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Nonlinear Partial Differential Equations

Asymptotic Behavior of Solutions and Self-Similar Solutions

BIRKHÄUSER

Progress in Nonlinear Differential Equations and Their Applications
Volume 79

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ISBN 978-0-8176-4173-3 e-ISBN 978-0-8176-4651-6
DOI 10.1007/978-0-8176-4651-6
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010928841

Mathematics Subject Classification (2010): Primary: 35B40, 35C06. Secondary: 35Q30, 76D05, 35K05, 35G55, 26D10, 42B20

Printed on acid-free paper

Birkhäuser is part of Springer Science+Business Media (www.birkhauser.com)

In memory of Professor Tetsuro Miyakawa
– with our profound admiration

Contents

Part II Useful Analytic Tools

Preface

The purpose of this book is to present typical methods (including rescaling methods) for the examination of the behavior of solutions of nonlinear partial differential equations of diffusion type. For instance, we examine such equations by analyzing special so-called self-similar solutions. We are in particular interested in equations describing various phenomena such as the Navier–Stokes equations. The rescaling method described here can also be interpreted as a renormalization group method, which represents a strong tool in the asymptotic analysis of solutions of nonlinear partial differential equations. Although such asymptotic analysis is used formally in various disciplines, not seldom there is a lack of a rigorous mathematical treatment. The intention of this monograph is to fill this gap. We intend to develop a rigorous mathematical foundation of such a formal asymptotic analysis related to self-similar solutions. A self-similar solution is, roughly speaking, a solution invariant under a scaling transformation that does not change the equation. For several typical equations we shall give mathematical proofs that certain self-similar solutions asymptotically approximate the typical behavior of a wide class of solutions.

Since nonlinear partial differential equations are used not only in mathematics but also in various fields of science and technology, there is a huge variety of approaches. Moreover, even the attempt to cover only a few typical fields and methods requires many pages of explanations and collateral tools so that the approaches are self-contained and accessible to a large audience. It is not our intention to survey many topics of nonlinear partial differential equations. Our aim in this book is to explain some asymptotic methods by studying typical examples.

Historically, partial differential equations were introduced soon after the notion of differentiation and integration was settled, with the purpose to model dynamical behavior of the motion of bodies such as a string or a membrane. A partial differential equation (PDE) is an equation describing a functional relation of a set of unknowns and their derivatives. Here the unknowns depend in general on several independent variables such as time and space. If the unknowns depend only on one variable, the equation is called an ordinary differential equation (ODE). Thus, compared with ODEs there is a much larger diversity of problems modeled by PDEs. In fact, various PDEs are proposed to model phenomena not only in physics, for example in mechanics, electromagnetics, and thermodynamics, but also in various other fields of science and technology such as social sciences and finance. On the other hand, PDEs do not only describe real-world phenomena, but also play an important role in the description of mathematical objects such as those, for example, in differential geometry and complex analysis. If a PDE is linear with respect to the unknowns and their derivatives, it is called a linear partial differential equation. Typical examples of linear PDEs are the heat equation, the Poisson equation, and the Laplace equation in electromagnetics. However, in the modeling of certain phenomena there appear several key PDEs that are not linear. PDEs of this type are called nonlinear partial differential equations. A typical example is given by the Navier–Stokes equations, which represent the fundamental equations of hydrodynamics. There is a huge variety of nonlinear PDEs, and so far it seems impossible to discuss fundamental problems in a unified way. Typical problems in mathematical analysis include a solvability problem—existence of solutions of a PDE—under suitable supplemental conditions such as initial or boundary conditions. For linear PDEs such problems can be discussed somewhat in a unified way. This, however, seems to be hopeless for the nonlinear case, since each nonlinear PDE has a special structure. So, we do not intend to establish a unified theory at the present stage. Rather we mostly study a specific class of nonlinear PDEs having a similar structure. (Note that the set of linear PDEs is a special class of PDEs.) Even for fundamental problems such as solvability, necessary prerequisites depend upon equations. From the applied point of view other problems such as profile and behavior of solutions, are also very important. Indeed, researchers in applied fields often conjecture the behavior of solutions by studying special solutions. However, there is a tendency among mathematical books treating PDEs in a rigorous way to spend many pages on solvability problems, and it is often difficult to explain the behavior of solutions.

The aim of this book is to study the behavior of solutions in a rigorous way by discussing typical examples without even assuming knowledge of functional analysis. For this purpose, the structure of this book differs essentially from the setup of usual mathematical textbooks. In the conventional style, authors explain fundamental universal theory for PDE analysis, such as elementary functional analysis, and discuss PDEs in that framework. This is a smart way to encode a lot of mathematical information in a small number of pages, which is also very efficient. In this book, however, we pursue a different way. We study directly the behavior of solutions of particular equations without preparing the fundamental theory. Instead, we discuss fundamental tools used in the analysis of these PDEs in the second part of this book. We hope that the reader will learn to deal with tools such as calculus inequalities during the study of PDEs. This more direct way should give students a strong motivation for the study of such fundamental tools and an idea of their usefulness for applications.

The book at hand consists of two parts. Part I includes Chapters 1, 2, and 3. Part II includes Chapters 4, 5, 6, and 7. In Part I we present a way to study the behavior of solutions of nonlinear PDEs of diffusion type using self-similar solutions. In Chapter 1 we show as a preliminary result by two methods that the large-time behavior of solutions of the heat equation is asymptotically self-similar. The first method relates to a representation formula of the solution. This argument is simple; however, it is restrictively applicable to nonlinear PDEs. The second method replaces the problem by the task of showing the convergence of a family of functions of rescaled solutions. This argument, however, applies to a wide range of problems.

In fact, in Chapter 2 we analyze in detail by the second method the two-dimensional vorticity equations (obtained from the Navier–Stokes equations). We shall prove that the vorticity, which is the solution of the vorticity equations, is asymptotically self-similar as time tends to infinity. Moreover, its behavior is proportional to the behavior of the Gauss kernel (also called the Gaussian vortex), provided that the total circulation is small. We present a proof that is more transparent than the ones given in the previous literature and that is based on an improvement of the fundamental $L^q - L^1$ estimate (Section 2.3) for the heat equation with transport term. We also complete the proof by giving an estimate (Section 2.5.2) for the family of rescaled functions (which is missing in the literature). Our purpose is to get a sharp result with a method as elementary as possible. For example, the estimates on the derivatives of the vorticity (Section 2.4.2) are new in the sense that they include the cases $p = 1$ and $p = \infty$. The proof is elementary in the sense that it does not use a complicated function space or interpolation of spaces.

As an application of the asymptotic behavior of the vorticity we discuss in Section 2.6 the formation of the Burgers vortex in three dimensions. A few years ago the convergence to the Gaussian vortex was proved without assuming that the total circulation is small. We include this beautiful result, which is based on relative entropy, in Section 2.8. In order to make this book self-contained we also give a proof of all key statements (except for the lemma in Section 2.5.2), including those in Part II by admitting the unique solvability of the vorticity equations as well as the solvability of the heat equation with transport term. We hope that the reader, while following the proofs, will learn about the significance of the calculus inequalities, provided in Chapter 6, in the analysis of these individual PDEs. Almost all inequalities invoked in Chapters 1 and 2 are proved in Part II, unless their proof is given in Chapters 1 and 2 already.

In Chapter 3 we first present a typical result of large-time asymptotic behavior of solutions for the porous medium equation, however, without giving a proof. Then, we present a method to analyze asymptotic behavior of solutions for the mean curvature flow equations near a singularity. These equations are often used to model the motion of phase boundaries such as antiphase grain boundaries. We show that the key monotonicity formula is also valid for the harmonic map flow equation and the semilinear heat equation. Furthermore, we give an elementary proof (Section 3.2.3) of the uniqueness of self-similar solutions of the mean curvature flow equations for axisymmetric surfaces. Finally, as an example of non-diffusion-type equations we mention a nonlinear Schrödinger equation and a generalized KdV equation. Also for these equations we present an existence result of self-similar solutions describing large-time behavior and behavior near a singularity, respectively. Here we just state the results without giving a proof. So, Chapter 3 is a collection of several different topics, while Chapter 2 is written toward one explicit goal.

In Part II we give explicit proofs for various important functional analytic statements invoked in Part I. In Chapter 4 we prove decay estimates for the heat equation and uniqueness of the solution, if the initial value is given by the Dirac delta distribution. We review several basic notions, such as the fundamental solution for the heat equation with transport term, and prove its unique existence. For the reader's convenience we give also a proof of integration by parts in unbounded multidimensional domains. In Chapter 5 we give a variant of the Ascoli–Arzelà theorem, which is a fundamental compactness result for families of functions. This variant applies also to families defined on a domain that is not necessarily compact. In Chapter 6 we prove several important inequalities. Except for the boundedness of singular integral operators, we present proofs based on estimates for the solution of the heat equation. Compared to other existing textbooks this approach is quite unusual. From these interesting applications we learn that estimates for the solution of the heat equation can be important in various situations, although they are rather elementary. Our intention is not to give the shortest proof. We rather try to explain variants of the proofs. In Chapter 7, we summarize basic knowledge on integration theory and on bounded linear operators.

The inequalities in Chapter 6 are very important in the analysis of nonlinear PDEs in general, i.e., also for PDEs not treated in this book. In mathematical analysis it is often crucial how to estimate various quantities. These inequalities are presented rather in textbooks on real analysis than in textbooks on PDEs. Even though these inequalities are classical results, we included their proofs in order to make this book self-contained. We often mention unsolved problems at the present stage in italics in order to animate further research. (In fact, a problem raised in the Japanese version published in 1999 has been solved.) In the approaches presented in Part I and Part II we usually proceed as follows: first we state what we want to show and discuss applications; then we give the technical details of the proof. We hope that the reader will be able to read results and proofs with a clear view why the corresponding problems are studied, although some of them look just technical. We also remark that the range of the topics treated in this book is too broad to give a complete list of references. We therefore just tried to give a list of typical references. However, we included "notes and comments" or "research history" in some chapters, which should help the reader to find further related literature. To shorten the description we often refer to a theorem, proposition, lemma, corollary, remark, or definition in a particular subsection just by its subsection number. For example, instead of writing "the theorem in §2.2.1" we often write "Theorem 2.2.1" if no confusion seems likely.

It is widely known that nonlinear analysis is significant for science and technology. As a very attractive topic, the analysis of nonlinear PDEs can be regarded as an important subfield of nonlinear analysis. However, to understand nonlinear PDEs in a rigorous mathematical way, it is often believed that a wide-ranging knowledge including Lebesgue integration theory, functional analysis, theory of distributions, real analysis, and the theory of ODEs is necessary. Of course, if one is familiar with these subjects, the description of results can be simplified and their treatment can be unified in an elegant way (in contrast to the approach presented in this book, where we tried not to use these theories). However, some readers might be interested in studying properties of solutions of nonlinear PDEs as soon as possible (before mastering these prerequisites). This book is written mainly for such readers. The layout is chosen in a way that the reader will gain necessary analytic knowledge and intuition naturally during the study of the behavior of solutions of PDEs. For this purpose several elementary facts such as differentiation under the integral sign are elaborately explained in Part II. As a consequence this requires a great deal of text on linear PDEs (although this is also useful in analyzing nonlinear PDEs). Very nonlinear structure is discussed mainly in Chapter 3.

The prerequisite to read Part I is only calculus including integration by parts in higher dimensions. If one reads Part II in a logically complete way, an elementary part of Lebesgue integration theory is necessary. Our hope is that the reader will see how mathematical theory taught in freshman and sophomore courses represents the basis for various theories with beautiful applications to PDEs.

For the reader who is interested in large-time asymptotic behavior of solutions of the heat and vorticity equations we suggest to read Sections 1.1, 2.1, 2.2, 2.6, 2.7.1, 2.8 first. For the reader who is interested in various applications of self-similar solutions we suggest to read Section 2.7.3 and Chapter 3. We hope these sections are useful to readers who are also interested in various other disciplines than mathematics such as, for instance, hydrodynamics and engineering.

The authors are grateful to Professor Haim Brezis for inviting them to write this book and for his patience.

The present book is based on the first two authors' book *Hisenkei Henbibun Hoteishiki* published in Japanese by Kyoritsu Shuppan in 1999. The book is not just a simple translation of the Japanese version. We expanded and revised several parts. However, the structure and the spirit are similar to the Japanese version.

The authors are grateful to Professor Tohru Ozawa and Professor Masao Yamazaki for valuable comments on the Japanese edition. They are also grateful to Professor Toshio Mikami and Professor Masayoshi Takeda for informative remarks on references of Section 6.1.5 of the Japanese edition. Furthermore, they are grateful to Professor Hisashi Naito and Ms. Yumiko Naito for their help on the translation into English.

The authors are grateful to Dr. Yasunori Maekawa and to Dr. Yukihiro Seki for their help in revising Chapter 2 and Chapter 3, respectively. They are grateful to Professor Marco Cannone, Professor Dongho Chae, Professor Yuki Naito, Professor Takayoshi Ogawa, Professor Gieri Simonett, Professor Michael Struwe, and Professor Fred Weissler for informative remarks. Finally, the authors would like to thank all colleagues, students, and readers of the Japanese edition who contributed with useful hints and comments to the success of this book.

March 2010

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Part I

Asymptotic Behavior of Solutions of Partial Differential Equations

1

Behavior Near Time Infinity of Solutions of the Heat Equation

Partial differential equations that include time derivatives of unknown functions are often called evolution equations. One important problem about evolution equations is to analyze the behavior of solutions at sufficiently large time. Such problems have been studied extensively from various points of view. Here, we are concerned with the *initial value problem* of the *heat equation*, which is a linear partial differential equation. It is not difficult to determine the asymptotic behavior of solutions of the heat equation near time infinity, and we introduce two methods to analyze its behavior. The first method is based on a representation formula of the solution of the equation directly; here we shall give a proof, which is short and easy. This method is sufficient to obtain the result for the heat equation; however, it may not apply to nonlinear problems in general, since we do not expect that solutions for nonlinear problems usually have a representation formula. The second method is based on a scaling transformation of the solution using the structure of the heat equation. By this method we shall give a proof of the behavior of solutions again. The proof by the second method is longer and it seems to be inefficient, but its idea can apply to nonlinear problems, which we study in Chapter 2 and in several parts of Chapter 3. To be familiar with the method, we give the proof for the heat equation, which is easier and more transparent to handle than nonlinear problems.

1.1 Asymptotic Behavior of Solutions Near Time Infinity

We consider the *heat equation*

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \qquad x \in \mathbb{R}^n, \quad t > 0. \tag{1.1}$$

Here we denote by $\frac{\partial u}{\partial t}$ the partial derivative with respect to the time variable t of a real-valued function $u = u(x,t)$, and by Δ the *Laplacian* i.e.,

M.-H. Giga et al., *Nonlinear Partial Differential Equations*,
Progress in Nonlinear Differential Equations and Their Applications 79,
DOI 10.1007/978-0-8176-4651-6_1,

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$$

where $x = (x_1, \ldots, x_n)$ is the coordinate expression of the spatial variable x.

In fields outside mathematics, Δu is sometimes denoted by $\nabla^2 u$.

We denote by $\frac{\partial u}{\partial x_j}$ the partial derivative of u with respect to the variable x_j, and by $\frac{\partial^2 u}{\partial x_j^2}$ the second partial derivative of u with respect to the variable x_j. The condition $x \in \mathbb{R}^n, t > 0$ in (1.1) means that equation (1.1) is satisfied for all x in n-dimensional Euclidean space $\mathbb{R}^n$ and all $t > 0$. In the following we use this convention. (We often abbreviate (1.1) by

$$\frac{\partial u}{\partial t} = \Delta u \qquad \text{in } \mathbb{R}^n \times (0, \infty),$$

without indicating x and t explicitly. Here we denote by $A \times B$ the product set

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

for sets A and B and by $(0, \infty)$ the half-open interval $\{t \in \mathbb{R} : t > 0\}$. The product set $\mathbb{R}^n \times (0, \infty)$ is naturally regarded as a subset of $\mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$.)

Physically, when heat conducts in n-dimensional space $\mathbb{R}^n$ in a homogeneous medium, the temperature distribution $u(x, t)$ at point x and time t is considered to satisfy the heat equation (1.1). (For simplicity here we set the density, the specific heat and the thermal conductivity of the medium to 1.) Thus the cases $n = 1$, 2, 3 are especially important. To understand the essence of the theory, the reader not familiar with n-dimensional space is recommended to read Chapter 1 by replacing n by 1, 2, or 3. We consider the problem of finding a (solution) u satisfying (1.1) and the condition for the initial temperature distribution $u(x, 0)$:

$$u(x, 0) = f(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

for a given real-valued function f on $\mathbb{R}^n$. This problem is called the *Cauchy problem* or the *initial value problem* for (1.1). The initial condition (1.2) is often written as

$$u|_{t=0} = f \qquad \text{on } \mathbb{R}^n.$$

We are interested in the temperature distribution when sufficient time has passed. Mathematically, this corresponds to studying the behavior of the solution u of the Cauchy problem (1.1), (1.2) when t is large enough.

Solutions of the initial value problem of the heat equation (1.1), (1.2) are represented by

$$u(x, t) = \int_{\mathbb{R}^n} G_t(x - y) f(y)\, dy, \qquad x \in \mathbb{R}^n, \quad t > 0, \tag{1.3}$$

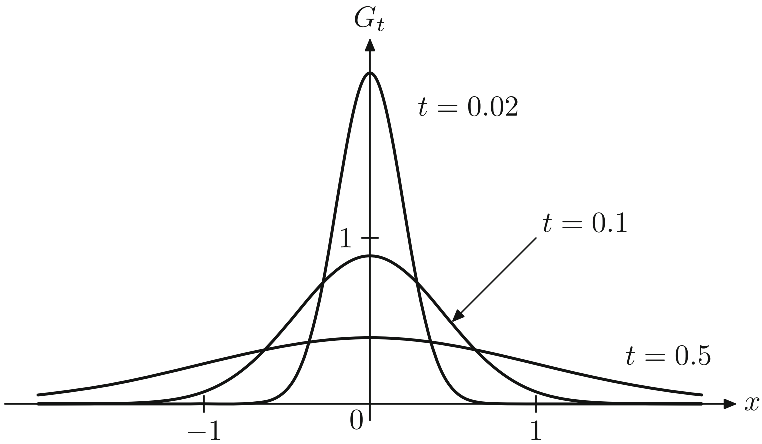

Figure 1.1. A few examples of the graph of $G_t(x)$ as a function of x for $n = 1$.

or

$$u(x_1, \ldots, x_n, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G_t(x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n, t) \cdot f(y_1, \ldots, y_n)\, dy_1 dy_2 \cdots dy_n$$

using the *Gauss kernel*

$$G_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right), \qquad x \in \mathbb{R}^n, \quad t > 0,$$

if the absolute value $|f(x)|$ of the function $f(x)$ does not grow too much at space infinity. See figure 1.1.

Throughout this book, G_t denotes the Gauss kernel and $\exp z$ denotes the exponential function e^z, where e is the base of the natural logarithm. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm $(x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ of x. The function u of (1.3) is differentiable to any order with respect to $x_1, x_2, \ldots, x_n$ and $t > 0$, satisfies the heat equation (1.1) (See Exercise 1.1 (i) and 7.2 and §4.1.6), and satisfies (1.2) in the sense of $\lim_{t\to 0} u(x,t) = f(x)$ (see §4.2) if, for example, f is continuous and f is equal to zero outside a large ball in $\mathbb{R}^n$. We denote the set of such functions f by $C_0(\mathbb{R}^n)$. See Figure 1.2. In Chapter 1, unless otherwise mentioned we assume that the initial data f is in $C_0(\mathbb{R}^n)$, so that the solution u of (1.1), (1.2) is represented by (1.3). Although it is important to examine whether there exist other solutions satisfying (1.1), (1.2), we do not consider such a problem in this section. This is postponed to §4.4. When $t \to \infty$ (i.e., t goes to infinity), to what function of x does $u(x,t)$

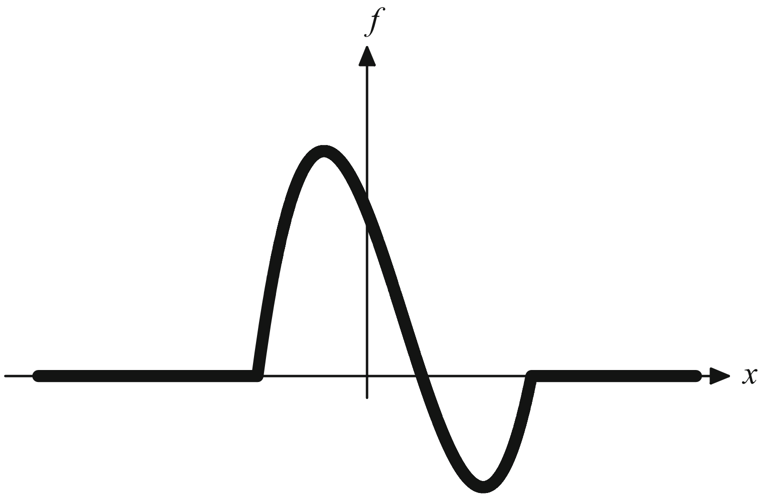

Figure 1.2. An example of the graph of $f \in C_0(\mathbb{R})$.

tend? We guess that $u(x,t)$ tends to zero as $t \to \infty$, since f is in $C_0(\mathbb{R}^n)$ and there is no heat source. In the following, we prove that this observation is correct.

In this chapter unless otherwise specified, f belongs to $C_0(\mathbb{R}^n)$, since then the integral of f can be regarded as an integral of a continuous function on a sufficiently large ball, which is finite, so that it is easy to handle (see Exercise 1.1 (ii)). Of course, we may consider more general f. Inequalities in §1.1.1–§1.1.3 are also valid if the integral is well defined. For example, we may consider f as a continuous function such that the Riemann integral on the right-hand side of the inequality is finite, or more generally we may consider a Lebesgue measurable function f with finite Lebesgue integral. In both cases, the inequalities in §1.1.1–§1.1.3 are valid. Here, to check the differentiability under the integral is a good exercise in analysis, but we do not check it in this chapter. Instead, see §4.1.4, §4.1.6, §7.2.1 and exercises at the end of Chapter 7.

1.1.1 Decay Estimate of Solutions

Proposition. *Let u be the solution (1.3) of the heat equation with initial data $f(\in C_0(\mathbb{R}^n))$. Then*

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \le \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |f(y)|\, dy, \qquad t > 0. \tag{1.4}$$

In particular, $\lim_{t\to\infty} \sup_{x\in\mathbb{R}^n} |u(x,t)| = 0$, i.e., $u(x,t)$ converges to 0 on $\mathbb{R}^n$ uniformly as $t \to \infty$.

Before giving the proof, we recall the notation sup, which represents the supremum of a set. For any subset A of $\mathbb{R}$, the real number M that satisfies the following two conditions is called the *supremum* of the set A, and is denoted by $\sup A$ (if A is bounded from above, the existence of such a number M follows from the definition of the real numbers):

(i) We have $a \le M$ for any element a of A (i.e., $a \in A$). (Such an M is called an *upper bound* of the set A.)
(ii) For any M' less than M (i.e., $M' < M$), there exists an element a' of A such that $a' > M'$.

In other words, M is the minimal upper bound of A. For a set A with no upper bound we set $\sup A = \infty$, and for the empty set A we set $\sup A = -\infty$. If A is the image of a real-valued function h defined on a set U, instead of writing $\sup A$ as $\sup\{h(x) : x \in U\}$ we may write

$$\sup_{x \in U} h(x) \quad \text{or simply} \quad \sup_U h.$$

Its value is called the *supremum* of the function h in U. If there exists $x_0 \in U$ satisfying

$$\sup_{x \in U} h(x) = h(x_0),$$

$\sup_U h$ is called the *maximum* of h in U and is denoted by $\max_U h$.

In general such an x_0 does not always exist, and even if it exists, showing its existence may not be easy. On the other hand, the supremum is always defined for any real-valued function, which makes it a convenient notion. If $\sup_U |h|$ is finite, h is said to be *bounded* in U.

Similarly, the *infimum* $\inf_U h$ of a function h on U is defined by

$$\inf_{x \in U} h(x) = -\sup_{x \in U}(-h(x)).$$

We sometimes write the range of the independent variables of a function h directly under "sup" or "inf" as in §1.3.3.

Proof of Proposition. By (1.3), for $t > 0$, we have

$$|u(x,t)| \le \int_{\mathbb{R}^n} G_t(x-y)|f(y)|\,dy \le \sup_{y \in \mathbb{R}^n} (G_t(x-y)) \int_{\mathbb{R}^n} |f(y)|\,dy.$$

Since we have

$$G_t(x-y) \le \frac{1}{(4\pi t)^{n/2}},$$

(1.4) follows. □

By this result we observe that u converges to 0 uniformly with order at least $t^{-n/2}$ as $t \to \infty$. We next ask whether the space integral of $|u|$ or its power also decays. For this purpose we first define the L^p-*norm* and L^∞-norm of continuous functions f on $\mathbb{R}^n$ as

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(y)|^p\,dy\right)^{1/p}, \qquad 1 \le p < \infty \quad (p \text{ a constant}),$$
$$\|f\|_\infty = \sup_{y\in\mathbb{R}^n} |f(y)|.$$

For simplicity, $\|f\|_\infty$ denotes $\|f\|_p$ with $p=\infty$. (This convention is natural by the fact of Exercise 2.3.) Although ∞ and $-\infty$ are not real numbers, we regard ∞, $-\infty$ as symbols that satisfy $-\infty < a < \infty$ for any real number a so that we are able to handle various inequalities in a synthetic way. Moreover, we use the convention $\frac{1}{\infty}=0$, $a+\infty=\infty$, since it is useful to shorten statements. For function $u(x,t)$ with variable $(x,t)\in\mathbb{R}^n\times(0,\infty)$, $\|u\|_p(t)$ denotes the L^p-norm of $u(x,t)$ as a function of x, i.e.,

$$\|u\|_p(t) = \begin{cases} \left(\int_{\mathbb{R}^n} |u(x,t)|^p\,dx\right)^{1/p}, & 1\le p<\infty, \quad t>0, \\ \sup_{x\in\mathbb{R}^n}|u(x,t)|, & p=\infty, \quad t>0. \end{cases}$$

A more general estimate than in §1.1.1 holds.

1.1.2 L^p-L^q Estimates

Theorem. *Let u be the solution (1.3) of the heat equation with initial data f, and let $1\le q\le p\le\infty$. Then*

$$\|u\|_p(t) \le \frac{1}{(4\pi t)^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}}\|f\|_q, \qquad t>0. \tag{1.5}$$

By this theorem the decay order of the spatial L^p-norm of u is estimated by a nonpositive power of t. When $p=\infty$ and $q=1$, (1.5) is nothing but (1.4).

Although the proof of this theorem is more complicated than that of (1.4), it can be proved easily using the *Young inequality* for convolutions (see §4.1.2).

We have studied the decay of the value of u. We ask whether the derivatives of u decay to 0 as $t\to\infty$.

1.1.3 Derivative L^p-L^q Estimates

Theorem. *Let u be the solution (1.3) of the heat equation with initial data f. For $1\le q\le p\le\infty$ there exists a constant $C=C(p,q,n)$ depending only on p, q, n such that*

$$\left\|\frac{\partial u}{\partial x_j}\right\|_p(t) \le \frac{C}{t^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{1}{2}}}\|f\|_q, \qquad j=1,\dots,n, \quad t>0, \tag{1.6}$$

$$\left\| \frac{\partial u}{\partial t} \right\|_p (t) \le \frac{C}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+1}} \|f\|_q, \qquad t > 0. \tag{1.7}$$

Moreover, for higher derivatives, there exists a constant $C = C(p, q, n, k, \alpha)$ *such that*

$$\|\partial_t^k \partial_x^\alpha u\|_p(t) \le \frac{C}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+k+\frac{|\alpha|}{2}}} \|f\|_q, \qquad t > 0. \tag{1.8}$$

(*Here* k *is a natural number or* 0 *and* α *is a multi-index* $(\alpha_1, \dots, \alpha_n)$; α_i $(1 \le i \le n)$ *is a natural number or* 0. *We use the convention*

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}, \quad \partial_{x_i} = \frac{\partial}{\partial x_i}, \quad \partial_t = \frac{\partial}{\partial t},$$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$\partial_{x_i}^0 u = u, \quad \partial_t^0 u = u.$$

In other words, $|\alpha|$ *is the order of the derivative* ∂_x^α *in the spatial direction.*)

We remark that one can choose the constants C in (1.6)–(1.8) independent of p and q.

By (1.6), (1.7), and (1.8), we observe that the power of t increases by 1/2 by differentiating once in the spatial variables, and that it increases by 1 by differentiating in the time variable. Since the proof of ((1.8) for general cases is somewhat time-consuming, we shall prove (1.6) only when $p = \infty$ and $q = 1$, and leave the remaining proof to the reader. (See §4.1.2 and Exercise 4.3.)

Proof of (1.6) (*In the case of* $p = \infty, q = 1$). Differentiating (1.3) under the integral, we have

$$\partial_{x_j} u(x,t) = \int_{\mathbb{R}^n} (\partial_{x_j} G_t)(x-y) f(y)\, dy.$$

The symbol $(\partial_{x_j} G_t)(x-y)$ is the quantity obtained by differentiating $G_t(x)$ in x_j and evaluating at $x - y$. A calculation shows that

$$\partial_{x_j} G_t(x) = \frac{1}{(4\pi t)^{n/2}} \left(-\frac{2x_j}{4t} \right) \exp\left(-\frac{|x|^2}{4t} \right).$$

The power of t seems to increase not by 1/2 but by 1. However, setting $z = |x|/(2t^{1/2})$, we have

$$\left| \frac{x_j}{2t} \exp\left(-\frac{|x|^2}{4t} \right) \right| \le \frac{1}{t^{\frac{1}{2}}} z \exp(-z^2).$$

Fortunately, $z \exp(-z^2)$ is a bounded function in $z \ge 0$. In fact, $z \exp(-z^2)$ attains its maximum $C_1 = 1/\sqrt{2e}$ at $z = 1/\sqrt{2}$. (See Exercise 1.2.) We obtain

$$\|\partial_{x_j} G_t\|_\infty \le \frac{1}{(4\pi t)^{\frac{n}{2}}} \frac{C_1}{t^{\frac{1}{2}}}.$$

Similarly as in the proof of (1.4), we have

$$\left\| \frac{\partial u}{\partial x_j} \right\|_\infty \le \|\partial_{x_j} G_t\|_\infty \int_{\mathbb{R}^n} |f(y)|\, dy \le \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} \|f\|_1, \qquad C = C_1 \frac{1}{(4\pi)^{\frac{n}{2}}},$$

which proves (1.6). □

We call the estimate in §1.1.1 a decay estimate by focusing at the behavior for large t; however, the estimate also shows a decrease in smoothness of u as t tends to 0. For this reason, in §1.1.2–§1.1.3 we call the estimate not a decay estimate but an L^p-L^q-estimate.

In the above we have observed decay orders of various norms for the solution of the initial value problem of the heat equation (1.1) with (1.2). How does $u(x,t)$ converges to 0 as t tends to infinity? We already know that the solution u decays as $\|u\|_\infty \le (4\pi t)^{-n/2}\|f\|_1$ by (1.4). So, if we can find a simple well-known (nonzero) function v such that the L^∞-norm $\|u - v\|_\infty(t)$ goes to 0 faster than $t^{-n/2}$ as $t \to \infty$, then we may say that u behaves like v for large t. In this situation v is called a *leading term* of the decay of u. What function is the leading term of the decay of the solution of the heat equation (1.1) with (1.2)? We would like to choose the function v as simple as possible. The next result states that we may choose v as a constant multiple of the Gauss kernel.

1.1.4 Theorem on Asymptotic Behavior Near Time Infinity

Theorem. *Let u be the solution* (1.3) *of the heat equation with initial data $f \in C_0(\mathbb{R}^n)$. Let $m = \int_{\mathbb{R}^n} f(y)\, dy$. Then*

$$\lim_{t\to\infty} t^{n/2}\|u - mg\|_\infty(t) = 0, \tag{1.9}$$

where $g(x,t) = G_t(x)$ is the Gauss kernel.

This theorem shows that u has a similar behavior as mg to that of $t \to \infty$ when $m \neq 0$. Therefore (1.9) is often called an asymptotic formula, and we write

$$u \sim mg \quad (t \to \infty).$$

This notation is good for intuitive understanding of the behavior of u; however, for a rigorous expression we need a formula like (1.9). When $m = 0$, (1.9) shows that $\|u\|_\infty(t)$ goes to zero faster than $t^{-n/2}$. In this case (1.9) does not give a leading term. We now prove the asymptotic formula (1.9).

1.1.5 Proof Using Representation Formula of Solutions

By $m = \int_{\mathbb{R}^n} f(y)\,dy$ we have

$$\begin{aligned}&(4\pi t)^{n/2}\{u(x,t) - mg(x,t)\}\\ &\quad = \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)\,dy - \exp\left(-\frac{|x|^2}{4t}\right)\int_{\mathbb{R}^n} f(y)\,dy\\ &\quad = \int_{\mathbb{R}^n} (h_\eta(x-y) - h_\eta(x)) f(y)\,dy,\end{aligned}$$

with

$$h_\eta(x) = \exp(-\eta|x|^2), \quad \eta = 1/(4t) \quad (>0),$$

which implies

$$\begin{aligned}&(4\pi t)^{n/2}|u(x,t) - mg(x,t)|\\ &\quad \le \int_{\mathbb{R}^n} |h_\eta(x-y) - h_\eta(x)|\,|f(y)|\,dy, \quad x \in \mathbb{R}^n, \quad t > 0.\end{aligned} \tag{1.10}$$

We use the integral form of the mean value theorem (see §1.1.6) to get

$$|h_\eta(x-y) - h_\eta(x)| \le |y| \int_0^1 |(\nabla h_\eta)(x - (1-\tau)y)|\,d\tau, \quad x, y \in \mathbb{R}^n, \tag{1.11}$$

where ∇ denotes the *gradient*, i.e.,

$$\nabla\varphi = (\partial_{x_1}\varphi, \ldots, \partial_{x_n}\varphi),$$

for a function φ on $\mathbb{R}^n$. Since

$$\nabla h_\eta(x) = -2\eta x h_\eta(x),$$

a similar argument as in the proof of (1.6) in §1.1.3 yields

$$|\nabla h_\eta(x)| \le 2\eta^{1/2} z \exp(-|z|^2) \le 2\eta^{1/2} C_1, \qquad C_1 = 1/\sqrt{2e}$$

with $z = \eta^{1/2}|x|$. By this inequality and (1.11) we get

$$|h_\eta(x-y) - h_\eta(x)| \le 2|y|\eta^{1/2} C_1.$$

Applying this to (1.10), we have

$$\begin{aligned}(4\pi t)^{n/2}|u(x,t) - mg(x,t)| &\le \int_{\mathbb{R}^n} 2\eta^{1/2} C_1 |y|\,|f(y)|\,dy\\ &= \frac{C_1}{t^{1/2}} \int_{\mathbb{R}^n} |y|\,|f(y)|\,dy.\end{aligned}$$

Since the right-hand side of this inequality is independent of x, taking the supremum of both sides yields

$$t^{n/2}\|u - mg\|_\infty(t) \le \frac{C_1}{(4\pi)^{\frac{n}{2}} t^{\frac{1}{2}}} \int_{\mathbb{R}^n} |y|\,|f(y)|\,dy, \quad t > 0. \tag{1.12}$$

Since the assumption $f \in C_0(\mathbb{R}^n)$ guarantees that

$$\int_{\mathbb{R}^n} |y||f(y)|\,dy$$

is finite, we can take the limit $t \to \infty$ in (1.12) to get the asymptotic formula (1.9). □

In the asymptotic formula (1.9) we have estimated the L^∞-norm of the difference between u and mg. A more general formula

$$\lim_{t\to\infty} t^{n(1-1/p)/2}\|u - mg\|_p(t) = 0, \qquad 1 \le p \le \infty,$$

can be proved by a similar argument (using §4.1.1); however, we do not carry this out here. (See [Giga Kambe 1988].)

1.1.6 Integral Form of the Mean Value Theorem

Theorem. *Assume that a function h is continuous in $\mathbb{R}^n$ up to all its first order partial derivatives $\partial_{x_i} h$ $(1 \le i \le n)$ i.e., h belongs to the C^1-class on $\mathbb{R}^n$. Then*

$$h(x) - h(x-y) = \int_0^1 \langle (\nabla h)(x - (1-\tau)y), y\rangle\,d\tau, \quad x, y \in \mathbb{R}^n, \tag{1.13}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^n$. Applying the Schwarz inequality on $\mathbb{R}^n$ $(|\langle a, b\rangle| \le |a|\,|b|,\ a,\ b \in \mathbb{R}^n)$ to the right-hand side yields

$$|h(x-y) - h(x)| \le |y| \int_0^1 |(\nabla h)(x - (1-\tau)y)|\,d\tau, \quad x, y \in \mathbb{R}^n.$$

These statements are also valid if $\mathbb{R}^n$ is replaced by a convex subset of $\mathbb{R}^n$.

This theorem is very useful, as is the differential form of the mean value theorem for a function of one variable:

"There exists $\theta \in (0,1)$ such that $h(x) - h(x-y) = h'(x - \theta y)y$," where h' denotes the derivative of h. The proof is elementary.

Proof. We set $F(s) = h(x - y + sy)$. The fundamental theorem of calculus yields

$$h(x) - h(x-y) = F(1) - F(0) = \int_0^1 F'(\tau)\,d\tau.$$

By the chain rule for the composition of functions, we have

$$F'(\tau) = \sum_{j=1}^{n} \frac{\partial h}{\partial x_j}(x-(1-\tau)y)y_j = \langle (\nabla h)(x-(1-\tau)y), y\rangle,$$

so that (1.13) follows. □

1.2 Structure of Equations and Self-Similar Solutions

The inequality (1.12) is stronger than the asymptotic formula (1.9) in the sense that the difference of u and mg is estimated by the integral involving the initial data and power of t. The proof in §1.1.5 was established using representation (1.3) of the solution directly, to get a stronger result. However, such a strategy is difficult to apply to nonlinear problems, whose solution cannot be expected to have an explicit representation formula in general. In the following, we shall give another proof that is based on structures of the equation. Although the proof is longer than that in §1.1.5 and looks inefficient, this strategy is often useful for nonlinear problems, as discussed in Chapter 2. The reason is that we do not need a representation formula of the solution if we obtain necessary estimates of a family of solutions by some other method. We shall give another proof for the heat equation, which is somewhat easier and simpler than the proof for nonlinear problems.

To begin with, we mention special solutions that reflect the structure of the heat equation (1.1).

1.2.1 Invariance Under Scaling

Proposition. *Assume that a real-valued function $u = u(x,t)$ satisfies the heat equation*

$$\partial_t u - \Delta u = 0$$

in an open set $Q \subset \mathbb{R}^n \times \mathbb{R}$, i.e.,

$$\partial_t u(x,t) - \Delta u(x,t) = 0, \quad (x,t) \in Q.$$

For any nonzero real number λ, we define the function u^λ by

$$u^\lambda(x,t) = u(\lambda x, \lambda^2 t).$$

(*We remark that u^λ is not u to the power λ.*) *Then the following properties hold:*

(i) *The function u^λ satisfies the heat equation in*

$$Q_\lambda = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : (\lambda x, \lambda^2 t) \in Q\}.$$

(ii) *For any real number μ, the function μu satisfies the heat equation in Q.*

Proof. Evidently the linearity of the heat equation yields (ii). To prove (i) we use the chain rule. A direct calculation shows that

$$\partial_t u^\lambda(x,t) = \partial_t\{u(\lambda x, \lambda^2 t)\} = \lambda^2(\partial_t u)(\lambda x, \lambda^2 t),$$

$$\partial_{x_j} u^\lambda(x,t) = \partial_{x_j}\{u(\lambda x, \lambda^2 t)\} = \lambda(\partial_{x_j} u)(\lambda x, \lambda^2 t) \quad (1 \le j \le n),$$

$$\Delta u^\lambda(x,t) = \sum_{j=1}^{n} \partial_{x_j}\partial_{x_j}\{u(\lambda x, \lambda^2 t)\} = \lambda^2(\Delta u)(\lambda x, \lambda^2 t),$$

which yields $\partial_t u^\lambda - \Delta u^\lambda = \lambda^2\{(\partial_t u)(\lambda x, \lambda^2 t) - (\Delta u)(\lambda x, \lambda^2 t)\} = 0$. Here we write $(\partial_t u)$, $(\partial_{x_j} u)$, (Δu) using parentheses, since we emphasize that we first differentiate $u(x,t)$ and then evaluate at $(\lambda x, \lambda^2 x)$. □

As we have seen in §1.1, the solution of the initial value problem of the heat equation (1.1) with (1.2) converges to 0 uniformly as $t \to \infty$. We next introduce a conserved quantity called total heat, which is invariant under evolution of time.

1.2.2 Conserved Quantity for the Heat Equation

Proposition. *Let u be the solution* (1.3) *of the heat equation with initial data $f \in C_0(\mathbb{R}^n)$. Then, for any $t > 0$,*

$$\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} f(x)\,dx.$$

This proposition can be proved directly using commutation of integrals (see §7.2.2) and the identity $\int_{\mathbb{R}^n} G_t(x)\,dx = 1$ (see §4.1.2). Indeed, for $t > 0$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} u(x,t)\,dx &= \int_{\mathbb{R}^n} \left\{\int_{\mathbb{R}^n} G_t(x-y) f(y)\,dy\right\} dx \\ &= \int_{\mathbb{R}^n} \left\{\int_{\mathbb{R}^n} G_t(x-y)\,dx\right\} f(y)\,dy = \int_{\mathbb{R}^n} f(y)\,dy.\end{aligned}$$

□

One can also prove the proposition without using the representation of solution (1.3). Indeed, we differentiate $\int u\,dx$ with respect to t to get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n} u(x,t)\,dx &= \int_{\mathbb{R}^n} \partial_t u(x,t)\,dx \\ &= \int_{\mathbb{R}^n} \Delta u\,dx = \int_{\mathbb{R}^n} \operatorname{div}(\nabla u)\,dx = 0.\end{aligned}$$

Hence $\int_{\mathbb{R}^n} u(x,t)\,dx$ is independent of $t > 0$. The last identity is derived by integration by parts; however, we should be careful, since $\mathbb{R}^n$ is unbounded. Here $|\nabla u|$ goes to zero fast enough at space infinity that the last identity is justified. (See §4.5.) Here, div denotes the *divergence* and is defined by

$$\operatorname{div} F = \sum_{j=1}^{n} \frac{\partial F^j}{\partial x_j} = \frac{\partial F^1}{\partial x_1} + \cdots + \frac{\partial F^n}{\partial x_n}$$

for a vector-valued function $F = (F^1, \ldots, F^n)$ on $\mathbb{R}^n$. Using equality $(*)$ in Exercise 7.3 with $p = 1$, one is able to show that $\int_{\mathbb{R}^n} u(x,t)\,dx \to \int_{\mathbb{R}^n} f(x)\,dx$ as $t \to 0$. This completes the proof of the proposition.

This proposition shows that even if $u(x,t)$ converges to 0 uniformly as $t \to \infty$, $\int_{\mathbb{R}^n} u(x,t)\,dx$ may not always converge to 0. Note that the statement "if an integrand converges uniformly, then the integral and limit operation is commutative" is valid, provided that the domain of integration is bounded or more generally of finite area.

Among the transformations of functions introduced in §1.2.1, is there any transformation that preserves the conserved quantity "total heat"?

1.2.3 Scaling Transformation Preserving the Conserved Quantity

Proposition. *Let u be a continuous function on $\mathbb{R}^n \times (0,\infty)$. Assume that $\int_{\mathbb{R}^n} u(x,t)\,dx$ is of nonzero finite value and is independent of $t > 0$. Suppose that $\mu = \mu(\lambda)$ is a positive function of $\lambda > 0$. Then, the integral $\int_{\mathbb{R}^n} \mu u^\lambda(x,t)\,dx$ is independent of $\lambda > 0$ if and only if μ is a positive constant multiple of λ^n.*

Proof. This can be proved by a direct calculation. Indeed, for $t > 0$ we have

$$\int_{\mathbb{R}^n} \mu u^\lambda(x,t)\,dx = \int_{\mathbb{R}^n} \mu u(\lambda x, \lambda^2 t)\,dx = \frac{\mu}{\lambda^n} \int_{\mathbb{R}^n} u(z, \lambda^2 t)\,dz \qquad (1.1)$$

(where we applied the change of variables $\lambda x = z$). Since $\int_{\mathbb{R}^n} u(z,\lambda^2 t)\,dz$ is independent of $\lambda^2 t$, μ/λ^n is a (positive) constant independent of λ if and only if $\int_{\mathbb{R}^n} \mu(\lambda) u^\lambda(x,t)\,dx$ is independent of λ. □

1.2.4 Summary of Properties of a Scaling Transformation

For a real-valued function $u = u(x,t)$ defined on $\mathbb{R}^n \times (0,\infty)$, we set

$$u_k(x,t) = k^n u(kx, k^2 t), \quad k > 0. \qquad (1.14)$$

As in §1.2.1, if u satisfies the heat equation (1.1),

$$\partial_t u - \Delta u = 0,$$

in $\mathbb{R}^n \times (0,\infty)$, then u_k also satisfies (1.1) in the same domain. Moreover, as in §1.2.2 and §1.2.3, if u decays at space infinity rapidly enough, then we have

$$\int_{\mathbb{R}^n} u_k(x,t)\,dx = \int_{\mathbb{R}^n} u(x,t)\,dx,$$

whose value is independent of $t > 0$.

The map producing u^λ or u_k from u by taking constant multiples of independent and/or dependent variables is called in general a *scaling transformation*; u^λ or u_k is called a *rescaled function* (or scaling transformation) of u. The scaling transformation from u to u_k preserves its total temperature, and the solvability property of the heat equation.

1.2.5 Self-Similar Solutions

If a solution u of the heat equation (1.1) on $\mathbb{R}^n \times (0,\infty)$ satisfies $u = u_k$ on $\mathbb{R}^n \times (0,\infty)$ for all $k > 0$, then u is called a *forward self-similar solution* (or simply a *self-similar solution*) of the heat equation. Here u_k is the function defined in (1.14). In other words a solution that is invariant under the scaling transformation $u \mapsto u_k$ is called a self-similar solution. Naively speaking, a scaling transformation corresponds to a change of unit of measurement. A rescaled function is "similar" to the original one. This is why we use the word "self-similar."

Example. The Gauss kernel $g(x,t)$ is a self-similar solution. Indeed, for $k > 0$, we have

$$g_k(x,t) = k^n g(kx, k^2 t) = \frac{k^n}{(4\pi k^2 t)^{n/2}} \exp\left(-\frac{|kx|^2}{4k^2 t}\right) = g(x,t)$$
$$x \in \mathbb{R}^n, t > 0.$$

It is easy to show that the Gauss kernel is a solution of the heat equation in $\mathbb{R}^n \times (0,\infty)$ by direct calculation (Exercise 1.1). This fact is also essentially invoked to show that (1.3) is a solution of (1.1) (see Exercise 7.2). As discussed at the end of §1.4.6, it turns out that a self-similar solution u satisfying $\|u\|_1(1) < \infty$ (i.e., $\|u\|_1(1)$ is finite) is a constant multiple of g.

1.2.6 Expression of Asymptotic Formula Using Scaling Transformations

Proposition. *The asymptotic formula* (1.9) *is equivalent to*

$$\lim_{k\to\infty} \|u_k - mg\|_\infty(1) = 0. \tag{1.15}$$

Here u_k is a rescaled function of u defined in (1.14). (*For simplicity, we impose similar assumptions as in Theorem 1.1.4.*)

Proof. It is sufficient to prove that for $k^2 = t$,

$$\|u_k - mg\|_\infty(1) = t^{n/2}\|u - mg\|_\infty(t).$$

In fact, since $g = g_k$, we have $v_k = u_k - mg$ with $v = u - mg$. Moreover,

$$\|v_k\|_\infty(1) = \sup_{x\in\mathbb{R}^n} k^n|v(kx, k^2)| = k^n \sup_{z\in\mathbb{R}^n} |v(z, k^2)| = t^{n/2}\|v\|_\infty(t)$$

by setting $z = kx$. We thus obtain the equivalence of (1.15) and (1.9). □

By this fact, it is important to study $\lim_{k\to\infty} u_k(x, 1)$ in order to understand the behavior of u at infinity of (x, t).

1.2.7 Idea of the Proof Based on Scaling Transformation

Formula (1.15), which is equivalent to the asymptotic formula (1.9), shows that the family of functions $\{u_k\}$ converges to the self-similar solution mg (at $t = 1$) as $k \to \infty$. We will prove (1.15) in another way different from §1.1.5. Roughly speaking, the strategy of the proof is divided into the following two steps.

The first step: compactness. We show that any subsequence of $\{u_k\}$ (as $k \to \infty$) has a convergent subsequence.

The second step: a characterization of limit functions. By analyzing properties of the limit functions, we show that they are unique independent of choice of subsequences, and equal mg. By the first step and this result, we conclude that $\{u_k\}$ converges to mg without taking subsequences (Exercise 1.4).

To implement this strategy we need to formulate the notion of convergence of sequences of functions. In §1.3 we shall formulate the notion of convergence of sequences of functions so that we can complete the first step. In §1.4 we shall implement the second step.

Once we have shown that u_k converges to a function U (in some sense, for example "pointwise convergence"), then U is scaling invariant, i.e., $U_k = U$. Indeed, for $h > 0$, we have

$$\begin{aligned} U_k(x, t) = k^n U(kx, k^2 t) &= \lim_{h\to\infty} k^n u_h(kx, k^2 t) = \lim_{h\to\infty} h^n k^n u(khx, k^2 h^2 t) \\ &= \lim_{\ell\to\infty} \ell^n u(\ell x, \ell^2 t) = U(x, t), \end{aligned}$$

where $\ell = kh$. The functions U_k and u_h are, respectively, rescaled functions of U and u defined by (1.14). If U is also a solution of the equation (1.1), then U is a self-similar solution. So it is natural to conjecture that $\{u_k\}$ converges to a self-similar solution as $k \to \infty$.

We do not estimate the difference of u_k and mg directly to prove that $\{u_k\}$ converges to mg. Instead, we prove that $\{u_k\}$ has a convergent subsequence and its limit is independent of the choice of subsequences. To implement the first step, estimates of solutions are useful. However, we need not necessarily use an explicit formula of the solution. Even for problems that we do not expect to have an explicit formula for their solutions, such as nonlinear problems, this strategy may be applicable if we can estimate the solutions in a certain way. Thus, this strategy based on scaling transformation is more broadly applicable than the method using directly an explicit formula of solutions as used in the proof in §1.1.5. In fact, we will prove the asymptotic formula of nonlinear problems in Chapters 2 and 3 using this idea.

1.3 Compactness

To discuss the convergence of sequences of functions, it is often useful to consider a set of functions (function spaces) having specific properties, so that each function is regarded as a point of the set and convergence of functions is regarded as the convergence of sequences of points in the set. In fact, the theory of general topology and functional analysis has been significantly developed to handle the convergence of sequences of functions synthetically. By interpreting the notion of convergence in an abstract way, not only does the whole outlook of the theory become better, but also the "individuality" of various types of convergence becomes clear. This is a great advantage of abstraction in mathematics.

If the notion of distance is defined in a set X, the distance function is used to define the notion of the convergence of sequences. In fact, whenever a sequence $\{x_j\}_{j=1}^{\infty}$ in X satisfies the property that the distance $d(x_j, x)$ between x_j and x converges to zero for $x \in X$ as $j \to \infty$, i.e.,

$$\lim_{j\to\infty} d(x_j, x) = 0,$$

we say that the sequence $\{x_j\}_{j=1}^{\infty}$ converges to x as j goes to infinity, and write $\lim_{j\to\infty} x_j = x$ or $x_j \to x$ $(j \to \infty)$. A set in which a metric d is defined is called a *metric space*. By this definition, notions of an open set and a compact set are defined in X similarly as in $\mathbb{R}^n$. An open ball $B_r(x)$ centered at $x \in X$ with radius r is defined by

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

(We may also write $B_r(x)$ as $B(x, r)$.) (The nonnegative (real-valued) function d defined on $X \times X$ is called a *metric* if (i) $d(x, y) = 0 \Leftrightarrow x = y$; (ii) (symmetry) $d(x, y) = d(y, x)$; (iii) (triangular inequality) $d(x, y) \le d(x, z) + d(z, y)$ for any x, y, $z \in X$.)

Example. Let X be a normed space equipped with norm $\|\cdot\|$. If we define $d(x, y) = \|x - y\|$, $x, y \in X$, then d is a metric in X, and X is regarded as a metric space equipped with the metric d. Especially, the Euclidean space $\mathbb{R}^n$

is a normed space with the norm $|\cdot|$, so that Euclidean space can be regarded as a metric space with the metric defined in this way.

A set X is called a *normed space* if X is a vector space in which a norm is defined. Here, $\|\cdot\|$ is called a *norm* in X if for each $x \in X$, a nonnegative real value $\|x\|$ is defined and satisfies

(i) $\|x\| = 0 \Leftrightarrow x = 0$;
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for any scalar α and $x \in X$;
(iii) (triangular inequality) $\|x + y\| \leq \|x\| + \|y\|$ for any x, $y \in X$.

There are various notions of convergence for a sequence of functions, such as pointwise convergence and uniform convergence; however, the notion of convergence is not necessarily regarded as convergence in a suitable metric space.

Next we consider a subset of a metric space. A set is called (sequentially) *compact* whenever any sequence in the set has a convergent subsequence. We give a definition in a formal way.

Definition. *A subset K of a metric space X is called relatively sequentially compact if any sequence in K contains a convergent subsequence in X. (Here, its limit may not belong to K but is required to belong to X.)*

When X is a metric space, this property is equivalent to saying that the closure $\overline{K}$ of K is compact (i.e., any open covering of K has a finite subcovering) in X as a topological space. Thus in the following we simply say that K is *relatively compact* in X.

When the metric space X is $\mathbb{R}^n$, K is *relatively compact* if and only if K is *bounded* (i.e., K is contained in a sufficiently large ball). This is well known as the *Bolzano–Weierstrass theorem.* (Therefore K is compact if and only if K is a bounded closed set.) We note that in general, boundedness is much easier to check than relative compactness. For a subset K of a general metric space X, the notion of boundedness can be defined similarly; however, in general, boundedness does not necessarily imply relative compactness, as mentioned later. What is a criterion of relative compactness for a family of functions? This question has been well studied throughout the development of functional analysis. We recall a classical result for a family of continuous functions, which is a clue to the later development.

1.3.1 Family of Functions Consisting of Continuous Functions

The formula (1.15) asserts that $\{u_k\}$ *converges uniformly* to mg in $\mathbb{R}^n$ only at $t = 1$. If one is able to prove that a subsequence of $\{u_k\}$ converges in $\mathbb{R}^n \times (0, \infty)$, then one concludes that the limit function satisfies (1.1). Thus to show the convergence of the subsequence, it is useful to complete the second step of §1.2.7. Unfortunately, uniform convergence of the sequence in $\mathbb{R}^n \times (0, \infty)$ cannot be expected, since convergence of a subsequence of $\{u_k\}$ may be slow around $t = 0$. So, we will prove the uniform convergence in $\mathbb{R}^n \times [\eta, 1/\eta]$ for any $\eta \in (0, 1)$. (Here, we cut near $t = \infty$ for simplicity.) However, $\mathbb{R}^n \times [\eta, 1/\eta]$ is unbounded, so it is noncompact. For this reason we shall use a modified version (§5.2.1) of the Ascoli–Arzelà theorem (§5.1.1) for the family $\{u_k\}$ of continuous functions defined on a compact set in order to prove the existence of a subsequence converging uniformly on a noncompact set.

The conventional version of the Ascoli–Arzelà theorem and the definition of compactness are explained in detail in several standard textbooks. The reader is referred to a recent book of J. Jost [Jost 2005] or [Yano 1997] for more comprehensive introductions to these topics.

Definition. *Assume that a sequence $\{M_j\}_{j=1}^{\infty}$ of compact subsets of a metric space M satisfies the following three conditions:*

(i) $M_j \subset M_{j+1}$ $j = 1, 2, \dots;$
(ii) $\bigcup_{j=1}^{\infty} M_j = M;$
(iii) *for any compact subset M_0 in M, there exists j_0 such that $M_0 \subset M_{j_0}$.*

Then $\{M_j\}_{j=1}^{\infty}$ is called an exhausting sequence of compact sets of M.

There exists an exhausting sequence of compact sets both of $\mathbb{R}^n$ and of $\mathbb{R}^n \times [\eta, 1/\eta]$. In fact, when $M = \mathbb{R}^n$, we can choose $\overline{B}_j$ as M_j, and when $M = \mathbb{R}^n \times [\eta, 1/\eta]$ we can choose $\overline{B}_j \times [\eta, 1/\eta]$ as M_j. Here, $\overline{B}_j$ denotes the closure of the open ball B_j centered at the origin with radius $j > 0$.

Let $C(M)$ be the set of all real-valued continuous functions on a metric space M. (if M is an interval $[a, b]$, $(a, b), \dots$, we may write $C[a, b]$, $C(a, b), \dots$ instead of $C([a, b])$, $C((a, b)), \dots$.) We next suppose that M has an exhausting sequence of compact sets. We consider the set of elements in $C(M)$ that converge to 0 at infinity, if it exists. That is to say, we set

$$C_\infty(M) = \left\{ h \in C(M) : \lim_{j\to\infty} \sup_{z \in M \setminus M_j} |h(z)| = 0 \right\}.$$

(Here $M \setminus M_j = \{z \in M : z \notin M_j\}$ denotes the complement of M_j in M. If $M \setminus M_j$ is the empty set, we use the convention that $\sup_{z \in M \setminus M_j} |h(z)| = 0$.) By condition (iii) of the definition of an exhausting sequence of compact sets, $C_\infty(M)$ is independent of the choice of $\{M_j\}_{j=1}^{\infty}$ (See Exercise 1.5). In particular, if M is compact, $C_\infty(M)$ is nothing but $C(M)$.

As usual we define the norm of $h \in C_\infty(M)$ as

$$\|h\|_{\infty,M} = \sup_{z \in M} |h(z)|.$$

By the condition on h at infinity and the boundedness of continuous functions on a compact set (the Weierstrass theorem), the value of $\|h\|_{\infty,M}$ is finite. It is easy to check that $\|h\|_{\infty,M}$ satisfies the conditions of a norm. Moreover, $C_\infty(M)$ becomes a complete normed space, i.e., a *Banach space*

(Exercise 1.6). Hence $C_\infty(M)$ becomes a complete metric space by defining distance as $d(h_1, h_2) = \|h_1 - h_2\|_{\infty,M}$ between h_1, $h_2 \in C_\infty(M)$. Of course, the sequence $\{h_j\}_{j=1}^\infty (\subset C_\infty(M))$ *converges uniformly* in M if and only if h_j converges in the metric space $C_\infty(M)$.

We say that a subset K of a normed space is *bounded* in X if K is contained in a sufficiently large ball in X. If $X = C_\infty(M)$, then $K \subset X$ is bounded if and only if

$$\sup_{h\in K} \|h\|_{\infty,M} < \infty.$$

If one regards K as a set of functions on M, the boundedness of K in X is called *uniform boundedness* of the family of functions in K. A bounded set in $C_\infty(M)$ is not necessarily relatively compact, as the following example shows. This phenomenon is different from the case $X = \mathbb{R}^n$.

Example 1. Let $M = [0, 1]$, $K = \{h_\ell\}_{\ell=1}^\infty$, and $h_\ell(z) = z^\ell$. Then K is bounded in $C_\infty(M)(= C(M))$, but is not relatively compact (Exercise 1.7).

We introduce the following notation. In the remaining part of §1.3, unless otherwise claimed, we denote by M a metric space having an exhausting sequence of compact sets $\{M_j\}_{j=1}^\infty$.

Definition. *A subset K in $C_\infty(M)$ is called equicontinuous if*

$$\lim_{y\to z} \sup_{h\in K} |h(z) - h(y)| = 0$$

holds for all $z \in M$.

The subset K in Example 1 is not equicontinuous, since the previous formula does not hold at $z = 1$. When M is compact, a subset K in $C(M)$ is bounded in $C(M)$ and equicontinuous if and only if K is relatively compact. (Ascoli–Arzelà theorem). If M is not compact, for relative compactness we need a condition on the decay at infinity in M.

Definition. *We say that a subset K in $C_\infty(M)$ has the equidecay property if*

$$\lim_{j\to\infty} \sup_{h\in K} \sup_{z\in M\setminus M_j} |h(z)| = 0$$

holds. This notion is independent of the choice of an exhausting sequence of compact sets $\{M_j\}_{j=1}^\infty$ as is the space $C_\infty(M)$.

Example 2. Let $M = \mathbb{R}$. For $\varphi \in C_0(\mathbb{R})$, we define $h_\ell \in C_\infty(M)$ as $h_\ell(z) = \varphi(z - \ell)$, $z \in M$ ($\ell = 1, 2, 3, \ldots$). Then, if $\varphi \not\equiv 0$, K is bounded and equicontinuous in $C_\infty(M)$, but it is not relatively compact in $C_\infty(M)$, nor does it satisfy the equidecay property (Exercise 1.7).

1.3.2 Ascoli–Arzelà-type Compactness Theorem

Theorem. *Let M be a metric space with an exhausting sequence of compact sets, and K a subset of $X = C_\infty(M)$. If K is bounded in X, equicontinuous, and if it has the equidecay property, then K is relatively compact in X. The converse is also true.*

If the metric space M is compact, this theorem is nothing but the *Ascoli–Arzelà theorem*, and the condition of equidecay is not required. When M has an exhausting sequence of compact sets, the proof can easily be obtained from the result when M is compact (See Chapter 5).

Using this theorem, we shall prove the relative compactness of the family of functions $\{u_k\}$ obtained by the scaling transformation (1.14) of the solution of the heat equation. To prove the asymptotic formula (1.15) it is enough to consider the behavior for large k, so we set $k \geq 1$.

1.3.3 Relative Compactness of a Family of Scaled Functions

Proposition. *Let $M = \mathbb{R}^n \times [\eta, 1/\eta]$ for $\eta \in (0,1)$. Assume that the solution u of the heat equation with initial data $f \in C_0(\mathbb{R}^n)$ is given by (1.3). Let u_k be defined by (1.14). Then $K = \{u_k : k \geq 1\}$ is relatively compact in $C_\infty(M)$.*

Proof. First we need to show that $K \subset C_\infty(M)$. Since u_k is continuous in M, $K \subset C(M)$ is clear. In part (iii) of the following proof, we will show that u_k belongs to $C_\infty(M)$.

If we show that K is bounded, equicontinuous, and if it has the equidecay property, the claim follows from the compactness theorem in §1.3.2. Here, we prove these using the L^p-L^q estimate obtained by the representation formula of the solution. Actually, as we will mention in §2.3, this type of estimate can be obtained through integration by parts without using the representation formula of the solution. In fact, with a larger constant in the right-hand side of (1.5), we are able to prove (1.5) by a method presented in §2.3.

(i) **Boundedness**. Using the decay estimate (1.4) of the solution,

$$\|u\|_\infty(t) \leq \frac{1}{(4\pi t)^{n/2}} \|f\|_1, \tag{1.a}$$

we obtain

$$\|u_k\|_\infty(t) = \sup_{x \in \mathbb{R}^n} k^n |u(kx, k^2 t)| = k^n \sup_{z \in \mathbb{R}^n} |u(z, k^2 t)|$$
$$\leq k^n \frac{1}{(4\pi k^2 t)^{n/2}} \|f\|_1 = \frac{1}{(4\pi t)^{n/2}} \|f\|_1.$$

If $(x,t) \in M$ such that $t \geq \eta$, then

$$\|u_k\|_{\infty,M} = \sup_{(x,t)\in M} |u_k(x,t)| \leq \frac{1}{(4\pi\eta)^{n/2}} \|f\|_1.$$

Since the right-hand side is independent of $k > 0$, K is bounded in $C_\infty(M)$.

(ii) **Equicontinuity**. We use the L^p-L^q estimate for derivatives of solutions (1.6) and (1.7) with $p = \infty$ and $q = 1$:

$$\|\partial_{x_j} u\|_\infty(t) \leq \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} \|f\|_1, \quad (j = 1, \dots, n), \tag{1.b}$$

$$\|\partial_t u\|_\infty(t) \leq \frac{C}{t^{\frac{n}{2}+1}} \|f\|_1. \tag{1.c}$$

Here, C is a constant depending only on the dimension n. Similarly to the proof of (i), we obtain

$$\begin{aligned} \|\partial_{x_j} u_k\|_\infty(t) &= k^{n+1} \sup_{x\in\mathbb{R}^n} |(\partial_{x_j} u)(kx, k^2 t)| \\ &\leq \frac{Ck^{n+1}}{(k^2 t)^{\frac{n}{2}+\frac{1}{2}}} \|f\|_1 = \frac{C}{t^{\frac{n}{2}+\frac{1}{2}}} \|f\|_1. \end{aligned}$$

Since $(\partial_{x_j} u_k)(x,t) = k \cdot k^n (\partial_{x_j} u)(kx, k^2 t)$, the power of k increases by one compared with (i). Similarly, we have

$$\|\partial_t u_k\|_\infty(t) \leq \frac{C}{t^{\frac{n}{2}+1}} \|f\|_1.$$

By the last two inequalities we observe that

$$\|\partial_{x_j} u_k\|_{\infty,M} = \sup_{(x,t)\in M} |\partial_{x_j} u_k(x,t)|$$

and

$$\|\partial_t u_k\|_{\infty,M} = \sup_{(x,t)\in M} |\partial_t u_k(x,t)|$$

are estimated by a constant L that is independent of k. Using the integral form of the mean value theorem (§1.1.6) for (y,s), $(x,t) \in M$, we have

$$|u_k(y,s) - u_k(x,t)| \leq L(n+1)^{1/2} (|y-x|^2 + |t-s|^2)^{1/2},$$

which implies

$$\lim_{\substack{y\to x \\ s\to t}} \sup_{k\geq 1} |u_k(y,s) - u_k(x,t)| = 0.$$

Thus we obtain the equicontinuity of K. We note that $(n+1)^{1/2}$ in the previous inequality comes from the following calculation: the Euclidean norm $(\sum_{i=0}^n p_i^2)^{1/2}$ of $p = (p_0, \dots, p_n)$ is estimated by

$$\left(\sum_{i=0}^{n} p_i^2\right)^{1/2} \le \left(\sum_{i=0}^{n} L^2\right)^{1/2} = L(n+1)^{1/2},$$

provided that $\max_{1\le i\le n} |p_i| \le L$.

(iii) **Equidecay property**. We note that u_k is the solution of the heat equation with initial data

$$f_k(x) = k^n f(kx), \qquad x \in \mathbb{R}^n,$$

and $\|f_k\|_1 = \|f\|_1$. (In fact, noticing this property, the estimates of u_k in (i) and (ii) immediately follow from the derivative L^∞-L^1 estimate in §1.1.3.)

We define the *support* of f by

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}},$$

where the "overline" represents the set's closure. Since $f \in C_0(\mathbb{R}^n)$, taking an open ball B_{j_0} centered at the origin in $\mathbb{R}^n$ with sufficiently large radius $j_0 > 0$, we have $\operatorname{supp} f \subset B_{j_0}$, so that $\operatorname{supp} f_k \subset B_{j_0}$ for $k \ge 1$. Using the decay estimate (1.d) with respect to the space direction proved in §1.3.4 for $j \ge j_0, k \ge 1$, we have

$$\sup_{|x|\ge j} |u_k(x,t)| \le \frac{\|f_k\|_1}{(4\pi\eta)^{n/2}} \sup_{|x|\ge j} \exp\left(-\frac{\eta}{4}|x|^2 + \eta\frac{j_0}{2}|x|\right)$$

for $\eta \le t \le 1/\eta$. Since $\|f_k\|_1 = \|f\|_1$, and since $|x|^2 - 2j_0|x| \ge |x|^2/3$ for $|x| \ge j \ge 3j_0$, we obtain

$$\lim_{j\to\infty} \sup_{k\ge 1} \sup_{|x|\ge j} \sup_{\eta\le t\le 1/\eta} |u_k(x,t)| = 0,$$

which yields that K has the equidecay property. Here we observe that the essential part is to show that $\sup_{|x|\ge j} \sup_{\eta\le t\le 1/\eta} |u_k(x,t)|$ is bounded by a sequence of positive numbers that is independent of k and converges to zero as $j \to \infty$. In particular, each u_k belongs to $C_\infty(M)$. Finally, we remark that the condition $k \ge 1$ is used only in (iii). □

Remark. In the proof we have invoked estimates of a solution (1.a), (1.b), (1.c), and (1.d). We emphasize that once these estimates (with possibly larger constant C) are obtained in some way, the representation formula of the solution is unnecessary in order to prove Proposition 1.3.3. As we have noted, the aim of the latter part of this chapter is to give a proof for asymptotic formula (1.9) by a method not based on the representation formula of the solution directly, but based on scaling transformation. However, for estimates (1.a), (1.b), (1.c), and (1.d), we have cited (1.4), (1.6), (1.7), and Proposition 1.3.4, respectively, which are proved in this chapter using the representation formula of the solution, since we would like to avoid complicating the proof.

Of course such estimates can also be obtained without using the representation formula, and some of them are proved in the following chapters. We need some decay assumption at space infinity so that one can carry out integration by parts. Here are strategies to derive such estimates without applying the representation formula.

(a) An L^∞-L^1 estimate like (1.a) used in the proof (i) can also be obtained in §2.3.1 just by applying integration by parts. There, the estimates are stated only for two-dimensional space, but it is easily extended to general dimensions (see Exercise 2.7).

(b) The estimate (1.b) can be obtained by combining three estimates:
The L^2-L^1 estimate $\|u\|_2(t/3) \le \frac{C_1}{t^{n/4}}\|f\|_1$,
The spatial derivative L^2 estimate $\|\nabla u\|_2(2t/3) \le \frac{C_2}{t^{1/2}}\|u\|_2(t/3)$,
The L^∞-L^2 estimate $\|\partial_{x_j} u\|_\infty(t) \le \frac{C_3}{t^{n/4}}\|\partial_{x_j} u\|_2(2t/3)$.
These estimates can be obtained without using the representation formula. The above L^2-L^1 estimate can be obtained by the extended result (see Exercise 2.7) of §2.3.1. To get the spatial derivative L^2 estimate, the reader is referred to Exercise 2.8 (ii). Since $\partial_{x_j} u$ solves the heat equation with initial data $\partial_{x_j} u(2t/3)$, the last inequality follows from the extended result (see Exercise 2.7) of §2.3.1.

(c) Similarly as in (b), the estimate (1.c) follows by Exercise 2.8 (iii) instead of (ii). Here we invoked the property $\partial_t u = \Delta u$.

(d) If we use the method of §2.4.3, we are able to prove an estimate that is weaker than (1.d) but is still enough to deduce the equidecay property.

1.3.4 Decay Estimates in Space Variables

Proposition. *Let u be the solution of the heat equation given in (1.3) with initial data $f \in C_0(\mathbb{R}^n)$. Assume that there exists an open ball B_{j_0} centered at the origin with radius $j_0 > 0$ such that* $\operatorname{supp} f \subset B_{j_0}$. *Then for $\eta \in (0,1)$,*

$$|u(x,t)| \le \frac{\|f\|_1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{\eta}{4}|x|^2 + \eta\frac{j_0}{2}|x|\right), \quad x \in \mathbb{R}^n,\ \eta \le t \le 1/\eta, \tag{1.d}$$

holds.

Proof. We have

$$|u(x,t)| \le \sup_{|y|\le j_0} g(x-y,t)\|f\|_1$$

by estimating the representation of the solution

$$u(x,t) = \int_{B_{j_0}} g(x-y,t)f(y)\,dy = \int_{|y|\le j_0} g(x-y,t)f(y)\,dy$$

with the Gauss kernel g. Since

$$|x-y|^2 \geq (|x|-|y|)^2 = |x|^2 + |y|^2 - 2|x||y| \geq |x|^2 - 2|x|j_0$$

for $|y| \leq j_0$, if $\eta \leq t \leq 1/\eta$, we obtain

$$\sup_{|y|\leq j_0} g(x-y,t) \leq \frac{1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{\eta}{4}(|x|^2 - 2|x|j_0)\right),$$

which yields the desired estimate. □

Remark.

(i) Assume that $f \in C(\mathbb{R}^n)$ is an integrable function, i.e., we have $\|f\|_1 = \int_{\mathbb{R}^n} |f(x)|\,dx < \infty$. Moreover, assume that the solution u of the heat equation with the initial value f is given by (1.3). Then, for $j_0 > 0$ and $\eta \in (0,1)$, we have

$$\begin{aligned} |u(x,t)| \leq & \frac{\|f\|_1}{(4\pi\eta)^{n/2}} \exp\left(-\frac{\eta}{4}|x|^2 + \eta\frac{j_0}{2}|x|\right) \\ & + \frac{\|f\|_1}{(4\pi\eta)^{n/2}} \int_{|y|>j_0} |f(y)|\,dy, \quad x \in \mathbb{R}^n, \quad \eta < t < 1/\eta. \end{aligned}$$

One can similarly prove this inequality by dividing the domain of integration $\mathbb{R}^n$ into $|y| > j_0$ and $|y| \leq j_0$. The same conclusion is still valid even if f is merely Lebesgue integrable without assuming the continuity of f.

(ii) The same conclusion in Proposition 1.3.3 still holds under the assumption of (i). The boundedness and the equicontinuity can be derived from estimates (1.4), (1.6), and (1.7), which can be shown under the assumption of finiteness of $\|f\|_1$. To show that the set $\{u_k : k \geq 1\}$ has the equidecay property, we may use the inequality in (i) instead.

1.3.5 Existence of Convergent Subsequences

Theorem. *Let u_k be as in Proposition 1.3.3. Then for any subsequence $\{u_{k(\ell)}\}_{\ell=1}^\infty$ of $\{u_k : k \geq 1\}$, there exists a subsequence $\{u_{k(\ell(i))}\}_{i=1}^\infty$ of $\{u_{k(\ell)}\}_{\ell=1}^\infty$ satisfying the following properties:*

(i) *The sequence $\{u_{k(\ell(i))}\}_{i=1}^\infty$ converges to a continuous function U as $i \to \infty$ in $\mathbb{R}^n \times (0,\infty)$ pointwise.*

(ii) *For each $\eta \in (0,1)$, the convergence* (i) *for $\{u_{k(\ell(i))}\}_{i=1}^\infty$ is uniform in $\mathbb{R}^n \times [\eta, 1/\eta]$.*

Hereinafter, for simplicity of notation, $u_{k(\ell)}$ and $u_{k(\ell(i))}$ are denoted by $u_{k'}$ and $u_{k''}$, respectively. At a glance, this is obvious by the result in §1.3.3, but we should be careful, since the choice of the subsequence should be independent of

η. By Proposition 1.3.3, for each $\eta \in (0,1)$, $\{u_{k(\ell)}\}_{\ell=1}^{\infty}$ contains a subsequence converging uniformly in $\mathbb{R}^n \times [\eta, 1/\eta]$. Using the following lemma, we can choose a subsequence of $\{u_{k(\ell)}\}_{\ell=1}^{\infty}$ that converges on $\mathbb{R}^n \times [\eta, 1/\eta]$ uniformly and is independent of η. Since the uniform limit of a sequence of continuous functions is continuous, the limit U of $u_{k''}$ is continuous.

1.3.6 Lemma

Lemma. *Let $\{h_\ell\}_{\ell=1}^{\infty}$ be a sequence of functions that is defined on a set Y. Assume that $\{Y_j\}_{j=1}^{\infty}$ is an exhausting sequence of subsets of Y satisfying $\cup_{j=1}^{\infty} Y_j = Y$. Set $h_\ell^0 = h_\ell (\ell = 1, 2, 3, \dots)$. Let $\{h_\ell^j\}_{\ell=1}^{\infty}$ be a uniformly convergent subsequence of $\{h_\ell^{j-1}\}_{\ell=1}^{\infty}$ $(j \geq 1)$ in Y_j. Then there exists a subsequence $\{h_{\ell'}\}$ of $\{h_\ell\}$ such that $\{h_{\ell'}\}$ converges uniformly in each Y_j.*

The proof is based on a diagonal argument (see §5.2.2 and §5.2.4). In fact, $\{h_\ell^\ell\}_{\ell=1}^{\infty}$, which is a subsequence of $\{h_\ell\}$, converges uniformly in each Y_j.

We apply this lemma to the proof of §1.3.5 with $Y = \mathbb{R}^n \times (0, \infty)$, $Y_j = \mathbb{R}^n \times [1/(j+1), j+1]$, $\{h_\ell\} = \{u_{k'}\}$. Since by §1.3.3 the assumption of the lemma is satisfied, the result in §1.3.5 is proved.

Thus we have proved the compactness part which is the first step in §1.2.7.

1.4 Characterization of Limit Functions

We shall derive an equation that U satisfies, where U is a limit of a convergent subsequence of the family of rescaled functions $\{u_k : k \geq 1\}$ constructed by (1.14) from the solution u of the heat equation. To simplify descriptions, we use the following standard notation for families of functions. Let D be an open set in $\mathbb{R}^n$, and $r = 0, 1, 2, 3, \dots$. By $C^r(D)$ we denote the set of all (real-valued) functions of class C^r on D:

$$C^r(D) = \{\varphi \in C(D) : \partial_x^\alpha \varphi \in C(D) \text{ for a multi-index } \alpha \text{ satisfying } |\alpha| \leq r\}.$$

By definition $C^0(D) = C(D)$, i.e., $C^0(D)$ is the set of all continuous functions on D. If a function belongs to $C^r(D)$ for all r we say that it is a *smooth function* on D or of class C^∞. By $C^\infty(D)$ we denote the set of all such functions, i.e.,

$$C^\infty(D) = \bigcap_{r=1}^{\infty} C^r(D).$$

For the closure $\overline{D}$ of D, we define

$$C^r(\overline{D}) = \left\{ \varphi \in C(\overline{D}) : \begin{array}{c} \partial_x^\alpha \varphi \text{ is continuous on } D \\ \text{and can be extended continuously to } \overline{D} \\ \text{for every multi-index } \alpha \text{ with } |\alpha| \leq r \end{array} \right\},$$

$$C^\infty(\overline{D}) = \bigcap_{r=1}^{\infty} C^r(\overline{D}).$$

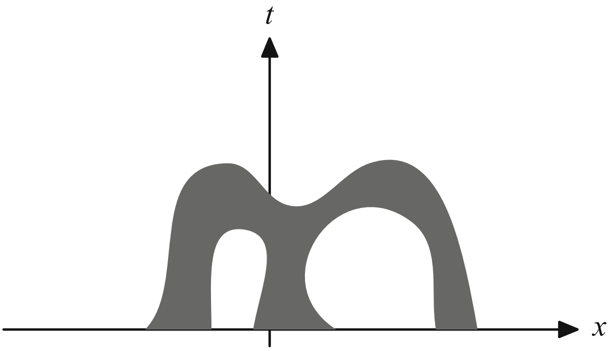

Figure 1.3. An example of a compact set in $\mathbb{R} \times [0, \infty)$.

If a function belongs to $C^r(\overline{D})$ we say that it is of class C^r on $\overline{D}$. If a function belongs to $C^\infty(\overline{D})$, we say that it is of class C^∞ on $\overline{D}$. (Here, a function Φ defined on a set W with $Y \subset W$ is called an *extension* of a function φ on Y if $\Phi(y) = \varphi(y)$ for all $y \in Y$.) For any subset Y in $\mathbb{R}^n$, by $C_0(Y)$ we denote the set of all continuous functions with compact support. We also define

$$C_0^\infty(\overline{D}) = C_0(\overline{D}) \cap C^\infty(\overline{D}), \quad C_0^\infty(D) = C_0(D) \cap C^\infty(D).$$

When $Y = \mathbb{R}^n$, $C_0(Y)$ agrees with $C_0(\mathbb{R}^n)$. If Y is an open set in $\mathbb{R}^n$ and a set Z contains Y, we can identify $C_0^\infty(Y)$ as a subset of $C_0^\infty(Z)$, since a function f in $C_0^\infty(Y)$ can be regarded as a function $C_0^\infty(Z)$ by setting $f(x) = 0$ for $x \in Z \setminus Y$. We use a similar identification for C_0. However, $C_0^\infty(Y)$ does not coincide with $C_0^\infty(Z)$. For example, $C_0^\infty(\mathbb{R}^n \times (0, \infty))$ does not coincide with $C_0^\infty(\mathbb{R}^n \times [0, \infty))$. See Figure 1.3. One should carefully distinguish between $[0, \infty) = \{t \geq 0\}$ and $(0, \infty) = \{t > 0\}$.

Now we study a limit of $f_k(x) = k^n f(kx)$ as $k \to \infty$, which is the initial data of u_k.

1.4.1 Limit of the Initial Data

Proposition. *For $f \in C_0(\mathbb{R}^n)$ we set $f_k(x) = k^n f(kx)$, $k \geq 1$. For any continuous function ψ on $\mathbb{R}^n$ (i.e., $\psi \in C(\mathbb{R}^n)$),*

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} f_k(x)\psi(x)\,dx = m\psi(0), \quad m = \int_{\mathbb{R}^n} f(x)\,dx,$$

holds. (The same convergence result still holds even if f is simply (Lebesgue) integrable in $\mathbb{R}^n$ and for any bounded $\psi \in C(\mathbb{R}^n)$.)

Remark. When $f \in C_0(\mathbb{R}^n)$, the value of the first integral in the proposition does not change if the domain of integration $\mathbb{R}^n$ is replaced by an open ball B_R such that $\operatorname{supp} f \subset B_R$, since $k \geq 1$. Of course, we may assume that $\psi \in C(B_R)$.

This proposition is easily proved in a similar way as it is proved that $u(x,t)$ defined by (1.3) converges to $f(x)$ as $t \to 0$, and we shall prove it in §4.2.5. In other words, f_k converges to the m multiple of the Dirac δ distribution (δ measure) in the sense of measures (or distributions). The Dirac δ distribution can be regarded as the map that evaluates a continuous function at $x = 0$, i.e.,

$$\delta : f \mapsto f(0)$$

for $f \in C(\mathbb{R}^n)$. When the initial data is not a function as in this case, how do we interpret the initial condition?

1.4.2 Weak Form of the Initial Value Problem for the Heat Equation

Multiplying the heat equation $\partial_t u - \Delta u = 0$ by $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, and then integrating on $\mathbb{R}^n \times (0,\infty)$, we obtain

$$0 = \int_0^\infty \int_{\mathbb{R}^n} \varphi(\partial_t u - \Delta u)\, dx\, dt.$$

Using integration by parts (§4.5.3), for $u \in C^\infty(\mathbb{R}^n \times [0,\infty))$ we have

$$\begin{aligned}\int_0^\infty \varphi \partial_t u\, dt &= [\varphi(x,t)u(x,t)]_{t=0}^\infty - \int_0^\infty u \partial_t \varphi\, dt \\ &= -\varphi(x,0)u(x,0) - \int_0^\infty u\partial_t\varphi\, dt,\\ \int_{\mathbb{R}^n} \varphi \Delta u\, dx = - \int_{\mathbb{R}^n} &\langle \nabla\varphi, \nabla u\rangle\, dx = \int_{\mathbb{R}^n} (\Delta\varphi) u\, dx,\end{aligned}$$

which yields[1]

$$0 = -\int_{\mathbb{R}^n} \varphi(x,0)u(x,0)\, dx - \int_0^\infty \int_{\mathbb{R}^n} (\partial_t\varphi + \Delta\varphi)u\, dx\, dt. \tag{1.16}$$

We do not carry out the justification of commutation of integrals in this chapter; in fact, the last equality is justified by Fubini's theorem in §7.2.2. Of course, for $u \in C^\infty(\mathbb{R}^n \times [0,\infty))$ it is sufficient to consider the Riemann integral. Here, by definition, φ is zero for large t and for large $|x|$; however, we note that $\varphi(x,0)$ may not be identically zero (but belongs to $C_0^\infty(\mathbb{R}^n)$). Conversely, if u is smooth in $\mathbb{R}^n \times (0,\infty)$ and continuous in $\mathbb{R}^n \times [0,\infty)$ (i.e., $u \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,\infty))$, and (1.16) holds for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, then u is a solution of the heat equation with initial data $u(x,0)$ (Exercise 1.8). Thus, we define weak solutions of the heat equation as follows.

[1] The equation (1.16) also holds if u is continuous in $\mathbb{R}^n \times [0,\infty)$ and smooth in $\mathbb{R}^n \times (0,\infty)$. In this case, we do not assume the continuity of $\partial_t u$ at $t = 0$; hence $\int_0^\infty \varphi\partial_t u\, dt$ is not necessarily finite. So, we replace the time interval of integration $(0,\infty)$ to (ε,∞) $(\varepsilon > 0)$ and integrate by parts then we obtain (1.16) by letting $\varepsilon \to 0$.

1.4.3 Weak Solutions for the Initial Value Problem

Definition. *Assume that u is locally integrable in $\mathbb{R}^n \times [0, \infty)$.*

(i) *Assume that f is locally integrable in $\mathbb{R}^n$. A function u is called a weak solution of the heat equation (1.1) with initial data f if for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0, \infty))$,*

$$0 = \int_{\mathbb{R}^n} \varphi(x, 0) f(x)\, dx + \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) u\, dx\, dt. \tag{1.17}$$

(ii) *Instead of (1.17), if*

$$0 = m\varphi(0, 0) + \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) u\, dx\, dt \tag{1.18}$$

holds for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0, \infty))$, then u is called a weak solution of the heat equation (1.1) with initial data $m\delta$ (m times the δ distribution). Here m is a real number.

We shall give a general definition of local integrability of functions in §4.1.1. We here describe the notion for the above functions u and f in the following way. First we remark that the function u defined in $\mathbb{R}^n \times [0, \infty)$ is *locally integrable* in $\mathbb{R}^n \times [0, \infty)$ if for any positive number R and T,

$$I(R, T) = \int_0^T \int_{|x| \le R} |u(x, t)|\, dx\, dt < \infty.$$

Of course, if $u \in C(\mathbb{R}^n \times [0, \infty))$, then u is locally integrable in $\mathbb{R}^n \times [0, \infty)$. If $u \in C(\mathbb{R}^n \times (0, \infty))$, u is not assumed to be continuous up to $t = 0$, so that u is not necessarily bounded in $B_R \times (0, T)$ hence $I(R, T)$ is not necessarily finite. We can interpret $I(R, T)$ as an improper Riemann integral for such functions. Weak solutions that appear in this book belong to $C(\mathbb{R}^n \times (0, \infty))$. We also note that f is locally integrable in $\mathbb{R}^n$ if and only if for any positive number $R > 0$ the integral of $|f|$ over B_R is finite.

Of course, by the arguments in §1.4.2, a classical solution[1] of (1.1) with initial data $f \in C_0(\mathbb{R}^n)$ is a weak solution. More generally, when the initial data f is a Radon measure μ, one obtains a definition of a weak solution with initial data μ by replacing the right-hand side of (1.17) by $\int_{\mathbb{R}^n} \varphi(x, 0)\, d\mu(x)$. From this point of view we can interpret (i) and (ii) synthetically by regarding $f(x)\, dx$ as $d\mu(x)$. But we wrote statements (i) and (ii) as above to avoid an unnecessarily difficult notion. (For the notion of measures see the book of W. Rudin [Rudin 1987].) In this definition we assume that the function u is locally integrable, so that the second term of the right-hand side of (1.17) is

[1] The function u is called a classical solution if $\partial_t^k \partial_x^\alpha u$ is continuous in $\mathbb{R}^n \times (0, \infty)$, $|\alpha| + 2k \le 2$, and u satisfies (1.1), and moreover, u is continuous in $\mathbb{R}^n \times [0, \infty)$ and satisfies (1.2).

finite; however, it is possible to consider u as a more general distribution. One of the reasons that it is called a weak solution is that we can check whether a function is a solution without assuming differentiability of the function. In §3.1 we mention a problem in which nondifferentiable weak solutions play a key role. But in the case of the heat equation a weak solution is smooth for $t > 0$. The problem is to show the convergence to the initial data.

Next, we would like to characterize the limit of subsequences of the family of rescaled functions $\{u_k : k \geq 1\}$ as $k \to \infty$. For this purpose, in the next theorem we consider a sequence of weak solutions of the heat equation, which is more general than what we need right now.

1.4.4 Limit of a Sequence of Solutions to the Heat Equation

Theorem. *Let $v_i \in C(\mathbb{R}^n \times [0,\infty))$ be a weak solution of the heat equation (1.1) with initial data $v_{i0} \in C(\mathbb{R}^n)$ $(i = 1, 2, \dots)$, and m a real number. Assume the following conditions:*

(i) (*The limit of the initial data*) *For all $\psi \in C_0^\infty(\mathbb{R}^n)$,*

$$\lim_{i\to\infty} \int_{\mathbb{R}^n} v_{i0}\psi \, dx = m\psi(0).$$

(ii) (*Uniform estimate*)

$$\sup_{i\geq 1} \sup_{t>0} \|v_i\|_1(t) < \infty.$$

(iii) (*Convergence*) *The function v_i converges to v in any compact subset of $\mathbb{R}^n \times (0,\infty)$ uniformly as $i \to \infty$. (Hence $v \in C(\mathbb{R}^n \times (0,\infty))$.)*

Then v is a weak solution of the heat equation (1.1) with the initial data $m\delta$.

Proof. Since v_i is a weak solution with initial data v_{i0}, for $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$ we have

$$0 = \int_{\mathbb{R}^n} \varphi(x,0)v_{i0}(x)\, dx + \int_0^\infty \int_{\mathbb{R}^n} (\partial_t\varphi + \Delta\varphi)v_i \, dx\, dt.$$

Since by (i) the first term of the right-hand side converges to $m\varphi(0,0)$ as $i \to \infty$, it is enough to prove that the second term of the right-hand side converges to

$$\int_0^\infty \int_{\mathbb{R}^n} (\partial_t\varphi + \Delta\varphi)v \, dx\, dt$$

as $i \to \infty$ and that v is locally integrable in $\mathbb{R}^n \times [0,\infty)$. First we prove the desired convergence. Set

$$F_i(t) = \int_{\mathbb{R}^n} (\partial_t\varphi(x,t) + \Delta\varphi(x,t))v_i(x,t)\, dx \quad (i = 1, 2, \dots),$$
$$F(t) = \int_{\mathbb{R}^n} (\partial_t\varphi(x,t) + \Delta\varphi(x,t))v(x,t)\, dx.$$

The supports of these functions are contained in an interval $[0, T)$ that is independent of i, since $\varphi(x,t)$ is identically zero as a function of the variable x for sufficiently large t. Moreover, since, by (iii), v_i converges to v uniformly in any compact subset of $\mathbb{R}^n \times (0,\infty)$, we can interchange integration and limit operation (see the proposition at the beginning of §7.1), i.e.,

$$\lim_{i\to\infty} F_i(t) = F(t),$$

at any point $t > 0$. Moreover, F_i and F are continuous on $(0, T)$,

$$|F_i(t)| \le \left(\sup_{\mathbb{R}^n \times (0,\infty)} |\partial_t \varphi + \Delta\varphi| \right) \int_{\mathbb{R}^n} |v_i(x,t)|\, dx,$$

and the right-hand side of this equality is bounded as a function of $t > 0$ and i by the uniform estimate of (ii). Therefore by the dominated convergence theorem (§7.1.1), we have

$$\lim_{i\to\infty} \int_0^T F_i(t)\, dt = \int_0^T F(t)\, dt.$$

On the other hand, since $\operatorname{supp} F_i$, $\operatorname{supp} F \subset [0, T)$, we get

$$\lim_{i\to\infty} \int_0^\infty F_i(t)\, dt = \int_0^\infty F(t)\, dt,$$

which is the desired convergence. By similar arguments, for any $R > 0$ and $T > 0$ we also have

$$\infty > \lim_{i\to\infty} \int_0^T \int_{B_R} |v_i(x,t)|\, dx\, dt = \int_0^T \int_{B_R} |v(x,t)|\, dx\, dt,$$

so that v is locally integrable in $\mathbb{R}^n \times [0,\infty)$. (The interchange of integration and limit operation used above may be proved by the theory of Riemann integration without Lebesgue integration theory.) (Since $v \in C(\mathbb{R}^n \times (0,\infty))$, the local integrability of v on $\mathbb{R}^n \times (0,\infty)$ is obvious. However, we need estimates near $t = 0$ to prove the local integrability on $\mathbb{R}^n \times [0,\infty)$ as mentioned above.) □

1.4.5 Characterization of the Limit of a Family of Scaled Functions

Theorem. *Assume that the solution of the heat equation with initial data $f \in C_0(\mathbb{R}^n)$ is given by (1.3). Let u_k be given by (1.14). Assume that a subsequence $\{u_{k''}\}$ of $\{u_k : k \ge 1\}$ converges to a continuous function U uniformly in each compact subset of $\mathbb{R}^n \times (0,\infty)$ as $k \to \infty$. Then U is a weak solution of the heat equation (1.1) with initial data $m\delta$, where $m = \int_{\mathbb{R}^n} f\, dx$. Moreover,*

$$\sup_{t>0} \|U\|_1(t) \le \|f\|_1. \tag{1.19}$$

Proof. We apply Theorem 1.4.4 as follows. Condition (i) in §1.4.4 follows from §1.4.1, (iii) is contained in the assumptions of §1.4.5, and (ii) is easily proved by the estimate

$$\|u_k\|_1(t) \le \|f_k\|_1 = \|f\|_1, \quad t > 0,$$

which is the L^p-L^q estimate in §1.1.2 with $p = q = 1$. Hence we can apply Theorem 1.4.4, so that U is a weak solution of the heat equation with initial data $m\delta$. By Fatou's lemma (§7.1.2), we obtain

$$\|U\|_1(t) = \int_{\mathbb{R}^n} \lim_{k'' \to \infty} |u_{k''}(x,t)|\, dx \le \varliminf_{k'' \to \infty} \|u_{k''}\|_1(t) \le \|f\|_1,$$

which yields (1.19). Here for a sequence $\{a_j\}_{j=1}^{\infty}$, $\varliminf_{j\to\infty} a_j$ is the *limit inferior*, which is defined by

$$\varliminf_{j\to\infty} a_j = \lim_{j\to\infty} \inf_{k\ge j} a_k.$$

(We remark that in this proof we use integrals only for continuous functions on an unbounded domain in $\mathbb{R}^n$. Therefore, it is sufficient to apply Fatou's lemma (§7.1.4) only for improper Riemann integrals.) □

1.4.6 Uniqueness Theorem When Initial Data is the Delta Function

Theorem. *Assume that the function $v \in C(\mathbb{R}^n \times (0,\infty))$ satisfies*

$$\sup_{t>0} \|v\|_1(t) < \infty.$$

Let m be a real number. Assume that v is a weak solution of the heat equation (1.1) with initial data $m\delta$. Then v is unique and $v = mg$ in $\mathbb{R}^n \times (0,\infty)$, where g is the Gauss kernel.

It is easy to prove that the Gauss kernel g is a weak solution of the heat equation with initial data δ (Exercise 1.9). (In the definition of weak solutions, we do not assume that u is continuous on $\mathbb{R}^n \times [0,\infty)$, but assume that u is locally integrable, so that we can handle g that is not continuous at $t = 0$.) We will prove the uniqueness in §4.4.1. The assumption in the theorem about the boundedness of $\|v\|_1(t)$ is a decay condition of the function v as $x \to \infty$. We can prove the uniqueness under weaker assumptions, but it cannot be removed completely.

A self-similar solution V of the heat equation satisfying $\|V\|_1(1) < \infty$ with initial data $m\delta$ is a weak solution (Exercise 1.10). Since $\|V\|_1(t) = \|V_k\|(1) = \|V\|(t)$ with $k^2 = t$, $t > 0$, we have $V = mg$ by the uniqueness theorem, where $m = \int_{\mathbb{R}^n} V(x,1)\, dx$.

1.4.7 Completion of the Proof of Asymptotic Formula (1.9) Based on Scaling Transformation

By §1.4.5 and §1.4.6, the limit U of $\{u_{k''}\}$ derived in §1.3.5 equals mg with

$$m = \int_{\mathbb{R}^n} f\,dx.$$

Since the limit U is independent of the choice of subsequences of $\{u_k\}$ in §1.3.5, by §1.3.5 and Exercise 1.4, for any $\eta \in (0,1)$, $\{u_k\}$ converges to mg uniformly in $\mathbb{R}^n \times [\eta, 1/\eta]$ as $k \to \infty$. Thus we have (1.15). By Proposition 1.2.6 we obtain the asymptotic formula (1.9). □

Remark. In fact, the asymptotic formula (1.9) still holds under the assumption that $f \in C(\mathbb{R}^n)$ is integrable in $\mathbb{R}^n$. This can be proved using the method of scaling transformation (§1.3.4). (Moreover, if f is assumed to be Lebesgue integrable, then the continuity assumption for f is unnecessary.) It is also possible to prove the asymptotic formula (1.9) for general integrable initial data f by modifying the proof of §1.1.5. Indeed, since

$$|h_\eta(x-y) - h_\eta(x)| \le 2|y|\eta^{1/2}C_1,$$

we conclude that

$$\begin{aligned}
&(4\pi t)^{n/2}|u(x,t) - mg(x,t)| \\
&\le \left(\int_{|y|\le R} + \int_{|y|>R}\right) |h_\eta(x-y) - h_\eta(x)|\,|f(y)|\,dy \\
&\le \int_{|y|\le R} \eta^{1/2}|y|\,|f(y)|\,dy + 2\int_{|y|>R} |f(y)|\,dy.
\end{aligned}$$

The right-hand side is independent of x. We send $t \to \infty$ first and then send $R \to \infty$ to get (1.9). This argument as well as (1.9) is found essentially, for example, in [Cazenave Dickstein Weissler 2003], where the relationship between large-time behavior of a solution of the heat equation and the asymptotic behavior at spatial infinity for the initial data is studied.

1.4.8 Remark on Uniqueness Theorem

In a similar way we can prove the uniqueness result in §1.4.6 when the initial data f of v is integrable. In particular, if $f \in C_0(\mathbb{R}^n)$, then

$$v(x,t) = \int_{\mathbb{R}^n} G_t(x-y)f(y)\,dy$$

is the unique weak solution (with initial data f) satisfying $\sup_{t>0} \|v\|_1(t) < \infty$.

Exercises 1

1.1 (i) (§1.1, §1.2.5) For the Gauss kernel $g(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$, calculate $\partial_t g$, $\partial_{x_i} g$, $\partial_{x_i}\partial_{x_j} g$ $(1 \le i,j \le n)$, and show that $\partial_t g(x,t) - \Delta g(x,t) = 0, x \in \mathbb{R}^n, t > 0$.
(ii) (§1.1) Show that if $f \in C_0(\mathbb{R}^n)$, then $\|f\|_p$ is finite for $1 \le p \le \infty$.

1.2 (§1.1.2) Let f be a function defined on $[0,\infty)$ of the form $f(s) = s^a e^{-s}$ with $a > 0$. Show that f is bounded on $[0,\infty)$ and that it attains its maximum value $(a/e)^a$ at $s = a$.

1.3 (§1.2.6) For a positive number k, set $t = k^2$. Show that

$$\|v_k\|_p(1) = t^{\frac{n}{2}(1-\frac{1}{p})}\|v\|_p(t).$$

Here $1 \le p \le \infty$ and $v_k(x,t) = k^n v(kx, k^2 t)$ with $k > 0$.

1.4 (§1.2.7) Consider a subset $A = \{a_k : k \ge 1\}$ in $\mathbb{R}$. (It is not necessarily a sequence.) Assume that there exists a convergent subsequence $\{a_{k(\ell(i))}\}_{i=1}^{\infty}$ of each subsequence $\{a_{k(\ell)}\}_{\ell=1}^{\infty}$ of the set A with $\lim_{\ell\to\infty} k(\ell) = \infty$. Moreover, its limit α is independent of the choice of the subsequence (independent of the choice of $k(\ell), \ell(i)$). Show that then there exists $\lim_{k\to\infty} a_k$, which equals α. (Here we assume $\lim_{i\to\infty} \ell(i) = \infty$.) (Even if A is merely a subset of a metric space, the same conclusion holds.)

1.5 (§1.3.1) Assume that the metric space M has an exhausting sequence of compact sets $\{M_j\}_{j=1}^{\infty}$. Show that $C_\infty(M)$ is independent of the choice of $\{M_j\}_{j=1}^{\infty}$.

1.6 (§1.3.1) Assume that a metric space M has an exhausting sequence of compact sets. Show that $C_\infty(M)$ is complete with respect to the norm $\|\cdot\|_{\infty,M}$. That is to say, show that any *Cauchy sequence* $\{f_j\}_{j=1}^{\infty}$ of $C_\infty(M)$ converges in $C_\infty(M)$. In other words, show that if

$$\lim_{j\to\infty} \sup_{\ell,m\ge j} \|f_\ell - f_m\|_{\infty,M} = 0,$$

then there exists $f \in C_\infty(M)$ such that $\lim_{j\to\infty} \|f_j - f\|_\infty = 0$.

1.7 (§1.3.1) Prove Example 1 and Example 2. Show that K in Example 1 is not equicontinuous.

1.8 (§1.4.2, §1.4.3) Show that if $u \in C(\mathbb{R}^n \times [0,\infty)) \cap C^\infty(\mathbb{R}^n \times (0,\infty))$ is a weak solution of heat equation with initial data $u(x,0)$, then u is a solution of the heat equation, i.e., u satisfies (1.1).
Hint: For $h \in C(\Omega)$, if

$$\int_\Omega h\varphi\, dx = 0$$

holds for all $\varphi \in C_0^\infty(\Omega)$, then $h \equiv 0$, i.e., h is identically zero on Ω, where Ω is an open set in $\mathbb{R}^m$.

1.9 (§1.4.6) Show that the Gauss kernel g is a weak solution of the heat equation with initial value δ.
Hint: It is sufficient to prove that $v_i(x,t) = g(x, t+1/i)$ satisfies the assumptions in §1.4.4.

1.10 (§1.4.6) Show that a self-similar solution u of the heat equation satisfying $\|u\|_1(1) < \infty$ is a weak solution of the heat equation with initial data $m\delta$, where $m = \int_{\mathbb{R}^n} u(x,1)\,dx$.
Hint: Set $v_i(x,t) = u(x, t+1/i)$.

Remark (For 1.8). We recall here the fundamental lemma of the calculus of variations, which is a more general result than that of the hint.

Lemma. *Let h be a locally integrable function in an open set Q in $\mathbb{R}^n$. If*

$$\int_Q h\varphi\,dx = 0$$

for all $\varphi \in C_0^\infty(Q)$, then h is zero almost everywhere in Q.

This lemma shows that a locally integrable function h that is zero in the sense of distributions is identically zero *almost everywhere* in Q, i.e., h equals zero outside some set of Lebesgue measure zero in Q.

2

Behavior Near Time Infinity of Solutions of the Vorticity Equations

The Navier–Stokes equations are famous as fundamental equations of fluid mechanics and have been well studied as typical nonlinear partial differential equations in mathematics. It is not too much to say that various mathematical methods for analyzing nonlinear partial differential equations have been developed through studies of the Navier–Stokes equations. There have been many studies of the behavior of solutions of the Navier–Stokes equations near time infinity. In this chapter, as an application of the previous section, we study the behavior of the vorticity of a two dimensional flow near time infinity. In particular, we study whether or not the vorticity converges to a self-similar solution.

The main purpose of this chapter is to show that the vorticity of a two-dimensional flow asymptotically converges to a constant multiple of the Gauss kernel (called the *Gaussian vortex*, which is self-similar) if the total circulation is sufficiently small. This result is applicable (as mentioned in §2.6) to the problem of the formation of the Burgers vortex in a three-dimensional flow, which is a very interesting topic in fluid mechanics. (*Very recently, the smallness assumption has been removed. We shall mention this improvement at the end of this chapter.*) This type of asymptotic behavior of the vorticity (§2.2.2) is proved in papers cited in §2.7.1. To estimate a limit of rescaled solutions is an important step in the proof, and it has not been mentioned in the literature so far. In this chapter we will present a new result concerning this step and complete the whole proof. Moreover, we give a clearer proof of the asymptotic formula (§2.4 and §2.5) by introducing recent improvements of the fundamental L^q-L^1 estimate (§2.3) of the linear heat equation with a convective term. The estimates of several quantities, including the derivatives of vorticities and velocities, are established by applying the fundamental L^q-L^1 estimate, in which various fundamental inequalities in calculus (§2.4) play essential roles. These inequalities are proved in Chapter 6. In this chapter, we often rewrite differential equations as integral equations. Such an operation is justified in Chapter 4. The existence and the uniqueness of solutions to the vorticity equations are stated in §2.2.1 without proofs. We admit these results

M.-H. Giga et al., *Nonlinear Partial Differential Equations*,
Progress in Nonlinear Differential Equations and Their Applications 79,
DOI 10.1007/978-0-8176-4651-6_2,

here. When we consider the vorticity equations it is useful to study the heat equation with a convective term, for it is considered a linearized version of the original equations. The existence of solutions to this linearized equation is again admitted in this chapter. Several properties of the fundamental solution to the heat equation with a convective term are presented in Lemma 2.5.2 without proof. They are effectively used to obtain the estimates for the limit of rescaled solutions. Throughout this chapter we try to establish sharp results as elementarily as possible. For example, an elementary proof is presented for the estimates of derivatives of the vorticity (§2.4.2), which gives new results for the cases $p = 1$ and $p = \infty$. In §2.1, we derive the vorticity equations from the Navier–Stokes equations, and in §2.7, we mention the history of research on the vorticity equations and related topics. This chapter intends to give an elementary approach without Lebesgue integrals or distribution theory, so the only prerequisite to reading it is a basic knowledge of differential and integral calculus for functions of several variables. For this reason some assumptions of the results are not optimal.

2.1 Navier–Stokes Equations and Vorticity Equations

We consider the *Navier–Stokes equations*, which are used to model the motion of incompressible viscous flows and which are the fundamental equations of fluid mechanics:

$$\rho_0 \left\{ \frac{\partial u^i}{\partial t}(x,t) + \sum_{j=1}^{n} u^j(x,t) \frac{\partial u^i}{\partial x_j}(x,t) \right\} - \nu \sum_{j=1}^{n} \frac{\partial^2 u^i}{\partial x_j^2}(x,t) + \frac{\partial p}{\partial x_i}(x,t) = 0$$

for $x \in \mathbb{R}^n$, $0 < t < T$, and $i = 1, 2, \ldots, n$, and

$$\sum_{j=1}^{n} \frac{\partial u^j}{\partial x_j}(x,t) = 0$$

for $x \in \mathbb{R}^n$ and $0 < t < T$.

Here we assume $T > 0$ or $T = \infty$, and n denotes an integer greater than or equal to 2. The vector

$$u(x,t) = (u^1(x,t),\ u^2(x,t), \ldots, u^n(x,t))$$

denotes the velocity vector of the fluid at a point $x \in \mathbb{R}^n$ and at time $t \in (0, T)$; $p(x,t)$ denotes the pressure of the fluid. Of course, $u^i(x,t)$ $(i = 1, \ldots, n)$ and $p(x,t)$ are real-valued functions, and ρ_0 and ν are positive constants that describe the density and the kinematic viscosity of the fluid, respectively. We note that the above system is the Navier–Stokes equations with no external force term. For given ρ_0, ν, and the initial velocity $u(x,0)$, the problem to find u and p satisfying the above Navier–Stokes equations is called the *initial* *value problem* for the Navier–Stokes equations. Here we have $n+1$ equations and $n+1$ unknown functions. Observe that by assuming an initial condition also for the pressure, the conditions are overdetermined and we cannot solve the initial value problem. Hence, we do not assign the initial value of the pressure. In physics one often adds the word "field" to describe physical quantities depending on x. For example, u is called the velocity vector field and p is called the pressure field. However, in this book we do not use this terminology.

We often express the Navier–Stokes equations in a concise form using notation of vector analysis:

$$\begin{aligned} \rho_0\{\partial_t u + (u,\nabla)u\} - \nu\Delta u + \nabla p = 0 &\quad \text{in } \mathbb{R}^n \times (0,T), \\ \operatorname{div}\ u = 0 &\quad \text{in } \mathbb{R}^n \times (0,T). \end{aligned}$$

Here, div and ∇ denote the divergence and the gradient with respect to the spatial variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, respectively. Moreover, (u, ∇) denotes the operator $\sum_{j=1}^n u^j \frac{\partial}{\partial x_j}$, and we assume that it acts on each element u^i of u. Namely, the ith component of $(u,\nabla)u$ is $\sum_{j=1}^n u^j \frac{\partial u^i}{\partial x_j}$; and the Laplacian Δu for the vector-valued function u is $(\Delta u^1, \Delta u^2, \ldots, \Delta u^n)$. The first equation describes the momentum conservation law, and the second equation describes the mass conservation law, which expresses the incompressibility.

Using a suitable scaling transformation for the dependent variables u and p, and independent variables x and t, we may assume that $\rho_0 = 1$, and $\nu = 1$. In fact, for example, if we set $\tilde{t} = (\rho_0\nu)^{-1/3}t$, $\tilde{x} = (\rho_0/\nu^2)^{1/3}x$, $\tilde{u} = (\rho_0^2/\nu)^{1/3}u$, and $\tilde{p} = (\rho_0/\nu^2)^{1/3}p$, then we obtain (at least formally) the Navier–Stokes equations for $\tilde{u}(\tilde{x},\tilde{t})$ and $\tilde{p}(\tilde{x},\tilde{t})$ on $\mathbb{R}^n \times (0,\tilde{T})$ with $\rho_0 = 1$ and $\nu = 1$. (We may obtain the Navier–Stokes equations with $\rho_0 = 1$ and $\nu = 1$ also by another transformation.) Here we set $\tilde{T} = (\rho_0\nu)^{-1/3}T$. Thus we assume that the positive constants ρ_0 and ν are 1, unless otherwise claimed. That is, we consider

$$\partial_t u + (u,\nabla)u - \Delta u + \nabla p = 0 \quad \text{in } \mathbb{R}^n \times (0,T), \tag{2.1}$$

$$\operatorname{div}\ u = 0 \quad \text{in } \mathbb{R}^n \times (0,T). \tag{2.2}$$

2.1.1 Vorticity

Let a set $v = (v^1, v^2, \ldots, v^n)$ of functions v^i $(i = 1, 2, \ldots, n)$ be an n-dimensional (real) vector-valued function defined on $\mathbb{R}^n$, namely, a *vector field* on $\mathbb{R}^n$. (Here and hereinafter, we simply call v a function, or $\mathbb{R}^n$*-valued function*, if we need to emphasize that v is vector-valued. Let curl be the differential operator that represents the *rotation*. (It is also expressed as rot.) That is, for spatial variables $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and for v whose component v^i is C^1 on $\mathbb{R}^n$, we define

$$\operatorname{curl}\ v = \begin{cases} \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}, & \text{if } \ n = 2, \\ \left(\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}, \ \frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1}, \ \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2} \right), & \text{if } \ n = 3. \end{cases}$$

If v denotes the velocity, then $\operatorname{curl} v$ is called the *vorticity*. In case of spatial dimension $n = 3$, if $v^3 \equiv 0$ and (v^1, v^2) depends only on (x_1, x_2), so that $v = (v^1(x_1, x_2), v^2(x_1, x_2), 0)$, then

$$\operatorname{curl} v = \left(0, 0, \ \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}\right).$$

Thus we may identify the third component with $\operatorname{curl}(v^1, v^2)$ for $n = 2$.

Next we consider $n = 3$ and $v = (0, 0, \varphi)$. Here we assume that $\varphi = \varphi(x_1, x_2)$ depends only on x_1 and x_2 (is independent of x_3) and that φ is continuously differentiable. In this case

$$\operatorname{curl} v = \left(\frac{\partial \varphi}{\partial x_2}, \ -\frac{\partial \varphi}{\partial x_1}, 0\right),$$

and we may identify this by $\nabla^\perp \varphi$. Here, we define the differential operator $\nabla^\perp$ by $\nabla^\perp \varphi = \left(\frac{\partial \varphi}{\partial x_2}, \ -\frac{\partial \varphi}{\partial x_1}\right)$. By definition, $\langle \ \nabla^\perp \varphi, \ \nabla \varphi \ \rangle = 0$, namely, $\nabla^\perp \varphi$ is perpendicular to $\nabla \varphi$, so we use the notation $\nabla^\perp$.

In the following, we explain a convenient formula for deriving the vorticity equations. The proofs are left to the reader as Exercise 2.1.

2.1.2 Vorticity and Velocity

Proposition. *Assume that $n = 2$ or $n = 3$. Let $v = (v^1, v^2, \ldots, v^n)$ be a vector field on $\mathbb{R}^n$. Assume that its components v^j $(j = 1, 2, \ldots, n)$ are continuous up to second order derivatives (i.e., $v^j \in C^2(\mathbb{R}^n)$). Then for $n = 3$ we have*

$$-\Delta v = \operatorname{curl}\operatorname{curl} v - \nabla \ \operatorname{div} \ v \qquad \text{in } \mathbb{R}^3; \tag{2.3.1}$$

for $n = 2$ we have

$$-\Delta v = \nabla^\perp \operatorname{curl} v - \nabla \ \operatorname{div} \ v \qquad \text{in } \mathbb{R}^2. \tag{2.3.2}$$

We assume that the velocity v and the pressure p (in the Navier–Stokes equations) are sufficiently smooth, and we write the vorticity as $\omega(x, t) = \operatorname{curl} u(x, t)$. For $n = 3$ vorticity ω is an $\mathbb{R}^3$-valued function; for $n = 2$ vorticity ω is a scalar real-valued function. By the above proposition and (2.2) we see that $-\Delta u(x, t) = \operatorname{curl} \omega(x, t)$ when $n = 3$; $-\Delta u(x, t) = \nabla^\perp \omega(x, t)$ when $n = 2$.

Applying curl to (2.1), for $n = 3$ we obtain

$$\partial_t \omega + (u, \nabla)\omega - (\omega, \nabla)u - \Delta \omega = 0 \qquad \text{in } \mathbb{R}^3 \times (0, T).$$

Here, we have used $\operatorname{curl}((u,\nabla)u) = (u,\nabla)\omega - (\omega,\nabla)u + \omega(\operatorname{div} u)$, $\operatorname{curl}(\Delta u) = \Delta\omega$, and $\operatorname{curl}(\nabla p) = 0$. In the case $n = 2$, using $\operatorname{curl}((u,\nabla)u) = (u,\nabla)\omega$, we obtain

$$\partial_t\omega + (u,\nabla)\omega - \Delta\omega = 0 \quad \text{in } \mathbb{R}^2 \times (0,T)$$

for u satisfying $\operatorname{div} u = 0$.

Hence from the Navier–Stokes equations, we obtain the following equation for the vorticity ω and the velocity u; in the case $n = 3$,

$$\begin{aligned} \partial_t\omega + (u,\nabla)\omega - (\omega,\nabla)u - \Delta\omega = 0 \qquad & \text{in } \mathbb{R}^3 \times (0,T), \\ -\Delta u = \operatorname{curl}\ \omega \qquad & \text{in } \mathbb{R}^3 \times (0,T). \end{aligned}$$

In the case $n = 2$,

$$\partial_t\omega + (u,\nabla)\omega - \Delta\omega = 0 \qquad \text{in } \mathbb{R}^2 \times (0,T), \tag{2.4}$$

$$-\Delta u = \nabla^\perp\omega \qquad \text{in } \mathbb{R}^2 \times (0,T). \tag{2.5}$$

2.1.3 Biot–Savart Law

In the sequel we assume that the dimension of the space is two. We set $E(x) = -\frac{1}{2\pi}\log|x|$, $x \in \mathbb{R}^2, x \neq 0$. This is called the *fundamental solution* of the Laplace operator. The next proposition is proved in §6.3.5.

Proposition. *For $f \in C_0^\infty(\mathbb{R}^2)$, $-\Delta(E * f) = f$ in $\mathbb{R}^2$.*

Here $*$ denotes *convolution*, i.e., $(E * f)(x) = \int_{\mathbb{R}^2} E(x-y)f(y)dy$. We use the same notation for the convolution $a * b = (a * b^1, a * b^2)$ of a scalar function $a = a(x)$ and an $\mathbb{R}^2$-valued function $b = (b^1(x), b^2(x))$ that are defined on $\mathbb{R}^2$.

For a smooth real-valued function w defined on $\mathbb{R}^2$ we set $v = E * (\nabla^\perp w)$. If the support of $\nabla^\perp w$ is compact, then by the above proposition, w satisfies

$$-\Delta v = \nabla^\perp w \quad \text{in } \mathbb{R}^2.$$

Conversely, v satisfying this equation is expressed by $v = E * (\nabla^\perp w)$, under a suitable decay condition on v at space infinity.[1] Thus $v = E * (\nabla^\perp w)$ is formally equivalent to $-\Delta v = \nabla^\perp w$, in this sense.

We define a vector field $\mathbf{K}$ (which is defined on the domain $\mathbb{R}^2$ excluding the origin) as

$$\mathbf{K}(x) = \frac{1}{2\pi}\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right), \quad x \in \mathbb{R}^2, \quad x \neq 0. \tag{2.6}$$

[1] To show this, use the fact that a bounded harmonic function on the whole plane is a constant. This statement is called Liouville's theorem.

Since

$$\nabla^{\perp} E(x) = \left(\frac{\partial E}{\partial x_2}(x), -\frac{\partial E}{\partial x_1}(x) \right) = \mathbf{K}(x), \quad x \in \mathbb{R}^2, \quad x \neq 0,$$

we obtain

$$E * (\nabla^{\perp} w) = (\nabla^{\perp} E) * w = \mathbf{K} * w \quad \text{in } \mathbb{R}^2$$

(at least for $w \in C_0^\infty \mathbb{R}^2$). (For the justification of the commutation of convolution and differential, see §4.1.4 and §6.3.5. See also §6.3.6.) The formula $v = \mathbf{K} * w$ determining v from w is called the *Biot–Savart law*. The function v obtained by this relation satisfies $\operatorname{div} v = \operatorname{div}(\mathbf{K} * w) = \operatorname{div} \nabla^{\perp}(E * w) = 0$ in $\mathbb{R}^2$.

Here and hereinafter we use the symbol $\mathbf{K}$ as defined in (2.6).

2.1.4 Derivation of the Vorticity Equations

We have obtained (2.4) and (2.5) from the two-dimensional Navier–Stokes equations of §2.1.2. Instead of (2.5) we consider the Biot–Savart law, which is formally equivalent to (2.5) (We note that (2.5) has been derived from the mass conservation law (2.2).) So we consider

$$\partial_t \omega + (u, \nabla)\omega - \Delta\omega = 0 \qquad \text{in } \mathbb{R}^2 \times (0, T), \tag{2.7}$$

$$u = \mathbf{K} * \omega \quad \text{in } \mathbb{R}^2 \times (0, T). \tag{2.8}$$

This system is called the two-dimensional *vorticity equations*. For a given function ω_0 on $\mathbb{R}^2$, the problem of finding a real-valued function $\omega = \omega(x, t)$ and an $\mathbb{R}^2$-valued function $u = (u^1(x, t), u^2(x, t))$ satisfying

$$\omega(x, 0) = \omega_0(x), \quad x \in \mathbb{R}^2, \tag{2.9}$$

and the vorticity equations is called the *initial value problem* for the vorticity equations. In this chapter, we analyze the asymptotic behavior of the vorticity near time infinity.

As stated above, the vorticity equations are derived from the Navier–Stokes equations. Conversely, we can also derive the Navier–Stokes equations from the vorticity equations by determining the pressure p suitably. (For example, see [Giga Miyakawa Osada 1988].) Hence the analysis for solutions of the vorticity equations is equivalent to the analysis for solutions of the Navier–Stokes equations.

2.2 Asymptotic Behavior Near Time Infinity

Consider the initial value problem of the vorticity equations (2.7), (2.8), and (2.9) in the plane. Similarly to the heat equation, if we assume that ω_0 does not grow at space infinity, it is known that problem (2.7), (2.8), and (2.9) has a unique global-in-time smooth solution. The existence and uniqueness problem has been well studied in various situations. In this chapter, we consider the problem in the case that the initial vorticity ω_0 is a continuous function with compact support. The existence and the uniqueness problems will be commented on in §2.7.2 together with the research history, but we will not give their proofs. In this chapter we focus on the asymptotic behavior of ω as $t \to \infty$, admitting the following unique existence theorem.

2.2.1 Unique Existence Theorem

Theorem. *For the initial vorticity $\omega_0 \in C_0(\mathbb{R}^2)$ there exists a unique pair of smooth functions (ω, u) satisfying (2.7), (2.8), and (2.9) in $\mathbb{R}^2 \times (0, \infty)$, and having the following properties:*

(i) *We have $\omega \in C(\mathbb{R}^2 \times [0, \infty))$ and ω satisfies the initial condition (2.9). Moreover, $\lim_{t\to 0} \|\omega - \omega_0\|_p(t) = 0$ for any p with $1 \le p \le \infty$.*

(ii) *For any t_0 and t_1 with $0 < t_0 < t_1$, $\sup_{t_0 \le t \le t_1} \|\partial_t^\ell \partial_x^\alpha \omega\|_p(t) < \infty$, where $1 \le p \le \infty$, α is an arbitrary multi-index, and $\ell = 0, 1, 2, \ldots$.*

(iii) *For any t_0 and t_1 with $0 < t_0 < t_1$, $\sup_{t_0 \le t \le t_1} \|\partial_t^\ell \partial_x^\alpha u\|_r(t) < \infty$, where $2 < r \le \infty$, α is an arbitrary multi-index, and $\ell = 0, 1, 2, \ldots$. (For a vector-valued function v, $\|v\|_p$ denotes $\| \, |v| \, \|_p$ and $\partial_t^\ell \partial_x^\alpha v$ denotes the vector with ith component $\partial_t^\ell \partial_x^\alpha v^i$, where v^i denotes the ith component of v.)*

The conditions (ii) and (iii) imply that for each $t > 0$, $\omega(x, t)$ and $u(x, t)$ decay at space infinity as functions of x. Moreover, $\|\omega - \omega_0\|_p(t) \to 0$ (as $t \to 0$) in (i) means the L^p-continuity of the map in t with values $\omega(x, t)$ (which is a function of x). This property is important and is also valid for solutions of the heat equation as mentioned in Exercise 7.3 and Theorem 4.2.1.

Remark. By (2.8) we have $u = \mathbf{K} * \omega$ (in $\mathbb{R}^2 \times (0, \infty)$), but for each $t > 0$, $\omega(x, t)$ is not compactly supported as a function of x. Thus there is a problem as to whether $\mathbf{K} * \omega$ is well defined. Fortunately, as remarked in §6.3.5, the property (ii) of the solution ω is sufficient to define (the components of) $\mathbf{K} * \omega$ as a smooth function on $\mathbb{R}^2 \times (0, \infty)$ satisfying

$$\partial_t^\ell \partial_x^\alpha (\mathbf{K} * \omega) = \mathbf{K} * (\partial_t^\ell \partial_x^\alpha \omega)$$

in $\mathbb{R}^2 \times (0, \infty)$, where α is an arbitrary multi-index and $\ell = 0 ,1 ,2, \ldots$.

Hereinafter, we simply define a *solution* of (2.7), (2.8), and (2.9) to be a pair of smooth solutions (ω, u) that satisfies (2.7), (2.8), and (2.9) on $\mathbb{R}^2 \times (0, \infty)$ and that satisfies properties (i), (ii), and (iii) of the unique existence theorem. The main purpose of this chapter is to establish the asymptotic behavior of ω as $t \to \infty$ for the vorticity equations, which is similar to Theorem 1.1.4. As we will see later, ω decays as $t \to \infty$. Our aim is to obtain the leading part of the decay.

2.2.2 Theorem for Asymptotic Behavior of the Vorticity

Theorem. *Let the pair of functions* (ω, u) *denote the solution of (2.7), (2.8), and (2.9) with initial vorticity* $\omega_0 (\in C_0(\mathbb{R}^2))$*. Furthermore, we set* $m = \int_{\mathbb{R}^2} \omega_0(y)dy$*. Then there exists a (small) constant* m_0 *such that for any* m *with* $|m| < m_0$*,*

$$\lim_{t\to\infty} t\|\omega - mg\|_\infty(t) = 0 \tag{2.10}$$

holds. Here $g(x,t) = G_t(x)$ *is the Gauss kernel.*

Remark. According to very recent results of Th. Gallay and C. E. Wayne [Gallay Wayne 2005], the smallness assumption $|m| < m_0$ can be removed. We shall discuss this topic in §2.8. Thus, the result exactly corresponds to Theorem 1.1.4 with $n = 2$ for the heat equation.

We can prove (2.10) by regarding the term $(u, \nabla)\omega$ of equation (2.7) as a perturbation of the heat equation and using the expression of the solution of the heat equation. However, to carry out this strategy we need the stronger assumption that

$$\|\omega_0\|_1 = \int_{\mathbb{R}^2} |\omega_0(y)|dy$$

is sufficiently small. We give an example to show that the latter assumption is actually stronger. Consider ω_0 with $\omega_0(x_1, x_2) = A\cos x_2 \sin x_1$, $|x_1| < \pi$, $|x_2| < \pi/2$, and $\omega_0(x_1, x_2) = 0$ otherwise. Although $m = 0$, $\|\omega_0\|_1$ can be chosen as large as one likes by choosing the constant A large. In this book we introduce the method of the scaling transformation to prove the above theorem. Just as for the heat equation we begin by studying the scaling invariance of the vorticity equations.

2.2.3 Scaling Invariance

Proposition. *Assume that the pair of functions* (ω, u) *satisfies (2.7) and (2.8) in* $\mathbb{R}^2 \times (0, \infty)$*. For* $k > 0$ *define* $(\omega_k, \overline{u}_k)$ *by*

$$\omega_k(x,t) = k^2\omega(kx, k^2t), \quad \overline{u}_k(x,t) = ku(kx, k^2t). \tag{2.11}$$

Then $(\omega_k, \overline{u}_k)$ *satisfies (2.7) and (2.8) in* $\mathbb{R}^2 \times (0,\infty)$*, by replacing* ω *by* ω_k *and* u *by* $\overline{u}_k$*.*

Proof. It is easy to show that $(\omega_k, \overline{u}_k)$ satisfies (2.7) by an argument similar to the heat equation (§1.2.1). We shall check how the Biot–Savart law (2.8) varies under the scaling transformation. For $(x,t) \in \mathbb{R}^2 \times (0,\infty)$ we calculate $(\mathbf{K} * \omega_k)(x,t)$ to get

$$\begin{aligned}(\mathbf{K} * \omega_k)(x,t) &= \int_{\mathbb{R}^2} \mathbf{K}(x-y)\omega_k(y,t)dy = k^2 \int_{\mathbb{R}^2} \mathbf{K}(x-y)\omega(ky, k^2t)dy \\ &= \int_{\mathbb{R}^2} \mathbf{K}\left(\frac{kx - z}{k}\right)\omega(z, k^2t)dz.\end{aligned}$$

Using the property $\mathbf{K}(\lambda x) = \lambda^{-1}\mathbf{K}(x)$ for $\lambda > 0$, we obtain

$$(\mathbf{K} * \omega_k)(x,t) = ku(kx, k^2 t) = \overline{u}_k(x,t),$$

which implies that $(\omega_k, \overline{u}_k)$ satisfies (2.8). □

Thus we obtain an invariance of the vorticity equations under the scaling transformation of (2.11). As mentioned in §1.2.1 for the heat equation, there are some other scaling transformations under which the heat equation is invariant. But observe that equations (2.7) and (2.8) are not invariant under such scaling transformations, since (2.7) includes the term $(u, \nabla)\omega$.

If a pair of functions (ω, u) satisfies the vorticity equations (2.7) and (2.8) on $\mathbb{R}^2 \times (0, \infty)$, and for any $k > 0$, $\omega = \omega_k$ and $u = \overline{u}_k$ hold on $\mathbb{R}^2 \times (0, \infty)$, then (ω, u) is called a *forward self-similar solution* of the vorticity equations (or simply a *self-similar solution*).

We next observe that (2.7), (2.8), and (2.9) have a conserved quantity similar to that of the heat equation.

2.2.4 Conservation of the Total Circulation

Proposition. *Assume that a pair of functions (ω, u) is the solution of (2.7), (2.8), and (2.9) with the initial vorticity $\omega_0 (\in C_0(\mathbb{R}^2))$. Then*

$$\int_{\mathbb{R}^2} \omega(x,t)dx = \int_{\mathbb{R}^2} \omega_0(x)dx$$

for all $t > 0$. In particular, for any ω_k $(k > 0)$ defined in (2.11), and for any $t > 0$,

$$\int_{\mathbb{R}^2} \omega_k(x,t)dx = \int_{\mathbb{R}^2} \omega_0(x)dx.$$

Proof. Formally, calculating similarly to the case of the heat equation in §1.2.2, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} \omega(x,t)dx = \int_{\mathbb{R}^2} \Delta\omega\, dx - \int_{\mathbb{R}^2} (u,\nabla)\omega\, dx = -\int_{\mathbb{R}^2} (u,\nabla)\omega\, dx.$$

Here we recall

$$\operatorname{div} u = \operatorname{div}(\mathbf{K} * \omega) = \operatorname{div} \nabla^{\perp}(E * \omega) = 0$$

to get $(u, \nabla)\omega = \operatorname{div}(u\omega)$. By integration by parts (§4.5.2) we now obtain

$$\int_{\mathbb{R}^2} (u,\nabla)\omega\, dx = \int_{\mathbb{R}^2} \operatorname{div}(u\omega)dx = 0.$$

Thus we have shown that $\int_{\mathbb{R}^2} \omega(x,t)dx$ is independent of t, and we formally obtain the first identity. Using (ii) and (iii) of the unique existence theorem, one can justify this calculation by Theorem 7.2.1. Since

$$\int_{\mathbb{R}^2} \omega_k(x,t)dx = k^2 \int_{\mathbb{R}^2} \omega(kx, k^2 t)dx = \int_{\mathbb{R}^2} \omega(y, k^2 t)dy,$$

for the rescaled ω_k and for $t > 0$, we get the latter identity. □

In fluid mechanics $\int_{\mathbb{R}^2} \omega(x,t)dx$ is called the *total circulation*. For this reason we used this word in the title of this section.

The Gauss kernel is a self-similar solution not only for the heat equation, but also for the vorticity equations, as we will see in §2.2.5.

2.2.5 Rotationally Symmetric Self-Similar Solutions

Lemma. *Assume that a smooth real-valued function $\rho(x)$ on $\mathbb{R}^2$ depends only on the length $|x| = \sqrt{x_1^2 + x_2^2}$ of $x = (x_1, x_2)$. (That is, it is invariant under rotations centered at the origin.) Assume that $E*\rho$ is defined as a C^1-function on $\mathbb{R}^2$ and that the vector field v is expressed by $v = \mathbf{K} * \rho = \nabla^{\perp}(E * \rho)$. Then*

$$(v, \nabla)\rho \equiv 0 \text{ in } \mathbb{R}^2.$$

In particular, for $m \in \mathbb{R}$, set $\omega = mg$ $(= mG_t)$ and $u = \mathbf{K} * \omega$. Then (ω, u) satisfies the vorticity equations (2.7) and (2.8) on $\mathbb{R}^2 \times (0, \infty)$. Hence $(mg, m\mathbf{K} * g)$ is a self-similar solution.

As we will mention in Proposition 6.3.5, if $\rho \in C_0^\infty(\mathbb{R}^2)$, then $E * \rho \in C^\infty(\mathbb{R}^2)$ and $\mathbf{K} * \rho = \nabla^{\perp}(E * \rho)$ in $\mathbb{R}^2$. The same properties hold for $E * \rho$, even if the support of ρ is not compact and its decay rate as $|x| \to \infty$ is fast like the Gauss kernel $G_t(x)$.

Proof. Since $\rho(x)$ is rotationally symmetric, $\rho(Qx) = \rho(x)$ for any 2×2 rotation matrix Q (i.e., $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\theta \in \mathbb{R}$). On the other hand, by a coordinate transformation of the integral, we obtain

$$\begin{aligned} (E * \rho)(Qx) &= \int_{\mathbb{R}^2} E(Qx - y)\rho(y)dy \\ &= \int_{\mathbb{R}^2} E(Qx - Qz)\rho(Qz)dz = \int_{\mathbb{R}^2} E(Q(x-z))\rho(Qz)dz. \end{aligned}$$

Since E and ρ are rotationally symmetric, $E*\rho$ is also rotationally symmetric, i.e., $(E * \rho)(Qx) = (E * \rho)(x)$. That is to say, $E * \rho$ is a function depending only on $|x|$.

Hence $\nabla\rho(x)$ and $\nabla(E*\rho)(x)$ are parallel to $x/|x|$ for $x \neq 0$. In particular, $\nabla\rho(x)$ is parallel to $\nabla(E*\rho)(x)$ for $x \neq 0$. On the other hand, for $h \in C^1(\mathbb{R}^2)$, $\operatorname{curl} h = \nabla^{\perp} h$ is orthogonal to the gradient vector ∇h of h. Hence $v = \mathbf{K} * \rho = \nabla^{\perp}(E * \rho)$ is orthogonal to $\nabla(E * \rho)$. Therefore, v is orthogonal to $\nabla\rho$, i.e.,

$$(v, \nabla)\rho = \langle v, \nabla\rho \rangle = \langle \nabla^{\perp}(E * \rho), \nabla\rho \rangle \equiv 0.$$

(By the assumption of the smoothness of ρ, we have $\nabla\rho(0) = 0$. Hence the above equality is still valid on all of $\mathbb{R}^2$ including $x = 0$.) For fixed $t > 0$, the Gauss kernel $g(x,t) = G_t(x)$ is a function depending only on $|x|$. So the first result implies that $(\omega, u) = (mg, m\mathbf{K} * g)$ satisfies (2.7) and (2.8), since mg is a solution of the heat equation. □

As in the case of the heat equation (§1.2.6), to prove the asymptotic formula (2.10), it suffices to prove that the rescaled functions $\{\omega_k\}$ uniformly converge to mg as $k \to \infty$ at $t = 1$, i.e., $\lim_{k\to\infty} \|\omega_k - mg\|_\infty(1) = 0$. The strategy of the proof is also similar to the case of the heat equation (§1.2.7), but each step, i.e., to show the "compactness" or the "characterization of the limit function," becomes more complicated. In §2.3 and §2.4, we shall prove estimates that play an important role in the proof of "compactness," and we will prove the "compactness" in the first part of §2.5. In §2.5.1 to §2.5.4, we give the "characterization of the limit function," and complete the proof of (2.10) in §2.5.5.

To prove the "compactness" we begin by deriving decay estimates for solutions of (2.7) and (2.8). Observe that a decay estimate derived from (2.7) will in general depend on u. Here, by the fact that the function u in $(u, \nabla)\omega$ in (2.7) depends on ω, it is necessary to provide a suitable estimate of u and ω. This is different from the case of the heat equation. If possible, we obtain a decay estimate of ω that is independent of u. As we prove in the next section, we fortunately obtain such an estimate from (2.7).

2.3 Global L^q-L^1 Estimates for Solutions of the Heat Equation with a Transport Term

First, since u satisfying (2.8) always satisfies $\operatorname{div} u = 0$, we consider (2.7) for a given u satisfying $\operatorname{div} u = 0$ in this section. We regard (2.7) as a linear equation with respect to ω. For an unknown function ω and a given coefficient v with $\operatorname{div} v = 0$, consider a heat equation with terms of first derivatives (which are also called transport terms) as

$$\partial_t \omega - \Delta\omega + (v, \nabla)\omega = 0. \qquad (\mathrm{H}_v)$$

Here, v is an $\mathbb{R}^2$-valued function $v(x,t) = (v^1(x,t), v^2(x,t))$ defined on $\mathbb{R}^2 \times (0,\infty)$. In this section (§2.3), we establish an L^q-L^1 estimate (independent of v) for the solution ω of this linear equation.

2.3.1 Fundamental L^q-L^r Estimates

Theorem. *Assume that the functions v^1, $v^2 \in C^\infty(\mathbb{R}^2 \times (0,\infty))$ satisfy $\operatorname{div} v = 0$ in $\mathbb{R}^2 \times (0,\infty)$, where $v = (v^1, v^2)$. Assume that $\omega \in C^\infty(\mathbb{R}^2 \times (0,\infty))$ satisfies (H_v) in $\mathbb{R}^2 \times (0,\infty)$. Moreover, they satisfy the following initial condition (I) and conditions (at space infinity) (a) and (b):*

(I) *Assume that the function* $\omega_0 \in C(\mathbb{R}^2)$ *satisfies* $\|\omega_0\|_1 < \infty$. *Assume that* $\omega \in C(\mathbb{R}^2 \times [0,\infty))$, $\omega(x,0) = \omega_0(x), x \in \mathbb{R}^2$, *and that* $\|\omega\|_1(t)$ *is continuous at* $t = 0$.

(a) *For any* t_0, t_1 $(0 < t_0 < t_1)$, $\sup_{t_0 \le t \le t_1} \|\partial_t^\ell \partial_x^\alpha \omega\|_p(t) < \infty$, *where* α *is a multi-index satisfying* $|\alpha| + 2\ell \le 2$, *and* $\ell = 0,\ 1,\ 1 \le p \le \infty$.

(b) $\|v\|_\infty(t) < \infty$ *for each* $t > 0$.

Then there exists a universal constant $\kappa > 0$ *that is independent of* v, ω, ω_0, t, q, *such that*

$$\|\omega\|_q(t) \le \frac{1}{(\kappa t)^{1-1/q}} \|\omega_0\|_1$$

holds for all $t > 0$ *and* q *with* $1 \le q \le \infty$.

In the case of $v \equiv 0$, this estimate corresponds to the L^p-L^q estimate (1.5) for the heat equation with $q = 1$ and $\kappa = 4\pi$ in two-dimensional space. The important aspect of this estimate lies in the fact that we may take κ independent of the special profile of v, provided that $\operatorname{div} v = 0$ even if v diverges to infinity as $t \to 0$. To prove this estimate, we first establish a quantitative estimate that implies that the L^r-norm of ω is nonincreasing as a function of t.

2.3.2 Change Ratio of L^r-Norm per Time: Integral Identities

Lemma. *Assume that the functions* v *and* ω *satisfy the assumptions of the theorem in §2.3.1 except condition (I). Then for* $r = 2^m$ $(m = 0, 1, 2, \ldots)$, $\|\omega\|_r^r(t)$ *is differentiable for* $t > 0$ *and*

$$\frac{d}{dt} \int_{\mathbb{R}^2} (\omega(x,t))^r dx = -4 \left(1 - \frac{1}{r}\right) \int_{\mathbb{R}^2} |\nabla((\omega(x,t))^{r/2})|^2 dx$$

for $t > 0$. *In particular, for* $m \ge 1$, $\|\omega\|_r(t)$ *is nonincreasing with respect to* t *for* $t > 0$. *Hence, for* $m \ge 1$, *if* $\|\omega\|_r(t)$ *is continuous at* $t = 0$, *then* $\|\omega\|_r(t) \le \|\omega\|_r(0)$ *holds for* $t \ge 0$.

Proof. When $r = 1$, this integral identity describes the conservation of the total circulation, and the proof is the same as in §2.2.4. We shall prove the identity for $r = 2^m$, $m \ge 1$. By assumption (a) in §2.3.1 we may differentiate under the integral sign (§7.2.1) to get

$$\frac{d}{dt} \int_{\mathbb{R}^2} \omega^r \, dx = \int_{\mathbb{R}^2} r\omega^{r-1} \partial_t \omega \, dx = \int_{\mathbb{R}^2} r\omega^{r-1} \Delta\omega \, dx - \int_{\mathbb{R}^2} r\omega^{r-1} (v, \nabla)\omega \, dx$$

for $t > 0$. Integrating by parts (§4.5.2) for the first term of the right hand side, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} r\omega^{r-1} \Delta\,\omega \, dx &= \int_{\mathbb{R}^2} \operatorname{div}(r\omega^{r-1}\nabla\omega) dx - \int_{\mathbb{R}^2} \langle r\nabla(\omega^{r-1}), \nabla\omega \rangle dx \\ &= -\int_{\mathbb{R}^2} r(r-1)\omega^{r-2} |\nabla\omega|^2 \, dx; \end{aligned}$$

moreover, using the chain rule for the composition of functions, we have

$$= -4\left(1 - \frac{1}{r}\right)\int_{\mathbb{R}^2} |\nabla(\omega^{r/2})|^2 dx.$$

Since $\operatorname{div} v = 0$, integrating by parts (§4.5.2), we obtain for the second term

$$\int_{\mathbb{R}^2} r\omega^{r-1}(v, \nabla)\omega dx = \int_{\mathbb{R}^2} (v, \nabla)\omega^r dx = \int_{\mathbb{R}^2} \operatorname{div}(v\omega^r) dx = 0.$$

We thus obtain the integral identity at least formally. In the above calculation, we impose the decay conditions (assumptions (a), (b) in §2.3.1) for v, ω, and $\nabla\omega$ at space infinity to justify the integration by parts. For details the reader is referred to the divergence theorem in the whole space in §4.5.2. □

Using this idea, we can prove the estimate corresponding to $q = 1$ in §2.3.1.

2.3.3 Nonincrease of L^1-Norm

Lemma. *Assume that the functions v and ω satisfy the assumption in §2.3.1. Then $\|\omega\|_1(t) \le \|\omega_0\|_1$ for all $t \ge 0$.*

Proof. Since $|\omega|$ is not differentiable by the operation $|\cdot|$, we cannot calculate $\frac{d}{dt}\int |\omega| dx$ directly. To overcome this difficulty we approximate $|\omega|$ by smooth functions as follows. Using the function

$$\psi_\varepsilon(z) = (z^2 + \varepsilon)^{1/2} - \varepsilon^{1/2}, \quad z \in \mathbb{R},\ \varepsilon > 0,$$

we calculate

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi_\varepsilon(\omega) dx \quad \text{for } t > 0.$$

Since $|\psi_\varepsilon(z)| \le |z|$, $z \in \mathbb{R}$, $\int_{\mathbb{R}^2} \psi_\varepsilon(\omega) dx$ is finite for any $t > 0$, provided $\|\omega\|_1(t) < \infty$. Similarly to §2.3.2, by integration by parts (§4.5.2), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} \psi_\varepsilon(\omega) dx &= \int_{\mathbb{R}^2} \psi'_\varepsilon(\omega)\Delta\omega\, dx - \int_{\mathbb{R}^2} \psi'_\varepsilon(\omega)(v, \nabla)\omega\, dx \\ &= -\int_{\mathbb{R}^2} \langle \nabla(\psi'_\varepsilon(\omega)), \nabla\omega\rangle dx + \int_{\mathbb{R}^2} \operatorname{div}(\psi'_\varepsilon(\omega)\nabla\omega) dx \\ &\quad - \int_{\mathbb{R}^2} \operatorname{div}(\psi_\varepsilon(\omega)v) dx \\ &= -\int_{\mathbb{R}^2} \psi''_\varepsilon(\omega)|\nabla\omega|^2 dx.\end{aligned}$$

Since ψ_ε is convex, so that $\psi''_\varepsilon > 0$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi_\varepsilon(\omega) dx \le 0.$$

For $0 < \delta < t$, integrating both sides on the interval (δ, t) and letting $\varepsilon \to 0$, we obtain $\|\omega\|_1(t) \le \|\omega\|_1(\delta)$. (The fact that limit and integration commute easily follows from $\|\omega\|_1(t) < \infty$. In fact, we may apply the dominated convergence theorem in §7.1.1.) By the assumption of continuity of $\|\omega\|_1(t)$ at $t = 0$, sending $\delta \to 0$ yields the desired result. □

In the next section we shall derive a system of differential inequalities for

$$y_r(t) = \|\omega\|_r^r(t), \quad r = 2^m, \ m = 0, 1, 2, \dots,$$

from the integral identity of the change ratio of the L^r-norm. Since the non-increasing property of $y_r(t)$ is not enough to yield the fundamental L^q-L^1 estimate in §2.3.1, we have to estimate the right hand side of the integral identity in Lemma 2.3.2 in a more precise way.

For this purpose, we use the *Nash inequality* in $\mathbb{R}^2$:

$$\|\varphi\|_2^2 \le C\|\varphi\|_1\|\nabla\varphi\|_2.$$

Here φ is a continuously differentiable function defined on $\mathbb{R}^2$ (namely, $\varphi \in C^1(\mathbb{R}^2)$), $\|\varphi\|_1 < \infty$, and C is a constant that is independent of φ. (The proof of this inequality is given in §6.1.2.) The Nash inequality is one of the important inequalities frequently used in the analysis of partial differential equations. Defining κ by $C = 1/\kappa^{1/2}$, we obtain

$$\|\nabla\varphi\|_2^2 \ge \kappa\|\varphi\|_2^4/\|\varphi\|_1^2.$$

(Take the constant C in the Nash inequality as the best possible constant obtained in [Carlen Loss 1993]. Then we may take κ as $\kappa = \pi(j_{1,1}/2)^2 \approx 3.670 \cdot \pi$, which is still smaller than 4π. Hence quantitatively, the fundamental L^q-L^1 estimate in §2.3.1 is still weaker than the L^q-L^1 estimate of the heat equation in the case $v = 0$ (§1.1.2). Here $j_{1,1}$ denotes the smallest positive zero of the Bessel function J_1.) Applying the Nash inequality to $\varphi = \omega^{r/2}$ $(r = 2^m, m = 1, 2, \dots)$, we obtain

$$\|\nabla(\omega^{r/2})\|_2^2(t) \ge \kappa\|\omega^{r/2}\|_2^4(t)/\|\omega^{r/2}\|_1^2(t) = \kappa\|\omega\|_r^{2r}(t)\|\omega\|_{r/2}^{-r}(t)$$

for $t > 0$ (provided that $\|\omega\|_{r/2}(t) \ne 0$). By the integral identity in Lemma 2.3.2, we now obtain

$$\frac{d}{dt}\|\omega\|_r^r(t) \le -4\kappa\left(1 - \frac{1}{r}\right)\|\omega\|_r^{2r}(t)\|\omega\|_{r/2}^{-r}(t), \quad t > 0.$$

We thus obtain the following system of differential inequalities.

2.3.4 Application of the Nash Inequality

Proposition. *Assume that v and ω satisfy the assumptions in Theorem 2.3.1 except for condition (I). For $r = 2^m$ $(m = 0, 1, 2, \dots)$ and $y_r(t) = \|\omega\|_r^r(t)$ the following differential inequalities hold:*

$$y_{r/2}^2 \frac{dy_r}{dt} \le -4\kappa \left(1 - \frac{1}{r}\right) y_r^2, \qquad t > 0, \quad (r = 2^m, m = 1, 2, 3, \dots),$$

where κ is a universal constant independent of v and ω.

Remark. Both the differential inequalities in the above proposition and the integral identities in Lemma 2.3.2 hold for any strictly positive t. So we need not assume condition (I) of Theorem 2.3.1.

This system of differential inequalities leads to a successive estimate for $y_{2^m}(t)$ by the following lemma. Note that this lemma itself is independent of the above proposition.

Lemma. *Let $r = 2^m$ $(m = 0, 1, 2, \dots)$ and $a > 0$. Assume that $y_r = y_r(t)$ is a positive function defined in $(0, \infty)$, belonging to $C^1(0, \infty)$ for $m \ge 1$, and satisfying*[1]

$$\frac{dy_r}{dt} \le -a \left(1 - \frac{1}{r}\right) \frac{y_r^2}{y_{r/2}^2}, \qquad t > 0 \quad (r = 2^m, m = 1, 2, \dots).$$

Moreover, assume that y_1 is bounded in $(0, \infty)$, namely, there exists a constant $M_1 > 0$ such that $y_1(t) \le M_1$ $(t > 0)$. Then the following two statements are valid:

(i) *The inequality*

$$y_r(t) \le M_r t^{1-r}, \qquad t > 0,\ r = 2^m,\ m = 0, 1, 2, \dots,$$

holds, where M_r is defined by $M_r = a^{-1} r M_{r/2}^2$ successively.

(ii) *If the inequality in (i) holds for $r = 2^m$ $(m = 0, 1, 2, \dots)$, then for sufficiently large m,*

$$(y_r(t))^{1/r} \le \frac{4}{a} M_1 t^{-1+1/r}, \quad t > 0.$$

Proof. (i) Let us prove the claim by an induction argument with respect to m. It is obvious in the case $m = 0$. We shall we prove $y_{2r}(t) \le M_{2r} t^{1-2r}$ with $r = 2^m$ under the assumption that the claim is valid for any positive integer less than or equal to m. Applying the assumption of the induction to the differential inequality, we obtain

$$\frac{dy_{2r}}{dt}(t) \le -a \left(1 - \frac{1}{2r}\right) \frac{y_{2r}^2(t)}{M_r^2} t^{2r-2}$$

for $t > 0$. Dividing both sides by $-y_{2r}^2$, we get

$$-\frac{dy_{2r}}{dt}(t) / y_{2r}^2(t) \ge a \left(1 - \frac{1}{2r}\right) M_r^{-2} t^{2r-2} \quad (> 0).$$

[1] $y_r^2(t)$ denotes $(y_r(t))^2$.

Integrating both sides over the interval $[s,t] \subset (0,\infty)$, we obtain

$$\frac{1}{y_{2r}(t)} - \frac{1}{y_{2r}(s)} \geq a\left(1 - \frac{1}{2r}\right) M_r^{-2} \int_s^t \tau^{2r-2} d\tau = \frac{a}{2rM_r^2}(t^{2r-1} - s^{2r-1}).$$

Recalling that $y_{2r}(s) \geq 0$, we obtain

$$\frac{1}{y_{2r}(t)} \geq \frac{a}{2rM_r^2}(t^{2r-1} - s^{2r-1}),$$

and letting $s \to 0$ results in

$$\frac{1}{y_{2r}(t)} \geq \frac{a}{2rM_r^2} t^{2r-1}, \quad t > 0.$$

Hence $y_{2r}(t) \leq M_{2r} t^{1-2r}$, and the proof is now complete.
(ii) Setting $\mu_r = M_r^{1/r}$, we obtain

$$(y_r(t))^{1/r} \leq \mu_r t^{-1+1/r}, \quad t > 0, \ m = 0, 1, 2, \ldots,$$

by (i). So we shall estimate μ_r by successive equalities to prove $\mu_r \leq 4\mu_1/a$ for large r. To obtain this estimate, for $r = 2^m$, we set $b_m = \log \mu_r$. Since $M_r^{1/r} = (r/a)^{1/r} M_{r/2}^{2/r}$, for b_m we obtain the following successive equalities:

$$\begin{aligned} b_0 &= \log \mu_1, \\ b_m &= b_{m-1} + \frac{1}{2^m} \log\left(\frac{2^m}{a}\right), \quad m = 1, 2, \ldots. \end{aligned}$$

We thus deduce

$$b_m = b_0 + \sum_{j=1}^m \left(\frac{j}{2^j}\log 2 - \frac{1}{2^j}\log a\right), \quad m = 1, 2, \ldots.$$

If j is sufficiently large, say $2^j/a > 1$, each summand is positive. Hence, for sufficiently large m, we obtain (say $2^m > a$)

$$\begin{aligned} b_m &\leq b_0 + \left(\sum_{j=1}^\infty \frac{j}{2^j}\right)\log 2 - \left(\sum_{j=1}^\infty \frac{1}{2^j}\right)\log a \\ &= b_0 + 2\log 2 - \log a = b_0 + \log(4/a). \end{aligned}$$

(Since the series or its derivative is a geometric series, it is easy to determine their values (Exercise 2.2).) Applying exp to both sides, for a sufficiently large r, we obtain $\mu_r \leq 4\mu_1/a$. The proof of (ii) is now complete. □

2.3.5 Proof of Fundamental L^q-L^1 Estimates

We are now in position to prove Theorem 2.3.1. By an application of the Nash inequality, we obtain a system of differential inequalities for y_r as in Proposition 2.3.4 with $y_r(t) = \|\omega\|_r^r(t)$ for $r = 2^m(m = 0, 1, 2, \dots)$. By the nonincrease of the L^1-norm, which is obtained in Lemma 2.3.3, we have $y_1(t) \le M_1$, $t \ge 0$ with $M_1 = \|\omega_0\|_1$. Note that $y_r(t)$ is positive for $t \ge 0$ unless $\omega_0 \equiv 0$. (Indeed, if $y_r(t_0) = 0$ for some $t = y_0 > 0$, then by the strong maximum principle (§2.3.8 and [Protter Weinberger 1967]) ω must be zero for $t \in [0, t_0]$.) The result for $\omega \equiv 0$ is trivial, so we may assume that $y_r(t) > 0$ for all $t \ge 0$. Using Lemma 2.3.4 with $a = 4\kappa$, for sufficiently large m we obtain

$$\|\omega\|_r(t) \le \frac{1}{\kappa t^{1-1/r}} \|\omega_0\|_1, \quad t > 0,$$

with $r = 2^m$. Since $\|\omega\|_\infty(t) = \lim_{r\to\infty} \|\omega\|_r(t)$ for $t > 0$ (Exercise 2.3), we get

$$\|\omega\|_\infty(t) \le \frac{1}{\kappa t} \|\omega_0\|_1, \quad t > 0.$$

By the Hölder inequality (§4.1.1), for q with $1 \le q \le \infty$,

$$\|\omega\|_q(t) \le \|\omega\|_1^{\frac{1}{q}}(t) \|\omega\|_\infty^{1-\frac{1}{q}}(t), \quad t > 0.$$

(This may also be derived by a direct calculation (Exercise 2.4).) Thus the nonincrease of the L^1-norm $\|\omega\|_1(t) \le \|\omega_0\|_1$ for $t \ge 0$ and the estimate of $\|\omega\|_\infty(t)$ imply

$$\|\omega\|_q(t) \le \frac{1}{(\kappa t)^{1-1/q}} \|\omega_0\|_1, \quad t > 0,$$

which yields the assertion. □

One may prove the fundamental L^q-L^1 estimate by the system of differential inequalities, nonincrease of the L^1-norm, and by a duality argument without using Lemma 2.3.4. We shall give the idea of this method of proof. Setting $r = 2$ in the system of differential inequalities in §2.3.4, and recalling that $\|\omega\|_1(t) \le \|\omega_0\|_1$ in §2.3.2, we obtain

$$\frac{dy_2}{dt} \le -2\kappa y_2^2 \|\omega_0\|_1^{-2}.$$

Since $y_2 \ge 0$, this inequality implies that $y_2(t)$ is nonincreasing with respect to t. Thus, if $y_2(t_1) = 0$ for some $t_1 \ge 0$, then $y_2(t) = 0$ for all $t \ge t_1$. If there exists no such t_1, then $y_2(t) > 0$ $(t > 0)$ and $\lim_{t\to\infty} y_2(t) = 0$. Let t_* be the minimum for such t_1 (admit $t_* = \infty$). If $t_* = 0$, by the continuity of $\|\omega\|_1(t)$ at $t = 0$, we obtain $\omega_0 \equiv 0$, so that $\omega \equiv 0$ by $\|\omega\|_1(t) \le \|\omega_0\|_1$. Thus, we may assume $t_* > 0$. Dividing both sides of the above differential inequality by y_2^2 for $0 < t < t_*$, and integrating over $(0, t)$, we obtain

$$y_2(t) \leq (2\kappa t)^{-1} \|\omega_0\|_1^2,$$

or the L^2-L^1 estimate

$$\|\omega\|_2(t) \leq (2\kappa t)^{-1/2} \|\omega_0\|_1, \quad t > 0.$$

Next we consider the "duality problem" for (H_v). Fix $t_0 > 0$ and consider

$$\begin{aligned} \partial_t \psi + (v, \nabla)\psi + \Delta\psi = 0 \qquad & \text{in } \mathbb{R}^2 \times (0, t_0), \\ \psi(x, t_0) = \psi_0(x), \qquad & x \in \mathbb{R}^2. \end{aligned}$$

Here we assume $\psi_0 \in C_0(\mathbb{R}^2)$ and $\|\psi_0\|_1 \leq 1$. Admitting the existence of a solution ψ (§4.4.4), we multiply both sides of the evolution equation by ω and integrate them over $\mathbb{R}^2 \times (0, t_0)$. Since $\operatorname{div} v = 0$ by assumption, integrating by parts yields

$$\int_{\mathbb{R}^2} \omega(x, t_0)\psi_0(x) dx = \int_{\mathbb{R}^2} \omega_0(x)\psi(x, 0) dx.$$

By the L^2-L^1 estimate for solutions of (H_v) obtained at the beginning of the proof, we observe that $\|\psi(x, 0)\|_2 \leq (2\kappa t_0)^{-1/2} \|\psi_0\|_1 \leq (2\kappa t_0)^{-1/2}$. We thus obtain

$$\begin{aligned} \sup_{\substack{\|\psi_0\|_1 \leq 1 \\ \psi_0 \in C_0(\mathbb{R}^2)}} \left| \int_{\mathbb{R}^2} \omega(x, t_0)\psi_0(x) dx \right| &\leq \|\omega_0\|_2 \ \|\psi(x, 0)\|_2 \\ &\leq (2\kappa t_0)^{-1/2} \|\omega_0\|_2. \end{aligned}$$

Here we used the *Schwarz inequality* (§4.1.1). The latter term is equal to $\|\omega\|_\infty(t_0)$ in view of the characterization of the norm by duality (Chapter 6 (6.8)). We thus obtain the L^∞-L^2 estimate

$$\|\omega\|_\infty(t_0) \leq (2\kappa t_0)^{-1/2} \|\omega_0\|_2.$$

For general $t > 0$ we set $t_0 = t/2$ and use the L^2-L^1 estimate and the L^∞-L^2 estimate to obtain

$$\|\omega\|_\infty(t) \leq (\kappa t)^{-1/2} \|\omega\|_2(t/2) \leq (\kappa t)^{-1} \|\omega_0\|_1.$$

As in the proof given in the first paragraph of this section, we obtain the fundamental L^q-L^1 estimate from this estimate by interpolating with the L^1-L^1 estimate.

By similar arguments to establish the fundamental L^q-L^1 estimates it is also possible to derive the following L^q-L^r estimate ($r = 2^m$) for ω, as in the case of the heat equation.

2.3.6 Extension of Fundamental L^q-L^1 Estimates

Theorem. *Assume that v and ω satisfy the same assumptions as in §2.3.1. Let κ be the universal constant in §2.3.1. Then for $\rho = 2^k$ $(k = 0, 1, 2, \dots)$ and q with $\rho \le q \le \infty$, we have*

$$\|\omega\|_q(t) \le \frac{1}{(\kappa t)^{1/\rho - 1/q}} \|\omega_0\|_\rho, \quad t > 0.$$

Here $\|\omega\|_\rho(t)$ is assumed to be continuous at $t = 0$.

Proof. Replacing $y_1 \le M_1$ by $y_\rho \le N_\rho$ with a constant N_ρ in Lemma 2.3.4, arguing as in the proof of the lemma for the differential inequality with $m \ge k + 1$ we obtain

$$y_s(t) \le N_s t^{1-s/\rho}, \quad s = 2^m \ge \rho = 2^k, \quad t > 0,$$

instead of (i). We thus define $N_s = a^{-1}\rho^{-1} s N_{s/2}^2$ inductively. Then for sufficiently large $s = 2^m$, instead of (ii), we obtain

$$(y_s(t))^{1/s} \le \left(\frac{4}{a}\right)^{1/\rho} N_\rho^{1/\rho} t^{-1/\rho + 1/s}, \quad t > 0$$

(Exercise 2.5).

Since we assume that $\|\omega\|_\rho(t)$ is continuous at $t = 0$, by §2.3.2 and §2.3.3 we obtain $\|\omega\|_\rho(t) \le \|\omega_0\|_\rho$, $t > 0$. We set

$$y_s(t) = \|\omega\|_s^s(t), \quad s = 2^m, \quad m = k, k+1, k+2, \dots, \quad t > 0,$$

$N_\rho = \|\omega_0\|_\rho^\rho$, and $a = 4\kappa$. Then we observe that similar inequalities as in Lemma 2.3.4 are valid for the above t_s. By similar arguments as in §2.3.5, we obtain $\|\omega\|_\infty(t) \le \frac{1}{(\kappa t)^{1/\rho}} \|\omega_0\|_\rho$ for $t > 0$. By the Hölder inequality (§4.1.1) for general q with $\rho \le q \le \infty$ we then conclude that

$$\begin{aligned} \|\omega\|_q(t) &\le \|\omega\|_\infty^{1-\rho/q}(t) \|\omega\|_\rho^{\rho/q}(t) \\ &\le \left(\frac{1}{(\kappa t)^{1/\rho}} \|\omega_0\|_\rho\right)^{1-\rho/q} \|\omega_0\|_\rho^{\rho/q} \\ &= \frac{1}{(\kappa t)^{1/\rho - 1/q}} \|\omega_0\|_\rho, \qquad t > 0. \end{aligned}$$

□

2.3.7 Maximum Principle

Proposition. *Assume that v and ω satisfy the same assumptions as in §2.3.1. Then*

$$\|\omega\|_\infty(t) \le \|\omega_0\|_\infty, \quad t \ge 0.$$

Here $\|\omega\|_\rho(t)$ is assumed to be continuous at $t = 0$ for sufficiently large ρ $(< \infty)$.

This is easy to prove. Indeed, if we set $q = \infty$ in §2.3.6 and send $\rho \to \infty$, then we get the desired result, since $\lim_{\rho\to\infty} \|\omega_0\|_\rho = \|\omega_0\|_\infty$ (Exercise 2.3).

This proposition is called the maximum principle, since it estimates the upper bound of $|\omega(x,t)|$ as a function of x on $t > 0$. If one assumes the boundedness of $\|\omega\|_\infty(t)$ and $\|v\|_\infty(t)$, we may prove the proposition without assuming $\operatorname{div} v = 0$. We shall discuss this property in the next section.

2.3.8 Preservation of Nonnegativity

Theorem. *Let T be a given positive number. Assume that the functions v^i and w are bounded on $\mathbb{R}^n \times (0,T)$, for $i = 1, \dots, n$. Moreover, assume that $w \in C(\mathbb{R}^n \times [0,T)) \cap C^2(\mathbb{R}^n \times (0,T))$ satisfies*

$$\partial_t w - \Delta w + (v, \nabla) w = 0 \quad \text{in } \mathbb{R}^n \times (0,T)$$

for $v = (v^1, \dots, v^n)$. Then we have the following properties:

(i) *If $w(\cdot,0)$ is nonnegative on $\mathbb{R}^n$, then w is also nonnegative on $\mathbb{R}^n \times [0,T)$. Namely, if $w(x,0) \ge 0$, $x \in \mathbb{R}^n$, then $w(x,t) \ge 0$, $x \in \mathbb{R}^n$, $t \in [0,T)$.*
(ii) $\sup_{x\in\mathbb{R}^n} w(x,t) \le \sup_{x\in\mathbb{R}^n} w(x,0)$ *for $t \in [0,T)$, and* $\inf_{x\in\mathbb{R}^n} w(x,t) \ge \inf_{x\in\mathbb{R}^n} w(x,0)$ *for $t \in [0,T)$.*
(iii) $\|w\|_\infty(t) \le \|w\|_\infty(0)$, *for $t \in [0,T)$.*

Proof. Property (iii) immediately follows from (ii). Property (ii) follows from (i). Indeed, we set $\sup_{x\in\mathbb{R}^n} w(x,0) = M_0$ and $\tilde{w} = -(w - M_0)$ and observe that $\tilde{w}$ satisfies $\partial_t \tilde{w} - \Delta \tilde{w} + (v,\nabla)\tilde{w} = 0$ in $\mathbb{R}^n \times (0,T)$. Thus by (i), we obtain $\sup_{x\in\mathbb{R}^n} w(x,t) \le M_0$. The claim for the infimum is proved by similar arguments.

It remains to prove (i).

We transform the dependent variable w to $u = e^{-t} w$. Then u satisfies

$$\partial_t u + u - \Delta u + (v,\nabla) u = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Let L be the operator acting on u which is defined by the left hand side of this equation. That is to say, the left hand side is denoted by Lu. We assume that w has a negative value at $(x_0,t_0) \in \mathbb{R}^n \times (0,T)$, and we shall derive a contradiction. By the definition of u, we get $(-\alpha =) u(x_0,t_0) < 0$. If there exists a point $(\hat{x},\hat{t})$ in $\mathbb{R}^n \times (0,t_0]$ at which $\inf_{\mathbb{R}^n \times [0,t_0]} u$ is attained, then we have

$$\partial_t u(\hat{x},\hat{t}) \le 0, \quad \nabla u(\hat{x},\hat{t}) = 0, \quad \Delta u(\hat{x},\hat{t}) \ge 0;$$

hence we obtain

$$Lu(\hat{x},\hat{t}) \le u(\hat{x},\hat{t}) \le u(x_0,t_0) = -\alpha < 0.$$

This contradicts $Lu = 0$ (in $\mathbb{R}^n \times (0,T)$). However, unfortunately, since $\mathbb{R}^n$ is unbounded, we do not know whether there exists a point at which $\inf_{\mathbb{R}^n \times [0,t_0]} u$

is attained. For this reason, we use the following trick. Let $A > 0$ and $\varepsilon > 0$ be constants to be determined later, and set

$$u_\varepsilon = u + \varepsilon(At + |x|^2).$$

Then we have

$$Lu_\varepsilon = Lu + \varepsilon(A + At + |x|^2 - 2n + 2 \langle v, x \rangle).$$

Since $Lu = 0$ in $\mathbb{R}^n \times (0,T)$, we choose $A > 0$ such that

$$A \geq \sup\{2n + 2|x| \, \|v\|_\infty(t) - |x|^2 : x \in \mathbb{R}^n, \ t \in [0,t_0]\},$$

and conclude that $Lu_\varepsilon \geq 0$ in $\mathbb{R}^n \times (0,t_0)$. (By the assumption of the boundedness of v, the above supremum is finite.) We fix such an A and take $\varepsilon > 0$ so small that $u_\varepsilon(x_0,t_0) = u(x_0,t_0) + \varepsilon(At_0 + |x_0|^2) \leq -\alpha/2 < 0$.

Since w is bounded on $\mathbb{R}^n \times [0,T)$, u is also bounded on $\mathbb{R}^n \times [0,T)$. Moreover, if

$$|x| > \varepsilon^{-1/2} \left(- \inf_{\mathbb{R}^n \times (0,T)} u \right)^{1/2} =: R, \ x \in \mathbb{R}^n,$$

then $u_\varepsilon \ (x,t) > 0 \ (t \in (0,T))$. Since the function u_ε is continuous in $\mathbb{R}^n \times [0,t_0]$, u_ε has a minimum value on $\overline{B}_R \times [0,t_0]$ (the Weierstrass theorem). This means that there exists a point $(x_1,t_1) \in \overline{B}_R \times [0,t_0]$ such that

$$u_\varepsilon(x_1,t_1) = \inf\{u_\varepsilon(x,t) : \ x \in \overline{B}_R, \ t \in [0,t_0]\}.$$

Note that $u_\varepsilon(x_1,t_1) \leq u_\varepsilon(x_0,t_0) < 0$. Since $u_\varepsilon(x,0) \geq 0 \ (x \in \mathbb{R}^n)$ by assumption and since $u_\varepsilon(x,t) \geq 0 \ (|x| = R, \ t \in [0,t_0])$ by the choice of R, we conclude that $|x_1| < R$ and $t_1 \in (0,t_0]$. Since u_ε attains its minimum in $B_R \times [0,t_0]$ at (x_1,t_1), we obtain

$$\partial_t u_\varepsilon(x_1,t_1) \leq 0, \quad \nabla u_\varepsilon(x_1,t_1) = 0, \quad \Delta u_\varepsilon(x_1,t_1) \geq 0.$$

Thus

$$(Lu_\varepsilon)(x_1,t_1) \leq u_\varepsilon(x_1,t_1) \ \leq u_\varepsilon(x_0,t_0) \leq -\alpha/2 < 0,$$

which contradicts $Lu_\varepsilon \geq 0$ in $\mathbb{R}^n \times (0,T)$. We thus conclude that w is nonnegative on $\mathbb{R}^n \times (0,T)$. □

Following the lines of the proof, we see that in order to prove (i) it suffices to assume that $Lw \geq 0$ (in $\mathbb{R}^n \times (0,T)$) instead of $Lw = 0$ (in $\mathbb{R}^n \times (0,T)$). It is known that w is positive on $t > 0$ unless w is identically zero under the situation of (i). This is called the strong maximum principle, which, however, is not discussed in this book. For details about the maximum principle and the strong maximum principle readers are referred to [Protter Weinberger 1967], [Kumanogo 1978]. Also in [Ito 1979] the strong maximum principle is discussed in detail, although the main theme is the construction of fundamental solutions of diffusion equations.

2.4 Estimates for Solutions of Vorticity Equations

Using the results of the previous section, we shall derive estimates for solutions of the vorticity equations (2.7) and (2.8). As in the case of the heat equation, it is important to derive estimates depending only on the L^1-norm of the initial value ω_0. Using the fundamental L^q-L^1 estimate in §2.3.1, we shall first derive estimates for the vorticity and the velocity of (2.7) and (2.8).

2.4.1 Estimates for Vorticity and Velocity

Theorem. *Let the initial vorticity ω_0 be in $C_0(\mathbb{R}^2)$ and let κ be the universal constant of §2.3.1. Then there exist positive constants $L_j(p)$ ($j = 1, 2$) depending only on p and satisfying the following properties for all solutions (ω, u) of (2.7), (2.8), and (2.9):*

(i) *We have $\|\omega\|_q(t) \leq \frac{1}{(\kappa t)^{1-1/q}}\|\omega_0\|_1$ for all $t > 0$ and q with $1 \leq q \leq \infty$.*

(ii) *For each p satisfying $2 < p < \infty$ define q by $1/p = 1/q - 1/2$. Then $\|u\|_p(t) \leq L_1(p)\|\omega\|_q$ for all $t > 0$. Moreover, for each p satisfying $2 < p \leq \infty$ the estimate*

$$\|u\|_p(t) \leq \frac{L_1(p)}{(\kappa t)^{\frac{1}{2}-\frac{1}{p}}}\|\omega_0\|_1$$

is valid for all $t > 0$.

(iii) *For each q satisfying $1 < q < \infty$ the estimate*

$$\|\nabla u\|_q(t) \leq L_2(q)\|\omega\|_q(t) \leq \frac{L_2(q)}{(\kappa t)^{1-\frac{1}{q}}}\|\omega_0\|_1$$

is valid for all $t > 0$.

(iv) *For each p satisfying $2 < p \leq \infty$ the convergence*

$$\lim_{t\to 0}\|u - u_0\|_p(t) = 0$$

is valid for $u_0 = \mathbf{K} * \omega_0$.

For an $\mathbb{R}^2$-valued function $v = (v^1, v^2)$ defined on $\mathbb{R}^2$, ∇v denotes the matrix $(\partial_{x_i} v^j)_{1\leq i,j\leq 2}$, whereas the expression $|\nabla v|$ denotes the Hilbert–Schmidt norm $(\sum_{j=1}^2 |\nabla v^j|^2)^{1/2}$. For p with $1 \leq p \leq \infty$, we define $\|\nabla v\|_p = \| \, |\nabla v| \, \|_p$.

Proof. The first estimate (i) is obvious by applying the fundamental L^q-L^1 estimate in §2.3.1 to (2.7). Estimates (ii) and (iii) are derived from (i) together with various fundamental estimates in differential and integral calculus discussed in Section 6. One will see the importance of such fundamental inequalities through the proof of the theorem.

We first note that the velocity is expressed by the Biot–Savart law (2.8) using the vorticity ω. If we write $x = (x_1, x_2)$ and $\mathbf{K} = (K_1, K_2)$, then we obtain

$$|K_i(x)| \le \frac{1}{2\pi}\frac{1}{|x|}, \quad i = 1, 2,$$

since $|x_1| \le |x|$, $|x_2| \le |x|$. Thus $u = (u^1, u^2)$ is also estimated by

$$|u^i(x)| \le \int_{\mathbb{R}^2} \frac{1}{2\pi|x-y|}|\omega(y)|dy, \quad i = 1, 2.$$

In the proof below we suppress the dependence of u and ω with respect to t unless it is necessary to clarify. In other words, we simply write $u(x)$ and $\omega(x)$ instead of $u(x,t)$ and $\omega(x,t)$, respectively. For a function f defined on $\mathbb{R}^2$ we define the operator I_1 by

$$(I_1(f))(x) = \frac{1}{|x|} * f = \int_{\mathbb{R}^2} \frac{1}{|x-y|} f(y)dy, \quad x \in \mathbb{R}^2.$$

Then we obtain

$$|u^i(x)| \le \frac{1}{2\pi}(I_1(|\omega|))(x), \quad x \in \mathbb{R}^2.$$

For this operator I_1 it is known that

$$\|I_1(f)\|_p \le C\|f\|_q, \quad 1/p = 1/q - 1/2,\ 1 < q < 2,$$

which is a special case of the *Hardy–Littlewood–Sobolev inequality* proved in §6.2.1. Here C is a constant depending only on p. The above inequality is at least valid for a continuous function f with $\|f\|_q < \infty$. For more details see §6.2 (especially the theorem and the remark in §6.2.1). We apply this inequality to $I_1(|\omega|)$ and observe that there exists a constant L_1 depending only on p such that

$$\|u\|_p \le L_1\|\omega\|_q, \quad 1/p = 1/q - 1/2,\ 1 < q < 2.$$

(By the unique existence theorem in §2.2.1, ω is continuous and satisfies $\|\omega\|_q < \infty$. Thus we may apply Theorem 6.2.1 to ω.) This inequality is considered as an estimate of the velocity by the vorticity. (Thus the Hardy–Littlewood–Sobolev inequality is considered as a generalization of the estimate for the velocity by the vorticity in a two-dimensional fluid.)

Combining this estimate and (i), we obtain (ii) for $2 < p < \infty$. Since we cannot remove the restriction of the index $1 < q < 2$ in the Hardy–Littlewood–Sobolev inequality, we have to prove the case $p = \infty$ separately.

Postponing the proof of (ii) in the case of $p = \infty$, we consider (iii). The operator that maps ω to $\partial_{x_j} u^i$ is a typical example of *singular integral operator* (§6.4.2). In this case, the L^p estimate of the singular integral operator is well known as the *Calderón–Zygmund inequality*; this will be discussed with a proof in §6.4. By the Calderón–Zygmund inequality (§6.4.2) the derivative of the velocity is estimated by the vorticity as

$$\|\nabla u\|_q \leq \overline{C}\|\omega\|_q, \quad 1 < q < \infty,$$

where $\overline{C}$ is a constant depending only on q. We remark that the above inequality is not valid for $q = 1$ and $q = \infty$. (By the unique existence theorem (§2.2.1), for a fixed $t > 0$, ω is C^1 on $\mathbb{R}^2$ as a function of x, and ω and $|\nabla\omega|$ are bounded and integrable on $\mathbb{R}^2$. Hence we obtain the last inequality by Theorem 6.4.2 and remark (i) afterward.) (Note that in the case of $q = 2$ it is easy to prove that

$$\|\nabla u\|_2^2 = \|\omega\|_2^2,$$

provided that the following integration by parts on $\mathbb{R}^2$ is justified:

$$\begin{aligned}\|\nabla u\|_2^2 &= \sum_{j=1}^{2}\int_{\mathbb{R}^2}\langle\nabla u^j, \nabla u^j\rangle dx = -\int_{\mathbb{R}^2}\langle u, \Delta u\rangle dx \\ &= \int_{\mathbb{R}^2}\langle u, (\nabla^\perp\mathrm{curl}\ - \nabla\mathrm{div}\ )u\rangle dx = \int_{\mathbb{R}^2}\langle u, \nabla^\perp\mathrm{curl}\ u\rangle dx \\ &= \int_{\mathbb{R}^2}(\mathrm{curl}\, u)(\mathrm{curl}\, u)dx = \|\omega\|_2^2.\end{aligned}$$

Here we have used the property $\Delta = -\nabla^\perp\mathrm{curl} + \nabla\mathrm{div}$ (§2.1.2), $\mathrm{div}\, u = 0$, and $\mathrm{curl}\, u = \omega$. Physically, up to constant multiples, $\|\nabla u\|_2^2$ and $\|\omega\|_2^2$ correspond to the enstrophy and the energy of vorticity, respectively.) In the case of $1 < q < \infty$, combining the Calderón–Zygmund inequality and (i), we obtain (iii) with $L_2 = \overline{C}$.

We now consider (ii) in the case of $p = \infty$. Sometimes the modulus of a function is estimated by the modulus of its derivatives. There are several types of such inequalities including the Sobolev inequality. Here, we use a special case of the *Gagliardo–Nirenberg inequality* (§6.1.1) of the form

$$\|u\|_\infty \leq \tilde{C}\|u\|_r^{1-2/r}\|\nabla u\|_r^{2/r}, \quad 2 < r < \infty.$$

(By Theorem 2.2.1, each component of u is C^1 and satisfies $\|u\|_r < \infty$; hence we may apply the above inequality to u.) In this inequality, it is always assumed that $\|u\|_r$ is finite. Here $\tilde{C}$ is a constant depending on r and independent of u. We shall discuss the Gagliardo–Nirenberg inequality in detail in §6.1. (Without the finiteness of $\|u\|_r$ this inequality fails in the case of a nonzero constant function. If $\|u\|_r$ is finite, the inequality is valid, since u vanishes identically when $|\nabla u| = 0$ on $\mathbb{R}^2$.) We choose an $r \in (2, \infty)$ in the Gagliardo–Nirenberg inequality and apply (ii) and (iii) to the right-hand side to get

$$\|u\|_\infty(t) \le \tilde{C}\left(\frac{L_1(r)}{(\kappa t)^{\frac{1}{2}-\frac{1}{r}}}\|\omega_0\|_1\right)^{1-\frac{2}{r}}\left(\frac{L_2(r)}{(\kappa t)^{1-\frac{1}{r}}}\|\omega_0\|_1\right)^{\frac{2}{r}}$$
$$= \frac{\tilde{C}(L_1(r))^{1-2/r}(L_2(r))^{2/r}}{(\kappa t)^{1/2}}\|\omega_0\|_1$$

for $t > 0$. The exponent of $1/t$ is calculated as

$$\left(\frac{1}{2}-\frac{1}{r}\right)\left(1-\frac{2}{r}\right)+\left(1-\frac{1}{r}\right)\frac{2}{r} = \frac{1}{2}.$$

We thus obtain (ii) in the case of $p = \infty$ with

$$L_1(\infty) = \tilde{C}(L_1(r))^{1-2/r}(L_2(r))^{2/r}.$$

Finally, we shall prove (iv). Since $u - u_0 = \mathbf{K} * (\omega - \omega_0)$, for p with $2 < p < \infty$, using the Hardy–Littlewood–Sobolev inequality, we obtain

$$\|u - u_0\|_p(t) \le L_1(p)\|\omega - \omega_0\|_q(t), \quad 1/p = 1/q - 1/2,\ 1 < q < 2.$$

On the other hand, by the unique existence theorem (§2.2.1 (i)), we have $\|\omega - \omega_0\|_q(t) \to 0$ $(t \to 0)$; hence (iv) follows for p with $2 < p < \infty$. In the case of $p = \infty$ (each component of) u_0 is C^1 at least for $\omega_0 \in C_0(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$ (Remark (ii) in §6.3.5), so we may argue similarly as in the proof of (ii). The Gagliardo–Nirenberg inequality yields

$$\|u - u_0\|_\infty(t) \le \tilde{C}\|u - u_0\|_r^{1-2/r}(t)\|\nabla(u - u_0)\|_r^{2/r}(t), \quad 2 < r < \infty,$$

for all $t > 0$. By the Calderón–Zygmund inequality (theorem and remark (i) in §6.4.2) we have

$$\|\nabla(u - u_0)\|_r(t) \le \overline{C}\|\omega - \omega_0\|_r(t).$$

We also have

$$\|u - u_0\|_r(t) \le L_1(r)\|\omega - \omega_0\|_q(t),\ 1/r = 1/q - 1/2,\ 1 < q < 2,$$

which is obtained from the Hardy–Littlewood–Sobolev inequality, and now observe that

$$\|u - u_0\|_\infty(t) \le \tilde{C}(L_1(r))^{1-2/r}\bar{C}^{2/r}\|\omega - \omega_0\|_q^{1-2/r}(t)\|\omega - \omega_0\|_r^{2/r}(t).$$

Using $\|\omega - \omega_0\|_s(t) \to 0$ $(t \to 0)$ $(1 \le s \le \infty)$ again (see the unique existence theorem (§2.2.1(i))), we conclude (iv) in the case of $p = \infty$ for $\omega_0 \in C_0(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$. For general $\omega_0 \in C_0(\mathbb{R}^2)$, u_0 may not be C^1, so additional work is necessary. However, by Remark (ii) in §6.4.2, the last inequality is still valid, so we can prove (iv) in the case of $p = \infty$. □

The Gagliardo–Nirenberg inequality, the Hardy–Littlewood–Sobolev inequality, and the Calderón–Zygmund inequality are valid in higher-dimensional spaces, under suitable corrections for the relation of the exponents. These general inequalities are discussed in Chapter 6.

For the second half of the proof of (ii) including the case $p = \infty$, it is possible to prove it by applying only the fundamental L^q-L^1 estimate §2.3.1. We shall give its proof only for the case of $p = \infty$. Using the operator I_1, we have

$$|u^i(x,t)| \leq \frac{1}{2\pi} I_1(|\omega(\cdot,t)|)(x), \quad i = 1,2, \quad t > 0.$$

For $A > 0$ we write

$$I_1(|\omega(\cdot,t)|)(x) = \int_{|x-y|\leq A} \frac{|\omega(y,t)|}{|x-y|} dy + \int_{|x-y|>A} \frac{|\omega(y,t)|}{|x-y|} dy.$$

We use the fundamental L^∞-L^1 estimate (§2.3.1) to obtain

$$\begin{aligned} I_1(|\omega(\cdot,t)|)(x) &\leq \|\omega\|_\infty(t) \int_{|x-y|\leq A} \frac{dy}{|x-y|} + \frac{1}{A}\|\omega\|_1(t) \\ &\leq \{2\pi A(\kappa t)^{-1} + A^{-1}\}\|\omega_0\|_1. \end{aligned}$$

We set $A = (\kappa t/2\pi)^{1/2}$ (so that $2\pi A(\kappa t)^{-1} = A^{-1}$). Thus we obtain

$$I_1(|\omega(\cdot,t)|)(x) \leq 2(2\pi/\kappa t)^{1/2}\|\omega_0\|_1, \quad t > 0.$$

Therefore, for $t > 0$, we get $|u^i(x,t)| \leq 2(2\pi\kappa t)^{-1/2}\|\omega_0\|_1$. In the case of $2 < p < \infty$, to estimate $I_1(|\omega\ (\cdot,t)|)(x)$ it is sufficient to use the *Young inequality* (§4.1.1).

The estimates of derivatives of the vorticity for $p = 1$ and $p = \infty$ in the corollary below are new. The key step for the proof is to establish a new Gronwall-type lemma.

2.4.2 Estimates for Derivatives of the Vorticity

Theorem. *Let the initial vorticity ω_0 be in $C_0(\mathbb{R}^2)$. Then there exists a positive constant W depending only on $\|\omega_0\|_1$ such that any solution (ω, u) of (2.7), (2.8), and (2.9) satisfies*

(i) $\|\nabla\omega\|_p(t) \leq \frac{W}{t^{\frac{3}{2}-\frac{1}{p}}}\|\omega_0\|_1$ *for* $t > 0$, $1 \leq p \leq \infty$,

(ii) $\|\partial_{x_i}\partial_{x_j}\omega\|_p(t) \leq \frac{W}{t^{2-\frac{1}{p}}}\|\omega_0\|_1$ *for* $t > 0$, $1 \leq p \leq \infty$, $1 \leq i,\ j \leq 2$,

(iii) $\|\partial_t\omega\|_p(t) \leq \frac{W}{t^{2-\frac{1}{p}}}\|\omega_0\|_1$ *for* $t > 0$, $1 \leq p \leq \infty$.

Moreover, the constant $W = W(\|\omega_0\|_1)$ may be chosen such that it is non-decreasing with respect to $\|\omega_0\|_1$.

Corollary. *Let the initial vorticity* ω_0 *be in* $C_0(\mathbb{R}^2)$. *Then there exist positive constants* W_1, W_2 *depending only on a multi-index* β, *a nonnegative integer* b, *and* $\|\omega_0\|_1$ *such that any solution* (ω, u) *of (2.7), (2.8), and (2.9) satisfies*

$$\|\partial_t^b \partial_x^\beta \omega\|_p(t) \le \frac{W_1}{t^{b+\frac{|\beta|}{2}+1-\frac{1}{p}}} \|\omega_0\|_1 \qquad \text{for } 1 \le p \le \infty,$$

$$\|\partial_t^b \partial_x^\beta u\|_p(t) \le \frac{W_2}{t^{b+\frac{|\beta|}{2}+\frac{1}{2}-\frac{1}{p}}} \|\omega_0\|_1 \qquad \text{for } 2 < p \le \infty.$$

Moreover, the constant W_j *may be chosen such that it is nondecreasing with respect to* $\|\omega_0\|_1$.

Note that in this corollary the estimates of derivatives for all orders are obtained. Especially, the estimates in the above theorem are special cases of this corollary for $2b + |\beta| \le 2$. These estimates are obtained using estimates for vorticities and velocities §2.4.1 and the L^p-L^q estimate for derivatives of solutions of the heat equation §1.1.3.

In the following we regard the nonlinear term $-(u, \nabla)\omega$ as a given function, and apply the results of the linear heat equations with inhomogeneous terms. This argument is called the perturbation argument, which is one of the standard methods for analyzing nonlinear partial differential equations.

Proof of Theorem. The basic idea of the proof is as follows: Since $\operatorname{div} u = 0$, we may rewrite the nonlinear term $(u, \nabla)\omega$ of (2.7) as $\operatorname{div}(u\omega)$. Here we consider equation (2.7) on $\mathbb{R}^2 \times (0, \infty)$, and write it as

$$\partial_t \omega - \Delta \omega = \operatorname{div}\ h_1, \quad h_1 = -u\omega.$$

We often suppress the x-dependence of functions of t and x for simplicity. For example, $f(t)$ denotes the function $f(\cdot, t)$ of x on $\mathbb{R}^2$. By the estimates for the vorticity and the velocity obtained in §2.3.3 and §2.4.1, h_1 satisfies

$$\|h_1\|_1(t) \le \|u\|_\infty(t)\|\omega\|_1(t) \le \frac{L_1(\infty)}{(\kappa t)^{1/2}} \|\omega_0\|_1^2, \quad t > 0,$$

where κ is the universal constant of §2.3.1. The function $\|h_1\|_1(t)$ is not always bounded near $t = 0$, but regarding h_1 as a known function and using Theorem 4.4.3, by the "variation-of-constant formula" as it is known for ordinary differential equations of first order, we obtain

$$\omega(t) = e^{t\Delta}\omega_0 + \int_0^t \operatorname{div}\ (e^{(t-s)\Delta} h_1(s))ds \qquad \text{in } \mathbb{R}^2,\ t > 0.$$

Hence ω is expressed by integrals. This is an equality of functions on $\mathbb{R}^2$ with parameter $t > 0$. As explained at the beginning of §4.3, $e^{t\Delta} f$ denotes the function of $x \in \mathbb{R}^2$ at $t > 0$ and it is the solution of the heat equation with initial value f. That is to say, $e^{t\Delta}$ is an operator with a parameter t that operates on functions on $\mathbb{R}^2$. As in §1.1, let G_t be the Gauss kernel. Then $(e^{t\Delta}f)(x) = (G_t * f)(x)$. (For a solution (ω, u) of (2.7), (2.8), and (2.9) it is easy to see that ω is a weak solution (its definition will be given in §4.3.4) of $\partial_t\omega - \Delta\omega = \operatorname{div} h_1$ with initial value ω_0, where the term $\operatorname{div} h_1$ is regarded as a known inhomogeneous term. Hence we can apply Theorem 4.4.3 to this equation.)

Using a positive constant $0 < \varepsilon < 1$, we rewrite this representation[1] for ω by

$$\omega(t) = e^{t\Delta}\omega_0 + \int_{t(1-\varepsilon)}^{t} e^{(t-s)\Delta}(\operatorname{div}\ h_1(s))ds + \int_0^{t(1-\varepsilon)} \operatorname{div}(e^{(t-s)\Delta}\ h_1(s))ds.$$

Here we use the commutativity of $e^{(t-s)\Delta}$ and div given in Proposition 4.1.6. Differentiating both sides with respect to the spatial variables, and taking the L^p-norm $(1 \le p \le \infty)$, we obtain[2]

$$\|\nabla\omega\|_p(t) \le \|\nabla(e^{t\Delta}\omega_0)\|_p + \int_{t(1-\varepsilon)}^{t} \|\nabla(e^{(t-s)\Delta}(\operatorname{div}\ h_1(s)))\|_p ds + \int_0^{t(1-\varepsilon)} \|\nabla(\operatorname{div}(e^{(t-s)\Delta}\ h_1(s)))\|_p ds.$$

Here we write each term of the right-hand side as $J_1(t)$, $J_2(t)$, and $J_3(t)$, respectively. (We divide the integral over the interval $(0, t)$ into integrals over $(0, t(1-\varepsilon))$ and $(t(1-\varepsilon), t)$, since this integral may be unbounded near $s = 0$ and $s = t$. For this reason, for the term with the interval of integration $(0, t(1-\varepsilon))$, we first take the convolution with the Gauss kernel and then differentiate with respect to spatial variables. For the term with the interval of integration $(t(1-\varepsilon), t)$ we differentiate h_1 with respect to the spatial variables. We later take ε small.) We shall estimate J_1, J_2, and J_3. First, by the L^p-L^1 estimate for derivatives of solutions of the heat equation §1.1.3, there exists a constant C_1 that is independent of ω_0 and t (by analyzing the constant that appears in the proof of the estimate in §1.1.3, we can take C_1 even independent of p) such that

$$J_1(t) \le C_1\|\omega_0\|_1 t^{\frac{1}{p}-\frac{3}{2}}, \quad t > 0.$$

Similarly, by §1.1.3, for the integrand of J_2 we have

$$\|\nabla(e^{(t-s)\Delta}(\operatorname{div}\ h_1(s)))\|_p \le \frac{C_1}{(t-s)^{1/2}}\|\operatorname{div}\ h_1\|_p(s), \quad 0 < s < t.$$

[1] The expression $e^{t\Delta}h$ for an $\mathbb{R}^n$-valued function $h = (h^1, \ldots, h^n)$ stands for the $\mathbb{R}^n$-valued function with $e^{t\Delta}h^i$ as the ith component.

[2] In these calculations we always use the property $\|\int f dt\|_p \le \int \|f\|_p dt$ for a function f of x and t. For the proof we refer to Exercise 6.5.

Moreover, using (ii) in §2.4.1 for $\operatorname{div} h_1 = -(u, \nabla)\omega$ we observe that

$$\begin{aligned}\|\operatorname{div}\ h_1\|_p(s) &\le \|u\|_\infty(s)\|\nabla\omega\|_p(s)\\ &\le L_1(\infty)(\kappa s)^{-1/2}\|\omega_0\|_1\|\nabla\omega\|_p(s), \quad 0 < s < t,\end{aligned}$$

for $t > 0$. We obtain

$$J_2(t) \le C_1 L_1(\infty)\kappa^{-1/2}\|\omega_0\|_1 \int_{t(1-\varepsilon)}^t \frac{1}{(t-s)^{1/2}}\frac{1}{s^{1/2}}\|\nabla\omega\|_p(s)ds.$$

For the integrand of J_3, using §1.1.3 as $q = 1$, and §2.4.1 we obtain

$$\|\nabla(\operatorname{div}(e^{(t-s)\Delta}\ h_1(s)))\|_p \le \frac{C_2}{(t-s)^{2-\frac{1}{p}}}\|u\omega\|_1(s) \le \frac{C_2 L_1(\infty)}{(t-s)^{2-\frac{1}{p}}(\kappa s)^{\frac{1}{2}}}\|\omega_0\|_1$$

for $0 < s < t$. Here C_j $(j = 2, 3)$ denotes a constant independent of $p, \omega_0, t,$ and s. If $1 \le p \le \infty$, then the integral

$$\int_0^{t(1-\varepsilon)} (t-s)^{-2+\frac{1}{p}}\ s^{-\frac{1}{2}}ds = A_\varepsilon t^{-\alpha}, \quad A_\varepsilon = \int_0^{1-\varepsilon} (1-\tau)^{-2+\frac{1}{p}}\tau^{-\frac{1}{2}}d\tau$$

converges. Hence, for $1 \le p \le \infty$ we obtain

$$J_3(t) \le C_2 L_1(\infty)\|\omega_0\|_1^2 A_\varepsilon \kappa^{-\frac{1}{2}} t^{-\alpha}, \quad t > 0.$$

By these estimates for J_1, J_2, and J_3, we obtain

$$\begin{aligned}&\|\nabla\omega\|_p(t)\\ &\quad \le \|\omega_0\|_1 \left\{(C_1 + W_1 A_\varepsilon)t^{-\alpha} + C_3\int_{t(1-\varepsilon)}^t \frac{1}{(t-s)^{1/2}}\frac{1}{s^{1/2}}\|\nabla\omega\|_p(s)ds\right\}.\end{aligned}$$

Here, we set

$$W_1 = C_2\kappa^{-1/2}L_1(\infty)\|\omega_0\|_1, \quad C_3 = C_1 L_1(\infty)\kappa^{-1/2}.$$

These constants are not only independent of ω, but also independent of ε except for A_ε. (A_ε diverges to infinity as $\varepsilon \to 0$.) We apply the following Gronwall-type lemma to the above estimate for $\|\nabla\omega\|_p(t)$ with $\psi(t) = \|\nabla\omega\|_p(t)$ to get (i). Here, by (ii) in §2.2.1, $\psi(t)$ is continuous in $t > 0$. However, the boundedness of $t^\alpha\psi(t)$ near $t = 0$ is not clear, so we cannot apply the next lemma directly. We regard $t = \eta > 0$ as an initial time and argue in the same way as above. We then apply the next lemma for $\psi_\eta(t) = \|\nabla\omega\|_p(t+\eta)$ to get

$$\|\nabla\omega\|_p(t+\eta) \le W t^{1/p-3/2}\|\omega_0\|_1.$$

Note that $t^\alpha\psi_\eta(t)$ is bounded near $t = 0$ by Theorem 2.1.1. Since W is independent of η we obtain (i).

Lemma. *Assume that ψ is a continuous function defined on $(0,T)$, where $0 < T \le \infty$ (it suffices to assume that ψ is locally integrable to prove the claim). Let α be a real number, γ, δ positive numbers, and $\gamma+\delta = 1$, $0 < \gamma < 1$. Assume that $t^\alpha \psi(t)$ is bounded near $t = 0$. Moreover, let b_ε be a positive number determined by a positive number $\varepsilon < 1$. (For the sake of simplicity, we assume that b_ε is nonincreasing with respect to ε.) Assume that there exists a constant σ such that*

$$0 \le \psi(t) \le \sigma \left(b_\varepsilon t^{-\alpha} + \int_{t(1-\varepsilon)}^{t} \frac{\psi(s)}{(t-s)^\gamma s^\delta} ds \right),$$

for any $0 < t < T$, $0 < \varepsilon < 1$, and that is independent of ε and t. Then there exists a constant C depending only on σ, α, δ, γ (and b_ε, which is a function of ε) such that

$$\psi(t) \le C\sigma t^{-\alpha}$$

for any $0 < t < T$. Moreover, we may take C nondecreasing with respect to σ.

Proof of Lemma. By the assumption, for $0 < t < T$, we have

$$\psi(t)t^\alpha \le \sigma \left(b_\varepsilon + t^\alpha \int_{t(1-\varepsilon)}^{t} \frac{\psi(s)s^\alpha}{(t-s)^\gamma s^{\delta+\alpha}} ds \right).$$

We consider

$$\varphi(t) = \sup_{0 \le \tau \le t} \tau^\alpha \psi(\tau).$$

Then, for $t > 0$, we have

$$\begin{aligned} \varphi(t) &\le \sigma \left(b_\varepsilon + t^\alpha \varphi(t) \int_{t(1-\varepsilon)}^{t} \frac{ds}{(t-s)^\gamma s^{\delta+\alpha}} \right) \\ &= \sigma \left(b_\varepsilon + \varphi(t) \int_{1-\varepsilon}^{1} \frac{d\tau}{(1-\tau)^\gamma \tau^{\delta+\alpha}} \right). \end{aligned}$$

The last equality is obtained by the coordinate transformation $s = t\tau$ and $\gamma + \delta = 1$. Since $0 < \gamma < 1$,

$$I(\varepsilon) = \int_{1-\varepsilon}^{1} \frac{1}{(1-\tau)^\gamma \tau^{\delta+\alpha}} d\tau$$

converges for $0 < \varepsilon < 1$. Since $I(\varepsilon)$ is an increasing function with respect to ε, for $\sigma > 0$, there exists a unique $\varepsilon > 0$ such that $I(\varepsilon) = \min(\frac{1}{2\sigma}, I(1))$. (I(1) can be ∞.) For such an $\varepsilon = \varepsilon(\sigma)$, we have

$$\varphi(t) \le \sigma b_{\varepsilon(\sigma)} + \frac{1}{2}\varphi(t), \quad \eta/(1-\varepsilon(\sigma)) < t < T.$$

Hence we obtain

$$\varphi(t) \leq 2\sigma b_{\varepsilon(\sigma)}, \quad 0 < t < T.$$

Since $\varepsilon(\sigma)$ is nonincreasing with respect to σ, $C = 2b_{\varepsilon(\sigma)}$ is nonincreasing with respect to σ. Therefore, the lemma is proved. □

Now we return to the proof of the theorem. First, as mentioned in Remark 2.2.1, since $\partial_{x_j} u = \mathbf{K} * \partial_{x_j}\omega$ $(j = 1, 2)$, by the estimate of $\|\nabla\omega\|_p$ in (i), similarly as in the proof of (ii) and (iii) of Theorem 2.4.1, we obtain

$$\|\nabla u\|_p(t) \leq L_1(p) W t^{1/p-1} \|\omega_0\|_1 \quad (2 < p \leq \infty), \quad t > 0.$$

To obtain the estimates for second derivatives of ω, by similar arguments as in (i), we differentiate twice the integral equation satisfied by ω, and then estimate it. This yields

$$\|\nabla\nabla\omega\|_p(t) \leq \|\nabla\nabla(e^{t\Delta}\omega_0)\|_p + \int_{t(1-\varepsilon)}^{t} \frac{C_1}{(t-s)^{1/2}} \|\nabla(u, \nabla)\omega\|_p(s)ds$$

$$+ \int_0^{t(1-\varepsilon)} \frac{C_2'}{(t-s)^{3/2-1/p}} \|h_1\|_1(s)ds$$

(C_2' is a constant independent of p, ω_0, t, s.)

Since $\|\nabla(u, \nabla)\omega\|_p \leq \|\nabla u\|_\infty \|\nabla\omega\|_p + \|u\|_\infty \|\nabla\nabla\omega\|_p$ holds, by substituting the estimates of $\|\nabla u\|_\infty(t)$, $\|u\|_\infty(t)$, $\|\nabla\omega\|_p(t)$, and $\|h_1\|_1(t)$, we obtain an inequality for $\|\nabla\nabla\omega\|_p(t)$, to which the Gronwall-type lemma is applicable. Hence we obtain (ii) from the lemma. Using (2.7), if we apply (i), (ii), and the estimate for $\|u\|_\infty$ in §2.4.1, then we obtain (iii) from $\partial_t\omega = \Delta\omega - (u, \nabla)\omega$. Hence Theorem 2.4.2 is proved. □

Next let us state the outline of the proof of the corollary. First we consider the case $b = 0$. Recalling (ii) of Theorem 2.4.2 and $\partial_x^\beta u = \mathbf{K} * (\partial_x^\beta \omega)$, we can obtain estimates not only for $\|u\|_p(t)$, $\|\nabla u\|_p(t)$, but also for $\|\partial_x^\beta u\|_p(t)$ $(2 < p \leq \infty, |\beta| = 2)$ (which are mentioned in the corollary), analogously to the calculation in the proof of (iii). By similar calculations as in the proof of the theorem, if we differentiate the integral equation of ω three times and use the estimates of $\|\partial_x^\beta u\|_\infty(t)$ for $|\beta| \leq 2$, then we obtain an estimate for $\psi(t) = \sum_{|\beta|=3} \|\partial_x^\beta \omega\|_p(t)$. By applying the Gronwall-type lemma, we obtain the estimate $\|\partial_x^\beta \omega\|_p(t)$ for $|\beta| = 3$, which is claimed in the corollary.

In general, once the claim in the corollary for $\|\partial_x^\beta \omega\|_p(t)$ $(1 \leq p \leq \infty, |\beta| = k \geq 1)$ is established, then by estimating $\mathbf{K} * (\partial_x^\beta \omega)$, we obtain the claim for $\|\partial_x^\beta u\|_p(t)$ $(2 < p \leq \infty, |\beta| = k)$. Next, by differentiating the integral equation of ω $k+1$ times and using the estimate for $\|\partial_x^\gamma u\|_\infty(t)$ for $|\gamma| \leq k$, we get the estimate in the corollary for $\|\partial_x^\mu \omega\|_p(t)$ $(1 \leq p \leq \infty, |\mu| = k+1)$ by the Gronwall-type lemma. Hence by induction with respect to k, we obtain the estimates in the corollary for $b = 0$.

For the estimate of $b > 0$, using $\partial_t \omega = \Delta\omega - (u, \nabla)\omega$, $\partial_t u = \mathbf{K} * \partial_t \omega$ repeatedly, we can replace the time derivative by spatial derivatives. Then from the estimates of ω and u for $b = 0$, we obtain the desired estimates for the time derivative.

Similarly to the case of the heat equation, we will obtain an estimate for the vorticity ω at space infinity. However, the proof is not so simple as in the case of the heat equation.

2.4.3 Decay Estimates for the Vorticity in Spatial Variables

Proposition. *Let the initial value ω_0 be in $C_0(\mathbb{R}^2)$. Then there exists a constant W' satisfying the following property: Assume that (ω, u) is a solution of (2.7), (2.8), and (2.9), and that the support of ω_0 $\operatorname{supp}\omega_0$ is contained in the open ball B_{R_0} with radius R_0. Then we have*

$$\sup_{|x|\geq R} |\omega(x,t)| \leq \frac{W'}{R} \|\omega_0\|_1 \left(\frac{1}{t^{1/2}} + 1 \right),$$

for all $R \geq \max(2R_0, 1)$, $t > 0$, where W' depends only on $\|\omega_0\|_1$ and is nondecreasing with respect to $\|\omega_0\|_1$.

Proof. The outline of the proof is as follows: First we multiply a suitable function to ω to construct a modified function ω_R that vanishes in $B_{R/2}$ and coincides with ω outside B_R. We calculate the equation that ω_R satisfies, and then establish estimates for $\|\omega_R\|_\infty$.

The First Step (Construction of $\boldsymbol{\omega_R}$)

First we choose a function $\theta \in C^\infty[0, \infty)$ satisfying $0 \leq \theta \leq 1$,

$$\theta(\rho) = \begin{cases} 0, & \rho \leq 1/2, \\ 1, & \rho \geq 1, \end{cases}$$

and $\theta' \geq 0$. Then we set

$$\varphi_R(x) = \theta\left(\frac{|x|}{R}\right), \quad x \in \mathbb{R}^2,$$

and define the function ω_R as

$$\omega_R(x,t) = \omega(x,t)\varphi_R(x), \quad x \in \mathbb{R}^2,\ t \geq 0.$$

(By definition, $\omega_R(x,t) = \omega(x,t)$ if $|x| \geq R$, $t > 0$. We may construct such a θ as in the first step of the proof in §4.4.2.) Since ω satisfies (2.7), ω_R satisfies

$$\begin{aligned} &\partial_t \omega_R - \Delta\omega_R + (u, \nabla)\omega_R = h_2, \\ &h_2 = ((u, \nabla)\varphi_R)\omega - 2\langle \nabla\varphi_R, \nabla\omega\rangle - (\Delta\varphi_R)\omega \end{aligned}$$

on $\mathbb{R}^2 \times (0,\infty)$. In the proof below we use the results on inhomogeneous heat equations with a transport term, in which h_2 is regarded as a given inhomogeneous term, while $(u,\nabla)\omega_R$ is not. This is because it seems difficult to derive the desired estimate if we apply the results on inhomogeneous heat equations without a transport term by regarding $(u,\nabla)\omega_R$ as a given function.

By the unique existence theorem (§2.2.1), $u \in C^\infty(\mathbb{R}^2 \times (0,\infty))$, and for any $t_1 > t_0 > 0$, multi-indices α, and $\ell = 0,1,2,\dots$, we have

$$\sup_{t_0 \le t \le t_1} \|\partial_x^\alpha \partial_t^\ell u\|_\infty(t) < \infty.$$

Then by Theorem 4.4.4 we may define an *evolution system* $U(t,s)$, $t > s$, that corresponds to $\partial_t - \Delta + (u,\nabla)$, for $s > 0$. Namely, when $f \in C(\mathbb{R}^2)$ is bounded and integrable, ($\|f\|_\infty < \infty$ and $\|f\|_1 < \infty$), then we may express a solution V of

$$\begin{cases} \partial_t V - \Delta V + (u,\nabla)V = 0 & \text{in } \mathbb{R}^2 \times (s,\infty), \\ V|_{t=s} = f & \text{in } \mathbb{R}^2, \end{cases}$$

by $V(x,t) = (U(t,s)f)(x)$. Moreover, h_2 is a C^∞ function on $\mathbb{R}^2 \times (0,\infty)$, and is zero on $|x| \ge R$. Hence if $1 \le p \le \infty$, $\|h_2\|_p(t)$ is finite for $t > 0$. Therefore $U(t,s)h_2(s)$ for $s > 0$ is well defined. Here $U(t,s)h_2(s)$ is a function of $t > s$ and the spatial variable x. Here and in the sequel, we suppress the x-dependence for simplicity. Similarly as above, for $\omega_R \in C(\mathbb{R}^2 \times (0,\infty))$, $\omega_R(t)$ denotes the function $\omega_R(x,t)$ of x on $\mathbb{R}^2$. Here, ω_R is also C^∞ on $\mathbb{R}^2 \times (0,\infty)$, and by the unique existence theorem (§2.2.1), for $0 < t_0 < t_1$, we have

$$\sup_{t_0 \le t \le t_1} \|\omega_R\|_p(t) < \infty, \qquad 1 \le p \le \infty.$$

Since we have L^∞-estimates for higher derivatives, we may use the evolution system $U(t,s)$. By Theorem 4.4.4 and (ii) of the remark afterward, ω_R is given by

$$\omega_R(t) = U(t,\varepsilon)\omega_R(\varepsilon) + \int_\varepsilon^t U(t,s)h_2(s)ds, \quad 0 < \varepsilon < t,$$

in $\mathbb{R}^2$. This is an equality as functions in $\mathbb{R}^2$ with the parameter t ($> \varepsilon$). (Since ∇u may diverge to infinity at $t = 0$, in order to use Theorem 4.4.4 we introduced $\varepsilon > 0$.)

The Second Step (Estimates for $\int_\varepsilon^t U(t,s)h_2(s)ds$)

First, by the Hölder inequality, for $1 \le q \le \infty$, h_2 is bounded by

$$\begin{aligned} \|h_2\|_q(t) \le \|u\|_\infty(t)\|\nabla\varphi_R\|_\infty\|\omega\|_q(t) + 2\|\nabla\varphi_R\|_\infty\|\nabla\omega\|_q(t) \\ + \|\Delta\varphi_R\|_\infty\|\omega\|_q(t). \end{aligned}$$

By the chain rule and using the constants C_θ and C'_θ, which depend only on θ, we have

$$\|\nabla\varphi_R\|_\infty \le \frac{C_\theta}{R}, \quad \|\Delta\varphi_R\|_\infty \le \frac{C'_\theta}{R^2}, \le \frac{C'_\theta}{R}, \quad R \ge 1.$$

Therefore, using the estimate for derivatives in §2.4.2 and the estimates for u and ω in §2.4.1, for $1 \le q \le \infty$, we obtain

$$\|h_2\|_q(t) \le \frac{W_1}{R}(t^{-\frac{1}{2}} + 1)t^{\frac{1}{q}-1}\|\omega_0\|_1, \quad t > 0, \ R \ge 1.$$

Here W_j $(j = 1, 2, 3)$ are constants, which are independent of R and have the same property as W in §2.4.2.

On the other hand, for p with $1 \le p \le \infty$, by recalling that $\|U(t,s)h_2(s)\|_p$ is continuous at $t = s$ as a function of t $(t \ge s)$ (§4.4.4), and using the fundamental L^q-L^1 estimate (§2.3.1) and its generalization (§2.3.6) we obtain

$$\|U(t,s)h_2(s)\|_\infty \le \frac{1}{\kappa(t-s)}\|h_2\|_1(s), \qquad t > s,$$

$$\|U(t,s)h_2(s)\|_\infty \le \frac{1}{[\kappa(t-s)]^{1/2}}\|h_2\|_2(s), \quad t > s.$$

Using the above estimates, we will estimate $\int_\varepsilon^t U(t,s)h_2(s)ds$. Since the integrand may be infinite at $s = 0$ and $s = t$, we divide the interval of integration. If $0 < \varepsilon < t/2$, we obtain

$$\begin{aligned}
&\left\|\int_\varepsilon^t U(t,s)h_2(s)ds\right\|_\infty \\
&\le \int_\varepsilon^{t/2} \|U(t,s)h_2(s)\|_\infty ds + \int_{t/2}^t \|U(t,s)h_2(s)\|_\infty ds \\
&\le \int_0^{t/2} \frac{1}{\kappa(t-s)}\|h_2\|_1(s)ds + \int_{t/2}^t \frac{1}{[\kappa(t-s)]^{1/2}}\|h_2\|_2(s)ds \\
&\le \frac{W_2}{R}\|\omega_0\|_1 \left\{\int_0^{t/2} \frac{1}{(t-s)}\left(\frac{1}{s^{1/2}} + 1\right) ds\right. \\
&\qquad\qquad \left. + \int_{t/2}^t \frac{1}{(t-s)^{1/2}}\left(\frac{1}{s^{1/2}} + 1\right)\frac{1}{s^{1/2}} ds\right\}.
\end{aligned}$$

Setting $s = t\tau$ and calculating the integral, we obtain

$$\begin{aligned}
&\int_0^{t/2} \frac{1}{(t-s)}\left(\frac{1}{s^{1/2}} + 1\right) ds \\
&= \frac{1}{t^{1/2}}\int_0^{1/2} \frac{1}{(1-\tau)\tau^{1/2}}d\tau + \int_0^{1/2} \frac{1}{1-\tau}d\tau = A_0 t^{-1/2} + A_1,
\end{aligned}$$

$$\int_{t/2}^{t} \frac{1}{(t-s)^{1/2}} \left(\frac{1}{s^{1/2}} + 1 \right) \frac{1}{s^{1/2}} ds$$

$$= \frac{1}{t^{1/2}} \int_{1/2}^{1} \frac{1}{(1-\tau)^{1/2}\tau} d\tau + \int_{1/2}^{1} \frac{1}{(1-\tau)^{1/2}\tau^{1/2}} d\tau = A_2 t^{-1/2} + A_3,$$

where A_0, A_1, A_2, A_3 are real numbers independent of t. Hence we obtain

$$\left\| \int_{\varepsilon}^{t} U(t,s) h_2(s) ds \right\|_{\infty} \le \frac{W_3}{R} \|\omega_0\|_1 \left(\frac{1}{t^{1/2}} + 1 \right),$$

for $t \ge 2\varepsilon > 0$, $R \ge 1$.

The Third Step (Estimate for $U(t,\varepsilon)\omega_R(\varepsilon)$)

By the maximum principle (§2.3.7), for $\varepsilon > 0$, we have

$$\|U(t,\varepsilon)\omega_R(\varepsilon)\|_{\infty} \le \|\omega_R(\varepsilon)\|_{\infty}, \quad t \ge \varepsilon.$$

(By property (§2.2.1) for ω, $\omega_R(\varepsilon)$ belongs to $C(\mathbb{R}^2)$ and $\|\omega_R(\varepsilon)\|_p < \infty$ $(1 \le p \le \infty)$. Moreover, by Theorem 4.4.4, $\|U(t,\varepsilon)\omega_R(\varepsilon)\|_p$ is continuous at $t = \varepsilon$ as a function of t $(t \ge \varepsilon)$. Hence we may apply §2.3.7.) On the other hand, if $R > 2R_0 > 0$, φ_R is zero on the ball B_{R_0}; hence from the assumption on the support of ω_0, we obtain $\omega_R(0) = 0$. By the continuity of $\omega(t)$ at $t = 0$ (§2.2.1 (i)),

$$\lim_{\varepsilon \to 0} \|\omega_R(\varepsilon)\|_{\infty} = \lim_{\varepsilon \to 0} \|\omega_R(\varepsilon) - \omega_R(0)\|_{\infty} \le \lim_{\varepsilon \to 0} \|\varphi_R\|_{\infty} \|\omega(\varepsilon) - \omega_0\|_{\infty} = 0$$

is valid for $R > 2R_0 > 0$. Hence we obtain

$$\lim_{\varepsilon \to 0} \|U(t,\varepsilon)\omega_R(\varepsilon)\|_{\infty} = 0, \quad t > 0,$$

for $R > 2R_0 > 0$.

The Final Step (Completion of the proof)

Taking $R \ge \max(2R_0, 1)$ and estimating the L^{∞}-norm of the formula for ω_R at the end of the first step, we obtain

$$\|\omega_R\|_{\infty}(t) \le \|U(t,\varepsilon)\omega_R(\varepsilon)\|_{\infty} + \frac{W_3}{R} \|\omega_0\|_1 \left(\frac{1}{t^{1/2}} + 1 \right), \quad t > 2\varepsilon,$$

by the second step. Using the result of the third step, as $\varepsilon \to 0$, we obtain

$$\|\omega_R\|_{\infty}(t) \le \frac{W_3}{R} \|\omega_0\|_1 \left(\frac{1}{t^{1/2}} + 1 \right), \quad t > 0.$$

Hence, recalling that $\sup_{|x| \ge R} |\omega(x,t)| \le \|\omega_R\|_{\infty}(t)$, we obtain the desired inequality. □

2.5 Proof of the Asymptotic Formula

Now we prove the asymptotic formula (2.10) in §2.2.2. Assume that the initial vorticity ω_0 is in $C_0(\mathbb{R}^2)$, and that (ω, u) is a solution of (2.7), (2.8), and (2.9). We consider the family $\{(\omega_k, \overline{u}_k)\}_{k\geq 1}$, which is rescaled as in (2.11). First we consider the "compactness" that is announced in §2.2.5. By Proposition 2.2.3, $(\omega_k, \overline{u}_k)$ satisfies (2.7) and (2.8), and its initial values are $\omega_k|_{t=0} = \omega_{0k}$, where we define $\omega_{0k}(x) = k^2\omega(kx), x \in \mathbb{R}^2$. In the estimates in §2.4.1, §2.4.2, and §2.4.3, set $\omega = \omega_k$ and $u = \overline{u}_k$. Since $\|\omega_{0k}\|_1 = \|\omega_0\|_1$, we may take coefficients W and W' independent of k. Hence using the Ascoli–Arzelà-type compactness theorem (§1.3.2), as in the case of the heat equation (§1.3.5), for any subsequence $\{\omega_{k(\ell)}\}_{\ell=1}^{\infty}$, $(\lim_{\ell\to\infty} k(\ell) = \infty)$ of $\{\omega_k\}_{k\geq 1}$, there exists a subsequence $\{\omega_{k(\ell(i))}\}_{i=1}^{\infty}$, $(\lim_{i\to\infty} \ell(i) = \infty)$ such that $\omega_{k(\ell(i))}$ converges pointwise to some function $\overline{\omega} \in C(\mathbb{R}^2 \times (0,\infty))$ on $\mathbb{R}^2 \times (0,\infty)$ as $i \to \infty$. Moreover, its convergence is uniform on $\mathbb{R}^2 \times [\eta, 1/\eta]$ for any $\eta \in (0,1)$.

For the limit function $\overline{\omega}$, we define $\overline{u} = \mathbf{K} * \overline{\omega}$. Then $\overline{\omega}$ is a "weak solution" of

$$\begin{cases} \partial_t\overline{\omega} - \Delta\overline{\omega} + (\overline{u}, \nabla)\overline{\omega} = 0, \\ \overline{u} = \mathbf{K} * \overline{\omega}, \\ \overline{\omega}|_{t=0} = m\delta, \quad m = \displaystyle\int_{\mathbb{R}^2} \omega_0 \, dx, \end{cases}$$

in $\mathbb{R}^2 \times (0,\infty)$, where δ denotes the Dirac δ distribution. We will state this fact in §2.5.1 in a precise form. In §2.5.4 we will prove the uniqueness of this limit function and we will characterize the limit function that is mentioned in the end of §2.2.5.

Before discussing the uniqueness, we will show that if $\overline{\omega}(x,t)$ is smooth on $x \in \mathbb{R}^2$, $t > 0$, and if $1 \leq p \leq \infty$, then for any multi-index β and $b = 0, 1, 2, \ldots$,

$$\sup_{t>0} t^{\frac{|\beta|}{2}+b+1-\frac{1}{p}} \|\partial_t^b \partial_x^\beta \overline{\omega}\|_p(t) < \infty. \tag{2.12a}$$

By Corollary 2.4.2, there exists a positive constant W such that for ω_k, we have

$$\begin{aligned} \sup_{t>0} t^{\frac{|\beta|}{2}+b+1-\frac{1}{p}} \|\partial_t^b \partial_x^\beta \omega_k\|_p(t) &\leq W(\|\omega_{0k}\|_1, \beta, b)\|\omega_{0k}\|_1 \\ &= W(\|\omega_0\|_1, \beta, b)\|\omega_0\|_1, \end{aligned}$$

where the right-hand side is independent of k. In particular, for any $b = 0, 1, 2, \ldots$, any multi-index β, and any $\eta > 0$, we have

$$\sup_{k\geq 1} \sup_{t\geq\eta} \|\partial_t^b \partial_x^\beta \omega_k\|_\infty(t) < \infty.$$

By these estimates, using the theorem on the convergence of higher derivatives (§5.2.5) that is obtained as an application of the Ascoli–Arzelà theorem, we see that $\overline{\omega} \in C^\infty(\mathbb{R}^2 \times (0,\infty))$ and $\partial_t^b \partial_x^\beta \omega_{k(\ell(i))}$ converges uniformly to $\partial_t^b \partial_x^\beta \overline{\omega}$

on any compact subset of $\mathbb{R}^2 \times (0, \infty)$ as $i \to \infty$. On the other hand, for any $t > 0$,

$$\|\partial_t^b \partial_x^\beta \overline{\omega}\|_p(t) \le \varliminf_{i\to\infty} \|\partial_t^b \partial_x^\beta \omega_{k(\ell(i))}\|_p(t)$$

(Exercise 2.6); hence by the estimates

$$t^{\frac{|\beta|}{2}+b+1-\frac{1}{p}} \|\partial_t^b \partial_x^\beta \omega_k\|_p(t) \le W(\|\omega_0\|_1, \beta, b)\|\omega_0\|_1 < \infty,$$

(2.12a) follows.

We will rigorously show that $(\overline{\omega}, \overline{u})$ is a weak solution of the vortex equation with initial value $m\delta$. We note that by (2.12a) and Remark (iv) in §6.3.5, $\overline{u} = \mathbf{K} * \overline{\omega}$ is defined as a smooth function on $\mathbb{R}^2 \times (0, \infty)$.

2.5.1 Characterization of the Limit Function as a Weak Solution

Theorem. *The function $\overline{\omega}$, which is defined by the limit of $\omega_{k(\ell(i))}$ as $i \to \infty$, satisfies*

$$0 = m\varphi(0,0) + \int_0^\infty \int_{\mathbb{R}^2} \{(\varphi_t + \Delta\varphi)\overline{\omega} + \langle \nabla\varphi, \overline{u}\,\overline{\omega}\rangle\} dx\, dt$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0, \infty))$, where we set $\overline{u} = \mathbf{K} * \overline{\omega}$, and $m = \int_{\mathbb{R}^2} \omega_0\, dx$.

Here the pair $(\overline{\omega}, \overline{u})$ with $\overline{u} = \mathbf{K} * \overline{\omega}$ is called a weak solution of (2.7) and (2.8) with initial value $m\delta$ if $(\overline{\omega}, \overline{u})$ satisfies the above integral equality for any $\varphi \in C_0^\infty(\mathbb{R}^2 \times [0, \infty))$. For $\overline{\omega}$ and $\overline{u} = (\overline{u}^1, \overline{u}^2)$, we assume that every term in the above integral equality makes sense and that $\overline{u} = \mathbf{K} * \overline{\omega}$ is well defined. For example, it suffices to assume that $\overline{\omega}$, $|\overline{u}|$, and $|\overline{u}\,\overline{\omega}|$ are locally integrable (§1.4.3) on $\mathbb{R}^2 \times [0, \infty)$, and that $\|\overline{\omega}\|_q(t) < \infty$, $t > 0$, $1 < q < \infty$. Let $g = G_t$. Then $(mg, \mathbf{K} * (mg))$ is a weak solution of (2.7) and (2.8) with initial value $m\delta$ by Lemma 2.2.5 and Exercise 1.9.

The basic idea of the proof is the same as in the case of the heat equation. Since $(\omega_k, \overline{u}_k)$ is a solution of the vortex equation (2.7), (2.8), and (2.9) with initial value ω_{0k}, for $\varphi \in C_0^\infty[\mathbb{R}^2 \times [0, \infty))$, by integration by parts it satisfies

$$0 = \int_{\mathbb{R}^2} \varphi(x,0)\omega_{0k}(x)dx + \int_0^\infty \int_{\mathbb{R}^2} (\varphi_t + \Delta\varphi)\omega_k\, dx\, dt + \int_0^\infty \int_{\mathbb{R}^2} \langle \nabla\varphi, \overline{u}_k\omega_k\rangle dx\, dt.$$

By the estimate $\sup_{t>0} \|\omega_k\|_1(t) \le \|\omega_0\|_1$ in §2.3.3, as $k \to \infty$, the first and second terms on the right-hand side converge to

$$m\varphi(0,0) + \int_0^\infty \int_{\mathbb{R}^2} (\varphi_t + \Delta\varphi)\overline{\omega}\, dx\, dt,$$

by similar arguments as in Proposition 1.4.1 and in the proof of §1.4.4. In this proof, ω_k simply denotes a subsequence $\omega_{k(\ell(i))}$ of ω.

Moreover, for each fixed $t > 0$, $\overline{u}_k$ converges uniformly to $\overline{u}$ for $x \in \mathbb{R}^2$. This fact will be established at the end of the proof. Set

$$F_k(t) = \int_{\mathbb{R}^2} \langle \nabla\varphi, \overline{u}_k \omega_k \rangle dx.$$

If $\overline{u}_k$ converges uniformly to $\overline{u}$, since the support of $\varphi(\cdot, t)$ is compact for each $t > 0$ and so the region of integration is actually bounded (hence we can interchange integrals and limits as in Proposition 7.1), we see that $F_k(t)$ converges to

$$F(t) = \int_{\mathbb{R}^2} \langle \nabla\varphi, \overline{u}\;\overline{\omega} \rangle dx$$

for each $t > 0$. On the other hand, by Theorem 2.4.1, we get

$$|F_k(t)| \le C_\varphi \|\overline{u}_k\|_\infty(t) \|\omega_k\|_1(t) \le C_\varphi \|\omega_0\|_1^2 L_1(\infty)(\kappa t)^{-1/2},$$

where κ is the universal constant of §2.3.1 and $C_\varphi = \sup\{|\nabla\varphi(x,t)| : x \in \mathbb{R}^2, t \ge 0\} < \infty$. Since $t^{-1/2}$ is integrable on neighborhoods of $t = 0$, and φ is zero for sufficiently large t, by the dominated convergence theorem (§7.1.1), we obtain

$$\lim_{k\to\infty} \int_0^\infty F_k(t)dt = \int_0^\infty F(t)dx.$$

Hence, we have proved

$$\lim_{k\to\infty} \int_0^\infty \int_{\mathbb{R}^2} \langle \nabla\varphi, \overline{u}_k \omega_k \rangle dx\, dt = \int_0^\infty \int_{\mathbb{R}^2} \langle \nabla\varphi, \overline{u}\,\overline{\omega} \rangle dx\, dt,$$

and $(\overline{\omega}, \overline{u})$ is a weak solution of (2.7) and (2.8) with initial value $m\delta$.

In the following, we show that $\overline{u}_k$ converges uniformly to $\overline{u}$ with respect to $x \in \mathbb{R}^2$ for each fixed $t > 0$. First set $v_k = \overline{u}_k - \overline{u}$, $w_k = \omega_k - \overline{\omega}$. We note that $v_k = \mathbf{K} * w_k$. By the Gagliardo–Nirenberg inequality (§6.1.1), the Calderón–Zygmund inequality (§6.4.2), and the Hardy–Littlewood–Sobolev inequality (§6.2.1) (by fixing $t > 0$) for $2 < p < \infty$ we have

$$\begin{aligned}\|v_k\|_\infty(t) &\le C\|\nabla v_k\|_p^{2/p}(t)\|v_k\|_p^{1-2/p}(t) \\ &\le C'\|w_k\|_p^{2/p}(t)\|w_k\|_q^{1-2/p}(t), \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{2}.\end{aligned}$$

(See (i) of the remark below.) Here the constants C and C' depend only on p and are independent of t. Using the Hölder inequality, we obtain

$$\|v_k\|_\infty(t) \le C'\|w_k\|_q^{1-2/p}(t)\{\|w_k\|_q^{q/p}(t)\|w_k\|_\infty^{1-q/p}(t)\}^{2/p}.$$

Since $q/p = 2/(p+2)$, the right-hand side of the last inequality is

$$C'\|w_k\|_q^{\frac{p}{2+p}}(t)\;\|w_k\|_\infty^{\frac{2}{2+p}}(t).$$

By (i) of Theorem 2.4.1, for $t > 0$, $1 \leq r \leq \infty$, we have $\|\omega_k\|_r(t) \leq (\kappa t)^{-1+1/r}\|\omega_0\|_1$, which leads to $\|\overline{\omega}\|_r(t) \leq (\kappa t)^{-1+1/r}\|\omega_0\|_1$ as in (2.12a). Hence $\|w_k\|_r(t) \leq 2(\kappa t)^{\frac{1}{r}-1}\|\omega_0\|_1$ and we obtain

$$\|v_k\|_\infty(t) \leq 2^{\frac{p}{2+p}} C'(\kappa t)^{\left(\frac{2-p}{2}\cdot\frac{1}{2+p}\right)}\|\omega_0\|_1^{\frac{p}{2+p}}\|w_k\|_\infty^{\frac{2}{2+p}}(t).$$

By the definition of the subsequence $\{\omega_{k(\ell(i))}\}_{i=1}^\infty$ (§2.5), for $\eta \in (0,1)$, w_k converges uniformly to 0 on $\mathbb{R}^2 \times [\eta, 1/\eta]$. Hence v_k converges uniformly to 0 on the same region. □

Remark.

(i) Since we do not use distribution theory or Lebesgue integrals, we need to check that $v_k(x,t)$ is C^1 on $\mathbb{R}^2$ as a function of x in the step using the Gagliardo–Nirenberg inequality. By Theorem 2.2.1, $\overline{u}_k$ is smooth on $\mathbb{R}^2 \times (0,\infty)$. Moreover, similarly to Remark 2.5.1, $\overline{u} = \mathbf{K} * \overline{\omega}$ is smooth on $\mathbb{R}^2 \times (0,\infty)$. Hence for each fixed $t > 0$, $v_k(x,t)$ is C^∞ on $\mathbb{R}^2$ as a function of x. On the other hand, the continuity of w_k and $\|w_k\|_q(t) < \infty$ $(1 < q < 2)$, which is assumed when we apply the Hardy–Littlewood–Sobolev inequality (see also Remark 6.2.1), is also obtained by the continuity of $\overline{\omega}$ and (2.12a) for $\overline{\omega}$ in the case $b = 0$ and $|\beta| = 0$. We can also check that the Calderón–Zygmund inequality is available in our case, since $\overline{\omega}$ is smooth and $\|\overline{\omega}\|_p(t)$ is bounded for any $t > 0$, $1 \leq p \leq \infty$ by (2.12a) with $b = 0$ and $|\beta| = 0$. Note that these justifications are not needed if we use distributions and Lebesgue integrals. The key fact in this proof is Theorem 2.4.1, in particular, the L^q-L^1 estimate for ω. Hence we need not assume that $\overline{\omega}$ and $\overline{u}$ are C^∞ on $t > 0$.

(ii) The function $\overline{u}$ is C^∞ on $\mathbb{R}^2 \times (0,\infty)$ and for any multi-index β, $b = 0$, 1, 2, ..., and $2 < p \leq \infty$ it satisfies

$$\sup_{t>0} t^{\frac{|\beta|}{2}+b+\frac{1}{2}-\frac{1}{p}}\|\partial_x^\beta \partial_t^b \overline{u}\|_p(t) < \infty. \tag{2.12b}$$

To prove (2.12b) we first note that $\overline{u} \in C^\infty(\mathbb{R}^2 \times (0,\infty))$, and that $\partial_x^\beta \partial_t^b \overline{u}_{k(\ell(i))}$ converges uniformly to $\partial_x^\beta \partial_t^b \overline{u}$ on any compact subset of $\mathbb{R}^2 \times (0,\infty)$ as $i \to \infty$. This can be verified by §5.2.5 and the fact that $\overline{u}_k$ converges pointwise to $\overline{u}$ on $\mathbb{R}^2 \times (0,\infty)$. We can check that the assumption required in §5.2.5 is satisfied if we use Corollary 2.4.2 and the estimates

$$\sup_{k\geq 1}\sup_{t>0} t^{\frac{|\beta|}{2}+b+\frac{1}{2}-\frac{1}{p}}\|\partial_x^\beta \partial_t^b \overline{u}_k\|_p(t) < \infty, \quad 2 < p \leq \infty,$$

which can be obtained as the estimates for ω_k. Therefore, as in the proof of the estimates for $\overline{\omega}$ in (2.12a), we obtain (2.12b) from the estimates for $\overline{u}_k$.

Next, we will prove the estimate of $\overline{\omega}$ by $|m|$. As mentioned in the proof of the theorem, we have

$$\sup_{t>0}(\kappa t)^{1-1/p}\|\overline{\omega}\|_p(t) \le \|\omega_0\|_1.$$

If we try to prove the uniqueness of the limit $\overline{\omega}$ under the assumption of the smallness for $|m|$ but not for $\|\omega_0\|_1$, it is needed to prove the better estimate

$$\sup_{t>0}(\kappa t)^{1-1/p}\|\overline{\omega}\|_p(t) \le |m|.$$

(The proof of uniqueness without this estimate is not known so far.) This estimate by $|m|$ is stronger than the estimate by $\|\omega_0\|_1$. Indeed, it claims that if $m = \int \omega_0 dx = 0$ then $\overline{\omega} \equiv 0$. The next section is devoted to the proof of the above estimate by $|m|$. This estimate was first established in the Japanese edition of this book, but recently was also obtained by [Gallay Wayne 2005] in an implicit way (see §2.8). In the proof below we use some results on fundamental solutions of parabolic operators that are generalizations of the heat operator $\partial_t - \Delta$.

2.5.2 Estimates for the Limit Function

Theorem. *Assume that the pair of functions $(\overline{\omega}, \overline{u})$ and m are given as in Theorem 2.5.1. Then we have*

$$\sup_{t>0}(\kappa t)^{1-1/p}\|\overline{\omega}\|_p(t) \le |m|, \quad 1 \le p \le \infty,$$

where κ is the universal constant in the fundamental L^q-L^1 estimate in §2.3.1.

Proof. **The First Step**

First we show that it is sufficient to prove the above theorem in the case $p = 1$ only. Since ω_k satisfies $(H_{\overline{u}_k})$ on $\mathbb{R}^2 \times (0,\infty)$ for $k \ge 1$, $\overline{\omega}$ satisfies $(H_{\overline{u}})$ on $\mathbb{R}^2 \times (0,\infty)$ for the limit $(\overline{\omega}, \overline{u})$ of any subsequence of $(\omega_k, \overline{u}_k)$. This is because, as mentioned in the paragraph containing (2.12a) of §2.5 and in (ii) of Remark 2.5.1, subsequences $\{\omega_{k(\ell(i))}\}$ and $\{\overline{u}_{k(\ell(i))}\}$ of $\{\omega_k\}$ and $\{\overline{u}_k\}$ converge to $\overline{\omega}$ and $\overline{u}$ respectively together with their higher derivatives uniformly on each compact set in $\mathbb{R}^2 \times (0,\infty)$. Moreover, by (2.12a) and (2.12b), $\overline{\omega}$ and $\overline{u}$ satisfy the assumption of the fundamental L^q-L^1 estimates in §2.3.1, except for condition (I). Hence the system of differential inequalities in Proposition 2.3.4 holds. So if we show that

$$\|\overline{\omega}\|_1(t) \le |m|, \quad t > 0,$$

then we can prove the estimate in the theorem for general p, similarly to §2.3.5 using Lemma 2.3.4.

The Second Step

For an $\mathbb{R}^2$-valued function v defined on $\mathbb{R}^2 \times (0, \infty)$, $\mathit{\Gamma}_v(x, t, y, s)$ $(x,\, y \in \mathbb{R}^2,$ $t > s \geq 0)$ denotes the *fundamental solution* of the operator $\partial_t - \Delta + (v, \nabla)$. (The definition and basic properties of the fundamental solution will be given in §4.4.5.) As in the proof of Theorem 2.4.2, for $t > 0$, $(\omega_k, \overline{u}_k)$ satisfies

$$\omega_k(t) = e^{t\Delta}\omega_{0k} - \int_0^t \operatorname{div}\, (e^{(t-s)\Delta}(\overline{u}_k \omega_k)(s))ds \quad \text{ in } \mathbb{R}^2.$$

By (iv) of Theorem 2.4.1, $\overline{u}_k$ is bounded near $t = 0$. Moreover, by (ii) of Theorem 2.4.1 we have $\sup_{0<t<T} \|\overline{u}_k\|_\infty(t)\infty$ for any $T > 0$. Since $\operatorname{div}\, \overline{u}_k = 0$ in $\mathbb{R}^2 \times (0, \infty)$, there exists a unique fundamental solution $\mathit{\Gamma}_{\overline{u}_k}(x, t, y, s)$ on $t > s \geq 0$ (by the unique existence theorem, Theorem 2 in §4.4.5). On the other hand, by (i) of Theorem 2.4.1, we have $\sup_{t>0} \|\omega_k\|_1(t) \leq \|\omega_{0k}\|_1 < \infty$; hence as in §4.4.5, by the lemma for the uniqueness in §4.4.4,

$$\omega_k(x, t) = \int_{\mathbb{R}^2} \mathit{\Gamma}_{\overline{u}_k}(x, t, y, 0)\omega_{0k}(y)dy, \quad t > 0, \quad x \in \mathbb{R}^2.$$

Since $\|\omega_k\|_1(t) \leq \|\omega_0\|_1$ for $t > 0$, by the following lemma the family of functions $\{\mathit{\Gamma}_{\overline{u}_k}(x, t, y, 0)\}_{k\geq 1}$ is uniformly bounded and equicontinuous as functions of $y \in \mathbb{R}^2$ for each $t > 0$, $x \in \mathbb{R}^2$. That is, for each $x \in \mathbb{R}^2$ and $t > 0$, we have

$$\sup_{k\geq 1} \sup_{y\in\mathbb{R}^2} |\mathit{\Gamma}_{\overline{u}_k}(x, t, y, 0)| < \infty,$$

$$\lim_{y'\to y} \sup_{k\geq 1} |\mathit{\Gamma}_{\overline{u}_k}(x, t, y', 0) - \mathit{\Gamma}_{\overline{u}_k}(x, t, y, 0)| = 0, \quad y \in \mathbb{R}^2.$$

Hence for each $R > 0$, by applying the Ascoli–Arzelà theorem (§5.1.1) to this family on a closed ball $\overline{B}_R$, $\{\mathit{\Gamma}_{\overline{u}_k}(x, t, y, 0)\}_{k\geq 1}$ contains a uniformly convergent subsequence on $\overline{B}_R$ as functions of y. In other words, there exist a subsequence $\{k_j\}$ of $\{k(\ell(i))\}_{i=1}^\infty$ and a continuous function $A_{t,x}(y)$ on $\overline{B}_R$ such that

$$\lim_{j\to\infty} \sup_{y\in B_R} |\mathit{\Gamma}_{\overline{u}_{k_j}}(x, t, y, 0) - A_{t,x}(y)| = 0, \quad t > 0,\ x \in \mathbb{R}^2.$$

(More precisely, we should write $\{k_j\}_{j=1}^\infty$ as $\{k(\ell(i(j)))\}_{j=1}^\infty$. For simplicity we abbreviate such a notation. We assume $k_j \to \infty$ as $j \to \infty$.)

Lemma. *Assume that* $v = (v^1, v^2)$ *with* $v^1,\, v^2 \in C^\infty(\mathbb{R}^2 \times (0, \infty))$ *satisfies* $\operatorname{div} v = 0$ *on* $\mathbb{R}^2 \times (0, \infty)$, *and that for each* $S > 0$, $\sup_{0<t<S} \|v\|_\infty(t) < \infty$. *Then there exists a unique fundamental solution* $\mathit{\Gamma}_v(x, t, y, s)$ *of* $\partial_t - \Delta + (v, \nabla)$ *(see the unique existence theorem, Theorem 2 in §4.4.5) that is continuous on* $\{(x, t, y, s) : x, y \in \mathbb{R}^2, 0 \leq s < t < \infty\}$. *Moreover, assume that* $\sup_{t_1\leq t\leq t_2} \|\partial_x^\alpha \partial_t^\ell v\|_\infty(t) < \infty$ *for each* $t_2 > t_1 > 0$ *and for any multi-index* α *and* $\ell = 0,\, 1,\, 2, \ldots$. *Then the following are valid:*

(i) *We have* $0 \le \Gamma_v(x,t,y,s) \le (\kappa(t-s))^{-1}$, $x,\ y \in \mathbb{R}^2$, $0 \le s < t$, *where* κ *is the universal constant in §2.3.1.*

(ii) *Assume that* $0 < t < T$, *and that the function* v *is given as* $v = \mathbf{K} * \omega$ *with a function* $\omega \in C(\mathbb{R}^2 \times (0,T))$. *Moreover, let* $\sup_{0<t<T} \|\omega\|_1(t) \le M_1$ *and let* $t_0 > 0$. *Then there exist a positive constant* C *depending only on* t_0 *and* M_1 *and a constant* $\mu \in (0,1)$ *such that*

$$|\Gamma_v(x,t,y,0) - \Gamma_v(x',t,y',0)| \le C(|x-x'|^2 + |y-y'|^2)^{\mu/2}, \qquad T > tt_0,\ x,x',y,y' \in \mathbb{R}^2.$$

The Third Step

If we replace k by k_j and take $j \to \infty$ in the expression of ω_k by the fundamental solution in the second step, then we obtain

$$\overline{\omega}(x,t) = m\ A_{t,x}(0), \quad t > 0,\ x \in \mathbb{R}^2.$$

In what follows we assume $k_j \ge 1$. Choosing the radius R such that $\operatorname{supp}\omega_0 \subset B_R$, we calculate

$$\omega_{k_j}(x,t) - mA_{t,x}(0) = \int_{B_R} \{\Gamma_{\overline{u}_{k_j}}(x,t,y,0) - A_{t,x}(y)\}\omega_{0k_j}(y)dy + \int_{B_R} A_{t,x}(y)\omega_{0k_j}(y)dy - mA_{t,x}(0).$$

Since $\operatorname{supp}\omega_{0k_j} \subset B_R$, from the properties of the limit of the initial value (see Proposition 1.4.1, Remark 1.4.1, and §4.2.5), the equality

$$\lim_{j\to\infty} \int_{B_R} A_{t,x}(y)\omega_{0k_j}(y)dy = mA_{t,x}(0), \quad t > 0, \quad x \in \mathbb{R}^2,$$

follows. On the other hand, by the uniform convergence of $\Gamma_{\overline{u}_{k_j}}$ and by $\|\omega_{0k_j}\|_1 = \|\omega_0\|_1$, which are obtained in the Second Step, for $t > 0$ and $x \in \mathbb{R}^2$, we obtain

$$\left|\int_{B_R} \{\Gamma_{\overline{u}_{k_j}}(x,t,y,0) - A_{t,x}(y)\}\omega_{0k_j}(y)dy\right| \le \sup_{y\in B_R} |\Gamma_{\overline{u}_{k_j}}(x,t,y,0) - A_{t,x}(y))|\ \|\omega_0\|_1 \to 0 \quad (j \to \infty).$$

Hence for $t > 0$, and $x \in \mathbb{R}^2$, we have shown that $\lim_{j\to 0}\omega_{k_j}(x,t) = mA_{t,x}(0)$, and then $\overline{\omega}(x,t) = mA_{t,x}(0)$ follows, since $\overline{\omega}$ is the limit of ω_{k_j}.

The Fourth Step

Since $\operatorname{div} \overline{u}_{k_j} = 0$, as in §4.4.5, it follows that

$$\int_{\mathbb{R}^2} \Gamma_{\overline{u}_{k_j}}(x,t,y,0)dx = 1, \quad y \in \mathbb{R}^2, \quad t > 0.$$

Moreover, since $\Gamma_{\overline{u}_{k_j}} \geq 0$, we have $A_{t,x}(0) \geq 0$. Hence, by Fatou's lemma (§7.1.2), we obtain

$$\int_{\mathbb{R}^2} A_{t,x}(0)dx \leq \varliminf_{j\to\infty} \int_{\mathbb{R}^2} \Gamma_{\overline{u}_{k_j}}(x,t,0,0)dx = 1.$$

Hence we obtain $\|\overline{\omega}\|_1(t) \leq |m|$ for $t > 0$. This completes the proof except for the proof of the lemma. □

Proof of Lemma. (i) As stated in §4.4.5, $\Gamma_v \geq 0$ is an important property of the fundamental solution, which follows from the nonnegativity-preserving principle in §2.3.8. By the definition of the fundamental solution and the assumption on v, the function w given by

$$w(x,t) = \int_{\mathbb{R}^2} \Gamma_v(x,t,y,s)f(y)dy, \quad t > s > 0, \ x \in \mathbb{R}^2,$$

for $f \in C_0(\mathbb{R}^2)$ satisfies (H_v) on $\mathbb{R}^2 \times (s,T)$. By the assumption on higher derivatives of v, if $s > 0$, then w coincides with the solution constructed in §4.4.4 (§4.4.5). Since we assumed $s > 0$, assumptions (I) and (a) in §2.3.1 are satisfied (ω has to be replaced by w) by Theorem 4.4.4. By the fundamental L^q-L^1 estimate (§2.3.1), we obtain

$$\|w\|_\infty(t) \leq (\kappa(t-s))^{-1}\|f_0\|_1, \qquad t > s > 0.$$

Now let $w_0 \in C_0(\mathbb{R}^2)$ be a given function satisfying $w_0 \geq 0$ and $w_0 \not\equiv 0$, and set $m = \int_{\mathbb{R}^2} w_0(y)dy$. For a given $y_0 \in \mathbb{R}^2$ and $k \geq 1$, set $w_{0k}(y) = k^2 w_0(k(y-y_0)+y_0)$. Then, since $\Gamma_v(x,t,y,s)$ is continuous with respect to y (Definition 4.4.5), we obtain

$$m\Gamma_v(x,t,y_0,s) = \lim_{k\to\infty} w_k(x,t)$$

and

$$w_k(x,t) = \int_{\mathbb{R}^2} \Gamma_v(x,t,y,s)w_{0k}(y)dy, \qquad s > 0,$$

by Remark 1.4.1, Proposition 1.4.1, and §4.2.5. On the other hand, since $w_k \geq 0$ by $\Gamma_v \geq 0$ and $w_{0k} \geq 0$, using the above fundamental L^q-L^1 estimate, we obtain

$$\int_{\mathbb{R}^2} \Gamma_v(x,t,y,s)w_{0k}(y)dy \leq \|w_k\|_\infty(t) \leq (\kappa(t-s))^{-1}m,$$
$$t > s > 0, \quad x \in \mathbb{R}^2.$$

Here we used $m = \int_{\mathbb{R}^2} w_{0k}(y)dy$. Therefore, by letting $k \to \infty$, for $t > 0$ and $x \in \mathbb{R}^2$, we obtain

$$0 \le \Gamma_v(x, t, y_0, s) \le (\kappa(t-s))^{-1}, \quad t > s > 0.$$

By the continuity of Γ_v in $s \in [0, t)$, for $s \ge 0$ we obtain inequality (i).

In order to prove (ii), we appeal to general results on elliptic and parabolic equations with discontinuous coefficients. This, however, exceeds the range of this book. For a proof the reader is referred to [Osada 1987] (see also [Giga Miyakawa Osada 1988]). There the structure of the Biot–Savart law and the Nash–Moser methods are effectively used to prove (ii). □

To continue the proof of Theorem 2.5.2, instead of (ii) of the lemma, it is sufficient to prove that if

$$\sup_{0<t<T} t^{\frac{1}{2}+\frac{|\alpha|}{2}} \|\partial_x^\alpha v\|_\infty(t) \le M_1, \quad |\alpha| \le 1,$$

then

$$|\partial_y^\alpha \Gamma_v(x, t, y, 0)| \le Ct^{-3/2}, \quad T > t > 0,\ |\alpha| = 1,\ x, y \in \mathbb{R}^2.$$

(Here C is a constant depending only on M_1.) However, it is *not known whether such an estimate is valid.* On the other hand, one can prove the estimate

$$|\partial_x^\alpha \Gamma_v(x, t, y, 0)| \le Ct^{-3/2}, \quad T > t > 0,\ |\alpha| = 1,\ x, y \in \mathbb{R}^2,$$

by similar arguments as in (i) of §2.4.2. In [Maekawa 2008b] under the assumption that $\sup_{t>0} t^{\frac{1}{2}} \|v\|_\infty(t) < \infty$ and div $v(t) = 0$ (but the special structure for the velocity v of $v = \mathbf{K} * \omega$ is not assumed there), the Hölder continuity in (ii) of the lemma is obtained by establishing pointwise Gaussian lower bounds for fundamental solutions.

Finally, we will prove that if $|m|$ is sufficiently small, the weak solution satisfying (2.12a) is unique. As in the case of the heat equation, the uniqueness of the weak solution shows that the limit function $\overline{\omega}$ agrees with mg, which is the weak solution with initial value $m\delta$. By this result, we can prove the asymptotic formula (2.10). As a first step to prove the uniqueness we see that $\overline{\omega}$ satisfies the following integral equation.

2.5.3 Integral Equation Satisfied by Weak Solutions

Proposition. *Assume that the pair of functions $(\overline{\omega}, \overline{u})$ is a weak solution of (2.7) and (2.8) with initial value $m\delta$, where $m \in \mathbb{R}$. Moreover, we assume that $\overline{\omega}$ and $\overline{u}$ are smooth on $\mathbb{R}^2 \times (0, \infty)$ and $\overline{\omega}$ satisfies (2.12a). Then $(\overline{\omega}, \overline{u})$ satisfies*

$$\overline{\omega}(t) = mG_t - \int_0^t \operatorname{div}\,(e^{(t-s)\Delta}(\overline{u}\,\overline{\omega})(s))ds \qquad \textit{in } \mathbb{R}^2$$

for $t > 0$.

Remark. If $\overline{\omega}$ satisfies (2.12a), the velocity $\overline{u}$ defined by $\overline{u} = \mathbf{K} * \overline{\omega}$ satisfies (2.12b) automatically. By Remark 6.3.5, $\overline{u}$ is smooth on $\mathbb{R}^2 \times (0, \infty)$, and satisfies $\partial_t^b \partial_x^\beta (\mathbf{K} * \overline{\omega}) = \mathbf{K} * (\partial_t^b \partial_x^\beta \overline{\omega})$. (Here $b = 0, 1, 2, \ldots$, and β is a multi-index.) Therefore by using the Calderón–Zygmund inequality, the Hardy–Littlewood–Sobolev inequality, and the Gagliardo–Nirenberg inequality as in the proof of (ii) and (iii) of Theorem 2.4.1, we obtain (2.12b) for $\overline{u} = \mathbf{K} * \overline{\omega}$.

Proof. First we set $h(s) = -\overline{u}(s)\overline{\omega}(s)$. Regarding h as a given function, we consider $\overline{\omega}$ as a weak solution of $\partial_t \overline{\omega} - \Delta \overline{\omega} = \operatorname{div} h$ on $\mathbb{R}^2 \times (0, \infty)$ with initial value $m\delta$ (see Definition 4.3.4). We shall apply Theorem 4.4.3. Since $\overline{u}$ and $\overline{\omega}$ are smooth in $\mathbb{R}^n \times (0, \infty)$, h is also smooth in $\mathbb{R}^n \times (0, \infty)$. By the estimate for the derivatives of $\overline{\omega}$ (2.12a) and the estimate for the derivatives of $\overline{u}$ (2.12b), we obtain

$$\sup_{\delta \le t \le T} \|\partial_x^\beta \partial_t^\ell h\|_\infty(t) < \infty, \quad 0 < \delta < T.$$

Moreover, $\sup_{t>0} t^{1/2} \|\overline{u}\|_\infty(t) < \infty$ by (2.12b) and $\sup_{t>0} \|\overline{\omega}\|_1(t) < \infty$ by (2.12a) imply $\sup_{t>0} t^{1/2} \|h\|_1(t) < \infty$. Since $\sup_{t>0} \|\overline{\omega}\|_1(t) < \infty$, we can apply Theorem 4.4.3, and the integral equality in the proposition is proved. □

Here we used the smoothness of the weak solution for $t > 0$ and estimate (2.12a). But for the proof of the proposition, instead of the smoothness for $\overline{\omega}$ and (2.12a), it is sufficient to assume the local integrability of $\overline{\omega}$ on $\mathbb{R}^2 \times (0, \infty)$ and $\sup_{t>0} t^{1-1/p} \|\overline{\omega}\|_p(t) < \infty$ for each p with $1 \le p \le \infty$.

This follows from the fact that a weak solution of this type is always smooth in $t > 0$ and satisfies (2.12a); see [Giga Miyakawa Osada 1988].

2.5.4 Uniqueness of Solutions of Limit Equations

Theorem. *For a positive constant c_0, there exists a (small) positive number m_0 such that the following statement is satisfied. Let $(\overline{\omega}, \overline{u})$ be a weak solution of (2.7) and (2.8) with initial value $m\delta$. Assume that $\overline{\omega}$ and $\overline{u}$ are smooth on $\mathbb{R}^2 \times (0, \infty)$ and satisfy (2.12a) and*

$$\sup_{t>0} t^{1/4} \|\overline{\omega}\|_{4/3}(t) \le c_0 |m|.$$

If $|m| < m_0$, then $\overline{\omega} = mg$ on $\mathbb{R}^2 \times (0, \infty)$. Here $g(x, t) = G_t(x)$ denotes the Gauss kernel.

Proof. First we assume that for $i = 1, 2$, (ω_i, u_i) are weak solutions of (2.7) and (2.8) with initial value $m\delta$ such that ω_i is smooth on $\mathbb{R}^2 \times (0, \infty)$ and satisfies (2.12a). We will show that $\omega_1 \equiv \omega_2$ on $\mathbb{R}^2 \times (0, \infty)$. By §2.5.3, for any $t > 0$,

$$\omega_i(t) = mG_t - \int_0^t \operatorname{div}\ (e^{(t-s)\Delta} (u_i \omega_i)(s)) ds, \quad i = 1, 2,$$

in $\mathbb{R}^2$. Set $w = \omega_1 - \omega_2$, $v = u_1 - u_2$. Then w satisfies

$$w(t) = \int_0^t \text{div}\ (e^{(t-s)\Delta} h_2(s))ds \quad \text{in } \mathbb{R}^2, \quad t > 0,$$
$$h_2 = -u_1 w - v\omega_2 \quad \text{in } \mathbb{R}^2 \times (0, \infty).$$

Using this integral equation for w, we estimate $\|w\|_{4/3}(t)$. By the L^p-L^q estimate for derivatives of the heat equation (§1.1.3), for p and q with $1 \le q \le p \le \infty$, we obtain

$$\|\text{div}\,(e^{(t-s)\Delta} h_2)\|_p \le \frac{C_1}{(t-s)^{\frac{1}{2}+\frac{1}{q}-\frac{1}{p}}} \|h_2\|_q(s), \quad 0 < s < t.$$

(Here and in the sequel, C_j, $j = 1, 2, 3$, are constants independent of s, t, ω_i, and u_i $(i = 1, 2)$.) On the other hand, for $v = \mathbf{K} * w$ and $u_i = \mathbf{K} * \omega_i$, using the Hardy–Littlewood–Sobolev inequality (§6.2.1), we obtain

$$\|v\|_r(t) \le L_1(r)\|w\|_{p_1}(t), \quad \|u_i\|_r(t) \le L_1(r)\|\omega_i\|_{p_1}(t),$$
$$1/r = 1/p_1 - 1/2,\ 1 < p_1 < 2,\ i = 1, 2,$$

for $t > 0$. Here $L_1 = L_1(r)$ is a constant depending only on r. Let us take the $L^{4/3}$-norm of both sides of the integral equation for w. Then using these inequalities and the Hölder inequality, i.e., $\|u_1 w\|_1 \le \|u_1\|_4 \|w\|_{4/3}$ and $\|v\omega_2\|_1 \le \|v\|_4 \|\omega_2\|_{4/3}$, we have

$$\|w\|_{4/3}(t) \le C_1 L_1(4) \int_0^t \frac{1}{(t-s)^{3/4}} \{\|\omega_1\|_{4/3}(s) + \|\omega_2\|_{4/3}(s)\} \|w\|_{4/3}(s) ds$$

for $t > 0$. If ω_i $(i = 1, 2)$ satisfies $\sup_{t>0} t^{1/4} \|\omega_i\|_{4/3}(t) \le c_0|m|$, then we obtain

$$\|w\|_{4/3}(t) \le C_1 L_1(4) \int_0^t \frac{1}{(t-s)^{3/4}} \frac{2c_0|m|}{s^{1/4}} \|w\|_{4/3}(s) ds$$

for $t > 0$. We set $t = \tau$ and multiply both sides by $\tau^{1/4}$. Then, by taking the supremum of both sides on $(0, t)$ with respect to τ, we obtain

$$\sup_{0<\tau<t} \tau^{1/4} \|w\|_{4/3}(\tau)$$
$$\le 2C_1 L_1(4) c_0 |m| \left\{ \sup_{0<\tau<t} \int_0^\tau \frac{\tau^{1/4}}{(\tau-s)^{3/4} s^{1/2}} ds \right\} \cdot \left\{ \sup_{0<\tau<t} \tau^{1/4} \|w\|_{4/3}(\tau) \right\}.$$

(By assumption, $\sup_{0<\tau<t} \tau^{1/4} \|w\|_{4/3}(\tau)$ is always finite. Hence the left-hand side of the inequality is finite.) Since

$$\int_0^\tau \frac{\tau^{1/4}}{(\tau-s)^{3/4} s^{1/2}} ds = \int_0^1 (1-\rho)^{-3/4} \rho^{-1/2} d\rho \ =: C_2$$

is a positive constant independent of τ, setting $C_3 = 2C_1L_1(4)c_0C_2$, we have

$$\sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau) \leq C_3|m| \sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau).$$

Let m_0 be a positive number such that $0 < m_0 < C_3^{-1}$. Then we have $1 > C_3|m|$ for $|m| < m_0$, which gives $\sup_{0<\tau<t} \tau^{1/4}\|w\|_{4/3}(\tau) = 0$. Therefore we obtain $\|w\|_{4/3}(t) \equiv 0$, $t > 0$, that is, ω_1 is identically equal to ω_2 on $\mathbb{R}^2 \times (0,\infty)$, and u_1 is identically equal to u_2.

On the other hand, $(mg, \mathbf{K} * (mg))$ is a weak solution of (2.7) and (2.8) with initial value $m\delta$ that satisfies (2.12a). (See the answer to Exercise 7.2.) Hence by the above uniqueness result (by setting $\omega_1 = \overline{\omega}$ and $\omega_2 = mg$), we obtain $\overline{\omega} = mg$. □

The uniqueness without the assumption of the smallness of $|m|$ was open for years. The difficulty is that the convective term $(\overline{u}, \nabla)\overline{\omega}$ cannot be regarded as small with respect to the diffusion term $\Delta\overline{\omega}$. Recently, Gallay and Wayne gave an affirmative answer to this uniqueness problem in [Gallay Wayne 2005]. The key idea there is to introduce the relative entropy as a Lyapunov function. The details will be discussed in §2.8.

2.5.5 Completion of the Proof of the Asymptotic Formula

Finally, we summarize the proof of the asymptotic formula (2.10) by rescaling methods. First assume that the initial vorticity ω_0 is in $C_0(\mathbb{R}^2)$, and (ω, u) is a solution of (2.7), (2.8), and (2.9). Let $\{(\omega_k, \overline{u}_k)\}_{k\geq 1}$ be the rescaled family. Then each subsequence $\{\omega_{k(\ell)}\}_{\ell=1}^{\infty}$ of $\{\omega_k\}_{k\geq 1}$ (under suitable choice of subsequence $\{\omega_{k(\ell(i))}\}_{i=1}^{\infty}$) converges uniformly on $\mathbb{R}^n \times [\eta, 1/\eta]$ for any $\eta \in (0,1)$. The limit function $\overline{\omega}$, together with the velocity $\overline{u} = K * \overline{\omega}$, should be a weak solution of (2.7) and (2.8) with initial value $m\delta$ that satisfies (2.12a) and (2.12b). (This result is obtained in the first part of §2.5 and §2.5.1.) Here $m = \int \omega_0 dx$. By §2.5.2 and §2.5.3, using the uniqueness of solutions §2.5.4, $\overline{\omega} = mg$ follows. (Here $g(x,t) = G_t(x)$ denotes the Gauss kernel.) Hence the limit $\overline{\omega}$ is independent of the choice of the subsequence of $\{\omega_k\}$. By Exercise 1.4, for any $\eta \in (0,1)$, $\{\omega_k\}$ converges uniformly to mg on $\mathbb{R}^n \times [\eta, 1/\eta]$ as $k \to \infty$. Hence we obtain $\lim_{k\to\infty} \|\omega_k - mg\|_\infty(1) = 0$, and by §1.2.6, we get (2.10). This completes the proof of the theorem of asymptotic behavior of vorticities (§2.2.2).

The assumption that the absolute value of the total circulation m is small is used only in the proof of the uniqueness of the limit equation. The main advantage of the rescaling method is that no matter how large $\|\omega_0\|_1$ is, we can prove show the asymptotic formula (2.10) if $|m|$ is sufficiently small.

Moreover, since the smallness assumption on $|m|$ is actually unnecessary in Theorem 2.5.4 (see §2.8), the asymptotic formula (2.10) is still valid without the smallness assumption.

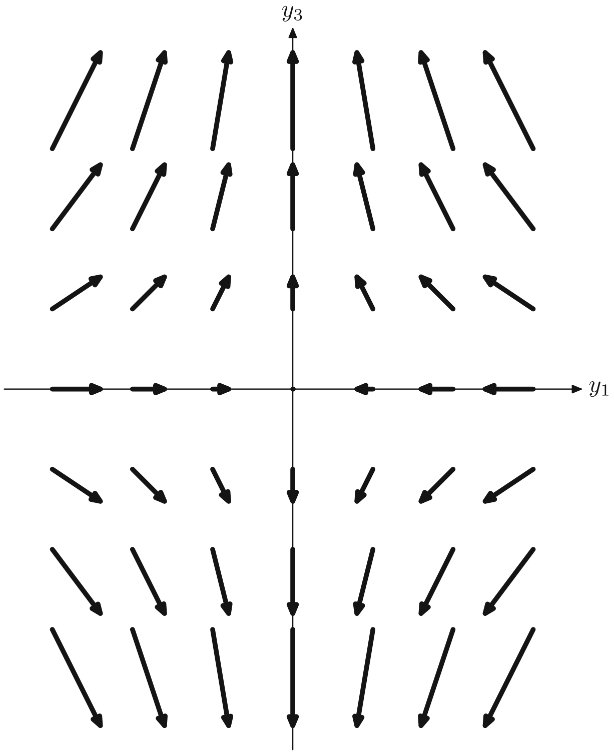

Figure 2.1. Vector field U at the cross section $y_2 = 0$ (in the case of $\alpha > 0$).

2.6 Formation of the Burgers Vortex

Let us apply the asymptotic formula (2.10) to a problem of fluid mechanics. We consider an incompressible viscous flow whose velocity field is expressed by a sum of

- an axially symmetric flow U without vortices,
- a two-dimensional flow V whose vortex vector is parallel to the symmetric axis of U.

Here and in the sequel, y_1, y_2, and y_3 denote spatial variables, τ denotes the time variable, and the y_3-axis is taken as the axis of the symmetry. Consider U as

$$U(y_1, y_2, y_3) = (-\alpha y_1, -\alpha y_2, 2\alpha y_3), \quad \alpha \in \mathbb{R}.$$

See Figure 2.1. If $\alpha > 0$, the flow concentrates on the y_3-axis of symmetry, and diverges to (plus and minus) infinity of the y_3-axis. Obviously, it satisfies

$$\operatorname{div} U = 0, \quad \operatorname{curl} U = 0, \quad \Delta U = 0.$$

Since $(U,\nabla)U = -\nabla P$, if we set $P(y_1,y_2,y_3) = -\frac{\alpha^2}{2}(y_1^2,y_2^2,4y_3^2)$, it is clear that the pair of functions (U,P) is a stationary solution (i.e., it is independent of time) of the Navier–Stokes equations

$$\frac{\partial u}{\partial \tau} - \nu \Delta u + (u,\nabla)u + \nabla p = 0, \quad \mathrm{div}\ u = 0 \tag{2.13}$$

in $\mathbb{R}^3 \times (0,\infty)$. In this section, we consider the Navier–Stokes equations with density $\varrho_0 = 1$ and viscosity $\nu > 0$. We assume that the unknown function $u = u(y_1,y_2,y_3,\tau)$ in (2.13) has the form

$$u = U + V. \tag{2.14}$$

Since V denotes the velocity vector that expresses the two-dimensional flow, it is given by

$$V(y_1,y_2,\tau) = (V^1(y_1,y_2,\tau), V^2(y_1,y_2,\tau), 0),$$

and its vorticity vector is expressed by

$$(0,0,\Omega(y_1,y_2,\tau)), \quad \Omega = \frac{\partial V^2}{\partial y_1} - \frac{\partial V^1}{\partial y_2}.$$

We are concerned with the behavior of $\Omega(y,\tau)$ as $\tau \to \infty$ in the case of $\alpha > 0$. (Here, we write $y = (y_1,y_2)$, and so $\Omega(y,\tau)$ denotes $\Omega(y_1,y_2,\tau)$. We use similar notation for other functions.) In the case of $\alpha = 0$, the pair of functions (Ω, V) with $\nu = 1$ satisfies the two-dimensional vorticity equations (2.7), (2.8), and (2.9) (see §2.1). Hence by the asymptotic formula (2.10), (if $|m|$ is sufficiently small), Ω asymptotically behaves like mg as $\tau \to \infty$. Here we set

$$m = \int_{\mathbb{R}^2} \Omega_0(y)dy, \qquad \Omega_0(y) = \Omega(y,0).$$

(V also expresses the two-dimensional vector field (V^1,V^2).) If ν is a general positive constant, using the scaling transformation ($\tilde{\tau} = \nu\tau$, $\tilde{\Omega} = \Omega/\nu$, $\tilde{V} = V/\nu$), we obtain the asymptotic formula $\Omega \sim mg^\nu$ $(\tau \to \infty)$, by considering the two-dimensional vorticity equations. Here we set $g^\nu(y,\tau) = \frac{1}{4\pi\nu\tau} e^{-|y|^2/4\nu\tau}$.

In the case of $\alpha > 0$, set

$$\overline{\Omega}_m(y) = \frac{m}{\pi \ell^2} e^{-|y|^2/\ell^2}, \quad \ell = \left(\frac{2\nu}{\alpha}\right)^{1/2},$$

for $y \in \mathbb{R}^2$. The above $\overline{\Omega}_m$ is called the *Burgers vortex*. In the following, we discuss the convergence of Ω to $\overline{\Omega}_m$ when τ goes to infinity.

2.6.1 Convergence to the Burgers Vortex

First we derive the equation for V from (2.13) and (2.14). Let U and P be as stated in the first part of this section. Since $(U,\nabla)U = -\nabla P$, $(V,\nabla)U = -\alpha V$, $(U,\nabla)V = -\alpha(y,\nabla)V$, we obtain

$$\frac{\partial V}{\partial \tau} - \nu \Delta V - \alpha(y, \nabla)V - \alpha V + (V, \nabla)V + \nabla(p - P) = 0, \ \operatorname{div} V = 0. \quad \text{(S)}$$

Considering V as a two-dimensional vector field and applying curl to both sides, we obtain

$$\frac{\partial \Omega}{\partial \tau} - \nu \Delta \Omega - \alpha(y, \nabla)\Omega - 2\alpha\Omega + (V, \nabla)\Omega = 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty) \qquad \text{(R)}$$

for $\Omega = \Omega(y, \tau)$, $V = V(y, \tau)$, $y \in \mathbb{R}^2$, and $\tau > 0$. Here we used

$$\operatorname{curl}((y, \nabla)V) = (y, \nabla)\Omega + \Omega, \quad y \in \mathbb{R}^2.$$

As in the calculation that leads to the vorticity equations from the Navier–Stokes equations, we obtain (R) and the Biot–Savart law $V = \mathbf{K} * \Omega$ on $\mathbb{R}^2 \times (0, \infty)$ from (S). Here and in the sequel, we assume $\nu = 1$. If we transform (R) with $V = \mathbf{K} * \Omega$ by

$$\begin{cases} x = e^{\alpha\tau} y, \ t = \displaystyle\int_0^\tau e^{2\alpha\sigma} d\sigma, \\ \omega(x, t) = e^{-2\alpha\tau}\Omega(y, \tau), \ u(x, t) = e^{-\alpha\tau} \ V(y, \tau), \end{cases}$$

then we obtain the equation for Ω. The pair of functions (ω, u) is a solution of the vorticity equations (2.7), (2.8), and (2.9) with initial value ω_0 given by $\omega_0(x) = \Omega_0(x)$, $x \in \mathbb{R}^2$.

Assume that $\Omega_0 \in C_0(\mathbb{R}^2)$ and take m_0 as in §2.2.2. Then by the asymptotic formula

$$\lim_{t \to \infty} t\|\omega - mg\|_\infty(t) = 0, \quad |m| < m_0,$$

we obtain

$$\lim_{\tau \to \infty} t(\tau)e^{-2\alpha\tau}\|\Omega - m\Omega_*\|_\infty(\tau) = 0, \quad t(\tau) = (e^{2\alpha\tau} - 1)/(2\alpha).$$

Here Ω_* is a function such that $g(x, t) = e^{-2\alpha\tau}\Omega_*(y, \tau)$, and since $t = t(\tau)$ and $x = e^{\alpha\tau}y$, we have

$$\Omega_*(y, \tau) = \frac{1}{\pi\ell^2(1 - e^{-2\alpha\tau})} \exp\left(-\frac{|y|^2}{\ell^2(1 - e^{-2\alpha\tau})}\right).$$

From $\lim_{\tau \to \infty} t(\tau)e^{-2\alpha\tau} = \ell^2/4$, we obtain

$$\lim_{\tau \to \infty} \|\Omega - m\Omega_*\|_\infty(\tau) = 0.$$

On the other hand, since

$$\lim_{\tau \to \infty} \|m\Omega_* - \overline{\Omega}_m\|_\infty(\tau) = 0,$$

we also obtain $\lim_{\tau\to\infty} \|\Omega - \overline{\Omega}_m\|_\infty(\tau) = 0$. Note that $\overline{\Omega}_m$ is a stationary (τ-independent) solution of (R). In the case that ν is a general positive constant, as in the case of $\alpha = 0$, we can reduce (R) to the case of $\nu = 1$, and similar results hold. Since the total circulation $\int_{\mathbb{R}^2} \omega(x,t)dx$ is conserved and is equal to $\int_{\mathbb{R}^2} \omega_0(x)dx$ (independent of t) (Proposition 2.2.4), for $\tau \geq 0$, it follows that $\int_{\mathbb{R}^2} \Omega(y,\tau)dy = m$. Summarizing the above arguments, we finally obtain the following theorem.

Theorem. *Let $\Omega_0 \in C_0(\mathbb{R}^2)$ and set $m = \int_{\mathbb{R}^2} \Omega_0(y)dy$. Assume that the pair of functions (Ω, V) satisfies* (R) *and the Biot–Savart law $V = \mathbf{K} * \Omega$ on $\mathbb{R}^2 \times (0,\infty)$. Moreover, assume that (i), (ii), and (iii) in Theorem 2.2.1 are satisfied by Ω, Ω_0, and V, instead of ω, ω_0, and u, respectively. Then there exists a (small) positive constant m_0 (that is independent of $\alpha > 0$, Ω_0, and ν) such that if $|m|/\nu < m_0$ then*

$$\lim_{\tau\to\infty} \|\Omega - \overline{\Omega}_m\|_\infty(\tau) = 0.$$

Moreover, the total circulation at $\tau \geq 0$ is $\int_{\mathbb{R}^2} \Omega(y,\tau)dy = m$.

Remark. Since (2.10) is still valid (see Remark 2.2.2) without the smallness assumption on $|m|$, the assertion of Theorem 2.6.1 is still valid without assuming that $|m|/\nu$ is small. Thus the Burgers vortex is stable under two-dimensional perturbations even if it is large. As for the stability under three-dimensional perturbations, the linear stability is observed numerically by [Schmid Rossi 2004]. Recently, it was rigorously proved by [Gallay Maekawa] that the Burgers vortex is locally stable under three-dimensional perturbations independent of the value of the total circulation.

2.6.2 Asymmetric Burgers Vortices

The Burgers vortex is a simple model of tubelike structures that are observed in concentrated vorticity fields, and it is considered to represent the balance between the stretching effect by the axisymmetric straining flow and the diffusion effect through the action of viscosity. In real flows or numerical observations, however, such vortex tubes are not purely axisymmetric and usually have an elliptic core region; see, for example, [Kida Ohkitani 1992]. To explain this phenomenon as proposed in [Robinson Saffman 1984], we instead of (2.14) postulate that the unknown velocity field is of the form

$$u = U_\lambda + V$$

with

$$U_\lambda = \left(-\frac{1+\lambda}{2} y_1,\ -\frac{1-\lambda}{2} y_2,\ y_3 \right),$$

where $\lambda \geq 0$ is a fixed parameter. Note that the case $\lambda = 0$ corresponds to (2.14). The equation for $\Omega = \frac{\partial V^2}{\partial y_1} - \frac{\partial V^1}{\partial y_2}$ then becomes

$$\frac{\partial \Omega}{\partial \tau} - \nu \Delta \Omega - \frac{1+\lambda}{2}\,\frac{\partial \Omega}{\partial y_1} - \frac{1-\lambda}{2}\,\frac{\partial \Omega}{\partial y_2} - \Omega + (V, \nabla)\Omega = 0. \qquad (\mathrm{R}')$$

A stationary solution for (R′) with $\lambda \neq 0$ is called an asymmetric (or nonaxisymmetric) Burgers vortex. Different from the (axisymmetric) Burgers vortex $\overline{\Omega}_m$, an explicit representation is no longer available for an asymmetric Burgers vortex. Several properties of nonaxisymmetric Burgers vortices are numerically studied in [Robinson Saffman 1984], [Moffatt Kida Ohkitani 1994], and [Prochazka Pullin 1998] by changing two parameters: the total circulation m and the asymmetry parameter λ. The first work in mathematical analysis on this problem was done by [Gallay Wayne 2006], [Gallay Wayne 2007]. In [Gallay Wayne 2007] the existence of an asymmetric Burgers vortex is proved for all m when $\lambda \in [0, 1/2)$ is sufficiently small. Moreover, it is shown that for these values of parameters the asymmetric Burgers vortex is locally stable under two-dimensional perturbations. In [Gallay Wayne 2006] the existence of an asymmetric Burgers vortex is proved when $|m|$ is sufficiently small depending on $\lambda \in [0, 1)$; moreover, its local stability is obtained under three-dimensional perturbations. The results in [Gallay Wayne 2006], [Gallay Wayne 2007] have been substantially extended by [Maekawa 2009a], [Maekawa 2009b]. In [Maekawa 2009a] the existence of an asymmetric Burgers vortex and its local stability under two-dimensional perturbations are obtained for sufficiently large $|m|$ when $\lambda \in [0, 1/2)$. In [Maekawa 2009b] it is proved that an asymmetric Burgers vortex exists for any m and $\lambda \in [0, 1)$. There seem to be no mathematical results on (R′) for $\lambda \geq 1$. In particular, it seems that there are no stationary solutions to (R′) that decay at spatial infinity if $\lambda \geq 1$.

In [Robinson Saffman 1984], [Moffatt Kida Ohkitani 1994], and [Prochazka Pullin 1998] it is observed that the isovorticity contour of an asymmetric Burgers vortex becomes more circular as $|m|$ is increasing. This mechanism is explained in [Moffatt Kida Ohkitani 1994] by deriving a formal asymptotic expansion at large $|m|$. This asymptotic expansion is rigorously proved by [Gallay Wayne 2007] for sufficiently small $\lambda \in [0, 1/2)$ by [Maekawa 2009a] for all $\lambda \in [0, 1/2)$ and finally by [Maekawa 2009b] for all $\lambda \in [0, 1)$.

2.7 Self-Similar Solutions of the Navier–Stokes Equations and Related Topics

In this section we present recent developments mainly on self-similar solutions of the Navier–Stokes equations. We start with a brief history on behavior of vorticities at time infinity. Next we introduce the existence problem of solutions to the Navier–Stokes equations or the vorticity equations. Finally, we present the mentioned results on self-similar solutions to the Navier–Stokes equations.

2.7.1 Short History of Research on Asymptotic Behavior of Vorticity

The asymptotic formula (2.10) shows that the solution of the two-dimensional vorticity equations asymptotically converges to the rotationally symmetric self-similar solution at time infinity. This formula was first obtained by [Giga Kambe 1988]. In this paper the authors directly estimated $\omega - mg$ using the integral equation. But their argument required the smallness assumption of $\|\omega_0\|_1$. This result was improved using rescaling arguments in [Carpio 1994]. The advantage of this rescaling method is that we can relax the condition of the smallness of $\|\omega_0\|_1$ to the smallness of $|m|$ as we have seen in §2.2.2. Also in [Carpio 1994] the estimates as in §2.4.1 play essential roles, but the author used slightly weaker estimates there. For example, instead of §2.4.1(i), it is used that

$$\|\omega\|_q(t) \leq \frac{C}{t^{1-\frac{1}{q}}}\|\omega_0\|_1,$$

where the constant C depends on $\|\omega_0\|_1$ nonincreasingly. This estimate is obtained in [Giga Miyakawa Osada 1988] by rewriting the Biot–Savart law to apply the results of [Osada 1987]. Later, the fundamental decay estimate in §2.3.1 was obtained by [Kato 1994, Ben-Artzi 1994]. In these works the estimates of §2.4.1 are also established. The contents of §2.3, except for a slight improvement in §2.3.6, is based on [Kato 1994]. (Another proof in §2.3.5 is due to [Ben-Artzi 1994].) The idea of the proof is based on the Nash–Moser method, which estimates fundamental solutions (in the case of the heat equation it is the Gauss kernel) of the second-order linear parabolic equations of divergence form (generalization of the heat equation) under an assumption that allows for singular coefficients in the equations. The fundamental work on this problem was done by [Nash 1958]. There are many references to the Nash–Moser method (see references below in this section). Here we only refer to the nice paper [Fabes Stroock 1986], since it is rather easy to read.

The method to obtain estimates of the velocity from the vorticity as in §2.4.1 is established in [Giga Miyakawa Osada 1988]. (Another proof in §2.4.1(ii) is given by [Ben-Artzi 1994].) The estimates of derivatives of vorticities §2.4.2 were obtained by [Kato 1994] in the case of $1 < p < \infty$. In this book we have proved them in a more elementary way, which covers the case $p = 1$ and $p = \infty$. In §2.4.3 we have proved the decay estimate at spatial infinity. We can also derive the same result using the pointwise estimate of fundamental solutions established in [Osada 1987] for general parabolic PDEs including (2.7). The method used in this book is more elementary, although the class of equations to which we can apply this argument will be restricted. Combining this with the estimate in [Kato 1994], we can improve the results in [Osada 1987]. For details, readers should refer to [Matsui Tokuno 1997]. In [Maekawa 2008a], spatial decay estimates for derivatives of solutions are obtained, which lead to the asymptotic behavior of derivatives of solutions by the above rescaling arguments.

In [Carpio 1994] there is no statement on the estimate of the limit function by the value $|m|$ as in §2.5.2, while one needs this estimate to prove the uniqueness of solutions to the limit equation. For this reason we have to use general results on parabolic PDEs by [Osada 1987] (§2.5.2). Since the details are complicated and beyond the scope of this book, we have omitted them. As mentioned at the end of §2.5.2, results in [Maekawa 2008b] simplify the proof of Lemma 2.5.2(ii) without assuming the special relation $v = \mathbf{K} * \omega$. The pointwise estimates for fundamental solutions by Gaussian-like functions from above and below are called the Aronson estimates. These estimates were at first obtained by [Aronson 1968] for second order parabolic PDEs of divergence form (but without transport terms). It is known that the Aronson estimates lead to the Hölder continuity of fundamental solutions; for example, see [Fabes Stroock 1986]. As for the equations (H_v), [Carlen Loss 1996] and [Matsui Tokuno 1997] obtained the pointwise Gaussian upper bounds for fundamental solutions, and in [Maekawa 2008b] the pointwise Gaussian lower bounds (and thus the Hölder continuity) are also established. For the Aronson estimates and the Hölder continuity of fundamental solutions to more general parabolic PDEs the reader is referred to [Fabes 1992], [Liskevich Samenov 2000], [Liskevich Zhang 2004], [Zhang 2004], [Zhang 2006], [Samenov 2006].

The uniqueness of solutions to the limit equations is essentially included in [Giga Miyakawa Osada 1988]. In [Giga Miyakawa Osada 1988] the time global solution of (2.7), (2.8), and (2.9) is constructed when the initial data is a general finite Radon measure. The uniqueness is proved in [Giga Miyakawa Osada 1988] under the assumption that the initial data is sufficiently regular in the sense that the singularity of Dirac delta type is sufficiently small. This result was slightly improved by [Kato 1994], but for a long time it remained an open problem whether the uniqueness of weak solutions holds when the total mass $|m|$ is large, even if the initial value is just $m\delta$. Recently the uniqueness of solutions with initial data as $m\delta$ was affirmatively proved in [Gallay Wayne 2005] and then in [Gallagher Gallay Lions 2005]. In [Gallay Wayne 2005] they introduced the relative entropy and showed that it is a Lyapunov function for the "flow" of the solution $\{\omega(t)\}_{t\geq 1}$, which leads to a characterization of solutions with initial data as $m\delta$. The details will be discussed in §2.8. In [Gallagher Gallay Lions 2005] another proof using the radial rearrangement of the vorticity is given. Using the results of [Gallay Wayne 2005], the uniqueness of solutions with general finite Radon measures as initial data was also proved by [Gallagher Gallay 2005].

In this chapter we have established the asymptotic formula (2.10) as elementarily as possible using scaling transformations. Readers can see how useful the detailed analysis of the linear equation (H_v) is for the study of the nonlinear equation (2.7). Throughout the chapter the initial data ω_0 has been assumed to be a continuous function with compact support. But this is just for simplicity and we can take the initial data in larger classes of functions. For example, if we define solutions appropriately, for any initial data

ω_0 belonging to $L^1(\mathbb{R}^2)$ we can prove the asymptotic formula (2.10) in §2.2.2 under the assumption $|m| < m_0$ (see [Carpio 1994]). If we use the argument by [Gallay Wayne 2005], then this smallness assumption $|m| < m_0$ is again removed.

The convergence to the Burgers vortex in §2.6 is obtained in [Kambe 1984], for rotationally symmetric Ω. In the case of the rotationally symmetric vortex, the result is reduced to the case of the heat equation (§2.2.5); hence, in order to obtain the desired convergence, the results in §1.1.4 are sufficient. Of course, we do not need to assume the smallness of $|m|$. By the proof using the expression of solutions (§1.1.5), we have $t\|u - mg\|_\infty(t) \le C\ t^{-1/2}, \quad t > 0$. In fact, we can show not only that $\Omega \to \overline{\Omega}_m$ $(\tau \to \infty)$, but more precisely,

$$\|\Omega - \overline{\Omega}_m\|_\infty(\tau) = O(e^{-\alpha\tau}) \quad (\tau \to \infty).$$

In the nonrotationally symmetric case, by [Giga Kambe 1988],

$$\|\Omega - \overline{\Omega}_m\|_\infty(\tau) = O(e^{-\alpha\sigma\tau}) \quad (\tau \to \infty) \tag{2.15}$$

is proved for $0 \le \sigma < 1$ under the assumption that $\|\Omega\|_1$ is sufficiently small. This shows that Ω converges exponentially to $\overline{\Omega}_m$. In [Gallay Wayne 2005] this exponential convergence of Ω to $\overline{\Omega}_m$ is also verified for any m without a smallness assumption.

The transformation (2.7) from (R) is due to [Lundgren 1982, Kambe 1983]. But note that mathematically it is considered a transformation by similarity variables as stated in §2.7.3.

2.7.2 Problems of Existence of Solutions

The first mathematical approach to the initial value problem of the Navier–Stokes equations for general initial data was developed by [Leray 1933, Leray 1934a, Leray 1934b]. In [Leray 1934b], it is proved that in $\mathbb{R}^3$ if the L^2-norm of the initial velocity is finite (in other words, if the initial kinetic energy is finite), then there exists a time-global weak solution of the Navier–Stokes equations. It is already known that if the initial velocity is sufficiently small in some sense or the spatial dimension is two, then the weak solution is smooth and unique. However, if the spatial dimension is three, *for general initial data the uniqueness and smoothness of weak solutions are still open.* For solvability problems including this famous open problem the reader is referred to the fundamental books [Ladyzhenskaya 1969, Temam 1977, Galdi 1994, Lions 1996], [Sohr 2001], [Chemin et al 2006] and the articles [Masuda 1985], [Yamazaki 1999], [Kozono 2002], [Cannone 2004], [Hishida 2008]. Note that analysis of the Navier–Stokes equations covers a very broad field with generalizations in many different directions. As variants here we just point out recent results on the Navier–Stokes equations with more general boundary conditions or in the time-dependent domain [Saal 2006], [Saal 2007a], [Saal 2007b]. In the 1960s Kato and Fujita considered a good successive approximation method to construct time local smooth solutions, which became a big milestone for solving the initial value problem of nonlinear partial differential equations; [Kato Fujita 1962, Kato 1996].

For the initial value problem of the two-dimensional vorticity equations, even if the initial vorticity is a continuous function with compact support, the L^2-norm of the initial velocity is not always finite. Hence we cannot directly obtain the existence of time global solutions of the vorticity equations from Leray's results. In general, we have at least three methods to prove the existence of time global solutions of evolution equations:

(i) Extend time local solutions globally in time.
(ii) Approximate the problem by a problem for which the existence of time global solutions is easily obtained.
(iii) Use a fixed-point theorem.

Each method requires an a priori estimate, that is, we have to estimate how large a solution can be if it exists.

Fortunately, in the case of two-dimensional vorticity equations, we can uniquely construct a time global smooth solution ω by a successive approximation and the method (i). Indeed, the maximal existence time T of the time local solution is estimated by the L^p-norm $(1 < p < 2)$ of the initial vorticity ω_0 as $T^{\frac{1}{p}-1} \leq C\|\omega_0\|_p$, where the constant C is independent of ω_0. Hence, if $\|\omega\|_p(t_0)$ is bounded (for example bounded by M) for a solution ω, then there exists a constant T_M, which depends only on M, such that the solution can be extended to the time $t_0 + T_M$. If the estimates as in §2.4.1 are valid for a time local solution, we can repeat this procedure, and the solution can be extended to any time as a smooth solution. In fact, in [Giga Miyakawa Osada 1988] a time global solution is constructed by this argument.

Finally, we consider the smoothness of weak solutions that is mentioned in the last part of §2.5.3. If $\|\omega\|_p(t_0) < \infty$ for each p with $1 \leq p \leq \infty$, then there exists a smooth time global solution with the initial data $\omega(t_0)$ at initial time t_0 that satisfies (i), (ii), and (iii) in §2.2.1. If we can show that this solution and the weak solution coincide (namely, if we have the uniqueness of weak solutions), then the weak solution is smooth and satisfies (for example) (i), (ii), and (iii) in §2.2.1. Thus it suffices to prove the uniqueness of weak solutions with L^p-initial data. For L^p-initial data, different from the case that the initial data is the δ measure, we have in fact a good estimate for weak solutions near the initial time, which yields the uniqueness of weak solutions without smallness assumptions for initial data. In [Leray 1934b] a similar method as above is used to estimate the set of the time at which the weak solution in $\mathbb{R}^3$ can be singular.

In the next section we consider self-similar solutions of the Navier–Stokes equations.

2.7.3 Self-Similar Solutions

As we have seen in this chapter, forward self-similar solutions play an important role in the asymptotic behavior of solutions. We can consider self-similar solutions to the Navier–Stokes equations as in the case of the vorticity equations. Assume that a pair of smooth functions (u, p) satisfies the Navier–Stokes equations

$$\partial_t u - \Delta u + (u, \nabla)u + \nabla p = 0, \quad \operatorname{div} u = 0$$

in $\mathbb{R}^n \times (0, \infty)$ $(n \geq 2)$. Here $u = (u^1, \ldots, u^n)$ is an $\mathbb{R}^n$-valued function. Then for each $\lambda > 0$ the pair of functions $(u^{(\lambda)}, p^{(\lambda)})$ rescaled by

$$u^{(\lambda)}(x,t) = \lambda u(\lambda x, \lambda^2 t), \; p^{(\lambda)}(\lambda, t) = \lambda^2 p(\lambda x, \lambda^2 t), \quad x \in \mathbb{R}^n, \quad t > 0, \tag{2.16}$$

is also a solution of the Navier–Stokes equations. In general, if a pair of functions (u, p) in $\mathbb{R}^n \times (0, \infty)$ (which is not necessarily a solution of the Navier–Stokes equations) satisfies

$$u^{(\lambda)}(x,t) = u(x,t), \quad p^{(\lambda)}(x,t) = p(x,t), \quad x \in \mathbb{R}^n, \quad t > 0, \; \lambda > 0,$$

then the pair (u, p) is called *forwardly self-similar*, and if (u, p) is a solution of the Navier–Stokes equations, it is called a *forward self-similar solution*. For example, let $g(x,t) = G_t(x)$ be the Gauss kernel and consider the associated velocity field $u = \mathbf{K} * g$. Then if we set the pressure field p as $p = E * \sum_{i,j=1}^{2} \partial_{x_i} \partial_{x_j} (u^i u^j)$, then (u, p) is a forward self-similar solution of the Navier–Stokes equations.

In general, if (u, p) is a self-similar solution, then by setting $\lambda = 1/\sqrt{t}$, it is expressed as

$$u(x,t) = \frac{1}{\sqrt{t}} u\left(\frac{x}{\sqrt{t}}, 1\right), \quad p(x,t) = \frac{1}{t} p\left(\frac{x}{\sqrt{t}}, 1\right).$$

Hence, (u, p) is forwardly self-similar if and only if it can be written in the form

$$u(x,t) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right), \quad p(x,t) = \frac{1}{t} P\left(\frac{x}{\sqrt{t}}\right),$$

using a pair of functions (U, P) on $\mathbb{R}^n$ (where U is an $\mathbb{R}^n$-valued function). Thus it will be useful to consider the equations for (U, P) instead of (u, p). To derive the equations for (U, P), first we transform the dependent variables as $\hat{u}(x,t) = \sqrt{t} u(x,t)$, $\hat{p}(x,t) = t p(x,t)$, and next transform the independent variables as $y = x/\sqrt{t}$, and set $\tilde{u}(y,t) = \hat{u}(\sqrt{t} y, t)$, $\tilde{p}(y,t) = \hat{p}(\sqrt{t} y, t)$. Then we easily see that (u, p) is forwardly self-similar if and only if $(\tilde{u}, \tilde{p})$ is independent of $t > 0$, that is, if it depends only on $y \in \mathbb{R}^n$. Now let us derive the equation that $(\tilde{u}, \tilde{p})$ satisfies when (u, p) is a solution of the Navier–Stokes equations in $\mathbb{R}^n \times (0, \infty)$. First, by the equalities

$$\partial_t \tilde{u}(y,t) = \frac{1}{2\sqrt{t}} u(\sqrt{t}y,t) + \frac{1}{2}\sum_{i=1}^{n} y_i(\partial_{x_i} u)(\sqrt{t}y,t) + \sqrt{t}\partial_t u(x,t),$$

$$\partial_{y_j}\tilde{u}(y,t) = t(\partial_{x_j} u)(\sqrt{t}y,t),\ \Delta\tilde{u}(y,t) = t^{3/2}(\Delta u)(\sqrt{t}y,t),$$

$$\partial_{y_j}\tilde{p}(y,t) = t^{3/2}(\partial_{x_j} p)(\sqrt{t}y,t),$$

if (u,p) satisfies the Navier–Stokes equations in $\mathbb{R}^n \times (0,\infty)$, then $(\tilde{u},\tilde{p})$ satisfies

$$t\partial_t\tilde{u} - \Delta\tilde{u} - \frac{1}{2}(y,\nabla)\tilde{u} - \frac{1}{2}\tilde{u} + (\tilde{u},\nabla)\tilde{u} + \nabla\tilde{p} = 0,\ \mathrm{div}\ \tilde{u} = 0, \quad t>0,\ y\in\mathbb{R}^n.$$

Moreover, since $t\partial_t = \partial_s$, if we transform as $s = \log t$ and set $w(y,s) = \tilde{u}(y,e^s)$, $q(y,s) = \tilde{p}(y,e^s)$, then (w,q) satisfies

$$\partial_s w - \Delta w - \frac{1}{2}(y,\nabla)w - \frac{1}{2}w + (w,\nabla)w + \nabla q = 0,\ \mathrm{div}\, w = 0,\ s\in\mathbb{R},\ y\in\mathbb{R}^n \quad (\mathrm{S}')$$

(conversely, if (w,q) satisfies (S′), then (u,p) satisfies the Navier–Stokes equations in $\mathbb{R}^n \times (0,\infty)$, which can be seen by performing the above calculation inversely). We sometimes call new variables (y,s,w,q) *similarity variables* with respect to the rescaling (2.16). Let us rewrite the above transformation

$$s = \log t,\ y = x/\sqrt{t},\ w(y,s) = \sqrt{t}u(x,t),\ q(y,s) = tp(x,t).$$

Note that (w,q) is related to (u,p) as $w(y,s) = e^{s/2}u(ye^{s/2},e^s)$, $q(y,s) = e^s p(ye^{s/2},e^s)$.

The equation (S′) is nothing but the equation (S) in §2.6.1 with $n=2$, $\alpha = 1/2$, and $\nu = 1$ under a suitable choice of p. The transformation from (S) to the Navier–Stokes equations (the vorticity equations) is essentially the same as the transformation by the above similarity variables.

We have now established the equations for (U,P). Indeed, since (U,P) is independent of s in the similarity variables, (u,p) is a forward self-similar solution if and only if $U = U(y)$ and $P = P(y)$ satisfy

$$-\Delta U - \frac{1}{2}(y,\nabla)U - \frac{1}{2}U + (U,\nabla)U + \nabla P = 0, \quad \mathrm{div}\, U = 0, \quad y\in\mathbb{R}^n, \quad (\mathrm{E})$$

in $\mathbb{R}^n$. This equation is just the one that stationary solutions of (S′) satisfy.

Are there any forward self-similar solutions except for $u = \mathbf{K} * g$? In fact, many special solutions are already known. We refer to [Okamoto 1997] for the construction of special solutions and their properties including backward self-similar solutions (we will mention backward self-similar solutions later).

Let us consider the forward self-similar solution $u = \mathbf{K} * g$. If it is regarded as the velocity field in $\mathbb{R}^3$, then its initial vorticity concentrates on an axis through the origin and is zero outside the axis. Can we construct a self-similar solution whose initial vorticity concentrates on half-lines through the origin, and is zero outside of them? This problem is studied in [Giga Miyakawa 1989], where small self-similar solutions are constructed by analyzing the vorticity equations directly instead of equation (E).

In [Carpio 1994], it is proved that if the initial vorticity is small, then the solution asymptotically converges to one of the above self-similar solutions.

The initial velocity u_0 of a self-similar solution is a function homogeneous of degree -1, i.e., $\lambda u_0(\lambda x) = u_0(x)$ $(\lambda > 0, x \in \mathbb{R}^n)$, which is easily seen if the initial vorticity is g. The L^p-norm of such a function is not finite except when it is identically zero. For example, $1/|x|$ does not satisfy $\||x|^{-1}\|_p < \infty$ for any p $(1 \leq p \leq \infty)$. So we cannot use classical existence theorems of solutions in L^p spaces, and we need to introduce alternative function spaces that include these homogeneous functions. This is the reason that Morrey spaces are used in the analysis of the Navier–Stokes equations in [Giga Miyakawa 1989]. After this work, the Navier–Stokes equations in Morrey spaces were studied also by [Kato 1994] and [Taylor 1992], and completed by [Kozono Yamazaki 1994]. In [Kozono Yamazaki 1995], relations with self-similar solutions are also considered. Because of the important applications to forward self-similar solutions, the Navier–Stokes equations have been studied in several function spaces that include functions homogeneous of degree -1 (other than the Morrey spaces). In [Cannone Meyer Planchon 1994], [Cannone 1995, Cannone 1997], and [Cannone Planchon 1996], Besov spaces are used to construct forward self-similar solutions. In [Meyer 1999] forward self-similar solutions are obtained in Lorentz spaces, and in [LeJan Sznitman 1997] the Navier–Stokes equations are studied by probabilistic arguments in pseudomeasure spaces that include self-similar solutions. A simpler proof of the construction of forward self-similar solutions in pseudomeasure spaces is given in [Cannone Planchon 2000], where harmonic analysis plays an essential role.

There are many studies concerning decay properties of solutions to the Navier–Stokes equations at time or space infinity. Here we refer only to [Miyakawa 1996, Miyakawa 1997, Miyakawa 1998].

Next we consider backward self-similar solutions. Let $u^{(\lambda)}$ and $p^{(\lambda)}$ be rescaled functions of u and p as in (2.16). But in this case we assume that u and p are defined in $\mathbb{R}^n \times (-\infty, 0)$. If

$$u^{(\lambda)}(x,t) = u(x,t),\ p^{(\lambda)}(x,t) = p(x,t), \quad x \in \mathbb{R}^n,\ t < 0,\ \lambda > 0,$$

holds, then (u,p) is called *backwardly self-similar*. Moreover, if (u,p) is a solution of the Navier–Stokes equations, it is called a *backward self-similar solution*. As with forward self-similar solutions, if (u,p) is backwardly self-similar, then we can write

$$u(x,t) = \frac{1}{\sqrt{-t}} U\left(\frac{x}{\sqrt{-t}}\right),\ p(x,t) = \frac{1}{-t} P\left(\frac{x}{\sqrt{-t}}\right).$$

By rewriting the Navier–Stokes equations in the similarity variables

$$y = x/\sqrt{-t},\ w(y,s) = \sqrt{-t}u(x,t),\ q(y,s) = tp(x,t),\ s = -\log(-t),$$

we see that the functions $w = w(y,s)$ $(= e^{-s/2}u(ye^{-s/2}, e^{-s}))$ and $q = q(y,s)$ $(= e^{-s}q(ye^{-s/2}, e^{-s}))$ satisfy, instead of (S′),

$$\frac{\partial w}{\partial s} - \Delta w + \frac{1}{2}(y,\nabla)w + \frac{1}{2}w + (w,\nabla)w + \nabla q = 0,\ \operatorname{div} w = 0,\ s \in \mathbb{R},\ y \in \mathbb{R}^n.$$

This is just the case of $\alpha = -1/2$ and $\nu = 1$ in the equation (S) in §2.6.1. The pair (u,p) is a backward self-similar solution if and only if $U = U(y)$ and $P = P(y)$ satisfy

$$-\Delta U + \frac{1}{2}(y,\nabla)U + \frac{1}{2}U + (U,\nabla)U + \nabla p = 0,\ \operatorname{div}\ U = 0, \quad y \in \mathbb{R}^n. \qquad \text{(L)}$$

This equation is called *Leray's equation*, since in [Leray 1934b] the auhtor suggested the idea of proving the existence of a solution (u,p) that diverges to infinity in finite time by constructing a backward self-similar solution. Let (U,P) be a smooth solution of (L) with $U(0) \neq 0$. For $T > 0$, set

$$u(x,t) = \frac{1}{\sqrt{T-t}}U\left(\frac{x}{\sqrt{T-t}}\right),\ p(x,t) = \frac{1}{T-t}P\left(\frac{x}{\sqrt{T-t}}\right).$$

Then (u,p) is a solution of the Navier–Stokes equations in the interval $(0,T)$, but $u(0,t)$ diverges to infinity as $t \to T$ (this is called "blowup" at time T). Usually weak solutions are constructed under the assumption that the initial data $u_0 = u(x,0)$ has the finite energy $\|u_0\|_2 < \infty$ and that they satisfy the energy inequality

$$\|u\|_2^2(t) + 2\int_0^t \|\nabla u\|_2^2(s)ds \leq \|u_0\|_2^2, \quad t > 0.$$

Are there any self-similar solutions satisfying the energy inequality? If such solutions exist, we can construct a weak solution that loses regularities in finite time. In the case of $n = 2$ every weak solution satisfying the energy inequality is shown to be smooth for all time, so the blowup does not occur. Hence there is no solution (U,P) of (L) with the above properties (we can prove this directly by multiplying both sides of (L) by U and performing integration by parts). When $n = 3$, by the energy inequality we have $\|U\|_2 < \infty$ and $\|\nabla U\|_2 < \infty$. Then by the Sobolev inequality we obtain $\|U\|_6 < \infty$ (moreover, the Hölder inequality yields $\|U\|_3 < \infty$). The problem is whether there exists (U,P) satisfying (L). For this problem, it is proved in [Necas Ru̇žička Šverák 1996] that any weak solution of (L) with $U \in L^3 \cap W^{1,2}_{\text{loc}}$ must be identically zero. Later in [Málek Nečas Pokorný Schonbek 1999] it is shown by another approach that any weak solution of (L) belonging to $W^{1,2}$ is a trivial function. This is extended by [Miller O'Leary Schonbek 2001], in which the nonexistence of pseudo (backward) self-similar solutions is obtained. Although backward self-similar solutions discussed in the above papers (if they exist) are assumed to decay at spatial infinity, [Tsai 1998] relaxed this condition and proved that any solution of (L) belonging to $W^{1,2}_{\text{loc}}$ is a constant function. The existence of backward self-similar solutions is discussed also for other equations related to the Navier–Stokes equations. For example, in [Guo Jiang 2006] it is proved that there are no backward self-similar solutions to the isothermal compressible Navier–Stokes equations. Moreover, [Chae 2007a] showed the nonexistence of self-similar blowing-up solutions to the three-dimensional Euler equations. Related to these results, recently [Chae 2007b] showed that asymptotically self-similar blowup does not occur for solutions to the Navier–Stokes equations or the Euler equations. See also [Chae preprint] for a simplified proof. These results are extended to cover equations in magnetohydrodynamics in [Chae 2008], [Chae 2009].

Hence Leray's idea of using backward self-similar solutions does not give the construction of blowup solutions. But it does not mean the nonexistence of blowup solutions. As for relations between the smoothness of solutions of the Navier–Stokes equations and backward self-similar solutions, we refer to [Kozono 1997], [Kozono Sohr 1996], [Kozono 1998], [Escauriaza Seregin Sverak 2003]. We also refer to [Cannone 2004], in which several topics related to the Navier–Stokes equations (including the topic of self-similar solutions) are discussed using tools of harmonic analysis.

In Section 3 we will see that backward self-similar solutions are deeply connected with blowup phenomena in some nonlinear partial differential equations.

2.8 Uniqueness of the Limit Equation for Large Circulation

In this section, we shall prove that a weak solution of (2.7)–(2.8) with initial data $m\delta$ is unique, provided that the vorticity ω satisfies the Gaussian estimate

$$\frac{C_1}{t}e^{-|x|^2/C_2 t} \le \omega(x,t) \le \frac{C_1'}{t}e^{-|x|^2/C_2' t}, \quad x \in \mathbb{R}^2, \quad t > 0, \tag{2.16}$$

with some positive constants C_1, C_2, C_1', C_2' independent of x, t. As in §2.4 this estimate yields (2.12a), (2.12b). Our main statement in this section is summarized as follows.

2.8.1 Uniqueness of Weak Solutions

Theorem. *Let the pair (ω, u) be a weak solution of (2.7)–(2.8) with initial data $m\delta$, where $m > 0$. Assume that ω and u are smooth in $\mathbb{R}^2 \times (0,\infty)$ and satisfy (2.16) (so that (2.12a) and (2.12b) hold). Then $\omega = mg$.*

As proved by H. Osada [Osada 1987] (see also [Giga Miyakawa Osada 1988]), $\Gamma_v(x,t,0,0)$ in Lemma 2.5.2 satisfies the Gaussian estimate (2.16) with constants depending only on $M_1 = \sup_{0<t<\infty} \|\omega\|_1(t)$ if $v = \mathbf{K} * \omega$. This estimate is inherited by the rescaled limit $(\bar{\omega}, \bar{u})$ of a subsequence of $\{(\omega_k, \bar{u}_k)\}$ as $k \to \infty$, so $\bar{\omega}$ satisfies (2.16). By the above uniqueness theorem one is able to conclude that $\bar{\omega} = mg$ without assuming that $|m|$ is small. We argue in the same way as in §2.5.5 and obtain (2.10) without assuming that $|m|$ is small.

We shall prove this theorem in the rest of this section.

2.8.2 Relative Entropy

The main idea of the proof is to use a relative entropy function with respect to g for (2.7)–(2.8) of the form

$$H(g, \omega)(t) = \int_{\mathbb{R}^2} \omega(x,t) \log \left(\frac{\omega(x,t)}{g(x,t)} \right) dx.$$

This quantity is monotone decreasing in time if (ω, u) is a solution of (2.7)–(2.8).

Theorem. *Let the pair* (ω, u) *be a smooth solution of (2.7)–(2.8) in* $\mathbb{R}^2 \times I$*, where* I *is an open interval in* $(0, \infty)$*. Assume that there exist positive constants* C_1, C_2, C_1', C_2' *that satisfy*

$$C_1 e^{-|x|^2/C_2} \le \omega(x,t) \le C_1' e^{-|x|^2/C_2'} \quad \text{for} \quad t \in I, \quad x \in \mathbb{R}^2. \tag{2.17}$$

Then

$$\frac{d}{dt} H(g, \omega)(t) = -\int_{\mathbb{R}^2} \left| \nabla \left(\frac{\omega}{g} \right) \right|^2 \frac{g^2}{\omega} dx, \quad t \in I. \tag{2.18}$$

If $H(g, \omega)(t)$ *is a constant on* I*, then* $\omega = \hat{m} g$ *in* $\mathbb{R}^2 \times I$ *with some constant* $\hat{m} > 0$.

Proof. As in §2.4.2, from the estimate (2.17) it follows that

$$\|\partial_t^b \partial_x^\beta \omega\|_p(t), \quad \|\partial_t^b \partial_x^\beta u\|_q(t)$$

are bounded on any compact time interval of I for all $p \in [1, \infty]$, $q \in (2, \infty]$, $b = 0, 1, 2 \ldots$, and all multi-indices β. These bounds and (2.17) justify all calculations below. For example, (2.17) guarantees that $H(g, \omega)(t)$ is a well-defined convergent quantity for $t \in I$.

We differentiate under the integral sign to observe that

$$\frac{d}{dt} H(g, \omega)(t) = \int \partial_t \omega \log \left(\frac{\omega}{g} \right) dx - \int \omega \frac{\partial_t g}{g} dx + \int \partial_t \omega \, dx;$$

all integration in this proof is over the whole plane, so we suppress the region of integration. We use (2.7) and $\partial_t g = \Delta g$ and observe that the last term vanishes by integration by parts, so that

$$\frac{d}{dt}H(g,\omega)(t) = \int \left(\Delta\omega \log\left(\frac{\omega}{g}\right) - \frac{\omega}{g}\Delta g\right) dx - \int \log\left(\frac{\omega}{g}\right) \operatorname{div}(u\omega)dx$$
$$=: I + II,$$

where we have invoked the property $\operatorname{div}(u\omega) = (u,\nabla)\omega$. Integrating by parts, we obtain

$$I = -\int \langle \nabla\omega, \nabla(\omega/g)\rangle/(\omega/g)dx + \int \langle \nabla(\omega/g), \nabla g\rangle dx$$
$$= \int \left\langle \nabla\left(\frac{\omega}{g}\right), \nabla\left(\frac{g}{\omega}\right)\right\rangle \omega\, dx = -\int \left|\nabla\left(\frac{\omega}{g}\right)\right|^2 \frac{g^2}{\omega} dx.$$

Again integrating by parts yields

$$II = \int u\omega\left(\frac{\nabla\omega}{\omega} - \frac{\nabla g}{g}\right) dx = \int \left((u,\nabla)\omega - ((u,\nabla)g)\frac{\omega}{g}\right) dx$$
$$= \int \operatorname{div}(u\omega)dx + \frac{1}{2t}\int \langle x, u(x,t)\rangle \omega(x,t)dx,$$

where the explicit form of $g = e^{-|x|^2/4t}(4\pi t)^{-1}$ is invoked. The first term vanishes by integration by parts. Since $u = \mathbf{K} * \omega$, the second term also vanishes by the next lemma. This implies (2.18). If H is constant in I, then by (2.18), $\omega = \hat{m}(t)g$ with $\hat{m}$ independent of x. Since $\int \omega\, dx$ is independent of t (§2.2.4), $\hat{m}$ is also independent of t. We thus conclude that $\omega = \hat{m}g$ for $t \in I$. □

Lemma. *Let ω and $\tilde{\omega}$ be functions on $\mathbb{R}^2$ such that $|\omega|^{2+\varepsilon}$, $|\tilde{\omega}|^{2+\varepsilon}$, $(|x|+1)\omega$, $(|x|+1)\tilde{\omega}$ are integrable on $\mathbb{R}^2$ for some $\varepsilon > 0$. Let B be the bilinear form defined by*

$$B(\omega,\tilde{\omega}) = \int \langle x, \nabla^{\perp}E * \omega\rangle \tilde{\omega}\; dx$$

Then $B(\omega,\tilde{\omega}) = -B(\tilde{\omega},\omega)$. In particular, $B(\omega,\omega) = 0$.

Proof. By definition

$$-2\pi B(\omega,\tilde{\omega}) = \iint \left\langle x, \frac{(x-y)^{\perp}}{|x-y|^2}\right\rangle \omega(y)\tilde{\omega}(x)dx\,dy,$$

where $x^{\perp} = (x_2, -x_1)$. (By our assumption the integrand is integrable on $\mathbb{R}^2 \times \mathbb{R}^2$, so we may change the order of integration by Fubini's theorem (§7.2.2).)

The right-hand side equals

$$\iint \frac{\langle x-y, (x-y)^{\perp}\rangle}{|x-y|^2}\omega(y)\tilde{\omega}(x)dx\,dy + \iint \frac{\langle y, (x-y)^{\perp}\rangle}{|x-y|^2}\omega(y)\tilde{\omega}(x)dx\,dy$$
$$= 0 + 2\pi B(\tilde{\omega},\omega).$$

The proof is now complete. □

2.8.3 Boundedness of the Entropy

Lemma. *Let the pair* (ω, u) *be a smooth solution of (2.7)–(2.8) in* $\mathbb{R}^2 \times (0, \infty)$ *satisfying the Gaussian estimate (2.16). Then*

$$H_0 = \lim_{t \to 0} H(g, \omega)(t) \quad \text{and} \quad H_\infty = \lim_{t \to \infty} H(g, \omega)(t)$$

exist as finite values.

Proof. By Theorem 2.8.2 the function $H(t) = H(g, \omega)(t)$ is nonincreasing on $(0, \infty)$. So it suffices to prove that $H(t)$ is bounded on $(0, \infty)$. By estimate (2.16),

$$\begin{aligned} H(t) &\le \int \omega \log(4\pi C_1' e^{-|x|^2/C_2' t} / e^{-|x|^2/4t}) dx \\ &\le \int \omega \left\{ \left(-\frac{|x|^2}{C_2' t} + \frac{|x|^2}{4t} \right) + \max(0, \log(4\pi C_1')) \right\} dx. \end{aligned}$$

Applying (2.16) and changing the variable of integration as $y = x/t^{1/2}$, we see that $H(t)$ is bounded from above, since $y^2 e^{-y^2/C_2'}$ is integrable on $\mathbb{R}^2$. Similarly, one is able to prove that $H(t)$ is bounded from below. □

2.8.4 Rescaling

We rescale (ω, u) as before by

$$\begin{aligned} \omega_k(x, t) &= k^2 \omega(kx, k^2 t), \\ \bar{u}_k(x, t) &= k u(kx, k^2 t), \end{aligned}$$

where (ω, u) is the solution of Theorem 2.8.1. If (ω, u) satisfies (2.12a),(2.12b), the rescaled pair (ω_k, u_k) satisfies (2.12a), (2.12b) with the same bound independent of $k > 0$. As argued at the beginning of §2.5, applying the Ascoli–Arzelà theorem (§5.2.5), we see that for any subsequence $\{\omega_{k(\ell)}\}_{\ell=1}^\infty$ $(k(\ell) \to \infty)$ (respectively, $k(\ell) \to 0$) there are a subsequence $\{\omega_{k(\ell(i))}\}$ and a limit σ_∞ (resp. σ_0) such that $\omega_{k(\ell(i))}$ converges to σ_∞ (resp. σ_0) locally uniformly in $\mathbb{R}^2 \times (0, \infty)$ with its derivatives. When $k \to 0$, differently from the case $k \to \infty$ we are unable to apply §2.4.3, so we do not claim uniform convergence in $\mathbb{R}^2 \times [\eta, 1/\eta]$ as $k \to 0$. Estimates (2.12a) hold for σ_0, σ_∞ and also (2.12b) holds for $u_0 = \mathbf{K} * \sigma_0, u_\infty = \mathbf{K} * \sigma_\infty$. As in §2.5.1, (σ_0, u_0) and $(\sigma_\infty, u_\infty)$ are smooth solutions of (2.7)–(2.8) in $\mathbb{R}^2 \times (0, \infty)$ satisfying (2.12a), (2.12b).

Proposition. *Let* σ_0 *and* σ_∞ *be functions defined as above. Then*

$$H(g, \sigma_0)(t) = H_0 \quad \text{and} \quad H(g, \sigma_\infty)(t) = H_\infty$$

for all $t \in (0, \infty)$*. In particular,* $\sigma_0 = m_0 g$, $\sigma_\infty = m_\infty g$ *with constants* m_0 *and* m_∞*.*

Proof. Since $g_k = g$, we easily see that

$$H(g, \omega_k)(t) = H(g, \omega)(k^2 t),$$

so that

$$\lim_{k\to 0} H(g, \omega_k)(t) = H_0, \quad \lim_{k\to\infty} H(g, \omega_k)(t) = H_\infty.$$

Since ω_k satisfies (2.16) with a constant independent of k, one is able to prove that

$$\lim_{k\to 0} H(g, \omega_k)(t) = H(g, \sigma_0)(t), \quad \lim_{k\to\infty} H(g, \omega_k)(t) = H(g, \sigma_\infty)(t)$$

by Lebesgue's dominated convergence theorem (§7.1.1); the way to estimate the integrand is the same as in §2.8.3. The last statement follows from Theorem 2.8.2. □

2.8.5 Proof of the Uniqueness Theorem

We are now in position to prove Theorem 2.8.1. Let (ω, u) be a (weak) solution of (2.7)–(2.8) satisfying (2.16) with initial data $m\delta$. Then one is able to prove that $\int \omega\, dx = m$ by Proposition 2.5.3. This implies that $\int \omega_k\, dx = m$, which yields $\int \sigma_0\, dx = \int \sigma_\infty\, dx = m$. Since $\sigma_\infty = m_\infty g$, $\sigma_0 = m_0 g$ in Proposition 2.8.4, we conclude that $m_\infty = m_0 = m$, since $\int g\, dx = 1$. Thus $H_0 = H_\infty$. By Theorem 2.8.2 this implies that $\omega = m' g$ with some $m' \in \mathbb{R}$. However, $\int \omega\, dx = m$, so $m' = m$. We now conclude that $\omega = mg$ and the assertion follows. □

Remark. The main result (Theorem 2.8.1) is due to [Gallay Wayne 2005], where the authors studied rescaled vorticity equations (R) in §2.6.1 and $V = \mathbf{K} * \mathit{\Omega}$ (with $\alpha = 1/2, \nu = 1$) for $(\mathit{\Omega}, V)$ with

$$\mathit{\Omega}(y, \tau) = e^\tau \omega(e^{\tau/2} y, e^\tau), \quad V(y, \tau) = e^{\tau/2} u(e^{\tau/2} y, e^\tau).$$

The quantity $H(g, \omega)$ is transformed as

$$H(\mathit{\Omega}) = \int \mathit{\Omega} \log(\mathit{\Omega}/e^{-|y|^2/4}) dy.$$

To prove Theorem 2.8.1 they instead studied the dynamical system (R) with $V = \mathbf{K} * \mathit{\Omega}$ for $\tau \in \mathbb{R}$ instead of the rescaled functions ω_k directly.

In their paper they further studied asymptotic expansions, not only for the leading term of ω as $t \to \infty$, but also the second term by spectral analysis for (R).

2.8.6 Remark on Asymptotic Behavior of the Vorticity

In §2.2.2 we estimate the difference only in the L^∞-norm. However, it is possible to replace this by the L^p-norm. We just state the results currently available (also for the velocity) without proofs.

Theorem. *For $\omega_0 \in L^1(\mathbb{R}^2)$ let (ω, u) be the solution of (2.7)–(2.9). Then it satisfies*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \|\omega - mg\|_p(t) = 0, \quad 1 \le p \le \infty, \tag{2.19}$$

$$\lim_{t\to\infty} t^{\frac{1}{2}-\frac{1}{p}} \|u - m\mathbf{K} * g\|_q(t) = 0, \quad 2 < g \le \infty, \tag{2.20}$$

for $m = \int_{\mathbb{R}^2} \omega_0 \, dx$.

The result (2.20) for u follows from ω as in the proof of Theorem (ii), (iii) of §2.4.1. Results (2.19) for general $p, 1 \le p < \infty$, follow from a Rellich-type compactness theorem instead of the Ascoli–Arzelà-type theorem. A full proof using (R) is given in [Gallay Wayne 2005]. Convergence of higher derivatives was also shown by [Maekawa 2008a].

Exercises 2

2.1 (§2.1.2) Prove formulas (2.3.1) and (2.3.2).

2.2 (§2.3.4) Calculate $\sum_{j=1}^{\infty} j/2^j$.

2.3 (§2.3.5, §2.3.7) For $f \in C(\mathbb{R}^n)$, show that $\lim_{r\to\infty} \|f\|_r = \|f\|_\infty$. Here we assume that there exists an r_0 $(1 \le r_0 < \infty)$ with $\|f\|_{r_0} < \infty$. (To show this, it suffices to assume (Lebesgue) measurability. We need not assume continuity.)

2.4 (§2.3.5) For $1 \le q \le \infty$, prove that $\|f\|_q \le \|f\|_1^{1/q} \|f\|_\infty^{1-1/q}$, where $f \in C(\mathbb{R}^n)$. (For a more general case, see Exercise 6.2.)

2.5 (§2.3.6) In Lemma 2.3.4, show that if $y_\rho \le N_\rho$, $\rho = 2^k$, then for sufficiently large m,

$$(y_s(t))^{1/s} \le \left(\frac{4}{a}\right)^{1/\rho} N_\rho^{1/\rho} t^{-1/\rho+1/s}, \quad t > 0,$$

where $s = 2^m \ge \rho$.

2.6 (§2.5) Assume that $\lim_{k\to\infty} f_k(x) = f(x)$, $x \in \mathbb{R}^n$, (i.e., that f_k converges pointwise to f on $\mathbb{R}^n$) and f_k, f are (Lebesgue) integrable (it may be assumed that the functions are continuous). Under these assumptions, show that

$$\|f\|_q \le \varliminf_{k\to\infty} \|f_k\|_q, \quad 1 \le q \le \infty.$$

(Hint: Use Fatou's lemma from §7.1.2.)

2.7 (§2.3) Extend the estimates in Theorem 2.3.6 to n-dimensional space to prove L^p-L^q estimates (like (1.5)) for the heat equation without using the representation formula.

2.8 (§2.3)

(i) Extend Lemma 2.3.2 to n-dimensional space (for Exercise 2.7). Prove in particular that

$$2\int_0^t \|\nabla\omega\|_2^2(s)ds \le \|\omega_0\|_2^2.$$

(ii) If $v = 0$, then

$$\|\nabla\omega\|_2(t) \le Ct^{-1/2}\|\omega_0\|_2, \quad t > 0,$$

with some constant C. (One may take $C = 1/\sqrt{2}$.)

(iii) If $v = 0$, then

$$\|\partial_x^\alpha \omega\|_2(t) \le C' t^{-1}\|\omega_0\|_2, \quad t > 0,\ |\alpha| = 2,$$

with some constant C'.

Hints:

(i) Integrate the identity in the lemma over the time interval $(0, t)$.

(ii) By scaling it suffices to prove the estimate only at $t = 1$. The estimate (i) implies that there is a set $J \subset (0, 1)$ whose Lebesgue measure is at least 1/2 such that $\|\nabla\omega\|_2(s) \le \|\omega_0\|_2$ for $s \in J$. Since $\|\nabla\omega\|_2(1) \le \|\nabla\omega\|_2(s)$ for the heat equation we have $\|\nabla\omega\|_2(1) \le \|\omega_0\|_2$. One may modify this argument in order to get $\|\nabla\omega\|_2(1) \le C\|\omega_0\|_2$ for any $C > 1/\sqrt{2}$.

(iii) Use (ii) twice to get $\|\partial_x^\alpha\omega\|_2(t) \le C(t/2)^{-1/2}\|\nabla\omega\|_2(t/2) \le C^2(t/2)^{-1}\|\omega_0\|_2$.

3

Self-Similar Solutions for Various Equations

We first present for the porous medium equation, a typical nonlinear degenerate diffusion equation, that its (forward) self-similar solution well describes asymptotic behavior of solutions, as is observed for the heat equation, without proof. We next explain that it is important to classify backward self-similar solutions in order to analyze behavior of solutions near singularities for the axisymmetric mean curvature flow equation as an example. In what follows, a self-similar solution is regarded as a stationary solution of the equation written with similarity variables. Convergence behavior of a solution of the equation to its stationary corresponds to the asymptotic behavior of the solution of the original equation near singularities. We give an outline of the proof of convergence and mention that a monotonicity formula plays a key role. Moreover, we give a simple proof of uniqueness of the stationary solutions, i.e., the backward self-similar solutions of the original equation. The proof is simpler and easier than that in the literature. We remark that the method using similarity variables is applicable, to some extent, to other diffusion equations such as semilinear heat equations and harmonic map flow equations. Finally, we note that the existence of forward self-similar solutions has also been proved for nonlinear equations of nondiffusion type.

3.1 Porous Medium Equation

The porous medium equation is proposed in order to describe the distribution of the density of a substance that flows through a uniformly distributed porous medium. For example, this equation may give a clue to the distribution of the density of water as it soaks into concrete. It is usually derived as follows. Let $\rho = \rho(x,t)$ (≥ 0) denote the density of the substance (water, for example) at time t and point $x \in \mathbb{R}^n$. (Physically, the cases $n = 1, 2, 3$ are important.) Moreover, $v = v(x,t)$ ($\in \mathbb{R}^n$) denotes the velocity vector of the substance and $p = p(x,t)$ ($\in \mathbb{R}$) denotes the pressure. By the mass conservation law we obtain

M.-H. Giga et al., *Nonlinear Partial Differential Equations*,
Progress in Nonlinear Differential Equations and Their Applications 79,
DOI 10.1007/978-0-8176-4651-6_3,

$$\partial_t \rho + \operatorname{div}(\rho v) = 0. \tag{3.1}$$

By Darcy's law,[1] which reflects the fact that the substance flows in a porous medium, we obtain

$$v = -\nabla p. \tag{3.2}$$

Assuming the constitutive law for pressures and densities[1]

$$p(\rho) = \rho^\gamma, \quad \gamma \geq 1, \tag{3.3}$$

by substituting (3.2) and (3.3) into (3.1), we obtain

$$\partial_t \rho - \frac{\gamma}{1+\gamma} \Delta \rho^{1+\gamma} = 0. \tag{3.4}$$

To simplify the equation (3.4), we shall take a constant C such that $C^\gamma = \frac{\gamma}{1+\gamma}$, and set $u = C\rho$, and then (3.4) is equivalent to

$$\partial_t u - \Delta u^m = 0, \tag{3.5}$$

where $m = \gamma + 1$. The assumption $\gamma \geq 1$ corresponds to $m \geq 2$. For $m = 1$, (3.5) is the heat equation. In this book, when $m > 1$, we call (3.5) the *porous medium equation*. The equation is also important for m satisfying $0 < m < 1$, since this describes plasma phenomena, for example. Since the properties of the solutions for $m < 1$ and for $m > 1$ are significantly different, we will discuss only the case for $m > 1$. Since u originally denotes a positive constant multiple of the density, we consider only nonnegative solutions.

Let us calculate self-similar solutions of the porous medium equation (3.5). As in the case of the heat equation, if u satisfies (3.5) in $\mathbb{R}^n \times (0, \infty)$ (and u and u^m are smooth), then

$$u_{\mu,\lambda}(x,t) = \mu u(\lambda x,\ \lambda^2 \mu^{m-1} t), \quad \lambda > 0,\ \mu > 0,$$

also satisfies (3.5) in $\mathbb{R}^n \times (0, \infty)$. Moreover, it can be shown that its total mass $\int_{\mathbb{R}^n} u(x,t)dx$ is conserved for evolution of time in the same way as for the heat equation in §1.2.2 (if integration by parts is justified). Since the total mass is conserved under the scaling transformation $u_{\mu,\lambda}$ with $\mu = \lambda^n$ above, we define the scaling transformation by

$$u_k(x,t) = k^n u(kx,\ k^{2+(m-1)n} t), \quad k > 0, \tag{3.6}$$

which preserves the total mass and is a generalization of the scaling transformation for the heat equation. Below, we shall consider only the case $m > 1$.

[1] In order to reflect physical phenomena, (3.2) and (3.3) have positive constant coefficients on the right-hand sides. Those multipliers can be normalized to one by changing scales as at the beginning of §2.1.

3.1.1 Self-Similar Solutions Preserving Total Mass

Let u be a function invariant under the scaling transformation (3.6) preserving total mass such that

$$u_k(x,t) = u(x,t), \quad x \in \mathbb{R}^n,\ t > 0,\ k > 0,$$

is satisfied. Then u can be expressed as

$$u(x,t) = t^{-\ell} w(t^{-\ell/n} x) \tag{3.7}$$

with

$$w(y) = u(y,1), \quad y \in \mathbb{R}^n, \quad k = t^{-\ell/n}, \quad \ell = \frac{n}{2+(m-1)n}.$$

Similarly to the Navier–Stokes equations in §2.7.3, a direct calculation shows that a function u invariant under the scaling transformation (3.6) is a solution of (3.5) if and only if w satisfies

$$\Delta w^m(y) + \frac{\ell}{n}\langle y, \nabla w(y)\rangle + \ell w(y) = 0, \quad y \in \mathbb{R}^n. \tag{3.8}$$

(This is a formal argument under the assumption that u and u^m are sufficiently smooth.) Now we shall choose the pressure $v = w^{m-1}$ as a dependent variable instead of density. If $v > 0$, then we obtain an equation for $v = v(y)$ from equation (3.8):

$$v^{\frac{1}{m-1}} \frac{m}{m-1} \left\{ \Delta v + \frac{1}{m-1}|\nabla v|^2 v^{-1} + \frac{\ell}{mn} v^{-1}\langle y, \nabla v\rangle + \frac{\ell(m-1)}{m} \right\} = 0 \tag{3.9}$$

for $y \in \mathbb{R}^n$. Let us find a nonnegative solution radially symmetric with respect to the origin and quadratic in $|y|$ near the origin. We in particular consider a solution of the form

$$\tilde{v}(y) = \left(\beta^2 - c^2|y|^2\right)_+, \quad y \in \mathbb{R}^n, \tag{3.10}$$

where $(a)_+ = \max(a,0)$ denotes the positive part of a. Here β and c are constants. Since $\Delta\tilde{v} = -2nc^2$ at $y \in \mathbb{R}^n$ with $\tilde{v}(y) > 0$, setting

$$c^2 = \frac{\ell(m-1)}{2mn}, \tag{3.11}$$

we have $\Delta\tilde{v} + \frac{\ell(m-1)}{m} = 0$. By a direct calculation we obtain

$$\begin{aligned} \frac{1}{m-1}|\nabla\tilde{v}|^2 + \frac{\ell}{mn}\langle y, \nabla\tilde{v}\rangle &= \frac{4c^4|y|^2}{m-1} - \frac{\ell 2c^2|y|^2}{mn} \\ &= 2c^2|y|^2\left(\frac{2c^2}{m-1} - \frac{\ell}{mn}\right) = 0. \end{aligned}$$

(The final equality is due to the choice of c in (3.11).) This shows that $\tilde{v}$ with (3.11) formally satisfies (3.9). A further discussion is necessary to conclude that $\tilde{v}$ satisfies (3.9) on the boundary of the ball where $\tilde{v} > 0$.

Definition. *Let $\tilde{w}$ be a function on $\mathbb{R}^n$ of the form $\tilde{w}(y) = (\beta^2 - c^2|y^2|)_+^{1/(m-1)}$ with $c^2 = \frac{\ell(m-1)}{2mn}$, $\ell = \frac{n}{2+(m-1)n}$. Take β^2 such that $\int_{\mathbb{R}^n} \tilde{w}(y)dy = 1$. For $L > 0$ we call*

$$V_L(x,t) = L^{\frac{1}{m-1}} \frac{1}{(Lt)^\ell} \tilde{w}\left(\frac{|x|}{(Lt)^{\ell/n}}\right), \quad x \in \mathbb{R}^n, \ t > 0,$$

a Barenblatt self-similar solution.

From the expression of V_L it is obvious that V_L is invariant under the scaling transformation (3.6) from the expression of V_L. As we have observed, V_L satisfies (3.5) at (x,t) where $V_L(x,t) > 0$. By the choice of β, we obtain

$$\int_{\mathbb{R}^n} V_L(x,t)dx = L^{1/(m-1)}$$

by a simple calculation; hence the total mass is conserved for $t > 0$. At least for $m \geq 2$, V_L is not differentiable on the boundary of the open set where $V_L > 0$. We shall extend the notion of a solution of (3.5) to such a nondifferentiable function. For this reason it is important to introduce the notion of a weak solution.

3.1.2 Weak Solutions

Let u be a locally integrable function on $\mathbb{R}^n \times [0,\infty)$.

(i) Let f be a locally integrable function on $\mathbb{R}^n$. A function u is said to be a *weak solution* of the porous medium equation (3.5) with initial value f if u satisfies

$$0 = \int_{\mathbb{R}^n} \varphi(x,0)f(x)dx + \int_0^\infty \int_{\mathbb{R}^n} (u\partial_t\varphi + u^m \Delta\varphi)dx\, dt$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$.

(ii) If u satisfies, instead of (i),

$$0 = \kappa\varphi(0,0) + \int_0^\infty \int_{\mathbb{R}^n} (u\partial_t\varphi + u^m \Delta\varphi)dx\, dt$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, then u is called a *weak solution* of (3.5) with initial value $\kappa\delta$, where $\kappa \in \mathbb{R}$.

It is not difficult to check that the function V_L is a weak solution with $\kappa = L^{1/(m-1)}$ in the sense of (ii). (Moreover, under suitable conditions it can be shown that there is no other solution except the Barenblatt self-similar solution.) Thereby the name "self-similar solution" has been justified.

This self-similar solution has some properties different from the Gauss kernel $g(x,t)$. The Gauss kernel $g(x,t)$ is positive everywhere in the whole space for $t > 0$. On the other hand, the set of points x with $V_L(x,t) > 0$ forms a ball, and its radius increases with evolution of time t. This is because the diffusion of the porous medium equation (3.5) ($m > 1$) is degenerate at (x,t) where $u(x,t) = 0$. In general, when the initial value is a continuous function with compact support, the support of the corresponding weak solution $u(\cdot,t)$ of (3.5) is bounded, so the support has the property of "finite propagation speed" in contrast to the heat equation. (Recall that for the heat equation, if the initial value is nonnegative and is not identically zero, then the support of the solution $u(\cdot,t)$ is the whole space $\mathbb{R}^n$ for $t > 0$, no matter how small the support of the initial value is.)

For the porous medium equation (3.5) we are able to obtain the asymptotic behavior of the solution as time tends to infinity by analyzing the compactness and characterization of the limit function of the family $\{u_k\}_{k\geq 1}$ obtained by scaling transformation (3.6) in a similar way as in Chapter 1. In this book we present only the results in the next section, whose proof is given in [Friedman Kamin 1980]. For surveys of mathematical analysis on the porous medium equation, the reader is referred to [Aronson 1986] and the books [Vázquez 2006] and [Vázquez 2007]. For example, in [Vázquez 2006] smoothing and decay estimates for the solution of (3.5) are extensively discussed.

3.1.3 Asymptotic Formula

Theorem. *Let u be a weak solution of (3.5) with nonnegative initial value $f \in C_0(\mathbb{R}^n)$ and suppose that u satisfies*

$$\int_0^T \int_{\mathbb{R}^n} (|u(x,t)|^2 + |\nabla(u^m(x,t))|^2) dx\ dt < \infty$$

for $T > 0$. (Such a weak solution is known to exist and to be nonnegative.) If one sets $L = (\int_{\mathbb{R}^n} f(x)\ dx)^{m-1}$, then

$$\lim_{t\to\infty} t^{\ell} \|u - V_L\|_\infty(t) = 0, \quad \ell = \frac{n}{2+(m-1)n}.$$

The assumption on the initial value can be slightly weakened if we modify a bit the presentation of the result (see [Friedman Kamin 1980]).

3.2 Roles of Backward Self-Similar Solutions

As mentioned in §2.7.3, backward self-similar solutions are considered to play an important role for the analysis of existence of singularities and behavior near singularities of solutions of evolution equations, in general. Here we investigate the role of backward self-similar solutions of the mean curvature flow equation for axisymmetric surfaces as an example.

3.2.1 Axisymmetric Mean Curvature Flow Equation

Let $\Gamma(t)$ be a smooth n-dimensional hypersurface in $\mathbb{R}^{n+1} (n \geq 2)$ depending on the time variable t. Assume that it divides $\mathbb{R}^{n+1}$ into two parts. The vector $\mathbf{n}$ denotes the unit normal vector field on the surface $\Gamma(t)$. Let $V = V(x,t)$ be the normal velocity (in the direction of $\mathbf{n}$) at x on $\Gamma(t)$. The *mean curvature flow equation* is an equation describing the motion of $\Gamma(t)$ and requires that V be equal to the (n times) mean curvature $H = H(x,t)$ of $\Gamma(t)$ (in the direction of $\mathbf{n}$). Namely,

$$V = H \quad \text{on} \quad \Gamma(t). \tag{3.12}$$

The mean curvature flow equation is often used to model the motion of phase boundaries separating two phases by thermodynamic effects. For example, it is used to describe a grain boundary motion of metal that consists of a huge number of crystals (grains).

If $\Gamma(t)$ is axially symmetric, say rotationally symmetric with respect to the x_1-axis, then $\Gamma(t)$ can be expressed by rotating a curve $\gamma(t)$ in the x_1-r-plane with respect to the x_1-axis, where r denotes the distance from the x_1-axis. Let $\hat{\mathbf{n}} = \hat{\mathbf{n}}(P,t)$ be the downward unit normal vector of $\gamma(t)$ at P in the x_1-r-plane; $\hat{\mathbf{n}}$ points in the direction in which r decreases. When the surface $\Gamma(t)$ is axially symmetric, the mean curvature flow equation is reduced to the equation

$$v = k + \frac{n-1}{r} \cos\theta \quad \text{on} \quad \gamma(t) \tag{3.13}$$

for $r > 0$. Here $v(P,t)$ is the normal velocity at the point P on $\gamma(t)$ in the direction of $\hat{\mathbf{n}}$, $k(P,t)$ is the curvature at P on $\gamma(t)$ in the direction of $\hat{\mathbf{n}}$, and $\theta(P,t)$ is the angle between the tangent vector at P on $\gamma(t)$ and the x_1-axis. The right-hand side of this equation is exactly the (n times) mean curvature of $\Gamma(t)$. For t assume that a point P on $\gamma(t)$ is expressed by $(x_1, u(x_1,t))$, so that $\gamma(t)$ is expressed by the graph of a nonnegative function u. Then we have

$$v = \frac{-\partial_t u}{(1 + (\partial_{x_1} u)^2)^{1/2}}, \quad k = -\frac{\partial_{x_1}^2 u}{(1 + (\partial_{x_1} u)^2)^{3/2}},$$
$$\cos\theta = \frac{1}{(1 + (\partial_{x_1} u)^2)^{1/2}},$$

in the region where $u > 0$ and u is smooth. Then (3.13) is reduced to

$$\partial_t u - \frac{\partial_{x_1}^2 u}{1 + (\partial_{x_1} u)^2} + \frac{n-1}{u} = 0. \tag{3.14}$$

For example, if $T > 0$, then

$$\hat{u}(t) = \sqrt{2(n-1)(T-t)} \tag{3.15}$$

satisfies (3.14) for $t < T$. Geometrically, a cylinder with radius $\hat{u}(t)$ satisfies (3.12) for $t < T$ and the radius becomes zero at time T, so that the cylinder shrinks to the x_1-axis.

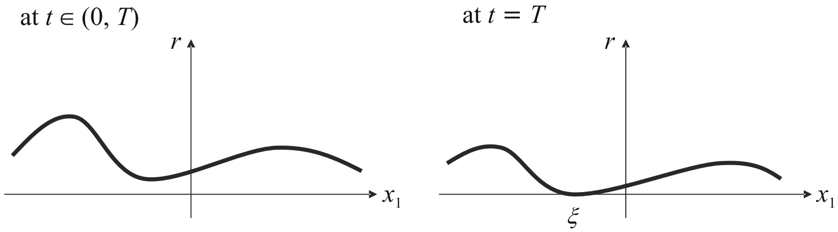

Figure 3.1. Two typical profiles of the graphs of $u(x_1, t)$ as a function of x_1.

3.2.2 Backward Self-Similar Solutions and Similarity Variables

The equation (3.14) is invariant under the scaling transformation

$$u_{(\lambda)}(x_1, t) = \frac{1}{\lambda} u(\lambda x_1, \lambda^2 t), \quad \lambda > 0,$$

i.e., if u solves (3.14), so does $u_{(\lambda)}$. This can be shown by direct calculations. Geometrically this property corresponds to invariance of the mean curvature flow equation (3.12) under dilation of surfaces and by dilation of the time variable. As in §2.7.3, u is said to be *backwardly self-similar* if $u_{(\lambda)}(x_1, t) = u(x_1, t)$ holds for all $x_1 \in \mathbb{R}$, $t < 0$, and $\lambda > 0$. If such a function u satisfies (3.14) in $\mathbb{R} \times (-\infty, 0)$, u is said to be a *backward self-similar solution*. For example, v in (3.15) is a backward self-similar solution if we replace $T - t$ by $-t$. (Here and in the sequel, we also call a solution a backward self-similar solution if it becomes backward self-similar by suitable translation in the time direction.) The function $\hat{u}$ defined by (3.15) is a backward self-similar solution that becomes zero at $t = T$.

How does the solution $u(x_1, t)$ generally behave when its local minima decrease and vanish at some time? Geometrically, we are going to investigate behaviors of the surface $\Gamma(t)$ at the time when the surface pinches off on the axis of rotation. See Figure 3.1. We shall introduce similarity variables as in §2.7.3. First we assume that one of the minima of $u(x_1, t)$ converges to zero as $t \to T$, and that the point at which the minimum is attained tends to the point $x_1 = \xi$ as $t \to T$. (The property that the point at which the minimum is attained tends to some point without oscillation is not trivial, since the equation (3.14) makes no sense at $t = T$. This is a typical property of one-dimensional second-order parabolic equations, which is proved in [Chen Matano 1989] using the nonincreasing property of the number of zeros for solutions of linear parabolic equations [Angenent 1988] together with reflection arguments.)

We have considered a scaling transformation around the origin

$$u_{(\lambda)}(x_1, t) = \frac{1}{\lambda} u(\lambda x_1,\ \lambda^2 t), \quad \lambda > 0.$$

Here we study a scaling transformation around point $(\xi, T) \in \mathbb{R} \times \mathbb{R}$,

$$u_{(\lambda)}^{\xi,T}(x_1, t) = \frac{1}{\lambda} u(\lambda(x_1 - \xi) + \xi, \lambda^2(t - T) + T)$$
$$x_1 \in \mathbb{R},\ t < T,\ \lambda > 0.$$

In particular, when $\xi = 0$ and $T = 0$, $u_{(\lambda)}^{\xi,T}$ agrees with $u_{(\lambda)}$. Note that $u_{(\lambda)}^{\xi,T}$ is a function obtained by magnifying u around $(x_1, t) = (\xi, T)$ when one takes λ small. Hence its limit as $\lambda \to 0$ can be understood to reflect the asymptotic behavior of u near $(x_1, t) = (\xi, T)$. If the limit u^∞ of $u_{(\lambda)}^{\xi,T}$ as $\lambda \to 0$ exists, then u^∞ satisfies

$$(u^\infty)_{(\lambda)}^{\xi,T}(x_1, t) = u^\infty(x_1, t),\ x_1 \in \mathbb{R}, \quad t < T,\ \lambda > 0.$$

This means that u^∞ is a backward self-similar solution up to translation in space variables. As in Chapter 1 and Chapter 2, an asymptotic formula for u around $(x_1, t) = (\xi, T)$ can be derived by showing compactness of the sequence $\{u_{(\lambda)}^{\xi,T}\}_{0<\lambda<1}$ and uniqueness of its limit as $\lambda \to 0$. However, here we will prove an asymptotic formula using similarity variables.

As in §2.7.3, we introduce similarity variables (z, τ, w) at $(x_1, t) = (\xi, T)$ with $T > 0$ for the variables (x_1, t, u) as follows:

$$\tau = -\log\sqrt{T - t}, \quad z = \frac{x_1 - \xi}{\sqrt{2(T - t)}} + \xi,$$
$$w(z, \tau) = \frac{1}{\sqrt{2(T - t)}} u(x_1, t).$$

Using these similarity variables, we have

$$w(z, \tau) = \frac{e^\tau}{\sqrt{2}} u(\sqrt{2} e^{-\tau}(z - \xi) + \xi,\ T - e^{-2\tau}).$$

If $u(x_1, t)$ is defined and positive in $J \times (0, T)$, where J is an open interval containing ξ, then so is w in the domain (connected open set)

$$\mathcal{W} = \left\{(z, \tau);\ z = \frac{1}{\sqrt{2}} e^\tau (x_1 - \xi) + \xi,\ x_1 \in J,\ \tau > -\frac{1}{2}\log T\right\}$$

in $\mathbb{R} \times \mathbb{R}$. See Figure 3.2. The cross-section

$$\left\{z \in \mathbb{R};\ z = \frac{1}{\sqrt{2}} e^{\tau_0}(x_1 - \xi) + \xi,\ x_1 \in J\right\}$$

at $\tau = \tau_0$ of $\mathcal{W}$ expands to the real line $\mathbb{R}$ as τ_0 tends to infinity.

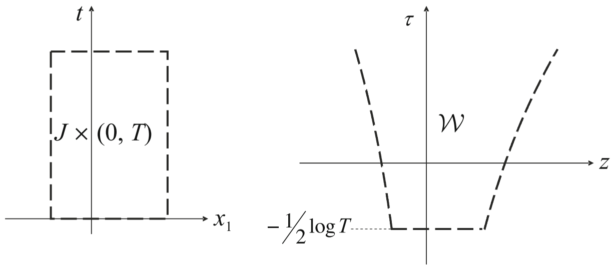

Figure 3.2. Domains of definition of u and w.

The constant $\sqrt{2}$ is just for computational convenience. Note that u is a backward self-similar solution if and only if w is independent of τ. Moreover, u satisfies (3.14) in $J \times (0,T)$ if and only if w satisfies

$$\partial_\tau w - \frac{\partial_z^2 w}{1 + (\partial_z w)^2} + z\partial_z w - w + \frac{n-1}{w} = 0 \tag{3.16}$$

in $\mathcal{W}$. The equation (3.16) is called an equation with similarity variables. The behavior of w as $\tau \to \infty$ corresponds to the behavior of u as $t \to T$. Since

$$w(z,\tau) = u_{(\lambda)}^{\xi,T}\left(z,\ T - \frac{1}{2}\right), \quad \lambda = \sqrt{2}e^{-\tau},$$

the function w represents magnification of $u(x_1,t)$ near $x_1 = \xi$ as τ becomes large. Hence, in order to determine the behavior of $u(x_1,t)$ near $(x_1,t) = (0,T)$, we shall analyze the behavior of w in (3.16) as $\tau \to \infty$.

Since the backward self-similar solution $\hat{u}$ corresponds to the constant function $\sqrt{n-1}$ in the equation (3.16), we expect that w converges to $\sqrt{n-1}$ as $\tau \to \infty$ if this self-similar solution exhibits typical behavior. To prove this convergence it is sufficient to show compactness of the sequence $\{w(\cdot,\tau)\}_{\tau \geq 1}$ and to characterize its possible limit functions as in Chapter 1. However, since in this problem it is difficult to characterize the limit functions of $\{w(\cdot,\tau)\}_{\tau\geq 1}$ directly, we will study the following two items:

(1) Compactness:
For a sequence $\{\tau_j\}$ such that $\lim_{j\to\infty} \tau_j = \infty$, set $w^j(z,\tau) = w(z,\tau+\tau_j)$ and show that $\{w^j\}_{j=1}^{\infty}$ contains a convergent subsequence $\{w^\ell\}$.

(2) Characterization of the limit function:
 (A) Show that the limit of $\{w^\ell\}$ is a solution of (3.16) that is independent of τ, that is, a stationary solution. (The limit is a function on $\mathbb{R}$ since the domain of the definition of w^ℓ converges to $\mathbb{R}$ as τ tends to infinity.) Furthermore, analyze properties of the limit function.
 (B) Show that the stationary solution obtained in (A) is nothing other than the constant $\sqrt{n-1}$.

Proving the above items (1) and (2) will provide an asymptotic formula (Theorem 3.2.4) of u near $x_1 = \xi$ as $t \to T$. To prove (1) we use a certain estimate based on structures of the equation. To prove (2)(A) it is sufficient to show that $\partial_\tau w$ diminishes as τ increases. (This in particular implies that w does not oscillate.) The proof of (2)(B) corresponds to the characterization of self-similar solutions. In the next section we prove (2)(B); however, for (1) and (2)(A) we present only outlines of the proofs, in §3.2.4 and §3.2.5, respectively. So far, the only known method to prove (2)(A) uses the monotonicity formula as discussed in this book.

3.2.3 Nonexistence of Nontrivial Self-Similar Solutions

Theorem. *If a positive function* $W \in C^2(\mathbb{R})$ *satisfies*

$$-\frac{\partial_z^2 W}{1 + (\partial_z W)^2} + z\,\partial_z W - W + \frac{n-1}{W} = 0 \tag{3.17}$$

in $\mathbb{R}$ *and*

$$z\,\partial_z W - W \leq 0 \tag{3.18}$$

in $\mathbb{R}$*, then* W *must be the constant function* $W \equiv \sqrt{n-1}$*. (This claim is valid for any real number* $n > 1$*.)*

If we express a backward self-similar solution by similarity variables, it solves (3.17). Hence this theorem asserts that a self-similar solution that is nonincreasing in time must be $\hat{u}$ (up to translation of time). (The condition (3.18) is equivalent to the property that the self-similar solution is nonincreasing in time t.) This theorem follows from the classification of the self-similar solutions (not necessarily axisymmetric) of equation (3.12) due to [Huisken 1993], whose proof is geometric and requires many pieces of geometric knowledge. An outline of a more analytic proof is given in [Altschuler Angenent Giga 1995]. Here we give an analytic and elementary proof, which improves the proof due to [Soner Souganidis 1993]. The last two articles impose the hypothesis that

$$\inf_{\mathbb{R}} W > 0 \tag{3.19}$$

for the infimum of W, but we do not assume (3.19). The supplemental additional conditions (3.18) and (3.19) are mostly satisfied if W is obtained as the limit of a solution w of equation (3.16) as $\tau \to \infty$ as far as the neck-pinching problem is considered. For simplicity of notation we denote $\partial_z W$ and $\partial_z^2 W$ by W' and W'', respectively.

Proof. Since W is a positive function, it is sufficient to derive that $W'' \leq 0$ on $\mathbb{R}$ (Exercise 3.2). Since

$$\frac{W''}{1 + (W')^2} \leq \frac{n-1}{W} \quad \text{in } \mathbb{R},$$

from (3.17) and (3.18), we have $\psi \le n-1$ with

$$\psi = \frac{WW''}{1+(W')^2}.$$

Examining ψ'', we will show that $\psi \le 0$.

The First Step

We shall derive a differential equation of second order that is satisfied by ψ. Since W is a solution of (3.17), we have

$$\psi = zWW' - W^2 + n - 1.$$

Differentiating both sides and expressing W'' by ψ, we obtain

$$\psi' = -WW' + z(W')^2 + z(1+(W')^2)\psi.$$

Differentiating again, we have

$$\begin{aligned}\psi'' &= -WW'' + 2zW'W''(1+\psi) + (1+(W')^2)\psi + z(1+(W')^2)\psi' \\ &= 2z\frac{W'}{W}(1+(W')^2)\psi(1+\psi) + z(1+(W')^2)\psi'.\end{aligned}$$

For the convenience of applying the maximum principle in the second step, we shall derive an expression for ψ'' such that the sign of the terms that do not contain ψ' can be easily checked. By (3.17) the above expression of ψ' yields

$$\psi' = \frac{W'}{W}(\psi - (n-1)) + z(1+(W')^2)\psi,$$

which implies

$$z(1+(W')^2)\psi = -\frac{W'}{W}(\psi - (n-1)) + \psi'. \tag{3.20}$$

Substituting this into the first term in the above expression of ψ'', we have

$$\psi'' = -\frac{2(W')^2}{W^2}(\psi-(n-1))(1+\psi) + z(1+(W')^2)\psi' + \frac{2W'}{W}(1+\psi)\psi'. \tag{3.21}$$

The Second Step

We next prove that ψ has no positive local maximum. By condition (3.18), we have $\psi \le n-1$ on $\mathbb{R}$. Hence, if ψ has a positive local maximum M at $z = z_0$, then ψ satisfies

$$\psi'' + b\psi' = \frac{2(W')^2}{W^2}(\psi - (n-1))(1+\psi) \ge 0$$

on a neighborhood I of z_0 by (3.21), where $b(z) = -z(1 + (W')^2) - 2(1 + \psi)W'/W$. The strong maximum principle (Exercise 3.1) yields that ψ identically equals a constant M on I. (In [Soner Souganidis 1993] an equation that ψ^2 satisfies is considered instead of (3.21).) Hence, equation (3.21) implies that $M = n-1$ or W' equals zero identically on I. However, if W' equals zero identically, so does ψ, which contradicts $M = \psi(z_0) > 0$. Thereby the case of $M = n - 1$ remains, but in this case ψ equals zero identically on I by (3.20), which also contradicts $\psi(z_0) > 0$. Therefore ψ has no positive local maximum.

The Third Step

We shall prove that $\psi \leq 0$ on $\mathbb{R}$. This property implies that $W'' \leq 0$ on $\mathbb{R}$, so that W is a constant function (Exercise 3.2) since $W > 0$. The only positive constant solution of (3.17), however, is $W = \sqrt{n-1}$. This completes the proof.

Unless $\psi \leq 0$ on $\mathbb{R}$, there is a point z_1 such that $\psi(z_1) > 0$ and $\psi'(z_1) \neq 0$. (Since ψ is not a positive constant by the second step, we are able to take such a z_1.) Since ψ has no positive local maximum, if $\psi'(z_1) > 0$, then ψ is nondecreasing in (z_1, ∞), that is, $\psi' \geq 0$ in (z_1, ∞). We thus obtain

$$\psi(z) \geq \psi(z_1) > 0, \quad z \in (z_1, \infty). \tag{3.22}$$

If $\psi'(z_1) < 0$, this inequality holds for $z \in (-\infty, z_1)$. In the following we discuss only (3.22), since the case of $\psi'(z_1) < 0$ can be discussed similarly.

Now we suppose that there exists $z_2 \in (z_1, \infty)$ with $W'(z_2) \geq 0$. Since W'' is positive on (z_1, ∞) by (3.22), W' is positive on (z_2, ∞). On the other hand, we know that $\psi \leq n - 1$ by (3.18), and that $\psi > 0$ and $\psi' \geq 0$ on (z_2, ∞). Then (3.21) yields that ψ'' is positive on $(\max(z_2, 0), \infty)$, but this contradicts the fact that ψ is nondecreasing on (z_2, ∞) and is bounded from above. Thus we may assume that W' is always negative on (z_1, ∞). Since the definition of ψ and (3.22) imply

$$W''(z) \geq \frac{\psi(z_1)}{W(z_1)} = c_0 > 0, \quad z \in (z_1, \infty), \tag{3.23}$$

integrating both sides of (3.23) on the interval (z_1, z) yields

$$W'(z) - W'(z_1) \geq c_0(z - z_1) > 0, \quad z \in (z_1, \infty). \tag{3.24}$$

Inequality (3.24) implies that W' is positive for sufficiently large z, which contradicts the hypothesis for W'. In this way, we are able to show that $\psi \leq 0$ on $\mathbb{R}$, which completes the proof. □

3.2.4 Asymptotic Behavior of Solutions Near Pinching Points

Suppose that a smooth axisymmetric closed surface $\Gamma(t)$ governed by the mean curvature flow equation (3.12) is given by

$$\Gamma(t) = \{(x_1, x_2, \dots, x_{n+1});\ r = u(x_1, t),\ a(t) \le x_1 \le b(t)\}$$

with

$$r = (x_2^2 + \cdots + x_{n+1}^2)^{1/2}.$$

(Hence u satisfies (3.14) in a region where $u > 0$.) Here $n \ge 2$ is an integer. We assume that

$$u(x_1, t) > 0, \quad a(t) < x_1 < b(t),$$
$$u(a(t), t) = 0, \quad u(b(t), t) = 0,$$

in the time interval $[0, T)$. Namely, we consider $\Gamma(t)$ as the surface obtained by rotating the graph of a function $r = u(x_1, t)$ with one space variable around the x_1-axis. Suppose that a point $\xi(t)$ at which $u(x_1, t)$ attains a local minimum moves continuously in t and that

$$\rho(t) = u(\xi(t), t)$$

converges to zero as $t \to T$. We also suppose that other local minima of u do not converge to zero as $t \to T$. In other words, a single neck of $\Gamma(t)$ pinches first at $t = T$. In this case, as mentioned in §3.2.2, the limit

$$\lim_{t \uparrow T} \xi(t) =: \xi(T)$$

exists. (Here $\lim_{t\uparrow T}$ denotes the limit as $t \to T$ with $t < T$, which is called the *left limit*. Similarly, $\lim_{t\downarrow T}$ denotes the limit as $t \to T$ with $t > T$, which is called the *right limit*. In this book we simply write them as $\lim_{t\to T}$, unless otherwise stated.) Moreover, we set

$$\varlimsup_{t\uparrow T} a(t) =: a(T), \quad \varliminf_{t\uparrow T} b(t) =: b(T).$$

Here $\varlimsup$ denotes the *limit superior*, and $\varliminf$ denotes the *limit inferior* defined by

$$\varlimsup_{t\uparrow T}\ a(t) = \lim_{t\uparrow T}\left(\sup_{t<s<T} a(s)\right), \quad \varliminf_{t\uparrow T}\ b(t) = \lim_{t\uparrow T}\left(\inf_{t<s<T} a(s)\right),$$

respectively. By monotonicity of motions near the axis (as explained later) $\lim_{t\uparrow T} a(t)$ and $\lim_{t\uparrow T} b(t)$ do exist. In this case, the asymptotic behavior of $\Gamma(t)$ near $x_1 = \xi(T)$ is described as follows.

Theorem. *Assume that $a(T) < \xi(T) < b(T)$. Then*

$$\lim_{t\uparrow T} \frac{u(\xi(T) + z\sqrt{2(T-t)}, t)}{\sqrt{2(T-t)}} = \sqrt{n-1}. \tag{3.25}$$

The convergence is uniform on every bounded interval $\{z \in \mathbb{R};\ |z| \le M\}$ with $M > 0$.

The assumption $a(T) < \xi(T) < b(T)$ is essential; if it does not hold, this asymptotic formula is no longer true. When $\xi(T)$ coincides with $b(T)$, the pinching behavior is essentially different from the one above. The existence of such a singularity was shown by [Altschuler Angenent Giga 1995], and later various explicit examples were constructed in [Angenent and Velázquez 1997].

Sketch of the proof. The asymptotic formula (3.25) is equivalent to

$$\lim_{\tau\to\infty} w(z,\tau) = \sqrt{n-1} \quad (\text{uniform convergence on } \{z \in \mathbb{R}; |z| \le M\}) \tag{3.26}$$

for a solution w of the equation (3.16) with respect to the similarity variables around $(\xi(T), T)$ with $T > 0$. To prove (3.26), as mentioned in §3.2.2, it suffices to prove (1) compactness and (2) characterization of the limit functions (A) and (B). Since we have already characterized the limit functions (2)(B) in §3.2.3, it remains to prove compactness (1), characterization of the limit function (2)(A), assumption (3.18) in §3.2.3, and the positivity of the limit function. Assumption $a(T) < \xi(T) < b(T)$ guarantees that $w(z,\tau)$ is well defined for $z \in \mathbb{R}$ with $|z| \le M$ for each $M > 0$, provided that τ is sufficiently large.

To prove (3.18) and (3.19) for the limit functions, we focus our attention on the following two properties:

(i) The monotonicity of motions near the axis of rotation

For sufficiently small $\mu > 0$, $\partial_t u(x_1,t) \le 0$, $t \in (0,T)$, $x_1 \in (a(t), b(t))$, provided that $u(x_1,t) \le \mu$. Namely, the mean curvature H (in the direction of the axis of rotation) is nonnegative near the axis, i.e.,

$$\frac{\partial_{x_1}^2 u}{1 + (\partial_{x_1} u)^2} - \frac{n-1}{u} \le 0.$$

By (3.12) this implies that $a(t)$ is nondecreasing and $b(t)$ is nonincreasing in time, so the limits $\lim_{t\uparrow T} a(t)$ and $\lim_{t\uparrow T} b(t)$ exist.

(ii) Estimates of the neck-shrinking rate

Let μ be the constant in (i). Then there exists a constant $\delta \in (0,1)$ such that

$$\frac{\partial_{x_1}^2 u}{1 + (\partial_{x_1} u)^2} \le (1-\delta)\frac{n-1}{u}$$

holds for (x_1,t) with $0 < u(x_1,t) \le \mu$. Hence, from equation (3.14), we obtain $\partial_t u \le -\delta(n-1)/u$. Setting $\rho(t) = u(\xi(t),t)$, since $\partial_{x_1} u(\xi(t),t) = 0$, we have

$$\partial_t \rho \le -\delta(n-1)/\rho,$$

which implies

$$\partial_t(\rho^2) \le -2\delta(n-1).$$

Integrating this over the interval (t, T) we obtain

$$\rho(t) \geq \sqrt{2\delta(n-1)(T-t)}, \quad (t \text{ is sufficiently close to } T \text{ with } 0 < t < T),$$

since $\rho(T) = 0$.

Both properties (i) and (ii) are proved in [Altschuler Angenent Giga 1995]. In the proof of (ii), the authors use an estimate of $|A|^2/H^2$ obtained by [Huisken 1990], where A represents the second fundamental form and H represents the mean curvature of a surface. Relation (i) is obtained by comparing the intersection numbers between the stationary solution and the curve $\gamma(t)$. The proofs of both (i) and (ii) become easier if the form of $\Gamma(t)$ is symmetric. The proof for such a symmetric surface is actually given in [Soner Souganidis 1993].

The inequality for u in property (i) is expressed by the similarity variables (z, τ, w) as follows:

$$\frac{\partial_z^2 w}{1 + (\partial_z w)^2} - \frac{n-1}{w} \leq 0$$

for (z, τ) with

$$0 < w(z, \tau) \leq \frac{\mu}{\sqrt{2(T-\tau)}}.$$

This property is inherited for the limit $W_* = \lim_{\tau \to \infty} w(z, \tau)$. We thereby see that

$$\frac{\partial_z^2 W_*}{1 + (\partial_z W_*)^2} - \frac{n-1}{W_*} \leq 0 \quad \text{in } \mathbb{R}.$$

We thus obtain (3.18) for W_* from (3.17). (Note that the domain of definition of W expands as τ increases and tends to $\mathbb{R}$ as $\tau \to \infty$, since $a(T) < \xi(T) < b(T)$.)

Properties (i) and (ii) are also important for obtaining an estimate for w satisfying (3.16). For example, property (ii) yields that

$$w(z, \tau) \geq \sqrt{2\delta(n-1)}, \quad |z| \leq M, \ \tau \geq \tau_M,$$

for any $M > 0$ provided that τ_M is taken large enough. (Hence we have a lower bound (3.19) even for the limit W_*.) Comparing w with a conelike surface $\tilde{u}(x_1, t) = c_0|x_1|$ and using (i), we obtain the upper estimate

$$\sup_{\substack{(z,\tau) \in \mathcal{W} \\ \tau \geq \tau_M}} |w(z, \tau)|/(1 + |z|) < \infty$$

for the solution w of (3.16); however, we do not prove it here. (It easily follows from the proof of Proposition 2.1 in [Soner Souganidis 1993] and Lemmas 5.11–5.13 in [Altschuler Angenent Giga 1995].) By the estimates of w and the arguments in the above articles, we are also able to obtain estimates for derivatives

$$\sup_{\tau \geq \tau_M} \sup_{|z| \leq M} |\partial_z w(z,t)| < \infty, \qquad \sup_{\tau \geq \tau_M} \sup_{|z| \leq M} |\partial_z^2 w(z,t)| < \infty,$$

which imply the higher-order derivative estimate

$$\sup_{\tau \geq \tau_M} \sup_{|z| \leq M} |\partial_\tau^k \partial_z^h w(z,\tau)| < \infty$$

for any $k, h = 0, 1, 2, \ldots$, since (3.16) is a parabolic equation. For estimates of higher-order derivatives of solutions of parabolic equations, the reader is referred to the standard book [Ladyženskaja Solonnikov Ural'ceva 1968].

Using these estimates on derivatives of w, we apply the convergence of the higher-order derivatives in §5.2.5 (obtained as an application of the Ascoli–Arzelà theorem in §5.1.1), and observe that for any sequence $\tau_j \to \infty$ there exists a subsequence $\tau_{j(\ell)}$ such that

$$w^\ell(z,\tau) := w(z,\ \tau + \tau_{j(\ell)})$$

converges to some function w_∞ as $\ell \to \infty$ with all its derivatives uniformly in $|z| \leq M$, $-1 \leq \tau \leq 1$, for any M. For each M, if ℓ is chosen sufficiently large, then w^ℓ satisfies (3.16) in $(-M, M) \times (-1, 1)$, so that w_∞ is a smooth solution of (3.16) in $\mathbb{R} \times (-1, 1)$. The question is whether w_∞ is independent of τ. If so, i.e., if (3.17) holds, then such a solution satisfies (3.18) as mentioned before and is equal to $w_\infty \equiv \sqrt{n-1}$ by §3.2.3. In particular, setting $\tau = 0$ in $w^\ell(z,\tau)$ yields

$$w(z, \tau_{j(\ell)}) \longrightarrow \sqrt{n-1} \quad (\ell \to \infty),$$

and the limit is independent of the choice of the subsequence $\{\tau_{j(\ell)}\}$. Therefore $w(z,\tau)$ converges to $\sqrt{n-1}$ as $\tau \to \infty$ without taking a subsequence (Exercise 1.4). Thus we have given a sketch of a proof for the asymptotic formula (3.25) except for the τ-independence of w_∞. □

To give a proof of the τ-independence of w_∞ it is convenient to find what is called a Lyapunov functional for (3.16), which is nonincreasing in τ along solutions $w(z,\tau)$. We will mention this as a monotonicity formula in the following section.

In terms of the asymptotic formula (3.25), the convergence of surfaces near pinching points looks similar to the convergence of a cylinder at first glance, so it is tempting to think that pinching points are not isolated. However, they are isolated unless the surface is a cylinder. (This is proved in [Dziuk Kawohl 1991] and [Soner Souganidis 1993] under a symmetry assumption with respect to the origin such that the unique pinching point is the origin. For a general setting see [Altschuler Angenent Giga 1995].) Since we are able to understand that the left-hand side of (3.25) magnifies u near $x_1 = \xi(T)$ as t tends to T, it does not contradict the fact that pinching points are isolated. What is the shape of $u(x_1, t)$ at $t = T$ near $x_1 = \xi(T)$? We may formally guess the asymptotic shape. However, there seems no proof available. Is it possible to prove it by extending ideas of the series of works including [Herrero Velázquez 1993], in which a detailed asymptotic formula for semilinear heat equations is obtained?

The asymptotic formula (3.25) was first proved in [Huisken 1993] for $n = 2$ and $H \geq 0$. It seems to be difficult to extend Huisken's proof to general dimensions. The asymptotic formula (3.25) in this book is due to [Altschuler Angenent Giga 1995].

3.2.5 Monotonicity Formula

We discuss the monotonicity formula, which plays an important role in showing that $\partial_\tau w$ tends to zero as τ tends to ∞ when w satisfies (3.16).

We first consider a system of ordinary differential equations:

$$\frac{dx}{dt} = -(\nabla F)(x), \quad t > 0, \ \ x(0) = x_0, \tag{3.27}$$

where $x(t) = (x_1(t), \ldots, x_m(t))$ is an $\mathbb{R}^m$-vector valued function and F is a smooth real-valued function on $\mathbb{R}^m$. This type of equation is called a *gradient system*, since the right-hand side consists of the gradient of F. For a solution x of (3.27) we have

$$\frac{d}{dt}F(x(t)) = \left\langle (\nabla F)(x(t)),\ \frac{dx}{dt} \right\rangle = -\left|\frac{dx}{dt}\right|^2 \tag{3.28}$$

by the chain rule. In particular, F decreases along the solution as time increases. This is often called a monotonicity formula. Integrating both sides over the interval $(0, T)$ and multiplying by -1, we have

$$F(x_0) - F(x(T)) = \int_0^T \left|\frac{dx}{dt}\right|^2 dt.$$

If F is bounded from below, then $\lim_{T\to\infty} F(x(T))$ exists (as a finite value) by the monotonicity of $F(x(t))$. Thus we have

$$\int_0^\infty \left|\frac{dx}{dt}\right|^2 dt < \infty, \tag{3.29}$$

so that the integral of the left-hand side is finite. Roughly speaking, the property (3.29) shows that $|dx/dt|$ approaches zero in some sense as t tends to ∞ (Exercise 3.3).

To extend this idea to (3.16), we first consider the equation

$$h_t = \frac{\partial_{x_1}^2 h}{1 + (\partial_{x_1} h)^2} \tag{3.30}$$

for a function $h = h(x_1, t)$ of $x_1 \in \mathbb{R}$ and $t > 0$. This equation is called the *curvature flow equation* or the *curve shortening equation*, which expresses that the graph of the function h moves so that the velocity in the normal direction equals its curvature. The right hand side is equal to

$$(1+(\partial_{x_1}h)^2)^{1/2}\ \partial_{x_1}\left(\frac{\partial_{x_1}h}{(1+(\partial_{x_1}h)^2)^{1/2}}\right), \tag{3.31}$$

which is the product of the infinitesimal length of the graph of h and the curvature. Recall that the curvature is obtained by "variation" of the length of curves. The length of the graph of h over an interval I is

$$L(h)=\int_I (1+(\partial_{x_1}h)^2)^{1/2}dx.$$

Since h depends on time, $L(h)$ is a function of time t. Its time derivative is

$$\frac{dL(h)}{dt}=\int_I \frac{\partial_{x_1}h}{(1+(\partial_{x_1}h)^2)^{1/2}}(\partial_{x_1}\partial_t h)dx,$$

and integration by parts yields

$$\frac{dL(u)}{dt}=-\int_I \partial_{x_1}\left\{\frac{\partial_{x_1}h}{(1+(\partial_{x_1}h)^2)^{1/2}}\right\}\partial_t h\ dx,$$

provided that $\partial_t h=0$ or $\partial_{x_1}h=0$ at the boundary of I. Since (3.30) and (3.31) yield

$$\frac{dL(h)}{dt}=-\int_I \frac{(\partial_t h)^2}{(1+(\partial_{x_1}h)^2)^{1/2}}dx, \tag{3.32}$$

$L(h)$ decreases along solutions as t increases, which corresponds to (3.28) for (3.27). For (3.30) we thus obtain a *functional* L, which is a function whose variables consist of functions.

For the more complicated equation (3.16) we consider the functional

$$F(w)=\int_I \sigma(z,w)(1+(\partial_z w)^2)^{1/2}dz$$

for a function $w=w(z,\tau)$, instead of L. Here we assume that w satisfies (3.16) in $I\times(0,\infty)$ for simplicity. We shall take a suitable function σ of two variables in order to obtain a nonincreasing property like (3.32). Differentiating $F(w)$ with respect to τ, we have

$$\begin{aligned}\frac{dF(w)}{d\tau}&=\int_I \frac{\partial\sigma}{\partial w}(z,w)(1+(\partial_z w)^2)^{1/2}\ \partial_\tau w\ dz\\&\quad+\int_I \sigma(z,w)\ \frac{\partial_z w}{(1+(\partial_z w)^2)^{1/2}}\partial_z\ (\partial_\tau w)\ dz.\end{aligned}$$

We integrate the second term by parts in a similar way as for the computation of dL/dt. The right-hand side of the above equality is reduced to

$$\int_I \partial_\tau w \left\{ \frac{\partial\sigma}{\partial w}(1+(\partial_z w)^2)^{1/2} - \frac{\partial\sigma}{\partial w}\frac{(\partial_z w)^2}{(1+(\partial_z w)^2)^{1/2}} \right.$$
$$\left. - \frac{\partial\sigma}{\partial z}\frac{\partial_z w}{(1+(\partial_z w)^2)^{1/2}} \right\} dz - \int_I \partial_\tau w \left\{ \frac{\partial}{\partial z}\frac{\partial_z w}{(1+(\partial_z w)^2)^{1/2}} \right\} \sigma\, dz,$$

provided that $\partial_\tau w = 0$ or $\partial_z w = 0$ on the boundary of I. If we choose z such that

$$\frac{\partial\sigma}{\partial w} = \left(\frac{n-1}{w} - w\right)\sigma, \quad \frac{\partial\sigma}{\partial z} = -z\sigma, \tag{3.33}$$

then we get

$$\frac{dF(w)}{d\tau} = -\int_I (\partial_\tau w)^2 \frac{\sigma}{(1+(\partial_z w)^2)^{1/2}} dz$$

by (3.16). Since (3.33) is a simple linear ordinary differential equation of first order, its solution is a constant multiple of

$$\sigma = w^{n-1}e^{-(w^2+z^2)/2}.$$

Hence we obtain the following *monotonicity formula* for (3.16).

Theorem (Monotonicity formula). *Set*

$$F(w) = \int_I w^{n-1}e^{-(w^2+z^2)/2}(1+(\partial_z w)^2)^{1/2} dz.$$

Suppose that $w = w(z,\tau)$ satisfies (3.16) in $I \times (0,\infty)$. If $\partial_\tau w = 0$ or $\partial_z w = 0$ on the boundary of I, then

$$\frac{dF(w)}{d\tau} = -\int_I (\partial_\tau w)^2 \ \frac{w^{n-1}e^{-(w^2+z^2)/2}}{(1+(\partial_z w)^2)^{1/2}} dz \tag{3.34}$$

for $\tau > 0$. (Even if I is equal to $\mathbb{R}$, the formula is still valid if the preceding integration by parts is justified.)

Integrating formula (3.34) on $(0,\tau_1)$ and multiplying it by (-1) gives us

$$\begin{aligned}(F(w))(0) &\geq (F(w))(0) - (F(w))(\tau_1) \\ &= \int_0^{\tau_1}\int_I (\partial_\tau w)^2 \ \frac{w^{n-1}e^{-(w^2+z^2)/2}}{(1+(\partial_z w)^2)^{1/2}} \ dz\, d\tau.\end{aligned}$$

Here $(F(w))(\tau)$ denotes the value of $F(w)$ at τ. Hence for the solution w of (3.16) we obtain

$$\int_0^\infty \left(\int_I (\partial_\tau w)^2 \ \frac{w^{n-1}e^{-(w^2+z^2)/2}}{(1+(\partial_z w)^2)^{1/2}} \ dz \right) d\tau < \infty,$$

which corresponds to (3.29) for (3.27). Here we note that even if the function $A = w^{n-1}e^{-(w^2+z^2)/2}/(1+(\partial_z w)^2)^{1/2}$ tends to zero as $\tau \to \infty$, there might be a chance that $|\partial_\tau w|$ may not be small as $\tau \to \infty$. We have to exclude such a situation. However, we do not carry this out here. This is discussed in [Altschuler Angenent Giga 1995, Soner Souganidis 1993]. For example, estimate (ii) in §3.2.4 plays an important role in showing that w in the function A does not tend to zero. Showing that A does not converge to zero as $\tau \to \infty$, we are able to claim that $|\partial_\tau w|$ converges to zero as $\tau \to \infty$. For general second-order parabolic equations of one spatial variable, the method of finding F, which is called a Lyapunov functional, is due to [Zelenyak 1968] and clearly discussed in the appendix of [Matano 1986].

When we try to prove τ-independence of w_∞ in §3.2.4, that is, to derive (3.34) so as to carry out (A) of (2) "Characterization of the limit function" in §3.2.2, Theorem 3.2.5 requires the conditions $\partial_\tau w = 0$ and/or $\partial_z w = 0$ at the boundary of I, which are not guaranteed in general. However, if the surface $\Gamma(t)$ is closed, then it is possible to obtain a monotonicity formula that plays the same role as (3.34), so that we can show that $\partial_\tau w$ tends to zero by similar arguments as above and we can carry out (2)(A). Here, we only mention the monotonicity formula for closed surfaces; we do not give a detailed proof of τ-independence of w_∞. (See [Altschuler Angenent Giga 1995, Soner Souganidis 1993] in §3.2.3.)

We shall consider the geometric meaning of $F(w)$ in the monotonicity formula (3.34) in order to expect a monotonicity formula for closed surfaces. Let $\hat{\Gamma}(\tau)$ be the closed surface obtained by rotating the graph $\hat{\gamma}(\tau)$ of $w(\tau)$ around the z-axis. Since

$$\int w^{n-1}(1 + (\partial_z w)^2)^{1/2} dz$$

expresses the area of $\hat{\Gamma}(\tau)$, we can interpret

$$F(w) = \int_{\hat{\Gamma}(\tau)} e^{-|y|^2/2} \, d\mathcal{H}^{n-1}(y),$$

where $d\mathcal{H}^{n-1}$ denotes the infinitesimal surface element of an $(n-1)$-dimensional surface and y is a point on $\hat{\Gamma}(\tau)$. We write the right-hand side as $\mathcal{F}(\hat{\Gamma})$ in the sense that it is defined by the surface $\hat{\Gamma}$. Here $\hat{\Gamma}$ is given by

$$\hat{\Gamma}(\tau) = \left\{ y = \frac{1}{\sqrt{2}} e^\tau (x - \zeta) + \zeta; \quad x \in \Gamma(T - e^{-2\tau}) \right\},$$

$$\zeta = (\xi, 0, \ldots, 0) \in \mathbb{R}^{n+1}, \quad \tau > -\frac{1}{2} \log T,$$

so that it is possible to derive the monotonicity formula

$$\frac{d}{d\tau} \mathcal{F}(\hat{\Gamma})(t) = -\int_{\hat{\Gamma}(\tau)} e^{-|y|^2/2} \hat{V}^2 \, d\mathcal{H}^{n-1}(y), \quad \tau > -\frac{1}{2} \log T, \tag{3.35}$$

for closed surfaces Γ from (3.34). Here $\hat{V}$ denotes the growth of the velocity of $\hat{\Gamma}(\tau)$ in the inner normal direction, so that

$$\hat{V}^2 = \frac{(\partial_\tau w)^2}{1 + (\partial_z w)^2}.$$

The formula (3.35) was first proved in [Huisken 1990].

Monotonicity formulas are basic tools in the study of the size of singular sets and the approximation problem for solutions of the mean curvature flow equation by inner translation layers of the Allen–Cahn equation. There is a local version [Ecker 2001]. The reader is referred to [Ecker 2004] for the role of the monotonicity formula in the study of regularity theory.

There are many useful self-similar solutions for the mean curvature flow equation including a shrinking sphere and Angenent's torus [Angenent 1992]. The reader is refereed to [Giga 2006, §1.7, §1.8] for this topic. We do not state any more about the mean curvature flow equation, but list some fundamental books and survey articles. For example, [Giga Chen 1996, Giga 1995, Giga 2000, Giga 2006] are devoted to the notion of extension of solutions after occurrence of singularities, which is called a level set approach. The book [Chou Zhu 2001] focuses on evolution of curves. The book [Ohta 1997] is a primer on the physical background and the derivation of the equation of surface motion.

3.2.6 The Cases of a Semilinear Heat Equation and a Harmonic Map Flow Equation

Analysis of singularities of solutions of nonlinear evolution equations using backward self-similar solutions became popular through the study of

$$\partial_t u - \Delta u = |u|^{p-1}u, \quad x \in \mathbb{R}^n,\ t > 0, \tag{3.36}$$

for a real-valued function $u = u(x,t)$, where p is a real number greater than one. The equation is called *semilinear* in the sense that the nonlinearity is so weak that the nonlinear term does not contain the highest-order derivatives of u in the equation. The porous medium equation and the mean curvature flow equation are not semilinear. The Navier–Stokes equations are semilinear. Equation (3.36) is an example of a semilinear heat equation with self-multiplication term.

For $T > 0$, consider

$$v(t) = k(T-t)^{-\beta}, \quad 0 < t < T,\ \beta = 1/(p-1),\ k = \beta^\beta, \tag{3.37}$$

which is a solution of (3.36) in $t < T$. ($-v$ is also a solution of (3.36).) This function v has the property that it diverges to infinity in finite time T, that is, it blows up. Even if a solution u of (3.36) evolves with nonconstant initial data it can diverge and blow up in finite time. The behavior of such a blowup solution can be analyzed well in a similar way to the analysis of the behavior of a solution of equation (3.14) near pinching points. Let us discuss the asymptotic behavior near blowup points for equation (3.36).

We now assume that a function u satisfies (3.36) in $Q_r(a,T) = B_r(a) \times (T - r^2, T)$, where $B_r(a)$ denotes the open ball in $\mathbb{R}^n$ centered at $a \in \mathbb{R}^n$ with radius r. Let p_S be the *Sobolev exponent* defined by

$$p_S = \begin{cases} \infty & \text{if} \quad n \leq 2, \\ (n+2)/(n-2) & \text{if} \quad n \geq 3. \end{cases}$$

This exponent relates to the Sobolev inequality $\|u\|_{p_S+1} \leq C\|\nabla u\|_2$ for $n \geq 3$ discussed in §6.1.1 (6.9).

Theorem (Asymptotic behavior near blowup points). *Suppose that*

$$\sup_{Q_r(a,T)} |u(x,t)|(T-t)^\beta < \infty. \tag{3.38}$$

Assume that $1 < p < p_S$. *Then*

$$\lim_{t \to T} u(a + z\sqrt{T-t},\ t)(T-t)^\beta = k, -k, \ \text{or } 0, \tag{3.39}$$

and the convergence is uniform in every bounded set with respect to z. *Moreover, when the limit is equal to zero,* u *is bounded in a neighborhood of the point* (a,T). *(Namely,* (a,T) *is not a blowup point of* u.*)*

Formula (3.39) is the result corresponding to (3.25). Hypothesis (3.38) is a restriction on the growth order of u as $t \to T$. If an initial data is bounded and a solution exists in $\mathbb{R}^n \times (0,T)$, then (3.38) holds for $1 < p < p_S$ (cf. [Giga Kohn 1989], [Giga Matsui Sasayama 2004a]). Moreover, the solution (3.37) satisfies (3.38), so that the result seems to be natural, but (3.38) may not be valid if $p \geq p_S$. However, this problem has not been completely solved. *For example, if $p = p_S$ and $n \geq 3$, it is unknown whether "blowup solutions" that do not satisfy (3.38) exist.* According to [Herrero Velázquez 1994], there exists a blowup solution that does not satisfy (3.38) for $p > p_{JL}$, where

$$p_{JL} = \begin{cases} \infty & \text{if} \quad n \leq 10, \\ 1 + 4/(n - 4 - 2\sqrt{n-1}) & \text{if} \quad n \geq 11, \end{cases}$$

which is often called the *Joseph–Lundgren exponent* . See §3.4.1 for further results.

The idea of the proof is similar to that in §3.2.2. Since we may assume $a = 0$ without loss of generality, we may introduce similarity variables centered at (a,T) with $a = 0$ and $T > 0$ in a similar way as in §2.7.3:

$$\tau = -\log(T-t), \quad z = \frac{x}{\sqrt{T-t}}, \quad w(z,\tau) = (T-t)^\beta u(x,t)$$

and then

$$w(z,\tau) = e^{-\tau\beta} u(ze^{-\tau/2}, T - e^{-\tau}).$$

Rewriting equation (3.36) using $w = w(z, \tau)$ with the independent variables (z, τ), we have

$$\partial_\tau w - \Delta w + \frac{1}{2}\,\langle z,\ \nabla w\rangle + \beta w - |w|^{p-1}w = 0,\ z \in \mathbb{R}^n,\ \tau > -\frac{1}{2}\log\, T. \quad (3.40)$$

We now present key results for (A) and (B) of (2) "Characterization of the limit function" in §3.2.2.

Theorem (Nonexistence of nontrivial self-similar solutions). *Assume that $1 < p \le p_S$. Then there exists no bounded stationary solution of (3.40) in $\mathbb{R}^n$ (solution of (3.40) satisfying $\partial_\tau w \equiv 0$) except the constant solutions $\pm k$, 0.*

Theorem (Monotonicity formula). *Let $p > 1$ and set*

$$E(w) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla w|^2 + \frac{\beta}{2}|w|^2 - \frac{1}{p+1}|w|^{p+1}\right) e^{-|z|^2/4} dz.$$

Suppose that $w = w(z, \tau)$ is a bounded solution of (3.40) in $\mathbb{R}^n \times (\tau_0, \infty)$, where $\tau_0 \in \mathbb{R}$. Then the following identity holds:

$$\frac{dE(w)}{d\tau} = -\int_{\mathbb{R}^n} (\partial_\tau w)^2 e^{-|z|^2/4} dz. \quad (3.41)$$

We invoke a certain integral identity for the proof of nonexistence of a nonconstant solution. The idea differs from that in §3.2.3. The restriction on p is essential, since it is known that there exists a nonconstant bounded solution for sufficiently large p $(> (n+2)/(n-2))$ by [Troy 1987]. In fact, it is proved by [Budd Qi 1989] that there are infinitely many bounded, radially symmetric, nonnegative solutions for p satisfying $p_S < p < p_{JL}$. If p satisfies $p_{JL} \le p < p_L$ with

$$p_L = \begin{cases} \infty & \text{if} \quad n \le 10, \\ 1 + 6/(n-10) & \text{if} \quad n \ge 11, \end{cases}$$

it is shown by [Lepin 1988], [Lepin 1990] that there exists at least one bounded radially symmetric, nonnegative solution besides the constant solutions. For $p > p_L$ it has been recently proved by N. Mizoguchi that there is no such solution; see [Mizoguchi 2004b] for a partial result. However, it is still an open problem whether there is a nonradial bounded solution of (3.40) for $p > p_L$. The exponent p_L is often called the *Lepin exponent*. A proof of the *monotonicity formula* is left as Exercise 3.4. This formula is also useful for regularity of a weak solution for $p = (n+2)/(n-2)$ [Chou Du Zheng 2007]. The results in this section are due to [Giga Kohn 1985, Giga Kohn 1987, Giga Kohn 1989]. After these works, a more detailed behavior near blowup points was obtained rigorously by [Filippas Kohn 1992] and [Herrero Velázquez 1993] using matched asymptotic expansions. For later progress we refer the reader to [Merle Zaag 1998] as well as §3.4.

We shall next derive a monotonicity formula for the *harmonic map flow equation*

$$\partial_t u - \Delta u = |\nabla u|^2 u, \quad x \in \mathbb{R}^n,\ t > 0, \tag{3.42}$$

with values in the m-dimensional unit sphere $S^m = \{y \in \mathbb{R}^{m+1}; |y| = 1\}$. Here $m \geq 1$ and $u = (u^1, \ldots, u^{m+1})$ is a function on $\mathbb{R}^n \times (0,T)$ with values in $\mathbb{R}^{m+1}$ such that $|u| = 1$. (Actually, if $|u_0| = 1$ for the initial value u_0, then $|u| = 1$ follows automatically.) Here $|\nabla u|^2$ denotes

$$|\nabla u|^2 = \sum_{i=1}^{m+1} |\nabla u^i|^2.$$

Equation (3.42) also falls into the category of semilinear equations. A stationary solution of equation (3.42) (a solution of (3.42) with $\partial_t u \equiv 0$) is a *harmonic map* from $\mathbb{R}^n$ to S^m. Harmonic maps have been actively studied in both geometry and analysis as a generalization of harmonic functions. For a geometric background the reader is referred for example to [Urakawa 1990].

Considering the energy

$$E(u) = \int_{\mathbb{R}^n} |\nabla u|^2 dx,$$

for a solution of equation (3.42), we obtain the monotonicity formula

$$\frac{dE(u)}{dt} = -\int_{\mathbb{R}^n} |\partial_t u|^2 dx, \quad t > 0. \tag{3.43}$$

Naturally, this is valid only for solutions with finite energy $(E(u))(t)$. Since $|u| \equiv 1$, if we note that

$$\langle u,\ \partial_t u \rangle_{\mathbb{R}^{m+1}} = \frac{1}{2}\frac{\partial}{\partial t}|u|^2 = 0$$

for the inner product on $\mathbb{R}^{m+1}$, the identity (3.43) is easily obtained by taking the inner product with $\partial_t u$ and (3.42) in $\mathbb{R}^{m+1}$ and integrating by parts in the spatial variables.

If the initial energy $(E(u))(0)$ is finite, then there exists a local-in-time smooth solution of equation (3.42). However, the derivatives of the solution may blowup in finite time. What is the asymptotic form of the blowup? A synthetic report on the occurrence of blowup, asymptotic behavior, and extension of a solution after blowup is available in survey notes [Struwe 1996]. Here we just present a monotonicity formula to analyze blowup behavior. We introduce similarity variables in a similar way as in §2.7.3:

$$\tau = -\log(T-t),$$

$$z = \frac{x}{\sqrt{T-t}}, \quad w(z,\tau) = u(x,t),$$

so that $w(z,\tau) = u(ze^{-\tau/2}, T - e^{-\tau})$. Rewriting (3.42) as an equation with the independent variables (z,τ) for $w = (w^1, \ldots, w^{m+1})$, we have

$$\partial_\tau w^i - \Delta w^i + \frac{1}{2}\langle z,\ \nabla w^i\rangle = |\nabla w|^2 w^i, \qquad i = 1, 2, \ldots, m+1. \tag{3.44}$$

Since $|w|^2 \equiv 1$, we have

$$\langle w,\ \partial_\tau\ w\rangle_{\mathbb{R}^{m+1}} = 0$$

for the inner product in $\mathbb{R}^{m+1}$, and hence, setting

$$\Psi(w) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla w|^2 e^{-|z|^2/4} dz,$$

we obtain the following *monotonicity formula*.

Theorem (Monotonicity formula). *Let u be a smooth solution such that $(E(u))(t) < \infty$. Then*

$$\frac{d}{d\tau}\Psi(w) = -\int_{\mathbb{R}^n} |\partial_\tau w|^2 e^{-|z|^2/4} dz, \quad t > 0. \tag{3.45}$$

The proof is left as Exercise 3.4. To analyze the behavior of a solution w of (3.44) as $\tau \to \infty$, it is useful to classify the stationary solutions of (3.44). However, they are not isolated from each other in contrast to the case of the semilinear heat equation (3.36). As a result, the asymptotic behavior is not as simple as (3.39) for (3.36). When $n = 2$, it is considered that w might tend to a harmonic map from $\mathbb{R}^2$ to S^m as $\tau \to \infty$. Indeed, if we take a suitable subsequence $\tau_j \to \infty$, one is able to prove the convergence, but it is unknown in general whether *it is convergent, without taking a subsequence, as* $\tau \to \infty$. Recently, this was proved by [Topping 2004] when the map is from S^2 to S^2. Less is known for higher-dimensional cases $n \geq 3$. See [Struwe 1996] and [Lin Wang 2008] concerning the behavior of solutions of the harmonic map flow equation (3.42), which includes what is mentioned above.

A monotonicity formula for solutions of a nonlinear equation was originally introduced in the study of size and asymptotic form of the singular set for minimal surfaces, which is treated in detail in [Simon 1983, Giusti 1984, Morgan 1991]. The monotonicity formula for the harmonic map flow equation was first proved in [Struwe 1988], where it was expressed not by similarity variables but by the original variables (x,t).

3.3 Nondiffusion-Type Equations

The structures of the equations that we have treated so far are of diffusion type like the heat equation even though they have nonlinearities. In this section we study the existence problem of forward self-similar solutions for a nonlinear Schrödinger equation and for generalized KdV equations, which are essentially different from the heat equation.

3.3.1 Nonlinear Schrödinger Equations

We consider a semilinear equation having power-like nonlinearity of the form

$$\sqrt{-1}\partial_t u + \Delta u = \gamma |u|^{p-1} u, \quad x \in \mathbb{R}^n,\ t > 0, \tag{3.46}$$

where $\gamma \in \mathbb{R}$ is a constant and $p > 1$. Here $\sqrt{-1}$ denotes the imaginary unit. The equation with $\gamma = 0$ is the Schrödinger equation appearing in quantum mechanics, so (3.46) is called a *nonlinear Schrödinger equation*. Equation (3.46) is semilinear and its nonlinear term consists of a power of the unknown function. This is a typical nonlinear Schrödinger equation. There is a large number of articles on the existence and blowup for this kind of equation. Concerning background on this equation, see [Ozawa 1997], [Ozawa 1998], [Tsutsumi 1995, Agemi Giga Ozawa 1997], [Cazenave 2003]. These are references for the situation of the mathematical theory before around 2000.

Equation (3.46) with $\gamma = 0$ is easily solved. Indeed, the solution u with initial value f is formally expressed as

$$u(x,t) = (G_{\sqrt{-1}t} * f)(x) = \frac{1}{(4\pi\sqrt{-1}t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4\sqrt{-1}t}} f(y)dy.$$

As in the case of the heat equation, we denote by $S(t)$ $(= e^{\sqrt{-1}t\Delta})$ the linear operator that maps t to the function u of x. Namely,

$$(S(t)f)(x) = (G_{\sqrt{-1}t} * f)(x).$$

The absolute value of $e^{-|x|^2/4\sqrt{-1}t}$ is equal to one, so it is not (absolutely) integrable. In contrast to the case of the heat equation, the integrand of $S(t)f$ is not (absolutely) integrable unless f is integrable on $\mathbb{R}^n$. There arises the problem how to interpret in general the integral expression. Here, we use the integral expression only for integrable functions f, and interpret, for other functions f, that $S(t)f$ is defined through the approximation of f by integrable functions. There are some estimates similar to those in §1.1.1 and §1.1.2 for the operator $S(t)$, but their exponents are drastically restricted.

L^r-$L^{r'}$ estimate

Lemma. *We have*

$$\|S(t)f\|_r \le (4\pi t)^{\frac{n}{2}\left(\frac{1}{r}-\frac{1}{r'}\right)} \|f\|_{r'},$$

where $1/r + 1/r' = 1$ *and* $2 \le r \le \infty$.

In the case of $r = \infty$ and $r' = 1$ the claim is proved similarly as in §1.1.1. For the case of $r = 2 = r'$, using Fourier transformation, we obtain $\|S(t)f\|_2 = \|f\|_2$. (In fact, $S(t)$ is a unitary operator on $L^2(\mathbb{R}^n)$, which is a Hilbert space consisting of the square integrable functions on $\mathbb{R}^n$.) For the remaining r, interpolating the results for $r' = 1$ and $r' = 2$ by the

Riesz–Thorin interpolation theorem (§6.2.4) yields the lemma. If one uses the Marcinkiewicz interpolation theorem (§6.2.4), then a constant multiple is needed on the right-hand side. There are many other important estimates for $S(t)$ such as Strichartz estimates, for which the reader is referred to the literature cited at the beginning of this subsection.

We shall now construct a self-similar solution of (3.46). Equation (3.46) is invariant under the scaling transformation defined by

$$u^{(\lambda)}(x,t) = \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t), \quad \lambda > 0,$$

which is like the semilinear heat equation. Hence we shall call a solution of (3.46) satisfying

$$u^{(\lambda)}(x,t) = u(x,t),\ \lambda > 0,\ x \in \mathbb{R}^n,\ t > 0,$$

a *forward self-similar solution.* Restricting the exponent of the nonlinear term, we are able to prove the existence of forward self-similar solutions.

Theorem (Existence of forward self-similar solutions). *Let the exponent p satisfy*

$$p_0 < p < \frac{n+2}{n-2},$$

where p_0 is the positive solution of the quadratic equation $\frac{n(p_0-1)}{2} = \frac{p_0+1}{p_0}$. Let φ be a finite linear combination of $P_k(x)|x|^{-q-k}$, where the real part of the complex number q is equal to $2/(p-1)$ and $P_k(x)$ is a homogeneous polynomial of x of degree k (including degree zero). Then $\|S(1)\varphi\|_{p+1}$ is finite, and if $\|S(1)\varphi\|_{p+1}$ is small enough, there exists a global-in-time solution of (3.46) with initial value φ that is a forward self-similar solution.

One constructs a forward self-similar solution by constructing a global-in-time solution for a homogeneous initial value. This method originates in the article [Giga Miyakawa 1989], where it is used to construct a forward self-similar solution of the Navier–Stokes equations (§2.7.3). This type of result for equation (3.46) is due to [Cazenave Weissler 1998a], where the stability of the self-similar solution is also discussed; there P_k is assumed to be harmonic. The result is extended by [Cazenave Weissler 1998b] in the form of the theorem above. By [Ribaud Youssfi 1998] the initial data is allowed to be any $C^n(\mathbf{R}^n\backslash\{0\}))$ *positively homogeneous* function of degree $(p-1)/2$. For further development see [Cazenave Weissler 2000], [Furioli 2001], [Miao Zhang Zhang 2003].

The number $(n+2)/(n-2)$ in the relation of the index also appears in the case of semilinear heat equations as the Sobolev critical exponent. As for p_0, we may interpret this as follows. The norm $\|f\|_{n(p-1)/2}$ is invariant under the transformation $f \mapsto f_\lambda = \lambda^{2/(p-1)} f(\lambda x)$. On the other hand, $\frac{p+1}{p}$ is the conjugate index of $p+1$, i.e., $\frac{1}{p+1} + \frac{p}{p+1} = 1$. As a result, we may interpret that $p = p_0$ is the value of p such that the L^q norm with the conjugate index

q of $p+1$ is invariant under the transformation $f \mapsto f_\lambda$. *The problem why p has to be greater than p_0 or whether the restriction is indeed necessary is still open.*

3.3.2 KdV Equation

The *KdV (Korteweg–de Vries) equation* is an equation derived in the nineteenth century to describe the behavior of water waves in a canal, and it is essentially of the form

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R},\ t > 0.$$

Here we consider its generalized version

$$\partial_t u + u^p\partial_x u + \partial_x^3 u = 0, \quad x \in \mathbb{R},\ t > 0, \tag{3.47}$$

where p is a positive integer. When an initial condition

$$u(x,0) = f(x)$$

is imposed, if f decays sufficiently rapidly as $x \to \pm\infty$, the initial value problem (3.47) is solvable globally in time for $p < 4$, even if the initial value f is large. When $p \geq 4$, the problem is globally solvable if the initial value is small in a suitable sense, but if not so, it had been unknown *whether the problem is globally solvable* and it had been conjectured through an experiment in numerical analysis that a local-in-time solution can blowup. As proved in [Martel Merle 2002], the blowup actually occurs for $p = 4$; the authors proved that the H^1 norm blows up for some initial data. The above solvability is discussed in detail in [Kato 1983].

Recently, in the case of $p \geq 4$, a backward self-similar blowup solution was constructed, so that at least the existence of a blowup solution was guaranteed. We first note that the equation is invariant under the scaling transformation

$$u_{(\lambda)}(x,t) = \lambda^{2/p}u(\lambda x, \lambda^2 t), \quad \lambda > 0.$$

As in §2.7.3, a backward self-similar solution of (3.47) is expressed as

$$u(x,t) = \frac{1}{(T-t)^{2/3p}}\varphi\left(\frac{x_* - x}{(T-t)^{1/3}}\right),$$

with a function φ on $\mathbb{R}$, $T > 0$, and $x_* \in \mathbb{R}$. The equation for $\varphi = \varphi(s)$ is

$$\varphi''' + \varphi^p\varphi' - \frac{2}{3p}\varphi - \frac{1}{3}s\varphi' = 0, \quad s \in \mathbb{R}. \tag{3.48}$$

Showing the existence of a solution decaying for large s, we can construct a blowup solution despite the fact that the initial value decays at space infinity. In the following we state a result on existence of self-similar solutions only for $p = 4$.

Theorem (Existence theorem for nontrivial backward self-similar solutions). *Assume that $p = 4$. Then there exist infinitely many smooth solutions φ (not identically zero) of equation (3.48) satisfying the following properties:*

(i) *$\varphi(s) > 0$ in $s > 0$ and its asymptotic form as $s \to \infty$ is*

$$\varphi(s) = cs^{-1/2}e^{-2s^{3/2}/(3\sqrt{3})}\left(1 - \frac{2}{3\sqrt{3}s^{3/2}} + o\left(\frac{1}{s^{3/2}}\right)\right).$$

Here c is a positive constant.

(ii) *The asymptotic form as $s \to -\infty$ is*

$$\varphi(s) = (-s)^{-1/2}\left[a\ \cos\left(\frac{(-s)^{-3/2}}{3\sqrt{3}}\right)\right.$$
$$\left. +b\ \sin\left(\frac{(-s)^{-3/2}}{3\sqrt{3}}\right) + o\left(\frac{1}{(-s)^{3/2}}\right)\right]^2.$$

Here a and b are real constants that do not vanish simultaneously.

Here, $o\left(\frac{1}{s^{3/2}}\right)$ means that the term multiplied by $s^{3/2}$ converges to zero as $s \to \infty$. This term is negligible compared with other terms in the asymptotic form. For this equation a nontrivial blowup solution as in Leray's proposal was constructed with a self-similar solution. *No result so far is available on the stability of this solution, that is, whether the solution of (3.47), when the initial value is perturbed slightly but suitably, blows up near the time T as the self-similar solution does.* The above existence theorem is due to [Bona Weissler 1999], in which the existence theorem for $p > 4$ is also shown. Note that this solution is not an H^1-solution and is different from those studied in [Martel Merle 2002]. For further developments see [Molinet Ribaud 2003].

Before finishing the explanation of self-similar solutions, we note a related topic. Recently the method to analyze asymptotic behavior of solutions with self-similar solutions is frequently seen as an example of the methods of using a *renormalization group*, because scaling transformation can be considered as an action of the multiplicative group of all positive real numbers. Stationary solutions of the equation written by similarity variables replacing the original equation are invariant solutions under this action. Although this may be thought of as just an issue of wording, many problems can be formulated from this point of view with this idea. Here, we mention only a related article on blowup for semilinear heat equations [Bricmont Kupiainen 1994]. Concerning the renormalization group method, the reader is referred to, for example, [Nishiura 1999], in which the singular perturbation method is introduced in detail for the equation of surface motion discussed in §3.2. This book also explains the method of matched asymptotic expansions.

3.4 Notes and Comments

It is by now well known that a solution of an initial value problem for nonlinear diffusion equations may cease to exist after finite time. This phenomenon was first observed for nonlinear heat equations typically of the form (3.36) in several pioneering works [Kaplan 1963], [Ito 1966] (see also [Ito 1990]), and [Fujita 1966]. It was rather surprising that there exists no nonnegative global-in-time solution of (3.36) if $p < p_F = 1 + 2/n$ (the *Fujita exponent*) no matter how small the initial data is, as shown in [Fujita 1966]. For the role of this exponent the reader is referred to the review paper [Levine 1990]. From the 1980s on, research has focused on behavior of blowup of solutions rather than its existence. The reader is referred to the review article [Ishige Mizoguchi 2004] and a recent book by P. Quittner and Ph. Souplet [Quittner Souplet 2007] for the development of the theory. This problem is related to combustion theory, where u^p of (3.36) is replaced by e^u; see [Bebernes Eberly 1989] for the basic theory of such an equation. In this section we give several comments on equation (3.36).

3.4.1 A Priori Upper Bound

In §3.2.6 we dealt with asymptotic behavior of solutions of the semilinear heat equation (3.36) and explained how condition (3.38) is useful in investigating blowup behavior. Nowadays it is standard to say that the blowup of a solution is of type I if it satisfies estimate (3.38) and of type II otherwise. The Sobolev exponent p_S plays a crucial role. As was stated in §3.2.6, it is shown in [Giga Kohn 1987] that the blowup of any nonnegative solution of (3.36) is of type I if $1 < p < p_S$. It is also shown in [Giga Kohn 1987] that for sign-changing solutions, the blowup is always of type I under the additional assumption that $p < (3n+8)/(3n-4)(< p_S)$ or $n = 1$. Later, the result was extended to all $p < p_S$ by [Giga Matsui Sasayama 2004a, Giga Matsui Sasayama 2004b]. These results are still valid when the original (3.36) is considered in a bounded convex domain with homogeneous Dirichlet conditions. For nonnegative solutions the convexity assumption turns out to be unnecessary [Polacik Quittner Souplet 2007].

On the other hand, if $p \geq p_S$, estimate (3.38) can fail to hold; type-II blowup can occur. This fact was first reported by [Herrero Velázquez 1994], whose proof is presented in [Herrero Velázquez unpublished]. Due to these articles, a type-II blowup solution does exist if $n \geq 11$ and $p > p_{JL} := 1 + 4/(n - 4 - 2\sqrt{n-1})$. The proof requires extremely difficult calculations. A shorter proof is available in [Mizoguchi 2004a]. The type-II blowup solutions constructed in these articles are from a category of positive radially symmetric functions. When $p_S \leq p < p_{JL}$, nonexistence of type-II blowup solutions in this category is proved in [Matano Merle 2004], in which sign-changing solutions are also taken into consideration except for $p = p_S$. When $p = p_S$, the formal analysis in [Filippas Herrero Velázquez 2000] suggests the existence of a sign-changing type-II blowup solution, and it is shown that if $n = 3$, a type-II blowup solution exists for a certain shrinking ball [Naito Suzuki 2007]. Type-II blowup (pinching) also exists for the mean curvature flow equation as indicated after Theorem 3.2.4.

3.4.2 Related Results on Forward Self-Similar Solutions

In §3.2 we explained how backward self-similar solutions play an important role to describe the singularity, such as finite-time blowup, of solutions of evolution equations. On the other hand, it is also useful to analyze forward self-similar solutions so as to investigate the asymptotic behavior of global solutions. There is a large number of articles on this topic. We present some results on forward self-similar solutions for a semilinear heat equation

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^n,\ t > 0,\ p > 1, \tag{3.49}$$

which has a long history of research. If u is a solution of (3.49), then so is $u_\lambda(x,t) := \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t)$ for any $\lambda > 0$. If a solution u of (3.49) satisfies

$$u(x,t) \equiv u_\lambda(x,t) \quad \text{in } \mathbb{R}^n \times (0,\infty), \tag{3.50}$$

for any $\lambda > 0$, then it is said to be a *forward self-similar solution* of (3.49). A forward self-similar solution is of the form

$$u(x,t) = t^{-1/(p-1)} v(x/\sqrt{t}), \quad x \in \mathbb{R}^n,\ t > 0, \tag{3.51}$$

where v is a solution of the semilinear elliptic equation

$$\Delta v + \frac{1}{2} x \cdot \nabla v + \frac{1}{p-1} v + v^p = 0 \quad \text{in } \mathbb{R}^n. \tag{3.52}$$

Conversely, if v is a solution of (3.52), the function u defined by (3.51) is a forward self-similar solution of (3.49). In general, initial data (at $t = 0$) of a self-similar solution, if it exists, should be of the form $A(x/|x|)|x|^{-2/(p-1)}$ with some function A defined on the unit sphere S^{n-1}. This fact is readily seen if one sets $t = 0$ and $\lambda = 1/|x|$ in (3.50). Let us consider the initial condition with nonnegative homogeneous initial data continuous outside the origin. In other words,

$$u(x,0) = \lambda a(x/|x|)|x|^{-2/(p-1)} \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{3.53}$$

where a is a nonnegative continuous function on S^{n-1} and $\lambda > 0$ is a parameter. The idea of constructing self-similar solutions by solving the Cauchy problem for homogeneous initial data goes back to the work of Giga and Miyakawa [Giga Miyakawa 1989] for the Navier–Stokes equations in vorticity form (§2.7.3). It is clear that u defined by (3.51) is a forward self-similar solution of (3.49) satisfying initial condition (3.53) if and only if v satisfies (3.52) and

$$\lim_{r\to\infty} r^{2/(p-1)} v(r\omega) = \lambda a(\omega), \quad \omega \in S^{N-1}. \tag{3.54}$$

For the existence problem of forward self-similar solutions, there are at least three approaches: ODE methods, variational methods, and the method using function spaces. The first approach has been developed by [Haraux Weissler 1982], [Peletier Terman Weissler 1986], [Weissler 1985a], [Naito 2006]. Since we discuss only radial solutions (radially symmetric solutions), problem (3.52) with (3.54) is reduced to the problem

$$v_{rr} + \left(\frac{N-1}{r} + \frac{r}{2}\right) v_r + \frac{1}{p-1} v + v^p = 0, \quad r > 0, \tag{3.55}$$

$$v'(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)} v(r) = \ell. \tag{3.56}$$

Here v in (3.52) is interpreted as a function of $r = |x|$ and is still denoted by v; $v_{rr} = v''$, $v_r = v'$ denote its derivatives. In these articles the Cauchy problem for (3.55) with initial condition $v(0) = \alpha$, $v'(0) = 0$ is studied in order to investigate the existence of a solution of (3.55)–(3.56).

The second approach was developed by Escobedo and Kavian [Escobedo Kavian 1987] and Weissler [Weissler 1985b]; in the latter article, variational methods are applied to radial functions. The problem is formulated as a minimization problem in a weighted Sobolev space and it is proved, for $p_F = 1 + 2/n < p < p_S$, that there exists a positive solution of (3.49) exhibiting exponential decay at infinity, hence showing existence of forward self-similar solutions with null data.

Kawanago [Kawanago 1996] studied the Cauchy problem for the semilinear heat equation (3.49) with initial data $\lambda\phi(x)$, $x \in \mathbb{R}^n$, where $\lambda > 0$ and $\phi \not\equiv 0$ is a nonnegative continuous function in $\mathbb{R}^n$. He showed that if $p_F < p < p_S$, then there exists $\lambda_0 > 0$ having the following property: If $\lambda < \lambda_0$, then the corresponding solution exists globally in time and tends to the Gauss kernel as time goes to infinity. If $\lambda > \lambda_0$, then the corresponding solution blows up in finite time, if $\lambda = \lambda_0$, then the corresponding solution exists globally in time and tends to a forward self-similar solution of (3.49) as time goes to infinity. Namely, the self-similar solution is a threshold solution for blowing-up solutions and global solutions converging to the Gauss kernel.

In [Naito 2004], the author considered the Cauchy problem (3.49) with initial data $u_0(x) = \lambda |x|^{-2/(p-1)}$, where $\lambda > 0$ is a parameter. A variational approach is used there to show multiple existence of self-similar solutions. In the low supercritical range $p_S < p < p_{JL}$, the Cauchy problem (3.55)–(3.56) is studied in [Souplet Weissler 2003, Naito 2006], where the number of positive radial solutions is discussed.

In [Souplet Weissler 2003] the case $p = p_S$ is also discussed. For the case $p = p_S$ the reader is referred to [Naito 2008] and references cited there. In [Naito 2008] the existence of more general self-similar solutions starting from (3.53) is also discussed.

A solution of problem (3.55)–(3.56) is a positive radial stationary solution of the Cauchy problem

$$\begin{cases} v_s = v_{rr} + \left(\dfrac{N-1}{r} + \dfrac{r}{2}\right) v_r + \dfrac{1}{p-1} v + v^p, \quad y \in \mathbb{R}^n, \ s > 0, \\ v(y,0) = \phi(y), \quad y \in \mathbb{R}^n, \end{cases} \tag{3.57}$$

where the initial function $\phi \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is assumed to be radial, nonnegative, and not identically zero and to satisfy

$$\lim_{|y|\to\infty} |y|^{2/(p-1)} \phi(y) = \ell$$

for some $\ell > 0$. Note that equation (3.57) is obtained from (3.49) by the change of variables

$$v(y,s) = (t+1)^{p-1} u(x,t), \quad x = (t+1)^{1/2} y, \quad t = e^{-s/2} - 1.$$

Recently, Naito [Naito in preparation] reported that a self-similar solution of (3.49) describes the large-time behavior of the solutions u of (3.49) such that $u(x,t_0) \le \phi(x)$, $x \in \mathbb{R}^n$, for some $t_0 > 0$.

Uniqueness of a radial positive solution of (3.52) exhibiting exponentially decay at infinity is proved in [Yanagida 1996, Dohmen Hirose 1998]. Moreover, it is proved in [Naito Suzuki 2000] that any solution v of (3.52) satisfying $\lim_{|x|\to\infty} |x|^{2/(p-1)} v(x) = 0$ must have radial symmetry. It is also shown there that there are nonradial self-similar solutions that have a decay as $|x|^{-2/(p-1)}$. Thus, in particular, the solution obtained in [Escobedo Kavian 1987] must be radially symmetric, and the initial data of the corresponding self-similar solution defined through (3.51) should be zero.

When initial data is not identically zero, the variational method would not work well, since the initial data does not belong to the weighted Sobolev space. Because initial data of the form (3.53) do not belong to $L^q(\mathbb{R}^n)$ for any $q \ge 1$, we are forced to work with other function spaces. Kozono and Yamazaki [Kozono Yamazaki 1994] introduced new function spaces of Besov type, which include such initial functions, based on Morrey spaces in place of L^q spaces. There the existence and uniqueness of solutions of (3.49) as well as of the Navier–Stokes equations with initial data belonging to these function spaces is discussed. Cazenave and Weissler [Cazenave Weissler 1998a] used other function spaces, which include self-similar solutions of (3.49) as well as of the nonlinear Schrödinger equation (§3.3.1), and discussed existence, uniqueness, and stability of self-similar solutions in a sufficiently narrow space. See also [Ribaud 1998, Snoussi Tayachi Weissler 1999] for related results. One is able to find recent progress on the stability properties in [Souplet 1999, Snoussi Tayachi Weissler 2001]. In [Souplet 1999], the author studied the Cauchy–Dirichlet problems and obtained a geometric necessary and sufficient condition on the domain under consideration for the null solution to be asymptotically stable in some Lebesgue spaces. In [Snoussi Tayachi Weissler 2001], the authors consider the general semilinear heat equation, where a general nonlinear term $g(u)$ satisfying some growth condition is added to the right-hand side of (3.49). It is proved that despite the fact that this equation has no self-similar structure, some global solutions are asymptotically self-similar solutions of the semilinear heat equation with $g = 0$.

For recent progress on this topic, the reader is referred to [Cazenave Dickstein Escobedo Weissler 2001], [Souplet Weissler 2003], [Benachour Karch Laurencot 2004], [Laurençot Vázquez 2007], and references therein. One finds both historical and up-to-date studies in the book [Quittner Souplet 2007]. It is a nice reference for superlinear elliptic and parabolic problems.

Exercises 3

3.1. (§3.2.3) Assume that $\psi \in C^2(I)$ satisfies $\psi'' + b\psi' \geq 0$ in an open interval I, where b is a bounded function on I. Show that if ψ achieves its maximum in I at a point $z_0 \in I$, then ψ is constant in I.

3.2. (§3.2.3) Assume that $w \in C^2(\mathbb{R})$ satisfies $w'' \leq 0$ in $\mathbb{R}$. Show that if w is positive in $\mathbb{R}$, then w is constant.

3.3. (§3.2.5) Assume that f is a nonnegative continuous function defined on $(0,\infty)$ and that the integral $\int_0^\infty f(t)dt$ is finite.

(i) Prove that $\lim_{n\to\infty} a_n = 0$ for $a_n = \int_n^{n+1} f(t)dt$.
(ii) Find an example of f not satisfying $\lim_{t\to\infty} f(t) = 0$.

3.4. (§3.2.6) Prove formulas (3.41) and (3.45). Here, one may freely use integration by parts.

Part II

Useful Analytic Tools

4

Various Properties of Solutions of the Heat Equation

Here we establish the tools used in Chapter 1 in order to analyze the asymptotic behavior of solutions for the heat equation. We start by deriving L^p-L^q estimates for solutions and their derivatives and the uniqueness theorem for weak solutions. For this purpose, we prepare the Young inequality for convolution, which has a wide range of applications. Furthermore, algebraic and commutativity properties, in particular concerning differentiation of convolutions, are stated. These properties turn out to be helpful in the proof of smoothness for $t > 0$ for the solution of the heat equation in Chapter 1. Next, we consider the continuity of the solution at time $t = 0$, in the case that the initial value is continuous. Continuity is proved by a fairly general method that applies to a large class of equations.

In the next step we derive a solution formula for the inhomogeneous heat equation. This formula is often used in Chapter 2. Here it is applied in order to prove uniqueness of (weak) solutions. We also give a result on unique solvability for heat equations including transport terms (first-order terms with unknown coefficients). Moreover, we discuss properties of the fundamental solution of the heat operator including transport terms (drift terms) that are used in §2.5.2. The section is closed by giving a sufficient condition for integration by parts on unbounded domains as applied in §1.2.2 and §2.3.

The properties discussed here are fundamental and typical tools in analysis. However, with regard to the fact that this monograph should serve as a textbook also for beginners, the results discussed are neither always optimal nor best possible. Rather the results are given in a form as general as required in the first part of the book.

4.1 Convolution, the Young Inequality, and L^p-L^q Estimates

We start with the essential estimate for convolution on which the L^p-L^q estimates are based. To this end, first let us recall the notion of convolution. Let

M.-H. Giga et al., *Nonlinear Partial Differential Equations*,

Progress in Nonlinear Differential Equations and Their Applications 79,
DOI 10.1007/978-0-8176-4651-6_4,

f and h be two functions on $\mathbb{R}^n$. We define the function $h * f$ on $\mathbb{R}^n$ by

$$(h * f)(x) = \int_{\mathbb{R}^n} h(x-y)f(y)dy, \quad x \in \mathbb{R}^n.$$

The expression $h*f$ is called *convolution* of h and f. Here, the value $(h*f)(x)$ is well defined, provided that the integral on the right-hand side is defined and finite. Hence, in general, suitable assumptions for h and/or f have to be given such that $h*f$ makes sense. For example, if f, $h \in C(\mathbb{R}^n)$ and the support of either f or h is compact, then $(h*f)(x)$ is defined for each $x \in \mathbb{R}^n$ and $h*f$ is defined as a continuous function on $\mathbb{R}^n$ (Exercise 7.1). Some properties of convolutions are discussed in §4.1.3, §4.1.4, and §4.1.6.

The *Young inequality* estimates $h*f$ in terms of h and f. This inequality is natural in the category of the Lebesgue integral. Readers not yet familiar with Lebesgue integration theory may consider f and h in $C(\mathbb{R}^n)$ such that either one of them belongs to $C_0(\mathbb{R}^n)$ as in §4.1.1. For the sake of simplicity of notation, some remarks on convolution for the Lebesgue integral are given in the end of §4.1.3. In §4.1 we often write $\int$ for $\int_{\mathbb{R}^n}$ when it is convenient.

The Young inequality provided in §4.1.1 is a very useful tool and therefore contained in many standard textbooks. For instance, we refer to an introductory book on real analysis [Folland 1999] or to the monograph [Kuroda 1980], which is a nice textbook on functional analysis for beginners. See also [Reed Simon 1975] or [Adams 1978] for further use of these inequalities.

4.1.1 The Young Inequality

Theorem. *Let* $1 \leq p, q, r \leq \infty$ *such that*

$$1/r = 1/p + 1/q - 1. \tag{4.1}$$

Then, for any $h \in L^p(\mathbb{R}^n)$ *and* $f \in L^q(\mathbb{R}^n)$ *we have that* $h*f \in L^r(\mathbb{R}^n)$ *and that*

$$\|h*f\|_r \leq \|h\|_p \; \|f\|_q. \tag{4.2}$$

Here $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, denotes the space of functions with integrable pth power on $\mathbb{R}^n$, i.e., the space of (Lebesgue) measurable functions f satisfying

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{1/p} < \infty.$$

Note that $f, h \in L^p(\mathbb{R}^n)$ are regarded as equal if they coincide for "*almost all*" points on $\mathbb{R}^n$ (almost all means except on sets with "Lebesgue measure" zero). Equipped with this equivalence relation, $L^p(\mathbb{R}^n)$ with $\|\cdot\|_p$ as a norm is a Banach space. (That is to say, Lebesgue defined a notion of an integral so that $L^p(\mathbb{R}^n)$ is complete.) When a function f satisfies $f \in L^1(\mathbb{R}^n)$, we call f *integrable* on $\mathbb{R}^n$. Moreover, $L^\infty(\mathbb{R}^n)$ denotes the space of all essentially bounded functions on $\mathbb{R}^n$, i.e., it contains the set of (Lebesgue) measurable functions satisfying

$$\|f\|_\infty := \inf\{M;\ \ |f(x)| \leq M \text{ for almost all } x \in \mathbb{R}^n\} < \infty.$$

With the same identification as before, $L^\infty(\mathbb{R}^n)$ with $\|\cdot\|_\infty$ as a norm forms a Banach space. Of course, if f is continuous, $\|f\|_p$ and $\|f\|_\infty$ agree with the definitions in §1.1.1. We may replace $\mathbb{R}^n$ by a domain Ω, or more generally, by a Lebesgue measurable set U. Then $L^p(\Omega)$ or $L^p(U)$, defined completely analogously, are Banach spaces too. Also the terminology remains the same, i.e., $f \in L^1(U)$ if and only if f is (Lebesgue) integrable on U. Furthermore, f is called *locally integrable* on U if $f \in L^1(K)$ for any compact subset $K \subset U$. (For an elementary introduction to Lebesgue integration theory see, e.g., [Folland 1999], [Rudin 1987], [Ito 1963], and [Kakita 1985].)

Before we turn to the proof of the Young inequality, we remark that the necessity of the relation of the indices p, q, and r can easily be seen by a scaling argument. In fact, for $\lambda > 0$ we set

$$h_\lambda(x) = h(\lambda x),\ \ f_\lambda(x) = f(\lambda x),\ \ x \in \mathbb{R}^n.$$

Then, by the substitution $\lambda x = z$, we obtain

$$\begin{aligned}\|h_\lambda\|_p &= \left(\int |h(\lambda x)|^p\ dx\right)^{1/p} \\ &= \left(\int |h(z)|^p\ dz \lambda^{-n}\right)^{1/p} = \|h\|_p \lambda^{-n/p}, \\ \|f_\lambda\|_q &= \|f\|_q\ \lambda^{-n/q}.\end{aligned}$$

Similarly,

$$\|h_\lambda * f_\lambda\|_r^r = \int \left|\int h(\lambda x - \lambda y) f(\lambda y) dy\right|^r\ dx = \|h * f\|_r^r\ \lambda^{-nr}\ \lambda^{-n}.$$

The Young inequality applied to f_λ and h_λ yields

$$\|h_\lambda * f_\lambda\|_r \leq \|h_\lambda\|_p\ \|f_\lambda\|_q.$$

Consequently, $\|h * f\|_r \lambda^{-n-n/r} \leq \|h\|_p \|f\|_q \lambda^{-n/p} \lambda^{-n/q}$. Now suppose that relation (4.1) does not hold. Then, by letting $\lambda \to \infty$ or $\lambda \to 0$ we can always achieve $h * f = 0$, which is meaningless. Hence, if (4.2) holds for all h and f, (4.1) should hold too. Equality (4.1) is called a *dimension balance relation*. A corresponding relation often appears in norm inequalities of this type such as the Hölder inequality (see the lines below).

Proof. We turn to the proof of (4.2). The most elementary method is based on the *Hölder inequality* (in the case of $p = 2$ it is called the *Schwarz inequality*) (Exercise 4.2), which is given by

$$\left| \int f_1(x) f_0(x) dx \right| \le \|f_1\|_p \|f_0\|_{p'},$$

$$1/p + 1/p' = 1,\ 1 \le p, p' \le \infty,\ f_1 \in L^p(\mathbb{R}^n),\ f_0 \in L^{p'}(\mathbb{R}^n).$$

Here p' is called the *conjugate exponent* of p. First we observe that for the case $r = \infty$, (4.2) immediately follows from the Hölder inequality. In case either $p = \infty$ or $q = \infty$, relation (4.1) implies that either $q = 1$ or $p = 1$ respectively and therefore that $r = \infty$. Hence, we may assume that $p, q, r < \infty$.

Let $0 \le \theta < 1$, to be determined later, and write $|f| = |f|^{1-\theta}\ |f|^{\theta}$. The Hölder inequality gives us

$$\begin{aligned} |(h * f)(x)| &\le \int |h(x-y)|\ |f(y)|^{1-\theta}\ |f(y)|^{\theta}\ dy \\ &\le \left(\int |h(x-y)|^p\ |f(y)|^{(1-\theta)p}\ dy \right)^{1/p} \left(\int |f(y)|^{\theta p'} dy \right)^{1/p'} \end{aligned}$$

for $x \in \mathbb{R}^n$. In the case of $p = 1$ we set $\theta = 0$. Then the latter term turns to 1. If $p > 1$, since $p' \ne \infty$, we determine θ through $p'\theta = q$. Then (4.1) implies $(1-\theta)p = p + q - pq = pq/r$. This implies

$$|(h * f)(x)| \le \|f\|_q^{q/p'} \left(\int |h(x-y)|^p\ |f(y)|^{pq/r}\ dy \right)^{1/p}.$$

Note that this inequality is also valid for $p = 1$ (i.e., $p' = \infty$), since $q/p' = 0$. Let q' be the conjugate exponent of q. Then by (4.1), $1/r = 1/p - 1/q'$. Thus, the conjugate exponent of r/p is q'/p. Next, by splitting $|h| = |h|^{p/r}|h|^{p/q'}$ and applying the Hölder inequality (for exponents q'/p and r/p), we obtain

$$\begin{aligned} &\int |h(x-y)|^p |f(y)|^{pq/r} dy \\ &\quad = \int |h(x-y)|^{p \cdot p/q'}\ |h(x-y)|^{p \cdot p/r}\ |f(y)|^{pq/r}\ dy \\ &\quad \le \left(\int |h(x-y)|^p\ dy \right)^{p/q'} \left(\int |h(x-y)|^p\ |f(y)|^q\ dy \right)^{p/r}. \end{aligned}$$

This yields

$$|(h * f)(x)| \le \|h\|_p^{p/q'}\ \|f\|_q^{q/p'} \left(\int |h(x-y)|^p\ |f(y)|^q\ dy \right)^{1/r}.$$

Taking the rth power on both sides, integrating over x, and interchanging the order of integration (§7.2.2) implies that

$$\|h * f\|_r^r \le \|h\|_p^{rp/q'+p}\ \|f\|_q^{rq/p'+q}.$$

Since by (4.1), $rp/q' + p = rp(1 - 1/q + 1/r) = r$ and $rq/p' + q = r$, we deduce (4.2). □

4.1.2 Proof of L^p-L^q Estimates

As an application of the Young inequality, we obtain the L^p-L^q estimates stated in §1.1.2. Assume that $u = G_t * f$ is the solution of the heat equation with initial value $f \in L^q(\mathbb{R}^n)$ given by (1.3) with the Gauss kernel G_t. By the Young inequality we immediately obtain

$$\|u\|_p(t) = \|G_t * f\|_p \le \|G_t\|_r \ \|f\|_q, \ t > 0,$$

for r with $1 \le r \le \infty$ and $1/p = 1/r + 1/q - 1$. It remains to calculate $\|G_t\|_r$. If $1 \le r < \infty$, for $t > 0$ we have

$$\begin{aligned}\int |G_t(x)|^r dx &= \int \frac{1}{(4\pi t)^{nr/2}} \ \exp\left(-\frac{r|x|^2}{4t}\right) dx \\ &= \frac{1}{(4\pi t)^{nr/2}} \left(\frac{4t}{r}\right)^{n/2} \int e^{-|z|^2} dz, \quad z = \left(\frac{r}{4t}\right)^{1/2} x.\end{aligned}$$

Since $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, we obtain $\int_{\mathbb{R}^n} e^{-|z|^2} dz = \pi^{n/2}$. Thus,

$$\|G_t\|_r^r = (4\pi t)^{\frac{n}{2}(1-r)} \ r^{-\frac{n}{2}}.$$

The relation $1/p = 1/r + 1/q - 1$ then implies

$$\|G_t\|_r = (4\pi t)^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} r^{-\frac{n}{2r}} \le (4\pi t)^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

Since, $r = \infty$ only in the case that $q = 1$ and $p = \infty$, (1.5) is nothing but (1.4), which has been proved in §1.1.1. Inequality (1.6) can be proved in a similar way (Exercise 4.3). □

4.1.3 Algebraic Properties of Convolution

Proposition. *Assume that* $1/r = 1/p + 1/q - 1$ *for* $1 \le p, q, r \le \infty$.

(i) (Commutativity) *If* $h \in L^p(\mathbb{R}^n)$ *and* $f \in L^q(\mathbb{R}^n)$, *then* $h * f = f * h$, *regarded as an equality in* $L^r(\mathbb{R}^n)$.
(ii) (Distributivity) *If* $h_i \in L^p(\mathbb{R}^n)$, $i = 1, 2$, *and* $f \in L^q(\mathbb{R}^n)$, *then* $(h_1 + h_2) * f = h_1 * f + h_2 * f$ *in* $L^r(\mathbb{R}^n)$.
(iii) (Associativity) *If* $h \in L^p(\mathbb{R}^n)$, $f \in L^q(\mathbb{R}^n)$, *and* $v \in L^s(\mathbb{R}^n)$, $1 \le s \le \infty$, *then* $v * (h * f) = (v * h) * f$ *in* $L^\rho(\mathbb{R}^n)$. *Here we assume that* $1/\rho = 1/r + 1/s - 1$, $1 \le \rho \le \infty$, *so that* $0 \le 1/p + 1/s - 1 \le 1$.

Outline of the proof.

(i) is obtained as a consequence of the translation invariance of the integral on $\mathbb{R}^n$. In fact, by the substitution $x - y = z$ we obtain

$$(h * f)(x) = \int h(x-y) f(y) dy = \int h(z) f(x-z) dz = (f * h)(x).$$

(ii) follows easily by the linearity of the integral.
(iii) is a consequence of Fubini's theorem (II) in §7.2.2. Indeed, interchanging the order of integration implies for almost all $x \in \mathbb{R}^n$ that

$$\int v(x-y)\left(\int h(y-z)f(z)dz\right)dy = \int\left(\int v(x-y)h(y-z)dy\right)f(z)dz.$$

By referring to the proof of the Young inequality one may easily check that the assumptions of Fubini's theorem are satisfied. □

Relations (i), (ii), and (iii) follow by fundamental properties of integrals. However, here we refer to the Lebesgue integral. Readers may consult [Ito 1963], [Jost 2005], [Rudin 1987] for the definition and properties of the Lebesgue integral. On the other hand, if the integrand is assumed to be continuous, one may check the relations in the sense of the Riemann integral.

We also remark that for $h, f \in L^1(\mathbb{R}^n)$, the Young inequality implies that $h * f \in L^1(\mathbb{R}^n)$ as well. In other words, convolution maps pairs of L^1 functions again into L^1. In the algebraic terminology this means that $L^1(\mathbb{R}^n)$ is a commutative algebra with convolution as its multiplication. Moreover, by the Young inequality, the convolution is a continuous operation. Hence $L^1(\mathbb{R}^n)$ is a commutative Banach algebra. But observe that there is no unit element in L^1 with respect to convolution (Exercise 4.1).

4.1.4 Interchange of Differentiation and Convolution

Proposition.

(I) *Assume f to be integrable on $\mathbb{R}^n$, i.e., $f \in L^1(\mathbb{R}^n)$.*
  (i) *Assume that h is a bounded and continuous function, i.e., $h \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then $h * f$ is bounded and continuous on $\mathbb{R}^n$.*
  (ii) *Assume that $h \in C^1(\mathbb{R}^n)$ and that for each $1 \le j \le n$ the function h and the derivative $\partial_{x_j} h$ are bounded on $\mathbb{R}^n$. Then $h * f$ is C^1 on $\mathbb{R}^n$ and*

$$(\partial_{x_j}(h * f))(x) = ((\partial_{x_j} h) * f)(x), \quad x \in \mathbb{R}^n.$$

(II) *Let $1 \le p \le \infty$, p' be the conjugate exponent of p, i.e., $1/p + 1/p' = 1$, and $f \in L^p(\mathbb{R}^n)$.*
  (i) *For $h \in C(\mathbb{R}^n) \cap L^{p'}(\mathbb{R}^n)$ the convolution $h * f$ is bounded and continuous on $\mathbb{R}^n$.*
  (ii) *Suppose that $h \in C^1(\mathbb{R}^n)$ and that for each $1 \le j \le n$ the quantities $\|\partial_{x_j} h\|_{p'}$ and $\|h\|_{p'}$ are finite. Then $h * f \in C^1(\mathbb{R}^n)$ and*

$$(\partial_{x_j}(h * f))(x) = ((\partial_{x_j} h) * f)(x), \quad x \in \mathbb{R}^n.$$

The proposition shows that a convolution is always as smooth as each of its factors. For example, if $h \in C^\infty(\mathbb{R}^n)$ and $\partial_x^\alpha h$ is bounded for each multi-index α, even if merely $f \in L^1(\mathbb{R}^n)$, as a consequence of (I) (ii) we have

$h*f \in C^\infty(\mathbb{R}^n)$. Observe that (I) is nothing but the special case $p=1$ of (II). However, we first prove (I), since (II) can be reduced to this case.

(I) (i) By assumption we have for the integrand of $h*f$ the estimate

$$|h(x-y)f(y)| \le \|h\|_\infty |f(y)|, \quad y \in \mathbb{R}^n.$$

Hence, the absolute value of the integrand is bounded from above by an integrable function $\|h\|_\infty |f(y)|$ that is independent of x. This shows that $h*f$ is bounded, and by Lebesgue's dominated convergence theorem (§7.1.1) and the continuity of h we obtain

$$\lim_{z\to x}(h*f)(z) = \int \lim_{z\to x} h(z-y)f(y)dy = (h*f)(x).$$

(ii) First note that $(\partial_{x_j}h)*f$ is continuous on $\mathbb{R}^n$ by (i). We fix $x^0 = (x_1^0,\dots,x_n^0) \in \mathbb{R}^n$ such that the component x_j^0 is contained in the open interval (a,b) and set

$$\begin{aligned}&\tilde{h}(x_j,y)\\ &\quad = f(y)h(x_1^0-y_1,\dots,x_{j-1}^0-y_{j-1},x_j-y_j,x_{j+1}^0-y_{j+1},\dots,x_n^0-y_n).\end{aligned}$$

We intend to apply the theorem on differentiation under the integral sign in §7.2.1. By $\|\partial_{x_j}h\|_\infty < \infty$ and $f \in L^1(\mathbb{R}^n)$ we have

$$\int_{(a,b)\times\mathbb{R}^n} \left|\frac{\partial\tilde{h}}{\partial x_j}\right| dx_j\ dy < \infty,$$

which shows that condition (ii) of Theorem in 7.2.1 is satisfied. Conditions (i) and (iii) are obvious. Furthermore, condition (iv) follows from the continuity of $(\partial_{x_j}h)*f$ and (i). Therefore, Theorem 7.2.1 implies that $h*f$ is C^1 with respect to x_j and that

$$(\partial_{x_j}(h*f))(x_1^0,\dots,x_j^0,\dots,x_n^0) = ((\partial_{x_j}h)*f)(x_1^0,\dots,x_j^0,\dots,x_n^0).$$

The right-hand side is continuous at $x=x^0$; hence $h*f$ is C^1 on $\mathbb{R}^n$.

(II) (i) We prove only the case $p=\infty$. The general case is left to the reader (Exercise 7.4). If $p=\infty$, the integrand of $(h*f)(x)$ cannot be estimated directly by an integrable function that is independent of x on $\mathbb{R}^n$. For this reason we consider the function f_R for $R>0$ defined by

$$f_R(x) = \begin{cases} f(x), & x \in B_R,\\ 0, & x \notin B_R.\end{cases}$$

Then we have

$$(h * f_R)(x) = \int_{B_R} h(x-y)f(y)dy \ , \ x \in \mathbb{R}^n.$$

For $R_0 > 0$ let $x \in B_{R_0}$. Since h is continuous on $\mathbb{R}^n$, h is bounded on B_{R+R_0} (the Weierstrass theorem). Furthermore, since $f \in L^\infty(\mathbb{R}^n)$, by the bounded convergence theorem (§7.1.1) $(h*f_R)(x)$ is continuous at $x \in B_{R_0}$. Since this argument holds for any $R_0 > 0$, we have $(h * f_R) \in C(\mathbb{R}^n)$.

Note that for $R > R_0$, $|x| \le R_0$, and $|y| \ge R$ the triangle inequality implies that $|x-y| \ge |x| - |y| \ge R - R_0$. For fixed R_0 we therefore obtain

$$\begin{aligned}
&\sup_{x \in \overline{B}_{R_0}} |(h * f_R)(x) - (h * f)(x)| \\
&\le \|f\|_\infty \sup_{x \in \overline{B}_{R_0}} \int_{\mathbb{R}^n \setminus B_R} |h(x-y)|dy \\
&\le \|f\|_\infty \int_{\mathbb{R}^n \setminus \overline{B}_{R-R_0}} |h(y)|dy \\
&= \|f\|_\infty \left(\|h\|_1 - \int_{\overline{B}_{R-R_0}} |h(y)|dy \right) \to 0 \quad (R \to \infty),
\end{aligned}$$

which shows that $h*f_R$ converges uniformly to $h*f$ on B_{R_0} as $R \to \infty$. But then $h * f$ is continuous on $\mathbb{R}^n$, since it is the uniform limit of the continuous family $\{h * f_R\}_{R>0}$ on B_{R_0} for each fixed $R_0 > 0$ (see answer of Exercise 1.6).

(ii) Again we fix $R_0 > 0$ and let $x \in B_{R_0}$. The continuity of h and $\partial_{x_j} h$ on $\mathbb{R}^n$ implies the boundedness on B_{R+R_0}. By similar arguments as in (I) (ii) for each $R > 0$, $h * f_R$ is C^1 on B_{R_0} and satisfies

$$(\partial_{x_j}(h * f_R))(x) = ((\partial_{x_j} h) * f_R)(x), \quad x \in B_{R_0}.$$

Analogously to (II) (i), $(\partial_{x_j} h) * f_R$ converges uniformly to $(\partial_{x_j} h) * f$ on B_{R_0} as $R \to \infty$. Clearly, $h * f_R$ converges uniformly to $h * f$ on B_{R_0} as $R \to \infty$, too. By the elementary result on the interchange of limit and differentiation obtained in §4.1.5, we see that $h * f$ is C^1 on B_{R_0} and that

$$((\partial_{x_j} h) * f)(x) = (\partial_{x_j}(h * f))(x), \quad x \in B_{R_0}.$$

By the fact that $R_0 > 0$ is arbitrary, (II) (ii) follows for the case that $p = \infty$. □

4.1.5 Interchange of Limit and Differentiation

Lemma. *Let f be a real-valued function defined on an open set Q in $\mathbb{R}^n$. Assume that for any $f_\varepsilon \in C^1(Q)$, $0 < \varepsilon < 1$,*

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x)$$

for any point $x \in Q$. Furthermore, assume that for each $1 \le j \le n$ the function $\partial_{x_j} f_\varepsilon$ converges uniformly to a function $h_j (\in C(Q))$ on Q for $\varepsilon \to 0$, i.e.,

$$\lim_{\varepsilon \to 0} \sup_{x \in Q} |\partial_{x_j} f_\varepsilon(x) - h_j(x)| = 0.$$

Then, f is C^1 on Q and

$$\partial_{x_j} f(x) = h_j(x), \quad x \in Q,\ 1 \le j \le n.$$

Proof. First we show that f is partially differentiable at each point $x = a$ $(= (a_1, \dots, a_n))$ in Q with respect to x_j. To this end, for small enough $|\sigma|$ the fundamental theorem of calculus yields

$$\begin{aligned} &f_\varepsilon(a_1, \dots, a_{j-1}, a_j + \sigma, a_{j+1}, \dots, a_n) \\ &\quad = \int_0^\sigma \partial_{x_j} f_\varepsilon(a_1, \dots, a_{j-1}, a_j + \tau, a_{j+1}, \dots, a_n) d\tau + f_\varepsilon(a). \end{aligned}$$

Since $\partial_{x_j} f_\varepsilon$ converges uniformly to h_j on Q as $\varepsilon \to 0$ we may interchange limit and integral (§7.1) to obtain

$$\lim_{\varepsilon \to 0} \int_0^\sigma (\partial_{x_j} f_\varepsilon)(\dots, a_j + \tau, \dots) d\tau = \int_0^\sigma h_j(\dots, a_j + \tau, \dots) d\tau.$$

The pointwise convergence of f_ε to f then implies

$$\begin{aligned} &f(a_1, \dots, a_{j-1}, a_j + \sigma, a_{j+1}, \dots, a_n) \\ &\quad = \int_0^\sigma h_j(a_1, \dots, a_{j-1}, a_j + \tau, a_{j+1}, \dots, a_n) d\tau + f(a). \end{aligned}$$

Since h_j is continuous, this formula shows that f is partially differentiable at $x = a$ with respect to x_j and that $\partial_{x_j} f(a) = h_j(a)$. Hence f is C^1 on Q. □

As an application of the results in §4.1.4, we can prove the differentiability of the solution $u = G_t * f$ of the heat equation in §1.1.

4.1.6 Smoothness of the Solution of the Heat Equation

Proposition. *Let* $1 \le p \le \infty$ *and* $f \in L^p(\mathbb{R}^n)$.

(i) *The function* $(G_t * f)$ *is* C^∞ *on* $\mathbb{R}^n$ *for* $t > 0$. *Moreover,*

$$\partial_x^\alpha (G_t * f) = (\partial_x^\alpha G_t) * f, \quad (x,t) \in \mathbb{R}^n \times (0,\infty),$$

for any multi-index α.

(ii) *Assume that* f *is* C^k *on* $\mathbb{R}^n$ *and that for any multi-index* α *with* $|\alpha| \le k$, $\|\partial_x^\alpha f\|_p < \infty$. *Then*

$$\partial_x^\alpha (G_t * f) = G_t * (\partial_x^\alpha f), \quad (x,t) \in \mathbb{R}^n \times (0,\infty).$$

(iii) *The function* $(x,t) \mapsto G_t * f(x)$ *is* C^∞ *on* $\mathbb{R}^n \times (0,\infty)$ *and*

$$\partial_t^k (G_t * f) = (\partial_t^k G_t) * f = (\Delta^k G_t) * f = \Delta^k (G_t * f)$$

for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$ *and* $k \in \mathbb{N}$.

Since $\|\partial_x^\alpha G_t\|_{p'} < \infty$, where $1/p + 1/p' = 1$ and $1 \le p' \le \infty$, (i) and (ii) follow immediately from §4.1.4. The first equality in (iii) is obtained as a consequence of differentiation under the integral sign given in §7.2.1. We remark that this argument is also used for the interchange of differentiation and convolution (Exercise 7.2). The second equality follows from $\partial_t G_t = \Delta G_t (t > 0)$ (Exercise 1.1 (i)). Here $\Delta^k = \underbrace{\Delta \ldots \Delta}_{k \text{ times}}$.

4.2 Initial Values of the Heat Equation

In Chapter 1 we derived a representation of the solution u of the heat equation in terms of the Gauss kernel given by $u = G_t * f$ with initial value f. A priori it is not clear whether and in what sense u converges to f as $t \to 0$. In fact, there are many sorts of convergence depending on the class of functions to which f belongs. In the sequel we discuss this problem for continuous f.

4.2.1 Convergence to the Initial Value

Theorem. *Assume that* f *is bounded and uniformly continuous on* $\mathbb{R}^n$. *Then* $G_t * f$ *converges uniformly to* f *as* $t \to 0$ $(t > 0)$, *that is,*

$$\lim_{t \to 0} \|G_t * f - f\|_\infty = 0.$$

Since a continuous compactly supported function is bounded and uniformly continuous, this theorem in particular applies to such initial values. On the other hand, if f is discontinuous at some point, $G_t * f$ cannot converge uniformly to f, in view of the fact that uniform convergence always implies that the limit function is continuous (answer of Exercise 1.6). The proof of the above theorem is elementary, and readers will easily find references (e.g., [Kuroda 1980], [Evans 1998], [John 1991]). Here, we give a proof not using ε-δ arguments. Before we start, let us recall the definition of uniform continuity.

4.2.2 Uniform Continuity

We set

$$\omega(\sigma) = \sup\{|f(x) - f(y)|; |x - y| \le \sigma;\ x, y \in K\}, \quad \sigma > 0.$$

A function f is *uniformly continuous* on a subset K in $\mathbb{R}^n$ if $\lim_{\sigma\to 0} \omega(\sigma) = 0$. Observe that ω is a nondecreasing function, but not necessarily continuous in $\sigma > 0$. By definition, $|f(x) - f(y)| \le \omega(|x - y|)$ $(x, y \in K)$.

In view of the wide range of applications, we prove a more general result than Theorem 4.2.1.

4.2.3 Convergence Theorem

Theorem. *Let K_t be an integrable function on $\mathbb{R}^n$ depending on a parameter $t > 0$ and satisfying the following conditions:*

(i) $\int_{\mathbb{R}^n} K_t(x)dx = 1$,
(ii) *For any* $\eta > 0$, $\lim_{t\downarrow 0} \int_{|x|\ge\eta} |K_t(x)|dx = 0$,
(iii) $c_0 := \overline{\lim}_{t\downarrow 0} \int_{\mathbb{R}^n} |K_t(x)|dx < \infty$.

Then, for any bounded uniformly continuous function f defined on $\mathbb{R}^n$ we have

$$\lim_{t\downarrow 0} \|K_t * f - f\|_\infty = 0.$$

Here $\overline{\lim}$ *denotes the limit superior, that is,*

$$\overline{\lim_{t\downarrow 0}}\ h(t) = \lim_{t\downarrow 0}\ \sup_{0<s<t} h(s),$$

for a function h defined on a neighborhood of 0.

In Fourier analysis a kernel K_t satisfying (i), (ii), and (iii) is often called a *summation kernel.* The region of large values of K_t concentrates at $x = 0$ as $t \to 0$.

Proof. Since $\int_{\mathbb{R}^n} K_t(x - y)dy = 1$ by (i), we have

$$\begin{aligned}(K_t * f)(x) - f(x) &= \int_{\mathbb{R}^n} K_t(x - y)f(y)dy - \int_{\mathbb{R}^n} K_t(x - y)f(x)dy \\ &= \int_{\mathbb{R}^n} K_t(x - y)(f(y) - f(x))dy\ ,\ x \in \mathbb{R}^n.\end{aligned}$$

Taking the absolute value and dividing the integral area $\mathbb{R}^n$ into the two parts $|x - y| \ge \eta$ and $|x - y| < \eta$ for $\eta > 0$ implies

$$|(K_t * f)(x) - f(x)| \leq \int_{|x-y|<\eta} |K_t(x-y)|\ |f(y) - f(x)| dy + \int_{|x-y|\geq\eta} |K_t(x-y)|\ |f(y) - f(x)| dy.$$

Since the distance between x and y is small in the first term on the right-hand side, we may employ the function ω defined in §4.2.2 to estimate this term as

$$\int_{|x-y|<\eta} |K_t(x-y)||f(y) - f(x)| dy \leq \int_{|x-y|<\eta} |K_t(x-y)|\omega(|x-y|) dy \leq \omega(\eta) \int_{\mathbb{R}^n} |K_t(z)| dz.$$

For the second term, using the boundedness of f, we obtain

$$\int_{|x-y|\geq\eta} |K_t(x-y)|\ |f(y) - f(x)| dy \leq \int_{|x-y|\geq\eta} |K_t(x-y)|\ 2\ \|f\|_\infty dy = 2\ \|f\|_\infty \int_{|z|\geq\eta} |K_t(z)| dz.$$

Thus, both terms can be estimated by quantities independent of x. This yields

$$\|K_t * f - f\|_\infty \leq \omega(\eta) \int_{\mathbb{R}^n} |K_t(z)| dz + 2\|f\|_\infty \int_{|z|\geq\eta} |K_t(z)| dz.$$

Taking $\overline{\lim}$ on both sides, assumptions (ii) and (iii) result in

$$\overline{\lim_{t\to 0}} \|K_t * f - f\|_\infty \leq \omega(\eta) c_0 + 0.$$

(Note that here the limit does not exist in general. Therefore, it is convenient to take the limit superior, which always exists in this situation.) Finally, we let $\eta \to 0$. Since the left-hand side is independent of η, and f is uniformly continuous, we obtain $\overline{\lim}_{t\to 0}\ \|K_t * f - f\|_\infty = 0$. Hence the assertion follows. □

We remark that for pointwise convergence it is sufficient to assume f to be continuous. In fact, if f is continuous at $\hat{x} \in \mathbb{R}^n$, by similar arguments we obtain that

$$|K_t * f(\hat{x}) - f(\hat{x})| \leq \sup_{|\hat{x}-y|\ <\eta} |f(\hat{x}) - f(y)| \int_{\mathbb{R}^n} |K_t(z)| dz + 2\|f\|_\infty \int_{|z|\geq\eta} |K_t(z)| dz.$$

Taking $\overline{\lim}_{t\to 0}$ gives us

$$\overline{\lim_{t\to 0}} |K_t * f(\hat{x}) - f(\hat{x})| \leq c_0 \sup_{|\hat{x}-y|\leq\eta} |f(\hat{x}) - f(y)|.$$

Letting $\eta \to 0$, the continuity of f at $\hat{x}$ implies that the right-hand side converges to 0. Hence we have proved the following corollary.

4.2.4 Corollary

Corollary. *For $t > 0$ let K_t be a summation kernel. If $f \in L^\infty(\mathbb{R}^n)$ is continuous at $\hat{x} \in \mathbb{R}^n$, then $\lim_{t\to 0}(K_t * f)(\hat{x}) = f(\hat{x})$.*

4.2.5 Applications of the Convergence Theorem 4.2.3

Proof of Theorem 4.2.1. According to Theorem 4.2.3, it is sufficient to prove that G_t is a summation kernel. As in the proof of the decay estimate in §4.2.1, by the facts $\|G_t\|_1 = 1$ and $G_t \geq 0$, properties (i) and (iii) are obvious. For any $\eta > 0$ we obtain

$$\begin{aligned} \int_{|x|\geq\eta} |G_t(x)|dx &= \frac{1}{(4\pi t)^{n/2}} \int_{|x|\geq\eta} \exp\left(-\frac{|x|^2}{4t}\right) dx \\ &= \pi^{-n/2} \int_{|z|\geq\eta/(2t^{1/2})} e^{-|z|^2} dz \to 0 \quad (t \to 0), \end{aligned}$$

where we substituted $z = x/(2t^{1/2})$. Hence property (ii) holds as well and Theorem 4.2.1 is proved. □

Proof of Proposition 1.4.1. We shall apply Corollary 4.2.4. We may assume $m \neq 0$. in fact, if $m = 0$, we pick $h \in C_0(\mathbb{R}^n)$ such that $\sigma := \int_{\mathbb{R}^n} h(x)dx \neq 0$. By the result for $m \neq 0$ we have $\int_{\mathbb{R}^n}(f + h)_k(x)\psi(x)dx \to \sigma\psi(0)$, $\int_{\mathbb{R}^n} h_k(x)\psi(x)dx \to \sigma\psi(0)$ $(k \to \infty)$. Consequently $\int_{\mathbb{R}^n} f_k(x)\psi(x)dx \to 0$. In the situation of Corollary 4.2.4 we set $f = \psi$, $K_t = f_k/m$, $t = 1/k$, and $\hat{x} = 0$. Suppose that $\psi \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. We have to prove that f_k/m is a summation kernel.[1] Since $f_k(x) = k^n f(kx)$, $f \in C_0(\mathbb{R}^n)$ (or more generally $f \in L^1(\mathbb{R}^n)$), and $k \geq 1$, we may calculate

$$\begin{aligned} \int_{\mathbb{R}^n} f_k(x)/m\, dx &= \int_{\mathbb{R}^n} k^n f(kx)dx \left\{\int_{\mathbb{R}^n} f(x)dx\right\}^{-1} = 1, \\ \int_{\mathbb{R}^n} |f_k(x)/m|dx &= \frac{1}{|m|}\int_{\mathbb{R}^n} k^n|f(kx)|dx = \frac{1}{|m|}\int_{\mathbb{R}^n} |f(x)|dx. \end{aligned}$$

Hence, we see that conditions (i) and (iii) of Theorem 4.2.3 are satisfied. Furthermore, for $\eta > 0$ we have that

$$\int_{|x|\geq\eta} |f_k(x)/m|dx = \frac{1}{|m|}\int_{|z|\geq k\eta} |f(z)|dz \to 0 \quad (1/k \to 0),$$

which shows that also condition (ii) is fulfilled. Hence Proposition 1.4.1 follows from Corollary 4.2.4. In analogy to the remark in §1.4.1 we note that for $f \in C_0(\mathbb{R}^n)$ it suffices to require $\psi \in C(\mathbb{R}^n)$ without assuming also the boundedness of ψ, since ψ is bounded on the support of the f_k $(k \geq 1)$. □

[1] If $f \in C_0^\infty(\mathbb{R}^n)$, $f \geq 0$, and $m > 0$, f_k/m is called a mollifier.

Observe that G_t and f_k are obtained by a scaling transformation from the functions G_1 and f, respectively. This shows that the results in §4.2.1 and §1.4.1 can be proved by a dilation argument without using §4.2.3 and §4.2.4 (Exercise 4.4).

4.3 Inhomogeneous Heat Equations

We consider the initial value problem for the heat equation with a heat source given by

$$\begin{aligned} \partial_t u(x,t) - \Delta u(x,t) &= h(x,t), \quad x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) &= f(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{4.3}$$

Here, f and h are known functions on $\mathbb{R}^n$ and $\mathbb{R}^n \times (0,\infty)$, respectively. If $h \not\equiv 0$, the first equation of (4.3) is called inhomogeneous.

In the first step we construct a solution formula for this equation by a formal discussion. We consider t as a parameter, and for a function f write the convolution $G_t * f$ of G_t and f in terms of the operator $e^{t\Delta}$. More precisely, we set

$$(e^{t\Delta} f)(x) = (G_t * f)(x), \quad x \in \mathbb{R}^n, \quad t > 0.$$

Since $u = G_t * f$ is a solution of the heat equation $\partial_t u - \Delta u = 0$, we have

$$\partial_t (e^{t\Delta} f) = \Delta e^{t\Delta} f.$$

(Recall that this formula is valid under suitable assumptions on f, as shown in Proposition 4.1.6.) Observe the analogy to the classical exponential function for $a \in \mathbb{R}$ given by

$$\partial_t e^{ta} = a e^{ta}.$$

This motivates the notation $e^{t\Delta}$. However, a rigorous justification is required. It is not obvious how to give a sense to $e^{t\Delta}$ for an unbounded operator such as Δ. For example, here the standard technique using the exponential series fails. The abstract theory dealing with such objects is called semigroup theory. A key generation theorem was established by K. Yosida and E. Hille around 1948. Since then, the theory has been applied to various fields. We refer to [Yosida 1964], [Tanabe 1975], [Goldstein 1985], [Engel Nagel 2000] for a comprehensive approach. In this book we will not give an introduction to semigroup theory. Here we just use the semigroup terminology. Our aim is to derive a representation of the solution of (4.3) with heat source in terms of the operator $e^{t\Delta}$. In fact, considering Δ as a number,

$$\partial_t u - \Delta u = h \tag{4.4}$$

is a linear first-order ordinary differential equation. The solution is

$$w = \int_0^t e^{(t-s)\Delta} \; h(s)ds, \tag{4.5}$$

by the variation of constants formula. In order to satisfy the initial condition we add $e^{t\Delta} f$ to (4.5). Since (4.3) is a linear equation,

$$u = e^{t\Delta} f + \int_0^t e^{(t-s)\Delta} \; h(s)ds \tag{4.6}$$

is the solution. In semigroup theory this formula is generalized from "numbers Δ" to a wide class of unbounded operators as the Laplacian Δ. Here we use (4.6) in order to show that the solution satisfies the initial condition (4.3) under certain assumptions on f and h.

4.3.1 Representation of Solutions

Next, we try to find out under what circumstances w in (4.5), obtained by the formal discussion above, is differentiable with respect to t. For simplicity we assume that $f = 0$. Differentiating w formally, we obtain

$$\partial_t w = e^{(t-t)\Delta} h(t) + \int_0^t \Delta e^{(t-s)\Delta} h(s)ds.$$

To ensure that the integral on the right-hand side is well defined, we have to check whether the L^∞-norm of its integrand is integrable with respect to s. If we merely assume h to be bounded and continuous, then by the L^∞-L^∞ estimate in §1.1.3 we have

$$\|\Delta(e^{(t-s)\Delta} h(s))\|_\infty \le \frac{C}{t-s} \|h\|_\infty(s).$$

Thus, the integrand is in general not integrable on the interval $(0, t)$ as a function with respect to s.

This formal calculation shows that it seems to be difficult to prove the differentiability of w with respect to t by just assuming h to be bounded and continuous with respect to (x, t). In fact, it is impossible. Additional assumptions are required such as Hölder continuity of h with respect to (x, t) (see [Ladyženskaja Solonnikov Ural'ceva 1968]). We will not prove such a precise result here. Instead we are content to give some sufficient conditions, which are easier to derive and to apply.

In the following we write $v(t) = v(\cdot, t)$, $t \in (0, \infty)$, for a function $v = v(x, t)$ defined on $\mathbb{R}^n \times (0, \infty)$. Moreover, $e^{(t-s)\Delta} v(s)$ denotes $G_{t-s} * v(s)$. Note that this notation is an abbreviation for $e^{(t-s)\Delta}(v(s))$. Analogously, $\Delta e^{(t-s)\Delta} v(s)$ and $e^{(t-s)\Delta} \Delta v(s)$ are abbreviations for $\Delta(e^{(t-s)\Delta} v(s))$ and $e^{(t-s)\Delta}((\Delta v)(s))$, respectively.

4.3.2 Solutions of the Inhomogeneous Equation: Case of Zero Initial Value

Proposition. *Assume that* $h \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *satisfies*

$$\sup_{0<t\le T} \|\partial_x^\alpha \partial_t^k h\|_\infty(t) < \infty$$

for any $T > 0$*, any multi-index* α*, and any* $k = 0, 1, 2, \ldots$. *Let* G_t *denote the Gauss kernel. We set*

$$w(x,t) = \int_0^t \int_{\mathbb{R}^n} G_{t-s}(x-y)h(y,s)dy\ ds,\ x \in \mathbb{R}^n,\ t > 0,$$

i.e.,

$$w(t) = \int_0^t e^{(t-s)\Delta} h(s)ds \text{ on } \mathbb{R}^n,\ t > 0.$$

Then, $w \in C^\infty(\mathbb{R}^n \times [0,\infty))$ *and for any* $T > 0$*, any multi-index* α*, and any* $k = 0, 1, 2, \ldots$,

$$\sup_{0<t\le T} \|\partial_x^\alpha \partial_t^k w\|_\infty(t) < \infty.$$

Furthermore, w *satisfies*

$$\begin{cases} \partial_t w - \Delta w = h, & x \in \mathbb{R}^n,\ t > 0, \\ w(x,0) = 0, & x \in \mathbb{R}^n, \end{cases}$$

and $\lim_{t\to 0} \|w\|_\infty(t) = 0$. *This implies that the initial value is approached uniformly in* $x \in \mathbb{R}^n$*, which in particular yields the pointwise continuity of* w *at* $t = 0$ *and* $w(x,0) = 0$ $(x \in \mathbb{R}^n)$.

Proof. First we cut off the singularity of the integrand. More precisely, we consider

$$w^\rho(t) = \int_0^{t-\rho} e^{(t-s)\Delta} h(s)ds$$

for $t > \rho > 0$. Then, we may differentiate under the integral sign (§7.2.1), which implies $w^\rho \in C^\infty(\mathbb{R}^n \times (\rho,\infty))$. Applying ∂_t to w^ρ with respect to t, we obtain[1]

$$\begin{aligned} \partial_t w^\rho(t) &= e^{\rho\Delta} h(t-\rho) + \int_0^{t-\rho} \frac{d}{dt} e^{(t-s)\Delta} h(s)ds \\ &= e^{\rho\Delta} h(t-\rho) + \int_0^{t-\rho} \Delta e^{(t-s)\Delta} h(s)ds \ (=: I_1^\rho + I_2^\rho). \end{aligned}$$

[1] For formal reasons we write the partial differential as d/dt in the term in which $e^{t\Delta}$ appears.

(Note that here we employed

$$\frac{d}{dt}\int_0^{\alpha(t)} F(t,s)ds = \alpha'(t)F(t,\alpha(t)) + \int_0^{\alpha(t)} \frac{\partial F}{\partial t}(t,s)ds.$$

This formula is easily obtained from the fundamental theorem of calculus and the chain rule under the assumption that $F, \partial F/\partial t, \alpha, \alpha'$ are continuous.) Our intention is to show that for each $t \in \mathbb{R}^n$, I_1^ρ and I_2^ρ converge uniformly to $I_1 = h(t)$ and

$$I_2 = \int_0^t e^{(t-s)\Delta}\Delta h(s)ds$$

as $\rho \to 0$, respectively.

According to properties of the convolution (§4.1.6) we have $\Delta e^{(t-s)\Delta}h = e^{(t-s)\Delta}\Delta h$ $(t > s)$, which implies

$$I_2 - I_2^\rho = \int_{t-\rho}^t e^{(t-s)\Delta}\ \Delta h(s)ds.$$

By §1.1.2 we have the estimate $\|e^{(t-s)\Delta}\ \Delta h(s)\|_\infty(t) \le \|\Delta h\|_\infty(s)$ $(t > s)$. Hence, we deduce

$$\|I_2 - I_2^\rho\|_\infty(t) \le \int_{t-\rho}^t \|\Delta h\|_\infty(s)ds \le \rho \sup_{0<s\le T} \|\Delta h\|_\infty(s), \quad \rho_0 \le t \le T,$$

for any ρ with $0 < \rho < \rho_0$ and for fixed ρ_0. In order to get a similar estimate for $I_1^\rho - I_1$ we write

$$\begin{aligned} I_1^\rho - I_1 &= e^{\rho\Delta}h(t-\rho) - h(t) \\ &= (e^{\rho\Delta} - I)h(t-\rho) + \{h(t-\rho) - h(t)\}. \end{aligned}$$

Here I denotes the identity operator. In view of Lemma 4.3.2, the first term on the right-hand side can be estimated as

$$\begin{aligned} (e^{\rho\Delta} - I)h(t-\rho) &= \int_0^\rho \frac{d}{ds}\ e^{s\Delta}h(t-\rho)ds \\ &= \int_0^\rho \Delta e^{s\Delta}h(t-\rho)ds = \int_0^\rho e^{s\Delta}\Delta h(t-\rho)ds. \end{aligned}$$

For $0 < \rho < \rho_0 \le t \le T$ we therefore obtain

$$\|(e^{\rho\Delta} - I)h(t-\rho)\|_\infty \le \int_0^\rho \|\Delta h(t-\rho)\|_\infty ds \le \rho \sup_{0<s\le T} \|\Delta h\|_\infty(s).$$

For the second term the integral form of the mean value theorem (§1.1.6) yields

$$\|h(t-\rho)-h(t)\|_\infty \le \rho \sup_{0<s\le T} \|\partial_t h\|_\infty(s).$$

Consequently,

$$\|I_1^\rho - I_1\|_\infty(t) \le \rho \left\{ \sup_{0<s\le T} \|\Delta h\|_\infty(s) + \sup_{0<s\le T} \|\partial_t h\|_\infty(s) \right\}$$

for all $0<\rho\le t\le T$. This implies that for $\rho\to 0$ and for any $\rho_0>0$, $I_1^\rho + I_2^\rho$ converges uniformly to I_1+I_2 on $\mathbb{R}^n\times[\rho_0,T]$. Thus, the limit function I_1+I_2 is continuous on this set. Since w^ρ converges uniformly to w on $\mathbb{R}^n\times[\rho_0,T]$ as well, the results in §4.1.5 imply that w is partially differentiable with respect to t and $\partial_t w$ is continuous on $\mathbb{R}^n\times[\rho_0,T]$. Hence, $\partial_t w = I_1+I_2$ on $\mathbb{R}^n\times[\rho_0,T]$. Since $\rho_0>0$ and $T>0$ have been arbitrary, this equality holds on $\mathbb{R}^n\times(0,\infty)$.

Very similar to the above argumentation for $\partial_t w$, it can be proved that

$$\partial_x^\alpha w^\rho(z) = \partial_x^\alpha \int_0^{t-\rho} e^{(t-s)\Delta} h(s)ds, \quad |\alpha|\le 2,$$

converges uniformly to

$$\int_0^t e^{(t-s)\Delta}\partial_x^\alpha h(s)ds$$

on $\mathbb{R}^n\times[\rho_0,T]$ as $\rho\to 0$. A repeated application of Lemma 4.1.5 then implies that $w(x,t)$ is twice partially differentiable with respect to x, and all partial derivatives (up to second order) are continuous on $\mathbb{R}^n\times[\rho_0,T]$. Consequently, w is C^2 on $\mathbb{R}^n\times(0,\infty)$ with respect to x. In particular, we have $\Delta w = I_2$, which implies

$$\partial_t w = I_1 + \Delta w = h + \Delta w, \quad x\in\mathbb{R}^n,\ t>0.$$

By virtue of the L^∞-L^∞ estimate $\|e^{t\Delta} f\|_\infty \le \|f\|_\infty$ derived in §1.1.2, we obtain

$$\|w\|_\infty(t) \le \int_0^t \|h\|_\infty(s)ds \le t \sup_{0<s\le T}\|h\|_\infty(s) \to 0 \quad (t\to 0).$$

Analogously, we may show that for any multi-index α and any nonnegative integer k, $\partial_x^\alpha\partial_t^k w^\rho$ converges uniformly to a continuous function on $\mathbb{R}^n\times[\rho_0,T]$. By §4.1.5 we therefore obtain that $w\in C^\infty(\mathbb{R}^n\times(0,\infty))$. Accordingly, it can also be shown that $\sup_{0<t\le T}\|\partial_x^\alpha\partial_t^k w\|_\infty(t)<\infty$. Thus, w is smooth at $t=0$, i.e., $w\in C^\infty(\mathbb{R}^n\times[0,\infty))$. This is the content of Exercise 4.5. □

Lemma. *Assume that $f:\mathbb{R}^n\to\mathbb{R}$ is bounded and uniformly continuous. For $0<\eta<t$ we set*

$$F_\eta = \int_\eta^t \frac{d}{d\tau} e^{\tau\Delta} f \; d\tau \ \text{ on } \mathbb{R}^n.$$

Then, F_η converges uniformly to $e^{t\Delta}f - f$ on $\mathbb{R}^n$ as $\eta \to 0$, i.e.,

$$\lim_{\eta\to 0} \|F_\eta - (e^{t\Delta}f - f)\|_\infty = 0.$$

If f is additionally bounded and C^1 on $\mathbb{R}^n$ and $\|\partial_{x_j} f\|_\infty$ is finite for each $1 \le j \le n$, then the integrand $(\frac{d}{d\tau}e^{\tau\Delta}f)(x)$ is integrable on $(0,t)$ for each x as a function in τ. (Note that the boundedness of the derivatives of f implies the uniform continuity of f on $\mathbb{R}^n$.) This integrability and the above convergence then yield

$$e^{t\Delta}f - f = \int_0^t \frac{d}{d\tau}e^{\tau\Delta}f\,d\tau = \int_0^t \Delta e^{\tau\Delta}f\,d\tau, \quad t > 0,$$

regarded as a function on $\mathbb{R}^n$.

Proof. Since G_t is smooth for $t > 0$ and since $f \in L^\infty(\mathbb{R}^n)$, $e^{t\Delta}f$ is smooth on $\mathbb{R}^n \times (0,\infty)$ (see Exercise 7.2). The fundamental theorem of calculus then implies $e^{t\Delta}f - e^{\eta\Delta}f = F_\eta$, $0 < \eta < t$, on $\mathbb{R}^n$. For bounded and uniformly continuous f, Theorem 4.2.1 gives us $\lim_{\eta\to 0}\|e^{\eta\Delta}f - f\|_\infty = 0$. Hence the first part follows.

Now assume that $f \in C^1(\mathbb{R}^n)$ and that $\|f\|_\infty$ and $\|\partial_{x_j}f\|_\infty$ are finite for $1 \le j \le n$. Proposition 4.1.6 yields

$$\frac{d}{d\tau}e^{\tau\Delta}f = \Delta e^{\tau\Delta}f = \sum_{j=1}^n \partial_{x_j}e^{\tau\Delta}\partial_{x_j}f.$$

The L^∞-L^∞ estimate for first-order derivatives obtained in §1.1.3 results in

$$\|\partial_{x_j}e^{\tau\Delta}\partial_{x_j}f\|_\infty \le \frac{C}{\tau^{1/2}}\|\partial_{x_j}f\|_\infty, \quad \tau > 0.$$

(Here C is a constant that is independent of τ and f.) Hence for each x, $(\frac{d}{d\tau}e^{\tau\Delta}f)(x)$ is integrable on the interval $(0,t)$ as a function in τ. □

The just proved lemma motivates the conjecture

$$e^{t\Delta}f - f = \int_0^t \frac{d}{d\tau}(e^{\tau\Delta}f)d\tau.$$

However, without any additional smoothness of f, by the singularity of $\partial_t G_t$ at $t = 0$, the integrand might not be integrable in general. Consequently, we considered the integral as the limit of F_η as $\eta \to 0$.

Since the heat equation is linear, the so-called *principle of superposition* applies. More precisely, this means that if w satisfies $\partial_t w - \Delta w = h$ and v satisfies $\partial_t v - \Delta v = 0$, then $w + v$ satisfies $\partial_t(w+v) - \Delta(w+v) = h$. By Proposition 4.3.2 and the fact that $e^{t\Delta}f$ is the solution of the heat equation with initial value f, the principle of superposition implies the following result.

4.3.3 Solutions of Inhomogeneous Equations: General Case

Corollary. *Assume the same hypotheses of Proposition 4.3.2. If* $f \in C_0(\mathbb{R}^n)$*, then* u *given by (4.6) is the solution of the initial value problem (4.3), which satisfies* $u \in C^\infty(\mathbb{R}^n \times (0,\infty)) \cap C(\mathbb{R}^n \times [0,\infty))$*.*

In §2.4.2, which differs from §4.3.2 and §4.3.3, we consider the case that the inhomogeneous term of (4.4) is not necessarily bounded near $t = 0$. Also in this case the above formula for the solution, in the form as it is used in §2.4.2, is still valid in many cases. Next we derive some results for the formula in this form of inhomogeneous term given in §2.4.2 with zero initial value.

4.3.4 Singular Inhomogeneous Term at $t = 0$

Theorem. *Assume that the function* $h^i \in C^\infty(\mathbb{R}^n \times (0,\infty))$ *satisfies*

$$\sup_{\delta \le t \le T} \|\partial_x^\alpha \partial_t^k h^i\|_\infty(t) < \infty$$

for any $T > 0$*, any* $\delta > 0$*, any multi-index* α*, and for any nonnegative integer* k*. (We emphasize that we do not assume the existence of a uniform bound in all these parameters, but merely the existence of some bound, which can depend on the parameters. So, for instance,* $\sup_{\delta \le t \le T} \|\partial_x^\alpha \partial_t^k h^i\|_\infty(t))$ *might grow as* $\delta \to 0$*.) Moreover, we assume that* $\sup_{t>0} t^{1/2}\|h^i\|_1(t) < \infty$*,* $i = 1, 2, \ldots, n$*. For* $h = (h^1, \ldots, h^n)$ *and* $t > 0$ *we set*

$$w(t) = \int_0^t \mathrm{div}\ (e^{(t-s)\Delta} h(s)) ds \qquad \text{in } \mathbb{R}^n.$$

Then $w \in C^\infty(\mathbb{R}^n \times (0,\infty))$*,*

$$\sup_{\delta \le t \le T} \|\partial_x^\alpha \partial_t^k w\|_\infty(t) < \infty,$$

and w *is a weak solution of*

$$\partial_t w - \Delta w = \mathrm{div}\ h \qquad \text{in } \ \mathbb{R}^n \times (0,\infty)$$

with zero initial value. Furthermore, we have $\sup_{0<t} \|w\|_1(t) < \infty$*.*

For a vector-valued function h the expression $e^{t\Delta}$ is to be understood as the matrix $e^{t\Delta} I$ with $I \in \mathbb{R}^{n \times n}$ the identity matrix, i.e., we have

$$e^{t\Delta} h = (e^{t\Delta} h^1, \ldots, e^{t\Delta} h^n).$$

In the above theorem w satisfies the differential equation in the usual sense. However, w is eventually not continuous at $t = 0$. Hence, we need to explain in which sense w satisfies the initial condition. Weak solutions for the inhomogeneous equation can be defined similarly to (1.17). A locally integrable function u on $\mathbb{R}^n \times [0,\infty)$ is called a *weak solution* of

$$\partial_t u - \Delta u = \text{div } \ h_2 + h_1 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

with initial value f if for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, u satisfies

$$0 = \int_{\mathbb{R}^n} \varphi(x,0) f(x) dx + \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) u \ dx \ dt + \int_0^\infty \int_{\mathbb{R}^n} (\varphi h_1 - \langle \nabla \varphi, \ h_2 \ \rangle) dx \ dt.$$

(The weak solution with initial value $m\delta$ for $m \in \mathbb{R}$ and δ the Dirac distribution is defined by replacing the first integral of the right-hand side by $m\varphi(0,0)$.) Here, h_2 is an $\mathbb{R}^n$-valued function and h_1 and h_2 are locally integrable on $\mathbb{R}^n \times [0,\infty)$, whereas f is locally integrable on $\mathbb{R}^n$.[1]

Proof. For $\sigma \in (0,1)$ we define

$$w_\sigma(t) = \int_\sigma^t e^{(t-s)\Delta}((\text{div}\, h)(s)) ds, \quad t > \sigma.$$

Then by Proposition 4.3.2 we have $w_\sigma \in C^\infty(\mathbb{R}^n \times [\sigma,\infty))$ and that w_σ satisfies

$$\begin{cases} \partial_t u - \Delta u = \text{div}\, h & \text{in } \mathbb{R}^n \times (\sigma,\infty), \\ u(x,\sigma) = 0, & x \in \mathbb{R}^n. \end{cases}$$

Multiplying the first equation by $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$ integrating over $\mathbb{R}^n \times (\sigma,\infty)$, and applying integration by parts, we obtain the weak form

$$0 = \int_\sigma^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) w_\sigma \ dx \ dt - \int_\sigma^\infty \int_{\mathbb{R}^n} \langle \nabla\varphi, \ h \rangle \ dx \ dt.$$

By the properties for convolution derived in §4.1.4 and §4.1.6 we also have

$$w_\sigma(t) = \int_\sigma^t \text{div}\, (e^{(t-s)\Delta} h(s)) ds, \quad t > \sigma.$$

Note that by assumption, $h^i(t) \in L^1(\mathbb{R}^n)$ for all $i = 1, 2, \ldots, n$ and $t > 0$. But then the L^1-L^1 estimate for $\partial_{x_j} e^{t\Delta}$ (§1.1.3) implies

$$\|w_\sigma\|_1(t) \le \int_0^t \frac{C}{(t-s)^{1/2}} \|h\|_1(s) ds, \quad t > \sigma,$$

[1] For $T > 0$ and a locally integrable function u on $\mathbb{R}^n \times [0,T)$ we may define the weak solution by replacing the time interval $(0,\infty)$ by $(0,T)$ and $[0,\infty)$ by $[0,T)$.

with a constant C depending only on the dimension. (Note that $\|h\|_1$ for a vector-valued function h is defined by $\| \, |h| \, \|_1$.) In view of $H := \sup_{t>0} t^{1/2} \|h\|_1(t) < \infty$, we can continue this estimate by

$$\begin{aligned}\|w_\sigma\|_1(t) &\leq \int_0^t \frac{CH}{(t-s)^{1/2}s^{1/2}} ds \\ &= CH \int_0^1 \frac{1}{(1-\tau)^{1/2}\tau^{1/2}} d\tau = C'H, \quad t > \sigma.\end{aligned}$$

The right-hand side is a constant that depends on the dimension n and H but is independent of t and σ.

Now suppose that w_σ converges uniformly to w as $\sigma \to 0$ on any compact subset of $\mathbb{R}^n \times (0, \infty)$. Then we define for $t \geq 0$ the function F_σ by

$$F_\sigma(t) = \begin{cases} \int_{\mathbb{R}^n} (\partial_t \varphi(x,t) + \Delta\varphi(x,t)) w_\sigma(x,t) dx, & t > \sigma, \\ 0, \quad 0 \leq t \leq \sigma. \end{cases}$$

Similarly to the proof of Theorem 1.4.4 it can be shown that F_σ converges pointwise to

$$F(t) = \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) w \; dx.$$

Observe that the uniform boundedness of $\sup_{t>\sigma} \|w_\sigma\|_1(t)$ for $0 < \sigma < 1$ implies that also $F_\sigma(t)$ is uniformly bounded for $0 < \sigma < 1$ on each interval $[0, T]$. The dominated convergence theorem (§7.1.1) therefore yields

$$\begin{aligned}\int_0^\infty F_\sigma(t) dt &= \int_\sigma^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) w_\sigma \; dx \; dt \\ &\to \int_0^\infty F(t) dt = \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) w \; dx \; dt\end{aligned}$$

as $\sigma \to 0$. Since for $t > 0$,

$$\left| \int_{\mathbb{R}^n} \langle \nabla\varphi(x,t), \; h(x,t) \rangle \; dx \right| \leq \frac{C_\varphi H}{t^{1/2}}, \quad C_\varphi = \sup_{\mathbb{R}^n \times [0,\infty)} |\nabla\varphi|,$$

again dominated convergence implies

$$\int_\sigma^\infty \int_{\mathbb{R}^n} \langle \nabla\varphi, h \rangle \; dx \; dt \to \int_0^\infty \int_{\mathbb{R}^n} \langle \nabla\varphi, h \rangle \; dx \; dt$$

as $\sigma \to 0$. Consequently, w is a weak solution of

$$\partial_t w - \Delta w = \text{div } \; h \text{ in } \mathbb{R}^n \times (0, \infty)$$

with zero initial value. Moreover, Fatou's lemma (§7.1.2) implies that $\|w\|_1(t) \leq \underline{\lim}_{\sigma \to 0} \|w_\sigma\|_1(t)$. Thus, we have $\|w\|_1(t) \leq C'H, \; t > 0$.

To complete the proof it remains to show that w_σ converges uniformly to w on any compact subset of $\mathbb{R}^n \times (0, \infty)$ and to prove the claimed estimate for higher derivatives $\partial_x^\alpha \partial_t^k w$. To this end, we prove for

$$w(t) - w_\sigma(t) = \int_0^\sigma \operatorname{div}\,(e^{(t-s)\Delta} h(s)) ds, \quad t > \sigma,$$

that

$$\lim_{\sigma \to 0} \sup_{\delta \le t \le T} \|\partial_x^\alpha \partial_t^k (w - w_\sigma)\|_\infty (t) = 0$$

for each $\delta \in (0, T)$. First observe that by applying Proposition 4.3.2 to w_σ we obtain

$$\sup_{\sigma < t \le T} \|\partial_x^\alpha \partial_t^k w_\sigma\|_\infty (t) < \infty.$$

Next we employ the L^∞-L^1 estimate for $\partial_{x_j} e^{t\Delta}$ (§1.1.3). Differentiating under the integral sign (§7.2.1) gives us

$$\begin{aligned} \|\partial_x^\alpha \partial_t^k (w - w_\sigma)\|_\infty (t) &\le \int_0^\sigma \|\partial_x^\alpha \partial_t^k \operatorname{div}\ (e^{(t-s)\Delta} h(s))\|_\infty ds \\ &\le \int_0^\sigma \frac{C_{\alpha,k}}{(t-s)^{k+(|\alpha|+1+n)/2}} \|h\|_1(s) ds \\ &\le C_{\alpha,k} H \int_0^\sigma \frac{1}{(t-s)^{k+(|\alpha|+1+n)/2} s^{1/2}} ds. \end{aligned}$$

Here $C_{\alpha,k}$ is a constant depending only on the dimension n (and on α, k, of course). If $t \ge \delta > 2\sigma$, we have $\delta - \sigma > \delta/2$. Thus, we obtain

$$\|\partial_x^\alpha \partial_t^k (w - w_\sigma)\|_\infty (t) \le C_{\alpha,k} H \left(\frac{2}{\delta}\right)^{k+(|\alpha|+1+n)/2} \int_0^\sigma \frac{1}{s^{1/2}} ds.$$

Since the right-hand side is independent of t,

$$\sup_{\delta \le t \le T} \|\partial_x^\alpha \partial_t^k (w - w_\sigma)\|_\infty (t) \to 0 \quad (\sigma \to 0)$$

follows. Consequently, $\sup_{\delta \le t \le T} \|\partial_x^\alpha \partial_t^k w\|_\infty (t) < \infty$. (The fact that $w \in C^\infty(\mathbb{R}^n \times (0, \infty))$ obviously follows from $w_\sigma \in C^\infty(\mathbb{R}^n \times [\sigma, \infty))$ and $w - w_\sigma \in C^\infty(\mathbb{R}^n \times [\delta, \infty))$ by differentiating under the integral sign.) □

Remark. If $\|h\|_1(t)$ is replaced by $\sup_{0<t<t_0} t^{1/2} \|h\|_1(t) =: H_0 < \infty$ in the assumption for some $0 < t_0 \le \infty$, the estimate for w in the assertion changes to $\sup_{0<t<t_0} \|w\|_1(t) \le C' H_0$. Here C' is again a constant depending only on the dimension n. Thus, we do not get just the boundedness of $\|w\|_1(t)$ on $(0, t_0)$, but also information on the specific form of the bound.

4.4 Uniqueness of Solutions of the Heat Equation

Concerning the initial value problem for the heat equation

$$\partial_t u - \Delta u = 0,\ t > 0, x \in \mathbb{R}^n; \quad u(x,0) = f(x),\ x \in \mathbb{R}^n,$$

$u = G_t * f$ is a solution if, e.g., $f \in C_0(\mathbb{R}^n)$. In fact, in §4.2.1 and §4.2.5 we proved that u approaches the initial value f as $t \to 0$. Furthermore, as a direct consequence of differentiation under the integral sign, it can be proved that u satisfies the equation for $t > 0$ (Exercise 7.2). It remains to show that there are no other solutions. It is well known that uniqueness for the above problem might fail if u is rapidly growing at space infinity. The problem of showing that there is just one solution is called the **uniqueness problem**. For continuous initial values, for example, the following growth condition is sufficient to guarantee uniqueness: For any $T > 0$,

$$\sup \left\{ \frac{\log(|u(x,t)| + 1)}{|x|^2 + 1}; x \in \mathbb{R}^n,\ 0 \le t < T \right\} < \infty$$

(see [Widder 1975]). From this condition we immediately see that uniqueness holds if u is bounded on $\mathbb{R}^n \times [0,T)$. The purpose of this section is to give a proof of a uniqueness theorem for weak solutions that covers also a class of discontinuous initial data (as in §1.4.6), such as for example $m\delta$. Recall that in §1.4.6 we assume that v satisfies the growth condition $\sup_{t>0} \|v\|_1(t) < \infty$. However, observe that this condition requires a sort of decay at space infinity.

4.4.1 Proof of the Uniqueness Theorem 1.4.6

We consider two weak solutions v_1 and v_2 and will show that $v_1 = v_2$. First note that by the principle of superposition and the definition of weak solutions, $w = v_1 - v_2$ is a weak solution of the heat equation with zero initial value. Hence, it suffices to prove the following uniqueness theorem.

4.4.2 Fundamental Uniqueness Theorem

Theorem. *If a function* $w \in C(\mathbb{R}^n \times (0,\infty))$ *satisfies* (i) $\sup_{t>0} \|w\|_1(t) < \infty$ *and* (ii) *for any* $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, $\int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) w\, dx\, dt = 0$, *then* $w \equiv 0$.

Remark. If ∞ is replaced by a finite $T > 0$ in each appearing time interval and condition (i) by $\sup_{0<t<T} \|w\|_1(t) < \infty$, we have that $w \equiv 0$ on $\mathbb{R}^n \times (0,T)$.

Proof. For functions h_1 and h_2 defined on $\mathbb{R}^n \times (0,\infty)$, we define the L^2-inner product by $\langle h_1, h_2 \rangle_2 = \int_0^\infty \int_{\mathbb{R}^n} h_1(x,t) h_2(x,t) dx\, dt$, and write (ii) as

$$\langle A\varphi, w \rangle_2 = 0,$$

with $A = \partial_t + \Delta$. We will show that $w = 0$ if this equality is satisfied for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$. This will follow if we can prove that the image of the operator A applied to $C_0^\infty(\mathbb{R}^n \times [0,\infty))$ is a dense subset of $L^2(\mathbb{R}^n \times [0,\infty))$. Then w is orthogonal to any function in $L^2(\mathbb{R}^n \times [0,\infty))$, which implies that $w = 0$. In other words, this means we need to show that for any ψ in an appropriate dense subset, the equation $A\varphi = \psi$ is solvable. For this, we will use the existence theorem for the inhomogeneous equation proved in §4.3.2. For a rigorous proof, however, we first have to introduce suitable classes for ψ and φ.

The First Step

First we prove that the equality (ii) still holds for $\varphi \in C^\infty(\mathbb{R}^n \times [0,\infty))$ satisfying

$$\begin{cases} \sup_{t>0} \|\partial_t\varphi\|_\infty(t) < \infty, \ \sup_{t>0} \|\partial_x^\alpha\varphi\|_\infty(t) < \infty \quad (|\alpha| \le 2) \quad \text{and} \\ \text{there exists } T' > 0 \text{ such that } \operatorname{supp}\varphi \subset \mathbb{R}^n \times [0, T'). \end{cases}$$

Note that for this purpose we will need condition (i).

Let us show that φ can be approximated by C^∞-functions with compact support in $\mathbb{R}^n \times [0,\infty)$. Pick a function $\theta \in C^\infty(\mathbb{R})$ such that $0 \le \theta \le 1$ and

$$\theta(\tau) = \begin{cases} 0, & \tau \ge 2, \\ 1, & \tau \le 1. \end{cases}$$

(For example,

$$q(s) = \begin{cases} e^{-1/s}, & s > 0, \\ 0, & s \le 0. \end{cases}$$

Then $q \in C^\infty(\mathbb{R})$, and we may set $\theta(\tau) = q(2-\tau)/\{q(2-\tau) + q(\tau-1)\}$.) Next, for $j = 1, 2, \dots$ we set

$$\theta_j(x) = \theta\left(|x|/j\right), \quad x \in \mathbb{R}^n.$$

This implies $\theta_j \in C_0^\infty(\mathbb{R}^n)$ and $\theta_j(x) = 1$ for $|x| \le j$. For $j \to \infty$, $\theta_j(x)$ converges to 1 pointwise for any $x \in \mathbb{R}^n$. Now we define φ_j as

$$\varphi_j(x,t) = \theta_j(x)\varphi(x,t), \quad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

Then $\varphi_j \in C_0^\infty(\mathbb{R}^n \times [0,\infty))$, and we have

$$\int_0^\infty \int_{\mathbb{R}^n} (\partial_t\varphi_j + \Delta\varphi_j) w \, dx \, dt = 0.$$

Thus, to prove (ii) for φ it remains to show that

$$\int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi_j) w \, dx \, dt \to \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi) w \, dx \, dt, \tag{4.7}$$

$$\int_0^\infty \int_{\mathbb{R}^n} (\Delta \varphi_j) w \, dx \, dt \to \int_0^\infty \int_{\mathbb{R}^n} (\Delta \varphi) w \, dx \, dt \tag{4.8}$$

as $j \to \infty$. By the definition of φ_j, obviously $(\partial_t \varphi_j) w \to (\partial_t \varphi) w$ pointwise on $\mathbb{R}^n \times (0, \infty)$ and we have

$$|(\partial_t \varphi_j) w|(x,t) \leq |w|(t,x) \sup_{t>0} \|\partial_t \varphi\|_\infty(t) \quad (x,t) \in \mathbb{R}^n \times (0,\infty).$$

Therefore, the assumptions on φ and w and the dominated convergence theorem (§7.1.1) imply (4.7). In order to see (4.8), we divide the integral into the three parts I, II and III according to

$$\int_0^\infty \int_{\mathbb{R}^n} (\Delta \varphi_j) w \, dx \, dt$$
$$= \int_0^\infty \int_{\mathbb{R}^n} \{(\Delta \varphi)\theta_j w + 2\langle \nabla \theta_j, \nabla \varphi \rangle w + \varphi(\Delta \theta_j) w\} dx \, dt.$$

Using (i) and $\sup_{t>0} \|\partial_x^\alpha \varphi\|(t) < \infty$ ($|\alpha| \leq 2$), analogously to the proof of (4.7) it follows that I converges to

$$\int_0^\infty \int_{\mathbb{R}^n} (\Delta \varphi) w \, dx \, dt$$

as $j \to \infty$. For the convergence of II, observe that

$$\partial_{x_\ell} \theta_j = \frac{1}{j} \theta' \left(\frac{|x|}{j} \right) \frac{x_\ell}{|x|}, \quad x \in \mathbb{R}^n, \quad \ell = 1, 2, \ldots, n.$$

This yields that $\|\partial_x^\beta \theta_j\|_\infty \leq C/j$ ($|\beta| = 1$) with C independent of j. By (i) and the assumptions on φ in the first step we obtain

$$|II| \leq \int_0^\infty \int_{\mathbb{R}^n} 2|\nabla \theta_j| \, |\nabla \varphi| \, |w| dx \, dt \leq \frac{2\sqrt{n}C}{j} \int_0^{T'} \int_{\mathbb{R}^n} |\nabla \varphi| \, |w| dx \, dt$$
$$\leq \frac{2\sqrt{n}C}{j} C'T' \sup_{s \geq 0} \|w\|_1(s) \to 0 \quad (j \to \infty).$$

Here C' satisfies $\sup_{t>0} \|\partial_x^\alpha \varphi\|_\infty(t) \leq C'$ for all α with $|\alpha| = 1$. In a very similar way it can be shown that $III \to 0$ as $j \to \infty$. Hence (4.8) follows.

The Second Step

For $T > 0$ and for $\psi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ with $\operatorname{supp} \psi \subset \mathbb{R}^n \times (0, T)$, we denote by φ the solution of

$$\begin{cases} \partial_t \varphi + \Delta\varphi = \psi, & t < T,\ x \in \mathbb{R}^n, \\ \varphi(x, T) = 0 & x \in \mathbb{R}^n. \end{cases}$$

By the substitution $\tau = T - t$ the above problem transforms to a standard inhomogeneous heat equation for the variables (x, τ) as treated in §4.3.2. Thus, by Proposition 4.3.2 there exists a solution $\varphi \in C^\infty(\mathbb{R}^n \times [0, T])$ to the above problem satisfying

$$\sup_{0<t<T} \|\partial_x^\alpha \partial_t^k \varphi\|_\infty(t) < \infty$$

for any multi-index α and for any nonnegative integer k. Moreover, representation (4.5) shows that there exists an $\varepsilon > 0$ such that φ is zero on $\mathbb{R}^n \times [T - \varepsilon, T]$. We extend φ as $\varphi(x, t) = 0$ for $t > T$. Then this extended φ satisfies all conditions in the first step. Therefore, we may substitute φ into (ii) to obtain

$$0 = \int_0^\infty \int_{\mathbb{R}^n} (\partial_t \varphi + \Delta\varphi) w \; dx \; dt = \int_0^T \int_{\mathbb{R}^n} \psi w \; dx \; dt.$$

Since this holds for all $\psi \in C_0^\infty(\mathbb{R}^n \times (0, T))$, the remark in Exercise 1.8 implies that w is identically zero on $\mathbb{R}^n \times (0, T)$. By the fact that $T > 0$ is arbitrary, w is identically zero on $\mathbb{R}^n \times (0, \infty)$. The proof is now complete.

□

4.4.3 Inhomogeneous Equation

Using the fundamental uniqueness theorem, we may prove that the solution of the inhomogeneous equation (4.3) is given by formula (4.6) under suitable assumptions on h and f. For example, it is easy to show that the solution w constructed in Proposition 4.3.2 and Theorem 4.3.4 is unique, provided $\sup_{t>0} \|w\|_1(t) < \infty$. However, here we just give a weaker version of these results as it is applied in §2.4.2 and §2.5.3.

Theorem. *Assume that the vector-valued function $h = (h^1, \ldots, h^n)$ satisfies the assumptions of Theorem 4.3.4. Then there exists a unique weak solution u of*

$$\partial_t u - \Delta u = \operatorname{div}\ h \qquad \textit{in } \mathbb{R}^n \times (0, \infty)$$

with initial value $f \in C_0(\mathbb{R}^n)$ satisfying $\sup_{t>0} \|u\|_1(t) < \infty$. Furthermore, u is given by

$$u(t) = e^{t\Delta} f + \int_0^t \operatorname{div}(e^{(t-s)\Delta} h(s)) ds, \quad t > 0.$$

If the initial value is $f = m\delta$ for $m \in \mathbb{R}$, there exists a unique weak solution u of the above inhomogeneous equation satisfying $\sup_{t>0} \|u\|_1(t) < \infty$ and

$$u(t) = mG_t + \int_0^t \operatorname{div}(e^{(t-s)\Delta}h(s))ds, \quad t > 0.$$

The solution u given by the above formula is exactly the weak solution considered in the theorem. This is a consequence of the principle of superposition of weak solutions (§4.3.4).[1] The uniqueness of weak solutions follows by the fundamental uniqueness theorem (§4.4.2).

If we assume the weaker condition $\sup_{0<t<t_0} \|u\|_1(t) < \infty$ for some $t_0 > 0$ instead of $\sup_{t>0} \|u\|_1(t) < \infty$, the remark after the fundamental uniqueness theorem implies that the result remains valid if $\mathbb{R}^n \times (0, \infty)$ is replaced by $\mathbb{R}^n \times (0, t_0)$, $t > 0$ by $0 < t < t_0$, and $T > 0$ by $0 < T < t_0$.

Next we consider existence and uniqueness for the initial value problem (H_v), which is studied in §2.4.3. Note that for $v \equiv 0$ the next theorem turns to a result for the standard heat equation.

4.4.4 Unique Solvability for Heat Equations with Transport Term

Theorem. *Assume that for given $T > s$, $v^i \in C^\infty(\mathbb{R}^n \times [s, T))$, $i = 1, 2, \dots, n$, and that for any multi-index α and $\ell = 0, 1, 2, \dots$,*

$$\sup_{s<t<T} \|\partial_x^\alpha \partial_t^\ell v^i\|_\infty(t) < \infty \quad (i = 1, 2, \dots, n).$$

Furthermore, we set $v = (v^1, \dots, v^n)$ and assume that $f \in C(\mathbb{R}^n)$ is bounded and integrable (i.e., $\|f\|_\infty < \infty$ and $\|f\|_1 < \infty$).

I. Homogeneous equation. *There exists a unique solution $w \in C(\mathbb{R}^n \times [s, T)) \cap C^\infty(\mathbb{R}^n \times (s, T))$ satisfying the initial value problem*

$$\begin{cases} \partial_t w - \Delta w + (v, \nabla) w = 0, & x \in \mathbb{R}^n,\ t \in (s, T), \\ w(x, s) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Furthermore, for any multi-index α and $\ell = 0, 1, 2, \dots$ there exists a constant C depending on v, ℓ, α, n, and $T - s$ such that

$$\|\partial_x^\alpha \partial_t^\ell w\|_p(t) \le \frac{C}{(t-s)^{\ell + \frac{|\alpha|}{2} + \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}} \|f\|_q, s < t < T,\ 1 \le q \le p \le \infty.$$

Concerning convergence to the initial value we have $\|w(t) - f\|_r \to 0$ $(t \to s)$ for each $1 \le r < \infty$.

By the existence and uniqueness the solution operator $U(t, s)$ that maps f to $w(\cdot, t)$ is well defined. The family $(U(t, s))_{0 \le s \le t \le T}$ of linear operators $U(t, s)$ is called a propagator or an evolution system.

---

[1] Note that $e^{t\Delta} f$ is a classical solution of the heat equation with initial value f (Exercise 7.2 and §4.2.1). Hence it is also a weak solution (§1.4.2 and §1.4.3). By Exercise 1.9, mG_t is the weak solution to the initial value $m\delta$.

II. Inhomogeneous equation. *Assume that h and v^i are given as above and set*

$$w(t) = \int_s^t U(t,\tau)h(\tau)d\tau, \quad s < t < T.$$

Then $w \in C^\infty(\mathbb{R}^n \times [s,T))$, and for any multi-index α and $k = 0, 1, 2, \ldots,$ w satisfies

$$\sup_{s<t<T} \|\partial_x^\alpha \partial_t^k w\|_\infty(t) < \infty$$

and

$$\begin{cases} \partial_t w - \Delta w + (v, \nabla)w = h, & x \in \mathbb{R}^n, \ t \in (s,T), \\ w(x,s) = 0, & x \in \mathbb{R}^n. \end{cases}$$

Moreover, we have $\lim_{t\to s} \|w\|_\infty(t) = 0.$

Observe that in the case $v = 0$ the evolution system is given by $U(t,s) = e^{(t-s)\Delta}$. Also note that the estimate for the derivatives of w is weaker than the L^p-L^q estimate for the derivatives of the solution of the heat equation (§1.1.3). This is because here the constant C depends on $T - s$ and since the estimate above does not directly imply the decay of solutions in time. Moreover, the constant C depends on v. Thus, the estimate is also weaker than the one in Theorem 2.3.1, even if we assume $\operatorname{div} v = 0$.

Proof.

(I) Existence. The proof is based on the method of successive approximation. This procedure is often used for existence proofs of local solutions for ordinary differential equations. In the proof we will be brief in details. Without loss of generality, we may set $s = 0$. We define w_j inductively by

$$\begin{cases} w_{j+1}(t) = w_0(t) - \int_0^t e^{(t-\tau)\Delta}((v,\nabla)w_j)(\tau)d\tau, & j = 0, 1, 2, \ldots, \\ w_0(t) = e^{t\Delta} f. \end{cases}$$

For $0 < T' < T$ let $X = C([0,T']; L^1(\mathbb{R}^n))$, i.e., X is the space of all continuous functions defined on $[0,T']$ with values in $L^1(\mathbb{R}^n)$. By the completeness of $L^1(\mathbb{R}^n)$ it can be shown by similar arguments as in the proof of the completeness of $C[0,T']$ (Exercise 1.6) that with $\sup_{0\le t\le T'} \|\cdot\|_1(t)$ as a norm, X is complete as well. Now, Exercise 7.3 implies $e^{t\Delta} f \in X$. Furthermore, it can be shown that w_j is smooth on $\mathbb{R}^n \times (0,T)$. Setting $(v,\nabla)w_j = \operatorname{div}(vw_j) - w_j \operatorname{div} v$, we obtain the representation

$$\int_0^t e^{(t-\tau)\Delta}((v,\nabla)w_j)(\tau)d\tau = \int_0^t \operatorname{div}(e^{(t-\tau)\Delta}(vw_j)(\tau))d\tau - \int_0^t e^{(t-\tau)\Delta}(w_j \operatorname{div} v)(\tau)d\tau.$$

If $w_j \in X$, it is not difficult to show that $w_{j+1} \in X$ as well using $(*)$ in Exercise 7.3 and the L^1-L^1 estimate for the derivatives of the solution of the heat equation (§1.1.3). (Here we omit the details.) Let us show that $\{w_j\}$ is a Cauchy sequence in X. Set $U_{j+1} = w_{j+1} - w_j U_{j+1}$. Then, we have

$$U_{j+1}(t) = -\int_0^t \operatorname{div}(e^{(t-\tau)\Delta}(vU_j)(\tau))d\tau + \int_0^t e^{(t-\tau)\Delta}(U_j \operatorname{div} v)(\tau)d\tau, \quad j = 1, 2, \dots .$$

To show that $\{w_j\}$ is a Cauchy sequence in X, it is sufficient to show that $\sum_{j=1}^{\infty} \sup_{0 \le t \le T'} \|U_j\|_1(t)$ converges to zero. Applying the L^1-L^1 estimate for derivatives (§1.1.3) and the L^1-L^1 estimate (§1.1.2) to the above representation, we obtain

$$\|U_{j+1}\|_1(t) \le C_1 \int_0^t \frac{1}{(t-\tau)^{1/2}} \|U_j\|_1(\tau)d\tau + C_2 \int_0^t \|U_j\|_1(\tau)d\tau.$$

Here the constants C_1, C_2 depend only on n, $\sup_{0<t<T} \|v\|_\infty(t)$, and $\sup_{0<t<T} \|\nabla v\|_\infty(t)$. Moreover, for $0 \le \tau < t \le T$ we have $1 \le T^{1/2} (t-\tau)^{-1/2}$. This yields

$$\|U_{j+1}\|_1(t) \le K \int_0^t \frac{1}{(t-\tau)^{1/2}} \|U_j\|_1(\tau)d\tau$$

for $0 \le t < T$ with $K := C_1 + C_2 T^{1/2}$. So, successively we obtain

$$\begin{aligned}
&\|U_{j+1}\|_1(t) \\
&\quad \le K^2 \int_0^t \left\{ \int_0^{t_1} \|U_{j-1}\|_1(t_2) \frac{dt_2}{(t_1-t_2)^{1/2}} \right\} \frac{dt_1}{(t-t_1)^{1/2}} \\
&\quad \le K^3 \int_0^t \left\{ \int_0^{t_1} \left\{ \int_0^{t_2} \|U_{j-2}\|_1(t_3) \frac{dt_3}{(t_2-t_3)^{1/2}} \right\} \frac{dt_2}{(t_1-t_2)^{1/2}} \right\} \\
&\qquad \times \frac{dt_1}{(t-t_1)^{1/2}} \\
&\quad \le K^{j+1} \int_0^t \left\{ \int_0^{t_1} \cdots \left\{ \int_0^{t_j} \|w_0\|_1(t_{j+1}) \frac{dt_{j+1}}{(t_j-t_{j+1})^{1/2}} \right\} \right. \\
&\qquad\qquad \left. \cdots \frac{dt_2}{(t_1-t_2)^{1/2}} \right\} \frac{dt_1}{(t-t_1)^{1/2}}.
\end{aligned}$$

Once again the L^1-L^1 estimate (§1.1.2) implies $\|w_0\|_1(t) \le \|f\|_1$ $(t > 0)$, which gives us

$$\|U_j\|_1(t) \le K^j \|f\|_1 I_j(t),$$

where

$$I_j(t) := \int_0^t \left\{ \int_0^{t_1} \cdots \left\{ \int_0^{t_{j-1}} \frac{dt_j}{(t_{j-1}-t_j)^{1/2}} \right\} \cdots \frac{dt_2}{(t_1-t_2)^{1/2}} \right\}$$
$$\times \frac{dt_1}{(t-t_1)^{1/2}}, \quad 0 < t \le T.$$

Calculating $I_j(t)$ we deduce that

$$I_j(t) = t^{j/2} B(1/2,1) B(1/2,3/2) \dots B(1/2, j/2+1/2),$$

where $B(p,q) = \int_0^1 (1-\tau)^{p-1}\tau^{q-1} dt$ is the beta function and where we used the relation $\int_0^t \tau^\alpha (t-\tau)^{-1/2} d\tau = t^{\alpha+1/2} B(1/2, \alpha+1)$ $(\alpha > -1)$. Employing relation $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ between the beta and gamma functions (§6.2.5), this can be expressed as

$$I_j(t) = t^{j/2} (\Gamma(1/2))^j \Gamma(1)/\Gamma((j+2)/2).$$

By the fact that $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$ we arrive at

$$\|U_j\|_1(t) \le \|f\|_1 A^j / \Gamma((j+2)/2),$$

with $A := K\sqrt{\pi}T^{1/2}$, for $0 \le t < T$ and $j = 1, 2, 3, \dots$. Finally, taking advantage of the relation $\Gamma(p+1) = p\Gamma(p)$, we see that $\sum_{j=1}^{\infty} A^j/\Gamma((j+2)/2)$ converges for any fixed A. Thus, $\{w_j\}_{j=0}^{\infty}$ is a Cauchy sequence in X. Since X is complete, there is a unique limit in X, which we denote by w. Since T' was arbitrary, w is defined on $[0, T)$, and satisfies the integral equation

$$w(t) = e^{t\Delta} f - \int_0^t \operatorname{div} \left(e^{(t-\tau)\Delta} (vw)(\tau) \right) d\tau$$
$$+ \int_0^t e^{(t-\tau)\Delta} (w \operatorname{div}\ v)(\tau) d\tau, \quad 0 < t < T. \tag{4.9}$$

(Here we skip the details on the interchangeability of integrals and limits, which is valid in this situation.) We just outline the remaining steps.

a) Successively for w_j it can be shown that

$$\sup_{0<t<T} \|\partial_x^\alpha \partial_t^\ell w_j\|_p(t) t^{\ell + \frac{|\alpha|}{2} + \frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right)} \le C_3 \|f\|_q,$$
$$\ell = 0, 1, 2, \dots,\ 1 \le q \le p \le \infty,\ j = 0, 1, 2, \dots,$$

where α is a multi-index; its totality is written by $\mathbb{N}_0^n = \overbrace{\mathbb{N}_0 \times \cdots \times \mathbb{N}_0}^{n}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Here C_3 is independent of j, f, q, and p. Analogously to the proof in §2.4.2 this estimate can be easily derived by dividing the interval of integration $(0, t)$ into $(0, t/2)$ and $(t/2, t)$.

b) The estimate in (a) and Theorem 5.2.5 then imply that $w \in C^\infty(\mathbb{R}^n \times (0,T))$ and that w_j and its derivatives converge uniformly to w on any compact subset in $\mathbb{R}^n \times (0,T)$.
c) By Fatou's lemma (§7.1.2), the estimate in (a) is also valid for w in the case that $p < \infty$. (The case $p = \infty$ is obvious.)
d) Since w satisfies the integral equation (4.9), similarly to Proposition 4.3.2 we may show that w satisfies $\partial_t w - \Delta w + (v, \nabla) w = 0$ in $\mathbb{R}^n \times (0,T)$.
e) The continuity of w at $t = 0$ is a consequence of representation (4.9) and §4.2.4. The L^p-continuity follows from the fact that $w \in X$ is the limit of the approximating sequence $\{w_j\}$ in $C([0,T']; L^p(\mathbb{R}^n))$. ($w_j \in C([0,T']; L^p(\mathbb{R}^n))$ follows from $(*)$ of Exercise 7.3.)

(II) Uniqueness. Also here we may assume $s = 0$ without loss of generality. Let w_1 and w_2 be two solutions satisfying the assumptions in I and whose initial values are f. Since the equation is linear, we may assume $f = 0$ considering $w = w_1 - w_2$. Applying the nonnegativity-preserving principle (§2.3.8) to w and $-w$, it is obvious that $w \equiv 0$. However, here we give a uniqueness proof based on the integral representation (4.9).[1] By the properties of w_1 and w_2, for each $\eta > 0$ we have $\sup_{\eta<t<T} \|\partial_x^\alpha \partial_t^\ell w\|_\infty(t) < \infty$ and $\sup_{0<t<T} \|w\|_1(t) < \infty$ for each $\alpha \in \mathbb{N}_0^n$ and $\ell = 0, 1, 2, \ldots$. Hence, by a similar argument as in Theorem 4.3.4,

$$\tilde{w}(t) = -\int_0^t e^{(t-\tau)\Delta}((v,\nabla)w)(\tau)d\tau, \quad 0 < t < T,$$

is a weak solution of $\partial_t \tilde{w} - \Delta \tilde{w} = h, h = -(v,\nabla)w$ with zero initial value. (Here the key is to consider h as a known function.) It is easy to show that $\sup_{0<t<T} \|\tilde{w}\|_1(t) < \infty$. The fundamental uniqueness theorem (§4.4.2) and the remark thereafter then imply that $w = \tilde{w}$. Consequently, we have

$$\begin{aligned} w(t) &= -\int_0^t e^{(t-\tau)\Delta}((v,\nabla)w)(\tau)\ d\tau \\ &= -\int_0^t \operatorname{div}(e^{(t-\tau)\Delta}(vw)(\tau))d\tau + \int_0^t e^{(t-\tau)\Delta}(w \operatorname{div}\ v)(\tau)d\tau \end{aligned}$$

on $\mathbb{R}^n$ for $0 < t < T$. Applying the L^1-L^1 estimates from §1.1.2 and §1.1.3, we may estimate the L^1-norm of w as

$$\|w\|_1(t) \le C_1 \int_0^t \frac{1}{(t-\tau)^{1/2}} \|w\|_1(\tau)d\tau + C_2 \int_0^t \|w\|_1(\tau)d\tau$$

[1] The advantage of using the integral representation lies in the fact that we can obtain uniqueness also for v that are unbounded near initial time as given in the lemma at the end of this section.

for $0 < t < T$. Here C_j, $j = 1, 2$, depends only on $\sup_{0<t<T} \|v\|_\infty(t)$ and $\sup_{0<t<T} \|\nabla v\|_\infty(t)$. Setting $t = s$ and taking the supremum over $s \in (0, t)$ we obtain

$$J(t) \leq J(t)(2C_1 t^{1/2} + C_2 t), \quad J(t) := \sup_{0<s<t} \|w\|_1(s).$$

Choosing $\varepsilon > 0$ such that $2C_1\varepsilon^{1/2} + C_2\varepsilon < 1$, we see that $J(t) = 0$ on $t \leq \varepsilon$. Thus, $w(t) = 0$ follows on this interval. Next, consider $t = \varepsilon$ as initial time. By the same method we may show that $w(t) = 0$ on $\varepsilon \leq t \leq 2\varepsilon$. Repeating this argument yields $w(t) = 0$ on $0 \leq t < T$[1]. Hence, uniqueness is proved.

(II) is proved analogously by replacing $e^{(t-s)\Delta}$ by $U(t, s)$ in Proposition 4.3.2. □

Remark.

(i) Note that we proved the uniqueness of solutions of the integral equation (4.9). By similar arguments, we may prove the uniqueness of the following integral equation in the case of $\operatorname{div} v = 0$.

Lemma. *Let* $v^i \in C^1(\mathbb{R}^n \times (0, T))$, $i = 1, 2, \ldots, n$, *and set* $v = (v^1, \ldots, v^n)$. *We assume* $\operatorname{div} v = 0$ *on* $\mathbb{R}^n \times (0, T)$ *and that there exists a* $\beta \in [0, 1/2)$ *such that* $\sup_{0<t<T} t^\beta \|v\|_\infty(t) < \infty$. *For the initial value we suppose that* $f \in L^1(\mathbb{R}^n)$. *Then we have* $w \in C(\mathbb{R}^n \times (0, T))$ *satisfying* $\sup_{0<t<T} \|w\|_1(t) < \infty$ *and*

$$w(t) = e^{t\Delta} f - \int_0^t \operatorname{div}\,(e^{(t-\tau)\Delta}(vw)(\tau))d\tau, \quad 0 < t < T. \tag{4.10}$$

Furthermore, w *is unique (if it exists).*

It is an open problem whether w is unique in the lemma, *under the assumption* $\sup_{0<t<T} t^{1/2}\|v\|_\infty(t) < \infty$, which is the version of $\beta = 1/2$ in the lemma. In this case, we cannot consider that "the term vw is small compared with $e^{t\Delta}f$".

In the proof of (I) of the theorem, we need estimates of higher derivatives of v and w to show that the solution w satisfies the integral equation (4.9), since we use the integral expression of solutions with known transport terms similarly to §4.3.4. But in fact, we do not need such an estimate if we use the theory of evolution equations [Tanabe 1975].

---

[1] Similarly as in the proof of existence, it can be shown that $\|w\|_1(t) \leq K\int_0^t (t-\tau)^{-1/2}\|w\|_1(\tau)d\tau$, $K = C_1 + C_2 T^{1/2}$, $0 < t < T$. Iterating implies $\|w\|_1(t) \leq A^j(\sup_{0<\tau<T} \|w\|_1(\tau))/\Gamma((j+2)/2)$, $j = 1, 2, 3, \ldots$, $0 < t < T$, where $A := K\sqrt{\pi}T^{1/2}$. Letting $j \to \infty$ yields $w \equiv 0$ on $\mathbb{R}^n \times (0, T)$.

(ii) Observe that in the proof of uniqueness for the integral representation of solutions with inhomogeneous terms we use only

$$\sup_{\eta<t<T} \|\partial_x^\alpha \partial_t^\ell w\|_\infty(t) < \infty.$$

Here α is a multi-index, $\ell = 0, 1, 2, \ldots, s < \eta < T$, $w \in C(\mathbb{R}^n \times [s, T)) \cap C^\infty(\mathbb{R}^n \times (s, T))$, and $\sup_{s<t<T} \|w\|_1(t) < \infty$. Then for

$$\tilde{w} = \int_s^t U(t, \tau)h(\tau)d\tau \quad \text{on } \mathbb{R}^n, \quad s < t < T,$$

satisfying $\sup_{s<t<T} \|\tilde{w}\|_1(t) < \infty$, we obtain $\tilde{w} = w$ if w satisfies

$$\begin{cases} \partial_t w - \Delta w + (v, \nabla)w = h, & x \in \mathbb{R}^n,\ t \in (s, T), \\ w(x, s) = 0, & x \in \mathbb{R}^n. \end{cases}$$

Furthermore, for an initial value $f \in C(\mathbb{R}^n)$ with $\|f\|_1 < \infty, \|f\|_\infty < \infty$, the function w, satisfying the conditions above and

$$\begin{cases} \partial_t w - \Delta w + (v, \nabla)w = h, & x \in \mathbb{R}^n,\ t \in (s, T), \\ w(x, s) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

is uniquely represented as

$$w(t) = U(t, s)f + \int_s^t U(t, \tau)h(\tau)d\tau, \quad x \in \mathbb{R}^n,\ t \in (s, T),$$

by the principle of superposition. This fact is used in the proof of §2.4.3.

4.4.5 Fundamental Solutions and Their Properties

Here we give the definition of the fundamental solution of (H_v) that appears in §2.5.2 and discuss some of its properties. For a more detailed approach we refer to [Ladyženskaja Solonnikov Ural'ceva 1968], [Ito 1979], [John 1991], [Friedman 2005]. Assume that T and s_0 are given real numbers such that $T > s_0 \geq 0$, and let $v = (v^1, \ldots, v^n)$ be an $\mathbb{R}^n$-valued function defined on $\mathbb{R}^n \times (s_0, T)$.

Definition (Fundamental solution). *Assume that $\varGamma_v(x, t, y, \tau)$ is continuous on $\{(x, t, y, \tau);\ x, y \in \mathbb{R}^n, T > t > \tau \geq s_0\}$, integrable on $\mathbb{R}^n$ as a function of y, and that $\sup_{s_0<\tau<t<T} \int_{\mathbb{R}^n} |\varGamma_v(x, t, y, \tau)|dx < \infty$. The function $\varGamma_v$ is called a fundamental solution of the operator $\partial_t - \Delta + (v, \nabla)$ if for any bounded continuous function f on $\mathbb{R}^n$,*

$$w(x, t) = \int_{\mathbb{R}^n} \varGamma_v(x, t, y, \tau)f(y)dy \tag{4.11}$$

satisfies

$$\partial_t w - \Delta w + (v, \nabla)w = 0 \quad \text{in } \mathbb{R}^n \times (\tau, T)$$

in the weak sense, $w \in C(\mathbb{R}^n \times [\tau, T))$, and $w|_{t=\tau} = f$ (on $\mathbb{R}^n$).

In the case that v is identically zero we set $\Gamma_0(x,t,y,\tau) = G_{t-\tau}(x-y)$, which is then the fundamental solution of $\partial_t - \Delta$. (See §4.1.6, §4.2.4, and Exercise 7.2.)

We give a sufficient condition for the existence of such a fundamental solution.

Theorem (Unique existence theorem 1). *Let* $s_0 \geq 0$, $v^i \in C^1(\mathbb{R}^n \times (0,T))$, $i = 1,2,\ldots,n$, *and* $v = (v^1,\ldots,v^n)$. *For each* $T' \in (0,T)$ *assume that* $\sup_{0<t\leq T'} \|v\|_\infty(t) < \infty$, $\sup_{0<t\leq T'} \|\nabla v\|_\infty(t) < \infty$. *Then there exists a unique fundamental solution* Γ_v. *(Here* $T = \infty$ *is permitted.)*

In case of $s_0 > 0$ this theorem is a special version of [Ito 1979, Chapters 1, 2] (see §2.3.8). In case of $s_0 = 0$, $v^i(x,t)$ is not always C^1 at $t = s_0$. However, Γ_v is continuous at $\tau = 0$; hence w given by (4.11) is also continuous at $\tau = 0$. (This property is not used in this monograph. However, it can be easily verified by integral equation (4.12) given below, for example.) The following theorem is used in Lemma 2.5.2.

Theorem (Unique existence theorem 2). *Let* $s_0 \geq 0$, $v^i \in C^1(\mathbb{R}^n \times (0,T))$, $i = 1,2,\ldots,n$, *and* $v = (v^1,\ldots,v^n)$. *For each* $T' \in (0,T)$ *assume that* $\sup_{0<t\leq T'} \|v\|_\infty(t) < \infty$, $\operatorname{div} v = 0$ *in* $\mathbb{R}^n \times (0,T)$. *Then, there exists a unique fundamental solution* Γ_v. *(Here* $T = \infty$ *is permitted.)*

We sketch the proofs of these theorems. Similarly as in the case of the Gauss kernel, this fundamental solution $\Gamma_v(x,t,y,s)$ gives rise to a weak solution of the equation $\partial_t z - \Delta z + (v,\nabla)z = 0$ in $\mathbb{R}^n \times (s,T)$ with initial value $\delta(\cdot - y)$ at time $t = s$ (see Exercise 1.9). Indeed, fixing y and s and setting $z(x,t) = \Gamma_v(x,t,y,s)$, for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [s,T))$, z satisfies

$$0 = \varphi(y,s) + \int_s^T \int_{\mathbb{R}^n} z\{(\partial_t \varphi + \Delta\varphi) + \operatorname{div}(v\varphi)\}dx\, dt.$$

Supposing that v and z are sufficiently smooth for $t > s$ so that we may apply the representation formula in §4.4.3, we observe that z satisfies

$$z(x,t) = G_{t-s}(x-y) - \int_s^t \int_{\mathbb{R}^n} G_{t-\tau}(x-\zeta)(v(\zeta,\tau),\nabla)z(\zeta,\tau)d\zeta\, d\tau \quad (4.12)$$

for $x \in \mathbb{R}$ and $T > t > s$. The fundamental solution z is obtained by solving the integral equation (4.12) through successive approximation. The standard way is to set $u(x,t) = z(x,t) - G_{t-s}(x-y)$ and rewrite (4.12) as[1]

$$u(x,t) = -\int_s^t \int_{\mathbb{R}^n} G_{t-\tau}(x-\zeta)(v(\zeta,\tau),\nabla_\zeta)(u(\zeta,\tau) + G_{\tau-s}(\zeta-y))d\zeta\, d\tau,$$

[1] ∇_ζ and ∇_x denote gradients with respect to ζ and x, respectively.

and then determine u successively. This yields a solution u by

$$u(x,t) = \int_s^t \int_{\mathbb{R}^n} G_{t-\tau}(x-\zeta) K(\zeta,\tau,y,s) d\zeta \, d\tau,$$

$$K(\zeta,\tau,y,s) := \sum_{\ell=0}^{\infty} J_\ell(\zeta,\tau,y,s),$$

where

$$J_0(x,t,y,s) := -(v(x,t),\nabla_x) G_{t-s}(x-y),$$

$$J_\ell(x,t,y,s) := \int_s^t \left(\int_{\mathbb{R}^n} J_0(x,t,\zeta,\tau) J_{\ell-1}(\zeta,\tau,y,s) d\zeta \right) d\tau, \ \ \ell = 1,2,\ldots.$$

We have thus solved the integral equation (4.12). We also remark that assumptions $\sup_{0<t\le T'} \|\nabla v\|_\infty(t) < \infty$ and $\operatorname{div} v = 0$ [Ito 1979] are not required for the existence, but for the uniqueness of the solution of the integral equation (4.12). (See the proof of uniqueness in §4.4.4 (I).)

Multiplying both sides of (4.12) by a bounded continuous integrable function $f = f(y)$ on $\mathbb{R}^n$ and then integrating with respect to y, we obtain in view of (4.11) that

$$w(t) = e^{(t-s)\Delta} f - \int_s^t e^{(t-\tau)\Delta}((v,\nabla)w(\tau)) d\tau \quad \text{ in } \mathbb{R}^n, \ T > t > s.$$

By the assumptions on v in the unique existence theorem 2, we have $\sup_{0<t<T} \|w\|_1(t) < \infty$ (this follows by properties 1° and 3° below). Hence, if $\operatorname{div} v = 0$ and if $\tilde{w} \in C(\mathbb{R}^n \times (0,T))$ with $\sup_{0<t<T} \|\tilde{w}\|_1(t) < \infty$ satisfies the integral equation

$$\tilde{w}(t) = e^{(t-s)\Delta} f - \int_s^t \operatorname{div}(e^{(t-\tau)\Delta}(v\tilde{w})(\tau)) d\tau, \ T > t > s,$$

then by $(v,\nabla)w = \operatorname{div}(vw)$ the uniqueness of solutions of integral equation (4.10) (Lemma 4.4.4) implies $w \equiv \tilde{w}$. In other words,

$$\tilde{w}(x,t) = \int_{\mathbb{R}^n} \Gamma_v(x,t,y,s) f(y) dy, \ 0 \le s < t < T, \ x \in \mathbb{R}^n.$$

If v satisfies the assumptions of Theorem 4.4.4 for each $s \in (0,T)$, the fundamental solution exists for $s_0 > 0$. On the other hand, $U(t,s)f$ satisfies the integral equation (4.9), and its solution is unique by the proof in §4.4.4. By the fact that w also satisfies (4.9), we obtain

$$U(t,s)f = \int_{\mathbb{R}^n} \Gamma_v(x,t,y,s) f(y) dy \quad \text{ in } \mathbb{R}^n, \ s_0 < s < t < T.$$

Consequently, the solution (4.11) given in terms of the fundamental solution coincides with solutions obtained by different methods. It is also possible to reconstruct the fundamental solution if $U(t,s)$ is given. In fact, choose $f \in C_0(\mathbb{R}^n)$ satisfying $f \geq 0$, $\int_{\mathbb{R}^n} f dx = 1$, and $y_0 \in \mathbb{R}^n$. Next, consider $U(t,s)f_k$ for the scaled function $f_k(x) = k^n f(k(x-y_0)+y_0)$ and let $k \to \infty$. Then $U(t,s)f_k$ converges to the fundamental solution $\Gamma_v(x,t,y_0,s)$ according to Proposition 1.4.1. This idea is also used in the proof of (i) in Lemma 2.5.2.

We list the basic properties of the fundamental solution used in §2.5.2.[1]

1° (Positivity) $\Gamma_v(x,t,y,s) \geq 0,\ x,y \in \mathbb{R}^n,\ T > t > s \geq s_0$. (In fact it is known that $\Gamma_v(x,t,y,s) > 0$.)
2° $\int_{\mathbb{R}^n} \Gamma_v(x,t,y,s)dy = 1,\ x \in \mathbb{R}^n,\ T > t > s > s_0$.
3° If $\operatorname{div} v = 0$, then $\int_{\mathbb{R}^n} \Gamma_v(x,t,y,s)dx = 1,\ y \in \mathbb{R}^n,\ T > t > s \geq s_0$.

Here the positivity 1° follows from the preservation of nonnegativity (§2.3.8). Let us prove this under the assumption that w given by (4.11) is bounded on $\mathbb{R}^n \times (\tau, T)$. For f in (4.11), choose the scaled f_k as defined above. In view of §2.3.8, we obtain $w \geq 0$ on $\mathbb{R}^n \times [\tau, T)$. Letting $k \to \infty$ yields $\Gamma_v(x,t,y_0,\tau) \geq 0$. Here we assumed $x \in \mathbb{R}^n$, and $T > t > \tau \geq s_0,\ y_0 \in \mathbb{R}^n$. Property 2° follows from the fact that the solution is identically 1 if the initial value equals 1. Concerning 3°, observe that for w given by (4.11) Lemma 2.3.2 implies that $\int_{\mathbb{R}^n} w(x,t)dx$ is independent of t. Consequently,

$$\int_{\mathbb{R}^n} f(y)dy = \int_{\mathbb{R}^n} w(x,t)dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \Gamma_v(x,t,y,s)f(y)dy \right) dx.$$

Interchanging the order of integration and bringing all terms over to the left-hand side yields

$$\int_{\mathbb{R}^n} \left(1 - \int_{\mathbb{R}^n} \Gamma_v(x,t,y,s)dx \right) f(y)dy = 0.$$

Since $f \in C_0(\mathbb{R}^n)$ is arbitrary, the fundamental lemma of the calculus of variations (remark in Exercise 1.8) implies 3°.

4.5 Integration by Parts

In Proposition 1.2.2 we proved that

$$\frac{d}{dt} \int_{\mathbb{R}^n} u(x,t)dx = 0$$

for a solution u of the heat equation. In the proof we applied integration by parts in the form

[1] For v we assume the assumptions in the unique existence theorem 1 and/or 2 to hold.

$$\int_{\mathbb{R}^n} \Delta u \, dx = \int_{\mathbb{R}^n} \operatorname{div}(\nabla u) dx = 0 \tag{4.13}$$

for u supported in unbounded domains. This relation will be verified now for a fairly general class of functions u under the assumption that ∇u is small at space infinity. But we emphasize that this is no restriction on the solution of the heat equation. Since the time variable t will not play a role in the sequel, we consider functions $u \in C^2(\mathbb{R}^n)$. Moreover, we assume $\int_{\mathbb{R}^n} |\Delta u| dx < \infty$. Under these assumptions we have

$$\int_{\mathbb{R}^n} \Delta u \, dx = \lim_{R \to \infty} \int_{B_R} \Delta u \, dx,$$

where $B_R \subset \mathbb{R}^n$ denotes the open ball centered at the origin with radius R. Integration by parts on bounded domains (§4.5.3) yields in view of $\nabla 1 \equiv 0$ that

$$\int_{B_R} \Delta u \, dx = \int_{\partial B_R} \langle \mathbf{n}, \nabla u \rangle d\sigma - \int_{B_R} \langle \nabla 1, \nabla u \rangle dx = \int_{\partial B_R} \langle \mathbf{n}, \nabla u \rangle d\sigma,$$

where $\mathbf{n}$ denotes the outer unit normal vector at the boundary ∂B_R of B_R, and $d\sigma$ denotes the infinitesimal surface element of ∂B_R.

Consequently, if there exists at least one subsequence $R_j \to \infty$ such that

$$\int_{\partial B_{R_j}} |\langle \mathbf{n}, \nabla u \rangle| d\sigma \le \int_{\partial B_{R_j}} |\nabla u| d\sigma \to 0,$$

we obtain (4.13). The aim of the next subsections is to establish this rigorously. The integration by parts applied in §1.2.2 will then be a consequence of the following propositions.

4.5.1 An Example for Integration by Parts in the Whole Space

Proposition. *Let $u \in C^2(\mathbb{R}^n)$ be such that $\|\Delta u\|_1 < \infty$ and $\|\nabla u\|_1 < \infty$. Then*

$$\int_{\mathbb{R}^n} \Delta u \, dx = 0.$$

Proof. Note that Cavalieri's principle (or coarea formula) implies that

$$\int_{\mathbb{R}^n} |\nabla u| dx = \int_0^\infty \left(\int_{\partial B_R} |\nabla u| d\sigma \right) dR.$$

We set

$$J(R) = \int_{\partial B_R} |\nabla u| d\sigma.$$

The assumption $\|\nabla u\|_1 < \infty$ then implies that $J \in L^1(0,\infty)$.[1] We show that $J(R)$ is continuous in $R > 0$. In fact, by a coordinate transformation we may write

[1] $L^1(0,\infty) = L^1((0,\infty))$.

$$J(R) = R^{n-1} J_0(R), \quad J_0(R) = \int_{\partial B_1} |\nabla u|(Ry) d\sigma_y.$$

Then it remains to show that $J_0(R)$ is pointwise continuous in each $R_0 > 0$.[2] In view of $u \in C^2(\mathbb{R}^n)$, we obtain that $|\nabla u|(Ry)$ converges uniformly to $|\nabla u|(R_0 y)$ on $\{|y| = 1\}$ as $R \to R_0$. Since in this situation integral and limit commute (§7.1), the continuity of $R \mapsto J_0(R)$ in each $R_0 > 0$ follows. Hence, there exists a subsequence $R_j \to \infty$ such that

$$J(R_j) \to 0,$$

since otherwise it would yield $J \notin L^1(0, \infty)$ (Exercise 4.6). The arguments in the beginning of this section then imply (4.13). □

Since the solution u in Proposition 1.2.2 satisfies the assumptions above for $t > 0$ (see §1.1.3), Proposition 4.5.1 applies to the proof in §1.2.2. Observe that by the same arguments we may prove Proposition 1.2.2 for f provided that $\|f\|_1$ is finite. Moreover, similarly to §4.5.1, the following theorem can be proved.

4.5.2 A Whole Space Divergence Theorem

Proposition. *Let F be a C^1 vector field in $\mathbb{R}^n$. Assume that $\|F\|_1 < \infty$ and $\|\operatorname{div} F\|_1 < \infty$. Then*

$$\int_{\mathbb{R}^n} \operatorname{div} F \, dx = 0.$$

By this proposition the integration by parts used in §2.3 is verified. Finally, we recall integration by parts on bounded domains.

4.5.3 Integration by Parts on Bounded Domains

Proposition. *Let $D \subseteq \mathbb{R}^n$ be a bounded domain of class C^1. For $f, h \in C^1(\overline{D})$ we have*

$$\int_D \frac{\partial f}{\partial x_j} h \, dx = \int_{\partial D} f h \, n_j \, d\mathcal{H}^{n-1} - \int_D f \, \frac{\partial h}{\partial x_j} dx \quad (1 \leq j \leq n).$$

Here n_j denotes the jth component of the outer unit normal vector $\mathbf{n}$ at the boundary ∂D of D, and $d\mathcal{H}^{n-1}$ denotes the infinitesimal surface element of ∂D ($\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure). (In the case that D is a sphere, we write $d\sigma$ instead of $d\mathcal{H}^{n-1}$.)

[2] The subscript y in $d\sigma_y$ indicates that y is the integration variable.

Recall that D is of class C^1 if and only if for each $x \in \partial D$, $\mathbf{n}(x)$ is well defined and continuous. In particular, D is located on one side of its boundary ∂D. Obviously B_R is of class C^1 (in fact of class C^∞). Thus, this proposition applies to B_R.

The first place integration by parts of the above type usually appears is in elementary calculus courses of vector analysis (for example the Gauss divergence theorem). So it gives an impression that integration by parts needs vector analysis. However, integration by parts in multiple integrals is based on integration by parts for functions of one variable and formulated without the terminology of vector analysis. We give the above description for practical use. The above proposition can be proved in a similar spirit or it can be regarded as an easy consequence of the Gauss divergence theorem.

The more concise formulation used in vector analysis is then obtained as follows. We set $f = f_j \in C^1(\overline{D})$, $j = 1, 2, \ldots, n$, and $F = (f_1, \ldots, f_n)$. Summing up the above formula from $j = 1$ to n implies

$$\int_D h \operatorname{div} F dx = \int_{\partial D} \langle F, \mathbf{n} \rangle h d\mathcal{H}^{n-1} - \int_D \langle F, \nabla h \rangle dx.$$

If $\langle F, \mathbf{n} \rangle h$ is zero on ∂D, we have

$$\int_D h \operatorname{div} F dx = - \int_D \langle F, \nabla h \rangle dx.$$

These formulas of integration by parts are very important in the analysis of partial differential equations in general.

Exercises 4

4.1 (§4.1.1) There exists no $h \in L^1(\mathbb{R}^n)$ satisfying $h * f = f$ for any $f \in C_0(\mathbb{R}^n)$.

4.2 (§4.1.1) Prove the Hölder inequality.

4.3 (§1.1.3, §4.1.2) Prove L^p-L^q estimates for derivatives (1.6).

4.4 (§1.4.1, §4.2.5) Prove the convergence in §1.4.1 without using §4.2.4. Instead, prove it directly using coordinate transformations for the integral variables.

4.5 (§4.3.2) Prove the estimate $\sup_{0<t\le T} \|\partial_x^\alpha \partial_t^k w\|_\infty(t) < \infty$ for w in Proposition 4.3.2.

4.6 (§4.5.1) Suppose that J is an integrable continuous nonnegative function on $(0, \infty)$. Then prove that there exists a sequence $\{R_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} J(R_j) = 0$ as $j \to \infty$.

5

Compactness Theorems

In this section we prove the Ascoli–Arzelà-type compactness theorem introduced in §1.3.2. The theorem is fundamental since a variety of compactness results on various function spaces follows. Here we give a detailed proof, since the case that the domain of definition of functions is not compact is usually not contained in elementary course books.

The proof is elementary and standard and based on fundamental arguments, such as the diagonal argument. In §1.3.2 the domain of definition of functions is supposed to form a metric space. This assumption can be relaxed to a topological space.

5.1 Compact Domains of Definition

First we treat the case that the domain of definition is compact. There are several different ways to prove the *Ascoli–Arzelà theorem*. Here we prefer a direct proof that does not require a new concept.

5.1.1 Ascoli–Arzelà Theorem

Theorem. *Let M be a compact set (more precisely, compact topological space) and K a subset of $C(M)$, where $C(M)$ denotes the space of continuous functions on M. (Note that in the present case we may identify $C(M)$ with $C_\infty(M)$.) The set K is relatively compact in $C(M)$ if and only if K is bounded and equicontinuous in $C(M)$.*

Proof. We concentrate on the essential part, i.e., that boundedness and equicontinuity imply relative compactness. The converse direction is left to the reader (Exercise 5.1). If M is empty, then K is also empty, so we may assume that M is nonempty.

M.-H. Giga et al., *Nonlinear Partial Differential Equations*,
Progress in Nonlinear Differential Equations and Their Applications 79,
DOI 10.1007/978-0-8176-4651-6_5,

The First Step

Let A be an (at most) countable nonempty set in M, and $\{f_j\}_{j=1}^{\infty}$ a sequence in K. We show that there exists a subsequence of $\{f_j\}$ that converges at each point x in A. (In this step we use only the boundedness of K.)

Let $A = \{x_\ell\}_{\ell=1}^{\infty}$. (If A is finite, we set $x_\ell := x_m$ for $\ell \geq m$ and a certain $m \in \mathbb{N}$.) Since K is bounded, $\{f_j(x_1)\}_{j=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$. Hence there exists a subsequence $\{f_j^1\}$ of $\{f_j\}$ such that the sequence $\{f_j^1(x_1)\}_{j=1}^{\infty}$ converges to a limit, which we denote by $\tilde{f}(x_1)$. Similarly, for $k = 2, 3, \ldots$, there exists a subsequence $\{f_j^k\}$ of $\{f_j^{k-1}\}$ such that the sequence $\{f_j^k(x_k)\}_{j=1}^{\infty}$ converges to a limit $\tilde{f}(x_k)$. We set

$$g_j = f_j^j.$$

Since $\{f_j^j(x_k)\}_{j=1}^{\infty}$ is a subsequence of $\{f_j^k(x_k)\}_{j=1}^{\infty}$ for $j > k$, we obtain

$$\lim_{j\to\infty} f_j^j(x_k) = \lim_{j\to\infty} f_j^k(x_k) = \tilde{f}(x_k).$$

This yields

$$\lim_{j\to\infty} g_j(x_k) = \tilde{f}(x_k)$$

for all $k = 1, 2, \ldots$. The family $\{g_j\}_{j=1}^{\infty}$ is the desired subsequence of $\{f_j\}_{j=1}^{\infty}$. (Observe that $\{f_j^j\}$ represents the diagonal elements of the double sequence $\{f_\ell^k\}$, which justifies the terminology *diagonal argument*.)

The Second Step

For each natural number k there exist $N(k)$ points x_i^k, $1 \leq i \leq N(k)$, in M and open neighborhoods $V_{x_i^k}$ of each point satisfying the properties

$$\sup_{f\in K}\sup_{y\in V_{x_i^k}} |f(x_i^k) - f(y)| \leq \frac{1}{k}, \qquad \bigcup_{i=1}^{N(k)} V_{x_i^k} = M.$$

This easily follows from the equicontinuity of K (the definition in §1.3.1 is still valid if M is a topological space) and the compactness of M. In fact, the equicontinuity implies that for any point $x \in M$, there exists an open neighborhood V_k of x (which depends on k) such that

$$\sup_{y\in V_x}\sup_{f\in K} |f(x) - f(y)| \leq \frac{1}{k}.$$

Obviously, $\{V_x\}_{x\in M}$ is an open covering of M. Since M is compact, it is already covered by finitely many neighborhoods $\{V_{x_i^k}\}_{i=1}^{N(k)}$ of certain x_i^k, $1 \leq i \leq N(k)$. Hence the second step follows.

The Third Step

Let A be the set

$$A = \{x_i^k;\ 1 \le i \le N(k), \quad k = 1, 2, 3, \dots\},$$

where the x_i^k are defined in the second step. Assume that $\{h_j\}_{j=1}^\infty$ is a sequence in K converging pointwise on A. We show that $\{h_j\}_{j=1}^\infty$ converges uniformly on M.

By the assumption, for any $x \in A$, $\{h_j(x)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. In other words, we have

$$\lim_{\ell \to \infty} \sup_{j,m \ge \ell} |h_j(x) - h_m(x)| = 0. \tag{5.1}$$

For $z \in M$, we write

$$h_j(z) - h_m(z) = h_j(z) - h_j(x_i^k) + h_j(x_i^k) - h_m(x_i^k) + h_m(x_i^k) - h_m(z).$$

Taking the absolute value and using the triangle inequality, we obtain

$$\begin{aligned} &|h_j(z) - h_m(z)| \\ &\quad \le |h_j(z) - h_j(x_i^k)| + |h_j(x_i^k) - h_m(x_i^k)| + |h_m(x_i^k) - h_m(z)| \\ &\quad \le 2 \sup_{f \in K} |f(z) - f(x_i^k)| + |h_j(x_i^k) - h_m(x_i^k)|. \end{aligned}$$

Next we fix k and pick x_i^k and $V_{x_i^k}$ given in the second step such that $z \in V_{x_i^k}$. This yields

$$|h_j(z) - h_m(z)| \le \frac{2}{k} + |h_j(x_i^k) - h_m(x_i^k)| \le \frac{2}{k} + \sup_{1 \le i \le N(k)} |h_j(x_i^k) - h_m(x_i^k)|.$$

Taking the supremum with respect to $z \in V_{x_i^k}$ we arrive at

$$\sup_{z \in M} |h_j(z) - h_m(z)| \le \frac{2}{k} + \sup_{1 \le i \le N(k)} |h_j(x_i^k) - h_m(x_i^k)|.$$

Passing to the upper limit (§3.2.4 and §4.2.3) with respect to j and m, by (5.1), we obtain

$$\overline{\lim_{j,m \to \infty}} \sup_{z \in M} |h_j(z) - h_m(z)| = \lim_{\ell \to \infty} \sup_{j,m \ge \ell} \sup_{z \in M} |h_j(z) - h_m(z)| \le \frac{2}{k}.$$

Since $k \in \{1, 2, \dots\}$ was arbitrary, we have shown that

$$\overline{\lim_{j,m \to \infty}} \sup_{z \in M} |h_j(z) - h_m(z)| = 0$$

and therefore

$$\overline{\lim_{j,m\to\infty}} \|h_j - h_m\|_{\infty,M} = 0.$$

Hence, $\{h_j\}_{j=1}^\infty$ is a Cauchy sequence in $C(M)$ with respect to the L^∞-norm. Since for compact M the space $C(M)$ equipped with the L^∞-norm is complete (Exercise 1.6), $\{h_j\}_{j=1}^\infty$ converges to a function h in $C(M)$ with respect to that norm. (More precisely, $\lim_{j\to\infty} \|h_j - h\|_{\infty,M} = 0$.) In other words, $\{h_j\}_{j=1}^\infty$ converges uniformly to h in M. Hence we have proved the assertion of the third step. (Note that in Exercise 1.6, M is assumed to be a metric space. However, for the completeness of $C(M)$ it is sufficient to assume that M is a compact topological space.)

The Fourth Step

The relative compactness of K is now an easy consequence of the previous steps. In fact, let A be the countable subset of M that is defined in the third step. For a sequence $\{f_j\}_{j=1}^\infty$ in K, let $\{g_j\}_{j=1}^\infty \subset K$ be the subsequence constructed in the first step converging at each point of A. By the outcome of the third step, $\{g_j\}_{j=1}^\infty$ converges uniformly to a certain g in $C(M)$. Thus, $\{f_j\}_{j=1}^\infty$ contains the subsequence $\{g_j\}_{j=1}^\infty$ converging in $C(M)$, and therefore K is relatively compact in $C(M)$. □

Let $\{x_k\}$ be a bounded sequence in a Banach space X that is continuously embedded in a Banach space Y. It is of fundamental importance in which cases there exists a subsequence of $\{x_k\}$ that converges in the weaker norm of the space Y, or in other words, under which circumstances the embedding of X into Y is compact. The above theorem plays a key role in answering this sort of question.

5.1.2 Compact Embeddings

Let X and Y be Banach spaces, and let T be a *bounded linear operator* from X to Y. In other words, T maps any bounded subset of X to a bounded subset in Y. (Note that if T is linear, the boundedness is equivalent to the continuity of T.) If T maps any bounded subset of X even to a relatively compact subset in Y, we call T a *compact operator.* If $X \subset Y$ and the identity operator from X to Y is compact, then we call X *compactly embedded* in Y.

Example 1. Let M be a compact subset in $\mathbb{R}$. For $0 < \nu < 1$, we set

$$C^\nu(M) = \left\{ f : M \to \mathbb{R};\ f \in L^\infty(M),\ [f]_\nu := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\nu} < \infty \right\}.$$

Then, equipped with the norm $\|f\|_{C^\nu} = \sup_M |f| + [f]_\nu$ the space $C^\nu(M)$ is a Banach space (Exercise 5.2). $C^\nu(M)$ is called a *Hölder space.*

Set $X = C^\nu(M)$ and $Y = C(M)$. By virtue of the fact that any bounded set in X is bounded and equicontinuous in Y, the Ascoli–Arzelà theorem yields that the embedding $X \subset Y$ is compact (Exercise 5.3). More generally, it can be shown that the embedding $C^\nu(M) \subset C^\mu(M)$ for $0 \le \mu < \nu \le 1$ is compact.

Example 2. Let D be a bounded open set in $\mathbb{R}^n$ and let

$$C^k(\overline{D}) = \{f \in C(\overline{D}); \partial_x^\alpha f \text{ extends continuously to } \overline{D} \text{ for } |\alpha| \le k\}$$

for $k = 1, 2, \ldots$ as defined in §1.4. Equipped with the norm

$$\|f\|_{C^k} = \sup_{|\alpha| \le k} \sup_D |\partial_x^\alpha f|$$

this space is a Banach space. Here α is the multi-index introduced in §1.1.3 and $\partial_x^\alpha f$ denotes the partial derivative of f of order $|\alpha|$. Suppose that the boundary of D is sufficiently smooth. Then, by the Ascoli–Arzelà theorem (and since limit and derivative commute according to §4.1.5), we may show that $C^k(\overline{D})$ is compactly embedded in $C^h(\overline{D})$ if $k > h \ge 0$. (Observe that $\overline{D}$ is compact.) The reader is encouraged to verify this result at least for the easiest case in which D is convex and $k = 1$, $h = 0$, and $k = 2$, $h = 1$ (Exercise 5.4). For any positive noninteger ν $(0 < \nu < \infty)$, we also define the Hölder space of order ν by

$$C^\nu(\overline{D}) := \{f \in C^k(\overline{D}) : \ k < \nu < k+1, \ [\partial_x^\alpha f]_{\nu-k} < \infty, \ |\alpha| \le k\},$$

which forms a Banach space if equipped with the norm

$$\|f\|_{C^\nu} := \|f\|_{C^k} + \sum_{|\alpha|=k} [\partial_x^\alpha f]_{\nu-k}.$$

It is easy to see that $C^\nu(\overline{D}) \subset C^\mu(\overline{D})$ for $0 \le \mu \le \nu < \infty$. It can be proved again by employing the Ascoli–Arzelà theorem that this embedding is compact for $0 \le \mu < \nu < \infty$.

Finally, we remark that many results on compact embeddings in the framework of L^p and Sobolev spaces are based on the idea of the Ascoli–Arzelà theorem as well.

5.2 Noncompact Domains of Definition

In this section we prove the *Ascoli–Arzelà-type compactness theorem* stated in §1.3.2. (The converse direction is again left to the reader (Exercise 5.5).)

5.2.1 Ascoli–Arzelà-Type Compactness Theorem

Theorem. *Let M be a topological space such that there exists an exhausting sequence of compact sets (see the Definition in §1.3.1), and let K be a subset of $C_\infty(M)$. If K is bounded, equicontinuous, and having the equidecay property, then K is relatively compact in $C_\infty(M)$ and conversely.*

5.2.2 Construction of Subsequences

Let an exhausting sequence of compact sets $\{M_j\}_{j=1}^\infty$ of M and a sequence $\{f_m\}_{m=1}^\infty \subset K$ be given. Since M_1 is compact, by the theorem in §5.1.1, we may choose a subsequence $\{f_m^1\}$ such that f_m^1 uniformly converges to an f_1 in M_1. Similarly, if $\{f_m^{k-1}\}$ uniformly converges to f^{k-1} on M_{k-1}, then we may choose a subsequence $\{f_m^k\}$ of $\{f_m^{k-1}\}$ such that f_m^k uniformly converges to f^k on M_k, using the Ascoli–Arzelà theorem in §5.1.1. So, inductively we have constructed subsequences $\{f_m^k\}$ for $k = 1, 2, \ldots$, where we set $f_m^0 = f_m$. Since the limit functions satisfy $f^j = f^k$ $(j \geq k)$ on M_k, there exists a function f such that $f = f^k$ on M_k. Next, we may employ again the diagonal argument. More precisely, we set $g_m = f_m^m$. Then, by construction, $\{g_m\}_{m=1}^\infty$ converges uniformly to f on each M_k. So far we have not used the equidecay property for K. The remaining question is whether $\{g_m\}_{m=1}^\infty$ converges uniformly to f on $\bigcup_{k=1}^\infty M_k = M$. The following proposition gives a sufficient condition by involving the equidecay property of K. This will complete the "if-part" of Theorem 5.2.1.

5.2.3 Equidecay and Uniform Convergence

Proposition. *Let M be a topological space such that there exists an exhausting sequence of compact sets. Suppose that on each M_j the sequence $\{h_m\}_{m=1}^\infty \subset C_\infty(M)$ converges uniformly to a function $h : M \to \mathbb{R}$. If $H = \{h_m\}_{m=1}^\infty$ has the equidecay property, then $\{h_m\}_{m=1}^\infty$ converges uniformly to h on M. Moreover, $h \in C_\infty(M)$.*

Proof. First we show that $h \in C_\infty(M)$. Since h_m converges uniformly to h on M_j, h is continuous on each M_j. Since $M = \bigcup_{j=1}^\infty M_j$, we have $h \in C(M)$. Hence, if we show that

$$\sup_{x \in M \setminus M_j} |h(x)| \to 0 \quad (j \to \infty),$$

we obtain $h \in C_\infty(M)$. Obviously, h_m converges pointwise to h on M. This implies by the lower semicontinuity of sup (Exercise 5.6) that

$$\begin{aligned}\sup_{x \in M \setminus M_j} |h(x)| &= \sup_{x \in M \setminus M_j} |\lim_{m\to\infty} h_m(x)| \leq \varliminf_{m\to\infty} \sup_{x \in M \setminus M_j} |h_m(x)| \\ &\leq \sup_{f \in H} \sup_{x \in M \setminus M_j} |f(x)|.\end{aligned}$$

The equidecay property of H yields

$$\sup_{f \in H} \sup_{x \in M \setminus M_j} |f(x)| \to 0 \quad (j \to \infty),$$

which shows that $h \in C_\infty(M)$.

Next, we show that the convergence of $\{h_m\}_{m=1}^{\infty}$ to h is uniform on M. By the above estimate for h we have

$$\begin{aligned}\sup_M |h_m(x) - h(x)| &\le \sup_{M_j} |h_m(x) - h(x)| + \sup_{x\in M\setminus M_j} (|h_m(x)| + |h(x)|)\\ &\le \sup_{M_j} |h_m(x) - h(x)| + 2 \sup_{f\in H} \sup_{x\in M\setminus M_j} |f(x)|.\end{aligned}$$

Letting $m \to \infty$, the uniform convergence of h_m to h on each M_j gives us

$$\overline{\lim_{m\to\infty}} \|h_m - h\|_{\infty,M} \le 2 \sup_{f\in H} \sup_{x\in M\setminus M_j} |f(x)|.$$

By the equidecay property, the right-hand side of the above inequality converges to 0 for $j \to \infty$, and the proof is now complete. □

5.2.4 Proof of Lemma 1.3.6

We used the equidecay property only to show the uniform convergence of the subsequence on M. In order to obtain a subsequence converging uniformly on each M_j, the diagonal argument in §5.2.2 is sufficient. This proves Lemma 1.3.6.

The diagonal argument can also be applied to derive convergence of derivatives.

5.2.5 Convergence of Higher Derivatives

Theorem. *Let f_ℓ, $\ell = 1, 2, \dots,$ be C^∞ functions defined on an open set D in $\mathbb{R}^n$. Assume that $\sup_{\ell\ge 1} \sup_{x\in K} |\partial_x^\alpha f_\ell|$ is finite for each multi-index α and for each compact subset K in D. Then we have:*

(i) *There exist a subsequence $\{f_{\ell(i)}\}_{i=1}^{\infty}$ of $\{f_\ell\}_{\ell=1}^{\infty}$ and an $f \in C^\infty(D)$ such that for any multi-index α, $\partial_x^\alpha f_{\ell(i)}$ converges uniformly to $\partial_x^\alpha f$ on any compact subset in D as $i \to \infty$.*

(ii) *Assume that $\{f_\ell\}_{\ell=1}^{\infty}$ converges pointwise to a function $\tilde{f}$ on D as $\ell \to \infty$. Then $\tilde{f} \in C^\infty(D)$, and for each multi-index α, $\partial_x^\alpha f_\ell$ converges uniformly to $\partial_x^\alpha \tilde{f}$ on each compact subset in D as $\ell \to \infty$.*

Proof. Let $B_r(a)$ be the open ball centered at $a \in \mathbb{R}^d$ with radius $r > 0$. Let $\mathcal{B}$ denote the collection of all $B_r(a)$ with rational radius r and center $a = (a_1, \dots, a_d)$ such that each a_i is rational for $1 \le i \le d$ and the closure $\overline{B_r(a)}$ is contained in D. Note that a sequence of smooth functions uniformly converges on each element of $\mathcal{B}$ if and only of it converges uniformly on each compact subset. By Exercise 5.4, for each $B_r(a) \in \mathcal{B}$ and each multi-index α any subsequence of $\{\partial_x^\alpha f_\ell\}$ contains a subsequence that converges uniformly to a certain $g_{\alpha,r,a}$ depending on α, r, a. First we fix α. Since $\mathcal{B}$ is countable, by applying the diagonal argument as used in §5.2.2, we can show that for each α any subsequence of $\{\partial_x^\alpha f_\ell\}$ contains a subsequence converging uniformly to a $g_\alpha \in C(D)$ on each $B_r(a)$ in $\mathcal{B}$ (or equivalently, on each compact subset).

Next, we apply the diagonal argument with respect to the multi-index α. Let $\{f_\ell^1\}$ be the convergent subsequence of $\{f_\ell\}_{\ell=1}^\infty$ and let $\{\partial_x^\alpha f_\ell^2\}$ be the convergent subsequence of $\{\partial_x^\alpha f_\ell^1\}$ for $|\alpha| = 1$. More generally, let $\{\partial_x^\alpha f_\ell^{k+1}\}$ be the convergent subsequence of $\{\partial_x^\alpha f_\ell^k\}$ for $|\alpha| = k$. Thus, inductively we have constructed a double sequence $\{f_\ell^k\}$. Now, set $h_\ell = f_\ell^\ell$. Then for each α, $\partial_x^\alpha h_\ell$ converges uniformly to g_α on each compact subset of D. Since limit and differential commute by Lemma 4.1.5, we have $\partial_x^\alpha g_0 = g_\alpha$, where g_0 expresses g_α for $|\alpha| = 0$. Hence, the subsequence $\{h_\ell\}$ of $\{f_\ell\}$ and g_0 represent the desired subsequence $\{f_{\ell(i)}\}$ and f in (i), respectively.

For (ii), observe that now the limit of any convergent subsequence of $\{f_\ell\}_{\ell=1}^\infty$ is independent of the choice of subsequence. Consequently, f_ℓ converges uniformly to $\tilde{f} \in C^\infty(D)$ on each compact subset of D (see Exercise 1.4). For the same reason, $\{\partial_x^\alpha f_\ell\}_{\ell=1}^\infty$ converges uniformly to $\partial_x^\alpha \tilde{f}$ on each compact subset of D. □

Exercises 5

5.1 (§5.1.1) Let M be a compact set and K a subset of $C(M)$. Show that if K is relatively compact in $C(M)$, then K is bounded and equicontinuous in $C(M)$.

5.2 (§5.1.2) Let M be a compact subset in $\mathbb{R}^n$, and $0 < \nu < 1$. Show that the Hölder space $C^\nu(M)$ with norm $\|f\|_{C^\nu}$ is a Banach space.

5.3 (§5.1.2) Let M and ν be as in Exercise 5.2. Show that $C^\nu(M)$ is compactly embedded in $C(M)$.

5.4 (§5.1.2) Let D be a bounded convex open set in $\mathbb{R}^n$. Show that $C^1(\overline{D})$ is compactly embedded in $C(\overline{D})$. Furthermore, show that $C^2(\overline{D})$ is compactly embedded in $C^1(\overline{D})$.

5.5 (§5.2.1) Let M be a topological space with an exhausting sequence of compact sets. Show that any relatively compact subset K in $C_\infty(M)$ is bounded, equicontinuous, and with the equidecay property in $C_\infty(M)$.

5.6 (§5.2.3) Let h_m, $m = 1, 2, \dots$, be real-valued functions defined on a set Z. Show that the supremum is lower semicontinuous, i.e., that

$$\sup_{x\in Z} \varliminf_{m\to\infty} h_m(x) \le \varliminf_{m\to\infty} \sup_{x\in Z} h_m(x).$$

6

Calculus Inequalities

In this section we introduce the Nash inequality and its generalized version, the Gagliardo–Nirenberg inequality. Roughly speaking, these inequalities provide estimates for an integral of a function by its derivatives, a tool that is very helpful not only in the analysis of the vorticity equations as demonstrated in Chapter 2, but in the analysis of nonlinear PDE in general.

There exist various methods to prove these inequalities. Here we will present an approach that relies on estimates for the solution of the heat equation (§1.1.2, §1.1.3). Here the special case of the Gagliardo–Nirenberg inequality known as the Sobolev inequality appears to be exceptional. We will present two methods of proof. The first one is based on the Hardy–Littlewood–Sobolev inequality (§6.3.3), whereas the second one is based on merely the Hölder inequality (§6.3.4). The Hardy–Littlewood–Sobolev inequality, in turn, will be obtained by applying the Marcinkiewicz interpolation theorem and again estimates for the solution of the heat equation. In order to keep our approach self-contained, we give also a proof of this result. In fact, the Marcinkiewicz interpolation theorem is a fairly general result with a wide range of applications. Here it will also be applied in order to prove the Calderón–Zygmund inequality.

In this section a number of integral operators will appear. Note that it is one of our main intentions to keep the book readable to readers not necessarily familiar with Lebesgue integration theory. Therefore, the integrals in the main statements are always to be understood as Riemann integrals. This is the reason that we will frequently require the functions that appear to be continuous or, depending on the situation, even C^k-functions.

6.1 The Gagliardo–Nirenberg Inequality and the Nash Inequality

The simplest method to estimate a function by its derivative is to use the fundamental theorem of calculus. That is, a C^1-function u of one variable

M.-H. Giga et al., *Nonlinear Partial Differential Equations*,
Progress in Nonlinear Differential Equations and Their Applications 79,
DOI 10.1007/978-0-8176-4651-6_6,

with compact support can be represented by

$$u(x) = \int_{-\infty}^{x} \frac{du}{dx}(z)dz, \quad x \in \mathbb{R}.$$

Hence we can estimate

$$|u(x)| \le \int_{-\infty}^{x} \left| \frac{du}{dx}(z) \right| dz \le \left\| \frac{du}{dx} \right\|_1, \quad x \in \mathbb{R},$$

which yields

$$\|u\|_\infty \le \left\| \frac{du}{dx} \right\|_1. \tag{6.1}$$

Using this inequality, we further obtain

$$\|u\|_2^2 = \int_{-\infty}^{\infty} u^2 dx \le \|u\|_1 \|u\|_\infty \le \|u\|_1 \left\| \frac{du}{dx} \right\|_1.$$

The Gagliardo–Nirenberg inequality can be regarded as a generalized version of these two inequalities.

6.1.1 The Gagliardo–Nirenberg Inequality

Theorem. *Assume that* $1 \le p, q, r \le \infty$ *and* $\sigma \in [0, 1]$ *such that*

$$\frac{1}{p} = \sigma \left(\frac{1}{r} - \frac{1}{n} \right) + (1 - \sigma) \frac{1}{q}. \tag{6.2}$$

Here n *is a natural number that expresses the dimension of space. We also assume, if* $n \ge 2$*, that*

$$p \ne \infty \quad \text{or} \quad r \ne n. \tag{6.3}$$

Then there exists a positive constant $C = C(p, q, r, n)$ *depending only on* p, q, r, n *such that*

$$\|u\|_p \le C \|u\|_q^{1-\sigma} \|\nabla u\|_r^{\sigma}, \tag{6.4}$$

for all $u \in C_0^1(\mathbb{R}^n)$ $(= C^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n))$*. If* $n = 1$, $p = \infty$, $r = 1$*, inequality (6.4) holds with* $C = 1$*.*

Furthermore, the following holds: If $p \ge q$, $r < \infty$*, inequality (6.4) holds for any* $u \in C^1(\mathbb{R}^n)$ *with* $\|u\|_q < \infty$ *(for* $n = 1$, $q = \infty$, $r = 1$*, instead of* $\|u\|_q < \infty$*, we assume that* $u \in C_\infty(\mathbb{R})$*.) If* $p \ge q$, $r = \infty$*, inequality (6.4) holds for any* $u \in C^1(\mathbb{R}^n)$ *with* $\|u\|_q < \infty$ *and* $\partial_{x_\ell} u \in C_\infty(\mathbb{R}^n)$, $\ell = 1, 2, \dots, n$*. If* $p < q$*, inequality (6.4) holds for any* $u \in C^1(\mathbb{R}^n)$ *with* $\|u\|_s < \infty$ *for some* $1 \le s < \infty$ *such that* $1/s \ge 1/r - 1/n$*. (Observe that the right hand side of (6.4) may attain* $+\infty$*.)*

Inequality (6.4) is called the *Gagliardo–Nirenberg inequality.* In the next subsection we will see that the *Nash inequality* is a special version of this inequality. We remark that (6.2) plays the same role as the balancing relation (4.1) in the Young inequality (§4.1.1). In the case of $\sigma = 1$, inequality (6.4) is called the *Sobolev inequality.* In the case of $\sigma = 0$, we have $p = q$, and (6.4) is obvious.

Remark. For readers familiar with Lebesgue integration theory and differentiation in the sense of distributions we make the following remark: In fact, inequality (6.4) holds for any u satisfying $u \in L^q(\mathbb{R}^n)$, $\partial_{x_\ell} u \in L^r(\mathbb{R}^n)$, $\ell = 1, 2, \ldots, n$, if $p \geq q$, $r < \infty$, $q < \infty$. In particular, (6.4) implies $u \in L^p(\mathbb{R}^n)$. Here $\partial_{x_\ell} u$ expresses the derivative in the sense of distributions.

6.1.2 The Nash Inequality

Theorem. *There exists a positive constant $C = C(n)$ such that for any $u \in C^1(\mathbb{R}^n)$ satisfying $\|u\|_1 < \infty$ we have that*

$$\|u\|_2^2 \leq C\|u\|_1^{\frac{4}{n+2}} \|\nabla u\|_2^{2-\frac{4}{n+2}}. \tag{6.5}$$

Note that for $p = 2$, $\sigma = 1 - 2/(n+2)$, $q = 1$, and $r = 2$ in (6.4) we arrive at (6.5). Nash proved this inequality in order to derive estimates for the fundamental solution of diffusion equations with discontinuous coefficients. His original proof [Nash 1958] relies on Fourier transformation. His paper had a deep impact on studies of nonlinear elliptic and parabolic equations.

In his paper Nash even obtained an explicit value for C in inequality (6.5), which is

$$C = 2\left(\frac{|S^{n-1}|}{n(2\pi)^n}\right)^{\frac{2}{n+2}}.$$

Here $|S^{n-1}|$ denotes the surface of the n-dimensional unit ball. (In terms of the gamma function it can be written as $|S^{n-1}| = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$. See §6.3.1 for a proof.) In [Carlen Loss 1993] the optimal constant (i.e., the smallest possible constant) is obtained for which (6.5) is valid.

First we will prove the Nash inequality (6.5) by utilizing estimates for the solution of the heat equation and its differentials. Of course, the Nash inequality is automatically proved if we have proved the Gagliardo–Nirenberg inequality. However, the proof of the Nash inequality is easier, and therefore we will prove this inequality first.

6.1.3 Proof of the Nash Inequality

The First Step

First we assume that the support of $u \in C^1(\mathbb{R}^n)$ is compact. Let $e^{t\Delta}u = G_t * u$ be the solution of the heat equation with initial value u. Adopting the notation in §4.3, by Lemma 4.3.2, for each $t > 0$ the equation

$$e^{t\Delta}u - u = \int_0^t \left(\frac{d}{d\tau} e^{\tau\Delta}u\right) d\tau = \int_0^t \Delta e^{\tau\Delta}u \; d\tau \tag{6.6}$$

holds, regarded as an equality of functions on $\mathbb{R}^n$. Hence, for $t > 0$ we obtain

$$\|u\|_2^2 = \int_{\mathbb{R}^n} u \; u \; dx = \int_{\mathbb{R}^n} u \; e^{t\Delta}u \; dx - \int_0^t \int_{\mathbb{R}^n} u \; \Delta e^{\tau\Delta}u \; dx \; d\tau.$$

(As in Chapter 4, the integrand of the second term on the right-hand side is continuous and compactly supported in spatial variables, which allows for interchanging the order of integration (see §7.2.2).) We denote the first term and the second term on the right-hand side by J_1 and J_2, respectively. By the Hölder inequality we have that

$$|J_1| \le \|u\|_1 \|e^{t\Delta}u\|_\infty, \quad t > 0.$$

Applying (1.4) then yields

$$|J_1| \le (4\pi t)^{-n/2} \|u\|_1^2, \quad t > 0.$$

For the second term we use integration by parts (§4.5.3) and the fact that $\nabla e^{t\Delta}u = e^{t\Delta}\nabla u$ (§4.1.6). This gives us

$$-\int_{\mathbb{R}^n} u\Delta e^{\tau\Delta}u \; dx = \int_{\mathbb{R}^n} \langle \nabla u, \nabla e^{\tau\Delta}u\rangle dx = \int_{\mathbb{R}^n} \langle \nabla u, e^{\tau\Delta}\nabla u\rangle dx.$$

Thus, we obtain

$$|J_2| \le \int_0^t \|\nabla u\|_2 \|e^{\tau\Delta}\nabla u\|_2 d\tau, \quad t > 0,$$

since $|\langle \; \nabla u, \; e^{\tau\Delta}\nabla u \; \rangle| \le |\nabla u| \; |e^{\tau\Delta}\nabla u|$ by the Schwarz inequality (§4.1.1). From (1.5) we infer that $\|e^{\tau\Delta}\nabla u\|_2 \le \|\nabla u\|_2$, which implies that

$$|J_2| \le \int_0^t \|\nabla u\|_2^2 d\tau = t\|\nabla u\|_2^2, \quad t > 0.$$

Summarizing, we deduce

$$\|u\|_2^2 \le (4\pi t)^{-n/2} \|u\|_1^2 + t\|\nabla u\|_2^2 \tag{6.7}$$

for all $t > 0$. In order to prove (6.5) it remains to choose t in a suitable way. In fact, if we set $(4\pi t)^{-n/2}\|u\|_1^2 = t\|\nabla u\|_2^2$ or equivalently

$$t^{1+n/2} = \frac{\|u\|_1^2}{\|\nabla u\|_2^2 (4\pi)^{n/2}},$$

then we see that the two terms on the right-hand side of (6.7) coincide. More precisely, (6.7) turns into

$$\|u\|_2^2 \le 2\left(\frac{\|u\|_1^2}{\|\nabla u\|_2^2(4\pi)^{n/2}}\right)^{\frac{2}{n+2}}\|\nabla u\|_2^2$$

$$= 2\left(\frac{1}{2^n\pi^{n/2}}\right)^{\frac{2}{n+2}}\|u\|_1^{\frac{4}{n+2}}\|\nabla u\|_2^{2-\frac{4}{n+2}}.$$

This proves (6.5). However, the constant C is larger than the constant that is obtained by the arguments of Nash in [Nash 1958].

The Second Step

Next we prove (6.5) for $u \in C^1(\mathbb{R}^n)$ that are not necessarily compactly supported. We just assume that the norms $\|u\|_2$, $\|u\|_1$, and $\|\nabla u\|_2$ are finite. As in the proof of Theorem 4.4.2, we approximate u by cutting off large values of $|x|$. For this purpose pick $\theta \in C^\infty(\mathbb{R})$ such that $\theta(y) = 1$ for $y \le 1$, $\theta(y) = 0$ for $y \ge 2$, and $0 \le \theta \le 1$. Next we define functions θ_j, $j = 1, 2, \ldots$, on $\mathbb{R}^n$ such that $\theta_j(x) = \theta(|x|/j)(x \in \mathbb{R}^n)$, and set

$$u_j(x) := \theta_j(x)u(x), \quad x \in \mathbb{R}^n.$$

Then we have that

$$\lim_{j\to\infty}\|u_j - u\|_2 = 0, \quad \lim_{j\to\infty}\|u_j - u\|_1 = 0, \quad \lim_{j\to\infty}\|\nabla(u_j - u)\|_2 = 0$$

(see Exercise 6.1). This in particular implies that

$$\lim_{j\to\infty}\|u_j\|_2 = \|u\|_2, \quad \lim_{j\to\infty}\|u_j\|_1 = \|u\|_1, \quad \lim_{j\to\infty}\|\nabla u_j\|_2 = \|\nabla u\|_2.$$

On the other hand, by the first step we know that (6.5) is satisfied for u_j with the same constant $C > 0$ for all $j = 1, 2, \ldots$. Letting $j \to \infty$ we see that (6.5) holds for all $u \in C^1(\mathbb{R}^n)$, provided that $\|u\|_2$, $\|u\|_1$, and $\|\nabla u\|_2$ are finite.

The Third Step

Finally, we prove (6.5) for general u satisfying $\|u\|_1 < \infty$. Observe that in the case that $\|\nabla u\|_2 = \infty$, relation (6.5) is obvious. Hence, we may assume that $\|\nabla u\|_2$ is finite. For a positive number t, we define the function u^t on $\mathbb{R}^n$ as

$$u^t := e^{t\Delta}u = G_t * u.$$

Then by §4.1.6, $u^t \in C^\infty(\mathbb{R}^n)$. Furthermore, since $\|u\|_1$ is finite, by the estimates derived in §1.1.2 we obtain that

$$\|u^t\|_1 \le \|u\|_1 < \infty, \quad \|u^t\|_2 \le (4\pi t)^{-\frac{n}{4}}\|u\|_1 < \infty$$

for $t > 0$. In view of $\nabla u^t = e^{t\Delta}\nabla u$ (§4.1.6), once more by (1.5) we have that

$$\|\nabla u^t\|_2 \le \|\nabla u\|_2 < \infty, \quad t > 0.$$

By the finiteness of $\|u^t\|_1$, $\|u^t\|_2$, and $\|\nabla u^t\|_2$, the second step now implies that

$$\|u^t\|_2^2 \le C\|u^t\|_1^{\frac{4}{n+2}} \|\nabla u^t\|_2^{2-\frac{4}{n+2}}, \quad t > 0.$$

Thus, the above estimates for u^t yield

$$\|u^t\|_2^2 \le C\|u\|_1^{\frac{4}{n+2}} \|\nabla u\|_2^{2-\frac{4}{n+2}}, \quad t > 0.$$

Since u is continuous, u^t converges pointwise to u in $\mathbb{R}^n$ as $t \to 0$. (Note that this does not follow directly from the corollary in §4.2.4, since u may be unbounded. But the pointwise convergence can be shown, for instance, using similar arguments as in Exercise 4.4.) By an application of Fatou's lemma (§7.1.2), i.e., $\|u\|_2^2 \le \underline{\lim}_{t\to 0} \|u^t\|_2^2$, we then arrive at (6.5) in view of the fact that the right-hand side is independent of t. Hence, the Nash inequality is proved. □

Remark. As mentioned earlier, the Nash inequality is a special case of the Gagliardo–Nirenberg inequality in §6.1.1. In fact, if ∇u is realized in the sense of distributions, the Nash inequality holds under the assumption that $u \in L^1(\mathbb{R}^n)$ and $\nabla u \in L^2(\mathbb{R}^n)$. The proof is the same in the first and second steps. The third step is also analogous except for the argument that u_t converges pointwise to u on $\mathbb{R}^n$ as $t \to 0$. Here one has to use general theory for the Lebesgue integral. As pointed out in Exercise 7.3, for $u \in L^1(\mathbb{R}^n)$ we have that $\lim_{t\to 0} \|u^t - u\|_1 = 0$. Using this fact we may choose a suitable subsequence u^{t_j} of u^t converging to u almost everywhere as $t_j \to 0$. (For example, see [Ito 1963, Theorem 22.2], [Rudin 1987, Theorem 3.12].) Hence, again by Fatou's lemma, applied to u^{t_j} instead of u^t, inequality (6.5) follows.

6.1.4 Proof of the Gagliardo–Nirenberg Inequality (Case of $\sigma < 1$)

Here we present a proof of the Gagliardo–Nirenberg inequality (6.4) by modifying the proof of the Nash inequality in a suitable way. But note that this method excludes the case $\sigma = 1$, which is the special case known as the Sobolev inequality. We will give a proof of the Sobolev inequality ($\sigma = 1$) at the end of this section.

In the proof we make use of the following characterization by *duality* of the L^p-norm of u:

$$\|u\|_p = \sup\left\{ \int_{\mathbb{R}^n} u\ (x)\ \varphi\ (x)\ dx;\ \|\varphi\|_{p'} \le 1,\ \varphi \in C_0^\infty(\mathbb{R}^n) \right\}. \tag{6.8}$$

Here p' is the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. Let us show that (6.8) is valid for all $u \in L^p(\mathbb{R}^n)$ and $1 \le p \le \infty$. By the Hölder inequality it is clear that the right-hand side of (6.8) is smaller than the left hand side. To see the converse direction we choose a suitable function $\varphi \in L^{p'}(\mathbb{R}^n)$. First note that for the case $u \equiv 0$ the formula is obvious. Hence we may assume that $\|u\|_p \neq 0$. Therefore the function $\varphi = v/\|v\|_{p'}$, where $v = |u|^{p-2}u$, is defined almost everywhere for $1 \leq p < \infty$. Observe that φ satisfies $\varphi \in L^{p'}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} u(x)\varphi\,(x)\,dx = \|u\|_p$, and that $\|\varphi\|_{p'} = 1$. Next we approximate φ by elements of $C_0^\infty(\mathbb{R}^n)$. Recall from Exercise 7.3 that $C_0^\infty(\mathbb{R}^n)$ is *dense* in $L^{p'}(\mathbb{R}^n)$ for $1 \leq p' < \infty$, that is, the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the topology of $L^{p'}(\mathbb{R}^n)$ is exactly the space $L^{p'}(\mathbb{R}^n)$. Therefore (6.8) is proved for the case that $1 < p' < \infty$.

In this book we will not carry out the proof of (6.8) for the cases $p' = 1$ or $p' = \infty$. Note that in the case $p' = \infty$ the space $C_0^\infty(\mathbb{R}^n)$ is not dense in $L^{p'}(\mathbb{R}^n)$, whereas the problem in the case $p = \infty$ is to give a suitable definition of v. We recommend to the reader to prove this as an exercise in Lebesgue integration theory and functional analysis.

Now we prove the Gagliardo–Nirenberg inequality for the case that $0 < \sigma < 1$. As in the proof of the Nash inequality, the essential work is done if we have proved (6.4) for C^1-functions with compact support.

The First Step ($p > q$ and $p \geq r$)

Employing (6.6) and (6.8) we obtain for $t > 0$ that

$$\begin{aligned}\|u\|_p &= \sup\left\{\int_{\mathbb{R}^n} u\left(e^{t\Delta}\varphi - \int_0^t \Delta e^{\tau\Delta}\varphi\,d\tau\right)dx;\ \|\varphi\|_{p'} \leq 1,\ \varphi \in C_0^\infty(\mathbb{R}^n)\right\}\\ &\leq \sup\{I_1(\varphi) + I_2(\varphi);\ \|\varphi\|_{p'} \leq 1,\ \varphi \in C_0^\infty(\mathbb{R}^n)\},\end{aligned}$$

where

$$I_1(\varphi) = \left|\int_{\mathbb{R}^n} u\ e^{t\Delta}\varphi\,dx\right|, \quad I_2(\varphi) = \left|\int_0^t \int_{\mathbb{R}^n} u\ \Delta e^{\tau\Delta}\varphi\,dx\,d\tau\right|.$$

To I_1 we apply the Hölder inequality (§4.1.1, Exercise 4.2), which yields

$$I_1(\varphi) \leq \|u\|_q \|e^{t\Delta}\varphi\|_{q'}, \quad t > 0.$$

Furthermore, since $q' > p'$, we may apply the $L^{q'}$-$L^{p'}$ estimate (1.5) for the solution of the heat equation to the result

$$I_1(\varphi) \leq C_1\ t^{-\frac{n}{2}\left(\frac{1}{p'}-\frac{1}{q'}\right)} \|u\|_q \|\varphi\|_{p'}, \quad t > 0.$$

Observe that here and in the sequel the constants C_j, $j = 1, 2, \ldots$, depend only on p, q, r, and the dimension n. In order to estimate I_2 we use integration by parts (§4.5.3) and the Hölder inequality. This gives us

$$I_2(\varphi) = \left|\int_0^t \int_{\mathbb{R}^n} \langle \nabla u,\ \nabla e^{\tau\Delta}\varphi\rangle\,dx\,d\tau\right| \leq \int_0^t \|\nabla u\|_r \|\nabla e^{\tau\Delta}\varphi\|_{r'} d\tau$$

for $t > 0$. By virtue of $r' \geq p'$ the $L^{r'}$-$L^{p'}$ estimate (1.8) for derivatives of the solution of the heat equation then implies

$$\|\nabla e^{\tau\Delta}\varphi\|_{r'} \leq C_2 \tau^{-\left(\frac{1}{2}+\alpha\right)} \|\varphi\|_{p'}$$

with

$$\alpha = \frac{n}{2}\left(\frac{1}{p'} - \frac{1}{r'}\right) = \frac{n}{2}\left(\frac{1}{r} - \frac{1}{p}\right).$$

A necessary and sufficient condition for the right-hand side to be integrable over τ on the interval $(0, t)$ is

$$\frac{1}{2} + \alpha < 1,$$

which is equivalent to

$$\frac{1}{p} > \frac{1}{r} - \frac{1}{n}.$$

By our assumptions $p > q$ and $0 < \sigma < 1$, this condition is satisfied and we obtain for $t > 0$ that

$$\begin{aligned} I_2(\varphi) &\leq C_2 \|\nabla u\|_r \|\varphi\|_{p'} \int_0^t \tau^{-\left(\frac{1}{2}+\alpha\right)} d\tau \\ &= C_3 \|\nabla u\|_r \|\varphi\|_{p'} t^{\frac{1}{2}-\alpha}, \end{aligned}$$

where $C_3 = C_2 / \left(\frac{1}{2} - \alpha\right)$. Summarizing then results in

$$\|u\|_p \leq C_1 \; t^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u\|_q + C_3 t^{\frac{1}{2}-\alpha} \|\nabla u\|_r, \quad t > 0.$$

Analogously to the proof of the Nash inequality, we choose t suitably so that the first term is equal to the second term of the right-hand side. This implies (6.4).

The Second Step (General exponents)

Note that the following two cases are not treated in the first step:

(i) $q < p < r$;
(ii) $r \leq p \leq q$.

Observe also that in view of condition (6.2), it is impossible to apply the method in the first step to the cases $p < r$ and $p \leq q$. Here we have to argue in a different way.

In the case of (i) the Hölder inequality implies that

$$\|u\|_p \leq \|u\|_q^{\rho} \|u\|_r^{1-\rho}, \quad \frac{1}{p} = \frac{\rho}{q} + \frac{1-\rho}{r}, \quad 0 \leq \rho \leq 1$$

(Exercise 6.2). By $r > q$ we have that

$$\frac{1}{r} - \frac{1}{n} < \frac{1}{r} < \frac{1}{q}.$$

Hence, there exists a $\sigma_1 \in (0,1)$ such that

$$\frac{1}{r} = \sigma_1 \left(\frac{1}{r} - \frac{1}{n} \right) + (1 - \sigma_1) \frac{1}{q}.$$

Since $r > q$, we may apply the estimate obtained in the first step, which gives us

$$\|u\|_r \le C_4 \|u\|_q^{1-\sigma_1} \|\nabla u\|_r^{\sigma_1}.$$

Applying this to the estimate above results in

$$\|u\|_p \le C_5 \|u\|_q^{\rho} \|u\|_q^{(1-\sigma_1)(1-\rho)} \|\nabla u\|_r^{\sigma_1(1-\rho)}$$

with $C_5 = C_4^{1-\rho}$. Setting $\rho = (1/p - 1/r) \,/\, (1/q - 1/r)$ and $\sigma = \sigma_1(1-\rho)$, we finally arrive at (6.4).

To prove (6.4) for the second case (ii) we make use of estimate (6.4) for the case $\sigma = 1$. As mentioned earlier, this is the case of the *Sobolev inequality*

$$\|u\|_{r_*} \le C_6 \|\nabla u\|_r, \tag{6.9}$$

where $1 \le r < \infty$ and $1 < r_* < \infty$ such that $1/r_* = 1/r - 1/n$. Observe that estimate (6.9) is an obvious consequence of (6.1) if $n = 1$. For $n \ge 2$ (6.9) will be derived in Section §6.3 in an independent way, which relies on an application of the Hardy–Littlewood–Sobolev inequality (§6.2.1). This method, however, requires $r > 1$. For $r = 1$ we present a proof based on the Hölder inequality (§6.3.4).

This is essentially due to the fact that the method used in the first step is not applicable to the case $\sigma = 1$.

First suppose that $p < q$. By relation (6.2) we have that $r_* \le p < q$ for $1/r_* = 1/r - 1/n$. Similarly to (i) the Hölder inequality leads to

$$\|u\|_p \le \|u\|_q^{1-\sigma} \|u\|_{r_*}^{\sigma}, \quad \frac{1}{p} = \frac{1-\sigma}{q} + \frac{\sigma}{r_*}.$$

Therefore, by applying estimate (6.9) we immediately obtain (6.4) for this case. For the case $p = q$ first assume that $r_* < \infty$. Then, since $r < p = r_*$, as in the case $p < q$ estimate (6.4) is again a consequence of (6.9). On the other hand, $r_* = \infty$ implies that $n = 1$ (hence $r = 1$) by condition (6.3). But then (6.4) follows easily from (6.1).

The Third Step

In order to prove (6.4) for functions that are not necessarily compactly supported we can adapt the second and third step in §6.1.3. More precisely, again we approximate an arbitrary C^1-function by functions with compact support. Here we also assume that the case $\sigma = 1$ is proved for the case of compactly supported C^1-functions, i.e., from now on we suppose that $0 < \sigma \le 1$.

Observe that in the case $\|\nabla u\|_r = \infty$ the assertion is obvious. Therefore, we may suppose $\|\nabla u\|_r < \infty$.

We start with the case $p \geq q$ and $p \geq r$ and assume that $u \in C^1(\mathbb{R}^n)$ satisfies $\|u\|_p < \infty$ and $\|u\|_q < \infty$. Next we set $u_j = \theta_j u$ with θ_j as defined in the second part of §6.1.3. (Note: if $q = \infty$ we assume $u \in C_\infty(\mathbb{R}^n)$, whereas for $r = \infty$ we assume that $\partial_{x_\ell} u \in C_\infty(\mathbb{R}^n)$ for $\ell = 1, \ldots, n$.) By $0 < \sigma \leq 1$ and $p \geq q$ it follows that $1/r - 1/n \leq 1/p$. If $p \geq q$ then we have $q < \infty$ except for the case that $p = q = \infty$ and $r = n = 1$.

(i) The case $p < \infty$: Exercise 6.1 (i), (iii) imply that

$$\lim_{j\to\infty} \|\nabla(u_j - u)\|_r = 0, \quad \lim_{j\to\infty} \|u_j - u\|_q = 0, \quad \lim_{j\to\infty} \|u_j - u\|_p = 0.$$

Since (6.4) is valid for the u_j, $j = 1, 2, \ldots$, passing to the limit yields the assertion for u.

(ii) The case of $p = \infty$: First exclude the case $r = n = 1$. Then $r > n$ in view of assumption (6.3). If $r < \infty$ then we have $\|\nabla(u_j - u)\|_r \to 0$, $\|u_j - u\|_q \to 0$, and $\|u\|_\infty = \lim_{j\to\infty} \|u_j\|_\infty$ again by Exercise 6.1 (i) and (iii). (Observe that we may not assume $\|u_j - u\|_\infty \to 0$ in this case.) Thus, again passing to the limit yields (6.4). Recall that for $r = \infty$ we assumed that $\partial_{x_\ell} u \in C_\infty(\mathbb{R}^n)$. By virtue of the fact that $q < \infty$, $\|u_j - u\|_q \to 0$, $\lim_{j\to\infty} \|u_j\|_\infty = \|u\|_\infty$, and that

$$\begin{aligned} \|\nabla(u_j - u)\|_\infty &\leq \|(\nabla\theta_j)u\|_\infty + \|(1 - \theta_j)\nabla u\|_\infty \\ &\leq \frac{C}{j}\|u\|_\infty + \|(1 - \theta_j)\nabla u\|_\infty \to 0 \quad \text{if} \quad j \to \infty, \end{aligned}$$

the assertion follows analogously. There remains the case $p = \infty$ and $r = n = 1$. Here a direct proof is possible. In fact, by the fundamental theorem of calculus we obtain that $u(x) - u(a) = \int_a^x \frac{du}{dx}(z)\, dz$. This yields $|u(x) - u(a)| \leq \left\|\frac{du}{dx}\right\|_1$. The assumption $\|u\|_q < \infty$ if $q < \infty$ or $u \in C_\infty(\mathbb{R})$ if $q = \infty$ then implies the existence of a sequence $a_j \to -\infty$ $(j \to \infty)$ satisfying $u(a_j) \to 0$ (see Exercise 4.6). This gives us estimate (6.1), i.e., we have that $\|u\|_\infty \leq \left\|\frac{du}{dx}\right\|_1$, which implies the validity of (6.4) also in this case.

By defining $u^t = e^{t\Delta}u$ as in the third step of §6.1.3, also here we may omit the assumption $\|u\|_p < \infty$. This is again due to the fact that $\|u^t\|_p < \infty$, $t > 0$, in view of (1.5) and since $p \geq q$. Hence, in the case $p \geq q$, $p \geq r$, and $0 < \sigma \leq 1$, estimate (6.4) is valid for functions with noncompact support.

By the above fact the case $p \geq q$ and $p < r$ can be treated analogously to the case (i) in the second step in §6.1.4.

For the case $p < q$ (hence $r < p$) observe that the Hölder inequality $\|u\|_p \leq \|u\|_q^{1-\sigma}\|u\|_{r_*}^\sigma$ with $1/r_* = 1/r - 1/n$, which we used in (ii) of the second step in this subsection, still holds in this case. In view of $r_* \geq s$

and $\|u\|_s < \infty$ we then again may apply the Sobolev inequality in order to obtain (6.4).

By the arguments above we see that the validity of (6.4) for C^1-functions with compact support implies the assertions for the general case as stated in the second part of the theorem. In fact, we have proved slightly more. By similar arguments as used at the end of the third step in the proof of the Nash inequality, the computations above show that for any $0 < \sigma \le 1$ the general case for functions with noncompact support is implied by the validity of (6.4) for $C_0^\infty(\mathbb{R}^n)$-functions.

6.1.5 Remarks on the Proofs

In the original papers [Gagliardo 1959] and [Nirenberg 1959] of Gagliardo and of Nirenberg respectively, inequality (6.4) is proved in an elementary way by merely using the fundamental theorem of calculus and the Hölder inequality. Their method also works for the case $\sigma = 1$, i.e., it includes a proof of the Sobolev inequality. In §6.3.4 we will present a proof in case of $\sigma = 1$ and $r = 1$ by the above idea. We also remark that even a more general inequality of type

$$\| \, |x|^\alpha u \, \|_p \le C \| \, |x|^\beta u \, \|_q^{1-\sigma} \| \, |x|^\gamma |\nabla u| \, \|_r^\sigma$$

is known. However, then the balancing relation (6.2) also involves the additional parameters α, β, γ, and becomes therefore much more complicated. (see [Caffarelli Kohn Nirenberg 1984]).

The method for the proof presented in §6.1.4 is well known in the field of probability theory (for the proof of the Nash inequality see, e.g., Remark II.3.3 (a) in [Varopoulas Saloff-Coste Coulhon 1992]). In the field of semigroup theory, it is a common method to apply the Nash inequality (or the Gagliardo–Nirenberg inequality) to obtain L^p-L^q estimates for semigroups generated by differential operators such as for $e^{t\Delta}$. In this sense, the L^p-L^q estimates for $e^{t\Delta}$ are equivalent to inequality (6.5) (see, e.g., [Carlen Kusuoka Stroock 1987]). We also would like to mention that the idea to employ a representation for $\|u\|_p$ in the first step of the proof of the Gagliardo–Nirenberg inequality is similar to Lemma 1.5.3 of [Fukushima Oshima Takeda 1994]. However, the representation for $\|u\|_p$ used there differs from ours. Our proof of (6.4) can be regarded as a simplified version of the one in [Maremonti 1998].

6.1.6 A Remark on Assumption (6.3)

In the statement (and in the proof) of the Gagliardo–Nirenberg inequality, we have seen that the case $p = \infty$, $r = n > 1$, $\sigma = 1$ is excluded (see (6.3)). This is due to the fact that

$$\|u\|_\infty \le C \|\nabla u\|_n$$

is not valid in that case. However, if we replace the L^∞-norm by the BMO seminorm

$$\|u\|_{\mathrm{BMO}} = \sup_{x\in\mathbb{R}^n} \sup_{\rho>0} \left(\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |u - u_\#| dx \right),$$

then we have that

$$\|u\|_{\mathrm{BMO}} \le C\|\nabla u\|_n.$$

Here $B_\rho(x)$ is the open ball centered at x with radius ρ, and $|B_\rho(x)|$ denotes its volume (with respect to the Lebesgue measure). Moreover, by $u_\#$ we denote the mean of u in $B_\rho(x)$ given by

$$u_\#(x) = \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} u(y)\ dy.$$

The *bounded mean oscillation seminorm* (*BMO seminorm*) and the above inequality were introduced first in [John Nirenberg 1961]. For $\|\cdot\|_{\mathrm{BMO}}$, (ii) and (iii) of the definition of a norm in §1.3 hold. However, $\|x\| = 0$ does not imply $x = 0$ in general, since $\|c\|_{\mathrm{BMO}}$ for any constant function c is zero. Therefore it is called a seminorm. The reader is referred to §6.5 for further development of the critical case of the Sobolev inequality discussed in this subsection.

6.2 Boundedness of the Riesz Potential

Let $0 < \alpha < n$. The *Riesz potential* $I_\alpha(f)$ of a function f is defined by

$$I_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy = \frac{1}{|x|^{n-\alpha}} * f, \quad x \in \mathbb{R}^n.$$

For example, if $f \in C_0(\mathbb{R}^n)$, by similar arguments as in the proposition in §4.1.4 (Exercise 7.1) it can be shown that $I_\alpha(f)$ is well defined as a continuous function on $\mathbb{R}^n$. It is of main interest under what circumstances the Riesz potential gives rise to a bounded operator from L^q to L^r. The next result, the so-called the *Hardy–Littlewood–Sobolev inequality* , answers this question.

6.2.1 The Hardy–Littlewood–Sobolev Inequality

Theorem. *Let $0 < \alpha < n$. Let $1 < p, r < \infty$ satisfy*

$$\frac{1}{r} = \frac{n-\alpha}{n} + \frac{1}{p} - 1 = \frac{1}{p} - \frac{\alpha}{n}.$$

Then there exists a constant $C = C(\alpha, p)$ such that for every $f \in C_0(\mathbb{R}^n)$ we have

$$\|I_\alpha(f)\|_r \le C\|f\|_p, \quad f \in C_0(\mathbb{R}^n).$$

Hence the operator I_α extends uniquely to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ (See §7.3).

Remark. It is a good exercise in Lebesgue integration theory to show that the extended operator, here and hereinafter also denoted by I_α, can be expressed as a Lebesgue integral through $|x|^{\alpha-n} * f$. If f is continuous on $\mathbb{R}^n$, another good exercise is to prove that I_α can be expressed as a Riemann integral.

If we set $q = n/(n-\alpha)$, we see that the balancing relation for the exponents is exactly the one for the Young inequality in §4.1.1. So, at first glance, one might think that the Hardy–Littlewood–Sobolev inequality is a consequence of the Young inequality. But observe that h in (4.2) here is given by $h(x) = |x|^{\alpha-n} = |x|^{n/q}$. Therefore h is obviously not $L^q(\mathbb{R}^n)$-integrable, which means that the Young inequality is not applicable in this situation.

There are many different ways to prove the Hardy–Littlewood–Sobolev inequality. Here again we prefer a method based on the estimates for the solution of the heat equation (§1.1.2).

The proof requires some further preparations related to Lebesgue integration theory. The integrals in §6.2.2–§6.2.4 should be interpreted as Lebesgue integrals. For a Lebesgue measurable function f on $\mathbb{R}^n$ we define its *distribution function* by

$$m_f(\lambda) = |\{x \in \mathbb{R}^n;\ |f(x)| > \lambda\}|.$$

Here $|A|$ denotes the Lebesgue measure of a set A in $\mathbb{R}^n$. By definition it follows that for $f \in L^\infty(\mathbb{R}^n)$ we have that $m_f(\lambda) = 0$ for $\lambda > \|f\|_\infty$. Conversely, if there exists a λ_0 such that $m_f(\lambda) = 0$ for all $\lambda > \lambda_0$, then $\|f\|_\infty \leq \lambda_0$. Thus we see that we can gain some knowledge on f by properties of its distribution function and conversely. In particular, we have the following relations.

6.2.2 The Distribution Function and L^p-Integrability

Proposition. *For $p > 0$ and $|f|^p \in L^1(\mathbb{R}^n)$ we have*

(i) $m_f(\lambda) \leq \lambda^{-p} \int_{\mathbb{R}^n} |f(x)|^p dx, \qquad \lambda > 0;$
(ii) $\int_{\mathbb{R}^n} |f(x)|^p dx = p \int_0^\infty t^{p-1}\, m_f(t) dt.$

For $p = 2$ the inequality in (i) is called the *Chebyshev inequality.* It is easy to see that it remains valid if $\mathbb{R}^n$ is replaced by any open set $\Omega \subseteq \mathbb{R}^n$. The Chebyshev inequality is one of the fundamental inequalities in probability theory.

Proof.

(i) For $\lambda > 0$ we set $F_\lambda = \{x \in \mathbb{R}^n; |f(x)| > \lambda\}$. Since $|f(x)| > \lambda$ on F_λ, we obtain

$$\int_{F_\lambda} \lambda^p\, dx \leq \int_{F_\lambda} |f(x)|^p\, dx \leq \int_{\mathbb{R}^n} |f(x)|^p\, dx.$$

Observe that the left-hand side equals $\lambda^p m_f(\lambda)$. This proves (i).

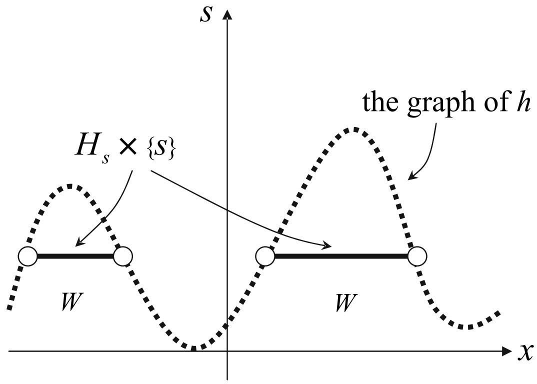

Figure 6.1. An example of a set W for $n = 1$.

(ii) For $h \in L^1(\mathbb{R}^n)$ satisfying $h \geq 0$ we have that

$$\int_{\mathbb{R}^n} h(x)\ dx = \int_{\mathbb{R}^n} \left(\int_0^{h(x)} 1 dy \right) dx.$$

Hence, by Fubini's theorem (§7.2.2), $\int_{\mathbb{R}^n} h(x)\ dx$ coincides with the $(n+1)$-dimensional Lebesgue measure $|W|$ of the set

$$W = \{(x, s) \in \mathbb{R}^{n+1};\ 0 \leq s < h(x)\}.$$

(For $n = 1$ see Figure 6.1 for a sketch of W.) If we set

$$H_s = \{x \in \mathbb{R}^n;\ h(x) > s\},$$

we see that W can also be represented by

$$W = \{(x, s) \in \mathbb{R}^{n+1};\ x \in H_s,\ s \geq 0\}.$$

See Figure 6.1. Applying Fubini's theorem once more yields

$$|W| = \int_0^\infty |H_s| ds, \quad \text{hence} \quad \int_{\mathbb{R}^n} h(x)\ dx = \int_0^\infty |H_s| ds.$$

Graphically this means that in the first representation we cut W into columns and in the second into rows. Now set $h = |f|^p$. In view of

$$H_s = \{x \in \mathbb{R}^n;\ |f(x)|^p > s\} = \{x \in \mathbb{R}^n;\ |f(x)| > s^{1/p}\} = F_{s^{1/p}},$$

this results in

$$\int_{\mathbb{R}^n} |f(x)|^p dx = \int_0^\infty m_f(s^{1/p}) ds.$$

By substituting $t = s^{1/p}$ we arrive at (ii). □

6.2.3 Lorentz Spaces

For a positive number q the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ satisfying

$$|f|_{q,\infty} := \sup_{\lambda>0} \lambda \, m_f(\lambda)^{1/q} < \infty$$

is denoted by $L^{q,\infty}(\mathbb{R}^n)$ and is called *Lorentz space.* By part (i) of Proposition 6.2.2 we know that

$$|f|_{q,\infty} \le \|f\|_q;$$

hence obviously $L^q(\mathbb{R}^n) \subset L^{q,\infty}(\mathbb{R}^n)$ for $1 \le q < \infty$. Observe that $|f|_{q,\infty}$ satisfies

$$\text{(i)} \quad |f|_{q,\infty} = 0 \quad \Leftrightarrow \quad f = 0 \text{ almost everywhere;}$$
$$\text{(ii)} \quad |\alpha f|_{q,\infty} = |\alpha| \; |f|_{q,\infty}, \; \alpha \in \mathbb{R},$$

whereas it does not satisfy the triangle inequality $|f+g|_{q,\infty} \le |f|_{q,\infty} + |g|_{q,\infty}$ for $f, g \in L^{q,\infty}(\mathbb{R}^n)$ in general. Therefore, $|\cdot|_{q,\infty}$ is not a norm. On the other hand, for $q > 1$ it can be shown that $|\cdot|_{q,\infty}$ is equivalent to a norm (see Exercise 6.3). Moreover, $L^{q,\infty}(\mathbb{R}^n)$ is complete with respect to this norm; hence it is a Banach space.

Remark. For the sake of simplicity we have introduced the distribution function and Lorentz spaces on $\mathbb{R}^n$ only. But note that the definitions work equally well for any domain $\Omega \subset \mathbb{R}^n$. In fact, they work for general measure spaces in the same way as for the definition of L^p-spaces. Since we did not use any specific features of $\mathbb{R}^n$ in the proof, Proposition 6.2.2 still holds in such a setting.

A crucial feature of Lorentz spaces is that they are strictly larger than the corresponding L^q-spaces. Indeed, they contain some important singular functions. For instance, the function $1/\sqrt{x}$ does not belong to $L^2(0,1)$, but by definition, it belongs to $L^{2,\infty}(0,1)$ (Exercise 6.4). By similar arguments it can be shown that for any domain $\Omega \subset \mathbb{R}^n$ and $q \ge 1$, $L^{q,\infty}(\Omega)$ is strictly larger than $L^q(\Omega)$.

Next we will prove the *Marcinkiewicz interpolation theorem.* This result will be the essential ingredient not only in the proof of the Hardy–Littlewood–Sobolev inequality, but also in the proof of the Calderón–Zygmund inequality, given in §6.4.

6.2.4 The Marcinkiewicz Interpolation Theorem

Theorem. *Let $1 \le p_i \le q_i < \infty$ for $i = 1, 2$ so that $q_1 \ne q_2$, and let $1 \le p, q < \infty$ be such that*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2} \tag{6.10}$$

for some $\theta \in (0,1)$*. (In other words,* $1/p$*, respectively* $1/q$*, lies on the line connecting* $1/p_1$ *and* $1/p_2$*, respectively* $1/q_1$ *and* $1/q_2$*, where the quotient of the distances is* $\theta/(1-\theta)$*.) Furthermore, suppose that* T *is a linear operator from* $L^{p_i}(\mathbb{R}^n)$ *to* $L^{q_i,\infty}(\mathbb{R}^d)$ *and that there exist constants* M_i *such that*

$$|Tf|_{q_i,\infty} \leq M_i \|f\|_{p_i}, \quad f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n), \tag{6.11}$$

for $i = 1,2$*. Then* T *extends to a bounded linear operator from* $L^p(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^d)$*. In particular, there exists a* $C > 0$ *such that*

$$\|Tf\|_q \leq C M_1^{\theta} M_2^{1-\theta} \|f\|_p, \quad f \in L^p(\mathbb{R}^n). \tag{6.12}$$

Here C *depends only on* p_i*,* q_i*,* $i = 1,2$*, and* p*,* q*.*

The result in which assumption (6.11) is replaced by the stronger one

$$\|Tf\|_{q_i} \leq M_i \|f\|_{p_i},$$

and which was obtained already in the first half of the twentieth century, is known as the *Riesz–Thorin theorem* (then (6.12) holds with $C = 1$ with no restriction on p_i and q_i. In particular, $p_i > q_i$ is allowed). Since $L^{q,\infty}$ is strictly larger than L^q, including even singular functions such as $1/|x|^{n/q}$, the Marcinkiewicz interpolation theorem is an essential improvement of the result of Riesz–Thorin. Analogously to §6.2.2, it will be proved by real-analytic arguments. We first give a detailed proof in the case $p_i = q_i$, $i = 1,2$ (hence $p = q$ by (6.10)), since it is easy to understand the arguments. (Logically speaking, it is enough to give a proof for the general case.) The assumption $p_i \leq q_i$ cannot be removed, but q_i and also p_i are allowed to be ∞ [Folland 1999].

Proof. Without loss of generality we may assume $p_1 < p < p_2$. For $s > 0$ we first split $f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ into a part with absolute value larger than s and a part with absolute value smaller than s, i.e., we set

$$f^s(x) = \begin{cases} f(x), & |f(x)| > s, \\ 0, & |f(x)| \leq s, \end{cases}$$

$$f_s(x) = \begin{cases} 0, & |f(x)| > s, \\ f(x), & |f(x)| \leq s. \end{cases}$$

Then we have

$$f(x) = f^s(x) + f_s(x), \quad x \in \mathbb{R}^n.$$

Since T is linear, we obtain that $Tf = Tf^s + Tf_s$. Consequently,

$$|Tf(y)| \leq |Tf^s(y)| + |Tf_s(y)|, \; y \in \mathbb{R}^d. \tag{6.13}$$

Observe that for measurable functions g, g_1, g_2 satisfying $|g| \leq |g_1| + |g_2|$ we always have that

$$|g(x)| > t \quad \Rightarrow \quad |g_1(x)| > t/2 \text{ or } |g_2(x)| > t/2$$

for each $t > 0$ and $x \in \mathbb{R}^n$. This implies that

$$\chi_{\{|g|>t\}}(x) \leq \chi_{\{|g_1|>t/2\}}(x) + \chi_{\{|g_2|>t/2\}}(x), \quad x \in \mathbb{R}^n, \ t > 0,$$

where χ_B denotes the characteristic function of a set $B \subset \mathbb{R}^n$, i.e.,

$$\chi_B(x) = \begin{cases} 1, \, x \in B, \\ 0, \, x \notin B. \end{cases}$$

Thus, for the distribution function we obtain

$$m_g(t) \leq m_{g_1}(t/2) + m_{g_2}(t/2), \quad t > 0. \tag{6.14}$$

Applying (6.14) to (6.13) results in

$$m_{Tf}(t) \leq m_{Tf^s}(t/2) + m_{Tf_s}(t/2), \quad t > 0. \tag{6.15}$$

Now we apply assumption (6.11) for $i = 1$ on Tf^s and for $i = 2$ on Tf_s. This gives us

$$\begin{aligned} m_{Tf^s}(t/2) &\leq (2M_1/t)^{q_1} \ \|f^s\|_{p_1}^{q_1}, \\ m_{Tf_s}(t/2) &\leq (2M_2/t)^{q_2} \ \|f_s\|_{p_2}^{q_2} \quad (t, s > 0). \end{aligned} \tag{6.16}$$

By the validity of (6.16) for all $s, t > 0$, we may regard s as a function depending on t, say $s = g(t)$, with g chosen suitably in the sequel. The task is now reduced to an estimate of $\|Tf\|_q$ by utilizing relations (6.15) and (6.16). Employing §6.2.2 (ii) we obtain

$$\int_{\mathbb{R}^d} |(Tf)(y)|^q dy = q \ \int_0^\infty t^{q-1} m_{Tf}(t) dt.$$

Inserting (6.15) and (6.16) into this equality we arrive at

$$\begin{aligned} \frac{1}{q} \int_{\mathbb{R}^d} |(Tf)(y)|^q dy \leq (2M_1)^{q_1} \int_0^\infty t^{q-1-q_1} \left(\int_{|f(x)|>g(t)} |f(x)|^{p_1} dx \right)^{\frac{q_1}{p_1}} dt \\ + (2M_2)^{q_2} \int_0^\infty t^{q-1-q_2} \left(\int_{|f(x)|\leq g(t)} |f(x)|^{p_2} dx \right)^{\frac{q_2}{p_2}} dt. \end{aligned} \tag{6.17}$$

The terms on the right-hand side of (6.17) will now be estimated by choosing g and substituting t suitably.

The case $p_i = q_i$:

Here we set $g(t) = t/A$ with a positive constant A to be defined later. Then the substitution $t = As$ turns (6.17) into[1]

$$\frac{1}{q}\int_{\mathbb{R}^d}|Tf|^q dy \le (2M_1)^{q_1}\ A^{q-q_1}\int_0^\infty s^{q-1-q_1}\left(\int_{|f|>s}|f|^{p_1}dx\right)ds$$
$$+ (2M_2)^{q_2}A^{q-q_2}\int_0^\infty s^{q-1-q_2}\left(\int_{|f|\le s}|f|^{p_2}dx\right)ds. \quad (6.18)$$

Analogously to the proof of §6.2.2 (ii) we set

$$W = \{(x,t)\in\mathbb{R}^{n+1}\ ;\ 0\le t<|f(x)|\},\quad H_s = \{x\in\mathbb{R}^n; |f(x)|>s\},$$

which means $W = \{(x,s)\in\mathbb{R}^{n+1}; x\in H_s,\ s\ge 0\}$. Interchanging the integrals (§7.2.2) then implies

$$\int_0^\infty s^{p-1-p_1}\left(\int_{|f|>s}|f|^{p_1}dx\right)ds = \int\int_W s^{p-1-p_1}|f(x)|^{p_1}ds\ dx$$
$$= \int_{\mathbb{R}^n}|f(x)|^{p_1}\left(\int_0^{|f(x)|}s^{p-1-p_1}ds\right)dx$$
$$= \frac{1}{p-p_1}\int_{\mathbb{R}^n}|f|^p dx.$$

In exactly the same way we obtain

$$\int_0^\infty s^{p-1-p_2}\left(\int_{|f|\le s}|f|^{p_2}dx\right)ds = \int_{\mathbb{R}^n}|f(x)|^{p_2}\left(\int_{|f(x)|}^\infty s^{p-1-p_2}ds\right)dx$$
$$= \frac{1}{p_2-p}\int_{\mathbb{R}^n}|f(x)|^p dx.$$

From $p_i = q_i$ $(i = 1,\, 2)$ we get $p = q$, and therefore by (6.18),

$$\frac{1}{p}\int_{\mathbb{R}^d}|Tf|^p dy \le \left[\frac{(2M_1)^{p_1}A^{p-p_1}}{p-p_1} + \frac{(2M_2)^{p_2}A^{p-p_2}}{p_2-p}\right]\int_{\mathbb{R}^n}|f(x)|^p dx.$$

Since this expression is valid for every $A > 0$ we can minimize the expression in brackets with respect to this variable. By differentiating with respect to A we find that the zero of the derivative will be attained in

---

[1] In the sequel, if no confusion seems likely, we omit the variable of integration. Furthermore, $|f| > s$ is an abbreviation for $\{x\in\mathbb{R}^n; |f(x)|>s\}$, and in the same sense we use $|f|\le s$.

$$A = 2M_1^{\frac{-p_1}{p_2-p_1}} M_2^{\frac{p_2}{p_2-p_1}}.$$

Inserting this, we find that the expression in brackets is the product of $\frac{1}{p-p_1} + \frac{1}{p_2-p}$ and

$$2^p M_1^{\frac{p_1(p_2-p)}{p_2-p_1}} M_2^{\frac{p_2(p-p_1)}{p_2-p_1}} = 2^p M_1^{\theta p} M_2^{(1-\theta)p}.$$

Therefore we have proved that

$$\|Tf\|_p^p \le C^p M_1^{\theta p} M_2^{(1-\theta)p} \|f\|_p^p \tag{6.19}$$

with

$$C = 2\left(\frac{p}{p-p_1} + \frac{p}{p_2-p}\right)^{1/p}.$$

Since $L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, the operator T extends uniquely to a bounded operator on $L^p(\mathbb{R}^n)$ (see §7.3), i.e., (6.12) is valid for every $f \in L^p(\mathbb{R}^n)$.

The case of general exponents:

Again it is sufficient to consider $f \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n)$. First suppose that $q_1 < q_2$.

Now we set $g(t) = (t/A)^{1/\mu}$ with constants $A, \mu > 0$ again to be defined later. By the substitution $t = As^\mu$, the first integral on the right-hand side of (6.17) takes the form

$$\begin{aligned} J_1 &:= \int_0^\infty t^{q-1-q_1} \left(\int_{|f|>s} |f|^{p_1} dx\right)^{\frac{q_1}{p_1}} dt \\ &= \mu A^{q-q_1} \int_0^\infty s^{(q-1-q_1)\mu+\mu-1} \left(\int_{\mathbb{R}^n} \chi_W(x,s)|f(x)|^{p_1} dx\right)^{\frac{q_1}{p_1}} ds \\ &= \mu A^{q-q_1} \int_0^\infty \left|\int_{\mathbb{R}^n} s^{((q-q_1)\mu-1)\frac{p_1}{q_1}} \chi_W |f(x)|^{p_1} dx\right|^{\frac{q_1}{p_1}} ds. \end{aligned}$$

By the assumption $q_1/p_1 \ge 1$ and the integral form of the *Minkowski inequality* (Exercise 6.5) we obtain

$$\begin{aligned} &\int_0^\infty \left|\int_{\mathbb{R}^n} s^{((q-q_1)\mu-1)\frac{p_1}{q_1}} \chi_W |f(x)|^{p_1} dx\right|^{\frac{q_1}{p_1}} ds \\ &\le \left(\int_{\mathbb{R}^n} \left|\int_0^\infty s^{(q-q_1)\mu-1} \chi_W |f(x)|^{q_1} ds\right|^{\frac{p_1}{q_1}} dx\right)^{\frac{q_1}{p_1}} \\ &= \left(\int_{\mathbb{R}^n} \left(\int_0^{|f(x)|} s^{(q-q_1)\mu-1} ds\right)^{\frac{p_1}{q_1}} |f(x)|^{p_1} dx\right)^{\frac{q_1}{p_1}}. \end{aligned}$$

Note that the last equality follows by the definition of the set W. Calculating the inner integral then results in

$$J_1 \leq \mu\ A^{q-q_1} \frac{1}{(q-q_1)\mu} \left(\int_{\mathbb{R}^n} |f|^{(q-q_1)\mu \frac{p_1}{q_1} + p_1} dx \right)^{\frac{q_1}{p_1}}.$$

The fact that we would like to have an estimate by the L^p norm of f shows us how we have to choose μ, namely as

$$\mu = \frac{q_1}{p_1} \frac{p-p_1}{q-q_1}.$$

Then

$$J_1 \leq A^{q-q_1} \frac{1}{q-q_1} \|f\|_p^{pq_1/p_1}$$

follows.

By a very similar calculation for

$$J_2 := \int_0^\infty t^{q-1-q_2} \left(\int_{|f| \leq s} |f|^{p_2} dx \right)^{\frac{q_2}{p_2}} dt,$$

we obtain

$$J_2 \leq A^{q-q_2} \frac{1}{q_2-q} \left(\int_{\mathbb{R}^n} |f|^{(q-q_2)\mu \frac{p_2}{q_2} + p_2} dx \right)^{\frac{q_2}{p_2}}.$$

At this point condition (6.10) comes into play. Indeed, this relation gives us

$$\frac{\theta}{1-\theta} = \frac{\frac{1}{q} - \frac{1}{q_2}}{\frac{1}{q_1} - \frac{1}{q}} = \frac{\frac{1}{p} - \frac{1}{p_2}}{\frac{1}{p_1} - \frac{1}{p}}, \quad \text{hence} \quad \mu = \frac{q_1}{p_1} \frac{p-p_1}{q-q_1} = \frac{q_2}{p_2} \frac{p-p_2}{q-q_2}.$$

Therefore, J_2 turns into

$$J_2 \leq A^{q-q_2} \frac{1}{q_2-q} \|f\|_p^{pq_2/p_2}.$$

Combining the estimates for J_1 and J_2, we infer from (6.17) that

$$\frac{1}{q} \|Tf\|_q^q \leq \frac{(2M_1)^{q_1} A^{q-q_1}}{q-q_1} \|f\|_p^{pq_1/p_1} + \frac{(2M_2)^{q_2} A^{q-q_2}}{q_2-q} \|f\|_p^{pq_2/p_2}.$$

Again by minimizing with respect to A we see that the right-hand side is minimal for

$$A = 2(M_1^{-q_1} M_2^{q_2} \|f\|_p^{p(\frac{q_2}{p_2} - \frac{q_1}{p_1})})^{\frac{1}{q_2-q_1}}.$$

This implies that

$$\|Tf\|_q \leq C'\ M_1^\theta\ M_2^{1-\theta} \|f\|_p, \quad C' = 2\left(\frac{q}{q-q_1} + \frac{q}{q_2-q} \right)^{1/q},$$

which proves (6.12) for the case $q_1 < q_2$.

For the case $q_1 > q_2$ we just have to interchange the roles of q_1 and q_2 in (6.16). This means here that we employ

$$m_{Tf^s}(t/2) \le (2M_2/t)^{q_2} \|f^s\|_{p_2}^{q_2}, \quad m_{Tf_s}(t/2) \le (2M_1/t)^{q_1} \|f_s\|_{p_1}^{q_1}.$$

Then by arguments analogous to the case $q_1 < q_2$, estimate (6.12) follows.[1]

□

Remark. In the proof of the Marcinkiewicz theorem we use only measure-space-theoretic properties of $\mathbb{R}^n$ and $\mathbb{R}^d$. Hence the theorem is valid if we replace $\mathbb{R}^n$ and $\mathbb{R}^d$ by arbitrary domains $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^d$, respectively. In fact, we may replace $\mathbb{R}^n$ and $\mathbb{R}^d$ by general measure spaces. Furthermore, for the operator T we do not really need its linearity. More precisely, it is sufficient to assume the following subadditivity:

$$|Tf(y)| \le |Tf_1(y)| + |Tf_2(y)|, \quad y \in \mathbb{R}^n,$$
$$f = f_1 + f_2,\ f_1, f_2 \in L^{p_1}(\mathbb{R}^n) \cap L^{p_2}(\mathbb{R}^n).$$

An application of this remark is given by the following integral estimate. The proof is left as Exercise 6.6.

Proposition. *Assume that* $1 \le r, p, q \le \infty$ *satisfy* $1 < q < r < \infty$ *and* $2/r = n(1/q - 1/p)$. *Then there exists a positive constant* $C = C(p, q, n)$ *such that*

$$\int_0^\infty \|e^{t\Delta} f\|_p^r dt \le C\|f\|_q^r$$

holds for any $f \in L^q(\mathbb{R}^n)$.

The importance of this proposition as well as its applications was pointed out by [Weissler 1981, Giga 1986].

6.2.5 Gauss Kernel Representation of the Riesz Potential

For $0 < \alpha < n$ we define the operator $(-\Delta)^{-\alpha/2}$ by

$$(-\Delta)^{-\alpha/2} f = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1}(e^{t\Delta} f) dt, \quad f \in C_0(\mathbb{R}^n).$$

The convergence of the integral is an easy consequence of the decay estimate (1.5) for $e^{t\Delta} f$. We will see later why this operator is denoted by $(-\Delta)^{-\alpha/2}$.

Lemma. *We have that*

$$(-\Delta)^{-\alpha/2} f = C(n, \alpha) I_\alpha(f) \quad \textit{with}$$
$$C(n,\alpha) = \Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right) \Big/ \left(\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi^{n/2}\right) \tag{6.20}$$

for $f \in C_0(\mathbb{R}^n)$.

[1] Alternatively, one could just replace μ by $-\mu$ in the definition of the function g.

Here Γ is Euler's gamma function, which is defined by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} dt,$$

for $z > 0$. We give a formal proof of (6.20). The justification of the following calculation is left as Exercise 6.7. Observe that by

$$\begin{aligned}\int_0^\infty t^{\frac{\alpha}{2}-1} e^{t\Delta} f\, dt &= \int_0^\infty t^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} G_t(x-y) f(y) dy \right) dt \\ &= \int_{\mathbb{R}^n} \left(\int_0^\infty t^{\frac{\alpha}{2}-1} G_t(x-y) dt \right) f(y) dy, \quad f \in C_0(\mathbb{R}^n),\end{aligned} \tag{6.21}$$

we obtain

$$\begin{aligned}\int_0^\infty t^{\frac{\alpha}{2}-1} G_t(x) dt &= \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} t^{\frac{\alpha}{2}-1} dt \\ &= \frac{1}{\pi^{n/2} 4^{\alpha/2}} \int_0^\infty \tau^{\frac{n}{2}-\frac{\alpha}{2}-1} e^{-\tau} d\tau \, \frac{1}{|x|^{n-\alpha}}.\end{aligned}$$

The substitution $\tau = |x|^2/(4t)$ then yields

$$\int_0^\infty t^{\frac{\alpha}{2}-1} G_t(x) dt = \frac{\Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)}{2^\alpha \pi^{n/2} |x|^{n-\alpha}},$$

which proves (6.20).

6.2.6 Proof of the Hardy–Littlewood–Sobolev Inequality

Here we prove the boundedness of the Riesz potential by utilizing relation (6.20) and the Marcinkiewicz interpolation theorem. The proof presented here is based on [Varopoulas Saloff-Coste Coulhon 1992, Proposition II.2.6]. The idea of this proof goes back to [Yoshikawa 1971].

First we show the boundedness of the operator

$$T_\alpha(f) = \int_0^\infty t^{\frac{\alpha}{2}-1} e^{t\Delta} f\, dt.$$

from L^p to the corresponding Lorentz space.

As mentioned in §4.1.6, note that if $f \in L^p(\mathbb{R}^n)$, the expression $e^{t\Delta} f$ is smooth for $t > 0$ and $x \in \mathbb{R}^n$. Thus, here we may regard $T_\alpha(f)(x)$ as an improper Riemann integral for $f \in L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$.

Lemma. *Suppose that* α, p, r *satisfy* $0 < \alpha < n$, $1 \le p < \infty$, $1 < r < \infty$, *and*

$$\frac{1}{r} = \frac{1}{p} - \frac{\alpha}{n}.$$

Then there exists a constant C, depending only on p, α, and n, such that

$$|T_\alpha(f)|_{r,\infty} \leq C\|f\|_p, \quad f \in L^p(\mathbb{R}^n).$$

Proof. For $S > 0$ we set

$$F^S = \int_0^S t^{\frac{\alpha}{2}-1} e^{t\Delta} f \, dt, \quad F_S = \int_S^\infty t^{\frac{\alpha}{2}-1} e^{t\Delta} f \, dt.$$

This implies

$$|(T_\alpha(f))(x)| \leq |F^S(x)| + |F_S(x)|, \quad x \in \mathbb{R}^n, \ S > 0.$$

(Note that the integrals $\int_0^S$ and $\int_S^\infty$ are understood in the sense of $\lim_{\eta\to 0}\int_\eta^S$ and $\lim_{\eta\to\infty}\int_S^\eta$ exist. Thus F^S, F_S are realized as the pointwise limits of a continuous function with respect to x, hence they are well defined.) We infer from (6.14) that

$$m_{T_\alpha(f)}(t) \leq m_{F^S}(t/2) + m_{F_S}(t/2), \quad t > 0.$$

By the L^∞-L^p estimate in §1.1.2, F_S can be estimated as

$$\begin{aligned}
\|F_S\|_\infty &\leq \int_S^\infty t^{\frac{\alpha}{2}-1} \|e^{t\Delta} f\|_\infty \, dt \\
&\leq C \int_S^\infty t^{\frac{\alpha}{2}-1} t^{-\frac{n}{2p}} \, dt \, \|f\|_p = C' S^{-\frac{n}{2r}} \|f\|_p,
\end{aligned}$$

with $C' = 2rC/n$. Choosing S such that $t/4 = C' S^{-\frac{n}{2r}} \|f\|_p$, we see by the estimate above that $m_{F_S}(t/2) = 0$ for $t > 0$. Furthermore, the Chebyshev inequality (§6.2.2) implies that

$$m_{F^S}(t/2) \leq (t/2)^{-p} \|F^S\|_p^p.$$

Employing the estimate $\|e^{t\Delta} f\|_p \leq \|f\|_p$ (§1.1.2), we obtain

$$\|F^S\|_p \leq \|f\|_p \int_0^S t^{\frac{\alpha}{2}-1} \, dt = \frac{2}{\alpha} \|f\|_p S^{\frac{\alpha}{2}}.$$

By our choice of S, we have that $S^{\frac{\alpha p}{2}} = (4C' \, \|f\|_p/t)^{\frac{\alpha p}{2}\cdot\frac{2r}{n}}$. This yields

$$m_{F^S}(t/2) \leq C'' \, t^{-p} \, t^{-\frac{\alpha p r}{n}} \|f\|_p^{p+\frac{\alpha p}{n} r},$$

with $C'' = 2^p \left(\frac{2}{\alpha}\right)^p (4C')^{\alpha p r/n}$. Since by definition $1 + \alpha r/n = r/p$, we have that

$$m_{F^S}(t/2) \leq C'' t^{-r} \|f\|_p^r, \quad t > 0.$$

Thus, we obtain

$$m_{T_\alpha(f)}(t) \leq C'' t^{-r} \|f\|_p^r, \quad t > 0,$$

and the claim follows. □

6.2.7 Completion of the Proof

Let p, r satisfy the conditions in the lemma in §6.2.6. It is clear that we can find p_i, r_i $(i = 1, 2)$, and $\theta \in (0, 1)$ satisfying $1 < p_i, r_i < \infty$, $r_1 \neq r_2$, and

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}, \quad \frac{1}{r_i} = \frac{1}{p_i} - \frac{\alpha}{n} \quad (i = 1, 2).$$

By §6.2.6, T_α is a bounded linear operator from $L^{p_i}(\mathbb{R}^n)$ to $L^{r_i,\infty}(\mathbb{R}^n)$. The Marcinkiewicz interpolation theorem (§6.2.4) shows that T_α extends to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$. Relation (6.20) then yields the Hardy–Littlewood–Sobolev inequality given in §6.2.1.

6.3 The Sobolev Inequality

Next we will prove the Sobolev inequality (6.9) for C^∞-functions with compact support in dimension $n \geq 2$. (The case $n = 1$ follows trivially from (6.1).) As mentioned in §6.1.1 this inequality is equivalent to the Gagliardo–Nirenberg inequality for $\sigma = 1$. After some preparations in §6.3.3 we give a proof for the case $r > 0$ by employing the Hardy–Littlewood–Sobolev inequality. This idea was originally used by Sobolev. But, as pointed out in §6.1.5, the Sobolev inequality can also be proved using the fundamental theorem of calculus and the Hölder inequality only. Since it can be important to know different approaches to the same inequality, we also present a proof by this method. One advantage of this approach is that it admits a proof in both cases $r > 1$ and $r = 1$. In §6.3.4, we will demonstrate this for the case $r = 1$ and outline how the general case $r > 1$ can be reduced to this one.

As in Remark 6.1.4, we emphasize that by a density argument we may omit the assumption of compact support. In fact, inequality (6.9) holds for all u such that $\nabla u \in L^r(\mathbb{R}^n)$.

First, we recall some basic facts about the operator $(-\Delta)^{-1}$ defined in §6.2.5. If $n \geq 3$, by setting $\alpha = 2 < n$ this operator can be expressed by (6.20). In fact, as proved in §6.2.6, it is a bounded linear operator from suitable $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$. The reason we write $(-\Delta)^{-1}$ will be clear after the next paragraph.

6.3.1 The Inverse of the Laplacian ($n \geq 3$)

Proposition. *Let $n \geq 3$. For any $f \in C_0^\infty(\mathbb{R}^n)$, we have*

(i) $(-\Delta)^{-1}\Delta f = -f$,
(ii) $(-\Delta)^{-1} f = E * f$,

where $E(x) = 1/((n-2)|S^{n-1}||x|^{n-2})$ for $x \in \mathbb{R}^n \setminus \{0\}$. Here $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ denotes the area of the $(n-1)$-dimensional unit sphere.

Proof.

(i) Since $\Gamma(1) = 1$, we have that

$$(-\Delta)^{-1}\Delta f = \int_0^\infty e^{t\Delta}\Delta f\ dt = \lim_{m\to\infty}\int_0^m e^{t\Delta}\Delta f\ dt.$$

By the L^∞-L^1 estimate in §1.1.2, there exists a constant $C > 0$ such that $\| e^{t\Delta}\Delta f\|_\infty \le \frac{C}{t^{n/2}}\|\Delta f\|_1$. Thus the convergence above is uniform in $x \in \mathbb{R}^n$. Moreover, by the equality $e^{t\Delta}\Delta f = \Delta e^{t\Delta} f$ (see Proposition 4.1.6) and (6.6) we obtain for $m > 0$ that

$$\int_0^m e^{t\Delta}\Delta f\ dt = \int_0^m \Delta\, e^{t\Delta} f\ dt = \int_0^m \frac{d}{dt}\, e^{t\Delta} f\ dt = e^{m\Delta} f - f.$$

From the L^∞-L^1 estimate we infer

$$\|e^{m\Delta} f\|_\infty \le \frac{C}{m^{n/2}}\|f\|_1 \to 0 \quad (m\to\infty).$$

Therefore the claim follows.

(ii) This is obtained by setting $\alpha = 2$ in the formula in §6.2.5. In fact, we have that

$$\begin{aligned} C(n,2) &= \frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma(1)2^2\pi^{n/2}} = \frac{2\Gamma\left(\frac{n}{2}\right)}{(n-2)}\cdot\frac{1}{2^2\pi^{n/2}} \\ &= \frac{1}{(n-2)|S^{n-1}|}, \end{aligned}$$

where we used relation $z\,\Gamma(z) = \Gamma(z+1)$ for the gamma function. □

Remark. It is a well-known fact that $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$. Nevertheless, we give a simple proof here. By introducing polar coordinates, $\int_{\mathbb{R}^n} e^{-|x|^2}dx$ $(=: J)$ is expressed by

$$J = |S^{n-1}|\int_0^\infty r^{n-1}e^{-r^2}dr.$$

On the other hand, since $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$, we have $J = \pi^{n/2}$. This results in

$$|S^{n-1}| = \pi^{n/2}/\int_0^\infty r^{n-1}e^{-r^2}dr.$$

Another change of variables implies

$$\int_0^\infty r^{n-1}e^{-r^2}dr = \int_0^\infty s^{\frac{n-1}{2}}\ e^{-s}\ \frac{1}{2s^{1/2}}ds = \frac{\Gamma(n/2)}{2}.$$

Hence we obtain that $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$.

6.3.2 The Inverse of the Laplacian ($n = 2$)

Proposition. *Let $n = 2$. For $x \in \mathbb{R}^2 \setminus \{0\}$ set $E(x) = -\frac{1}{2\pi}\log|x|$. Then $E * \Delta f = -f$ holds for $f \in C_0^\infty(\mathbb{R}^2)$.*

Note that this proposition can be proved by employing direct methods (Exercise 6.8). However, here we employ methods relying on the expression $(-\Delta)^{-\alpha/2}$. Observe also that in the two-dimensional case, expression (6.20) of $(-\Delta)^{-1}$ is a priori not well defined, since the integration over t may diverge. To overcome this difficulty we prove $E * \Delta f = -f$ by letting the parameter α in $(-\Delta)^{-\alpha/2}$ tend to 2 from below.

Lemma.

(i) $\lim_{\alpha\uparrow 2} \|(-\Delta)^{-(\alpha/2)}\Delta f + f\|_\infty = 0 \qquad (f \in C_0^\infty(\mathbb{R}^2))$.
(ii) $\lim_{\alpha\uparrow 2}((-\Delta)^{-(\alpha/2)}h)(x) = (E * h)(x)$, $x \in \mathbb{R}^2$, *holds for* $h \in C_0^\infty(\mathbb{R}^2)$ *satisfying* $\int_{\mathbb{R}^2} h(x)\,dx = 0$.

Setting $h = \Delta f$, integration by parts (§4.5.1) yields $\int_{\mathbb{R}^2} h(x)dx = 0$. This shows that the proposition is reduced to the assertions in the lemma.

Proof of the lemma.

(i) Pick $\varepsilon > 0$. We split the integration over t into two parts:

$$\begin{aligned}(-\Delta)^{-\alpha/2}\Delta f &= \frac{1}{\Gamma(\alpha/2)}\left\{\int_\varepsilon^\infty t^{\frac{\alpha}{2}-1}(e^{t\Delta}\Delta f)dt + \int_0^\varepsilon t^{\frac{\alpha}{2}-1}(e^{t\Delta}\Delta f)dt\right\}\\ &= \frac{1}{\Gamma(\alpha/2)}(J_1 + J_2).\end{aligned}$$

By Proposition 4.1.6 and integration by parts we obtain

$$\begin{aligned}J_1 &= \int_\varepsilon^\infty t^{\frac{\alpha}{2}-1}(\Delta e^{t\Delta} f)dt = \int_\varepsilon^\infty t^{\frac{\alpha}{2}-1}\frac{d}{dt}(e^{t\Delta} f)dt\\ &= -\left(\frac{\alpha}{2}-1\right)\int_\varepsilon^\infty t^{\frac{\alpha}{2}-2}(e^{t\Delta} f)dt - \varepsilon^{\frac{\alpha}{2}-1}(e^{\varepsilon\Delta} f)\end{aligned}$$

on $\mathbb{R}^2$. Observe that there is no contribution of the value $t = \infty$ by virtue of the uniform boundedness of $\|e^{t\Delta} f\|_\infty$ in $t > 0$ (§1.1.2). Let n be the space dimension. Now, employing the L^∞-L^1 estimate in §1.1.2, i.e.,

$$\|e^{t\Delta} f\|_\infty \le C_1 t^{-n/2}\|f\|_1, \qquad C_1 = (4\pi)^{-n/2},$$

we can estimate

$$\begin{aligned}\left\|\int_\varepsilon^\infty t^{\frac{\alpha}{2}-2}(e^{t\Delta} f)dt\right\|_\infty &\le C_1\,\|f\|_1\int_\varepsilon^\infty t^{\frac{\alpha}{2}-2-\frac{n}{2}}\,dt\\ &= C_1\,\varepsilon^{\frac{\alpha}{2}-1-\frac{n}{2}}\,\|f\|_1\,\frac{1}{\frac{n}{2}+1-\frac{\alpha}{2}}.\end{aligned}$$

The latter term is bounded as $\alpha \uparrow 2$. This implies

$$\lim_{\alpha \uparrow 2} \|J_1 + e^{\varepsilon\Delta} f\|_\infty = 0.$$

On the other hand, we have that

$$\|J_2\|_\infty \leq \int_0^\varepsilon t^{\frac{\alpha}{2}-1}\, dt \|\Delta f\|_\infty \leq \frac{2}{\alpha}\varepsilon^{\frac{\alpha}{2}} \|\Delta f\|_\infty \to \varepsilon\|\Delta f\|_\infty \quad (\alpha \uparrow 2).$$

This gives us

$$\overline{\lim_{\alpha \uparrow 2}} \|(-\Delta)^{-\alpha/2}\Delta f + f\|_\infty \leq \|f - e^{\varepsilon\Delta} f\|_\infty + \varepsilon\|\Delta f\|_\infty.$$

Letting $\varepsilon \to 0$, we obtain $\|f - e^{\varepsilon\Delta} f\|_\infty \to 0$ in view of Theorem 4.2.1. Hence (i) is proved.

(ii) Expressing $(-\Delta)^{-\alpha/2}$ in terms of the Riesz potential as given in §6.2.5 for $0 < \alpha < 2$, we have that

$$(-\Delta)^{-\alpha/2} h = C(2,\alpha) I_\alpha(h) \quad \text{ on } \mathbb{R}^2.$$

In view of $\int_{\mathbb{R}^2} h(x)\, dx = 0$, we obtain that

$$C(2,\alpha)I_\alpha(h) = E_\alpha * h, \quad E_\alpha(x) = \frac{\Gamma(1-\alpha/2)}{2^\alpha \pi \Gamma(\alpha/2)} \left(\frac{1}{|x|^{2-\alpha}} - 1 \right).$$

Well known properties of the gamma function imply that

$$\Gamma\left(1 - \frac{\alpha}{2}\right) = \frac{2}{2-\alpha} \Gamma\left(2 - \frac{\alpha}{2}\right),$$

and $\Gamma\left(2 - \frac{\alpha}{2}\right) \to \Gamma(1) = 1$ if $\alpha \uparrow 2$. Hence, $\Gamma\left(1 - \frac{\alpha}{2}\right)$ tends to infinity as $\alpha \uparrow 2$ with the principal term $\frac{1}{2-\alpha}$. Set $2 - \alpha = \delta$. If $x \neq 0$, we obtain

$$\begin{aligned}(|x|^{-\delta} - 1)/\delta &= \frac{1}{\delta}\{\exp(-\delta \log |x|) - \exp(-0 \log |x|)\} \\ &\to -\log |x| \quad (\delta \to 0).\end{aligned}$$

This implies that

$$\lim_{\alpha \uparrow 2} E_\alpha(x) = -\frac{1}{2\pi} \log |x|$$

for any $x \in \mathbb{R}^2$ with $x \neq 0$. Thus, we have proved that

$$\lim_{\alpha \uparrow 2} ((-\Delta)^{-\alpha/2} h)(x) = (E * h)(x)$$

if we can show that taking the limit and integration commute. By the equality

$$((-\Delta)^{-\alpha/2}h)(x) = \int_{\mathbb{R}^2} E_\alpha(x-y)h(y)dy = \int_{\mathbb{R}^2} E_\alpha(y)h(x-y)dy$$

and since the support of h is compact, this is a consequence of the dominated convergence theorem (§7.1.1). In order to apply this result, it remains to show that $|E_\alpha(y)|$ is bounded from above by a locally integrable function independent of $\alpha \in (1,2)$. In fact, if $y \neq 0$, we apply the mean value theorem in §1.1.6 to $|y|^{-\delta}$ as a function of δ. This yields

$$|y|^{-\delta} - 1 = -\delta \log|y| \int_0^1 \exp(-\delta\tau \log|y|)d\tau,$$

from which we may conclude that

$$|(|y|^{-\delta} - 1)/\delta| \le \begin{cases} |\log|y||\ |y|^{-1}, & 0 < |y| \le 1, \\ \log|y|, & |y| \ge 1, \end{cases}$$

for $0 < \delta < 1$. As a consequence we obtain the estimate

$$\sup_{1<\alpha<2} |E_\alpha(y)| \le \begin{cases} C_2|\log|y||\ |y|^{-1}, & 0 < |y| \le 1, \\ C_2 \log|y|, & |y| \ge 1, \end{cases}$$

$$C_2 = \sup_{1<\alpha<2} \frac{\Gamma(2-\alpha/2)}{2^{\alpha-1}\pi\Gamma(\alpha/2)}.$$

Since the right-hand side of the above inequality is locally integrable, the application of the dominated convergence theorem is justified. □

6.3.3 Proof of the Sobolev Inequality ($r > 1$)

Next we prove (6.9) (i.e., inequality (6.4) of Theorem 6.1.1 with $\sigma = 1$) for $u \in C_0^\infty(\mathbb{R}^n)$ in the case of $r > 1$ (see Remark 6.1.4). We apply Proposition 6.3.1 if $n \ge 3$, and Proposition 6.3.2 if $n = 1$. For $u \in C_0^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ we have that

$$u(x) = -\int_{\mathbb{R}^n} E(x-y)\Delta u(y)dy.$$

By properties (i) and (ii) in §6.3.5 we then obtain

$$u(x) = \int_{\mathbb{R}^n} \langle\ (\nabla E)(x-y),\ \nabla u(y)\ \rangle dy.$$

(This follows via integration by parts. But observe that E is not continuous for $y = x$, i.e., the method in §4.5.5 cannot be applied directly.) In view of

$$|\nabla E(x)| \le \frac{C}{|x|^{n-1}} \quad (C \text{ is a constant}) \tag{6.22}$$

(see Exercise 6.9 (i)) and by employing the Riesz potential I_1, we have that

$$|u(x)| \leq \int_{\mathbb{R}^n} \frac{C}{|x-y|^{n-1}} |\nabla u(y)| dy = C\ I_1(|\nabla u|). \tag{6.23}$$

Thus, the Hardy–Littlewood–Sobolev inequality implies (6.9) for the case $r > 1$ and $n \geq 2$. □

6.3.4 An Elementary Proof of the Sobolev Inequality ($r = 1$)

The method used in §6.3.2 for the proof of (6.9) does not apply to the case $r = 1$. For this case and $n \geq 2$ we will present a direct proof for functions $u \in C_0^1(\mathbb{R}^n)$ based on the Hölder inequality and the fundamental theorem of calculus.

Employing the latter result, we obtain that

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n) ds \quad (1 \leq i \leq n).$$

This yields

$$|u(x)| \leq \int_{-\infty}^{\infty} |\partial_{x_i} u(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)| ds \quad (1 \leq i \leq n).$$

Forming the product over $i = 1, \ldots, n$ and taking the $1/(n-1)$th power gives us[1]

$$|u(x)|^{\frac{n}{n-1}} \leq \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |\partial_{x_i} u| dx_i \right)^{\frac{1}{n-1}}.$$

Integrating this inequality with respect to the first variable x_1 and then applying the Hölder inequality in the form

$$\|f_1 \cdots f_m\|_1 \leq \prod_{i=1}^{m} \|f_i\|_{p_i}, \quad 1 \leq p_i \leq \infty, \quad \sum_{i=1}^{m} \frac{1}{p_i} = 1,$$

results in

$$\begin{aligned}
&\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}} dx_1 \\
&\quad \leq \left(\int_{-\infty}^{\infty} |\partial_{x_1} u| dx_1 \right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left\{ \int_{-\infty}^{\infty} |\partial_{x_i} u| dx_i \right\}^{\frac{1}{n-1}} dx_1 \\
&\quad \leq \left(\int_{-\infty}^{\infty} |\partial_{x_1} u| dx_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\partial_{x_i} u| dx_i dx_1 \right)^{\frac{1}{n-1}}.
\end{aligned}$$

[1] Here $\prod_{i=1}^{m} a_i$ denotes the product $a_1 a_2 \cdots a_m$.

Iterating this procedure with respect to variables $x_2, \ldots, x_n$ gives us

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \le \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\partial_{x_i} u| dx \right)^{\frac{1}{n-1}}.$$

Next we apply the fact that the geometric mean does not exceed the arithmetic mean, i.e., we use the inequality

$$\left(\prod_{i=1}^{n} a_j \right)^{1/n} \le \frac{1}{n} \sum_{j=1}^{n} a_j, \qquad a_j \ge 0,$$

in order to obtain

$$\|u\|_{n/(n-1)} \le \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\partial_{x_i} u| dx \right)^{1/n} \le \frac{1}{n} \int_{\mathbb{R}^n} \sum_{i=1}^{n} |\partial_{x_i} u| dx$$
$$\le \frac{\sqrt{n}}{n} \|\nabla u\|_1.$$

Hence the proof of (6.9) for the case $r = 1$ and $n \ge 2$ is complete. Note that the case $r > 1$ is obtained as a consequence of the case $r = 1$. Indeed, choosing $s > 1$ such that

$$r^* = \frac{sn}{n-1} = \frac{(s-1)r}{r-1}$$

and applying (6.9) on $u = |v|^s$, formally we deduce

$$\|v\|_{r^*}^s \le \frac{s\sqrt{n}}{n} |v|^{s-1} |\nabla v| \|_1.$$

The Hölder inequality applied to the right-hand side then yields

$$\|v\|_{r^*}^s \le C \|v\|_{r^*}^{s-1} \|\nabla v\|_1$$

with C depending only on r, n. Consequently, (6.9) holds for general $r > 1$. The only problem in this argument lies in the fact that $|v|$ might not be differentiable at 0. But this difficulty can be overcome by approximating $|v|$ by $\sqrt{|v|^2 + \varepsilon^2}$. Since this is analogous to the proof of the lemma in §2.3.3, we omit the details at this point. □

Next we collect some elementary properties of the *Newton potential* $E * f$ of f.

6.3.5 The Newton Potential

Proposition. *Suppose that $n \ge 2$, $f \in C_0^\infty(\mathbb{R}^n)$, and let α be a multi-index. For $x \in \mathbb{R}^n$ we set*

$$E(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & n = 2, \\ |x|^{2-n} / ((n-2)|S^{n-1}|), & n \ge 3. \end{cases}$$

Then the following properties hold:

(i) $E * f \in C^\infty(\mathbb{R}^n)$.
(ii) $\partial_x^\alpha(E * f) = E * \partial_x^\alpha f$ *on* $\mathbb{R}^n$.
(iii) $\partial_{x_j}(E * f) = (\partial_{x_j} E) * f$ *on* $\mathbb{R}^n$, $1 \le j \le n$.
(iv) $-\Delta(E * f) = f$ *on* $\mathbb{R}^n$.
(v) *If* $x \notin \operatorname{supp} f$ *then* $(\partial_x^\alpha(E * f))(x) = ((\partial_x^\alpha E) * f)(x)$.
(vi) *Assume that* $n = 2$ *and* $|\alpha| \ge 1$. *Let* B_R *be an open ball centered at the origin with radius* R *so that* $\operatorname{supp} f \subset B_R$. *Then there exists a constant* $C = C(R, f, n, \alpha)$ *independent of* x *such that*

$$|\partial_x^\alpha(E * f)(x)| \le \frac{C}{|x|^{n-2+|\alpha|}}, \quad |x| \ge 2R.$$

Proof.

(i), (ii) By Exercise 7.1, $E * f$ is continuous. We express the Newton potential as

$$(E * f)(x) = \int_{\mathbb{R}^n} f(x-y)E(y)dy.$$

Next we pick $j \in \{1, \ldots, n\}$ and fix all variables in the vector $x = (x_1, \ldots, x_j, \ldots, x_n) \in \mathbb{R}^n$ except for x_j. Furthermore, we set

$$h(x_j, y) = f(x_1 - y_1, \ldots, x_j - y_j, \ldots, x_n - y_n)E(y), \;\; y = (y_1, \ldots, y_n).$$

Let (a, b) be a bounded open interval containing x_j^0. Theorem 7.2.1 now implies that $H(x_j) = \int_{\mathbb{R}^n} h(x_j, y)dy$ is C^1 on (a, b) and $H'(x_j) = \int_{\mathbb{R}^n} \frac{\partial h}{\partial x_j}(x_j, y)dy$. Let us check that h satisfies the assumptions of Theorem 7.2.1. In fact, §7.2.1 (i) is obvious. Taking $R > 0$ such that $\operatorname{supp} f \subset Q_R := [-R, R]^n, (a, b) \subset [-R, R]$, we obtain

$$\begin{aligned}
&\int_a^b \int_{\mathbb{R}^n} \left| \frac{\partial h}{\partial x_j}(x_j, y) \right| dy\, dx_j \\
&\quad \le \int_{-R}^R \left(\int_{Q_{2R}} \left| \frac{\partial f}{\partial x_j}(x-y) \right| |E(y)| dy \right) dx_j \\
&\quad \le \|\partial_{x_j} f\|_\infty \int_{-R}^R \left(\int_{Q_{2R}} |E(y)| dy \right) dx_j.
\end{aligned}$$

This yields §7.2.1 (ii), since the right-hand side of the above inequality is finite by the local integrability of E. Relation §7.2.1 (iii) is obvious, since E is locally integrable, whereas §7.2.1 (iv) is an immediate consequence of the continuity of $\partial_{x_j} f * E$ (see Exercise 7.1). Hence $f * E$ is partially differentiable with respect to x_j at any point $(x_1, \ldots, x_j^0, \ldots, x_n)$ and we have $\partial_{x_j}(f * E) = (\partial_{x_j} f) * E$ on $\mathbb{R}^n$. This shows in particular that $E * f \in C^1(\mathbb{R}^n)$ and the validity of relation (ii) for the case $|\alpha| = 1$. An easy induction argument with respect to $|\alpha|$ yields the general case.

(iii) Now we write $(E * f)(x) = \int_{\mathbb{R}^n} E(x-y)f(y)dy$. Again we fix all variables except for x_j and set

$$h(x_j, y) = E(x_1 - y_1, \ldots, x_j - y_j, \ldots, x_n - y_n)f(y), \; y = (y_1, \ldots, y_n).$$

In a similar way as before, Theorem 7.2.1 yields that $H(x_j) = \int_{\mathbb{R}^n} h(x_j, y)dy$ is C^1 on an interval (a, b) such that $x_j^0 \in (a, b)$ and we have that $H'(x) = \int_{\mathbb{R}^n} \frac{\partial h}{\partial x_j}(x_j, y)dy$. In fact, h is C^1 with respect to x_j except on the segment

$$\Sigma := \{y = (y_1, \ldots, y_j, \ldots, y_n); \quad y_i = x_i (j \neq i), \quad y_j \in (a, b)\}.$$

On the other hand, the Lebesgue measure of Σ in $\mathbb{R}^n$ is zero.This implies assumption (i) of §7.2.1. Since we have

$$|\partial_{x_j} E(x)| \leq \frac{C_1}{|x|^{n-1}} \quad (x \in \mathbb{R}^n)$$

(see Exercise 6.9 (i)), by Exercise 6.9 (ii), $\partial_{x_j} E$ is locally integrable. The fact that $f \in C_0(\mathbb{R}^n)$ then implies that (iv) of Theorem 7.2.1 is fulfilled (see Exercise 7.1). Here C_k $(k = 1, 2, 3)$ denote constants independent of x. The assertion §7.2.1 (iii) is obvious by the local integrability of E. Finally, we have to check §7.2.1 (ii). Pick R such that $\operatorname{supp} f \subset Q_R$ and $(a, b) \subset [-R, R]$, $\{x_i; \; i \neq j\} \subset [-R, R]$. (Here x_i is the ith coordinate of x.) Employing the estimate for $\partial_{x_j} E$, we see that

$$\begin{aligned} \int_a^b \int_{\mathbb{R}^n} \left| \frac{\partial h}{\partial x_j}(x_j, y) \right| dy dx_j &\leq \int_{-R}^R \left(\int_{Q_R} \frac{C_1}{|x-y|^{n-1}} |f(y)| dy \right) dx_j \\ &\leq C_1 \|f\|_\infty \int_{-R}^R \left(\int_{Q_{2R}} \frac{1}{|y|^{n-1}} dy \right) dx_j \\ &\leq 2C_1 R \|f\|_\infty \int_{Q_{2R}} \frac{dy}{|y|^{n-1}} < \infty. \end{aligned}$$

Note that in the last inequality we used the fact that

$$x + Q_R = \{x + z; \; z \in Q_R\} \subset Q_{2R}.$$

Hence the assertion follows from Theorem 7.2.1.

(iv) This is a consequence of (ii) and Propositions 6.3.1 and 6.3.2.

(v) Assume that $x \notin \operatorname{supp} f$. Let (a, b) be a small neighborhood of x_j, the jth component of x, such that the distance between the segment Σ and $\operatorname{supp} f$ is positive. Here we set

$$h(x_j, y) = \partial_x^\alpha E(x_1 - y_1, \ldots, x_j - y_j, \ldots, x_n - y_n)f(y),$$

where we fixed again the remaining components of x. By construction, h and $\partial_{x_j} h$ are bounded and continuous on $(a, b) \times \operatorname{supp} f$. Relation (v) is again obtained as a consequence of Theorem 7.2.1 and induction with respect to $|\alpha|$.

(vi) Observe that we have $|\partial_x^\alpha E(x)| \le \frac{C_2}{|x|^{n-2+|\alpha|}}$, $x \in \mathbb{R}^n$ (Exercise 6.9 (i)). Relation (v) now implies that

$$\begin{aligned}|\partial_x^\alpha (E * f)(x)| &\le \int_{B_R} |\partial_x^\alpha E(x-y)|\, |f(y)| dy \\ &\le \int_{B_R} \frac{C_2}{|x-y|^{n-2+|\alpha|}} |f(y)| dy \\ &\le \frac{C_2 C_3}{(|x|-R)^{n-2+|\alpha|}} \quad (x \in \mathbb{R}^n)\end{aligned}$$

with $C_3 = \int_{B_R} |f(y)| dy$. If $|x| \ge 2R$ we have that $|x| - R \ge |x|/2$. This proves (vi). □

Remark. The above proposition, establishing differentiability and representations of derivatives of $E * f$, can be generalized to f that are not necessarily compactly supported. Indeed, the assertion remains true if f and its derivatives decay sufficiently fast for large x.

(i) For example, if f is continuous, bounded, and integrable on $\mathbb{R}^n$, then for $1 \le j \le n$, $(\partial_{x_j} E) * f$ is continuous on $\mathbb{R}^n$.
(ii) If $f \in C^1(\mathbb{R}^n)$ and f and $|\nabla f|$ are bounded and integrable on $\mathbb{R}^n$, then we have $(\partial_{x_j} E) * f \in C^1(\mathbb{R}^n)$ and $\partial_{x_i}((\partial_{x_j} E) * f) = (\partial_{x_i} E) * (\partial_{x_j} f)$ on $\mathbb{R}^n$ for $1 \le i,\ j \le n$. This follows by similar arguments as in the proof of Proposition (II) in §4.1.4 (see Exercise 7.4).
(iii) Iteratively it can be shown that $f \in C^r(\mathbb{R}^n)$ such that $\partial_x^\alpha f$ is bounded and integrable on $\mathbb{R}^n$ for every multi-index α with $|\alpha| \le r$ implies that $\partial_{x_j} E * f \in C^r(\mathbb{R}^n)$ and that $\partial_x^\alpha((\partial_{x_j} E) * f) = (\partial_{x_j} E) * (\partial_x^\alpha f)$ on $\mathbb{R}^n$.
(iv) Similarly the following can be shown: Let $f \in C(\mathbb{R}^n \times (t_0, t_1))$ and $\ell, r \in \mathbb{N} \cup \{0\}$. Suppose that $\partial_t^b \partial_x^\alpha f \in C(\mathbb{R}^n \times (t_0, t_1))$ and

$$\sup_{t_0<t<t_1} \|\partial_t^b \partial_x^\alpha f\|_\infty(t) < \infty, \qquad \sup_{t_0<t<t_1} \|\partial_t^b \partial_x^\alpha f\|_1(t) < \infty,$$

for all $0 \le b \le \ell$ and all multi-indices α with $|\alpha| \le r$. Then $(\partial_{x_j} E) * f$ is ℓ-times partially differentiable with respect to t and r-times partially differentiable with respect to x. Furthermore, we have that $\partial_t^b \partial_x^\alpha((\partial_{x_j} E) * f) = (\partial_{x_j} E) * (\partial_t^b \partial_x^\alpha f) \in C(\mathbb{R}^n \times (t_0, t_1))$ on $\mathbb{R}^n \times (t_0, t_1)$ for all $0 \le b \le \ell$ and $|\alpha| \le r$.

6.3.6 Remark on Differentiation Under the Integral Sign

A straightforward calculation shows that $\Delta E(x) = 0$ $(x \ne 0)$. But observe that the formal calculation

$$\Delta(E * f) = (\Delta E) * f = 0$$

is wrong. In fact, it contradicts relation (iv) of the last section. In this situation differentiation and integration do not commute, since the second-order derivatives of E are not locally integrable. In other words, for $x \in \operatorname{supp} f$ and $|\alpha| = 2$ the equality $\partial_x^\alpha (E * f)(x) = ((\partial_x^\alpha E) * f)(x)$ for convolutions does not hold in the classical sense. However, if we regard ΔE as a "distribution," it can be shown that $-\Delta E = \delta$, where δ denotes the Dirac delta distribution. Then we obtain

$$\Delta(E * f) = (\Delta E) * f = (\Delta E) * f = -\delta * f = -f.$$

This calculation is compatible with (iv). In the theory of distributions, convolutions and differentials always commute. The theory of distributions will not be treated within this book. For the interested reader we therefore refer to the monographs [Kakita 1985], [Schwartz 1966], [Treves 1967].

6.4 Boundedness of Singular Integral Operators

In §2.4.1 we estimated the L^q-norm of first-order derivatives of the velocity by the L^q-norm of the vorticity ($1 < q < \infty$). This estimate for singular integral operators of this form is usually called the Calderón–Zygmund inequality. There are many generalized versions known in the literature. Here we present a proof of this inequality for arbitrary dimension $n \geq 2$ in the form that we used in §2.4.1.

6.4.1 Cube Decomposition

Let K_0 be a closed cube in $\mathbb{R}^n$ (note that throughout this book by a "cube" we always mean a cube whose edges are parallel to the coordinate axis with length strictly larger than 0) and let f be a nonnegative integrable function on K_0 (i.e., $f \geq 0$ on K_0 and $f \in L^1(K_0)$). Depending on the values of f, we divide K_0 into smaller cubes. To this end, pick t greater than the mean value of f in K_0. Then we have

$$\int_{K_0} f(x)\ dx \leq t\ |K_0|.$$

(Here $|K_0| := \int_{K_0} dx$ denotes the volume of K_0.) We divide K_0 into 2^n congruent closed cubes such that the edges of the smaller cubes are half the size of the edges of K_0. Observe that the new cubes do not intersect, except at the boundary. We call the smaller cubes children of K_0, and K_0 the parent of the smaller cubes. Each child K' of K_0 on which

$$\int_{K'} f(x)\ dx \leq t\ |K'|,$$

is satisfied, we divide again into 2^n children (grandchildren of K_0 so to speak) and iterate this procedure. (Note that this procedure has to be carried out infinitely many times, since each child satisfying the above inequality has at least one grandchild satisfying this inequality as well.) On the other hand, for small cubes that are not divided we always have

$$\int_K f(x)\ dx > t\ |K|. \tag{6.24}$$

Let $\mathcal{K}$ denote the set of all cubes obtained by the above procedure and on which (6.24) is satisfied. By construction $\mathcal{K}$ is at most countable. Let $\tilde{K}$ be the parent of the element K of $\mathcal{K}$. Since $\tilde{K} \notin \mathcal{K}$, we obtain

$$\int_K f(x)\ dx \le \int_{\tilde{K}} f(x)\ dx \le t|\tilde{K}|.$$

On the other hand, by virtue of $|\tilde{K}| = 2^n|K|$, this inequality and (6.24) imply

$$t < \frac{1}{|K|} \int_K f(x)\ dx \le 2^n t. \tag{6.25}$$

Next we set

$$G = K_0 \setminus \bigcup_{K \in \mathcal{K}} K.$$

(In other words, G is the set of points not contained in the union of all sets K of the class $\mathcal{K}$.) In what follows, we denote by $L(Q)$ the length of the edges of a cube Q and set for simplicity $L := L(K_0)$. By definition, for any point $x \in G$ there exists a sequence $\{Q_m\}_{m=1}^{\infty}$ of closed cubes $Q_m = Q_m(x)$ such that $L(Q_m) = L/2^m$, $x \in Q_m$, and such that

$$\int_{Q_m(x)} f(y)\ dy \le t\ |Q_m(x)|. \tag{6.26}$$

In particular, $L(Q_m) \to 0$ as $m \to \infty$. Lebesgue's differentiation theorem (see next claim) therefore yields

$$\lim_{m\to\infty} \frac{1}{|Q_m(x)|} \int_{Q_m(x)} f(y)\ dy = f(x),$$

for almost every point x in G. Hence, by (6.26), we obtain $f(x) \le t$ for almost all x in G. In what follows we write for short

$$f(x) \le t, \quad \text{a.a. } x \in G. \tag{6.27}$$

Summarizing, we see that on G the function f is bounded from above by t, whereas outside G the mean value of f is bounded from above as in (6.25).

Theorem (Lebesgue's differentiation theorem). *Assume that the function f is integrable on the closed set (more generally on a Lebesgue measurable set) $X \subseteq \mathbb{R}^n$. For each point x in X, let $Q_m = Q_m(x)$ be a sequence of closed cubes such that $x \in Q_m$ for all $m \in \mathbb{N}$ and $L(Q_m) \to 0$ if $m \to \infty$. Then the mean value of f on $Q_m(x)$ converges to $f(x)$ as $m \to \infty$ for almost all $x \in X$, i.e.,*

$$\lim_{m\to\infty} \frac{1}{|Q_m(x)|} \int_{Q_m(x)\cap X} f(y)\,dy = f(x), \quad a.a.\ x \in X. \tag{6.28}$$

In case f is continuous at the interior point $x \in X$, it is easy to show that the mean value of f in $Q_m(x)$ converges to $f(x)$ as $m \to \infty$. However, if f is only integrable, the convergence of (6.28) is not valid for each $x \in X$ in general. It is just valid for "almost all $x \in X$". The proof requires the covering theorem. Here we do not give a proof of this result. Instead, the interested reader is recommended to consult [Yosida 1976], [Rudin 1987, Theorem 8.8].

For later purpose we summarize the properties of the cubes obtained by the procedure above.

Lemma. *Let K_0 be a closed cube in $\mathbb{R}^n$ and $f \in L^1(K_0)$ such that $f \geq 0$ on K_0. Assume that $t > 0$ satisfies*

$$\int_{K_0} f(x)\,dx \leq t|K_0|.$$

Then there exists an at most countable sequence $\{K_j\}_{j=1}^{\infty}$ of closed cubes satisfying

$$t < \frac{1}{|K_j|} \int_{K_j} f(x)\,dx \leq 2^n t, \quad K_j \subset K_0 \quad (j = 1, 2, 3, \ldots),$$

$$f(x) \leq t, \quad a.a.\ x \in G := K_0 \setminus \bigcup_{j=1}^{\infty} K_j,$$

$$\mathrm{int}\ K_i \cap \mathrm{int}\ K_j = \phi \quad (i \neq j).$$

Here int A *denotes the interior of the set A. (It is possible that $\{K_j\}$ is empty or finite.)*

For a clear statement of the next result we recall some facts and notation from §6.3.

Let $x \in \mathbb{R}^n$. The function $E(x)$ was defined by $E(x) = -\frac{1}{2\pi}\log|x|$ if $n = 2$, and by $E(x) = |x|^{2-n}/(|S^{n-1}|(n-2))$ if $n \geq 3$. Pick $i, j \in \{1, \ldots, n\}$ and set $w := E * f = f * E$ for $f \in C_0^\infty(\mathbb{R}^n)$. The operator that maps f to $\partial_{x_i}\partial_{x_j} w$ is a linear operator from $C_0^\infty(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ (see Proposition 6.3.5 (i)). Next we will prove the L^p $(1 < p < \infty)$ boundedness of this operator, which is known as the *Calderón–Zygmund inequality* .

6.4.2 The Calderón–Zygmund Inequality

Theorem. *Assume that $1 < p < \infty$ and $n \geq 2$. Then there exists a constant C depending only on p, n such that for any $f \in C_0^\infty(\mathbb{R}^n)$,*

$$\|\partial_{x_i}\partial_{x_j}(E * f)\|_p \leq C\|f\|_p \quad (1 \leq i,\ j \leq n). \tag{6.29}$$

Hence, for each $i, j \in \{1, \dots, n\}$ the operator defined by

$$T : f \mapsto \partial_{x_i}\partial_{x_j}(E * f)$$

extends uniquely to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ (see §7.3).

Remark.

(i) Let $f \in C^1(\mathbb{R}^n)$ be such that f and $|\nabla f|$ are bounded and integrable on $\mathbb{R}^n$. In view of Remark (ii) in §6.3.5, we obtain $(\partial_{x_j}E) * f \in C^1(\mathbb{R}^n)$ $(1 \leq j \leq n)$. Moreover, Tf is represented through the integral

$$Tf = \partial_{x_i}((\partial_{x_j}E) * f) \qquad \text{on } \mathbb{R}^n.$$

As mentioned in Remark 6.2.1, this is a nontrivial result and a good exercise in Lebesgue integration theory. Observe that in the case $n = 2$ the Calderón–Zygmund inequality implies the following estimate for the operator $\mathbf{K}$ defined in (2.6), which we frequently applied in Chapter 2. Indeed, for $1 < q < \infty$ the L^q-boundedness of the operator T yields the existence of a constant $\overline{C}$ depending only on q such that

$$\|\nabla(\mathbf{K} * f)\|_q \leq \overline{C}\|f\|_q$$

for any $f \in C^1(\mathbb{R}^2)$ such that f and $|\nabla f|$ are bounded and integrable on $\mathbb{R}^2$.

(ii) Suppose that $2 < r < \infty$ and again $f \in C^1(\mathbb{R}^2)$ such that f and $|\nabla f|$ are bounded and integrable on $\mathbb{R}^2$. Combining the Gagliardo–Nirenberg inequality (§6.1.1), the Hardy–Littlewood–Sobolev inequality (§ 6.2.1), and the Calderón–Zygmund inequality in the form given in (i) for the operator $\mathbf{K}$, we deduce that

$$\begin{aligned}\|\mathbf{K} * f\|_\infty &\leq \tilde{C}\|\mathbf{K} * f\|_r^{1-2/r}\|\nabla(\mathbf{K} * f)\|_r^{2/r} \\ &\leq \tilde{C}C^{1-2/r}\overline{C}^{2/r}\|f\|_q^{1-2/r}\|f\|_r^{2/r}, \quad 1/q = 1/r + 1/2,\end{aligned}$$

with a constant $\tilde{C} > 0$ independent of f. This estimate is used in the proof of (ii) of Theorem in §2.4.1. Note that without the assumption on the derivative of f, $\mathbf{K} * f$ is not C^1 on $\mathbb{R}^2$ in general. In this case ∇ has to be realized as a differential in the sense of distributions. Nevertheless, the estimate

$$\|\mathbf{K} * h\|_\infty \le \tilde{C} C^{1-2/r} \bar{C}^{2/r} \|h\|_q^{1-2/r} \|h\|_r^{2/r},$$

where $1/q = 1/r + 1/2$ and $2 < r < \infty$, can be proved without distribution theory for any bounded integrable function $h \in C(\mathbb{R}^2)$ by approximating h through the functions $f_j := G_{1/j} * h$ $(j = 1, 2, \ldots)$. By the assumption of h, we have $\|h\|_q < \infty$ and $\|h\|_r < \infty$. Hence, employing Exercise 7.3, we obtain $\|f_j - h\|_q \to 0$, and $\|f_j - h\|_r \to 0$ $(j \to \infty)$. On the other hand, by $\|h\|_1 < \infty$, §4.1.6, and §1.1.3, we have that $f_j \in C^\infty(\mathbb{R}^2)$ and that $\partial_x^\alpha f_j$ is bounded and integrable on $\mathbb{R}^2$ for any multi-index α. This implies

$$\|\mathbf{K} * f_j\|_\infty \le \tilde{C} C^{1-2/r} \bar{C}^{2/r} \|f_j\|_q^{1-2/r} \|f_j\|_r^{2/r}.$$

Thus, by taking the limit $j \to \infty$, the desired estimate for h follows, provided we can show that $\mathbf{K} * f_j$ converges (pointwise) to $\mathbf{K} * h$ on $\mathbb{R}^2$ for almost all $x \in \mathbb{R}^2$. Employing the method applied at the end of §2.4.1 (there it was applied to the operator I_1), we obtain $\|\mathbf{K} * h\|_\infty \le 2\pi\|h\|_\infty + \|h\|_1 < \infty$. On the other hand, exchanging the order of integration (§7.2.2) yields $\mathbf{K} * f_j = G_{1/j} * (\mathbf{K} * h)$. By Remark 6.3.5, $\mathbf{K} * h$ is continuous. But then Corollary 4.2.4 implies that $\mathbf{K} * f_j$ converges pointwise to $\mathbf{K} * h$ as $j \to \infty$ on $\mathbb{R}^2$.

(iii) Inequality (6.29) is subtle in the following sense: For example, in the case $n = 2$ and $i = j = 1$ we have

$$\partial_{x_1}\partial_{x_1} E(x) = -\frac{1}{2\pi} \frac{-x_1^2 + x_2^2}{|x|^4} =: K(x), \tag{6.30}$$

and we see that $\partial_{x_1}\partial_{x_1} E$ is not contained in $L^1(\mathbb{R}^2)$. In spite of this fact it seems that inequality (6.29) might be obtained by regarding K as an element of $L^1(\mathbb{R}^2)$ and by applying the Young inequality to $K * f$ (except for the cases $p = 1$ and $p = \infty$). Note that $K(x)$ as defined in (6.30) satisfies the following properties for $n = 2$:

(i) $K(\lambda x) = \lambda^{-n} K(x)$ $(\lambda > 0)$ (positive homogeneity of order n),
(ii) $\int_{|x|=1} K(x) d\sigma = 0$.

Furthermore, K is smooth except at $x = 0$. These properties also hold for general dimension n and for $\partial_x^\alpha E$ with $|\alpha| = 2$. However, for general K satisfying (i) and (ii) it is a priori not clear how to define $Tf = K * f$, in view of the fact that it is not integrable near $x = 0$. An operator T defined with a nonintegrable K satisfying (i) and (ii) is called a *singular integral operator*. Due to their significance, singular integral operators have been extensively studied in the literature (see, e.g., [Stein 1993]). However, this will not be a topic of the monograph at hand. Here we just consider the special case of inequality (6.29). The proof of (6.29) presented in this book will be based on real-analytic methods as prepared in the previous section.

The boundedness of singular integral operators on $L^p(\mathbb{R}^n)$ was first proved in [Calderon Zygmund 1952]. We also refer to [Tanabe 1981] for a precise proof of the Calderón–Zygmund inequality and the Marcinkiewicz interpolation theorem. The proof of the Sobolev inequality and the properties of the Newton potential given in §6.3.5 presented here are essentially the same as given in [Gilbarg Trudinger 1983], which is a well-known and famous textbook on elliptic equations.

The proof of (6.29) can be outlined as follows: First we prove the inequality for $p = 2$ (§6.4.3). Next we show that

$$|\partial_{x_i}\partial_{x_j}(E * f)|_{1,\infty} \leq C\|f\|_1,$$

where $\|\cdot\|_{1,\infty}$ denotes the norm in the Lorentz space $L^{1,\infty}(\mathbb{R}^n)$ (§6.4.4) (at this point we emphasize that the corresponding estimate in L^1, i.e., if we replace $\|\cdot\|_{1,\infty}$ by $\|\cdot\|_1$, does not hold). This is the most intricate part of the proof. Here we employ the dividing procedure for cubes introduced in §6.4.1. Once the estimate in the Lorentz norm is derived, we can apply the Marcinkiewicz interpolation theorem in order to obtain (6.29) for $1 < p \leq 2$. The case $p > 2$ then follows by a duality argument.

6.4.3 L^2 Boundedness

Proposition. *Let $n \geq 2$. Set $w = E * f$ for $f \in C_0^\infty(\mathbb{R}^n)$. Then we have*

$$\sum_{1\leq i,j\leq n} \|\partial_{x_i}\partial_{x_j} w\|_2^2 = \|f\|_2^2.$$

In particular, for fixed $i, j \in \{1, \ldots, n\}$ the operator T defined through $Tf = \partial_{x_i}\partial_{x_j} w$ extends to a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ (§7.3).

Proof. Take an open ball B_R that contains the support of f and that is centered at the origin. By part (i) of the proposition in §6.3.5 we know that $w \in C^\infty(\mathbb{R}^n)$. Furthermore, the propositions in §6.3.1, §6.3.2, and §6.3.5 (iv) yield $-\Delta w = -\Delta(E * f) = -E * \Delta f = f$. Employing integration by parts twice and the fact that $f = 0$ on ∂B_R, we therefore obtain

$$\begin{aligned} &\sum_{1\leq i,j\leq n} \int_{B_R} |\partial_{x_i}\partial_{x_j} w|^2\, dx \\ &\quad = -\sum_{1\leq i,j\leq n} \int_{B_R} (\partial_{x_i}\partial_{x_i}\partial_{x_j} w)\partial_{x_j} w\, dx \\ &\quad\quad + \sum_{j=1}^{n} \int_{\partial B_R} \left(\frac{\partial}{\partial\nu}(\partial_{x_j} w)\right) \partial_{x_j} w\, d\sigma \end{aligned}$$

$$= \int_{B_R} |\Delta w|^2 \, dx - \int_{\partial B_R} \Delta w \frac{\partial w}{\partial \nu} d\sigma + \int_{\partial B_R} \left\langle \nabla w, \ \frac{\partial}{\partial \nu} \ (\nabla w) \right\rangle d\sigma$$
$$= \int_{B_R} |f|^2 \, dx + \int_{\partial B_R} \left\langle \nabla w, \ \frac{\partial}{\partial \nu} \ (\nabla w) \right\rangle d\sigma.$$

Here $\partial/\partial\nu$ denotes the differential in the outer normal direction at ∂B_R, and σ the surface measure on ∂B_R. Now we fix $R_0 > 0$ such that $\operatorname{supp} f \subseteq B_{R_0}$. Part (vi) of Proposition 6.3.5 then implies the existence of constants C_1, C_2 independent of x such that

$$|\partial_{x_i} w(x)| \leq \frac{C_1}{|x|^{n-1}}, \ |\partial_{x_i}\partial_{x_j} w(x)| \leq \frac{C_2}{|x|^n}, \quad x \in \mathbb{R}^n \backslash B_{2R_0}, \ 1 \leq i, j \leq n.$$

Thus, for $R \geq 2R_0$ we deduce

$$\left| \int_{\partial B_R} \langle \nabla w, \ \frac{\partial}{\partial \nu} \ (\nabla w) \rangle d\sigma \right| \leq \frac{C_2}{R^{n-1}} \frac{C_4}{R^n} R^{n-1} |S^{n-1}| \to 0 \quad (R \to \infty).$$

This results in

$$\sum_{1 \leq i,j \leq n} \int_{\mathbb{R}^n} |\partial_{x_i}\partial_{x_j} w(x)|^2 \, dx = \int_{\mathbb{R}^n} |f(x)|^2 \, dx. \qquad \square$$

6.4.4 Weak L^1 Estimate

Proposition. *Let $n \geq 2$, $f \in C_0^\infty(\mathbb{R}^n)$, and $w = E * f$. Then there exists a constant $C = C(n)$ depending only on n such that*

$$|\partial_{x_i}\partial_{x_j} w|_{1,\infty} \leq C\|f\|_1, \quad f \in C_0^\infty(\mathbb{R}^n) \quad (1 \leq i, \ j \leq n).$$

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$. For fixed $i, j \in \{1, \ldots, n\}$ and $Tf = \partial_{x_i}\partial_{x_j} w$ it is sufficient to show that the distribution function m_{Tf} of Tf satisfies

$$m_{Tf}(t) \leq \frac{C\|f\|_1}{t}, \quad t > 0.$$

For this purpose, fix $t > 0$. Next let $K_0 \subseteq \mathbb{R}^n$ be a closed cube containing the support of f such that

$$\int_{K_0} |f(x)| dx \leq t|K_0|.$$

Observe that in view of the compact support of f this can always be achieved by choosing K_0 large enough. By decomposing $\mathbb{R}^n$ according to the results stated in Lemma 6.4.1, we next will split f into a "good" part g and a "bad" part b. Let $\{K_\ell\}_{\ell=1}^\infty$ be the family of closed cubes in $\mathbb{R}^n$ enjoying the properties of Lemma 6.4.1 for $|f|$. We set

$$g(x) = \begin{cases} f(x), & x \in G := K_0 \backslash \bigcup_{\ell=1}^{\infty} K_\ell, \\ \dfrac{1}{|K_\ell|} \displaystyle\int_{K_\ell} f(y) dy, & x \in K_\ell, \quad \ell = 1, 2, \ldots, \end{cases}$$

and $b(x) = f(x) - g(x)$. First we observe that $g = b = 0$ on $\mathbb{R}^n \backslash K_0$ and that $g, b \in L^2(\mathbb{R}^n)$. Furthermore, Lemma 6.4.1 implies

$$|g(x)| \leq 2^n t, \quad \text{a.a. } x \in K_0,$$
$$b(x) = 0, \quad x \in G,$$
$$\int_{K_\ell} b \, dx = 0, \quad \ell = 1, 2, 3, \ldots.$$

Since T is linear and bounded on $L^2(\mathbb{R}^n)$ (§6.4.3), we may write $Tf = Tg + Tb$. Hence, similarly as in (6.14), we obtain

$$m_{Tf}(t) \leq m_{Tg}(t/2) + m_{Tb}(t/2).$$

This gives us the possibility to estimate the term related to g and the term related to b separately, which will be done in (a) and (b) respectively.

(a) Estimate of Tg.

The L^2-boundedness of T implies $\|Tg\|_2 \leq \|g\|_2$. Applying the Chebyshev inequality (§6.2.2) then gives us

$$m_{Tg}(t/2) \leq \left(\frac{2}{t}\right)^2 \int_{\mathbb{R}^n} |Tg|^2 dx \leq \left(\frac{2}{t}\right)^2 \int_{K_0} |g|^2 dx.$$

The fact that $|g| \leq 2^n t$ yields

$$m_{Tg}(t/2) \leq \frac{2^{n+2}}{t} \int_{K_0} |g| \, dx.$$

By the definition of g it is clear that $\|g\|_1 \leq \|f\|_1$. Consequently,

$$m_{Tg}(t/2) \leq 2^{n+2} \|f\|_1 / t.$$

(b) Estimate of Tb.

(The First Step: Decomposition of b)

Here we first introduce a further decomposition of b as $b(x) = \sum_{\ell=1}^{\infty} b_\ell(x)$ a.a. $x \in \mathbb{R}^n$, where

$$b_\ell(x) = \begin{cases} b(x), & x \in K_\ell, \\ 0, & x \notin K_\ell, \end{cases}$$

for $\ell = 1, 2, 3, \dots$. (Note that the decomposition of f as $f = g + \sum_{\ell=1}^{\infty} b_\ell$ with g and b_ℓ, $\ell = 1, 2, 3, \dots$, as above is called the *Calderón–Zygmund decomposition.*) Since f and g are supported in K_0 and bounded, b is so as well. By the bounded convergence theorem (see §7.1.1) this implies that $\lim_{m\to\infty} \|\sum_{\ell=1}^{m} b_\ell - b\|_2 = 0$, i.e., b can be regarded as the limit of the series $\sum_{\ell=1}^{\infty} b_\ell$ in the L^2-sense. Thus, by the boundedness of T in $L^2(\mathbb{R}^n)$, also $A_k := \sum_{\ell=1}^{k} Tb_\ell$ converges to Tb as $k \to \infty$ in the L^2-sense. By general facts of integration theory (see for example [Ito 1963, Theorem 22.2], [Rudin 1987, Theorem 3.12]) this in particular yields the existence of a subsequence of A_k converging pointwise to Tb for almost all $x \in \mathbb{R}^n$.

(The Second Step: Approximation of b_ℓ)

In order to estimate Tb, we intend to employ the integral representation for the operator T. To this end we will approximate $b_\ell \in L^\infty(K_\ell)(\subset L^2(K_\ell))$ in $L^2(K_\ell)$ by elements in $C_0^\infty(\operatorname{int} K_\ell)$ with vanishing mean values on K_ℓ. For the construction of such a sequence first we may choose $\{b_{\ell m}\}_{m=1}^{\infty} \subseteq C_0^\infty\ (\mathbb{R}^n)$ satisfying supp $b_{\ell m} \subset$ int K_ℓ and

$$\lim_{m\to\infty} \|b_{\ell m} - b_\ell\|_2 = 0 \tag{6.31}$$

(Exercise 7.3). In particular, this implies that

$$\lim_{m\to\infty} \int_{K_\ell} b_{\ell m}\ dx = \int_{K_\ell} b_\ell\ dx = 0. \tag{6.32}$$

Now observe that

$$b_{\ell m} - \frac{1}{|K_\ell|} \int_{K_\ell} b_{\ell m}\ dx$$

has vanishing mean value in K_ℓ. Therefore, by (6.32), it converges to b_ℓ on $L^2(K_\ell)$. However, the support of this function is not compact in $\operatorname{int} K_\ell$. To make this function compactly supported we apply a cut-off technique as follows. We choose $\psi_{\ell m} \in C_0^\infty(\operatorname{int} K_\ell)$ satisfying $\psi_{\ell m} = 1$ on the support of $b_{\ell m}$, $0 \le \psi_{\ell m} \le 1$, and $\lim_{m\to\infty} \psi_{\ell m}(x) = 1$ for all $x \in$ int K_ℓ. Note that such a $\psi_{\ell m}$ is easily constructed by employing the function $\theta(\tau) = q(2-\tau)/(q(2-\tau)+q(\tau-1))$ as defined in the proof of Theorem 4.4.2. In fact, we may set $\theta_\varepsilon(\tau) = \theta(2+(\tau+\varepsilon-2)/\varepsilon)$ for $0 < \varepsilon < 1$ and $\tau \in \mathbb{R}$, where $\theta \in C^\infty(\mathbb{R})$ is a function satisfying $\theta(\tau) = 0$ for $\tau \ge 2$, $\theta(\tau) = 1$ for $\tau \le 1$, and $0 \le \theta(\tau) \le 1$ for $\tau \in \mathbb{R}$. This implies that $\theta_\varepsilon(\tau) = 0$ for $\tau \ge 2-\varepsilon$, $\theta_\varepsilon(\tau) = 1$ for $\tau \le 2-2\varepsilon$, and $0 \le \theta_\varepsilon \le 1$. Let $(x_{\ell 1}, \dots, x_{\ell n})$ denote the center of K_ℓ, and L_ℓ the length of each edge of K_ℓ. Since the support of $b_{\ell m}$ is compact in K_ℓ, by choosing $1 > \varepsilon_\ell(m) > 0$ suitably, we obtain

$$\operatorname{supp} b_{\ell m} \subset \Big\{ x = (x_1, \dots, x_n) \in K_\ell; \\ |x_i - x_{\ell i}| \le \frac{L_\ell}{2}(1 - \varepsilon_\ell(m)),\ i = 1, 2, \dots, n \Big\}.$$

Moreover, observe that we may choose the $\varepsilon_\ell(m)$ in a way such that $\varepsilon_\ell(m) \to 0$ $(m \to \infty)$. Now set

$$\psi_{\ell m}(x) = \prod_{i=1}^{n} \theta_{\varepsilon_\ell(m)} \left(\frac{4|x_i - x_{\ell i}|}{L_\ell} \right), \quad x \in K_\ell.$$

Then we have that $\operatorname{supp} \psi_{\ell m} \subset \operatorname{int} K_\ell$ and that $\psi_{\ell m} = 1$ on $\operatorname{supp} b_{\ell m}$. By virtue of $\varepsilon_\ell(m) \to 0$ $(m \to \infty)$, we also see that $\lim_{m\to\infty} \psi_{\ell m}(x) = 1$, $x \in \operatorname{int} K_\ell$. The smoothness of $\psi_{\ell m}$ and the property that $0 \le \psi_{\ell m} \le 1$ are obvious by the definition of θ_ε.

Utilizing the function $\psi_{\ell m}$, we set

$$\widetilde{b_{\ell m}}(x) = b_{\ell m}(x) - \left(\int_{K_\ell} b_{\ell m}(y)dy \right) \psi_{\ell m}(x) / \int_{K_\ell} \psi_{\ell m}(y)dy.$$

By definition we therefore obtain $\int_{K_\ell} \widetilde{b_{\ell m}}(x)dx = 0$ and $\widetilde{b_{\ell m}} \in C_0^\infty(\operatorname{int} K_\ell)$. Furthermore, (6.31) and (6.32) imply that

$$\lim_{m\to\infty} \|\widetilde{b_{\ell m}} - b_\ell\|_2 = 0.$$

Thus, we may proceed under the assumption that $b_{\ell m} \in C_0^\infty(\operatorname{int} K_\ell)$ satisfies (6.31) and

$$\int_{K_\ell} b_{\ell m}(x)dx = 0. \tag{6.33}$$

(The Third Step: Estimate of $Tb_{\ell m}$ outside K_ℓ)

For $b_{\ell m}$ we may write

$$(Tb_{\ell m})(x) = \partial_{x_i}\partial_{x_j} \left(\int_{K_\ell} E(x-y) b_{\ell m}(y)dy \right), \quad x \in \mathbb{R}^n.$$

If $x \notin K_\ell$, we can interchange differentiation and integration (§6.3.5 (v)) to obtain

$$(Tb_{\ell m})(x) = \int_{K_\ell} (\partial_{x_i}\partial_{x_j} E)(x-y) b_{\ell m}(y)dy.$$

Let $\overline{y}$ denote the center of K_ℓ. Since $b_{\ell m}$ has vanishing mean value, the integral $Tb_{\ell m}$ can be represented as

$$(Tb_{\ell m})(x) = \int_{K_\ell} \{(\partial_{x_i}\partial_{x_j} E)(x-y) - (\partial_{x_i}\partial_{x_j} E)(x-\overline{y})\} b_{\ell m}(y)dy.$$

By the integral form of the mean value theorem (§1.1.6), and the fact that $|\partial_x^\alpha E(x)| \le C_0 |x|^{-n+2-|\alpha|}$ (Exercise 6.9 (i)), we obtain

$$\begin{aligned} &|\partial_{x_i}\partial_{x_j} E(x-y) - \partial_{x_i}\partial_{x_j} E(x-\overline{y})| \\ &\quad \le |y-\overline{y}| \int_0^1 |(\nabla \partial_{x_i}\partial_{x_j} E)(x-y+\theta(y-\overline{y}))| d\theta \\ &\quad \le |y-\overline{y}| C_1 (\operatorname{dist}(x, K_\ell))^{-n-1} \end{aligned}$$

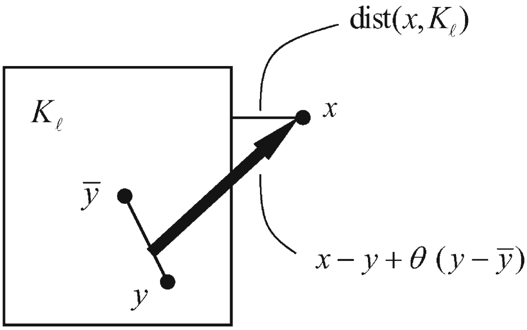

Figure 6.2. Estimates by $\text{dist}(x, K_\ell) : |x - y + \theta(y - \bar{y})| \geq \text{dist}(x, K_\ell)$.

(see Figure 6.2.). Here $\text{dist}\,(x, K_\ell) := \inf\{|x - y|; y \in K_\ell\}$ denotes the distance between x and K_ℓ and δ_ℓ the diameter of K_ℓ (which is twice the distance of $\overline{y}$ to a vertex of K_ℓ). Hence, for $x \notin K_\ell$ we deduce the estimate

$$|(Tb_{\ell m})(x)| \leq C_1 \delta_\ell (\text{dist}\,(x, K_\ell))^{-n-1} \int_{K_\ell} |b_{\ell m}| dy, \tag{6.34}$$

with a constant $C_1 > 0$ depending only on the dimension n. Next, let B^ℓ denote the ball centered at $\overline{y}$ with radius δ_ℓ. We integrate (6.34) over $\mathbb{R}^n \backslash B^\ell$. Since $\text{dist}\,(x, K_\ell) \geq |x - \overline{y}| - \delta_\ell/2$ for $x \in \mathbb{R}^n \backslash B^\ell$, we obtain

$$\int_{\mathbb{R}^n \backslash B^\ell} |(Tb_{\ell m})(x)| dx \leq C_1 \delta_\ell \int_{|x| \geq \delta_\ell} \frac{1}{(|x| - \delta_\ell/2)^{n+1}} dx \int_{K_\ell} |b_{\ell m}(y)| dy.$$

Introducing polar coordinates, the inner integral can be estimated as

$$\begin{aligned}
\int_{|x| \geq \delta_\ell} \frac{1}{(|x| - \delta_\ell/2)^{n+1}} dx &= |S^{n-1}| \int_{\delta_\ell}^{\infty} \frac{r^{n-1}}{(r - \delta_\ell/2)^{n+1}} dr \\
&= |S^{n-1}| \int_{\delta_\ell/2}^{\infty} \frac{(\rho + \delta_\ell/2)^{n-1}}{\rho^{n+1}} d\rho \\
&\leq |S^{n-1}| \int_{\delta_\ell/2}^{\infty} \frac{(2\rho)^{n-1}}{\rho^{n+1}} d\rho = 2^{n-1} |S^{n-1}| \frac{2}{\delta_\ell},
\end{aligned}$$

and we conclude with

$$\int_{\mathbb{R}^n \backslash B^\ell} |(Tb_{\ell m})(x)| dx \leq C_2 \int_{K_\ell} |b_{\ell m}(y)| dy.$$

(The Fourth Step: Estimate of Tb)

Observe that $\|b_{\ell m} - b_\ell\|_1 \leq |K_\ell|^{1/2}\|b_{\ell m} - b_\ell\|_2 \to 0$ as $m \to \infty$. This implies that a suitable subsequence of $Tb_{\ell m}$ converges to Tb_ℓ a.a. $x \in \mathbb{R}^n$. Fatou's lemma (§7.1.2) then yields that

$$\int_{\mathbb{R}^n \setminus B^\ell} |Tb_\ell| dx \leq C_2 \int_{K_\ell} |b_\ell| dy.$$

Let $\{A_{k(i)}\}$ be the subsequence of $\{A_k\}$ converging to Tb a.a. $x \in \mathbb{R}^n$, which exists according to the first step. Summing up the inequality above from $\ell = 1$ to $\ell = k(i)$, taking the limit $i \to \infty$, and using Fatou's lemma again, we obtain

$$\int_{G^*} |Tb| dx \leq C_2 \sum_{\ell=1}^{\infty} \int_{K_\ell} |b_\ell| dy = C_2 \int_{K_0} |b| dy, \; G^* = \mathbb{R}^n \setminus F^*, \; F^* = \bigcup_{\ell=1}^{\infty} B^\ell.$$

Note that $\int_{K_0} |g| dy \leq \int_{K_0} |f| dy$ implies that $\int_{K_0} |b| dy \leq 2 \int_{K_0} |f| dy$. This gives us

$$\int_{G^*} |Tb| dx \leq 2C_2 \|f\|_1.$$

Employing the Chebyshev inequality (§6.2.2) for $p = 1$, and for $\mathbb{R}^n$ replaced by G^*, we arrive at

$$|\{x \in G^*; \; |(Tb)(x)| > t/2\}| \leq \frac{C_3}{t} \|f\|_1, \; C_3 = 4C_2. \tag{6.35}$$

(See Remark 6.2.3.) Thus, the estimate for the distribution function of Tb on G^* is proved.

On F^* the estimate is proved as follows. Since $|B^\ell| = |S^{n-1}|\delta_\ell^n/n$ and $|K_\ell| = (\delta_\ell/n^{1/2})^n$, we obtain that

$$|F^*| \leq \sum_{\ell=1}^{\infty} |B^\ell| = |S^{n-1}| n^{n/2-1} \sum_{\ell=1}^{\infty} |K_\ell|.$$

By the definition of K_ℓ we have

$$t < \frac{1}{|K_\ell|} \int_{K_\ell} |f| dx.$$

This implies that

$$\sum_{\ell=1}^{\infty} |K_\ell| \leq \frac{1}{t} \sum_{\ell=1}^{\infty} \int_{K_\ell} |f| dx \leq \frac{1}{t} \|f\|_1.$$

Consequently,

$$|\{x \in F^*; \; |(Tb)(x)| > t/2\}| \leq |F^*| \leq \frac{C_4}{t} \|f\|_1. \tag{6.36}$$

Estimates (6.35) and (6.36) now result in $m_{Tb}(t/2) \leq C_5 \|f\|_1/t$. Combining (a) and (b), we finally arrive at $m_{Tf}(t) \leq C_6 \|f\|_1/t$; hence the proposition follows. □

6.4.5 Completion of the Proof

We are now in position to complete the proof of the Calderón–Zygmund inequality (§6.4.2). For $n \geq 2$ and fixed $i, j \in \{1, \ldots, n\}$ we still set $Tf = \partial_{x_i}\partial_{x_j}(E * f)$. By the results in the previous paragraphs T is a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. (See §7.3 and Exercise 7.3. Note that $L^{1,\infty}(\mathbb{R}^n)$ is not a normed space. However, it is complete as a pseudonormed space with the pseudonorm $|f|_{1,\infty}$. We refer to, for example, [Bergh Löfström 1976] for basic facts on pseudonormed spaces, in particular on Lorentz spaces. Furthermore, observe that the assertion in the extension theorem in §7.3 is still valid for complete pseudonormed spaces Y. Hence, we may apply this result on $Y = L^{1,\infty}(\mathbb{R}^n)$.) The Marcinkiewicz interpolation theorem (§6.2.4) then implies

$$\|Tf\|_p = \|\partial_{x_i}\partial_{x_j}(E * f)\|_p \leq C(n,p)\|f\|_p, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{6.37}$$

for $1 < p \leq 2$. The case $p > 2$ follows by a duality argument. In fact, for f, $g \in C_0^\infty(\mathbb{R}^n)$ the Hölder inequality gives us

$$\begin{aligned}\int_{\mathbb{R}^n} (Tf)g\,dx &= \int_{\mathbb{R}^n} (E * f)\partial_{x_i}\partial_{x_j} g\,dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} E(x-y)f(y)(\partial_{x_i}\partial_{x_j} g)(x)dx\,dy \\ &= \int_{\mathbb{R}^n} f\ Tg\ dy \leq \|f\|_p\|Tg\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1.\end{aligned}$$

By the duality characterization (6.8) for $p > 2$ we further know that

$$\begin{aligned}\|Tf\|_p &= \sup\left\{\int_{\mathbb{R}^n} (Tf)gdx; \|g\|_{p'} \leq 1,\ g \in C_0^\infty(\mathbb{R}^n)\right\} \\ &\leq \sup\{\|f\|_p\|Tg\|_{p'}; \|g\|_{p'} \leq 1,\ g \in C_0^\infty(\mathbb{R}^n)\}.\end{aligned}$$

The fact that $1 < p' < 2$ and relation (6.37) then result in

$$\|Tf\|_p \leq C(n,p')\|f\|_p.$$

This proves (6.29) for $1 < p < \infty$ and $f \in C_0^\infty(\mathbb{R}^n)$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ (Exercise 7.3), by §7.3 the operator T extends uniquely to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. □

6.5 Notes and Comments

We first give comments on §6.1.6, where the critical case of the Sobolev inequality is discussed. There are inequalities asserting that exponential integrability of u is controlled by the L^n-norm of the gradient of u. Such an inequality is often called a *Trudinger–Moser inequality*. Here is a typical form. There exist positive constants α and C such that

$$\int_{\mathbb{R}^n} (\exp(\alpha|u(x)|^{n'}) - \sum_{j=0}^{n-2}(\alpha|u(x)|^{n'})^j)\ dx \le C\|u\|_n^n \tag{6.38}$$

for all $u \in C_0^1(\mathbb{R}^n)$ with $\|u\|_n < \infty,\ \|\nabla u\|_n \le 1,\ n \ge 2$, where $n' = n/(n-1)$, the conjugate exponent of n. This type of inequality was first obtained by [Trudinger 1967] and improved by [Moser 1970]. The version (6.38) is a special case of the inequality given in [Ozawa 1995], where n is replaced by a general exponent with necessary modification. The proof is based on the Gagliardo–Nirenberg inequality

$$\|u\|_p \le Cp^{1-1/n}\|u\|_n^{n/p}\|\nabla u\|_n^{1-n/p}, \tag{6.39}$$

where the dependence of the constant in (6.4) with respect to p is explicit. In [Ozawa 1995] inequality (6.39) is obtained by proving the Hardy–Littlewood–Sobolev inequality (§6.2.1) with explicit dependence of the constant with respect to r and p. Like the Sobolev inequality, the Trudinger inequality has substantial applications to nonlinear partial differential equations. We give only an example where it is used for the study of equations of chemotaxis [Nagai Senba Yoshida 1997]. The Trudinger–Moser inequality can be extended in Lorentz–Zygmund-type spaces. For such developments the reader is referred to [Edmunds Gurka Opic 1995], [Edmunds Hurri-Syrjanen 2001], [Mizuta Shimomura 1998].

There is another development for the critical case of the Sobolev inequality. The first example is provided by [Brezis Gallouet 1980]. The *Brezis–Gallouet inequality* is of the form

$$\|u\|_\infty \le C[1 + \|\nabla u\|_2\{\log(\|\Delta u\|_2 + e)\}^{1/2}]$$

for $u \in C_0^1(\mathbb{R}^2)$. A similar critical inequality for higher-dimensional space with general exponent (instead of L^2) is due to [Brezis Wainger 1980]. The *Brezis–Wainger inequality* is of the form

$$\|u\|_\infty \le C[1 + \|(-\Delta)^{n/2p}u\|_p\{1 + \log(e + \|(I - \Delta)^{s/2}u\|_p)\}^{1-1/p}]$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ with $\|(-\Delta)^{n/2p}u\|_p < \infty$, $\|(I - \Delta)^{s/2}u\|_p < \infty$, where $s > n/p$. These inequalities can be considered a variant of the Gagliardo–Nirenberg inequality in the sense that the dependence with respect to one norm is logarithmic instead of powerlike. For a further development of the logarithmic Sobolev inequality the reader is refereed to [Ogawa Taniuchi 2004] (and [Kozono Ogawa Taniuchi 2002]) for inequalities in Besov spaces and to [Ogawa 2003] for inequalities in Lizorkin–Triebel spaces. Note that these results include the Beale–Kato–Majda inequality [Beale Kato Majda 1984] in their study of the Euler equations. As carried out in these works, such estimates provide a nice regularity criterion for several evolution equations including the Navier–Stokes equations and the harmonic map flow equations.

We next give comments on §6.2 and §6.3. There are various ways to prove the Hardy–Littlewood–Sobolev inequality. A standard proof is to estimate its kernel and apply the Marcinkiewicz interpolation theorem; see, e.g., [Folland 1999], [Ozawa 1995]. The proof given in [Reed Simon 1975] uses the Hunt interpolation theorem as well as the Marcinkiewicz interpolation theorm and it is in some sense the shortest one. There is a method using maximum functions. For example, see [Ziemer 1989]. This book also contains a proof of the Sobolev inequality based on the isoperimetric inequality. The proof of the Marcinkiewicz interpolation theorem is found in standard textbooks on analysis, for example [Folland 1999], where a proof of the *Riesz–Thorin theorem* is given. For Lorentz spaces, see [Bergh Löfström 1976]. To construct new function spaces by interpolating two function spaces is very important for an effective use of interpolation theorems. For this direction, see also [Butzer Berens 1967, Triebel 1978, Muramatu 1985, Komatsu 1978]. Sobolev spaces, which are not introduced in this book, are treated in many elementary textbooks on partial differential equations. Moreover, textbooks on interpolation theory also treat them. There are famous books [Adams 1978, Mazja 1985] mainly treating Sobolev spaces. Rellich's theorem, which is a compactness result for Sobolev spaces corresponding to the Ascoli–Arzelà theorem (Section 5) for continuous functions, is very important. Besides the Sobolev spaces there are many further important function spaces. For a comprehensive overview we refer to [Triebel 1983, Triebel 1992].

We further give some remarks on §6.4. There is a large branch within analysis that is concerned with the treatment of singular integral operators and that goes far beyond the discussions in §6.4. This branch is called harmonic analysis. It includes the theory of Fourier multipliers, which is closely related to the theory of singular integrals. For a comprehensive introduction to harmonic analysis we refer to the books [Stein 1993] [Torchinsky 1986], [García-Cuerva Rubio de Francia 1985], and [Stein 1970] as well as to [Stein Weiss 1971].

Many of the results on multipliers and singular integral operators have counterparts in a Banach-space-valued setting such as $L^p(\mathbb{R}^n, X)$, i.e., if the image space $\mathbb{C}$ or $\mathbb{R}$ is replaced by a Banach space X. The value of the X-valued versions of these results lies in their importance for the treatment of linear and nonlinear partial differential equations. In particular, the notion of strong solutions in recent years has become significant for the treatment of quasilinear problems. In this context one aims for solutions in anisotropic Sobolev spaces such as $W^{1,p}((0,T), L^p(\mathbb{R}^n)) \cap L^p((0,T), W^{2,p}(\mathbb{R}^n))$ for the heat equation in $\mathbb{R}^n$. In this context the X-valued versions ($X = L^p(\mathbb{R}^n)$ in the case of the heat equation) of the results on multipliers and singular integral operators serve as a powerful tool.

Inter alia there are X-valued versions of the Marcinkiewicz interpolation theorem and of the Calderón–Zygmund inequality (see [Torchinsky 1986], [Hieber 1999]). For X-valued multiplier theorems and their relation to X-valued singular integral operators we refer to the orignal paper [Weis 2001] and to the booklet [Denk Hieber Prüss 2003].

Exercises 6

6.1 (§6.1.3) Let $\theta \in C^\infty(\mathbb{R})$ be such that $\theta(y) = 0$ for $y \geq 2$, $\theta(y) = 1$ on $y \leq 1$, and $0 \leq \theta \leq 1$ on $\mathbb{R}$. For a natural number j we define $\theta_j(x) = \theta(|x|/j)$ $(x \in \mathbb{R}^n)$, and for $u \in C^1(\mathbb{R}^n)$ we set $u_j = \theta_j u$.

- (i) For $\|u\|_p < \infty$ show that $\lim_{j\to\infty} \|u_j - u\|_p = 0$ if $p \in [1, \infty)$, and that $\|u\|_\infty = \lim_{j\to\infty} \|u_j\|_\infty$ if $p = \infty$. Furthermore, if $u \in C_\infty(\mathbb{R}^n)$ and $\|u\|_\infty < \infty$, show that $\lim_{j\to\infty} \|u_j - u\|_\infty = 0$.
- (ii) If $\|u\|_p, \|\nabla u\|_p < \infty$ for $p \in [1, \infty)$ and $\|\nabla u\|_p < \infty$, then $\lim_{j\to\infty} \|\nabla(u_j - u)\|_p = 0$.
- (iii) If $\|u\|_p < \infty$ and $\|\nabla u\|_r < \infty$ for $1 \leq r \leq p < \infty$ satisfying $1/r - 1/p \leq 1/n$, then $\lim_{j\to\infty} \|\nabla(u_j - u)\|_r = 0$. (The assertion remains valid if $p = \infty$, $n < r < \infty$.)

6.2 (§6.1.4) For $u \in L^q(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ with $1 \leq q < r \leq \infty$, show that $u \in L^p(\mathbb{R}^n)$ for $q \leq p \leq r$ and that

$$\|u\|_p \leq \|u\|_q^\rho \|u\|_r^{1-\rho}, \quad \frac{1}{p} = \frac{\rho}{q} + \frac{1-\rho}{r}, \quad 0 \leq \rho \leq 1.$$

6.3 (§6.2.3) For a Lebesgue integrable function f on $\mathbb{R}^n$, we define

$$\|f\|_{q,\infty} := \sup\left\{ |E|^{\frac{1}{q}-1} \int_E |f(x)|dx : E \subseteq \mathbb{R}^n \text{ Lebesgue measurable}, \ |E| < \infty \right\},$$

where $1 < q < \infty$. Show that there exist positive constants C_1 and C_2, independent of f, such that

$$C_1 \|f\|_{q,\infty} \leq |f|_{q,\infty} \leq C_2 \|f\|_{q,\infty}.$$

Moreover, show that $\|f\|_{q,\infty}$ is a norm in $L^{q,\infty}(\mathbb{R}^n)$.

6.4 (§6.2.3) Show that $1/\sqrt{x} \in L^{2,\infty}(0,1)$, but that $1/\sqrt{x} \notin L^2(0,1)$.

6.5 (§6.2.4) Prove the integral form of the *Minkowski inequality*:

$$\int_\Omega \left| \int_U f(x,y)dx \right|^r dy \leq \left(\int_U \left(\int_\Omega |f(x,y)|^r dy \right)^{1/r} dx \right)^r.$$

Here $1 \le r < \infty$, whereas Ω and U are (Lebesgue) measurable sets on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. (For students not yet familiar with measurable sets, the assertion can be proved under the relaxed assumption that Ω and U are open sets, and that f is continuous on $\overline{U} \times \overline{\Omega}$.)

6.6 (§6.2.4) Prove Proposition 6.2.4.

6.7 (§6.2.5) Prove (6.21).

6.8 (§6.3.2) Prove Proposition 6.3.2 directly.

6.9 (§6.3.3, §6.3.5, §6.4.4)

(i) Let α be a multi-index, $n \ge 2$, and assume for $n = 2$ that $|\alpha| \ge 1$. Show that

$$\sup_{x \in \mathbb{R}^n, x \neq 0} |\partial_x^\alpha E(x)| \; |x|^{n-2+|\alpha|} < \infty.$$

(ii) For $1 \le j \le n$ prove that $\partial_{x_j} E$ is locally integrable on $\mathbb{R}^n$.

7

Convergence Theorems in the Theory of Integration

This section gives a summary of some elementary facts used frequently throughout this book, and can be regarded as an appendix. In particular, we consider sufficient conditions for the interchange of integration and limit operations. In detail, we discuss a result on uniform convergence, the dominated convergence theorem, the bounded convergence theorem, Fatou's lemma, and the monotone convergence theorem from the points of view of both Lebesgue integration theory and Riemann integration theory. Note that these are well-known results; hence we will be brief in details. For the proof of the monotone convergence theorem and Fubini's theorem we merely refer to the appropriate literature.

We also present a theorem for differentiation under the integral sign, which is based on the interchange of the order of integration. This theorem allows for an elegant differentiation under the integral sign for integrals including an unbounded function. It is in particular applied in §6.3.5. Since this result seems not to be contained in many elementary textbooks on integration theory, we give its proof here.

Finally, we recall that a linear operator in a normed space Y that is bounded on a dense subspace extends uniquely to a bounded linear operator on Y. This elementary functional-analytic fact, for instance, is used in Chapter 6.

7.1 Interchange of Integration and Limit Operations

From many calculations in the previous chapters it can be seen that the question of interchangeability of integration and limit operations is of fundamental importance in the analysis of differential equations. Since integration can also be regarded as a limiting process, this problem reduces to the question of interchangeability of two limit operations. Among many sufficient conditions guaranteeing the valdity of the interchange of integration and limit operations,

M.-H. Giga et al., *Nonlinear Partial Differential Equations*,
Progress in Nonlinear Differential Equations and Their Applications 79,
DOI 10.1007/978-0-8176-4651-6_7,

the most elementary one is the condition of uniform convergence of function sequences.

Proposition. *For natural numbers* $m = 1, 2, \ldots,$ *let* f_m *be (real-valued) continuous functions defined on a closed ball* $\overline{B}_R \subset \mathbb{R}^n$ *centered at the origin with radius* R *such that* f_m *converges uniformly to* f *on* $\overline{B}_R$*. (Observe that by this assumption* f *is continuous on* $\overline{B}_R$*; see the answer to Exercise 1.6.) Then we have*

$$\lim_{m\to\infty} \int_{B_R} f_m(x)dx = \int_{B_R} f(x)dx.$$

The statement still holds if B_R *is replaced by any compact subset of* $\mathbb{R}^n$*.*

We may easily prove this result by

$$\begin{aligned}\left|\int_{B_R} f_m(x)dx - \int_{B_R} f(x)dx\right| &\le \int_{B_R} |f_m(x) - f(x)|dx \\ &\le \left(\sup_{\overline{B}_R} |f_m - f|\right) |B_R| \to 0 \quad (m \to \infty).\end{aligned}$$

Instead of a sequence of natural numbers m, we may also consider a continuous parameter $t \in \mathbb{R}$. In fact, assuming that $f(\cdot, t)$ converges uniformly to f as $t \to t_0$ on $\overline{B}_R$, we obtain in the same way that

$$\lim_{t\to t_0} \int_{B_R} f(x,t)dx = \int_{B_R} f(x)dx.$$

(Here also $t_0 = \infty$ is allowed.)

On the other hand, note that in the proposition above, the finiteness of the integration area, i.e., $|B_R| < \infty$, is essential. In fact, B_R can in general not be replaced by $\mathbb{R}^n$. This follows, for example, from the discussion in §1.2.2, which shows that even if a function $u(\cdot, t)$ converges uniformly to 0 on $\mathbb{R}^n$ as $t \to \infty$,

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} u(x,t)dx = 0 \;\left(= \int_{\mathbb{R}^n} \lim_{t\to\infty} u(x,t)dx\right)$$

might not be true in general. Thus, one seeks a sufficient condition such that integration and passing to the limit can be interchanged, even in the case of unbounded integration areas or sequences of unbounded functions. This problem is the subject of the next subsection.

7.1.1 Dominated Convergence Theorem

For simplicity we will restrict ourselves to the case of $\mathbb{R}^n$. First we discuss the case of Lebesgue integrals (Lebesgue's *dominated convergence theorem*). Here the required conditions are easily stated, but nevertheless widely applicable.

Theorem (The case of the Lebesgue integral). *For $m = 1, 2, \dots,$ let $f_m(x)$ be real-valued integrable functions on $\mathbb{R}^n$ (that is, $f_m \in L^1(\mathbb{R}^n)$) such that*

$$\lim_{m\to\infty} f_m(x) = f(x) \tag{7.1}$$

for each point $x \in \mathbb{R}^n$. If there exists a function $g \in L^1(\mathbb{R}^n)$ such that for all $m = 1, 2, \dots,$

$$|f_m(x)| \leq g(x) \tag{7.2}$$

for each $x \in \mathbb{R}^n$, then f is integrable on $\mathbb{R}^n$ and we have

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} f_m(x)dx = \int_{\mathbb{R}^n} f(x)dx. \tag{7.3}$$

As before we may replace the natural m by a real parameter t, and $m \to \infty$ by $t \to t_0$ in the statement of the theorem. Then (7.3) still follows from (7.1) and (7.2). This is due to the fact that in $\mathbb{R}^n$ convergence is equivalent to sequential convergence, that is, the equivalence of $\lim_{t\to t_0} F(t) = \alpha$ and $\lim_{m\to\infty} F(t_m) = \alpha$ for any sequence $\{t_m\}$ converging to t_0. In this monograph we mainly use this theorem in the convergence form, i.e., in the case that $t \to t_0$ with a real parameter t. In the statement we may replace "for each $x \in \mathbb{R}^n$" by "for almost all $x \in \mathbb{R}^n$ (in the sense of the Lebesgue measure theory)." Furthermore, the theorem also applies without any change to complex-valued functions. As an application of this result we obtain the *bounded convergence theorem* for integrals over bounded sets.

Theorem (Bounded convergence theorem). *Let Ω be a bounded open set in $\mathbb{R}^n$. (Then Ω is in particular Lebesgue measurable with finite Lebesgue measure $|\Omega|$.) For $m = 1, 2, \dots,$ we assume that $h_m(x)$ are real-valued integrable functions on Ω satisfying*

$$\lim_{m\to\infty} h_m(x) = h(x)$$

for each $x \in \Omega$. Then, if there exists a constant M such that

$$|h_m(x)| \leq M \quad (x \in \Omega,\ m = 1, 2, \dots), \tag{7.4}$$

then the function h is also integrable on Ω and we have

$$\lim_{m\to\infty} \int_{\Omega} h_m(x)dx = \int_{\Omega} h(x)dx.$$

The existence of M in (7.4) is equivalent to

$$\sup_{m\geq 1} \sup_{x\in\Omega} |h_m(x)| < \infty,$$

i.e., it is equivalent to the *uniform boundedness* of the h_m on Ω with respect to m. The bounded convergence theorem is readily obtained by setting

$$f_m(x) = \begin{cases} h_m(x), & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases} \quad g(x) = \begin{cases} M, & x \in \Omega, \\ 0, & x \notin \Omega, \end{cases}$$

and applying the dominated convergence theorem. Indeed, since Ω is bounded, g is integrable; hence the assumptions of the dominated convergence theorem are satisfied. We remark that also here the assumption on the finiteness of the measure of Ω is essential, as pointed out in §1.2.2 and the discussions above §7.1.1. Observe that the proposition at the beginning of §7.1 is a special case of the bounded convergence theorem.

Finally, note that (7.2) in general cannot be dropped in order to obtain (7.3). However, under certain circumstances it can be weakened, as the following celebrated result (*Fatou's lemma*) on "lower semicontinuity of integrals" shows.

7.1.2 Fatou's Lemma

Lemma. *Assume that for* $m = 1, 2, \dots$, *the functions* h_m *are (Lebesgue) integrable on* $\mathbb{R}^n$ *such that* $h_m(x) \geq 0$ *for* $x \in \mathbb{R}^n$. *Then we have*

$$\int_{\mathbb{R}^n} \varliminf_{m\to\infty} h_m(x)dx \leq \varliminf_{m\to\infty} \int_{\mathbb{R}^n} h_m(x)dx, \tag{7.5}$$

where $\varliminf_{m\to\infty} h_m(x)$ *denotes the limit inferior, which is defined as*

$$\varliminf_{m\to\infty} h_m(x) = \lim_{m\to\infty} \inf_{k\geq m} h_k(x).$$

(Note that $\varliminf_{m\to\infty} h_m(x) = -\varlimsup_{m\to\infty}(-h_m(x))$.*)*

Observe that by no means is the existence of the limits above assumed. In fact, if the left-hand side of the inequality (7.5) is infinity, the right-hand side is so as well. The dominated convergence theorem can be obtained as a consequence of Fatou's lemma. Indeed, by setting

$$h_m = g + f_m,$$

an application of Fatou's lemma shows that $(-\infty < -\int g \leq)\int f \leq \varliminf_{m\to\infty} \int f_m$. Analogously, setting $h_m = g - f_m$, Fatou's lemma implies $(\infty > \int g \geq) \int f \geq \varlimsup_{m\to\infty} \int f_m$. Hence we obtain (7.3).

On the other hand, Fatou's lemma is a direct consequence of the *monotone convergence theorem* (see next section) by setting $g_m(x) = \inf_{k\geq m} h_k(x)$.

7.1.3 Monotone Convergence Theorem

Theorem. *For* $m = 1, 2, \dots$, *we assume that* g_m *are real-valued (Lebesgue) integrable functions on* $\mathbb{R}^n$ *such that* $g_{m+1}(x) \geq g_m(x) \geq 0$ *for each* $x \in \mathbb{R}^n$. *Then we have*

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} g_m(x)dx = \int_{\mathbb{R}^n} \lim_{m\to\infty} g_m(x)dx.$$

In analogy to the dominated convergence theorem, we remark that it suffices to assume that $g_{m+1}(x) \geq g_m(x) \geq 0$ for almost all $x \in \mathbb{R}^n$. Moreover, also here nothing is assumed on convergence of g_m, i.e., g_m may in particular tend to infinity.

The monotone convergence theorem essentially follows from the definition of the Lebesgue integral. So, since we do not give an introduction to Lebesgue integration theory in this book, we forbear from giving a proof here.

Also note that a measurable function in §7.1.2 and §7.1.3 may be interpreted as an almost everywhere pointwise limit of a sequence of continuous functions. In particular, continuous functions are measurable.

7.1.4 Convergence for Riemann Integrals

Except for the case of uniform convergence, results concerning commutativity of integrals and limit operations are usually considered in the framework of Lebesgue integration theory. To students this might give the impression that in order to understand such convergence theorems, a study of Lebesgue integration theory is unavoidable. However, there are convergence theorems corresponding to §7.1.1–§7.1.3 also for Riemann integrals. Particularly in Chapter 1, Lebesgue integrability is not essential. For example, by the assumption that the functions and integrands are continuous, the dominated convergence theorem in §7.1.1, Fatou's lemma in §7.1.2, and the monotone convergence theorem in §7.1.3 can be proved in the framework of improper Riemann integrals. (For the dominated convergence theorem in §7.1.1 we assume that Ω is a bounded closed set, where the volume is taken in the sense of Riemann integrals.) On the other hand, the continuity assumption for the functions that appear may be too strong, since sometimes we have to deal naturally with unbounded or discontinuous functions. This motivates the statement of a version of the dominated convergence theorem also for Riemann integrals. Here we restrict ourselves to the case $n = 1$. This suffices for the application in §1.4.4.

Theorem. *Assume that f_m, f, g are continuous on $\mathbb{R}$ except for finitely many points. Suppose also that (7.1) and (7.2) hold except for finitely many points. Then we have (7.3).*

This theorem can be regarded as an extension of Arzelà's theorem. (Note that the assumption "except for finitely many points" can be relaxed to "except for a set of Lebesgue measure zero," which includes in particular countable sets, and that it holds also in higher dimensions.) For Arzelà's theorem and its extension we refer to [Kodaira 1976 1977 1979, II Theorem 5.10, 5.12, IV Theorem 8.9, 8.10] and [Fujita 1981, pp. 1–3]. Compared to Lebesgue's dominated convergence theorem, direct proofs of Arzelà-type convergence theorems are much more intricate. Moreover, these results are included in the result of Lebesgue. This is the reason why they usually do not appear in elementary calculus courses. On the other hand, these are important tools suitable for undergraduate students still not familiar with Lebesgue integration theory. This motivates the study also of Arzelà-type convergence theorems. However, it should be mentioned that the assumptions for Lebesgue's result are handier and much easier to state. And another important point is that there are no assumptions on the limit function, which is usually required for Arzelà-type results.

7.2 Commutativity of Integration and Differentiation

We consider a sufficient condition for "differentiation under the integral sign," which is a helpful tool for the differentiation of parameter-dependent integrals. The commutativity of integration and differentiation is often justified by Lebesgue's dominated convergence theorem. The following result, however, will be derived in a different way. It is obtained as a consequence of the commutativity of the order of integration (i.e., Fubini's theorem; see §7.2.2) and the fundamental theorem of calculus. The result applies directly to the case that singularities of the integrand are moving with respect to parameters; see Proposition 6.3.5.

7.2.1 Differentiation Under the Integral Sign

Theorem. *Let the function $h = h(x, y)$ be defined on $(a, b) \times \mathbb{R}^n$ and assume that it satisfies the following properties.*

(i) *For almost all $y \in \mathbb{R}^n$, $h(x, y)$ is C^1 on (a, b) with respect to x.*
(ii) *The derivative $\frac{\partial h}{\partial x}(x, y)$ is integrable on $(a, b) \times \mathbb{R}^n$, i.e., $\int_{(a,b)\times\mathbb{R}^n} |\frac{\partial h}{\partial x}(x, y)| dx\ dy < \infty$.*
(iii) *The function $h(c, y)$ is integrable on $\mathbb{R}^n$ with respect to y at least at one point $c \in (a, b)$, i.e., $\int_{R^n} |h(c, y)| dy < \infty$.*
(iv) *The function $U(x) = \int_{\mathbb{R}^n} \frac{\partial h}{\partial x}(x, y) dy$ is continuous on the interval (a, b).*

Then $H(x) = \int_{\mathbb{R}^n} h(x, y) dy$ is C^1 on (a, b) and its differential is given by

$$H'(x) = \int_{\mathbb{R}^n} \frac{\partial h}{\partial x}(x, y) dy \ (= U(x)). \tag{7.6}$$

This theorem is easily proved as follows. First, assumption (i) and the fundamental theorem of calculus imply for almost all y that

$$h(x, y) - h(c, y) = \int_c^x \frac{\partial h}{\partial x}(\xi, y) d\xi, \quad x \in (a, b). \tag{7.7}$$

Assumption (ii) now allows for an application of Fubini's theorem. Hence, interchanging the order of integration, we obtain for the right-hand side of (7.7) that

$$\int_{\mathbb{R}^n} \left(\int_c^x \frac{\partial h}{\partial x}(\xi, y) d\xi \right) dy = \int_c^x \left(\int_{\mathbb{R}^n} \frac{\partial h}{\partial x}(\xi, y) dy \right) d\xi.$$

This and (iii) also imply that two of the three terms in (7.7) are integrable with respect to y. Thus, also the third term, that is, $h(x, y)$, is integrable and we obtain by integrating equation (7.7) that

$$H(x) - H(c) = \int_c^x U(\xi) d\xi, \quad x \in (a, b). \tag{7.8}$$

By (iv), $U(x)$ is continuous. Hence, (7.8) and once again the fundamental theorem of calculus imply that H is C^1 and $H'(x) = U(x)$. This yields (7.6).

For a corresponding version of the above result in the framework of Riemann integrals, see the discussion at the end of the next section.

7.2.2 Commutativity of the Order of Integration

Theorem (Fubini's theorem). *Let f be a real-valued function on $\mathbb{R}^m \times \mathbb{R}^n$.*

(I) *Assume that f is integrable on $\mathbb{R}^m \times \mathbb{R}^n$, i.e.,*

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} |f(x, y)| dx\ dy < \infty. \tag{7.9}$$

Then we have

(i) *For almost all x we have $\int_{\mathbb{R}^n} |f(x, y)| dy < \infty$, and for almost all y we have $\int_{\mathbb{R}^m} |f(x, y)| dx < \infty$.*

(ii) *$\int_{\mathbb{R}^n} |f(x, y)| dy$ is integrable on $\mathbb{R}^m$ with respect to x and $\int_{\mathbb{R}^m} |f(x, y)| dx$ is integrable on $\mathbb{R}^n$ with respect to y.*

Furthermore, we may interchange the order of integration, i.e.,

$$\begin{aligned}\int_{\mathbb{R}^m \times \mathbb{R}^n} f(x, y) dx\ dy &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x, y) dy \right) dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x, y) dx \right) dy.\end{aligned} \tag{7.10}$$

(II) *Existence of either one of the integrals*

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x, y)| dy \right) dx, \quad \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} |f(x, y)| dx \right) dy$$

implies the existence of the other integral and the validity of (7.9) and (7.10).

The proof of this theorem is too long to be given here.

There is also a counterpart for Riemann integrals of this result. Indeed, assuming f to be continuous and replacing $\mathbb{R}^m$ and $\mathbb{R}^n$ by closed rectangles in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, the result can be proved in an elementary way in the framework of Riemann integration theory. (In this case, "almost all" in (i) should be replaced by "all.")

Note that then the theorem in §7.2.1 also becomes a result in the framework of Riemann integrals by the usual changes in the assumptions. More precisely, we have to replace $\mathbb{R}^n$ by a closed rectangle in $\mathbb{R}^n$, the interval (a, b) by the closed interval $[a, b]$, integrability by continuity, and "almost all" in (i) by "all." A more comprehensive approach to Fubini's theorem for Riemann integrals is given in [Kodaira 1976 1977 1979].

7.3 Bounded Extension

In Chapter 6 we several times employed the fact that a densely defined bounded operator extends boundedly to the closure of the dense subset. For instance, in §6.2.1 it was applied in order to show that the Riesz potential I_α, initially defined on the dense subspace $C_0(\mathbb{R}^n)$ (see Exercise 7.3), extends to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$. In detail this functional-analytic fact on bounded extensions reads as follows.

Theorem (Extension theorem). *Let X be a normed space with norm $\|\cdot\|_X$, and Y a Banach space with norm $\|\cdot\|_Y$. Assume that X_0 is a dense subspace of X (i.e., the closure of X_0 equals X) and that T is a linear operator from X_0 to Y. Assume furthermore that there exists a $C_0 > 0$ independent of f such that*

$$\|Tf\|_Y \leq C_0 \|f\|_X, \quad f \in X_0. \tag{7.11}$$

Then there exists a unique bounded linear operator $\hat{T}$ from X to Y satisfying

$$\|\hat{T}h\|_Y \leq C_0 \|h\|_X, \quad h \in X, \tag{7.12}$$

$$\hat{T}f = Tf, \quad f \in X_0. \tag{7.13}$$

(The operator $\hat{T}$ is called the extension of T and often denoted by T as well.)

The proof is a good and elementary exercise in functional analysis and therefore left to the reader.

The content of this section is discussed more in detail in pertinent textbooks on integration theory, as e.g. in [Ito 1963], [Rudin 1987]. The monotone convergence theorem in §7.1.3 e.g. is a special case of [Ito 1963, Theorem 13.2], [Rudin 1987, Lebesgue's Monotone Convergence Theorem 1.26]. Fubini's Theorem in §7.2.2 is a special case of [Ito 1963, Theorem 15.3], [Rudin 1987, Theorem 7.8] whereas the extension theorem in §7.3 is given in [Ito 1963, Theorem 25.2]; see also [Yosida 1964].

Exercises 7

7.1. (§4.1, §6.2, §7.1)

(i) Assume that $f \in C_0(\mathbb{R}^n)$ and that h is locally integrable on $\mathbb{R}^n$, i.e., $h \in L^1(B_R)$ for any ball B_R. Show that $(h * f)(x)$ is continuous with respect to $x \in \mathbb{R}^n$. (Hint: See the proof of (I) (i) in Proposition 4.1.4.)

(ii) Assume that $f \in C_0(\mathbb{R}^n)$ and that $h \in C(\mathbb{R}^n)$. Show that $(h * f)(x)$ is continuous with respect to $x \in \mathbb{R}^n$ using Proposition 7.1 only.

7.2. (§1.1, §4.1.6) Let $1 \le p \le \infty$. For $f \in L^p(\mathbb{R}^n)$ set $u(x,t) = (G_t * f)(x)$, $x \in \mathbb{R}^n$, $t > 0$, where $G_t(x)$ is the Gauss kernel. Show that u is partially differentiable infinitely many times as a function of (x,t). Moreover, show that u satisfies the heat equation (1.1) in $t > 0$. (In case of problems with general dimension, start with $n = 1$.)

7.3. (§6.1.4) Show that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, by employing

$$\lim_{t \downarrow 0} \|G_t * f - f\|_p = 0, \quad f \in L^p(\mathbb{R}^n), \quad 1 \le p < \infty, \tag{$*$}$$

and Exercise 7.2. (Hence, $C_0(\mathbb{R}^n)$, which contains $C_0^\infty(\mathbb{R}^n)$ as a subspace, is also dense in $L^p(\mathbb{R}^n)$. The reader may find the proof of $(*)$ in [Kuroda 1980]. In [Kuroda 1980], the author uses the continuity of parallel transformations with respect to the L^p-norm.) Show generally that $C_0^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any open set Ω in $\mathbb{R}^n$ under the assumption that $1 \le p < \infty$.

7.4. (§4.1.4) Let $1 \le p \le \infty$ and let p' be the conjugate index of p. Moreover, assume that $f \in L^p(\mathbb{R}^n)$. Show that $h * f$ is bounded and continuous if h is continuous on $\mathbb{R}^n$ and $\|h\|_{p'}$ is finite. Moreover, if h is C^1 on $\mathbb{R}^n$ and for each j $(1 \le j \le n)$, $\|\partial_{x_j} h\|_{p'}$ is finite, then $h * f$ is C^1 on $\mathbb{R}^n$ and satisfies

$$(\partial_{x_j}(h * f))(x) = ((\partial_{x_j} h) * f)(x), \quad x \in \mathbb{R}^n.$$

7.5. Show that $C^\infty[0,1]$ is not dense in the Hölder space $C^\mu[0,1]$ $(0 < \mu < 1)$, but that $C^\infty[0,1]$ is dense in $C[0,1]$.

Answers to Exercises

Chapter 1

1.1 (i) We calculate $\partial_t g(x,t) = (-\frac{n}{2t} + \frac{|x|^2}{4t^2})g(x,t)$, $\partial_{x_i} g(x,t) = -\frac{x_i}{2t} g(x,t)$, $\partial_{x_i}\partial_{x_j} g(x,t) = (-\frac{\delta_{ij}}{2t} + \frac{x_i x_j}{4t^2})g(x,t)$ $(1 \le i,j \le n)$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \ne j$. From this we also see that

$$\Delta g = \sum_{i=1}^{n} \partial_{x_i}\partial_{x_i} g(x,t) = \left(-\frac{1}{2t}\sum_{i=1}^{n}\delta_{ii} + \frac{1}{4t^2}\sum_{i=1}^{n} x_i^2\right) g(x,t)$$
$$= \left(-\frac{n}{2t} + \frac{|x|^2}{4t^2}\right) g(x,t) = \partial_t g,$$

i.e., $\partial_t g - \Delta g = 0$.

(ii) Since $f \in C_0(\mathbb{R}^n)$, f is identically zero outside a ball B_R. Since f is continuous, f is bounded in $\overline{B}_R$ (the Weierstrass theorem). We set $M = \sup_{\overline{B}_R} |f| < \infty$. This implies $\|f\|_\infty = M < \infty$. For $1 \le p < \infty$ we have

$$\|f\|_p^p = \int_{B_R} |f(x)|^p dx \le M^p \int_{B_R} 1 dx < \infty.$$

(Using spherical coordinates, $\int_{B_R} 1dx$ can be explicitly calculated. Its finiteness can easily be seen from $\int_{B_R} 1dx \le \int_K 1dx = (2R)^n$, where we used $B_R \subset K = [-R,R] \times \cdots \times [-R,R]$.)

1.2 Since $f'(s) = (a-s)s^{a-1}e^{-s}$, f is increasing on $0 < s < a$, and is decreasing on $a < s < \infty$. Hence f achieves its maximum on $[0,\infty)$ at $a = s$. Moreover, since $f \ge 0$, f is bounded on $[0,\infty)$. Hence the maximum value of f is $f(a) = (a/e)^a$.

1.3 First we consider the case $1 \le p < \infty$. By definition,

$$\|v_k\|_p(1) = \left(\int_{\mathbb{R}^n} (k^n |v(kx,k^2)|)^p dx\right)^{1/p}.$$

Observe that $y = kx$ is a homothety transformation of $\mathbb{R}^n$, hence its Jacobian is k^n. This implies

$$\left(\int_{\mathbb{R}^n} (k^n|v(kx,k^2)|)^p dx\right)^{1/p} = k^n \left(\int_{\mathbb{R}^n} |v(y,k^2)|^p \frac{dy}{k^n}\right)^{1/p}$$
$$= k^{n-\frac{n}{p}} \|v\|_p(k^2).$$

Setting $k^2 = t$, we obtain the desired equality.

For the case $p = \infty$ we have $\|v_k\|_\infty(1) = \sup_{x\in\mathbb{R}^n} k^n|v(kx,k^2)| = t^{\frac{n}{2}}\|v\|_\infty(t)$. Hence, also here we have $\|v_k\|_\infty(1) = t^{n/2}\|v\|_\infty(t)$.

1.4 We prove the claim by contradiction. Suppose that a_k does not converge to α as $k \to \infty$. This implies the existence of a positive constant ε and a subsequence $\{a_{k(\ell)}\}_{\ell=1}^\infty$ such that the distance between α and $a_{k(\ell)}$ is greater than ε for all ℓ. But by the assumption, there exists a subsequence of $a_{k(\ell)}$ that converges to α. This contradicts the fact that the distance between $a_{k(\ell)}$ and α is greater than ε. Hence a_k converges to α as $k \to \infty$.

1.5 Let $\{M_j\}_{j=1}^\infty$ and $\{N_j\}_{j=1}^\infty$ be two exhausting sequences of M. For $f \in C(M)$ we set $a_j = \sup\{|f(x)|;\ x \in M\backslash M_j\}$, $b_j = \sup\{|f(x)|;\ x \in M\backslash N_j\}$. It suffices to prove that $\lim_{j\to\infty} a_j = 0$ if and only if we have $\lim_{j\to\infty} b_j = 0$. By the definition of exhausting sequences of compact sets, for each M_j there exists a natural number $i = i(j)$ such that $M_j \subset N_{i(j)}$. By the choice of $N_{i(j)}$ we have $b_{i(j)} \le a_j$. Moreover, we may assume that $i(j) \to \infty$ $(j \to \infty)$. Hence, if $\lim_{j\to\infty} a_j = 0$, then $\lim_{j\to\infty} b_{i(j)} = 0$. Since b_j is nonincreasing, this shows that $\lim_{j\to\infty} b_j = 0$. So we have proved that $\lim_{j\to\infty} a_j = 0$ yields $\lim_{j\to\infty} b_j = 0$. By interchanging the roles of M_j and N_j we may prove that the converse is also true.

1.6 Since the sequence $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $C_\infty(M)$, for each $x \in M$, $\{f_j(x)\}_{j=1}^\infty$ is a Cauchy sequence of real numbers. By the completeness of the real number field $\mathbb{R}$, the limit of $f_j(x)$ as $j \to \infty$ exists, which we denote by $f(x)$.

(i) "f_j converges uniformly to f on M." We will show that

$$\lim_{j\to\infty} \|f - f_j\|_{\infty,M} = 0.$$

By the definition of f and interchanging supremum and limit, we obtain

$$\sup_{x\in M} |f(x) - f_j(x)| = \sup_{x\in M} \lim_{\ell\to\infty} |f_\ell(x) - f_j(x)|$$
$$\le \varliminf_{\ell\to\infty} \sup_{x\in M} |f_\ell(x) - f_j(x)|$$
$$= \varliminf_{\ell\to\infty} \|f_\ell - f_j\|_{\infty,M}.$$

(This interchanging property of supremum and limit is called lower semicontinuity of sup, and is left as Exercise 5.6. The proof is very easy.) Since $\{f_\ell\}_{\ell=1}^{\infty}$ is a Cauchy sequence of $C_\infty(M)$, we also have $\overline{\lim}_{\ell,m\to\infty} \|f_\ell - f_m\|_{\infty,M} = 0$. Taking the upper limit on both sides of the above inequality in j, we obtain $\overline{\lim}_{j\to\infty} \|f - f_j\|_{\infty,M} \le \overline{\lim}_{j\to\infty} \underline{\lim}_{\ell\to\infty} \|f_\ell - f_j\|_{\infty,M} = 0$. Since $\|f - f_j\|_{\infty,M} \ge 0$, this shows that $\{f_j\}$ converges uniformly to f on M.

(ii) Since the uniform limit of continuous functions is continuous, we obtain $f \in C(M)$. This fact can be found in every fundamental textbook of elementary calculus. For the reader's convenience we give a proof here.
Assume that $\|f - f_j\|_{\infty,M} \to 0$ $(j \to \infty)$, $f_j \in C(M)$. For $x, y \in M$, we may estimate

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f_j(y) + f_j(y) - f_j(x) + f_j(x) - f(x)| \\ &\le |f(y) - f_j(y)| + |f_j(y) - f_j(x)| + |f_j(x) - f(x)| \\ &\le 2\|f - f_j\|_{\infty,M} + |f_j(y) - f_j(x)|. \end{aligned}$$

Taking first the upper limit on both sides as $y \to x$, the continuity of f_j yields $\overline{\lim}_{y\to x} |f(y) - f(x)| \le 2\|f - f_j\|_{\infty,M}$. Letting then $j \to \infty$ yields $\overline{\lim}_{y\to x} |f(y) - f(x)| = 0$. Hence, f is continuous in x.

(iii) "$f \in C_\infty(M)$." In order to prove this, we approximate f by f_ℓ and use similar arguments as in (ii). First we estimate as

$$\begin{aligned} |f(x)| &= |f(x) - f_\ell(x) + f_\ell(x)| \\ &\le |f(x) - f_\ell(x)| + |f_\ell(x)| \\ &\le \|f - f_\ell\|_{\infty,M} + |f_\ell(x)|. \end{aligned}$$

Taking first the supremum over $M\backslash M_j$ and then taking the limit superior as $j \to \infty$, the fact that $f_\ell \in C_\infty(M)$ implies

$$\begin{aligned} \overline{\lim_{j\to\infty}} \sup_{M\backslash M_j} |f| &\le \|f - f_\ell\|_{\infty,M} + \lim_{j\to\infty} \sup_{M\backslash M_j} |f_\ell| \\ &= \|f - f_\ell\|_{\infty,M}. \end{aligned}$$

Since $\{f_\ell\}$ converges uniformly to f as $\ell \to \infty$, we obtain

$$\overline{\lim_{j\to\infty}} \sup_{M\backslash M_j} |f| = 0.$$

This shows that $\lim_{j\to\infty} \sup_{M\backslash M_j} |f| = 0$. By virtue of $f \in C(M)$ we get $f \in C_\infty(M)$. Thus, $C_\infty(M)$ is complete.

1.7 Example 1. Since $\|h_\ell\|_{\infty,M} = 1$ for $\ell = 1, 2, \ldots,$ K is bounded in $C(M)$. Assuming that K is relatively compact, $h_\ell(z) = z^\ell$ has a convergent subsequence in $C(M)$, i.e., there exist $f \in C(M)$ and a subsequence $\{h_{\ell(i)}\}_{i=1}^{\infty}$ of h_ℓ satisfying $\lim_{i\to\infty} \|h_{\ell(i)} - f\|_{\infty,M} = 0$. In particular, $h_{\ell(i)}$ converges pointwise to f on $M = [0,1]$. However, its limit satisfies $f(z) = 0$ at any $z \in [0,1)$ and $f(1) = 1$; hence it is discontinuous. This contradicts the fact that the uniform limit of continuous functions is continuous. Therefore, K cannot be relatively compact. If a bounded set K is equicontinuous, the Ascoli-Arzelà theorem implies that K is relatively compact in $C(M)$, which again contradicts the above result. Therefore, K is not equicontinuous. The fact that K is not equicontinuous also easily follows from $\sup_{\ell\geq 1} |h^\ell(z) - h^\ell(1)| \geq 1$ ($z \in [0,1)$).

Example 2. For $h_\ell \in K$, $\|h_\ell\|_{\infty,M} = \|\varphi\|_{\infty,M}$ is independent of ℓ; hence K is bounded.

(i) Equicontinuity: Since φ is a continuous function with compact support, it is uniformly continuous (§4.2.2). Setting $\omega(\sigma) = \sup\{|\varphi(x) - \varphi(y)|;\ |x-y| \leq \sigma,\ x, y \in \mathbb{R}\}$, we have $\omega(\sigma) \to 0$ ($\sigma \to 0$). Next, we estimate

$$\sup_{h\in K} |h(z) - h(y)| = \sup_{\ell\geq 1} |\varphi(z-\ell) - \varphi(y-\ell)| \\ \leq \omega(|z-y|).$$

This yields $\overline{\lim}_{y\to z} \sup_{h\in K} |h(z) - h(y)| \leq \overline{\lim}_{y\to z} \omega(|z-y|) = 0$, which shows the equicontinuity of K.

(ii) "K is not relatively compact." Set $h_\ell(x) = \varphi(x-\ell)$. If K is relatively compact, then h_ℓ has a convergent subsequence in $C_\infty(M)$. More precisely, there exist $f \in C_\infty(M)$ and a subsequence $\{h_{\ell(i)}\}_{i=1}^{\infty}$ of h_ℓ satisfying $\lim_{i\to\infty} \|h_{\ell(i)} - f\|_{\infty,M} = 0$. In particular, $h_{\ell(i)}$ converges pointwise to f on $\mathbb{R}$. Since $\varphi \in C_0(\mathbb{R})$, the limit is zero. In other words, $f \equiv 0$. On the other hand, $\|h_{\ell(i)}\|_{\infty,M} = \|\varphi\|_{\infty,M} \neq 0$ contradicts the fact that $h_{\ell(i)}$ converges uniformly to f. Hence K is not relatively compact.

(iii) "K does not have the equidecay property." Let $\{M_j\}_{j=1}^{\infty}$ be an exhausting sequence of compact sets of $\mathbb{R}$. Since $\varphi \not\equiv 0$, there exists $x_0 \in \mathbb{R}$ such that $|\varphi(x_0)| > 0$. Choosing a suitable large natural number ℓ_0, we may assume that $x_0 + \ell_0 \notin M_j$. This implies

$$\sup_{\ell\geq 1} \sup_{x\in M\setminus M_j} |h_\ell(x)| = \sup_{\ell\geq 1} \sup_{x\in M\setminus M_j} |\varphi(x-\ell)| \\ \geq |\varphi(x_0 + \ell_0 - \ell_0)| = |\varphi(x_0)|.$$

Thus, we obtain $\overline{\lim}_{j\to\infty} \sup_{h\in K} \sup_{M\setminus M_j} |h| \geq |\varphi(x_0)| > 0$, which shows that K is not equidecay.

1.8 First we prove the fact mentioned in the Hint by a contradiction argument. Assume that $h \not\equiv 0$. Replacing h by $-h$ if necessary, we may assume that there exists an $x_0 \in \Omega$ such that $h(x_0) = a > 0$. By the continuity of h there exists a small open ball $B_r(x_0)$ centered at x_0 and with radius r such that $h(x) \geq a/2,\ x \in B_r(x_0),\ \overline{B_r(x_0)} \subset \Omega$. Next, we set

$$q(s) = \begin{cases} e^{-1/s}, & s > 0, \\ 0, & s \leq 0, \end{cases}$$

and $\varphi(x) = q(r^2 - |x - x_0|^2)$. Then $\varphi \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\varphi \subset \overline{B_r(x_0)} \subset \Omega$; hence we have $\varphi \in C_0^\infty(\mathbb{R}^n)$. Since φ is nonnegative, we obtain for the integral of the product of φ and h that

$$0 = \int_\Omega h\ \varphi\ dx \geq \frac{a}{2} \int_{B_r(x_0)} \varphi\ dx > 0,$$

which is a contradiction. Thus, $h \equiv 0$.
The fundamental lemma of the calculus of variations, mentioned as a remark, can be found, e.g., in [Kakita 1985]. There many types of proof are given, which are essentially based on two ideas: either exhausting the space with allowed functions φ or approximating h by smooth functions. Here we present a proof using the latter method. Utilizing the above q for x_0 and r with $x_0 \in Q$, $\overline{B_r(x_0)} \subset Q$, we set $\Phi(x;\ x_0,\ r) = q(r^2 - |x - x_0|^2)$. Furthermore, for $\psi \in C_0^\infty(Q)$ we set $\varphi = (G_t * \psi)\Phi$, where $*$ is the convolution as defined in §2.1.3 and §4.1. Then, $\varphi \in C_0^\infty(Q)$. By assumption we have $0 = \int_Q h\ \varphi\ dx = \int_Q (\Phi h)(G_t * \psi) dx$. Fubini's theorem (§7.2.2) now implies $0 = \int_Q (G_t * \Phi h)\psi dx$. Note that $G_t * \Phi h$ is continuous with respect to x for $t > 0$ as a consequence of Lebesgue's dominated convergence theorem (Exercise 7.2). Since $\psi \in C_0^\infty(Q)$ is arbitrary, by the Hint of Exercise 1.8 we have $(G_t * (\Phi h))(x) = 0,\ x \in Q$. Next observe that Φh is an integrable function on $\mathbb{R}^n$ and that (*) of Exercise 7.3 yields $\lim_{t \downarrow 0} \|G_t * (\Phi h) - \Phi h\|_1 = 0$. This implies that $(G_{t_j} * (\Phi h))(x)$ converges to $(\Phi h)(x)$ for almost all x as $t_j \to 0$ for suitable $t_j \to 0$. (See [Rudin 1987, Theorem 3.12], [Ito 1963, Theorem 22.2].) On the other hand, since $G_t * (\Phi h) \equiv 0$ on Q, Φh is zero *almost everywhere* on Q. (In other words, $(\Phi h)(x) = 0$ for *almost all* $x \in Q$.) In view of the fact that Φ is positive on $B_r(x_0)$, h is zero almost everywhere on a neighborhood $B_r(x_0)$ of x_0. Since $x_0 \in Q$ was arbitrary, h is zero almost everywhere on Q.
Now, if $u \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap (C(\mathbb{R}^n \times [0, \infty))$ is a weak solution with initial value $u(x, 0)$, we may reverse integration by parts in §1.4.2 to obtain the result that

$$0 = \int_0^\infty \int_{\mathbb{R}^n} \varphi(\partial_t u - \Delta u) dx\ dt$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times [0, \infty))$. Applying the Hint yields $\partial_t u - \Delta u = 0$ on $\mathbb{R}^n \times (0, \infty)$. (Note that it is sufficient to prove the above equality for $\varphi \in C_0^\infty(\mathbb{R}^n \times (0, \infty))$.)

1.9 Since $v_i(x,t) = g(x, t+1/i)$ is a solution of the heat equation with initial value $v_{i0}(x) = g(x, 1/i)$ (Exercise 1.1), it is a weak solution of (1.1) with initial value v_{i0}. Thus, to solve the exercise it suffices to verify the assumptions of Theorem 1.4.4. Assumption (iii) is obvious since $g(x,t)$ is continuous on $t > 0$, and therefore uniformly continuous on any compact subset of $\mathbb{R}^n \times (0, \infty)$. The assumption (ii) is also obvious, since $\|G_t\|_1 = 1$. Finally, to see assumption (i) we employ §4.2.4 with $\hat{x} = 0$ and $K_t = G_t$. Note that (i) also can be proved directly; see Exercise 4.4.

1.10 Since $v_i(x,t) = u(x, t+1/i)$ is a solution of the heat equation with initial value $v_{i0}(x) = u(x, 1/i)$, it is a weak solution of (1.1) with initial value v_{i0}. As in Exercise 1.9 it therefore suffices to verify the assumptions of the theorem in §1.4.4 for $m = \int_{\mathbb{R}^n} u(x,1)dx$. First we show (i), i.e., "the convergence to the initial value." By the self-similarity of u we have $v_{i0}(x) = u(x, 1/i) = k^n u(kx, 1)$, $k^2 = i$, $k > 0$, and $\|u\|_1(1) < \infty$. Hence, (i) follows from Proposition 1.4.1.
Next we show (ii), i.e., "the uniform estimate." By the L^1-L^1 estimate (§1.1.2), we have $\|v_i\|_1(t) \le \|v_{i0}\|_1$. (Observe that the support of v_{i0} is not compact. However, the L^1-L^1 estimate is valid for $v_{i0} \in L^1(\mathbb{R}^n)$.) Set $i = k^2$, $k > 0$. Then the self-similarity and

$$\|v_{i0}\|_1 = \int_{\mathbb{R}^n} |k^n u(kx, 1)| dx = \|u\|_1(1)$$

imply that $\sup_{i \ge 1} \sup_{t > 0} \|v_i\|_1(t) \le \|u\|_1(1) < \infty$. By the continuity of u on $t > 0$, (iii) can be obtained similarly as in Exercise 1.9. From Theorem §1.4.4 we then infer that u is a weak solution of the heat equation with initial value $m\delta$.

Chapter 2

2.1 Let $n = 3$ and $i = 1, 2, 3$. The ith component of $\nabla \operatorname{div} v$ is given by

$$(\nabla \operatorname{div} v)^i = \partial_{x_i} \sum_{j=1}^{3} \partial_{x_j} v^j,$$

whereas the ith component of $\operatorname{curl}\operatorname{curl} v$ is given by

$$\begin{aligned}(\operatorname{curl}\operatorname{curl} v)^i &= \partial_{x_{i+1}} (\operatorname{curl} v)^{i+2} - \partial_{x_{i+2}} (\operatorname{curl} v)^{i+1} \\ &= \partial_{x_{i+1}} (\partial_{x_i} v^{i+1} - \partial_{x_{i+1}} v^i) - \partial_{x_{i+2}} (\partial_{x_{i+2}} v^i - \partial_{x_i} v^{i+2}) \\ &= -\partial^2_{x_{i+1}} v^i - \partial^2_{x_{i+2}} v^i + \partial_{x_{i+1}} \partial_{x_i} v^{i+1} + \partial_{x_{i+2}} \partial_{x_i} v^{i+2}.\end{aligned}$$

Here the indices are modulo 3. This implies

$$-\Delta v^i = (\operatorname{curl}\operatorname{curl} v)^i - \partial_{x_i}\operatorname{div} v \qquad (1 \le i \le 3),$$

which proves (2.3a). In the case that $n = 2$ we regard v as a three-component vector, where v^1 and v^2 are functions depending only on (x_1, x_2), and where $v^3 = 0$. Then we obtain

$$\operatorname{curl}\operatorname{curl} v = \operatorname{curl}(0, 0, \partial_{x_1} v^2 - \partial_{x_2} v^1) = (\nabla^\perp(\partial_{x_1} v^2 - \partial_{x_2} v^1), 0).$$

Thus, (2.3b) follows from (2.3a).

2.2 Differentiating both sides of the geometric series $\sum_{j=0}^\infty x^j = \frac{1}{1-x}$, $|x| < 1$, we obtain $\sum_{j=0}^\infty jx^{j-1} = \frac{1}{(1-x)^2}$, since it is termwise differentiable under summation for $|x| < 1$. Setting $x = 1/2$, by $\sum_{j=0}^\infty jx^j = x/(1-x)^2$ we obtain $\sum_{j=0}^\infty j2^{-j} = 2$. Of course, the assertion also follows easily from $\sum_{j=0}^\infty jx^j - \sum_{j=0}^\infty (j-1)x^j = \sum_{j=0}^\infty x^j$.

2.3 Suppose that $\|f\|_\infty > M > 0$. Then $\{x \in \mathbb{R}^n : |f(x)| \ge M\}$ has positive Lebesgue measure. Hence, for sufficiently large R, the set $F = \{x \in B_R : |f(x)| \ge M\}$ has finite positive Lebesgue measure. This yields

$$\|f\|_r \ge \left(\int_F |f(x)|^r \, dx \right)^{1/r} \ge M|F|^{1/r}.$$

(Here $|F|$ denotes the Lebesgue measure of F.) Letting $r \to \infty$ we obtain $\varliminf_{r\to\infty} \|f\|_r \ge M$. Since M was an arbitrary positive constant such that $M < \|f\|_\infty$, we have shown that $\varliminf_{r\to\infty} \|f\|_r \ge \|f\|_\infty$ for the case $\|f\|_\infty \le \infty$.

Hence it remains to show that $\varlimsup_{r\to\infty} \|f\|_r \le \|f\|_\infty$ for $\|f\|_\infty < \infty$. If $\|f\|_\infty = 0$, then $f = 0$ almost everywhere on $\mathbb{R}^n$. Hence $\|f\|_r = 0$ for $r \ge 0$ and the inequality is obvious. Therefore, we may assume $\|f\|_\infty > 0$. Since $\|f\|_{r_0} < \infty$, for any $\varepsilon > 0$, there exists a sufficiently large R such that

$$\|f\|_{r_0}^{r_0} = \int_{B_R} |f|^{r_0} dx + \int_{\mathbb{R}^n \setminus B_R} |f|^{r_0} dx \le \int_{B_R} |f|^{r_0} dx + \varepsilon.$$

By this choice of R for $r \ge r_0$ we obtain

$$\|f\|_r^r \le \|f\|_\infty^r |B_R| + \int_{\mathbb{R}^n \setminus B_R} |f|^r dx \le \|f\|_\infty^r |B_R| + \|f\|_\infty^{r-r_0} \varepsilon.$$

This implies

$$\varlimsup_{r\to\infty} \|f\|_r \le \|f\|_\infty \varlimsup_{r\to\infty} (|B_R| + \|f\|_\infty^{-r_0} \varepsilon)^{1/r} = \|f\|_\infty.$$

2.4 We have

$$\|f\|_q = \left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{1/q}$$
$$\leq \left(\int_{\mathbb{R}^n} \|f\|_\infty^{q-1}|f(x)|dx\right)^{1/q} \leq \|f\|_\infty^{1-1/q}\|f\|_1^{1/q}.$$

2.5 (i) We will show inductively that $y_s(t) \leq N_s t^{1-s/\rho}$, $s = 2^m \geq \rho = 2^k$, $t > 0$, where N_s is defined by $N_s = sN_{s/2}^2/\rho a$, $s = 2^m$, $m+1 \geq k$. We employ induction with respect to $m = k, k+1, \ldots$. If $m = k$, the desired result is obvious by the assumption. Suppose that the desired result is true up to m and set $s = 2^m \geq 2^k = \rho$. Substituting the induction hypothesis into the differential inequality, we obtain

$$\frac{dy_{2s}}{dt}(t) \leq -a\left(1 - \frac{1}{2s}\right)\frac{y_{2s}^2(t)}{N_s^2}t^{2s/\rho-2}$$

for $t > 0$. Dividing this inequality by $-y_{2s}^2$ yields

$$-\frac{dy_{2s}}{dt}(t)/y_{2s}^2(t) \geq a\left(1 - \frac{1}{2s}\right)N_s^{-2}t^{2s/\rho-2}$$
$$\geq a\left(1 - \frac{\rho}{2s}\right)N_s^{-2}t^{2s/\rho-2} \quad (> 0).$$

Hence, similarly to the proof of Lemma 2.3.4 we deduce

$$\frac{1}{y_{2s}(t)} \geq \frac{a\rho}{2sN_s^2}t^{2s/\rho-1} = \frac{1}{N_{2s}}t^{2s/\rho-1},$$

which implies (i).

(ii) For sufficiently large $s = 2^m$ we will show that

$$(y_s(t))^{1/s} \leq (4/a)^{1/\rho}\, N_\rho^{1/\rho}t^{-1/\rho+1/s}$$

for $t > 0$, $\rho = 2^k$. Set $\nu_s = (N_s)^{1/s}$. Then (i) implies that $(y_s)^{1/s} \leq \nu_s t^{-1/\rho+1/s}$, $s \geq \rho$. Thus, in order to obtain (i) it suffices to show that $\nu_s \leq \nu_\rho(4/a)^{1/\rho}$ for sufficiently large s. Set $c_m = \log \nu_s$, $s = 2^m$. By the successive relations $(N_s)^{1/s} = (s/\rho a)^{1/s}(N_{s/2})^{2/s}$, we have $c_m = c_k + \sum_{j=k+1}^m \frac{1}{2^j}((j-k)\log 2 - \log a)$. Similarly to Lemma 2.3.4 (ii) we estimate c_m for large m, obtaining result

$$c_m \leq c_k + \frac{1}{2^k}\log\frac{4}{a}.$$

Applying the exponential function to both sides implies $\nu_s \leq \nu_\rho(4/a)^{1/\rho}$. Hence, (ii) is proved.

2.6 First consider the case $1 \leq q < \infty$. Then $|f_k(x)|^q$ converges pointwise to $|f(x)|^q$ as $k \to \infty$. Fatou's lemma (§7.1.2) implies

$$\int_{\mathbb{R}^n} |f(x)|^q dx \leq \varliminf_{k\to\infty} \int_{\mathbb{R}^n} |f_k(x)|^q dx.$$

Since $t \mapsto t^{1/q}$ is continuous in $t \geq 0$, taking the $1/q$ power on both sides of this inequality yields

$$\|f\|_q \leq \left(\varliminf_{k\to\infty} \|f_k\|_q^q\right)^{1/q} = \varliminf_{k\to\infty} \|f_k\|_q.$$

For the case $q = \infty$, it is sufficient to prove that $M \leq \varliminf_{k\to\infty} \|f_k\|_\infty$ for each M satisfying $\|f\|_\infty > M > 0$. Suppose the result does not hold. Then, $\varliminf_{k\to\infty} \|f_k\|_\infty < M$. Hence for sufficiently large k we have $\|f_k\|_\infty \leq M$. Since f is the pointwise limit of f_k, we have $\|f\|_\infty \leq M$. This contradicts the definition of M. Hence, we have shown that $M \leq \varliminf_{k\to\infty} \|f_k\|_\infty$ for each M with $\|f\|_\infty > M > 0$. This implies $\|f\|_\infty \leq \varliminf_{k\to\infty} \|f_k\|_\infty$.

2.7 and 2.8 Answers are omitted.

Chapter 3

3.1 Let M be the maximum value of ψ on I. We will show that assuming $\psi \not\equiv M$ on I leads to a contradiction. By the continuity of ψ there exists a open interval $J = (x_0, x_1)$, $(\overline{J} \subset I)$, such that

$$\text{(i)} \qquad \psi(x) < M,\ x \in [x_0, x_1),\ \psi(x_1) = M$$

or

$$\text{(ii)} \qquad \psi(x) < M,\ x \in (x_0, x_1],\ \psi(x_0) = M.$$

We may concentrate on the second case, since the proof for case (i) is analogous.

Recall that $\rho(x) = e^{\alpha(x-x_0)} - 1$ satisfies $\rho'' + b\rho' > 0$ on I if $\alpha > \sup_I |b|$. We fix such an $\alpha > 0$ and choose $\varepsilon > 0$ such that $\psi(x_1) + \varepsilon\rho(x_1) < M$. Then set $\varphi = \psi + \varepsilon\rho$. By the signature of ρ and the definition of ε we obtain $\varphi(x) < M$, $x < x_0$, $\varphi(x_0) = M$, and $\varphi(x_1) < M$. Taking a subinterval $[y_0, x_1] \subset I$ such that $y_0 < x_0$, we observe that the maximum point a of φ in $[y_0, x_0]$ is an interior point of $[y_0, x_0]$. (The existence of the maximum follows from the Weierstrass theorem.) On the other hand, by the definition of ρ and since $\psi'' + b\psi' \geq 0$, we deduce $\varphi''(a) + b\varphi'(a) > 0$. Thus, at the maximum point we have $\varphi'(a) = 0$ and $\varphi''(a) \leq 0$, which contradicts the above inequality. Therefore, case (ii) cannot occur. Similarly, the case (i) cannot occur. This implies $\psi \equiv M$.

Remark. This result is the strong maximum principle for the case of one variable. For a general version of the strong maximum principle we refer to the standard textbook [Protter Weinberger 1967].

3.2 Assume that there exist x_0 and x_1 such that $w(x_0) < w(x_1)$ and $x_0 < x_1$. The assumption $w''(x) \leq 0$ implies that w is concave. Consequently, we have

$$w(x) \leq \frac{w(x_1) - w(x_0)}{x_1 - x_0}(x - x_0) + w(x_0)$$

for $x < x_0$. However, this contradicts the fact that $w(x) > 0$ for $x < x_0$. Similarly, there exists no pair of x_0 and x_1 such that $w(x_0) > w(x_1)$ and $x_0 < x_1$. Hence w is a constant function.

3.3 (i) We remark that

$$\infty > \int_0^\infty f(t)dt = \sum_{n=0}^\infty a_n.$$

Since $a_n \geq 0$ we obtain $\lim_{n\to\infty} a_n = 0$. (Elementary exercise: if $\sum_{n=0}^\infty a_n < \infty$ and $a_n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.)

(ii) An example of such an f is constructed as follows. Set

$$h(t) = \begin{cases} \sin t, & 0 \leq t \leq \pi, \\ 0, & t < 0 \quad \text{or} \quad \pi < t, \end{cases}$$

and $h_j(t) = h(j^2 t - 2j^3\pi)$ for $j \geq 1$. Then we obtain $\operatorname{supp} h_j = [2j\pi,\ 2j\pi + \pi/j^2]$, $\max h_j = 1$, and

$$\int_0^\infty h_j(t)dt = \frac{1}{j^2}\int_0^\pi \sin t dt = \frac{2}{j^2}.$$

We set $f(t) = \sum_{j=1}^\infty h_j(t)$. Since $\operatorname{supp} h_j$ are disjoint sets for $j \geq 1$, the above summation is finite for $t \in [0, \infty)$. The nonnegativity and continuity of h_j also imply that f is nonnegative and continuous. Moreover, for any t we can choose a sufficiently large j such that t is to the left of $\operatorname{supp} h_j$. Hence, by $\max h_j = 1$ there exists a number s such that $s > t$ and $f(s) \geq 1$. This yields $\lim_{t\to\infty} f(t) \neq 0$. On the other hand, we have

$$\int_0^\infty f(t)dt = \sum_{j=1}^\infty \int_{\operatorname{supp} h_j} h_j\ dt = 2\sum_{j=1}^\infty \frac{1}{j^2} < \infty.$$

3.4 (3.41) Differentiating $E(w)$ with respect to τ, we obtain

$$\frac{dE(w)}{d\tau} = \int_{\mathbb{R}^n} (\langle\ \nabla w,\ \nabla\partial_\tau w\ \rangle + \beta w \partial_\tau w - |w|^{p-1} w \partial_\tau w) e^{-|z|^2/4} dz.$$

In view of

$$\int_{\mathbb{R}^n} \langle \nabla w, \nabla \partial_\tau w \rangle e^{-|z|^2/4} dz = -\int_{\mathbb{R}^n} \Delta w \partial_\tau w \; e^{-|z|^2/4} dz + \int_{\mathbb{R}^n} \frac{1}{2} \langle \; z, \; \nabla w \; \rangle \partial_\tau w \; e^{-|z|^2/4} dz,$$

we obtain

$$\begin{aligned} \frac{dE(w)}{d\tau} &= -\int_{\mathbb{R}^n} \partial_\tau w \left(\Delta w - \frac{1}{2} \langle z, \nabla w \rangle - \beta w + |w|^{p-1} w \right) \cdot e^{-|z|^2/4} dz \\ &= -\int_{\mathbb{R}^n} (\partial_\tau w)^2 e^{-|z|^2/4} dz. \end{aligned}$$

3.5 (3.45) Differentiating $\Psi(w)$ with respect to τ implies

$$\begin{aligned} \frac{d\Psi(w)}{d\tau} &= \int_{\mathbb{R}^n} \sum_{i=1}^{m+1} \langle \; \nabla w^i, \; \nabla \partial_\tau w^i \; \rangle e^{-|z|^2/4} dz \\ &= -\sum_{i=1}^{m+1} \int_{\mathbb{R}^n} \left(\Delta w^i - \frac{1}{2} \langle \; z, \; \nabla w^i \; \rangle \right) \cdot \partial_\tau w^i \; e^{-|z|^2/4} dz. \end{aligned}$$

Since $|w|^2 \equiv 1$, we have $\langle w, \partial_\tau w \rangle_{\mathbb{R}^{m+1}} = 0$. This yields

$$\begin{aligned} \frac{d\Psi(w)}{d\tau} &= -\sum_{i=1}^{m+1} \int_{\mathbb{R}^n} \left(\Delta w^i - \frac{1}{2} \; \langle z, \nabla w^i \rangle + |\nabla w|^2 w^i \right) \cdot \partial_\tau w^i e^{-|z|^2/4} dz \\ &= -\int_{\mathbb{R}^n} |\partial_\tau w|^2 e^{-|z|^2/4} dz. \end{aligned}$$

Chapter 4

4.1 Let $r > 0$ be a positive number, and set $Q = \mathbb{R}^n \setminus \overline{B}_r$. For $f \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} f \subset Q$, by the assumption we have $(h * f)(0) = f(0) = 0$. Since $h * f = f * h$, we obtain $\int_Q \varphi(y) h(y) dy = 0$ for arbitrary $\varphi(y) = f(-y)$ in $C_0^\infty(Q)$. By the fundamental lemma of calculus of variations (Exercise 1.8), h is almost everywhere zero on Q. Since $r > 0$ is arbitrary, h is almost everywhere zero on $\mathbb{R}^n$. Hence $h * f = 0$ for $f \in C_0^\infty(\mathbb{R}^n)$. Thus, there exists no $h \in L^1(\mathbb{R}^n)$ such that $h * f = f$ for every $f \in C_0^\infty(\mathbb{R}^n)$.

4.2 First recall that for real numbers $1 < p, p' < \infty$ satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ the following Young inequality holds:

$$ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad a > 0, \; b > 0.$$

(Equality holds if and only if $a^p = b^{p'}$.) (If $p = 2$, it is just the relation between geometric and arithmetic means.)
This inequality follows from the fact that the logarithmic function $\log x$ is strictly concave, since $(\log x)'' = -x^{-2} < 0$. In fact, for $0 \le \lambda \le 1$ we have $\lambda \log x + (1-\lambda) \log y \le \log(\lambda x + (1-\lambda) y)$, and equality holds if and only if $x = y$. Substituting $\lambda = 1/p$, $x = a^p$, and $y = b^{p'}$ into the above inequality, we obtain Young's inequality.
The Hölder inequality is obvious for $p = 1$ or $p = \infty$, hence we assume $1 < p,\, p' < \infty$. If $\|f_1\|_p = 0$ or $\|f_0\|_{p'} = 0$, then $f_1 f_0$ is also zero and we see that also in this case the Hölder inequality holds. So, we may assume $\|f_1\|_p \neq 0$ and $\|f_0\|_p \neq 0$. Integrating Young's inequality for $a = |f_1(x)|/\|f_1\|_p$, $b = |f_0(x)|/\|f_0\|_{p'}$, we obtain

$$\begin{aligned}
&\frac{1}{\|f_1\|_p \|f_0\|_{p'}} \int |f_1(x) f_0(x)| dx \\
&\quad \le \frac{1}{p} \frac{1}{\|f_1\|_p^p} \int |f_1(x)|^p dx + \frac{1}{p'} \frac{1}{\|f_0\|_{p'}^{p'}} \int |f_0(x)|^{p'} dx \\
&\quad = \frac{1}{p} + \frac{1}{p'} = 1,
\end{aligned}$$

and the Hölder inequality is proved.
We remark that the Hölder inequality is valid not only for integrals over $\mathbb{R}^n$, but also for integrals over an open set Ω in $\mathbb{R}^n$, or even more generally, for a measure space X with measure μ. In fact, we have

$$\left| \int_X f_1 f_0 d\mu \right| \le \int_X |f_1 f_0| d\mu \le \left(\int_X |f_1|^p d\mu \right)^{1/p} \left(\int_X |f_0|^{p'} d\mu \right)^{1/p'}$$

for all μ-integrable functions f_0, f_1 on X. The proof works completely analogously to the above by replacing dx by $d\mu$.

4.3 Applying the Young inequality to $\partial_{x_j} u = \partial_{x_j}(G_t * f) = (\partial_{x_j} G_t) * f$, we obtain $\|\partial_{x_j} u\|_p(t) \le \|\partial_{x_j} G_t\|_r \|f\|_q$ for $1 \le r \le \infty$ with $1/p = 1/r + 1/q - 1$. If $r < \infty$, the substitution $z = (r/4t)^{1/2} x$ yields $\|\partial_{x_j} G_t\|_r^r = \int |x_j/2t|^r |G_t(x)|^r dx$. Then, similarly as in §4.1.2 we obtain

$$\|\partial_{x_j} G_t\|_r^r = \frac{1}{(4\pi t)^{nr/2}} \left(\frac{4t}{r} \right)^{n/2} \left(\frac{1}{rt} \right)^{r/2} \int |z_j|^r e^{-|z|^2} dz.$$

By Exercise 1.2 we have

$$\begin{aligned}
|z_j|^r e^{-z_j^2} &= |z_j|^r e^{-z_j^2/2} \cdot e^{-z_j^2/2} \\
&\le 2^{r/2} \left(\sup_{s>0} s^{r/2} e^{-s} \right) e^{-z_j^2/2} \\
&\le 2^{r/2} (r/2e)^{r/2} e^{-z_j^2/2}.
\end{aligned}$$

Hence, $\int |z_j|^r e^{-z^2} dz$ is finite. Therefore, there exists a constant C depending only on n and r such that $\|\partial_{x_j} G_t\|_r^r \le C t^{-r\sigma}$ $(t > 0)$, where $\sigma = 1/2 + n/2r'$. This implies $\|\partial_{x_j} u\|_p(t) \le C t^{-\sigma} \|f\|_q$ $(t > 0)$. (The case $r = \infty$ was shown already in §1.1.3.)

4.4 A simple substitution gives us

$$\int_{\mathbb{R}^n} f_k(x)\psi(x)dx = \int_{\mathbb{R}^n} k^n f(kx)\psi(x)dx = \int_{\mathbb{R}^n} f(y)\psi\left(\frac{y}{k}\right) dy$$

for $k \ge 1$. We have $|\psi(y/k)| \le \sup\{|\psi(y)| : y \in \operatorname{supp}\ f\} = c_0 < \infty$, which yields $|f(y)\psi(y/k)| \le c_0 |f(y)|,\ y \in \operatorname{supp}\ f$. The right-hand side represents an integrable function independent of k. Lebesgue's dominated convergence theorem (§7.1.1) therefore implies

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(x)\psi(x)dx = \int_{\mathbb{R}^n} f(y) \lim_{k\to\infty} \psi\,(y/k)\,dy = \psi(0) \int_{\mathbb{R}^n} f(y)dy.$$

Note that here the boundedness of $\operatorname{supp} f$ and f allows for an application of the dominated convergence theorem. In the case that $f \in L^1(\mathbb{R}^n)$ and ψ is bounded we may estimate $|f(y)\psi(y/k)| \le (\sup_{\mathbb{R}^n} |\psi|)|f(y)|$. Thus, also in this case Lebesgue's dominated convergence theorem yields the assertion.

4.5 Similarly to the proof in §4.3.2 we obtain

$$\partial_t^k w^\rho(t) = e^{\rho\Delta} \sum_{h=0}^{k} \Delta^h \partial_t^{k-h} h(t-\rho) + \int_0^{t-\rho} e^{(t-s)\Delta} \Delta^k h(s)ds.$$

(Note that here $\Delta^0 = \partial_t^0 = I$ with I the identity operator.) From the estimate $\|e^{\rho\Delta} f\|_\infty \le \|f\|_\infty$ (see §1.1.2) we infer that

$$\begin{aligned}\|\partial_x^\alpha \partial_t^k w^\rho(t)\|_\infty \le &\sum_{h=0}^{k} \|\Delta^h \partial_t^{k-h} \partial_x^\alpha h(t-\rho)\|_\infty \\ &+ \int_0^{t-\rho} \|\Delta^k \partial_x^\alpha h(s)\|_\infty ds.\end{aligned}$$

By assumption, $c_j = \sup_{|\sigma|+2\ell\le j} \sup_{0\le t\le T} \|\partial_x^\sigma \partial_t^\ell h\|_\infty(t)$ is finite. For $\rho \le t \le T$ this gives us $\|\partial_x^\alpha \partial_t^k w^\rho(t)\|_\infty \le (k+1)c_{2k+|\alpha|} + Tc_{2k+|\alpha|}$. Thus, $\partial_x^\alpha \partial_t^k w^\rho$ converges uniformly on $\mathbb{R}^n \times [\rho_0, T]$, $\rho_0 > 0$ as $\rho \to 0$. This implies $\sup_{0<t\le T} \|\partial_x^\alpha \partial_t^k w\|_\infty(t) < \infty$.

4.6 Suppose the assertion does not hold. Then, $\underline{\lim}_{R\to\infty} J(R) = c_0 > 0$. Hence, we have $J(R) \ge c_0/2$ for $R \ge R_0$ and sufficiently large R_0. This contradicts $\int_1^\infty J\ dt < \infty$.

Chapter 5

5.1 For $f \in C(M)$ we define an open ball in $C(M)$ centered at f with radius $\varepsilon > 0$ by $B(f,\varepsilon) = \{h \in C(M);\ \|f-h\|_{\infty,M} < \varepsilon\}$. Since $\overline{K}$ is compact and $\overline{K} \subset \bigcup_{f\in K} B(f,\varepsilon)$, there exist suitable $f_1,\dots,f_{N(\varepsilon)} \in C(M)$ such that $\overline{K} \subset \bigcup_{i=1}^{N(\varepsilon)} B(f_i,\varepsilon)$. Since $B(f_i,\varepsilon)$ are bounded sets, K is bounded, too.

For the equicontinuity of K pick $\varepsilon > 0$ and let $\{f_i\}_{i-1}^{N(\varepsilon)}$ be as constructed above. For $z \in M$, since $f_i \in C(M)$, $1 \le i \le N(\varepsilon)$, there exists a suitable neighborhood V_z^i of z such that $|f_i(z) - f_i(y)| \le \varepsilon$ for $y \in V_z^i$. Since $\{f_1,\dots,f_{N(\varepsilon)}\}$ is finite, we observe that $V_z = \bigcap_{i=1}^{N(\varepsilon)} V_z^i$ is still a neighborhood of z. This implies $|f_i(z) - f_i(y)| \le \varepsilon$ for all i with $1 \le i \le N(\varepsilon)$ and $y \in V_z$. By construction, for any $f \in K$, there exists an i with $1 \le i \le N(\varepsilon)$ such that $\|f - f_i\|_{\infty,M} < \varepsilon$. Thus, for $y \in V_z$ we deduce $|f(z) - f(y)| \le |f(z) - f_i(z)| + |f_i(z) - f_i(y)| + |f_i(y) - f(y)| \le 2\|f - f_i\|_{\infty,M} + |f_i(z) - f_i(y)| \le 2\varepsilon + \varepsilon$ (i.e., V_z is independent of f). In other words, for any given ε we can find a neighborhood V_z of z as above such that $\sup_{f\in K} |f(z) - f(y)| \le 3\varepsilon$ for all $y \in V_z$. This shows that $\lim_{y\to z} \sup_{f\in K} |f(z) - f(y)| = 0$; hence K is equicontinuous.

5.2 We will show only the completeness of $C^\nu(M)$. Let $\{f_j\}_{j=1}^\infty$ be a Cauchy sequence in $C^\nu(M)$. By the definition of the norm $\|f\|_{C^\nu}$, the sequence $\{f_j\}_{j=1}^\infty$ is also a Cauchy sequence in $C(M)$. By the completeness of $C(M)$ (Exercise 1.6), $\{f_j\}_{j=1}^\infty$ converges uniformly to an element f in $C(M)$. Furthermore, the lower semicontinuity of the supremum (Exercise 5.6) implies

$$[f]_\nu = \sup_{x\neq y} \frac{|\lim_{j\to\infty}(f_j(x) - f_j(y))|}{|x-y|^\nu} \le \varliminf_{j\to\infty} [f_j]_\nu .$$

Since $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence in $C^\nu(M)$, it is bounded, i.e., $\sup_{j\ge1} \|f_j\|_{C^\nu} < \infty$. Hence $[f]_\nu$ is finite, implying $f \in C^\nu(M)$.

Again by the lower semicontinuity of the supremum we have

$$\varlimsup_{j\to\infty} [f - f_j]_\nu = \varlimsup_{j\to\infty} [\lim_{\ell\to\infty} f_\ell - f_j]_\nu \le \varlimsup_{j\to\infty} \varliminf_{\ell\to\infty} [f_\ell - f_j]_\nu .$$

The fact that $\{f_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to $\|\cdot\|_{C^\nu}$ shows that the right-hand side is zero. Since $\|f - f_j\|_{\infty,M} \to 0$ as $j \to \infty$ we obtain $\lim_{j\to\infty} \|f - f_j\|_{C^\nu} = 0$. Thus, $C^\nu(M)$ is complete.

5.3 Let K be a bounded subset of $C^\nu(M)$, i.e., $A = \sup_{f\in K} \|f\|_{C^\nu} < \infty$. The fact that K is bounded as a subset of $C(M)$ is obvious from the definition of the norm $\|\cdot\|_{C^\nu}$. The uniform boundedness in $C^\nu(M)$ implies $\sup_{f\in K} |f(y) - f(z)| \le A|y-z|^\nu$. Consequently, $\varlimsup_{y\to z} \sup_{f\in K} |f(y) - f(z)| \le \varlimsup_{y\to z} A|y-z|^\nu = 0$, and we see that K is equicontinuous. By the Ascoli-Arzelà theorem for compact M, K is relatively compact in $C(M)$. Hence $C^\nu(M)$ is compactly embedded in $C(M)$.

5.4 Let K be a bounded subset in $C^1(\overline{D})$, i.e.,

$$A = \sup_{f \in K} \sup_{|\alpha| \leq 1} \sup_{D} |\partial_x^\alpha f| < \infty.$$

The fact that K is bounded as a subset of $C(\overline{D})$ is obvious. For $z \in \overline{D}$, take a ball $B_r(z)$ centered at z with radius r. Then, by the convexity of D, the line segment connecting any point y with z in $B_r(z) \cap \overline{D}$ is contained in $B_r(z) \cap \overline{D}$. Hence, the integral form of the mean value theorem gives us

$$|f(y) - f(z)| \leq |y - z| \int_0^1 |\nabla f(\tau y + (1-\tau)z)| d\tau \leq \sqrt{n} A |y - z|,$$

for all $y \in \overline{D} \cap B_r(z)$. (See §1.1.6.) This yields

$$\varlimsup_{\substack{y \to z \\ y \in \overline{D}}} \sup_{f \in K} |f(y) - f(z)| \leq \sqrt{n} A \varlimsup_{y \to z} |y - z| = 0,$$

which shows the equicontinuity of K. By the Ascoli–Arzelà theorem for compact domains, K is relatively compact in $C(\overline{D})$. Thus, $C^1(\overline{D})$ is compactly embedded in $C(\overline{D})$.
Let K be a bounded set in $C^2(\overline{D})$. For a subsequence $\{f_k\}_{k=1}^\infty$ in K, consider $\{\partial_{x_j} f_k\}_{k=1}^\infty$ ($j = 1, 2, \ldots, n$). By similar arguments as above, the boundedness of K in $C^2(\overline{D})$, $\{f_k\}_{k=1}^\infty$, and $\{\partial_{x_j} f_k\}_{k=1}^\infty$ are equicontinuous. According to the Ascoli–Arzelà theorem we may choose a suitable subsequence $\{f_{k(i)}\}_{i=1}^\infty$ such that $f_{k(i)}$ and $\partial_{x_j} f_{k(i)}$ converge uniformly to continuous functions h_0 and h_j ($j = 1, \ldots, n$) on $\overline{D}$ as $i \to \infty$, respectively. By interchanging limits and differentials (§4.1.5), h_0 is C^1 and $h_j = \partial_{x_j} h_0$. Hence, we obtain $\|f_{k(i)} - h_0\|_{C^1} \to 0$ ($i \to \infty$). This shows that K is relatively compact in $C^1(\overline{D})$.

5.5 Let K be relatively compact in $C_\infty(M)$ and $\varepsilon > 0$. Then, analogously to Exercise 5.1 there exists a finite set $\{f_i\}_{i=1}^{N(\varepsilon)}$ such that $\overline{K} \subset \bigcup_{i=1}^{N(\varepsilon)} B(f_i, \varepsilon)$. Hence, the boundedness and the equicontinuity in $C_\infty(M)$ follow completely analogously to those in Exercise 5.1. For the equidecay property note that we may assume that

$$\sup_{1 \leq i \leq N(\varepsilon)} \sup_{x \in M \setminus M_j} |f_i(x)| \leq \varepsilon$$

by choosing j sufficiently large. Now, for $f \in K$ we choose f_i such that $f \in B(f_i, \varepsilon)$. This implies

$$\begin{aligned} \sup_{x \in M \setminus M_j} |f(x)| &\leq \sup_{x \in M \setminus M_j} |f(x) - f_i(x)| + \sup_{x \in M \setminus M_j} |f_i(x)| \\ &\leq \|f - f_i\|_{\infty, M} + \varepsilon < 2\varepsilon. \end{aligned}$$

Thus, for each $\varepsilon > 0$ there exists $j = j(\varepsilon)$ such that

$$\sup_{f\in K} \sup_{x\in M\setminus M_j} |f(x)| \leq 2\varepsilon.$$

Since $\{M_j\}_{j=1}^{\infty}$ is an increasing sequence,

$$\lim_{j\to\infty} \sup_{f\in K} \sup_{x\in M\setminus M_j} |f(x)| = 0,$$

which proves the equidecay property of K in $C_\infty(M)$.

5.6 The fact that $h_m(z) \leq \sup_Z h_m$ is obvious by the definition of the supremum. This implies $\underline{\lim}_{m\to\infty} h_m(z) \leq \underline{\lim}_{m\to\infty} \sup_Z h_m$, valid for arbitrary $z \in Z$. Taking the supremum on the left-hand side with respect to z shows that $\sup_Z \underline{\lim}_{m\to\infty} h_m(z) \leq \underline{\lim}_{m\to\infty} \sup_Z h_m$.

Chapter 6

6.1 (i) Assume that $1 \leq p < \infty$. Since $u_j(x) = u(x)$ for $|x| \leq j$, we have

$$\|u_j - u\|_p^p \leq \int_{|x|\geq j} |\theta_j(x) - 1|^p |u(x)|^p dx \leq \int_{|x|\geq j} |u(x)|^p dx.$$

Since $\|u\|_p^p < \infty$, the right-hand side converges to 0 as $j \to \infty$. For $u \in C_\infty(\mathbb{R}^n)$ we obtain

$$\|u_j - u\|_\infty \leq \sup_{|x|\geq j} |u(x)| \to 0 \quad (j \to \infty).$$

The norm $\|u_j\|_\infty$ is nondecreasing with respect to j, which implies $\|u_j\|_\infty \leq \|u\|_\infty$. On the other hand, by Exercise 2.6, $\|u\|_\infty \leq \underline{\lim}_{j\to\infty} \|u_j\|_\infty$. Thus, $\|u\|_\infty = \lim_{j\to\infty} \|u_j\|_\infty$. Note that here $u \in C_\infty(\mathbb{R}^n)$ is not required in this paragraph.

(ii) Note that

$$\nabla u_j(x) = \theta_j(x)\nabla u(x) + \frac{1}{j}\theta'(|x|/j)\frac{x}{|x|}u(x).$$

The triangle inequality for the L^p-norm yields

$$\|\nabla(u_j - u)\|_p \leq \|(\theta_j - 1)\nabla u\|_p + \frac{1}{j}\|\theta'(|x|/j)u(x)\|_p.$$

For the first term we can proceed analogously to (i) in view of $\|\nabla u\|_p < \infty$. For the second term we use the fact that $\sup_{\mathbb{R}} |\theta'| < \infty$, $\|u\|_p < \infty$. Therefore, this term converges to 0 as $j \to \infty$, which proves the assertion.

(iii) Similarly as in (ii) it is sufficient to prove that

$$j^{-1}\|\theta'(|x|/j)u(x)\|_r \to 0 \quad (j \to \infty)$$

in the estimate of $\|\nabla(u_j - u)\|_r$. By the Hölder inequality (Exercise 4.2) we obtain

$$\|\theta'(|x|/j)u\|_r \le \left(\int_{|x|\ge j} |u|^p dx\right)^{1/p} \|\theta'(|x|/j)\|_\rho$$

for $1/\rho + 1/p = 1/r$, $p < \infty$. By assumption we have $\rho \ge n$. Hence, a change of the variables of integration gives us that $\|\theta'(|x|/j)\|_\rho j^{-1} \to 0$ $(j \to \infty)$ if $\rho > n$, or $\|\theta'(|x|/j)\|_\rho j^{-1}$ is bounded as $j \to \infty$ if $\rho = n$. If $p < \infty$, the fact that $\int_{|x|\ge j} |u|^p dx \to 0$ $(j \to \infty)$ then implies $j^{-1}\|\theta'(|x|/j)u\|_r \to 0$ $(j \to \infty)$. (Even for $p = \infty$ if $r = \rho > n$, then the last convergence follows.)

6.2 The relation of the indices can be written as $1 = \rho p/q + (1-\rho)p/r$. Thus, by the Hölder inequality we obtain

$$\begin{aligned}\|u\|_p^p &= \int_{\mathbb{R}^n} |u|^{\rho p}|u|^{(1-\rho)p} dx \le \| \, |u|^{\rho p}\|_{q/\rho p} \| \, |u|^{(1-\rho)p}\|_{r/((1-\rho)p)} \\ &\le \|u\|_q^{\rho p}\|u\|_r^{(1-\rho)p}.\end{aligned}$$

6.3 First suppose that $|f|_{q,\infty} < \infty$. Then, by definition we have $m_f(\lambda) \le |f|_{q,\infty}^q \lambda^{-q}$, $\lambda > 0$. For a measurable set E $(|E| < \infty)$ in $\mathbb{R}^n$ we consider the distribution function $m_f(\lambda, E) = |\{x \in E;\ |f(x)| > \lambda\}$ of f in E. We obtain $m_f(\lambda, E) \le \min(m_f(\lambda), |E|) \le \min(|f|_{q,\infty}^q \lambda^{-q}, |E|)$, $\lambda > 0$. By replacing $\mathbb{R}^n$ by E we have according to Proposition 6.2.2(ii) that

$$\int_E |f(x)| dx = \int_0^\infty m_f(\lambda, E) d\lambda.$$

Splitting this integral at $\beta > 0$ gives us

$$\begin{aligned}\int_E |f(x)| dx &= \int_0^\beta m_f(\lambda, E) d\lambda + \int_\beta^\infty m_f(\lambda, E) d\lambda \\ &\le |E| \int_0^\beta d\lambda + |f|_{q,\infty}^q \int_\beta^\infty \lambda^{-q} d\lambda \\ &= |E|\beta + |f|_{q,\infty}^q \frac{1}{q-1}\beta^{1-q}.\end{aligned}$$

Now we set $\beta = |f|_{q,\infty}|E|^{-1/q}$. This yields

$$\int_E |f(x)| dx \le \left(1 + \frac{1}{q-1}\right) |f|_{q,\infty}|E|^{1-1/q},$$

hence $\|f\|_{q,\infty} \le \left(1 + \frac{1}{q-1}\right)|f|_{q,\infty}$.

Conversely, suppose that $\|f\|_{q,\infty} < \infty$. We set $E = \{x \in \mathbb{R}^n;\ |f(x)| > \lambda\} \cap B_r$. (Here B_r denotes an open ball centered at the origin with radius r.) Similarly to Proposition 6.2.2(i) we deduce $\lambda|E| \le \int_E |f(x)|dx$ for $\lambda, r > 0$. This implies $\lambda|E|^{1/q} \le \|f\|_{q,\infty}$ for $\lambda, r > 0$. Letting $r \to \infty$ we obtain $|E| \to m_f(\lambda)$ and therefore $|f|_{q,\infty} \le \|f\|_{q,\infty}$. By definition it is clear that $\|f\|_{g,\infty}$ is a norm, so the proof is left to the reader.

6.4 For $f(x) = 1/\sqrt{x}$ we calculate $\int_\varepsilon^1 |f(x)|^2 dx = \int_\varepsilon^1 dx/x = [\log x]_\varepsilon^1$, $\varepsilon > 0$. This yields $\int_0^1 |f(x)|^2 dx = \lim_{\varepsilon\to 0} \int_\varepsilon^1 |f(x)|^2 dx = \infty$. Therefore, $f \notin L^2(0,1)$. On the other hand, $m_f(\lambda) = |\{x \in (0,1) :\ |f(x)| > \lambda\}| = |(0,\ \min(\lambda^{-2},\ 1)| \le \lambda^{-2}$. So, $|f|_{2,\infty} \le 1 < \infty$, which shows that $f \in L^{2,\infty}(0,1)$.

6.5 The case $r = 1$ is obvious by Fubini's theorem (§7.2.2). Let $r > 1$. Without loss of generality we may assume that $f \ge 0$ (and $f \not\equiv 0$). Consider a sequence of integrable functions $\{f_j\}$ ($f_j \le f$) with $\int_\Omega |\int_U f_j(x,y)dx|^r dy < \infty$ and that converges almost everywhere monotonically to f on $\Omega \times U$. Thus, we may also assume that $\int_\Omega |\int_U f(x,y)dx|^r dy < \infty$. Fubini's theorem implies

$$\begin{aligned}\int_\Omega \left|\int_U f(x,y)dx\right|^r dy &= \int_\Omega \left\{\left|\int_U f(x,y)dx\right|^{r-1} \cdot \int_U f(z,y)dz\right\} dy \\ &= \int_U \left\{\int_\Omega \left|\int_U f(x,y)dx\right|^{r-1} f(z,y)dy\right\} dz.\end{aligned}$$

Applying the Hölder inequality to the y-integral, we obtain

$$\begin{aligned}&\int_\Omega \left|\int_U f(x,y)dx\right|^{r-1} f(z,y)dy \\ &\quad \le \left(\int_\Omega |\int_U f(x,y)dx|^{(r-1)r'} dy\right)^{1/r'} \left(\int_\Omega f(z,y)^r dy\right)^{1/r},\end{aligned}$$

where $1/r + 1/r' = 1$. Inserting this, we arrive at

$$\begin{aligned}&\int_\Omega \left|\int_U f(x,y)dx\right|^r dy \\ &\quad \le \left(\int_\Omega \left|\int_U f(x,y)dx\right|^r dy\right)^{1-1/r} \int_U \left(\int_\Omega f(z,y)^r dy\right)^{1/r} dz.\end{aligned}$$

Dividing both sides by $\left(\int_\Omega |\int_U f(x,y)dx|^r dy\right)^{1-1/r}$ and taking the rth power, we obtain the integral form of the Minkowski inequality.

6.6 We set $(Tf)(t) = \|e^{t\Delta} f\|_p$. By the L^p-L^q estimate (§1.1.2) and since $t^{-\alpha} \in L^{1/\alpha,\infty}(0,\infty)$ ($0 < \alpha \le 1$), we obtain $|Tf|_{L^{r_i,\infty}(0,\infty)} \le C\|f\|_{q_i}$, where $2/r_i = n(1/q_i - 1/p)$, $1 \le q_i$, $r_i \le \infty$, $i = 1, 2$. (Here we set

$L^{\infty,\infty}(0,\infty) = L^\infty(0,\infty)$.) For given $1 < q < r < \infty$ we choose q_1 and q_2 with $1 < q_1 < q < q_2 < \infty$ and $\infty > r_i > q_i$ $i = 1, 2$, which is always possible. The Marcinkiewicz interpolation theorem then implies $\|Tf\|_{L^r(0,\infty)} \leq C\|f\|_q$ $(f \in L^q(\mathbb{R}^n))$.

6.7 By the estimates in §1.1.2 we have

$$\left\| \int_{\mathbb{R}^n} G_t(x-y)|f(y)|dy \right\|_\infty \leq \|f\|_\infty,$$

$$\left\| \int_{\mathbb{R}^n} G_t(x-y)|f(y)|dy \right\|_\infty \leq \frac{\|f\|_1}{(4\pi t)^{n/2}}, \quad t > 0.$$

This implies

$$\int_0^1 t^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} G_t(x-y)|f(y)|dy \right) dt \leq \|f\|_\infty \int_0^1 t^{\frac{\alpha}{2}-1} dt,$$

$$\int_1^\infty t^{\frac{\alpha}{2}-1} \left(\int_{\mathbb{R}^n} G_t(x-y)|f(y)|dy \right) dt \leq \frac{\|f\|_1}{(4\pi)^{n/2}} \int_1^\infty t^{\frac{\alpha}{2}-\frac{n}{2}-1} dt.$$

Since $f \in C_0(\mathbb{R}^n)$ and $0 < \alpha < n$, the right-hand sides are finite. Hence, by Fubini's theorem (§7.2.2 (II)) we may interchange the order of integration.

6.8 Integration by parts yields

$$\begin{aligned} \int_{|y|\geq\varepsilon} \Delta_y f(x-y)E(y)dy = & \int_{|y|\geq\varepsilon} f(x-y)\Delta_y E(y)dy \\ & + \int_{|y|=\varepsilon} \frac{\partial f(x-y)}{\partial \nu_y} E(y)d\sigma_y \\ & - \int_{|y|=\varepsilon} f(x-y)\frac{\partial E}{\partial \nu_y}(y)d\sigma_y \end{aligned}$$

for $\varepsilon > 0$. Here Δ_y and $\partial/\partial\nu_y$ are Laplacian with respect to y and the outer normal derivative with respect to $|y| \geq \varepsilon$ respectively, whereas $d\sigma_y$ is the line element of the circle with radius ε. Since $\Delta E(y) = 0$, $y \neq 0$, the first term of the right hand side is zero. Moreover, we have

$$\left| \int_{|y|=\varepsilon} \frac{\partial f}{\partial \nu_y}(x-y)E(y)d\sigma_y \right| \leq \sup_{\mathbb{R}^2} |\nabla f| \int_0^{2\pi} |E(\varepsilon\eta)|\varepsilon d\theta \to 0$$

for $\varepsilon \to 0$, where $\eta = (\cos\theta, \sin\theta)$. Next, observe that

$$\frac{\partial E}{\partial \nu_y}(x) = \frac{1}{2\pi} \left(\frac{\partial}{\partial r} \log r \right) |_{r=|x|} = \frac{1}{2\pi|x|}.$$

By the fact that

$$\int_{|y|=\varepsilon} f(x-y)\frac{\partial E}{\partial \nu_y}(y)d\sigma_y = \int_0^{2\pi} f(x_1-\varepsilon\cos\theta_1,\ x_2-\varepsilon\sin\theta_1)\frac{\varepsilon}{2\pi\varepsilon}d\theta,$$

the continuity of f at x shows that

$$\int_{|y|=\varepsilon} f(x-y)\frac{\partial E}{\partial \nu_y}(y)d\sigma_y \to f(x) \quad (\varepsilon \to 0).$$

This yields

$$\begin{aligned}(E * \Delta f)(x) &= (\Delta f * E)(x)\\ &= \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} \Delta_y f(x-y)E(y)dy = -f(x).\end{aligned}$$

6.9 (i) This can be proved by a direct calculation of $\partial_x^\alpha E$. However, we will use the following scaling method, since it is more transparent. For $h\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ we define the scaled function h_λ by $h_\lambda(x) = h(\lambda x)$ for $\lambda > 0$. If there exists a d such that $h_\lambda(x) = \lambda^d h(x)$, $\lambda > 0$, $x \in \mathbb{R}^n$ $(x \neq 0)$, h is called *positively homogeneous* of degree d. If h is positively homogeneous of degree d, then $\partial_x^\alpha h$ is positively homogeneous of degree $d - |\alpha|$. In fact, $\lambda^d \partial_x^\alpha h = \partial_x^\alpha(h_\lambda) = \lambda^{|\alpha|}(\partial_x^\alpha h)_\lambda$.

In the case of $n \geq 3$, E is obviously positively homogeneous of degree $2-n$. In case of $n = 2$, since $E(\lambda x) = E(x) - \frac{1}{2\pi}\log\lambda$, $\partial_{x_j}E$ is positively homogeneous of degree -1. Hence, $\partial_x^\alpha E$ is positively homogeneous of degree $2-n-|\alpha|$. (If $n = 2$, we assume $|\alpha| \geq 1$.)

Now, consider $x \in \mathbb{R}^n$ lying on a sphere centered at the origin, i.e., $|x| = r$ for fixed $r > 0$. Since $\partial_x^\alpha E$ is positively homogeneous of degree $2-n-|\alpha|$, we obtain

$$\partial_x^\alpha E(x) = \partial_x^\alpha E\left(|x|\frac{x}{|x|}\right) = |x|^{2-n-|\alpha|}\partial_x^\alpha E\left(\frac{x}{|x|}\right).$$

The continuity of $\partial_x^\alpha E$ on the unit sphere $|x| = 1$ and the Weierstrass theorem imply

$$C = \sup_{\substack{x\in\mathbb{R}^n\\ x\neq 0}} |\,(\partial_x^\alpha E)\left(\frac{x}{|x|}\right)| < \infty.$$

Hence we obtain (i).

(ii) By (i) there exists a constant C' independent of x such that $|\partial_{x_j}E(x)| \leq C/|x|^{n-1}$. This implies

$$\begin{aligned}\int_{B_R} |\partial_{x_j}E(x)|dx &\leq C\int_{B_R}\frac{dx}{|x|^{n-1}}\\ &= C|S^{n-1}|\int_0^R r^{1-n+n-1}dr = C|S^{n-1}|R,\end{aligned}$$

which is finite. Thus, $\partial_{x_j}E$ is locally integrable on $\mathbb{R}^n$.

Chapter 7

7.1 (i) Assume that R is a real number such that $\operatorname{supp} f \subset B_R$. If $x \in B_R$, $f(x-y)$ is zero on $y \notin B_{2R}$. Thus, we have $(h * f)(x) = \int_{B_{2R}} f(x-y)h(y)dy$ for $x \in B_R$. The integrand is estimated as $|f(x-y)h(y)| \le \|f\|_\infty |h(y)|$. Since $h \in L^1(B_{2R})$, by the dominated convergence theorem (§7.1.1) we obtain $\lim_{z \to x}(h*f)(z) = (h*f)(x)$ for $x \in B_R$. Since R is arbitrary, $(h*f)(x)$ is continuous with respect to $x \in \mathbb{R}^n$.

(ii) Let R be as in (i). Since $h(x-y)$ is continuous as a function of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, it is uniformly continuous as a function on $\overline{B}_R \times \overline{B}_R$. Hence, $\lim_{z \to x} h(z-y)f(y) = h(x-y)f(y)$ uniformly for $y \in B_R$ if $x \in B_R$. Since $(h*f)(x) = \int_{B_R} h(x-y)f(y)dy$, Proposition 7.1 implies the continuity of $(h*f)(x)$ in $x \in B_R$. Hence, $h*f$ is continuous on $\mathbb{R}^n$.

7.2 (Sketch)
First we show that $W_t * f$ is continuous as a function of (x,t) on $\mathbb{R}^n \times (0,\infty)$, where $W_t = \partial_x^\alpha \partial_t^k G_t$ and where α is a multi-index and $k = 0, 1, \ldots$. We choose a suitable polynomial $P_{k,\alpha}$ such that W_t is expressed as

$$W_t(x) = t^{-k-|\alpha|/2-n/2} P_{k,\alpha}(x/t^{1/2}) \exp\{-|x|^2/(4t)\}.$$

By Exercise 1.2 there exists a constant $C = C(k,\alpha,n) > 0$ such that $|W_t(x)| \le C t^{-k-|\alpha|/2-n/2} \exp\{-|x^2|/(8t)\}$, $t > 0$, $x \in \mathbb{R}^n$. (Remark: using this estimate the result in §1.1.3 follows immediately from the Young inequality.) By estimating the right-hand side, for each $R > 0$ and $b > a > 0$, there exist constants A_0, $A_1 > 0$ such that

$$|W_t(x-y)| \le A_0 \exp(-A_1|y|^2) =: A(y), \quad x \in B_R, t \in (a,b), y \in \mathbb{R}^n.$$

Since $A \in L^{p'}(\mathbb{R}^n)$, Af is integrable on $\mathbb{R}^n$ by the Hölder inequality. The dominated convergence theorem (§7.1.1) then implies $W_t * f \in C(B_R \times (a,b))$, i.e., $W_t * f \in C(\mathbb{R}^n \times (0,\infty))$.
Next we show that $W_t * f$ is C^1 with respect to $t > 0$ for each $x \in B_R$, and that $\partial_t(W_t * f) = (\partial_t W) * f$. To this end we set $h(t,y) = W_t(x-y)f(y)$ and apply Theorem 7.2.1. By the results above, we obtain $|W_t(x-y)|$, $|\partial_t W_t(x-y)| \le A(y)$, $t \in (a,b)$, $y \in \mathbb{R}^n$, for suitable A_0 and A_1. Using these estimates, (ii) and (iii) in §7.2.1 can be proved. The results above also show that (iv) is valid. Since (i) is obvious, $W_t * f$ is C^1 with respect to t and we have $\partial_t(W_t * f) = (\partial_t W) * f$. By very similar arguments it follows that $W_t * f$ is C^1 with respect to x_j as well and that $\partial_{x_j}(W_t * f) = (\partial_{x_j} W_t) * f$. Thus, we obtain $G_t * f \in C^\infty(\mathbb{R}^n \times (0,\infty))$ and $\partial_x^\alpha \partial_t^k (G_t * f) = (\partial_x^\alpha \partial_t^k G_t) * f$.
The fact $\partial_t G_t = \Delta G_t$ $(t > 0)$ is proved in Exercise 1.1. This yields $\partial_t(G_t * f) = (\partial_t G) * f = (\Delta G_t) * f = \Delta(G_t * f)$, and therefore u satisfies the heat equation (1.1) for $t > 0$.

7.3 By Exercise 7.2 we have $G_t * f \in C^\infty(\mathbb{R}^n)$; hence the fact that $C^\infty(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ follows from ($*$). Similarly as in the first step of the proof in §4.4.2 we choose $\theta_j \in C_0^\infty(\mathbb{R}^n)$, $j = 1, 2, \dots$, satisfying $\theta_j(x) = 1$ $(|x| \le j)$, $0 \le \theta_j \le 1$, $\theta_j(x) = 0$ $(|x| \ge 2j)$. Since $f_j = \theta_j f \in C_0^\infty(\mathbb{R}^n)$ for $f \in C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and

$$\|f_j - f\|_p^p \le \int_{\mathbb{R}^n\setminus B_j} |f|^p dx \to 0 \quad (j \to \infty),$$

we see that $C_0^\infty(\mathbb{R}^n)$ is dense in $C^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ with respect to the L^p-norm. Therefore $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.
We extend $f \in L^p(\Omega)$ by zero outside Ω. Then $f \in L^p(\mathbb{R}^n)$. Next we use the density of $C_0^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ in order to construct a sequence $\{h_j\}_{j=1}^\infty \subset C_0^\infty(\Omega(R))$ such that $\int_{\Omega(R)} |h - h_j|^p dx \to 0$ as $j \to \infty$, for $h \in C_0^\infty(\mathbb{R}^n)$. Here we set $\Omega(R) = \Omega \cap B_R$ and choose R such that $\operatorname{supp} h \subset B_R$. To do so, we determine $\varphi_j \in C_0^\infty(\Omega(R))$ with $\lim_{j\to\infty} \varphi_j(x) = 1$ and $0 \le \varphi_j(x) \le 1$ for $x \in \Omega(R)$ and set $h_j = \varphi_j h$. The dominated convergence theorem then yields the desired property of the h_j. Let us show that such φ_j exist. Similarly as in the first step of the proof in §4.4.2, we pick $\theta \in C_0^\infty[0, \infty)$ satisfying $\theta(\tau) = 0$ for $\tau \le 1/2$, $\theta(\tau) = 1$ for $\tau \ge 1$, and $0 \le \theta \le 1$. Next, we define ρ_0 by $\rho_0(x) = \operatorname{dist}(x, \partial(\Omega(R)))$ for $x \in \Omega(R)$ and $\rho_0(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega(R)$. Note that ρ_0 is bounded and uniformly continuous on $\mathbb{R}^n$, however not smooth in general. Fortunately, by §4.2.1 we have $\|G_t * \rho_0 - \rho_0\|_\infty \to 0$ as $t \to 0$. Hence, for each j we may choose a $t > 0$ such that $\|\rho_j - \rho_0\|_\infty \le 1/(4j)$, where $\rho_j = G_t * \rho_0$. Using this ρ_j we define $\varphi_j(x) = \theta(j\rho_j(x))$. Then, $\varphi_j \in C_0^\infty(\Omega(R))$ and we may easily check that $\lim_{j\to\infty} \varphi_j(x) = 1$ for $x \in \Omega(R)$ and $0 \le \varphi_j \le 1$.

7.4 Let f_R be defined as in the proof of (II) of Proposition 4.1.4. Then we have $f_R \in L^1(\mathbb{R}^n)$. Assume that $R > R_0 > 0$. Since h is bounded on B_{R+R_0}, the dominated convergence theorem (§7.1.1) implies that $(h * f_R)(x)$ is continuous with respect to $x \in B_{R_0}$. Hence $h * f_R$ is continuous on $\mathbb{R}^n$. Similarly to the proof of (II) of Proposition 4.1.4, we will show that $h * f_R$ converges uniformly to $h * f$. To this end, we estimate $h * f_R - h * f$ in a slightly different way. By the Hölder inequality we have

$$\sup_{x\in\overline{B}_{R_0}} |(h * (f_R - f))(x)| \le \|f\|_p \sup_{x\in\overline{B}_{R_0}} \left(\int_{R^n\setminus B_R} |h(x-y)|^{p'} dy\right)^{1/p'}$$

$$\le \|f\|_p \left(\int_{R^n\setminus \overline{B}_{R-R_0}} |h(y)|^{p'} dy\right)^{1/p'} .$$

In view of $\|h\|_{p'} < \infty$, the latter term converges to zero as $R \to \infty$. As the uniform limit of continuous functions, $h * f$ is continuous on B_{R_0}. Since R_0 is an arbitrary, $h * f$ is continuous on $\mathbb{R}^n$. The boundedness of

$h * f$ is a consequence of the Young inequality (§4.1.1). (It can also be obtained directly using the Hölder inequality.)

The second statement is proved by similar arguments to those in the proof of (II)(ii) of Proposition 4.1.4.

7.5 Suppose there exist $f_j \in C^\infty[0,1]$ and $f \in C^\mu[0,1]$ such that $\|f - f_j\|_{C^\mu} \to 0$ as $j \to \infty$. For a function u defined on $[0,1]$ we set $A(\tau, u) = \sup\{|u(x) - u(0)|/|x|^\mu : 0 < x \le \tau\}$, $\tau \in (0,1]$. We simply write $A(\tau, f_j)$ for $A_j(\tau)$ and $A(\tau, f)$ for $A(\tau)$. Then we obtain

$$|A_j(\tau) - A(\tau)| \le \sup\left\{\frac{|(f_j(x) - f(x)) - (f_j(0) - f(0))|}{|x|^\mu} : 0 < x \le \tau\right\}.$$

By the definition of $[f_j - f]_\mu$ (see Example 1 §5.1.2 with $M = [0,1]$), we therefore have

$$\sup_{0<\tau\le 1} |A_j(\tau) - A(\tau)| \le [f_j - f]_\mu.$$

The assumption that $\|f - f_j\|_{C^\mu} \to 0$ as $j \to \infty$ implies $[f_j - f]_\mu \to 0$; hence A_j converges uniformly to A on $(0,1]$. If $f_j \in C^\infty[0,1]$, then A_j is continuous in $(0,1]$, and by the mean value theorem (§1.1.6) we obtain $|A_j(\tau)| \le \|f_j'\|_\infty \tau^{1-\mu}$ for $\tau > 0$. In particular, A_j extends continuously to $\tau = 0$ and we have $A_j(0) = 0$. Since A is the uniform limit of A_j on $(0,1]$, we have $\lim_{\tau\to 0} A(\tau) = 0$. However, for $f(x) = x^\mu$ we have $f \in C^\mu[0,1]$ but $A(\tau) \equiv 1$. Thus, a sequence $\{f_j\}_{j=1}^\infty \subset C^\infty[0,1]$ satisfying $\|f_j - f\|_{C^\mu} \to 0$ cannot exist. In other words, x^μ does not belong to the closure of $C^\infty[0,1]$ with respect to the C^μ-norm.

In fact, it is known that the closure $h^\mu[0,1]$ of $C^\infty[0,1]$ with respect to the C^μ-norm is

$$h^\mu[0,1] = \left\{f \in C^\mu[0,1]; \quad \lim_{y\to x} \frac{|f(x) - f(y)|}{|x-y|^\mu} = 0, \quad x \in [0,1]\right\}.$$

Next, let $\tilde{f} \in C_0(\mathbb{R})$ be an extension of $f \in C[0,1]$ on $\mathbb{R}$. By §4.2.1 we have $\|G_t * \tilde{f} - \tilde{f}\|_\infty \to 0$ as $t \to 0$. Clearly, $G_t * \tilde{f} \in C^\infty(\mathbb{R})$ (§4.1.6). For $f_t(x) = (G_t * \tilde{f})(x)$, $x \in [0,1]$, we therefore obtain $f_t \in C^\infty[0,1]$ and $\sup_{0\le x\le 1} |f_t(x) - f(x)| \to 0$ as $t \to 0$. This shows that $C^\infty[0,1]$ is dense in $C[0,1]$.

Comments on Further References

In [Barenblatt 1979, Barenblatt 1996] a formal aspect of asymptotic analysis is described in detail to study the behavior of solutions using self-similar solutions. As discussed in §3.2.6, to give a precise description of the solution near blowup time, analyzing the asymptotic behavior near blowup time is not sufficient. We need to match approximate solutions apart from blowup points. The method of matched asymptotic expansions first divides the domain into several regions and then constructs an approximate solution in each divided region. The important step is that one has to match approximate solutions on the boundary of divided regions so that there is no jump. This method was originally developed in fluid mechanics. As in [Barenblatt 1979, Barenblatt 1996], this method is often used formally. Only recently has it been applied rigorously to many problems. For blow-up problems the reader is referred to [Herrero Velázquez 1993]. Very recently, we were informed of two nice survey papers [Eggers Fontelos 2009], [Bernoff Witelski 2009] on self-similarity closely related to the present book. In [Eggers Fontelos 2009] the authors emphasized how useful similarity variables are in analysis on singularities of solutions of partial differential equations. In [Bernoff Witelski 2009] the authors discussed a methodology for identifying self-similar solutions and determining their stability. The authors are grateful to Professor Robert V. Kohn for informing them of these two papers.

Asymptotic analysis is an important method for analyzing the behavior of solutions of partial differential equations of any type. For example, it is very important in the analysis of linear equations [Fujiwara 1976 1977]. It also includes a singular perturbation method in the analysis of the profile of solutions of reaction-diffusion equations. This method is considered one of asymptotic analysis. For this method, see [Nishiura 1999], where several matched asymptotic expansions are discussed.

There are many elementary textbooks on partial differential equations (PDE) with various goals. We just point out a few books that are relatively easy to read but still contain a great deal on nonlinear partial differential equations. The book of M. Taylor [Taylor 1996] is a self-contained book covering a wide range of topics on PDEs. It also includes the De Giorgi–Nash–Moser theory for elliptic equations (related to Chapter 2 of our book). This is a very important reference for the applicability of basic theory as well as its significance as a fundamental tool for the study of PDEs. The book of L.C. Evans [Evans 1998] is shorter but the contents are very rich. Different from [Taylor 1996], this book focuses on typical important examples for applications instead of developing a general theory. It provides numerous typical methods to analyze PDEs with emphasis on nonlinear problems. There are several elementary books whose major goal is to study nonlinear PDEs, for example [Logan 1994], [Roubicek 2005]. However, the goals of these books are quite different from ours.

There are elementary books on the analysis on partial differential equations also published in Japanese, for example, [Ikawa 1996, Murata Kurata 1997, Ikawa 1997, Matano Jimbo 1997, Kaneko 1998]. But there are only a few descriptions of analysis of nonlinear equations. There is no overlap between these and the present book except for elementary facts, for example the expression of the solution of the heat equation (§4.1.6, Exercise 7.2), and the expression of the inverse operator of the Laplacian by E (§6.3.5). Among them, the book [Murata Kurata 1997] is most closely related to this book. In fact, there is a deep relation between [Murata Kurata 1997, Corollary 3.68 §3.3 (e)] and the fundamental decay estimates in §2.3.1 in this book. (But results in §2.3.1 are not directly derived from Corollary 3.68.) The elementary proof of the Sobolev inequality (§6.3.4) is also given in [Murata Kurata 1997, Theorem 3.23 §3.2].

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Glossary

Sets and General Topology

$x \in A$	x is an element of A
$A \subset B$	A is a subset of B (possibly, $A = B$)
$A \times B$	Cartesian product of A and B (§1.1)
$B \backslash A$	complementary set of A in B (§1.3.1)
$\overline{A}$	closure of A
$\operatorname{int} A$	interior of A
∂A	boundary of A
$P \Rightarrow Q$	If the claim P is valid, then the claim Q is valid
$P \Leftrightarrow Q$	The claim P and the claim Q are equivalent

Real Numbers

$\mathbb{R}$	the field of real numbers, real line
$a \leq b;\ a \geq b$	$a \leqq b$ ($a = b$ or $a < b$); $a \geqq b$ ($a = b$ or $a > b$)
$\|a\|; a_+\ (a \in \mathbb{R})$	absolute value of a; positive part of a (§3.1.1)
$\sup; \max$	supremum; maximum (§1.1.1)
$\inf;\ \min$	infimum; minimum (§1.1.1)
$\lim$	limit
$a_j \to a$ as $j \to \infty$	$\lim_{j\to\infty} a_j = a$
$\overline{\lim}$	limit superior (§3.2.4, §4.2.3)
$\underline{\lim}$	limit inferior (§1.4.5, §3.2.4, §7.1.2)
$[a, \infty); (a, \infty); (a, b)$	$\{t \in \mathbb{R} :\ t \geq a\}$; $\{t \in \mathbb{R} :\ t > a\}$; $\{t \in \mathbb{R} : a < t < b\}$

$0 \le a < \infty$; $-\infty < a < \infty$	a is finite and nonnegative; a is a real number
$a =: b$, $b := a$	define b by a; set the value of b by a
$\sum$; $\prod$	summation symbol; product symbol (§6.3.4)

Functions

e^z, $\exp z$	exponential function
$\log t$	logarithmic function
$\cos\theta$; $\sin\theta$	cosine function; sine function
G_t	the Gauss kernel (§1.1)
$\Gamma(p)$; $\mathrm{B}(p,q)$	gamma function (§4.4.4, §6.2.5); beta function (§4.4.4)

Euclidean Spaces

$\mathbb{R}^n = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n}$	n-dimensional Euclidean space
$\langle\ a,\ b\ \rangle$, $a, b \in \mathbb{R}^n$	standard inner product of a and b in $\mathbb{R}^n$, i.e., $\langle\ a,\ b\ \rangle := \sum_{i=1}^n a_i b_i$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$
$\lvert x\rvert,\ x \in \mathbb{R}^n$	Euclidean norm (length) of x, i.e., $\lvert x\rvert = \langle\ x, x\ \rangle^{1/2}$
B_R	an open ball with radius R centered at the origin in $\mathbb{R}^n$
S^{n-1}	$(n-1)$-dimensional unit sphere ($S^{n-1} = \partial B_1$)
$\lvert S^{n-1}\rvert$	area of $(n-1)$-dimensional unit sphere (§6.3.1)
a.a.	almost all (§6.4.1); almost everywhere
dist (x, A)	distance between a point x and a set A (§6.4.4)

Operators

$\partial_t = \frac{\partial}{\partial t}$	partial differential operator in the direction of t
$\partial_{x_j} = \frac{\partial}{\partial x_j}$	partial differential operator in the direction of x_j
∂_x^α	partial differential operator with respect to x of order $\lvert\alpha\rvert$ (§1.1.3)
∇	gradient (§1.1.5)

div	divergence (§1.2.2)
Δ	Laplacian (§1.1)
curl	rotation (§2.1.1)
$\nabla^{\perp}$	(§2.1.1)
(u, ∇) $(= \sum_{i=1}^{n} u^i \partial_{x_i})$ $u = (u^1, \dots, u^n)$	differentiation in the direction of u (§2.1)
$\operatorname{supp} f$	support of f (§1.3.3)
$f * g$	convolution of f and g (§2.1.3, §4.1)
$\int_Q f(x)dx$	integral of f over Q
$\int_{\partial B_R} f d\sigma$, $\int_{\lvert x\rvert=R} f(x)d\sigma$	surface integral of f over the sphere of radius R
$e^{t\Delta} f$	$G_t * f$ (§2.4.2, §4.3)
u^{λ}, u_k, ω_k, $u_{(\lambda)}, u^{(\lambda)}$	scaling transformation of u, ω
$f \equiv 0$	the function f equals zero identically

Function Spaces

$C(Y)$	the space of all real-valued continuous functions on Y (§1.3.1)
$C_0(Y)$	$\{f \in C(Y); \operatorname{supp} f \text{ is compact } \}$ (§1.4)
$C_\infty(M)$	the space of all $f \in C(M)$ that converge to zero at infinity (§1.3.1)
$C^\infty(Y)$	the set of all smooth functions on Y (§1.4)
$C_0^\infty(Y)$	$C_0(Y) \cap C^\infty(Y)$ (§1.4)
$C^r(Q)$ (Q is an open set in $\mathbb{R}^n$)	the set of all h whose derivative $\partial_x^\alpha h$ is continuous on Q for $\lvert\alpha\rvert \le r$ (r is a natural number) (§1.4)
$h \in C^r(Q)$	h is C^r on Q
$h \in C^\infty(Q)$	h is C^∞ on Q
$L^p(\Omega)$ ($\Omega \subset \mathbb{R}^n$)	the space of all measurable functions with pth integrable power on Ω (§4.1.1)
$L^{q,\infty}(\mathbb{R}^n)$	Lorentz space on $\mathbb{R}^n$ (§6.2.3)
$m_f(\lambda)$	distribution function of f (§6.2.1)
$\lVert h\rVert_p$	L^p-norm of h (§1.1.1)

Index