Theory of Causal Differential Equations

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Theory of Causal Differential Equations

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Preface

The problems of modern society are both complex and inter-disciplinary. Despite the apparent diversity of problems, however, often tools developed in one context are adaptable to an entirely different situation. For example, consider the well known Lyapunov's second method. This interesting and fruitful technique has gained increasing significance and has given decisive impetus for modern development of stability theory of discrete and dynamic system. It is now recognized that the concept of Lyapunov function and theory of differential inequalities can be utilized to investigate qualitative and quantitative properties of a variety of nonlinear problems. Lyapunov function serves as a vehicle to transform a given complicated system into a simpler comparison system. Therefore, it is enough to study the properties of the simpler system to analyze the properties of the complicated system via an appropriate Lyapunov function and the comparison principle.

It is in this perspective, the present monograph is dedicated to the investigation of the theory of causal differential equations or differential equations with causal operators, which are nonanticipative or abstract Volterra operators. As we shall see in the first chapter, causal differential equations include a variety of dynamic systems and consequently, the theory developed for CDEs (Causal Differential Equations) in general, covers the theory of several dynamic systems in a single framework. Also, many of the same tools which are employed for ODEs (Ordinary Differential Equations) are applicable to CDEs including the method of Lyapunov Functions, for not only the development of stability theory but also other qualitative properties of solutions of CDEs.

It is Volterra who used causal operators implicitly in his work on integral equations and Tonelli gave a sharp definition. It is Tychonoff who made a significant contribution in developing the theory of functional equations involving causal operators. At the end of the last century, the functional equations of a variety of types such as delay differential equations, functional differential equations and integrodifferential equations, etc., were developed and studied and all these are special cases of causal differential equations .

The first book dealing with functional equations with causal operators is of Corduneanu, which offers some basic theory and paves the way for future development. Recently, a lot of important research has been done in this area and there exists sufficient amount of literature to warrant assembling the existing results in a unified way. It is now widely recognized that this important branch of nonlinear analysis merits further study in order to explore and appreciate the intricacies and advantages involved in the investigation of such dynamic systems.

It is with this spirit we see the importance of the present monograph. As a result, we provide a systematic account of recent developments, describe the present state of the useful theory, show the essential unity achieved and initiate several new extensions to other types of CDEs such as CDEs in infinite dimensional spaces with or without memory, fractional CDEs and set differential equations with causal operators. We hope that this book would motivate scientists to investigate CDEs and their applicability to real world problems and becomes instrumental for further advancement of this interesting area of nonlinear analysis. One can extend CDEs to cover the case of dynamic systems on time scales, impulsive systems and hybrid systems to name a few.

In Chapter 1, we collect preliminary material providing necessary tools, some relevant basic concepts and useful results. Defining causal operators, we give examples of several dynamic systems that are included under the banner of causal systems. Also, we list some useful fixed point theorems, define measures of noncompactness and nonconvexity and indicate necessary preliminaries of a Banach space.

Chapter 2 is devoted to the investigation of fundamental theory of causal differential equations (CDEs) such as comparison principles, existence, uniqueness and continuous dependence of solutions on initial data, existence of extremal solutions, global existence and Nagumo type uniqueness results. Existence of Euler solutions via nonsmooth analysis is also discussed. We present some basic theory for differential and integral equations of Sobolev type and finally inequalities results for causal differential systems.

Chapter 3 contains the method of lower and upper solutions including monotone iterative technique, quasilinearization and their generalizations. Chapter 3, therefore, provides a rich source of methods of finding approximating sequences which are monotone and converge to the extremal solutions/unique solution of the causal differential system.

Chapter 4 introduces the stability theory via Lyapunov method by employing Lyapunov functions, Lyapunov functionals and functions on product spaces. Stability theory in terms

of two measures is indicated and is useful for covering many stability concepts in a single frame work. The method of vector Lyapunov functions and cone valued Lyapunov functions is described to provide weaker sufficient conditions for stability theory.
Finally Chapter 5 deals with several new topics that are initiated in the context of CDEs. They are

(i) CDEs in a Banach Space,
(ii) CDEs with fractional derivatives,
(iii) CDEs with memory,
(iv) causal set differential equations,
(v) CDEs with retardation and anticipation.

Few basic results are proved in each case, enough to provide initial apparatus for further study.
Some important features of the monograph are as follows:

(1) It is the first book that attempts to describe the theory of CDEs as an independent discipline;
(2) It incorporates the recent general theory of CDEs showing the interconnections between various dynamic systems and CDEs;
(3) It introduces several new areas of study by providing the initial apparatus for further advancement;
(4) It is a timely introduction to a subject that is broad enough to cover a variety of dynamic systems in a single setup that follows the present trend of studying analysis and dynamic systems in a general framework.

The monograph will be very useful to those scientists and doctoral students who work in nonlinear analysis in general. It is a good reference book to researchers in several disciplines where real world problems are considered and to graduate students in those disciplines. In fact, the concept of causal operators is prevalent in the engineering literature and thus engineering researchers may find the monograph useful.

Contents

Chapter 1

Preliminaries

1.1 Introduction

This chapter is essentially introductory in nature. Its main purpose is to introduce some relevant basic concepts, present some known results from standard books and sketch some useful results that are not so well known. Section 1.2 deals with the notion of causal operators and provides several possible examples that may be treated as causal operators so that one can formulate causal functional equations and causal differential equations.

In Section 1.3, we assemble some known basic results including comparison results. The notion of Dini derivatives and their properties are given. In Section 1.4, we list some well known fixed point results as well as one not so well known contraction result where domain and range of the operator are different but connected.

Necessary preliminaries of a Banach space are described in Section 1.5, while Section 1.6 is devoted to the notion of directional derivatives and their properties. In Section 1.7, we define the concept of the measure of noncompactness and provide some of its useful properties. Section 1.8 introduces the measure of nonconvexity. Finally Section 1.9 is devoted to notes and comments.

1.2 Causal Operators

This monograph is primarily a brief account of the investigation of equations with causal or nonanticipatory or Volterra operators. We shall use the word causal to denote such operators. The causal operators can be described by several functions or functionals that occur in the formulation of many dynamic systems as well as functional systems including discrete systems. Therefore, the study of the theory of causal systems becomes very important. This is because a single result involving causal operators covers interesting corresponding

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results from many categories of dynamic systems, thus avoiding duplication and monotony of repetition. Moreover, in this general set up, one can visualize how far we can go and where we get stumbling blocks, if at all. The investigation of the theory of causal systems, dynamic or otherwise, is an important branch of nonlinear analysis.

Let us first define the concept of the causal operator. Let $E = E(J,X)$ where J is an appropriate time interval, X represents either finite or infinite dimensional space, depending on the requirement of the context, so that E is a function space. The operator $Q : E \to E$ is said to be a causal or nonanticipatory operator if the following property is satisfied:

(*) for each couple of elements x,y of E such that $x(s) = y(s)$ for $0 \leq t_0 \leq s \leq t$, we also have $(Qx)(s) = (Qy)(s)$ for $0 \leq t_0 \leq s \leq t, t < T$, T being arbitrary. Of course, the definition needs a slight modification when E is a space of measurable functions on $[t_0,T)$, $t_0 \geq 0$, requiring property (*) to be valid almost everywhere on $[t_0,T]$. We wish to point out that for causal operators, a notation identical with what is encountered for a general equation with memory can be stated as follows. A representation of the form

$$(Qx)(t) = Q(t,x_t)$$

where for each $t \in [t_0,T)$, $Q(t,x_t)$ is a functional on E which takes values in X, for each t, while the whole family of functionals, $t \in [t_0,T)$, define the operator from $E = C([t_0,T),X)$ to itself. Sometimes, this description is more clear in defining the operator Q. Many operators on function spaces have been defined and investigated in connection with their use in the theory of functional or functional differential equations.

The sum and product of two causal operators $Q : E \to E$ and $P : E \to E$ are causal. Another property of causal operators is related to the convergence of a sequence of such operators. For illustration, let us take $E = C([t_0,T),\mathbb{R}^n)$ as the underlying space. Let $\{Q_n\}$ be a sequence of causal operators on E such that

$$\lim_{n\to\infty} (Q_n x)(t) = (Qx)(t), \tag{1.1}$$

for each $(t,x) \in [t_0,T) \times E$. The question is whether we can infer that the limit $Q : E \to E$ is also a causal operator. The answer is yes because the causality of $\{Q_n\}$ implies that

$$(Q_n x)(s) = (Q_n y)(s), \quad s \in [t_0,T).$$

If we let $n \to \infty$ on both sides, in the above relation and use (1.1) for each fixed $s \in [t_0,T)$, we obtain the causality of Q.

The problem of invertibility of causal operators is not true in general. If $E = C([0,\infty),\mathbb{R})$ and Q, the operator defined by

$$(Qx)(t) = x(t/2), \quad t \in [0,\infty),$$

is continuous and has inverse in E, namely,

$$(Q^{-1}y)(t) = y(2t), \quad t \in [0, \infty). \tag{1.2}$$

Also, this operator is continuous, i.e., if $y_n \to y$ uniformly on any interval $[0, T]$, then $(Q^{-1}y_n) \to (Q^{-1}y)$ on any bounded interval of $\mathbb{R}_+$. But, as seen from (1.2) Q^{-1} is not a causal operator. Hence, we cannot take it for granted the causality of the inverse of a causal operator.

However, with reference to the classical examples of causal operators, there are many circumstances where the inverse exists and is a causal operator. An example which plays an important role in the study of linear integral equations (of Volterra) of the second kind

$$x(t) = \int_{t_0}^{t} k(t,s)x(s)ds + f(t) \tag{1.3}$$

is provided by the operator

$$(Lx)(t) = x(t) - \int_{t_0}^{t} k(t,s)x()sds$$

so that (1.3) can be written as

$$(Lx)(t) = f(t), \quad t \in [t_0, T],$$

choosing $E = C([t_0, T], \mathbb{R}^n)$ and assuming that the kernel $k(t,s)$ is a continuous $n \times n$ matrix valued function on $t_0 \le s \le t \le T$. This operator L is invertible on the space E and expressing $x = (L^{-1}f)$, we get the solution of (1.3) in terms of the resolvent formula given by

$$(L^{-1}f)(t) = x(t) = f(t) + \int_{t_0}^{t} R(t,s)f(s)ds,$$

where $R(t,s)$ is the resolvent kernel associated with $k(t,s)$ obtained by the method of successive approximations. Clearly L^{-1} is causal.

Next, we shall provide examples of dynamic equations that can be included in causal differential equations of the form

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0. \tag{1.4}$$

Clearly, the IVP

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{1.5}$$

can be considered as a causal differential equation by identifying $f(t, x(t))$ with $(Qx)(t)$.

The next example is difference-differential IVP

$$\left.\begin{aligned} &x'(t) = f(t, x(t), x(t-\tau)), \quad \tau > 0, \\ &x_{t_0} = \phi_0(s), \quad t_0 - \tau \le s \le t_0, \end{aligned}\right\} \tag{1.6}$$

where ϕ_0 is the initial function at $t = t_0$. Clearly (1.6) is a causal differential equation since, we can make the identification

$$f(t,x(t),x(t-\tau)) \equiv (Qx)(t)$$

and because of the finite memory that is involved, we need the information on $x(t)$ for $[t_0-\tau,t_0]$ given as the initial function $\phi_0(s)$.

More general than (1.6) is known as the functional differential equation, given by

$$x'(t) = f(t,x_t), \quad x_{t_0} = \phi_0(s), \tag{1.7}$$

where $x_t = x_t(s) = x(t+s)$, $-\tau \le s \le 0$. (1.7) is also known as delay-differential equation and is clearly a causal differential equation with finite memory.

Consider the next IVP given by an integro-differential equation

$$x'(t) = \int_{t_0}^{t} k(t,s,x(s))ds, \quad x(t_0) = x_0. \tag{1.8}$$

One can incorporate finite memory and write (1.8) as

$$\left.\begin{array}{l} x'(t) = \int_{t-\tau}^{t} k(t,s,x(s))ds, \quad \tau > 0, \\ x(t_0+s) = x_{t_0} = \phi_0(s), \quad -\tau \le s \le 0,\ t_0 \ge 0. \end{array}\right\} \tag{1.9}$$

A more general problem than (1.8) is given by

$$\left.\begin{array}{l} x'(t) = f(t,x(t)) + \int_{t_0}^{t} k(t,s,x(s))ds, \\ x(t_0) = x_0, \end{array}\right\} \tag{1.10}$$

and if we incorporate delay, we get

$$\left.\begin{array}{l} x'(t) = f(t,x(t)) + \int_{t-\tau}^{t} k(t,s,x(s))ds \\ x(t_0+s) = x_{t_0} = \phi_0(s), \quad -\tau \le s \le 0,\ t_0 \ge 0 \end{array}\right\} \tag{1.11}$$

or in general

$$\left.\begin{array}{l} x'(t) = f(t,x_t) + \int_{\tau-z}^{t} k(t,s,x(s))ds \\ x(t_0+s) = x_{t_0} = \phi_0(s), \quad -\tau \le s \le 0,\ t_0 \ge 0. \end{array}\right\} \tag{1.12}$$

The very general integro-differential equation can be written as

$$\left.\begin{array}{l} x'(t) = f(t,x(t), \int_{t_0}^{t} k(t,s,x(s))ds \\ x(t_0) = x_0 \end{array}\right\} \tag{1.13}$$

which can be extended with memory as

$$\left.\begin{array}{l} x'(t) = f(t,x(t), \int_{t-\tau}^{t} k(t,s,x(s))ds \\ x_{t_0} = \phi_0(s), \end{array}\right\} \tag{1.14}$$

or more generally,

$$\left.\begin{array}{l} x'(t) = f(t,x(t),x_t,\int_{t-\tau}^{t} k(t,s,x(s))ds \\ x_{t_0} = \phi_0(s). \end{array}\right\} \tag{1.15}$$

All equations (1.8)–(1.15) are examples of causal differential equations. In all of these equations where x_t occurs, x_t represents the graph of x in $[t-\tau,t]$ shifted to the interval $[-\tau,0]$. We could have delay or memory of Volterra type, namely, x_t is the graph of x on $[t_0-\tau,t]$, in which case, an IVP could be of the form, for example,

$$x'(t) = f(t,x_t) + \int_{t_0-\tau}^{t} k(t,t,x(s))ds, \quad x_{t_0} = \phi_0(s).$$

This remark applies to (1.9), (1.11), (1.13) etc.

Other possibilities also exist. For example, the IVP

$$x'(t) = f\left(t, \max_{t_0\leq s\leq t} x(s)\right), \quad x(t_0) = x_0$$

or

$$x'(t) = f(t,x[t]), \quad x(t_0) = x_0$$

where the notation $x([t])$ represents the maximum value of $x(t)$ in each interval $[t_n,t_{n+1}]$, $n = 1,2,\ldots$. Another possibility is difference equation

$$x_{n+1} - x_n = \sum_{i=1}^{n} k(i+1,i,x_i), \quad x_{n_0} = x_0.$$

Thus, we see that the IVP for causal differential equation or causal equation may include a variety of problems, that we normally study independently in different branches of nonlinear analysis. However, all of these can be incorporated as special cases of causal differential or causal functional equations in finite or infinite dimensional spaces.

The equations with causal operators also include integral equations of the type

$$x(t) = h(t) + \int_{t_0}^{t} k(t,s,x(s))ds.$$

The functional differential equation can also contain advance argument as well as delay, such as

$$x'(t) = f(t,x(t),x(t-\tau_1),x(t+\tau_2)), \quad \tau_1,\tau_2 > 0.$$

A typical example with infinite delay is

$$x'(t) = \int_{-\infty}^{t} k(t,s,x(s))ds$$

and in general, functional differential equations with unbounded delay are also examples of causal differential equations.

1.3 Known Basic Results

We list, in this section, certain basic comparison results, that concern with estimating a function which satisfies a differential inequality by the extremal solutions of the corresponding differential equation. Sometimes, it is enough to have the differential inequality satisfied relative to only Dini derivatives. We adopt the following notation for Dini derivatives:

$$D^{+}u(t) = \limsup_{h \to 0^{+}} \frac{1}{h}[u(t+h) - u(t)];$$

$$D_{+}u(t) = \liminf_{h \to 0^{+}} \frac{1}{h}[u(t+h) - u(t)];$$

$$D^{-}u(t) = \limsup_{h \to 0^{-}} \frac{1}{h}[u(t+h) - u(t)];$$

$$D_{-}u(t) = \liminf_{h \to 0^{-}} \frac{1}{h}[u(t+h) - u(t)],$$

where $u \in C((t_0, t_0 + a), \mathbb{R})$. When $D^{+}u(t) = D_{+}u(t)$, the right derivative is denoted by $u'_{+}(t)$. Similarly, $u'_{-}(t)$ denotes the left derivative when $D^{-}u(t) = D_{-}u(t)$.

The following results are useful in the sequel. Let us begin with a simple result.

Lemma 1.3.1. Suppose $m(t)$ is continuous on (a,b). Then $m(t)$ is nondecreasing (nonincreasing) on (a,b) if and only if $D^{+}m(t) \geq 0 (\leq 0)$ for every $t \in (a,b)$, where

$$D^{+}m(t) = \limsup_{\delta \to 0^{+}} \frac{1}{\delta}[m(t+\delta) - m(t)].$$

Proof. The condition is obviously necessary. Let us prove that it is sufficient. Assume first that $D^{+}m(t) > 0$ on (a,b). If there exists two points $\alpha, \beta \in (a,b)$, $\alpha < \beta$, such that $m(\alpha) > m(\beta)$, then there exists a μ with $m(\alpha) > \mu > m(\beta)$ and some points $t \in [\alpha, \beta]$ such that $m(t) > \mu$. Let $\zeta = \sup\{t; m(t) > \mu, t \in [\alpha, \beta]\}$. Clearly, $\zeta \in (\alpha, \beta)$ and $m(\zeta) = \mu$. Therefore, for every $t \in (\zeta, \beta)$, we have

$$\frac{m(t) - m(\zeta)}{t - \zeta} < 0$$

which implies $D^{+}m(\zeta) \leq 0$. This is a contradiction and therefore the proof is complete.

Lemma 1.3.2. Let $v, w \in C([t_0, T], \mathbb{R})$ and for some fixed Dini derivative, $Dv(t) \leq w(t)$, $t \in [t_0, T]$. Then $D_{-}v(t) \leq w(t)$, $t \in [t_0, T]$.

Proof. Define the function

$$m(t) = v(t) - \int_{t_0}^{t} w(s)ds.$$

It then follows, from the assumption, that

$$Dm(t) = Dv(t) - w(t) \leq 0, \quad t \in [t_0, T].$$

Hence by Lemma 1.3.1, $m(t)$ is nonincreasing in t on $[t_0, T]$. Consequently,

$$D_{-}m(t) = D_{-}v(t) - w(t) \leq 0, \quad t \in [t_0, T],$$

and the lemma is proved.

The existence theorem together with the extension result imply the following.

Theorem 1.3.1. Let $g \in C(E, \mathbb{R})$, where E is an open (t,u)-set in $\mathbb{R}^2$ and $(t_0, u_0) \in E$. Then, the IVP

$$u' = g(t,u), \quad u(t_0) = u_0, \tag{1.16}$$

has extremal solutions (that is, minimal and maximal solutions) which can be extended up to the boundary of E.

The next lemma is useful in certain situations.

Lemma 1.3.3. Let the hypotheses of Theorem 1.3.1 hold and let $[t_0, T)$ be the largest interval of existence of maximal solution $r(t)$ of IVP (1.16). Suppose that $[t_0, t_1]$ is a compact subinterval of $[t_0, T)$. Then there is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the maximal solution $r(t, \varepsilon)$ of IVP

$$u' = g(t,u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon, \tag{1.17}$$

exists over $[t_0, t_1]$ and

$$r(t) = \lim_{\varepsilon \to 0} r(t, \varepsilon)$$

uniformly on $[t_0, t_1]$.

Lemma 1.3.4. Let $g \in C[[t_0, t_0 + a] \times \mathbb{R}, \mathbb{R}]$ and nondecreasing in u for each $t \in [t_0, t_0 + a]$. Assume that

$$g(t, 0) \equiv 0,$$

$$|g(t,u)| \leq M \text{ on } [t_0, t_0 + a] \times \mathbb{R},$$

and $u(t) \equiv 0$ is the unique solution of

$$u' = g(t,u), \quad u(t_0) = 0$$

on $[t_0, t_0 + a]$. Then, the successive approximations

$$u_0(t) = M(t - t_0),$$

$$u_{n+1}(t) = \int_{t_0}^{t} g(s, u_n(s))ds$$

are well defined;

$$0 \le u_{n+1}(t) \le u_n(t) \text{ on } [t_0, t_0 + a],$$

and

$$\lim_{n\to\infty} u_n(t) \equiv 0 \text{ uniformly on } [t_0, t_0 + a].$$

Moreover, for $n \ge 1$, the maximal solution $r_n(t)$ of

$$u' = g(t,u) + kg(t, u_{n-1}(t)), \quad u_n(t_0) = 0, \quad k > 0,$$

exists on $[t_0, t_0 + a]$, and

$$\lim_{n\to\infty} r_n(t) \equiv 0 \text{ uniformly on } [t_0, t_0 + a].$$

One of the results that is widely used is the following comparison theorem:

Theorem 1.3.2. Let E be an open (t,u)-set in $\mathbb{R}^2$ and let $g \in C[E, \mathbb{R}]$. Suppose that $[t_0, t_0 + a)$ is the largest interval in which the maximal solution $r(t)$ of (1.16) exists. Let $m \in C[(t_0, t_0 + a), \mathbb{R}]$, $(t, m(t)) \in E$ for $t \in [t_0, t_0 + a)$, $m(t_0) \le u_0$, and for a fixed Dini derivative,

$$Dm(t) \le g(t, m(t)),$$

$t \in [t_0, t_0 + a)$. Then,

$$m(t) \le r(t), \quad t \in [t_0, t_0 + a),$$

where $r(t) = r(t, t_0, u_0)$ is the maximal solution of IVP (1.16) existing on $[t_0, t_0 + a)$.

To give another comparison theorem that, in certain situations, is more useful than Theorem 1.3.2, we require the following result:

Theorem 1.3.3. Let E be the product space $[t_0, t_0 + a) \times \mathbb{R}^2$ and $g \in C[E, \mathbb{R}]$. Assume that g is nondecreasing in v for each t and u. Suppose that $r(t)$ is the maximal solution of the differential equation

$$u' = g(t, u, u), \quad u(t_0) = u_0 \ge 0$$

existing on $[t_0, t_0 + a)$, and

$$r(t) \ge 0, \quad t \in [t_0, t_0 + a).$$

Then, the maximal solution $r_1(t)$ of

$$u' = g_1(t, u), \quad u(t_0) = u_0 \ge 0,$$

where $g_1(t,u) = g(t,u,r(t))$, exists on $[t_0,t_0+a)$ and

$$r(t) = r_1(t), \quad t \in [t_0,t_0+a).$$

Theorem 1.3.4. Let the hypothesis of Theorem 1.3.3 hold; $m \in C[[t_0,t_0+a),\mathbb{R}]$ such that $(t,m(t),v) \in E$, $t \in [t_0,t_0+a)$, and $m(t_0) \leq u_0$. Assume that for a fixed Dini derivative the inequality

$$Dm(t) \leq g(t,m(t),v)$$

is satisfied for $t \in [t_0,t_0+a)$. Then, for all $v \leq r(t)$, $t \in [t_0,t_0+a)$, we have

$$m(t) \leq r(t), \quad t \in [t_0,t_0+a).$$

1.4 Fixed Point Theorems and Auxiliary Results

In this section, we shall list some fixed point theorems and other auxiliary results that we need in the course of our discussion. One of the most useful tools in proving existence and uniqueness of solutions for a variety of equations is what is known as Banach Contraction Principle which is stated below.

Theorem 1.4.1. Let (E,d) be a complete metric space and $T : E \to E$ is a contraction mapping, that is, for every $x,y \in E$,

$$d(Tx,Ty) \leq \alpha d(x,y), \quad 0 < \alpha < 1.$$

Then, there exists a unique fixed point x of T in E such that $x = Tx$.

Banach Contraction principle provides an abstract setting for the classical method of iterations or successive approximations. Another fixed point result which has many applications in the theory of functional equations is Schauder's fixed point theorem, which we state next.

Theorem 1.4.2. Let E be a Banach space and $B \subset E$, a convex, closed bounded set. If $T : E \to E$ is a continuous operator such that $TB \subset B$ and T is relatively compact, then T has a fixed point.

A more general result known as Tychonoff's fixed point theorem is in the context of Fréchet space. Before stating the theorem, let us define Fréchet space.

A Fréchet space is a linear space endowed with an invariant metric with respect to translation and is complete with respect to this metric. It is usual to define a Fréchet space by means of the use of seminorms. A seminorm on the linear space X is a map from X into $\mathbb{R}_+$, say $x \to |x|$, such that the following holds:

(i) $|x| \geq 0$;

(ii) $|\lambda x| = |\lambda||x|$;

(iii) $|x+y| \leq |x| + |y|$.

The only difference with respect to a norm consists in the fact that $|x| = 0$ does not necessarily imply $x = 0$.

Theorem 1.4.3. Let F be a Fréchet space whose distance function is constructed by means of a sufficient, countable family of seminorms, say $\{|x_k|; k \geq 1\}$, i.e., from $|x|_k = 0$, $k \geq 1$ one derives $x = 0$. If $B \subset F$ is a closed, convex set and $T : B \to B$ is a continuous operator such that TB is relatively compact, then T has at least one fixed point in B.

An extension of Banach contraction principle is the following result. Let E be a Banach space and $E_0 = C([a,b],E)$ where $[a,b]$ is any closed interval in $\mathbb{R}$. Suppose that T is an operator defined from E_0 into E. We shall say $\phi \in E_0$ is a fixed point of T if $T\phi = \phi(c)$ for some fixed $c \in [a,b]$. The operator T is said to be a contraction if

$$|T\phi - T\psi|_E \leq \alpha|\phi - \psi|_{E_0},$$

for all $\phi, \psi \in E_0$ and $0 < \alpha < 1$, where $|\phi|_{E_0} = \max_{a \leq s \leq b} |\phi(s)|_E$. Now, we can state the general fixed point result.

Theorem 1.4.4. Suppose that $T : E_0 \to E$ and that T is a contraction. Then, given $\phi_0 \in E_0$, every sequence of iterates $\{\phi_n\}$ satisfying $T\phi_n = \phi_{n+1}(c)$ fpr a given $c \in [a,b]$ and $|\phi_{n+1} - \phi_n|_{E_0} = |\phi_{n+1}(c) - \phi_n(c)|$, converges to a fixed point ϕ^* of T.

Let us now state the well-known Ascoli-Arzela criterion which is helpful in determining the compactness of a set in $C([t_0,T],\mathbb{R}^n)$.

Theorem 1.4.5. Let $M \subset E = C([t_0,T],\mathbb{R}^n)$, $t_0 \geq 0$. Then M is relatively compact with respect to uniform convergence on $[t_0,T]$ if and only if

(i) M is bounded in E, that is, there exists a $N > 0$ such that $|x(t)| \leq N$, $t \in [t_0,T]$, for each $x \in M$;

(ii) M is equicontinuous, which means that for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that for $t,s \in [t_0,T]$,

$$|x(t) - x(s)| < \varepsilon, \text{ for } |t-s| < \delta,$$

for every $x \in M$.

One simple situation in which equicontinuity takes place is when all functions in M satisfy the Lipschitz condition

$$|x(t) - x(s)| \leq L|t-s| \text{ on } [t_0,T],$$

with the same constant $L > 0$. Sometimes, the supremum norm needs to be replaced by an equivalent norm by introducing a weighted norm, namely, if $g \in C([t_0, T], \mathbb{R})$ is positive, define

$$|x|_g = \left(\sup \frac{|x(t)|}{g(t)} : t \in [t_0, T]\right).$$

Proof of Theorem 1.4.4. Let $\phi_0 \in E_0$ be given. By hypothesis $T\phi_0 \in E$. Suppose $T\phi_0 = x_1$. Choose $\phi_1 \in E_0$ such that $x_1 = \phi_1(c)$ and $|\phi_1(c) - \phi_0(c)|_E = |\phi_1 - \phi_0|_{E_0}$. Defining ϕ_0 inductively so that

$$T\phi_n = x_{n+1} = \phi_{n+1}(c)$$

and

$$|\phi_{n+1}(c) - \phi_n(c)|_E = |\phi_{n+1} - \phi_n|_{E_0}, \text{ for } n = 1, 2, \ldots,$$

we claim the sequence $\{\phi_n\}$ is Cauchy in E_0. First we observe

$$|\phi_n - \phi_{n+1}|_{E_0} = |\phi_n(c) - \phi_{n+1}(c)|_E = |T\phi_{n-1} - T\phi_n| \leq \alpha|\phi_{n-1} - \phi_n|_{E_0}.$$

By induction one can easily verify

$$|\phi_n - \phi_{n+1}|_{E_0} \leq \alpha^n|\phi_0 - \phi_1|_{E_0}.$$

If $m \geq n$, by the triangle inequality, we get

$$\begin{aligned}
|\phi_m - \phi_n|_{E_0} &\leq |\phi_m - \phi_{m-1}|_{E_0} + |\phi_{m+1} - \phi_{m-2}|_{E_0} + \cdots + |\phi_{n+1} - \phi_n| \\
&\leq \alpha^{m-1}|\phi_0 - \phi_1|_{E_0} + \alpha^{m-2}|\phi_0 - \phi_1|_{E_0} + \cdots + \alpha^n|\phi_0 - \phi_1|_{E_0} \\
&= (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)|\phi_0 - \phi_1|_{E_0} \\
&\leq \alpha^n(1-\alpha)^{-1}|\phi_0 - \phi_1|_{E_0}.
\end{aligned}$$

Hence as $m, n \to \infty$, $|\phi_m - \phi_n|_{E_0} \to 0$. This shows that $\{\phi_n\}$ is a Cauchy sequence, and the fact that E_0 is complete implies that $\{\phi_n\}$ converges to a limit ϕ^* in E_0. That is, there exists $\phi^* \in E_0$ such that

$$\lim_{n\to\infty} \phi_n = \phi^*.$$

Therefore

$$T\phi^* = T\left(\lim_{n\to\infty} \phi_n\right) = \lim_{n\to\infty}(T\phi_n) = \lim_{n\to\infty} \phi_{n+1}(c) = \phi^*(c).$$

The proof is complete.

1.5 Preliminaries in a Banach Space

Let E be a linear space (vector space) over the field ϕ (of real numbers or complex numbers) and let p be a function from E into $\mathbb{R}_+$, the set of non-negative real numbers. Then p is a norm on E if

(i) $p(ax) = |ap(x)|$ for all $x \in E$ and $a \in \phi$;
(ii) $p(x+y) \le p(x) + p(y)$ for all $x,y \in E$;
(iii) $p(x) = 0$ if and only if $x = 0$ where 0 is the null element of E.

In this case, we write $p(\cdot) = |\cdot|$ and say that $(E, |\cdot|)$ is a normed linear space over ϕ. It is easily seen that $\rho(x,y) = |x-y|$, $x,y \in E$, defines a metric on E and that a sequence $\{x_n\} \subset E$ converges in this metric topology to $x \in E$ if and only if $\lim_{n\to\infty} |x_n - x| = 0$. We shall generally refer to this metric topology as the norm (uniform or strong) topology. A normed linear space E over the field of real numbers is said to be a real Banach space if it is a complete metric space when equipped with the metric $\rho(x,y)$. We shall be working mostly with real Banach spaces.

A mapping $f : E \to \mathbb{R}$ is said to be a linear functional on E if $f(x+y) = f(x) + f(y)$ and $f(ax) = af(x)$ for all $x,y \in E$ and $a \in \mathbb{R}$, $\mathbb{R}$ being the field of real numbers. A linear functional f on E is bounded if there is an $M \ge 0$, such that $|f(x)| \le M$ for all $x \in E$ with $|x| \le 1$. The dual space E^* of E is defined to be a class of all continuous linear functionals on E and for each $x^* \in E^*$, we define

$$|x^*| = \sup\{|x^*(x)| : x \in E, |x| \le 1\}.$$

It is easy to show that this is the norm on E^* and with this norm, it is clear that E^* is a normed linear space. Also, since $\mathbb{R}$ is complete, it follows that E^* is complete and thus is a Banach space. Note that continuity and boundedness of a linear functional are equivalent concepts, i.e., a linear functional $f \in E^*$ if and only if it is bounded on E.

One of the fundamental results in functional analysis is the following theorem due to Hahn-Banach which assures that a linear functional on a linear subspace of a linear space that is bounded by a seminorm can always be extended to the entire space in such a manner that its seminorm boundedness is preserved. However, such an extension, in general, is not unique.

Theorem 1.5.1. (Hahn-Banach) Suppose that E is a linear space over the field $\mathbb{R}$ and that $p : E \to \mathbb{R}$ is such that $p(\lambda x) = \lambda p(x)$, $p(x+y) \le p(x) + p(y)$ for all $\lambda \ge 0$ and $x,y \in E$. Assume that Ω is a linear subspace of E and f is a linear functional from Ω into $\mathbb{R}$ such that

$f(x) \le p(x)$ for all $x \in \Omega$. Then there is a linear functional $g : E \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in \Omega$ and $g(x) \le p(x)$ for all $x \in E$.

If E is a normed linear space, we have the following version of Theorem 1.5.1.

Theorem 1.5.2. Let Ω be a linear subspace of the normed linear space E over $\mathbb{R}$ and let $y^* \in \Omega^*$. Then there is an $x^* \in E^*$ such that $|x^*| = |y^*|$ and $x^*(x) = y^*(x)$ for all $x \in \Omega$.

Some important consequences of Hahn-Banach theorem are as follows.

Corollary 1.5.1. If $x \in E$ and $x \neq 0$, then there is an $x^* \in E^*$ such that $|x^*| = 1$ and $x^*(x) = |x|$.

Corollary 1.5.2. Let Ω be a subspace of E and $x \in E$ with $d(x,\Omega) = d > 0$, where $d(x,\Omega) = \inf_{y\in\Omega} |x-y|$. Then there is an $x^* \in E^*$ with $d|x^*| = 1, x^*(x) = 1$ and $x^*(y) = 0$ for all $y \in \Omega$.

Now, let $x \in E$, E a normed linear space, and define the mapping $\tilde{x} : E^* \to \mathbb{R}$ by

$$\tilde{x}(x^*) = x^*(x), \quad x^* \in E^*.$$

Clearly $\tilde{x}$ is linear and $|\tilde{x}(x^*)| \le |x^*||x|$ for all $x^* \in E^*$. Hence $\tilde{x}$ is a bounded linear functional on E^* and $\tilde{x}$ is a member of the dual space of E^*, which is denoted by E^{**} and called the bidual or second dual of E. By Corollary 1.5.1, it follows that $|\tilde{x}| = |x|$ and the mapping $\tau : E \to E^{**}$ defined by $\tau x = \tilde{x}$ is linear and norm preserving. This mapping τ is called the canonical embedding of E into E^{**}. A Banach space E is said to be reflexive if $E^{**} = \{\tau x : x \in E\}$. Note that any finite dimensional Banach space is reflexive.

In a reflexive Banach space, the following result concerning weak convergence is true.

Theorem 1.5.3. If E is a reflexive Banach space, any norm bounded sequence in E has a weakly convergent subsequence.

If $x^* \in E^*$ and $\{x_n^*\}$ is a sequence in E^*, then $\{x_n^*\}$ is said to be weak* convergent to x^* if

$$\lim x_n^*(x) = x^*(x), \text{ for all } x \in E.$$

In this case, we say that $\{x_n^*\}$ converges to x^* in weak*-topology.

The next result concerning weak*-convergence is also valid.

Theorem 1.5.4. Let E be a Banach space and $\{x_n^*\}$ be a bounded sequence in E^*. Then $\{x_n^*\}$ has a weak*-convergent subsequence.

We list below the definitions of some useful concepts in terms of weak topology. Let E_w denote the space E when endowed with weak topology generated by the continuous linear functional on E.

Definition.

(i) A subset A of E_w is totally bounded if and only if for all $x^* \in E^*$ and $\varepsilon > 0$, the set A can be covered by a finite number of x^*-balls of radius ε;

(ii) If $\{y_n\}$ is a sequence in E, then $\{y_n\}$ is weakly Cauchy if given $\varepsilon > 0$, $x^* \in E^*$, there exists $N = N(x^*, \varepsilon)$ such that $n, m \geq N$ implies $|x^*(y_n - y_m)| < \varepsilon$;

(iii) E is weakly complete if every Cauchy sequence converges weakly to a point in E.

With these definitions, we can now state that a set A is compact in a weak topology (or weakly compact) if and only if it is weakly complete and totally bounded.

Let $x(t)$ be a function mapping some interval $I \subset \mathbb{R}$ into E. The function $x(t)$ is said to be strongly continuous at $t_0 \in I$ if

$$\lim_{t \to t_0} |x(t) - x(t_0)| = 0,$$

that is, the convergence of $x(t)$ to $x(t_0)$ is in the norm topology on E. If $x : I \to E$ is continuous at each point of I then we say that x is continuous on I and write $x \in C[I, E]$. $x'(t)$ is said to be the strong derivative of $x(t)$ if

$$\lim_{h \to 0} \left| \frac{1}{h}[x(t+h) - x(t)] - x'(t) \right| = 0.$$

The Riemann integral of $x(t)$ can be similarly defined. Some useful properties of the integral are given in the following lemmas.

Lemma 1.5.1. Let E be a Banach space and $x(t)$ be an integrable function from I into E. Then

$$\frac{1}{b-a} \int_a^b x(s)ds \in \overline{co}(\{x'(s) : s \in [a, b]\})$$

for all $a, b \in I$, with $a < b$, where $\overline{co}(A)$ is the closed convex hull of A.

Lemma 1.5.2. If $\{x_n\}$, $n = 1, 2, \ldots$, is a sequence of continuous functions from I into E such that $\lim_{n \to \infty} x_n(t) = x(t)$ uniformly for $t \in I$, then

$$\lim_{n \to \infty} \int_a^b x_n(s)ds = \int_a^b x(s)ds \text{ for all } [a, b] \subset I.$$

A useful form of the Ascoli-Arzela theorem for a family of functions from I into X is as follows.

Theorem 1.5.5. Let F be an equicontinuous family of functions from I into X. Let $\{x_n(t)\}$ be a sequence in F such that, for each $t \in I$, the set $\{x_n(t) : n \geq 1\}$ is relatively compact in X, i.e., the closure of the set $\{x_n(t) : n \geq 1\}$ is compact. Then there is a subsequence $\{x_{n_k}(t)\}$ which converges uniformly on I to a continuous function $x(t)$.

The following is a useful extension theorem due to Dugundji.

Theorem 1.5.6. Suppose that E_1 and E_2 are two Banach spaces, $\Omega \subset E_1$ and $f : \Omega \to E_2$ is a continuous mapping. Then there is a continuous extension $\tilde{f} : E_1 \to E_2$ of f such that $\tilde{f}(E_1) \subset co(f(\Omega))$.

The next result extends the mean value theorem to functions with values in a Banach space in terms of both strong and weak topologies.

Theorem 1.5.7. Let $u \in C[J,X]$, where J is an interval and X is a real Banach space. Suppose that $a,b \in J$, $a < b$, and there is an at most countable subset Γ of $[a,b]$ such that $u'_+(t)$ exists for all $t \in [a,b] - \Gamma$. Then the following relation holds:

$$u(b) - u(a) \in (b-a)\overline{co}(\{u'_+(t) : [a,b] - \Gamma\}),$$

where $\overline{co}(A)$ is a closed convex hull of A.

Theorem 1.5.8. (Krein-S̃mulian Theorem) Let $(E, |\cdot|)$ be a Banach space over ϕ and suppose $K \subset E^*$ is convex. Then the following statements are equivalent:

(i) K is weak*-closed;
(ii) for each $a \geq 0$ the set $K \cap aB_1^*$ is weak*-closed, where $B_1^* = \{x^* \in E^* : |x^*| \leq 1\}$.

Theorem 1.5.9. (Eberlein-S̃mulian Theorem) Let $(E, |\cdot|)$ be a Banach space over ϕ and suppose $K \subset E$ is weakly closed in E. Then the following statements are equivalent:

(i) K is weakly compact;
(ii) K is weakly sequentially compact.

1.6 Directional Derivatives

When we use $|x|$ or $|x|^2$ as a measure in estimates later, we need to assume conditions in terms of their one-sided directional derivatives. Here, we define and list several properties of such derivatives.

Let $x,y \in E$, E being a real Banach space with norm $|\cdot|$. Define

$$[x,y]_h = \frac{1}{h}(|x+hy| - |x|)$$

for any $h \in \mathbb{R}$. Then we have the following result.

Lemma 1.6.1.

(i) The limits $\lim_{h \to 0^+} [x,y]_h = [x,y]_+$ and $\lim_{h \to 0^-} [x,y]_h = [x,y]_-$ exist; and
(ii) $[x,y]_+$ is upper semicontinuous and $[x,y]_-$ is lower semicontinuous.

Proof. Let us first show that $[x,y]_h$ is monotone nondecreasing in h. Suppose $0 < h_1 < h_2$ and let $\beta \in (0,1)$ be such that $h_1 = (1-\beta)h_2$. Since

$$x + h_1 y = x + (1-\beta)h_2 y = \beta x + (1-\beta)(x + h_2 y),$$

we have

$$[x,y]_{h_1} = \frac{1}{h_1}(|x+h_1y|-|x|) = \frac{1}{h_1}\{|\beta x+(1-\beta)(x+h_2y)|-|x|\}$$

$$\leq \frac{1}{h_1}\{\beta|x|+(1-\beta)|x+h_2y|-|x|\} = \frac{|x+h_2y|-|x|}{h_2} = [x,y]_{h_2}.$$

Similarly, one gets $[x,y]_{h_1} \leq [x,y]_{h_2}$ if $h_1 < h_2 < 0$. If $h_1 < 0 < h_2$, we let $h = \min(-h_1,h_2)$ and note that

$$2|x| = |x+hy+x-hy| \leq |x+hy|+|x-hy|.$$

This implies that $[x,y]_{-h} \leq [x,y]_h$, which in turn yields

$$[x,y]_{h_1} \leq [x,y]_{-h} \leq [x,y]_h \leq [x,y]_{h_2},$$

proving that $[x,y]_h$ is monotone. If $-1 \leq h_1 \leq h_2 \leq 1$, the monotone property of $[x,y]$ gives

$$[x,y]_{-1} \leq [x,y]_{h_1} \leq [x,y]_{h_2} \leq [x,y]_1$$

and hence the limits

$$[x,y]_+ = \lim_{h\to 0^+}[x,y]_h,$$

and

$$[x,y]_- = \lim_{h\to 0^-}[x,y]_h$$

exist. To prove that $[x,y]_+$ is upper semicontinuous; let $\{x_n\},\{y_n\}$ be two sequences in E such that $\lim_{n\to\infty}x_n = x$ and $\lim_{n\to\infty}y_n = y$. Then for $h > 0$

$$[x_n,y_n]_+ \leq \frac{1}{h}\{|x_n+hy_n|-|x_n|\} \text{ for all } n \geq 1.$$

Letting $n \to \infty$, we get

$$\limsup_{n\to\infty}[x_n,y_n]_+ \leq \frac{1}{h}\{|x+hy|-|x|\}$$

for all $h > 0$. We now let $h \to 0^+$, obtaining

$$\limsup_{n\to\infty}[x_n,y_n]_+ \leq [x,y]_+. \tag{1.18}$$

Since $[x,y]_- = -[x,-y]_+$, by definition, lower semicontinuity of $[x,y]_-$ follows from (1.18) thus proving the lemma.

Some properties of $[x,y]_\pm$ are listed in the following lemma.

Lemma 1.6.2. Let $[x,y]_\pm$ be defined as in Lemma 1.6.1. Then,

(i) $[x,y]_- \leq [x,y]_+$;

(ii) $|[x,y]_\pm| \leq |y|$;

(iii) $|[x,y]_\pm - [x,z]_\pm| \le |y-z|$;

(iv) $[x,y]_+ = -[x,-y]_- = -[x,y]_-$;

(v) $[sx,ry]_\pm = r[x,y]_\pm$ for $r,s \ge 0$;

(vi) $[x,\alpha x]_\pm = \alpha|x|, \quad \alpha \in \mathbb{R}$;

(vii) $[x,y+z]_+ \le [x,y]_+ + [x,z]_+$ and $[x,y+z]_- \ge [x,y]_- + [x,z]_-$;

(viii) $[x,y+z]_+ \ge [x,y]_+ + [x,z]_-$ and $[x,y+z]_- \le [x,y]_- + [x,z]_+$;

(ix) $[x,y+\alpha x]_\pm = [x,y]_\pm + \alpha|x|, \quad \alpha \in \mathbb{R}$;

(x) if $x:[a,b] \to E$ such that $x'_\pm(t)$ (the right and left derivatives of $x(t)$) exists for some $t \in (a,b)$ and $m(t) = |x(t)|$, then $m'_\pm(t) = [x(t),x'_+(t)]_+$.

1.7 Measure of Noncompactness

Let A be a bounded subset in a Banach space E. The diameter of A is defined by

$$\text{dia}(A) = \sup\{|x-y| : x,y \in A\}.$$

Clearly, $0 \le \text{dia}(A) < \infty$. Kuratowski's measure of noncompactness of A is defined by

$$\alpha(A) = \inf\{d > 0 : A \text{ is covered by a finite number of sets with diameter} \le d\}.$$

In particular, given $d > \alpha(A)$, there exists a finite number of sets $S_1, S_2, \ldots, S_n \subset A$ such that $\text{dia}(S_i) \le d$ and $\cup_{i=1}^n S_i = A$. In other words, $\alpha(A)$ can be regarded as a measure of the extent to which A is not compact. Note also that $\alpha(A) \le \text{dia}(A)$ and $\alpha(A) \le 2d$ if $\sup_{x\in A}|x| \le d$. The various properties of α that will be useful later are listed in the following theorem.

Theorem 1.7.1. Let A,B be bounded subsets of E. Then

(i) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact, where $\overline{A}$ denotes the closure of A;

(ii) $\alpha(A) = \alpha(\overline{A})$;

(iii) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in \mathbb{R}$ where $\lambda A = \{\lambda x : x \in A\}$;

(iv) $\alpha(A \cup B) = \max(\alpha(A), \alpha(B))$;

(v) $\alpha(A) \le \alpha(B)$ if $A \subset B$;

(vi) $\alpha(A+B) \le \alpha(A) + \alpha(B)$ where $A+B = \{x+y : x \in A \text{ and } y \in B\}$; in particular, if $A = \{x_n\}$, $B = \{y_n\}$ are two countable sets of points in E, then

$$\alpha(\{x_n\}) - \alpha(\{y_n\}) \le \alpha(\{x_n - y_n\});$$

(vii) α is continuous with respect to the Hausdorff metric;

(viii) $\alpha(A) = \alpha(co(A))$ where $co(A)$ is the convex hull of A;

(ix) if $\{A_n\}$ is a family of nonempty bounded subsets of E such that $A_{n+1} \subset A_n$ for $n = 1,2,\ldots$, and $\lim_{n\to\infty} \alpha(A_n) = 0$, then $\cap_{n=1}^\infty \overline{A_n}$ is nonempty and compact.

Also, using the definitions of $\alpha(\cdot)$, it is easy to show that $\alpha(A\times B) = \max(\alpha(A),\alpha(B))$ where $A\times B$ is the Cartesian product of two bounded subsets A,B of Banach spaces E,F respectively, with $|(x,y)| = \max(|x|,|y|)$, $x\in A$, $y\in B$.

While considering bounded subsets of $C(I,E)$, I being any compact subset of the real line, it is convenient to use the following notation: for any set $H\subset C[I,E]$, $H(t)$ and $H(I)$ denote the sets given by $\{\phi(t) : \phi\in H\}$ and $\cup_{t\in I}\{\phi(t) : \phi\in H\}$ respectively. A useful property of measure of noncompactness α is in the following result, which, in some sense, also provides a generalization of the theorem of Ascoli-Arzela.

Theorem 1.7.2. If $H=\{x_k\}$, where $x_k\in C[I,E]$, is any bounded equicontinuous family of functions, then

$$\sup_{t\in I}\alpha(H(t)) = \alpha(H).$$

The proof of Theorem 1.7.2, is an immediate consequence of

Lemma 1.7.1. If $H\subset C[I,E]$ is a bounded, equicontinuous set, then

(a) $\alpha(H) = \alpha(H(I))$;

(b) $\alpha(H(I)) = \sup_{t\in I}\alpha(H(t))$.

1.8 The Measure of Nonconvexity

As we have seen, the measure of noncompactness which was introduced by Kuratowski has now become an important tool in nonlinear analysis. Following Kuratowski, we shall introduce a measure of nonconvexity which has many properties in common with the measure of noncompactness and therefore we may now have convexity where we previously had compactness in the statement of some results.

Let E be a Banach space (with norm $|\cdot|$) and A is a subset in E. Denote by $co(A)$ the convex hull of A. We say that A is α-measurable with measure $\alpha(A)$ if

$$\alpha(A) = \sup_{b\in co(A)}\inf_{a\in A}|b-a| < \infty. \tag{1.19}$$

Alternatively, if $H(X,Y)$ denotes the Hausdorff distance between two subsets X and Y,

$$\alpha(A) = H(A,co(A)). \tag{1.20}$$

Clearly, a bounded set is α-measurable.

From the definition, the following properties of α can be derived in a straightforward manner.

$$\alpha(A) = 0 \text{ iff } \overline{A} \text{ (the closure of } A\text{) is convex;} \tag{1.21}$$

$$\alpha(\lambda A) = |\lambda|\alpha(A) \text{ for } \lambda \in \mathbb{R}^1 \text{ (where } \lambda A = \{\lambda a | a \in A\}); \tag{1.22}$$

$$\alpha(A+B) \le \alpha(A) + \alpha(B); \tag{1.23}$$

$$|\alpha(A) - \alpha(B)| \le \alpha(A-B); \tag{1.24}$$

$$\alpha(\overline{A}) = \alpha(A); \tag{1.25}$$

$$\alpha(A) \le \text{diam}(A) \text{ (the diameter of } A); \tag{1.26}$$

$$|\alpha(A) - \alpha(B)| \le 2H(A,B). \tag{1.27}$$

Note that all of these properties are shared by the measure of noncompactness γ. Recall $\gamma(A) = \inf\{d > 0 | A$ can be covered by a finite number of sets of diameter $\le d\}$. α is not monotone in the sense that $\alpha(A) \le \alpha(B)$ if $A \subset B$. If it did, then every closed set would be convex which is not true. Unfortunately $\alpha(A)$ measures only the nonconvexity of $\overline{A}$ and not A itself if A is not closed.

As a consequence of (1.27) and a similar inequality for γ, $|\gamma(A) - \gamma(B)| \le H(A,B)$, the measures α and γ are continuous with respect to the Hausdorff metric, that is,

Proposition 1.8.1. Let A_n be a sequence of subsets of E such that A_n approaches a subset A_∞ in the Hausdorff metric. Then

(i) if A_n are α-measurable,

$$\lim_{n\to\infty} \alpha(A_n) = \alpha(A_\infty); \tag{1.28}$$

(ii) if A_n are bounded

$$\lim_{n\to\infty} \gamma(A_n) = \gamma(A_\infty). \tag{1.29}$$

Proposition 1.8.2. (Kuratowski) Let (X,ρ) be a complete metric space and let $A_0 \supset A_1 \supset \dots$ be a decreasing sequence of nonempty, closed subsets of E. Assume $\gamma(A_n) \to 0$. Then if we write $A_\infty = \cap_{n\ge 0} A_n$, A_∞ is a nonempty compact set and A_n approaches A_∞ in the Hausdorff metric.

Proposition 1.8.3. Let $A_0 \supset A_1 \supset \dots$ be a decreasing sequence of closed bounded subsets of E. Let $A_\infty = \cap_{n\ge 0} A_n$. Then A_∞ is nonempty, convex and compact and A_n converges to A_∞ in the Hausdorff metric iff $\alpha(A_n) \to 0$ and $\gamma(A_n) \to 0$.

Proof. Suppose $\gamma(A_n) \to 0$. It follows from Proposition 1.8.2 that A_n converges to the nonempty compact set A_∞ in the Hausdorff metric. If, in addition, $\alpha(A_n) \to 0$ then in view of (1.28), $\alpha(A_\infty) = 0$. Since A is also closed, A_∞ is convex by (1.21).

Suppose $A_n \to A_\infty$ in the Hausdorff metric and $\alpha(A_\infty) = \gamma(A_\infty) = 0$. Then by (1.28) and (1.29), $\alpha(A_n) \to 0$ and $\gamma(A_n) \to 0$.

Proposition 1.8.4. Let $A_0 \supset A_1 \supset \ldots$ be a decreasing sequence of closed, bounded subsets of E such that $\alpha(A_n) \to 0$ and $\gamma(A_n) \to 0$. Suppose T is a continuous map of $A_0 \to A_0$ such that

$$Tx \in A_n, \text{ if } x \in A_n, \ n = 0, 1, \ldots \tag{1.30}$$

Then there exists an $x \in A_\infty = \cap_{n \geq 0} A_n$ such that

$$Tx = x, \tag{1.31}$$

Proof. The result is a corollary of the Schauder principle since, from Proposition 1.8.3, A_∞ is nonempty, convex and compact and T maps A_∞ into itself.

Closely associated with the notion of measure of noncompactness is the concept of k-set-contraction. Let (X_1, d_1) and (X_2, d_2) be metric spaces and suppose $T : X_1 \to X_2$ is a continuous map. We say T is a k-set-contraction if given any bounded set A in X_1, $T(A)$ is bounded and $\gamma_2(T(A)) \leq k\gamma_1(A)$ where γ_i denotes the measure of noncompactness in X_i, $i = 1, 2, \ldots$

Proposition 1.8.5. Let C be a closed, bounded, convex set and $T : C \to C$ a k-set-contraction, $k < 1$. Then T has a fixed point, i.e., a point x satisfying (1.31).

The above generalization of the Schauder principle was further extended by introducing a comparison function ψ which has the following properties:

(i) ψ maps a conical segment of regular cone in a partially ordered space into itself;
(ii) ψ is monotone;
(iii) ψ is upper semi-continuous from the right;
(iv) $\psi(x) = x$ iff $x = \theta$ (the zero of the space).

Then Darbo's condition $\gamma(T(A)) \leq k\gamma(A)$, $k < 1$, is replaced by the weaker condition

$$\gamma(T(A)) \leq \psi(\gamma(A)). \tag{1.32}$$

Definition 1.8.1. A function $\psi : [0, \infty) \to [0, \infty)$ is a comparison function if

(i) $\psi(t) < t$ for $t > 0$,
(ii) $\psi(0) = 0$, ψ is upper semi-continuous from the right.

Proposition 1.8.6. Let ψ be a comparison map and let $S_0, S_1, \ldots$ be a sequence of nonnegative real numbers such that $S_n \leq \psi(S_{n-1})$, $n - 1, 2, \ldots$ Then the sequence S_n converges to zero.

Proof. Since $S_n \le \psi(S_{n-1}) \le S_{n-1}$, the sequence S_n converges monotonically. Suppose $S_\infty > 0$. Then $\psi S_\infty < S_\infty \le S_n$, $n = 1,2,\ldots$. But this contradicts the upper semi-continuity from the right.

Proposition 1.8.7. Let $\psi : [0,a) \to [0,a)$ be nonincreasing, upper semicontinuous from the right, and $\psi(t) = t$ iff $t = 0$. Then ψ has an extension to $[0,\infty)$ which is a comparison function.

Proof. Since the interval $[0,a]$ is a segment of the regular cone (of nonnegative real numbers) it follows that if $t \le \psi(t)$ then $t \le t_0$ where t_0 is the maximal solution of $\psi(t) = t$. By assumption $t_0 = 0$. Thus $t \le \psi(t)$ iff $t = 0$. If we define $\phi(t) = \psi(a)$, $t \ge a$ then ϕ is a comparison function.

Definition 1.8.2. Let $(X_1, |\cdot|_1)$ and $(X_2, |\cdot|_2)$ be Banach spaces and suppose $T : X_1 \to X_2$ is a continuous map. We say that T is a ψ-set-contraction with respect to convexity (compactness) if given any α_1-measurable (bounded) set A in X_1, $T(A)$ is α_2-measurable (bounded) and

$$\alpha_2(T(A)) \le \psi(\alpha_1(A)) \tag{1.33}$$

$$(\gamma_2(T(A)) \le \psi(\gamma_1(A))) \tag{1.34}$$

where $\alpha_i(\gamma_i)$ denotes the measure of nonconvexity (noncompactness) in X_i, $i = 1,2$. We say that T is a contraction if $|Tx - Ty|_2 \le \psi(|x-y|_1)$ for every $x,y \in X_1$. The following result is a generalization of a similar result due to Darbo in regard to relating the notion of k-contraction i.e. ψ-contraction with $\psi(t) = kt$, to the notion of k-set-contraction.

Proposition 1.8.8. Let $(X_1, |\cdot|_1)$ and $(X_2, |\cdot|_2)$ be Banach spaces. Let T be a ψ-contraction, then

(i) T is a ψ-set-contraction with respect to compactness;
(ii) $H(TA, TB) \le \psi(H(A,B))$ whenever $H(A,B) < \infty$;
(iii) if for every α-measurable set A, $co(TA) \subset \overline{T(\overline{co}(A))}$ (where $\overline{co}(X)$ denotes the convex closure of X), then T is ψ-set-contraction with respect to convexity.

Proof.

(i) Let A be a bounded set in X_1 and suppose $\gamma_1(A) = d$. Then given $\varepsilon > 0$, we can write $A = \cup_{j=1}^m S_j$, $\operatorname{diam}(S_j) \le d + \varepsilon$. Thus $T(A) = \cup_{j=1}^m T(S_j)$ and since T is a ψ-contraction, $\operatorname{diam}(T(S_j)) \le \psi(d+\varepsilon)$. Let ε_i be a sequence of positive numbers converging to zero such that $\psi(d+\varepsilon_i)$ converges and let $b = \lim \psi(d+\varepsilon_i)$. Then by upper semi-continuity from the right, $b \le \psi(d)$. Hence $\gamma_2(TA) \le \psi(d)$.

(ii) Let A and B be sets such that $H(A,B) = d < \infty$. Let $b \in B$. Then $\inf\{|Tb - Ta|_2, a \in A\} \leq \inf\{\psi(|b-a|_1), a \in A\} \leq \psi d$, by the upper semi-continuity from the right of the function ψ. Similarly $\inf\{|Ta - Tb|_2, b \in B\} \leq \psi d$. Thus $H(TA, TB) \leq d$.

(iii) Let A be an α-measurable set in X_1. Then from (ii), $\alpha(TA) = H(TA, \overline{co}(TA)) \leq H(TA, \overline{T(\overline{co}(A))}) = H(TA, T(\overline{co}(A))) \leq \psi(H(A, \overline{co}A)) = \psi\alpha(A)$.

Theorem 1.8.1. Let A be a closed subset of a Banach space and T a map from A onto itself. If T is set contractive with respect to convexity (compactness) then A is convex (compact). In particular, the set of fixed points of a set contractive, with respect to convexity (compactness) map of a closed subset of a Banach space $\mathscr{B}$ into $\mathscr{B}$ is convex (compact).

Proof. Set $m = \alpha(T(A)) = \alpha(A)$ $(m = \gamma(T(A)) = \gamma(A))$. Then $m \leq \psi(m)$. If $m > 0$ then $\psi(m) < m$. But this is impossible. Clearly $m = 0$.

Theorem 1.8.2. Let C be a closed, bounded set and $T : C \to C$ a ψ_1-set-contraction with respect to convexity and a ψ_2-set-contraction with respect to compactness. The set of fixed points of T is nonempty, convex, and compact.

Proof. Let $C_0 = C$, and $C_{n+1} = \overline{T(C_n)}$. Then $C_{n+1} \subset C_n$. Let $s_n = \gamma(C_n)$, $t_n = \alpha(C_n)$, then it follows from Proposition 1.8.6 that $s_n \to 0$ and $t_n \to 0$. By Proposition 1.8.4, the set $F(T)$ of fixed points of T is nonempty and, by Theorem 1.8.1, it is also convex and compact.

1.9 Notes and Comments

The material concerning causal operators detailed in Sec. 1.2 is taken from Corduneanu [1] and [2]. See Karakostas [3] for topological dynamics generated by causal operators. The comparison results listed in Sec. 1.3 are from Lakshmikantham and Leela [4]. For the proofs of the fixed point theorems listed in Sec. 1.4, see Edwards [5], Tychonoff [6], Deimling [7], Kantorovich and Arilov [8], Bernfeld, Lakshmikantham and Reddy [9]. The contents of the remaining sections are adapted from Lakshmikantham and Leela [10] except Sec. 1.8, which is taken from Eisenfeld and Lakshmikantham [11]. See Tonelli [12] and Turinici [13].

Chapter 2

Basic Theory

2.1 Introduction

This chapter is devoted to the basic theory of causal differential equations (CDE) and therefore forms a basis for the remaining chapters. We begin Section 2.2 with causal differential inequalities and prove fundamental results involving strict and nonstrict functional and causal differential inequalities, which are useful to prove existence of extremal solutions and important comparison results. Section 2.3 deals with local existence results, including the existence of maximal and minimal solution of CDEs, which are required for later discussion. While proving the important comparison results in this section, we realize that when imposing various conditions on the causal functions involved, it is enough to suppose such conditions on a suitable subset of the function space, which is a definite advantage in investigating the qualitative properties of solutions of CDEs.

In Section 2.4, we discuss global existence by utilizing two different approaches, one is based on the use of Tychonoff's fixed point theorem and the other on a direct approach employing the comparison results. Section 2.5 starts with the simple existence and uniqueness results under Lipschitz condition, then by using the general uniqueness conditions, we prove the convergence of successive approximations that guarantees simultaneously the existence and uniqueness. Section 2.6 employs Nagumo and Krasnoselskii-Krein type conditions to achieve the convergence and successive approximations. In view of the singularity involved in this type of conditions, the proofs differ substantially. The continuous dependence with respect to the initial data and parameters are studied in Section 2.7.

In Section 2.8, we investigate the existence of Euler solutions, first without demanding continuity on the functions involved and then show that the Euler solution is actually the usual solution of CDE under continuity assumption. The flow invariance results are considered, in Section 2.9 utilizing the concepts of nonsmooth analysis. In Section 2.10, we indicate

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the extension of the scalar results on inequalities to systems of causal differential inequalities employing the necessary notions. In Section 2.11, we present nonlinear variation of parameters. Section 2.12, 2.13 deal with integral and differential equations of Sobolev type which are new types of dynamic systems which are not yet well investigated. Finally, notes and comments are given in Section 2.14.

2.2 Causal Functional and Differential Inequalities

Recall that an operator $Q : E \to E$, $E = C([t_0,T],\mathbb{R}^n)$ is said to be causal (or non-anticipatory) if, for any $x,y \in E$ such that $x(s) = y(s)$, we have $(Qx)(s) = (Qy)(s)$, $t_0 \le s < T$. Let us consider the causal functional equation

$$x(t) = (Qx)(t), \quad x(t_0) = x_0, \tag{2.1}$$

where the causal operator $Q : E \to E$ is continuous and $x(t_0) = x_0$, $t_0 \ge 0$, denotes the initial value for any $x \in E$. A basic result in causal functional inequalities for the scalar case is the following.

Theorem 2.2.1: Assume that $Q : E \to E$ is a continuous causal operator where $E = C([t_0,T],\mathbb{R})$ and let $v,w \in E$ satisfy

$$v(t) \le (Qv)(t), \quad w(t) \ge (Qw)(t), \quad t_0 \le t < T. \tag{2.2}$$

Suppose that Q is semi-nondecreasing i.e.

$$x(t_1) = y(t_1), \quad x(t) < y(t), \quad t_0 \le t < t_1 < T$$

implies

$$(Qx)(t_1) = (Qy)(t_1) \text{ and } (Qx)(t) \le (Qy)(t) \quad t_0 \le t < t_1 < T.$$

Then,

$$v(t) < w(t), \quad t_0 \le t < T \tag{2.3}$$

whenever

$$v(t_0) < w(t_0) \tag{2.4}$$

provided one of the inequalities in (2.2) is strict.

Proof. Suppose that the claim (2.3) is false and $w(t) > (Qw)(t)$. Then because of the continuity of the functions involved and (2.4), there would exist a $t_1 > t_0$ such that

$$v(t_1) = w(t_1), \quad v(t) < w(t), \quad t_0 \le t < t_1. \tag{2.5}$$

Since Q is assumed to be semi-nondecreasing, we have

$$v(t_1) \le (Qv)(t_1) \le (Qw)(t_1) < w(t_1).$$

This contradicts $v(t_1) = w(t_1)$ and therefore, proves the claim (2.3). The proof is complete.

Remark 2.2.1. We have utilized the semi-nondecreasing nature of the causal operator Q in Theorem 2.2.1. This is because the causal operators include ordinary differential equations as well, where we do not require any monotone character. To incorporate this aspect, we have coined a weaker notion of semi-nondecreasing.

The next result is for nonstrict inequalities which demands a one-sided Lipschitz condition.

Theorem 2.2.2. Suppose that the assumptions of Theorem 2.2.1 hold. Assume further that

$$(Qx)(t) - (Qy)(t) \le L \max_{t_0 \le s \le t} [x(s) - y(s)] \tag{2.6}$$

whenever $x(s) \ge y(s)$ for $t_0 \le s \le t$ and $0 < L < 1$. Then

$$v(t_0) \le w(t_0) \text{ implies } v(t) \le w(t), \quad t_0 \le t < T. \tag{2.7}$$

Proof. Set $w_\varepsilon(t) = w(t) + \varepsilon$, $\varepsilon > 0$ being arbitrary and small. Then we have

$$w_\varepsilon(t_0) \ge w(t_0) + \varepsilon \ge v(t_0) + \varepsilon > v(t_0)$$

and

$$w_\varepsilon(t) \ge w(t), \quad t_0 \le t < T.$$

Now, utilizing the one-sided Lipschitz condition (2.6), we get

$$w_\varepsilon(t) \ge (Qw)(t) + \varepsilon \ge (Qw_\varepsilon)(t) - L\varepsilon + \varepsilon > (Qw)(t)$$

for $t_0 \le t < T$, because of the condition $0 < L < 1$. Now, applying Theorem 2.2.1 to $v(t)$ and $w_\varepsilon(t)$, we find that

$$v(t) < w_\varepsilon(t), \quad t_0 \le t < T.$$

Since $\varepsilon > 0$ is arbitrary, it follows by taking $\varepsilon \to 0$, that

$$v(t) \le w(t), \quad t_0 \le t < T,$$

proving the stated conclusion (2.7).

We shall now consider the fundamental result on causal differential inequalities. For this purpose, consider the IVP for the causal differential equation

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \tag{2.8}$$

where $Q : E \to E$ is a continuous causal operator. Relative to (2.8) we can prove the following basic inequalities result.

Theorem 2.2.3. Suppose that $v,w \in C([t_0,T],\mathbb{R})$ and

$$v'(t) \le (Qv)(t), \quad w'(t) \ge (Qw)(t), \quad t_0 \le t < T. \tag{2.9}$$

Then, $v(t_0) < w(t_0)$ implies

$$v(t) < w(t), \quad t_0 \le t < T, \tag{2.10}$$

provided one of the inequalities in (2.9) is strict and the causal operator Q is semi-nondecreasing.

Proof. Suppose that the conclusion (2.10) is false and $w'(t) > (Qw)(t)$. Then, the continuity of v,w and the fact $v(t_0) < w(t_0)$ yield that there exists a $t_1 > t_0$ such that

$$v(t_1) = w(t_1), \quad v(t) < w(t) \text{ on } t_0 \le t < t_1. \tag{2.11}$$

The semi-nondecreasing nature of Q and (2.11) give

$$(Qv)(t_1) \le (Qw)(t_1). \tag{2.12}$$

In view of (2.11), we get for small $h > 0$,

$$v(t_1-h) - v(t_1) < w(t_1-h) - w(t_1)$$

and hence (2.9) and (2.12) show that

$$(Qv)(t_1) \ge v'(t_1) \ge w'(t_1) > (Qw)(t_1) \ge (Qv)(t_1).$$

This is a contradiction and therefore the claim (2.10) is valid. The proof is complete.

As before, for nonstrict differential inequalities, we require a one-sided Lipschitz condition.

Theorem 2.2.4. Under the assumptions of Theorem 2.2.3 and the one-sided Lipschitz condition (2.6) of Theorem 2.2.2,

$$v(t) \le w(t), \quad t_0 \le t < T, \tag{2.13}$$

provided $v(t_0) \le w(t_0)$.

Proof. We set $w_\varepsilon(t) = w(t) + \varepsilon \exp(2L(t-t_0))$ for small $\varepsilon > 0$ so that we have

$$w_\varepsilon(t_0) > w(t_0) \text{ and } w_\varepsilon(t) \ge w(t).$$

Now use the one-sided Lipschitz condition

$$(Qw_\varepsilon)(t) - (Qw)(t) \le L \max_{t_0 \le s \le t} (w_\varepsilon(s) - w(s)) = L\varepsilon \exp(2L(t-t_0))$$

to obtain

$$\begin{aligned} w_\varepsilon'(t) &\ge w'(t) + 2\varepsilon L \exp(2L(t-t_0)) \\ &\ge (Qw)(t) + 2\varepsilon L \exp(2L(t-t_0)) \\ &\ge (Qw_\varepsilon)(t) - L\varepsilon \exp(2L(t-t_0)) + 2\varepsilon L \exp(2L(t-t_0)) \\ &> (Qw_\varepsilon(t)). \end{aligned}$$

Applying Theorem 2.2.3 to $v(t)$ and $w_\varepsilon(t)$, we have

$$v(t) < w_\varepsilon(t), \quad t_0 \le t < T$$

which yields as $\varepsilon \to 0$, $v(t) \le w(t)$, $t_0 \le t < T$. The proof is complete.
As an application of Theorem 2.2.1, consider the example

$$x(t) = (Qx)(t) = h(t) + \int_{t_0}^{t} K(t,s,x(s))ds,$$

where $K \in C([t_0,T) \times \mathbb{R},\mathbb{R})$, $h \in C([t_0,T),\mathbb{R})$. Assume $K(t,s,x)$ is monotone nondecreasing in x for each (t,s) and

$$x(t) \le h(t) + \int_{t_0}^{t} K(t,s,x(s))ds = (Qx)(t),$$

$$y(t) \ge h(t) + \int_{t_0}^{t} K(t,s,y(s))ds = (Qy)(t),$$

one of the inequalities being strict. Then $x(t_0) < y(t_0)$ implies $x(t) < y(t)$, $t \in [t_0,T)$. If one assumes further

$$|K(t,s,x) - K(t,s,y)| \le L|x-y|, \quad 0 < L < 1,$$

which leads to

$$|(Qx)(t) - (Qy)(t)| \le L \max_{t_0 \le s \le t} |x(s) - y(s)|$$

and then Theorem 2.2.2 is valid, which gives a result concerning non-strict inequalities.

2.3 Existence and Extremal Solutions

We shall consider the causal functional equation and causal differential equation respectively given by

$$x(t) = (Qx)(t), \quad x(t_0) = x_0 \tag{2.14}$$

and

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0 \tag{2.15}$$

where $Q : E \to E$, $E = C([t_0,T],\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$. The problem (2.15) can be reduced to

$$x(t) = x_0 + \int_{t_0}^{t} (Qx)(s)ds. \tag{2.16}$$

We shall discuss the existence problem for both (2.14) and (2.15). However, we will first deal with the functional equation with causal operator (2.14) by proving the following existence theorem.

Theorem 2.3.1. Assume that the causal operator $Q: E \to E$ in equation (2.14) is continuous and compact, as well as the property of the fixed initial value $x(t_0) = x_0 \in \mathbb{R}^n$. Then there exists a solution of (2.14) on $C([t_0, t_0+\delta], \mathbb{R}^n)$ for some $\delta > 0$ and $t_0 + \delta \leq T$.

Proof. For any $x \in C([t_0, T], \mathbb{R}^n)$, consider the initial value $x(t_0) = x_0$ and the functional equation

$$x(t) - x_0 = (Qx)(t) - x_0 \tag{2.17}$$

which is equivalent to (2.14). In view of the compactness assumption of Q, if we fix an $r > 0$, one can write

$$|(Qx)(t) - x_0| \leq \beta$$

for $t_0 \leq t \leq t_0 + \delta$ and for all $x \in C([t_0, T], \mathbb{R}^n)$ satisfying

$$|x(t) - x_0| \leq r, \quad t \in [t_0, T].$$

If r is fixed, then δ depends on β. We can assume without loss of generality that $\beta \leq r$. Then (2.17) shows that

$$|(Qx)(t) - x_0| \leq \beta \leq r,$$

for all $x(t)$ with $|x(t) - x_0| \leq r$, on $[t_0, t_0+\delta]$. This means that in the space $E = C([t_0, t_0 + \delta], \mathbb{R}^n)$, the ball of radius r with center x_0 is taken into itself by Q. Since Q is assumed to be continuous and compact, Schauder's fixed point theorem applies in the Banach space $C([t_0, t_0+\delta], \mathbb{R}^n)$ and the ball $|x(t) - x_0| \leq r$ is clearly convex and closed. Hence there exists at least one solution $x(t)$ of (2.14) on the interval $[t_0, t_0+\delta]$. The proof is complete.

We shall next prove the existence result for IVP (2.15) which is equivalent to (2.16).

Theorem 2.3.2. Consider the causal differential equation (2.15) with the initial condition. Suppose that Q is a causal operator on $C([t_0, T], \mathbb{R}^n)$, continuous and takes bounded sets into bounded sets. Then, there exists a solution $x(t)$ of (2.15) on $[t_0, t_0+\delta]$, $t_0 + \delta \leq T$.

Proof. Since the IVP (2.15) is equivalent to the functional equation (2.16), it is enough to prove that the operator W given by

$$(Wx)(t) = x_0 + \int_{t_0}^{t} (Qx)(s)ds \tag{2.18}$$

satisfies the conditions of Theorem 2.3.1. The continuity and causality of the operator W is clear. Since the convergence in E is uniform convergence, the continuity of Q implies the continuity of W. To prove the compactness of W in E, suppose that $x \in B \subset E$ where B is a bounded set in E, with Q bounded in E. In view of the assumption that Q takes bounded sets into bounded sets, for each $x \in B$ we have

$$|(Qx)(t)| \leq K, \quad t \in [t_0, T].$$

Therefore, for each $x \in B$, one has

$$|W(x)(t)| \leq |x_0| + K(t - t_0), \quad t \in [t_0, T]$$

and

$$|(Wx)(t) - (Wx)(s)| \leq K|t - s|, \quad t, s \in [t_0, T].$$

These inequalities show that the subset of E consisting of those functions that are of the form $(Wx)(t)$, $x \in B$, satisfies the conditions of Ascoli-Arzela Theorem and hence W is compact. Hence, by Theorem 2.3.1, there exists a solution $x(t)$ of (2.16) on $[t_0, t_0 + \delta]$ and equivalently of (2.15). Hence the theorem is proved.

We shall now discuss the existence of extremal solutions for the IVP (2.15) i.e. maximal and minimal solutions for (2.15). We will do this for the scalar case only.

Definition. Let $r(t)$ be a solution of IVP (2.15) on $[t_0, T]$. Then $r(t)$ is said to be the maximal solution of (2.15) if for every solution $x(t)$ of (2.15) existing on $[t_0, T)$, we have $x(t) \leq r(t)$, $t_0 \leq t < T$.

Similarly, if $p(t)$ is a solution of (2.15) and for every solution $x(t)$ of (2.15) existing on $[t_0, T)$, we have $x(t) \geq p(t)$, $t_0 \leq t < T$, then $p(t)$ is said to be the minimal solution of (2.15).

Theorem 2.3.3. Let the assumptions of Theorem 2.3.2 hold and suppose further that Q is semi-nondecreasing on $[t_0, T]$. Then, there exists extremal solutions for IVP (2.15).

Proof. We shall indicate the proof of existence for maximal solution only. Consider, for some arbitrary small $\varepsilon > 0$, the IVP

$$x'(t) = (Qx)(t) + \varepsilon, \quad x(t_0) = x_0 + \varepsilon \tag{2.19}$$

so that we have the corresponding operator equation equivalent to (2.19)

$$(Wx)(t) = x_0 + \varepsilon + \int_{t_0}^{t} (Qx)(s)ds + \varepsilon(t - t_0).$$

By Theorem 2.3.2, there exists an $\eta > 0$ such that there is a solution $x(t, \varepsilon) \equiv x_\varepsilon(t)$ on $[t_0, t_0 + \eta]$ for IVP (2.19). Let $0 < \varepsilon_2 < \varepsilon_1 \leq \varepsilon$. Then,

$$x(t_0, \varepsilon_2) < x(t_0, \varepsilon_1,)$$

$$x'(t, \varepsilon_2) \leq (Qx_{\varepsilon_2})(t) + \varepsilon_2,$$

$$x'(t, \varepsilon_1) > (Qx_{\varepsilon_1})(t) + \varepsilon_2.$$

An application of Theorem 2.2.3 yields

$$x(t, \varepsilon_2) < x(t, \varepsilon_1), \quad t \in [t_0, t_0 + \eta].$$

The family of functions $\{x(t,\varepsilon)\}$ are equicontinuous and uniformly bounded on $[t_0, t_0+\eta]$. Arguing as in Theorem 2.3.2, it follows from Ascoli-Arzela Theorem that there exists a decreasing sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, and the uniform limit

$$r(t) = \lim_{n\to\infty} x(t,\varepsilon_n)$$

exists on $[t_0, t_0+\eta]$. It can be easily shown that $r(t)$ is a solution of (2.15). To show that $r(t)$ is the desired maximal solution of (2.15) on $[t_0, t_0+\eta]$, let $x(t)$ be any solution of (2.15) defined on $[t_0, t_0+\eta]$. Then by Theorem 2.2.3, it follows that for any $\varepsilon > 0$,

$$x(t) < x(t,\varepsilon), \quad t_0 \le t \le t_0+\eta.$$

The uniqueness of the maximal solution shows that $x(t,\varepsilon)$ tends uniformly to $r(t)$ on $[t_0, t_0+\eta]$ and therefore the proof is complete.

An immediate consequence of Theorem 2.3.3 is the following comparison theorem.

Theorem 2.3.4. Assume that the conditions of Theorem 2.3.3 hold and suppose that $m \in C^1([t_0,T],\mathbb{R}_+)$ satisfies

$$m'(t) \le (Qm)(t), \quad m(t_0) \le x_0.$$

Then, we have

$$m(t) \le r(t), \quad t \in [t_0,T],$$

where $r(t)$ is the maximal solution of (2.15) existing on $[t_0,T]$.

Proof. Let $x(t,\varepsilon) \equiv x_\varepsilon(t)$, for $\varepsilon > 0$, be any solution of

$$x'(t,\varepsilon) = (Qx_\varepsilon)(t)+\varepsilon, \quad x(t_0) = x_0+\varepsilon.$$

Then, by Theorem 2.2.3, we get

$$m(t) < x(t,\varepsilon), \quad t \in [t_0,T]$$

from which it follows that $m(t) \le r(t)$ on $[t_0,T]$ and the proof is complete.

Another variant of comparison theorem is the following result.

Theorem 2.3.5. Consider the IVP (2.15) where the causal operator Q satisfies the inequality

$$|(Qx)(t)| \le g\left(t, \max_{t_0\le s\le t} |x(s)|\right) \quad t \in [t_0,T)$$

where $g \in C([t_0,T)\times\mathbb{R}_+,\mathbb{R}_+)$ and $g(t,u)$ is monotone nondecreasing in u for each t. Suppose that $r(t)$ is the maximal solution of the scalar initial value problem

$$u' = g(t,u), \quad u(t_0) = u_0 \ge 0 \tag{2.20}$$

existing on $[t_0, T)$. Then

$$|x(t)| = |x(t, t_0, x_0)| \le r(t) \tag{2.21}$$

whenever $|x_0| \le u_0$ and $r(t, t_0, u_0) = r(t)$ is the maximal solution of the scalar IVP (2.20) and $x(t)$ is any solution of the causal differential equation (2.15).

Proof. Let $m(t) = |x(t, t_0, x_0)|$. Then,

$$\begin{aligned} D_- m(t) &= \liminf_{h \to 0^-} \frac{1}{h}[m(t+h) - m(t)] \\ &\le |x'(t)| = |(Qx)(t)| \\ &\le g\left(t, \max_{t_0 \le s \le t} m(s)\right). \end{aligned}$$

To prove the stated inequality (2.21), it is enough to show that

$$m(t) < u(t, \varepsilon) \tag{2.22}$$

where $u(t, \varepsilon)$ is any solution of

$$u'(t) = g(t, u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon,$$

$\varepsilon > 0$ being sufficiently small, since $\lim_{\varepsilon \to 0} u(t, \varepsilon) = r(t)$.

If (2.22) is not true, there exists a $t_1 > t_0$ such that

$$m(t_1) = u(t_1, \varepsilon) \text{ and } m(t) < u(t, \varepsilon), \text{ on } [t_0, t_1).$$

This yields

$$D_- m(t_1) \ge u'(t_1, \varepsilon) = g(t_1, u(t_1, \varepsilon)) + \varepsilon. \tag{2.23}$$

Since $g(t, u) \ge 0$, $u(t, \varepsilon)$ is nondecreasing in t and this implies

$$\max_{t_0 \le s \le t_1} m(s) = u(t_1, \varepsilon) = m(t_1)$$

and we get the inequality

$$D_- m(t_1) \le g\left(t_1, \max_{t_0 \le s \le t_1} m(s)\right) = g(t_1, u(t_1, \varepsilon))$$

which contradicts (2.23). Hence the theorem is proved.

Remark. We observe that in the proof of Theorem 2.3.5, we get the contradiction at t_1 where $\max_{t_0 \le s \le t_1} m(s) = m(t_1)$ and therefore, it is enough to assume that

$$|(Qx)(t)| \le g\left(t, \max_{t_0 \le s \le t} |x(s)|\right) \tag{2.24}$$

holds only on

$$\Omega = \{x \in C([t_0, T], \mathbb{R}^n) : \max_{t_0 \le s \le t} |x(s)| = |x(t)|\},$$

instead of requiring the inequality (2.24) for all $x \in C([t_0,T],\mathbb{R}^n)$. Hereafter, we will employ this observation and modify the assumption (2.24) to hold only for $x \in \Omega$, whenever we refer to Theorem 2.3.5.

Let us apply existence result of theorem 2.3.1 to the Volterra integral equation

$$x(t) = f(t) + \int_{t_0}^{t} K(t,s,x(s))ds = (Qx)(t), \quad t \in [t_0,T),$$

where f and K satisfy the following assumptions:

(i) $f \in C([t_0,T,\mathbb{R}^n])$;

(ii) $K \in C([t_0,T] \times [t_0,T] \times \mathbb{R}^n, \mathbb{R}^n)$.

Then, there exists a $\delta > 0$ such that $t_0 + \delta \leq T$ and the integral equation $x(t) = (Qx)(t)$ admits a solution in $C([t_0,t_0+\delta],\mathbb{R}^\ltimes)$.

The proof is a consequence of the fact that the operator Q satisfies the hypotheses of Theorem 2.3.1. Only compactness of operator Q needs a little explanation, since continuity and causality are obvious.

Assume that $x(t) \in B \subset C([t_0,t_0+\delta],\mathbb{R}^n) = C$, with set B being bounded in C. This means that we can find $M > 0$ such that $|x(t)| \leq M$, $t \in [t_0,T]$ and $x \in B$. The set $J \times J \times B$, where $J = [t_0,T]$, is compact, i.e. it is bounded and closed in $\mathbb{R}^{n+2}$. This implies boundedness of K on that set, $|K(t,s,x)| \leq A$ for some $A > 0$, as well as its uniform continuity. One can derive for $x \in B$,

$$|(Qx)(t)| \leq \sup|f(t)| + A(T-t_0), \quad t \in [t_0,T]$$

and

$$|(Qx)(t+h)-(Qx)(t)| \leq |f(t+h)-f(t)|$$

$$+\int_{t_0}^{t} |K(t+h,s,x(s)) - K(t,s,x(s))|ds + \int_{t}^{t+h} |K(t+h,s,x(s))|ds.$$

the first inequality proves the uniform boundedness of the set $\{(Qx) : x \in B\}$, while the second leads to the conclusion that the set is uniformly equicontinuous on $[t_0,T]$. Ascoli-Arzela theorem can be applied to get compactness of the operator Q.

2.4 Global Existence

Consider the IVP for causal differential equation

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \tag{2.25}$$

where $Q \in E = C([t_0,\infty),\mathbb{R}^n)$ is a continuous causal operator. We shall first employ Tychonoff's fixed point theorem to get global existence of solutions of (2.25).

Theorem 2.4.1. Let $Q \in E$ and satisfy the estimate

$$|(Qx)(t)| \le g(t,|x(t)|), \quad x \in \Omega, \tag{2.26}$$

where $\Omega = \{x \in E : \max_{t_0 \le s \le t} |x(s)| = |x(t)|\}$ and $g \in C([t_0,\infty)\times\mathbb{R}_+,\mathbb{R}_+)$, $g(t,u)$ is monotone nondecreasing in u for each $t \in [t_0,\infty)$. Assume that, for every $u_0 > 0$, the scalar differential equation

$$u' = g(t,u), \quad u(t_0) = u_0 \ge 0, \tag{2.27}$$

has a solution $u(t)$ existing on $[t_0,\infty)$. Then for every x_0 such that $|x_0| \le u_0$, there exists a solution $x(t)$ of (2.25) on $[t_0,\infty)$ satisfying

$$|x(t)| \le u(t), \quad t \in [t_0,\infty).$$

Proof. Consider the space E, the topology on E being that induced by the family of pseudonorms $\{p_n(x)\}_{n=1}^{\infty}$, where for $x \in E$,

$$p_n(x) = \sup_{t_0 \le t < n} |x(t)|.$$

A fundamental system of neighborhoods is then given by $\{V_n(x)\}_{n=1}^{\infty}$ where

$$V_n(x) = \{x \in E : p_n(x) \le 1\}.$$

Under this topology, E becomes a complete, locally convex linear space. Now, define a subset E_0, $E_0 \subset E$ as follows:

$$E_0 = \{x \in \Omega : |x(t)| \le u(t),\ t \ge t_0\}, \tag{2.28}$$

where $u(t)$ is a solution of (2.27) existing on $[t_0,\infty)$. It is clear that, in the topology of E, the set E_0 is closed, convex and bounded. Consider the integral operator defined by

$$(Tx)(t) = x_0 + \int_{t_0}^{t} (Qx)(s)ds, \tag{2.29}$$

whose fixed point corresponds to the solution of (2.25). This operator T is compact in the topology of E and therefore closure of TE_0 is compact in view of the boundedness of E_0. To prove the theorem, it remains to be shown that $TE_0 \subset E_0$. To this end, we observe that for any $x \in E_0$

$$\begin{aligned} |(Tx)(t)| &\le |x_0| + \int_{t_0}^{t} |(Qx)(s)|ds \\ &\le |x_0| + \int_{t_0}^{t} g(s,|x(s)|)ds \\ &\le |x_0| + \int_{t_0}^{t} g(s,u(s))ds, \end{aligned} \tag{2.30}$$

because of (2.29), (2.26) and (2.28). Here, we have used the monotone nature of $g(t,u)$, the definition of set E_0 and the fact that $u(t)$ is a solution of (2.27) with $|x_0| \le u_0$. It then follows from (2.30) that

$$|(Tx)(t)| \le u(t),$$

proving that $TE_0 \subset E_0$ and the proof is complete.

A direct proof of global existence is given by the following result.

Theorem 2.4.2. Assume that $Q \in E$ and is smooth enough to guarantee local existence of solutions of (2.25) for any (t_0, x_0) and satisfies the inequality (2.26) for $x \in \Omega$, where $g \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, $g(t,u)$ is nondecreasing in u for each $t \in [t_0, \infty)$. Suppose further that the maximal solution $r(t)$ of the scalar equation (2.27) exists on $[t_0, \infty)$. Then the largest interval of existence of any solution $x(t)$ of (2.25) such that $|x_0| \le u_0$ is $[t_0, \infty)$.

Proof. Let $x(t)$ by any solution of (2.25) with $|x_0| \le u_0$, which exists on $[t_0, \beta)$, for $t_0 < \beta < \infty$ and such that the value of β cannot be increased. Define

$$m(t) = |x(t)|, \quad t_0 \le t < \beta. \tag{2.31}$$

Then, using the assumption (2.26), we obtain

$$D^+ m(t) \le |x'(t)| = |(Qx)(t)| \le g(t, |x(t)|)$$

i.e. we have the differential inequality

$$D^+ m(t) \le g(t, m(t)), \quad m(t_0) \le u_0.$$

Hence, by Theorem 2.3.5, we have

$$m(t) \le |x(t)| \le r(t), \quad t_0 \le t < \beta,$$

where $r(t)$ is the maximal solution of (2.27). For any t_1, t_2 such that $t_0 \le t_1 < t_2 < \beta$,

$$\begin{aligned} |x(t_1) - x(t_2)| &= \left| \int_{t_1}^{t_2} (Qx)(s)ds \right| \\ &\le \int_{t_1}^{t_2} g(s, m(s))ds \\ &\le \int_{t_1}^{t_2} g(s, r(s))ds \\ &= r(t_2) - r(t_1). \end{aligned} \tag{2.32}$$

Here we used (2.26), (2.31) and the monotone character of $g(t,u)$. Since $\lim_{t\to\beta^-} r(t)$ exists and is finite, taking limits as $t_1, t_2 \to \beta^-$ and using Cauchy criterion for convergence, it follows that $\lim_{t\to\beta^-} x(t)$ exists. We define

$$x(\beta) = \lim_{t\to\beta^-} x(t)$$

and consider the IVP

$$x'(t) = (Qx)(t)$$

with initial value $x(\beta)$. By assumed local existence, we find that $x(t)$ can be continued beyond β, contradicting our assumption about its interval of existence. Hence, every solution of (2.25) with $|x_0| \leq u_0$ exists on $[t_0, \infty)$ and the proof is complete.

Corollary 2.4.1. Assume that the causal operator Q in IVP (2.15) satisfies

$$|(Qx)(t)| \leq \lambda(t)|x(t)|, \quad x \in \Omega, \tag{2.33}$$

in place of (2.26), where $\lambda(t) \geq 0$ is continuous on $[t_0, \infty)$, $g(u) \geq 0$ is continuous for $u \geq 0$, $g(0) = 0$, $g(u) > 0$ for $u > 0$ and $g(u)$ is nondecreasing in u. If $g(u)$ satisfies

$$\int_{u_0}^{\infty} \frac{du}{g(u)} = \infty \tag{2.34}$$

for $u_0 > 0$, then for every $x_0 \in \mathbb{R}^n$, there exists a solution of (2.25) for $t \geq t_0$.

Proof. The result follows from Theorem 2.4.1 if we show that the scalar differential equation

$$u' = \lambda(t)g(u), \quad u(t_0) = u_0 > 0 \tag{2.35}$$

has a solution existing for $t \geq t_0$. If we write

$$G(u) = \int_{u_0}^{u} \frac{du}{g(u)} = \int_{t_0}^{t} \lambda(s)ds,$$

it is easily seen that the function $G(u)$ is strictly increasing in u and hence, its inverse exists. In view of the assumptions concerning g, the domain of the inverse function is $[0, \infty)$ and therefore, the solution $u(t)$ of (2.35) is defined for $t \geq t_0$.

2.5 Existence and Uniqueness

Consider the IVP for causal differential equation

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \tag{2.36}$$

where $Q \in E = C(J, \mathbb{R}^n)$, $Q : E \to E$ and $J = [t_0, t_0 + a]$. This IVP is equivalent to

$$x(t) = x_0 + \int_{t_0}^{t} (Qx)(s)ds, \quad t \in J. \tag{2.37}$$

We shall first discuss the uniqueness result for (2.36) under Lipschitz condition and apply the contraction mapping theorem to get local existence and uniqueness.

Theorem 2.5.1. Suppose that Q satisfies

$$|(Qx)(t) - (Qy)(t)| \leq L|x(t) - y(t)|, \quad x, y \in \Omega, \; L > 0, \tag{2.38}$$

where

$$\Omega = \{x, y \in E : \max_{t_0 \le s \le t} |x(s) - y(s)| = |x(t) - y(t)|\}.$$

Then, there exists a unique solution $x(t)$ of IVP (2.36) on J if $a < \frac{1}{L}$.

Proof. Define the metric

$$|x - y|_0 = \max_{t_0 \le t \le t_0 + a} |x(s) - y(s)|$$

for all $x, y \in E$. For any $x \in E$, define operator T on J by the relation

$$Tx(t) = x_0 + \int_{t_0}^{t} (Qx)(s)ds,$$

so that $Tx \in E$. Using (2.38), we get for $x, y \in \Omega$,

$$\begin{aligned} |Tx(t) - Ty(t)| &\le \int_{t_0}^{t} |(Qx)(s) - (Qy)(s)|ds \\ &\le L \int_{t_0}^{t} \max_{t_0 \le s \le t} |x(s) - y(s)|ds \\ &\le aL|x - y|_0 \end{aligned} \tag{2.39}$$

which implies

$$|Tx - Ty|_0 \le La|x - y|_0. \tag{2.40}$$

Then, if $La < 1$, T is a contraction and the contraction mapping principle assures that there exists a unique fixed point of T, say x^* which shows that $x^*(t)$ is the unique solution of IVP (2.36) on J. The proof is complete.

If, on the other hand, we use the weighted metric,

$$|x - y|_* = \max_{t_0 \le s \le t_0 + a} |x(s) - y(s)|e^{-\lambda s}, \quad \lambda > 0,$$

and λ to be chosen later suitably, we get from (2.39)

$$|Tx(t) - Ty(t)| \le L \int_{t_0}^{t} \max_{t_0 \le s \le t} |x(s) - y(s)|e^{-\lambda s} e^{\lambda s} ds \le L|x - y|_* \int_{t_0}^{t} e^{\lambda s} ds$$

so that

$$e^{-\lambda t}|Tx(t) - Ty(t)| \le Le^{-\lambda t}|x - y|_* \frac{e^{\lambda t}}{\lambda} = \frac{L}{\lambda}|x - y|_*$$

and

$$|Tx - Ty|_* \le \frac{1}{2}|x - y|_*$$

by choosing $\lambda = 2L$. Now, the contraction mapping principle shows that there exists a unique fixed point of T, say x^*, which is the unique solution of IVP (2.36). This approach avoids restriction $La < 1$.

We shall next discuss the convergence of successive approximations for IVP (2.36) under a condition more general than (2.38) and the proof is very instructive.

Theorem 2.5.2. Assume that the following hypotheses hold:

(i) $Q \in E = C(\mathbb{R}_0, \mathbb{R}^n)$ where $\mathbb{R}_0 = \{(t,x) : t \in J, |x - x_0|_0 \leq b\}$ and $|(Qx)(t)| \leq M_0$ on $\mathbb{R}_0$;

(ii) $g \in C(J \times [0,2b], \mathbb{R}_+)$, $0 \leq g(t,u) \leq M_1$ on $J \times [0,2b]$, $g(t,0) \equiv 0$, $g(t,u)$ is nondecreasing in u for each t and $u(t) \equiv 0$ is the unique solution of the scalar IVP

$$u' = g(t,u), \quad u(t_0) = 0, \quad t \in J; \tag{2.41}$$

(iii) whenever $x, y \in \Omega$,

$$|(Qx)(t) - (Qy)(t)| \leq g(t, |x(t) - y(t)|) \text{ on } \mathbb{R}_0. \tag{2.42}$$

Then the successive approximations defined by

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} (Qx_n)(s)ds, \quad n = 0,1,2,..., \tag{2.43}$$

on $[t_0, t_0 + \alpha]$ where $\alpha = \min(a, b/M)$, $M = \max\{M_0, M_1\}$, converge uniformly to the unique solution $x(t)$ of IVP (2.36).

Proof. It is easy to see, by induction, that the successive approximations (2.43) are defined and continuous on $[t_0, t_0 + \alpha]$ and

$$|x_n(t) - x_0| \leq b, \quad n = 0,1,2,\ldots$$

We shall now define the successive approximations for IVP (2.41) as follows:

$$\begin{cases} u_0 = M(t - t_0) \\ u_{n+1}(t) = \int_{t_0}^{t} g(s, u_n(s))ds, \quad t \in [t_0, t_0 + \alpha]. \end{cases} \tag{2.44}$$

An easy induction proves that the successive approximations (2.44) are well defined and satisfy

$$0 \leq u_{n+1}(t) \leq u_n(t), \quad \text{on } [t_0, t_0 + \alpha].$$

Since $|u_n'(t)| \leq M_1$, we conclude by Ascoli-Arzela Theorem and the monotonocity of the sequence $\{u_n(t)\}$ that

$$\lim_{n \to \infty} u_n(t) = u(t), \quad t \in [t_0, t_0 + \alpha]$$

uniformly. It is clear that $u(t)$ satisfies (2.41) and hence by (ii), $u(t) \equiv 0$ on $[t_0, t_0 + \alpha]$ by Lemma 1.3.4. Now,

$$|x_1(t) - x_0| \leq \int_{t_0}^{t} (Qx_0)(s)ds \leq M(t - t_0) \equiv u_0(t).$$

Assume that for some fixed integer k,

$$|x_k(t) - x_{k-1}(t)| \leq u_{k-1}(t).$$

Since

$$|x_{k+1}(t)-x_k(t)| \le \int_{t_0}^{t} |(Qx_k)(s)-(Qx_{k-1})(s)|ds,$$

using the nondecreasing nature of $g(t,u)$ in u and the assumption (iii), we get

$$|x_{k+1}(t)-x_k(t)| \le \int_{t_0}^{t} g(s,u_{k-1}(s))ds \equiv u_k(t),$$

in view of (2.44) and Theorem 2.3.5. Thus, by principle of induction, the inequality

$$|x_{k+1}(t)-x_n(t)| \le u_n(t), \quad t \in [t_0, t_0+\alpha] \tag{2.45}$$

is true for all n. Also,

$$\begin{aligned} |x'_{n+1}(t)-x'_n(t)| &\le |(Qx_n)(t)-(Qx_{n-1})(t)| \\ &\le g(t,|x_n(t)-x_{n-1}(t)|) \\ &\le g(t,u_{n-1}(t)), \end{aligned} \tag{2.46}$$

because of (2.45) and the nondecreasing character of $g(t,u)$. Let $n \le m$. Then one can easily obtain using (2.46),

$$\begin{aligned} |x'_n(t)-x'_m(t)| &= |Qx_{n-1}(t)-(Qx_{m-1})(t)| \\ &\le |(Qx_n)(t)-(Qx_{n-1})(t)|+|(Qx_m)(t)-(Qx_{m-1})(t)| \\ &\quad +|(Qx_n)(t)-(Qx_m)(t)| \\ &\le g(t,u_{n-1}(t))+g(t,u_{m-1}(t))+g(t,|x_n(t)-x_m(t)|). \end{aligned}$$

Since $u_{n+1}(t) \le u_n(t)$, it follows that

$$D^+|x_n(t)-x_m(t)| \le g(t,|x_n(t)-x_m(t)|)+2g(t,u_{n-1}(t))$$

where D^+ is the Dini derivative. An application of Theorem 1.4.1 in [4] gives

$$|x_n(t)-x_m(t)| \le r_n(t), \quad t \in [t_0,t_0+\alpha],$$

where $r_n(t)$ is the maximal solution of

$$v'_n = g(t,v_n)+2g(t,u_{n-1}(t)), \quad v_n(t_0)=0,$$

for each n. Since $g(t,u_{n-1}(t)) \to 0$, as $n \to \infty$, uniformly on $[t_0,t_0+\alpha]$, it follows by Lemma 1.3.4 that $r_n(t) \to 0$ uniformly on $[t_0,t_0+\alpha]$. This implies that $x_n(t)$ converges uniformly to $x(t)$ and it is now easy to show that $x(t)$ is a solution of (2.36) by standard arguments. To show that the solution is unique, let $y(t)$ be another solution of IVP (2.36) existing on $[t_0,t_0+\alpha]$. Define $m(t)=|x(t)-y(t)|$ and note $m(t_0)=0$. Then,

$$D^+m(t) \le |x'(t)-y'(t)| = |(Qx)(t)-(Qy)(t)|$$

$$\leq g(t,|x(t)-y(t)|) = g(t,m(t)),$$

whenever $x,y \in \Omega$, using assumption (iii). Again, applying Theorem 2.3.5, we get

$$m(t) \leq r(t), \quad t \in [t_0, t_0+\alpha],$$

where $r(t)$ is the maximal solution of IVP (2.41). But by assumption (ii), $r(t) \equiv 0$ and this proves that $x(t) \equiv y(t)$. Hence the limit of the successive approximations is the unique solution of IVP (2.36) and the proof is complete.

Let us now obtain an error estimate between the solution and an approximate solutions of IVP (2.36).

Definition. A function $y_\varepsilon(t) = y(t,\varepsilon)$, $\varepsilon > 0$, is said to be an approximate solution of IVP (2.36) with $y(t_0,\varepsilon) = y_0$, if $y(t,\varepsilon)$ satisfies the inequality

$$|y'(t,\varepsilon) - (Qy_\varepsilon)(t)| \leq \varepsilon, \quad t_0 \leq t \leq t_0+\alpha.$$

We can prove the following result which provides the desired estimate.

Theorem 2.5.3. Assume that Q in (2.36) satisfies

$$|(Qx)(t) - (Qy_\varepsilon)(t)| \leq g(t,|x(t)-y(t,\varepsilon)|)$$

where $g \in C([t_0,t_0+\alpha] \times \mathbb{R}_+, \mathbb{R}_+)$. Then

$$|x(t)-y(t,\varepsilon)| \leq r(t,t_0,u_0,\varepsilon), \quad t \in [t_0,t_0+\alpha],$$

where $r(t,t_0,u_0,\varepsilon)$ is the maximal solution of (2.41) with $u_0 = |x_0 - y_0|$.

Proof. As before let $m(t) = |x(t)-y(t,\varepsilon)|$. We obtain,

$$\begin{aligned} D^+m(t) &\leq |x'(t)-y'(t,\varepsilon)| \\ &\leq |x'(t)-(Qx)(t)| + |(Qy_\varepsilon)(t) - y'(t,\varepsilon)| \\ &\quad + |(Qx)(t)-(Qy_\varepsilon)(t)| \\ &\leq \varepsilon + g(t,|x(t)-y(t,\varepsilon)|) \\ &= g(t,m(t)) + \varepsilon, \quad \text{on } [t_0,t_0+\alpha]. \end{aligned}$$

This yields, by Theorem 2.3.5, the estimate

$$|x(t)-y(t,\varepsilon)| \leq r(t,t_0,|x_0-y_0|,\varepsilon), \quad t \in [t_0,t_0+\alpha].$$

If $g(t,u) = Lu$, then it is easy to get

$$|x(t)-y(t,\varepsilon)| \leq |x_0-y_0| \exp(L(t-t_0)) + \frac{\varepsilon}{L}(e^{L(t-t_0)}-1)$$

since the RHS is the solution of $u' = Lu$, $u_0 = |x_0-y_0|$.

2.6 Nagumo-Type Conditions

We continue to consider IVP (2.36) and prove the convergence of successive approximations to the unique solution when Nagumo-type conditions are assumed for the causal operator Q.

Theorem 2.6.1. Suppose that the hypothesis (i) and (ii) of Theorem 2.5.2 hold. Suppose further that for $x,y \in \Omega$,

$$|(Qx)(t)-(Qy)(t)| \le \frac{|x(t)-y(t)|}{t-t_0}, \quad t \neq t_0. \tag{2.47}$$

Then, the conclusion of Theorem 2.5.2 is true.

Proof. It is easy to show that the sequence defined by (2.43) is uniformly bounded and equi-continuous on $[t_0, t_0+\alpha]$ and hence, there exists uniformly convergent subsequences. Suppose that

$$x_n(t)-x_{n-1}(t) \to 0 \text{ as } n \to \infty,$$

then (2.43) implies that the limit of any such subsequence is the unique solution of (2.36). It then follows that a selection of the subsequence is unnecessary and that the full sequence $\{x_n(t)\}$ also converges uniformly on $[t_0, t_0+\alpha]$ to the unique solution.

To prove the conclusion of the theorem, it is therefore sufficient to show that $m(t) \equiv 0$, where

$$m(t) = \limsup_{n\to\infty} |x_n(t)-x_{n-1}(t)|. \tag{2.48}$$

We shall first show that $m(t)$ is continuous for $t_0 \le t \le t_0+\alpha$. Since we have $|(Qx)(t)| \le M_0$ on $\mathbb{R}_0$, we see that

$$|x_n(t_1)-x_{n-1}(t_1)| \le |x_n(t_2)-x_{n-1}(t_2)| + 2M_0|t_1-t_2|$$

$$\le m(t_2) + 2M_0|t_1-t_2| + \varepsilon$$

for large n, if $\varepsilon > 0$. Hence, we have

$$m(t_1) \le m(t_2) + 2M_0|t_1-t_2| + \varepsilon.$$

As t_1, t_2 can be interchanged and $\varepsilon > 0$ is arbitrary, we obtain

$$|m(t_1)-m(t_2)| \le 2M_0|t_1-t_2|,$$

which proves the continuity of $m(t)$. The assumption (2.47) together with the relation (2.43) yields

$$|x_{n+1}(t)-x_n(t)| \le \int_{t_0}^{t} |(Qx_n)(s)-(Qx_{n-1})(s)|ds$$

$$\leq \int_{t_0}^{t} (s-t_0)^{-1} |x_n(s) - x_{n-1}(s)| ds,$$

for $t \neq t_0$, $t \in [t_0, t_0+\alpha]$ and for those $x_n(t)$ which belong to Ω for a fixed t in $[t_0, t_0+\alpha]$, there is a sequence of integers $n_1 < n_2 < \ldots$ such that

$$|x_{n+1}(t) - x_n(t)| \to m(t)$$

as $n = n_k \to \infty$ and that

$$m^*(s) = \lim_{n=n_k \to \infty} |x_n(s) - x_{n-1}(s)|$$

exists uniformly on $[t_0, t_0+\alpha]$. Thus,

$$m(t) \leq \int_{t_0}^{t} (s-t_0)^{-1} m^*(s) ds. \tag{2.49}$$

Since g is assumed to be monotone nondecreasing in u and $m^*(s) \leq m(s)$, we obtain from (2.49) the inequality

$$m(t) \leq \int_{t_0}^{t} (s-t_0)^{-1} m(s) ds, \quad t \neq t_0. \tag{2.50}$$

Now,

$$\frac{m(t)}{t-t_0} = \frac{1}{t-t_0} \limsup_{n\to\infty} \int_{t_0}^{t} |x_n(s) - x_{n-1}(s)| ds,$$

and therefore as $t \to t_0$,

$$\begin{aligned}
\lim_{t\to t_0} \left(\frac{m(t)}{t-t_0} \right) &= \lim_{t\to t_0} \left[\frac{1}{t-t_0} \limsup_{n\to\infty} \int_{t_0}^{t} |x_n(s) - x_{n-1}(s)| ds \right] \\
&= \limsup_{n\to\infty} \lim_{t\to t_0} \left[\frac{1}{(t-t_0)} \int_{t_0}^{t} |x_n(s) - x_{n-1}(s)| ds \right] \\
&= \limsup_{n\to\infty} [|(Qx_{n-1})(t_0) - (Qx_{n-2})(t_0)|] = 0.
\end{aligned}$$

Setting $\psi(t) = \frac{m(t)}{t-t_0}$ and noting $\psi(t_0) = 0$, it is enough to show that $\psi(t) \equiv 0$. If this is not true, let

$$\beta = \max_{t_0 \leq t \leq t_0+\alpha} \psi(t) = \psi(t_1).$$

Then we get from (2.50) that

$$\beta \leq (t_1-t_0)^{-1} \int_{t_0}^{t_1} \frac{m(s)}{s-t_0} ds = (t_1-t_0)^{-1} \int_{t_0}^{t} \psi(s) ds$$

$$< \beta (t_1-t_0)^{-1} \int_{t_0}^{t_1} ds = \beta.$$

This is a contradiction and hence the proof is complete.

The next result is a generalization of Nagumo's Theorem known as Krasnoselskii-Krein's Theorem.

Theorem 2.6.2. Assume that hypothesis (i) and (ii) of Theorem 2.5.2 hold. Assume also that for $x, y \in \Omega$,

$$|(Qx)(t) - (Qy)(t)| \le \frac{K|x(t) - y(t)|}{t - t_0}, \quad t \neq t_0, K > 1 \tag{2.51}$$

and

$$|(Qx)(t) - (Qy)(t)| \le C|x(t) - y(t)|^{\alpha}, \quad 0 < \alpha < 1, \quad K(1 - \alpha) < 1. \tag{2.52}$$

Then the conclusion of Theorem 2.5.2 is valid.

Proof. Let $x(t), y(t)$ be solutions of (2.36) and $m(t) = |x(t) - y(t)|$. Then using (2.52), we have

$$m(t) \le \int_{t_0}^{t} C|x(s) - y(s)|^{\alpha} ds = \int_{t_0}^{t} C(m(s))^{\alpha} ds.$$

Set

$$R(t) = \int_{t_0}^{t} C(m(s))^{\alpha} ds,$$

so that

$$R'(t) = C(m(t))^{\alpha} \le C(R(t))^{\alpha}.$$

It is easy to see that $\frac{d}{dt}(R^{1-\alpha}(t)) \le C(1-\alpha) \le C$ and hence $(R(t))^{1-\alpha} \le C(t - t_0)$. This shows that

$$m(t) \le R(t) \le C_1(t - t_0)^{\frac{1}{1-\alpha}} \text{ for some } C_1.$$

Setting

$$\psi(t) = \frac{m(t)}{(t - t_0)^K},$$

the following inequality

$$0 \le \psi(t) \le C_1(t - t_0)^{\frac{1}{1-\alpha} - K} = C_1(t - t_0)^{\frac{1 - K(1-\alpha)}{1-\alpha}}$$

is satisfied. Since $K(1-\alpha) < 1$, it follows that $\psi(t_0) = 0$. It is now enough to show that $\psi(t) \equiv 0$ to prove the theorem. If not,

$$\begin{aligned}
\beta &= \max_{t_0 \le t \le t_0 + \alpha} \psi(t) = \psi(t_1) = \frac{m(t_1)}{(t_1 - t_0)^K} \\
&< (t_1 - t_0)^{-K} \int_{t_0}^{t_1} \frac{K|x(s) - y(s)|}{s - t_0} ds \\
&= (t_1 - t_0)^{-K} \int_{t_0}^{t_1} K\psi(s)(s - t_0)^{K-1} ds \\
&< \beta(t_1 - t_0)^{-K} K \int_{t_0}^{t_1} (s - t_0)^{K-1} ds < \beta,
\end{aligned}$$

using (2.51). This contradiction proves the uniqueness of solutions directly. In order to show the convergence of successive approximations, we can proceed as in Theorem 2.6.1 with minor modifications and therefore we do not give the proof. Note that, in this case, $K > 1$ where as in Nagumo's Theorem, $0 < K \leq 1$.

Consider the example

$$\left.\begin{aligned} &x'(t) = f(t,x) + \textstyle\int_{t_0}^{t} K(t,s,x(s))ds \equiv (Qx)(t), \\ &x(t_0) = x_0, \end{aligned}\right\} \tag{2.53}$$

where f and K satisfy

$$|f(t,x) - f(t,y)| \leq \frac{1}{2}\frac{(x-y)}{t-t_0}, \quad t \neq t_0,$$

and

$$|K(t,s,x) - K(t,s,y)| \leq \lambda(t,s)\frac{|x-y|}{t-t_0}, \quad t \neq t_0,$$

$\lambda(t,s)$ being continuous. Then,

$$\int_{t_0}^{t} |K(t,s,x(s)) - K(t,s,y(s))|ds \leq \int_{t_0}^{t} \frac{\lambda(t,s)}{(s-t_0)} \max_{t_0 \leq s \leq t} |x(s) - y(s)|ds.$$

When $x, y \in \Omega$, we have

$$\max_{t|0 \leq s \leq t} |x(s) - y(s)| = |x(t) - y(t)|$$

and if $\int_{t_0}^{t} \lambda(t,s) \leq \frac{1}{2}$, then we get

$$|(Qx)(t) - (Qy)(t)| \leq \frac{|x(t) - y(t)|}{t - t_0}, \quad t \neq t_0$$

which is the Nagumo condition for the causal operator Q. Theorem 2.6.1. can be applied and the uniqueness of solutions of (2.53) follows.

One can, similarly arrive at Krasnoselskii-Krein type conditions by imposing suitable conditions on f and K. We leave the details to the reader.

2.7 Continuous Dependence Relative to Initial Data

We shall consider the problem of continuity of solutions of the IVP

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \tag{2.54}$$

with respect to the initial values t_0, x_0. For this purpose, we need the following result.

Lemma 2.7.1. Let $Q : E \to E = C([t_0, T], \mathbb{R}^n)$ and let

$$G(t,r) = \max_{|x(t)-x_0| \leq r} |(Qx)(t)|.$$

Assume that $r^*(t,t_0,0)$ is the maximal solution of

$$u' = G(t,u), \quad u(t_0) = 0.$$

Let $x(t) = x(t,t_0,x_0)$ be any solution of (2.54). Then

$$|x(t) - x_0| \le r^*(t,t_0,0), \quad t \in [t_0,T].$$

Proof. Define $m(t) = |x(t) - x_0|$. Then,

$$D^+m(t) \le |x'(t)| = |(Qx)(t)| \le \max_{|x(t)-x_0|} |(Qx)(t)| \le G(t,m(t)).$$

This implies by Theorem 1.3.2 that

$$|x(t) - x_0| \le r^*(t,t_0,0) \text{ on } [t_0,T]$$

proving the lemma.

Theorem 2.7.1. Let $Q : E \to E$ and satisfy

$$|(Qx)(t) - (Qy)(t)| \le g(t,|x(t) - y(t)|), \quad x,y \in \Omega, \tag{2.55}$$

where $g \in C([t_0,T] \times \mathbb{R}_+,\mathbb{R}_+)$. Assume that $u(t) \equiv 0$ is the unique solution of the scalar differential equation

$$u' = g(t,u), \quad u(t_0) = u_0, \tag{2.56}$$

with $u_0 = 0$. Then, if the solutions $u(t,t_0,u_0)$ of (2.56) through (t_0,u_0) are continuous with respect to (t_0,u_0), then the solutions $x(t,t_0,x_0)$ of (2.54) are unique and continuous with respect to the initial values (t_0,x_0).

Proof. Since the uniqueness follows from Theorem 2.5.2, we have to prove the continuity part only. To this end, let $x(t) = x(t,t_0,x_0)$, $y(t) = y(t,t_0,y_0)$ be the solutions of (2.54) through (t_0,x_0), (t_0,y_0) respectively. Defining $m(t) = |x(t) - y(t)|$, we obtain from (2.55),

$$D^+m(t) \le g(t,m(t))$$

and by comparison Theorem 1.3.2, we have

$$m(t) \le r(t,t_0,|x_0 - y_0|), \quad t \in [t_0,T],$$

where $r(t,t_0,u_0)$, with $u_0 = |x_0 - y_0|$, is the maximal solution of (2.56). Since the solutions $u(t,t_0,u_0)$ of (2.56) are assumed to be continuous with respect to initial values, it follows that

$$\lim_{x_0 \to y_0} r(t,t_0,|x_0 - y_0|) = r(t,t_0,0),$$

and by hypothesis, $r(t,t_0,0) \equiv 0$. This, in view of definition of $m(t)$, shows that

$$\lim_{x_0 \to y_0} x(t,t_0,x_0) = y(t,t_0,y_0)$$

and the continuity of solutions relative to x_0 follows.

We shall next prove the continuity with respect to t_0. If $x(t,t_0,x_0)$, $y(t,t_1,x_0)$, $t_1 > t_0$, are any two solutions of (2.54) through (t_0,x_0), (t_1,x_0) respectively, then as before, we obtain the inequality

$$D^+ m(t) \leq g(t,m(t))$$

where $m(t) = |x(t,t_0,x_0) - y(t,t_1,x_0)|$. Also $m(t_1) = |x(t_1,t_0,x_0) - x_0|$. Hence, by Lemma 2.7.1,

$$m(t_1) \leq r^*(t_1,t_0,0)$$

and as a result

$$m(t) \leq \tilde{r}(t), \quad t > t_1,$$

where

$$\tilde{r}(t) = r(t,t_1,r^*(t_1,t_0,0))$$

is the maximal solution of (2.56) through $(t_1, r^*(t_1,t_0,0))$. Since $r^*(t_0,t_0,0) = 0$, we have

$$\lim_{t_1 \to t_0} \tilde{r}(t,t_1,r^*(t_1,t_0,0)) = \tilde{r}(t,t_0,0)$$

and by hypothesis, $\tilde{r}(t,t_0,0) \equiv 0$. This proves desired continuity of solutions relative to t_0 and the proof is complete.

We shall now prove the continuous dependence of solutions of (2.54) with respect to a parameter.

Theorem 2.7.2. Suppose that $Q : E^* \to E$, where E^* is an open set in $\tilde{E} = E \times \mathbb{R}_+$ that contains the parameter μ and for $\mu = \mu_0$, let $x_0(t) = x(t,t_0,x_0,\mu_0)$ be the solution of

$$x'(t) = (Qx,\mu)(t), \quad x(t_0) = x_0, \tag{2.57}$$

existing on $[t_0,T]$. Assume that

$$\lim_{\mu \to \mu_0} (Qx,\mu)(t) = (Qx,\mu_0)(t) \tag{2.58}$$

uniformly in $(t,x(t))$ and

$$|(Qx,\mu)(t) - (Qy,\mu)(t)| \leq g(t,|x(t)-y(t)|), \tag{2.59}$$

for $x, y \in \Omega$, where $g \in C([t_0, T] \times \mathbb{R}_+, \mathbb{R}_+)$. Suppose that $u(t) \equiv 0$ is the unique solution of (2.56) with $u(t_0) = 0$. Then, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that, whenever $|\mu - \mu_0| < \delta(\varepsilon)$, the IVP

$$x'(t) = (Qx, \mu)(t), \quad x(t_0) = x_0 \tag{2.60}$$

admits a unique solution $x(t) = x(t, t_0, x_0, \mu)$ satisfying

$$|x(t) - x_0(t)| < \varepsilon, \quad t \in [t_0, T].$$

Proof. The uniqueness of solutions is obvious from Theorem 2.5.2. From the assumption that $u(t) \equiv 0$ is the only solution of (2.56), it follows by Lemma 1.3.3 that given any compact interval $[t_0, t_0 + a] \subset [t_0, T]$ and any $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon) > 0$ such that the maximal solution $r(t, t_0, 0, \eta)$ of

$$u' = g(t, u) + \eta$$

exists on $[t_0, T]$ and satisfies

$$r(t, t_0, 0, \eta) < \varepsilon, \quad t \in [t_0, T].$$

Also, because of (2.58), given $\eta > 0$, there exists a $\delta = \delta(\eta) > 0$ such that, whenever $|\mu - \mu_0| < \delta$, we have

$$|(Qx, \mu)(t) - (Qx, \mu_0)(t)| < \eta.$$

Now let $\varepsilon > 0$ be given and define

$$m(t) = |x(t) - x_0(t)|,$$

where $x(t)$, $x_0(t)$ are the solutions of (2.57), (2.60) respectively. Then, using assumption (2.59), we get

$$D^+ m(t) \leq g(t, m(t)) + \eta$$

and by comparison Theorem 1.3.2,

$$m(t) \leq r(t, t_0, 0, \eta).$$

Hence, whenever $|\mu - \mu_0| < \delta$,we have

$$|x(t) - x_0(t)| < \varepsilon, \quad t \in [t_0, T].$$

Clearly, δ depends on ε since η does. The proof is complete.

2.8 Existence of Euler Solutions

We consider the IVP for causal differential equation

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \tag{2.61}$$

where $Q : E \to E$, $E = C([t_0,T],\mathbb{R}^n)$. Let

$$\pi = \{t_0,t_1,\ldots,t_N = T\} \tag{2.62}$$

be a partition of $[t_0,T]$ and consider the interval $[t_0,t_1]$. Note that the right side of the causal differential equation

$$x'(t) = (Qx_0)(t_0), \quad x(t_0) = x_0$$

is a constant on $[t_0,t_1]$ and hence, the IVP (2.61) clearly has a unique solution $x(t) = x(t,t_0,x_0)$ on $[t_0,t_1]$. We define the node $x_1 = x(t_1)$ and iterate next by considering on $[t_1,t_2]$, the IVP

$$x'(t) = (Qx_1)(t_1), \quad x(t_1) = x_1.$$

The next node is $x_2 = x(t_2) = x(t_2,t_1,x_1)$ and proceed this way till the entire arc $x_\pi = x_\pi(t)$ has been defined on $[t_0,T]$. We employ the notation x_π to emphasize the role played by the particular partition π in finding the arc x_π which is the Euler polygonal arc corresponding to the partition π. The diameter μ_π of the partition π is given by

$$\mu_\pi = \max[t_i - t_{i-1} \ : \ 1 \le i \le N]. \tag{2.63}$$

Definition 2.8.1. By an Euler solution of (2.61), we mean any arc $x = x(t)$ which is the uniform limit of Euler polygonal arcs x_π, corresponding to some sequence π_j such that $\pi_j \to 0$ i.e. as the diameter $\mu_{\pi_j} \to 0$ as $j \to \infty$.

Clearly, the corresponding number N_j of the position points in π_j and the nodes also go to ∞. Also, the Euler arc satisfies the initial condition $x(t_0) = x_0$.

We can now prove the following result on the existence of Euler solution for IVP (2.61).

Theorem 2.8.1. Assume that

(i) for $x \in \Omega = \{x \in E : \max_{t_0 \le s \le T} |x(s)| = x(t)\}$ and $t \in [t_0,T]$,

$$|(Qx)(t)| = g(t,|x(t)|),$$

where $g \in C([t_0,T] \times \mathbb{R}_+,\mathbb{R}_+)$, $g(t,u)$ is nondecreasing in (t,u);

(ii) the maximal solution $r(t) = r(t,t_0,u_0)$ of the scalar differential equation

$$u' = g(t,u), \quad u(t_0) = u_0, \tag{2.64}$$

exists on $[t_0,T]$. Then

(a) there exists at least one Euler solution $x(t) = x(t,t_0,x_0)$ of the IVP (2.61) which satisfies the Lipschitz condition;

(b) any Euler solution $x(t)$ of (2.61) satisfies the relation

$$|x(t) - x_0| \leq r(t,t_0,u_0) - u_0, \quad t \in [t_0,T], \tag{2.65}$$

where $u_0 = |x_0|$.

Proof. Let π be the partition of $[t_0,T]$ defined by (2.62) and let $x_\pi = x_\pi(t)$ denote the corresponding arc with nodes of x_π represented by $x_0,x_1,\ldots,x_N$. Let us set $x_\pi = x_i(t)$ on $t_i \leq t \leq t_{i+1}$, $i = 0,1,\ldots,N-1$ and observe that $x_i(t_i) = x_i$, $i = 0,1,2,\ldots,N$. On the interval (t_i,t_{i+1}), we have

$$|x'_\pi(t)| = |(Qx_i)(t_i)| \leq g(t_i,|x_i|). \tag{2.66}$$

On $[t_0,t_1]$, we obtain

$$\begin{aligned} |x_i(t) - x_0| &= |x_0 + \int_{t_0}^{t} (Qx_0)(t_0)ds - x_0| \leq \int_{t_0}^{t} |(Qx_0)(t_0)|ds \\ &\leq \int_{t_0}^{t} g(t_0,|x_0|)ds \leq \int_{t_0}^{t} g(s,r(s))ds \\ &= r(t,t_0,|x_0|) - |x_0| \\ &\leq r(T,t_0,|x_0|) - |x_0| \equiv M, \text{ say.} \end{aligned}$$

Here we have employed the properties of the norm and the integral, monotone character of $g(t,u)$ in u and the fact that $r(t,t_0,u_0) \geq 0$ is nondecreasing in t. Similarly, we get, on $[t_1,t_2]$,

$$\begin{aligned} |x_2(t) - x_0| &= |x_1 + \int_{t_1}^{t} (Qx_1)(t_1)ds - x_0| \\ &= |x_0 + \int_{t_0}^{t_1} (Qx_0)(t_0)ds + \int_{t_1}^{t} (Qx_1)(t_1)ds - x_0| \\ &\leq \int_{t_0}^{t_1} |(Qx_0)(t_0)|ds + \int_{t_1}^{t} |(Qx_1)(t_1)|ds \\ &\leq \int_{t_0}^{t_1} g(s,r(s))ds + \int_{t_1}^{t} g(s,r(s))ds \\ &= \int_{t_0}^{t} g(s,r(s))ds \leq r(T,t_0,|x_0|) - |x_0| = M. \end{aligned}$$

Proceeding in this way, we obtain on $[t_i,t_{i+1}]$,

$$|x_i(t) - x_0| \leq r(T,t_0,|x_0|) - |x_0| = M.$$

Hence, it follows that

$$|x_\pi(t) - x_0| \leq M, \quad \text{on } [t_0,T].$$

Also, from (2.66), we have

$$|x'_\pi(t)| \le g(T, r(T)) = r'(T, t_0, |x_0|) \equiv K, \text{ (say).}$$

Consequently, using similar arguments, we can find for $t_0 \le s \le t \le T$,

$$\begin{aligned} |x_\pi(t) - x_\pi(s)| &\le \int_{t_0}^{t} |(Qx_\pi)(z)|dz - \int_{t_0}^{s} |(Qx_\pi)(z)|dz \\ &\le \int_{t_0}^{t} g(z, r(z))dz - \int_{t_0}^{s} g(z, r(z))dz \\ &= \int_{s}^{t} g(z, r(z))dz = r(t) - r(s) \\ &= r'(\sigma)|t - s| \le K(t - s) \end{aligned}$$

for some σ, $s \le \sigma \le t$, proving $x_\pi(t)$ satisfies Lipschitz condition with constant K on $[t_0, T]$. Now, let π_j be a sequence of partitions of $[t_0, T]$ such that $\pi_j \to 0$, i.e., $\mu_{\pi_j} \to 0$ and therefore $N_j \to \infty$. Then, the corresponding polygonal arcs x_π on $[t_0, T]$ all satisfy

$$x_{\pi_j}(t_0) = x_0, \quad |x_{\pi_j}(t) - x_0| \le M \text{ and } |x'_{\pi_j}(t)| \le K.$$

Hence the family $\{x_{\pi_j}\}$ is equicontinuous and uniformly bounded, and as a consequence, Ascoli-Arzela Theorem guarantees the existence of a subsequence which converges uniformly to a continuous function $x(t)$ on $[t_0, T]$ and that is also absolutely continuous on $[t_0, T]$. Thus, by definition, $x(t)$ is an Euler solution of the IVP (2.61) on $[t_0, T]$ and the claim of the theorem follows. The inequality (2.65) in part (b) is inherited by $x(t)$ from the sequence of polygonal arcs generating it when we identify T with t. Hence the proof is complete.

If (Qx) in (2.61) is assumed continuous, then one can show that $x(t)$ actually satisfies (2.61).

Theorem 2.8.2. Under the assumptions of Theorem 2.8.1, if we also suppose that Q is continuous, then $x(t)$ is a solution of IVP (2.61).

Proof. Let x_{π_j} denote a sequence of polygonal arcs for IVP (2.61) converging uniformly to an Euler solution $x(t)$ on $[t_0, T]$. Clearly, the arcs $x_{\pi_j}(t)$ all lie in $\overline{B}(x_0, M) = \{x \in E : |x - x_0| \le M\}$ and satisfy Lipschitz condition with some constant K. Since a continuous function is uniformly continuous on compact sets, for any given $\varepsilon > 0$, one can find a $\delta > 0$ such that

$$|x - x^*| < \delta, \quad |t - t^*| < \delta \text{ implies } |(Qx)(t) - (Qx^*)(t^*)| < \varepsilon$$

for $t, t^* \in [t_0, T]$, $x, x^* \in \{x_{\pi_j}\}$. Let j be sufficiently large so that the particular diameter μ_{π_j} satisfies $\mu_{\pi_j} < \delta$ and $K\mu_{\pi_j} < \delta$ for any t which is not one of the infinitely many points at which x_{π_j} is a node, we have $x'_{\pi_j}(t) = (Qx_{\pi_j})(\tilde{t})$ for some $\tilde{t}$ within $\mu_{\pi_j} < \delta$ of t. Since $|x_{\pi_j}(t) - x_{\pi_j}(\tilde{t})| \leq K\mu_{\pi_j} < \delta$, we get

$$|x'_{\pi_j}(t) - (Qx_{\pi_j})(t)| = |(Qx_{\pi_j})(\tilde{t}) - (Qx_{\pi_j})(t)| < \varepsilon.$$

It follows for any $t \in [t_0, T]$, we obtain

$$\left| x_{\pi_j}(t) - x_{\pi_j}(t_0) - \int_{t_0}^{t} (Qx_{\pi_j})(s)ds \right|$$

$$= \left| x_{\pi_j}(t_0) + \int_{t_0}^{t} x'_{\pi_j}(s)ds - x_{\pi_j}(t_0) - \int_{t_0}^{t} (Qx_{\pi_j})(s)ds \right|$$

$$= \left| \int_{t_0}^{t} x'_{\pi_j}(s)ds - \int_{t_0}^{t} (Qx_{\pi_j})(s)ds \right|$$

$$\leq \int_{t_0}^{t} |x'_{\pi_j}(s) - (Qx_{\pi_j})(s)| ds$$

$$\leq \varepsilon(t - t_0) < \varepsilon(T - t_0).$$

Letting $j \to \infty$, we have from this inequality,

$$\left| x(t) - x_0 - \int_{t_0}^{t} (Qx)(s)ds \right| < \varepsilon(T - t_0).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$x(t) = x_0 + \int_{t_0}^{t} (Qx)(s)ds, \quad t \in [t_0, T],$$

which implies that $x(t)$ is continuously differentiable and therefore,

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \quad t \in [t_0, T].$$

The proof is complete.

Remark 2.8.1. One can extend the notion of Euler solution of (2.61) from the interval $[t_0, T]$ to $[t_0, \infty)$, if we define Q and g on $[t_0, \infty)$ instead of $[t_0, T]$ and assume that the maximal solution $r(t)$ exists on $[t_0, \infty)$. Then we can show that an Euler solution exists on every compact interval $[t_0, T]$, $t_0 < T < \infty$.

2.9 Flow Invariance

Consider the IVP

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0 \in F \tag{2.67}$$

where F is a closed set in $\mathbb{R}^n$ and $Q : E \to E = C([t_0,\infty),\mathbb{R}^n)$.

Definition. The set F is said to be flow invariant with respect to Q if every solution $x(t)$ of (2.67) on $[t_0,\infty)$ is such that $x(t) \in F$ for $t_0 \le t < \infty$.

A set B is called a distance set if for each $x \in \mathbb{R}^n$, there corresponds a point $y \in B$ such that $d(x,B) = |x-y|$.

A function $g \in C[\mathbb{R}_+^2,\mathbb{R}_+]$ is said to be a uniqueness function if the following holds: If $m \in C[\mathbb{R}_+,\mathbb{R}_+]$ is such that $m(t_0) \le 0$ and $D^+m(t) \le g(t,m(t))$ whenever $m(t) > 0$, then $m(t) \le 0$ for $t_0 \le t < \infty$.

We shall first prove a result on flow invariance for set F.

Theorem 2.9.1. Let $F \subset \mathbb{R}^n$ be closed and distance set. Suppose further that, for each t,

(i) $\lim_{h\to 0} \frac{1}{h} d(x+h(Qx)(t),F) = 0, \quad t \ge t_0$ and $x \in \partial F$;

(ii) $|(Qx)(t)-(Qy)(t)| \le g(t,|x(t)-y(t)|), x,y \in \Omega, x \in \mathbb{R}^n - F, y \in \partial F$ and $t \ge t_0$, g being the uniqueness function.

Then F is flow invariant with respect to Q.

Proof. Let $x(t)$ be a solution of (2.67) for $t \ge t_0$. Assume that $x(t) \in F$ for $t_0 \le t < t_0 + a < \infty$, $[t_0,t_0+a)$ is the maximal interval of existence i.e. $x(t)$ leaves the set F at $t = t_0+a$ for the first time. Let $x(t_1) \notin F$, $t_1 \in (t_0+a,\infty)$ and let $y_0 \in \partial F$ be such that

$$d(x(t_1),F) = |x(t_1)-y_0|.$$

Set for $t \in [t_0,\infty)$, $m(t) = d(x(t),F)$ and $v(t) = |x(t)-y_0|$. For $h > 0$ sufficiently small, we have, letting $x = x(t_1)$,

$$\begin{aligned} m(t_1+h) &\le |x(t_1+h) - y_0 - h(Qy_0)(t_1)| + d(y_0+h(Qy_0)(t_1),F) \\ &\le |x+h(Qx)(t_1) - y_0 - h(Qy_0)(t_1)| + d(y_0+h(Qy_0)(t_1),F) \\ &\quad + |x(t_1+h) - x - h(Qx)(t_1)| \\ &\le |x-y_0| + hg(t_1,|x-y_0|) + o(h). \end{aligned}$$

Setting $m(t_1) \equiv v(t_1) > 0$, we obtain

$$D^+m(t_1) \le g(t_1,m(t_1)).$$

This implies, in view of the fact that g is a uniqueness function and $m(t_0) = 0$, that $m(t) \leq 0$, $t_0 \leq t < \infty$. This contradicts $d(x(t_1), F) = m(t_1) > 0$. The proof is complete.

Next we consider weak flow invariance of Q. The system (F, Q), where $F \subset \mathbb{R}^n$ is closed, is said to be weakly flow invariant provided that for all $x_0 \in F$, there exists an Euler solution $x(t)$ of (2.67) on $[t_0, \infty)$ such that $x_0 = x(t_0)$ and $x(t) \in F$, $t \geq t_0$. In order to prove weak invariance, we have to employ the notion of proximal normal.

Let $F \subset \mathbb{R}^n$ be a closed set. Assume that for any $x \in \mathbb{R}^n$ such that x and F are disjoint and for any $s \in F$, there exists a $z \in \mathbb{R}^n$ such that $x = s + z$. Then $x - s$ is called the Hukuhara difference. Suppose now that for any $x \in \mathbb{R}^n$, there is an element $s \in F$ whose distance to x is minimal, i.e.,

$$|x - s| = \inf_{s_0 \in F} |x - s_0|,$$

then s is called a projection of x onto F. The set of all such elements is denoted by $\mathrm{Proj}_F(x)$. The element $x - s$ will be called the proximal normal direction to F at s. Any nonnegative multiple $\xi = t(x - s)$, $t \geq 0$, is called the proximal normal to F at s. The set of all ξ obtained in this way is said to be the proximal normal cone to F at s and it is denoted by $N_F^p(s)$.

We can now prove the following result which offers sufficient conditions for the weak invariance of the system (F, Q) in terms of proximal normal.

Theorem 2.9.2. Let Q satisfy the assumptions of Theorem 2.8.1. Let A be an open set containing $x(t)$ for all $t \in [t_0, T]$. Suppose that for any $(t, z) \in (t_0, T) \times A$, the proximal aiming condition is satisfied i.e., there exists $s \in \mathrm{Proj}_A(z)$ such that

$$\langle (Qz)(t), z - s \rangle \leq 0,$$

where $\langle \cdot, \cdot \rangle$ is the inner product. Then we have

$$d(x(t), F) \leq d(x(t_0), F), \quad t \in [t_0, T].$$

Proof. Let x_π be one polygonal arc in the sequence converging uniformly to x, as per definition of the Euler solution. As usual, denote its node at t_i by x_i, $i = 0, 1, \ldots, N$ and $x_0 = x(t_0)$. We may suppose that $x_\pi(t)$ lies in set A for all $t \in [t_0, T]$. Accordingly, there exists for each i, a point $s_i \in \mathrm{Proj}_A(x_i)$ such that

$$\langle (Qx_i)(t_i), x_i - s_i \rangle \leq 0.$$

Letting K be the apriori bound on $|x_\pi'|$, we calculate

$$\begin{aligned} d^2(x_1,F) &\le |x_1 - s_0|^2, \quad (\text{since } x_0 \in F) \\ &= |x_1 - x_0|^2 + |x_0 - s_0|^2 + 2\langle x_1 - x_0, x_0 - s_0\rangle \\ &\le K^2(t_1 - t_0)^2 + d^2(x_0,F) + 2\int_{t_0}^{t_1} \left\langle x_\pi'(t), x_0 - s_0\right\rangle ds \\ &= K^2(t_1 - t_0)^2 + d^2(x_0,F) + 2\int_{t_0}^{t_1} \langle (Qx_0)(t_0), x_0 - s_0\rangle ds \\ &\le K^2(t_1 - t_0)^2 + d^2(x_0,F), \end{aligned}$$

since the inner product in the integral term is ≤ 0. The same estimates apply at any node x_i and hence

$$d^2(x_i,F) \le d^2(x_{i-1},F) + K^2(t_i - t_{i-1})^2.$$

Repeating this recursively, we get

$$\begin{aligned} d^2(x_i,F) &\le d^2(x_0,F) + K^2 \sum_{l=1}^{i} (t_l - t_{l-1})^2 \\ &\le d^2(x_0,F) + K^2 \mu_\pi \sum_{l=1}^{i} (t_l - t_{l-1}) \\ &\le d^2(x_0,F) + K^2 \mu_\pi (T - t_0). \end{aligned}$$

Consider now the sequence x_{π_j} of polygonal arcs converging to x. Since the above estimate holds at every node and since $\mu_{\pi_j} \to 0$, same K applying to each x_{π_j}, we can deduce that in the limit,

$$d(x(t),F) \le d(x(t_0),F), \quad t \in [t_0,T]$$

as claimed. The proof is complete.

2.10 Systems of Causal Differential Inequalities

As we have seen in earlier sections, most of the results relative to causal differential equations which depend on causal differential inequalities are proved only for the scalar case. This includes the existence of extremal solutions as well. If we wish to extend such results to systems of causal differential equations, we need to first prove the corresponding results for systems of causal differential inequalities. For this purpose, we shall make use of vectorial inequalities freely, with the understanding that the inequalities are component-wise, i.e., the vectorial inequality $x \le y$, $x, y \in \mathbb{R}^n$ implies that $x_i \le y_i$ holds for each i, $1 \le i \le n$. We need the following notion of quasi-semi monotonicity relative to the causal operator. The operator $Q : E \to E = C([t_0, T], \mathbb{R}^n)$ is said to be quasi-semi monotone nondecreasing, if for any $x, y \in E$, $x \le y$ and $x_i = y_i$, and for any fixed i, $1 \le i \le n$, we have

$$(Q_i x)(t) \le (Q_i y)(t), \text{ for each } i. \tag{2.68}$$

and Q_i is semi-nondecreasing for each i.

This concept is to be imposed on Q when we deal with systems of causal differential inequalities and existence of extremal solutions. As an extension to the case of systems, we shall indicate the proof of one result corresponding to Theorem 2.2.3.

Theorem 2.10.1. Suppose that $v, w \in C^1([t_0, T], \mathbb{R}^n)$, $Q \in C([t_0, T], \mathbb{R}^n)$ and

$$v'(t) \le (Qv)(t), \quad w'(t) \ge (Qw)(t), \quad t \in [t_0, T], \tag{2.69}$$

where the inequalities are component-wise. Assume further that Q is quasi-semi monotone nondecreasing in x. Then $v(t_0) < w(t_0)$ implies

$$v(t) < w(t), \quad t \in [t_0, T], \tag{2.70}$$

provided one of the inequalities in (2.69) is strict.

Proof. If (2.70) is not true, then the initial condition $v(t_0) < w(t_0)$, together with continuity of v, w yields that there exists an index i, $1 \le i \le n$ and a $t_1 > t_0$ such that

$$\left.\begin{aligned} v_i(t_1) = w_i(t_1),\ v_j(t_1) < w_j(t_1), \quad i \ne j, \quad 1 \le j \le n \\ \text{and } \ v_j(t) < w_j(t), \quad 1 \le j \le n, \quad t_0 < t < t_1 \end{aligned}\right\} \tag{2.71}$$

Suppose that the second inequality in (2.69) is strict. Using (2.71), it follows that for index i,

$$v_i'(t_1) \ge w_i'(t_1) \tag{2.72}$$

and therefore,

$$(Q_i v)(t_1) \ge v_i'(t_1) \ge w_i'(t_1) > (Q_i w)(t_1) \ge (Q_i v)t_1,$$

using (2.69), (2.71), (2.72) and the quasi-semimonotone nondecreasing character of Q. This is a contradiction and hence the conclusion (2.70) is true for $t \in [t_0, T]$. The proof is complete.

For nonstrict inequalities, we state the following result.

Theorem 2.10.2. Suppose that the conditions of Theorem 2.10.1 hold. Assume further that for each i,

$$|(Q_i x)(t) - (Q_i y)(t)| \le L_i \max_{t_0 \le s \le t} |x_i(s) - y_i(s)|, \quad 0 < L_i < 1.$$

Then $v(t_0) \leq w(t_0)$ implies $v(t) \leq w(t)$, $t_0 \leq t \leq T$.

For the proof, we follow the proof of Theorem 2.2.2 with $w_{i\varepsilon}(t) = w_i(t) + \varepsilon_i$, for each i, $\varepsilon_i > 0$ and proceed with suitable modifications to get the strict inequality

$$w'_{i\varepsilon}(t) > (Q_i w_{i\varepsilon})(t), \quad t_0 \leq t \leq T.$$

Now, using the arguments of Theorem 2.10.1, we can prove

$$v(t) < w_\varepsilon(t), \quad t \in [t_0, T]$$

and since $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) > 0$ is arbitrary, we get for $t \in [t_0, T]$

$$v(t) \leq w(t), \quad t \in [t_0, T].$$

2.11 Nonlinear Variation of Parameters

We shall develop, in this section, the nonlinear variation of parameters formula and for this purpose, we need to prove the differentiability of solutions of causal differential system relative to the initial values. Let us consider the causal differential system

$$u'(t) = (Qu)(t), \quad u(t_0) = u_0,$$

where Q is smooth enough to guarantee existence, uniqueness and continuous dependence with respect to the initial values and parameters. Before we proceed, we state the following integral mean value theorem which is needed.

Theorem 2.11.1. Let the Frechet derivative Q_u of Q exist and be continuous. Then for $u_1, u_2 \in J = [t_0, t_0 + \eta]$, we have

$$(Qu_1)(t) - (Qu_2)(t) = \int_0^1 Q[\lambda u_1 + (1-\lambda)u_2]_u(t)(u_1 - u_2)(t)d\lambda.$$

We begin with the following theorem, which establishes the continuity and differentiability of the solutions with respect to initial values.

For convenience, we rewrite the causal differential system in the form

$$\begin{cases} u'(t) = (Q_{t_0}u)(t) \\ u(t_0) = u_0 \end{cases} \tag{2.73}$$

where $Q_{t_0} \in C[E, E]$ denotes a causal operator.

Theorem 2.11.2. Let $u(t, t_0, u_0)$ be the unique solution of (2.73) existing on some interval $J_0 = [t_0, t_0 + \eta]$. Assume that the Frechet derivative $(Q_{t_0}u)(_u) \equiv L(t, t_0, u_0)$ exists and is continuous on E. Then,

(a) $\phi(t,t_0,u_0) = \frac{\partial u(t,t_0,u_0)}{\partial u_0}$ exists and is a solution of

$$z'(t) = L(t,t_0,u_0)(z) \text{ such that } \phi(t_0,t_0,u_0) = I, \tag{2.74}$$

where $L(t,t_0,u_0)$ is a linear operator and I is the identity matrix;

(b) $\psi(t,t_0,u_0) = \frac{\partial u(t,t_0,u_0)}{\partial t_0}$ exists and is a solution of

$$\begin{aligned} y'(t) &= L(t,t_0,u_0)(y) - (\hat{Q}_{t_0}u)(t), \\ y(t_0) &= -(Q_{t_0}u)(t_0) \end{aligned} \tag{2.75}$$

where $(\hat{Q}_{t_0})(t)$ is the term in $(Qu)(t)$ that depends on the initial time t_0;

(c) the functions $\phi(t,t_0,u_0)$ and $\psi(t,t_0,u_0)$ satisfy the relation

$$\psi(t,t_0,u_0) + \phi(t,t_0,u_0)(Q_{t_0}u)(t_0) = \int_{t_0}^{t} R(t,s;t_0,u_0)(\hat{Q}_{t_0}u)(s)ds, \tag{2.76}$$

where $R(t,s;t_0,u_0)$ is the solution of the IVP

$$\frac{\partial R(t,s;t_0,u_0)}{\partial s} + L(t,t_0,u_0)(R(t,s;t_0,u_0)) = 0, \tag{2.77}$$

$R(t,t;t_0,u_0) = I$, $t_0 \le s \le t$ and $R(t,t_0;t_0,u_0) = \phi(t,t_0,u_0)$.

Proof. Under the assumptions on Q, it is clear that solutions $u(t,t_0,u_0)$ of (2.73) exist, are unique and continuous in t,t_0 and u_0 on some interval. Consequently, the operator $L(t,t_0,u_0)$ is continuous in t,t_0, and u_0 on that interval. Therefore, the solutions of the linear initial-value problems (2.74) and (2.75) exist and are unique over the same interval. To prove (a), let $e_k = (e_k^1, e_k^2, \ldots, e_k^n)$ be the vector such that $e_k^j = 0$ if $j \neq k$ and $e_k^k = 1$. Then for some k, $\tilde{u}(t,h) = u(t,t_0,u_0 + e_k h)$ is defined on J_0 and $\lim_{h\to 0}\tilde{u}(t,h) = u(t,t_0,u_0)$ uniformly on J_0. Let $u(t) = u(t,t_0,u_0)$ and $u(t,h) = \tilde{u}(t,h) - u(t)$. Then differentiating $u(t,h)$ with respect to t and using Theorem 2.11.1, it follows that

$$\begin{aligned} \frac{d}{dt}u(t,h) &= \tilde{u}'(t,h) - u'(t) \\ &= (Q_{t_0}\tilde{u} - Q_{t_0}u)(t) \\ &= \int_0^1 [Q_{t_0}(\lambda\tilde{u} + (1-\lambda)u)]_u(t)d\lambda(\tilde{u}(t,h) - u(t)) \\ &\equiv L(t,t_0,u_0,h)(\tilde{u}(t+h) - u(t)). \end{aligned}$$

Dividing by h, $h \neq 0$,

$$\frac{u'(t,h)}{h} = L(t,t_0,u_0,h)\frac{u(t,h)}{h},$$

and since

$$\frac{u(t_0,h)}{h} = \frac{u(t_0,t_0,u_0 + e_k h) - u(t_0,t_0,u_0)}{h} = e_k,$$

it is clear that $\frac{u(t,h)}{h}$ is a solution of the following initial-value problem

$$\begin{cases} z'(t) = L(t,t_0,u_0,h)z, \\ z(t_0) = e_k, \end{cases} \tag{2.78}$$

where $L(t,t_0,u_0,h) = \int_0^1 [Q_{t_0}(\lambda\tilde{u} + (1-\lambda)u)]_u(t)d\lambda$. Since $\lim_{h\to 0}\tilde{u}(t,h) = u(t)$ uniformly on J_0, continuity of $(Q_{t_0}u)_u$ implies that $\lim_{h\to 0} L(t,t_0,u_0,h) = L(t,t_0,u_0)$ uniformly on J_0. Also observe that $L(t,t_0,u_0,h)$ is linear and hence we conclude that (2.78) admits a unique solution, which is continuous with respect to h for fixed t,t_0,u_0.

Next, consider the family of initial value problems defined by (2.78), with a small parameter h, for $k = 1,2,\ldots,n$. Since the solutions corresponding to this family of initial value problems are all continuous functions of h for fixed t,t_0,u_0, it follows that $\lim_{h\to 0}\frac{u(t,h)}{h} = \frac{\partial}{\partial u_0}u(t_0,u_0)$, which is a solution of (2.74) with $\frac{\partial}{\partial u_0}u(t_0,t_0,u_0) = I$. Also, in view of assumptions on $L(t,t_0,u_0)$, it is clear that $\frac{\partial}{\partial u_0}u(t,t_0,u_0)$ is also continuous with respect to its arguments.

To prove (b), define $\hat{u}(t,h) = u(t,t_0+h,u_0)$. Then, differentiating with respect to t we have

$$\begin{aligned} u(t,h) &= \hat{u}(t,h) - u(t) \\ u'(t,h) &= (Q_{t_0+h}\hat{u})(t) - (Q_{t_0}u)(t) \\ &= (Q_{t_0+h}\hat{u})(t) - (Q_{t_0+h}u)(t) - (Q_{t_0}u)(t_0+h) \\ &= \int_0^1 [Q_{t_0+h}(\lambda\hat{u} + (1-\lambda)u)]_u(t)d\lambda(\hat{u}(t,h) - u(t)) - (Q_{t_0}u)(t_0+h) \\ &\equiv L(t,t_0,u_0,h)(\hat{u}(t,h) - u(t)) - (\hat{Q}_{t_0}u)(t). \end{aligned}$$

It is clear that $\frac{u(t,h)}{h}$ is solution of the following initial value problem

$$\begin{cases} y'(t) = L(t,t_0,u_0,h)(z) - (\hat{Q}_{t_0}y)(t) \\ y(t_0+h) = \frac{u(t_0+h,h)}{h} = -\frac{1}{h}\int_{t_0}^{t_0+h}(Q_{t_0}u)(s)ds \end{cases} \tag{2.79}$$

where $L(t,t_0,u_0,h) = \int_0^1 [Q_{t_0+h}(\lambda\hat{u} + (1-\lambda)u)]_u(t)d\lambda$.

Noting that $\lim_{h\to 0}\frac{1}{h}(Q_{t_0}u)(s)ds = (Q_{t_0}u)(t_0)$ and using an argument similar to the argument in the proof of (a), we see that $\frac{\partial}{\partial t_0}u(t,t_0,u_0)$ exists, is continuous in its arguments, and is a solution of (2.75).

The result in (c) follows from the fact that $\phi(t,t_0,u_0)$ and $\psi(t,t_0,u_0)$ are solutions of (2.74) and (2.75), respectively, and the fact that $R(t,s;ty_0,u_0)$ is the solution of the IVP (2.77), which is a linear equation. observe that (2.74) is the homogeneous linear equation corresponding to (2.75).

Having established the continuity and differentiability of the solutions of (2.73) with respect to initial values, we now proceed to obtain the nonlinear variation of parameters formula for the solutions $r(t,t_0,u_0)$ of the perturbed system

$$\begin{cases} r'(t) = (Q_{t_0}r)(t) + (P_{t_0}r)(t) \\ r(t_0) = u_0, \end{cases} \tag{2.80}$$

where $P_{t_0} \in C[E,E]$.

Theorem 2.11.3. Suppose the hypotheses of Theorem 2.11.2 hold. Let $r(t,t_0,u_0)$ be any solution of (2.80) existing on J_0. Then $r(t,t_0,u_0)$ satisfies the integral equation

$$\begin{aligned} r(t,t_0,u_0) &= u(t,t_0,u_0) + \int_{t_0}^{t}\int_{s}^{t} R(s,t;t_0,u_0)(\hat{Q}_s u)(\sigma)d\sigma ds \\ &\quad + \int_{t_0}^{t} \phi(t,s,r(s))(P_{t_0}r)(s)ds \end{aligned} \tag{2.81}$$

where $R(s,t;t_0,u_0)$ is the solution of the IVP (2.77).

Proof. Setting $p(s) = u(t,s,r(s))$ where $r(s) = r(s,t_0,u_0)$, we have

$$\begin{aligned} p'(s) &= \frac{\partial u(t,s,r(s))}{\partial t_0} + \frac{\partial u(t,s,r(s))}{\partial u_0} r'(s) \\ &= \psi(t,s,r(s)) + \phi(t,s,r(s))[(Q_{t_0}r)(s) + (P_{t_0}r)(s)]. \end{aligned}$$

Integrating from t_0 to t, we have

$$\begin{aligned} p(t) - p(t_0) &= \int_{t_0}^{t} [\psi(t,s,r(s)) + \phi(t,s,r(s))(Q_{t_0}r)(s)]ds \\ &\quad + \int_{t_0}^{t} \phi(t,s,r(s))(P_{t_0}r)(s)ds \\ &= \int_{t_0}^{t}\int_{s}^{t} R(t,s;t_0,u_0)(\hat{Q}_s u)(\sigma)d\sigma ds \\ &\quad + \int_{t_0}^{t} \phi(t,s,r(s))(P_{t_0}r)(s)ds. \end{aligned}$$

Thus, using the fact that

$$u(t,t,r(t)) = r(t,t_0,u_0) \text{ and } u(t,t_0,r(t_0)) = u(t,t_0,u_0),$$

we have

$$\begin{aligned} r(t,t_0,u_0) = u(t,t_0,u_0) &+ \int_{t_0}^{t}\int_{s}^{t} R(s,t;t_0,u_0)(\hat{Q}_s u)(\sigma)d\sigma ds \\ &+ \int_{t_0}^{t} \phi(t,s,r(s))(P_{t_0}r)(s)ds, \end{aligned}$$

completing the proof.

2.12 Integral Equations of Sobolev Type

We have so far investigated causal functional and differential equations utilizing the general concept of causal operator that includes several popularly known dynamic equations and we shall continue to study the same in the entire monograph. However, we shall discuss in the next two sections what is known as Sobolev type Volterra integral and differential equations in a special form since these special cases themselves are very new types of dynamic systems which are not yet known and popular. Moreover, using the causal operator framework creates more confusion and the results would not be clearer.

In this section we shall consider Volterra type Sobolev integral equations and the next section contains differential equations. We shall indicate how one could employ the causal operator for these equations that we plan to consider so that it can pave the way for further work in this area. For example,

$$u(t,x) = (Qu)(t,x),$$

$$u'(t,x) = (Qu)(t,x), \quad u(t_0,x) = u_0(t_0,x), \quad \frac{d}{dt} =' $$

where

$$u(t,x) = (Qu)(t,x) = u_0(t,x) + \int_{t_0}^{t} K(t,x,s,u(s,x),u(x,s))ds$$

and

$$u'(t,x) = (Qu)(t,x) = f(t,x,u(t,x),u(x,t)), \quad u(t_0,x) = u_0(t_0,x).$$

Consider the following system of integral equations:

$$u(t,x) = u_0(t,x) + \int_{t_0}^{t} K(t,x,s,u(s,x),u(x,s))ds, \tag{2.82}$$

where $u_0(t,x) \in C[J \times J, \Omega]$, $K \in C[J \times J \times J \times \Omega \times \Omega, \Omega]$, $J = [t_0, t_0 + a] \subset \mathbb{R}$ and Ω is an open subset of $\mathbb{R}^n$. For convenience, we list these needed assumptions:

(A1) $|K(t,x,s,u,v)| \leq M$ for all $(t,x,s,u,v) \in J \times J \times J \times \Omega \times \Omega$;

(A2) $\lim_{\substack{t_1 \to t_2 \\ x_1 \to x_2}} [\sup\{\int_I |K(t_1,x_1,s,u(s,x_1),u(x_1,s)) - K(t_2,x_2,u(s,x_2),u(x_2,s))|ds :$ $u(s,x),u(x,s) \in C[J \times J, B]\}] = 0$ for every set $B \subseteq \Omega$ and for every interval $I \subseteq J$.

We also use $B_\varepsilon(u(t_0,x_0))$ to denote the ball in $\mathbb{R}^n$ of radius ε centered at $u(t_0,x_0)$.

We now prove the following existence result.

Theorem 2.12.1. Let $K \in C[J \times J \times J \times \Omega \times \Omega, \Omega]$, $u_0(t,x) \in C[J \times J, \Omega]$, and suppose the conditions (A1) and (A2) are satisfied. Then there exists a solution $u(t,x)$ for the problem (2.82) on $J_0 \times J_0$ where $J_0 = [t_0, t_0 + \gamma]$ for some $\gamma > 0$.

Proof. Set $\eta = \sup\{u \in B_\varepsilon(u_0(t_0,x_0)) \subseteq \Omega : \varepsilon > 0\}$. Since $u_0(t,x)$ is uniformly continuous on $J \times J$, there exists a $\delta_1 > 0$ such that

$$|u_0(t,x) - u_0(t_0,x_0)| < \eta/2 \text{ whenever } |t-t_0| < \delta_1 \text{ and } |x-x_0| < \delta_1. \tag{2.83}$$

Let $\gamma = \min\{a, \delta_1, \eta/2M\}$, and let $J_0 = [t_0, t_0+\gamma]$. Define $A \subseteq C[J_0 \times J_0, \Omega]$ by

$$A = \{\phi(t,x) \in C[J_0 \times J_0, \Omega] : \sup_{t,x \in J_0} |\phi(t,x) - u_0(t,x)| \le \eta/2\}.$$

We note that $\phi(x,t)$ also satisfies $|\phi(x,t) - u_0(x,t)| < \eta/2$. In other words, A can be defined as

$$A = \left\{ \phi \in [J_0 \times J_0, \Omega] : \sup_{t,x \in J_0} |\phi(t,x) - u_0(t,x)| \le \eta/2, \sup_{t,x \in J_0} |\phi(x,t) - u_0(x,t)| \le \eta/2 \right\}.$$

Clearly A is closed, bounded, and convex.

For any ϕ in A define the function $T\phi$ by the relation

$$(T\phi)(t,x) = u_0(t,x) + \int_{t_0}^{t} K(t,x,s,\phi(s,x),\phi(x,s))ds. \tag{2.84}$$

We now apply the Schauder fixed-point theorem to assert the existence of a fixed point of T in A. If $\phi \in A$, then

$$|(T\phi)(t,x) - u_0(t,x)| \le \int_{t_0}^{t} |K(t,x,s,\phi(s,x),\phi(x,s))|ds \le M(t-t_0) \le \eta/2.$$

Thus $TA \subseteq A$.

We now prove TA is uniformly bounded and equicontinuous.

For any $\phi \in A$, we have, for $(t,x) \in J_0 \times J_0$,

$$|(T\phi)(t,x) - u_0(t_0,x_0)| \le |(T\phi)(t,x) - u_0(t,x) + u_0(t,x) - u_0(t_0,x_0)| \le \eta$$

by (2.83) and (2.84), which shows that TA is uniformly bounded.

For $t_1, t_2, x_1, x_2 \in J_0$, $t_1 \ge t_2$ and $\phi \in A$, we get

$$|(T\phi)(t_1,x_1) - (T\phi)(t_2,x_2)| \le |u_0(t_1,x_1) - u_0(t_2,x_2)|$$

$$+ \int_{t_0}^{t_1} |K(t_1,x_1,s,\phi(s,x_1),\phi(x_1,s)) - K(t_2,x_2,s,\phi(s,x_2),\phi(x_2,s))|ds$$

$$+ \int_{t_2}^{t_1} |K(t_1,x_1,s,\phi(s,x_1),\phi(x_1,s))|ds = I_1 + I_2 + I_3, \text{ say.}$$

Since $u_0(t,x)$ is uniformly continuous on $J_0 \times J_0$ by a proper choice of δ (say δ_1^*), I_1 can be made less than $\varepsilon/3$. I_2 can be made less than $\varepsilon/3$ by the assumption A_2 and by a proper choice of δ_2^*; I_3 can be made less than $\varepsilon/3$ by assumption A_1 and by a proper choice of δ_3^*.

Consequently, if $\max(|t_1-t_2|,|x_1-x_2|)<\min(\delta_1^*,\delta_2^*,\delta_3^*)$, then

$$|(T\phi)(t_1,x_1)-(T\phi)(t_2,x_2)|<\varepsilon.$$

This implies that the set TA is an equicontinuous family and therefore the closure of $\{TA\}$ is compact.

Let $\{u_n(t,x)\}\subseteq A$ be a sequence converging to $u(t,x)$. It is easy to see that $u_n(x,t)$ is also a sequence converging to $u(x,t)$. Since K is continuous, we have

$$K(t,x,s,u_n(t,x),u_n(x,t))\to K(t,x,s,u(t,x),u(x,t)).$$

Using the bounded convergence theorem, it then follows that

$$\int_{t_0}^{t} K(t,x,s,u(s,x),u(x,s))ds=\lim_{n\to\infty}\int_{t_0}^{t} K(t,x,s,u_n(s,x),u_n(x,s))ds.$$

Hence $Tu_n\to Tu$, which shows T is continuous. By the Schauder fixed-point theorem, T has a fixed point in A. Hence the proof is complete.

We shall next develop the theory of integral inequalities.

Theorem 2.12.2. Assume that $K\in C[J\times J\times J\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n]$; $u,v\in C[J\times J,\mathbb{R}^n]$; $K(t,x,s,u,v)$ is monotone nondecreasing in u,v for each $(t,x,s)\in J\times J\times J$; and for $(t,x)\in J\times J$,

$$u(t,x)\le u_0(t,x)+\int_{t_0}^{t} K(t,x,s,u(s,x),u(x,s))ds \tag{2.85}$$

$$v(t,x)\ge u_0(t,x)+\int_{t_0}^{t} K(t,x,s,v(s,x),v(x,s))ds. \tag{2.86}$$

Then $u(t_0,x)<v(t_0,x)$ implies $u(t,x)<v(t,x)$ for $(t,x)\in J\times J$, provided that one of the inequalities (2.85) and (2.86) is strict.

Proof. Suppose that one of the inequalities (2.85) and (2.86) is strict. Then if the conclusion is not true, the set

$$Z=\cup_{i=1}^{n}\{(t,x)\in J\times J: u_i(t,x)\ge v_i(t,x) or u_i(x,t)\ge v_i(x,t)\}$$

is nonempty. Let Z_t be the projection of Z on the t-axis, and let $t_1=\inf Z_t$. Clearly $t_1>t_0$. It follows that there is an index j, $1\le j\le n$, such that

$$u_i(s,x)<v_i(s,x) \text{ for } t_0<s<t_1,\ t_0\le x\le t_0+a, \tag{2.87}$$

$$u_i(x,s)<v_i(x,s) \text{ for } t_0\le x\le t_0+a,\ t_0<s<t_1, \tag{2.88}$$

for $i=1,2,\ldots,n$, and either

$$u_j(t_1,x)\le v_j(t_1,x),$$

or

$$u_j(x,t_1) \leq v_j(x,t_1),$$

for all $x \in J$. Consequently, there is an $x_1 \in J$ such that either $u_j(t_1,x_1) = v_j(t_1,x_1)$ or $u_j(x_1,t_1) = v_j(x_1,t_1)$.

Consider the case $u_j(t_1,x_1) = v_j(t_1,x_1)$. Using now (2.85), (2.86), the fact that K is nondecreasing, and the relations (2.87), (2.88), we get

$$\begin{aligned} u_j(t_1,x_1) &\leq u_{j_0}(t_1,x_1) + \int_{t_0}^{t_1} K_j(t_1,x_1,s,u(s,x_1),u(x_1,s))ds \\ &\leq u_{j_0}(t_1,x_1) + \int_{t_0}^{t_1} K_j(t_1,x_1,s,v(s,x_1),v(x_1,s))ds \\ &< v_j(t_1,x_1), \end{aligned}$$

which leads to a contradiction.

To consider the other situation, we first observe that the inequalities (2.85) and (2.86) can also be written in the form

$$u(x,t) \leq u_0(x,t) + \int_{x_0=t_0}^{x} K(x,t,s,u(s,t),u(t,s))ds, \tag{2.89}$$

$$v(x,t) \geq v_0(x,t) + \int_{x_0=t_0}^{x} K(x,t,s,v(s,t),v(t,s))ds. \tag{2.90}$$

As before, using (2.89), (2.90) together with (2.87), (2.88) and the monotonicity of K, we obtain

$$\begin{aligned} u_j(x_1,t_1) &\leq u_{0_j}(x_1,t_1) + \int_{x_0}^{x_1} K_j(x_1,t_1,s,u(s,t_1),u(t_1,s))ds \\ &\leq u_{0_j}(x_1,t_1) + \int_{x_0}^{x_1} K_j(x_1,t_1,s,v(s,t_1),v(t_1,s))ds < v_j(x_1,t_1). \end{aligned}$$

This contradicts the case $u_j(x_1,t_1) = v_j(x_1,t_1)$. Consequently Z is empty and the proof is complete.

If one of the inequalities (2.85), (2.86) is not assumed strict, the conclusion of Theorem 2.12.2 fails to hold. However, if K satisfies a one-sided Lipschitz condition, we get the following result.

Theorem 2.12.3. Let the assumption (i) of Theorem 2.12.2 hold. Suppose further that

$$K(t,x,s,u_1,v_1) - K(t,x,s,u_2,v_2) \leq A[(u_1-u_2)+(v_1-v_2)] \tag{2.91}$$

whenever $u_1 \geq u_2$, $v_1 \geq v_2$, where A is an $n \times n$ matrix such that $a_{ij} \geq 0$, $i \neq j$. Then $u(t_0,x) \leq v(t_0,x)$ for $x \in J$ implies $u(t,x) \leq v(t,x)$ on $J \times J$.

Proof. Let $\tilde{v}(t,x) = v(t,x) + \varepsilon e^{2A(t+x)}$, where $\varepsilon > 0$ is a sufficiently small vector. Then by (2.87),

$$\tilde{v}(t,x) \geq u_0(t,x) + \int_{t_0}^{t} K(t,x,s,v(s,x),v(x,s))ds + \varepsilon e^{2A(t+x)}.$$

Because of the condition on the matrix A, it is clear that $\varepsilon e^{2A(t_0+x)} > 0$. Consequently, we have

$$\tilde{v}(t,x) > u_0(t,x) + \int_{t_0}^{t} K(t,x,s,\tilde{v}(s,x),\tilde{v}(x,s))ds. \tag{2.92}$$

By Theorem 2.12.2, we now get

$$u(t,x) < \tilde{v}(t,x) = v(t,x) + \varepsilon e^{2A(t+x)} \text{ on } J \times J.$$

Taking the limit as $\varepsilon \to 0$, we conclude that $u(t,x) \leq v(t,x)$ on $J \times J$, which proves the stated result.

We only prove the existence of maximal solution for (2.82). The existence of minimal solution can be proved similarly.

Theorem 2.12.4. Let K and u_0 be as in Theorem 2.12.1 and suppose $K(t,x,s,u,v)$ is monotone nondecreasing in u and v for each $(t,x,s) \in J \times J \times J$. Then there exists a $\gamma > 0$ so that the maximal solution to (2.82) exists on $[t_0, t_0 + \gamma] \times [t_0, t_0 + \gamma]$.

Proof. Let η and δ_1 be as in Theorem 2.12.1. Choose $\delta = \min\{a, \delta_1, \eta/4M\}$. As before, set

$$A = \left\{ \phi(t,x) \in C[J_0 \times J_0, \Omega] : \sup_{(t,x) \in j_0 \times J_0} |\phi(t,x) - u_0(t,x)| \leq \eta/2 \right\},$$

where $J_0 = [t_0, t_0 + \gamma]$. We define T_n as $T_n\phi = T\phi + \varepsilon/n$, for $n = 1, 2, \ldots$, where $\varepsilon > 0$ is arbitrarily small vector and T is the same map defined in (2.84). The continuity of T_n follows from the continuity of T. Further,

$$\begin{aligned} |(T_n\phi)(t,x) - u_0(t,x)| &= \frac{\varepsilon}{n} + \int_{t_0}^{t} K(t,x,s,\phi(s,x),\phi(x,s))ds \\ &\leq \left|\frac{\varepsilon}{n}\right| + M(t - t_0) \leq \frac{\eta}{4} + \frac{\eta}{4} \leq \frac{\eta}{2}. \end{aligned}$$

Thus $T_n \subseteq A$ for each n.

As in Theorem 2.12.1, it can be shown that T_nA is equicontinuous at each $(t,x) \in J_0 \times J_0$. Thus T_n has a fixed point ϕ_n. Let $m > n$. Consider

$$\begin{aligned} \phi_n(t,x) &= u_0(t,x) + \frac{\varepsilon}{n} + \int_{t_0}^{t} K(t,x,s,\phi_n(s,x),\phi_n(x,s))ds \\ &> u_0(t,x) + \frac{\varepsilon}{m} + \int_{t_0}^{t} K(t,x,s,\phi_n(s,x),\phi_n(x,s))ds, \\ \phi_m(t,x) &= u_0(t,x) + \frac{\varepsilon}{m} + \int_{t_0}^{t} K(t,x,s,\phi_m(s,x),\phi_m(x,s))ds. \end{aligned}$$

Also $\phi_n(t_0,x) = u_0(t_0,x) + \varepsilon/n > u_0(t_0,x) + \varepsilon/m = \phi_m(t_0,x)$. Thus by Theorem 2.12.2, $\phi_n(t,x) > \phi_m(t,x)$ for $(t,x) \in J_0 \times J_0$ which shows that $\{\phi_n\}$ is monotone in n.
Now consider $\{\phi_n(t,x)\}$. Since

$$\begin{aligned}
|\phi_n(t,x) - u_0(t_0,x_0)| &= |T_n\phi_n(t,x) - u_0(t_0,x_0)| \\
&= |(T\phi_n)(t,x) + \varepsilon/n - u_0(t_0,x_0)| \\
&= |(T\phi_n)(t,x) - u_0(t,x)| \\
&\quad + |u_0(t,x) - u_0(t_0,x_0)| + |\varepsilon/n| \\
&< \eta/2 + \eta/4 + \eta/4 = \eta.
\end{aligned}$$

This proves $\{\phi_n\}$ is uniformly bounded. Also,

$$\begin{aligned}
|\phi_n(t_1,x_1) - \phi_n(t_2,x_2)| &= |(T_n\phi_n)(t_1,x_1) - (T_n\phi_n)(t_2,x_2)| \\
&= |(T\phi_n)(t_1,x_1) - (T\phi_n)(t_2,x_2)|.
\end{aligned}$$

This shows that $\{\phi_n\}$ is equicontinuous at each $(t,x) \in J_0 \times J_0$. Hence by Ascoli's theorem there exists a uniformly convergent subsequence $\{\phi_{n_k}\}$ of $\{\phi_n\}$. The monotonicity of the sequence now implies that the whole sequence $\{\phi_n\}$ converges uniformly to ψ. Consequently as $n \to \infty$, the bounded convergence theorem gives

$$\int_{t_0}^{t} K(t,x,s,\phi_n(s,x),\phi_n(x,s))ds \to \int_{t_0}^{t} K(t,x,s,\psi(s,x),\psi(x,s))ds,$$

and this shows ψ is a fixed point of T.
If $u(t,x)$ is any other fixed point of T, then

$$\begin{aligned}
u(t,x) &= u_0(t,x) + \int_{t_0}^{t} K(t,x,s,u(s,x),u(x,s))ds, \\
\phi_n(t,x) &= u_0(t,x) + \frac{\varepsilon}{n} + \int_{t_0}^{t} K(t,x,s,\phi_n(s,x),\phi_n(x,s))ds \\
&> u_0(t,x) + \int_{t_0}^{t} K(t,x,s,\phi_n(s,x),\phi_n(x,s))ds,
\end{aligned}$$

and $u(t_0,x) = u_0(t_0,x) < u_0(t_0,x) + \varepsilon/n = \phi_n(t_0,x)$. Thus by Theorem 2.12.2, $u(t,x) < \phi_n(t,x)$ for $(t,x) \in J_0 \times J_0$. This implies that $u(t,x) \le \lim_{n\to\infty} \phi_n(t,x) = \psi(t,x)$ for $(t,x) \in J_0 \times J_0$. Thus ψ is the maximal solution to (1.1) on $J_0 \times J_0$.
Finally, we give a comparison theorem.

Theorem 2.12.5. Let $m \in C[J \times J, \Omega]$, $K \in C[J \times J \times J \times \Omega \times \Omega, \Omega]$, and $K(t,x,s,u,v)$ be monotone nondecreasing in u,v for each $(t,x,s) \in J \times J \times J$ and for $(t,x) \in J \times J$. Let

$$m(t,x) \le u_0(t,x) + \int_{t_0}^{t} K(t,x,s,m(s,x),m(x,s))ds.$$

Let $r(t,x)$ be the maximal solution of the equation (2.82) on $j \times J$. Then $m(t,x) \le r(t,x)$ on $J \times J$.

Proof. Let $u(t,x,\varepsilon)$ be any solution of the integral equation

$$u(t,x,\varepsilon) \equiv u_0(t,x) + \varepsilon + \int_{t_0}^{t} K(t,x,s,u(s,x,\varepsilon),u(x,s,\varepsilon))ds$$

for sufficiently small $\varepsilon > 0$. Then by Theorem 2.12.2 we have

$$m(t,x) < u(t,x,\varepsilon) \text{ on } J \times J.$$

Since $\lim_{\varepsilon \to 0} u(t,x,\varepsilon) = r(t,x)$, where $r(t,x)$ is the maximal solution of (2.82), the stated result follows.

2.13 Differential Equations of Sobolev Type

In an embedding method for solving linear Fredholm integral equations introduced by Sobolev [14], the solution of the following differential equation with initial value for the resolvent kernel is involved

$$K_x(t,y,x) = K(t,x,x)K(x,y,x),$$

$$K(t,y,0) = \phi(t,y), \quad 0 \le t,\, y,x \le a.$$

This differential equation is unusual and this is the basis for the study of integral and differential equations of Sobolev type in a similar form. As pointed out in Section 2.12, we shall concentrate to discuss the IVPs for different equations of Sobolev type.
We consider equations of the form

$$u'(t,x) = f(t,x,u(t,x),u(x,t)), \quad u(t_0,x) = u_0(x), \left(' = \frac{d}{dt}\right), \tag{2.93}$$

where $u_0 \in C[J,\mathbb{R}^n]$, $J = [t_0,t_0+a]$ and $f \in C[J \times J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n]$. We need the following assumptions:

(A1)

$$|f(t,x,u,v)| \le M \text{ for all } (t,x,u,v) \in J \times J \times \mathbb{R}^n \times \mathbb{R}^n;$$

(A2)

$$\lim_{x_1 \to x_2} \left(\sup_{\phi} \left\{ \int_I |f(s,x_1,\phi(s,x_1),\phi(x_1,s)) - f(s,x_2,\phi(s,x_2),\phi(x_2,s))|ds : \right.\right.$$

$$\left.\left. \phi \in C[J \times J,\mathbb{R}^n] \right\} \right) = 0;$$

(A3)

$$\lim_{x_1\to x_2}\left(\sup_{\phi}\left(\sup_{\psi}\left\{\int_I |f(s,x_1,\phi(s,x_1),\psi(x_1,s))-f(s,x_2,\phi(s,x_2),\psi(x_2,s))|ds:\right.\right.\right.$$

$$\left.\left.\left.\phi,\psi\in C[J\times J,\mathbb{R}^n]\right\}\right)\right)=0;$$

(A4)

$$|f(t,x,u,v)-f(t,x,\overline{u},v)|\le L|u-\overline{u}|.$$

We now prove the following existence result.

Theorem 2.13.1. Suppose that $u_0\in C[J,\mathbb{R}^n]$, $f\in C[J\times,J\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n]$ satisfying the assumptions (A1) and (A2). Then a solution to (2.93) exists on $[t_0,t_0+\alpha]$ for some $\alpha>0$.

Proof. Since u_0 is continuous on J, $u_0(J)$ is bounded and uniformly continuous. Thus $\exists N>0$ such that

$$|u_0(x)-u_0(\overline{x})|<N \text{ for every } x,\overline{x}\in J.$$

Let $\alpha=\min\{a,N/M\}$ and let $J_\alpha=[t_0,t_0+\alpha]$. Define $A\subseteq C[J_0\times J_0,\mathbb{R}^n]$ by

$$A=\{\phi\in C[J_\alpha\times J_\alpha,\mathbb{R}^n]:\sup_{t,x\in J_\alpha}|\phi(t,x)-u_0(x)|\le N\}.$$

Clearly A is closed, bounded and convex.

For any $\phi\in A$, define the function $T\phi$ by

$$(T\phi)(t,x)=u_0(x)+\int_{t_0}^{t}|f(s,x,\phi(s,x),\phi(x,s))|ds.$$

Then

$$|(T\phi)(t,x)-u_0(x)|\le\int_{t_0}^{t}|f(s,x,\phi(s,x),\phi(x,s))|ds\le\alpha M\le N.$$

Thus $TA\subseteq A$. Also $|(T\phi)(t,x)|\le\sup_{x\in J_\alpha}|u_0(x)|+N$. Thus TA is uniformly bounded. We show that TA is equicontinuous. Let $\varepsilon>0$ be given, and let $t_1,x_1,t_2,x_2\in J_\alpha$. Then

$$\begin{aligned}&|(T\phi)(t_1,x_1)-(T\phi)(t_2,x_2)|\\ &\le|u_0(x_1)-u_0(x_2)|+\int_{t_1}^{t_2}|f(s.x_2,\phi(s,x_2),\phi(x_2,s))|ds\\ &\quad+\int_{t_0}^{t_1}|f(s,x_2,\phi(s,x_2),\phi(x_2,s))-f(s,x_1,\phi(s,x_1),\phi(x_1,s))|ds\\ &=I_1+I_2+I_3.\end{aligned}$$

Since $u_0(x)$ is uniformly continuous, we can choose δ_1 so that $|x_1-x_2|<\delta_1\Rightarrow I_1<\varepsilon/3$. Also $I_2\le(t_2-t_1)M$; thus if $|t_2-t_1|<\varepsilon/3M=\delta_2$, $I_2<\varepsilon/3$. Now using (A2) we choose δ_3 so that $|x_1-x_2|<\delta_3\Rightarrow I_3<\varepsilon/3$.

Thus if $\max\{|t_1-t_2|,|x_1-x_2|\}<\min\{\delta_1,\delta_2,\delta_3\}=\delta$,

$$|(T\phi)(t_1,x_1)-(T\phi)(t_2,x_2)|<\varepsilon.$$

Thus TA is equicontinuous and $\overline{TA}$ is compact.

Now let $\{\phi_n\}\in A$ be a sequence converging to ψ. Since f is continuous,

$$\int_{t_0}^{t} f(s,x,\phi_n(s,x),\phi_n(x,s))ds \to \int_{t_0}^{t} f(s,x,\psi(s,x),\psi(x,s))ds.$$

Thus $T\phi_n\to T\psi$, and therefore T is continuous. Now applying the Schauder fixed point theorem, the proof is complete.

Our next result provides conditions for the extension of solutions to (2.93).

Theorem 2.13.2. Let $u_0\in C[J,\mathbb{R}^n]$ and $f\in C[J\times J\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n]$, and suppose that assumptions (A1), (A3) and (A4) hold. Then any solution u of (2.93) which exists on $J_\alpha\times J_\alpha$ can be extended to $J_\beta\times J_\beta$, where $\beta=\min(2\alpha,\alpha)$.

Proof. Let u be a solution of (2.93). Let $\gamma=\min(a/2,\alpha)$. Restrict u to $J_\gamma\times J_\gamma$.

Consider the equation

$$U'(t,x)=F(t,x,U(t,x),U(x,t)),\quad U(t_0,x)=U_0(x), \tag{2.94}$$

where

$$U_0(x)=(u(t_0+\gamma,x),u_0(x+\gamma)),$$

and

$$F(t,x,U,W)=(f(t+\gamma,x,u_1,w_2),f(t,x+\gamma,w_1,u_2)),$$

where

$$U=(u_1,u_2),\quad W=(w_1,w_2)\text{ with }u_i,w_i\in\mathbb{R}^n\text{ for }i=1,2.$$

It is clear that $U_0\in C[J_\gamma,\mathbb{R}^{2n}]$ and that $F\in C[J_\gamma\times J_\gamma\times\mathbb{R}^{2n}\times\mathbb{R}^{2n},\mathbb{R}^{2n}]$, and it is easy to verify that

$$|U_0(x)-U_0(\overline{x})|\le\sqrt{10}N\text{ and }|F(t,x,V,W)|\le\sqrt{2}M$$

for $t,x,\overline{x}\in J_\gamma$. Thus there exists a solution $U(t,x)=(u_1(t,x),u_2(t,x))$ to (2.94) on $J_\beta\times J_\beta$, where $\beta=\min(\gamma,\sqrt{5}N/M)=\gamma$.

Note that $u_1'(t,x)=f(t+\gamma,x,u_1(t,x),u_2(x,t))$, $u_2'(t,x)=f(t,x+\gamma)$, $u_2(x,t),u_1(x,t)$ and $u_1(x_0,s)\equiv u(x_0+\gamma,s)$.

Now let $m(t)=|u_2(t,x_0)-u(t,x_0+\gamma)|$. Then $m(0)=0$, and by assumption (A4)

$$m'(t)\le|f(t,x_0+\gamma,u_2(t,x_0),u_1(x_0,t))-f(t,x_0+\gamma,u(t,x_0+\gamma),u(x_0+\gamma,t))|\le Lm(t).$$

Thus $m(t) \equiv 0$, and so $u_2(t,x_0) \equiv u(t,x_0+\gamma)$.

Now consider the equation

$$\tilde{u}'(t,x) = \tilde{f}(t,x,\tilde{u}(t,x),\tilde{u}(x,t)), \quad \tilde{u}(t_0,x) = u_2(t_0+\gamma,x), \tag{2.95}$$

where $\tilde{f}$ is defined by $\tilde{f}(t,x,u,w) = f(t+\gamma,x+\gamma,u,w)$. Using Theorem 2.13.1, we conclude that there exists a solution $\tilde{u}(t,x)$ to (2.95) on $J_\gamma \times J_\gamma$, and using (A4), as above we find that $\tilde{u}(t,x_0) \equiv u_1(t,x_0+\gamma)$.

Now define the function $\overline{u}(t,x)$ on $J_\beta \times J_\beta$ as follows:

$$\overline{u}(t,x) = \begin{cases} u(t,x), & t,x \in J_\gamma, \\ u_1(t-\gamma,x), & t \in [t_0+\gamma,t_0+\beta],\ x \in J_\gamma, \\ u_2(t,x-\gamma), & x \in [t_0+\gamma,t_0+\beta],\ t \in J_\gamma, \\ \tilde{u}(t-\gamma,x-\gamma), & t,x \in [t_0+\gamma,t_0+\beta]. \end{cases}$$

We need only establish that $\overline{u}$ is an extension of u. We verify one case: suppose that $t \in [t_0+\gamma,t_0+\beta]$, $x \in J_\gamma$; then

$$\overline{u}_1'(t,x) = u_1'(t-\gamma,x) = f(t,x,u_1(t-\gamma,x),u_2(x,t-\gamma)) = f(t,x,\overline{u}(t,x),\overline{u}(x,t)).$$

The other cases are similar. Thus $\overline{u}$ extends the solution u to $J_\beta \times J_\beta$.

Remark. The above theorem can be used to extend solution of (2.93) to $J \times J$ as long as $a < +\infty$. One can easily see that if $\alpha \geq a/2$, then the value of β in Theorem 2.13.2 is a. For values of $\alpha < a/2$ one needs only to repeat the above argument a finite number of times to extend u to $J \times J$.

Corollary 2.13.1. Let u_0 and f be as in Theorem 2.13.2. Then solutions to (2.93) can be extended to $[t_0,t_0+a] \times [t_0,t_0+a]$ as long as a is finite.

We shall next develop the theory of differential inequalities. Consider the following system of differential inequalities:

$$D_-u(t,x) \leq f(t,x,u(t,x),u(x,t)), \tag{2.96}$$

$$D_-v(t,x) \geq f(t,x,v(t,x),v(x,t)), \tag{2.97}$$

where

$$D_-v(t,x) = \liminf_{h\to 0^-}\left[\frac{v(t+h,x)-v(t,x)}{h}\right].$$

Definition 2.13.1. A function $f(t,x,u,v) \in C[J \times J \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ is said to be quasimonotone nondecreasing where $u,v \in C[J \times J, \mathbb{R}^n]$ whenever $f_i(t,x,u,v) \leq f_i(t,x,\overline{u},v)$, where $u_i \leq \overline{u_i}$ and $u_j = \overline{u_j}$ for every $i,j = 1,2,\ldots,n$.

Theorem 2.13.3. Let f be (i) quasimonotone nondecreasing in $u(t,x)$ and nondecreasing in $u(x,t)$ on $J\times J$. Then if further if one of the inequalities above is strict and $u(t_0,x)<v(t_0,x)$, then $u(t,x)<v(t,x)$ on $J\times J$.

Proof. If the conclusion is not true, consider the set $Z(t,x)=\{(t,x)|u(t,x)\geq v(t,x),u(x,t\geq v(x,t))\}$, which is nonempty. Let Z_t be the projection of Z on the t axis. Let $t_1=\inf Z_t$. Certainly $t_1>t_0$. It follows that there is an index j, $1\leq j\leq n$, such that for $i=1,2,\ldots,n$

$$u_i(s,x)<v_i(s,x) \text{ for } s,x\in[t_0,t_1]\times[t_0,t_0+a],$$

$$u_i(x,s)<v_i(x,s) \text{ for } x,s\in[t_0,t_0+a]\times[t_0,t_1],$$

and either

$$u_j(t_1,x)\leq v_j(t_1,x)$$

or

$$u_j(x,t_1)\leq v_j(x,t_1)$$

$\forall x\in J$. Consequently there is an $x_1\in J$ such that either

$$u_j(t_1,x_1)=j_j(t_1,x_1) \tag{2.98}$$

or

$$u_j(x_1,t_1)=j_j(x_1,t_1). \tag{2.99}$$

Let x_1 be the minimum value of x for which (2.98) or (2.99) happens. Certainly $x_1>t_0$. If (2.98) happens, then

$$D_-u_j(t_1,x_1)=\liminf_{h\to 0^-}\frac{u_j(t_1+h,x_1)-u_j(t_1,x_1)}{}$$

$$>\liminf_{h\to 0^-}\frac{v_j(t_1),+h,x_1)-v_j(t_1,x_1)}{h}=D_-v_j(t_1,x_1).$$

But by hypothesis

$$D_-u_j(t_1,x_1)\leq f_j(t_1,x_1,u(t_1,x_1),u(x_1,t_1))$$

$$\leq f_j(t_1,x_1,v(t_1,x_1),v(x_1,t_1))<D_-v_j(t_1,x_1),$$

which leads to a contradiction.

If (2.99) happens, we have $u_j(x_1,t_1)=v_j(x_1,t_1)$. Let $x_1=\tilde{t}$, $t_1=\tilde{x}$, i.e., $u_j(\tilde{t},\tilde{x})=v_j(\tilde{t},\tilde{x})$ and $u_j(\tilde{t}+h,\tilde{x})<v_j(\tilde{t}+h,\tilde{x})$ for $h<0$, by definition of x_1 and t_1. Therefore

$$D_-u_j(\tilde{t},\tilde{x})=\liminf_{h\to 0^-}\frac{u_j(\tilde{t}+h,\tilde{x})-u_j(\tilde{t},\tilde{x})}{h}$$

$$> \liminf_{h\to 0^-} \frac{v_j(\tilde{t}+h,\tilde{x}) - v_j(\tilde{t},\tilde{x})}{h} = D_- v_j(t_1,x_1).$$

But by hypothesis

$$D_- u_j(\tilde{t},\tilde{x}) \le f_j(\tilde{t},\tilde{x}.u(\tilde{t},\tilde{x}),u(\tilde{x},\tilde{t}))$$

$$\le f_j(\tilde{t},\tilde{x},v(\tilde{t},\tilde{x}),v(\tilde{x},\tilde{t})) < D_- v_j(\tilde{t},\tilde{x}),$$

whence a contradiction, and the theorem is complete.

If one of the inequalities (2.96), (2.97) is not assumed strict, the conclusion of theorem 2.13.3 fails to hold. However, if f satisfies a one-sided Lipschitz condition, we get the following result.

Theorem 2.13.4. Let the assumption (i) of Theorem 2.13.3 hold. Suppose further that

$$f(t,x,v_1,w_1) - f(t,x,v_2,w_2) \le L[(v_1 - v_2) + (w_1 - w_2)]. \tag{2.100}$$

Whenever $v_1 \ge v_2$, $w_1 \ge w_2$. Then $u(t_0,x) \le v(t_0,x)$ for $x \in J$ implies $u(t,x) \le v(t,x)$ on $J \times J$.

Proof. Let $\tilde{v}(t,x) = v(t,x) + \varepsilon e^{3L(t+x)}$ where $\varepsilon > 0$ is a sufficiently small vector in $\mathbb{R}^n$. then $\tilde{v}'(t,x) = v'(t,x) + 3\varepsilon L e^{3L(t+x)}$. That is,

$$\tilde{v}'(t,x) = f(t,x,v(t,x),v(x,t)) + 3\varepsilon L e^{3L(t+x)}$$

$$\ge f(t,x,\tilde{v}(t,x),\tilde{v}(x,t)) + \varepsilon L e^{3L(t+x)}.$$

Consequently, we have

$$\tilde{v}'(t,x) > f(t,x,\tilde{v}(t,x),\tilde{v}(x,t)). \tag{2.101}$$

We now get $u(t,x) < \tilde{v}(t,x)$ on $J \times J$. Taking the limit as $\varepsilon \to 0$. we conclude $u(t,x) \le v(t,x)$ on $J \times J$, which provides the stated result.

2.14 Notes and Comments

The basic results presented in this chapter are new in the general setup of causal differential equations and causal functional equations. For special cases of Sec. 2.2, see Caljuk [15], Gripenberg, Londen and Staffans [16], Lakshmikantham and Mohana Rao [17], Lakshmikantham and Leela [4], Lakshmikantham, Leela and Martynyuk [18], Mamedov, Asherov and Atdaev [19], MeNabb and Weir [20], Nohel [21], Sumin [22, 23], Volterra [24] and Zhukovskii [25].

For existence results, see Corduneanu [2]. See also Corduneanu [26]-[1]. For existence results for functional differential equations and integro-differential equations, see Driver [27], Azbelev [28]-[29], Azbelev et al [30]-[31], Brandi and Ceppitelli [32], Burton [33, 34], Hale [35], Miller [36], Hara and Miyazaki [37], Kolmanovskii Myshkis [38], Kwapisz [39]-[40], Lakshmikantham and Mahana Rao [17], Li [41]-[42], Meehan and ORegan [43], Oguztorelli [44], oRegan [45], Staffans [46] and Zhivotovskii [47]. Euler solutions and flow invariance are based on the corresponding results of Clarke, Ledyaev, Stern and Wolenski [48] for ordinary differential equations. For the results related to nonlinear variations of parameters, see Drici, McRae and Vasundhara Devi [49]. The results related to Sobolev integral and differential equations see Vatsala and Vaughn [50], Lakshmikantham, Vatsala and Vaughn [51]. See also Lakshmikantham and Lord [52]. See also Lakshmikantham and Mahana Rao [17] for special cases.

For allied results, see Azbelev et al [53]-[31] and Corduneanu [54]-[55]. See also Buhgeim [56], Christyakov and Simonov [57], Ceppitelli and Faina [58], Corduneanu and Li [59], Corduneanu and Mahdavi [60], Gao et al [61], Kurbatov [62], Mahdavi [63]-[64], Li and Mahdavi [65], Myshkis [66], Neustadt [67], Marcelli and Salvadori [68], Rugh [69], Sandberg [70]-[71] and Schetzen [72].

Chapter 3

Theoretical Approximation Methods

3.1 Introduction

This chapter introduces the theoretical methods that are constructive. We first prove in Section 3.2, an existence result in a special closed set generated by the lower and upper solutions. Next in Section 3.3, we describe a constructive technique that offers monotone sequences which converge to the extremal solutions. This technique is very important because the iterates are solutions of a certain causal differential equation which can be computed explicitly and the method can be applied to many nonlinear problems. In Section 3.4, the monotone iterative technique is extended to causal differential equations where the right hand side is the sum of the two functions, one of which is monotone nondecreasing and the other is monotone nonincreasing. The results obtained include several special cases and hence, very valuable.

Section 3.5 deals with periodic boundary value problems (PBVP) for causal differential equations. Since these problems do not follow the techniques of IVP, we need to develop the required technology appropriately. Therefore the necessary causal differential inequalities theorem for PBVP is proved and then deduce the corresponding linear causal differential inequality result that is employed in the process of developing monotone iterative technique for PBVP.

Section 3.6 is devoted to the development of the method of quasilinearization, which not only offers monotone sequences that converge uniformly to the solution of IVP for causal differential equations but also shows that the convergence is quadratic. The advantage of this method is familiar to those analysts who employ numerical methods for real world problems. Section 3.7 develops the extensions of generalized quasilinearization method and considers various results in the set up. In Section 3.8, we consider Newton's method for causal equations and explore its connection to the method of quasilinearization. We

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compare and contrast Newton versus quasilinearization methods.

Finally, Section 3.9 gives notes and comments.

3.2 Method of Lower and Upper Solutions

Consider the IVP for causal differential equation

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0 \quad t_0 \geq 0, \tag{3.1}$$

where the causal operator $Q : E \to E$ is continuous, $E = C(J, \mathbb{R})$, $J = [t_0, T]$.

Definition. Let $v, w \in C(J, \mathbb{R})$. v, w are lower and upper solution of (3.1) if they satisfy the inequalities

$$\left.\begin{array}{l} v'(t) \leq (Qv)(t), \quad v(t_0) \leq x_0, \\ w'(t) \geq (Qw)(t), \ w(t_0) \geq x_0 \end{array}\right\} \tag{3.2}$$

respectively, for $t \in J$.

When we know the existence of lower and upper solutions of (3.1) such that $v(t) \leq w(t)$, $t \in J$, then we can prove the existence of a solution of the IVP (3.1) in the closed set

$$\tilde{\Omega} = \{x \in E : v(t) \leq x(t) \leq w(t), \ t \in J\}.$$

Theorem 3.2.1. Let $v, w \in C(J, \mathbb{R})$ be lower and upper solutions of IVP (3.1) satisfying $v(t) \leq w(t)$, $t \in J$. Suppose also that the operator Q is bounded on $\tilde{\Omega}$. Then, there exists a solution $x(t)$ of (3.1) in the closed set $\tilde{\Omega}$, i.e. $v(t) \leq x(t) \leq w(t)$, $t \in J$.

Proof. Let $P \in C(J, \mathbb{R})$ be defined by

$$(Px)(t) = \max[v(t), \min(x(t), w(t))].$$

Then $(QPx)(t)$ defines a continuous extension of Q on E which is also bounded since Q is assumed to be bounded on $\tilde{\Omega}$. Hence there exists a solution of IVP

$$x'(t) = (QPx)(t), \quad x(t_0) = x_0$$

on J. For any $\varepsilon > 0$, consider

$$\left.\begin{array}{r} w_\varepsilon(t) = w(t) + \varepsilon(1+t), \\ v_\varepsilon(t) = v(t) - \varepsilon(1+t). \end{array}\right\}$$

We then have $v_\varepsilon(t_0) < x_0 < w_\varepsilon(t_0)$, since $v(t_0) \leq x_0 \leq w(t_0)$. We wish to show that

$$v_\varepsilon(t) < x(t) < w_\varepsilon(t), \text{ on } J. \tag{3.3}$$

If this is not true, then there exists a $t_1 \in (t_0, T]$ at which $x(t_1) = w_\varepsilon(t_1)$ and $v_\varepsilon(t) < x(t) < w_\varepsilon(t)$, $t_0 \leq t < t_1$. Then

$$x(t_1) > w(t_1) \text{ and } (Px)(t_1) = w(t_1).$$

Moreover,

$$v(t_1) \le (Px)(t_1) \le w(t_1).$$

Hence,

$$w'(t_1) \ge (QPx)(t_1) = x'(t_1).$$

Since $w'_\varepsilon(t_1) > w'(t_1)$, we have $w'_\varepsilon(t_1) > x'(t_1)$. However, with $x(t_1) = w_\varepsilon(t_1)$ and $x(t) < w_\varepsilon(t)$, $t_0 \le t < t_1$, we have $w'_\varepsilon(t_1) \le x'(t_1)$, which is a contradiction to $w'_\varepsilon(t_1) > x'(t_1)$. Hence for all $t \in J$, $x(t) < w_\varepsilon(t)$ and consequently (3.3) holds on J. Letting $\varepsilon \to 0$ we get $v(t) \le x(t) \le w(t)$ on J. The proof is complete.

Next we shall present a simple result giving conditions that guarantee the existence of lower and upper solutions.

Theorem 3.2.2. Suppose that $(Qx)(t)$ is nonincreasing in $x \in C(J, \mathbb{R})$. Then there exists lower and upper solutions v_0, w_0 for the IVP (3.1) such that $v_0(t) \le w_0(t)$ on J.

Proof. Let $y(t)$ be the solution of

$$y'(t) = (Q0)(t), \quad y(0) = y_0.$$

Define $v_0(t) = -R_0 + y(t)$ and $w_0(t) = R_0 + y(t)$. Choose $R_0 > 0$ sufficiently large so that $v_0(t) \le 0 \le w_0(t)$ on J. Since Q is nonincreasing, this implies that

$$w'_0(t) = y'(t) = (Q0)(t) \ge (Qw_0)(t)$$

and

$$v'_0(t) = y'(t) = (Q0)(t) \le (Qv_0)(t), \quad t \in J.$$

The functions $v_0(t)$, $w_0(t)$ are desired lower and upper solutions of (3.1).

Remark. If Q is assumed to be bounded on the sector $\{x \in E : v_0(t) \le x(t) \le w_0(t),\ t \in J\}$, then by Theorem 3.2.1 there exists a solution $x(t)$ of (3.1) lying in the sector. The uniqueness of $x(t)$ is a consequence of nonincreasing nature of Q.

3.3 Monotone Iterative Technique

The results of Section 3.2 offer theoretical existence results in a sector, or a closed set. We shall now describe a constructive method that yields monotone sequences that converge to solutions of (3.1). Since each member of these sequences happens to be the solution of a certain linear differential equation which can be explicitly computed, the advantage and importance of the technique needs no special emphasis. Moreover, these ideas and methods can successfully be employed to generate two-sided bounds on solutions of IVP from which qualitative and quantitative behavior can also be investigated. Furthermore, one can apply these techniques to a variety of problems generalizing the ideas involved. Let us first prove a simple result to bring out the ideas clearly.

Theorem 3.3.1. Let $Q: E \to E$, $E = C([0,T],\mathbb{R})$, $v_0, w_0 \in E$ be lower and upper solutions of IVP (3.1) such that $v_0(t) \le w_0(t)$ on $J = [0,T]$. Suppose that $(Qx)(t) + Mx(t)$, $M > 0$ be nondecreasing function in $x \in E$, i.e., for any $x, y \in E$, we have

$$(Qx)(t) + Mx(t) \ge (Qy)(t) + My(t), \tag{3.4}$$

whenever, $x, y \in \Omega$ where

$$\Omega = \{x, y \in E : \max_{0 \le s \le t}(x(s) - y(s)) = x(t) - y(t),\ x(t) \ge y(t)\}. \tag{3.5}$$

Then, there exists monotone sequences $\{v_n\}$, $\{w_n\}$ such that $v_n(t) \to v(t)$, $w_n(t) \to w(t)$ uniformly on J and v, w are minimal and maximal solutions of IVP (3.1) on the sector $\tilde{\Omega}$, where

$$\tilde{\Omega} = \{x \in E : v_0(t) \le x(t) \le w_0(t),\ t \in J\},$$

with

$$v_0 \le v_1 \le v_2 \le \cdots \le v_n \le w_n \le \cdots \le w_2 \le w_1 \le w_0, \text{ on } J.$$

Proof. For any $\eta \in E$ such that $v_0(t) \le \eta(t) \le w_0(t)$ on J, consider the linear differential equation

$$\left.\begin{aligned} x'(t) &= (Q\eta)(t) - M[x(t) - \eta(t)] \\ x(0) &= x_0, \quad v_0(0) \le x_0 \le w_0(0). \end{aligned}\right\} \tag{3.6}$$

It is clear that for every such η, there exists a unique solution for IVP (3.6). Define a mapping A by $A\eta = x$. This mapping will be used to define the sequences $\{v_n\}$, $\{w_n\}$. Let us now prove that

(a) $v_0 \le Av_0$, $w_0 \ge Aw_0$;

(b) A is a monotone operator on the sector

$$[v_0, w_0] = \{x \in E : v_0(t) < x(t) < w_0(t),\ t \in J\}.$$

To prove (a), set $Av_0 = v_1$, where v_1 is the unique solution of (3.6) with $\eta = v_0$. Setting $p = v_1 - v_0$, we see that $p(0) \ge 0$ and

$$p' = v_1' - v_0' \ge (Qv_0)(t) - M(v_1 - v_0) - (Qv_0)(t) = -Mp.$$

This shows that $p(t) \geq p(0)e^{-Mt} \geq 0$ and hence $v_0 \leq v_1$ on J or equivalently, $v_0 \leq Av_0$. In a similar way, we can prove that $w_0 \geq Aw_0$.

To prove (b), let $\eta_1, \eta_2 \in [v_0, w_0]$ such that $\eta_1 \leq \eta_2$ and (3.4) is satisfied. Suppose that $x_1 = A\eta_1, x_2 = A\eta_2$ and set $p = x_2 - x_1$, so that $p(0) = 0$ and

$$p' = (Q\eta_2)(t) - M[x_2(t) - \eta_2(t)] - (Q\eta_1)(t) + M[x_1(t) - \eta_1(t)]$$

$$\geq -M(\eta_2 - \eta_1) - M(x_2 - \eta_2) + M(x_1 - \eta_1) = -Mp.$$

Here in using the monotone character of $(Qx)(t) + Mx(t)$, we have utilized $\eta_1, \eta_2 \in \Omega$. As before, the foregoing inequality implies $x_2 \geq x_1$, which in turn yields $A\eta_2 \geq A\eta_1$, proving (b).

We now define the sequences $\{v_n\}$, $\{w_n\}$ by

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}$$

and conclude from the previous arguments that on J,

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0.$$

Since it is easy to show that the sequences $\{v_n\}$, $\{w_n\}$ are uniformly bounded and equicontinuous, the fact that they are also monotone leads to the fact that the entire sequence $\{v_n\}$, $\{w_n\}$ converge uniformly and monotonically on J to v, w respectively. It is easy to show that v, w are solutions of (3.1) in view of the fact that v_n, w_n satisfy for $t \in J$,

$$v_n'(t) = (Qv_{n-1})(t) - M(v_n(t) - v_{n-1}(t)), \quad v_n(0) = x_0,$$

$$w_n'(t) = (Qw_{n-1})(t) - M(w_n(t) - w_{n-1}(t)), \quad w_n(0) = x_0.$$

To prove that v and w are extremal solutions of (3.1), we have to show that if $x(t)$ is any solution of (3.1) such that $v_0(t) \leq x(t) \leq w_0(t)$, $t \in J$, then

$$v_0(t) \leq v(t) \leq x(t) \leq w(t) \leq w_0, \quad t \in J.$$

Suppose that for some n, $v_n \leq x \leq w_n$ on J and set $p = x - v_{n+1}$ so that $p(0) = 0$ and

$$p' = (Qx)(t) - (Qv_n)(t) + M(v_{n+1} - v_n)$$

$$\geq -M(x - v_n) + M(v_{n+1} - v_n) = -Mp.$$

Here, we have used the monotone character of $(Qx)(t) + Mx(t)$ with the condition $x, v_n \in \Omega$. This implies as before $v_{n+1} \leq x$ on J. Similarly, $x \leq w_{n+1}$ on J and hence $v_{n+1} \leq x \leq w_{n+1}$ on J.

Since $v_0 \le x \le w_0$ on J, this proves by induction that on J, $v_n \le x \le w_n$ for all n. Taking the limit as $n \to \infty$, we conclude that $v \le x \le w$, proving that v, w are extremal solutions of (3.1). The proof is complete.

We observe that the special case when (Qx) is monotone nondecreasing is covered by Theorem 3.3.1. To see this, it is enough to take $M = 0$ in (3.4). However, the other case, when (Qx) is monotone nonincreasing is not covered by Theorem 3.3.1 and is of particular interest. We shall next discuss this important special case. We find that under somewhat special conditions, we shall show that when (Qx) is nonincreasing, a single iteration procedure yields an alternating sequence which forms two monotone sequences bounding the solution from above and below. The iteration scheme in this case is simply either

$$v'_{n+1}(t) = (Qv_n)(t), \quad v_{n+1}(0) = x_0, \tag{3.7}$$

or

$$w'_{n+1}(t) = (Qw_n)(t). \quad w_{n+1}(0) = x_0. \tag{3.8}$$

Theorem 3.3.2. Suppose that (Qx) is nonincreasing in x, then either

(i) the iterates $v_n(t)$ given by (3.7) and the unique solution $x(t)$ of (3.1) satisfy for $t \in J$,

$$v_0 \le v_2 \le \cdots \le v_{2n} \le x(t) \le v_{2n+1} \le \cdots \le v_3 \le v_1 \tag{3.9}$$

provided $v_2(t) \ge v_0(t)$ on J. Furthermore, the alternating sequences $\{v_{2n}\}, \{v_{2n+1}\}$ converge uniformly and monotonically to $p(t), r(t)$ respectively and $p(t) \le x(t) \le r(t)$ on J; or

(ii) the iterates $w_n(t)$ given by (3.8) and the unique solution $x(t)$ of (3.1) satisfy for $t \in J$,

$$w_1 \le w_3 \le \cdots \le w_{2n+1} \le x(t) \le w_{2n} \le \cdots \le w_2 \le w_0 \tag{3.10}$$

provided $w_2(t) \le w_0(t)$ on J. Moreover, the alternating sequences $\{w_{2n+1}\}$, $\{w_{2n}\}$ converge uniformly and monotonically to $p^*(t), r^*(t)$ respectively and $p^*(t) \le x(t) \le r^*(t)$ on J.

In fact, since the extremal solutions of (3.1) are unique, $p^*(t) = p(t)$ and $r^*(t) = r(t)$, $t \in J$.

Proof. By Theorem 3.2.2, there exists lower and upper solutions v_0, w_0 and a unique solution $x(t)$ of (3.1) such that $v_0 \le x \le w_0$ on J. We shall only prove the case (i) since the proof of (ii) follows similar arguments.

Assuming that $v_0 \le v_2$ on J, we shall first show that

$$v_0(t) \le v_2(t) \le x(t) \le v_3(t) \le v_1(t) \text{ on } J. \tag{3.11}$$

Setting $p = v_1 - v_0$, we find that

$$p' = v_1' - v_0' \geq (Qv_0)(t) - (Qv_0)(t) = 0, \quad p(0) \geq 0$$

and therefore, $p(t) \geq p(0) \geq 0$ i.e., $v_1 \geq v_0$ on J. Now letting $p = x - v_1$, we get

$$p' = x' - v_1' = (Qx)(t) - (Qv_0)(t) \leq 0 \text{ and } p(0) = 0.$$

This implies $x(t) \leq v_1(t)$ on J. By using similar arguments, we can show successively

$$v_2(t) \leq x(t), \quad v_3(t) \leq v_1(t), \text{ and } x(t) \leq v_3(t), \quad t \in J.$$

Consequently, we have proved that (3.11) holds for $t \in J$.

To prove (3.9), we use the induction principle, i.e. assume that (3.9) is true for some n and show that it holds for $(n+1)$. Consider $p = v_{2n+2} - v_{2n+1}$. Then by using (3.7) and the monotone character of Q, we have

$$p' = v_{2n+2}' - v_{2n+1}' = (Qv_{2n+1})(t) - (Qv_{2n})(t) \leq 0$$

and $p(0) = 0$. This shows $p(t) \leq 0$ and hence $v_{2n+2}(t) \leq v_{2n+1}(t)$. By repeating similar arguments we can get

$$v_0 \leq v_2 \leq \cdots \leq v_{2n} \leq v_{2n+2} \leq x \leq v_{2n+3} \leq v_{2n+1} \leq \cdots \leq v_3 \leq v_1$$

on J. Since (3.9) is true for $n = 1$, it follows by induction that (3.9) is true for all n. It is easy to conclude that the sequences $\{v_{2n}\}, \{v_{2n+1}\}$ converge uniformly and monotonically to $p(t), r(t)$ respectively and that $p(t) \leq x(t) \leq r(t)$ on J. This proves (i) and the proof of Theorem 3.3.2 is complete.

Corollary 3.3.1. In addition to the assumptions of Theorem 3.3.2, suppose that

$$(Qu_1)(t) - (Qu_2)(t) \geq -M(u_1(t) - u_2(t))$$

wherever $u_1(t) \geq u_2(t)$, i.e. $u_1, u_2 \in \Omega$. Then $p(t) = r(t) = x(t)$ on J.

We note that in the proof of Theorem 3.3.2, p and r are indeed quasi solutions since $p'(t) = (Qr)(t)$, $r'(t) = (Qp)(t)$ on J.

3.4 Generalized Monotone Iterative Technique

We shall devote this section to proving general results relative to monotone iterative technique which contain as special case, several important results of interest. We need the following definition which characterizes lower and upper solutions of various types. We consider

$$x'(t) = (Px)(t) + (Qx)(t), \quad x(0) = x_0 \tag{3.12}$$

where $P, Q : E \to E = C(J, \mathbb{R})$, $J = [0, T]$.

Definition. Relative to the IVP for causal differential equation (3.12), the functions $\alpha, \beta \in C^1(J, \mathbb{R})$ are said to be

(a) natural lower and upper solutions of (3.12) if

$$\begin{cases} \alpha'(t) \leq (P\alpha)(t) + (Q\alpha)(t), \ \alpha(0) \leq x_0 \\ \beta'(t) \geq (P\beta)(t) + (Q\beta)(t), \ \beta(0) \geq x_0; \end{cases} \tag{3.13}$$

(b) coupled lower and upper solutions of type I for (3.12) if

$$\begin{cases} \alpha'(t) \leq (P\alpha)(t) + (Q\beta)(t), \ \alpha(0) \leq x_0 \\ \beta'(t) \geq (P\beta)(t) + (Q\alpha)(t), \ \beta(0) \geq x_0; \end{cases} \tag{3.14}$$

(c) coupled lower and upper solutions of type II for (3.12) if

$$\begin{cases} \alpha'(t) \leq (P\beta)(t) + (Q\alpha)(t), \ \alpha(0) \leq x_0 \\ \beta'(t) \geq (P\alpha)(t) + (Q\beta)(t), \ \beta(0) \geq x_0; \end{cases} \tag{3.15}$$

(d) coupled lower and upper solutions of type III for (3.12) if

$$\begin{cases} \alpha'(t) \leq (P\beta)(t) + (Q\beta)(t), \ \alpha(0) \leq x_0 \\ \beta'(t) \geq (P\alpha)(t) + (Q\alpha)(t), \ \beta(0) \geq x_0. \end{cases} \tag{3.16}$$

Whenever $\alpha(t) \leq \beta(t)$, $t \in J$ and the operators P, Q are monotone in the sense that (Px) is nondecreasing and (Qx) is nonincreasing, then the lower and upper solutions defined in (3.13) and (3.16) also satisfy (3.15). Hence it is enough to consider only the cases (3.14) and (3.15). We are now in a position to prove the first main result.

Theorem 3.4.1. Assume that the following hypotheses hold:

(i) $\alpha_0, \beta_0 \in C^1(J, \mathbb{R})$ are the coupled lower and upper solutions of type I for IVP (3.12) with $\alpha_0(t) \leq \beta_0(t)$ on J;

(ii) the operators P, Q in (3.12) are such that $P, Q : E \to E$, (Px) is nondecreasing in x and (Qx) is nonincreasing in x.

Then there exists monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ such that

$$\alpha_n(t) \to p(t), \quad \beta_n(t) \to r(t)$$

uniformly on J and p, r are coupled minimal and maximal solutions of IVP (3.12), i.e., p, r satisfy

$$p'(t) = (Pp)(t) + (Qr)(t), \quad \alpha_0(0) \leq p(0) \leq \beta_0(0),$$

$$r'(t) = (Pr)(t) + (Qp)(t), \quad \alpha_0(0) \leq r(0) \leq \beta_0(0).$$

Proof. Consider the following linear causal differential equations

$$\alpha'_{n+1}(t) = (P\alpha_n)(t) + (Q\beta_n)(t), \quad \alpha_{n+1}(0) = x_0, \tag{3.17}$$

$$\beta'_{n+1}(t) = (P\beta_n)(t) + (Q\alpha_n)(t), \quad \beta_{n+1}(0) = x_0. \tag{3.18}$$

Clearly, there exists unique solutions $\alpha_{n+1}(t)$ and $\beta_{n+1}(t)$ on J, for the IVPs (3.17) and (3.18) respectively. Now we wish to prove that

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_1 \leq \beta_0 \tag{3.19}$$

on J. Setting $n = 0$ in (3.17) and taking $p = \alpha_0 - \alpha_1$, we obtain

$$p' = \alpha'_0 - \alpha'_1 \leq (P\alpha_0)(t) + (Q\beta_0)(t) - (P\alpha_0)(t) - (Q\beta_0)(t) = 0$$

and $p(0) \leq 0$. This implies that $p(t) \leq 0$ on J, which gives $\alpha_0(t) \leq \alpha_1(t)$ on J. Similarly it can be shown that $\beta_1 \leq \beta_0$ on J.
Now set $p = \alpha_1 - \beta_1$. Then, by using (3.17), we have

$$p' = \alpha'_1 - \beta'_1 = (P\alpha_0)(t) + (Q\beta_0)(t) - (P\beta_0)(t) - (Q\alpha_0)(t) \leq 0$$

using the monotone nature of the operators P, Q and the fact $\alpha_0 \leq \beta_0$ on J. Thus, we have $p'(t) \leq 0$, $p(0) \leq 0$ which yields $p(t) \leq 0$, i.e., $\alpha_1(t) \leq \beta_1(t)$ on J. Now assume that for some integer $k > 1$

$$\alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1} \text{ on } J.$$

We shall show that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k, \text{ on } J.$$

Consider $p = \alpha_k - \alpha_{k+1}$ on J. Then, by (3.17) and the monotone nature of P and Q, we obtain

$$p' = \alpha'_k - \alpha'_{k+1} = (P\alpha_{k-1})(t) + (Q\beta_{k-1})(t) - (P\alpha_k)(t) - (Q\beta_k)(t)$$

$$\leq (P\alpha_k)(t) + (Q\beta_k)(t) - (P\alpha_k)(t) - (Q\beta_k)(t) \leq 0.$$

Since $p(0) = 0$, this implies $p(t) \leq 0$ or equivalently, $\alpha_k \leq \alpha_{k+1}$ on J. Similarly, we can show that $\beta_{k+1} \leq \beta_k$ on J, using (3.18) and the monotone properties of P, Q. To prove $\alpha_{k+1} \leq \beta_{k+1}$, set $p = \alpha_{k+1} - \beta_{k+1}$ to see that $p(0) = 0$ and

$$p' = \alpha'_{k+1} - \beta'_{k+1} = (P\alpha_k)(t) + (Q\beta_k)(t) - (P\beta_k)(t) - (Q\alpha_k)(t)$$

$$\leq (P\beta_k)(t) + (Q\alpha_k)(t) - (P\beta_k)(t) - (Q\alpha_k)(t) \leq 0,$$

since $\alpha_k \le \beta_k$. That yields $p(t) \le 0$ i.e., $\alpha_{k+1} \le \beta_{k+1}$. Now, by induction principle, we have (3.19) for all n.

Clearly, the constructed sequences $\{\alpha_n\}, \{\beta_n\}$ are uniformly bounded and equicontinuous. Since they are monotone sequences, we find that not only some subsequences but the entire sequences $\{\alpha_n\}, \{\beta_n\}$ converge uniformly and monotonically to p, r on J respectively. It is easy to see that from (3.17) and (3.18) that p, r are coupled solutions.

To show that p, r are coupled minimal and maximal solutions of IVP (3.12), let $x(t)$ be any solution of (3.12) such that $\alpha_0 \le x \le \beta_0$ on J. Suppose that for some k, $\alpha_k \le x \le \beta_k$ on J. Setting $p = \alpha_{k+1} - x$, we obtain $p(0) = 0$ and

$$p' = \alpha'_{k+1} - x' = (P\alpha_k)(t) + (Q\beta_k)(t) - (Px)(t) - (Qx)(t) \le 0,$$

using the monotone nature of P, Q and the assumption $\alpha_k \le x \le \beta_k$ on J. That implies $p(t) \le 0$, proving that $\alpha_{k+1} \le x$ on J. Similarly, it can be shown that $x \le \beta_{k+1}$ on J and hence, by principle of induction, $\alpha_n \le x \le \beta_n$ holds for all n. Taking limit as $n \to \infty$, we have $p \le x \le r$, completing the proof that p, r are coupled minimal and maximal solutions of (3.12) since, from (3.17), (3.18) we get

$$p'(t) = (Pp)(t) + (Qr)(t), \quad p(0) = x_0,$$

$$r'(t) = (Pr)(t) + (Qp)(t), \quad r(0) = x_0$$

respectively on J.

Corollary 3.4.1. If in addition to the assumptions of Theorem 3.4.1, we suppose that for $u_1 \ge u_2$, $u_1, u_2 \in \Omega$, we have

$$(Pu_1)(t) - (Pu_2)(t) \le N_1(u_1(t) - u_2(t)), \quad N_1 > 0,$$

$$(Qu_1)(t) - (Qu_2)(t) \ge -N_2(u_1(t) - u_2(t)), \quad N_2 > 0,$$

then $p(t) = x(t) = r(t)$ on J.

Proof. Since $p \le r$ on J, it is enough to show that $r \le p$. Consider $y = r - p$. Then, $y(0) = 0$ and

$$\begin{aligned} y' = r' - p' &= (Pr)(t) + (Qp)(t) - (Pp)(t) - (Qr)(t) \\ &\le N_1(r-p) + N_2(r-p) \\ &= (N_1 + N_2)y. \end{aligned}$$

Hence $y(t) \le 0$ on J, proving that $r \le p$ on J. hence $r = x = p$ on J, completing the proof.

Remark 3.4.1. Following the proof of theorem 3.4.1, there are several interesting remarks to be made which indicate many ramifications and provide useful special cases:

(1) In Theorem 3.4.1, suppose that $(Qx) = 0$. Then α_0, β_0 are natural lower and upper solutions of (3.12) and with (Px) nondecreasing, we get the monotone sequences $\{\alpha_n\}, \{\beta_n\}$ converging to minimal and maximal solutions of (3.12) respectively, lying in the sector $[\alpha_0, \beta_0]$.

(2) However if (Px) is not nondecreasing and $(Qx) = 0$, we can assume that $(Px) + Mx$ is nondecreasing in x for some $M > 0$ and still come to the same conclusion as above, since the IVP

$$x'(t) = (\tilde{P}x)(t), \quad x(t_0) = x_0$$

satisfies the conditions of Theorem 3.3.1.

Also, when (Px) is not nondecreasing, we consider the IVP

$$x'(t) = (\tilde{P}x)(t) - Mx(t), \quad x(t_0) = x_0, \tag{3.20}$$

where $(\tilde{P}x) = (Px) + Mx, M > 0$ is nondecreasing. (3.20) is same as (3.12) with $(Qx) = 0$. We see that it can also be seen as (3.12) with (Px) replaced by $(\tilde{P}x)$ and (Qx) replaced by $-Mx$. Hence we get the same conclusions as of Theorem 3.4.1, since $(\tilde{P}x)$ is nondecreasing and $-Mx$ is nonincreasing in x.

(3) If $(Px) = 0$ in Theorem 3.4.1, we obtain the result for nonincreasing (Qx) and α_0, β_0 are coupled lower and upper solutions of the IVP $x'(t) = (Qx)(t)$ with nonincreasing (Qx). In this case, the monotone iterates $\{\alpha_n\}, \{\beta_n\}$ converge to p, r respectively which satisfy

$$p'(t) = (Qr)(t), \quad r'(t) = (Qp)(t), \quad r(t_0) = x_0 = p(t_0).$$

(4) If in (3) above, we suppose that Qx is not nonincreasing and there exists a $N > 0$ such that $(\tilde{Q}x) = (Qx) - Nx$ is nonincreasing we can consider the IVP

$$x'(t) = (Qx)(t) = (\tilde{Q}x)(t) + Nx(t), \quad x(t_0) = x_0$$

which is the same as IVP (3.12) with (Px) replaced by Nx which is nondecreasing and (Qx) replaced by $(\tilde{Q}x)$ which is nonincreasing. Hence the case then reduces to Theorem 3.4.1 and the conclusion of Theorem 3.4.1 remains valid.

(5) Suppose (Px) is nondecreasing but (Qx) is not nonincreasing. Then consider the IVP

$$x'(t) = (\tilde{P}x)(t) + (\tilde{Q}x)(t), \quad x(t_0) = x_0, \tag{3.21}$$

where $(\tilde{P}x) = (Px) + Nx$, $N > 0$ is nondecreasing and $(\tilde{Q}x) = (Qx) - Nx$, $N > 0$, is nonincreasing. This results in Theorem 3.4.1 with $(Px), (Qx)$ replaced by $(\tilde{P}x), (\tilde{Q}x)$ respectively and the conclusion of Theorem 3.4.1 holds. Note that $(\tilde{P}x)(t) + (\tilde{Q}x)(t) = (Px)(t) + (Qx)(t)$ and hence, (3.21) is the same as (3.12).

(6) If (Px) is not nondecreasing but (Qx) is nonincreasing, then consider the IVP

$$x'(t) = (\tilde{P}x)(t) + (\tilde{Q}x)(t), \quad x(t_0) = x_0, \tag{3.22}$$

where $(\tilde{P}x) = (Px) + Mx$, $M > 0$ is nondecreasing and $(\tilde{Q}x) = (Qx) - Mx$, $M > 0$ is nonincreasing. This results in Theorem 3.4.1 and so, the conclusion of Theorem 3.4.1 is valid. Again note that IVP (3.22) is the same as (3.12) since $(\tilde{P}x)(t) + (\tilde{Q}x)(t) = (Px)(t) + (Qx)(t)$.

(7) If (Px) is not nondecreasing and (Qx) is not nonincreasing, then for $M > 0, N > 0$, such that $(\tilde{P}x) = (Px) + Mx$ is nondecreasing and $(\tilde{Q}x) = (Qx) - Nx$ is nonincreasing, we get the context of Theorem 3.4.1 with $(Px), (Qx)$ replaced by $(\tilde{P}x), (\tilde{Q}x)$ respectively and hence the conclusion of Theorem 3.4.1 remains valid.

Next, we shall consider the case of the coupled lower and upper solutions of type II for IVP (3.12). Here, we need not assume the existence of coupled lower and upper solutions, since it can be established with the given assumptions.

Theorem 3.4.2. Assume that the hypothesis (ii) of Theorem 3.4.1 holds. Then, for any solution $x(t)$ of (3.12) with $\alpha_0 \le x \le \beta_0$ on J, we have the iterates α_n, β_n satisfying, for $t \in J$,

$$\left.\begin{aligned} &\alpha_0 \le \alpha_2 \le \cdots \le \alpha_{2n} \le x \le \alpha_{2n+1} \le \cdots \le \alpha_3 \le \alpha_1, \\ &\beta_1 \le \beta_3 \le \cdots \le \beta_{2n+1} \le x \le \beta_{2n} \le \cdots \le \beta_2 \le \beta_0, \end{aligned}\right\} \tag{3.23}$$

provided $\alpha_0 \le \alpha_2$ and $\beta_2 \le \beta_0$ on J, where the iterates are given by

$$\left.\begin{aligned} &\alpha'_{n+1}(t) = (P\beta_n)(t) + (Q\alpha_n)(t),\ \alpha_{n+1}(0) \le x_0, \\ &\beta'_{n+1}(t) = (P\alpha_n)(t) + (Q\beta_n)(t),\ \beta_{n+1}(0) \ge x_0,\ t \in J. \end{aligned}\right\} \tag{3.24}$$

Moreover, the monotone sequences $\{\alpha_{2n}\}, \{\alpha_{2n+1}\}, \{\beta_{2n}\}, \{\beta_{2n+1}\}$ converge uniformly to $p, r, r*, p*$ respectively and they satisfy

$$r'(t) = (Pr^*)(t) + (Qp)(t),$$

$$p'(t) = (Pp^*)(t) + (Qr)(t),$$

$$r^{*\prime}(t) = (Pr)(t) + (Qp^*)(t),$$

$$p^{*\prime}(t) = (Pp)(t) + (Qr^*)(t),$$

for $t \in J$ and $p \le x \le r$, $p^* \le x \le r^*$, $t \in J$, $r(0) = p(0) = p^*(0) = r^*(0) = x_0$.

Proof. In view of hypothesis (ii), it is easy to construct coupled lower and upper solutions of (3.12) following the method of Theorem 3.2.2. Hence, we proceed further assuming such coupled lower and upper solutions α_0, β_0 exist. Assume that $\alpha_0 \leq \alpha_2$ and $\beta_2 \leq \beta_0$, on J. We show that

$$\left.\begin{aligned} &\alpha_0 \leq \alpha_2 \leq x \leq \alpha_3 \leq \alpha_1, \\ &\beta_1 \leq \beta_3 \leq x \leq \beta_2 \leq \beta_0, \text{ on } J. \end{aligned}\right\} \tag{3.25}$$

Set $p = x - \alpha_1$, so that by (3.24) and (3.12)

$$p' = x' - \alpha_1' = (Px)(t) + (Qx)(t) - (P\beta_0)(t) - (Q\alpha_0)(t) \leq 0.$$

Here we have used the fact $\alpha_0 \leq x \leq \beta_0$ on J, x being any solution of (3.12) and the monotone nature of the operators P, Q. Also, $p(0) = 0$. Hence $p(t) \leq 0$ on J, i.e., $x \leq \alpha_1$ on J.

We shall next show that $\alpha_3 \leq \alpha_1$, $\beta_1 \leq x$ and $\alpha_2 \leq x$, by considering the differences $p = \alpha_3 - \alpha_1$, $p = \beta_1 - x$ and $p = \alpha_2 - x$ respectively and showing in each case $p'(t) \leq 0, t \in J$. In fact, for $p = \alpha_3 - \alpha_1$, we have $p(0) = 0$ and using (3.23),

$$p'(t) = \alpha_3'(t) - \alpha_1'(t) = (P\beta_2)(t) + (Q\alpha_2)(t) - (P\beta_0)(t) - (Q\alpha_0)(t) \leq 0$$

because of the assumptions $\alpha_0 \leq \alpha_2$, $\beta_2 \leq \beta_0$ and the monotone nature of P, Q. Hence, $p(t) \leq 0, t \in J$ and thus $\alpha_3(t) \leq \alpha_1(t)$. Similarly, the difference $p = \beta_1 - x$ leads to $p(0) = 0$, and

$$p'(t) = \beta_1'(t) - x'(t) = (P\alpha_0)(t) + (Q\beta_0)(t) - (Px)(t) - (Qx)(t) \leq 0$$

because of the fact that $\alpha_0 \leq x \leq \beta_0$ (3.24) and monotone character of P, Q, thus proving $p(t) \leq 0, t \in J$, i.e., $\beta_1(t) \leq x(t)$.

The difference $p = \alpha_2 - x$ leads to $p(0) = 0$ and

$$p'(t) = \alpha_2'(t) - x'(t) = (P\beta_1)(t) + (Q\alpha_1)(t) - (Px)(t) - (Qx)(t) \leq 0$$

since $\beta_1 \leq x \leq \alpha_1$, (Px) is nondecreasing and (Qx) is nonincreasing. Using similar arguments we can show that for each of the following $\beta_3 - x$, $x - \beta_2$, and $\beta_1 - \beta_3$ has their derivatives less than 0, $t \in J$ thus proving $\beta_3 \leq x, x \leq \beta_2$ and $\beta_1 \leq \beta_3$. Combining all these arguments, we now have the desired relation (3.25).

Now, assume for $n > 2$, the following inequalities hold:

$$\left.\begin{aligned} &\alpha_{2n-4} \leq \alpha_{2n-2} \leq x \leq \alpha_{2n-1} \leq \alpha_{2n-3} \\ &\beta_{2n-3} \leq \beta_{2n-1} \leq x \leq \beta_{2n-2} \leq \beta_{2n-4}, \quad t \in J. \end{aligned}\right\} \tag{3.26}$$

By employing arguments similar to our earlier discussion, it can be shown that for $t \in J$,

$$\left.\begin{aligned} &\alpha_{2n-2} \leq \alpha_{2n} \leq x \leq \alpha_{2n+1} \leq \alpha_{2n-1} \\ &\beta_{2n-1} \leq \beta_{2n+1} \leq x \leq \beta_{2n} \leq \beta_{2n-2}. \end{aligned}\right\} \tag{3.27}$$

In fact, using the differences $\alpha_{2n-2}-\alpha_{2n}$, $\beta_{2n}-\beta_{2n-2}$, $\alpha_{2n+1}-\alpha_{2n-1}$ and $\beta_{2n-1}-\beta_{2n+1}$ successively to show that each of these is negative for $t\in J$, since in each case we get $p(0)=0$ and $p'(t)\le 0$, in view of the relations in (3.26), (3.24) and the monotone character of P,Q.

To prove $\alpha_{2n}\le x$ and $x\le\beta_{2n}$, consider the relations

$$p'(t)=\alpha_{2n}'(t)-x'(t)=(P\beta_{2n-1})(t)+(Q\alpha_{2n-1})(t)-(Px)(t)-(Qx)(t)\le 0$$

and

$$p'(t)=x'(t)-\beta_{2n}'=(Px)(t)+(Qx)(t)-(P\alpha_{2n-1})(t)-(Q\beta_{2n-1})(t)\le 0$$

together with $p(0)=0$ in each case, where the monotonicity of P,Q and the inequalities $x\le\alpha_{2n-1}$, $\beta_{2n-1}\le x$ of (3.26) are used. Hence, as before, we can conclude that (3.27) holds whenever we assume (3.26) to be true.

Now, with the principle of induction, the two chains of inequalities in (3.23) are established for $t\in J$ and all n. By employing reasoning similar to that of Theorem 3.4.1, we arrive at

$$\lim_{n\to\infty}\alpha_{2n}=p,\quad \lim_{n\to\infty}\alpha_{2n+1}=r,$$

$$\lim_{n\to\infty}\beta_{2n+1}=p^*,\quad \lim_{n\to\infty}\beta_{2n}=r^*,\ \text{on } J.$$

Thus,

$$\lim_{n\to\infty}[(P\beta_{2n})(t)+(Q\alpha_{2n})(t)]=(Pr^*)(t)+(Qp)(t),$$

$$\lim_{n\to\infty}[(P\beta_{2n+1})(t)+(Q\alpha_{2n+1})(t)]=(Pp^*)(t)+(Qr)(t),$$

$$\lim_{n\to\infty}[(P\alpha_{2n})(t)+(Q\beta_{2n})(t)]=(Pp)(t)+(Qr^*)(t),$$

$$\lim_{n\to\infty}[(P\alpha_{2n+1})(t)+(Q\beta_{2n+1})(t)]=(Pr)(t)+(Qp^*)(t),$$

on J. It is now easy to obtain the relations satisfied by r,p,r^*, and p^* as stated in the conclusion of Theorem 3.4.2. The proof is complete.

Theorem 3.4.2 also contains several special cases as in Theorem 3.4.1. But to avoid monotony of ideas, we have not stated all those special cases. It is easy for the reader to recognize the various special cases.

3.5 Monotone Technique for PBVPs

We have seen that the monotone iterative technique is an effective and flexible mechanism to provide constructive existence results in a closed set, generated by the lower and upper solutions for the IVP of causal differential equations. In this section, we shall extend the technique for periodic boundary value problems. We need the following lemma before we proceed further.

Lemma 3.5.1. Let $m \in C^1[J,\mathbb{R}]$ be such that

$$m'(t) \le -Mm(t) - (Lm)(t), \quad m(0) \le m(2\pi), \quad t \in J = [0,2\pi], \tag{3.28}$$

where $M > 0$ and $L \in C[E,E]$ is a positive linear operator, that is, $Lm \ge 0$ wherever $m \ge 0$. Then, $m(t) \le 0$, $t \in J$ provided one of the following conditions hold:

(a) $2\pi e^{2M\pi}(Le^{-M})(2\pi) \le 1$;
(b) $2\pi(M + (L1)(2\pi)) \le 1$.

Proof. Suppose (a) holds. Set $v(t) = m(t)e^{Mt}$ so that inequality (3.28) reduces to

$$v'(t) \le -e^{Mt}(L(ve^{-M}))(t). \tag{3.29}$$

It is enough to prove $v(t) \le 0$ for $t \in J$.
If this is not true, then we have the following cases:

(A) $v(t) \ge 0$ for $t \in J$ and $v(t) \not\equiv 0$;
(B) there exists $t_1, t_2 \in J$ such that $v(t_1) > 0$ and $v(t_2) < 0$.

In case (A) we have $v(0) \le v(2\pi)$ and from (3.29) we also have that $v'(t) \le 0$ on J. Since $v(0) \le v(2\pi)$ and $v(t)$ is nonincreasing on J, $v(t) \equiv C > 0$. Hence, $m(t) = Ce^{-Mt}$, which implies $m(0) \ge m(2\pi)$. In view of(3.28), we conclude that $C = 0$, and we get a contradiction to (A).
In case (B), we have two situations:

(i) $v(2\pi) \ge 0$ and
(ii) $v(2\pi) < 0$.

When $v(2\pi) \ge 0$, it is clear that $v(0) < v(2\pi)$. Suppose that $v(t_2) = -\lambda$ where $\min_{0\le t\le 2\pi} v(t) = -\lambda$, $\lambda > 0$. Then, $t_2 \in [0,2\pi)$, and using the mean-value theorem on $[t_2, 2\pi]$, we get

$$v'(t_0) = \frac{v(2\pi)+\lambda}{2\pi - t_2} > \frac{\lambda}{2\pi}, \tag{3.30}$$

for some $t_0 \in (t_2, 2\pi)$. On the other hand, since $v(s) \geq -\lambda$ for $s \in [0, 2\pi]$ we have from (3.29) and condition (a):

$$\begin{aligned} v'(t_0) &\leq -e^{Mt_0}(L(ve^{-M}))(t_0) \\ &\leq e^{Mt_0}(L(\lambda e^{-M}))(t_0) \\ &\leq \lambda e^{2\pi M}(L(e^{-M}))(2\pi) \leq \frac{\lambda}{2\pi} \end{aligned}$$

This is a contradiction to (3.30). When $v(2\pi) < 0$, we also have $v(0) < 0$ and there exists a $t^* \in (0, 2\pi)$ such that $v(t^*) = 0$ and $v(t) < 0$ for $t \in (t^*, 2\pi]$. It is clear that $\min_{0 \leq t \leq t^*} v(t) < 0$. Let $-\lambda = \min_{0 \leq t \leq t^*} v(t) = v(t_2)$, where $t_2 \in [0, t^*)$ and $\lambda > 0$. We can repeat the argument employed above in the interval $[t_2, t^*]$ and obtain a contradiction. This completes the proof of the lemma when condition (a) holds.

If (b) holds, we proceed starting directly from (3.28). Again we have the two cases (A) and (B) relative to $m(t)$. When (A) holds, since $m(0) \leq m(2\pi)$, it is clear that

$$m(t_0) = \max_{0 \leq t \leq 2\pi} m(t) > 0 \text{ with } t_0 \in (0, 2\pi].$$

Therefore, in view of (3.28)

$$0 = m'(t_0) \leq -Mm(t_0) - (Lm)(t_0) \leq -Mm(t_0) < 0,$$

which is a contradiction.

When case (B) holds, we argue exactly as before and find that condition (B) leads to a contradiction. The proof of the lemma is complete.

Let us begin with the definition of the upper and lower solutions for the PBVP involving causal operators,

$$\begin{aligned} u'(t) &= (Qu)(t), \\ u(0) &= u(2\pi), \end{aligned} \tag{3.31}$$

where $Q \in C[E, E]$, $E = C[J, \mathbb{R}]$.

Definition 3.5.1. $\alpha, \beta \in C^1[J, \mathbb{R}]$ are said to be lower and upper solutions of (3.31) respectively, if

$$\alpha'(t) \leq (Q\alpha)(t), \quad \alpha(0) \leq \alpha(2\pi),$$

and

$$\beta'(t) \geq (Q\beta)(t), \quad \beta(0) \geq \beta(2\pi).$$

Now, we are in a position to develop the monotone iterative technique (MIT) for (3.31) and we proceed to do so in the following theorem.

Theorem 3.5.1. Let $Q \in C[E, E]$. Assume that for $t \in J$,

(i) $\alpha, \beta \in C^1[J, \mathbb{R}]$ are lower and upper solutions of (3.31) respectively;

(ii) whenever $\alpha(t) \leq v(t) \leq u(t) \leq \beta(t)$, $(Qu)(t) - (Qv)(t) \geq -M(u(t)-v(t)) - (L(u-v))(t)$, where $M > 0$ is a positive constant and $L \in C[E,E]$ is a positive linear operator such that $e^{2M\pi}|L| < M$, where $|L| = \sup_{|u|=1}\left[\frac{|Lu|}{|u|}\right]$.

Then, there exists monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ with $\alpha_0 = \alpha$, $\beta_0 = \beta$ such that $\lim_{n\to\infty}\alpha_n(t) = p(t)$, $\lim_{n\to\infty}\beta_n(t) = r(t)$ where p, r are minimal and maximal solutions of the PBVP (3.31), respectively, satisfying $\alpha(t) \leq p(t) \leq r(t) \leq \beta(t)$ on J.

Proof. For any $\eta \in C[J, \mathbb{R}]$, $\alpha(t) \leq \eta(t) \leq \beta(t)$, $t \in J$, consider the linear PBVP

$$u'(t) + Mu(t) = -(Lu)(t) + \sigma_\eta(t), \quad u(0) = u(2\pi), \tag{3.32}$$

where $\sigma_\eta(t) = (Q\eta)(t) + M\eta(t) + (L\eta)(t)$.

Now,

$$u(t) = e^{-Mt}\left[\frac{1}{e^{2M\pi}-1}\int_0^{2\pi}[\sigma_\eta(s) - (Lu)(s)]e^{Ms}ds + \int_0^t [\sigma_\eta(s) - (Lu)(s)]e^{Ms}ds\right]$$

$$\equiv (Su)(t).$$

To show S is a contraction, consider

$$\begin{aligned}|(Su)(t) - (Sv)(t)| &= \left| e^{-Mt}\frac{1}{e^{2M\pi}-1}\int_0^{2\pi}(L(u-v))(s)e^{Ms}ds \right. \\ &\quad \left. + e^{-Mt}\int_0^t (L(u-v))(s)e^{Ms}ds \right| \\ &\leq \frac{|u-v|}{e^{2M\pi}-1}e^{2M\pi}|L|\int_0^{2\pi}e^{Ms}ds \\ &\leq \frac{e^{2M\pi}}{M}|L|\,|u-v|.\end{aligned}$$

Thus from our hypothesis, we have that S is a contraction, and hence has a unique fixed point. Therefore, the linear PBVP (3.32) has a unique solution.

Next, we need to show that any solution $u(t)$ of (3.32) satisfies $u(t) \in [\alpha(t), \beta(t)]$, $t \in J$. We have

$$\alpha'(t) \leq -M\alpha(t) - (L\alpha)(t) + \sigma_\alpha(t)$$

and

$$u'(t) = -Mu(t) - (Lu)(t) + \sigma_\eta(t) \geq -Mu(t) - (Lu)(t) + \sigma_\alpha(t).$$

Setting $p = \alpha - u$, we have

$$p'(t) \leq -Mp(t) - (Lp)(t)$$

and $p(0) \leq p(2\pi)$. Now, Lemma 3.5.1 gives $p(t) \leq 0$ and hence $\alpha(t) \leq u(t)$, $t \in J$. Similarly, we can show that $u(t) \leq \beta(t)$, $t \in J$. Hence, we have $\alpha(t) \leq u(t) \leq \beta(t)$, $t \in J$.
Next we want to show that

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0 \quad t \in J,$$

where

$$\alpha_n' + M\alpha_n = -(L\alpha_n) + (Q\alpha_{n-1}) + M\alpha_{n-1} + (L\alpha_{n-1}),$$

$$\beta_n' + M\beta_n = -(L\beta_n) + (Q\beta_{n-1}) + M\beta_{n-1} + (L\beta_{n-1}).$$

First we show that $\alpha_0 \equiv \alpha \leq \alpha_1$. Now

$$\alpha_0' \leq (Q\alpha_0) \text{ and } \alpha_1' + M\alpha_1 = -(L\alpha_1) + M\alpha_0 + (L\alpha_0).$$

Let $p = \alpha_0 - \alpha_1$. Then $p' = \alpha_0' - \alpha_1' = -Mp - (Lp)$. and $p(0) \leq p(2\pi)$. Hence, by Lemma 3.5.1, $\alpha_0(t) \leq \alpha_1(t)$, $t \in J$.
Assume $\alpha_{k-1}(t) \leq a_k(t)$, $t \in J$. Let $p = \alpha_k - \alpha_{k+1}$. Using hypothesis (ii) and simplifying, we obtain

$$\begin{aligned} p' &= \alpha_k' - \alpha_{k+1}' \\ &= -M\alpha_k + (Q\alpha_{k-1}) + M\alpha_{k-1} + (La_{k-1}) - (L\alpha_k) \\ &\quad + M\alpha_{k+1} - (Q\alpha_k) - (M\alpha_k) - (L\alpha_{k+1}) + (L\alpha_{k+1}) \\ &\leq -M\alpha_k + M\alpha_{k-1} - (L\alpha_k) - (L\alpha_k) + M\alpha_{k+1} - M\alpha_k \\ &\quad - (L\alpha_k) + (L\alpha_{k+1}) + M(\alpha_k - \alpha_{k-1}) + (L(\alpha_k - \alpha_{k-1})) \\ &= -Mp - (Lp) \end{aligned}$$

and $p(0) = p(2\pi)$. Again, using Lemma 3.5.1, we get $\alpha_k(t) \leq \alpha_{k+1}(t)$, $t \in J$. Thus, by induction, we have

$$\alpha(t) = \alpha_0(t) \leq \alpha_1(t) \leq \alpha_k(t), \quad t \in J.$$

Similarly, we can show that

$$\beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0 = \beta.$$

We next show that $\alpha_n(t) \leq \beta_n(t)$, $t \in J$. Letting $p = \alpha_n - \beta_n$, and proceeding as before we arrive at

$$p' \leq -Mp - (Lp) \text{ and } p(0) = p(2\pi),$$

which yields $\alpha_n \leq \beta_n$, $t \in J$, $n = 1, 2, \ldots$, from Lemma 3.5.1. Hence, we have

$$\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \beta_n \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta_0, \quad t \in J.$$

It then follows, using standard arguments, that $\lim_{n\to\infty}\alpha_n(t) = p(t)$ and $\lim_{n\to\infty}\beta_n(t) = r(t)$ uniformly on J, and $p(t)$ and $r(t)$ are solutions of the PBVP (3.31).
To show that $p(t)$ and $r(t)$ are extremal solutions of (3.31), let $u(t)$ be any solution of (3.31) such that $u(t) \in [\alpha(t), \beta(t)]$, and suppose for some $k > 0$, $\alpha_{k-1}(t) \le u(t) \le \beta_{k-1}(t)$, $t \in J$. Let $p = \alpha_k - u$. Then

$$p' = \alpha_k' - u' = -M\alpha_k + (Q\alpha_{k-1}) + M\alpha_{k-1} + (L\alpha_{k-1}) - (L\alpha_k) - (Qu).$$

Since $\alpha_{k-1} \le u$, we have from the hypothesis (ii) of the theorem that

$$(Qu) - (Q\alpha_{k-1}) \ge -M(u - \alpha_{k-1}) + (L(u - \alpha_{k-1})).$$

Substituting the above inequality in $p'(t)$ we get,

$$p' \le -Mp - (Lp).$$

Also $p(0) = p(2\pi)$. Now applying Lemma 3.5.1 we get

$$\alpha_k(t) \le u(t).$$

Similarly, $u(t) \le \beta_k(t)$. Thus, from the induction principle, it follows that

$$\alpha_n \le u \le \beta_n, \text{ for all } n, t \in J.$$

Now taking limits as $n \to \infty$, we obtain

$$p(t) \le u(t) \le r(t).$$

Hence $p(t)$ and $r(t)$ are extremal solutions of the PBVP (3.31).
Remark 3.5.1. Theorem 3.5.1 and Lemma 3.5.1 hold for any linear operator L and a constant $M > 0$ such that $M < \frac{1}{2\pi}$ and $e^{2M\pi}|L| < M$. But, it should be observed that even if $M > \frac{1}{2\pi}$ the Theorem 3.5.1 holds if L satisfies the condition (a) in Lemma 3.5.1.

3.6 The Method of Quasilinearization (MQL)

If we utilize the technique of lower and upper solutions coupled with the method of quasilinearization and employ the idea of Newton-Fourier, it is possible to construct concurrently upper and lower bounding monotone sequences whose elements are the solutions of linear initial value problems. Compared to monotone iterative technique, the method of quasilinearization has the advantage that the sequences converge quadratically to the unique solution.
We consider the IVP for causal differential equation

$$x'(t) = (Qx)(t), \quad x(0) = x_0, \quad t \in J = [0, T], \tag{3.33}$$

where $Q : E \to E$ is a continuous operator, $E = C(J, \mathbb{R})$. We shall prove the following simple result to bring out the ideas involved.

Theorem 3.6.1. Assume that

(i) $\alpha_0, \beta_0 \in C(J, \mathbb{R})$ satisfy the inequalities

$$\alpha_0'(t) \le (Q\alpha_0)(t), \quad \beta_0'(t) \ge (Q\beta_0)(t), \quad \alpha_0 \le \beta_0, \quad t \in J;$$

(ii) the second Frechet derivative of Q exists and satisfies $Q_{xx}(x) \ge 0$ and $Q_x(x)y$ is semi-nondecreasing in y for each x.

Then, there exists monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ which converge uniquely and quadratically to the unique solution $x(t)$ of (3.33) on J.

Proof. In view of assumption (ii), we obtain for $x \ge y$, $x, y \in C(J, \mathbb{R})$,

$$(Qx)(t) \ge (Qy)(t) + Q_x(y)(x-y)(t) \tag{3.34}$$

and for $x, y \in \Omega$,

$$(Qx)(t) - (Qy)(t) \le L(x(t) - y(t)), \quad t \in J, \ L > 0. \tag{3.35}$$

Since Q satisfies one-sided Lipschitz condition, it follows that (3.33) possesses the unique solution $x(t)$ on J. Define the iterates $\{\alpha_n(t)\}, \{\beta_n(t)\}$ as follows:

$$\alpha_{n+1}'(t) = (Q\alpha_n)(t) + (Q_x\alpha_n)(\alpha_{n+1} - \alpha_n)(t), \quad \alpha_{n+1}(0) = x_0, \tag{3.36}$$

$$\beta_{n+1}'(t) = (Q\beta_n)(t) + (Q_x\beta_n)(\beta_{n+1} - \beta_n)(t), \quad \beta_{n+1}(0) = x_0, \tag{3.37}$$

with $\alpha_0(0) \le x_0 \le \beta_0(0)$, $t \in J$. We shall show that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_2 \le \beta_1 \le \beta_0, \text{ on } J. \tag{3.38}$$

Let us first prove that

$$\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0, \text{ on } J. \tag{3.39}$$

Set $p = \alpha_0 - \alpha_1$ so that $p(0) \le 0$ and by (3.36)

$$p' \le (Q\alpha_0)(t) - [(Q\alpha_0)(t) + (Q_x\alpha_0)(\alpha_1 - \alpha_0)(t)] \le (Q_x\alpha_0)p.$$

By Theorem 2.2.1, it follows that $p(t) \le 0$ on J which implies $\alpha_0(t) \le \alpha_1(t)$ on J. Similarly, it can be shown that $\beta_1 \le \beta_0$ on J. Now set $p = \alpha_1 - \beta_1$ so that $p(0) = 0$ and by (3.34), (3.36), and (3.37),

$$p' = [(Q\alpha_0) + (Q_x\alpha_0)(\alpha_1 - \alpha_0)(t)] - [(Q\beta_0) + (Q_x\beta_0)(\beta_1 - \beta_0)(t)]$$

$$\le (Q_x(\alpha_0))(\alpha_0 - \beta_0) + Q_x(\alpha_0)[\alpha_1 - \alpha_0] - Q_x(\beta_0)[\beta_1 - \beta_0]$$

which in view of hypothesis (ii) regarding $Q_x(x)y$ seminondecreasing in y for each x, yields

$$p' \leq Q_x(\alpha_0)p.$$

This shows that by Theorem 2.2.1, $p(t) \leq 0$ which gives $\alpha_1 \leq \beta_1$ on J. This establishes (3.39). Assuming, that for some integer k, the relation

$$\alpha_{k-1} \leq \alpha_k \leq \beta_k \leq \beta_{k-1}$$

is true, we can prove, by arguments similar to those in getting (3.39) that

$$\alpha_k \leq \alpha_{k+1} \leq \beta_{k+1} \leq \beta_k$$

and thus, by the principle of induction, (3.38) is true for all n.

Therefore, the iteration scheme described by (3.36) and (3.37) yield the monotone sequences $\{\alpha_n\}, \{\beta_n\}$ satisfying (3.38). Employing standard arguments, it is now easy to prove that the entire sequences $\{\alpha_n\}, \{\beta_n\}$ converge uniformly and monotonically to the unique solution $x(t)$ of (3.33) on J.

We shall next show that the convergence is quadratic. For this purpose, consider

$$p_{n+1} = x - \alpha_{n+1} \geq 0, \quad q_{n+1} = \beta_{n+1} - x \geq 0, \text{ on } J$$

and note that $p_{n+1}(0) = q_{n+1}(0) = 0$, $n \geq 1$. Now,

$$\begin{aligned} p'_{n+1} &= x'(t) - \alpha'_{n+1}(t) = (Qx)(t) - [(Q\alpha_n)(t) + (Q_x\alpha_n)(\alpha_{n+1} - \alpha_n)(t)] \\ &= Q_x(\eta)p_n - Q_x(\alpha_n)(p_n - p_{n+1}) \\ &\leq [Q_x(x) - Q_x(\alpha_n)]p_n + Q_x(\alpha_n)p_{n+1} \\ &= Q_{xx}(\sigma)p_n^2 + Q_x(\alpha_n)p_{n+1}, \end{aligned}$$

where $\alpha_n \leq \eta \leq x$ and $\alpha_n \leq \sigma \leq \beta_n$. Thus we get

$$p'_{n+1} \leq Mp_{n+1} + Np_n^2, \text{ on } J,$$

where $|Q_x(x)| \leq M$, $|Q_{xx}(x)| \leq N$ on J. Consequently, applying Gronwall inequality, treating Np_n^2 as forcing term, it follows that

$$0 \leq x(t) - \alpha_{n+1}(t) \leq N\int_0^t [x_n(s) - \alpha_n(s)]^2 e^{M(t-s)} ds,$$

which yields the desired estimate

$$\max_J |x(t) - \alpha_{n+1}(t)| \leq \frac{N}{M}e^{MT} \max_J |x(t) - \alpha_n(t)|^2.$$

In a similar way, one can obtain, after some computation, the estimate

$$\max_J |\beta_{n+1} - x(t)| \leq \frac{3}{2}\frac{N}{M}e^{MT} \max_J |\beta_n(t) - x(t)|^2 + \frac{1}{2}\frac{N}{M}e^{MT} \max_J |x(t) - \alpha_n(t)|^2.$$

The proof is complete.

3.7 Extension of Quasilinearization

We have seen in Section 3.6 that when the operator Q in (3.33) is convex, one can obtain lower and upper bounds simultaneously, that converge quadratically to the unique solution of (3.33). We have a similar conclusion in case of Q being concave. The question remains whether we can obtain corresponding results when Q admits a decomposition into a difference of two convex or concave functions or equivalently, Q admits a splitting into convex and concave parts. The answer is affirmative and we shall consider such a question in this section.

Let $\alpha_0, \beta_0 \in C^1(J,\mathbb{R})$, $J=[0,T]$ and define the sector

$$S = \{(t,x) \in J \times E : \alpha_0(t) \leq x(t) \leq \beta_0(t)\}$$

where $E = C(J,\mathbb{R})$. We shall consider the IVP

$$x'(t) = (Px)(t) + (Qx)(t), \quad x(0) = x_0, \tag{3.40}$$

where $P,Q : E \to E$ are continuous. Let us consider the case when the lower and upper solutions of (3.40) are natural.

Theorem 3.7.1. Assume the following hypotheses:

(H1) $\alpha_0, \beta_0 \in C^1(J,\mathbb{R})$, $\alpha_0(t) \leq \beta_0(t)$ and for $t \in J$,

$$\alpha_0'(t) \leq (P\alpha_0)(t) + (Q\alpha_0)(t),$$

$$\beta_0'(t) \geq (P\beta_0)(t) + (Q\beta_0)(t),$$

with $\alpha_0(0) \leq x_0 \leq \beta_0(0)$;

(H2) $P,Q \in C(J,\mathbb{R})$ and for $(t,x) \in S$, the Frechet derivatives P_x, Q_x, P_{xx} and Q_{xx} exist, are continuous and satisfy

$$P_{xx}(x) \geq 0, \quad Q_{xx}(x) \leq 0.$$

Then there exists monotone sequences $\{\alpha_n\}, \{\beta_n\}$ which converge uniformly to the unique solution of (3.40) and the convergence is quadratic.

Proof. We note that hypothesis (H2) yields the following inequalities:

$$(Px)(t) \geq (Py)(t) + (P_x y)(x-y)(t), \tag{3.41}$$

$$(Qx)(t) \geq (Qy)(t) + (Q_x x)(x-y)(t), \tag{3.42}$$

for $x \geq y$. It is also clear that P,Q satisfy, for any u_1, u_2 such that $\alpha_0(t) \leq u_2(t) \leq u_1(t) \leq \beta_0(t)$, $t \in J$,

$$-L(u_1(t) - u_2(t)) \leq (Pu_1)(t) - (Pu_2)(t) \leq L(u_1(t) - u_2(t)), \tag{3.43}$$

$$-L(u_1(t)-u_2(t)) \le (Qu_1)(t)-(Qu_2)(t) \le L(u_1(t)-u_2(t)), \tag{3.44}$$

for some $L > 0$.
Consider the IVPs

$$\begin{aligned} u'(t) &= F(t,\alpha_0,\beta_0;u) \\ &\equiv (P\alpha_0)(t)+(P_u\alpha_0)(u-\alpha_0)(t)+(Q\alpha_0)(t)+(Q_u\beta_0)(u-\alpha_0)(t), \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} v'(t) &= G(t,\alpha_0,\beta_0;v) \\ &\equiv (P\beta_0)(t)+(P_u\alpha_0)(v-\beta_0)(t)+(Q\beta_0)(t)+(Q_u\beta_0)(v-\beta_0)(t), \end{aligned} \tag{3.46}$$

with $\alpha_0(0) \le u_0$, $v_0 \le \beta_0(0)$, where $u(0) = u_0$, $v(0) = v_0$. The inequalities (3.41), (3.42) together with hypothesis (H1) imply

$$\begin{aligned} \alpha_0'(t) &\le (P\alpha_0)(t)+(Q\alpha_0)(t) \equiv F(t,\alpha_0,\beta_0;\alpha_0) \\ \beta_0'(t) &\ge (P\beta_0)(t)+(Q\beta_0)(t) \\ &\ge (P\alpha_0)(t)+(P_u\alpha_0)(\beta_0-\alpha_0)(t)+(Q\alpha_0)(t)+(Q_u\beta_0)(\beta_0-\alpha_0)(t) \\ &\equiv F(t,\alpha_0,\beta_0;\beta_0). \end{aligned}$$

Hence, by Theorem 3.2.1 and the fact that (3.45) is a linear IVP, it follows that there exists a unique solution $\alpha_1(t)$ of (3.45) such that $\alpha_0(t) \le \alpha_1(t) \le \beta_0(t)$ on J. Similarly using (3.41), (3.42) and (H1), we obtain

$$\begin{aligned} \alpha_0' &\le (P\alpha_0)(t)+(Q\alpha_0)(t) \\ &\le (P\beta_0)(t)+(P_u\alpha_0)(\alpha_0-\beta_0)(t)+(Q\beta_0)(t)+(Q_u\beta_0)(\alpha_0-\beta_0)(t) \\ &\equiv G(t,\alpha_0,\beta_0;\alpha_0), \\ \beta_0' &\ge (P\beta_0)(t)+(Q\beta_0)(t) \equiv G(t,\alpha_0,\beta_0;\beta_0), \end{aligned}$$

and therefore, as before, there exists a unique solution $\beta_1(t)$ of (3.46) such that $\alpha_0(t) \le \beta_1(t) \le \beta_0(t)$ on J.
Since $\alpha_1'(t) = F(t,\alpha_0,\beta_0;\alpha_1)$, we get by using (3.45),

$$\begin{aligned} \alpha_1' &= (P\alpha_0)(t)+(P_u\alpha_0)(\alpha_1-\alpha_0)(t)+(Q\alpha_0)(t)+(Q_u\beta_0)(\alpha_1-\alpha_0)(t) \\ &\le (P\alpha_1)(t)+(Q\alpha_1)(t)+(Q_u\alpha_1)(\alpha_0-\alpha_1)(t)+(Q_u\beta_0)(\alpha_1-\alpha_0)(t) \\ &\le (P\alpha_1)(t)+(Q\alpha_1)(t)+[Q_u\beta_0-Q_u\alpha_1](\alpha_1-\alpha_0)(t) \\ &\le (P\alpha_1)(t)+(Q\alpha_1)(t) \end{aligned}$$

because of the fact $Q_u(u)$ is nonincreasing and $\alpha_1(t) \le \beta_0(t)$. Similarly, using (3.46), since $\beta_1' = G(t, \alpha_0, \beta_0; \beta_1)$, we obtain

$$\begin{aligned}
\beta_1' &= (P\beta_0)(t) + (P_u\alpha_0)(\beta_1 - \beta_0)(t) + (Q\beta_0)(t) + (Q_u\beta_0)(\beta_1 - \beta_0)(t) \\
&\ge (P\beta_1)(t) + (Q\beta_1)(t) + (P_u\beta_1)(\beta_0 - \beta_1)(t) + (P_u\alpha_0)(\beta_1 - \beta_0)(t) \\
&\ge (P\beta_1)(t) + (Q\beta_1)(t) + [P_u\alpha_0 - P_u\beta_1](\beta_1 - \beta_0)(t) \\
&\ge (P\beta_1)(t) + (Q\beta_1)(t)
\end{aligned}$$

because $P_u(u)$ is nondecreasing and $\alpha_0(t) \le \beta_1(t) \le \beta_0(t)$. It then follows from Theorem 3.2.1, (3.43) and (3.44) that $\alpha_1(t) \le \beta_1(t)$ on J. As a result, we get

$$\alpha_0(t) \le \alpha_1(t) \le \beta_1(t) \le \beta_0(t) \text{ on } J. \tag{3.47}$$

Next we consider the IVPs

$$u'(t) = F(t, \alpha_1, \beta_1; u), \quad u(0) = u_0 \tag{3.48}$$

$$v'(t) = G(t, \alpha_1, \beta_1; v), \quad v(0) = v_0. \tag{3.49}$$

We observe that

$$\begin{aligned}
\alpha_1'(t) &\le (P\alpha_1)(t) + (Q\alpha_1)(t) \equiv F(t, \alpha_1, \beta_1; \alpha_1), \\
\beta_1'(t) &\ge (P\beta_1)(t) + (Q\beta_1)(t) \\
&\ge (P\alpha_1)(t) + (P_u\alpha_1)(\beta_1 - \alpha_1)(t) + (Q\alpha_1)(t) + (Q_u\alpha_1)(\beta_1 - \alpha_1)(t) \\
&\equiv F(t, \beta_1, \alpha_1; \beta_1),
\end{aligned}$$

in view of (3.41) and (3.42). Consequently, by Theorem 3.2.1 and the fact that (3.48), (3.49) are linear, we obtain, as before, a unique solution $\alpha_2(t)$ of (3.48) such that

$$\alpha_1(t) \le \alpha_2(t) \le \beta_1(t) \le \beta_0(t) \text{ on } J.$$

Similarly, since

$$\begin{aligned}
\alpha_1'(t) &\le (P\alpha_1)(t) + (Q\alpha_1)(t) \\
&\le (P\beta_1)(t) + (P_u\alpha_1)(\alpha_1 - \beta_1)(t) + (Q\beta_1)(t) + (Q_u\alpha_1)(\alpha_1 - \beta_1)(t) \\
&\equiv G(t, \alpha_1, \beta_1; \alpha_1), \\
\beta_1'(t) &\ge (P\beta_1)(t) + (Q\beta_1)(t) \equiv G(t, \alpha_1, \beta_1; \alpha_1),
\end{aligned}$$

we find that there exists a unique solution $\beta_2(t)$ of (3.49) satisfying

$$\alpha_1(t) \le \beta_2(t) \le \beta_1(t) \le \beta_0(t) \text{ on } J.$$

In view of the fact

$$\alpha_2'(t) = F(t, \alpha_1, \beta_1; \alpha_2),$$
$$\beta_2'(t) = G(t, \alpha_1, \beta_1; \beta_2),$$

we have, as before, using (3.41) and (3.42),

$$\alpha_2' \le (P\alpha_2)(t) + (Q\alpha_2)(t),$$
$$\beta_2' \ge (P\beta_2)(t) + (Q\beta_2)(t),$$

which yields by Theorem 3.2.1, (3.43) and (3.44)

$$\alpha_2(t) \le \beta_2(t) \text{ on } J.$$

It therefore follows that

$$\alpha_0(t) \le \alpha_1(t) \le \alpha_2(t) \le \beta_2(t) \le \beta_1(t) \le \beta_0(t) \text{ on } J$$

because of (3.47). This process can be continued successively to arrive at

$$\alpha_0(t) \le \alpha_1(t) \le \cdots \le \alpha_n(t) \le \beta_n(t) \le \cdots \le \beta_1(t) \le \beta_0(t) \text{ on } J,$$

where the elements of the monotone sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ are the unique solutions of the linear IVPs

$$\alpha_{n+1}'(t) = F(t, \alpha_n, \beta_n; \alpha_{n+1}), \quad \alpha_{n+1}(0) = u_0,$$
$$\beta_{n+1}'(t) = G(t, \alpha_n, \beta_n; \beta_{n+1}), \quad \beta_{n+1}(0) = v_0.$$

Employing the standard arguments, it is easy to conclude that the sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ converge uniformly to the unique solution of (3.40) on J.

We shall now show that the convergence of the sequences $\{\alpha_n(t)\}, \{\beta_n(t)\}$ to the unique solution of (3.40) is indeed quadratic. To do this, consider

$$p_n(t) = u(t) - \alpha_n(t) \ge 0, \quad q_n = \beta_n(t) - u(t) \ge 0$$

and note that $p_n(0) = q_n(0) = 0$.

Using the definitions of $\alpha_n(t), \beta_n(t)$ and the mean value theorem together with hypothesis (H2), we obtain successively,

$$\begin{aligned}
p_n'(t) &= (Pu)(t) + (Qu)(t) + [(P\alpha_{n-1})(t) + (P_u\alpha_{n-1})(\alpha_n - \alpha_{n-1})(t) \\
&\quad + (Q\alpha_{n-1})(t) + (Q_u\beta_{n-1})(\alpha_n - \alpha_{n-1})(t)] \\
&= [(Pu\xi) - (P_u\alpha_{n-1}) + (Q_u\sigma) - (Q_u\beta_{n-1})]p_{n-1}(t) \\
&\quad [(P_u\alpha_{n-1}) + (Q_u\beta_{n-1})]p_n(t) \\
&\le [(P_u u) - (P_u\alpha_{n-1}) + (Q_u u) - (Q_u\beta_{n-1})]p_{n-1}(t) \\
&\quad + [(P_u\alpha_{n-1}) + (Q_u\beta_{n-1})]p_n(t) \\
&= (P_{uu}\varepsilon_1)p_{n-1}^2(t) - (Q_{uu}\sigma_1)(\beta_{n-1} - \alpha_{n-1})p_{n-1}(t) \\
&\quad + [(P_u\alpha_{n-1}) + (Q_u\beta_{n-1})]p_n(t)
\end{aligned}$$

where $\alpha_{n-1} < \xi, \sigma < u$, $\alpha_{n-1} < \varepsilon_1 < u$ and $\alpha_{n-1} < \sigma_1 < \beta_{n-1}$. But,

$$\begin{aligned} &-Q_{uu}(\sigma_1)[\beta_{n-1} - \alpha_{n-1}]p_{n-1}(t) \\ &\leq N_2[q_{n-1}(t) + p_{n-1}(t)]p_{n-1}(t) \\ &\leq N_2[p_{n-1}^2(t) + p_{n-1}(t)q_{n-1}(t)] \\ &\leq \frac{3}{2}N_2 p_{n-1}^2(t) + \frac{1}{2}N_2 q_{n-1}^2(t). \end{aligned}$$

Thus,

$$p_n'(t) \leq Mp_n(t) + (N_1 + \frac{3}{2}N_2)p_{n-1}^2(t) + \frac{1}{2}N_2 q_{n-1}^2(t),$$

where

$$|P_{uu}(u)| \leq N_1, \quad |Q_{uu}(u)| \leq N_2, \quad |P_u(u)| \leq M_1,$$

$$|Q_u(u)| \leq M_2 \text{ and } M = M_1 + M_2.$$

Now, Gronwall inequality implies

$$0 < p_n(t) \leq \int_0^t e^{M(t-s)}[(N_1 + \frac{3}{2}N_2)p_{n-1}^2(s) + \frac{1}{2}N_2 q_{n-1}^2(s)]ds$$

which yields, for $t \in J$,

$$\max_J |u(t) - \alpha_n(t)| \leq \frac{e^{MT}}{M}\left[\left(N_1 + \frac{3}{2}N_2\right)\max_J |u(t) - \alpha_{n-1}(t)|^2 + \frac{1}{2}N_2 \max_J |\beta_{n-1}(t) - u(t)|^2\right].$$

Similarly, we get with some computation, an estimate relative to q_n given by

$$\max_J |\beta_n(t) - u(t)| \leq \frac{e^{MT}}{M}\left[\left(\frac{3}{2}N_1 + N_2\right)\max_J |\beta_{n-1}(t) - u(t)|^2 + \frac{1}{2}N_1 \max_J |\alpha_{n-1}(t) - u(t)|^2\right].$$

These estimates on $p_n(t)$ and $q_n(t)$ establish the quadratic convergence of the iterates $\{\alpha_n(t)\}, \{\beta_n(t)\}$ to the unique solution of IVP (3.40).

By choosing lower and upper solutions differently with suitable linear problems, one can show that the same conclusion as in Theorem 3.7.1 is valid. Moreover, Theorem 3.7.1 includes several special cases. For example, Q is not convex but $(Q+S)$ is a convex causal operator, is one special case. In this case, we write

$$u'(t) = ((Qu) + (Su) - (Su))(t) = [(\tilde{Q}u) + (-Su)](t)$$

where $(\tilde{Q}u) = [(Qu) + (Su)]$ is convex and $-Su$ is concave. Also, another special case is when (Qu) is not concave but $(Qu + Su)$ is concave with (Su) being concave, so that

$$u'(t) = [(-Su) + (\tilde{Q}u)](t),$$

where $\tilde{Q}u = (Qu + Su)$ and $Pu = -Su$. Theorem 3.7.1 is applicable in all such cases and we get quadratic convergence of iterates.

3.8 Newton's Method Versus Quasilinearization

In this section we compare and contrast Newton's method and the method of quasilinearization. Some of the several possible situations are explored providing a preliminary discussion.

Let us consider the problem of finding a root of scalar causal functional equation

$$0 = (Qx)(t) \tag{3.50}$$

where $Q : E \to E$, $E = C([0,T],\mathbb{R})$ which means that we are seeking functions $x(t)$ or constant valued functions $x(t)$ that makes $(Qx)(t) = 0$. For example, if

$$(Qx)(t) = \int_{t_0}^{t} k(t,s,x(s))ds,$$

then the above problem (3.50) reduces to finding $x(t)$ which will make the integral to be zero. It is possible for $x(t)$ to be just a constant function or a nonconstant function. Thus, it follows that finding the zeroes of $(Qx)(t)$ implies finding a function $x^*(t)$, constant or otherwise such that $(Qx^*)(t) = 0$ and in the example considered above, $\int_{t_0}^{t} k(t,s,x^*(s))ds = 0$.

On the other hand, let us also consider the IVP for causal differential equation

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0,\ t_0 \geq 0, \tag{3.51}$$

on the interval $J = [0,T]$, where $Q : E \to E, E = C([0,T],\mathbb{R})$. If we are looking for constants functions as solutions of this IVP, then the problem is identical to finding roots of the causal equation (3.50), that are constant valued functions.

As is well known, Newton's method is the best known procedure for finding the solution of functional equation (3.50). Although, it may not be the best method for the given problem, its simplicity and rapid convergence of the iterates often motivate one to employ this method for solving (3.50). The iteration formula of Newton's method is

$$x_{n+1} - x_n = -(Qx_n)/(Q_x x_n), \quad n \geq 0. \tag{3.52}$$

For (3.52) to hold, one needs to assume that Q is continuously Fréchet differentiable and $(Q_x x) \neq 0$, in a neighborhood of an isolated zero of (3.50), namely in the sector $[\alpha_0, \beta_0] = \{x \in E : \alpha_0 < x < \beta_0\}$.

There are three possibilities:

(a) $(Q_x x) > 0$;

(b) $(Q_x x < 0)$;

(c) $(Q_x x \neq 0)$;

or equivalently, (Qx) is monotone increasing or (Qx) is monotone decreasing or (Qx) is not monotone on $[\alpha_0, \beta_0]$. In Newton's method, we require only $(Q_x x \neq 0)$ and the iterative scheme need not produce a sequence of monotone iterates. However, when (Qx) is monotone, it is possible to develop monotone iterative technique for IVP (3.51) when suitable lower and upper solutions are assumed to exist.

Let us consider the following two situations for (3.50):

(1) $(Q_x x) < 0$, $0 < (Q\alpha_0)$, $0 > (Q\beta_0)$, $\alpha_0 < \beta_0$ and $(Q_{xx}x) < 0$;
(2) $(Q_x x) > 0$, $0 < (Q\beta_0)$, $0 > (Q\alpha_0)$, $\alpha_0 < \beta_0$ and $(Q_{xx}x) > 0$.

Comparing this to IVP (3.51), we have the following two cases:

(1*) $(Q_x x)(t) \leq 0$, $\alpha_0'(t) \leq (Q\alpha_0)(t)$, $\beta_0' \geq (Q\beta_0)(t)$, $\alpha_0(t) \leq \beta_0(t)$, $(Q_{xx}x)(t) \leq 0$ on J;
(2*) $(Q_x x)(t) \geq 0$, $\alpha_0'(t) \leq (Q\beta_0)(t)$, $\beta_0'(t) \geq (Q\alpha_0)(t)$, $\alpha_0(t) \leq \beta_0(t)$, $(Q_{xx}x)(t) \geq 0$ on J.

For IVP (3.51), the cases (1*) and (2*) lead us to the method of generalized quasilinearization and we require the conditions in (1*), (2*) to hold only in the closed set

$$[\alpha_0, \beta_0] = \{x \in E : \alpha_0(t) \leq x(t) \leq \beta_0(t),\ t \in J\}.$$

One of the tools in order to obtain monotone sequences from the iteration schemes in the method of quasilinearization is the appropriate comparison result. In case of IVP (3.51), one needs the simple linear differential inequality

$$p'(t) \leq (Q_x \eta)p(t), \quad p(0) < 0,\ t \in J \tag{3.53}$$

where $\eta = \eta(t)$ is such that $\alpha_0(t) \leq \eta(t) \leq \beta_0(t)$, which implies that $p(t) \leq 0$ on J.
This conclusion does not require $(Q_x \eta)$ to be either only positive or only negative. On the other hand, the corresponding comparison result for (3.50) would lead to

$$0 < (Q_x \eta)p, \tag{3.54}$$

where $\alpha_0 < \eta < \beta_0$. In order to conclude $p < 0$ or $p > 0$, we certainly require to assume that $(Q_x \eta) < 0$ or $(Q_x \eta) > 0$. We therefore see that there is a similarity between generalized quasilinearization method and Newton's method, when we write (3.52) in the form

$$0 = (Qx_n) + (Q_x x_n)(x_{n+1} - x_n) \tag{3.55}$$

and the corresponding iterative scheme for (3.51) which is given by

$$\left.\begin{aligned} x_{n+1}'(t) &= (Qx_n)(t) + (Q_x x_n)(x_{n+1} - x_n)(t) \\ x_{n+1}(0) &= x_0. \end{aligned}\right\} \tag{3.56}$$

Thus the procedure in the proofs for both the methods would hold good, with appropriate modifications. In both cases, we obtain quadratic convergence of the monotone sequences to the unique solution of each problem.

The discussion described above generates a variety of results, some are known and some unknown. We shall attempt to consider possible solutions with a view to compare and contrast both of these important methods.

We shall begin by considering the case of natural lower and upper solutions and a result relative to (3.50) under the assumption (1).

Theorem 3.8.1. Suppose that the conditions listed in (1) hold. Then there exist monotone sequences $\{\alpha_n\},\{\beta_n\}$ satisfying

$$\alpha_0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < r < \beta_n < \cdots < \beta_2 < \beta_1 < \beta_0, \tag{3.57}$$

which are generated by the iteration scheme

$$\left.\begin{aligned} 0 &= (Q\alpha_n) + (Q_x\beta_n)(\alpha_{n+1}-\alpha_n) \\ 0 &= (Q\beta_n) + (Q_x\beta_n)(\beta_{n+1}-\beta_n) \end{aligned}\right\} \tag{3.58}$$

where r is the isolated zero of (3.50). Moreover, the sequences converge to the unique solution r of (3.50) quadratically.

Proof. We have

$$0 < (Q\alpha_0) - [(Q\alpha_0) + (Q_x\beta_0)(\alpha_1-\alpha_0)] = -(Q_x\beta_0)(\alpha_1-\alpha_0),$$

which implies the relation $\alpha_0 < \alpha_1$, because of the assumption $(Q_x\beta_0) < 0$. Similarly, we can get $\beta_1 < \beta_0$. Next, we shall show that $\alpha_1 < r < \beta_1$.

Since we have successively

$$\begin{aligned} 0 &= (Qr) - [(Q\alpha_0) + (Q_x\beta_0)(\alpha_1-\alpha_0)] \\ &= (Q - x\sigma)(r-\alpha_0) - (Q_x\beta_0)(\alpha_1-\alpha_0) \\ &> (Q_x\beta_0)(r-\alpha_0-\alpha_1+\alpha_0) \\ &= (Q_x\beta_0)(r-\alpha_1), \end{aligned}$$

where $\alpha_0 < \sigma < \beta_0$ and the decreasing nature of $(Q_x x)$ is used.

We arrive at $\alpha_1 < r$. A similar argument shows that $r < \beta_1$. Thus, we have

$$\alpha_0 < \alpha_1 < r < \beta_1 < \beta_0. \tag{3.59}$$

Repeating the above proof inductively, we can show that (3.57) holds. Then, the boundedness of monotone sequences $\{\alpha_n\},\{\beta_n\}$ show that they converge to $\tilde{\alpha},\tilde{\beta}$ respectively and it is easy to see that $(Q\tilde{\alpha}) = 0$ and $(Q\tilde{\beta}) = 0$. Since r is the unique solution in $[\alpha_0,\beta_0]$, it follows that $\tilde{\alpha} = r = \tilde{\beta}$.

To show that the sequences $\{\alpha_n\}, \{\beta_n\}$ converge quadratically to r, using the standard computation, we arrive at

$$0 < r - \alpha_{n+1} < M_1[(\beta_n - r)(r - \alpha_n)] < \frac{M_1}{2}[(\beta_n - r)^2 + (r - \alpha_n)^2],$$

where $|(Q_x\sigma)/(Q_{xx}\beta_n)| \le M_1$. A similar estimate holds for $\beta_{n+1} - r$ and hence, combining the two estimates, we get

$$|r - \alpha_{n+1}| + |\beta_{n+1} - r| \le M[(r - \alpha_n)^2 + (\beta_n - r)^2]$$

for some constant $M > 0$. The proof is therefore complete.

The next result is relative to IVP (3.51) corresponding to the case (1*) except that we do not need the condition $(Q_x x) < 0$.

Theorem 3.8.2. Assume that the conditions listed in (1*) hold without $(Q_x x) < 0$. Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ defined by the scheme

$$\left.\begin{array}{l} \alpha'_{n+1}(t) = (Q\alpha_n)(t) + (Q_x\beta_n)(\alpha_{n+1} - \alpha_n)(t),\ \alpha_{n+1}(0) = x_0, \\ \beta'_{n+1}(t) = (Q\beta_n)(t) + (Q_x\beta_n)(\beta_{n+1} - \beta_n)(t),\ \ \beta_{n+1}(0) = x_0 \end{array}\right\}$$

Which converge uniformly to the unique solution of (3.51) and the convergence is quadratic.

We can prove a result similar to that of Theorem 3.8.2 in case of (1*), provided we replace the condition $(Q_{xx}x) \le 0$ by $(Q_{xx}x) \ge 0$. This is stated below.

Theorem 3.8.3. Assume that (1*) holds with $(Q_{xx}x) \le 0$ replaced by $(Q_{xx}x) \ge 0$ and without $(Q_x x) \le 0$. Then the same conclusion of Theorem 3.8.2 is true.

In this case, the iterative scheme is

$$\alpha'_{n+1} = (Q\alpha_n)(t) + (Q_x\alpha_n)(\alpha_{n+1} - \alpha_n)(t), \quad \alpha_{n+1}(0) = x_0,$$

$$\beta'_{n+1} = (Q\beta_n)(t) + (Q_x\alpha_n)(\beta_{n+1} - \beta_n)(t), \quad \beta_{n+1}(0) = x_0.$$

We do not offer the proofs of Theorem 3.8.2 and 3.8.3 since they are special cases of results of earlier sections. The cases of coupled lower and upper solutions (2), (2*) maybe considered similarly.

3.9 Notes and Comments

The method of lower and upper solutions coupled with monotone iterative technique and quasilinearization, including generalizations and extensions are theoretical approximation results that are very useful in applications and are popular in several disciplines. Therefore, this entire chapter is dedicated to such results.

The contents of Sec. 3.5 are taken from Drici, McRae and Vasundhara Devi [73] and the rest of the results are new in the present general framework and are adapted from Lakshmikantham and Köksal [74], Lakshmikantham and Vatsala [75], Ladde, Lakshmikantham and Vatsala [76], Lakshmikantham and Zhang [77], Vasundhara Devi and Vatsala [78].

Chapter 4

Stability Theory

4.1 Introduction

This chapter is devoted essentially to the theory of stability and boundedness of Causal differential equations, using Lyapunov's method. The concepts of Lyapunov stability have given rise to several notions that are important in applications. For example, other than usual stability concepts originated by Lyapunov, we have partial stability, conditional stability, perfect stability and eventual stability of asymptotically invariant sets, to name a few. Corresponding to these, notions of boundedness have been formulated and sufficient conditions are provided. In order to unify a variety of known concepts of stability and boundedness, it is found beneficial to employ two different measures and obtain criteria in terms of two measures.

In Section 4.2, we prove necessary comparison results in terms of Lyapunov functions and other relevant theorems. Section 4.3 offers various stability and boundedness concepts in terms of two measures and show these definitions unify various known stability concepts. In Section, 4.5, stability criteria in terms of a Lyapunov function are given. Here it becomes necessary to choose minimal classes of functions relative to which the generalized derivative of Lyapunov functions has to satisfy certain conditions. Section 4.6 uses Lyapunov functionals to offer stability criteria.

In Section 4.7, we present Lyapunov functions on product spaces unifying the results of stability theory. The stability criteria in terms of two different measures are given in Section 4.8, while in Section 4.9, the method of vector Lyapunov functions is employed. In order, not to duplicate the results in each of these situations, we have only presented some typical results which cover a variety of known results. Finally, in Section 4.10, we give notes and comments.

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4.2 Comparison Theorems via Lyapunov Functions

In order to investigate the theory of stability for causal differential equations, we need comparison results in terms of Lyapunov-like functions.
Consider the causal differential system

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0 \; t_0 \geq 0, \tag{4.1}$$

where $Q : E \to E = C([t_0,\infty),\mathbb{R}^n)$. We assume the existence and uniqueness of solutions $x(t)$ of (4.1). When we utilize Lyapunov-like functions, it is necessary to select some classes of functions relative to which the generalized derivative of Lyapunov function has to satisfy suitable conditions. We define the following sets:

$$E_A = \{x \in E : V(s,x(s))A(s) \leq V(t,x(t))A(t),\ t_0 \leq s \leq t\},$$
$$E_1 = \{x \in E : V(s,x(s)) \leq V(t,x(t)),\ t_0 \leq s \leq t\},$$

where

(i) $A(t) > 0$ is continuously differentiable on $\mathbb{R}_+$ and $A(t_0) = 1$;
(ii) $V \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ is a Lyapunov function.

We now prove the following comparison results.
Theorem 4.2.1. Suppose that the following hypotheses hold;

(i) $V \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$, $V(t,x)$ is locally Lipschitzian in x;
(ii) for $t \geq t_0$ and $x \in E_1$,

$$D^+V(t,x(t)) \leq g(t,V(t,x(t))),$$

where

$$D^+V(t,x(t)) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x(t)+h(Qx)(t)) - V(t,x(t))]$$

and $g \in C(\mathbb{R}_+^2,\mathbb{R}_+)$;
(iii) $r(t) = r(t,t_0,u_0)$ is the maximal solution of the scalar differential equation

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0, \tag{4.2}$$

existing on $[t_0,\infty)$.

Then, if $x(t) = x(t,t_0,x_0)$ is any solution of IVP (4.1) existing on $[t_0,\infty)$, $V(t_0,x_0) \leq u_0$ implies

$$V(t,x(t)) \leq r(t), \quad t \geq t_0.$$

Proof. Let $x(t) = x(t,t_0,x_0)$ be any solution of IVP (4.1) for $t \geq t_0$. Define

$$m(t) = V(t,x(t))$$

so that $m(t_0) = V(t_0,x_0) \leq u_0$. For some sufficiently small $\varepsilon > 0$, consider the differential equation

$$u' = g(t,u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon, \tag{4.3}$$

whose solutions $u(t,\varepsilon) = u(t,t_0,u_0,\varepsilon)$ exist as far as $r(t)$ exists. To prove the conclusion of Theorem 4.2.1, it is enough to show that

$$m(t) = V(t,x(t)) < u(t,\varepsilon), \quad t \geq t_0.$$

Suppose this is not true. Then, there exists a $t_1 > t_0$ such that

$$m(t) < u(t,\varepsilon), \quad t_0 \leq t < t_1$$

and

$$m(t_1) = u(t_1,\varepsilon).$$

It then follows that

$$D^+ m(t_1) \geq u'(t_1,\varepsilon) = g(t_1,u(t_1,\varepsilon)) + \varepsilon. \tag{4.4}$$

From the assumptions on g, the solutions $u(t,\varepsilon)$ are nondecreasing in t. Since $m(t) = V(t,x(t))$, we get

$$V(s,x(s)) \leq u(t_1,\varepsilon), \text{ for } t_0 \leq s \leq t_1.$$

Consequently, $x(t) \in E_1$. Using the fact that $V(t,x)$ is assumed to be Lipschitzian in x, the standard computation yields

$$\begin{aligned}
m(t+h) - m(t) &= V(t+h,x(t+h)) - V(t,x(t)) \\
&\leq V(t+h,x(t+h)) - V(t+h,x(t)+h(Qx)(t)) \\
&\quad +V(t+h,x(t)+h(Qx)(t)) - V(t,x(t)) \\
&\leq L|x(t+h) - x(t) - h(Qx)(t)| \\
&\quad +V(t+h,x(t)+h(Qx)(t)) - V(t,x(t)).
\end{aligned}$$

This shows that

$$\begin{aligned}
D^+ m(t) &= \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x(t)+h(Qx)(t)) - V(t,x(t))] \\
&\leq g(t,V(t,x(t))) = g(t,m(t)), \quad t_0 \leq t \leq t_1 < \infty.
\end{aligned}$$

hence at $t = t_1$, we get

$$D^+ m(t_1) \leq g(t_1, m(t_1)) < g(t_1, u(t_1, \varepsilon)) + \varepsilon$$

which contradicts (4.4). Hence $m(t) < u(t, \varepsilon)$, which as $\varepsilon \to 0$ yields the desired estimate

$$m(t) = V(t, x(t)) \leq r(t), \quad t \geq t_0,$$

and the proof is complete.

Corollary 4.2.1. Let V satisfy the conditions of Theorem 4.2.1 with $g(t,u) \equiv 0$ and $x(t) \in E_1$. Then

$$V(t, x(t)) \leq V(t_0, x_0), \quad t \geq t_0,$$

where $x(t)$ is any solution of IVP (4.1) or equivalently,

$$V(t_2, x(t_2)) \leq V(t_1, x(t_1)), \quad t_0 \leq t_1 \leq t_2 < \infty.$$

Theorem 4.2.2. Suppose that the hypothesis of theorem 4.2.1 hold except that the inequality in (ii) is replaced by

$$A(t)D^+ V(t, x(t)) + V(t, x(t))A'(t) \leq g(t, V(t, x(t))A(t)),$$

for $t \geq t_0$ and $x \in E_A$. Then, $A(t_0)V(t_0, x_0) \leq u_0$ implies

$$V(t, x(t))A(t) \leq r(t), \quad t \geq t_0.$$

Proof. Define $L(t, x(t)) = V(t, x(t))A(t)$. Let $t \geq t_0$ and $x(t) \in E_A$. Then it is easy to see that

$$\begin{aligned} D^+ L(t, x(t)) &\leq A(t)D^+ V(t, x(t)) + V(t, x(t))A(t) \\ &\leq g(t, L(t, x(t))). \end{aligned}$$

Then, by Theorem 4.2.1, it follows that

$$V(t, x(t))A(t) = L(t, x(t)) \leq r(t), \quad t \geq t_0.$$

To prove a general comparison theorem, we need the following result.

Lemma 4.2.1. Let $g_0, g \in C(\mathbb{R}^2_+, \mathbb{R})$ be such that

$$g_0(t, u) \leq g(t, u). \tag{4.5}$$

Then, the left maximal solution $\eta(t, \tau_0, v_0)$ of the IVP

$$v' = g_0(t, v), \quad v(\tau_0) = v_0, \tag{4.6}$$

and the right maximal solution $r(t) = r(t, t_0, u_0)$ of IVP (4.3) satisfy the relation

$$r(t, t_0, u_0) \leq \eta(t, \tau_0, v_0), \quad t \in [t_0, \tau_0], \tag{4.7}$$

where $r(\tau_0, t_0, u_0) \leq v_0$.

Proof. It is known that $\lim_{\varepsilon\to 0} u(t,\varepsilon) = r(t,t_0,u_0)$ and $\lim_{\varepsilon\to 0} v(t,\varepsilon) = \eta(t,\tau_0,v_0)$, where $u(t,\varepsilon)$ is any solution of

$$u' = g(t,u)+\varepsilon, \quad u(t_0) = u_0+\varepsilon$$

existing for $t \geq t_0$ and $v(t,\varepsilon)$ is any solution of

$$v' = g_0(t,v)+\varepsilon, \quad v(\tau_0) = v_0$$

existing to the left of τ_0 and $\varepsilon > 0$ is sufficiently small. Note that (4.7) follows if we first prove the inequality $u(t,\varepsilon) < v(t,\varepsilon)$, $t_0 \leq t \leq \tau_0$. Since $g_0 \leq g$ and $r(\tau_0,t_0,u_0) \leq v_0$, it is easy to see that, for sufficiently small $\delta > 0$, we have

$$u(t,\varepsilon) < v(t,\varepsilon), \quad \tau_0-\delta \leq t < \tau_0,$$

and in particular,

$$u(\tau_0-\delta,\varepsilon) < v(\tau_0-\delta,\varepsilon).$$

We claim that

$$u(t,\varepsilon) < v(t,\varepsilon), \quad t_0 \leq t < \tau_0-\delta.$$

If this is not true, there exists a $t^* \in [t_0, \tau_0-\delta)$ such that

$$u(t^*,\varepsilon) = v(t^*,\varepsilon), \quad u(t,\varepsilon) < v(t,\varepsilon), \quad t^* < t \leq \tau_0-\delta.$$

This leads to the contradiction

$$g(t^*,u(t^*,\varepsilon))+\varepsilon = u'(t^*,\varepsilon) \leq v'(t^*,\varepsilon) = g_0(t^*,v(t^*,\varepsilon))+\varepsilon.$$

Hence, $u(t,\varepsilon) < v(t,\varepsilon)$, $t_0 \leq t \leq \tau_0-\delta$ and the proof of the lemma is complete.

We are now in position to prove the following comparison result, which plays an important role in the study of causal differential inequalities.

Theorem 4.2.3. Let $m \in C(\mathbb{R}_+,\mathbb{R}_+)$, $Q : E \to E = C(\mathbb{R}_+,\mathbb{R})$ and satisfies

$$D^+m(t) \leq (Qm)(t)+g(t,m(t)), \quad t \in I_0, \tag{4.8}$$

where

$$I_0 = \{t \geq t_0 : m(s) \leq \eta(s,t,m(t)), \quad t_0 \leq s \leq t\},$$

$\eta(t,\tau_0,v_0)$ being the left maximal solution of (4.6) existing on $[t_0,\tau_0]$. Assume that

$$g_0(t,u) \leq (Qu)(t)+g(t,u) \tag{4.9}$$

and $r(t)$ is the maximal solution of IVP

$$u'(t) = (Qu)(t) + g(t, u(t)), \quad u(t_0) = u_0, \tag{4.10}$$

existing on $[t_0, \infty)$. Then

$$m(t_0) \leq u_0 \text{ implies } m(t) \leq r(t), \quad t \geq t_0. \tag{4.11}$$

Proof. Since it is known that $\lim_{\varepsilon \to 0} u(t, \varepsilon) = r(t)$ where $u(t, \varepsilon)$ is any solution of

$$u'(t) = (Qu)(t) + g(t, u(t)) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon,$$

for $\varepsilon > 0$ being sufficiently small, on any compact interval $[t_0, \tau_0] \subset [t_0, \infty)$, it is enough to prove that

$$m(t) < u(t, \varepsilon), \quad t_0 \leq t \leq \tau_0.$$

If this is not true, then there exists a $t^* \in (t_0, \tau_0]$ such that

$$m(t^*) = u(t^*, \varepsilon), \quad m(s) < u(s, \varepsilon), \quad t_0 \leq s < t^*.$$

Thus implies that

$$D^+ m(t^*) \geq u(t^*, \varepsilon) = (Qu)(t^*) + g(t^*, u(t^*, \varepsilon)) + \varepsilon. \tag{4.12}$$

Consider now the left maximal solution $\eta(s, t^*, m(t^*))$, $t_0 \leq s \leq t^*$, of

$$u' = g_0(t, u), \quad u(t^*) = m(t^*).$$

By Lemma 4.2.1,

$$r(s, t_0, u_0) \leq \eta(s, t^*, m(t^*)), \quad t_0 \leq s \leq t^*.$$

Since

$$r(t^*, t_0, u_0) = \lim_{\varepsilon \to 0} u(t^*, \varepsilon) = m(t^*) = \eta(t^*, t^*, m(t^*))$$

and $m(s) \leq u(s, \varepsilon)$, $t_0 \leq s \leq t^*$, it follows that

$$m(s) \leq r(s, t_0, u_0) \leq \eta(s, t^*, m(t^*)), \quad t_0 \leq s \leq t^*.$$

This inequality implies that $t^* \in I_0$ and as a result (4.8) yields

$$D^+ m(t^*) \leq (Qm)(t^*) + g(t^*, m(t^*))$$

which contradicts (4.12). Thus, $m(t) \leq r(t)$, $t \geq t_0$ and the proof is complete.

4.3 Definitions of Stability and Boundedness

Let $x(t,t_0,x_0)$ be any solution of the causal differential system

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \quad t_0 \geq 0, \tag{4.13}$$

where $Q \in C(E,\mathbb{R}^n)$, $E = C([t_0,\infty),\mathbb{R}^n)$. Let

$$S\rho = \{x \in \mathbb{R}^n : |x| < \rho\}. \tag{4.14}$$

Assume that (4.13) admits the trivial solution $x(t) \equiv 0$ through $(t_0,0)$. We now list a few definitions concerning the stability of the trivial solution.

Definition 4.3.1. The trivial solution $x \equiv 0$ of (4.13) is

(S1) equistable, if for each $\varepsilon > 0$, $t_0 \in J = [t_0,\infty)$, there exists a positive function $\delta = \delta(t_0,\varepsilon)$ that is continuous in t_0 for each ε such that the inequality $|x_0| \leq \delta$ implies

$$|x(t,t_0,x_0)| < \varepsilon, \quad t \geq t_0;$$

(S2) uniformly stable if δ in (S1) is independent of t_0;

(S3) quasi-equi asymptotically stable if, for each $\varepsilon > 0$, $t_0 \in J$, there exists positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\varepsilon)$ such that, for $t \geq t_0 + T$ and $|x_0| < \delta_0$

$$|x(t,t_0,x_0)| < \varepsilon;$$

(S4) quasi uniformly asymptotically stable if the numbers δ_0 and T in (S3) are independent of t_0;

(S5) equi-asymptotically stable if (S1) and (S3) hold simultaneously;

(S6) uniformly asymptotically stable if (S2) and (S4) hold together;

(S7) quasi-equi asymptotically stable if, for each $\varepsilon > 0$, $\alpha > 0$, and $t_0 \in J$, there exists a positive number $T = T(t_0,\varepsilon,\alpha)$ such that $|x_0| \leq \alpha$ implies

$$|x(t,t_0,x_0)| < \varepsilon, \quad t \geq t_0 + T;$$

(S8) quasi uniformly asymptotically stable if the T in (S7) is independent of t_0;

(S9) completely stable if (S1) holds and (S7) is verified for all α, $0 \leq \alpha < \infty$;

(S10) uniformly completely stable if (S2) holds and (S8) is verified for all α, $0 \leq \alpha < \infty$.

Remark 4.3.1. Sometimes the notion of quasi-asymptotic stability may be relaxed somewhat as in (S7) and (S8). Clearly the ε, α given in the preceding definitions must be less than ρ of (4.14), and therefore the concepts (S1)-(S8) are of local nature. If, on the other hand, $\rho = \infty$, so that $S_\rho = \mathbb{R}^n$, the corresponding concept of stability would be of global character. These considerations lead to (S9) and (S10). We note further that the definitions

(S7) and (S8) may hold even when $Q(0) \not\equiv 0$. In other words, the assumption about the existence of the trivial solution is not necessary.

In characterizing Lyapunov functions, it is convenient to introduce certain classes of monotone functions.

Definition 4.3.2.

(i) A function $\phi(r)$ is said to belong to the class $\mathscr{K}$ if $\phi \in C([0,\rho),\mathbb{R}_+)$, $\phi(0) = 0$, and $\phi(r)$ is strictly monotone increasing in r;

(ii) a function $\sigma(t)$ is said to belong to the class $\mathscr{L}$ if $\sigma \in C(J,\mathbb{R}_+)$, $\sigma(t)$ is monotone decreasing in t, and $\sigma(t) \to 0$ as $t \to \infty$;

(iii) a function $\phi(t,r)$ is said to belong to the class $\mathscr{K}\mathscr{K}$ if $\phi \in C(J \times [0,\rho),\mathbb{R}_+)$, $\phi \in \mathscr{K}$ for each $t \in J$, and ϕ is monotone increasing in t for each $r > 0$ and $\phi(t,r) \to \infty$ as $t \to \infty$ for each $r > 0$.

Definition 4.3.3.

(i) A function $V(t,x)$ with $V(t,0) \equiv 0$ is said to be positive definite (negative definite) if there exists a function $\phi \in \mathscr{K}$ such that the relation

$$V(t,x) \geq \phi(|x|), \quad (\leq -\phi(|x|))$$

is satisfied for $(t,x) \in J \times S_\rho$;

(ii) a function $V(t,x) \geq 0$ is said to be decrescent if a function $\phi \in \mathscr{K}$ exists such that

$$V(t,x) \leq \phi(|x|), \quad (t,x) \in J \times s_\rho.$$

To use the method of Lyapunov, which attempts to make statements about the stability properties directly by using suitable functions, we need to study the scalar differential equation

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0, \quad t_0 \geq 0, \tag{4.15}$$

where $g \in C(J \times \mathbb{R}_+,\mathbb{R})$. We suppose that $g(t,0) \equiv 0$ so that $u \equiv 0$ is a solution of (4.15) through $(t_0,0)$. Furthermore, this assumption also implies that the solutions $u(t) = u(t,t_0,u_0)$ of (4.15) are nonnegative for $t \geq t_0$ so as to assure that $g(t,u(t))$ is defined.

Corresponding to the stability definitions (S1)-(S8), we designate by (S1*)-(S8*) the concepts concerning the stability of the solution $u \equiv 0$ of (4.15).

Definition 4.3.4. The trivial solution $u \equiv 0$ of (4.15) is said to be (S1*) equistable if, for each $\varepsilon > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0,\varepsilon)$ that is continuous in t_0 for each ε such that

$$u(t,t_0,u_0) < \varepsilon, \quad t \geq t_0,$$

provided

$$u_0 \le \delta.$$

The definitions (S2*)-(S8*) may be formulated similarly.

To the different types of stability, there correspond different types of boundedness. Some important types are defined in the following:

Definition 4.3.5. The differential system (4.13) is said to be

(B1) equibounded if, for each $\alpha \ge 0$, $t_0 \in J$, there exists a positive function $\beta = \beta(t_0, \alpha)$, which is continuous in t_0 for each α, such that the inequality

$$|x_0| \le \alpha$$

implies

$$|x(t, t_0, x_0)| < \beta, \quad t \ge t_0;$$

(B2) uniform bounded if the β in (B1) is independent of t_0;

(B3) quasi-equi-ultimately bounded if, for each $\alpha \ge 0$ and $t_0 \in J$, there exist positive numbers N and $T = T(t_0, \alpha)$ such that the inequality

$$|x_0| \le \alpha$$

implies

$$|x(t, t_0, x_0)| < N, \quad t \ge t_0 + T;$$

(B4) quasi-uniform-ultimately bounded if the T in (B3) is independent of t_0;

(B5) equi-ultimately bounded if (B1) and (B3) hold at the same time;

(B6) uniform-ultimately bounded if (B2) and (B4) hold simultaneously;

(B7) equi-Lagrange stable if (B1) and (S7) hold simultaneously;

(B8) uniform-Lagrange stable if (B2) and (S8) hold simultaneously;

Proposition 4.3.1. If $f(t, 0) = 0$, $t \in J$, and β occurring in (B1) and (B2) has the property that $\beta \to 0$ as $\alpha \to 0$, then the definitions (B1), (B2) imply the definitions (S1), (S2) respectively.

The proof of the statement is obvious.

Proposition 4.3.2. Quasi-equi-ultimate boundedness implies equi-boundedness if

$$|(Qx)(t)| \le g(t, |x(t)|) \tag{4.16}$$

where $g \in C(J \times \mathbb{R}_+, \mathbb{R}_+)$.

Proof. Consider the function $m(t) = |x(t,t_0,x_0)|$, where $x(t,,t_0,x_0)$ is any solution of (4.13). Then,

$$D^+m(t) \le |x'(t,t_0,x_0)| = |f(t,x(t,t_0,x_0))| \le g(t,m(t)),$$

using assumption (4.16). By comparison Theorem 1.3.2, we have

$$|x(t,t_0,x_0)| \le r(t,t_0,\alpha), \quad t \ge t_0, \tag{4.17}$$

whenever $|x_0| \le \alpha$, where $r(t,t_0,\alpha)$ is the maximal solution of

$$u' = g(t,u), \quad u(t_0) = \alpha. \tag{4.18}$$

By the quasi-equi-ultimate boundedness, given $\alpha \ge 0$ and $t_0 \in J$, there exist two positive numbers N and $T = T(t_0,\alpha)$ such that the inequality $|x_0| \le \alpha$ implies

$$|x(t,t_0,x_0)| < N, \quad t \ge t_0 + T.$$

Since $g(t,u) \ge 0$, the solution $r(t,t_0,\alpha)$ of (4.18) is monotonic nondecreasing in t, and therefore we have, from (4.17), that

$$|x(t,t_0,x_0)| \le r(t_0+T,t_0,\alpha), \quad t \in [t_0,t_0+T].$$

It then follows that

$$|x(t,t_0,x_0)| \le \max[N, r(t_0+T,t_0,\alpha)], \quad t \ge t_0,$$

and this proves (B1).

Analogous to the group of definitions (B1)-(B8), we can define the concepts of boundedness and Lagrange stability with respect to the scalar differential equation (4.15) and designate them $(B1^*) - (B8^*)$.

4.4 Definitions Relative to Two Measures

The concepts of Lyapunov stability have given rise to several new notions that are important in applications. For example, partial stability, conditional stability, eventual stability and boundedness to name a few. In order to unify a variety of known concepts of stability and boundedness, it is beneficial to employ two different measures and obtain criteria in terms of two measures.

Consider the differential system (4.13) where the operator Q is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $x(t) = x(t,t_0,x_0)$ of (4.13). Let us first define the following classes of functions for future use:

$$\mathscr{CK} = \{a \in C[\mathbb{R}_+^2, \mathbb{R}_+] : a(t,s) \in \mathscr{K} \text{ for each } t\},$$

$$\Gamma = \{h \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+] : \inf_{(t,x)} h(t,x) = 0\},$$

$$\Gamma_0 = \{h \in \Gamma : \inf_x h(t,x) = 0 \text{ for each } t \in \mathbb{R}_+\}.$$

We shall now define various stability concepts for the system (4.13) in terms of two measures h_0, $h \in \Gamma$.

Definition 4.4.1. The differential system (4.13) is said to be

(S1) (h_0,h)-equi-stable if, for each $\varepsilon > 0$, $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta = \delta(t_0,\varepsilon)$ that is continuous in t_0 for each ε such that $h_0(t_0,x_0) < \delta$ implies $h(t,x(t)) < \varepsilon$, $t \geq t_0$, where $x(t) = x(t,t_0,x_0)$ is any solution of system (4.13);

(S2) (h_0,h)-uniformly stable if the δ in (S1) is independent of t_0;

(S3) (h_0,h)-equi-attractive, if for each $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exist positive constants $\delta_0 = \delta(t_0)$ and $T = T(t_0\varepsilon)$ such that $h_0(t_0,x_0) < \delta_0$ implies $h(t,x(t)) < \varepsilon$, $t \geq t_0 + T$;

(S4) (h_0,h)-uniformly attractive, if (S3) holds with δ_0 and T being independent of t_0;

(S5) (h_0,h)-equi-asymptotically stable if (S1) and (S3) hold simultaneously;

(S6) (h_0,h)-uniformly asymptotically stable if (S2) and (S4) hold together;

(S7) (h_0,h)-equi-attractive in the large if for each $\varepsilon > 0$, $\alpha > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive number $T = T(t_0,\varepsilon,\alpha)$ such that $h_0(t_0,x_0) < \alpha$ implies that $h(t,x(t)) < \varepsilon$, $t \geq t_0 + T$;

(S8) (h_0,h)-uniformly attractive in the large if the constant T in (S7) is independent of t_0.

Remark 4.4.1. Sometimes the notion of attractivity may be relaxed somewhat as in (S7) and (S8) and the corresponding concepts of stability would be of global character.

A few choices of the two measures (h_0,h) given below will demonstrate the generality of the Definition 4.4.1. Furthermore, the concepts in terms of two measures (h_0,h) enable us to unify a variety of stability notions found in the literature, which would otherwise be treated separately. It is easy to see that Definition 4.4.1 reduces to

(1) the well known stability of the trivial solution $x(t) \equiv 0$ of (4.13) or equivalently, of the invariant set $\{0\}$, if $h(t,x) = h_0(t,x) = |x|$;

(2) the stability of the prescribed motion $x_0(t)$ of (4.13) if $h(t,x) = h_0(t,x) = |x - x_0(t)|$;

(3) the partial stability of the trivial solution of (4.13) if $h(t,x) = |x|_s$, $1 \leq s < n$ and $h_0(t,x) = |x|$;

(4) the stability of asymptotically invariant set $\{0\}$, if $h(t,x) = h_0(t,x) = |x| + \sigma(t)$, where $\sigma \in \mathscr{L}$;

(5) the stability of the invariant set $A \subset \mathbb{R}^n$ if $h(t,x) = h_0(t,x) = d(x,A)$, where $d(x,A)$ is the distance of x from the set A;

(6) the stability of conditionally invariant set B with respect to A, where $A \subset B \subset \mathbb{R}^n$, if $h(t,x) = d(x,B)$, $h_0(t,x) = d(x,A)$;

(7) the conditional stability of the trivial solution of (4.13) if $h_0(t,x) = |x| + d(x,M)$, where M is the k-dimensional manifold containing the origin;

(8) the orbital stability of periodic solution of (4.13) if $h(t,x) = h_0(t,x) = d(x,C)$, where C is the closed orbit in the phase space.

We recall that the set $\{0\}$ is said to be asymptotically invariant relative to (4.13) if given $\varepsilon > 0$, there exists a $\tau(\varepsilon) > 0$ such that $x_0 = 0$ implies $|x(t,t_0,0)| < \varepsilon$ for $t \geq t_0 \geq \tau(\varepsilon)$. Recall also that $x = 0$ is said to be conditionally stable if given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$x_0 \in \{x : |x| < \delta\} \cap M \text{ implies } x(t) \in \{x : |x| < \varepsilon\},\ t \geq t_0.$$

We remark that when we wish to discuss the notion indicated in (4), we need to restrict the initial time t_0 to a suitable subset of $\mathbb{R}_+$ so that it is possible to have $h_0(t_0,x_0) < \delta$. Similarly, when we intend to consider the concept defined in (7), we choose the initial data x_0 to be in the manifold M in order that $h_0(t_0,x_0) < \delta$ implies $x_0 \in S(h_0,\delta) \cap M$, where $S(h_0,\delta) = \{x \in \mathbb{R}^n : h_0 = |x| + d(x,M) < \delta\}$. We note further that several other combinations of choices are possible for h_0, h is addition to those given in (1) to (8). The following definition will be useful in the sequel

Definition 4.4.2. Let $h_0, h \in \Gamma$. Then we say that

(i) h_0 is finer than h if there exists a $\rho > 0$ and a function $\varphi \in \mathscr{CK}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \varphi(t, h_0(t,x))$;

(ii) h_0 is uniformly finer than h if in (i) φ is independent of t;

(iii) h_0 is asymptotically finer than h if there exists a $\rho > 0$ and a function $\varphi \in \mathscr{KL}$ such that $h_0(t,x) < \rho$ implies $h(t,x) \leq \varphi(h_0(t,x),t)$.

4.5 Stability Criteria-Lyapunov Functions

In order to discuss the stability properties of IVP (4.13), let us assume that the solutions $x(t) = x(t,t_0,x_0)$ of (4.13) exist and are unique for $t \geq t_0$. We shall give sufficient conditions for the stability of the zero solution of (4.13) in terms of Lyapunov functions and we assume that (4.13) admits a trivial solution. Let us start first proving a simple result.

Theorem 4.5.1. Assume that

(i) $V \in C(\mathbb{R}_+ \times S(\rho), \mathbb{R}_+)$ and $V(t,x)$ is locally Lipschitzian in x;

(ii) for $t \geq t_0, x \in E_1, D^+V(t,x) \leq 0$;

(iii) $a, b \in \mathscr{K}$ are such that on $\mathbb{R}_+ \times S(\rho), V(t,x)$ satisfies

$$b(|x|) \leq V(t,x) \leq a(|x|),$$

i.e. V is positive definite and decrescent.

Then the trivial solution of (4.13) is stable.

Proof. Let $\varepsilon > 0$ and $t_0 \geq 0$ be given. Choose $\delta = \delta(\varepsilon) > 0$ such that $a(\delta) < b(\varepsilon)$. Then, we claim that with this δ, stability of the trivial solution of (4.13) follows. If it is not true, there would exist a solution $x(t)$ of (4.13) and $t_2 > t_1 > t_0$ satisfying

$$|x(t_1)| = \delta,\ |x(t_2)| = \varepsilon \text{ and } \delta \leq |x(t)| \leq \varepsilon,\ t_1 \leq t \leq t_2. \tag{4.19}$$

Then, we get from (ii) and Corollary 4.2.1, the estimate

$$V(t_2, x(t_2)) \leq V(t_1, x(t_1))$$

and therefore, (4.19) and assumption (ii), together with the choice of δ yield

$$b(\varepsilon) = b(|x(t_2)|) \leq V(t_2, x(t_2)) \leq V(t_1, x(t_1)) \leq a(|x(t_1)|) = a(\delta) < b(\varepsilon).$$

This contradiction proves stability of the trivial solution of (4.13), completing the proof.

The next result gives simple criterion for asymptotic stability of the trivial solution of (4.13).

Theorem 4.5.2. Let the assumptions of Theorem 4.5.1 hold except that condition (ii) is replaced by

(ii*) $D^+V(t,x)A(t) + V(t,x)A'(t) \leq 0$ for $t \geq t_0$, $x \in E_A$, where $A(t)$ is continuously differentiable for $t \geq t_0$ with $A(t_0) = 1, A(t) \geq 1$ and $A(t) \to \infty$ as $t \to \infty$.

Then the trivial solution of (4.13) is asymptotically stable.

Proof. By Theorem 4.2.2, we get

$$V(t, x(t))A(t) \leq V(t_0, x_0), \quad t \geq t_0, \tag{4.20}$$

and therefore, we have the stability of the trivial solution of (4.13). We only have to prove quasi-asymptotic stability. For this purpose, let $\varepsilon = \rho$ so that $\delta_0 = \delta(t_0, \rho)$. Choose $|x_0| < \delta_0$. Then, in view of (ii*), (iii) and (4.20) it follows that given any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a $T = T(t_0, \varepsilon) > 0$ satisfying

$$b(|x(t)|) \leq V(t, x(t)) \leq V(t_0, x_0)A^{-1}(t) \leq a(|x_0|)A^{-1}(t) < a(\delta_0)A^{-1}(t) < \varepsilon$$

for $t \geq t_0 + T$, where $A^{-1}(t) = \frac{1}{A(t)} \to 0$ as $t \to \infty$. Hence, the zero solution of (4.13) is quasi-asymptotically stable, thus establishing the asymptotic stability of the trivial solution of (4.13).

Consider the example

$$x'(t) = -ax(t) + \int_{t_0}^{t} k(t,s)x(s)ds, \quad a > 0 \tag{4.21}$$

where $k \in C(\mathbb{R}_+^2, \mathbb{R}_+)$. Take

$$L(t,x) = A(t)V(t,x) = e^{\delta t}x^2, \quad \delta > 0.$$

Then, the set $E_A = \{x \in C(\mathbb{R}_+, \mathbb{R}) : x^2(s)e^{\delta s} \leq x^2(t)e^{\delta t}\ t \geq t_0\}$. Hence, for $x \in E_A$,

$$x(s) \leq x(t)\exp\left(\frac{\delta}{2(t-s)}\right)$$

and

$$D^+L(t,x(t)) = \delta e^{\delta t}x^2(t) + 2x(t)e^{\delta t}\left[-ax(t) + \int_{t_0}^{t} k(t,s)x(s)ds\right] \leq 0,$$

when we assume that the kernel $k(t,s)$ in (4.21) satisfies the condition

$$\int_{t_0}^{t} k(t,s)\exp\left(\frac{\delta}{2}(t-s)\right)ds \leq \frac{2a-\delta}{2}. \tag{4.22}$$

Now, applying Theorem 4.5.2, it follows that the zero solution of (4.21) is exponentially asymptotically stable. Since δ is arbitrary, on letting $\delta \to 0$, the condition (4.22) reduces to

$$\int_{t_0}^{t} k(t,s)ds \leq a,$$

which is a sufficient condition for uniform stability of the zero solution of (4.21).

4.6 Stability Criteria-Lyapunov Functionals

As in differential equations with delay, one can utilize Lyapunov functions and Lyapunov functionals in the study of stability theory of causal differential equations. In this section, we employ Lyapunov functionals for discussing stability theory in the context of causal differential equations, given by IVP (4.13).

Let $V \in C(\mathbb{R}_+ \times E, \mathbb{R}_+)$ be any Lyapunov functional. We define its generalized derivative

$$D^+V(t,x(t)) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h, x(t+h,t,x)) - V(t,x)], \tag{4.23}$$

where $x(t+h,t,x)$ is the solution of IVP (4.13) through (t,x). Thus, the generalized derivative of a Lyapunov functional is defined along the solutions $x(t,t_0,x_0)$ of (4.13). We shall provide a result parallel to the original Lyapunov's second theorem on uniform asymptotic stability, in this set up.

Theorem 4.6.1. Assume that

(i) $V \in C(\mathbb{R}_+ \times E, \mathbb{R}_+)$ and $D^+V(t,x(t)) \le -C(|x(t)|), \quad C \in \mathscr{K}$;
(ii) $b(|x(t)|) \le V(t,x(t)) \le a(|x(t)|), \quad a,b \in \mathscr{K}$.

then the trivial solution $x \equiv 0$ of (4.13) is uniformly asymptotically stable.

Proof. Let $\varepsilon > 0$ and $t_0 \ge 0$ be given. Choose a $\delta = \delta(\varepsilon) > 0$ such that

$$a(\delta) < b(\varepsilon). \tag{4.24}$$

With this δ, ε, uniform stability of the trivial solution of (4.13) is valid. If not, there would exist a $t_1 > t_0$ and a solution $x(t,t_0,x_0)$ of (4.13) satisfying

$$|x(t_1,t_0,x_0)| = \varepsilon, \quad |x(t,t_0,x_0)| \le \varepsilon, \quad t_0 \le t \le t_1. \tag{4.25}$$

From the condition (i), it follows that

$$V(t,x(t,t_0,x_0)) \le V(t_0,x_0), \quad t_0 \le t \le t_1. \tag{4.26}$$

Now, using (4.24), (4.25), (4.26) and condition (ii), we get

$$b(\varepsilon) = b(|x(t_1,t_0,x_0)|) \le V(t_1,x(t_1,t_0,x_0)) \le V(t_0,x_0) \le a(|x_0|) < a(\delta) < b(\varepsilon)$$

which is a contradiction. Hence uniform stability follows.

To prove uniform asymptotic stability, set $\varepsilon = \rho$ and designate $\delta_0 = \delta(\rho)$ so that

$$|x_0| < \delta_0 \text{ implies } |x(t,t_0,x_0)| < \rho, \quad t \ge t_0.$$

In view of uniform stability, it is enough to show that there exists a t^*, $t_0 < t^* < t_0 + T$, where $T = 1 + \frac{a(\delta_0)}{c(\delta)}$ and $|x_0| < \delta_0$ and $|x(t^*,t_0,x_0)| < \delta(\varepsilon)$. Here $\delta = \delta(\varepsilon)$ corresponds to $\varepsilon > 0$ for uniform stability. If not, let $\delta \le |x(t,t_0,x_0)|$, $t \in [t_0, t_0 + T]$. Then, by condition (i), we have

$$V(t,x(t,t_0,x_0)) \le V(t_0,x_0) - \int_{t_0}^{t} C(|x(s,t_0,x_0)|)ds,$$

for $t \in [t_0, t_0 + T]$. As a result,

$$\begin{aligned} 0 &\le V(t_0+T, x(t_0+T,t_0,x_0)) \\ &\le V(t_0,x_0) - \int_{t_0}^{t_0+T} C(|x(s,t_0,x_0)|)ds \\ &\le a(\delta_0) - C(\delta)T < 0, \end{aligned}$$

by the definition of T. This contradiction shows that there exists a $t^* > t_0$ such that $|x(t^*,t_0,x_0)| < \delta$. This implies, by stability, that

$$|x_0| < \delta \text{ and } |x(t,t_0,x_0)| < \varepsilon,\ t \ge t_0 + T,$$

and the proof is complete.

4.7 Lyapunov Functions on Product Spaces

If we examine the literature where the Lyapunov functionals constructed for all the examples, relative to differential equations with delay and integro-differential equations, we find that we have inadvertently employed a combination of a Lyapunov function and a functional in such a way that the corresponding derivative of the Lyapunov function can be estimated suitably without demanding a minimal class of functions or prior knowledge of solutions. This observation leads us to consider the method of Lyapunov functions on product spaces to investigate stability properties of solutions of causal differential equations.

Consider the causal differential system (4.13) and let $x(t) = x(t,t_0,x_0)$ be any solution of IVP (4.13) existing on $J = [t_0,\infty)$. We wish to utilize Lyapunov functions on the product space $\mathbb{R}^n \times E$ and develop the stability theory for (4.13). In order to preserve clarity, we shall use the notation x_t to denote the Volterra operator which represents $x(t)$ from t_0 to t.

If $V \in C(\mathbb{R}_+ \times \mathbb{R}^n \times E, \mathbb{R}_+)$, then we define

$$\begin{aligned} D^+V(t,x(t),x_t) = {} & \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x(t)+h(Qx)(t),x_{t+h}(t,x)) \\ & -V(t,x,x_t(t,x))]. \end{aligned} \tag{4.27}$$

Also, if we assume that V is locally Lipschitzian in x and $x(t) = x(t,t_0,x_0)$ is a solution of (4.13), then (4.27) is equivalent to

$$D^+V(t,x(t),x_t) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,x(t+h),x_{t+h}(t_0,x_0)) - V(t,x(t),x_t)].$$

We shall now give sufficient conditions guaranteeing uniform asymptotic stability for zero solution of (4.13).

Theorem 4.7.1. Assume there exists function $V(t,x(t),x_t)$ such that

(i) $V \in C(\mathbb{R}_+ \times \mathbb{R}^n \times E, \mathbb{R}_+)$ with

$$D^+V(t,x(t),x_t) \le 0;$$

(ii) $b(|x(t)|) \le V(t,x(t),x_t) \le a(|x(t)|)$, $a,b \in \mathscr{K}$.

Then the zero solution of (4.13) is uniformly stable.

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \ge 0$ be given. Choose $\delta = \delta(\varepsilon) > 0$ such that

$$a(\delta) < b(\varepsilon). \tag{4.28}$$

Let $x(t,t_0,x_0) = x(t)$ be a solution of (4.13) with initial values (t_0,x_0) existing for all $t \ge t_0$. We claim that the zero solution of (4.13) is uniformly stable with this $\delta > 0$. If this is not true, then there exists a $t_1 > t_0$ such that

$$|x_0| < \delta \text{ and } |x(t_1,t_0,x_0)| = \varepsilon. \tag{4.29}$$

It then follows from (i) that

$$V(t,x(t),x_t) \le V(t_0,x_0). \tag{4.30}$$

Hence the condition (ii), relations (4.29) and (4.30) lead to

$$\begin{aligned} b(\varepsilon) \le b(|x(t_1)|) \le V(t_1,x(t_1),x_{t_1}) &\le V(t_0,x_0) \\ &\le a(|x_0|) < a(\delta) < b(\varepsilon), \end{aligned}$$

which is a contradiction and this completes the proof.

Theorem 4.7.2. Suppose that the function $V(t,x(t),x_t)$ of Theorem 4.7.1 satisfies (ii) and in place of (i), the inequality

$$D^+V(t,x(t),x_t) \le -c(|x(t)|), \quad c \in \mathscr{K}. \tag{4.31}$$

Then the zero solution of (4.13) is uniformly asymptotically stable.

Proof. By Theorem 4.7.1, the zero solution is uniformly stable. Thus, taking $\varepsilon = \rho$, there exists a $\delta_0 = \delta_0(\rho)$ such that

$$|x_0| < \delta_0 \text{ implies } |x(t)| < \rho, \quad t \ge t_0.$$

From assumption (4.31), we have

$$V(t,x(t),x_t) \le V(t_0,x_0,x_0) - \int_{t_0}^{t} c(|x(s)|)ds.$$

Now, let $0 < \eta < \rho$ and $T(\eta) = \frac{a(\delta_0)}{c(\delta(\eta))} + 1$, where $\delta(\eta) > 0$ corresponds to η in uniform stability. We claim that with $|x_0| < \delta_0$, there exists a $t_1 \in [t_0, t_0 + T]$ such that

$$|x(t_1)| < \delta(\eta), \quad t_1 \in [t_0, t_0 + T]$$

and hence it follows from uniform stability of the zero solution of (4.13) that

$$|x(t)| < \eta, \quad t \ge t_1.$$

Suppose that it is not true. Then we get

$$|x(t)| \ge \delta(\eta), \text{ for all } t \in [t_0, t_0 + T].$$

Therefore, at $t_1 = t_0 + T$, it follows that

$$\begin{aligned} 0 < b(\eta) \le b(|x(t_1)|) &\le V(t_1,x(t_1),x_{t_1}) \\ &\le V(t_0,x_0,x_0) - \int_{t_0}^{t_1} c(|x(s)|)ds \\ &\le a(\delta_0) - c(\delta(\eta))T < 0, \end{aligned}$$

in view of the choice of T. This contradiction proves that there exists a $t^* \in [t_0, t_0 + T]$ such that

$$x(t^*) < \delta(\eta) \text{ and } |x(t)| < \eta, \quad t \geq t_0 + T$$

whenever $|x_0| < \delta_0$. This implies uniform asymptotic stability and the proof is complete.
Consider the example

$$u'(t) = \alpha(t)u + \int_0^t a(t,s)u(s)ds, \quad t \geq 0, \tag{4.32}$$

where we assume that $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is continuous and the integral $\int_t^\infty |a(z,t)|dz$ is defined and finite for all $t \geq 0$. Assume also that there is a real number β such that

$$\int_0^t |a(t,s)|ds + \int_t^\infty |a(z,t)|dz + 2|\alpha(t)| \leq -\beta. \tag{4.33}$$

Then the zero solution of (4.32) is stable if $\alpha(t) < 0$.
Suppose that $\alpha(t) < 0$ and consider the Lyapunov functional on the product space $\mathbb{R} \times C$ given by

$$V(t,u,u(\cdot)) = u^2 + \int_0^t \int_t^\infty |a(z,s)|dzu^2(s)ds.$$

Then the time derivative of V along the solutions of (4.32) is given by

$$\begin{aligned} V'(t,u,u(\cdot)) &= 2\alpha(t)u^2 + 2\int_0^t |a(t,s)||u(s)||u|ds \\ &\quad + \int_t^\infty |a(z,t)|dzu^2 - \int_0^t |a(t,s)|u^2(s)ds. \end{aligned}$$

Since $2|u(s)||u| \leq u^2(s) + u^2$, it follows that

$$\begin{aligned} V'(t,u,u(\cdot)) &\leq 2\alpha(t)u^2 + \int_0^t |a(t,s)|[u^2(s) + u^2]ds \\ &\quad + \int_t^\infty |a(z,t)|dzu^2 - \int_0^t |a(t,s)|u^2(s)ds \\ &= \left[2\alpha(t) + \int_0^t |a(t,s)|ds + \int_0^t |a(z,t)|dz\right]u^2. \end{aligned}$$

Thus, in view of (4.33), we get

$$V'(t,u,u(\cdot)) \leq -\beta u^2.$$

Since V is positive definite and $V'(t,u,u(\cdot)) \leq 0$, it follows that the zero solution of (4.32) is stable.

4.8 Stability in Terms of two Measures

In this section, we shall consider a more general case and develop the basic Lyapunov theory in terms of two different measures introduced in Section 4.4.

Theorem 4.8.1. Assume that

(i) $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, $h \in \Gamma$, $V(t,x)$ is locally Lipschitzian in x and h-positive definite;

(ii) $D^+V(t,x) \leq 0$, $(t,x) \in S(h,\rho)$, where $S(h,\rho) = \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,\ h(t,x < \rho,\ \rho > 0)\}$ and $x \in E_1$.

Then

(A) if, in addition, $h_0 \in \Gamma$, h_0 is finer than h and $V(t,x)$ is h_0-weakly decrescent, then the system (4.13) is (h_0,h)-equistable;

(B) if, in addition, $h_0 \in \Gamma$, h_0 is uniformly finer than h, and $V(t,x)$ is h_0-decrescent, then the system (4.13) is (h_0,h)-uniformly stable.

Proof. Let us first prove (A). Since $V(t,x)$ is h_0-weakly decrescent, then for $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$, there exist constant $\delta_0 = \delta_0(t_0) > 0$ and function $a \in \mathscr{CK}$ such that

$$V(t_0,x_0) \leq a(t_0, h_0(t_0,x_0)), \text{ provided } h_0(t_0,x_0) < \delta_0. \tag{4.34}$$

The fact that $V(t,x)$ is h-positive definite implies that there exist constant $\rho_0 \in (0,\rho)$ and function $b \in \mathscr{K}$ such that

$$b(h(t,x)) \leq V(t,x), \text{ whenever } h(t,x) \leq \rho_0. \tag{4.35}$$

Also, by the assumption that h_0 is finer than h, there exist constant $\delta_1 = \delta_1(t_0) > 0$ and function $\varphi \in \mathscr{CK}$ such that

$$h(t_0,x_0) \leq \varphi(t_0, h_0(t_0,x_0)), \text{ if } h_0(t_0,x_0) < \delta_1, \tag{4.36}$$

where δ_1 is chosen so that $\varphi(t_0,\delta_1) < \rho_0$.

Let $\varepsilon \in (0,\rho_0)$ and $t_0 \in \mathbb{R}_+$ be given. By the assumption on a, there exists a $\delta_2 = \delta_2(t_0,\varepsilon) > 0$ that is continuous in t_0 such that

$$a(t_0,\delta_2) < b(\varepsilon). \tag{4.37}$$

Choose $\delta(t_0) = \min\{\delta_0, \delta_1, \delta_2\}$. Then $h_0(t_0,x_0) < \delta$ implies, by (4.34)-(4.37), that

$$b(h(t_0,x_0)) \leq V(t_0,x_0) \leq a(t_0, h_0(t_0,x_0)) < b(\varepsilon),$$

which in turn yields that $h(t_0,x_0) < \varepsilon$. We now claim that for every solution $x(t) = x(t,t_0,x_0)$ of (4.13) with $h_0(t_0,x_0) < \delta$,

$$h(t,x(t)) < \varepsilon, \quad t \geq t_0. \tag{4.38}$$

If this is not true, then there would exist a $t_1 > t_0$ such that

$$h(t_1,x(t_1)) = \varepsilon \text{ and } h(t,x(t)) < \varepsilon, \quad t \in [t_0,t_1), \tag{4.39}$$

for some solution $x(t) = x(t,t_0,x_0)$ of (4.13). Set $m(t) = V(t,x(t))$ for $t \in [t_0,t_1]$. Since $V(t,x)$ is locally Lipschitzian in x, it follows from Corollary 4.2.1 that $m(t)$ is nonincreasing in $[t_0,t_1]$. Thus it follows from (4.34)-(4.37) that

$$b(\varepsilon) = b(h(t_1,x(t_1))) \leq V(t_1,x(t_1)) \leq V(t_0,x_0) < b(\varepsilon),$$

which is a contradiction. Hence (4.38) is true and the system (4.13) is (h_0,h)-equistable.
To prove (B), note that if $V(t,x)$ is h_0-decrescent and h_0 is uniformly finer than h, then the functions a and φ in (4.34) and (4.36) are independent of t. Consequently, it is easily seen that the constant δ can be chosen to be independent of t_0. Hence the system (4.13) is (h_0,h)-uniformly stable.
We next prove a result on (h_0,h)-uniform asymptotic stability.
Theorem 4.8.2. Assume that

(i) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h;
(ii) $V \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+]$, $V(t,x)$ is locally Lipschitzian in x, h-positive definite, h_0-decrescent and

$$D^+V(t,x) \leq -C(h_0(t,x)),\ (t,x) \in S(h,\rho),\ C \in \mathscr{K}.$$

Then the system (4.13) is (h_0,h)-uniformly asymptotically stable.
Proof. Since $V(t,x)$ is h-positive definite and h_0-decrescent, there exist constants $0 < \rho_0 \leq \rho$, $0 < \delta_0$ and functions $a, b \in \mathscr{K}$ such that

$$b(h(t,x)) \leq V(t,x), \quad (t,x) \in S(h,\rho_0) \tag{4.40}$$

and

$$V(t,x) \leq a(h_0(t,x)),\ \text{if } h_0(t,x) < \delta_0. \tag{4.41}$$

It follows from Theorem 4.8.1 that the system (4.13) is (h_0,h)-uniformly stable. If we let $\varepsilon = \rho_0$, then there exists $\delta_1 = \delta_1(\rho) > 0$ such that

$$h_0(t_0,x_0) < \delta_1 \text{ implies } h(t,x(t)) < \rho_0,\ t \geq t_0,$$

where $x(t) = x(t,t_0,x_0)$ is any solution of (4.13).
Let $0 < \varepsilon < \rho_0$ and $\delta = \delta(\varepsilon)$ be the same δ as in the definition for (h_0,h)-uniform stability. Assume that $h_0(t_0,x_0) < \delta^* = \min\{\delta_0, \delta_1\}$. Set $T = T(\varepsilon) = \frac{a(\delta^*)}{C(\delta)} + 1$. To prove (h_0,h)-uniform asymptotic stability, it is enough to show that there exists a $t^* \in [t_0, t_0+T]$, such that

$$h_0(t^*,x(t^*)) < \delta$$

. If this is not true, then there exists a solution $x(t) = x(t,t_0,x_0)$ of (4.13) with $h_0(t_0,x_0) < \delta^*$ such that

$$h_0(t,x(t)) \geq \delta, \quad t \in [t_0,t_0+T]. \tag{4.42}$$

Let $m(t) = V(t,x(t))$. Then it follows from condition (ii) that

$$D^+m(t) \leq -C(h_0(t,x(t))), \quad t \geq t_0$$

which implies by (4.41) that

$$\int_{t_0}^{t_0+T} C(h_0(s,x(s)))ds \leq m(t_0) \leq a(\delta^*).$$

On the other hand, from (4.42), we obtain

$$\int_{t_0}^{t_0+T} C(h_0(s,x(s)))ds \geq C(\delta)T > a(\delta^*),$$

which is a contradiction. Thus the proof of the theorem is complete.

4.9 Vector Lyapunov Functions

In 1960, Bellman and Matrosov independently introduced the concept of vector (or several) Lyapunov functions and developed the method of vector Lyapunov functions to study stability theory. This method has more flexibility compared to utilizing a single Lyapunov function. For example, each of the components of the vector Lyapunov function need not be positive definite and decrescent and this offers more flexibility in constructing and discovering Lyapunov functions for a given problem. Since the introduction of this new method, it has been applied to a variety of problems. In particular, its application for the study of large scale dynamic systems has become very effective and profitable, since several Lyapunov functions appear naturally in dealing with a large scale system and its many subsystems. We shall extend in this section, the method of vector Lyapunov functions to the causal differential system

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \quad t_0 \geq 0, \tag{4.43}$$

utilizing the vector Lyapunov function on a product space i.e. $V \in C(\mathbb{R}_+ \times \mathbb{R}^n \times E, \mathbb{R}^N_+)$. Of course, we need to first prove the following comparison result in terms of several Lyapunov functions.

Theorem 4.9.1. Assume that the vector Lyapunov function V is locally Lipschitzian in x and satisfies the vectorial inequality

$$D^+V(t,x(t),x_t) \leq g(t,V(t,x(t),x_t)) \tag{4.44}$$

where $g \in C(\mathbb{R}_+ \times \mathbb{R}^N_+, \mathbb{R}^N)$ and $D^+V(t,x(t),x_t)$ is given by (4.27). Suppose further that $g(t,u)$ is quasi-monotone nondecreasing in u and $r(t) = r(t,t_0,u_0)$ is the maximal solution of the system

$$u' = g(t,u), \quad u(t_0) = u_0 \geq 0, \tag{4.45}$$

existing on $[t_0,\infty)$. Then $V(t_0,x_0) \leq u_0$ implies

$$V(t,x(t),x_t) \leq r(t,t_0,u_0), \quad t \geq t_0, \tag{4.46}$$

where $x(t) = x(t,t_0,x_0)$ is any solution of the causal differential system (4.43) and vectorial inequalities and to be understood component-wise.

Proof. Set $m(t) = V(t,x(t),x)t)$ with $m(t_0) = u_0$. It is easy to get the following differential inequality

$$D^+m(t) \leq g(t,m(t)), \quad t \geq t_0,$$

utilizing the condition (4.44). By Theorem 2.10.1, with $v(t) = m(t)$, $w(t) = u(t,\varepsilon)$, where $u(t,\varepsilon)$ is any solution of

$$u' = g(t,u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon,$$

for small, arbitrary $\varepsilon > 0$, we get

$$m(t) < u(t,\varepsilon), \quad t \geq t_0,$$

which yields, as $\varepsilon \to 0$, the desired estimate (4.46) and the proof is complete.

We shall now give a result that gives the sufficient conditions for the stability properties of the trivial solution of (4.43) in terms of vector Lyapunov functions.

Theorem 4.9.2. Assume that

(i) $g \in C(\mathbb{R}_+ \times \mathbb{R}^N_+, \mathbb{R}^N)$, $g(t,0) \equiv 0$ and $g(t,u)$ is quasi-monotone decreasing in u;

(ii) $V \in C(\mathbb{R}_+ \times \mathbb{R}^N \times E, \mathbb{R}^n_+)$, V is locally Lipschitzian and

$$b(|x|) \leq V_0(t,x(t),x_t) \leq a(|x|), \quad a,b \in \mathscr{K},$$

where

$$V_0(t,x(t),x_t) = \sum_{i=1}^{N} V_i(t,x(t),x_t);$$

(iii) $D^+V(t,x(t),x_t) \leq g(t,V(t,x(t),x_t))$.

Then, the stability properties of the trivial solution of (4.45) imply the corresponding stability properties of the trivial solution of (4.43).

Proof. We shall prove only equi-asymptotic stability of the trivial solution of (4.43). For this purpose, let us first prove equistability. Let $0 < \varepsilon < \rho$ and $t_0 \geq 0$ be given. Assume that the trivial solution of (4.45) is equiasymptotically stable. Then, it is equistable. Hence, given $b(\varepsilon) > 0$ and $t_0 \geq 0$, there exists a $\delta_1 = \delta_1(t_0, \varepsilon)$ such that

$$\sum_{i=1}^{N} u_{0_i} < \delta_1 \text{ implies } \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon), \tag{4.47}$$

for $t \geq t_0$, where $u(t, t_0, u_0)$ is any solution of (4.45). Choose $u_0 = V(t_0, x_0, x_0)$ and a $\delta = \delta(t_0, \varepsilon) > 0$ satisfying

$$a(\delta) < \delta_1. \tag{4.48}$$

Let $|x_0| < \delta$. Then, we claim that $|x(t)| < \varepsilon, t \geq t_0$. If this is not true, there exists a solution $x(t)$ of (4.43) and a $t_1 > t_0$ such that

$$|x(t_1)| = \varepsilon \text{ and } |x(t)| \leq \varepsilon < \rho, \quad t_0 \leq t \leq t_1. \tag{4.49}$$

Hence, we have by Theorem 4.9.1,

$$V(t, x(t, x_t)) \leq r(t, t_0, u_0), \quad t_0 \leq t \leq t_1, \tag{4.50}$$

where $r(t) = r(t, t_0, u_0)$ is the maximal solution of (4.45). Since

$$V_0(t_0, x_0, x_0) \leq a(|x_0|) < a(\delta) < \delta_1,$$

the relations (4.47)-(4.49) yield

$$b(\varepsilon) \leq V_0(t_1, x(t_1), x_{t_1}) \leq r_0(t_1, t_0, u_0) < b(\varepsilon)$$

where

$$r_0(t, t_0, u_0) = \sum_{i=1}^{N} r_i(t, t_0, u_0).$$

This contradiction proves equistability.

Suppose next that the trivial solution of (4.45) is quasi-equi asymptotically stable. Set $\varepsilon = \rho$ and designate by $\delta_0 = \hat{\delta}(t_0, \rho)$. Let $0 < \eta < \rho$. Then, given $b(\eta)$ and $t_0 \geq 0$, there exists a $\delta_1^* = \hat{\delta}_1(t_0, \eta) > 0$ and $T = T(t_0, \eta) > 0$ satisfying

$$\sum_{i=1}^{N} u_{i_0} < \delta_1^* \text{ implies } \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon), \tag{4.51}$$

for $t \geq t_0 + T$.

Choosing $u_0 = V(t_0, x_0, x_0)$ as before, we find $\delta_0^* > 0$ such that $a(\delta_0^*) < \delta_1^*$. Let $\delta_0 = \min(\delta_1^*, \delta_0^*)$ and $|x_0| < \delta_0$. This implies $|x(t)| < \rho, t \geq t_0$ and therefore the estimate (4.50) holds for $t \geq t_0$. Suppose now that there is a sequence $\{t_k\}$, $t_k \geq t_0 + T$, $t_k \to \infty$ as $k \to \infty$ and $\eta \leq |x(t_k)|$ with $|x_0| < \delta_0$. In view of (4.51) this leads to a contradiction

$$b(\eta) \leq V_0(t_k, x(t_k), x_{t_k}) \leq r_0(t_k, t_0, u_0) < b(\eta).$$

Hence the trivial solution of (4.43) is equiasymptotically and the proof is complete.

4.10 Cone-valued Lyapunov Functions

The method of vector Lyapunov functions, though flexible and effective, has an unpleasant requirement of quasi-monotonicity of the comparison system. In the case of comparison systems being linear, this quasi-monotonicity requirement reduces to requiring all off-diagonal elements of the comparison matrix involved in the linear differential system to be non-negative. However, in applications, one encounters comparison systems which satisfy stability conditions without being quasi-monotone. Hence, the application potential of this useful and effective method of vector Lyapunov functionsis diminished. It was observed later in 1974, that this difficulty is due to the choice of the cone relative to the comparison system, namely, $\mathbb{R}_+^N$, the cone of non-negative elements in $\mathbb{R}^N$ and a possible approach to overcome this limitation is to choose an appropriate cone other than $\mathbb{R}_+^N$ to work in a given situation depending on the problem at hand. This observation gave rise to the development of differential inequalities through arbitrary cones and the method of cone valued Lyapunov functions.

In order to consider differential inequalities in a Banach space X in general, we need to introduce the concept of a cone which induces a partial ordering in X.

A proper subset K of X is said to be a cone if K is closed, convex with $K+K \subset K$, $\lambda K \subset K$, $\lambda \geq 0$ and $K \cap \{-K\} = \theta$, where θ denotes the null element of the Banach space X. Let $\overline{K}$, K° denote the closure and interior of K respectively. Assume K° to be nonempty. The cone induces the order relations in X defined by

$$x \leq y \text{ if and only if } y - x \in K,$$

$$x < y \text{ if and only if } y - x \in K^\circ,$$

for $x, y \in X$. Let K^* be the set of all continuous linear functionals ϕ on X such that $\phi(u(t)) \geq 0$ for all $u(t) \in K$, $t \in J = [t_0, T)$ and let K_0^* be the set of all continuous linear functionals on X such that $\phi(u(t)) > 0$ for all $u(t) \in K^\circ$.

Let us consider the causal differential system

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \quad t \geq 0, \tag{4.52}$$

where $Q \in C(\mathbb{R}_+ \times E, \mathbb{R}^n)$, $E = C([t_0, T), \mathbb{R}^n)$ and assume that the operator Q is smooth enough to guarantee the existence and uniqueness of solutions $x(t, t_0, x_0) = x(t)$, $t \geq t_0$, for IVP (4.52). We say that $Q : E \to E$ is nondecreasing if

$$u \leq v \text{ implies } (Qu)(t) \leq (Qv)(t),\ t \in J = [t_0, T),$$

where $u \le v$ means $u(t) \le v(t)$, for $u, v \in E$, $t \in J$.
To prove a basic result on differential inequalities in a cone, we need the following result.
Lemma 4.10.1. Let K be a cone set with nonempty interior K°. Then,

(i) $x \in K$ is equivalent to $\phi(x) = (\phi, x) \ge 0$ for all $\phi \in K^*$;
(ii) $x \in \partial K$ implies that there exists a $\phi \in K_0^*$ such that $\phi(x) \equiv 0$.

We can now state the basic inequalities result in cone K.
Theorem 4.10.1. Let K be a cone in E with nonempty interior K°. Assume that

(i) $u, v \in C^1(J, E)$, $Q \in C(B, E)$ with $B \equiv B[u_0, b] = \{u \in E : |u - u_0| \le b\}$ and Q is nondecreasing;
(ii) $u'(t) - (Qu)(t) < v'(t) - (Qv)(t)$, $t \in (t_0, T)$.

Then $u(t_0) < v(t_0)$ implies that $u(t) < v(t)$, $t \in J$.
Proof. Suppose that the assertion of the theorem is false. Then, there exists a $t_1 > t_0$ such that

$$v(t_1) - u(t_1) \in \partial K \text{ and } v(t) - u(t) \in K^\circ,\ t \in [t_0, t_1).$$

By Lemma 4.10.1 , there exists a $\phi \in K_0^*$ such that

$$\phi(v(t_1) - u(t_1)) = 0.$$

Setting $m(t) = \phi(v(t) - u(t))$, we see that

$$m(t_1) = 0 \text{ and } m(t) > 0 \text{ for } t_0 \le t < t_1.$$

Consequently, $m'(t_1) \le 0$. Further, at $t = t_1$, we have $u(t_1) = v(t_1)$ and $u(t) < v(t)$, $t_0 \le t < t_1$. Thus $u(t) \le v(t)$ on $[t_0, t_1]$.
Using the nondecreasing nature of Q, we have

$$(Qu)(t) \le (Qv)(t), \quad t_0 \le t \le t_1 < T.$$

Hence, from (ii), we get

$$m'(t_1) = \phi(v'(t_1) - u'(t_1)) > \phi((Qv)(t_1) - (Qu)(t_1)) \ge 0.$$

This contradiction proves the theorem.
Using Theorem 4.10.2, w can prove the existence of the maximal solution of (4.52) relative to the cone K (see Section 5.5).
Relative to the system (4.52), we require the comparison differential system

$$w' = g(t, w), \quad w(t_0) = w_0 \ge 0, \tag{4.53}$$

where $g \in C(\mathbb{R}_+ \times K, \mathbb{R}^N)$, $K \subset \mathbb{R}^N$ is a cone. We need the following notion to define the concepts of stability relative to the solutions of (4.53) in terms of suitable two measures $H_0, H \in \Sigma$, where

$$\Sigma = \{H \in C(K, \mathbb{R}_+) : H(0) = 0 \text{ and } H(w) \text{ is increasing relative to cone } K\}.$$

Definition 4.10.1. Let $H_0, H \in \Sigma$. Then, we say that the comparison system (4.53) is (H_0, H)-equistable, if given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon) > 0$ that is continuous in t_0 for each ε such that

$$H_0(w_0) < \delta \text{ implies } H(w(t)) < \varepsilon, \quad t \geq t_0,$$

where $w(t) = w(t, t_0, w_0)$ is any solution of (4.53).
Other definitions of (H_0, H)-stability can be formulated, based on the Definition 4.10.1 and the corresponding (h_0, h)-stability definitions in Section 4.4.
Let us now introduce the cone-valued Lyapunov-like functions. The following comparison theorem plays an important role when we employ cone-valued Lyapunov-like functions.
Theorem 4.10.2. Assume that

(i) $V \in C(\mathbb{R}_+ \times \mathbb{R}^n \times e, K)$, V is locally Lipschitzian relative to the cone K and

$$D^+V(t, x(t), x_t) \leq g(t, V(t, x(t)x_t)),$$

where $D^+V(t, x(t), x_t)$ is as defined in Section 4.7;
(ii) $g \in C(\mathbb{R}_+ \times K, \mathbb{R}^N)$ and $g(t, w)$ is quasi-monotone nondecreasing in w relative to K for each $t \in \mathbb{R}_+$, that is, if $u \leq v$ and $\phi(u) = \phi(v)$ for some $\phi \in K_0^*$, then $\phi(g(t, u)) \leq \phi(g(t, v))$, $t \in \mathbb{R}_+$, $u, v \in K$.

If $r(t) = r(t, t_0, w_0)$ is the maximal solution of (4.53) relative to cone K and $x(t) = x(t, t_0, x_0)$ is any solution of (4.52) such that $V(t_0, x_0, x_0) \leq w_0$, then on the common interval of existence, we have

$$V(t, x(t), x_t) \leq r(t, t_0, w_0.)$$

Corollary 4.10.1. In Theorem 4.10.2, the function $g(t, w) \equiv 0$ is admissible to yield

$$V(t, x(t), x_t) \leq V(t_0, x_0, x_0).$$

The proof of Theorem 4.10.2 follows from Theorem 5.5.5 (see Section 5.5 for inequalities in cones and existence of extremal solutions relative to cones). This is because, we can immediately obtain, defining $m(t) = V(t, x(t), x_t)$, the differential inequality

$$D^+m(t) \leq g(t, m(t))$$

from which the stated result follows, using Theorem 5.5.5.

We are now in a position to prove (h_0,h)-stability results for the system (4.52), utilizing the corresponding (H_0,H)-stability notions relative to the comparison system (4.53). First we give two simple results, parallel to original Lyapunov theory, in terms of cone valued Lyapunov functions.

Theorem 4.10.3. Assume that the following hypotheses hold:

(i) $h_0,h \in \Gamma$ and $h(t,x) \leq \phi(h_0(t,x))$ if $h_0(t,x) < \rho_0$ for some $\rho_0 > 0$, where $\phi \in \mathscr{K}$;
(ii) $H_0,H \in \Sigma$ and $H(w) \leq \psi(H_0(w))$ if $H_0(w) < \lambda$ for some $\lambda > 0$, where $\psi \in \mathscr{K}$;
(iii) $V \in C(\mathbb{R}_+ \times \mathbb{R}^n \times E, K)$ and V is locally Lipschitzian with respect to come K;
(iv) there exists a $\rho > 0$ such that $\phi(\rho_0) < \rho$ satisfying

$$b(h(t,x(t))) \leq H(V(t,x(t),x_t)) \text{ if } h(t,x(t)) < \rho,$$

$$H_0(V(t,x(t),x_t)) \leq a(t,h_0(t,x(t))) \text{ if } h_0(t,x(t)) < \rho_0,$$

where $b \in \mathscr{K}$ and $a \in C\mathscr{K} = \{a \in C(\mathbb{R}_+^2,\mathbb{R}_+) : a(t,w) \in \mathscr{K} \text{ for } t \in \mathbb{R}_+\}$;
(v) $D^+V(t,x(t),x_t) \leq 0$ in $S(h,\rho) = \{(t,x(t)) : h(t,x(t)) < \rho\}$.

Then, the causal differential system (4.52) is (h_0,h)-equistable. If in addition, a in (iv) is such that $a \in \mathscr{K}$ i.e. $a(t,w)$ is independent of t, then (4.52) is (h_0,h)-uniformly stable.

Proof. Let $0 < \varepsilon < \min(\rho,\lambda)$ and $t_0 \in \mathbb{R}_+$ be given. Let $w_0 = V(t_0,x_0,x_0)$. Choose $\delta = \delta(t_0,\varepsilon) < \min(\rho_0,\lambda_0)$ with $a(t_0,\lambda_0) \leq \lambda$ and

$$\psi(a(t_0,\delta)) < b(\varepsilon). \tag{4.54}$$

Let $h_0(t_0,x_0) < \delta$ and note that

$$\begin{aligned} b(h(t_0,x_0)) &\leq H(V(t_0,x_0,x_0)) \leq \psi(H_0(V(t_0,x_0,x_0))) \\ &\leq \psi(a(t_0,h_0(t_0,x_0))) \leq \psi(a(t_0,\delta)) \\ &< b(\varepsilon), \end{aligned} \tag{4.55}$$

which implies that $h(t_0,x_0) < \varepsilon$. We claim that with this δ, the system (4.52) is (h_0,h)-equistable. If this is not true, because of (4.55), there exists a $t_1 > t_0$ and a solution $x(t) = x(t,t_0,x_0)$ of (4.52) with $h_0(t_0,x_0) < \delta$ such that

$$h(t_1,x(t_1)) = \varepsilon \text{ and } h(t,x(t)) \leq \varepsilon, \quad t_0 \leq t \leq t_1. \tag{4.56}$$

Hence, condition (iv) yields, by Corollary 4.10.1, the estimate

$$V(t,x(t),x_t) \leq V(t_0,x_0,x_0), \quad t_0 \leq t \leq t_1. \tag{4.57}$$

Consequently, using (iv), (4.54), (4.56) and (4.57), we get

$$\begin{aligned} b(\varepsilon) &= b(h(t_1,x(t_1))) \leq H(V(t_1,x(t_1),x_{t_1})) \\ &\leq \psi(H_0(V(t_0,x_0,x_0))) \leq \psi(a(t_0,h_0(t_0,x_0))) \\ &\leq \psi(a(t_0,\delta)) < b(\varepsilon), \end{aligned}$$

which is a contradiction. Hence, (h_0,h)-equistability of (4.52) follows. If $a \in \mathscr{K}$ in condition (iv), then it is easy to see that λ_0 and consequently δ, are independent of t_0. As a result, (h_0,h)-uniform stability of (4.52) follows. The proof is complete.

The next result provides criteria for (h_0,h)-uniform asymptotic stability.

Theorem 4.10.4. Let the hypotheses (i)-(iv) of Theorem 4.10.3 hold with $a \in \mathscr{K}$. Suppose further that

$$D^+V(t,x(t),x_t) \leq -\tilde{C}(h_0(t,x(t))), \tag{4.58}$$

for $(t,x(t)) \in S(h,\rho)$, where $\tilde{C} \in C(\mathbb{R}_+,K)$ and $\tilde{C}(t)$ is increasing in t relative to K with $\tilde{C}(0) = 0$. Then, the system (4.52) is (h_0,h)-uniformly asymptotically stable.

Proof. Since $\tilde{C} \in \mathscr{K}$, it follows that

$$D^+V(t,x(t),x_t) \leq 0$$

in $S(h,\rho)$ and this implies by Theorem 4.10.3, that the system (4.52) is (h_0,h)-uniformly stable. Let $\varepsilon = \varepsilon_0 = \min(\rho,\lambda)$ and designate by $\delta_0 = \delta(\varepsilon_0)$ so that we have

$$h_0(t_0,x_0) < \delta_0 \text{ implies } h(t,x(t)) < \varepsilon_0, \quad t \geq t_0. \tag{4.59}$$

Now, let $h_0(t_0,x_0) < \delta_0$ and for any $\varepsilon < \varepsilon_0$, choose a $T = T(\varepsilon) > 0$ such that

$$H(\tilde{C}(\delta)T) \geq \psi(a(\delta_0)), \tag{4.60}$$

where $\delta = \delta(\varepsilon)$ corresponds to ε in (h_0,h)-uniform stability.

To prove (h_0,h)-uniform asymptotic stability, it is sufficient to show that there exists a $t^* \in [t_0,t_0+T]$ with $h_0(t^*,x(t^*)) < \delta$, where $x(t)$ is any solution of (4.52) with $h_0(t_0,x_0) < \delta_0$. If this is not true, then we have

$$h_0(t,x(t)) \geq \delta, \quad t \in [t_0,t_0+T]. \tag{4.61}$$

Setting

$$m(t) = V(t,x(t),x_t) + \int_{t_0}^{t} \tilde{C}(h_0(s,x(s)))ds,$$

we obtain, from (4.59), the estimate

$$m(t) \leq m(t_0) = V(t_0,x_0,x_0) \tag{4.62}$$

for $t \in [t_0, t_0+T]$. Hence, using the assumptions (ii), (iv) and the relations (4.61) and (4.62), it follows that

$$\begin{aligned} H(\tilde{C}(\delta)T) &= H\left(\tilde{C}(\delta)\int_{t_0}^{t_0+T} ds\right) \\ &\le H\left(\int_{t_0}^{t_0+T} \tilde{C}(h_0(s,x(s))ds)\right) \\ &\le H(m(t_0+T)) \le H(V(t_0,x_0,x_0)) \\ &\le \psi(H_0(V(t_0,x_0,x_0))) \\ &\le \psi(a(h_0(t_0,x_0))) < \psi(a(\delta_0)). \end{aligned}$$

This is a contradiction and therefore, we have (h_0,h)-uniform asymptotic stability of (4.52), and the proof is complete.

Let us next discuss (h_0,h)-equi asymptotic stability which is in spirit of Marachkov's Theorem.

Theorem 4.10.5. Let the assumptions (i)-(iv) of Theorem 4.10.3 hold. Suppose further that

$$D^+V(t,x(t),x_t) \le -\tilde{C}(h(t,x(t))) \tag{4.63}$$

for $(t,x) \in S(h,\rho)$, where $\tilde{C}$ is the same function defined in Theorem 4.10.4. Assume also that $h(t,x)$ is locally Lipschitzian in x and $D^+h(t,x)$ is bounded above or below in $S(h,\rho)$ and $H(w) \to \infty$ as $|w| \to \infty$. Then, the system (4.52) is (h_0,h)-equiasymptotically stable.

Proof. As in Theorem 4.10.4, since $a \in C\mathscr{K}$, we arrive at

$$h_0(t_0,x_0) < \delta_0 = \delta_0(t_0,\varepsilon_0) \text{ implies } h(t,x(t)) < \varepsilon_0, \quad t \ge t_0,$$

where $x(t) = x(t,t_0,x_0)$ is any solution of (4.52). To prove the claim of the theorem, it is enough to show that $h(t,x(t)) \to 0$ as $t \to \infty$. We shall first note that $\lim_{t\to\infty} \inf h(t,x(t)) = 0$. If not, there exists a $T > t_0$ and an $\eta > 0$ such that

$$h(t,x(t)) \ge \eta, \quad t \ge T.$$

Hence, (4.63) yields, as before

$$H(\tilde{C}(\eta)(t-T)) \le H\left(\int_T^t \tilde{C}(h(s,x(s)))ds\right) \le \psi(a(\delta_0))$$

which, in view of the assumption on H, leads to a contradiction.

Suppose that $\limsup_{t\to\infty} h(t,x(t)) \ne 0$. Then, for any $\eta > 0$ there exist divergent sequences $\{t_n\}, \{t_n^*\}$ such that $t_i < t_i^* < t_{i+1}$, $i = 1,2,\ldots$, and

$$\left.\begin{aligned} &h(t_i,x(t_i)) = \tfrac{\eta}{2},\ h(t_i^*,x(t_i^*)) = \eta, \text{ and,} \\ &\tfrac{\eta}{2} < h(t,x(t)) < \eta,\ t \in (t_i,t_i^*). \end{aligned}\right\} \tag{4.64}$$

Indeed, one could also have, instead of (4.64),

$$\left.\begin{array}{l} h(t_i,x(t_i)) = \eta,\ h(t_i^*,x(t_i^*)) = \frac{\eta}{2} \text{ and} \\ \frac{\eta}{2} < h(t,x(t)) < \eta,\ t \in (t_i,t_i^*). \end{array}\right\} \tag{4.65}$$

Suppose that $D^+h(t,x) \le M$. Then, using (4.64), we get

$$t_i^* - t_i \ge \frac{\eta}{4M} = \gamma > 0.$$

In view of (4.63), it then follows, for large n,

$$H\left(\tilde{C}\left(\frac{\eta}{2}\right)\gamma n\right) \le \psi(a(\delta_0)).$$

Since $H(w) \to \infty$ as $|w| \to \infty$, this leads to a contradiction as $n \to \infty$. Thus, $h(t,x(t)) \to 0$ as $t \to \infty$.

the case $D^+h(t,x)$ bounded below can be proved similarly using (4.65). Hence the proof is complete.

Remark. The assumption (4.63) of Theorem 4.10.5 can be generalized to

$$D^+V(t,x(t),x_t) \le -\tilde{C}(w(t,x(t))),$$

in $S(h,\rho)$, where $w(t,x)$ is h-positive definite and w satisfies similar conditions as h. Then, the conclusion of Theorem 4.10.5 remains true.

Employing the comparison Theorem 4.10.2, we shall now consider a general set of criteria for (h_0,h)-stability properties which unify several stability concepts in a single set up.

Theorem 4.10.6. Assume that:

(A0) $h_0, h \in \Gamma$ and h_0 is uniformly finer than h;

(A1) $V \in C(\mathbb{R}_+ \times \mathbb{R}^n \times E, K)$ and V is locally Lipschitzian relative to K;

(A2) $H_0, H \in \Sigma$ and H_0 is finer than H;

(A3) $g \in C(\mathbb{R}_+ \times K, \mathbb{R}^N)$ and for $(t,x) \in S(h,\rho)$,

$$D^+V(t,x(t),x_t) \le g(t,V(t,x(t),x_t))$$

where $g(t,w)$ is quasi-monotone nondecreasing in w relative to K for each $t \in \mathbb{R}_+$;

(A4) $b(h(t,x(t))) \le H(V(t,x(t),x_t))$ if $h(t,x(t)) < \rho$ and

$$H_0(V(t,x(t),x_t)) \le a(h_0(t,x(t))) \text{ if } h_0(t,x(t)) < \rho_0$$

where $a,b \in \mathscr{K}$.

Then, the (H_0, H)-stability properties of the comparison system (4.53) imply the corresponding (h_0, h)-stability properties of the system (4.52).

Proof. We shall prove only (h_0, h)-equistability, since based on this proof, one can construct proofs of other (h_0, h)-stability properties.

Since H_0 is finer than H, there exists a $\lambda > 0$ and a $\psi \in \mathscr{K}$ such that

$$H(w) \leq \psi(H_0(w)) \text{ if } H_0(w) < \lambda. \tag{4.66}$$

Let $0 < \varepsilon < \min(\rho, \lambda)$ and $t_0 \in \mathbb{R}_+$ be given. Suppose that the comparison system (4.53) is (H_0, H)-equistable. Then, given $b(\varepsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exist a $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ with $\delta_1 < \min(\lambda, \psi^{-1}(b(\varepsilon)))$ such that for $t \geq t_0$

$$H_0(w_0) < \delta_1 \text{ implies } H(w(t)) < b(\varepsilon), \tag{4.67}$$

where $w(t) = w(t, t_0, w_0)$ is any solution of the system (4.53). Also, h_0 is finer than h implies that there exists a $\phi \in \mathscr{K}$ such that

$$h(t,x) \leq \phi_0(h_0(t,x)) \text{ if } h_0(t,x) < \rho_0, \tag{4.68}$$

with $\phi(\rho_0) < \rho$. Choose $w_0 = V(t_0, x_0, x_0)$ and $\delta < \min(\rho_0, \lambda_0)$, where $a(\lambda_0) \leq \lambda$ such that

$$a(\delta) < \delta_1. \tag{4.69}$$

Now, let $h_0(t_0, x_0) < \delta$ and note that

$$\begin{aligned} b(h(t_0,x_0)) &\leq H(V(t_0,x_0,x_0)) \leq \psi(H_0(V(t_0,x_0,x_0))) \\ &\leq \psi(a(h_0(t_0,x_0))) \leq \psi(a(\delta)) \\ &< \psi(\delta_1) < b(\varepsilon), \end{aligned}$$

so that

$$h(t_0,x_0) < \varepsilon. \tag{4.70}$$

We claim that with this δ, it follows that

$$h_0(t_0,x_0) < \delta \text{ implies } h(t,x(t)) < \varepsilon,\ t \geq t_0$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (4.52). If this is not true, because of (4.70), there exists a $t_1 > t_0$ and a solution $x(t) = x(t, t_0, x_0)$ of (4.52) with $h_0(t_0, x_0) < \delta$ such that

$$h(t_1, x(t_1)) = \varepsilon \text{ and } h(t, x(t)) < \varepsilon, \quad t_0 \leq t \leq t_1, \tag{4.71}$$

which shows that $(t, x(t)) \in S(h, \rho)$ for $t_0 \leq t \leq t_1$. Hence by Theorem 4.10.2, we get

$$V(t, x(t), x_t) \leq r(t, t_0, w_0), \quad t_0 \leq t \leq t_1 \tag{4.72}$$

where $r(t,t_0,w_0)$ is the maximal solution of (4.53) relative to cone K. Since $H(w)$ is nondecreasing in w with respect to K, we obtain using (A4), (4.71) and (4.72),

$$b(\varepsilon) = b(h(t_1,x(t_1))) \le H(V(t_1,x(t_1),x_{t_1})) \le H(r(t_1,t_0,w_0)). \tag{4.73}$$

But

$$H_0(w_0) = H_0(V(t_0,x_0,x_0)) \le a(h_0(t_0,x_0)) < a(\delta) < \delta_1,$$

and hence, by using (4.67),

$$H(r(t_1,t_0,w_0)) < b(\varepsilon),$$

which contradicts (4.73). It therefore follows that the system (4.52) is (h_0,h)-equistable. The proof is complete.

4.11 Notes and Comments

Usually the norm is employed to estimate the functions that are involved in the evolution process to obtain qualitative properties such as global existence and bounds on solutions. This, however, is not conducive to provide best possible sufficient conditions, since the estimated functions are always nonnegative. For example, in Section 2.4, we have assumed $|(Qx)(t)| \le g(t, \max_{t_0 \le s \le t} |x(t)|)$ and $g(t,u) \ge 0$ and thus the solutions of the comparison equation are non-decreasing. In order to avoid this problem, directional derivatives given in Section 1.6 can be utilized to give sufficient conditions as

$$(x,(Qx))_\pm \le g(t,|x|)$$

0r equivalently,

$$< x,(Qx) > \le g(t,|x|)$$

Where $< \cdot >$ is the inner product. In these estimates, the function g need not be positive. These are all special cases of Lyapunov functions because we can take $V(t,x) = |x|^2$ or $V(t,x) = |x|$ that result in the directional derivatives. Hence, employing Lyapunovlike functions is very useful not only to develop stability theory but to investigate other qualitative and quantitative properties of solutions via comparison principle.

All the results of this chapter are new in the given setup and are adapted from Lakshmikantham and Leela [4], Lakshmikantham et al [18]-[79], [80]-[81]. For special cases, see also Driver [27], Hale [35], Corduneanu [2], Corduneanu and Lakshmikantham [82], Drici et al. [83] where Lyapunov functions and functionals are used.

Chapter 5

Miscellaneous Topics in Causal Systems

5.1 Introduction

This chapter deals with extensions and generalizations of causal differential equations to other important areas of nonlinear analysis. We begin with the set differential equations with causal operators naming them as causal set differential equations (CSDE). Set differential equations in a metric space has gained much attention recently due to its applicability to multivalued differential inclusions and fuzzy differential equations and its inclusion of ordinary differential systems as a special case. The generalization of this dynamic system to include causal differential equations would cover a wider variety of situations and therefore, it would initiate an interesting and useful branch of nonlinear analysis that requires further investigation.

The first two sections, 5.2 and 5.3, cover the basic results including stability results in terms of Lyapunov functions. Necessary preliminaries are provided in these two sections. Sections 5.4 and 5.5 contain the extension of causal differential equations to a Banach space setting so that it covers extensions of CDEs to special Banach spaces. Some fundamental results are investigated including global existence, showing possible differences under different sets of conditions. Section 5.6 deals with the extension to fractional causal differential equations. Since the theory of ordinary fractional differential equations is very new, this generalization should create a great interest for young researchers to obtain further fruitful results in this hybrid dynamic system.

Section 5.7 is dedicated to causal differential equations with memory. We provide some basic simple results covering this generalization. The next two sections deal with causal differential equations with retardation (memory) and anticipation. This area of investigation is also very new and important because there are several possible approaches to follow and several ways of formulating the problem. It is a very fruitful area of investiga-

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tion because of its applicability to decision theory, chaotic epidemics and wavelet theory. Section 5.10 contains neutral differential equations with causal operators on a semi axis. Finally, Section 5.11 deals with notes and comments.

5.2 Causal Set Differential Equations

The study of set differential equations (SDE) in a metric space is interesting due to its applicability to multivalued differential inclusions and fuzzy differential equations and its inclusion of ordinary differential systems as a special case [84].

A combination of these two concepts leads to a set differential equation with causal operators. In this section, using this setup, we obtain some basic results on existence, uniqueness, and continuous dependence of solutions with respect to initial values.

Let $K_c(\mathbb{R}^n)$ denote the collection of all nonempty compact and convex subsets of $\mathbb{R}^n$. Define the Hausdorff metric

$$D[A,B] = \max\left[\sup_{x\in B} d(x,A), \sup_{y\in A} d(y,B)\right],$$

where A, B are bounded sets in $\mathbb{R}^n$ and $d(x,A) = \inf[d(x,y) : y \in A]$. We observe that $K_c(\mathbb{R}^n)$ is a complete metric space.

Suppose that the space $K_c(\mathbb{R}^n)$ is equipped with the natural algebraic operations of addition and nonnegative scalar multiplication. Then, $K_c(\mathbb{R}^n)$ becomes a semilinear metric space, which can be embedded as a complete cone into a corresponding Banach space.

We note that the Hausdorff metric satisfies the following properties:

$$D[A+C, B+C] = D[A,B] \quad D[A,B] = D[B,A]$$
$$D[\lambda A, \lambda B] = \lambda D[A,B] \qquad D[A,B] \le D[A,C] + D[C,B]$$

for all $A, B, C \in K_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}_+$.

Given any two sets $A, B \in K_c(\mathbb{R}^n)$ if there exists set $C \in K_c(\mathbb{R}^n)$ satisfying $A = B + C$, then $A - B$ is defined as the Hukuhara difference of the sets A and B.

The mapping $F : I \to K_c(\mathbb{R}^n)$ has a Hukuhara derivative $D_H F(t_0)$ at a point $t_0 \in I$, if

$$\lim_{h\to 0^+} \frac{F(t_0+h) - F(t_0)}{h}, \quad \lim_{h\to 0^+} \frac{F(t_0) - F(t_0-h)}{h}$$

exists in the topology of $K_c(\mathbb{R}^n)$ and are equal to $D_H F(t_0)$. Here I is any interval in $\mathbb{R}$.

Now we can consider the set differential equation

$$D_H U = F(t,U), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n), \quad t_0 \ge 0, \tag{5.1}$$

where $F \in C[\mathbb{R}_+ \times K_c(\mathbb{R}^n), K_c(\mathbb{R}^n)]$.

Definition 5.2.1. The mapping $U \in C^1[J, K_c(\mathbb{R}^n)]$, $J = [t_0, t_0 + a]$, is said to be a solution of (5.1) on J if it satisfies (5.1) on J.

Since $U(t)$ is continuously differentiable, we have

$$U(t) = U_0 + \int_{t_0}^{t} D_H U(s) ds, \quad t \in J.$$

Hence, we can associate with the IVP (5.1). The Hukuhara integral

$$U(t) = U_0 + \int_{t_0}^{t} F(s, U(s)) ds, \quad t \in J.$$

The following properties are useful tools in proving theorems in the SDE setup. If $F : [t_0, T] \to K_c(\mathbb{R}^n)$ is integrable, we have

$$\int_{t_0}^{t_2} F(t)dt = \int_{t_0}^{t_1} F(t)dt + \int_{t_1}^{t_2} F(t)dt, \quad t_0 \le t_1 \le t_2 \le T,$$

$$\int_{t_0}^{T} \lambda F(t)dt = \lambda \int_{t_0}^{T} F(t)dt, \quad \lambda \in \mathbb{R}_+.$$

Also, if $F, G : [t_0, T] \to K_c(\mathbb{R}^n)$ are integrable, then $D[F(\cdot), G(\cdot)] : [t_0, T] \to \mathbb{R}$ is integrable and

$$D\left[\int_{t_0}^{t} F(s)ds, \int_{t_0}^{t} G(s)ds\right] \le \int_{t_0}^{t} D[F(s), G(s)]ds.$$

We observe that

$$D[A, \theta] = |A| = \sup_{a \in A} |a|$$

for $A \in K_c(\mathbb{R}^n)$, where θ is the zero element of $\mathbb{R}^n$, which is regarded as a one-point set.

We shall now extend certain basic results to SDEs with causal or nonanticipative maps of Volterra type, since such equations provide a unified treatment of the basic theory of SDEs, SDEs with delay and set integrodifferential equations which in turn include ordinary dynamic systems of the corresponding types.

Let $E = C[[t_0, T], K_c(\mathbb{R}^n)]$ with norm

$$D_0[U, \theta] = \sup_{t_0 \le t \le T} D[U(t), \theta].$$

Definition 5.2.2. Suppose that $Q \in C[E, E]$, then Q is said to be a causal map or a nonanticipative map if $U(s) = V(s)$, $t_0 \le s \le t \le T$, where $U, V \in E$, then $(QU)(s) = (QV)(s)$, $t_0 \le s \le t$.

We define the IVP for an SDE with causal map or CSDE using the Hukuhara derivative as follows:

$$D_H U(t) = (QU)(t), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n). \tag{5.2}$$

Before we proceed to prove an existence and uniqueness result for (5.2), we need the following comparison result.

Theorem 5.2.1. Assume that $m \in C[J,\mathbb{R}_+]$, $g \in C[J\times\mathbb{R}_+,\mathbb{R}_+]$ and for $t \in J = [t_0,T]$,

$$D_- m(t) \leq g(t,|m|_0(t)),$$

where $|m|_0(t) = \sup_{t_0\leq s\leq t}|m(s)|$. Suppose that $r(t) = r(t,t_0,w_0)$ is the maximal solution of the scalar differential equation

$$w' = g(t,w), \quad w(t_0) = w_0 \geq 0, \tag{5.3}$$

existing on J. Then, $m(t_0) \leq w_0$ implies $m(t) \leq r(t)$, $t \in J$.

Next we obtain an estimate of the distance between any two solutions of (5.2) in terms of the maximal solution of (5.3) utilizing Theorem 5.2.1.

We define $D_0[U,V](t) = \max_{t_0\leq s\leq t} D[U(s),V(s)]$.

Theorem 5.2.2. Let $Q \in C[E,E]$ be a causal map such that for $t \in J$,

$$D[(QU)(t),(QV)(t)] \leq g(t,D_0[U,V](t)), \tag{5.4}$$

where $g \in C[J\times\mathbb{R}_+,\mathbb{R}_+]$. Suppose further that the maximal solution $r(t,t_0,w_0)$ of the differential equation (5.3) exists on J. Then, if $U(t)$, $V(t)$ are any two solutions of (5.2) satisfying $U(t_0) = U_0$, $V(t_0) = V_0$, $U_0,V_0 \in K_c(\mathbb{R}^n)$ on J, respectively,

$$D[U(t),V(t)] \leq r(t,t_0,w_0), \quad t \in J,$$

provided that $D[U_0,V_0] \leq w_0$.

Proof. Set $m(t) = D[U(t),V(t)]$. Then $m(t_0) = D[U_0,V_0] \leq w_0$. Now for small $h > 0$, $t \in J$, consider $m(t+h) = D[U(t+h),V(t+h)]$. Using the property of the Hausdorff metric D, we successively get the following relations:

$$\begin{aligned} m(t+h) &\leq D[U(t+h),U(t)+h(QU)(t)] + D[U(t)+h(QU)(t),V(t+h)] \\ &\leq D[U(t+h),U(t)+h(QU)(t)] + D[U(t)+h(QU)(t),V(t)+h(QV)(t)] \\ &\quad + D[V(t)+h(QV)(t),V(t+h)] \\ &\leq D[U(t+h),U(t)+h(QU)(t)] + D[U(t),U(t)+h(QV)(t)] \\ &\quad + D[U(t)+h(QV)(t),V(t)+h(QV)(t)] + D[V(t)+h(QV)(t),V(t+h)]. \end{aligned}$$

Next, using the property of the Hausdorff metric D and the fact that Hukuhara differences $U(t+h)-U(t)$ and $V(t+h)-V(t)$ exist for small $h > 0$, we arrive at

$$m(t+h) \leq D[U(t)+Z(t,h),U(t)+h(QU)(t)] + D[h(QU)(t),h(QV)(t)]$$

$$+D[U(t),V(t)] + D[V(t)+h(QV)(t),V(t)+Y(t,h)],$$

where $U(t+h) = U(t) + Z(t,h)$ and $V(t+h) = V(t) + Y(t,h)$. Again the property of Hausdorff metric Dgives

$$m(t+h) \leq D[Z(t,h), h(QU)(t)] + D[h(QU)(t), h(QV)(t)]$$

$$+D[U(t),V(t)] + D[h(QV)(t), Y(t,h)].$$

Since the Hukuhara differences exist, we can replace $Z(t,h) and Y(t,h)$ with $U(t+h) - U(t)$ and $V(t+h) - V(t)$, respectively. This gives, on subtracting $m(t)$ and dividing both sides with $h > 0$,

$$\frac{m(t+h) - m(t)}{h} \leq D\left[\frac{U(t+h) - U(t)}{h}, (QU)(t)\right] + D[(QU)(t), (QV)(t)]$$

$$+D\left[(QV)(t), \frac{V(t+h) - V(t)}{h}\right].$$

Now taking limit supremum as $h \to 0^+$ and using the fact that $U(t)$ and $V(t)$ are solutions of (5.2) along with the assumption (5.4) we obtain

$$D^+m(t) \leq D[(QU)(t), (QV)(t)] \leq g(t, D_0[U,V](t)) = g(t, |m|_0(t)), \quad t \in J.$$

Theorem 5.2.1 now guarantees the stated conclusion and the proof is complete.

Corollary 5.2.1. Let $Q \in C[E,E]$ be a causal map such that

$$D[(QU)(t), \theta] \leq g(t, D_0[U, \theta](t)),$$

where $g \in C[J \times \mathbb{R}_+, \mathbb{R}_+]$. Also, suppose that $r(t,t_0,w_0)$ is the maximal solution of the scalar differential equation (5.3). Then, if $U(t,t_0,U_0)$ is any solution of (5.2) through (t_0,U_0) with $U_0 \in K_c(\mathbb{R}^n), D[U_0, \theta] \leq w_0$ implies $D[U(t), \theta] \leq r(t,t_0,w_0)$, $t \in J$.

We begin by providing a local existence result using successive approximations.

Theorem 5.2.3. Assume that

(a) $Q \in C[B,E]$ is a causal map, where $B = B(U_0,b) = \{U \in E : D_0[U,U_0] \leq b\}$ and $D_0[(QU), \theta](t) \leq M_1$ on B;

(b) $g \in C[J \times [0,2b], \mathbb{R}_+]$, $g(t,w) \leq M_2$ on $J \times [0,2b]$, $g(t,0) \equiv 0$, $g(t,w)$ is nondecreasing in w for each $t \in J$ and $w(t) = 0$ is the only solution of

$$w' = g(t,w), \quad w(t_0) = 0 \text{ on } J; \tag{5.5}$$

(c) $D[(QU)(t), (QV)(t)] \leq g(t, D_0[U,V](t))$ on B.

Then, the successive approximations defined by

$$U_{n+1}(t) = U_0 + \int_{t_0}^{t} (QU_n)(s)ds, \quad n = 0,1,2,\ldots, \tag{5.6}$$

exists on $J_0 = [t_0, t_0 + \eta)$, where $\eta = \min[T - t_0, b/M]$ and $M = \max(M_1, M_2)$, and converge uniformly to the unique solution $U(t)$ of (5.2).

Proof. For $t \in J_0$, we have, by induction, using properties of the Hausdorff metric D, and the integral,

$$\begin{aligned} D[U_{n+1}(t), U_0] &= D\left[U_0 + \int_{t_0}^{t} (QU_n)(s)ds, U_0\right] = D\left[\int_{t_0}^{t} (QU_n)(s)ds, \theta\right] \\ &\leq \int_{t_0}^{t} D[(QU_n)(s), \theta]ds, \\ &\leq \int_{t_0}^{t} D_0[QU_n, \theta](t)ds \leq M_1(t - t_0) \leq M(t - t_0) \leq b, \end{aligned}$$

which shows the successive approximations are well defined on J_0.

Next, we define successive approximations for the problem (5.5) as follows:

$$w_0(t) = M(t - t_0),$$

$$w_{n+1}(t) = \int_{t_0}^{t} g(s, w_n(s))ds, \quad t \in J_0,\ n = 0,1,2,\ldots.$$

Then,

$$w_1(t) = \int_{t_0}^{t} g(s, w_0(s))ds \leq M_2(t - t_0) \leq M(t - t_0) = w_0(t).$$

Assume, for some $k > 1$, $t \in J_0$, that

$$w_k(t) \leq w_{k-1}(t).$$

Then, using monotonicity of g, we get

$$w_{k+1}(t) = \int_{t_0}^{t} g(s, w_k(s))ds \leq \int_{t_0}^{t} g(s, w_{k-1}(s))ds = w_k(t).$$

Hence, the sequence $\{w_k(t)\}$ is monotone decreasing.

Since $w_k'(t) = g(t, w_{k-1}(t)) \leq M_2$, $t \in J_0$, we conclude by Ascoli-Arzela theorem and the monotonicity of the sequence $\{w_k(t)\}$ that

$$\lim_{t \to \infty} w_n(t) = w(t)$$

uniformly on J_0. Since $w(t)$ satisfies (5.5), we get from condition (b) that $w(t) \equiv 0$ on J_0. Observing that for each $t \in J_0$, $t_0 \leq s \leq t$,

$$\begin{aligned} D[U_1(s), U_0] &= D\left[U_0 + \int_{t_0}^{s} (QU_0)(\xi)d\xi, U_0\right] = D\left[\int_{t_0}^{s} (QU_0)(\xi)d\xi, \theta\right] \\ &\leq \int_{t_0}^{s} D[(QU_0)(\xi), \theta]d\xi \leq D_0[(QU_0), \theta](s - t_0) \\ &\leq D_0[(QU_0), \theta](t - t_0) \leq M_1(t - t_0) \leq M(t - t_0) = w_0(t), \end{aligned}$$

which implies that $D_0[U_1,U_0](t) \le w_0(t)$. We assume, for some $k > 1$,

$$D_0[U_k,U_{k-1}](t) \le w_{k-1}(t), \quad t \in J_0.$$

Consider, for any $t \in J_0$, $t_0 \le s \le t$,

$$\begin{aligned} D[U_{k+1}(s),U_k(s)] &\le \int_{t_0}^{s} D[(QU_k)(\xi),(QU_{k-1})(\xi)]d\xi \le \int_{t_0}^{s} g(\xi,D_0[U_k,U_{k-1}](\xi))d\xi \\ &\le \int_{t_0}^{s} g(\xi,w_{k-1}(\xi))d\xi \le \int_{t_0}^{t} g(\xi,w_{k-1}(\xi))d\xi = w_k(t). \end{aligned}$$

which further gives

$$D_0[U_{k+1},U_k](t) \le w_k(t), \quad t \in J_0.$$

Thus, we consider that

$$D_0[U_{n+1},U_n](t) \le w_n(t), \tag{5.7}$$

for $t \in J_0$ and for all $n = 0,1,2,\ldots$.
We claim that $\{U_n(t)\}$ is a Cauchy sequence. To show this, let $n \le m$. Setting $v(t) = D[U_n(t),U_m(t)]$ and using (5.6), we get

$$\begin{aligned} D^+v(t) &\le D[D_HU_n(t),D_HU_m(t)](t) = D[(QU_{n-1})(t),(QU_{m-1})(t)] \\ &\le D[(QU_{n-1})(t),(QU_n)(t)] + D[(QU_n)(t),(QU_m)(t)] \\ &\quad + D[(QU_m)(t),(QU_{m-1})(t)] \le g(t,D_0[U_{n-1},U_n](t)) + g(t,D_0[U_n,U_m](t)) \\ &\quad + g(t,D_0[U_{m-1},U_m](t)) \le g(t,w_{n-1}(t)) + g(t,|v|_0(t)) + g(t,w_{n-1}(t)) \\ &= g(t,|v|_0(t)) + 2g(t,w_{n-1}(t)). \end{aligned}$$

The above inequalities yield, on using Theorem 5.2.1, the estimate

$$v(t) \le r_n(t), \quad t \in J_0,$$

where $r_n(t)$ is the maximal solution of

$$r_n' = g(t,r_n) + 2g(t,w_{n-1}(t)), \quad r_n(t_0) = 0,$$

for each n. Since as $n \to \infty$, $2g(t,w_{n-1}(t)) \to 0$ uniformly on J_0, it follows by Lemma 1.3.3 that $r_n(t) \to 0$, as $n \to \infty$ uniformly on J_0. This implies from (5.7) that $U_n(t)$ converges uniformly to $U(t)$ on J_0 and clearly $U(t)$ is a solution of (5.2).
To prove uniqueness, let $V(t)$ be another solution of (5.2) on J_0. Set $m(t) = D[U(t),V(t)]$. Then, $m(t_0) = 0$ and

$$D^+m(t) \le g(t,|m|_0(t)), \quad t \in J_0.$$

Since $m(t_0) = 0$, it follows from Theorem 5.2.1 that

$$m(t) \le r(t,t_0,0), \quad t \in J_0,$$

where $r(t,t_0,0)$ is the maximal solution of (5.5). The assumption (b) now shows that $U(t) = V(t)$, $t \in J_0$, proving uniqueness.

Assuming local existence, we next discuss a global existence result.

Theorem 5.2.4. Let $Q \in C[E,E]$ be a causal map such that

$$D[(QU)(t),\theta] \le g(t,D_0[U,\theta](t)),$$

where $g \in C[\mathbb{R}_+^2,\mathbb{R}_+]$, $g(t,w)$ is nondecreasing in w for each $t \in \mathbb{R}_+$ and the maximal solution $r(t) = r(t,t_0,w_0)$ of (5.3) exists on $[t_0,\infty)$. Suppose further that Q is smooth enough to guarantee the local existence of solutions of (5.2) for any $(t_0,U_0) \in \mathbb{R}_+ \times K_c(\mathbb{R}^n)$. Then, the largest interval of existence of any solution $U(t,t_0,U_0)$ of (5.2) is $[t_0,\infty)$, whenever $D[U_0,\theta] \le w_0$.

Proof. Suppose that $U(t) = U(t,t_0,U_0)$ is any solution of (5.2) existing on $[t_0,\beta)$, $t_0 < \beta < \infty$ with $D[U_0,\theta] \le w_0$, and the value of β cannot be increased. Define $m(t) = D[U(t),\theta]$ and note that $m(t_0) \le w_0$. Then, it follows that

$$D^+m(t) \le D[D_H U(t),\theta] \le D[(QU)(t),\theta] \le g(t,D_0[U,\theta](t)).$$

Using Theorem 5.2.1, we obtain

$$m(t) \le r(t), \quad t_0 \le t < \beta.$$

For any t_1,t_2 such that $t_0 < t_1 < t_2 < \beta$, using the assumptions and properties of Hausdorff metric D,

$$\begin{aligned} D[U(t_1),U(t_2)] &= D\left[\int_{t_0}^{t_1} (QU)(s)ds, \int_{t_0}^{t_2} (QU)(s)ds\right] \\ &\le \int_{t_1}^{t_2} D[(QU)(s),\theta]ds \le \int_{t_1}^{t_2} g(s,D_0[U,\theta](s))ds. \end{aligned}$$

Employing the estimate above and the monotonicity of $g(t,w)$, we find

$$D[U(t_1),U(t_2)] \le \int_{t_1}^{t_2} g(s,r(s))ds = r(t_2) - r(t_1).$$

Since $\lim_{t\to\beta^-} r(t,t_0,w_0)$ exists, taking the limit as $t_1,t_2 \to \beta^-$, we get that $\{U(t_n)\}$ is a Cauchy sequence and therefore $\lim_{t\to\beta^-} U(t,t_0,U_0) = U_\beta$ exists. We then consider the IVP

$$D_H U(t) = (QU)(t), \quad U(\beta) = U_\beta.$$

As we have assumed the local existence, we note that $U(t,t_0,U_0)$ can be continued beyond β, contradicting our assumption that β cannot be increased. Thus, every solution

$U(t,t_0,U_0)$ of (5.2) such that $D[U_0,\theta] \leq w_0$ exists globally on $[t_0,\infty)$ and hence the proof is complete.

Next, we discuss the continuous dependence of solutions with respect to initial values.

Lemma 5.2.1. Let $Q \in C[E,E]$ be a causal map and let

$$G(t,k(t)) = \sup[D[(QU)(t),\theta] : D[U(t),U_0] \leq k(t)].$$

Assume that $r^*(t,t_0,0)$ is the maximal solution of

$$w' = G(t,w), \quad w(t_0) = 0, \text{ on } J.$$

Let $U(t) = U(t,t_0,0)$ be the solution of (5.2). Then,

$$D[U(t),U_0] \leq r^*(t,t_0,0), \quad t \in J.$$

Proof. Set $m(t) = D[U(t),U_0]$, $t \in J$. Then,

$$\begin{aligned} m(t+h) - m(t) &= D[U(t+h),U_0] - D[U(t),U_0] \\ &= D[U(t+h),U(t)+h(QU)(t)] + D[U(t)+h(QU)(t),U(t)]. \end{aligned}$$

Hence,

$$\frac{m(t+h)-m(t)}{h} \leq D\left[\frac{U(t+h)-U(t)}{h},(QU)(t)\right] + D[(QU)(t),\theta],$$

$$D^+m(t) \leq D[(QU)(t),\theta] \leq \sup[D[(QU)(t),\theta] : D[U(t),U_0] \leq m(t)] \leq G(t,m(t)).$$

This implies by Theorem 1.3.2 that

$$D[U(t),U_0] \leq r^*(t,t_0,0), \quad t \in J.$$

Theorem 5.2.5. Assume that

(a) assumptions (a), (b), and (c) of Theorem 5.2.3 hold;

(b) the solutions $w(t,t_0,w_0)$ of (5.3) through every point (t_0,w_0) are continuous with respect to (t_0,w_0).

Then, the solution $U(t) = U(t,t_0,U_0)$ of (5.2) is continuous with respect to (t_0,U_0).

Proof. Let $U(t) = U(t,t_0,U_0)$, $V(t) = V(t,t_0,V_0)$, $U_0,V_0 \in K_c(\mathbb{R}^n)$ be two solutions of (5.2). Then, defining $m(t) = D[U(t),V(t)]$, we get from Theorem 5.2.3 the estimate

$$D[U(t),V(t)] \leq r(t,t_0,D[U_0,V_0]), \quad t \in J.$$

Since $\lim_{U_0 \to V_0} r(t,t_0,D[U_0,V_0]) = r(t,t_0,0)$ uniformly on J and by hypothesis $r(t,t_0,0) \equiv 0$, consequently $\lim_{U_0 \to V_0} U(t,t_0,U_0) = V(t,t_0,V_0)$ uniformly and hence $U(t,t_0,U_0)$ is continuous with respect to U_0.

To prove continuity with respect to t_0, we let $U(t) = U(t,t_0,U_0)$, $V = V(t,\tau_0,U_0)$ be two solutions of (5.2) with $\tau_0 > t_0$. Again, setting $m(t) = D[U(t),V(t)]$ and noting that $m(\tau_0) = D[U(\tau_0),U_0]$, using Lemma 5.2.1, we get

$$m(\tau_0) \le r^*(\tau_0,t_0,U_0).$$

Hence, using Theorem 5.2.2, we obtain

$$m(t) = \tilde{r}(t), \quad t \ge \tau_0,$$

where $\tilde{r}(t,\tau_0,r^*(\tau_0,t_0,0))$ is the maximal solution of (5.2) through $(\tau_0,r^*(\tau_0,t_0,0))$. Since $r^*(t,t_0,0) = 0$, we have

$$\lim_{\tau_0\to t_0} \tilde{r}(t,\tau_0,r^*(\tau_0,t_0,0)) = \tilde{r}(t,t_0,0),$$

uniformly on J. By hypothesis, $\tilde{r}(t,t_0,0) \equiv 0$ which proves the continuity of $U(t,t_0,U_0)$ relative to t_0.

5.3 Comparison Results and Stability Theory

In this section, we first prove some basic comparison results, which are used subsequently to establish stability properties of CSDEs, that is causal set differential equations. We begin with some definitions. Let $E = C[[t_0,\infty),K_c(\mathbb{R}^n)]$ with norm

$$\sup_{t\in[t_0,\infty)} \frac{D[U(t),\theta]}{h(t)} < \infty,$$

where θ is the zero element of $\mathbb{R}^n$, which is regarded as a point set and $h : [t_0,\infty) \to \mathbb{R}_+$ is a continuous map. E equipped with such a norm is a Banach space.

Definition 5.3.1. Let $Q \in C[E,E]$. Q is said to be a causal map or nonanticipative map if $U(s) = V(s)$, $t_0 \le s \le t < \infty$, and $U,V \in E$ then $(QU)(s) = (QV)(s)$, $t_0 \le s \le t < \infty$.

Consider the initial value problem (IVP) for CSDEs defined using the Hukuhara derivative:

$$D_H U(t) = (QU)(t), \quad U(t_0) = U_0 \in K_c(\mathbb{R}^n). \tag{5.8}$$

In order to use the method of Lyapunov function (MLF), it is necessary to select minimal subsets of E over which the derivative of the Lyapunov function can be conveniently estimated. For that purpose, let $L \in C[\mathbb{R}_+ \times B,\mathbb{R}_+]$, where $B = B(\theta,b) = \{U \in K_c(\mathbb{R}^n) : D[U,\theta] \le b\}$. Define the following sets:

$$E_\alpha = \{U \in K_c(\mathbb{R}^n) : L(s,U(s))\alpha(s) \le L(t,U(t))\alpha(t),\ t_0 \le s \le t\},$$

$$E_I = \{U \in K_c(\mathbb{R}^n) : L(s,U(s)) \le L(t,U(t)),\ t_0 \le s \le t\},$$

$$E_0 = \{U \in K_c(\mathbb{R}^n) : L(s,U(s)) \le f(L(t,U(t))),\ t_1 \le s \le t,\ t_1 \ge t_0\},$$

where

(i) $\alpha(t) > 0$ is a continuous function on $\mathbb{R}_+$,

(ii) $f(r)$ is a continuous on $\mathbb{R}_+$, nondecreasing in r and $f(r) > r$ for $r > 0$.

We now prove the comparison results,

Theorem 5.3.1. Let $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ and let $L(t,U)$ be locally Lipschitzian in U, i.e., for $U,V \in B$, $t \in \mathbb{R}_+$, and $K > 0$, $|L(t,U) - L(t,V)| \leq KD(U,V)$.

(i) Assume that for $t \geq t_0$ and $U \in E_1$,

$$D_-L(t,U(t)) \leq g(t,L(t,U(t))) \tag{5.9}$$

where $D_-L(t,U(t)) = \liminf_{h\to 0^-} \frac{1}{h}[L(t+h,U(t)+h(QU)(t)) - L(t,U(t))]$, and $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$.

(ii) Let $r(t) = r(t,t_0,w_0)$ be the maximal solution of

$$w' = g(t,w), \quad w(t_0) = w_0 \geq 0, \tag{5.10}$$

existing on $t_0 \leq t < \infty$.

Let $U(t,t_0,U_0)$ be any solution of (5.8) such that $U(t,t_0,U_0) \in B$ for $t \in [t_0,t_1]$ and let $L(t_0,U_0) \leq w_0$. Then $L(t,U(t,t_0,U_0)) \leq r(t)$ for all $t \in [t_0,t_1]$.

Proof. Let $U(t,t_0,U_0)$ be any solution of (5.8) such that $U(t,t_0,U_0) \in B$ for $t \in [t_0,t_1]$. Define $m(t) = L(t,U(t))$, $t \in [t_0,t_1]$. For sufficiently small $\varepsilon > 0$, consider the differential equation

$$w' = g(t,w) + \varepsilon = g_\varepsilon(t,w), \quad w(t_0) = w_0 + \varepsilon,$$

whose solutions $w(t,\varepsilon) = w(t,t_0,w_0,\varepsilon)$ exist as far as $r(t)$ exists to the right of t_0. Since the continuity of $w(t)$ implies that $\lim_{\varepsilon\to 0} w(t,\varepsilon) = r(t)$, it is sufficient to show that

$$m(t) < w(t,\varepsilon), \quad t \in [t_0,t_1]. \tag{5.11}$$

Suppose that (5.11) is not true. then there exists $t_2 \in (t_0,t_1)$ such that

(a) $m(t) \leq w(t,\varepsilon)$, $t_0 \leq t \leq t_2$, and

(b) $m(t_2) = w(t_2,\varepsilon)$.

It then follows from (a) and (b) that

$$D_-m(t_2) \geq \liminf_{h\to 0^-} \frac{w(t_2+h,\varepsilon) - w(t_2,\varepsilon)}{h} = D_-w(t_2,\varepsilon) = g(t_2,w(t_2,\varepsilon)) + \varepsilon. \tag{5.12}$$

From the assumption on g, the solutions $w(t,\varepsilon)$ are increasing functions of t. Since, $m(t) = L(t,U(t))$ and using (a) and (b), we have

$$L(s,U(s)) \leq L(t_2,U(t_2)), \quad t_0 \leq s \leq t_2.$$

Consequently, $U(t,t_0,U_0) \in E_I$, $t_0 \le t \le t_2$. Since $L(t,U)$ is Lipschitzian in U and satisfies condition (i), we have

$$\begin{aligned} m(t+h) - m(t) &= L(t+h, U(t+h)) - L(t, U(t)) \\ &= L(t+h, U(t+h)) - L(t+h, U(t) + h(QU)(t)) \\ &\quad + L(t+h, U(t) + h(QU)(t)) - L(t, U(t)) \\ &\ge -KD[U(t+h), U(t) + h(QU)(t)] \\ &\quad + L(t+h, U(t) + h(QU)(t)) - L(t, U(t)), \end{aligned}$$

which, upon taking the lim inf as $h \to 0^-$ and using the fact that $D_H U(t)$ exists and is equal to $(QU)(t)$ yields

$$D_- m(t) \le D_- L(t, U(t)) \le g(t, L(t, U(t))) = g(t, m(t)).$$

Therefore, it follows, for $t = t_2$, that

$$D_- m(t_2) \le g(t_2, m(t_2)) = g(t_2, w(t_2, \varepsilon)),$$

which is a contradiction to (5.12). Hence the proof of the theorem is complete.

Corollary 5.3.1. Let $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ and let $L(t,U)$ be locally Lipschitzian in U. Assume that

$$D_- L(t, U(t)) \le 0 \text{ for } t \ge t_0 \text{ and } U \in E_0.$$

Let $U(t) = U(t,t_0,U_0)$ be any solution of (5.8), then $L(t,U(t)) \le L(t_0,U_0)$, $t \ge t_0$.

Proof. Proceeding as in the previous theorem with $g(t,w) = 0$, we have

$$L(s, U(s)) \le L(t_2, U(t_2)), t_2 \in (t_0, t_1), \quad t_2 \in (t_0, t_1),\ t_0 \le s \le t_2.$$

Since $L(t_2, U(t_2)) = w(t_2, \varepsilon) = L(t_0, U_0) + \varepsilon(t_2 - t_0) + \varepsilon > 0$, we have $L(s, U(s)) \le f(L(t_2, U(t_2)))$ for $t_0 \le s \le t_2$. The rest of the proof is similar to that of Theorem 5.3.1.

Theorem 5.3.2. Assume the hypotheses of Theorem 5.3.1 hold, except for inequality (5.9), which is replaced by

$$\alpha(t) D_- L(t, U(t)) + L(t, U(t)) D_- \alpha(t) \le w(t, L(t, U(t)) \alpha(t)), \tag{5.13}$$

for $t > t_0$, $U \in E_\alpha$, where $\alpha(t) > 0$ is continuous on $\mathbb{R}_+$ and $D_-\alpha(t) = \liminf_{h \to 0^-} \frac{\alpha(t+h) - \alpha(t)}{h}$. Then $\alpha(t_0) L(t_0, U_0) \le w_0$ implies that $\alpha(t) L(t, U(t)) \le r(t)$, $t \ge t_0$.

Proof. Let $P(t, U(t)) = L(t, U(t)) \alpha(t)$. Let $t \ge t_0$ and using $U \in E_\alpha$. For sufficiently small $h > 0$, we have

$$\begin{aligned} &P(t+h, U(t) + h(QU)(t)) - P(t, U(t)) \\ &= L(t+h, U(t) + h(QU)(t)) \alpha(t+h) - L(t, U(t)) \alpha(t) \\ &= L(t+h, U(t) + h(QU)(t)) (\alpha(t+h) - \alpha(t)) \\ &\quad + [L(t+h, U(t) + h(QU)(t)) - L(t, U(t))] \alpha(t), \end{aligned}$$

from which it follows,

$$D_-P(t,U(t)) = L(t,U(t))D_-\alpha(t) + \alpha(t)D_-L(t,U(t))$$

$$\leq w(t, L(t,U(t))\alpha(t)) = w(t, P(t,U(t))),$$

for $t \in (t_0, t_1]$ and $U \in E_1$, where E_1, in this case, is to be defined with $P(t,U(t))$ replacing $L(t,U(t))$ in the definition of set E_1. Since $P(t,U)$ is locally Lipschitzian in U, then all the assumptions of Theorem 5.3.2 are satisfied with $P(t,U(t))$ replacing $L(t,U(t))$. Hence, the conclusion of the theorem follows from the proof of Theorem 5.3.1.

To prove a general comparison result in terms of Lyapunov-like functions, we need the following known result.

Lemma 5.3.1. Let $g_0, g \in C[\mathbb{R}^2_+, \mathbb{R}]$ be such that

$$g_0(t,w) \leq g(t,w), \quad (t,w) \in \mathbb{R}^2_+. \tag{5.14}$$

Then the right maximal solution $r(t,t_0,w_0)$ of (5.10) and the left maximal solution $\eta(t,T_0,v_0)$ of

$$v' = g_0(t,v), \quad v(T_0) = v_0, \tag{5.15}$$

satisfy the relation

$$r(t,t_0,w_0) \leq \eta(t,T_0,v_0), \quad t \in [t_0, T_0],$$

whenever $r(T_0,t_0,w_0) \leq v_0$.

Theorem 5.3.3. Assume that

(i) $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ and $L(t,u)$ is locally Lipschitzian in U,

(ii) $g_0, g \in C[\mathbb{R}^2_+, \mathbb{R}]$ are such that $g_0(t,w) \leq g(t,w)$, $t, w \in \mathbb{R}^2_+$, and $\eta(t,T_0,v_0)$ is the left maximal solution of (5.15) existing on $t_0 \leq t \leq T_0$ and $r(t,t_0,w_0)$ the right maximal solution of (5.10) existing on $[t_0, \infty)$;

(iii) $D_-L(t,U(t)) \leq g(t, L(t,U(t)))$ on Ω, where

$$\Omega = \{U \in E : L(s,U(s)) \leq \eta(s,t,L(t,U(t))),\ t_0 \leq s \leq t\}.$$

Then we have

$$L(t,U(t,t_0,U_0)) \leq r(t,t_0,w_0), \quad t \geq t_0, \tag{5.16}$$

whenever $L(t_0,U_0) \leq w_0$.

Proof. Set $m(t) = L(t,U(t,t_0,U_0))$, $t \geq t_0$, so that $m(t_0) = L(t_0,U_0) \leq w_0$. Let $w(t,\varepsilon)$ be any solution of

$$w' = g(t,w) + \varepsilon, \quad w(t_0) = w_0 + \varepsilon,$$

for sufficiently small $\varepsilon > 0$. Then since $r(t,t_0,w_0) = \lim_{\varepsilon\to 0^+} w(t,\varepsilon)$, it is enough to prove that $m(t) < w(t,\varepsilon)$ for $t \geq t_0$. If this is not true, there exists a $t_1 > t_0$ such that $m(t_1) = w(t_1,\varepsilon)$ and $m(t) < w(t,\varepsilon)$ for $t_0 < t < t_1$. This implies that

$$D_- m(t_1) \geq w'(t,\varepsilon) = g(t_1, m(t_1)) + \varepsilon. \tag{5.17}$$

Now consider the left maximal solution $\eta(s,t_1,m(t_1))$ of (5.15) with $v(t_1) = m(t_1)$ on the interval $t_0 < t < t_1$. By Lemma 5.3.1, we have

$$r(s,t_0,w_0) \leq \eta(s,t_1,m(t_1)), \quad s \in [t_0,t_1].$$

Since

$$r(t_1,t_0,w_0) = \lim_{\varepsilon\to 0^+} w(t,\varepsilon) = m(t_1) = \eta(t_1,t_1,m(t_1))$$

and $m(s) \leq w(s,\varepsilon)$ for $t_0 < s \leq t_1$, it follows that

$$m(s) \leq r(s,t_0,w_0) \leq \eta(s,t_1,m(t_1)), \quad s \in [t_0,t_1].$$

This inequality implies that hypothesis (iii) holds for $U(s,t_0,U_0)$ on $t_0 < s \leq t_1$ and hence, standard computation yields

$$D_- m(t_1) \leq g(t_1, m(t_1)),$$

which contradicts (5.17). Thus $m(t) \leq r(t,t_0,w_0)$, $t \geq t_0$, and the proof is complete.

In order to discuss the stability properties of (5.8), let is assume that the solutions of (5.8) exist and are unique for all $t \geq t_0$. In addition, in order to match the behavior of solutions of (5.8) with those of the corresponding ordinary differential equation with causal map, we assume that $U_0 = V_0 + W_0$, so the Hukuhara difference $U_0 - V_0 = W_0$ exists. Consequently, in what follows, we consider the solutions $U(t) = U(t,t_0,U_0 - V_0) = U(t,t_0,W_0)$. This we have the initial value problem

$$D_H U(t) = (QU)(t), \quad U(t_0) = W_0. \tag{5.18}$$

To illustrate the idea mentioned above, we present a simple example in $K_c(\mathbb{R}^n)$. Consider

$$D_H U(t) = -\int_0^t U(s)ds, \quad U(0) = U_0 \in K_c(\mathbb{R}^n).$$

Then using interval methods, we get

$$u_1' = -\int_0^t u_2(s)ds,$$

$$u_2' = -\int_0^t u_1(s)ds,$$

where $U(t) = [u_1(t), u_2(t)]$ and $U_0 = [u_{10}, u_{20}]$. Clearly, this yields

$$u_1^{(4)} = u_1, \quad u_1(0) = u_{10},$$

$$u_2^{(4)} = u_2, \quad u_2(0) = u_{20},$$

whose solutions are given by

$$u_1(t) = \left(\frac{u_{10} - u_{20}}{2}\right)\left(\frac{e^t + e^{-t}}{2}\right) + \left(\frac{u_{10} + u_{20}}{2}\right)\cos(t)$$

$$u_2(t) = \left(\frac{u_{20} - u_{10}}{2}\right)\left(\frac{e^t + e^{-t}}{2}\right) + \left(\frac{u_{10} + u_{20}}{2}\right)\cos(t).$$

That is, for $t \geq 0$,

$$U(t, t_0, U_0) = \left[-\frac{1}{2}(u_{20} - u_{10}), \frac{1}{2}(u_{20} - u_{10})\right]\left(\frac{e^t + e^{-t}}{2}\right)$$

$$+ \left[\frac{1}{2}(u_{10} + u_{20}), \frac{1}{2}(u_{10} + u_{20})\right]\cos(t) \quad t \geq 0.$$

Then choosing

$$V_0 = \left[-\frac{1}{2}(u_{20} - u_{10}), \frac{1}{2}(u_{20} - u_{10})\right],$$

we obtain

$$U(t, t_0, W_0) = \left[\frac{1}{2}(u_{10} + u_{20}), \frac{1}{2}(u_{10} + U_{20})\right]\cos(t), \quad t \geq 0,$$

which implies the stability of the trivial solution of the initial value problem.

Next, we give an example which illustrates that one can get asymptotic stability as well in SDE with causal maps. Consider the following differential equation

$$D_H U(t) = -aU - b\int_0^t U(s)ds, \quad U(0) = U_0 \in K_c(\mathbb{R}^n), \tag{5.19}$$

$a, b > 0$. As before we take $U(t) = [u_1(t), u_2(t)]$ and $U_0 = [u_{10}, u_{20}]$. Then equation (5.19) reduces to

$$u_1' = -au_2 - b\int_0^t u_2(s)ds,$$

$$u_2' = -au_1 - b\int_0^t u_1(s)ds,$$

and

$$u_1^{(4)} = a^2 u_1'' + 2ab u_1' + b^2 u_1, \quad u_1(0) = u_{10},$$

$$u_2^{(4)} = a^2 u_2'' + 2ab u_2' + b^2 u_2, \quad u_2(0) = u_{10},$$

from which, by choosing $a = 1$ and $b = 2$, we obtain

$$
\begin{aligned}
u_1(t) &= \frac{1}{6}(u_{10} - u_{20})e^{-t} + \frac{1}{3}(u_{10} - u_{20})e^{2t} \\
&\quad + e^{-\frac{1}{2}t}\left[\frac{1}{2}(u_{10} + u_{20})\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{1}{2\sqrt{7}}(u_{10} + u_{20})\sin\left(\frac{\sqrt{7}}{2}t\right)\right], \\
u_2(t) &= \frac{1}{6}(u_{20} - u_{10})e^{-t} + \frac{1}{3}(u_{20} - u_{10})e^{2t} \\
&\quad + e^{-\frac{1}{2}t}\left[\frac{1}{2}(u_{10} + u_{20})\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{1}{2\sqrt{7}}(u_{10} + u_{20})\sin\left(\frac{\sqrt{7}}{2}t\right)\right].
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
U(t, t_0, U_0) &= (u_{20} - u_{10})\left[-\frac{1}{6}, \frac{1}{6}\right]e^{-t} + (u_{20} - u_{10})\left[-\frac{1}{3}, \frac{1}{3}\right]e^{2t} \\
&\quad + (u_{20} + u_{10})\left[\frac{1}{2}, \frac{1}{2}\right]e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{7}}{2}t\right) \\
&\quad - (u_{20} + u_{10})\left[\frac{1}{2\sqrt{7}}, \frac{1}{2\sqrt{7}}\right]e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{7}}{2}t\right),\ t \geq 0.
\end{aligned}
$$

Now, choosing $u_{10} = u_{20}$, we eliminate the undesirable terms and, therefore we get asymptotic stability of the zero solution of (5.19).

We are now in a position to give sufficient conditions for the stability, and the asymptotic and uniform asymptotic stability of the zero solution of (5.18).

Theorem 5.3.4. Assume that there exists functions $L(t, U(t))$ and $g(t, w)$ satisfying the following conditions

(i) $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t, 0) \equiv 0$;

(ii) $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$ where $B = B(\theta, \rho) = \{U \in K_c(\mathbb{R}^n) : D[U, \theta] \leq \rho\}$, $L(t, \theta) \equiv 0$, and $L(t, U)$ is positive definite and locally Lipschitzian in U;

(iii) for $t > t_0$ and $U \in E_1$, $D_-L(t, U(t)) \leq g(t, L(t, U(t)))$.

Then the stability of the zero solution of (5.10) implies the stability of the zero solution of (5.18).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Since $L(t, U)$ is positive definite, it follows that there exists a function $b \in \mathscr{K}$ such that

$$b(D[U, \theta]) \leq L(t, U) \text{ for } (t, U) \in \mathbb{R}_+ \times B. \tag{5.20}$$

Suppose that the zero solution is stable. Then given $b(\varepsilon) > 0$, $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that whenever $w_0 < \delta$, we have

$$w(t) < b(\varepsilon), \quad t \geq t_0, \tag{5.21}$$

where $w(t,t_0,w_0)$ is any solution of (5.10). Choose $w_0 = L(t_0,W_0)$. Since $L(t,U(t))$ is continuous and $L(t,\theta) \equiv 0$, there exists a positive function $\delta_1 = \delta_1(t_0,\varepsilon) > 0$ such that $D[W_0,\theta] \leq \delta_1$ and $L(t_0,W_0) \leq \delta$ hold simultaneously.

We claim that if $D[W_0,\theta] \leq \delta_1$, then $D[U(t),\theta] < \varepsilon$ for all $t \geq t_0$. Suppose this not true. Then there exists a solution $U(t) = U(t,t_0,W_0)$ satisfying the properties $D[U(t_2),\theta] = \varepsilon$ and $D[U(t),\theta] < \varepsilon$ for $t_0 < t < t_2$, $t_2 \in (t_0,t_1)$. Together with (5.20), this implies that

$$L(t_2,U(t_2)) \geq b(\varepsilon). \tag{5.22}$$

Furthermore, $U(t) \in B$ for $t \in [t_0,t_2]$. Hence, the choice of $w_0 = L(t_0,W_0)$ and condition (iii) give, as a consequence of Theorem 5.3.1, the estimate

$$L(t,U(t)) \leq r(t), \quad t \in [t_0,t_2],$$

where $r(t) = r(t,t_0,w_0)$ is the maximal solution of the comparison problem. Now from equations (5.20), (5.22), we have

$$b(\varepsilon) \leq L(t_2,U(t_2)) \leq r(t_2) < b(\varepsilon),$$

which is a contradiction. Therefore the proof of the theorem is complete.

The following theorem provides sufficient conditions for asymptotic stability:

Theorem 5.3.5. Assume that

(i) there exist functions $L(t,U)$, $g(t,w)$ satisfying the conditions of Theorem 4.3;

(ii) there exists a function $\alpha(t)$ such that $\alpha(t) > 0$ is continuous for $t \in \mathbb{R}_+$ and $\alpha(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Further, assume that relation (5.13) holds for $t > t_0$, $U \in E_\alpha$. Then, if the zero solution of (5.3) is stable, then the zero solution of (5.18) is asymptotically stable.

Proof. Let $), \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Set $\alpha_0 = \min_{t\in\mathbb{R}_+} \alpha(t)$, then $\alpha_0 > 0$ follows from assumption (ii). Since $L(t,U)$ is positive definite, there exists a $b \in \mathscr{K}$ such that (5.20) holds. Define

$$\varepsilon_1 = \alpha_0 b(\varepsilon). \tag{5.23}$$

Then, the stability of the zero solution of (5.10) implies that, given $\varepsilon_1 > 0$ and a $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\varepsilon_1,t_0)$ such that $w_0 < \delta$ implies that

$$w(t,t_0,w_0) < \varepsilon_1, \quad t \geq t_0, \tag{5.24}$$

where $w(t,t_0,w_0)$ is any solution of (5.10). Choose $w_0 = L(t_0,W_0)$. Then proceeding as in the proof of Theorem with ε_1 instead of $b(\varepsilon)$, we can prove that the zero solution of (5.18) is stable.

Let $U(t,t_0,W_0)$ be any solution of (5.18) such that $D[W_0,\theta] \leq \delta_0$, where $\delta_0 = \delta(t_0, 1/2\rho)$. Since the zero solution of (5.18) is stable, it follows that $D[U(t),\theta] < 1/2\rho$, $t \geq t_0$. Since $\alpha(t) \to \infty$ as $t \to \infty$, there exists a number $T = T(t_0,\varepsilon) > 0$ such that

$$b(\varepsilon)\alpha(t) > \varepsilon_1, \quad t \geq t_0 + T. \tag{5.25}$$

Now from Theorem 5.3.4 and relation (5.20), we get

$$\alpha(t)b(D[U(t),\theta]) \leq \alpha(t)L(t,U(t)) \leq r(t), \quad t \geq t_0, \tag{5.26}$$

where $U(t) = U(t,t_0,W_0)$ is any solution of (5.18) such that $D[W_0,\theta] \leq \delta_0$.
If the zero solution of (5.18) is not asymptotically stable, then there exists a sequence $\{t_k\}$, $t_k \geq t_0 + T$ and $t_k \to \infty$ as $k \to \infty$ such that $D[U(t_k),\theta] \geq \varepsilon$ for some solution $U(t)$ satisfying $D[W_0,\theta] \leq \delta_0$. The relations (5.24) and (5.26) yield that $b(\varepsilon)\alpha(t_k) \geq \varepsilon_1$, a contradiction to (5.25). Thus, the zero solution of (5.18) is asymptotically stable.
The next theorem gives sufficient conditions for the uniform asymptotic stability of (5.18).
Theorem 5.3.6. Assume there exists a function $L(t,U)$ satisfying the following properties:

(i) $L \in C[\mathbb{R}_+ \times B, \mathbb{R}_+]$, where $B = B(\theta,\rho) = \{U \in K_c(\mathbb{R}^n) : D[U,\theta] \leq \rho\}$, $L(t,U)$ is positive definite, decrescent and locally Lipschitzian in U;
(ii) $D_-L(t,U(t)) \leq -c(D[U(t),\theta])$ for $t > t_0$, $U \in E_0$ and $c \in \mathscr{K}$.

Then the zero solution of (5.18) is uniformly asymptotically stable.
Proof. Since $L(t,U)$ is positive definite and decrescent, there exist $a,b \in \mathscr{K}$ such that

$$b(D[U,\theta]) \leq L(t,U) \leq a(D[U,\theta]) \tag{5.27}$$

for $(t,U) \in \mathbb{R}_+ \times B$. Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbb{R}_+$ be given. Choose $\delta = \delta(\varepsilon) > 0$ such that

$$a(\delta) < b(\varepsilon). \tag{5.28}$$

We claim that if $D[W_0,\theta] \leq \delta$, $D[U(t),\theta] < \varepsilon$ for all $t \geq t_0$, where $U(t) = U(t,t_0,W_0)$ is any solution of (5.18). Suppose this is not true. Then there exists a solution $U(t)$ of (5.18) with $D[W_0,\theta] \leq \delta$ and $t_2 > t_0$, such that $D[U(t_2,t_0,W_0),\theta] = \varepsilon$ and $D[U(t,t_0,W_0),\theta] \leq \varepsilon$ for $t \in [t_0,t_2]$. Thus, in view of (5.27), we have

$$L(t_2,U(t_2)) \geq b(\varepsilon). \tag{5.29}$$

It is clear that, since $\varepsilon < \rho$, $U(t) \in B$. By our choice of $w_0 = L(t_0,W_0)$ and by the condition that $D_-L(t,U(t)) \leq 0$ for $t > t_0$, $U \in E_0$, and by Corollary 5.3.1, we have the estimate

$$L(t,U(t)) \leq L(t_0,W_0), \quad t \in [t_0,t_2]. \tag{5.30}$$

Now the relations (5.27) to (5.30) leads to the contradiction $b(\varepsilon) \leq L(t_2, U(t_2)) \leq a(D[W_0, \theta]) \leq a(\delta) < b(\delta)$.
This proves uniform stability. Now let $U(t) = U(t, t_0, W_0)$ be any solution of (5.18) such that $D[W_0, \theta] \leq \delta_0$, where $\delta_0 = \delta(\rho/2)$, δ being the same as before. It then follows from uniform stability that $D[U(t), \theta] \leq \rho/2$ for $t \geq t_0$, and hence $U(t) \in B$ for all $t > t_0$. Let $0 < \eta < \delta_0$ be given. Clearly, we have $b(\eta) \leq a(\delta_0)$. In view of the assumptions of $f(r)$, there exists a $\beta = \beta(\eta) > 0$ such that

$$f(r) > r + \beta \text{ if } b(\eta) \leq r \leq a(\delta_0). \tag{5.31}$$

Furthermore, there exists a positive integer $N = N(\eta)$ such that

$$b(\eta) + N\beta > a(\delta_0). \tag{5.32}$$

If we have, for some $t \geq t_0$, $L(t, U(t)) \geq b(\eta)$, it follows from (5.27) that there exists a $\delta_2 = \delta(\eta) > 0$, such that $D[U(t), \theta] \geq \delta_2$. This in turn implies that

$$c(D[U(t), \theta]) \geq c(\delta_2) = \delta_3, \tag{5.33}$$

where $\delta_3 = \delta_3(\eta)$. We construct $N+1$ numbers $t_k = t_k(t_0, \eta)$ such that $t_0(t_0, \eta) = t_0$ and $t_{k+1}(t_0, \eta) = t_k(t_0, \eta) + \beta/\delta_3$. By letting $T(\eta) = N\beta/\delta_3$, we have $t_k(t_0, \eta) = t_0 + T(\eta)$.
Now to prove uniform asymptotic stability, we still have to prove $D[U(t), \theta]$ for all $t \geq t_0 + T(\eta)$. It is therefore sufficient to show that

$$L(t, U(t)) < b(\eta) + (N-k)\beta, \quad t \geq t_k,\ k = 0, 1, 2, \ldots, N. \tag{5.34}$$

Now we prove (5.34) by induction. For $k = 0$, $t \geq t_0$, we have, using (5.27),

$$L(t, U(t)) \leq L(t_0, U_0) \leq a(\delta_0) < b(\eta) + N\beta. \tag{5.35}$$

Suppose we have, for some k,

$$L(s, U(s)) < b(\eta) + (N-k)\beta, \quad s \geq t_k,$$

and, if possible, assume that for $t \in [t_k, t_k + 1]$,

$$L(t, U(t)) \geq b(\eta) + (N-k-1)\beta.$$

It then follows that

$$a(\delta_0) \geq a(D[U(s), \theta]) \geq L(s, U(s)) \geq b(\eta) + N\beta - (k+1)\beta \geq b(\eta).$$

Therefore from (5.31), we conclude that

$$f(L(s, U(s))) \geq L(s, U(s)) + \beta > b(\eta) + (N-k)\beta > L(s, U(s))$$

for $t_k < s < t$, $t \in [t_k, t_{k+1}]$. In turn, this implies that $U(t) \in E_0$ for $t_k < s < t \in [t_k, t_{k+1}]$. Hence, we obtain from assumption (ii) and (5.35) that

$$\begin{aligned} L(t_{k+1}, U(t_{k+1})) &\leq L(t_k, U(t_k)) - \int_{t_k}^{t_{k+1}} c(D[U(s), \theta])ds \\ &< \beta(\eta) + (N-k)\beta - \delta_3(t_{k+1} - t_k) \\ &< b(\eta) + (N-k)\beta. \end{aligned}$$

This contradiction shows that there exists $t^* \in [t_k, t_{k+1}]$ such that

$$L(t^*, U(t^*)) < b(\eta) + (N-k-1)\beta. \tag{5.36}$$

Now we show that (5.36) implies that

$$L(t, U(t)) < b(\eta) + (N-k-1)\beta, \quad t \geq t^*.$$

If not trues, then there exists $t_1 > t^*$ such that $L(t_1, U(t_1)) = b(\eta) + (N-k-1)$ or for some small $h > 0$, $L(t_1 + h, U(t_1 + h)) < b(\eta) + (N-k-1)\beta$, which implies that

$$D_{-}L(t_1, U(t_1)) \geq 0. \tag{5.37}$$

As we did before, we can show that $U(t) \in B$, for $t^* \leq s \leq t_1$, and $D_{-}L(t_1, U(t_1)) - \delta_3 < 0$. This contradicts (5.37), and hence

$$L(t, U(t)) < b(\eta) + (N-k-1)\beta, \quad t \geq t_{k+1}.$$

This completes the proof of the theorem.

Our final stability result is a general result, which offers various stability criteria in a single set-up. The proof of this theorem, which can be obtained by using the comparison result Theorem 5.3.3 is omitted.

Theorem 5.3.7. Assume that there exists a function $L(t, U)$ satisfying properties (i), (ii), and (iii) of Theorem 5.3.3. Then the stability properties of the zero solution of (5.10) imply the corresponding properties of the zero solution of (5.18).

We now show that this theorem unifies the various stability results discussed earlier. To that end, consider the following special cases:

(a) Suppose $g_0(t, w) \equiv 0$. Then $\eta(s, T_0, v_0) = v_0$, and hence Ω reduces to E_1.

(b) Suppose $g_0(t, w) = -[\alpha'(t)/\alpha(t)]w$, where $\alpha(t) > 0$ is continuously differentiable on $\mathbb{R}_+$ and $\alpha(t) \to \infty$ as $t \to \infty$. Let $g(t, w) = g_0(t, w) + [1/\alpha(t)]g_1(t, \alpha(t)w)$ with $g_1 \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$, then $\eta(s, T_0, v_0) = v_0[\alpha(T_0)/\alpha(s)]$. Thus $\Omega = E_\alpha$.

(c) Let $g_0 = g = -c(w)$, $c \in \mathscr{K}$. Then it is easy to show that $\eta(s, T_0, v_0) = \phi^{-1}[\phi(v_0) - (s - T_0)]$, $t_0 \leq s \leq T_0$ where $\phi(w) = \phi(w_0) + \int_{w_0}^{w} \frac{ds}{c(s)}$ and ϕ^{-1} is the inverse function of ϕ. Since $\eta(s, T_0, v_0)$ is increasing in s to the left of T_0, on choosing a fixed $s_0 \leq T_0$ and defining $f(r) = \eta(s_0, T_), v_0)$, it is clear that $f(r) > r$ for $r > 0$. Thus $f(r)$ is continuous and increasing in r. Hence, $\Omega = E_0$.

5.4 Causal Differential Equations in a Banach Space

Let X be a Banach space with the norm $|\cdot|$, $J=[t_0,T]$, $E=C(J,X)$ and $B=B[u_0,b]=[u\in E:|u-u_0|_E\le b]$. We define $|u-v|_E(t)=\max_{t_0\le s\le t}|u(s)-v(s)|$. In this section we shall extend the basic results such as existence, uniqueness, global existence, to causal differential equations in a Banach space. Then introducing the causal differential inequalities in cones, we study the existence of extremal solutions.

Consider the Cauchy problem

$$\begin{cases} u'(t)=(Qu)(t), \\ u(t_0)=u_0, \end{cases} \tag{5.38}$$

where $Q:B\to E$.

Let us begin by proving a local existence result using successive approximations.

Theorem 5.4.1. Assume that

(a) $Q\in C[B,E]$ is a causal map where $B=B(u_0,b)=\{u\in E:|u-u_0|_0(t)\le b\}$ and $|Qu|_0(t)\le M_1$, on B;

(b) $g\in C[J\times[0,2b],\mathbb{R}_+]$, $g(t,w)\le M_2$ on $J\times[0,2b]$, $g(t,0)\equiv 0$, $g(t,w)$ is nondecreasing in w for each $t\in J$ and $w(t)=0$ is the only solution of

$$w'=g(t,w),\quad w(t_0)=0 \text{ on } J; \tag{5.39}$$

(c) $|(Qu)(t)-(Qv)(t)|\le g(t,|u-v|_0(t))$, on B.

Then, the successive approximations defined by

$$u_{n+1}(t)=u_0+\int_{t_0}^{t}(Qu_n)(s)ds,\quad n=0,1,2,\ldots \tag{5.40}$$

exist on $J_0=[t_0,t_0+\eta)$, where $\eta=\min\{T-t_0,b/M\}$ and $M=\max\{M_1,M_2\}$, and converge uniformly to the unique solution $u(t)$ of (5.38).

Proof. For $t\in J_0$ we have, by induction

$$\begin{aligned} |u_{n+1}(t)-u_0| &= \left|\int_{t_0}^{t}(Qu_n)(s)ds\right| \\ &\le \int_{t_0}^{t}|(Qu_n)(s)|ds \\ &\le \int_{t_0}^{t}|Qu_n|_0(t)ds \\ &\le M_1(t-t_0) \\ &\le M(t-t_0) \\ &\le b \end{aligned}$$

which shows the successive approximations are well defined on J_0.

Next, we define the successive approximations for problem (5.39) as follows:

$$\begin{aligned} w_0(t) &= M(t-t_0) \\ w_{n+1} &= \int_{t_0}^{t} g(s, w_n(s))ds, \quad n = 0,1,2,\ldots \end{aligned}$$

Then,

$$w_1(t) = \int_{t_0}^{t} g(s, w_0(s))ds \le M_2(t-t_0) \le M(t-t_0) = w_0(t).$$

Assume for some $k > 1$, $t \in J_0$, that

$$w_k(t) \le w_{k-1}(t).$$

Then, using the monotonicity of g, we get

$$w_{k+1}(t) = \int_{t_0}^{t} g(s, w_k(s))ds \le \int_{t_0}^{t} g(s, w_{k-1}(s)) = w_k(t).$$

Hence, the sequence $\{w_k(t)\}$ is monotone decreasing.

Since $w_k'(t) = g(t, w_{k-1}(t)) \le M_2$, $t \in J_0$, we conclude by the Ascoli-Arzela theorem and the monotonicity of the sequence $\{w_k(t)\}$ that

$$\lim_{k\to\infty} w_k(t) = w(t)$$

uniformly on J_0. Since $w(t)$ satisfies (5.39), we get from condition (b), that $w(t) \equiv 0$ on J_0.

Observing that for each $t \in J_0$, $t_0 \le s \le t$,

$$\begin{aligned} |u_1(s) - u_0| &= \left| \int_{t_0}^{s} (Qu_0)(\zeta) d\zeta \right| \\ &\le \int_{t_0}^{s} |(Qu_0)(\zeta)| d\zeta \\ &\le |Qu_0|_0 (s-t_0) \le |Qu_0|_0 (t-t_0) \\ &\le M_1(t-t_0) \le M(t-t_0) = w_0(t), \end{aligned}$$

which implies that $|u_1 - u_0|_0(t) \le w_0(t)$. We assume for some $k > 1$,

$$|u_k - u_{k-1}|_0(t) \le w_{k-1}(t), \quad t \in J_0.$$

Consider, for any $t \in J_0$, $t_0 \le s \le t$,

$$\begin{aligned} |u_{k+1}(s) - u_k(s)| &\le \int_{t_0}^{s} |(Qu_k)(\zeta) - (Qu_{k-1})(\zeta)| d\zeta \\ &\le \int_{t_0}^{s} g(\zeta, |u_k - u_{k-1}|_0(\zeta)) d\zeta \\ &\le \int_{t_0}^{s} g(\zeta, w_{k-1}(\zeta)) d\zeta \\ &\le \int_{t_0}^{t} g(\zeta, w_{k-1}(\zeta)) d\zeta = w_k(t), \end{aligned}$$

which further gives

$$|u_{k+1} - u_k|_0(t) \leq w_k(t), \quad t \in J_0.$$

Thus, we conclude that

$$|u_{n+1} - u_n|_0(t) \leq w_n(t) \tag{5.41}$$

for $t \in J_0$ and for all $n = 0, 1, 2, \ldots$. We claim that $u_n(t)$ is a Cauchy sequence. To show this, let $n \leq m$. Setting $v(t) = |u_n(t) - u_m(t)|$ and using (5.40), we get

$$\begin{aligned}
D^+ v(t) &\leq |u_n'(t) - u_m'(t)| \\
&= |(Qu_{n-1})(t) - (Qu_{m-1})(t)| \\
&\leq |(Qu_{n-1})(t) - (Qu_n)(t)| + |(Qu_n)(t) - (Qu_m)(t)| \\
&\quad + |(Qu_m)(t) - (Qu_{m-1})(t)| \\
&\leq g(t, |u_{n-1} - u_n|_0(t)) + g(t, |u_n - u_m|_0(t)) \\
&\quad + g(t, |u_{m-1} - u_m|_0(t)) \\
&\leq g(t, w_{n-1}(t)) + g(t, |v|_0(t)) + g(t, w_{n-1}(t)) \\
&= g(t, |v|_0(t)) + 2g(t, w_{n-1}(t)).
\end{aligned}$$

The above inequalities yield, on using Theorem 5.2.1, the estimate

$$v(t) \leq r_n(t), \quad t \in J_0,$$

where the maximal solution of

$$r_n' = g(t, r_n) + 2g(t, w_{n-1}(t)), \quad r_n(t_0) = 0$$

for each n. Since as $n \to \infty$, $2g(t, w_{n-1}(t)) \to 0$ uniformly on J_0, it follows by Lemma 1.3.3 that $r_n(t) \to 0$, as $n \to \infty$ uniformly on J_0. This implies from (5.41) that $u_n(t)$ converges uniformly to $u(t)$ is a solution of (5.38).

To prove uniqueness, let $v(t)$ be another solution of (5.38) on J_0. Set $m(t) = |u(t) - v(t)|$. Then $m(t_0) = 0$ and

$$D^+ m(t) \leq g(t, |m|_0(t)), \quad t \in J_0.$$

Since $m(t_0) = 0$, it follows from Theorem 5.2.1 that

$$m(t) \leq r(t, t_0, 0), \quad t \in J_0,$$

where $r(t, t_0, 0)$ is the maximal solution of (5.39). The assumption (b) now shows that $u(t) = v(t)$, $t \in J_0$, proving uniqueness.

Having discussed the situation relative to the method of successive approximations, we now proceed to investigate another type of approximate solution for the problem (5.38). Since the existence and uniqueness of solutions of (5.38) is guaranteed when Q satisfies a local Lipschitzian condition, one can utilize this fact to construct a type of approximate solution if it is possible to approximate a continuous function Q by a sequence of locally Lipschitzian functions. The following result exploits this idea.

Theorem 5.4.2. Assume that $Q \in C[B,E]$, $|Qu| \le M$ on B and $\eta = \min\{T - t_0, b/(M+1)\}$. Let $\{\varepsilon_n\}$ be a sequence such that $0 \le \varepsilon_n \le 1$, $\lim_{n\to\infty} \varepsilon_n = 0$. Then, for each positive integer n, the Cauchy problem (3.1) has an ε-approximate solution $u_n(t)$ on $[t_0, t_0+\eta]$ satisfying

(i) $u_n(t)$ is continuously differentiable on $[t_0, t_0+\eta]$ and

(ii) $|u_n' - (Qu_n)(t)| \le \varepsilon_n$ for $t_0 \le t \le t_0 + \eta$.

To prove this theorem, we first need the following known result which shows that a continuous function $F(x)$ can be approximated by locally Lipschitzian functions.

Lemma 5.4.1. Let $F \in C[\Omega, X]$, where $\Omega \subset X$ is open. Then, for each $\varepsilon > 0$, there exists a locally Lipschitzian function $F_\varepsilon(x) : \Omega \to X$ such that $|F(x) - F_\varepsilon(x)| < \varepsilon$ on Ω.

Let us now prove Theorem 5.4.2.

Proof. By Dugundji's extension theorem, Theorem 1.5.8, Q has a continuous extension $\tilde{Q} : E \to E$ such that $|\tilde{Q}u|_E \le M$ on E. Also, there exists, for every $0 < \varepsilon_n < 1$, a function $\tilde{Q}_{\varepsilon_n} : E \to E$ which is locally Lipschitizian in u satisfying

$$|\tilde{Q}_{\varepsilon_n} u - \tilde{Q}u|_E \le \varepsilon_n.$$

In particular, we have $|\tilde{Q}_{\varepsilon_n} u - \tilde{Q}u|_E \le \varepsilon_n$ and $|\tilde{Q}_{\varepsilon_n} u|_E \le M+1$ on B. Let $u_n(t)$ be the unique solution of

$$u'(t) = (\tilde{Q}_{\varepsilon_n} u)(t), \quad u(t_0) = u_0,$$

which exists on $[t_0, t_0+\eta]$, $\eta = \min[T - t_0, b/(m+1)]$.

Hence, we have

$$|u_n'(t) - (Qu_n)(t)| = |(\tilde{Q}_{\varepsilon_n} u_n)(t) - (Qu_n)(t)| \le \varepsilon_n$$

for $t_0 \le t \le t_0 + \eta$. The theorem is proved.

Next we prove a local existence result using a compactness condition.

Theorem 5.4.3. Assume that

(i) $|Qu|_E \le M$ on B and $\eta = \min\{T - t_0, b/(M+1)\}$,

(ii) (a) $\alpha((QA)(t)) \le g(t, \sup_{t_0 \le s \le t} \alpha(A(s)))$ for every bounded set $A(s) \subset B$,

(b) Q is uniformly continuous on B,

(iii) $g \in C[[t_0,T] \times [0,2b],\mathbb{R}]$, $g(t,0) \equiv 0$, and $w(t) \equiv 0$ is the unique solution of (5.39).

Then the Cauchy problem (5.40) has a solution on $[t_0,t_0+\eta]$.

Proof. By Theorem 5.4.2, we have approximate solutions $\{u_n(t)\}$ on $[t_0,t_0+\eta]$ such that

$$u_n'(t) = (Qu_n)(t) + y_n(t), \quad u_n(t_0) = u_0,$$

and $|y_n(t)| \leq \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. Since $\{u_n(t)\}$ is equicontinuous and uniformly bounded, it is enough to show, to use Ascoli-Arzela's theorem, that the set $\{u_n(t)\} = \{u_n(t) : n \geq 1\}$ is relatively compact, that is, $\alpha(\{u_n(t)\}) = 0$ on $[t_0,t_0+\eta]$.

Let $m(t) = \alpha(\{u_n(t) : n \geq k\})$, and note that $m(t_0) = \alpha(\{u_n(t_0) : n \geq k\}) = \alpha(\{u_0\}) = 0$.

First we show $m(t)$ is continuous as follows. By property (vi) on Theorem 1.7.1, we have, for $t_0 \leq s \leq t \leq t_0+\eta$,

$$\begin{aligned} |m(t) - m(s)| &= |\alpha(\{u_n(t) : n \geq k\}) - \alpha(\{u_n(s) : n \geq k\})| \\ &\leq |\alpha(\{u_n(t) - u_n(s) : n \geq k\})| \\ &= \left| \alpha\left(\left\{ \int_s^t ((Qu_n)(\zeta) + y_n(\zeta))d\zeta : n \geq k \right\} \right) \right|, \end{aligned}$$

and setting $A = \{x_n(t) : n \geq k\}$, where $x_n(t) = \int_s^t ((Qu_n)(\zeta) + y_n(\zeta))d\zeta$, we have $\sup_{x_n \in A} |x_n|_E \leq (M+1)|t-s|$ and $\alpha(A) \leq 2(M+1)|t-s|$. Hence,

$$|m(t) - m(s)| \leq 2(m+1)|t-s|,$$

which implies $m(t)$ is continuous.

Next, we show that $D_{-}m(t) \leq g(t,m(t))$ on $[t_0,t_0+\eta]$, where $D_{-}m(t) = \liminf_{h \to 0^+} 1/h[m(t) - m(t-h)]$.

By property (vi) of Theorem 1.7.1, we obtain

$$\begin{aligned} \frac{1}{h}[m(t) - m(t-h)] &= \frac{1}{h}[\alpha(\{u_n(t) : n \geq k\}) - \alpha(\{u_n(t-h) : n \geq k\})] \\ &\leq \frac{1}{h}[\alpha(\{u_n(t) - u_n(t-h) : n \geq k\})]. \end{aligned}$$

By the Mean-Value Theorem 1.5.5 and properties (ii) and (vii) of Theorem 1.7.1, we have, with $J_h \equiv [t-h,t]$,

$$\begin{aligned}
\frac{1}{h}[m(t)-m(t-h)] &\leq \frac{1}{h}[\alpha(\{h\overline{\text{co}}(\{u_n'(\zeta) : \zeta \in [t-h,t]\}) : n \geq k\})] \\
&= \alpha(\{u_n'(\zeta) : \zeta \in [t-h,t] : n \geq k\}) \\
&= \alpha(\{u_n'(\zeta) : n \geq k, \zeta \in [t-h,t]\}) \\
&= \alpha\left(\bigcup_{\zeta \in J_h} \{u_n'(\zeta) : n \geq k\}\right) \\
&= \alpha\left(\bigcup_{t \in J_h} \{(Qu_n)(t) + y_n(t) : n \geq k\}\right) \\
&\leq \alpha\{((Qu_n)(J_h) + y_n(J_h) : n \geq k)\} \\
&\leq \alpha(\{(Qu_n)(J_h) + 2\varepsilon_k : n \geq k\}),
\end{aligned}$$

where $(Qu_n)(J_h) = \bigcup_{\zeta \in J_h}(Qu_n)(\zeta)$ and $y_n(J_h) = \bigcup_{\zeta \in J_h} y_n(\zeta)$.
Then it follows that

$$D_-m(t) \leq \liminf_{h \to 0^+} \alpha(\{(Qu_n)(J_h) + 2\varepsilon_k : n \geq k\}).$$

The equicontinuity of $\{u_n(t)\}$ and the uniform continuity of Q imply that

$$\lim_{h \to 0^+} (Qu_n)(J_h) = (Qu_n)(t) \text{ for } n \geq k,$$

with respect to the Hausdorff metric. Also, in view of assumption (ii), we have

$$\liminf_{h \to 0^+}(\alpha\{(Qu_n)(J_h) : n \geq k\}) = \alpha(\{Qu_n(t) : n \geq k\}) \leq g\left(t, \sup_{t_0 \leq t \leq t} \alpha(\{u_n(s) : n \geq k\})\right).$$

Therefore,

$$D_-m(t) \leq g\left(t, \sup_{t_0 \leq s \leq t} \alpha(\{u_n(s) : n \geq k\})\right) + 2\varepsilon_k.$$

Then by the basic comparison Theorem 1.3.2,

$$m(t) = \alpha(\{u_n(t) : n \geq k\}) \leq r_k(t,t_0,0), \quad t \in [t_0, t_0+\eta],$$

where $r_k(t,t_0,0)$ is the maximal solution of

$$\begin{cases} u'(t) = g(t,u) + 2\varepsilon_k \\ u(t_0) = 0. \end{cases}$$

By Lemma 1.3.3 and condition (iii), we have

$$\lim_{k \to \infty} r_k(t,t_0,0) = r(t,t_0,0),$$

where $r(t,t_0,0) \equiv 0$ is the maximal solution of (5.39) on $[t_0, t_0+\eta]$. Thus $\alpha\{u_n(t) : n \geq k\} = 0$ on $[t_0, t_0+\eta]$, proving the theorem.

5.5 Global Existence and Inequalities in Cones

We shall continue to consider the IVP for causal differential equation (5.38). We shall begin by proving a global existence theorem in a direct way and then through a Lyapunov function. We explain the different approaches. Then we discuss causal differential inequalities in cones and utilizing these results prove existence of extremal solutions.

Theorem 5.5.1. Let $Q \in C[E,E]$ be a causal map such that

$$|(Qu)(t)| \leq g(t, |u|_0(t)), \tag{5.42}$$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t,w)$ is nondecreasing in w for each $t \in \mathbb{R}$ and the maximal solution $r(t) = r(t,t_0,w_0)$ of (5.39) exists on $[t_0,\infty)$. Suppose further that Q is smooth enough to guarantee the local existence of solutions of (5.38) for any $(t_0,u_0) \in \mathbb{R}_+ \times X$. Then, the largest interval of existence of any solution $u(t,t_0,u_0)$ of (5.38) is $[t_0,\infty)$, whenever $|u_0| \leq w_0$.

Proof. Suppose that $u(t) = u(t,t_0,u_0)$ is any solution of (5.38) existing on $[t_0,\beta)$, $t_0 < \beta < \infty$ with $|u_0| \leq w_0$, and the value of β cannot be increased. Define $m(t) = |u(t)|$ and note that $m(t_0) \leq w_0$. Then it follows that,

$$D^+ m(t) \leq |u'(t)| = |(Qu)(t)| \leq g(t, |u|_0(t)).$$

Using Theorem 1.3.2, we obtain

$$m(t) \leq r(t), \quad t_0 \leq t < \beta. \tag{5.43}$$

For any t_1, t_2 such that $t_0 < t_1 < t_2 < \beta$,

$$\begin{aligned} |u(t_1) - u(t_2)| &\leq \left| \int_{t_0}^{t_1} (Qu)(s)ds - \int_{t_0}^{t_2} (Qu)(s)ds \right| \\ &\leq \int_{t_1}^{t_2} |(Qu)(s)|ds \\ &\leq \int_{t_1}^{t_2} g(s, |u|_0(s))ds. \end{aligned}$$

Employing the estimate (5.43) and the monotonicity of $g(t,w)$, we find

$$|u(t_1) - u(t_2)| \leq \int_{t_1}^{t_2} g(s, r(s))ds = r(t_2) - r(t_1).$$

Since $\lim_{t\to\beta_-} r(t,t_0,w_0)$ exists, taking the limits as $t_1,t_2 \to \beta^-$, we get that $\{u(t_n)\}$ is a Cauchy sequence and therefore $\lim_{t\to\beta^-} u(t,t_0,u_0) = u_\beta$ exists. We then consider the IVP

$$u'(t) = (Qu)(t), \quad u(\beta) = u_\beta.$$

As we have assumed the local existence, we note that $u(t,t_0,u_0)$ can be continued beyond β, contradicting our assumption that β cannot be increased. Thus every solution $u(t,t_0,u_0)$ of (5.38) such that $|u_0| \le w_0$ exists globally on $[t_0,\infty)$ and hence the proof.

Remark. Clearly, Q is bounded on bounded sets, if Q satisfies condition (5.42), which need not be true if we relax (5.42) to

$$((Qx),x) \le g(t,|x|)|x|. \tag{5.44}$$

This is also the case when one relaxes (5.42) to a more general condition by means of Lyapunov like function. If we need only global existence we could also remove the restriction of monotony on $g(t,u)$. This is the motivation for the next result.

Theorem 5.5.2. Assume that

(i) $Q \in C[\mathbb{R}_+ \times E, E]$, Q is bounded on bounded sets and for any $(t_0,x_0) \in \mathbb{R}_+ \times E$ there exists a local solution for the problem (5.38).

(ii) $V \in C[\mathbb{R}_+ \times E, \mathbb{R}_+]$, V is locally Lipschitzian in x, $V(t,x) \to \infty$ as $|x| \to \infty$ uniformly for $[0,T]$ for every $T > 0$ and for $(t,x) \in \mathbb{R}_+ \times E$

$$D^+V(t,x) \equiv \lim_{h\to 0} \frac{1}{h}[V(t+h,x+h(Qx)) - V(t,x)] \le g(t,V(t,x)),$$

where $g \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$;

(iii) the maximal solution $r(t) = r(t,t_0,u_0)$ of (5.39) exists on $[t_0,\infty)$ and is positive if $u_0 > 0$.

Then for every $x_0 \in E$ such that $V(t_0,x_0) \le u_0$, the problem (5.38) has a solution $x(t)$ on $[t_0,\infty)$ which satisfies the estimate

$$V(t,x(t)) \le r(t), \quad t \ge t_0. \tag{5.45}$$

Proof. Let S denote the set of all functions x defined on $I_x = [t_0,c_x)$ with values in E such that $x(t)$ is a solution of (5.38) on I_x and $V(t,x(t)) \le r(t)$, $t \in I_x$. We define a partial order $\le$ on S as follows: the relation $x \le y$ implies $I_x \subseteq I_y$ and $y(t) \equiv x(t)$ on I_x. We shall first show that S is nonempty. By (i), there exists a solution $x(t)$ of (5.38) defined on $I_x = [t_0,c_x)$. Setting $m(t) = V(t,x(t))$ for $t \in I_x$ and using assumption (ii), it is easy to obtain the differential inequality

$$D^+m(t) \le g(t,m(t)). \quad t \in I_x.$$

Now, by Theorem it follows that

$$V(t,x(t)) \le r(t), \quad t \in I_x, \tag{5.46}$$

where $r(t)$ is the maximal solution of (5.39). This shows that $x \in S$ and so S is nonempty.

If $(x_\beta)_\beta$ is a chain $(S,\leq)$, then there is a uniquely defined map y on $I_y = [t_0, \sup_\beta c_{x_\beta})$ that coincides with x_β on I_{x_β}. Clearly $y \in S$ and hence y is an upper bound of $(x_\beta)_\beta$ in $(S,\leq)$. Then Zorn's lemma assures the existence of a maximal element z in $(S,\leq)$. The proof of the theorem is complete if we show that $c_z = \infty$. Suppose that it is not true, so that $c_z < \infty$. Since $r(t)$ is assumed to exist on $[t_0,\infty)$, $r(t)$ is bounded on I_z. Since $V(t,x) \to \infty$ as $|x| \to \infty$ uniformly in t on $[t_0,c_z]$, the relation $V(t,z(t)) \leq r(t)$ on I_z implies that $|z(t)|$ is bounded on I_z. By (i), this shows that there is an $M > 0$ such that

$$|(Qz)(t)| \leq M, \quad t \in I_z.$$

We then have, for all $t_1,t_2 \in I_z$, $t_1 \leq t_2$,

$$|z(t_2)-z(t_1)| \leq \int_{t_1}^{t_2} |(Qz)(s)|ds \leq M(t_2-t_1),$$

which shows that z is Lipschitzian on I_2 and consequently has a continuous extension z_0 on $[t_0,c_z]$. By continuity, we get

$$z_0(c_z) = x_0 + \int_{t_0}^{c} ((Qz_0)s)ds.$$

This implies that $z_0(t)$ is a solution of (5.38) on $[t_0,c_z]$ and, clearly, $V(t,z_0(t)) \leq r(t)$, $t \in [t_0,c_z]$. Consider the problem

$$x' = (Qx)(t), \quad x(c_0) = z_0(c_z).$$

By the assumed local existence there exists a solution $x_0(t)$ on $[c_z, c_z+\delta]$, $\delta > 0$. Define

$$z_1(t) = \begin{cases} z_0(t) \text{ for } t_0 \leq t \leq c_z \\ x_0(t) \text{ for } c_z \leq t < c_z+\delta. \end{cases}$$

Clearly $z_1(t)$ is a solution of (5.38) on $[t_0,c_z+\delta)$ and, by repeating the arguments that were used to obtain (5.46), we get

$$V(t,z_1(t)) \leq r(t), \quad t \in [t_0, c_z+\delta).$$

This contradicts the maximality of z and hence $c_z = \infty$. The proof is complete.

In order to develop the theory of differential inequalities in a Banach space X, we need to introduce the concept of a cone which induces a partial ordering in X.

A proper subset K of X is said to be a cone if $\lambda K \subset K$, $\lambda \geq 0$, $K+K \subset K$, $K = \overline{K}$ and $K \cap \{-K\} = \theta$, where θ denotes the null element of the Banach space X, and $\overline{K}$ denotes the closure of K. Let K° denote the interior of K and assume that K° is nonempty. The cone induces the order relations in X defined by

$$x \leq y \text{ iff } y-x \in K,$$
$$x < y \text{ iff } y-x \in K^\circ, \quad x,y \in X.$$

Let K^* be the set of all continuous linear functionals c on X such that $c(u(t)) \geq 0$ for all $u(t) \in K$ and $t \in J$, and let K_0^* be the set of all continuous linear functionals on X such that $c(u(t)) > 0$ for all $u(t) \in K^\circ$. We say $u \leq v$ on E if $u(t) \leq v(t)$ for $u, v \in E$, $t \in J$. A function Q from E into E is said to be nondecreasing if $u \leq v$ implies $Qu \leq Qv$.

To prove a basic result, on differential inequalities, we need the following lemma.

Lemma 5.5.1. Let K be a cone with nonempty interior K°. Then

(i) $x \in K$ is equivalent to $cx \geq 0$ for all $c \in K^*$;

(ii) $x \in \partial K$ implies that there exists a $c \in K_0^*$ such that $cx \equiv 0$.

Theorem 5.5.3. Let K be a cone in X with nonempty interior K°. Assume that

(i) $u, v \in C^1[J, X]$, $Q \in C[B, E]$, $B = B(u_0, b)$ and Q is nondecreasing;

(ii) $u'(t) - (Qu)(t) < v'(t) - (Qv)(t), \quad t \in (t_0, T)$.

Then, $u(t_0) < v(t_0)$ implies that $u(t) < v(t)$, $t \in J$.

Proof. Suppose that the assertion of the theorem is false. Then, there exists a $t_1 > t_0$ such that

$$v(t_1) - u(t_1) \in \partial K \text{ and } v(t) - u(t) \in K^\circ, \quad t \in [t_0, t_1).$$

Thus, by Lemma 5.4.2, there exists a $c \in K_0^*$ such that $c(v(t_1) - u(t_1)) = 0$.

Setting $m(t) = c(v(t) - u(t))$, we see that $m(t) > 0$ for $t_0 \leq t < t_1$ and $m(t_1) = 0$. Consequently, $m'(t_1) \leq 0$. Further, at $t = t_1$, we have $u(t_1) = v(t_1)$ and $u(t) < v(t)$, $t_0 \leq t < t_1$. Thus, $u(t) \leq v(t)$, $t_0 \leq t \leq t_1$.

Using the nondecreasing nature of Q, we have $(Qu)(t) \leq (Qv)(t)$, $t_0 \leq t \leq t_1 < T$. Hence from (ii), we get

$$m'(t_1) = c(v'(t_1) - u'(t_1)) > c((Qv)(t_1) - (Qu)(t_1)) \geq 0.$$

This contradiction proves the theorem.

Using Theorem 5.4.3, we can prove the existence of the maximal solution of (5.38) relative to the cone K.

Theorem 5.5.4. Let K be a cone in X with nonempty interior K°. Suppose that

(i) $Q \in C[B, E]$ and Q is nondecreasing;

(ii) Q is uniformly continuous on B (and hence, we may assume that $|Qu|_E \leq M$ on B);

(iii) $g \in C[J \times \mathbb{R}_+, \mathbb{R}]$ with $g(t, 0) \equiv 0$, and the only solution of the scalar differential equation (5.39) is the trivial solution;

(iv) $\alpha((QA)(t)) \leq g(t, \sup_{t_0 \leq s \leq t} \alpha((A)(S)))$, $t \in J$, where $A(s)$ is bounded subsets of B and α is the measure of noncompactness.

Then, there exists a maximal solution of (5.38) relative to K on $[t_0, t_0+\eta]$ where $\eta = \min\{T-t_0, b/(M+1))\}$.

Proof. Let $y_0 \in K^\circ$ with $|y_0| = 1$. Consider the system

$$\begin{cases} u'(t) = (Qu)(t) + \frac{1}{n}y_0, \\ u(t_0) = u_0 + \frac{1}{n}y_0, \end{cases} \tag{5.47}$$

for each integer n.

Consider

$$\left|(Qu)(t) + \frac{1}{n}y_0\right| \leq |(Qu)(t)| + \frac{1}{n}|y_0| \leq |Qu|_E + \frac{1}{n} \leq M + \frac{1}{n}.$$

Applying Theorem 5.4.3, we conclude that there exists a solution $u_n(t)$ of (5.47) for each n and a solution $u(t)$ of (5.38) on $[t_0, t_0+\eta]$. The above inequality implies the equicontinuity of the family $\{u_n(t)\}$. Noting that $\alpha(\{u)n(t_0)\}) = \alpha(\{u_0 + (1/n)y_0\}) = 0$, by property (iv) of Theorem 1.7.1, we conclude, as in Theorem 5.4.3, that the set $\{u_n(t)\}$ is relatively compact for each $J \in [t_0, t_0+\eta]$. We then apply Ascoli-Arzela's theorem to obtain a subsequence of $\{u_n(t)\}$ which converges uniformly to a continuous function $r(t)$ on $[t_0, t_0+\eta]$, using Theorem 5.4.5, the comparison theorem on cones, the sequence $\{u_n(t)\}$ is monotone and hence $r(t)$ is a solution of (5.38) on $[t_0, t_0+\eta]$.

Now let $u(t)$ be any solution of (5.38) on $[t_0, t_0+\eta]$. Then $u'(t) - (Qu)(t) = 0 < (1/n)y_0 = u_n'(t) - (Qu_n)(t)$ and $u(t_0) = u_0 < u_0 + (1/n)y_0 = u_n(t_0)$.

Then, by Theorem 5.4.5, we get $u(t) \leq u_n(t)$ for $t \in [t_0, t_0+\eta]$.

Therefore,

$$u(t) \leq \lim_{n\to\infty} u_n(t) \equiv r(t), \quad t \in [t_0, t_0+\eta].$$

This shows that $r(t)$ is the desired maximal solution and the proof is complete.

Theorem 5.5.5. Suppose that the assumptions of Theorem 5.4.6 are satisfied. Let $m \in C^1[[t_0, t_0+\eta], X]$ and

$$m'(t) \leq (Qm)(t), \quad t \in [t_0, t_0+\eta].$$

If $m(t_0) \leq u_0$, then $m(t) \leq r(t)$, $t \in [t_0, t_0+\eta]$, where $r(t)$ is the maximal solution of (5.38).

Proof. Let $u_n(t)$ be a solution of

$$\begin{cases} u'(t) = (Qu)(t) + \frac{1}{n}y_0, \\ u(t_0) = u_0 + \frac{1}{n}y_0, \end{cases}$$

for each positive integer n, where $y_0 \in K^\circ$ with $|y_0| = 1$.

Note that $u(t_0) = u_0 < u_0 + (1/n)y_0 = u_n(t_0)$ and $u_n'(t) - (Qu_n)(t) = (1/n)y_0 > \theta \geq m'(t) - (Qm)(t)$, for $t \in [t_0, t_0+\eta]$. By Theorem 5.4.5, $m(t) < u_n(t)$, for each n and $t \in [t_0, t_0+\eta]$. Hence, $m(t) \leq \lim_{n\to\infty} u_n(t) \equiv r(t)$, $t \in [t_0, t_0+\eta]$.

Corollary 5.5.1. Let the hypothesis of Theorem 5.5.4 hold and let $(Q\theta)(0) \equiv \theta$. Then the maximal solution $r(t)$ of (1) such that $r(t_0) = u_0 \in K$ remains in K for $t \in [t_0, t_0 + \eta]$.
Proof. The proof follows by choosing $m(t) \equiv \theta$ in Theorem 5.4.7.
Let E be a real Banach space and let $|\cdot|$ denote the norm in E. We let $B = \{x \in E : |x| \leq b\}$ denote the ball of radius b and let $\mathbb{R}_0 = [t_0, t_0 + a] \times B$ where $t_0 \geq 0$, $a > t_0$.
Consider the causal differential equation

$$x'(t) = (Qx)(t), \quad x(t_0) = x_0, \tag{5.48}$$

where $Q : E \to E$. There are several known results which guarantee the existence of solutions to (5.48). We mention in particular those given in [6]. One of the conditions given there is:

(I) f is uniformly continuous in $\mathbb{R}_0$.

Another is a compactness condition which is similar to the convexity condition II stated below.
For any subset $A \subset B$ and for small $h > 0$ set

$$A_h(Q) = \{y | y = x + h(Qx) : x \in A\}.$$

We introduce a (comparison) scalar equation

$$u' = g(t, u), \quad u(t_0) = 0 \tag{5.49}$$

where $g \in C[[t_0, t_0 + a] \times \mathbb{R}^+, \mathbb{R}]$. Assume that $u \equiv 0$ is the unique solution of (5.49). then the convexity condition on Q is

(II) $\liminf_{h \to 0^+} \{h^{-1}[\alpha(A_h)(Q) - \alpha(A)]\} \leq g(t, \alpha(A))$

for any subset $A \subset B$.
We also require the following condition on a set $A \subset B$:

(III) The set of solutions $x(t, x_0)$, $x_0 \in A$ of (5.48) exists and is equicontinuous.

Condition III is satisfied when A is pre-compact.
Theorem 5.5.6. Let $A \subset B$ have convex closure and let condition I, II, and III be satisfied for (5.38). Then the set

$$x(t, t_0, A) = \{x(t, t_0, x_0) | x_0 \in A\}$$

has convex closure for $t \in [t_0, t_0 + a]$.

Proof. Set $m(t) = \alpha(x(t,A))$ where α is the measure of nonconvexity and $x(t,A) = x(t,t_0,A)$. Our claim is then $m(t) = 0$. Now $m(t+h) - m(t) = \alpha(x(t+h,A)) - \alpha(x(t,A)) = [\alpha(x(t+h),A) - \alpha(A_h(Q))] + [\alpha(A_h(Q)) - \alpha(x(t,A))]$. If we know that

$$\liminf_{h\to 0^+} h^{-1}[\alpha(x(t+h,A)) - \alpha(A_h(Q))] \le 0 \tag{5.50}$$

then it follows from condition II that $D_+m(t) \le g(t,m(t))$ where D_+ denotes a Dini derivative. It follows further from the theory of differential inequalities that $m(t) \equiv 0$. Thus it remains to verify (5.50).

By properties

$$h^{-1}[\alpha(x(t+h,A)) - \alpha(A_h(Q))] \le \alpha[h^{-1}(x(t+h,A) - A_h(Q))]$$

$$\le 2 \sup_{x_0\in A} |h^{-1}[x(t+h,x_0) - x(t,x_0)] - (Qx)(t)|.$$

Hence it suffices to show that

$$h^{-1}(x(t+h,x_0) - x(t,x_0)) \to (Qx)(t)$$

uniformly in x_0. Now

$$|h^{-1}(x(t+h,x_0) - x(t,x_0)) - (Qx)(t)|$$

$$\le h^{-1}\int_t^{t+h} |(Qx)(t+s) - (Qx)(t)|ds.$$

By the uniform continuity of Q and by the equicontinuity of $x(t,x_0)$ this last expression can be made arbitrarily small, independent of x_0, by taking h sufficiently small. This concludes the argument.

5.6 Fractional Causal Differential Equations

We begin with some definitions. Let $t_0 \ge 0$ and $T > t_0$ be arbitrary and let $E = C[[t_0,T],\mathbb{R}^n]$ be a function space. The map $Q : E \to E$ is said to be a causal or a nonanticipative map if $x,y \in E$ have the property that if $x(s) = y(s),\ t_0 \le s \le t$, then $(Qx)(s) = (Qy)(s),\ t_0 \le s \le t,\ t < T$. Next, we give the definition of and relation between the Riemann-Liouville and Caputo fractional differential equations. The Riemann-Liouville fractional differential equation is given by

$$D^q x = (Qx)(t),\ x(t_0) = x^0 = x(t)(t-t_0)^{1-q}\,|_{t=t_0},\ t_0 \le t < T, \tag{5.51}$$

where $0 < q < 1$ and $\Gamma(q)$ is the standard gamma function. The corresponding Volterra fractional integral equation is given by

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)}\int_{t_0}^t (t-s)^{q-1}(Qx)(s)\,ds,$$

where $x^0(t) = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$. The Caputo derivative has the main advantage that the initial condition of the corresponding initial-value problem has the same form as that of ordinary differential equations, and also the derivative of a constant is zero. Hence, it is convenient to use the Caputo fractional derivative.

The fractional differential equation of Caputo type is given by

$$\left.\begin{aligned} {}^cD^q x &= (Qx)(s) \\ x(t_0) &= x_0 \end{aligned}\right\} \tag{5.52}$$

where $0 < q < 1$. If $x \in C^q([t_0, t_0 + a], \mathbb{R}^n)$ satisfies (5.52), it also satisfies the Volterra fractional integral

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} (Qx)(s)\, ds, \tag{5.53}$$

and vice versa.

The relation between the two types of fractional derivatives is given by

$$ {}^cD^q x(t) = D^q(x(t) - x(t_0)).$$

Next, we state some results that are needed to prove our main theorems. These results are stated for fractional differential equations of Riemann-Liouville type, but they can be readily extended to those of Caputo type. Let $p = 1 - q$ and $C_p([t_0, T], \mathbb{R}) = \{u : u \in C((t_0, T], \mathbb{R})$ and $(t - t_0)^p u(t) \in C([t_0, T], \mathbb{R})\}$. Consider the initial-value problem (IVP)

$$D^q x = f(t, x), \quad x(t_0) = x^0 = x(t)\,(t - t_0)^{1-q}\,|_{t-t_0} \tag{5.54}$$

where $f \in C(R_0, \mathbb{R}^n)$, $R_0 = \{(t, x) :\ t_0 \le t \le t_0 + a$ and $|\ x - x^0(t)\ | \le b\}$, and $x^0(t) = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$.

Lemma 5.6.1. Let $m \in C_p([t_0, T], \mathbb{R})$ be locally Hölder continuous with exponent $\lambda > q$, and for any $t_1 \in (t_0, T]$,

$$m(t_1) = 0 \text{ and } m(t) \le 0 \text{ for } t_0 \le t \le t_1.$$

Then,

$$D^q m(t_1) \ge 0. \tag{5.55}$$

Lemma 5.6.2. Let $\{x_\varepsilon(t)\}$ be a family of continuous functions on $[t_0, T]$, for $\varepsilon > 0$, such that

$$\begin{aligned} D^q x_\varepsilon(t) &= f(t, x_\varepsilon(t)) \\ x_\varepsilon^0 &= x_\varepsilon(t)(t - t_0)^{1-q}\,|_{t=t_0}, \end{aligned}$$

and $|f(t,x_\varepsilon(t))| \leq M$ for $t_0 \leq t \leq T$. Then the family $\{x_\varepsilon(t)\}$ is equicontinuous on $[t_0,T]$.

Theorem 5.6.1. Assume that $m \in C_p([t_0,T],\ \mathbb{R}_+)$ is locally ̋continuous, $g \in C([t_0,T] \times \mathbb{R}_+, \mathbb{R})$ and

$$D^q m(t) \leq g(t,m(t)),\ t_0 \leq t \leq T.$$

Let $r(t)$ be the maximal solution of the IVP

$$D^q u(t) = g(t,u(t)),\ u(t)(t-t_0)^{1-q}\,|_{t=t_0} = u^0 \geq 0, \tag{5.56}$$

existing on $[t_0,T]$, such that $m^0 \leq u^0$, where $m^0 = m(t)(t-t_0)^{1-q}\,|_{t=t_0}$. Then, we have

$$m(t) \leq r(t),\ t_0 \leq t \leq T.$$

Lemma 5.6.3. Assume that $f \in C[\Omega,\mathbb{R}]$, where Ω is an open set in $\mathbb{R}^2$, $(t_0,x^0) \in \Omega$, with $x^0 = x(t)(t-t_0)^{1-q}\,|_{t=t_0}$. Suppose that $[t_0,t_0+a)$ is the largest interval of existence of the maximal solution $r(t)$ of the fractional differential equation (5.56). Assume that $[t_0,t_1]$ is a compact interval of $[t_0,t_0+a)$. Then, there is an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the maximal solution $r(t,\varepsilon)$ of

$$D^q x = f(t,x) + \varepsilon \text{ with initial value } x^0 + \varepsilon, \tag{5.57}$$

where $x^0 = x(t)(t-t_0)^{1-q}\,|_{t=t_0}$, exists on $[t_0,t_1]$, and $\lim\limits_{\varepsilon \to 0} r(t,\varepsilon) = r(t)$, uniformly on $[t_0,t_1]$.
We begin with the theory of fractional differential inequalities.

Theorem 5.6.2. Let $\alpha,\beta \in C^q(J,\mathbb{R}]$ be ̋continuous with exponent $\lambda > q$, such that

$$^cD^q\alpha(t) \leq (Q\alpha)(t), \tag{5.58}$$

$$^cD^q\beta(t) \geq (Q\beta)(t), \tag{5.59}$$

with one of the inequalities (5.58) or (5.59) being strict and $\alpha(t_0) < \beta(t_0)$. Then $\alpha(t) < \beta(t),\ t \in J$.

Proof. Suppose the conclusion does not hold. Then there exists a $t_1 > t_0$ such that $\alpha(t_1) = \beta(t_1)$ and $\alpha(t) < \beta(t),\ t_0 \leq t < t_1$. Now set $m(t) = \alpha(t) - \beta(t)$. Then $m(t_1) = 0$ and $m(t) < 0,\ t_0 \leq t < t_1$. Now, observe that $^cD^q m(t) = D^q[m(t) - m(t_0)]$, where $D^q m(t)$ is the Riemann-Liouville fractional derivative and also that $m(t_0) < 0$ implies $-D^q m(t_0) > 0$. Thus, by Lemma 5.6.3, we have $^cD^q m(t_1) \geq D^q m(t_1) \geq 0$. This yields

$$(Q\alpha)(t_1) \geq {}^cD^q\alpha(t_1) \geq {}^cD^q\beta(t_1) > (Q\beta)(t_1),$$

a contradiction. Here, we have used (5.59) with a strict inequality. The contradiction validates the conclusion and the proof is complete.

Having proved the basic result for strict differential inequalities, we are now in a position to prove it for nonstrict inequalities.

Theorem 5.6.3. Assume that the hypothesis of Theorem 5.6.2 holds with nonstrict inequalities. Further, assume that

$$(Qx)(t)-(Qy)(t)\le L\max_{t_0\le s\le t}\mid x(s)-y(s)\mid \text{ for } x\ge y.$$

Then, $\alpha(t)\le\beta(t)$ on J, provided $\alpha(t_0)\le\beta(t_0)$.

Proof. Set $\beta_\varepsilon(t)=\beta(t)+\varepsilon E_q(2L(t-t_0)^q)$. Then, $\beta_\varepsilon(t_0)=\beta(t_0)+\varepsilon>\alpha(t_0)$. Further,

$$\begin{aligned}{}^cD^q\beta_\varepsilon(t) &= {}^cD^q\beta(t)+2L\varepsilon E_q(2L(t-t_0)^q)\\ &\ge (Q\beta)(t)+2L\varepsilon E_q(2L(t-t_0)^q)\\ &\ge (Q\beta)_\varepsilon)(t)+L\varepsilon E_q(2L(t-t_0)^q),\end{aligned}$$

which gives

$${}^cD^q\beta_\varepsilon(t)>(Q\beta_\varepsilon)(t). \tag{5.60}$$

Now, applying Theorem 5.6.1 to (5.58) and (5.60), we obtain that $\alpha(t)<\beta_\varepsilon(t)$. Taking the limit as $\varepsilon\to 0$, we arrive at $\alpha(t)\le\beta(t)$, and the conclusion holds.

Next, we shall prove a general uniqueness theorem using successive approximations.

Theorem 5.6.4. Assume that

(1) $Q\in C[B,E]$ is a causal map where $B=B(x_0,b)=\{x\in E:\ \max_J\mid x(t)-x_0\mid\le b\}$, $J=[t_0,T]$ and $\mid(Qx)\mid\le M_0$ on B;

(2) $g\in C(J\times[0,2b],\mathbb{R}_+)$, $g(t,u)\le M_1$ on $J\times[0,2b]$, $g(t,0)\equiv 0$, $g(t,u)$ is nondecreasing in u for each $t\in J$, and $u(t)\equiv 0$ is the only solution of

$${}^cD^q u \;=\; g(t,u),\; u(t_0)=0 \text{ on } J; \tag{5.61}$$

and

(3) $\mid(Qx)(t)-(Qy)(t)\mid\le g(t,\mid x-y\mid_0(t))$ on B, where $\mid x-y\mid_0(t)=\max_{t_0\le s\le t}\mid x(t)-y(t)\mid$.

Then, the successive approximations defined by

$$x_{n+1}(t)=x_0+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}(Qx_n)(s)ds,\; n=0,1,2,..... \tag{5.62}$$

exist and are continuous on $I_0=[t_0,t_0+\alpha]$, with $\alpha=\min(T-t_0,(\frac{b\Gamma(1+q)}{M})^{\frac{1}{q}})$ and $M=\max\{M_0,M_1\}$, and converge uniformly to the unique solution $x(t)$ of (5.52).

Proof. By our choice of α, we have, for $t\in I_0$,

$$\begin{aligned}\mid x_1(t)-x_0\mid &\le \tfrac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\mid f(s,x_0)\mid ds.\\ &\le \tfrac{M(t-t_0)^q}{\Gamma(q+1)}\le\tfrac{M\alpha^q}{\Gamma(1+q)}\le b.\end{aligned}$$

Hence, using induction, one can show that the successive approximations are continuous and satisfy

$$| x_n(t) - x_0 | \leq b, \; n = 0,1,2,\dots \,. \tag{5.63}$$

Next, we shall define the successive approximations for the IVP (5.60) as follows:

$$\begin{aligned} u_0(t) &= \frac{M(t-t_0)^q}{\Gamma(1+q)}, \\ u_{n+1}(t) &= \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, u_n(s)) ds \;\; \text{on } I_0. \end{aligned} \tag{5.64}$$

Since $g(t,u)$ is assumed to be nondecreasing in u for each t, using induction we can show that the successive approximations (5.64) are well defined and satisfy

$$0 \leq u_{n+1}(t) \leq u_n(t) \text{ on } I_0.$$

Moreover, $| D^q u_{n+1}(t) | = g(t, u_{n-1}(t)) \leq M$, and equicontinuity follows from Lemma 5.6.3. Thus, using Ascoli-Arzela theorem and the monotonicity of the sequence $\{u_n(t)\}$, we obtain $\lim_{n\to\infty} u_n(t) = u(t)$, uniformly on I_0. Clearly, $u(t)$ satisfies (5.60). Hence, by assumption (b), $u(t) \equiv 0$ on $[t_0, t_0+\alpha] = I_0$.

We first note that $| x_1(t) - x_0 | \leq M\frac{(t-t_0)^q}{\Gamma(1+q)} \equiv u_0(t)$, which gives $| x_1 - x_0) |_0 (t) \leq u_0(t)$. Then, assuming $| x_k - x_{k-1} |_0 (t) \leq u_{k-1}(t)$ for some k, we have

$$| x_{k+1}(t) - x_k(t) | \;\; \leq \;\; \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} \, | (Qx_k)(s) - (Qx_{k-1}(s) | \, ds.$$

Using condition (c) and the monotone character of $g(t,u)$ in u, we get

$$\begin{aligned} | x_{k+1}(t) - x_k(t) | &\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} g(s, | x_k - x_{k-1} |_0) ds \\ &\equiv u_k(t). \end{aligned}$$

Hence, $| x_{k+1} - x_k |_0 (t) \leq u_k(t)$. Thus, by induction, the inequality $| x_{n+1} - x_n |_0 (t) \leq u_n(t)$ on I_0 holds for all n. Also,

$$\begin{aligned} | \; {}^cD^q x_{n+1}(t) - {}^cD^q x_n(t) \; | &= | \, (Qx_n)(t) - (Qx_{n-1})(t) \\ &\leq g(t, | x_n - x_{n-1} |_0 (t)) \leq g(t, u_n(t)). \end{aligned}$$

Let $n \leq m$. Then,

$$\begin{aligned} {}^cD^{+q} \, | x_n(t) - x_m(t) | &\leq | \; {}^cD^q x_n(t) - {}^cD^q x_m(t) \; | \\ &\leq g(t, u_{n-1}(t)) + g(t, u_{m-1}(t)) + g(t, | x_n - x_m |_0 (t)). \end{aligned}$$

Since $u_{n+1}(t) \leq u_n(t)$ for all n, it follows that

$${}^cD^{+q} \, | x_n(t) - x_m(t) | \;\; \leq \;\; g(t, | x_n - x_m |_0 (t)) + 2(g(t, u_{n-1}(t))),$$

where ${}^cD^{+q}$ is the Caputo Dini derivative corresponding to D^+. An application of Theorem 5.6.1 (adjusted to the case of Caputo derivative) gives $|\, x_n - x_m\,|_0\,(t) \le \gamma_n(t)$, on I_0, where $\gamma_n(t)$ is the maximal solution of the IVP

$$ {}^cD^q v = g(t,v) + 2g(t,u_{n-1}(t)),\ v(t_0) = 0 \text{ for each } n. $$

Since, as $n \to \infty$, $g(t,u_{n-1}(t)) \to 0$ uniformly on I_0, using Lemma 5.6.3, we can conclude that $\gamma_n(t) \to 0$, uniformly on I_0. This implies that $\{x_n(t)\}$ converges uniformly to $x(t)$. Now, using the Volterra fractional integral equation (5.53), we can conclude that $x(t)$ is a solution of the IVP (5.52).

To show that the solution $x(t)$ is unique, suppose $y(t)$ is another solution of the IVP (5.52) on I_0. Define $m(t) = |\, x(t) - y(t)\,|$. Then, $m(t_0) = 0$ and, by condition (c),

$$ {}^cD^{+q}m(t) \ \le |\ {}^cD^q x(t) - {}^cD^q m(t)\ | \ \le |\ (Qx)(t) - (Qy)(t)\ | \ \le\ g(t, |\, m\,|_0\,(t)). $$

Again, by Theorem 5.5.3, $m(t) \le r(t,t_0,0)$ on I_0, where $r(t)$ is the maximal solution of (5.61). But by assumption (c), $r(t) \equiv 0$. Hence, uniqueness follows and the proof is done.

Next, assuming local existence, we prove a global existence result.

Theorem 5.6.5. Let $Q \in C[E,E]$ be a causal map such that

$$ |\ (Qx)(t)\ | \le g(t, |\, x\,|_0\,(t)), \tag{5.65} $$

where $g \in C[\mathbb{R}_+^2, \mathbb{R}_+]$, $g(t,u)$ is nondecreasing in u for each $t \in \mathbb{R}_+$, and the maximal solution $r(t) = r(t,t_0,u_0)$ of the IVP

$$ {}^cD^q u = g(t,u),\ u(t_0) = u_0 \ge 0 \tag{5.66} $$

exists on $[t_0,\infty)$. Suppose Q is such that the local existence of solutions of (5.61) is guaranteed for any $(t_0,x_0) \in \mathbb{R}_+ \times B$. Then, the largest interval of existence of any solution $x(t,t_0,x_0)$ of (5.52) is $[t_0,\infty)$, whenever $|\, x_0\,| \le u_0$.

Proof. Suppose that $x(t) = x(t,t_0,x_0)$ is any solution of (5.52) existing on $[t_0,\beta)$, $t_0 < \beta < \infty$, with $|\, x_0\,| \le u_0$, and that the value of β cannot be increased. Define $m(t) = |\, x(t)\,|$. Then, it follows that

$$ {}^cD^q m(t) \le |{}^cD^q x(t)| = |(Qx)(t)| \le g(t, |x|_0(t)) = g(t, |m|_0(t)), $$

and,using Theorem 5.6.4,we can conclude that $m(t) \le r(t),\ \ t_0 \le t \le \beta$.

Also we have

$$ \begin{aligned} |{}^cD^g x(t) &= |(Qx)(t)| \\ &\le g(t, |x|_0(t)) \\ &\le g(t, |m|_0(t) \\ &\le g(t, r(t)) \\ &\le M \quad , \quad t_0 \le t \le \beta, \end{aligned} $$

since $g(t,u) \geq 0$ and $r(t,t_0,u_0)$ is non decreasing. Now, for any t_1,t_2 such that $t_0 < t_1 < t_2 < \beta$, we have

$$\begin{aligned}
|\, x(t_1) - x(t_2) \,| &= \left| \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} (t_1-s)^{q-1}(Qx)(s)ds - \frac{1}{\Gamma(q)} \int_{t_0}^{t_2} (t_2-s)^{q-1}(Qx)(s)ds \right| \\
&\leq \frac{1}{\Gamma(q)} \int_{t_0}^{t_1} |\, (t_1-s)^{q-1} - (t_2-s)^{q-1} \,| \ \ |\, (Qx)(s) \,|\, ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1} \ \ |\, (Qx)(s) \,|\, ds \\
&\leq \frac{M}{\Gamma(q)} \left[\int_{t_0}^{t_1} (t_1-s)^{q-1}ds - \int_{t_0}^{t_1} (t_2-s)^{q-1}ds + \int_{t_1}^{t_2} (t_2-s)^{q-1}ds \right] \\
&= \frac{2M(t_2-t_1)^q}{\Gamma(1+q)}.
\end{aligned}$$

Letting $t_1,t_2 \to \beta^-$ and using Cauchy criterion, we have that $\lim\limits_{t\to\beta^-} x(t,t_0,x_0)$ exists. Set

$$x(\beta,t_0,x_0) = \lim_{t\to\beta^-} x(t,t_0,x_0),$$

and consider the IVP

$${}^cD^q x = (Qx)(t), \ \ x(\beta) = x(\beta,t_0,x_0).$$

The solution $x(t,t_0,x_0)$ can be continued beyond β because of our assumption of local existence. Hence, the claim is true and the proof is complete.

5.7 Causal Differential Equations with Memory

This and the following two sections, we shall devote to extend the theory of causal differential equations with memory. As we shall see, this extension is natural and with appropriate modification, one can generalize almost all the results that we have discussed for the causal differential equations so far. However, we shall be content with providing some typical results and then investigate causal differential equations adding anticipation in addition to memory. Since the theory of functional differential equations is well studied subject, we begin from there so that similar framework can be utilized to discuss the causal differential equations with retardation and anticipation.

For causal operators, we can also utilize a notion identical to that encountered in the general functional differential equation of the type

$$x'(t) = f(t,x_t), \ x_{t_0} = \phi_0, \tag{5.67}$$

where $\phi_0 \in \mathscr{C}_0$ is the initial function, $\mathscr{C}_0 = C[[-\tau, 0], \mathbb{R}^n]$ and $f \in C[[t_0, T] \times \mathscr{C}_0,\ \mathbb{R}^n]$. The symbol x_t in (5.52) may be defined in two ways. For example, if $x \in C[[t_0 - \tau, T),\ \mathbb{R}^n],\ \tau > 0$, then for each $t \in [t_0, T)$

(i) x_t is the graph of x on $[t - \tau, t]$ shifted to the interval $[-\tau, 0]$;

(ii) x_t is the graph of x on $[t_0 - \tau, t]$.

The IVP (5.67) relates to the case (i) and the corresponding IVPs for functional differential equations is well studied area. Such equations are also called differential equations with delay or with retardation. In case (ii), the functional is known as Volterra operator, which is determined by t and the values of $x(s)$ on $t_0 - \tau \le s \le t$. Consider, for example, $Q(x)(t) \equiv Q(t, x_t)$, where for each $t \in [t_0, T)$, Q takes values in $\mathbb{R}^n$, whereas, the family $Q(t, x_t)$ of functionals for $t \in [t_0, T]$, defines the operator from $\mathscr{C}$ into itself. As an example, consider the operator given by the formula

$$Q(x)(t) = f(t) + \int_{t_0}^{t} K(t, s, x(s))ds,$$

then $$Q(t, x_t) = f(t) + \int_{t_0}^{t} K(t, s, x(s))ds.$$

One should note that $Q(t_0, x_{t_0}) = x_0 = f(t_0)$ is the initial value in this setup. This framework can also be employed to include the delay or the past history into the causal operators so that one can extend the study of differential equations involving causal operators with delay or past history. This leads to the IVP for such systems

In this framework, we have the IVP for such systems

$$\begin{aligned} x'(t) &= Q(t, x(t), x_t), \\ x_{t_0} &= \phi_0 \in \mathscr{C}_0 \end{aligned} \tag{5.68}$$

where Q takes values in $\mathbb{R}^n$ for each $t \in (t_0, T)$ and the family of functionals $\{Q_t = Q(t, x(t), x_t)\}_{t \in [t_0, T]}$ defines the operator from $E = C([t_0 - \tau, T], R^n)$ into itself. The investigation of the IVP (5.68) includes naturally not only IVPs with causal operators without delay but also several corresponding IVPs with delay. The examples may include the special cases

$$\begin{aligned} Q(t, x(t), x_t) &= f(t, x_t), \\ or &= f(t, x(t), x_t), \\ or &= \int_{t_0 - \tau}^{t} K(t, s, x(s))ds, \\ or &= f(t, x(t), x_t) + \int_{t_0 - \tau}^{t} K(t, s, x(s))ds, \\ or &= f(t, x(t), x_t), \int_{t_0 - \tau}^{t} K(t, s, x(s))ds). \end{aligned}$$

All of these, of course, need an initial function $\phi_0 \in C_0$. Moreover, the symbol x_t is now Volterra type, that is case (ii). Of course, instead of case (ii), one can also utilize case (i) for the symbol x_t, in which case, we need to replace $\int_{t_0-\tau}^{t} K(t,s,x(s))ds$ by $\int_{t-\tau}^{t} K(t,s,x(s))ds$ in the special cases. We prefer to use case (ii) because we are interested in the investigation of IVPs involving Volterra or causal operators. Once we have the results in the framework of causal operators with delay, parallel results can be obtained similarly employing suitable modification relative to the case (i). We should also note that, once the initial function ϕ_0 is fixed for the IVP (5.67), all the elements of the space E will have the same ϕ_0 on their tail end, namely, on $[t_0-\tau,t_0]$.

In this section, we shall consider the basic differential inequalities involving causal operators with memory or delay.

Theorem 5.7.1. Assume that

(A_1) for each $t \in (t_0,T)$, Q takes values in $\mathbb{R}$ and the family of causal operators $Q_t = Q(t,x(t),x_t)$, for $t \in [t_0,T)$ defines the map from $E = C[[t_0-\tau,T),\mathbb{R}]$ into itself; $Q(t,x(t),x_t)$ is semi nondecreasing in x_t, that is, if $x(t) = y(t)$ and $x(s) \le y(s)$, $t_0-\tau \le s < t$ for $t \in [t_0,T)$, then $Q(t,x(t),x_t) \le Q(t,y(t),y_t)$;

(A_2) $v,w \in C([t_0-\tau,T),\mathbb{R})$, v',w' exist for $t \in [t_0,T)$ and $v'(t) \le Q(t,v(t),v_t)$, $w'(t) \ge Q(t,w(t),w_t)$, $\quad t \in [t_0,T)$;

(A_3) $Q(t,x(t),x_t) - Q(t,y(t),y_t) \le \lambda(t) \max_{t_0-\tau\le s\le t}[x(s)-y(s)]$,

whenever $x_t \ge y_t$ and $\lambda(t) > 0$ is continuous for $t_0 \le t < T$.

Then we have

$$v(t) \le w(t),\ t \in [t_0,T) \tag{5.69}$$

provided that $v_{t_0} \le \phi_{t_0}$ on $[t_0-\tau,t_0]$.

Proof. Let us first consider the result for strict inequalities supposing any one of the inequalities in (A_2) is strict. Suppose that the corresponding conclusion $v(t) < w(t)$, $t \in [t_0,T)$ is false. Then there would exist a t_1 such that $t_0 < t_1 < T$ satisfying

$$v(t_1) = w(t_1),\ v(s) < w(s),\ t_0-\tau \le s < t_1, \tag{5.70}$$

because of the corresponding assumption $v_{t_0} < w_{t_0}$ on $[t_0-\tau,t_0]$. Using (5.70) we get, for small $h < 0$,

$$v(t_1+h) - v(t_1) < w(t_1+h) - w(t_1),$$

and therefore, it follows that

$$v'(t_1) \ge w'(t_1). \tag{5.71}$$

In view of (A_2), because of semimonotone property of Q, relations (5.70) and (5.71) we arrive at

$$Q(t_1,v(t_1),v_{t_1}) \geq v'(t_1) \geq w'(t_1) > Q(t_1,w(t_1),w_{t_1}) \geq Q(t_1,w(t_1),v_{t_1}),$$

which is a contradiction. Hence, $v(t) < w(t),\ t_0 \leq t < T$ is valid. Note that we have utilized only strict inequality for w in (A_2)

To prove the claim of Theorem 5.7.1, we let for $t_0 - \tau \leq t_0 \leq t < T$,

$$\tilde{\phi}(t) = \phi(t) + \varepsilon e^{2L(t)},$$

where $L(t) = \int_{t_0}^{t} \lambda(s)ds$ and $L(t) = 1$ on $[t_0 - \tau, t_0]$ and $\varepsilon > 0$ is small enough. Clearly $v_{t_0} \leq w_{t_0} < \tilde{w}_{t_0}$ and using (A_2),

$$\tilde{w}'(t) = w'(t) + 2\varepsilon\lambda(t)e^{2L(t)} \geq Q(t,w(t),w_t) + 2\varepsilon\lambda(t)e^{2L(t)}.$$

We shall next use (A_3), observing that $\tilde{w}_t > w_t$, because of the fact

$$\tilde{w}_t = w_t + \varepsilon e^{2L_t} \text{ and } e^{2L_t} \leq e^{2L(t)},$$
$$\tilde{w}'(t) \geq Q(t,\tilde{w}(t),\tilde{w}_t) + 2\varepsilon\lambda(t)e^{2L(t)} - \varepsilon\lambda(t)e^{2L(t)} > Q(t,\tilde{w}(t),\tilde{w}_t).$$

Now from the proof of the first part, it follows that $v(t) < \tilde{w}(t), t_0 \leq t < T$ and letting $\varepsilon \to 0$, we have the result $v(t) \leq w(t),\ t_0 \leq t < T$.

Remark 5.7.1. We note that instead of demanding the existence of v', w', we could only require $D_-v(t),\ D_-w(t)$ in (A_2) for the inequalities to hold. The proof requires only this. Here $D_-v(t) = \liminf_{h\to 0^-} \frac{1}{h}[v(t+h) - v(t)]$.

The following lemma is needed before we proceed further.

Lemma 5.7.1. Let $m \in [[t_0 - \tau, T], \mathbb{R}]$, and satisfy the inequality

$$D_-m(t) \leq g(t,|m_t|_0),\ |m_t|_0 = \max_{t_0-\tau\leq s\leq t<T} |m(s)|, \tag{5.72}$$

where $g \in C([t_0,T) \times \mathbb{R}_+, \mathbb{R}_+)$. Assume that the maximal solution $r(t) = r(t,t_0,u_0)$ of the scalar differential equation

$$u' = g(t,u),\ u(t_0) = u_0 \geq 0, \tag{5.73}$$

exists on $[t_0,T)$. Then, if $|m_{t_0}|_0 \leq u_0$, we have

$$m(t) \leq r(t),\ t \in [t_0,T). \tag{5.74}$$

To prove (5.74), it is enough to prove that

$$m(t) < u(t,t_0,u_0,\varepsilon),\ t \in [t_0,T), \tag{5.75}$$

where $u(t,t_0,u_0,\varepsilon)$ is any solution of

$$u' = g(t,u) + \varepsilon,\ u(t_0) = u_0 + \varepsilon,$$

$\varepsilon > 0$ being an arbitrary small quantity, since

$$\lim_{\varepsilon\to 0^+} u(t,t_0,u_0,\varepsilon) = r(t,t_0,u_0).$$

Since $|m_{t_0}|_0 < u_0+\varepsilon$, if (5.75) is not true, there would exist a $t_1,\ t_0 < t_1 < T$, such that

$$m(t_1) = u(t_1,t_0,u_0,\varepsilon),\ m(t) < u(t_1,t_0,u_0,\varepsilon),\ t_0 \le t < t_1.$$

Hence,

$$D_-m(t_1) \ge u'(t_1,t_0,u_0,\varepsilon) = g(t_1,u(t_1,t_0,u_0,\varepsilon))+\varepsilon. \tag{5.76}$$

Since $g(t,u) \ge 0,\ u(t_1,t_0,u_0,\varepsilon)$ is nondecreasing in t and this implies from the preceding considerations, that

$$|m_{t_1}|_0 = u(t_1,t_0,u_0,\varepsilon) = m(t_1).$$

Thus we are led to the inequality

$$D_-m(t_1) \le g(t_1,|m_{t_1}|_0) = g(t_1,u(t_1,t_0,u_0,\varepsilon)),$$

which is incompatible with (5.76). Thus the lemma follows.
Consider the IVP

$$x'(t) = Q(t,x(t),x_t),\ x_{t_0} = \phi_0, \tag{5.77}$$

where $\phi_0 \in C([t_0-\tau,t_0],\mathbb{R}^n)$. The operator Q takes values in $\mathbb{R}^n$ for each $t \in [t_0,t_0+a]$, and the family of causal operators $\{Q_t = Q(t,x(t),x_t)\}$ for $t \in [t_0,t_0+a]$ defines the map from $E = C([t_0-\tau,t_0+a],\mathbb{R}^n)$ into itself. Let us define $y \in C([t_0-\tau,t_0+a],\mathbb{R}^n)$ such that

$$y(t) = \left[\begin{array}{l}\phi_0(t),\ t_0-\tau \le t \le t_0,\\ \phi_0(t_0),\ t_0 \le t \le t_0+a,\end{array}\right.$$

and for $x \in E,\ |x|_0 = \max\limits_{t_0-\tau\le s\le t} |x(s)|$.
The following local existence result will now be proved.

Theorem 5.7.2. Let the map Q be as defined above. Suppose that Q is continuous and $|Q(t,x(t),x_t)| \le M$ on $R_0 = [[t_0,t_0+a]$ and $|x_t - y_t|_0 \le b]$. Then there exists a solution $x(t,t_0,\phi_0)$ of (5.77) on $t_0-\tau \le t \le t_0+\alpha$, where $\alpha = \min\left(a,\frac{b}{M}\right)$.

Proof. Let B be the Banach space $C([t_0-\tau,t_0+\alpha],\mathbb{R}^n)$ with norm as defined earlier. Let $S \subset B$ be defined by

$$S = \left[\begin{array}{l} x \in B : (i)\ x(s) = \phi_0(s),\ t_0-\tau \le s \le t_0;\\ \qquad (ii)\ |x(t_1)-x(t_2)| \le M|t_1-t_2|,\ t_1,t_2 \in [t_0,t_0+\alpha]\end{array}\right].$$

Since the members of S are uniformly bounded and equicontinuous on $[t_0-\tau,t_0+\alpha]$, the compactness of S follows. A straight forward computation shows that S is convex. Let us now define a mapping on S as follows. For any element $x \in S$, we let

(i) $T(x_{t_0}) = \phi_0$;

(ii) $T(x(t)) = \phi_0(t_0) + \int\limits_{t_0}^{t} Q(s, x(s), x_s)ds,\ t_0 \le t \le t_0 + \alpha.$

For every $x \in S$ and $t \in [t_0, t_0 + \alpha]$,

$$|x(t) - \phi_0(t_0)| \le M|t - t_0| \le M\alpha \le b,$$

and this yields that $|x_t - y_t|_0 \le b$. It therefore follows that

$$|T(x(t)) - \phi_0(t_0)| \le \int_{t_0}^{t} |Q(s, x(s), x_s)|ds \le M(t - t_0) \le M\alpha \le b,$$

which implies T is well defined on S, T maps S into itself and T is continuous. An application of Schauder's fixed point theorem shows the existence of at least one $x \in S$ such that

(i) $T(x_{t_0}) = x_{t_0}$;

(ii) $T(x(t)) = x(t), t_0 \le t \le t_0 + \alpha,$

which implies that

(i) $x_{t_0} = \phi_0,$

(ii) $x(t) = \phi_0(t_0) + \int\limits_{t_0}^{t} Q(s, x(s), x_s)ds,\ t_0 \le t \le t_0 + \alpha.$

Since $x \in S$, it follows that this $x \in S$ is also a solution of

$$x'(t) = Q(t, x(t), x_t),\ t_0 \le t \le t_0 + \alpha.$$

The proof is complete.

We shall next prove a global existence result.

Theorem 5.7.3. Let, for each $t \in (t_0, \infty)$, Q takes values in $\mathbb{R}^n$ and the family of causal operators $\{Q_t = Q((t, x(t), x_t)\}$, for $t \in [t_0, \infty)$ defines the map from $E = C([t_0 - \tau, \infty), \mathbb{R}^n)$ into itself. Suppose further that

$$|Q(t, x(t), x_t)| \le g(t, |x_t|_0) \tag{5.78}$$

where $g \in C[[t_0, \infty) \times \mathbb{R}_+, \mathbb{R}_+]$ and $g(t, u)$ is nondecreasing in u for each $t \in [t_0, \infty)$. Assume that the maximal solutions $r(t) = r(t, t_0, u_0)$ of the scalar differential equation

$$u' = g(t, u),\ u(t_0) = u_0 \ge 0,$$

exist on $[t_0, \infty]$. Then the largest interval of existence of any solution $x(t, t_0, \phi_0)$ of (5.77) is $[t_0, \infty)$.

Proof. Let $x(t,t_0,\phi_0)$ be any solution of (5.77) existing on some interval $[t_0-\tau,\beta)$, where $t_0<\beta<\infty$. Assume that the value of β cannot be increased. Define, for $t\in[t_0-\tau,\beta)$, $m(t)=|x(t,t_0,\phi_0)|$ so that $m_t=|x_t(t_0,\phi_0)|$. Using the assumption (5.78), it is easy to obtain the inequality

$$D_-m(t)\le g(t,|m_t|_0).$$

Choosing $|m_{t_0}|_0=|\phi_0|_0\le u_0$, we obtain from Lemma 5.6.3,

$$|x(t,t_0,\phi_0|\le r(t,t_0,u_0),\quad t_0\le t<\beta. \tag{5.79}$$

Since $g(t,u)\ge 0$, $r(t,t_0,u_0)$ is nondecreasing in t and therefore, it follows from (5.79) that

$$|x_t(t_0,\phi_0)|_0\le r(t,t_0,u_0),\quad t_0\le t<\beta. \tag{5.80}$$

For any t_1,t_2 such that $t_0<t_1<t_2<\beta$, we get

$$|x(t_2,t_0,\phi_0)-x(t_1,t_0,\phi_0|\le\int_{t_1}^{t_2}g(s,|x_s(t_0,\phi_0)|_0)ds,$$

which, in view of (5.80) and the monotonicity of $g(t,u)$ in u, implies

$$|x(t_2,t_0,\phi_0)-x(t_1,t_0,\phi_0)|\le\int_{t_1}^{t_2}g(s,r(s,t_0,u_0))ds=r(t_2,t_0,u_0)-r(t_1,t_0,u_0).$$

Letting $t_1,t_2\to\beta^-$, the foregoing relation shows that $\lim_{t\to\beta^-}x(t,t_0,\phi_0)$ exists, because of Cauchy's criterion for convergence. We now define $x(\beta,t_0,\phi_0)=\lim_{t\to\beta^-}x(t,t_0,\phi_0)$ and $\psi_0=x_\beta(t_0,\phi_0)$ as the new initial function at $t=\beta$. An application of local existence theorem shows that there exists a solution $x(t,t_0,\psi_0)$ of (5.77) on $[\beta,\beta+\alpha]$, $\alpha>0$. This means that the solution $x(t,t_0,\phi_0)$ can be continued to the right of β, which is contrary to the assumption that the value of β cannot be increased. Hence, the stated result follows.

5.8 Causal Differential Systems with Retardation and Anticipation

We investigate, in this section, the existence theory for functional differential equations with both retardation and anticipation as a first step by splitting the problem into two parts and using existing theory of functional differential equations with delay and the contraction mapping theorem for operators with PPF (past, present, and future) dependence.

The modeling of such equations serve to describe an organizational process. Anticipation represents future potential, not yet realized values of the states of the system, known at the present time. Hence the current evolution process of such systems takes into account past, present and future (PPF) states of dependence.

The investigation of the type of general functional differential systems with PPF dependence, has its own challenges and there may be several approaches to follow and several ways of posing the problem. However, recent practical applications in decision theory, chaotic epidemics and wavelet theory, to name a few, suggest the need for the development of some general theory, at least as a first step. With this motivation, we attempt to initiate the existence theory of systems with PPF dependence.

Let us now formulate the problem concerning the causal differential system involving both retardation and anticipation. Suppose that we are given

$$\left.\begin{aligned} x'(t) = Q(t,x_t,x^t), t \in [t_0,T],\ t_0 \geq 0, \\ x_{t_0} = \phi_0,\ x^T = \psi_0, \phi_0 \in C_1,\ \psi_0 \in C_2, \end{aligned}\right\} \tag{5.81}$$

where $C_1 = C([-h_1,0],R^n)$, $C_2 = C([0,h_2],R^n)$, $h_1,h_2 \geq 0$, $Q \in C([t_0,T] \times C_1 \times C_2,R^n)$, and

$$x_t(s) = x(t+s), -h_1 \leq s \leq 0, \quad x^T(\sigma) = x(T+\sigma), 0 \leq \sigma \leq h_2.$$

The function ϕ_0 is commonly known as delay or retardation or the past information. One may consider $\phi_0(s)$ defined on $t_0 - h_1 \leq s < t_0$ as the past, $\phi_0(t_0)$ as the present and ψ_0 is the potential future that one wishes to reach. From this point of view, we may consider (5.81) as the functional differential system with PPF (past, present and future) dependence. In fact, there exists a contraction mapping theorem where the contractive operator is of PPF dependence Theorem 1.4.4 and we shall be utilizing such a result.

We do not know how to prove directly an existence theorem for the posed problem (5.81). We plan to utilize the known theory of functional differential systems with retardation to build the theory for (5.81) by splitting suitably into two parts, since it is natural to employ, more often than not, existing information for dealing with totally new problems. For this purpose, we find that whenever a desired anticipation is involved, rarely one sits planning nothing and expects that future event happens by itself. Normally, one makes appropriate decisions to realize the desired future event. The decisions made continuously from the present, using past memory, hopefully, will show their effects in the future, but it is not guaranteed that the desired results occur. The desired future event depends, of course, on the past, the present, the dynamics, the desired future and the decisions one makes continuously. But the past and the present cannot be changed, since they have happened already, other factors should help to relate in some form to yield the desired outcome.

In order to solve the existence problem (5.81), we therefore choose a decision function $z \in C([t_0,T],R^n)$ such that $z(t_0) = \phi_0(t_0)$ (although this is not essential) and $z(T) = \psi_0(T)$, that is, $z^t = z(t+\sigma)$, $0 \le \sigma \le h_2$ is defined for $t \in [t_0, T]$ with $z^T = \psi_0$, as the tail end. With this decision function chosen, the system (5.81) becomes

$$\left.\begin{aligned} x'(t) &= Q(t, x_t, z^t) \equiv F(t, x_t) \\ x_{t_0} &= \phi_0, \end{aligned}\right\} \tag{5.82}$$

which is a functional differential system with only retardation. Because of the choice of $z(t)$, the system (5.82) employs the future information from $[t_0, T]$. We state the following known existence and uniqueness result suitably modified for our problem (5.82).

Theorem 5.8.1. Suppose that the functional f in (5.82) satisfies

$$|Q(t, \phi_1, z^t) - Q(t, \phi_2, z^t)| \le L|\phi_1 - \phi_2|_0,$$

where $|\phi_1 - \phi_2|_0 = \max_{-h_1 \le s \le 0} |\phi_1(s) - \phi_2(s)|$, for ϕ_1, $\phi_2 \in C_1$ and $t \in [t_0, T]$. If $0 < \alpha < \frac{1}{2L}$, then there exists a unique solution $x(t_0, \phi_0)\ (t)$ on $t_0 \le t \le t_0 + \alpha$, for every $z(t)$ chosen.
We shall next state a local existence result for (5.82) analogous to Theorem 6.1.1 in Volume II of [4].

Theorem 5.8.2. Let $F \in C([t_0, T] \times C_\rho, R^n)$ where $C_\rho = [\phi \in C_1 : |\phi|_0 < \rho]$. Then for every given initial function $\phi_0 \in C_\rho$ at $t = t_0$, there exists an $\eta > 0$ such that there is a solution $x(t_0, \phi_0)(t)$ of (5.82) existing on $[t_0, t_0 + \eta]$.
We therefore have by Theorem 5.8.2, local existence for a chosen $z(t)$. We shall next consider global existence of solutions of (5.82) on $[t_0, T]$ by utilizing Theorems 1.3.3 and 1.3.4 and Theorem 6.1.2 in Volume II of [4].

Theorem 5.8.3. Assume that the solutions of (5.82) exist locally for any choice of $z(t)$ satisfying

$$|z^t|_0 \le R(t), \quad t \in [t_0, T], \tag{5.83}$$

where $R(t) = R(t, t_0, u_0)$ is the maximal solution of

$$u' = g(t, u, u), \quad u(t_0) = u_0 \ge 0, \tag{5.84}$$

existing on $[t_0, T]$, $g \in C([t_0, T] \times R_+^2, R_+)$ and $g(t, u, v)$ is nondecreasing in (u, v) for $t \in [t_0, T]$. Suppose also that f in (5.82) has the estimate,

$$|Q(t, \phi, z^t)| \le g(t, |\phi(0)|, |z^t|_0), \tag{5.85}$$

for $\phi \in C_1$ such that $|\phi|_0 = |\phi(0)|$, and for any z satisfying (5.83). Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of

$$u' = g(t, u, R(t)), \quad u(t_0) = u_0 \ge 0 \tag{5.86}$$

existing on $[t_0,T]$. Then the largest interval of existence of any solution $x(t_0,\phi_0)(t)$ of (5.82) is $[t_0,T]$.

Proof. Let $|z^t|_0 \le R(t)$ on $[t_0,T]$ and $x(t_0,\phi_0)(t)$ be any solution of (5.82) existing on some interval $[t_0,t_1)$, $t_1 < T$. Suppose that the value of t_1 cannot be increased. Define for $t \in [t_0,t_1)$, $m(t) = |x(t_0,\phi_0)(t)|$ so that $m_t = |x_t(t_0,\phi_0)|$. Using (5.85), it is easy to get the differential inequality

$$D_-m(t) \le g(t,m(t),|z^t|_0).$$

Because of (5.83) and $g(t,u,v)$ is nondecreasing in v, we have

$$D_-m(t) \le g(t,m(t),R(t)), \quad t \in [t_0,T].$$

Choosing $|m_{t_0}| = |\phi_0|_0 \le u_0$, we obtain by Theorem 1.3.4,

$$m(t) \le r(t), \quad t \in [t_0,t_1). \tag{5.87}$$

For any t_2,t_3 such that $t_0 < t_2 < t_3 < t_1$, it follows that

$$|x(t_0,\phi_0)(t_3) - x(t_0,\phi_0)(t_2)| \le \int_{t_2}^{t_3} g(s,|x_s(t_0,\phi_0)|_0,R(s))ds.$$

In view of the monotone nature of $g(t,u,v)$ in u as well, using (5.87) we arrive at

$$|x(t_0,\phi_0)(t_3) - x(t_0,\phi_0)(t_2)| \le \int_{t_2}^{t_3} g(s,r(s),R(s))ds = r(t_3) - r(t_2). \tag{5.88}$$

Here we have used the fact that $r(t)$ is nondecreasing in t because $g(t,u,v) \ge 0$ and hence $m_t \le r(t)$ for $t \in [t_0,t_1)$ because of (5.84). Now letting $t_2,t_3 \to t_1$, we get from the relation (5.88) that $\lim_{t\to t_1^-} x(t_0,\phi_0)(t)$ exists, because of Cauchy criterion for convergence. We now define $x(t_0,\phi_0)(t_1) = \lim_{t\to t_1^-} x(t_0,\phi_0)(t)$ and consider $x_{t_1}(t_0,\phi_0) = \phi_1$ as the new initial function at $t = t_1$. An application of Theorem 5.8.2 yields that there exists a solution $x(t_1,\phi_1)(t)$ of (5.82) on $[t_2,t_1+\alpha_1]$, $\alpha_1 > 0$. This means that the solution $x(t_0,\phi_0)(t)$ can be continued to the right of t_1, which is a contradiction to our assumption that the value of $t_1 < T$ can not be increased. Hence the conclusion of the theorem follows.

Remark. If in Theorem 5.8.3, all the functions involved are assumed to exist on $[t_0,\infty)$, instead of $[t_0,T]$, then it is easy to see that the same proof shows that the solutions $x(t_0,\phi_0)(t)$ exist globally on $[t_0,\infty)$.

We wish to show that the solution $x(t_0,\phi_0)(t)$ of (5.82) which exists on $[t_0,T]$ for a chosen $z(t)$ satisfies the relation

$$x(t_0,\phi_0)(T) = \psi_0(T) = z(T), \tag{5.89}$$

so that we can claim that $x(t_0,\phi_0)(t)$ is the solution of the problem (5.81) for a chosen $z(t)$ on $[t_0,T]$. However, it is not, in general, necessary that (5.89) can happen for any $z(t)$. If, at least, one $z(t)$ exists satisfying (5.89), we have an existence result, because of the decision made, one could realize the future event. To achieve this second part of the problem, we need the following consideration.

Let, for some $\alpha > 0$ and $t \in [t_0, T]$,

$$\Omega = [\phi_1, \phi_2 \in C_1 : \max_{-h_1 \le s \le 0} |\phi_1(s) - \phi_2(s)| e^{\alpha(t+s)} = |\phi_1(0) - \phi_2(0)| e^{\alpha t}].$$

Let $z_1(t), z_2(t)$ be two different decision functions on $[t_0, T]$ as defined earlier, namely

$$z_1(t_0) = z_2(t_0) = \phi_0(t_0),\ z_1(T) = z_2(T) = \psi_0(T)$$

with the same ψ_0 as tail end. Let $x_1(t_0, \phi_0, z_1)(t)$, $x_2(t_0, \phi_0, z_2)(t)$ be the two solutions of (5.82) on $[t_0, T]$ corresponding to $z_1(t)$, $z_2(t)$, with the same initial function $\phi_0 \in C_1$. Suppose that the functional Q in (5.82) satisfies the condition, for ϕ_1, $\phi_2 \in \Omega$.

$$\begin{aligned} 2e^{\alpha t} \langle \phi_1(0) - \phi_2(0), Q(t, \phi_1, z_1^t) - Q(t, \phi_2, z_2^t) \rangle + e^{\alpha t} |\phi_1(0) - \phi_2(0)|^2 \\ \le c_1 e^{\alpha t} |z_1 - z_2|_0^2, \end{aligned} \tag{5.90}$$

where $c_1 > 0$, $\alpha > 0$ and $|z_1 - z_2|_0 = \max_{t_0 \le t \le T + h_2} |z_1(t) - z_2(t)|$.
Then taking $L(t, \phi_1(0) - \phi_2(0)) = |\phi_1(0) - \phi_2(0)|^2\ e^{\alpha t}$ for some $\alpha > 0$ and following the proof of Theorem 8.1.2 in [4], we obtain, for $t \in [t_0, T]$.

$$|x_1(t_0, \phi_1, z_1)(t) - x_2(t_0, \phi_0, z_2)(t)|^2 e^{\alpha t} \le c_1 |z_1 - z_2|_0^2 \int_0^t e^{\alpha s} ds. \tag{5.91}$$

From (5.91), it follows that

$$|x_1(t_0, \phi_0, z_1)(T) - x_2(t_0, \phi_0, z_2)(T)| \le \left(\frac{c_1}{\alpha}\right)^{\frac{1}{2}} |z_1 - z_2|_0. \tag{5.92}$$

We now define the operator S by $Sz = x(t_0, \phi_0, z)(T)$, where $S : E_0 \to E$, $E_0 = C([t_0, T + h_2], R^n)$ and $E = R^n$. Then we see that S satisfies the relation

$$|Sz_1 - Sz_2| \le \left(\frac{c_1}{\alpha}\right)^{\frac{1}{2}} |z_1 - z_2|_0.$$

If $\left(\frac{c_1}{\alpha}\right)^{\frac{1}{2}} < 1$, we can apply the fixed point theorem for PPF dependence (see Theorem 1.4.4) to conclude that there exists a $z \in E_0$ such that $Sz = z(T) = \psi_0(T)$. This implies that

$$x(t_0, \phi_0, z)(T) = \psi_0(T) = z(T). \tag{5.93}$$

The foregoing considerations prove the following result.

Theorem 5.8.4. Assume that the solutions of (5.82) exist and are unique for $[t_0, T]$. Suppose further that f satisfies condition (5.90). If $\left(\frac{c_1}{\alpha}\right)^{\frac{1}{2}} < 1$, then there exists a function $z \in C([t_0, T], R^n)$ such that (5.93) holds.

Example. Consider

$$x'(t) = -ax(t) + bx(t-h) + cx(t+h),$$

where $a,b,c,h > 0$ are constants. If $0 < \frac{\alpha}{2} \leq a - (be^{\alpha h} + c)$ and $\left(\frac{c}{\alpha}\right)^{\frac{1}{2}} < 1$ hold, then the conditions of the Theorem 5.8.4 are satisfied.

Example. An example of a functional differential equation corresponding to the above example is,

$$x'(t) = -ax(t) + b\int_{-h_1}^{0} x_t(s)ds + c\int_{0}^{h_2} x^t(\sigma)d\sigma,$$

where $a,b,c,h_1,h_2 > 0$ are constants. Here we need to choose

$$0 < \frac{\alpha}{2} \leq a - [\frac{b}{\alpha}(e^{\alpha h_1} - 1) + ch_2] \text{ and } \left(\frac{ch_2}{\alpha}\right)^{\frac{1}{2}} < 1$$

to make the hypothesis of Theorem to hold, and the needed computation is somewhat more complicated.

There are many interesting ways one can formulate the problem of PPF dependence (5.81). We shall state some typical formulations.

Consider, for example, extending (5.81) several times, namely, let $T_1 > T + h_2$ and on $[T_1, T_1 + h_3]$ provide a potential future function $\psi_1 \in C[T_1, T_1 + h_3], R^n)$. Here $z(t)$ can be chosen as $z(t) = \psi_0(t)$ on $[T, T + h_2]$ and arbitrary on $[T + h_2, T_1]$ such that $z(T_1) = \psi_1(T_1)$ and try to find one $z(t)$ satisfying

$$z(T_1) = \psi_1(T_1) = x(T, x_T, z(T_1))$$

where $x(T,x_T,z)(t)$ is the solution from (T,x_T), $x_T = (t_0, \phi_0)$. One can repeat the process several times to show that the evolution process meets the anticipated values several times.

Another type of functional differential system with retardation and anticipation could be as follows:

$$\left.\begin{aligned} x'(t) &= Q(t, x_t, x^t) \\ x_{t_0} &= \phi_0 \text{ and } x^{t_0} = \psi_0 \end{aligned}\right\} \tag{5.94}$$

where $\psi_0 \in C([t_0, T], R^n)$ or $\psi_0 \in C([t_0, \infty), R^n)$ such that $\lim_{t\to\infty} \psi_0(t) = 0$ in the latter case. Notice that here the anticipative function, that is, the decision function is given from the start $t = t_0$, either on finite interval $[t_0, T]$ or on infinite interval $[t_0, \infty)$. In this case, there is no need to choose $z(t)$ as before, since ψ_0 takes its place.

Yet another possibility is allowing for some time that the evolution process depends only on the past and the present and then subject the dynamics to utilize the decision function, that is, the problem is the same as (5.81) but $z(t)$ is chosen from $[T-h_2,T]$ such that $z(T-h_2)$ $= x(t_0,\phi_0)\ (T-h_2)$, $z(T)=\psi_0(T)$, with the tail end of ψ_0 as before.

Finally, let us consider the problem (5.94) with ψ_0 defined on the infinite interval $[t_0,\infty)$ and demand that the evolution process $x(t_0,\phi_0,\psi_0)\ (t)$ also has the same property as ψ_0, that is,

$$\lim_{t\to\infty} x(t_0,\phi_0,\psi_0)(t) = 0 = \lim_{t\to\infty}\psi_0(t). \tag{5.95}$$

We shall provide a set of sufficient conditions in terms of Lyapunov-like function to accomplish the conclusion of this problem.

Theorem 5.8.5. Assume that the solutions $x(t_0,\phi_0,\psi_0)(t)$ of (5.94) exist for $t\geq t_0$. Suppose further that

(i) there exists a $V\in C(R_+\times R^n,R_+)$ $V(t,x)$ is positive definite and $V(t,0)\equiv 0$;
(ii) for $\phi\in\Omega_0=[\phi\in C_1 : \max_{-h_1\leq s\leq 0} V(t+s,\phi(s))e^{\alpha(t+s)}$ $= V(t,\phi(0))\ e^{\alpha t}]$ for some $\alpha>0$,

$$D_-V(t,\phi(0))e^{\alpha t}+\alpha V(t,\phi(0))e^{\alpha t}$$

$$\leq g(t,V(t,\phi(0))e^{\alpha t},W(t,|\psi_0^t|_0),$$

where $W\in C[R_+^2,R_+]$, $g(t,u,v)\geq 0$ satisfies the assumptions of Theorem 1.4.4;
(iii) $W(t,|\psi_0^t|_0)\leq R(t)$, $t\geq t_0$, where $\psi_0^t=\psi_0(t+\sigma)$, $0\leq\sigma\leq h_2$;
(iv) the maximal solution $r(t)=r(t,t_0,u_0)$, $u_0\geq 0$, is, in addition, bounded on $[t_0,\infty)$.

Then every solution $x(t_0,\phi_0,\psi_0)\ (t)$ of (5.94) is such that (5.95) holds.

Proof. Let us first observe that the existence of solutions $x(t_0,\phi_0,\psi_0)\ (t)$ of (5.94) for all $t\geq t_0$, can be proved following the Remark after Theorem 5.8.3. Since we have assumed it, let us set

$$m(t)=V(t,x(t_0,\phi_0,\psi_0)(t))e^{\alpha t}$$

and employ the arguments as in Theorem 8.1.2 in [4] to obtain the differential inequality $D_-m(t)\leq g(t,m(t),W(t,|\psi_0^t|_0),\ t\geq t_0$. Then using (iii) and the monotone nature of g, we get

$$D_-m(t)\leq g(t,m(t),R(t)),\ \ t\geq t_0,$$

which yields,

$$m(t)\leq r(t,t_0,u_0),\ \ t\geq t_0,$$

provided $|m_{t_0}|_0 \leq u_0$. This then implies

$$V(t,x(t_0,\phi_0,\psi_0)(t))e^{\alpha t} \leq r(t,t_0,u_0) \leq N,\ t \geq t_0,$$

by assumption (iv), where $N > 0$ is the bound for $r(t,t_0,u_0)$. The positive definiteness of $V(t,x)$ immediately shows that (5.95) is true and the proof is complete.

As an example of (5.94) consider

$$x'(t) = -ax(t) + bx(t-h_1) + c(t)x(t+h_2)$$

$$x_{t_0} = \phi_0\ \ x^{t_0} = \psi_0$$

where $a,b,h_1,h_2 > 0$ are constants and $c(t) \geq 0$ is continuous on $[t_0,\infty)$. For $\phi \in \Omega_0$, with $V(t,x) = |x|$, we can compute the following estimate

$$\limsup_{h\to 0^+} \frac{1}{h}[|\phi(0) + hf(t,\phi,\psi_0)| - |\phi(0)|]e^{\alpha t} + \alpha e^{\alpha t}|\phi(0)|$$

$$\leq c(t)|\psi_0|_0 e^{\alpha t},\ \ t \geq t_0,$$

where $|\psi_0|_0 = \sup_{t_0 \leq t < \infty} |\psi_0(t)|$, choosing $0 < \alpha \leq (a - be^{\alpha h_1})$. We then get

$$|x(t_0,\phi_0,\psi_0)(t)|e^{\alpha t} \leq |\phi_0|_0 + \int_0^t c(s)e^{\alpha s}|\psi_0|_0 ds,\ t \geq t_0.$$

If $c(t)$ is such that $\int_0^\infty c(s)e^{\alpha s}ds \leq N$, we arrive at

$$e^{\alpha t}|x(t_0,\phi_0,\psi_0)(t)| \leq |\phi|_0 + |\psi_0|_0 N,\ \ t \geq t_0,$$

from which, it follows that $\lim_{t\to\infty} |x(t_0,\phi_0,\psi_0)(t) = 0$.

5.9 Monotone Iterative Technique

In this section, we employ the monotone iterative technique relative to the coupled lower and upper solutions to obtain minimal and maximal solutions of coupled type and under suitable conditions of uniqueness. It is then shown that the coupled extremal solutions yield the unique solution of the proposed problem. Since the modeling of such equations with PPF dependence serves to describe many practical problems as observed recently, the development of general theory is timely.

Let us consider the following functional differential equation with retardation and anticipation, given by

$$\begin{cases} x'(t) = Q(t,x(t),x_t,x^t),\ t \in I = [t_0,T], \\ \quad x_{t_0} = \phi_0,\ x^T = \psi_0, \quad t_0 \geq 0,\ t_0 < T, \end{cases} \tag{5.96}$$

where $\mathscr{C}_1 = C([-h_1,0],R)$, $\mathscr{C}_2 = C([0,h_2],R)$, $\phi_0 \in \mathscr{C}_1$, $\psi_0 \in \mathscr{C}_2$ and $f \in C(I \times R \times \mathscr{C}_1 \times \mathscr{C}_2, R)$, $h_1, h_2 > 0$. Here and in what follows, the symbols $x_t = x_t(s) = x(t+s)$, $-h_1 \le s \le 0$, $x^t = x^t(\sigma) = x(t+\sigma)$, $0 \le \sigma \le h_2$, representing retardation and anticipation, respectively. We plan to employ the monotone iterative technique for proving the existence of unique solution for (5.96) utilizing coupled lower and upper solutions of (5.96). Before we proceed further, we need to list the following known results relative to linear functional differential inequalities in a suitable form.

Lemma 5.9.1: Assume that

(i) $p \in C([t_0 - h_1, T + h_2], R)$, p is continuously differentiable on $I = [t_0, T]$ and

$$p'(t) \le -Mp(t) - N \int_{-h_1}^{0} p_t(s)ds, \ t \in I;$$

(ii) $p_{t_0}(s) \le 0, -h_1 \le s \le 0$, $p \in C^1([t_0 - h_1, t_0, R)$, $p'(s) \le \frac{\lambda}{T+h_1}$ where $\min_{[t_0-h_1,t_0]} p(s) = -\lambda$, $\lambda \ge 0$ and $[M + Nh_1]\ (T + h_1) \le 1$.

Then $p(t) \le 0$ on $t_0 \le t \le T$.

Lemma 5.9.2: Suppose that $p \in C([t_0 - h_1, T + h_2], R)$, $p'(s)$ exists and is continuous on I and

$$p'(t) \le -Lp(t) + N_1 \int_{-h_1}^{0} p_t(s)ds + N_2 \int_{0}^{h_2} p^t(\sigma)d\sigma, \ t \in I,$$

where $L, N_1, N_2 > 0$ satisfying $N_1\ h_1 + N_2 h_2 < L$. Then $p_{t_0} \le 0$, $p^T \le 0$ implies $p(t) \le 0$ on I.

Proof. If the conclusion is false, there exists a $t_1 \in (t_0, T)$ and an $\varepsilon > 0$ such that $p(t_1) = \varepsilon$, $p(t) \le \varepsilon$ on I. It then follows that

$$0 = p'(t_0) \le -L\varepsilon + N_1 \varepsilon h_1 + N_2 \varepsilon h_2 < 0,$$

by assumptions proving $p(t) \le 0$ on I.

Let us list the following assumptions relative to (5.96) for convenience.

(i) $\alpha_0, \beta_0 \in C^1(I, R)$ satisfying

$$\alpha_0'(t) \le f(t, \alpha_0(t), \alpha_{0t}, \beta_0^t), \ \alpha_{0t_0} = \phi_1, \ \alpha_0^T = \psi_1,$$

$$\beta_0'(t) \ge f(t, \beta_0(t), \beta_{0t}, \alpha_0^t), \ \beta_{0t_0} = \phi_2, \ \beta_0^T = \psi_2,$$

such that $\phi_1 \le \phi_0 \le \phi_2$, $\psi_1 \le \psi_0 \le \psi_2$, $\alpha_0(t) \le \beta_0(t)$ on I and $\phi_1, \phi_2 \in \mathscr{C}_1$, $\psi_1, \psi_2 \in \mathscr{C}_2$.

(ii) $Q(t,x,\phi,\psi)$ is nonincreasing in ψ for each (t,x,ϕ).

(iii) $Q(t,x,\phi,\xi) - Q(t,y,\psi,\xi) \ge -M(x-y) - N\int_{-h_1}^{0} (\phi - \psi)(s)ds$, for $\alpha_0(t) \le y \le x \le \beta_0(t)$, $\alpha_{0t} \le \psi \le \phi \le \beta_{0t}$, $\xi \in \mathscr{C}_2$ arbitrary and $M, N \ge 0$.

(iv) $\alpha_{0t_0} - \phi_0$, $\phi_0 - \beta_{0t_0}$ satisfying the assumptions (ii) of Lemma 5.9.1.

The type of coupled lower and upper solutions assumed in (i) are utilized profitably. We are now in a position to state and prove our main result.

Theorem 5.9.1: Suppose that assumptions (i) to (iv) are satisfied. Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ such that $\alpha_n(t) \to \rho(t)$, $\beta_n(t) \to r(t)$ uniformly as $n \to \infty$ on $[t_0 - h_1, T + h_2]$ and that (ρ, r) are coupled minimal and maximal solutions of (5.96). If, in addition,

(v) $Q(t,x,\phi_1,\psi_2) - Q(t,y,\phi_2,\psi_1) \le -L(x-y) + N_1 \int_{-h_1}^{0} (\phi_1 - \phi_2)(s)ds + N \int_0^{h_2} (\psi_1 - \psi_2)(\sigma)d\sigma$, where $L, N_1, N_2 > 0$, $\alpha_0(t) \le y \le x \le \beta_0(t)$, $\alpha_{0t} \le \phi_2 \le \phi_1 \le \beta_{0t}$, $\alpha_0^T \le \psi_2 \le \psi_1 \le \beta_0^T$ and $N_1 h_1 + N_2 h_2 < L$, holds, then $\rho(t) = r(t) = x(t)$ is the unique solution of (5.96) on I.

Proof: Consider the following linear problem for each $n = 1, 2, 3, \ldots,,$

$$\left.\begin{aligned} \alpha'_{n+1} &= f(t, \alpha_n, \alpha_{nt}, \beta_n^t) - M(\alpha_{n+1} - \alpha_n) - N \int_{-h_1}^{0} (\alpha_{(n+1),t} - \alpha_{n_t})(s)ds, \\ \beta'_{n+1} &= f(t, \beta_n, \beta_{nt}, \alpha_n^t) - M(\beta_{n+1} - \beta_n) - N \int_{-h_1}^{0} (\beta_{(n+1)t} - \beta_{nt})(s)ds, \end{aligned}\right\} \tag{5.97}$$

with $\alpha_{(n+1)t_0} = \phi_0$, $\beta_{(n+1)t_0} = \phi_0$ and α_{n+1}^T, β_{n+1}^T are chosen such that

$$\alpha_0^T \le \alpha_n^T \le \alpha_{n+1}^T \le \psi_0 \le \beta_{n+1}^T \le \beta_n^T \le \beta_0^T \tag{5.98}$$

and α_n^T, β_n^T converge uniformly to ψ_0 on $[0, h_2]$. (See remark.)

Clearly each linear problem has a unique solution on $[t_0 - h_1, T + h_2]$. We wish to show that

$$\alpha_0 \le \alpha_1 \le \alpha_2 \le \ldots \le \alpha_n \le \beta_n \le \ldots \le \beta_2 \le \beta_1 \le \beta_0 \text{ on } I. \tag{5.99}$$

We claim first that $\alpha_0 \le \alpha_1$ on I. For this purpose, set $p = \alpha_0 - \alpha_1$ so that it follows from (5.97), (5.98) and condition (i),

$$\begin{aligned} p' = \alpha'_0 - \alpha'_1 &\le f(t, \alpha_0, \alpha_{0t}, \beta_0^t) - f(t, \alpha_0, \alpha_{0t}, \beta_0^t) \\ &+ M(\alpha_1 - \alpha_0) + N \int_{-h_1}^{0} (\alpha_{1t} - \alpha_{0t})(s)ds \\ &\le -Mp - N \int_{-h_1}^{0} p_t(s)ds, \ t \in I \end{aligned}$$

and

$$p_{t_0} = \alpha_{0t_0} - \alpha_{1t_0} \le 0.$$

By Lemma 5.9.1, in view of assumption (iv), this implies $\alpha_0 \le \alpha_1$ on I. Similarly, we can show that $\beta_1 \le \beta_0$ on I.

Next we prove that $\alpha_1 \le \beta_1$ on I. Setting $p = \alpha_1 - \beta_1$, we obtain in view of (5.97), for $t \in I$,

$$p' = \alpha_1' - \beta_1' = f(t, \alpha_0, \alpha_{0t}, \beta_0^t) - M(\alpha_1 - \alpha_0) - N\int_{-h_1}^{0} (\alpha_{1t} - \alpha_{0t})(s)ds$$

$$-f(t, \beta_0, \beta_{0t}, \alpha_0^t) + M(\beta_1 - \beta_0) + N\int_{-h_1}^{0} (\beta_{1t} - \beta_{0t})(s)ds.$$

Since $f(t,x,\phi,\psi)$ is nonincreasing in ψ by (ii), $\alpha_0^t \le \beta_0^t$, assumption (iii) yields

$$p' \le M(\beta_0 - \alpha_0) + N\int_{-h_1}^{0} (\beta_{0t} - \alpha_{0t})(s)ds - M(\alpha_1 - \alpha_0) - N\int_{-h_1}^{0} (\alpha_{1t} - \alpha_{0t}(s)ds$$

$$+M(\beta_1 - \beta_0) + N\int_{-h_1}^{0} (\beta_{1t} - \beta_{0t})(s)ds$$

$$= -Mp - N\int_{-h_1}^{0} p_t(s)ds \text{ and } p_{t_0} = 0.$$

Thus we get using Lemma 5.9.1, $p(t) \le 0$ on I. As a result, it follows that

$$\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0 \text{ on } I. \tag{5.100}$$

Now suppose that for some $k > 1$, we have

$$\alpha_{k-1} \le \alpha_k \le \beta_k \le \beta_{k-1} \text{ on } I. \tag{5.101}$$

We shall show that

$$\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k \text{ on } I. \tag{5.102}$$

To do this, let $p = \alpha_k - \alpha_{k+1}$ so that $p_{t_0} = 0$ and

$$\begin{aligned} p' &= \alpha_k' - \alpha_{k+1}' \\ &= Q(t, \alpha_{k-1}, \alpha_{k-1t}, \beta_{k-1}^t) - M(\alpha_k - \alpha_{k-1}) - N\int_{-h_1}^{0} (\alpha_{kt} - \alpha_{k-1t})(s)ds \\ &\quad -Q(t, \alpha_k, \alpha_{kt}, \beta_k^t) + M(\alpha_{k+1} - \alpha_k) + N\int_{-h_1}^{0} (\alpha_{k+1t} - \alpha_{kt})(s)ds. \end{aligned}$$

Using monotone nature of Q and condition (iii), we have

$$p' \le M(\alpha_k - \alpha_{k-1} + N\int_{-h}^{0} (\alpha_{kt} - \alpha_{k-1t})(s)ds - M(\alpha_k - \alpha_{k-1})$$

$$-N\int_{-h_1}^{0} (\alpha_{kt} - \alpha_{k-1t})(s)ds + M(\alpha_{k+1} - \alpha_k) + N\int_{-h_1}^{0} (\alpha_{k+1t} - \alpha_{kt})(s)ds$$

$$= -Mp - N\int_{-h_1}^{0} p_t(s)ds \text{ and } p_{t_0} = 0.$$

This implies by Lemma 5.9.1 that $\alpha_k \leq \alpha_{k+1}$ on I. Similarly, we can show that $\beta_{k+1} \leq \beta_k$ on I. To prove $\alpha_{k+1} \leq \beta_{k+1}$ on I, consider $p = \alpha_{k+1} - \beta_{k+1}$ so that $p_{t_0} = 0$ and arguing as before, one can show that

$$p' \leq -Mp - \int_{-h_1}^{0} p_t(s)ds, \;\; p_{t_0} = 0,$$

which yields $\alpha_{k+1} \leq \beta_{k+1}$, on I. Thus we have (5.102) and therefore by induction, we see that (5.99) is valid on I. This together with (5.98) follows that (5.99) is also true on $[t_0, T + h_2]$.

Since the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are bounded by (5.99), employing the standard arguments, namely Ascoli-Arzela and Dini theorems, one can conclude that $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly on $[t_0, T]$, that is, $\alpha_n \to \rho$, $\beta_n \to r$ uniformly on $[t_0, T]$. Also, it is easy to show that (ρ, r) satisfy

$$\rho' = f(t, \rho, \rho_t, r^t), \;\; \rho_{t_0} = \phi_0,$$

$$r' = f(t, r, r_t, \rho^t), \;\; r_{t_0} = \phi_0,$$

with $\rho \leq r$ on I and $\rho^T = r^T$.

To show that (ρ, r) are coupled minimal and maximal solutions of (5.96), let $x(t)$ be any solution of (5.96) with $x_{t_0} = \phi_0$, $x^T = \psi_0$ such that $\alpha_0 \leq x \leq \beta_0$ on I. Then it is enough to show that $\rho \leq x \leq r$ since by definition of (ρ, r) we already have $\rho^T = x^T = r^T$. Setting $p = \alpha_1 - x$ so that $p_{t_0} = 0$ and

$$p' = \alpha_1' - x' = Q(t, \alpha_0, \alpha_{0t}, \beta_0^t) - M(\alpha_1 - \alpha_0) - N\int_{-h_1}^{0} (\alpha_{1t} - \alpha_{0t})(s)ds - Q(t, x, x_t, x^t)$$

$$\leq M(x - \alpha_0) + N\int_{-h_1}^{0} (x_t - \alpha_{0t})(s)ds - M(\alpha_1 - \alpha_0) - N\int_{-h_1}^{0} (\alpha_{1t} - \alpha_{0t}(s)ds$$

$$= -Mp - N\int_{-h_1}^{0} p_t(s)ds.$$

Thus we get from Lemma 5.9.1, $\alpha_1 \leq x$ on I. Similarly, $x \leq \beta_1$ on I. By proceeding similarly, it is easy to show that $\alpha_{n+1} \leq x \leq \beta_{n+1}$ on I. Hence (ρ, r) are coupled minimal and maximal solutions of (5.96).

If, in addition, condition (v) holds, since $\rho \leq r$, we let $p = r - \rho$ and find using $\rho^t \leq r^t$ and (v),

$$p' = r' - \rho' = Q(t, r, r_t, \rho^t) - Q(t, \rho, \rho_t, r^t)$$

$$\leq -Lp(t) + N_1\int_{h_1}^{0} p_t(s)ds + N_2\int_{0}^{h_2} p^t(\sigma)d\sigma,$$

and

$$p_{t_0} = 0, p^T = 0.$$

This implies by Lemma 5.9.2., $p(t) \leq 0$ on I, which means $\rho = r$ on I. Thus the common value $x = \rho = r$ is the unique solution of (5.96) with $x_{t_0} = \phi_0$, and $x^T = \psi_0$. The proof is therefore complete.

Remark. A simple choice of (5.98) would be to take for α_n^T, β_n^T, suitable translates of ψ_0 such that $\alpha_n^T = \psi_0 - \varepsilon_n$, $\beta_n^T = \psi_0 + \eta_n$, with $\alpha_n(T) = \psi_0(T) - \varepsilon_n$, $\beta_n(T) = \psi_0(T) + \eta_n$, for each n, where ε_n, $\eta_n > 0$ are monotone sequences tending to zero as $n \to \infty$. To make life simpler still, one can assume that $\alpha_0^T = \beta_0^T = \psi_0$. Note also that given any ϕ_0 with $\alpha_{0t_0} \leq \phi_0 \leq \beta_{0t_0}$, ψ_0 need to satisfy the inequality $\alpha_1(T) \leq \psi_0(T) \leq \beta_1(T)$ so that the choice (5.98) is possible. Recall that the method of lower and upper solutions provides existence results in the closed set generated by lower and upper solutions.

5.10 Neutral Differential Equations with Causal Operators on a Semi-Axis

Let us consider the functional differential equation

$$\frac{d}{dt}\left[\frac{dx(t)}{dt} - (Lx)(t)\right] = (Vx)(t), \quad t \in \mathbb{R}_+, \tag{5.103}$$

where $x \in \mathbb{R}^n$, $n \geq 1$ is an integer, and L, V are causal operators acting on the function space $C(\mathbb{R}_+, \mathbb{R}^n)$, consisting of all continuous maps from $\mathbb{R}_+$ into $\mathbb{R}^n$, the topology/convergence being defined by the family of semi-norms $\{|x|_k : k \geq 1\}$, with $|x_k| = \sup\{|x(t)| : 0 \leq t \leq k\}$, $k \geq 1$.

It is well known that the above topology/convergence means uniform convergence on any bounded interval $[0, T] \subseteq \mathbb{R}_+$.

An initial condition of the form

$$x(0) = x^0 \in \mathbb{R}^n, \quad \dot{x}(0) = v^0 \in \mathbb{R}^n, \tag{5.104}$$

must be associated with (5.103), if we expect to get a unique solution to the Cauchy problem (5.103), (5.104).

We will be interested here in finding global solutions to the problem, i.e. belonging to the space $C(\mathbb{R}_+, \mathbb{R}^n)$.

The space $C(\mathbb{R}_+, \mathbb{R}^n)$ contains as subspaces (closed or not) many usual spaces appearing in the theory of differential or integral equations. An example is the space $BC(\mathbb{R}_+, \mathbb{R}^n)$ consisting of all bounded continuous maps from $\mathbb{R}_+$ into $\mathbb{R}^n$ with the norm

$$|x| = \sup\{|x(t)| : t \in \mathbb{R}_+\}, \tag{5.105}$$

$BC(\mathbb{R}_+,\mathbb{R}^n)$ is a Banach space.

The main concern of this section is finding adequate conditions on the data, more precisely on the operators L and V, such that the existence of solution to the problem (5.103), (5.104) is guaranteed (on the semi-axis $\mathbb{R}_+$).

We shall briefly discuss in this section a connection between causal operators on function spaces and classical/integral operators of Volterra type, simply express by the formula

$$\int_0^t (Qx)(s)ds = \int_0^t K(t,s)x(s)ds, \quad t \in \mathbb{R}_+, \tag{5.106}$$

where Q stands for a linear causal operator on the space $C(\mathbb{R}_+,\mathbb{R}^n)$, and $K(t,s)$ denotes a matrix of type $n \times n$.

More precisely, the result described by (5.106), states that for each Q, there exists a measurable kernel $K(t,s)$, rendering the service described by (5.106). The formula (5.106) holds true for every $x \in C(\mathbb{R}_+,\mathbb{R}^n)$, and even in the most general case, $x \in L_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$. We notice that $K(t,s)$ must not be continuous. In order to assure the inclusion

$$\int_0^t K(t,s)x(s)ds \in C(\mathbb{R}_+,\mathbb{R}^n), \tag{5.107}$$

for each $x \in C(\mathbb{R}_+,\mathbb{R}^n)$, it suffices to deal with a locally integrable $K(t,s), (t,s) \in \Delta = \{(t,s) : 0 \le s \le t\}$ satisfying also the condition

$$\lim_{h\to 0}\left(\int_0^t |K(t+h,s)-K(t,s)|ds + \int_t^{t+h} |K(t+h,s)|ds\right) = 0 \tag{5.108}$$

for each $t \in \mathbb{R}_+$.

Let us notice that condition (5.108) on $K(t,s)$ is implied by (5.106), and the fact that Q is a linear operator acting on $C(\mathbb{R}_+,\mathbb{R}^n)$. Hence, it takes continuous maps into continuous ones.

The kernel $K(t,s)$ from (5.106) automatically verifies other properties, if it does satisfy extra conditions. For instance, one frequently encountered property is

$$\sup\left\{\int_0^t |K(t,s)|ds :\ t \in \mathbb{R}_+\right\} < \infty, \tag{5.109}$$

is implied by the requirement that the subspace $BC(\mathbb{R}_+,\mathbb{R}^n)$ must be left invariant by the operator V.

In general, the connection between the operator V, and the properties of the kernel $K(t,s)$, is not always easy to be established. The problem of clarifying such connections is of great significance for applications.

We shall formulate conditions for Q, by means of the relationship (5.106). In other words, by imposing the adequate conditions on the associated kernel $K(t,s)$. In special situations, when Q is chosen in a classical form, the connection may appear more transparent.

Let us return to equation (5.103) and make a few remarks that will help simplifying the coming considerations.

First, it is obvious that an additive constant to the operator L, i.e. a constant n-vector, does not change the equation (5.103). Hence, without loss of generality, we can assume

$$(Lx)(0) = \theta \in \mathbb{R}^n, \tag{5.110}$$

for any x in the space $C(\mathbb{R}_+, \mathbb{R}^n)$.

In case of classical Volterra operator

$$(Qx)(t) = f(t) + \int_0^t K(t,s,x(s))ds,$$

one obtains $(Qx)(0) = f(0) = \text{const}$, for any $x \in C(\mathbb{R}_+, \mathbb{R}^n)$, or in another underlying space. As mentioned above we can substitute θ in (5.110) by any constant $c \in \mathbb{R}^n$ without changing the equation.

By integrating both sides (5.103) from 0 to $t > 0$ we obtain the functional differential equation

$$\dot{x}(t) - (Lx)(t) = v^0 + \int_0^t (Vx)(s)ds, \tag{5.111}$$

if we take into account (5.104) and (5.110).

The following equation is related to the equation (5.111),

$$\dot{x}(t) - (Lx)(t) = f(t), \quad t \in \mathbb{R}_+. \tag{5.112}$$

The Cauchy problem for (5.112) with $x(0) = x^0 \in \mathbb{R}^n$ can be represented in case of linear and continuous operator L on $C(\mathbb{R}_+, \mathbb{R}^n)$, by an integral formula, involving the Cauchy operator associated to L. It is sort of variation of parameters formula (Lagrange), and it looks

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)f(s)ds, \quad t \in \mathbb{R}_+, \tag{5.113}$$

for any $f \in C(\mathbb{R}_+, \mathbb{R}^n)$. the Cauchy function (or kernel) $X(t,s)$ is defined by the formula

$$X(t,s) = I + \int_s^t \tilde{K}_0(t,u)du, \tag{5.114}$$

for each $(t,s) \in \Delta$,

$$\Delta = \{(t,s) : 0 \leq s \leq t\}, \tag{5.115}$$

where $\tilde{K}_0(t,s)$ is the resolvent kernel corresponding to $K_0(t,s)$, $(t,s) \in \Delta$, from the representation

$$\int_0^t (Lx)(s)ds = \int_0^t K_0(t,s)x(s)ds, \quad t \in \mathbb{R}_+. \tag{5.116}$$

In (5.116), $K_0(t,s)$ is measurable only, but the existence of $\tilde{K}_0(t,s)$is assured, for instance, if we accept the condition

$$K_0(t,s) \in L^\infty_{\text{loc}}(\Delta, \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)). \tag{5.117}$$

Let us apply formula (5.113) to the equation (5.111). We are led to the functional equation

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0 ds + \int_0^t X(t,s) \int_0^s (Vx)(u)du\, ds. \tag{5.118}$$

Since we already assumed L to be linear, there results that (5.118), which is an equivalent equation to the problem (5.103), (5.104) is also linear when V is linear. Otherwise, it is a nonlinear functional equation for $x(t)$.

In case V is a linear operator acting on $C(\mathbb{R}_+, \mathbb{R}^n)$, we can use the representation (5.106), and (5.118) leads to another equivalent form of the problem (5.103), (5.104):

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0 ds + \int_0^t X(t,s) \int_0^s K(s,u)x(u)du\, ds.$$

Interchanging the order of integration in the double integral, we obtain the functional differential equation

$$x(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0 ds + \int_0^t \left(\int_u^t X(t,s)K(s,u)ds \right) x(u)du,$$

which can be rewritten as

$$x(t) = f(t) + \int_0^t K_1(t,u)x(u)du, \quad t \in \mathbb{R}_+, \tag{5.119}$$

with

$$f(t) = X(t,0)x^0 + \int_0^t X(t,s)v^0 ds, \quad t \in \mathbb{R}_+, \tag{5.120}$$

and

$$K_1(t,u) = \int_u^t X(t,s)K(s,u)du, \quad 0 \le u \le t. \tag{5.121}$$

The equation (5.119) will be discussed in detail, and the existence result will be applied to the problem (5.103), (5.104).

Another case will be considered, when the operator V is not necessarily linear, but it is Lipschitzian continuous.

We have to examine the function (5.120) and the kernel $K_1(t,u)$ to see if we can construct a solution in $C(\mathbb{R}_+, \mathbb{R}^n)$.

First, the function $f(t)$ from (5.120) is the solution of the functional differential equation $\dot{x}(t) - (Lx)(t) = v^0$, with the initial condition $x(0) = x^0$. Hence, $f(t)$ is a continuously differentiable function on $\mathbb{R}_+$, with values in $\mathbb{R}^n$.

Second, the kernel $K_1(t,u)$ given by (5.121) is a locally bounded function on Δ. More precisely, we can infer

$$K_1(t,s) \in L^\infty_{\text{loc}}(\Delta, \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)). \tag{5.122}$$

Indeed, we shall admit that $K(t,s)$ belongs to L^∞_{loc}, as mentioned above. Furthermore, formula (5.114) shows us that $X(t,s)$ is also locally bounded on Δ. This fact is a consequence of property that states: any kernel $K(t,s)$, which is locally bounded on Δ, admits a resolvent $\tilde{K}(t,s)$, which is locally bounded on Δ.

Therefore, the integral equation (5.119), whose kernel $K_1(t,s)$ satisfies (5.122), admits a resolvent kernel $\tilde{K}_1(t,s)$, locally bounded on Δ, while its unique solution is represented by the resolvent formula

$$x(t) = f(t) + \int_0^t \tilde{K}_1(t,s) f(s)ds, \quad t \in \mathbb{R}_+, \tag{5.123}$$

for any function $f \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$. In particular (5.123) holds true when $f(t)$ is given by the formula (5.120).

In summarizing the discussion carried out above, we can state the following existence result for the problem (5.103), (5.104).

Theorem 5.10.1. Consider the neutral functional differential equation (5.103) with the initial conditions (5.104). Assume the following conditions are satisfied:

(i) the operators L and V are linear, continuous and causal, acting on the space $C(\mathbb{R}_+, \mathbb{R}^n)$;

(ii) the kernels $K(t,s)$ and $K_0(t,s)$, occurring in the representation (5.106) and (5.116) are locally bounded on Δ, defined by (5.115).

Then, there exists a unique solution $x(t)$, $t \in \mathbb{R}_+$, of the problem (5.103), (5.104), for arbitrary initial data $x^0, v^0 \in \mathbb{R}^n$. This solution is continuously differentiable on $\mathbb{R}_+$.

The proof is immediate if we rely on the discussion preceding the statement of Theorem 5.10.1, the equivalence (5.103), (5.104) with the equation (5.119) being the key ingredient.

Remark. The condition (5.108) is not the only condition that can be derived from the fact that Q is acting on $C(\mathbb{R}_+, \mathbb{R}^n)$.

Indeed, from (5.108) we read

$$\int_0^t K(t,s)x(s)ds \in AC_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n), \tag{5.124}$$

for every $x \in C(\mathbb{R}_+, \mathbb{R}^n)$. The space AC_{loc} in (5.124) is a subspace of $C(\mathbb{R}_+, \mathbb{R}^n)$, and the inclusion

$$\int_0^t K(t,s)x(s)ds \in C(\mathbb{R}_+, \mathbb{R}^n), \quad t \in \mathbb{R}_+, \tag{5.125}$$

tells us less than (5.124). Nevertheless, we prefer to use (5.125) instead of (5.124) for simplicity. For example, (5.125) implies

$$\lim_{h\to 0}\int_0^t |K(t+h,s) = K(t,s)|ds = 0, \quad t \in \mathbb{R}_+. \tag{5.126}$$

Remark. Considering the resolvent formula (5.123) for the solution of equation (5.119), we can obtain more information about the solution of (5.103), (5.104), making extra assumptions on the kernel $\tilde{K}_1(t,s)$. This kernel is determined, as shown by (5.121), by the properties of the operators L and Q.

For instance, if we assume that $\tilde{K}_1(t,s)$ satisfies the condition

$$\int_0^t |\tilde{K}_1(t,s)|ds \le M < \infty, \quad t \in \mathbb{R}_+, \tag{5.127}$$

and also

$$|X(t,0)| + \int_0^t |X(t,s)|ds \le N < \infty, \quad t \in \mathbb{R}_+, \tag{5.128}$$

then the solution of the problem (5.103), (5.104) will verify the inclusion

$$x(t) \in L^\infty(\mathbb{R}_+, \mathbb{R}^n), \quad x^0, v^0 \in \mathbb{R}^n. \tag{5.129}$$

The proof follows immediately from the formulas (5.120) and (5.123).

Further properties of the solution can be obtained by imposing various types of estimates on the kernels $K_1(t,s)$ or $\tilde{K}_1(t,s)$, as well as on $f(t)$.

The main problem is to establish the connection between the properties of the operators L and Q and the kernels occurring in the representation (5.106) and (5.116). This question is open.

We shall rewrite equation (5.118) in the form

$$x(t) = f(t) + \int_0^t X(t,s)\int_0^s (Qx)(u)ds\, ds, \quad t \in \mathbb{R}_+, \tag{5.130}$$

where $f(t)$ is given by (5.120). Equation (5.130), with $f(t)$ defined by (5.120), is equivalent to our problem. This is a functional integral equation and we shall treat it by the classical method of operation/successive approximations. This approach will lead to an existence and uniqueness result in the space $C(\mathbb{R}_+, \mathbb{R}^n)$. Of course, the operator Q is assumed to be acting on this space.

In order to simplify somewhat the procedure, we shall adopt a hypothesis which is part of the assumption (5.127). This hypothesis concerns only the linear operator L, which fully determines the Cauchy kernel $X(t,s)$, $0 \le s \le t$.

Namely, we assume in this section that

$$\int_0^t |X(t,s)|ds \le M < \infty, \quad (t,s) \in \Delta, \tag{5.131}$$

and we shall limit our consideration in regard to equation (5.130), only to those operators L for which (5.131) is satisfied.

Concerning the operator V, acting on the same space $C(\mathbb{R}_+,\mathbb{R}^n)$, we shall assume it verifies the Lipschitzian type condition

$$|(Qx)(t)-(Qy)(t)| \le \lambda(t)|x(t)-y(t)|, \quad t\in\mathbb{R}_+, \tag{5.132}$$

for any $x,y\in C(\mathbb{R}_+,\mathbb{R}^n)$. We also assume that $\lambda(t)$ is a nonnegative nondecreasing map from $\mathbb{R}_+$ into itself.

In order to prove the existence and uniqueness of a solution to (5.130), we construct the sequence of successive approximations $\{x_k(t):k\ge 0\}$, by letting $x_0(t)=f(t)$, and

$$x_{k+1}(t)=f(t)+\int_0^t X(t,s)\int_0^s (Qx_k(u))du\,ds, \tag{5.133}$$

for $k>1$, $t\in\mathbb{R}_+$.

We shall prove now that the sequence of successive approximations converges in $C(\mathbb{R}_+,\mathbb{R}^n)$. This means that the sequence converges uniformly on each bounded interval $[0,T]\subseteq\mathbb{R}_+$. The limit of this sequence

$$x(t)=\lim_{k\to\infty} x_k(t), \quad t\in\mathbb{R}_+, \tag{5.134}$$

will constitute the solution of our problem. As usual, if we subtract side by side the relationship (5.133) and the one corresponding to k instead of $k+1$, we find the following recurrent relation, valid for $k\ge 1$ and $t\in\mathbb{R}_+$:

$$x_{k+1}-x_k(t)=\int_0^t X(t,s)\int_0^s [(Qx_k)(u)-(Qx_{k-1})(u)]du\,ds. \tag{5.135}$$

Taking into account (5.131) and (5.132), we obtain from (5.135) the following recurrent inequality:

$$|x_{k+1}(t)-x_k(t)| \le M\sup_{0\le s\le t}\int_0^s \lambda(u)|x_k(u)-x_{k-1}(u)|du.$$

The above inequality leads immediately to

$$|x_{k+1}(t)-x_k(t)| \le M\int_0^t \lambda(s)\sup_{0\le u\le s}|x_k(u)-x_{k-1}(u)|du. \tag{5.136}$$

Let us denote

$$y_k(t)=\sup_{0\le s\le t}|x_k(s)-x_{k-1}(s)|, \tag{5.137}$$

and we rewrite (5.136) in the form

$$y_k(t)\le M\int_0^t \lambda(s)y_k(s)ds, \quad k\ge 1. \tag{5.138}$$

We had to keep in mind the fact that $\sup\{|x_k(t) - x_{k-1}(t)| : t \in [0,T]\}$ is nondecreasing in T.

Now by induction, the recurrent inequality (5.138) leads to, if one assumes $y_1(t) \le A$ on the interval $[0,T]$, $T > 0$ arbitrary,

$$y_{k+1}(t) \le A\frac{M^k}{k!}\left(\int_0^t \lambda(u)du\right)^k, \quad t \in [0,T]. \tag{5.139}$$

The inequality (5.139) obviously implies the uniform convergence of the sequence of successive approximations, on any finite interval of $\mathbb{R}_+$. Therefore, we have

$$\lim_{k\to\infty} x_k(t) = x(t) \in C(\mathbb{R}_+, \mathbb{R}^n). \tag{5.140}$$

The function $x(t)$ defined by (5.140) is a solution of equation (5.119), and this equation is equivalent to the problem. While (5.140) shows that $x(t)$ is a continuous solution, is has actually better regularity properties, as stipulated in Section 3.

Let us now formulate the main result of this section, related to our basic problem.

Theorem 5.10.2. Consider the initial value problem (5.103), (5.104), or equivalently the functional integral equation (5.119), under the following assumptions:

(i) The operator L is a linear continuous operator on the space $C(\mathbb{R}_+, \mathbb{R}^n)$.

(ii) The operator Q is also acting on the space $C(\mathbb{R}_+, \mathbb{R}^n)$, and verifies the Lipschitz condition (5.132), with $\lambda(t)$ nondecreasing on $\mathbb{R}_+$.

Then, there exists a unique solution $x(t) \in C(\mathbb{R}_+, \mathbb{R}^n)$, of (5.103), (5.104), or (5.119), and it is continuously differentiable on $\mathbb{R}_+$, as well as $\dot{x}(t) - (Lx)(t)$.

Once we know the solution of our problem does exist, we can think of obtaining further properties, of asymptotic nature.

Namely, we shall drop the assumption (5.127) on the resolvent kernel $\tilde{K}_1(t,s)$, and impose other conditions that can be verified more directly, on the operator Q. These conditions are

$$(Q\theta)(t) \equiv 0, \quad \lambda(t) \in L^1(\mathbb{R}_+, \mathbb{R}), \tag{5.141}$$

and they will help us to recognize the property of boundedness for the solution $x(t) \in BC(\mathbb{R}_+, \mathbb{R}^n)$.

Indeed, we see that the conditions of Theorem 5.10.2 are verified if we accept (5.141) and the Lipschitz condition with $\lambda(t)$ instead of the Lipschitz constant. But if we define $\tilde{\lambda}(t) = \sup\{\lambda(s) : 0 \le s \le t\}$, we find a function providing as $\lambda(t)$ does. Obviously, (5.141) implies $|(Qx)(t)| \le \lambda(t)|x(t)| \le \tilde{\lambda}(t)|x(t)|$. Hence, on behalf of Theorem 5.10.2 we have assured the existence and uniqueness of the solution.

We shall prove that the solution is actually in $BC(\mathbb{R}_+,\mathbb{R}^n)$, if we make the extra assumption (5.128) on $X(t,s)$.

From (5.128), (5.130), (5.131), we obtain the integral inequality

$$|x(t)| \leq N + M\int_0^t |(Vx)(s)|ds, \quad t \in \mathbb{R}_+, \tag{5.142}$$

$x(t)$ being the solution in $C(\mathbb{R}_+,\mathbb{R}^n)$ for our problem. the inequality (5.142) and Lipschitz condition imply

$$|x(t)| \leq N + M\int_0^t \lambda(s)|x(s)|ds, \quad t \in \mathbb{R}_+. \tag{5.143}$$

The inequality (5.143) is of Gronwall type, and yields

$$|x(t)| \leq M\exp\left(M\int_0^\infty \lambda(s)ds\right), \quad t \in \mathbb{R}_+,$$

which shows that $x(t) \in BC(\mathbb{R}_+,\mathbb{R}^n)$, as it follows from the second condition (5.141).

5.11 Notes and Comments

All the results starting from Sec. 5.2 to Sec. 5.7 are taken from Drici, McRae and Vasundhara Devi [49]-[73]. The contents of Sec. 5.8 and Sec. 5.9 are adapted from Gnana Bhaskar and Lakshmikantham [85]; see also Gnana Bhaskar, Lakshmikantham and Vasundhara Devi [86]. See Dubois [87] and [88] for the use of special equations with retardation and anticipation in industrial applications. Those problems have been open for a long time and need more investigation since they are useful in real world applications where decisions are to be made. There is a lot of scope for the development of this area. Moreover, we have only presented results that are available and a lot of scope exits for further research and advancement in several directions extending CDEs suitably.

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