation of Proposition 4.82 yields the following corollary.

Corollary 4.83. *If X is bounded, then the strongly extended well-posedness of (p^*) implies $e(\epsilon - argmin_{intC}(p), argmin_{intC}(p^*)) \to 0$ as $(\epsilon, p) \to (0, p^*)$. The converse holds provided $argmin_{intC}(p^*)$ is nonempty and compact.*

In the following theorem, we shall give a metric characterization of the extended well-posedness of problem (p^*).

Consider a real-valued function $c = c(t, s)$ defined for $t \geq 0, s \geq 0$ sufficiently small, such that

$$c(t, s) \geq 0, c(0, 0) = 0 \tag{4.66}$$

$$s_k \to 0, t_k \geq 0, c(t_k, s_k) \to 0 \text{ imply } t_k \to 0. \tag{4.67}$$

Theorem 4.84. *If problem* (p^*) *is well-posed in the extended sense, then there exists* c *satisfying (4.66), (4.67) and*

$$\{I(x,p) - V(p) - c(d(x, argmin_{intC}(p^*)), \rho(p,p^*))e\} \cap (-\ intC) = \emptyset, \quad \forall x \in X, p \in L. \tag{4.68}$$

Conversely, if $argmin_{intC}(p^*)$ *is nonempty and compact, and (4.66) and (4.67) hold for some* c *satisfying (4.68), then* (p^*) *is well-posed in the extended sense.*

Proof. Let (p^*) be well-posed in the extended sense. Consider

$$c(s,t) = \inf\{\inf_{y \in V(p)} \xi_e(I(x,p) - y) : \rho(p,p^*) = s, d(x, argmin_{intC}(p^*)) = t\}$$

where $s \geq 0, t \geq 0$ are suitably small. Let us first check that (4.66) and (4.67) hold. For any $s \geq 0, t \geq 0$ suitably small, for any x, p satisfying $\rho(p,p^*) = s, d(x, argmin_{intC}(p^*)) = t$, if $y \in V(p)$, then

$$I(x,p) - y \not\leq_{intC} 0,$$

therefore,

$$\xi_e(I(x,p) - y) \geq 0.$$

Thus,

$$\inf_{y \in V(p)} \xi_e(I(x,p) - y)) \geq 0.$$

Hence, $c(s,t) \geq 0$. Moreover, if $s = 0$, $t = 0$, it follows from the extended well-posedness of (p^*) that $argmin_{intC}(p^*)$ is nonempty and compact. Hence, $d(x, argmin_{intC}(p^*)) = 0$ implies $x \in argmin_{intC}(p^*)$. In addition, $\rho(p,p^*) = 0$ implies $p = p^*$. Consequently, the combination of $d(x, argmin_{intC}(p^*)) = 0$ and $\rho(p,p^*) = 0$ yields $I(x,p) \in V(p^*)$. Hence,

$$c(0,0) \leq \inf_{y \in V(p^*)} \xi_e(I(x,p) - y) \leq \xi_e(I(x,p) - I(x,p)) = 0.$$

This combined with the inequality $c(0,0) \geq 0$, which has been proved above, yields $c(0,0) = 0$. Thus, (4.66) is verified. Now we show that (4.67) holds. Suppose that $s_k \geq 0, t_k \geq 0$ and $c(s_k,t_k) \to 0$, then there exist p_k, x_k and $y_k \in V(p_k)$ such that

$$\xi_e(I(x_k,p_k) - y_k) \to 0, \rho(p_k,p^*) = s_k, d(x_k, argmin_{intC}(p^*)) = t_k. \tag{4.69}$$

By (4.69), there exist $\alpha_k > 0, \alpha_k \to 0$ such that

$$\xi_e(I(x_k,p_k) - y_k) \leq \alpha_k,$$

implying

$$I(x_k,p_k) \leq_C y_k + \alpha_k e.$$

Hence, $\{x_k\}$ is an asymptotically minimizing sequence corresponding to $\{p_k\}$. So

$$t_k = d(x_k, argmin_{intC}(p^*)) \to 0$$

by Proposition 4.82, i.e., (4.67) holds. By the definition of c, it is easy to see that (4.68) holds. This proves the first part. Now assume that $\{p_k\} \subset L$ with $p_k \to p^*$ and $\{x_k\}$ is an asymptotically minimizing sequence corresponding to $\{p_k\}$. That is, there exist $y_k \in V(p_k)$ and $\alpha_k > 0, \alpha_k \to 0$ such that

$$I(x_k, p_k) \leq_C y_k + \alpha_k e. \tag{4.70}$$

Moreover, (4.68) implies

$$I(x_k, p_k) - y_k - c(d(x_k, argmin_{intC}(p^*)), \rho(p_k, p^*))e \not\leq_{intC} 0. \tag{4.71}$$

The combination of (4.70) and (4.71) yields

$$\alpha_k e - c(d(x_k, argmin_{intC}(p^*)), \rho(p_k, p^*))e \not\leq_{intC} 0. \tag{4.72}$$

Arguing by contradiction, it is easy to see that (4.72) implies

$$c(d(x_k, argmin_{intC}(p^*)), \rho(p_k, p^*)) \to 0.$$

By (4.67), we have

$$d(x_k, argmin_{intC}(p^*)) \to 0.$$

Applying Proposition 4.82, we conclude that (p^*) is well-posed in the extended sense. ■

Recall that the Kuratowski measure of noncompactness of a subset A of X is defined by

$$\alpha(A) = \inf \{k > 0 : A \text{ has a finite cover of sets with diameter } < k\}.$$

In the following theorem, we need the condition:

$$\alpha(\cup\{\epsilon - argmin_{intC}(p) : \rho(p, p^*) < \epsilon\}) \to 0, \text{ as } \epsilon \to 0. \tag{4.73}$$

Definition 4.85. *Problem (p) is said to have the weak domination property if, for any $x \in X$, there exists $x' \in argmin_{intC}(p)$ such that $I(x', p) \leq_C I(x, p)$.*

Theorem 4.86. *If X is a complete metric space and $\forall x' \in X$, $\xi_e(I(x', .) - I(., .))$ is u.s.c. on $X \times \{p^*\}$, (4.73) holds and problem (p) enjoys the weak domination property, then problem (p^*) is well-posed in the strongly extended sense. Conversely, the strongly extended well-posedness of problem (p^*) implies (4.73).*

Proof. Let

$$T(\epsilon) = \cup\{\epsilon - argmin_{intC}(p) : \rho(p, p^*) < \epsilon\}.$$

Suppose that (4.73) holds. Then $cl(T(\epsilon))$ is nonempty, closed and increasing in ϵ (with respect to the relation of set inclusion). By (4.73), $\alpha(cl(T(\epsilon))) = \alpha(T(\epsilon)) \to 0$ as $\epsilon \to 0$. By the Kuratowski theorem ([126], p.318), we have $haus[cl(T(\epsilon)), T] \to 0$ as $\epsilon \to 0$, where

$$T = \cap\{cl(T(\epsilon)) : \epsilon > 0\} \tag{4.74}$$

is nonempty and compact.

Let us show that

$$T = argmin_{intC}(p^*). \tag{4.75}$$

It is obvious from (4.74) that

$$d(x, T(\epsilon)) = 0,\ \forall x \in T, \forall \epsilon > 0. \tag{4.76}$$

To prove (4.75), we need only to show that $T \subset argmin_{intC}(p^*)$ since $argmin_{intC}(p^*) \subset T$ holds automatically. Suppose that there exists $x \in T$ such that $x \notin argmin_{intC}(p^*)$. Then $\exists x' \in X$ with $J(x') - J(x) \leq_{intC} 0$. Hence, $\exists \delta > 0$ with $\xi_e(J(x') - J(x)) < -\delta$ or $\xi_e(I(x', p^*) - I(x, p^*)) < -\delta$. Since $\xi_e(I(x', .) - I(., .))$ is u.s.c. on $X \times \{p^*\}$, we deduce that $\exists \epsilon_0 > 0$ such that, when $d(x, u) < \epsilon_0$, $\rho(p, p^*) < \epsilon_0$, we have

$$\xi_e(I(x', p) - I(u, p)) < -\delta.$$

Namely,

$$I(x', p) - I(u, p) \leq_{intC} -\delta e.$$

So when $\epsilon <$ min (ϵ_0, δ), for any u with $d(u, x) < \epsilon_0$, $u \notin T(\epsilon)$ holds. In other words, $u \in T(\epsilon)$ implies $d(u, x) \geq \epsilon_0$, which contradicts (4.76). Hence, $T = argmin_{intC}(p^*)$.

Now we prove that problem (p^*) is well-posed in the strongly extended sense. Suppose that $p_k \to p^*, \{x_k\}$ is a weakly asymptotically minimizing sequence corresponding to $\{p_k\}$, namely,

$$[\inf_{y \in V(p_k)} \xi_e(y - I(x_k, p_k))] \geq 0.$$

Hence $\exists \epsilon_k > 0$, ϵ_k is decreasing and converges to 0 such that

$$y - I(x_k, p_k) + \epsilon_k e \not\leq_{intC} 0, \quad \forall y \in V(p_k).$$

Since (p_k) has weak domination property, we conclude that $x_k \in \epsilon_k - argmin_{intC}(p_k)$. Moreover, ϵ_k is monotone decreasing, we have a subsequence $\{p_{k_i}\}$ such that

$$\rho(p_{k_i}, p^*) \leq \epsilon_k$$

and

$$x_{k_i} \in \epsilon_k - argmin_{intC}(p_{k_i}) \subset T(\epsilon_k).$$

By (4.74) and (4.75), we know that

$$d(x_{k_i}, argmin_{intC}(p^*)) \to 0.$$

From the compactness of $argmin_{intC}(p^*)$, we deduce that (p^*) is well-posed in the strongly extended sense.

Now we prove the second part of the theorem. Assume that (p^*) is well-posed in the strongly extended sense. Consider the excess

$$q(\epsilon) = e(T(\epsilon), argmin_{intC}(p^*)), \quad \epsilon > 0.$$

We show that $q(\epsilon) \to 0$ as $\epsilon \to 0$. If not, there exist $\delta > 0$, $\epsilon_k \to 0$, $x_k \in T(\epsilon_k)$ such that

$$d(x_k, argmin_{intC}(p^*)) \geq \delta, \quad \forall k. \tag{4.77}$$

We can find $p_k \to p^*$ such that x_k is a weakly asymptotically minimizing sequence corresponding to p_k, and thus (4.77) contradicts the extended strong well-posedness of (p^*). Hence, $q(\epsilon) \to 0$ as $\epsilon \to 0$. So, we have

$$T(\epsilon) \subset \{u \in X : d(u, argmin_{intC}(p^*)) \leq q(\epsilon)\}.$$

Hence,

$$\alpha(T(\epsilon)) \leq 2q(\epsilon),$$

since

$$\alpha(argmin_{intC}(p^*)) = 0,$$

and (4.73) follows. ∎

Remark 4.87. If , for any $\lambda \in C^*, x' \in X$, $\lambda(I(x', .))$ is u.s.c. at p^* and $\lambda(I(., .))$ is l.s.c. on $X \times \{p^*\}$, then for any $x' \in X$, $\xi_e(I(x', .) - I(., .))$ is u.s.c. on $X \times \{p^*\}$.

Similar to the proof of the first part of Theorem 4.86, we can prove

Theorem 4.88. *If X is complete, and for any $x' \in X$, $\xi_e(I(x', .) - I(., .))$ is u.s.c. on $X \times \{p^*\}$ and (4.74) holds, then (p^*) is well-posed in the extended sense.*

Remark 4.89. In Theorem 4.88, we dropped the assumption that (p) has the weak domination property.

In the remainder of this section, we assume that $(X, \|\cdot\|)$ is a Banach space.

The following theorem is a new variant of Ekeland's variational principle for a vector-valued function without assuming that the function is C order bounded below.

Theorem 4.90. *Let $f : X \to Y$ be a vector-valued function, f is l.s.c. on X and $\xi_e(f)$ is bounded below on X. Let $\epsilon > 0$ and x^* satisfy*

$$f(x) - f(x^*) + \epsilon e \not\leq_C 0, \quad \forall x \in X.$$

Then, for any real number $\delta > 0$, $\exists x' \in X$ such that
(i) $f(x') \leq_C f(x^*)$,
(ii) $\|x' - x^*\| \leq \delta$,
(iii) $f(x) - f(x') + \frac{\epsilon}{\delta}\|x - x'\|e \not\leq_{intC} 0, \quad \forall x \in X\backslash\{x'\}$.

Proof. Let $X_1 = \{x \in X : f(x) \leq_C f(x^*)\}$. Then X_1 is a closed subset of X by the l.s.c. of f. It follows from the assumption on x^* that $\xi_e(f(x) - f(x^*)) + \epsilon \geq 0, \forall x \in X$, i.e., x^* is an ϵ-minimum of the scalar function $\xi_e(f(x) - f(x^*))$ on X_1. In addition, from the conditions of this theorem, we deduce that $\xi_e(f(x) - f(x^*))$ is l.s.c. and bounded below. Applying Theorem 4.1, we know that for any $\delta > 0$, there exists an $x' \in X_1$ such that (ii) holds, and $\xi_e(f(x) - f(x^*)) + \frac{\epsilon}{\delta}\|x - x'\| > 0, \forall x \in X_1\backslash\{x'\}$, i.e., $f(x) - f(x') + \frac{\epsilon}{\delta}\|x - x'\|e \not\leq_{intC} 0, \forall x \in X_1\backslash\{x'\}$, hence (iii) holds. Since $x' \in X_1$, (i) holds true automatically. ■

Let $C^{*0} = \{l \in C^* : \|l\| = 1\}$.

Proposition 4.91. *Assume that for all $p \in L$, $I(.,p)$ is Gateaux differentiable; $\forall x' \in X, \xi_e(I(x',.) - I(.,.))$ is u.s.c. on $X \times \{p^*\}; I(.,p)$ is l.s.c.; Assumption 4.72 holds; and*

(C_1) for any sequence $p_k \to p^*$ in P, if x_k is an asymptotically minimizing sequence corresponding to p_k and there exists $\lambda_k \in C^{*0}$ such that $\|\lambda_k(\nabla_x I(x_k, p_k))\| \to 0$, then $\{x_k\}$ has a convergent subsequence.

Then (p^*) is well-posed in the extended sense.

Proof. Given $p_k \to p^*$, let x_k be an asymptotically minimizing sequence corresponding to p_k. That is, $\exists y_k \in V(p_k), \alpha_k > 0, \alpha_k \to 0$ such that

$$I(x_k, p_k) \leq_C y_k + \alpha_k e.$$

It is easy to see that

$$I(x, p_k) - I(x_k, p_k) + 2\alpha_k e \not\leq_C 0, \quad \forall x \in X.$$

Noticing that $I(., p_k)$ is l.s.c. and Assumption 4.72 holds, applying Theorem 4.90 with $\epsilon = 2\alpha_k, \delta = \sqrt{2\alpha_k}$, we obtain $z_k \in X$ such that

$$I(z_k, p_k) \leq_C I(x_k, p_k) \text{ and } \|z_k - x_k\| \leq \sqrt{2\alpha_k} \tag{4.78}$$

and

$$I(x, p_k) - I(z_k, p_k) + \sqrt{2\alpha_k}\|x - z_k\|e \not\leq_{intC} 0, \quad \forall x \in X,$$

implying

$$(I(x,p_k)-I(z_k,p_k))/\|x-z_k\|+\sqrt{2\alpha_k}e \not\leq_{intC} 0, \quad \forall x\in X. \tag{4.79}$$

For any $d\in X\backslash\{0\}, t>0$, let $x=z_k+td$, substitute it into (4.79), let $t\to 0$ and apply the Gateaux differentiability of $I(.,p_k)$. We have

$$\nabla_x I(z_k,p_k)(d)+\sqrt{2\alpha_k}e \not\leq_{intC} 0, \quad \forall d\in X.$$

Applying the separation theorem for convex sets, we know there exists $\lambda_k \in C^{*0}$ such that

$$\lambda_k(\nabla_x I(z_k,p_k)(d))+\sqrt{2\alpha_k}\lambda_k(e)\geq 0, \quad \forall d\in X,$$

that is,

$$\lambda_k(\nabla_x I(z_k,p_k)(d))\geq -\sqrt{2\alpha_k}\lambda_k(e), \quad \forall d\in X.$$

Arguing by contradiction, we deduce that

$$\|\lambda_k(\nabla_x I(z_k,p_k))\|\leq \sqrt{2\alpha_k}\lambda_k(e)\leq \sqrt{2\alpha_k}\|e\|.$$

Hence,

$$\|\lambda_k(\nabla_x I(z_k,p_k))\|\to 0, \text{ as } k\to\infty.$$

Noticing that $\{z_k\}$ is still an asymptotically minimizing sequence corresponding to $\{p_k\}$ and applying (C_1), we know that $\{z_k\}$ has a subsequence $\{z_{k_i}\}$ converging to z^*. By (4.78), $\|z_k-x_k\|\to 0$ as $k\to\infty$. So $x_{k_i}\to z^*$. Let us show by contradiction that $z^*\in argmin_{intC}(X,J)$. Otherwise, $\exists x_0\in X$ with

$$I(x_0,p^*)-I(z^*,p^*)\leq_{intC} 0.$$

So $\exists\delta>0$ such that

$$I(x_0,p^*)-I(z^*,p^*)\leq_{intC} -\delta e,$$

that is,

$$\xi_e(I(x_0,p^*)-I(z^*,p^*))<-\delta.$$

It follows from the u.s.c. of $\xi_e(I(x_0,.)-I(.,.)$ on $X\times\{p^*\}$ that $\exists\epsilon_0>0$, when $\rho(p,p^*)<\epsilon_0$ and $\|u-z^*\|<\epsilon_0$,

$$\xi_e(I(x_0,p)-I(u,p))<-\delta,$$

that is,

$$I(x_0,p)-I(u,p)+\delta e\leq_{intC} 0.$$

Hence, when k is sufficiently large,

$$I(x_0,p_{k_i})-I(x_{k_i},p_{k_i})+\delta e\leq_{intC} 0,$$

which contradicts $x_{k_i}\in\alpha_{k_i}-argmin_{intC}(p_{k_i})$. So (p^*) is well-posed in the extended sense. ■

Similarly, we can prove the following two propositions.

Proposition 4.92. *Assume that for any $p \in L$, $I(.,p)$ is Gateaux differentiable; $\forall x' \in X, \xi_e(I(x',.) - I(.,.))$ is u.s.c. on $X \times \{p^*\}$; $I(.,p)$ is l.s.c.; Assumption 4.72 holds; and*

(C_2) for any sequence $p_k \to p^*$ in P, if x_k is a weakly asymptotically minimizing sequence corresponding to p_k and there exists $\lambda_k \in C^{*0}$ such that $\|\lambda_k(\nabla_x I(x_k, p_k))\| \to 0$, then $\{x_k\}$ has a convergent subsequence.

Moreover, $\forall p \in L$, problem (p) enjoys the weak domination property. Then (p^*) is well-posed in the strongly extended sense.

Proposition 4.93. *Assume that for any $p \in L$, $I(.,p)$ is Gateaux differentiable; $\forall x' \in X, \xi_e(I(x',.) - I(.,.))$ is u.s.c. on $X \times \{p^*\}$; $I(.,p)$ is l.s.c.; Assumption 4.72 holds; and*

(C_3) for any sequence $p_k \to p^*$ in P, if x_k is a strongly asymptotically minimizing sequence corresponding to p_k and there exists $\lambda_k \in C^{*0}$ such that $\|\lambda_k(\nabla_x I(x_k, p_k))\| \to 0$, then $\{x_k\}$ has a convergent subsequence.

Moreover, assume that

$$\|z_k - x_k\| \to 0, I(z_k, p_k) \leq_C I(x_k, p_k)$$

$$\text{and } x_k, p_k \text{ as in } (C_3) \text{ imply } d(I(z_k, x_k), V(p_k)) \to 0. \tag{4.80}$$

Then (p^*) is well-posed in the weakly extended sense.

Remark 4.94. (4.80) holds automatically when $Y = \mathbb{R}, C = \mathbb{R}_+$.

4.6 Extended Well-Posedness in Set-Valued Optimization

In this section, we investigate the extended well-posedness properties of set-valued optimization problems.

Let Z be a topological space and $F : X \rightrightarrows Z$ be a set-valued function. F is called strict if, for every $x \in X$, $F(x) \neq \emptyset$. Let $f : Y \to \mathbb{R}$ be a real function. Recall that f is monotone with respect to C if, for any $y_1, y_2 \in Y$ with $y_1 \leq_C y_2$, we have $f(y_1) \leq f(y_2)$. We assume that the parametric space (P, ρ) is a metric space, a point $p^* \in P$ is fixed and L is a closed ball in P with the center p^* and a positive radius.

We also make the following assumption.

Assumption 4.95. *Set-valued functions $J : X \rightrightarrows Y$, and $I : X \times L \rightrightarrows Y$ are strict and for any $p \in L$, there exist $\lambda \in C^* \backslash \{0\}$ and a real number a such that $\lambda(y) \geq a, \forall y \in I(X, p)$, where $J(x) = I(x, p^*), \forall x \in X$.*

Consider the following set-valued optimization problems:

$$(X, J) : \mathrm{Inf}_{intC} \bigcup_{x \in X} J(x)$$

and

$$(X, I(.,p)) : \mathrm{Inf}_{intC} \bigcup_{x \in X} I(x,p).$$

(X, J) is called the original problem, while $(X, I(.,p))$ is called the model perturbation of the original problem corresponding to the parameter $p \in L$. Let $V(p)$ denote the set of weak infima of $I(X,p)$, $\forall p \in L$. By $y \in V(p)$ we mean that the following conditions are satisfied:

(i) $y \in Y$;

(ii) There exists no $x \in X$ such that $[I(x,p) - y](-intC) \neq \emptyset$;

and

(iii) There exists a sequence $x_k \in X$ and $y_k \in I(x_k,p)$ such that $y_k \to y$.

Recall that an element $x \in X$ is called a weakly minimal solution to $(X, I(.,p))$ if there exists a $y \in I(x,p)$ such that $y \in V(p)$.

We denote by $argmin_{intC}(X, J)$ and $argmin_{intC}(X, I(.,p))$ the sets of the weakly minimal solutions of (X, J) and $(X, I(.,p))$, $\forall p \in L$, respectively.

Throughout this section, we always assume that $V(p) \neq \emptyset$ wherever it appears.

Definition 4.96. *(X, J) is said to be well-posed in the extended sense with respect to the embedding defined by I if*

(i) Assumption 4.95 holds;

(ii) $argmin_{intC}(X, J) \neq \emptyset$;

(iii)

$$\begin{aligned}&[\forall p_k \to p^* \text{ and } (x_k, y_k) \in X \times Min_C(I(x_k, p_k)) \text{ such that} \\ &y_k \leq_C z_k + \epsilon_k e, \text{ for some } \epsilon_k > 0, \epsilon_k \to 0 \\ &\qquad\qquad \text{and some } z_k \in V(p_k),]\end{aligned} \tag{4.81}$$

then there exist a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ and $x^* \in argmin_{intC}(X, J)$ such that $x_{k_i} \to x^*$.

The sequence (x_k, y_k) as in (4.81) is called an asymptotically minimizing sequence corresponding to sequence p_k.

Definition 4.97. *(X, J) is said to be well-posed in the strongly extended sense with respect to the embedding defined by I if*

(i) Assumption 4.95 holds;

(ii) $argmin_{intC}(X, J) \neq \emptyset$;

(iii)

$$[\forall p_k \to p^* \text{ and } (x_k, y_k) \in X \times Min_C(I(x_k, p_k)) \text{ such that}$$

$$(I(X,p_k) - y_k + \epsilon_k e) \cap (-C\backslash\{0\}) = \emptyset, \text{ for some } \epsilon_k > 0, \epsilon_k \to 0,] \quad (4.82)$$

then there exist a subsequence $\{x_{k_i}\}$ *of* $\{x_k\}$ *and* $x^* \in argmin_{intC}(X, J)$ *such that* $x_{k_i} \to x^*$.

The sequence (x_k, y_k) *as in (4.82) is called a strongly asymptotically minimizing sequence corresponding to sequence* p_k.

Clearly, an asymptotically minimizing sequence corresponding to p_k is a strongly asymptotically minimizing sequence corresponding to p_k.

Remark 4.98. (i) It is easy to see that the strongly extended well-posedness implies the extended well-posedness if $V(p)$ is nonempty, for all $p \in L$. However, the converse may not be true. The following example demonstrates this fact.

Example 4.99. Let $X = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $e = (1,1)$, $P = \mathbb{R}_+$, $p^* = 0$ and

$$I(x,p) = \begin{cases} (0,0) + [0,1] \times [0,1], & \\ & \text{if } x \in [0,1], \forall p; \\ (k, x-k) + (0,p) + [0,1] \times [0,1], & \\ & \text{if } x \in (k, k+1], k = 1, 2, \cdots, \forall p. \end{cases}$$

We show that the problem (X, J) is well-posed in the extended sense, but not well-posed in the strongly extended sense.

It is clear that $V(p) = \{(0,0)\}, \forall p$; $argmin_{intC}(X, J) = [0,1]$; and Assumption 4.95 holds. It is easy to see that for any sequences x_k, p_k, $y_k \in I(x_k, p_k)$ such that $p_k \to p^*$ and $y_k \leq_C (0,0) + \epsilon_k(1,1)$ with $\epsilon_k \to 0^+$, we have $x_k \in [0,1]$ when k is sufficiently large. Hence there exists a subsequence $\{x_{k_i}\}$ and $x^* \in [0,1]$ such that $x_{k_i} \to x^*$. So (X, J) is well-posed in the extended sense. However, for sequence $x_k = n$, $p_k = 1/k$, $y_k = (k, 1/k) \in I(x_k, p_k)$, it is easy to check that x_k is a strongly asymptotically minimizing corresponding to p_k and there exists no subsequence x_{k_i} and $x^* \in [0,1]$ such that $x_{k_i} \to x^*$. Hence (X, J) is not well-posed in the strongly extended sense.

(ii) When $Y = \mathbb{R}, C = \mathbb{R}_+$ and I, J are real-valued, the definition of the strongly extended well-posedness is the same as that of the extended well-posedness defined in [226].

(iii) If I, J are single-valued, then the extended (strongly extended) well-posedness here reduces to the extended (resp. strongly extended) well-posedness in [99].

(iv) If (X, J) is well-posed in the (strongly) extended sense, then the solution set $argmin_{intC}(X, J)$ is sequentially closed and compact.

For simplicity, we write $argmin_{intC}(p)$ instead of $argmin_{intC}(X, I(.,p))$; (p) instead of $(X, I(.,p))$. Thus, $(p^*) = (X, J)$.

Now we give some criteria and characterizations for these two types of well-posedness. First of all, we consider the following set-valued function:

$(\epsilon, p) \to M(\epsilon, p) = \epsilon - argmin_{intC}(p) = \{x \in X : \exists y \in \mathrm{Min}_C(I(x,p))$ with $(I(X,p) - y + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset\}$, for $\epsilon > 0, p \in L$.

Lemma 4.100. *Let $F : X \rightrightarrows Y$ be a set-valued function and there exist $\lambda \in C^*\backslash\{0\}$ and a real number a such that $\lambda(y) \geq a, \forall y \in F(X)$. Then for any $\epsilon > 0$, there exist an x^* and a $y^* \in F(x^*)$ such that $(F(X) - y^* + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset$.*

Proof. Let $s = \inf\{\lambda(y) : y \in F(X)\}$. By the assumption of Lemma 4.100, we have $s > -\infty$. Since $\lambda(e) > 0$, we deduce that, for any $\epsilon > 0$, there exist $x^* \in X$ and $y^* \in F(x^*)$ such that

$$\lambda(y^*) < s + \epsilon\lambda(e) \leq \lambda(y) + \epsilon\lambda(e), \quad \forall y \in F(X). \tag{4.83}$$

Now we show by contradiction that x^* and y^* are just what we want in Lemma 4.100.

Suppose that there exist $x_1 \in X$ and $y_1 \in F(x_1)$ such that

$$y_1 - y^* + \epsilon e \leq_{C\backslash\{0\}} 0.$$

Then

$$\lambda(y_1 - y^* + \epsilon e) \leq 0.$$

Therefore,

$$\lambda(y_1) \leq \lambda(y^*) - \epsilon\lambda(e),$$

which contradicts (4.83). The proof is complete. ∎

Proposition 4.101. *If Assumption 4.95 holds and, for any $p \in L, x \in X$, $I(x,p)$ enjoys the lower domination property, then $\forall\epsilon > 0, M(\epsilon, p) \neq \emptyset$.*

Proof. Let $p \in L$. It follows from Lemma 4.100 that, for any $\epsilon > 0$, there exist $x^* \in X$ and $y^* \in I(x^*, p)$ such that $(I(X,p) - y^* + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset$. Since $I(x^*, p)$ has the lower domination property, we conclude that there exists $y_1^* \in \mathrm{Min}_C(I(x^*, p))$ such that $y_1^* \leq_C y^*$. Therefore, $(I(X,p) - y_1^* + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset$. That is, $M(\epsilon, p) \neq \emptyset$. The proof is complete. ∎

In the following, we assume that (X, d) is a metric space.

Proposition 4.102. *Let Assumption 4.95 hold. If (p^*) is well-posed in the strongly extended sense, then*

$$M \text{ is upper semicontinuous at } (0, p^*). \tag{4.84}$$

Conversely, if M is u.s.c. at $(0, p^*)$ and $argmin_{intC}(p^*)$ is nonempty and compact, then (p^*) is well-posed in the strongly extended sense.

Proof. We prove (4.84) by contradiction.

Suppose that

$$\exists\delta > 0, (\epsilon_k, p_k) \to (0, p^*)(\epsilon_k > 0) \text{ and } x_k \in \epsilon_k - argmin_{intC}(p_k) \tag{4.85}$$

such that

$$d(x_k, argmin_{intC}(p^*)) \geq \delta. \tag{4.86}$$

By (4.85), $\exists y_k \in \text{Min}_C(I(x_k, p_k))$ with $(I(X, p_k) - y_k + \epsilon_k e) \cap (-C \backslash \{0\}) = \emptyset$, implying (x_k, y_k) is a strongly asymptotically minimizing sequence corresponding to p_k. Hence, there exist a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ and $x^* \in argmin_{intC}(p^*)$ such that $x_{k_i} \to x^*$, contradicting (4.86).

Now let us prove the second part. Let $p_k \to p^*, \epsilon_k \to 0(\epsilon_k > 0)$. If (x_k, y_k) is a strongly asymptotically minimizing sequence corresponding to p_k, then $x_k \in M(\epsilon_k, p_k)$. By the u.s.c. of M, we know that $\exists u_k \in argmin_{intC}(p^*)$ such that

$$x_k - u_k \to 0. \tag{4.87}$$

By the compactness of $argmin_{intC}(p^*)$, there exist a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ and $x^* \in argmin_{intC}(p^*)$ such that

$$u_{k_i} \to x^*. \tag{4.88}$$

The combination of (4.87) and (4.88) yields $x_{k_i} \to x^*$. Hence, (p^*) is strongly well-posed in the extended sense and the proof is complete. ■

Now we introduce the concept of the (strongly) extended stability of (p^*):

$$\begin{aligned}&[argmin_{intC}(p^*) \text{ is nonempty, for any sequence } p_k \to p^*, \text{ for any}\\ &\text{strongly asymptotically (resp. asymptotically) minimizing sequence}\\ &(x_k, y_k) \text{ corresponding to } p_k, \text{ we have}\\ &d(x_k, argmin_{intC}(p^*)) \to 0.]\end{aligned} \tag{4.89}$$

The following proposition establishes the relationship between the (strongly) extended stability of (p^*) and the (strongly) extended well-posedness of (p^*).

Proposition 4.103. *If (p^*) is well-posed in the (strongly) extended sense, then (4.89) holds. Conversely, (4.89) implies the well-posedness of (p^*) in the (resp. strongly) extended sense if $argmin_{intC}(p^*)$ is compact.*

Proof. We only prove the "strong" case since the proof of the other case is quite similar. We prove the first part of the proposition by contradiction. Suppose that (4.89) fails. Then $\exists\delta > 0, p_k \to p^*$ and a corresponding strongly asymptotically minimizing sequence (x_k, y_k) such that

$$d(x_k, argmin_{intC}(p^*)) \geq \delta. \tag{4.90}$$

By the strongly extended well-posedness of (p^*), there exist a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ and $x^* \in argmin_{intC}(p^*)$ such that $x_{k_i} \to x^*$, yielding $d(x_{k_i}, argmin_{intC}(p^*)) \to 0$, contradicting (4.90).

Conversely, for any $p_k \to p^*$ and a strongly asymptotically minimizing sequence (x_k, y_k) corresponding to p_k, it follows from (4.89) that

$$d(x_k, argmin_{intC}(p^*)) \to 0.$$

So $\exists u_k \in argmin_{intC}(p^*)$ such that

$$d(x_k, u_k) \to 0. \tag{4.91}$$

Now that $argmin_{intC}(p^*)$ is compact, there exist a subsequence $\{u_{k_i}\}$ of $\{u_k\}$ and $x^* \in argmin_{intC}(p^*)$ such that

$$u_{k_i} \to x^*. \tag{4.92}$$

The combination of (4.91) and (4.92) yields $d(x_{k_i}, x^*) \to 0$.

Hence, (p^*) is strongly extended well-posed and the proof is complete. ■

Now we consider the metric characterizations. First, we introduce the concept of the stability of (p^*):

Let $M^* = argmin_{intC}(p^*)$. (p^*) is said to be stable if $M^* \neq \emptyset$ and $\forall y_k \in J(x_k), \epsilon_k > 0, \epsilon_k \to 0, y_k - z_k \leq_C \epsilon_k e$, for some $\{z_k\} \subset V(p^*)$ implies $d(x_k, M^*) \to 0$.

A function $c : D \to \mathbb{R}$ is called forcing iff $0 \in D \subset [0, +\infty), c(0) = 0, c(t) \geq 0, \forall t \in D$ and $a_k \in D, c(a_k) \to 0$ implies $a_k \to 0$.

Proposition 4.104. *If*

$$\begin{aligned}[M^* \neq \emptyset \text{ and } (J(x) - V(p^*) - c[d(x, M^*)]e) \cap -intC) = \emptyset, \quad \forall x \in X \\ \text{and some forcing function } c], \end{aligned} \tag{4.93}$$

then (p^*) is stable. Conversely, if J is u.s.c. on X, $\forall x \in X, J(x)$ enjoys the lower domination property and (p^*) is stable, then (4.93) holds true.

Proof. Assume that (4.93) holds. Let $x_k \in X, y_k \in J(x_k), \epsilon_k \to 0, \epsilon_k > 0$ and $y_k \leq_C z_k + \epsilon_k e$, for some $\{z_k\} \subset V(p^*)$. Then

$$\xi_e(y_k - z_k) \leq \epsilon_k. \tag{4.94}$$

By (4.93), we have $y_k - z_k - c[d(x_k, M^*)]e \notin -\text{int}C$, implying

$$\xi_e(y_k - z_k) - c[d(x_k, M^*)] \geq 0. \tag{4.95}$$

The combination of (4.94) and (4.95) yields $c[d(x_k, M^*)] \to 0$, implying $d(x_k, M^*) \to 0$, by the forcing property of c.

Conversely, if J is u.s.c. on X, we first show that M^* is closed. In fact, suppose that $x_k \in M^*, x_k \to x^*$. Then $\exists y_k \in J(x_k)$ with

$$(J(X) - y_k) \cap (-intC) = \emptyset. \tag{4.96}$$

Suppose that $x^* \notin M^*$. Then $\forall y \in J(x^*), \exists z_y \in J(X\backslash\{x^*\})$ such that $z_y - y \in -intC$ (Otherwise, $\exists y^{*\prime} \in J(x^*)$ such that $(J(X\backslash\{x^*\} - y^{*\prime}) \cap -intC = \emptyset$, by the lower domination property of $J(x^*), \exists y^* \in \mathrm{Min}_C(J(x^*))$ with $y^* \leq_C y^{*\prime}$, hence, $(J(X) - y^*) \cap (-intC) = \emptyset$, implying $x^* \in M^*$). So there exist an open set U_y such that $y \in U_y$ and

$$z_y - U_y \subset -intC, \quad \forall y \in J(x^*). \tag{4.97}$$

Obviously, $J(x^*) \subset \bigcup_{y \in J(x^*)} U_y = U$. Due to the upper semicontinuity of J at x^* and $x_k \to x^*$, it follows that $J(x_k) \subset U$, when k is sufficiently large, implying $y_k \in U$, when k is sufficiently large. Due to (4.97), for every k sufficiently large, $\exists z_k \in J(X\backslash\{x^*\})$ such that $z_k - y_k \in -intC$, which contradicts (4.96).

Now we define

$$c(t) = \inf\{\inf_{z \in J(x), y \in V(p^*)} \xi_e(z-y) : d(x, M^*) = t\}, \quad \forall t \geq 0.$$

Since $\forall x \in X, \forall z \in J(x), \forall y \in V(p^*), z - y \notin -intC$, implying $\xi_e(z-y) \geq 0$, we have

$$c(t) \geq 0, \quad \forall t \geq 0. \tag{4.98}$$

As M^* is closed, so $x^* \in M^*$ if $d(x^*, M^*) = 0$. Thus $\inf_{z \in J(x^*), y \in V(p^*)} \xi_e(z-y) \leq 0$, yielding

$$c(0) \leq 0. \tag{4.99}$$

(4.98) and (4.99) jointly imply $c(0) = 0$. Suppose that $a_k \geq 0$ and $c(a_k) \to 0$. By the definition of c, it follows that $\exists \epsilon_k \to 0, \epsilon_k \geq 0$ and $x_k \in X, z_k \in J(x_k), y_k \in V(p^*)$ such that $d(x_k, M^*) = a_k$ and

$$\xi_e(z_k - y_k) \leq \epsilon_k. \tag{4.100}$$

By (4.100), we have $z_k - y_k \leq_C \epsilon_k e, \forall k \in N$. By the stability of (p^*), we know that $a_k = d(x_k, M^*) \to 0$. Finally, from the definition of $c(t)$, we have

$$\xi_e(z-y) \geq c[d(x, M^*)], \quad \forall z \in J(x), \forall y \in V(p^*),$$

i.e.,

$$(J(X) - V(p^*) - c[d(x, M^*)]e) \cap (-intC) = \emptyset.$$

The proof is complete. ■

Consider a real-valued function: $c = c(t, s)$ defined for $t \geq 0, s \geq 0$ sufficiently small such that

$$c(t,s) \geq 0, c(0,0) = 0, \tag{4.101}$$

$$s_k \to 0, t_k \geq 0, c(t_k, s_k) \to 0 \text{ implies } t_k \to 0. \tag{4.102}$$

The following theorem is a metric characterization of the extended well-posedness of (p^*).

Theorem 4.105. *If* (p^*) *is well-posed in the extended sense, then*

$$[(I(x,p) - V(p) - c[d(x, argmin_{intC}(p^*)), \rho(p,p^*)]e) \cap (-intC) = \emptyset, \\ \forall x \in X, \forall p \in L \text{ and some } c \text{ verifying } (4.101) \text{ and } (4.102).] \tag{4.103}$$

Conversely, if $V(p) \neq \emptyset$*, Assumption 4.95 holds and* $argmin_{intC}(p^*)$ *is nonempty and compact and (4.103) holds for some c verifying (4.101) and (4.102), then* (p^*) *is well-posed in the extended sense.*

Proof. Let (p^*) be well-posed in the extended sense.
Consider

$$c(t,s) = \inf\{\inf_{z \in I(x,p), y \in V(p)} \xi_e(z-y) : \rho(p,p^*) = s, d(x, argmin_{intC}(p^*)) = t\},$$

for $t \geq 0, s \geq 0$ sufficiently small. It is easy to see that $c(t,s) \geq 0$. We conclude that $c(0,0) = 0$ since $argmin_{intC}(p^*)$ is closed. Now let $s_k \geq 0, t_k \geq 0$ with $c(t_k, s_k) \to 0$ and $s_k \to 0$. Then $\exists x_k \in X, p_k \in L, z_k \in I(x_k, p_k), y_k \in V(p_k), \epsilon_k > 0, \epsilon_k \to 0$ such that

$$\xi_e(z_k - y_k) \leq \epsilon_k, \tag{4.104}$$

$$\rho(p_k, p^*) = s_k, d(x_k, argmin_{intC}(p^*)) = t_k. \tag{4.105}$$

By (4.100), we have

$$z_k - y_k \leq_C \epsilon_k e. \tag{4.106}$$

It follows from (4.105), (4.106) and the extended well-posedness of (p^*) that $t_k \to 0$. In addition, by the definition of $c(t,s)$, we have

$$\xi_e(z-y) \geq c[d(x, argmin_{intC}(p^*)), \rho(p,p^*)], \quad \forall z \in I(x,p), \forall y \in V(p),$$

implying

$$(I(x,p) - V(p) - c[d(x, argmin_{intC}(p^*), \rho(p,p^*))]e) \cap (-intC) = \emptyset.$$

Conversely, if $p_k \to p^*, x_k \in X, y_k \in I(x_k,p_k), \epsilon_k \to 0 (\epsilon_k > 0), z_k \in V(p_k)$ such that $y_k \leq_C z_k + \epsilon_k e$. By (4.103), we have $\epsilon_k \geq \xi_e(y_k - z_k) \geq c[d(x_k, argmin_{intC}(p^*)), \rho(p_k,p^*)]$. It follows from (4.102) that we deduce $d(x_k, argmin_{intC}(p^*)) \to 0$. Applying the compactness of $argmin_{intC}(p^*)$, we conclude that (p^*) is well-posed in the extended sense. The proof is complete.■

When p is near p^*, we need the following condition:

$$\alpha(\cup\{\epsilon - argmin_{intC}(p) : \rho(p,p^*) < \epsilon\}) \to 0, \text{ as } \epsilon \to 0, \tag{4.107}$$

where $\alpha(\cdot)$ is the Kuratowski measure of the noncompactness of a set.

Definition 4.106. *([143]) I is said to be compact on* $X \times \{p^*\}$ *if, for any* $x \in X$, $\forall\{(x_k, p_k)\}$ *with* $(x_k, p_k) \to (x, p^*), \forall y_k \in I(x_k, p_k)$, *there exist a subsequence* $\{y_{k_i}\}$ *and* $y \in I(x, p^*)$ *such that* $y_{k_i} \to y$.

Theorem 4.107. *Assume that X is a complete metric space and Assumption 4.95 holds.* $argmin_{intC}(p^*) \neq \emptyset, V(p) \neq \emptyset, \forall p \in L$. $\forall p \in L, x \in X, I(x,p)$ *is externally stable.* $I(.,.)$ *is compact on* $X \times \{p^*\}$, $I(x,.)$ *is lower semicontinuous at* p^* *for all* $x \in X$. *Then* (p^*) *is well-posed in the strongly extended sense if (4.107) holds. Conversely, this type of strong well-posedness implies (4.107).*

Proof. Let, for any $\epsilon > 0$,

$$T(\epsilon) = \cup\{\epsilon - argmin_{intC}(p) : \rho(p, p^*) \leq \epsilon\}.$$

It follows from Proposition 4.101 that $T(\epsilon) \neq \emptyset$. Besides, $\alpha(clT(\epsilon)) = \alpha(T(\epsilon)) \to 0$, as $\epsilon \to 0$.

By the Kuratowski theorem ([126], p.318), we have

$$\text{haus}[clT(\epsilon), T] \to 0 \text{ as } \epsilon \to 0,$$

where

$$T = \cap\{clT(\epsilon) : \epsilon > 0\}$$

is nonempty and compact. Now we prove that $T = argmin_{intC}(p^*)$. Let $\forall x^* \in T$. Then

$$d(x^*, T(\epsilon)) = 0, \forall \epsilon > 0.$$

Given $\epsilon_k > 0, \epsilon_k \to 0, \forall k, \exists u_k \in T(\epsilon_k)$ such that $d(x^*, u_k) < 1/k$. Hence, by the compactness of I at (x^*, p^*), $\exists p_k \to p^*, y_k \in I(u_k, p_k)$ such that there exist a subsequence $\{y_{k_i}\}$ and $y^* \in I(x^*, p^*)$ such that $y_{k_i} \to y^*$. We claim that $y^* \in V(p^*)$. Otherwise, $\exists x \in X$, $z \in I(x, p^*)$ and $\delta > 0$ with $z - y^* \leq_C -\delta e$. From the lower semicontinuity of $I(x,.)$ at p^* and $p_{k_i} \to p^*$, we deduce that $\exists z_{k_i} \in I(x, p_{k_i})$ such that $z_{k_i} - y_{k_i} \leq_C -\delta/2e$, when i is sufficiently large, which is impossible since $u_{k_i} \in T(\epsilon_{k_i})$, $y_{k_i} \in I(u_{k_i}, p_{k_i})$ and $\epsilon_{k_i} \to 0$. Hence, $x^* \in argmin_{intC}(p^*)$. The opposite inclusion $argmin_{intC}(p^*) \subset T$ is obvious.

Now let $p_k \to p^*$ and (x_k, y_k) be a strongly asymptotically minimizing sequence corresponding to p_k. Then, by taking a subsequence, we can find a decreasing $\epsilon_k > 0$ such that

$$(I(x_k, p_k) - y_k + \epsilon_k e) \cap (-C \backslash \{0\}) = \emptyset.$$

A further subsequence verifies $\rho(p_{k_i}, p^*) \leq \epsilon_k$, yielding (since ϵ_k decreases)

$$(I(x_{k_i}, p_{k_i}) - y_{k_i} + \epsilon_k e) \cap (-C \backslash \{0\}) = \emptyset.$$

Hence,

$$x_{k_i} \in \epsilon_k - argmin_{intC}(p_{k_i}) \subset T(\epsilon_k).$$

So

$$d(x_{k_i}, argmin_{intC}(p^*)) \to 0.$$

Consequently, the strong extended well-posedness of (p^*) holds true by the compactness of $argmin_{intC}(p^*)$.

The proof of the second part of this theorem is the same as that of Theorem 4.86. The proof is complete. ■

In what follows, we present a variant of Ekeland's variational principle for set-valued functions and derive necessary condition for an approximate solution to a set-valued optimization problem based on a kind of generalized derivative for set-valued functions defined by Chen and Jahn [37] (see also Definition 2.40 in Chapter 2). Finally, we introduce a condition and provide sufficient conditions for the extended and strongly extended well-posednesses of set-valued optimization problems.

From now on, we assume that X and Y are both Banach spaces.

We need the following Ekeland's variational principle for set-valued functions.

Lemma 4.108. *Let $F : X \rightrightarrows Y$ be strict, compact-valued and upper semicontinuous. Suppose that there exist $\lambda \in C^*\backslash\{0\}$, $a \in I\!R$ such that $\lambda(y) \geq a, \forall y \in F(X)$. Given $\epsilon > 0$, $x^* \in X$ and $y^* \in F(x^*)$ such that $((F(X) - y^*) + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset$, then, for any $\delta > 0$, there exist $x_\epsilon \in X$ and $y_\epsilon \in Min_C(F(x_\epsilon))$ such that*

(i) $y_\epsilon \leq_C y^*$;

(ii) $\|x_\epsilon - x^*\| \leq \delta$;

(iii) $(F(X) - y_\epsilon + \epsilon/\delta\|x - x_\epsilon\|e) \cap (-C\backslash\{0\}) = \emptyset$.

The proof is almost the same as that of Theorem 4.36 though the conditions are slightly weaker.

In the following, we will introduce a kind of epiderivatives for scalar set-valued functions and give a necessary optimality condition for an approximate solution to a set-valued optimization problem.

Definition 4.109. *Let $S \subset X$ be a nonempty set and $F : S \rightrightarrows I\!R$ be a set-valued function. Let $(x^*, y^*) \in S \times F(x^*)$ be given. The modified generalized contingent epideriative $D_mF(x^*, y^*) : X \to I\!R \cup \{-\infty\} \cup \{+\infty\}$ of F at (x^*, y^*) is defined as follows: for any $h \in X$,*

$$D_mF(x^*, y^*)(h) = \begin{cases} inf\{t : (h,t) \in T(epi(F), (x^*, y^*))\}, \\ \quad if\ \exists t \in I\!R \text{ such that } (h,t) \in T(epi(F), (x^*, y^*)), \\ +\infty, \text{ otherwise}, \end{cases}$$

where

$$epi(F) = \{(x, y) \in X \times I\!R : y \in F(x) + I\!R_+\}.$$

Remark 4.110. We have slightly modified the definition of the generalized contingent epiderivative of F at $(x^*, y^*) \in Gr(F)$ given in Definition 2.40 (when Y reduces to $\mathbb{R}$ and C reduces to $\mathbb{R}_+$). One advantage of this modification is that $D_m F(x^*, y^*)(h)$ is a finite real number or $-\infty$ or $+\infty$ for any $h \in X$ when $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, while the Definition 2.40 may lead to $D_g F(x^*, y^*)(h) = \emptyset$, for some $h \in X$.

Proposition 4.111. *Let $F : S \rightrightarrows \mathbb{R}$ be a set-valued function, $(x^*, y^*) \in S \times F(x^*)$, $f : S \to \mathbb{R}$ be locally Lipschitz near x^*, the direction derivative $f'(x^*, h)$ exists for all $h \in T(S, x^*)$. Then*

$$D_m(F+f)(x^*, y^*+f(x^*))(h) = D_m F(x^*, y^*)(h)+f'(x^*, h), \forall h \in X. \quad (4.108)$$

Proof. We prove (4.108) in three cases.

(i) If $h \notin T(S, x^*)$, then (4.108) holds automatically.

(ii) Suppose that $h \in T(S, x^*)$ and

$$\forall t \in \mathbb{R}, (h, t) \notin T(epi(F), (x^*, y^*)). \quad (4.109)$$

In this case, we assert that

$$\forall t \in \mathbb{R}, (h, t) \notin T(epi(F + f), (x^*, y^* + f(x^*)).$$

Otherwise, $\exists (h, t') \in T(epi(F + f), (x^*, y^* + f(x^*))$. So $\exists \lambda_k \in \mathbb{R}_+, \lambda_k \to +\infty, x_k \in S, y_k \geq z_k + f(x_k)$ with $z_k \in F(x_k)$ such that

$$h = \lim_{k \to +\infty} \lambda_k (x_k - x^*),$$

$$\begin{aligned} t' &= \lim_{k \to +\infty} \lambda_k (y_k - y^* - f(x^*)) \\ &= \lim_{k \to +\infty} \lambda_k (y_k - f(x_k) - y^*) + \lim_{k \to +\infty} \lambda_k (f(x_k) - f(x^*)) \\ &= \lim_{k \to +\infty} \lambda_k (y_k - f(x_k) - y^*) + f'(x^*, h). \end{aligned}$$

As

$$(x_k, y_k - f(x_k)) \in epi(F),$$

so we have

$$(h, t' - f'(x^*, h)) \in T(epi(F), (x^*, y^*)),$$

contradicting (4.109). Thus, we have proved that if $\forall t \in \mathbb{R}$, $(h, t) \notin T(epi(F), (x^*, y^*))$, then (4.108) holds.

(iii) $\exists t' \in \mathbb{R}$ such that $(h, t') \in T(epi(F), (x^*, y^*))$.

As shown in case (ii) (with F replaced by $F + f$, f replaced by $-f$, y^* replaced by $y^* + f(x^*)$ and $f(x^*)$ replaced by $-f(x^*)$), we can prove that there exists $t \in \mathbb{R}$ such that

$$(h,t) \in T(epi(F+f),(x^*,y^*+f(x^*)).$$

Let

$$(h,t) \in T(epi(F+f),(x^*,y^*+f(x^*)).$$

Then $\exists\lambda_k \in \mathrm{IR}_+, \lambda_k \to +\infty, x_k \in S, y_k \geq z_k + f(x^*)$ with $z_k \in F(x_k)$ such that

$$h = \lim_{k\to+\infty} \lambda_k(x_k - x^*),$$

$$t = \lim_{k\to+\infty} \lambda_k(y_k - y^* - f(x^*)) = t - f'(x^*,h).$$

So

$$(h, t - f'(x^*,h)) \in T(epi(F),(x^*,y^*)).$$

Thus,

$$t - f'(x^*,h) \geq D_m F(x^*,y^*)(h),$$

implying

$$D_m(F+f)(x^*,y^*+f(x^*))(h) \geq D_m F(x^*,y^*)(h) + f'(x^*,h). \tag{4.110}$$

On the other hand, noticing that $F = (F+f)+(-f)$, we have

$$D_m F(x^*,y^*)(h) \geq D_m(F+f)(x^*,y^*+f(x^*)) - f(x^*,h),$$

yielding

$$D_m(F+f)(x^*,y^*+f(x^*))(h) \leq D_m F(x^*,y^*)(h) + f'(x^*,h). \tag{4.111}$$

From (4.110) and (4.111), we derive (4.108). The proof is complete. ∎

Lemma 4.112. *Let $F : X \rightrightarrows Y$ be a set-valued function, $(x^*,y^*) \in X \times F(x^*)$ be such that $(F(X) - y^*) \cap (-intC) = \emptyset$. Then*

$$D_m\eta(x^*,0)(h) \geq 0, \quad \forall h \in X, \tag{4.112}$$

where $\eta(x) = \xi_e(F(x) - y^*) = \{\xi_e(y - y^*) : y \in F(x)\}$.

Proof. We argue by contradiction. Suppose that $\exists h^* \in X$ such that

$$D_m\eta(x^*,0)(h^*) < 0.$$

Then $\exists t^* < 0$ such that

$$(h^*,t^*) \in T(epi(\eta),(x^*,0)),$$

i.e., $\exists\lambda_k \in \mathrm{IR}_+, \lambda_k \to +\infty, x_k \in X, y_k \geq z_k$ with $z_k \in \eta(x_k)$ such that

$$h^* = \lim_{k\to+\infty} \lambda_k(x_k - x^*), t^* = \lim_{n\to+\infty} \lambda_k(y_k - 0).$$

So $z_k < 0$ when k is sufficiently large. Hence, $\exists v_k \in F(x_k)$ such that $z_k = \xi_e(v_k - y^*) < 0$, implying that $v_k - y^* \leq_{intC} 0$ contradicting (4.112). The proof is complete. ∎

Lemma 4.113. *Let $F : X \rightrightarrows Y$ be strict, compact-valued and upper semicontinuous. Suppose that there exists $\lambda \in C^*\backslash\{0\}$ and $a \in I\!R$ such that $\lambda(y) \geq a, \forall y \in F(X)$. Let $\epsilon > 0$ and $(x^*, y^*) \in Gr(F)$ satisfy*

$$(F(X) - y^*) + \epsilon e) \cap (-C\backslash\{0\}) = \emptyset.$$

Then, for any $\delta > 0$, there exist $x_\epsilon \in X$ and $y_\epsilon \in Min_C(F(x_\epsilon))$ such that
(i) $y_\epsilon \leq_C y^*$;
(ii) $\|x_\epsilon - x^*\| \leq \delta$;
(iii') $D_m\eta(x_\epsilon, 0)(h) + \epsilon/\delta\|h\| \geq 0, \forall h \in X$, where

$$\eta(x) = \xi_e(F(x) - y_\epsilon) = \{\xi_e(y - y_\epsilon) : y \in F(x)\}.$$

Proof. It follows from Lemma 4.108 that (i) and (ii) hold. Due to (iii) in Lemma 4.108, we know that

$$(x_\epsilon, y_\epsilon) \in X \times H(x_\epsilon)$$

satisfies

$$(H(X) - y_\epsilon) \cap (-C\backslash\{0\}) = \emptyset,$$

where

$$H(x) = F(x) + \epsilon/\delta\|x - x_\epsilon\|e.$$

By Lemma 4.112,

$$D_m\xi_e(H)(x_\epsilon, 0)(h) \geq 0, \forall h \in X,$$

where

$$\xi_e(H(x)) = \xi_e(F(x) - y_\epsilon) + \epsilon/\delta\|x - x_\epsilon\| = \eta(x) + \epsilon/\delta\|x - x_\epsilon\|.$$

It follows from Proposition 4.111 that

$$D_m\xi_e(H)(x_\epsilon, 0)(h) = D_m\eta(x_\epsilon, 0)(h) + \epsilon/\delta\|h\|.$$

Hence,

$$D_m\eta(x_\epsilon, 0)(h) + \epsilon/\delta\|h\| \geq 0, \quad \forall h \in X.$$

The proof is complete. ■

Proposition 4.114. *(p^*) is well-posed in the strongly extended sense if the following conditions hold:*

(i) $I(., p)$ is strict, compact-valued, upper semicontinuous and Assumption 4.95 holds when p is near p^*;

(ii) $I(.,.)$ is compact on $X \times \{p^*\}$ and $I(x, .)$ is lower semicontinuous at $p^*, \forall x \in X$;

(iii) $argmin_{intC}(p^*) \neq \emptyset$;

(iv) $\forall p_k \to p^*, (x'_k, y'_k)$ is a strongly asymptotically minimizing sequence corresponding to p_k and $D_m\eta_k(x'_k, 0)(h) + \epsilon_k\|h\| \geq 0, \forall h \in X$, for some $\epsilon_k \to 0 (\epsilon_k > 0)$, where $\eta_k(x) = \xi_e(I(x, p_k) - y'_k), \forall x \in X$, then there exists a convergent subsequence $\{x'_{k_i}\}$.

Proof. Let $p_k \to p^*$ and (x_k, y_k) be a strongly asymptotically minimizing sequence corresponding to p_k. Then $\exists \epsilon_k > 0, \epsilon_k \to 0$ such that

$$(I(X, p_k) - y_k + \epsilon_k e) \cap (-C\backslash\{0\}) = \emptyset. \tag{4.113}$$

Applying Lemma 4.113 (setting $\delta = \sqrt{\epsilon_k}$), it follows from (i) that $\exists x_k' \in X, y_k' \in I(x_k', p_k)$ such that

(a) $\|x_k - x_k'\| \leq \sqrt{\epsilon_k}$;

(b) $y_k' \leq_C y_k$;

(c) $D_m\eta_k(x_k', 0)(h) + \sqrt{\epsilon_k}\|h\| \geq 0, \quad \forall h \in X.$

By (b) and (4.113), we have

$$(I(X, p_k) - y_k' + \epsilon_k e) \cap (-C\backslash\{0\}) = \emptyset.$$

So $\{(x_k', y_k')\}$ is also a strongly asymptotically minimizing sequence corresponding to p_k. This fact combined with (iv) and (c) yields that there exist a subsequence $\{x_{k_i}'\}$ and $x^* \in X$ such that $x_{k_i}' \to x^*$.

This fact together with (a) implies $x_{k_i} \to x^*$.

Now let us show $x^* \in argmin_{intC}(p^*)$.

Indeed, by $y_{k_i} \in I(x_{k_i}, p_{k_i})$ and (ii), there exists a subsequence $\{y_{k_{i_l}}\}$ and $y^* \in I(x^*, p^*)$ such that $y_{k_{i_l}} \to y^*$. We show that $y^* \in V(p^*)$. Otherwise, $\exists x \in X$ and $z \in I(x, p^*), \delta > 0$ such that

$$z - y^* \leq_C -\delta e.$$

As $I(x, .)$ is l.s.c. at p^*, so $\exists z_{k_{i_l}} \in I(x, p_{k_{i_l}})$ such that $z_{k_{i_l}} \to z$. Hence,

$$z_{k_{i_l}} - y_{k_{i_l}} \leq_C -\delta/2e,$$

when l is large enough, contradicting (4.113). The proof is complete. ∎

Proposition 4.115. *(p^*) is well-posed in the extended sense if the following conditions hold:*

(i) $I(., p)$ is strict, compact-valued, upper semicontinuous and Assumption 4.95 holds when p is near p^*;

(ii) $argmin_{intC}(p) \neq \emptyset, \forall p \in L$;

(iii) $I(., .)$ is compact on $X \times \{p^*\}$ and $I(x, .)$ is lower semicontinuous at $p^*, \forall x \in X$.

(iv) $\forall p_k \to p^*, (x_k', y_k')$ is an asymptotically minimizing sequence corresponding to p_k and

$$D_m\eta_k(x_k', 0)(h) + \epsilon_k\|h\| \geq 0, \quad \forall h \in X,$$

for some $\epsilon_k > 0, \epsilon_k \to 0$, where

$$\eta_k(x) = \xi_e(I(x, p_k) - y_k'), \quad \forall x \in X,$$

then there exists a convergent subsequence $\{x_{k_i}'\}$. Moreover, $\|z_k - x_k'\| \to 0, y_k \in I(z_k, p_k), y_k \leq_C y_k'$ implies $d(y_k, V(p_k)) \to 0$.

Since the proof is almost the same as that of Proposition 4.114, we omit it.

5

Vector Minimax Inequalities

Pioneer work of minimax theorems and minimax inequalities belongs to Fan [67, 69]. Many applications of minimax theorems and minimax inequalities are found in optimization theory, game theory and mathematical economics. Nieuwenhuis [152] published the first work of minimax theorems for vector-valued functions in 1983. In this direction, several interesting results appeared in Ferro [71, 72] and Tanaka [187, 188]. In Li, Chen and Lee [134], minimax inequalities for set-valued functions were considered. In this chapter, we consider minimax inequalities for set-valued functions, and vector-valued functions.

5.1 Minimax Inequalities for Set-Valued Functions

Let X and Z be two metric spaces. Let $C \subset \mathbb{R}^\ell$ be a pointed, closed and convex cone with nonempty interior $intC$.

Lemma 5.1. *Let X_0 and Z_0 be compact subsets of X and Z, respectively. Let $F : X_0 \times Z_0 \rightrightarrows \mathbb{R}^\ell$ be a continuous set-valued function and, for each $(x, z) \in X_0 \times Z_0$, let $F(x, z)$ be a compact set. Then*

$$\Gamma(x) = Min_{intC} \cup_{z \in Z_0} F(x, z) \text{ and } L(z) = Max_{intC} \cup_{x \in X_0} F(x, z)$$

are u.s.c. on X_0 and Z_0, respectively.

Proof. First, we prove that Γ is u.s.c. on X_0. Since F is continuous and Z_0 is compact, $\cup_{z \in Z_0} F(x, z)$ is compact for ant $x \in X_0$. Thus $\Gamma(x)$ is compact-valued for any $x \in X_0$. Suppose that $\Gamma(x)$ is not u.s.c. at $x_0 \in X_0$. Then there exists $\epsilon > 0$ and, for any $1/k > 0$, $k = 1, 2, \cdots$, there exist $x_k \in B(x, 1/k)$ and $y_k \in \Gamma(x_k)$ such that

$$y_k \notin B(\Gamma(x_0), \epsilon). \tag{5.1}$$

Then, there exists $z_k \in Z_0$ such that $y_k \in F(x_k, z_k)$. By the compactness of Z_0, we can assume, without loss of generality, that $z_k \to z_0 \in Z_0$. Obviously, $x_k \to x_0$. Since X and Z are two metric spaces, by the compactness of X_0 and Z_0, we know that $X_0 \times Z_0$ is a compact set in $X \times Z$. Therefore, $F(X_0, Z_0)$ is a compact set. Since $\{y_k\} \subset F(X_0, Z_0)$, we can assume, without loss of generality, that $y_k \to y_0$ as $k \to \infty$. By the upper semicontinuity of $F(x, z)$, we have $y_0 \in F(x_0, z_0)$. Obviously, by (5.1), $y_0 \notin \Gamma(x_0)$. Then, there exist $z_0^* \in Z_0$ and $y_0^* \in F(x_0, z_0^*)$ such that

$$y_0 - y_0^* \in intC.$$

Take any sequence $\{z_k^*\} \in Z_0$ with the limit z_0^*. Since $F(x, z)$ is l.s.c. on $X_0 \times Z_0$, there exists $y_k^* \in F(x_k, z_k^*)$ such that $y_k^* \to y_0^*$. Hence, when n is large enough,

$$y_k - y_k^* \in intC,$$

which contradicts the assumption $y_k \in \Gamma(x_k)$. Then, $\Gamma(x)$ is u.s.c. on X_0. ■

Proposition 5.2. *[94] Let $X_0 \subset X$, $Z_0 \subset Z$ be nonempty convex subsets, and let $A \subset X_0 \times Z_0$ be a subset such that*

(**i**) *for each $z \in Z_0$, the set $\{x \in X_0 : (x, z) \in A\}$ is closed in X_0;*
(**ii**) *for each $x \in X_0$, the set $\{z \in Z_0 : (x, z) \notin A\}$ is convex or empty.*

Suppose that there exist a subset B of A and a compact convex subset K of X_0 such that B is closed in $X \times Z_0$ and such that

(**iii**) *for each $z \in Z_0$, the set $\{x \in K : (x, z) \in B\}$ is nonempty and convex.*

Then, there exists a point $x_0 \in K$ such that $\{x_0\} \times Z_0 \subset A$.

It is obvious that if $\ell = 1$ and $C = \mathbb{R}_+$, then, the symbols Max_C and Max_{intC} have the same meaning, so do Min_C and Min_{intC}. In the remainder of the section, we use the symbols max and min instead of Max_C (Max_{intC}) and Min_C (Min_{intC}) when $\ell = 1$ and $C = \mathbb{R}_+$, respectively.

Proposition 5.3. *Let $X_0 \subset X$ and $Z_0 \subset Z$ be two nonempty compact and convex sets. Assume that $F : X_0 \times Z_0 \rightrightarrows \mathbb{R}$ is a continuous set-valued function and that, for each $(x, z) \in X_0 \times Z_0$, $F(x, z)$ is a compact set and F satisfies (i) to (iii) below:*

(**i**) *for each $x \in X_0$, $F(x_0, \cdot)$ is $\mathbb{R}_+$-concave on Z_0;*
(**ii**) *for each $z \in Z_0$, $F(\cdot, z)$ is naturally quasi $\mathbb{R}_+$-convex on X_0;*
(**iii**) *for each $t \in Z_0$, there exists $x_t \in X_0$ such that*

$$max\ F(x_t, t) \leq max \cup_{z \in Z_0} min \cup_{x \in X_0} F(x, z).$$

Then,

$$min \cup_{x \in X_0} max \cup_{z \in Z_0} F(x, z) = max \cup_{z \in Z_0} min \cup_{x \in X_0} F(x, z). \qquad (5.2)$$

Proof. Since

$$\max \cup_{v\in Z_0} F(x,v) \geq \min \cup_{u\in X_0} F(u,z), \text{ for all } x \in X_0 \text{ and } z \in Z_0,$$

we have

$$\min \cup_{x\in X_0} \max \cup_{z\in Z_0} F(x,z) \geq \max \cup_{z\in Z_0} \min \cup_{x\in X_0} F(x,z).$$

Now, we prove that the converse inequality holds. Indeed, choose any real number t such that

$$\max \cup_{z\in Z_0} \min \cup_{x\in X_0} F(x,z) < t,$$

and let

$$A = B = \{(x,z) \in X_0 \times Z_0 : \forall y \in F(x,z), \quad y \leq t\}.$$

Now we prove that A and B satisfy conditions (i) and (ii) of Proposition 5.2 with $K = X_0$.

First, we show that (i) holds. Indeed, for each $z \in Z_0$, let

$$x_k \in \{x \in X_0 : (x,z) \in A\}$$

and

$$x_k \longrightarrow x_0.$$

By the lower semicontinuity of $F(\cdot,z)$, for any $y_0 \in F(x_0,z)$, there exists $y_k \in F(x_k,z)$ such that $y_k \to y_0$. Since $(x_k,z) \in A$ for any k, we have that $y_k \leq t$. Thus

$$x_0 \in \{x \in X_0 : (x,z) \in A\},$$

and hence $\{x \in X_0 : (x,z) \in A\}$ is closed.

Second, we show that (ii) holds. Indeed, since, for each $x \in X_0$,

$$\{z \in Z_0 : (x,z) \notin A\} = \{z \in Z_0 : \exists y_0 \in F(x,z) \text{ such that } y_0 > t\},$$

by the $\mathbb{R}_+$-concavity of $F(x,\cdot)$, we see that $\{z \in Z_0 : (x,z) \notin A\}$ is convex.

We show that B is closed. Let $(x_k,z_k) \in B$ and $(x_k,z_k) \to (x_0,z_0)$. Since F is l.s.c. at (x_0,z_0), for any $y \in F(x_0,z_0)$, there exists $y_k \in F(x_k,z_k), \forall k \in N$ such that $y_k \to y$. Since $(x_k,y_k) \in B, y_k \leq t$. Then, $y \leq t$ and $(x_0,z_0) \in B$; i.e., B is closed in $X_0 \times Z_0$.

Now, we show that (iii) in Proposition 5.2 holds. For any $z \in Z_0$, let

$$x_1, x_2 \in \{x \in X_0 : (x,z) \in B\} = \{x \in X_0 : \forall y \in F(x,z), y \leq t\}.$$

By the natural quasi $\mathbb{R}_+$-convexity of $F(\cdot,z)$, for any $y_0 \in F(\lambda x_1 + (1-\lambda)x_2, z), \lambda \in [0,1]$, there exists $y^* \in co\{F(x_1,z), F(x_2,z)\}$ such that $y_0 \in y^* - \mathbb{R}_+$. Then $y_0 \leq t$ and $\lambda x_1 + (1-\lambda)x_2 \in \{x \in X : (x,z) \in B\}$, i.e., $\{x \in X_0 : (x,z) \in B\}$ is a convex set. Obviously, for each $z \in Z_0$, $\{x \in X_0 : (x,z) \in B\}$ is nonempty by the assumption about t and the assumption (iii).

Then, by Proposition 5.2, there exists $x_0 \in X_0$ such that $\{x_0\} \times Z_0 \subset A$; i.e., $y \leq t$ for any $y \in F(x_0, z)$ and $z \in Z_0$. Thus, we have

$$\max \cup_{z \in Z_0} F(x_0, z) \leq t.$$

By the definition of t, we know that (5.2) holds. ■

Remark 5.4. In the assumption (iii) and (4.1.1) of Proposition 5.3, max and min exist. As for the righthand side of (iii), one could say that $\cup_{x \in X_0} F(x, z)$ is compact, since X_0 is compact and $F(\cdot, z)$ is compact-valued and u.s.c.. So, there exists min. By Lemma 5.1, $\min \cup_{x \in X_0} F(x, z)$ is u.s.c. in z. Therefore, $\cup_{z \in Z_0} \min \cup_{x \in Z_0} F(x, z)$ is compact and there exists a maximal point.

Remark 5.5. If $F(x, z)$ is a single-valued function, condition (iii) always holds. Proposition 5.3 is a generalization of the minimax theorem for single-valued functions. When $F(x, z)$ is a single-valued function, Proposition 5.3 reduces to Theorem 4 in Ha [94].

In the sequel, we need the following lemma.

Lemma 5.6. *[4] The convex hull of a compact subset of a finite dimensional space is compact.*

Now we present two types of minimax theorems for set-valued functions.

Theorem 5.7. *Let X and Z be two metric spaces. Let X_0 and Z_0 be compact convex subsets of X and Z, respectively. Let $F : X_0 \times Z_0 \rightrightarrows I\!R^\ell$ be a continuous set-valued function with compact values; for each $x \in X_0$, let $F(x, \cdot)$ be naturally quasi C-convex on Z_0 and, for each $z \in Z_0$, let $F(\cdot, z)$ be C-concave on X_0, where C is a closed, convex and pointed cone in $I\!R^\ell$. Suppose that $F(x, z)$ fulfills the following hypotheses:*

(H_1) there exists $t_0 \in Z_0$ such that

$$Max_{intC} \cup_{x \in X_0} F(x, t_0) \subset Max_{intC} \cup_{x \in X_0} F(x, t) - C, \quad \forall t \in Z_0;$$

(H_2) for each $u \in X_0$, there exists $t_u \in Z_0$ such that

$$Max_C \cup_{x \in X_0} Min_{intC} \cup_{z \in Z_0} F(x, z) - F(u, t_u) \subset C.$$

Then,

$$\begin{aligned} &Max_{intC} \cup_{x \in X_0} F(x, t_0) \\ &\subset Max_C \{co(\cup_{x \in X_0} Min_{intC} \cup_{z \in Z_0} F(x, z))\} - C \end{aligned} \tag{5.3}$$

Proof. Set

$$\Gamma(x) = \mathrm{Min}_{intC} \cup_{z \in Z_0} F(x, z).$$

By Lemma 5.1, $\Gamma(x)$ is u.s.c. on X_0 and, for each $x \in X_0$, $\Gamma(x)$ is a nonempty compact set. Since X_0 is compact, by Lemma 5.1 and Lemma 5.6, $\Gamma(X_0)$ and $co(\Gamma(X_0))$ are compact sets. Then $co(\Gamma(X_0)) - C$ is a closed and convex set.

Suppose that $\alpha \in \mathbb{R}^\ell$ and $\alpha \notin co(\Gamma(X_0)) - C$. By the separation theorem for convex sets, there exists a nontrivial linear continuous function $l : \mathbb{R}^\ell \to \mathbb{R}$ and $\delta \in \mathbb{R}, \epsilon > 0$ such that

$$l(\alpha) \geq \delta + \epsilon > \delta \geq l(\beta), \quad \forall \beta \in co(\Gamma(X_0)) - C.$$

Then, for any $y \in co(\Gamma(X_0))$ and $s \in C$, we have

$$l(s) \geq l(y - \alpha).$$

Thus,

$$l(s) \geq 0, \quad \forall s \in C.$$

Since C is a cone, taking $s = 0$, we have

$$l(\alpha) \geq \delta + \epsilon > \delta \geq l(\beta), \quad \forall \beta \in co(\Gamma(X_0)). \tag{5.4}$$

Consider the set-valued function

$$G = l(F) : X_0 \times Z_0 \rightrightarrows \mathbb{R}.$$

Obviously, for each $x \in X_0$, $l(F(x, \cdot))$ is naturally quasi-$\mathbb{R}_+$-convex on Z_0 and, for each $z \in Z_0$, $l(F(\cdot, z))$ is $\mathbb{R}_+$-concave on X under the assumed conditions. By hypothesis (H_2), we have that, for each $u \in X_0$, there exists $t_u \in Z_0$ such that

$$\max l(F(u, t_u)) \leq \max \cup_{x \in X_0} \min \cup_{z \in Z_0} l(F(x, z)).$$

Then, by Proposition 5.3, we have

$$\min \cup_{z \in Z_0} \max \cup_{x \in X_0} G(x, z) = \max \cup_{x \in X_0} \min \cup_{z \in Z_0} G(x, z). \tag{5.5}$$

Since $G(x, \cdot) = l(F(x, \cdot))$ is continuous for each $x \in X_0$ and Z_0 is compact, there exist $z_0 \in Z_0$ and $y_0 \in F(x, z_0)$ such that

$$l(y_0) = \min \cup_{z \in Z_0} l(F(x, z)).$$

Since $l(s) \geq 0, \forall s \in C$, arguing by contradiction, we have that

$$y_0 \in \Gamma(x) = \mathrm{Min}_{intC} \cup_{z \in Z_0} F(x, z).$$

Hence, by (5.4) for each $x \in X_0$,

$$\min \cup_{z \in Z_0} G(x, z) = l(y_0) \leq \delta < \delta + \epsilon \leq l(\alpha).$$

Then,

$$\max \cup_{x\in X_0} \min \cup_{z\in Z_0} G(x,z) \le \delta < \delta + \epsilon \le l(\alpha).$$

By (5.5),

$$\min \cup_{z\in Z_0} \max \cup_{x\in X_0} G(x,z) < l(\alpha).$$

By Lemma 5.1, $\max \cup_{x\in X_0} G(x,\cdot)$ is u.s.c. on Z_0. Thus, by the compactness of Z_0, there exists $z' \in Z_0$ such that

$$\max \cup_{x\in X_0} G(x,z') < l(\alpha);$$

i.e.,

$$l(y) < l(\alpha), \quad \forall y \in F(x,z') \text{ and } x \in X_0.$$

By $l(s) \ge 0, \forall s \in C$, we have

$$\alpha - y \notin -C, \quad \forall y \in F(x,z') \text{ and } x \in X_0;$$

that is,

$$\alpha \notin \mathrm{Max}_{intC} \cup_{x\in X_0} F(x,z') - C. \tag{5.6}$$

Thus if

$$\alpha \in \mathrm{Max}_{intC} \cup_{x\in X_0} F(x,t_0),$$

by hypothesis (H_1), we have that

$$\alpha \in \mathrm{Max}_{intC} \cup_{x\in X_0} F(x,t) - C, \quad \forall t \in Z_0,$$

which contradicts (5.6). Thus

$$\alpha \in \mathrm{Max}_{intC} \cup_{x\in X_0} F(x,t_0)$$

implies

$$\alpha \in co\left(\cup_{x\in X_0} \mathrm{Min}_{intC} \cup_{z\in Z_0} F(x,z)\right) - C.$$

Since

$$co\left(\cup_{x\in X_0} \mathrm{Min}_{intC} \cup_{z\in Z_0} F(x,z)\right)$$

is compact, by the domination property of a compact set, we have

$$\begin{aligned} &co\left(\cup_{x\in X_0} \mathrm{Min}_{intC} \cup_{z\in Z_0} F(x,z)\right) - C \\ &\subset \mathrm{Max}_C \left\{co\left(\cup_{x\in X_0} \mathrm{Min}_{intC} \cup_{z\in Z_0} F(x,z)\right)\right\} - C. \end{aligned}$$

Thus, (5.3) holds. ■

Remark 5.8. (**i**) Hypothesis (H_1) controls the change of $\mathrm{Max}_{intC} \cup_{x\in X_0} F(x,z)$ when z varies. Obviously, this condition holds if F is a scalar real set-valued function;

(**ii**) If F is a single-valued function, then hypothesis (H_2) always holds.

Theorem 5.9. *Let X_0 and Z_0 be compact subsets in X and Z, respectively. Let $F : X_0 \times Z_0 \rightrightarrows I\!R^\ell$ be a continuous set-valued function with compact values; for each $x \in X_0$, let $F(x, \cdot)$ be C-convex on Z_0 and, for each $z \in Z_0$, let $F(\cdot, z)$ be naturally quasi C-concave on X_0. Suppose that $F(x, z)$ fulfills the following hypotheses:*

(H_3) *there exists $x_0 \in X_0$ such that*

$$Min_{intC} \cup_{z \in Z_0} F(x_0, z) \subset Min_{intC} \cup_{z \in Z_0} F(x, z) + C, \quad \forall x \in X_0;$$

(H_4) *for each $t \in Z_0$, there exists $x_t \in X_0$ such that*

$$F(x_t, t) - Min_C \cup_{z \in X_0} Max_{intC} \cup_{x \in X_0} F(x, z) \subset C.$$

Then

$$Min_{intC} \cup_{z \in Z_0} F(x_0, z) \subset Min_C \{co\,(\cup_{z \in Z_0} Max_{intC} \cup_{x \in X_0} F(x, z))\} + C.$$

Proof. Set

$$L(z) = \mathrm{Max}_{intC} \cup_{x \in X_0} F(x, z).$$

Thus, $co(L(Z_0)) + C$ is a closed and convex set. Suppose that $\alpha \in \mathbb{R}^\ell$ and $\alpha \notin co(L(Z_0)) + C$. By a similar method to that used in the proof of Theorem 5.7, there exists a nontrivial continuous linear function $l : \mathbb{R}^\ell \to \mathbb{R}$ such that

$$l(\alpha) \leq \delta < \delta + \epsilon \leq l(\beta), \quad \beta \in co(L(Z_0)),$$

$$l(s) \geq 0, \quad \forall s \in C.$$

Consider the set-valued function

$$G = l(-F) : X_0 \times Z_0 \rightrightarrows \mathbb{R}.$$

Thus, $G = -l(F)$ satisfies the conditions of Proposition 5.3. We have that

$$\min \cup_{x \in X_0} \max \cup_{z \in Z_0} G(x, z) = \max \cup_{z \in Z_0} \min \cup_{x \in X_0} G(x, z). \tag{5.7}$$

Since $l(F(\cdot, z))$ is continuous for each $z \in Z_0$ and X_0 is compact, there exist $x_0 \in X_0$ and $y_0 \in F(x_0, z)$ such that

$$l(y_0) = \max \cup_{x \in X_0} l(F(x, z)).$$

Since $l(x) \geq 0, \forall s \in C$, we have

$$y_0 \in L(z) = \mathrm{Max}_{intC} \cup_{x \in X_0} F(x, z).$$

Thus, by (5.7), we have that

$$\max \cup_{x \in X_0} \min \cup_{z \in Z_0} l(F(x, z)) > l(\alpha),$$

and there exists $x' \in X_0$ such that

$$\alpha \notin \mathrm{Min}_{intC} \cup_{z \in Z_0} F(x', z) + C. \tag{5.8}$$

Thus, if

$$\alpha \in \mathrm{Min}_{intC} \cup_{z \in Z_0} F(x_0, z),$$

by hypothesis (H_3) we have that

$$\alpha \in \mathrm{Min}_C \cup_{z \in Z_0} F(x, z) + C, \quad \forall x \in X_0,$$

which contradicts (5.8). Thus,

$$\mathrm{Min}_{intC} \cup_{z \in Z_0} F(x_0, z) \subset \mathrm{Min}_C \{co\,(\cup_{z \in Z_0} \mathrm{Max}_{intC} \cup_{x \in X_0} F(x, z))\} + C,$$

and this completes the proof. ■

In the sequel, we consider minimax theorems for set-valued functions in a general scheme. Let X, Z and Y be real locally convex spaces. Let $C \subset Y$ be a pointed, closed and convex cone such that $intC \neq \varnothing$, and let Y^* denote the topological dual space of Y.

Lemma 5.10. *$C \subset Y$ is a closed and convex cone if and only if there exists a subset $\Gamma \subset Y^* \backslash \{0\}$ such that*

$$C = \{y \in Y : f(y) \leq 0, \forall f \in \Gamma\}. \tag{5.9}$$

Proof. Assume that C be a closed and convex cone. Let $\Gamma = -C^* \backslash \{0\}$. Using the standard separation theorem for convex cones, it is not hard to verify that (5.9) holds. Conversely, if there exists $\Gamma \subset Y^* \backslash \{0\}$ such that (5.9) holds, then, it is obvious that C is a closed and convex cone. ■

Let us recall the nonlinear scalarization function $\xi_{ea} : Y \rightarrow \mathbb{R}$, which is defined in Chapter 1 by

$$\xi_{ea}(y) = \min\{t \in \mathbb{R} : y \in a + te - C\}, \quad \forall y \in Y,$$

where $e \in intC$ and $a \in Y$.

The function ξ_{ea} is continuous and strictly monotone, and many other important properties can be found in Chapter 1.

Note that Proposition 5.3 can be established, when condition (i) in Proposition 5.3 is replaced by the assumption that $F(x, \cdot)$ is properly quasi $\mathbb{R}_+$-convex on Z_0.

Lemma 5.11. *Let $F : X_0 \times Z_0 \rightrightarrows Y$ be a set-valued function, and let, for each $x \in X_0$, $F(x, \cdot)$ be naturally quasi C-convex on Z_0. Suppose that, for each $z \in Z_0$, $-F(\cdot, z)$ is properly quasi C-convex on X_0. Then $\xi_{ea}(F(x, \cdot))$ is naturally quasi $\mathbb{R}_+$-convex on Z_0 and $-\xi_{ea}(F(\cdot, z))$ is properly quasi $\mathbb{R}_+$-convex X_0.*

Proof. Take any $z_1, z_2 \in Z_0$, $\lambda \in (0,1)$ and $y \in F(x, \lambda z_1 + (1-\lambda)z_2)$. By naturally quasi C-convexity of $F(x,\cdot)$, there exists $y_i \in F(x,z_1) \cup F(x,z_2)$ and $\alpha_i \geq 0, i = 1, 2, \cdots, n$, and $c \in C$ such that $\sum_{i=1}^{n} \alpha_i = 1$ and $y = \sum_{i=1}^{n} \alpha_i y_i - c$. Therefore,

$$\xi_{ea}(y) = \xi_{ea}(\sum_{i=1}^{n} \alpha_i y_i - c).$$

By the properties of ξ_{ea},

$$\xi_{ea}(y) \in \sum_{i=1}^{n} \alpha_i \xi_{ea}(y_i) - \mathbb{R}_+ \subset co\{\xi_{ea}(F(x,z_1)) \cup \xi_{ea}(F(x,z_2))\} - \mathbb{R}_+.$$

Thus, for each $x \in X_0$, $\xi_{ea}(F(x,\cdot))$ is naturally quasi $\mathbb{R}_+$-convex. By the monotonicity of ξ_{ea} and properly quasi C-convexity of $-F(\cdot,z)$, it is clear that $-\xi_{ea}(F(\cdot,z))$ is properly quasi $\mathbb{R}_+$-convex for any $z \in Z$. The proof is complete. ∎

Theorem 5.12. *Let X_0 and Z_0 be compact and convex subsets in X and Z, respectively, and let $e \in intC$. Suppose that the following conditions are satisfied:*

(**i**) *$F : X_0 \times Z_0 \rightrightarrows Y$ is a continuous set-valued function with compact-values;*
(**ii**) *for each $x \in X_0, -F(x,\cdot)$ is properly quasi C-convex on Z_0;*
(**iii**) *for each $z \in Z_0$, $F(\cdot,z)$ is naturally quasi C-convex on X_0;*
(**iv**) *for any $u \in X_0$, there exists $v \in Z_0$ such that*

$$F(u,v) \subset Max_C \cup_{x\in X_0} Min_{intC} \cup_{z\in Z_0} F(x,z) - C.$$

Then

$$\begin{aligned} &Min_C \cup_{x\in X_0} Max_{intC} \cup_{z\in Z_0} F(x,y) \\ &\subset Max_C \cup_{z\in Z_0} Min_{intC} \cup_{x\in X_0} F(x,y) + Y\backslash(C\backslash\{0\}). \end{aligned} \tag{5.10}$$

Proof. Set

$$L(x) = \mathrm{Max}_{intC} \cup_{z\in Z_0} F(x,z),$$

and let

$$y_0 \in \mathrm{Min}_C \cup_{x\in X_0} \mathrm{Max}_{intC} \cup_{z\in Z_0} F(x,z) = \mathrm{Min}_C L(X_0).$$

By the definition of minimal points, we have

$$(L(X_0) - y_0) \cap (-C) = \{0\},$$

that is

$$(L(X_0)\backslash\{y_0\}) \cap (y_0 - C) = \varnothing.$$

By Proposition 1.54, we have

$$\xi_{ey_0}(y) > 0, \quad \forall y \in L(X_0)\backslash\{y_0\}, \tag{5.11}$$

and

$$\xi_{ey_0}(y_0) = 0. \tag{5.12}$$

Let $x_0 \in X_0$. By the continuity of ξ_{ey_0} and the compactness of X_0, there exist $z_x \in Z_0$ and $y_1 \in F(x, z_x)$ such that

$$\max \cup_{z\in Z_0} \xi_{ey_0}(F(x,\cdot)) = \xi_{ey_0}(y_1).$$

By the properties of ξ_{ey_0}, we have

$$y_1 \in L(x).$$

From (5.11) and (5.12),

$$\max \cup_{z\in Z_0} \xi_{ey_0} F(x,\cdot)) \geq 0. \tag{5.13}$$

Since x is any element of X_0, (5.13) implies that

$$\min \cup_{x\in X_0} \max \cup_{z\in Z_0} \xi_{ey_0}(F(x,\cdot)) \geq 0. \tag{5.14}$$

Consider the set-valued function G:

$$G = \xi_{ey_0}(F) : X_0 \times Z_0 \rightrightarrows \mathbb{R}.$$

From Lemma 5.11 and (5.11), Proposition 5.3 holds for G. We have

$$\min \cup_{x\in X_0} \max \cup_{z\in Z_0} G(x,z) = \max \cup_{z\in Z_0} \min \cup_{x\in X_0} G(x,z).$$

So, there exist $x_0 \in X_0$, $z_0 \in Z_0$ and $y_2 \in F(x_0, z_0)$ such that

$$\begin{aligned}
&\min \cup_{x\in X_0} \max \cup_{z\in Z_0} G(x,z) \\
&= \max \cup_{z\in Z_0} G(x_0,z) \\
&= \max \cup_{z\in Z_0} \min \cup_{x\in X_0} G(x,z) \\
&= \min \cup_{x\in X_0} G(x,z_0) \\
&= \xi_{ey_0}(y_0).
\end{aligned} \tag{5.15}$$

Therefore, by the strict monotonicity of ξ_{ey_0}, we have

$$y_2 \in \mathrm{Max}_{intC} \cup_{z\in Z_0} F(x_0,z) = L(x_0), \tag{5.16}$$

and

$$y_2 \in \mathrm{Min}_{intC} \cup_{x\in X_0} F(x,z_0).$$

From (5.14) and (5.15), we get $\xi_{ey_0}(y_2) \geq 0$.

Suppose $y_0 = y_2$. Then

$$y_0 \notin y_2 + C\backslash\{0\}. \tag{5.17}$$

If $y_0 \neq y_2$, by (5.11) and (5.16), we get $\xi_{ey_0}(y_2) > 0$, from Proposition 1.54, we have $y_2 \in y_0 - C$, that is

$$y_0 \notin y_2 + C\backslash\{0\}. \tag{5.18}$$

¿From (5.17) and (5.18), we get

$$\begin{aligned} y_0 &\in y_2 + Y\backslash(C\backslash\{0\}) \\ &\subset \text{Min}_{intC} \cup_{x\in X_0} F(x, z_0) + Y\backslash(C\backslash\{0\}) \\ &\subset \cup_{z\in Z_0}\text{Min}_{intC} \cup_{x\in X_0} F(x, z) + Y\backslash(C\backslash\{0\}). \end{aligned}$$

Since $F(\cdot,\cdot)$ is continuous and X_0 and Z_0 are compact, by the domination property, we have

$$\cup_{z\in Z_0}\text{Min}_{intC} \cup_{x\in X_0} F(x, z) \subset \text{Max}_C \cup_{z\in Z_0} \text{Min}_{intC} \cup_{x\in X_0} F(x, z) - C.$$

Thus,

$$\begin{aligned} y_0 &\in \text{Max}_C \cup_{z\in Z_0} \text{Min}_{intC} \cup_{x\in X_0} F(x, z) - C + Y\backslash(C\backslash\{0\}) \\ &= \text{Max}_C \cup_{z\in Z_0} \text{Min}_{intC} \cup_{x\in X_0} F(x, z) + Y\backslash(C\backslash\{0\}). \end{aligned}$$

Hence, the inclusion (5.10) holds. This completes the proof. ■

5.2 Minimax Inequalities for Vector-Valued Functions

In this section we establish several types of minimax theorems for vector-valued functions.

First, we assume that X and Z are metric spaces.

Lemma 5.13. *Let X_0 and Z_0 be nonempty compact convex subsets in X and Z, respectively. Let $f : X_0 \times Z_0 \to I\!R^\ell$ be a continuous vector-valued function. Then, $\pi(z) = co(Max_{intC} f(X_0, z))$ and $\varphi(x) = co(Min_{intC} f(x, Z_0))$ are u.s.c. on Z_0 and X_0, respectively.*

Proof. By the continuity of f, the compactness of X_0 and Lemma 5.6, $\pi(z)$ is compact-valued on Z_0. Suppose that $\pi(z)$ is not u.s.c. at $z_0 \in Z_0$. Then $\exists\epsilon > 0$ and $\forall 1/n$, $n = 1, 2, \cdots$, there exist $z_k \in B(z_0, 1/k)$ and $\pi(z_k)$ such that

$$y_k \notin B(\pi(z_0), \epsilon). \tag{5.19}$$

Since $f(X_0, Z_0)$ is compact, $co(f(X_0, Z_0))$ is compact. Obviously,

$$y_k \in co(F(X_0, Z_0)).$$

Then, we can assume that $y_k \to y_0$. By (5.19), we have

$$y_0 \notin \pi(z_0) \tag{5.20}$$

Since $y_k \in \pi(z_k)$, there exist $y_k^i \in \mathrm{Max}_{intC} f(X_0, z_k)$ and $\lambda_k^i \geq 0$, $i = 0, 1, \cdots, \ell$, such that

$$\sum_{i=0}^{\ell} \lambda_k^i = 1 \text{ and } y_k = \sum_{i=0}^{\ell} \lambda_k^i y_k^i.$$

By Lemma 5.1, $\mathrm{Max}_{intC} f(X_0, z)$ is u.s.c. on Z_0. Since

$$\{y_k^i\} \subset \cup_{z \in Z_0} \mathrm{Max}_C f(X_0, z) \text{ and } \lambda_k^i \in [0, 1],\ i = 0, 1, 2, \cdots, \ell,$$

without loss of generality, we can assume that $\lambda_k^i \to \lambda_0^i$, $n \to \infty$, and $y_k^i \to y_0^i$, $n \to \infty$, $i = 0, 1, 2, \cdots, \ell$. Obviously,

$$\sum_{i=0}^{\ell} \lambda_0^i = 1 \text{ and } y_0 = \sum_{i=0}^{\ell} \lambda_0^i y_0^i.$$

By Lemma 5.1 $y_0^i \in \mathrm{Max}_{intC} f(X_0, z_0)$. Therefore, $y_0 \in \pi(z_0)$, which contradicts (5.20), and thus $\pi(z)$ is u.s.c. on Z_0.

By a similar method, we can prove that $\varphi(x)$ is u.s.c. on X_0. ∎

Theorem 5.14. *Let X_0 and Z_0 be nonempty compact convex sets in X and Z, respectively. Let $f : X_0 \times Z_0 \to I\!R^\ell$ be a vector-valued function such that*

- **(i)** $B_x = \{z \in Z_0 : f(x, z) \notin co(Max_{intC} f(X_0, z)) + C\}$ *is convex or empty for all* $x \in X_0$*;*
- **(ii)** $f(\cdot, \cdot)$ *is continuous on* $X_0 \times Z_0$*;*
- **(iii)** *for each* $z \in Z_0$, $f(\cdot, z)$ *is* C*-concave.*

Then, there exists $x_0 \in X_0$ such that

$$Min_{intC} f(x_0, Z_0) \subset Min_C \cup_{z \in Z_0} co(Max_{intC} f(Z_0, z)) + C.$$

Moreover, if

$$Max_C \cup_{u \in X_0} Min_{intC} f(u, Z_0) \subset Min_{intC} f(x, Z) + C, \quad \forall x \in X_0,$$

then we have that,

$$Max_C \cup_{x \in X_0} Min_{intC} f(x, Z_0) \subset Min_C \cup_{z \in Z_0} co(Max_{intC} f(X_0, z)) + C. \tag{5.21}$$

Proof. Suppose that

$$M(z) = \mathrm{Max}_{intC} f(X_0, z)$$

and

$$A = B = \{(x, z) \in X_0 \times Z_0 : f(x, z) \in co(M(z)) + C\}.$$

Now, we show that A and B satisfy the conditions of Proposition 5.2 with $K = X_0$. In fact, for any $z \in Z_0$, let

$$x_k \in \{x \in X_0 : (x, z) \in A\} \text{ and } x_k \to x_0.$$

Then

$$f(x_k, z) \in co(M(z)) + C.$$

By the compactness of X_0 and the continuity of $f(\cdot, z)$, the set $co(M(z))$ is compact. Then

$$f(x_0, z) \in co(M(z)) + C,$$

i.e.,

$$x_0 \in \{x \in X_0 : (x, z) \in A\},$$

and hence, $\{x \in X_0 : (x, z) \in A\}$ is closed. By condition (i), we have that, for any $x \in X_0$,

$$\{z \in Z_0 : (x, z) \notin A\} = \{z \in Z_0 : f(x, z) \notin co(\mathrm{Max}_{intC} f(X_0, z)) + C\}$$

is convex or empty.

We prove that B is a closed subset. Now let $(x_k, z_k) \in B$ and $(x_k, z_k) \to (x_0, z_0)$. Then, $f(x_k, z_k) \to f(x_0, z_0)$ since f is continuous on $X_0 \times Z_0$. Since $(x_k, z_k) \in B$,

$$f(x_k, z_k) \in co(M(z_k)) + C.$$

Then, there exist $y_k \in co(M(z_k))$ and $c_k \in C$ such that

$$f(x_k, z_k) = y_k + c_k.$$

By the compactness of Z_0 and Lemma 5.13, we have that $co(M(Z_0))$ is compact. Therefore, we can assume that $y_k \to y_0$ and $y_0 \in co(M(z_0)))$. So

$$c_k = f(x_k, z_k) - y_k \to f(x_0, z_0) - y_0 \in C.$$

Set $c_0 = f(x_0, z_0) - y_0$. Then

$$f(x_0, z_0) \in co(M(z_0)) + C$$

and $(x_0, z_0) \in B$, i.e., B is closed on $X_0 \times Z_0$.

Finally, we show that, for any $z \in Z_0$, $\{x \in X_0 : (x, z) \in B\}$ is convex and nonempty. Indeed, for any $z \in Z_0$, by the domination property of the compact set $f(X_0, z)$, there exists $x_0 \in X_0$ such that

$$f(x_0, z) \in \mathrm{Max}_{intC} f(X_0, z).$$

Thus,

$$f(x_0, z) \in co(M(z)) + C,$$

i.e.,

$$\{x \in X_0 : f(x, z) \in co(M(z)) + C\} \neq \varnothing.$$

Let $x_1, x_2 \in \{x \in X_0 : (x, z) \in B\}$. Then

$$f(x_1, z), f(x_2, z) \in co(M(z)) + C. \tag{5.22}$$

Since $f(\cdot, z)$ is C-concave, for $\lambda \in (0, 1)$,

$$\lambda f(x_1, z) + (1 - \lambda) f(x_2, z) \in f(\lambda x_1 + (1 - \lambda)x_2, z) - C.$$

Therefore, by (5.22), we have that

$$f(\lambda x_1 + (1 - \lambda)x_2, z) \in co(M(z)) + C,$$

$$\lambda x_1 + (1 - \lambda)x_2 \in \{x \in X_0 : (x, z) \in B\}.$$

Thus, by Proposition 5.2, there exists $x_0 \in X_0$ such that $\{x_0\} \times Z_0 \subset A$, that is

$$f(x_0, z) \in co(\mathrm{Max}_{intC} f(X_0, z)) + C, \quad \forall z \in Z_0.$$

Thus, by the domination property, Lemma 5.13 and the compactness of Z_0, we have that

$$f(x_0, z) \in \mathrm{Min}_C \cup_{t \in Z_0} co(\mathrm{Max}_{intC} f(X_0, t)) + C, \quad \forall z \in Z_0,$$

$$\mathrm{Min}_{intC} f(x, Z_0) \subset \mathrm{Min}_C \cup_{t \in Z_0} co(\mathrm{Max}_{intC} f(X_0, t)) + C.$$

Moreover, if

$$\mathrm{Max}_C \cup_{u \in X_0} \mathrm{Min}_{intC} f(u, Z_0) \subset \mathrm{Min}_{intC} f(x, Z_0) + C, \quad x \in X_0,$$

then

$$\mathrm{Max}_C \cup_{x \in X_0} \mathrm{Min}_{intC} f(x, Z_0) \subset \mathrm{Min}_C \cup_{t \in Z_0} co(\mathrm{Max}_{intC} f(X_0, t)) + C,$$

and this completes the proof of this theorem. ■

Theorem 5.15. *Let X_0 and Z_0 be nonempty compact convex sets in X and Z, respectively. Let $f : X_0 \times Z_0 \longrightarrow I\!R^\ell$ be a vector-valued function such that*

(i) $B_z = \{x \in X_0 : f(x, z) \notin co(Min_{intC} f(x, Z_0)) - C\}$ *is convex or empty for all* $z \in Z_0$*;*
(ii) $f(\cdot, \cdot)$ *is continuous on* $X_0 \times Z_0$*;*
(iii) *for each* $x \in X_0$, $f(x, \cdot)$ *is* C*-convex.*

Then, there exists $z_0 \in Z_0$ *such that*

$$Max_{intC} f(X_0, z_0) \subset Max_C \cup_{x \in X_0} co(Min_{intC} f(x, Z_0)) - C.$$

Moreover, if

$$Min_C \cup_{t \in Z_0} Max_{intC} f(X_0, t) \subset Max_{intC} f(X_0, z) - C, \quad \forall z \in Z_0,$$

then, we have that

$$Min_C \cup_{t \in Z_0} Max_{intC} f(X_0, t) \subset Max_C \cup_{x \in X_0} co(Min_{intC} f(x, Z_0)) - C.$$

Proof. Suppose that

$$Q(x) = \mathrm{Min}_{intC} f(x, Z_0)$$

and

$$A = B = \{(x, z) \in X_0 \times Z_0 : f(x, z) \in co(Q(x)) - C\}.$$

By a method similar to that used in the proof of Theorem 5.14, we can prove that A and B satisfy the conditions of Proposition 5.2 with $K = X_0$. Thus, by Proposition 5.2, there exists $z_0 \in Z_0$ such that $X_0 \times \{z_0\} \subset A$, that is,

$$f(x, z_0) \in co(\mathrm{Min}_{intC} f(x, Z_0)) - C, \quad \forall x \in X_0.$$

By Lemma 5.13 and the compactness of X_0, we have that

$$f(x, z_0) \in \mathrm{Max}_C \cup_{u \in X_0} co(\mathrm{Min}_{intC} f(u, Z_0)) - C, \quad \forall x \in X_0,$$

$$\mathrm{Max}_{intC} f(X_0, z_0) \subset \mathrm{Max}_C \cup_{u \in X_0} co(\mathrm{Min}_{intC} f(u, Z_0)) - C.$$

Moreover, if

$$\mathrm{Min}_C \cup_{t \in Z_0} \mathrm{Max}_{intC} f(X_0, t) \subset \mathrm{Max}_{intC} f(X_0, z) - C, \quad \forall z \in Z_0,$$

then we have that

$$\mathrm{Min}_C \cup_{t \in Z_0} \mathrm{Max}_{intC} f(X_0, t) \subset \mathrm{Max}_C \cup_{x \in X_0} co(\mathrm{Min}_{intC} f(x, Z_0)) - C,$$

and this completes the proof of this theorem. ■

Example 5.16. Let $X_0 = [0, 1], Z_0 = [0, 1], \ell = 2, C = \mathbb{R}^2_+$. Let $f : X_0 \times Z_0 \to \mathbb{R}^\ell$, $f(x, z) = (xz, xz)^\top$, for every $x, z \in [0, 1]$. We have that

(i)

$$B_x = \{z \in Z_0 : f(x, z) \notin co(\mathrm{Max}_{intC} f(X_0, z)) + C\} = \begin{cases} (0, 1], & x \neq 1, \\ \varnothing, & x = 1, \end{cases}$$

is convex or empty for any $x \in [0, 1]$;

(ii) $f(\cdot, \cdot)$ is continuous on $X_0 \times Z_0$;

(iii) for each $z \in Z_0$, $f(\cdot, z)$ is C-convex;

(iv) $\mathrm{Max}_C \cup_{x \in X_0} \mathrm{Min}_{intC} f(x, Z_0) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$ and $\mathrm{Min}_{intC} f(x, Z_0) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$ for all $x \in X_0$.

Then

$$\mathrm{Max} \cup_{u \in X_0} \mathrm{Min}_{intC} f(u, Z_0) \subset \mathrm{Min}_{intC} f(x, Z_0) + C, \quad \forall x \in X_0.$$

From the result of Theorem 5.14, we have that

$$\mathrm{Max}_C \cup_{x \in X_0} \mathrm{Min}_{intC} f(x, Z_0) \subset \mathrm{Min}_C \cup_{t \in Z_0} co(\mathrm{Max}_{intC} f(X_0, t)) + C.$$

In fact, we can verify that

$$\mathrm{Max}_C \cup_{x\in X_0} \mathrm{Min}_{intC} f(x, Z_0) = \{(0,0)^\top\},$$

$$\mathrm{Min}_C \cup_{t\in Z_0} co(\mathrm{Max}_{intC} f(X_0, t)) = \{(0,0)^\top\}.$$

Thus,

$$\{(0,0)^\top\} \subset \{(0,0)^\top\} + C.$$

In what follows, we assume that X, Z and Y are real Hausdorff topological vector spaces. Let $C \subset Y$ be a closed, convex and pointed cone such that $intC \neq \varnothing$. We will establish a minimax theorem for vector-valued functions as a special case of Theorem 5.12.

Theorem 5.17. *Let X_0 and Z_0 be compact and convex subsets in X and Z, respectively, and let $e \in intC$. Suppose that the following conditions are satisfied:*

(i) *$f : X_0 \times Z_0 \to Y$ is a continuous vector-valued function;*
(ii) *for each $x \in X_0$, $-f(x, \cdot)$ is properly quasi C-convex on Z_0;*
(iii) *for each $z \in Z_0$, $f(\cdot, z)$ is naturally quasi C-convex on X_0.*

Then

$$\begin{aligned}&Min_C \cup_{x\in X_0} Max_{intC} \cup_{z\in Z_0} f(x,z)\\ &\subset Max_C \cup_{z\in Z_0} Min_{intC} \cup_{x\in X_0} f(x,z) + Y\backslash(C\backslash\{0\}).\end{aligned}$$

Proof. Since f is a vector-valued function, (iv) of Theorem 5.12 holds. Then, the conclusion follows readily from Theorem 5.12. ∎

6

Vector Network Equilibrium Problems

The earliest network equilibrium model was proposed by Wardrop [197] for a transportation network. Since then, many other equilibrium models have also been proposed in the economics literature (see Nagurney [149]). Until only recently, all these equilibrium models are based on single cost or utility function. Recently, equilibrium models based on multicriteria consideration or vector-valued cost functions have been proposed. In Chen and Yen [44], a multicriteria traffic equilibrium model was proposed and the relationship between this model and the vector variational inequality problem was considered under a singleton assumption. Other papers that consider multicriteria equilibrium models can be found in Brenninger-Göthe et al [21], Chen, Goh and Yang [30], Dial [56], Goh and Yang [85], Leurent [131], and Yang and Goh [214]. In particular, the multicriteria network equilibrium model was formulated as a vector variational inequality problem in Goh and Yang [85] via a vector optimization approach, but without the singleton assumption.

In this chapter, we consider weak vector network equilibrium, vector network equilibrium and dynamic vector equilibrium problems. We establish their relations with vector variational inequalities and vector optimization problems.

6.1 Weak Vector Equilibrium Problem

Consider a transportation network $G = (\mathcal{N}, \mathcal{A})$ where $\mathcal{N}$ denotes the set of nodes and $\mathcal{A}$ denotes the set of arcs. Let $\mathcal{I}$ be the set of origin-destination (O-D) pair and P_i, $i \in \mathcal{I}$ be the set of paths joining O-D pair i. For a given path $k \in P_i$, let h_k denote the traffic flow on this path and $h = (h_1, h_2, \cdots, h_M) \in \mathbb{R}^M$, where $M = \sum_{i \in \mathcal{I}} |P_i|$. The path flow vector h induces a flow v_a on each arc $a \in \mathcal{A}$ given by

$$v_a = \sum_{i \in \mathcal{I}} \sum_{k \in P_i} \delta_{ak} h_k,$$

where $\Delta = [\delta_{ak}] \in \mathbb{R}^{|\mathcal{A}| \times M}$ is the arc path incidence matrix with $\delta_{ak} = 1$ if the arc belongs to path k and 0 otherwise. Let $v = [v_a : a \in \mathcal{A}] \in \mathbb{R}^{|\mathcal{A}|}$ be the vector of arc flow. Succinctly

$$v = \Delta h. \tag{6.1}$$

We will assume that the demand of traffic flow is fixed for each O-D pair, i.e., $\sum_{k \in P_i} h_k = d_i$, where d_i is a given demand of each O-D pair i. A flow $h \geq 0$ satisfying the demand is called a feasible flow. Let $\mathcal{H} = \{h : h \geq 0, \sum_{k \in P_i} h_k = d_i, \forall i \in \mathcal{I}\}$ be the set of feasible flows. $\mathcal{H}$ is clearly a closed and convex set. Let $t_a : \mathbb{R}^{|\mathcal{A}|} \to R^\ell$ be a vector-valued cost function for the arc a and it is in general a function of all the arc flows, and let metric $t(v) = [t_a(v) : a \in \mathcal{A}] \in \mathbb{R}^{\ell \times |\mathcal{A}|}$. The vector-valued cost function along the path k, we denote τ_k, $\tau_k : \mathbb{R}^M \to \mathbb{R}^\ell$ is assumed to be the sum of all the arc cost along this path, thus

$$\tau_k(h) = \sum_{a \in \mathcal{A}} \delta_{ak} t_a(v).$$

Let $T(h) = [\tau_k(h) : k \in P_i, i \in \mathcal{I}] \in \mathbb{R}^{\ell \times \mathcal{M}}$. Succinctly

$$T(h) = t(v)\Delta. \tag{6.2}$$

In this section, we consider an equilibrium problem defined on transportation network with vector-valued cost functions. In this model, the cost space is ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, with the ordering cone C, a pointed, closed and convex cone with nonempty interior $intC$.

Definition 6.1. *Given a flow h, we say that a path $p \in P_i$ for an O-D pair i is a weakly minimal one if there does not exist another path $p' \in P_i$ such that $\tau_{p'}(h) - \tau_p(h) \leq_{intC} 0$.*

Let $\Gamma_i(h) = \{\tau_p(h) : p \in P_i\}$ denote the (discrete) set of vector costs for all paths for O-D pair i, and

$$\mathcal{I}_i(h) = \{k \in P_i \mid \tau_k(h) - \tau_p(h) \not\geq_{intC} 0, \ \forall p \in P_i\} \subseteq P_i$$

denote the set of all weakly minimal paths for O-D pair i.

We define the weakly minimal frontier for O-D pair i to be the set of weakly minimal points in the cost-space of O-D pair i:

$$\text{Min}_{intC}(\Gamma_i(h)) = \{\xi \in \mathbb{R}^\ell \mid \xi = \tau_p(h) \ \text{ where } p \in \mathcal{I}_i(h)\}.$$

Note that $\text{Min}_{intC}(\Gamma_i(h))$ is a discrete set because it is a subset of the discrete set $\mathcal{I}_i(h)$.

The following weak vector equilibrium principle is a generalization of the well-known Wardrop's equilibrium principle (see Wardrop [197]):

Definition 6.2. *A flow $h \in H$ is said to be in weak vector equilibrium if*

$$\forall i \in \mathcal{I}, \forall k, l \in P_i, \quad \tau_k(h) \geq_{intC} \tau_l(h) \Longrightarrow h_k = 0. \tag{6.3}$$

A flow h in weak vector equilibrium is often referred to as a weak vector equilibrium flow.

In terms of the weakly minimal frontier for O-D pair i, the weak vector equilibrium principle can be stated in an equivalent form:

Definition 6.3. *(Equivalent weak vector equilibrium principle) The path flow vector h is in weak vector equilibrium if*

$$\forall i \in \mathcal{I},\ \forall p \in P_i,\ h_p = 0 \ \ \textit{whenever} \ \ \tau_p(h) \notin Min_{intC}(\Gamma_i(h)). \tag{6.4}$$

These definitions are natural generalizations of the Wardrop equilibrium principle for a scalar valued cost, in which case, a strict inequality $>$ is used in (6.3). The motivation for both the scalar and the vector cost cases is provided by the fact that an user will not choose to travel on a path if it is cheaper (both in the scalar and the vector sense) to travel on another path that links the same origin and destination.

We shall investigate weak vector equilibrium flows by virtue of linear scalarization function and nonlinear scalarization function, respectively.

Linear Scalarization Approach

Let us first introduce the concept of a parametric equilibrium flow.

Definition 6.4. *(Weak parametric equilibrium principle) Let a parameter $\lambda \in C^*$ be given. A path flow vector h is in weak λ-equilibrium if*

$$\begin{aligned}&\forall i \in \mathcal{I},\ \forall p \in P_i,\ h_p = 0 \ \ \textit{whenever} \ \ \exists\ e_i \in Min_{intC}(\Gamma_i(h)),\\ &\textit{such that } \lambda^\top \tau_p(h) > \lambda^\top e_i.\end{aligned}$$

Note that a parametric equilibrium flow is based on a scalar cost, as in the case of Wardrop's equilibria. In the case of scalarization for vector optimization, it is known that certain convexity assumption is necessary before the scalar optimal solution is necessarily a weakly minimal solution for the vector problem. In the present context, however, the set of concern $\Gamma_i(h)$ is discrete and hence convexity has no meaning. To get around this, we make the following assumption.

Assumption 6.5.

$$Min_{intC}(\Gamma_i(h)) \subseteq Min_{intC}(co(\Gamma_i(h))),$$

where $co(\Gamma_i(h))$ is the convex hull of the discrete set $\Gamma_i(h)$.

The following result establishes relationships between a weak vector equilibrium flow and a parametric equilibrium flow.

We need the following scalarization result, which is just Theorem 3.4.2 of [176].

Lemma 6.6. *Let $A \subset I\!R^\ell$ be a nonempty and convex set and $a^* \in Min_{intC}A$. Then, there exists $\lambda \in C^*\backslash\{0\}$ such that*

$$\lambda^\top a^* = \min_{a \in A} \lambda^\top a.$$

Theorem 6.7. *(i) If h is in weak vector equilibrium and Assumption 6.5 holds, then there exists $\lambda \in C^* \setminus \{0\}$ such that the path flow h is in weak λ-equilibrium;*

(ii) If h is in weak λ-equilibrium for some $\lambda \in C^*\backslash\{0\}$, then h is in weak vector equilibrium.

Proof. (i) Let h be in weak vector equilibrium. Then, for $k \in P_i$,

$$\begin{aligned} h_k > 0 &\Rightarrow \tau_k(h) \in \mathrm{Min}_{intC}(\Gamma_i(h)) \\ &\Rightarrow \tau_k(h) \in \mathrm{Min}_{intC}(\mathrm{co}(\Gamma_i(h)) \quad \text{by Assumption 6.5,} \\ &\Rightarrow \exists \lambda \in C^*\backslash\{0\} \ \text{ s.t. } \lambda^\top \tau_k(h) = \min_{\eta \in \mathrm{co}(\Gamma_i(h))} \lambda^\top \eta \\ &\qquad \text{by Lemma 6.6,} \\ &\Rightarrow \exists \lambda \in C^*\backslash\{0\} \ \text{ s.t. } \lambda^\top \tau_k(h) = \min_{\eta \in \Gamma_i(h)} \lambda^\top \eta \\ &\qquad \text{since } \Gamma_i(h) \subset \mathrm{co}(\Gamma_i(h)) \text{ and } \tau_k(h) \in \Gamma_i(h). \end{aligned}$$

Hence h is in weak λ-equilibrium.

(ii) Let $\lambda \in C^*\backslash\{0\}$ and let h be in weak λ-equilibrium. Suppose that h is not in weak vector equilibrium, then by Definition 6.3, there exists $i \in \mathcal{I}, p \in P_i$ such that,

$$h_p > 0 \quad \text{and} \quad \tau_p(h) \notin \mathrm{Min}_{intC}(\Gamma_i(h)).$$

Thus

$$h_p > 0 \quad \text{and } \lambda^\top \tau_p(h) > \lambda^\top e_i, \quad \text{for some } \ e_i \in \mathrm{Min}_{intC}(\Gamma_i(h)).$$

Hence h is not in weak λ-equilibrium, a contradiction. ■

For $\lambda \in C^*$, we define the minimum scalarized cost for O-D pair i as:

$$u_i(\lambda) = \min_{p \in P_i} \lambda^\top \tau_p(h). \tag{6.5}$$

Lemma 6.8. *If $\lambda \in C^*\backslash\{0\}$, then $u_i(\lambda) = \lambda^\top e_i$ for some $e_i \in Min_{intC}(\Gamma_i(h))$.*

Proof. From (6.5), let $p \in P_i$ be such that $u_i(\lambda) = \lambda^\top \tau_p(h)$. Choose $e_i := \tau_p(h)$. Suppose now that $e_i \notin \mathrm{Min}_{intC}(\Gamma_i(h))$, then there exists $\bar{p} \in P_i$, such that $\tau_p(h) \geq_{intC} \tau_{\bar{p}}(h)$. Since $\lambda \in C^*\backslash\{0\}$, $\lambda^\top \tau_p(h) > \lambda^\top \tau_{\bar{p}}(h)$, a contradiction. Therefore $e_i \in \mathrm{Min}_{intC}(\Gamma_i(h))$. ■

Theorem 6.9. *(i) Let $\lambda \in C^*$. Then h is in weak λ-equilibrium if the following condition holds:*

$$\forall i \in \mathcal{I}, \forall p \in P_i, \; h_p = 0 \text{ whenever } \lambda^\top \tau_p(h) > u_i(\lambda); \tag{6.6}$$

(ii) If $\lambda \in C^*\backslash\{0\}$ and h is in weak λ-equilibrium, then condition (6.6) holds.

Proof. (i) If there exists $e_i \in \mathrm{Min}_{intC}(\Gamma_i(h))$ such that $\lambda^\top \tau_p(h) > \lambda^\top e_i$, say $e_i = \tau_q(h)$ for some $q \in P_i$. Then $\lambda^\top \tau_p(h) > \lambda^\top \tau_q(h),\ q \in P_i$. Thus clearly

$$\lambda^\top \tau_p(h) > u_i(\lambda) = \min_{p \in P_i} \lambda^\top \tau_p(h),$$

by (6.6), $h_p = 0$, so h is in weak λ-equilibrium.

(ii) Let h be a weak λ-equilibrium flow and $\lambda \in C^*\backslash\{0\}$. If $\lambda^\top \tau_p(h) > u_i(\lambda)$, by Lemma 6.8, there exists $e_i \in \mathrm{Min}_{intC}(\Gamma_i(h))$ such that $u_i(\lambda) = \lambda^\top e_i$. Thus

$$\lambda^\top \tau_p(h) > \lambda^\top e_i, \quad \text{where} \quad e_i \in \mathrm{Min}_{intC}(\Gamma_i(h)).$$

By Definition 6.4, $h_p = 0$ and hence (6.6) holds. ∎

Next, we discuss relations between a weak vector equilibrium flow and a solution of a vector optimization problem.

Definition 6.10. *We say that*

(i) the vector cost function t_a is separable if t_a is a function of v_a only, i.e.,

$$t_a(v) = t_a(v_a), \; \forall a \in \mathcal{A}.$$

(ii) the vector cost function t_a is integrable if

$$\partial t_a^k / \partial v_{a'} = \partial t_{a'}^k / \partial v_a, \; \forall a, \; a' \in \mathcal{A}, \; \forall k = 1, \cdots, \ell.$$

Clearly, a separable cost is also integrable.

If the cost t_a is integrable, then for $\lambda \in C^*$

$$\partial(\lambda^\top t_a)/\partial v_{a'} = \partial(\lambda^\top t_{a'})/\partial v_a, \qquad \forall a, a' \in \mathcal{A},$$

and, by Theorem 4.1.6 of [157], there exists a real-valued potential function, denoted by $\oint^v \lambda^\top t(z) dz$ such that

$$\frac{\partial}{\partial v} \oint^v \lambda^\top t(w) dw = \lambda^\top t(v). \tag{6.7}$$

For $\lambda \in C^*$, consider the following (scalar) optimization problem P(λ):

$$\min \oint^v \lambda^\top t(w) dw \tag{6.8}$$

$$\text{subject to} \quad \sum_{p \in P_i} h_p = d_i, \; \forall i \in \mathcal{I} \tag{6.9}$$

$$h_p \geq 0, \quad \forall p \in P_i, \; \forall i \in \mathcal{I}, \tag{6.10}$$

and the definitional constraint:

$$v_a = \sum_{i \in \mathcal{I}} \sum_{p \in P_i} \delta_{ap} h_p, \ a \in \mathcal{A}. \tag{6.11}$$

Definition 6.11. *The cost matrix $t(v)$ is said to be C-monotone if*

$$(t(v_1) - t(v_2))(v_1 - v_2) \geq_C 0, \quad \textit{for } v_1, v_2 \in I\!R^{|\mathcal{A}|}.$$

Lemma 6.12. *The cost function of $P(\lambda)$ is convex if the cost matrix $t(v)$ is C-monotone.*

Proof. For a given $\lambda \in C^*$, we have

$$\sum_{k=1}^{\ell} \lambda_k (t^k(v_1) - t^k(v_2))(v_1 - v_2) \geq 0, \quad \text{for every } v_1, v_2 \in \mathrm{I\!R}^{|\mathcal{A}|}.$$

Thus

$$(\lambda^\top t(v_1) - \lambda^\top t(v_2))(v_1 - v_2) \geq 0, \quad \text{for every } v_1, v_2 \in \mathrm{I\!R}^{|\mathcal{A}|}.$$

By (6.7), the gradient of the function $\oint^v \lambda^\top t(w) dw$ is monotone, so the cost function of P(λ) is convex. ■

Let $\Gamma \in \mathrm{I\!R}^{M \times \mathcal{I}}$ denote the path-origin incidence matrix for the network with entry $[\Gamma]_{pi} = \gamma_{pi} = 1$ if $p \in P_i$ and $\gamma_{pi} = 0$ otherwise. Its transpose $\Gamma^\top$ is the origin-path incidence matrix.

Theorem 6.13. *Assume that the cost function t_a is integrable and the cost matrix $t(v)$ is C-monotone, and let $\lambda \in C^* \backslash \{0\}$. Then h is in weak λ-equilibrium if and only if h is a solution of $P(\lambda)$.*

Proof. Let the Lagrangian of $P(\lambda)$ be defined by

$$L = \oint^v \lambda^\top t(w) dw - u(\lambda)^\top (\Gamma^\top h - d).$$

Since the problem is convex by Lemma 6.12, the sufficient and necessary optimality conditions for P(λ) are given by

$$\frac{\partial L}{\partial h} = \lambda^\top t(v) \Delta - u(\lambda)^\top \Gamma^\top = \lambda^\top T(h) - u(\lambda)^\top \Gamma^\top \geq 0 \tag{6.12}$$

$$\frac{\partial L}{\partial h} h = (\lambda^\top T(h) - u(\lambda)^\top \Gamma^\top) h = 0; \tag{6.13}$$

$$\frac{\partial L}{\partial u} = h^\top \Gamma - d^\top = 0^\top. \tag{6.14}$$

Note that $u(\lambda)$ is given by (6.5). (6.12) is equivalent to (6.5), (6.13) says that h is in weak λ-equilibrium and (6.14) is the definitional constraint (6.11). It follows that h is in weak λ-equilibrium. ■

Now consider the following network vector optimization problem (NVO):

$$\begin{aligned} \text{Min}_C \quad & F(v) \\ \text{subject to} \quad & \sum_{p \in P_i} h_p = d_i, \ \forall i \in \mathcal{I} \\ & h_p \geq 0, \quad \forall p \in P_i, \ \forall i \in \mathcal{I}, \end{aligned}$$

and the definitional constraint (6.14), where

$$F(v) = \left(\oint^v t^1(w)dw, \cdots, \oint^v t^\ell(w)dw \right)^\top,$$

and t^k is the k^{th} row of the cost matrix $t(v)$. A vector v (or its corresponding h) is said to be a weakly minimal solution of (NVO) if there exists no other feasible v' such that $F(v') \leq_{intC} F(v)$.

In the special case where t_a is separable, the cost function $F(v)$ of problem (NVO) reduces to

$$F(v) = \left(\sum_{a \in \mathcal{A}} \int_0^{v_a} t_a^1(w)dw, \cdots, \sum_{a \in \mathcal{A}} \int_0^{v_a} t_a^\ell(w)dw \right)^\top. \tag{6.15}$$

Theorem 6.14. *Assume that the cost function t_a is integrable and the cost matrix $t(v)$ is C-monotone. If further Assumption 6.5 holds and h is in weak vector equilibrium, then h is a weakly minimal solution of (NVO).*

Proof. Let h be in weak vector equilibrium. By Theorem 6.7 (i), there exists $\lambda \in C^* \setminus \{0\}$ such that h is in weak λ-equilibrium. By Theorem 6.13, h is also a solution of P(λ). Since

$$\begin{aligned} \lambda^\top F(v) &= \lambda^\top \left(\oint^v t^1(w)dw, \cdots, \oint^v t^\ell(w)dw \right)^\top \\ &= \oint^v \sum_{k=1}^{\ell} \lambda_k t^k(w)dw \\ &= \oint^v \lambda^\top t(w)dw, \end{aligned} \tag{6.16}$$

which is the cost function of P(λ), it follows from $\lambda \in C^* \setminus \{0\}$ that a solution to P(λ) is also a weakly minimal solution of (NVO). ■

Next, necessary and sufficient optimality conditions of weak vector traffic equilibrium in terms of vector variational inequalities are given.

Theorem 6.15. *Let Assumption 6.5 hold , the cost function t_a be integrable and the cost matrix $t(v)$ be C-monotone. If h is in weak vector equilibrium, then h is a solution of the following (WVVI) of finding $h \in \mathcal{H}$:*

$$T(h)(g-h) \not\leq_{intC} 0, \ \forall g \in \mathcal{H}.$$

Proof. If h is in weak vector equilibrium, by Theorem 6.14, h is a weakly minimal solution of (NVO). A necessary condition for h to be a weakly minimal solution of (NVO) is that

$$\frac{\partial F(h)}{\partial h}(g-h) \not\leq_{intC} 0, \ \forall g \in \mathcal{H}.$$

From (6.1) and (6.15), we have

$$\frac{\partial F}{\partial h} = \frac{\partial F}{\partial v}\frac{\partial v}{\partial h} = \begin{pmatrix} t^1(v) \\ \vdots \\ t^r(v) \end{pmatrix} \Delta = t(v)\Delta = T(h).$$

Thus the conclusion follows. ∎

We may now establish a sufficient condition for a flow h to be in weak vector equilibrium.

Theorem 6.16. *$h \in \mathcal{H}$ is in weak vector equilibrium if h solves the (WVVI) of finding $h \in \mathcal{H}$:*

$$T(h)(\bar{h}-h) \not\leq_{intC} 0, \ \forall \bar{h} \in \mathcal{H}. \tag{6.17}$$

Proof. Let h satisfy (6.17). Choose $\bar{h}$ to be such that

$$\bar{h}_j = \begin{cases} h_j, & \text{if } j \neq k \text{ or } j, \\ 0, & \text{if } j = k, \\ h_k + h_j, & \text{if } j = j. \end{cases} \tag{6.18}$$

Clearly, $\bar{h} \in \mathcal{H}$ since $\forall i \in \mathcal{I}$, $\sum_{j \in P_i} h_j = \sum_{j \in P_i} \bar{h}_j = d_i$. Now

$$\begin{aligned} T(h)(\bar{h}-h) &= \sum_{i \in I} \sum_{j \in P_i} (\bar{h}_j - h_j)\tau_j(h) \\ &= (\bar{h}_k - h_k)\tau_k(h) + (\bar{h}_j - h_j)\tau_j(h) \\ &= h_k(\tau_j(h) - \tau_k(h)) \not\leq_{intC} 0. \end{aligned} \tag{6.19}$$

If

$$\tau_k(h) - \tau_j(h) \geq_{intC} 0, \tag{6.20}$$

then (6.19) and (6.20) together imply that $h_k = 0$ since C is a pointed cone. ∎

Nonlinear Scalarization Approach

In this subsection, we assume that $C = \mathbb{R}^\ell_+$. Choose any $a \in \mathbb{R}^\ell$ and $e \in int\mathbb{R}^\ell_+$. By using the nonlinear scalarization function ξ_{ea}, define a function $\xi^k_{ea} : \mathbb{R}^M \to \mathbb{R}$ by:

$$\xi^k_{ea}(h) = \xi_{ea}(\tau_k(h)), \quad k \in P_i, i \in \mathcal{I}.$$

The vector-valued function $\bar{\xi}_{ea} : \mathcal{H} \to \mathbb{R}^M$ and the scalar-valued function $u^i_{ea} : \mathcal{H} \to \mathbb{R},\ i \in \mathcal{I}$ are defined, respectively, by

$$\bar{\xi}_{ea}(h) = [\xi^k_{ea}(h) : k \in P_i,\ i \in \mathcal{I}] \tag{6.21}$$

and

$$u^i_{ea}(h) = \min_{k \in P_i} \xi_{ea}(\tau_k(h)), \quad i \in \mathcal{I}. \tag{6.22}$$

Definition 6.17. *The path flow $h \in \mathcal{H}$ is said to be in ξ_{ea}-equilibrium if there exist $e \in int\mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$ such that*

$$\forall i \in \mathcal{I}, \forall k, l \in P_i, \quad \xi_{ea}(\tau_k(h)) > \xi_{ea}(\tau_l(h)) \Longrightarrow h_k = 0. \tag{6.23}$$

Consider the following vector optimization problem (VO):

$$(VO) \qquad \operatorname*{Min}_{x \in X}{}_C\, f(x),$$

where $f : \mathbb{R}^M \to \mathbb{R}^\ell$, $X \subset \mathbb{R}^M$ is a possibly finite set. Note that neither f nor X is required to be convex.

We have the following non-convex scalarization theorem.

Theorem 6.18 (Non-convex Scalarization Theorem, Gerth and Weidner [78]). *Let $A \subset \mathbb{R}^\ell$ be a $\mathbb{R}^\ell_+$ order lower bounded subset. Then $y^* \in Min_{int\mathbb{R}^\ell_+} A$ if and only if, for some $a \in \mathbb{R}^\ell$ and $e \in intR^\ell_+$,*

$$\xi_{ea}(y^*) = \min \xi_{ea}(A).$$

We may now use Theorem 6.18 to establish an equivalent condition for a weak vector equilibrium in terms of a scalar variational inequality.

Theorem 6.19. *The path flow $h \in \mathcal{H}$ is in weak vector equilibrium if and only if h is in ξ_{ea}-equilibrium for some $e \in int\mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$.*

Proof. ($\Longleftarrow$)

Assume that h is in ξ_{ea}-equilibrium for some $e \in int\mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$, i.e., (6.23) holds. Now if $\tau_k(h) > \tau_l(h)$, for some path $l \in P_i$, then by the strict monotonicity of the ξ_{ea} function and (6.23), we conclude that $h_k = 0$, i.e., h is in weak vector equilibrium.

($\Longrightarrow$)

Conversely, let the set $K_i \subset \mathbb{R}^\ell$ be defined by $K_i = \{\tau_k(h) : k \in P_i\}$. If for all $e \in int\mathbb{R}^\ell_+$, $a \in \mathbb{R}^\ell$, h is not in ξ_{ea}-equilibrium, then there exists $k \in P_i$, $i \in \mathcal{I}$ such that

$$h_k > 0 \text{ and } \xi_{ea} \circ \tau_k(h) \; > \; \xi_{ea} \circ \tau_p(h), \tag{6.24}$$

where $\tau_p(h) \in \text{Min}_{int\mathbb{R}^\ell_+} K_i$. Theorem 6.18 implies that $\tau_k(h) \notin \text{Min}_{int\mathbb{R}^\ell_+} K_i$. By the domination theorem in vector optimization, $\exists \tau_p(h) \in \text{Min}_{int\mathbb{R}^\ell_+} K_i$ such that $\tau_k(h) > \tau_p(h)$, yet $h_k > 0$, i.e., h is not in weak vector equilibrium. ■

Remark 6.20. It is important to note that the set K_i in the above proof is a discrete set, in which convexity has no meaning. The converse proof would not have worked if we had used the linear scalarization instead, since this would have required the set K_i to be infinite and cone convex.

The problem of finding a ξ_{ea}-equilibrium for given $e \in \mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$ is still not directly solvable. We now reduce the ξ_{ea}- equilibrium to a scalar variational inequality and consequently well-known techniques for solving variational inequalities can be applied accordingly.

Theorem 6.21. *The path flow $h \in \mathcal{H}$ is in ξ_{ea}-equilibrium if and only if there exist $e \in int\mathbb{R}^\ell_+$ and $a \in \mathbb{R}^\ell$ such that h solves the following (scalar) variational inequality:*

$$\bar{\xi}_{ea}(h)^\top(\bar{h} - h) \geq 0, \quad \forall \bar{h} \in \mathcal{H}, \tag{6.25}$$

where $\bar{\xi}_{ea}(h) = [\xi^k_{ea}(h) : k \in P_i, i \in \mathcal{I}]$ and $\xi^k_{ea}(h) = \xi_{ea}(\tau_k(h))$.

Proof. ($\Longleftarrow$)

Assume that h solves the variational inequality (6.25). Choose the special $\bar{h}$ defined by (6.18), then

$$\begin{aligned} \bar{\xi}_{ea}(h)^\top(\bar{h} - h) &= \sum_{i\in\mathcal{I}} \sum_{j\in P_i} (\bar{h}_j - h_j)\xi^j_{ea}(h) \\ &= (\bar{h}_k - h_k)\xi^k_{ea}(h) + (\bar{h}_l - h_l)\xi^l_{ea}(h) \\ &= h_k(\xi^l_{ea}(h) - \xi^k_{ea}(h)) \\ &= h_k(\xi_{ea}(\tau_l(h)) - \xi_{ea}(\tau_k(h))) \\ &\geq 0. \end{aligned} \tag{6.26}$$

Thus if $\xi_{ea}(\tau_k(h)) - \xi_{ea}(\tau_l(h)) > 0$, (6.26) and $h_k \geq 0$ implies that $h_k = 0$, i.e., h is in weak vector equilibrium.

($\Longrightarrow$)

Conversely, we assume that $h \in \mathcal{H}$ is in ξ_{ea}-equilibrium and define,

$$P_i^1 := \{k \in P_i : \xi_{ea} \circ \tau_k(h) = u_{ea}^i(h)\},$$
$$P_i^2 := \{k \in P_i : \xi_{ea} \circ \tau_k(h) > u_{ea}^i(h)\}. \tag{6.27}$$

Then for any $\bar{h} \in \mathcal{H}$, we have

$$\begin{aligned}
\bar{\xi}_{ea}(h)^\top(\bar{h} - h) &= \sum_{i\in\mathcal{I}} \sum_{k\in P_i} \xi_{ea}^k \circ \tau_k(h)(\bar{h}_k - h_k) \\
&= \sum_{i\in\mathcal{I}} \left\{ \sum_{k\in P_i^1} u_{ea}^i(h)(\bar{h}_k - h_k) + \sum_{k\in P_i^2} u_{ea}^i(h)\bar{h}_k \right\} \\
&= \sum_{i\in\mathcal{I}} u_{ea}^i(h) \sum_{k\in P_i} (\bar{h}_k - h_k) \\
&= \sum_{i\in\mathcal{I}} u_{ea}^i(h)(d_i - d_i) \\
&= 0,
\end{aligned}$$

i.e., h solves the variational inequality (6.25). ■

Corollary 6.22. *Let $D \subset \mathbb{R}^\ell$ be a base of $\mathbb{R}^\ell_+$. Then the path flow $h \in \mathcal{H}$ is in weak vector equilibrium if and only if there exists a $d \in D \cap int\mathbb{R}^\ell_+$ such that h solves*

$$\bar{\xi}_{d0}(h)^\top(\bar{h} - h) \geq 0, \quad \forall \bar{h} \in \mathcal{H}. \tag{6.28}$$

Proof. Since $\xi_{e0}(y)$ is positively homogeneous for $\alpha > 0$ we have $\xi_{e0}(\alpha y) = \alpha\xi_{e0}(y)$. Since D is a base, for $e \in int\mathbb{R}^\ell_+$, there exist $\alpha_1 > 0$ and $d \in D$ such that $e = \alpha_1 d$, and we have $\xi_{e0}(y) = \frac{1}{\alpha_1}\xi_{d0}(y)$. Thus, by Theorem 6.19 and Theorem 6.21, the result of this Corollary holds. ■

6.2 Vector Equilibrium Problem

In this section, we consider an equilibrium problem defined on transportation networks with vector-valued cost functions. In this model, the cost space is again ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, with the ordering cone C, a pointed, closed and convex cone with nonempty interior int$C$.

Definition 6.23. *Given a flow h, we say that a path $p \in P_i$ for an O-D pair i is a minimal one if there does not exist another path $p' \in P_i$ such that $\tau_{p'}(h) - \tau_p(h) \leq_{C\setminus\{0\}} 0$.*

Let $\Gamma_i(h) = \{\tau_p(h) : p \in P_i\}$ denote the (discrete) set of vector costs for all paths for O-D pair i, and

$$\mathcal{I}_i'(h) = \{k \in P_i \ | \ \tau_k(h) - \tau_p(h) \not\geq_{C\setminus\{0\}} 0, \ \forall p \in P_i\} \subseteq P_i$$

denote the set of all minimal paths for O-D pair i.

We define the minimal frontier for O-D pair i to be the set of minimal points in the cost-space of O-D pair i:

$$\mathrm{Min}_C(\Gamma_i(h)) = \{\xi \in \mathbb{R}^\ell \mid \xi = \tau_p(h) \ \text{ where } p \in \mathcal{I}'_i(h)\}.$$

Note that $\mathrm{Min}_C(\Gamma_i(h))$ is a discrete set because it is a subset of $\mathcal{I}'_i(h)$ and $\mathcal{I}'_i(h)$ is a discrete set.

The following vector equilibrium principle is a generalization of the well-known Wardrop's equilibrium principle (see Wardrop [197]):

Definition 6.24. *A flow $h \in \mathcal{H}$ is said to be in vector equilibrium if*

$$\forall i \in \mathcal{I}, \forall k, l \in P_i, \quad \tau_k(h) \geq_{C\setminus\{0\}} \tau_l(h) \Longrightarrow h_k = 0.$$

A flow h in vector equilibrium is often referred to as a vector equilibrium flow.

In terms of the minimal frontier for O-D pair i, the vector equilibrium principle can be stated in an equivalent form:

Definition 6.25. *(Equivalent vector equilibrium principle) The path flow vector h is in vector equilibrium if:*

$$\forall i \in \mathcal{I}, \ \forall p \in P_i, \ h_p = 0 \ \ \textit{whenever} \ \ \tau_p(h) \notin \mathit{Min}_C(\Gamma_i(h)). \tag{6.29}$$

Definition 6.26. *(Parametric equilibrium principle) Let a parameter $\lambda \in C^*$ be given. A path flow vector h is in λ-equilibrium if*

$$\forall i \in \mathcal{I}, \ \forall p \in P_i, \ h_p = 0 \ \ \textit{whenever} \ \ \exists \ e_i \in \mathit{Min}_C(\Gamma_i(h)),$$
$$\textit{such that } \lambda^\top \tau_p(h) > \lambda^\top e_i.$$

Assumption 6.27. $\mathit{Min}_C(\Gamma_i(h)) \subseteq \mathit{Min}_C(co(\Gamma_i(h)))$.

We need the following scalarization result, which is just Theorem 3.4.2 of [176].

Lemma 6.28. *Let $A \subset \mathbb{R}^\ell$ be a nonempty and convex set and $a^* \in \mathit{Min}_C A$. Then, there exists $\lambda \in \mathit{int} C^*$ such that*

$$\lambda^\top a^* = \min_{a \in A} \lambda^\top a.$$

The following result establishes relationships between a vector equilibrium flow and a parametric equilibrium flow.

Theorem 6.29. *(i) If h is in vector equilibrium and Assumption 6.27 holds, then there exists $\lambda \in C^* \setminus \{0\}$ such that the path flow h is in λ-equilibrium;*
(ii) If h is in λ-equilibrium for some $\lambda \in$ intC^*, then h is in vector equilibrium.

Proof. (i) Similar to the proof of Theorem 6.7 (i), but using Lemma 6.28 instead.

(ii) Let $\lambda \in \text{int } C^*$ and let h be in λ-equilibrium. Suppose that h is not in vector equilibrium, then by Definition 6.24, there exists $i \in \mathcal{I}, p \in P_i$ such that,

$$h_p > 0 \ \text{ and } \ \tau_p(h) \notin \text{Min}_C(\Gamma_i(h)).$$

Thus

$$h_p > 0 \ \text{ and } \lambda^\top \tau_p(h) > \lambda^\top e_i, \quad \text{for some } \ e_i \in \text{Min}_C(\Gamma_i(h)).$$

Hence h is not in λ-equilibrium, a contradiction. ■

Lemma 6.30. *Let $u_i(\lambda)$ be defined. If $\lambda \in$ int C^*, then $u_i(\lambda) = \lambda^\top e_i$ for some $e_i \in Min_C(\Gamma_i(h))$.*

Proof. From (6.5), let $p \in P_i$ be such that $u_i(\lambda) = \lambda^\top \tau_p(h)$. Choose $e_i := \tau_p(h)$. Suppose now that $e_i \notin \text{Min}_C(\Gamma_i(h))$, then there exists $\bar{p} \in P_i$, such that $\tau_p(h) \geq_{C\setminus\{0\}} \tau_{\bar{p}}(h)$. Since $\lambda \in intC^*$, $\lambda^\top \tau_p(h) > \lambda^\top \tau_{\bar{p}}(h)$, a contradiction. Therefore $e_i \in \text{Min}_C(\Gamma_i(h))$. ■

Theorem 6.31. *(i) Let $\lambda \in C^*$. Then h is in λ-equilibrium if the following condition holds:*

$$\forall i \in \mathcal{I}, \forall p \in P_i, \ h_p = 0 \ \textit{whenever} \ \lambda^\top \tau_p(h) > u_i(\lambda); \tag{6.30}$$

(ii) If $\lambda \in$ int C^* and h is in λ-equilibrium, then condition (6.30) holds.

Proof. (i) If there exists $e_i \in \text{Min}_C(\Gamma_i(h))$ such that $\lambda^\top \tau_p(h) > \lambda^\top e_i$, say $e_i = \tau_q(h)$ for some $q \in P_i$, then $\lambda^\top \tau_p(h) > \lambda^\top \tau_q(h), \ q \in P_i$. Thus, clearly,

$$\lambda^\top \tau_p(h) > u_i(\lambda) = \min_{p \in P_i} \lambda^\top \tau_p(h),$$

by (6.30), $h_p = 0$, so h is in λ-equilibrium.

(ii) Let h be a λ-equilibrium flow. By Lemma 6.30, there exists $e_i \in \min_C(\Gamma_i(h))$ such that $u_i(\lambda) = \lambda^\top e_i$. Suppose that $\lambda^\top \tau_p(h) > u_i(\lambda)$. Then

$$\lambda^\top \tau_p(h) > \lambda^\top e_i.$$

By Definition 6.26, $h_p = 0$ and hence (6.30) holds. ■

We discuss relations between a vector equilibrium flow and a solution of a vector optimization problem.

The following theorems can be similarly proved as Theorems 6.13 and 6.14, respectively.

Theorem 6.32. *Assume that the cost function t_a is integrable and the cost matrix $t(v)$ is monotone, and let $\lambda \in intC^*$. h is in λ-equilibrium if and only if h is a solution of $P(\lambda)$.*

Theorem 6.33. *Assume that the cost function t_a is integrable and the cost matrix $t(v)$ is monotone. If further Assumption 6.27 holds and h is in vector equilibrium, then h is a weakly minimal solution of (NVO).*

The following theorem follows directly from Theorem 6.33.

Theorem 6.34. *Let Assumption 6.27 hold, the cost function t_a be integrable and the cost matrix $t(v)$ be monotone. If h is in vector equilibrium, then h is a solution of the following weak vector variational inequality problem of finding $h \in \mathcal{H}$ such that*

$$T(h)(g-h) \not\leq_{intC} 0, \ \forall g \in \mathcal{H}.$$

Theorem 6.35. *Let $Min_{intC}(\Gamma_i(h))$ be a singleton. If h is an equilibrium flow, then h satisfies the SVVI of finding $h \in \mathcal{H}$:*

$$T(h)(\bar{h}-h) \geq_{C\setminus\{0\}} 0, \quad \forall \bar{h} \in \mathcal{H}.$$

Proof. See the proof of Proposition 6.3 of [44]. ■

We may now establish a sufficient condition for a flow h to be in vector equilibrium.

Theorem 6.36. *$h \in \mathcal{H}$ is in vector equilibrium if h solves the (VVI) of finding $h \in \mathcal{H}$ such that*

$$T(h)(\bar{h}-h) \not\leq_{C\setminus\{0\}} 0, \ \forall \bar{h} \in \mathcal{H}. \tag{6.31}$$

Proof. Let h satisfy (6.31). Choose $\bar{h}$ to be such that

$$\bar{h}_j = \begin{cases} h_j, & \text{if } j \neq k \text{ or } j, \\ 0, & \text{if } j = k, \\ h_k + h_j, & \text{if } j = j. \end{cases}$$

Clearly, $\bar{h} \in \mathcal{H}$ since $\forall i \in \mathcal{I}$, $\sum_{j \in P_i} \bar{h}_j = \sum_{j \in P_i} h_j = d_i$. Now

$$\begin{aligned} T(h)(\bar{h}-h) &= \sum_{i\in\mathcal{I}} \sum_{j\in P_i} (\bar{h}_j - h_j)\tau_j(h) \\ &= (\bar{h}_k - h_k)\tau_k(h) + (\bar{h}_j - h_j)\tau_j(h) \\ &= h_k(\tau_j(h) - \tau_k(h)) \not\leq_{C\setminus\{0\}} 0. \end{aligned} \tag{6.32}$$

If

$$\tau_k(h) - \tau_j(h) \geq_{C\setminus\{0\}} 0, \tag{6.33}$$

then (6.32) and (6.33) together imply that $h_k = 0$ since C is a pointed cone. Thus, h is in vector equilibrium. ■

6.3 Dynamic Vector Equilibrium Problem

Let $\Omega = [0, t_f]$ be the time period under consideration. For $i \in \mathcal{I}$ and a given path $k \in P_i$, let $h_k(t)$ denote the traffic flow on this path at time $t \in \Omega$ and $M = \sum_{i \in \mathcal{I}} |P_i|$. Then, at time t,

$$h(t) = [h_k(t) : k \in P_i, i \in \mathcal{I}]$$

is a M-dimensional column vector.

We only take account of the functional setting for the set of flow trajectories. This set is assumed to be a reflexive Banach space $L^p(\Omega, \mathbb{R}^M)$ with $p > 1$. The dual space of $L^p(\Omega, \mathbb{R}^M)$ is $L^q(\Omega, \mathbb{R}^M)$, where $1/p + 1/q = 1$. On $L^q(\Omega, \mathbb{R}^M) \times L^p(\Omega, \mathbb{R}^M)$, we define the canonical bilinear form by

$$\langle G, h \rangle = \int_\Omega G(t)h(t)dt, \quad G \in L^q(\Omega, \mathbb{R}^M), h \in L^p(\Omega, \mathbb{R}^M).$$

For $i \in \mathcal{I}$, the demand $d_i(t) \geq 0$ on this O-D pair i depends on the time $t \in \Omega$. At time $t \in \Omega$, let

$$d(t) = [d_i(t) : i \in \mathcal{I}].$$

Also, for technical reasons, we think of the demand trajectories in $L^p(\Omega, \mathbb{R}^{|\mathcal{I}|})$.

A flow trajectory $h \in L^p(\Omega, \mathbb{R}^M)$ satisfying the demand

$$\sum_{k \in P_i} h_k(t) = d_i(t) \text{ a.e. on } \Omega, \ \forall i \in \mathcal{I}$$

is called a feasible path flow. Let $\mathcal{H}$ be the set of feasible path flows, i.e.,

$$\mathcal{H} = \{h \in L^p(\Omega, \mathbb{R}^M) | \ h(t) \geq 0 \text{ and } \sum_{k \in P_i} h_k(t) = d_i(t) \text{ a.e. on } \Omega, \ \forall i \in \mathcal{I}\}.$$

A path flow vector $h(t)$ induces an arc flow column vector $v(t) = [v_a(t) : a \in \mathcal{A}]$, where, for each arc $a \in \mathcal{A}$,

$$v_a(t) = \sum_{i \in \mathcal{I}} \sum_{k \in P_i} \delta_{ak} h_k(t)$$

where $\Delta = [\delta_{ak}] \in \mathbb{R}^{|\mathcal{A}| \times M}$ is the arc path incidence matrix. Hence

$$v(t) = \Delta h(t).$$

Let $\mathcal{V}$ be the set of feasible arc flows, i.e.,

$$\mathcal{V} = \{v \in L^p(\Omega, \mathbb{R}^{|\mathcal{A}|}) \mid v(t) \geq 0, v(t) = \Delta h(t) \text{ and } \sum_{k \in P_i} h_k(t) = d_i(t) \text{ a.e. on } \Omega, \forall i \in \mathcal{I}\}.$$

Let $(\mathbb{R}^\ell, C)$ be an ordered space with the ordering cone C and, for each t, $c_a(v(t)) \in \mathbb{R}^\ell$ be a vector cost functional on arc a (arc weight); let $c(v(t)) = [c_a(v(t)) : a \in \mathcal{A}]$ be an $\ell \times |\mathcal{A}|$-matrix. The vector weight along a path $k \in P_i$ is assumed to be the sum of all the arc weights along this path; thus

$$\tau_k(h(t)) = \sum_{a \in \mathcal{A}} \delta_{ak} c_a(v(t)) \in \mathbb{R}^\ell.$$

Set

$$T(h(t)) = c(v(t))\Delta$$

which is an $\ell \times M$ matrix with columns given by $\tau_k(h(t))$.

So we know that, for each $h \in \mathcal{H}$, $T(h(\cdot))$ is a functional from Ω to $\mathbb{R}^{\ell \times M}$. We assume that, for all $h \in \mathcal{H}$, $T(h(\cdot))$ is in $L^q(\Omega, \mathbb{R}^{\ell \times M})$ where $1/p+1/q=1$. Define a multi-cost path functional $U : L^p(\Omega, \mathbb{R}^M) \to L(L^p(\Omega, \mathbb{R}^M), \mathbb{R}^\ell)$ by

$$\langle U(h), \bar{h} \rangle = \int_\Omega T(h(t))\bar{h}(t)dt, \quad h, \bar{h} \in L^p(\Omega, \mathbb{R}^M).$$

And define a multi-cost arc functional $S : L^p(\Omega, \mathbb{R}^{|\mathcal{A}|}) \to L(L^p(\Omega, \mathbb{R}^{|\mathcal{A}|}), \mathbb{R}^\ell)$ by

$$\langle S(v), \bar{v} \rangle = \int_\Omega \sum_{a \in \mathcal{A}} c_a(v(t))\bar{v}_a(t)dt, \quad v, \bar{v} \in L^p(\Omega, \mathbb{R}^{|\mathcal{A}|}).$$

Assumption 6.37. *T is one-to-one, that is, if $h_1, h_2 \in \mathcal{H}$ and $T(h_1) = T(h_2)$, then $h_1(t) = h_2(t)$ a.e. on Ω.*

Note: It can be shown that if S is one-to-one and Δ is a square and nonsingular matrix, then Assumption 6.37 holds.

Proposition 6.38. *If the Assumption 6.37 holds, then the multi-cost path functional U is one-to-one on $\mathcal{H}$.*

Proof. Proving U is one-to-one on $\mathcal{H}$ is equivalent to show that if, for $h_1, h_2 \in \mathcal{H}$ and $h_1(t) \neq h_2(t)$ *a.e.* on Ω, then $U(h_1) \neq U(h_2)$. Suppose $U(h_1) = U(h_2)$. Then, from the definition of functional U, we have

$$\int_\Omega (T(h_1(t)) - T(h_2(t)))\bar{h}(t)dt = 0, \quad \forall \bar{h} \in L^p(\Omega, \mathbb{R}^M).$$

From the Hahn-Banach theorem,

$$T(h_1(t)) - T(h_2(t)) = 0, \quad a.e. \ on \ \Omega.$$

From Assumption 6.37,

$$h_1(t) - h_2(t) = 0, \quad a.e. \ on \ \Omega.$$

■

Definition 6.39. *Given an $h \in \mathcal{H}$, we say that a path $k \in P_i$ for an O-D pair i is a minimal one if there does not exist another path $k' \in P_i$ such that $\tau_k(h(t)) - \tau_{k'}(h(t)) \geq_{C\setminus\{0\}} 0$, a.e. on Ω.*

Given an $h \in \mathcal{H}$, let $\Gamma_i(h) = \{\tau_k(h(t)) : k \in P_i\}$ denote the (discrete) set of vector cost functionals of all paths for O-D pair i, and

$$\mathcal{I}''_i(h) = \{k \in P_i \mid \tau_k(h(t)) - \tau_{k'}(h(t)) \not\geq_{C\setminus\{0\}} 0, \text{ a.e. on } \Omega \ \forall k' \in P_i\} \subseteq P_i$$

denote the set of all minimal paths for O-D pair i.

We define the minimal frontier for O-D pair i to be the set of minimal points in the cost-space of O-D pair i:

$$\mathrm{Min}_C\,(\Gamma_i(h)) = \{\tau_p(h) \ \in \mathbb{R}^\ell \mid p \in \mathcal{I}''_i(h)\}.$$

Note that $\mathrm{Min}_C\,(\Gamma_i(h))$ is a discrete set because $\mathcal{I}''_i(h)$ is a discrete set.

Definition 6.40 (Dynamic vector equilibrium principle). *A continuous path flow vector $h \in \mathcal{H}$ is said to be in dynamic vector equilibrium if, $\forall i \in \mathcal{I}, \forall k, k' \in P_i$,*

$$h_k(t) = 0 \text{ whenever } \tau_k(h(t)) - \tau_{k'}(h(t)) \geq_{C\setminus\{0\}} 0, \text{ a.e. on } \Omega. \tag{6.34}$$

A flow h in dynamic vector equilibrium is often referred to as a dynamic vector equilibrium flow.

Remark 6.41. (i) If $\ell = 1$, (6.34) reduces to the dynamic (scalar) Wardrop's principle in Daniele et al [51].
(ii) The dynamic vector equilibrium principle can be stated in an equivalent form as: the path flow vector h is in dynamic vector equilibrium if, $\forall i \in \mathcal{I}$, $\forall p \in P_i$,

$$h_p(t) = 0 \text{ whenever } \ \tau_p(h(t)) \notin \mathrm{Min}_C\,(\Gamma_i(h))\,, \text{ a.e. on } \Omega.$$

Definition 6.42 (Dynamic weak vector equilibrium principle). *A continuous path flow vector $h(t) \in \mathcal{H}$ is said to be in dynamic weak vector equilibrium if, $\forall i \in \mathcal{I}, \forall k, k' \in P_i$,*

$$h_k(t) = 0 \text{ whenever } \tau_k(h(t)) - \tau_{k'}(h(t)) \geq_{intC} 0, \text{ a.e. on } \Omega. \tag{6.35}$$

A flow h in dynamic weak vector equilibrium is often referred to as a dynamic weak vector equilibrium flow.

The following are infinite dimensional versions of the assumptions used in [85].

Assumption 6.43. *Let $h \in \mathcal{H}$. Assume that*

$$Min_C(\Gamma_i(h)) \subset Min_C(co(\Gamma_i(h))), \text{ a.e. on } \Omega.$$

Remark 6.44. Assumption 6.43 is equivalent to assert that there exists a null set E_1 such that, for any $t \in \Omega \setminus E_1$,

$$\mathrm{Min}_C(\Gamma_i(h)) \subset \mathrm{Min}_C(co(\Gamma_i(h))).$$

Definition 6.45. *We say that the vector cost function c_a is conservative if $\partial c_a^k/\partial v_{a'} = \partial c_{a'}^k/\partial v_a$, a.e. on Ω, $\forall a, a' \in \mathcal{A}, \forall k = 1, \dots, \ell$*

Assumption 6.46. *The cost c_a is conservative for all $a \in \mathcal{A}$.*

Assumption 6.47. *Each row $c^k(v(t))$ of the cost matrix $c(v(t))$ is monotone, i.e., for all $k = 1, 2, \dots, \ell$, $v_1(t), v_2(t) \in I\!R^{|\mathcal{A}|}$,*

$$(c^k(v_1(t)) - c^k(v_2(t)))(v_1(t) - v_2(t)) \geq 0, a.e. \text{ on } \Omega.$$

Remark 6.48. Assumption 6.47 is to say that there exists a null set E_2 such that, for any $t \in \Omega \setminus E_2$, $c^k(v(t))$ is monotone, $k = 1, 2, \dots, \ell$.

In the following, an infinite dimensional (WVVI) problem is established as a necessary condition of a dynamic weak vector equilibrium flow.

Proposition 6.49 (Necessary condition). *If Assumptions 6.43, 6.46 and 6.47 hold and h is in dynamic weak vector equilibrium, then h is a solution of the following (WVVI) of finding $h \in \mathcal{H}$ such that:*

$$\langle U(h), g - h \rangle \not\leq_{intC} 0, \quad \forall g \in \mathcal{H}. \tag{6.36}$$

Proof. Since h is in dynamic weak vector equilibrium, then there exists a null set E_3 such that, for any $t \in \Omega \setminus E_3$,

$$\forall i \in \mathcal{I}, \forall k, k' \in P_i, \; h_k(t) = 0 \text{ whenever } \tau_k(h(t)) - \tau_{k'}(h(t)) \in intC,$$

and $h(t) \geq 0$ and $h_i(t) = d_i(t), \forall i \in \mathcal{I}$. So, for any $t \in \Omega \setminus (E_1 \cup E_2 \cup E_3)$, $h(t)$ is in weak vector equilibrium, and all the assumptions of Theorem 6.15 are satisfied. Hence, $T(h(t))(g(t) - h(t)) \not\leq_{intC} 0$ holds, for every $g \in H(t)$, where

$$H(t) = \{g \in \mathrm{IR}^M | g \geq 0 \text{ and } \sum_{p \in P_i} g_p = d_i(t), \forall i \in \mathcal{I}\}.$$

Since the union of finitely many null sets is a null set, $E_1 \cup E_2 \cup E_3$ is a null set. So,

$$T(h(t))(g(t) - h(t)) \not\leq_{intC} 0$$

a.e. on Ω, for any $g \in \mathcal{H}$. That is to say,

$$\langle U(h), g - h \rangle = \int_\Omega T(h(t))(g(t) - h(t))dt \not\leq_{intC} 0.$$

∎

Proposition 6.50 (Sufficient condition). *The flow $h \in \mathcal{H}$ is in dynamic vector equilibrium if h solves the following (VVI) of finding $h \in H$ such that*

$$\langle U(h), \bar{h} - h \rangle \not\leq_{C\setminus\{0\}} 0, \quad \forall \bar{h} \in \mathcal{H}. \tag{6.37}$$

Proof. Let $h \in \mathcal{H}$ satisfy (6.37). Choose $\bar{h} \in L^p(\Omega, \mathbb{R}^M)$ such that

$$\bar{h}_j(t) = \begin{cases} h_j(t), & \text{if } j \neq k \text{ or } k' \\ 0, & \text{if } j = k \\ h_k(t) + h_{k'}(t), & \text{if } j = k' \end{cases}$$

a.e. on Ω. Clearly, $\bar{h} \in \mathcal{H}$, since $h(t) \geq 0$ and $\Delta\bar{h}(t) = \Delta h(t) == d(t)$ a.e. on Ω.

Now,

$$\begin{aligned} &\langle U(h), \bar{h} - h \rangle \\ &= \int_\Omega T(h(t))(\bar{h}(t) - h(t))dt \\ &= \int_\Omega \sum_{i\in\mathcal{I}} \sum_{j\in P_i} (\bar{h}_j(t) - h_j(t))\tau_j(\bar{h}(t))dt \\ &= \int_\Omega (\bar{h}_k(t) - h_k(t))\tau_k(h(t)) + (\bar{h}_{k'}(t) - h_{k'}(t))\tau_{k'}(h(t))dt \\ &= \int_\Omega h_k(t)(\tau_{k'}(h(t)) - \tau_k(h(t))dt \not\leq_{C\setminus\{0\}} 0. \end{aligned} \tag{6.38}$$

If

$$\tau_k(h(t)) - \tau_{k'}(h(t)) \geq_{C\setminus\{0\}} 0, \ \textit{a.e.} \text{ on } \Omega,$$

then (6.38) implies that $h_k(t) = 0$ a.e. on Ω. Thus h is in vector dynamic equilibrium. ■

Proposition 6.51 (Sufficient condition). *The flow $h \in \mathcal{H}$ is in dynamic weak vector equilibrium if h solves the WVVI(6.36)*

Proof: The proof is similar to that Proposition 6.50 and omitted. ■

We apply the results in Chapter 3 to establish the existence of a dynamic weak vector equilibrium flow.

Proposition 6.52. *Suppose the multi-cost arc functional S is C-monotone and v-hemi-continuous, then there exists a path flow $h \in \mathcal{H}$, which is in dynamic weak vector equilibrium.*

Proof. Note that

$$\mathcal{H} = \{h \in L^p(\Omega, \mathbb{R}^M) |\ h(t) \geq 0 \ and \ \sum_{k \in P_i} h_k(t) = d_i(t) \ a.e. \ on \ \Omega, \ \forall i \in \mathcal{I}\}.$$

It is clear that $\mathcal{H}$ is bounded, convex and closed, i.e., compact in the weak topology of $L^p(\Omega, \mathbb{R}^M)$.

For any $h, \bar{h} \in \mathcal{H}$, set $v = \Delta h, \bar{v} = \Delta \bar{h}$. Then, from the C-monotonicity of the multi-cost arc functional S,

$$\begin{aligned}
&\langle U(h) - U(\bar{h}), h - \bar{h} \rangle \\
&= \int_\Omega (T(h(t)) - T(\bar{h}(t)))(h(t) - \bar{h}(t))dt \\
&= \int_\Omega (c(v(t))\Delta - c(\bar{v}(t))\Delta)(h(t) - \bar{h}(t))dt \\
&= \int_\Omega (c(v(t)) - c(\bar{v}(t)))(\Delta h(t) - \Delta \bar{h}(t))dt \\
&= \int_\Omega (c(v(t)) - c(\bar{v}(t)))(v(t) - \bar{v}(t))dt \\
&= \langle S(v) - S(\bar{v}), v - \bar{v} \rangle \geq_C 0,
\end{aligned}$$

i.e., the multi-cost path functional U is C-monotone on $\mathcal{H}$. Similarly,

$$\langle S(v + t\bar{v}), \bar{v} \rangle = \langle U(h + t\bar{h}), \bar{h} \rangle.$$

So, from the v-hemi-continuity of S,

$$\lim_{t \to 0^+} \langle U(h + t\bar{h}), \bar{h} \rangle = \lim_{t \to 0^+} \langle S(v + t\bar{v}), \bar{v} \rangle = \langle S(v), \bar{v} \rangle = \langle U(h), \bar{h} \rangle.$$

So, from the v-hemi-continuity of S, we have the multi-cost path functional U is v-hemi-continuous. Then, by Theorem 3.14, the WVVI (6.36) has one solution $h \in \mathcal{H}$. So by Proposition 6.51, h is in dynamic weak vector equilibrium.■

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