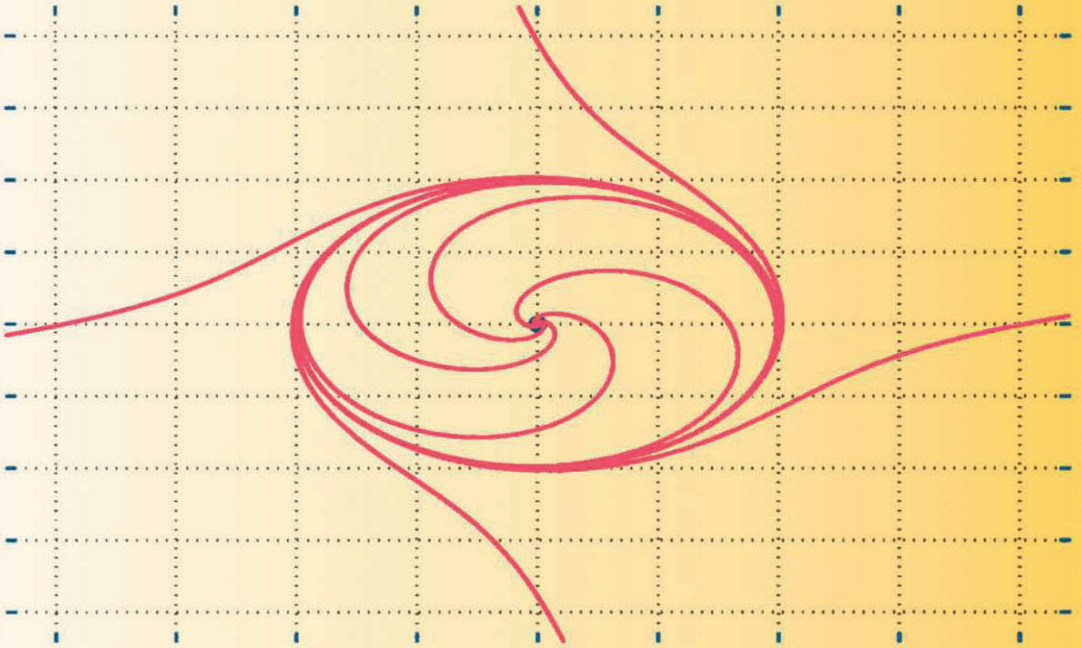
J. David Logan

A FIRST COURSE IN DIFFERENTIAL EQUATIONS



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A First Course in Differential Equations

With 55 Figures

Springer

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Dedicated to—

Reece Charles Logan,
Jaren Logan Golightly

Contents

Preface

There are many excellent texts on elementary differential equations designed for the standard sophomore course. However, in spite of the fact that most courses are one semester in length, the texts have evolved into calculus-like presentations that include a large collection of methods and applications, packaged with student manuals, and Web-based notes, projects, and supplements. All of this comes in several hundred pages of text with busy formats. Most students do not have the time or desire to read voluminous texts and explore internet supplements. The format of this differential equations book is different; it is a one-semester, brief treatment of the basic ideas, models, and solution methods. Its limited coverage places it somewhere between an outline and a detailed textbook. I have tried to write concisely, to the point, and in plain language. Many worked examples and exercises are included. A student who works through this primer will have the tools to go to the next level in applying differential equations to problems in engineering, science, and applied mathematics. It can give some instructors, who want more concise coverage, an alternative to existing texts.

The numerical solution of differential equations is a central activity in science and engineering, and it is absolutely necessary to teach students some aspects of scientific computation as early as possible. I tried to build in flexibility regarding a computer environment. The text allows students to use a calculator or a computer algebra system to solve some problems numerically and symbolically, and templates of MATLAB and Maple programs and commands are given in an appendix. The instructor can include as much of this, or as little of this, as he or she desires.

For many years I have taught this material to students who have had a standard three-semester calculus sequence. It was well received by those who appreciated having a small, definitive parcel of material to learn. Moreover, this text gives students the opportunity to start reading mathematics at a slightly higher level than experienced in pre-calculus and calculus. Therefore the book can be a bridge in their progress to study more advanced material at the junior–senior level, where books leave a lot to the reader and are not packaged in elementary formats.

Chapters 1, 2, 3, 5, and 6 should be covered in order. They provide a route to geometric understanding, the phase plane, and the qualitative ideas that are important in differential equations. Included are the usual treatments of separable and linear first-order equations, along with second-order linear homogeneous and nonhomogeneous equations. There are many applications to ecology, physics, engineering, and other areas. These topics will give students key skills in the subject. Chapter 4, on Laplace transforms, can be covered at any time after Chapter 3, or even omitted. Always an issue in teaching differential equations is how much linear algebra to cover. In two extended sections in Chapter 5 we introduce a moderate amount of matrix theory, including solving linear systems, determinants, and the eigenvalue problem. In spite of the book's brevity, it still contains slightly more material than can be comfortably covered in a single three-hour semester course. Generally, I assign most of the exercises; hints and solutions for selected problems are given in Appendix D.

I welcome suggestions, comments, and corrections. Contact information is on my Web site: `http://www.math.unl.edu/~dlogan`, where additional items may be found.

I would like to thank John Polking at Rice University for permitting me to use his MATLAB program pplane7 to draw some of the phase plane diagrams and Mark Spencer at Springer for his enthusiastic support of this project. Finally, I would like to thank Tess for her continual encouragement and support for my work.

David Logan
Lincoln, Nebraska

To the Student

What is a course in differential equations about? Here are some informal, preparatory remarks to give you some sense of the subject before we take it up seriously.

You are familiar with algebra problems and solving algebraic equations. For example, the solutions to the quadratic equation

$$x^2 - x = 0$$

are easily found to be $x = 0$ and $x = 1$, which are numbers. A differential equation (sometimes abbreviated DE) is another type of equation where the unknown is not a number, but a function. We will call it $u(t)$ and think of it as a function of time. A DE also contains derivatives of the unknown function, which are also not known. So a DE is an equation that relates an unknown function to some of its derivatives. A simple example of a DE is

$$u'(t) = u(t),$$

where $u'(t)$ denotes the derivative of $u(t)$. We ask what function $u(t)$ solves this equation. That is, what function $u(t)$ has a derivative that is equal to itself? From calculus you know that one such function is $u(t) = e^t$, the exponential function. We say this function is a solution of the DE, or it solves the DE. Is it the only one? If you try $u(t) = Ce^t$, where C is any constant whatsoever, you will also find it is a solution. So differential equations have lots of solutions (fortunately we will see they are quite similar, and the fact that there are many allows some flexibility in imposing other desired conditions).

This DE was very simple and we could guess the answer from our calculus knowledge. But, unfortunately (or, fortunately!), differential equations are usually more complicated. Consider, for example, the DE

$$u''(t) + 2u'(t) + 2u(t) = 0.$$

This equation involves the unknown function and both its first and second derivatives. We seek a function for which its second derivative, plus twice its first derivative, plus twice the function itself, is zero. Now can you quickly guess a function $u(t)$ that solves this equation? It is not likely. An answer is

$$u(t) = e^{-t}\cos t.$$

And,

$$u(t) = e^{-t}\sin t$$

works as well. Let's check this last one by using the product rule and calculating its derivatives:

$$\begin{aligned} u(t) &= e^{-t}\sin t, \\ u'(t) &= e^{-t}\cos t - e^{-t}\sin t, \\ u''(t) &= -e^{-t}\sin t - 2e^{-t}\cos t + e^{-t}\sin t. \end{aligned}$$

Then

$$\begin{aligned} & u''(t) + 2u'(t) + 2u(t) \\ = \ & -e^{-t}\sin t - 2e^{-t}\cos t + e^{-t}\sin t + 2(e^{-t}\cos t - e^{-t}\sin t) + 2e^{-t}\sin t \\ = \ & 0. \end{aligned}$$

So it works! The function $u(t) = e^{-t}\sin t$ solves the equation $u''(t) + 2u'(t) + 2u(t) = 0$. In fact,

$$u(t) = Ae^{-t}\sin t + Be^{-t}\cos t$$

is a solution regardless of the values of the constants A and B. So, again, differential equations have lots of solutions.

Partly, the subject of differential equations is about developing methods for finding solutions.

Why differential equations? Why are they so important to deserve a course of study? Well, differential equations arise naturally as *models* in areas of science, engineering, economics, and lots of other subjects. Physical systems, biological systems, economic systems—all these are marked by change. Differential equations model real-world systems by describing how they change. The unknown function $u(t)$ could be the current in an electrical circuit, the concentration of a chemical undergoing reaction, the population of an animal species in an ecosystem, or the demand for a commodity in a micro-economy. Differential equations are laws that dictate change, and the unknown $u(t)$, for which we solve, describes exactly how the changes occur. In fact, much of the reason that the calculus was developed by Isaac Newton was to describe motion and to solve differential equations.

For example, suppose a particle of mass m moves along a line with constant velocity V_0. Suddenly, say at time $t = 0$, there is imposed an external resistive force F on the particle that is proportional to its velocity $v = v(t)$ for times $t > 0$. Notice that the particle will slow down and its velocity will change. From this information can we predict the velocity $v(t)$ of the particle at any time $t > 0$? We learned in calculus that Newton's second law of motion states that the mass of the particle times its acceleration equals the force, or $ma = F$. We also learned that the derivative of velocity is acceleration, so $a = v'(t)$. Therefore, if we write the force as $F = -kv(t)$, where k is the proportionality constant and the minus sign indicates the force opposes the motion, then

$$mv'(t) = -kv(t).$$

This is a differential equation for the unknown velocity $v(t)$. If we can find a function $v(t)$ that "works" in the equation, and also satisfies $v(0) = V_0$, then we will have determined the velocity of the particle. Can you guess a solution? After a little practice in Chapter 1 we will be able to solve the equation and find that the velocity decays exponentially; it is given by

$$v(t) = V_0 e^{-kt/m}, \quad t \geq 0.$$

Let's check that it works:

$$mv'(t) = mV_0 \left(-\frac{k}{m}\right) e^{-kt/m} = -kV_0 e^{-kt/m} = -kv(t).$$

Moreover, $v(0) = V_0$. So it does check. The differential equation itself is a *model* that governs the dynamics of the particle. We set it up using Newton's second law, and it contains the unknown function $v(t)$, along with its derivative $v'(t)$. The solution $v(t)$ dictates how the system evolves.

In this text we study differential equations and their applications. We address two principal questions. (1) How do we find an appropriate DE to model a physical problem? (2) How do we understand or solve the DE after we obtain it? We learn modeling by examining models that others have studied (such as Newton's second law), and we try to create some of our own through exercises. We gain understanding and learn solution techniques by practice.

Now we are ready. Read the text carefully with pencil and paper in hand, and work through all the examples. Make a commitment to solve most of the exercises. You will be rewarded with a knowledge of one of the monuments of mathematics and science.

1
Differential Equations and Models

In science, engineering, economics, and in most areas where there is a quantitative component, we are greatly interested in describing how systems evolve in time, that is, in describing a system's **dynamics**. In the simplest one-dimensional case the state of a system at any time t is denoted by a function, which we generically write as $u = u(t)$. We think of the dependent variable u as the state variable of a system that is varying with time t, which is the independent variable. Thus, knowing u is tantamount to knowing what state the system is in at time t. For example, $u(t)$ could be the population of an animal species in an ecosystem, the concentration of a chemical substance in the blood, the number of infected individuals in a flu epidemic, the current in an electrical circuit, the speed of a spacecraft, the mass of a decaying isotope, or the monthly sales of an advertised item. Knowledge of $u(t)$ for a given system tells us exactly how the state of the system is changing in time. Figure 1.1 shows a **time series plot** of a generic state function. We always use the variable u for a generic state; but if the state is "population", then we may use p or N; if the state is voltage, we may use V. For mechanical systems we often use $x = x(t)$ for the position.

One way to obtain the state $u(t)$ for a given system is to take measurements at different times and fit the data to obtain a nice formula for $u(t)$. Or we might read $u(t)$ off an oscilloscope or some other gauge or monitor. Such curves or formulas may tell us *how* a system behaves in time, but they do not give us insight into *why* a system behaves in the way we observe. Therefore we try to formulate explanatory models that underpin the understanding we seek. Often these models are dynamic equations that relate the state $u(t)$ to its rates of

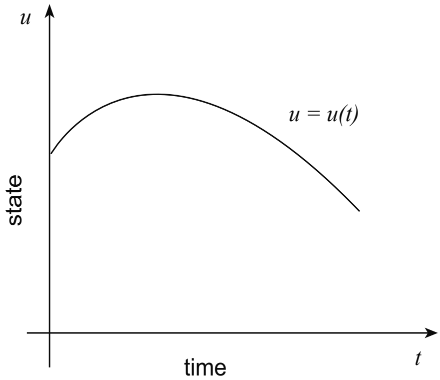

Figure 1.1 Time series plot of a generic state function $u = u(t)$ for a system.

change, as expressed by its derivatives $u'(t), u''(t), ...$, and so on. Such equations are called **differential equations** and many laws of nature take the form of such equations. For example, Newton's second law for the motion for a mass acted upon by external forces can be expressed as a differential equation for the unknown position $x = x(t)$ of the mass.

In summary, a differential equation is an equation that describes how a state $u(t)$ changes. A common strategy in science, engineering, economics, etc., is to formulate a basic principle in terms of a differential equation for an unknown state that characterizes a system and then solve the equation to determine the state, thereby determining how the system evolves in time.

1.1 Differential Equations

1.1.1 Equations and Solutions

A **differential equation** (abbreviated **DE**) is simply an equation for an unknown state function $u = u(t)$ that connects the state function and some of its derivatives. Several notations are used for the derivative, including

$$u', \frac{du}{dt}, \dot{u}, ...$$

The *overdot* notation is common in physics and engineering; mostly we use the simple *prime* notation. The reader should be familiar with the definition of the derivative:

$$u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}.$$

For small h, the difference quotient on the right side is often taken as an approximation for the derivative. Similarly, the second derivative is denoted by

$$u'', \frac{d^2u}{dt^2}, \ddot{u}, \ldots$$

and so forth; the nth derivative is denoted by $u^{(n)}$. The first derivative of a quantity is the "rate of change of the quantity" measuring how fast the quantity is changing, and the second derivative measures how fast the rate is changing. For example, if the state of a mechanical system is position, then its first derivative is velocity and its second derivative is acceleration, or the rate of change of velocity. Differential equations are equations that relate states to their rates of change, and many natural laws are expressed in this manner. The order of the highest derivative that occurs in the DE is called the **order** of the equation.

Example 1.1

Three examples of differential equations are

$$\begin{aligned} \theta'' + \sqrt{\frac{g}{l}}\sin\theta &= 0, \\ Lq'' + Rq' + \frac{1}{C}q &= \sin\omega t, \\ p' &= rp(1 - \frac{p}{K}). \end{aligned}$$

The first equation models the angular deflections $\theta = \theta(t)$ of a pendulum of length l; the second models the charge $q = q(t)$ on a capacitor in an electrical circuit containing an inductor, resistor, and a capacitor, where the current is driven by a sinusoidal electromotive force operating at frequency ω; in the last equation, called the logistics equation, the state function $p = p(t)$ represents the population of an animal species in a closed ecosystem; r is the population growth rate and K represents the capacity of the ecosystem to support the population. The unspecified constants in the various equations, l, L, R, C, ω, r, and K are called **parameters**, and they can take any value we choose. Most differential equations that model physical processes contain such parameters. The constant g in the pendulum equation is a **fixed parameter** representing the acceleration of gravity on earth. In mks units, $g = 9.8$ meters per second-squared. The unknown in each equation, $\theta(t)$, $q(t)$, and $p(t)$, is the state function. The first two equations are *second-order* and the third equation is *first-order*. Note that all the state variables in all these equations depend on time t. Because time dependence is understood we often save space and drop that dependence when writing differential equations. So, for example, in the first equation θ means $\theta(t)$ and θ'' means $\theta''(t)$.

In this chapter we focus on first-order differential equations and their origins. We write a generic first-order equation for an unknown state $u = u(t)$ in the form

$$u' = f(t, u). \tag{1.1}$$

When we have solved for the derivative, we say the equation is in **normal** form. There are several words we use to classify DEs, and the reader should learn them. If f does not depend *explicitly* on t (i.e., the DE has the form $u' = f(u)$), then we call the DE **autonomous**. Otherwise it is **nonautonomous**. For example, the equation $u' = -3u^2 + 2$ is autonomous, but $u' = -3u^2 + \cos t$ is nonautonomous. If f is a linear function in the variable u, then we say (1.1) is **linear**; else it is **nonlinear**. For example, the equation $u' = -3u^2 + 2$ is nonlinear because $f(t, u) = -3u^2 + 2$ is a quadratic function of u, not a linear one. The general form of a **first-order linear equation** is

$$u' = p(t)u + q(t),$$

where p and q are known functions. Note that in a linear equation both u and u' occur alone and to the first power, but the time variable t can occur in any manner. Linear equations occur often in theory and applications, and their study forms a significant part of the subject of differential equations.

A function $u = u(t)$ is a **solution**[1] of the DE (1.1) on an interval $I : a < t < b$ if it is differentiable on I and, when substituted into the equation, it satisfies the equation identically for all $t \in I$; that is,

$$u'(t) = f(t, u(t)), \quad t \in I.$$

Therefore, a function is a solution if, when substituted into the equation, every term cancels out. In a differential equation the solution is an unknown state function to be found. For example, in $u' = -u + e^{-t}$, the unknown is a function $u = u(t)$; we ask what function $u(t)$ has the property that its derivative is the same as the negative of the function, plus e^{-t}.

Example 1.2

This example illustrates what we might expect from a first-order linear DE. Consider the DE

$$u' = -u + e^{-t}.$$

[1] We are overburdening the notation by using the same symbol u to denote both a variable and a function. It would be more precise to write "$u = \varphi(t)$ is a solution," but we choose to stick to the common use, and abuse, of a single letter.

The state function $u(t) = te^{-t}$ is a solution to the DE on the interval $I : -\infty < t < \infty$. (Later, we learn how to find this solution). In fact, for any constant C the function $u(t) = (t + C)e^{-t}$ is a solution. We can verify this by direct substitution of u and u' into the DE; using the product rule for differentiation,

$$u' = (t + C)(-e^{-t}) + e^{-t} = -u + e^{-t}.$$

Therefore $u(t)$ satisfies the DE regardless of the value of C. We say that this expression $u(t) = (t+C)e^{-t}$ represents a **one-parameter family** of solutions (one solution for each value of C). This example illustrates the usual state of affairs for any first-order linear DE—there is a one-parameter family of solutions depending upon an arbitrary constant C. This family of solutions is called a **general solution**. The fact that there are many solutions to first-order differential equations turns out to be fortunate because we can adjust the constant C to obtain a specific solution that satisfies other conditions that might apply in a physical problem (e.g., a requirement that the system be in some known state at time $t = 0$). For example, if we require $u(0) = 1$, then $C = 1$ and we obtain a **particular solution** $u(t) = (t+1)e^{-t}$. Figure 1.2 shows a plot of the one-parameter family of solutions for several values of C. Here, we are using the word parameter in a different way from that in Example 1.1; there, the word parameter refers to a physical number in the equation itself that is fixed, yet arbitrary (like resistance in a circuit).

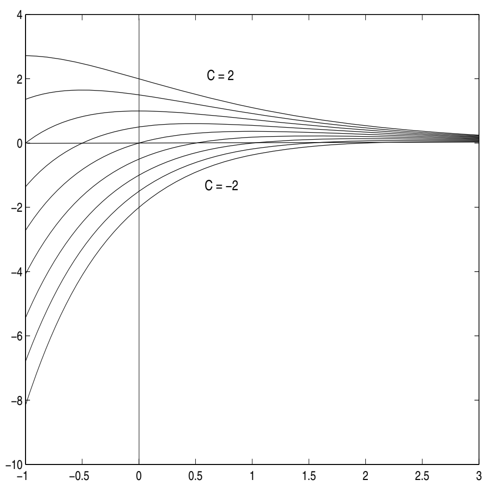

Figure 1.2 Time series plots of several solutions to $u' = e^{-t} - u$ on the interval $-1 \leq t \leq 3$. The solution curves, or the one-parameter family of solutions, are $u(t) = (t + C)e^{-t}$, where C is an arbitrary constant, here taking several values between -2 and 2.

An **initial value problem** (abbreviated **IVP**) for a first-order DE is the problem of finding a solution $u = u(t)$ to (1.1) that satisfies an **initial condition** $u(t_0) = u_0$, where t_0 is some fixed value of time and u_0 is a fixed state. We write the IVP concisely as

$$\text{(IVP)} \qquad \begin{cases} u' = f(t, u), \\ u(t_0) = u_0. \end{cases} \tag{1.2}$$

The initial condition usually picks out a specific value of the arbitrary constant C that appears in the general solution of the equation. So, it selects one of the many possible states that satisfy the differential equation. The accompanying graph (figure 1.3) depicts a solution to an IVP.

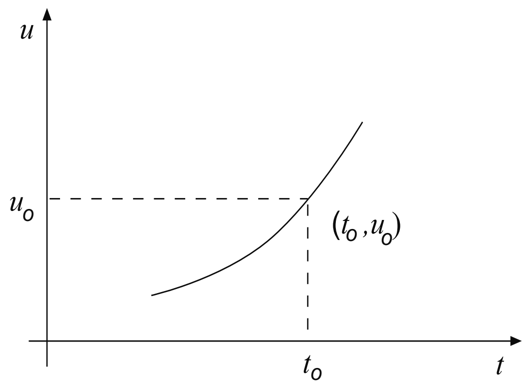

Figure 1.3 Solution to an initial value problem. The fundamental questions are: (a) is there a solution curve passing through the given point, (b) is the curve the only one, and (c) what is the interval (α, β) on which the solution exists.

Geometrically, solving an initial value problem means to find a solution to the DE that passes through a specified point (t_0, u_0) in the plane. Referring to Example 1.2, the IVP

$$u' = -u + e^{-t}, \quad u(0) = 1$$

has solution $u(t) = (t+1)e^{-t}$, which is valid for all times t. The solution curve passes through the point $(0, 1)$, corresponding to the initial condition $u(0) = 1$. Again, the initial condition selects one of the many solutions of the DE; it fixes the value of the arbitrary constant C.

There are many interesting mathematical questions about initial value problems:

1. **(Existence)** Given an initial value problem, is there a solution? This is the question of existence. Note that there may be a solution even if we cannot find a formula for it.
2. **(Uniqueness)** If there is a solution, is the solution unique? That is, is it the only solution? This is the question of uniqueness.
3. **(Interval of existence)** For which times t does the solution to the initial value problem exist?

Obtaining resolution of these theoretical issues is an interesting and worthwhile endeavor, and it is the subject of advanced courses and books in differential equations. In this text we only briefly discuss these matters. The next three examples illustrate why these are reasonable questions.

Example 1.3

Consider the initial value problem

$$u' = u\sqrt{t-3}, \quad u(1) = 2.$$

This problem has no solution because the derivative of u is not defined in an interval containing the initial time $t = 1$. There cannot be a solution curve passing through the point $(1, 2)$.

Example 1.4

Consider the initial value problem

$$u' = 2u^{1/2}, \quad u(0) = 0.$$

The reader should verify that both $u(t) = 0$ and $u(t) = t^2$ are solutions to this initial value problem on $t > 0$. Thus, it does not have a unique solution. More than one state evolves from the initial state.

Example 1.5

Consider the two similar initial value problems

$$\begin{aligned} u' &= 1 - u^2, \quad u(0) = 0, \\ u' &= 1 + u^2, \quad u(0) = 0. \end{aligned}$$

The first has solution

$$u(t) = \frac{e^{2t} - 1}{e^{2t} + 1},$$

which exists for every value of t. Yet the second has solution

$$u(t) = \tan t,$$

which exists only on the interval $\frac{-\pi}{2} < t < \frac{\pi}{2}$. So the solution to the first initial value problem is defined for all times, but the solution to the second "blows up" in finite time. These two problems are quite similar, yet the times for which their solutions exist are quite different.

The following theorem, which is proved in advanced books, provides partial answers to the questions raised above. The theorem basically states that if the right side $f(t, u)$ of the differential equation is nice enough, then there is a unique solution in a neighborhood the initial value.

Theorem 1.6

Let the function f be continuous on the open rectangle $R : a < t < b, c < u < d$ in the tu-plane and consider the initial value problem

$$\begin{cases} u' = f(t, u), \\ u(t_0) = u_0, \end{cases} \tag{1.3}$$

where (t_0, u_0) lies in the rectangle R. Then the IVP (1.3) has a solution $u = u(t)$ on some interval (α, β) containing t_0, where $(\alpha, \beta) \subset (a, b)$. If, in addition, the partial derivative[2] $f_u(t, u)$ is continuous on R, then (1.3) has a unique solution.

The **interval of existence** is the set of time values for which the solution to the initial value problem exists. Theorem 1.6 is called a *local* existence theorem because it guarantees a solution only in a small neighborhood of the initial time t_0; the theorem does not state how large the interval of existence is. Observe that the rectangle R mentioned in the theorem is open, and hence the initial point cannot lie on its boundary. In Example 1.5 both right sides of the equations, $f(t, u) = 1 - u^2$ and $f(t, u) = 1 + u^2$, are continuous in the plane, and their partial derivatives, $f_u = -2u$ and $f_u = 2u$, are continuous in in the plane. So the initial value problem for each would have a unique solution regardless of the initial condition.

In addition to theoretical questions, there are central issues from the viewpoint of *modeling* and *applications*; these are the questions we mentioned in the "To the Student" section.

1. How do we determine a differential equation that models, or governs, a given physical observation or phenomenon?

[2] We use subscripts to denote partial derivatives, and so $f_u = \frac{\partial f}{\partial u}$.

2. How do we find a solution (either analytically, approximately, graphically, or numerically) $u = u(t)$ of a differential equation?

The first question is addressed throughout this book by formulating model equations for systems in particles dynamics, circuit theory, biology, and in other areas. We learn some basic principles that sharpen our ability to invent explanatory models given by differential equations. The second question is one of developing methods, and our approach is to illustrate some standard analytic techniques that have become part of the subject. By an **analytic method** we mean manipulations that lead to a formula for the solution; such formulas are called **analytic solutions** or **closed-form** solutions. For most real-world problems it is difficult or impossible to obtain an analytic solution. By a **numerical solution** we mean an approximate solution that is obtained by some computer algorithm; a numerical solution can be represented by a data set (table of numbers) or by a graph. In real physical problems, numerical methods are the ones most often used. **Approximate solutions** can be formulas that approximate the actual solution (e.g., a polynomial formula) or they can be numerical solutions. Almost always we are interested in obtaining a graphical representation of the solution. Often we apply **qualitative methods**. These are methods designed to obtain important information from the DE without actually solving it either numerically or analytically. For a simple example, consider the DE $u' = u^2 + t^2$. Because $u' > 0$ we know that all solution curves are increasing. Or, for the DE $u' = u^2 - t^2$, we know solution curves have a horizontal tangent as they cross the straight lines $u = \pm t$. Quantitative methods emphasize understanding the underlying model, recognizing properties of the DE, interpreting the various terms, and using graphical properties to our benefit in interpreting the equation and plotting the solutions; often these aspects are more important than actually learning specialized methods for obtaining a solution formula.

Many of the methods, both analytic and numerical, can be performed easily on computer algebra systems such as Maple, Mathematica, or MATLAB, and some can be performed on advanced calculators that have a built-in computer algebra system. Although we often use a computer algebra system to our advantage, especially to perform tedious calculations, our goal is to understand concepts and develop technique. Appendix B contains information on using MATLAB and Maple.

1.1.2 Geometrical Interpretation

What does a differential equation $u' = f(t, u)$ tell us geometrically? At each point (t, u) of the tu-plane, the value of $f(t, u)$ is the slope u' of the solution curve $u = u(t)$ that goes through that point. This is because

$$u'(t) = f(t, u(t)).$$

This fact suggests a simple graphical method for constructing approximate solution curves for a differential equation. Through each point of a selected set of points (t, u) in some rectangular region (or window) of the tu-plane we draw a short line segment with slope $f(t, u)$. The collection of all these line segments, or mini-tangents, form the **direction field**, or **slope field**, for the equation. We may then sketch solution curves that fit this direction field; the curves must have the property that at each point the tangent line has the same slope as the slope of the direction field. For example, the slope field for the differential equation $u' = -u + 2t$ is defined by the right side of the differential equation, $f(t, u) = -u + 2t$. The slope field at the point $(2, 4)$ is $f(2, 3) = -3 + 2 \cdot 4 = 5$. This means the solution curve that passes through the point $(2, 4)$ has slope 5. Because it is tedious to calculate several mini-tangents, simple programs have been developed for calculators and computer algebra systems that perform this task automatically for us. Figure 1.4 shows a slope field and several solution curves that have been fit into the field.

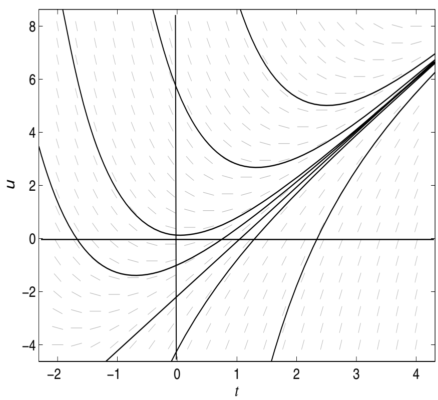

Figure 1.4 The slope field in the window $-2 \leq t \leq 4$, $-4 \leq u \leq 8$, with several approximate solution curves for the DE $u' = -u + 2t$.

Notice that a problem in differential equations is just opposite of that in differential calculus. In calculus we know the function (curve) and are asked to find the derivative (slope); in differential equations we know the slopes and try to find the state function that fits them.

Also observe that the simplicity of autonomous equations (no time t dependence on the right side)

$$u' = f(u)$$

shows itself in the slope field. In this case the slope field is independent of time, so on each horizontal line in the tu plane, where u has the same value, the slope field is the same. For example, the DE $u' = 3u(5-u)$ is autonomous, and along the horizontal line $u = 2$ the slope field has value 18. This means solution curves cross the line $u = 2$ with a relatively steep slope $u' = 18$.

EXERCISES

1. Verify that the two differential equations in Example 1.5 have solutions as stated.

2. From the set of commonly encountered functions, guess a nonzero solution $u = u(t)$ to the DE $u' = u^2$.

3. Show that $u(t) = \ln(t + C)$ is a one-parameter family of solutions of the DE $u' = e^{-u}$, where C is an arbitrary constant. Plot several members of this family. Find and plot a particular solution that satisfies the initial condition $u(0) = 0$.

4. Find a solution $u = u(t)$ of $u' + 2u = t^2 + 4t + 7$ in the form of a quadratic function of t.

5. Find value(s) of m such that $u = t^m$ is a solution to $2tu' = u$.

6. Plot the one-parameter family of curves $u(t) = (t^2 - C)e^{3t}$, and find a differential equation whose solution is this family.

7. Show that the one-parameter family of straight lines $u = Ct + f(C)$ is a solution to the differential equation $tu' - u + f(u') = 0$ for any value of the constant C.

8. Classify the first-order equations as linear or nonlinear, autonomous or nonautonomous.

   a) $u' = 2t^3u - 6$.

   b) $(\cos t)u' - 2u \sin u = 0$.

   c) $u' = \sqrt{1 - u^2}$.

   d) $7u' - 3u = 0$.

9. Explain Example 1.4 in the context of Theorem 1.6. In particular, explain why the theorem does not apply to this initial value problem. Which hypothesis fails?

10. Verify that the initial value problem $u' = \sqrt{u}$, $u(0) = 0$, has infinitely many solutions of the form

$$u(t) = \begin{cases} 0, & t \leq a \\ \frac{1}{4}(t-a)^2, & t > a, \end{cases}$$

where $a > 0$. Sketch these solutions for different values of a. What hypothesis fails in Theorem 1.6?

11. Consider the linear differential equation $u' = p(t)u + q(t)$. Is it true that the sum of two solutions is again a solution? Is a constant times a solution again a solution? Answer these same questions if $q(t) = 0$. Show that if u_1 is a solution to $u' = p(t)u$ and u_2 is a solution to $u' = p(t)u + q(t)$, then $u_1 + u_2$ is a solution to $u' = p(t)u + q(t)$.

12. By hand, sketch the slope field for the DE $u' = u(1 - \frac{u}{4})$ in the window $0 \leq t \leq 8$, $0 \leq u \leq 8$ at integer points. What is the value of the slope field along the lines $u = 0$ and $u = 4$? Show that $u(t) = 0$ and $u(t) = 4$ are constant solutions to the DE. On your slope field plot, draw in several solution curves.

13. Using a software package, sketch the slope field in the window $-4 \leq t \leq 4$, $-2 \leq u \leq 2$ for the equation $u' = 1 - u^2$ and draw several approximate solution curves. Lines and curves in the tu plane where the slope field is zero are called **nullclines**. For the given DE, find the nullclines. Graph the locus of points where the slope field is equal to -3.

14. Repeat Exercise 13 for the equation $u' = t - u^2$.

15. In the tu plane, plot the nullclines of the differential equation $u' = 2u^2(u - 4\sqrt{t})$.

16. Using concavity, show that the second-order DE $u'' - u = 0$ cannot have a solution (other than the $u = 0$ solution) that takes the value zero more than once. (Hint: construct a contradiction argument—if it takes the value zero twice, it must have a negative minimum or positive maximum.)

17. For any solution $u = u(t)$ of the DE $u'' - u = 0$, show that $(u')^2 - u^2 = C$, where C is a constant. Plot this one parameter-family of curves on a uu' set of axes.

18. Show that if u_1 and u_2 are both solutions to the DE $u' + p(t)u = 0$, then u_1/u_2 is constant.

19. Show that the linear initial value problem

$$u' = \frac{2(u-1)}{t}, \quad u(0) = 1,$$

has a continuously differentiable solution (i.e., a solution whose first derivative is continuous) given by

$$u(t) = \begin{cases} at^2 + 1, & t < 0, \\ bt^2 + 1, & t > 0, \end{cases}$$

for any constants a and b. Yet, there is no solution if $u(0) \neq 1$. Do these facts contradict Theorem 1.6?

1.2 Pure Time Equations

In this section we solve the simplest type of differential equation. First we need to recall the fundamental theorem of calculus, which is basic and used regularly in differential equations. For reference, we state the two standard forms of the theorem. They show that differentiation and integration are inverse processes.

Fundamental Theorem of Calculus I. If u is a differentiable function, the integral of its derivative is

$$\int_a^b \frac{d}{dt} u(t)dt = u(b) - u(a).$$

Fundamental Theorem of Calculus II. If g is a continuous function, the derivative of an integral with variable upper limit is

$$\frac{d}{dt} \int_a^t g(s)ds = g(t),$$

where the lower limit a is any number.

This last expression states that the function $\int_a^t g(s)ds$ is an antiderivative of g (i.e., a function whose derivative is g). Notice that $\int_a^t g(s)ds + C$ is also an antiderivative for any value of C.

The simplest differential equation is one of the form

$$u' = g(t), \tag{1.4}$$

where the right side of the differential equation is a given, known function $g(t)$. This equation is called a **pure time equation**. Thus, we seek a state function u whose derivative is $g(t)$. The fundamental theorem of calculus II, u must be an **antiderivative** of g. We can write this fact as $u(t) = \int_a^t g(s)ds + C$, or using the indefinite integral notation, as

$$u(t) = \int g(t)dt + C, \tag{1.5}$$

where C is an arbitrary constant, called the **constant of integration**. Recall that antiderivatives of a function differ by an additive constant. Thus, all solutions of (1.4) are given by (1.5), and (1.5) is called the general solution. The fact that (1.5) solves (1.4) follows from the fundamental theorem of calculus II.

Example 1.7

Find the general solution to the differential equation

$$u' = t^2 - 1.$$

Because the right side depends only on t, the solution u is an antiderivative of the right side, or

$$u(t) = \frac{1}{3}t^3 - t + C,$$

where C is an arbitrary constant. This is the general solution and it graphs as a family of cubic curves in the tu plane, one curve for each value of C. A particular antiderivative, or solution, can be determined by imposing an initial condition that picks out a specific value of the constant C, and hence a specific curve. For example, if $u(1) = 2$, then $\frac{1}{3}(1)^3 - 1 + C = 2$, giving $C = \frac{8}{3}$. The solution to the initial value problem is then $u(t) = \frac{1}{3}t^3 - t + \frac{8}{3}$.

Example 1.8

For equations of the form $u'' = g(t)$ we can take two successive antiderivatives to find the general solution. The following sequence of calculations shows how. Consider the DE

$$u'' = t + 2.$$

Then

$$\begin{aligned} u' &= \frac{1}{2}t^2 + 2t + C_1; \\ u &= \frac{1}{6}t^3 + t^2 + C_1 t + C_2. \end{aligned}$$

Here C_1 and C_2 are two arbitrary constants. For second-order equations we always expect two arbitrary constants, or a two-parameter family of solutions. It takes two auxiliary conditions to determine the arbitrary constants. In this example, if $u(0) = 1$ and if $u'(0) = 0$, then $c_1 = 1$ and $c_2 = 1$, and we obtain the particular solution $u = \frac{1}{6}t^3 + t^2 + 1$.

Example 1.9

The autonomous equation

$$u' = f(u)$$

cannot be solved by direct integration because the right side is not a known function of t; it depends on u, which is the unknown in the problem. Equations with the unknown u on the right side are not pure time equations.

Often it is not possible to find a simple expression for the antiderivative, or indefinite integral. For example, the functions $\frac{\sin t}{t}$ and e^{-t^2} have no simple analytic expressions for their antiderivatives. In these cases we must represent the antiderivative of g as

$$u(t) = \int_a^t g(s)ds + C$$

with a variable upper limit. Here, a is any fixed value of time and C is an arbitrary constant. We have used the dummy variable of integration s to avoid confusion with the upper limit of integration, the independent time variable t. It is really not advisable to write $u(t) = \int_a^t g(t)dt$.

Example 1.10

Solve the initial value problem

$$\begin{aligned} u' &= e^{-t^2}, \quad t > 0 \\ u(0) &= 2. \end{aligned}$$

The right side of the differential equation has no simple expression for its antiderivative. Therefore we write the antiderivative in the form

$$u(t) = \int_0^t e^{-s^2} ds + C.$$

The common strategy is to take the lower limit of integration to be the initial value of t, here zero. Then $u(0) = 2$ implies $C = 2$ and we obtain the solution to the initial value problem in the form of an integral,

$$u(t) = \int_0^t e^{-s^2} ds + 2. \tag{1.6}$$

If we had written the solution of the differential equation as

$$u(t) = \int e^{-t^2} dt + C,$$

in terms of an indefinite integral, then there would be no way to use the initial condition to evaluate the constant of integration, or evaluate the solution at a particular value of t.

We emphasize that integrals with a variable upper limit of integration define a function. Referring to Example 1.10, we can define the special function "erf" (called the **error function**) by

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds.$$

The factor $\frac{2}{\sqrt{\pi}}$ in front of the integral normalizes the function to force $\operatorname{erf}(+\infty) = 1$. Up to this constant multiple, the erf function gives the area under a bell-shaped curve $\exp(-s^2)$ from 0 to t. In terms of this special function, the solution (1.6) can be written

$$u(t) = 2 + \frac{\sqrt{\pi}}{2} \operatorname{erf}(t).$$

The erf function, which is plotted in figure 1.5, is an important function in probability and statistics, and in diffusion processes. Its values are tabulated in computer algebra systems and mathematical handbooks.

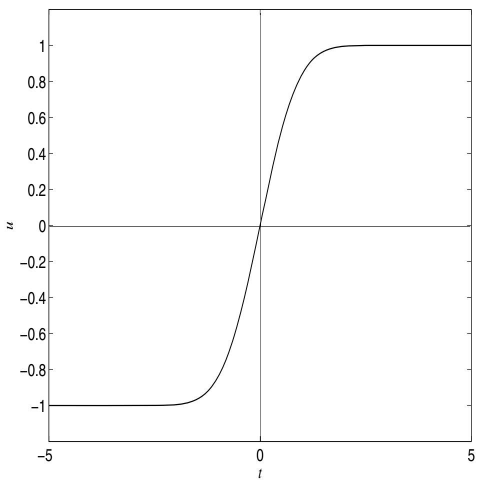

Figure 1.5 Graph of the erf function.

Functions defined by integrals are common in the applied sciences and are equally important as functions defined by simple algebraic formulas. To the point, the reader should recall that the natural logarithm can be defined by the integral

$$\ln t = \int_1^t \frac{1}{s} ds, \quad t > 0.$$

One important viewpoint is that differential equations often define special functions. For example, the initial value problem

$$u' = \frac{1}{t}, \quad u(1) = 0,$$

can be used to define the natural logarithm function $\ln t$. Other special functions of mathematical physics and engineering, for example, Bessel functions, Legendre polynomials, and so on, are usually defined as solutions to differential equations. By solving the differential equation numerically we can obtain values of the special functions more efficiently than looking those values up in tabulated form.

We end this section with the observation that one can find solution formulas using computer algebra systems like Maple, MATLAB, Mathematica, etc., and calculators equipped with computer algebra systems. Computer algebra systems do symbolic computation. Below we show the basic syntax in Maple, Mathematica, and on a TI-89 that returns the general solution to a differential equation or the solution to an initial value problem. MATLAB has a special add-on symbolic package that has similar commands. Our interest in this text is to use MATLAB for scientific computation, rather than symbolic calculation. Additional information on computing environments is in Appendix B.

The general solution of the first-order differential equation $u' = f(t, u)$ can be obtained as follows:

```
deSolve(u'=f(t,u),t,u)                       (TI-89)
dsolve(diff(u(t),t)=f(t,u(t)),u(t));         (Maple)
DSolve[u'[t]==f[t,u[t]], u[t], t]            (Mathematica)
```

To solve the initial value problem $u' = f(t, u)$, $u(a) = b$, the syntax is.

```
deSolve(u'= f(t,u) and u(a)=b, t, u)                 (TI-89)
dsolve(diff(u(t),t) = f(t,u(t)), u(a)=b, u(t));      (Maple)
DSolve[u'[t]==f[t,u[t]], u[a]==b, u[t], t]           (Mathematica)
```

EXERCISES

1. Using antiderivatives, find the general solution to the pure time equation $u' = t\cos(t^2)$, and then find the particular solution satisfying the initial condition $u(0) = 1$. Graph the particular solution on the interval $[-5, 5]$.

2. Solve the initial value problem $u' = \frac{t+1}{\sqrt{t}}, \quad u(1) = 4$.

3. Find a function $u(t)$ that satisfies the initial value problem $u'' = -3\sqrt{t}$, $u(1) = 1$, $u'(1) = 2$.

4. Find all state functions that solve the differential equation $u' = te^{-2t}$.

5. Find the solution to the initial value problem $u' = \frac{e^{-t}}{\sqrt{t}}$, $u(1) = 0$, in terms of an integral. Graph the solution on the interval $[1, 4]$ by using numerical integration to calculate values of the integral.

6. The differential equation $u' = 3u + e^{-t}$ can be converted into a pure time equation for a new dependent variable y using the transformation $u = ye^{3t}$. Find the pure time equation for y, solve it, and then determine the general solution u of the original equation.

7. Generalize the method of Exercise 6 by devising an algorithm to solve $u' = au + q(t)$, where a is any constant and q is a given function. In fact, show that
$$u(t) = Ce^{at} + e^{at} \int_0^t e^{-as} q(s) ds.$$
Using the fundamental theorem of calculus, verify that this function does solve $u' = au + q(t)$.

8. Use the chain rule and the fundamental theorem of calculus to compute the derivative of $\text{erf}(\sin t)$.

9. **Exact equations**. Consider a differential equation written in the (non-normal) form $f(t, u) + g(t, u)u' = 0$. If there is a function $h = h(t, u)$ for which $h_t = f$ and $h_u = g$, then the differential equation becomes $h_t + h_u u' = 0$, or, by the chain rule, just $\frac{d}{dt} h(t, u) = 0$. Such equations are called **exact** equations because the left side is (exactly) a total derivative of the function $h = h(t, u)$. The general solution to the equation is therefore given implicitly by $h(t, u) = C$, where C is an arbitrary constant.

   a) Show that $f(t, u) + g(t, u)u' = 0$ is exact if, and only if, $f_u = g_t$.

   b) Use part (a) to check if the following equations are exact. If the equation is exact, find the general solution by solving $h_t = f$ and $h_u = g$ for h (you may want to review the method of finding potential functions associated with a conservative force field from your multivariable calculus course).

      i. $u^3 + 3tu^2 u' = 0$.

      ii. $t^3 + \frac{u}{t} + (u^2 + \ln t)u' = 0$.

      iii. $u' = -\frac{\sin u - u \sin t}{t \cos u + \cos t}$.

10. An **integral equation** is an equation where the unknown $u(t)$ appears under an integral sign. Use the fundamental theorem of calculus to show that the integral equation

$$u(t) + \int_0^t e^{-p(t-s)} u(s) ds = A; \qquad p, A \quad \text{constants},$$

can be transformed into an initial value problem for $u(t)$.

11. Show that the integral equation

$$u(t) = e^{-2t} + \int_0^t s u(s) ds$$

can be transformed into an initial value problem for $u(t)$.

12. Show, by integration, that the initial value problem (1.3) can be transformed into the integral equation

$$u(t) = u_0 + \int_0^t f(s, u(s)) ds.$$

13. From the definition of the derivative, a difference quotient approximation to the first derivative is $u'(t) \cong \frac{u(t+h)-u(t)}{h}$. Use Taylor's theorem to show that an approximation for the second derivative is

$$u''(t) \cong \frac{u(t+h) - 2u(t) + u(t-h)}{h^2}.$$

(Recall that Taylor's expansion for a function u about the point t with increment h is

$$u(t+h) = u(t) + u'(t)h + \frac{1}{2}u''(t)h^2 + \cdots.$$

Use this and and a similar formula for $u(t-h)$.)

1.3 Mathematical Models

By a **mathematical model** we mean an equation, or set of equations, that describes some physical problem or phenomenon that has its origin in science, engineering, or some other area. Here we are interested in differential equation models. By **mathematical modeling** we mean the process by which we obtain and analyze the model. This process includes introducing the important and relevant quantities or variables involved in the model, making model-specific assumptions about those quantities, solving the model equations by some method, and then comparing the solutions to real data and interpreting the results. Often the solution method involves computer simulation. This comparison may lead to revision and refinement until we are satisfied that the model accurately describes the physical situation and is predictive of other similar observations. Therefore the subject of mathematical modeling involves physical intuition, formulation of equations, solution methods, and analysis. Overall, in mathematical modeling the overarching objective is to make sense of the world as we observe it, often by inventing caricatures of reality. Scientific exactness is sometimes sacrificed for mathematical tractability. Model predictions depend strongly on the assumptions, and changing the assumptions changes the model. If some assumptions are less critical than others, we say the model is robust to those assumptions.

The best strategy to learn modeling is to begin with simple examples and then graduate to more difficult ones. The reader is already familiar with some models. In an elementary science or calculus course we learn that Newton's second law, force equals mass times acceleration, governs mechanical systems such as falling bodies; Newton's inverse-square law of gravitation describes the motion of the planets; Ohm's law in circuit theory dictates the voltage drop across a resistor in terms of the current; or the law of mass action in chemistry describes how fast chemical reactions occur. In this course we learn new models based on differential equations. The importance of differential equations, as a subject matter, lies in the fact that differential equations describe many physical phenomena and laws in many areas of application. In this section we introduce some simple problems and develop differential equations that model the physical processes involved.

The first step in modeling is to select the relevant variables (independent and dependent) and parameters that describe the problem. Physical quantities have **dimensions** such as time, distance, degrees, and so on, or corresponding **units** such as seconds, meters, and degrees Celsius. The equations we write down as models must be dimensionally correct. Apples cannot equal oranges. Verifying that each term in our model has the same dimensions is the first task in obtaining a correct equation. Also, checking dimensions can often give us insight into what a term in the model might be. We always should be aware of the dimensions of the quantities, both variables and parameters, in a model, and we should always try to identify the physical meaning of the terms in the equations we obtain.

All of these comments about modeling are perhaps best summarized in a quote attributed to the famous psychologist, Carl Jung: "Science is the art of creating suitable illusions which the fool believes or argues against, but the wise man enjoys their beauty and ingenuity without being blind to the fact they are human veils and curtains concealing the abysmal darkness of the unknowable."

When one begins to feel too confident in the correctness of the model, he or she should recall this quote.

1.3.1 Particle Dynamics

In the late 16th and early 17th centuries scientists were beginning to quantitatively understand the basic laws of motion. Galileo, for example, rolled balls down inclined planes and dropped them from different heights in an effort to understand dynamical laws. But it was Isaac Newton in the mid-1600s (who developed calculus and the theory of gravitation) who finally wrote down a basic law of motion, known now as **Newton's second law**, that is in reality a differential equation for the state of the dynamical system. For a particle of mass m moving along a straight line under the influence of a specified external force F, the law dictates that "mass times acceleration equals the force on the particle," or

$$mx'' = F(t, x, x') \quad \text{(Newton's second law)}.$$

This is a second-order differential equation for the unknown location or position $x = x(t)$ of the particle. The force F may depend on time t, position $x = x(t)$, or velocity $x' = x'(t)$. This DE is called the **equation of motion** or the **dynamical equation** for the system. For second-order differential equations we impose *two* initial conditions, $x(0) = x_0$ and $x'(0) = v_0$, which fix the initial position and initial velocity of the particle, respectively. We expect that if the initial position and velocity are known, then the equation of motion should determine the state for all times $t > 0$.

Example 1.11

Suppose a particle of mass m is falling downward through a viscous fluid and the fluid exerts a resistive force on the particle proportional to the square of its velocity. We measure positive distance downward from the top of the fluid surface. There are two forces on the particle, gravity and fluid resistance. The gravitational force is mg and is positive because it tends to move the mass in a positive downward direction; the resistive force is $-ax'^2$, and it is negative because it opposes positive downward motion. The net force is then $F = mg - ax'^2$, and the equation of motion is $mx'' = mg - a(x'^2)^2$. This second-order equation can immediately be reformulated as a first-order differential equation for the velocity $v = x'$. Clearly

$$v' = g - \frac{a}{m}v^2.$$

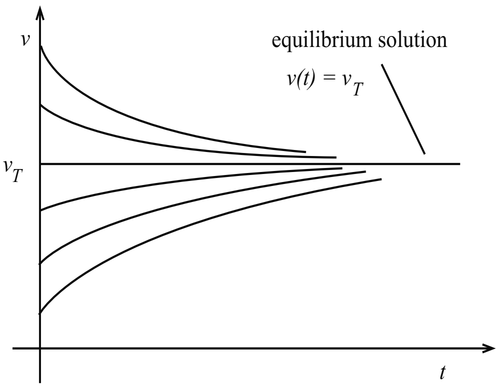

Figure 1.6 Generic solution curves, or time series plots, for the model $v' = g - (a/m)v^2$. For $v < v_T$ the solution curves are increasing because $v' > 0$; for $v > v_T$ the solution curves are decreasing because $v' < 0$. All the solution curves approach the constant terminal velocity solution $v(t) = v_T$.

If we impose an initial velocity, $v(0) = v_0$, then this equation and the initial condition gives an initial value problem for $v = v(t)$. Without solving the DE we can obtain important qualitative information from the DE itself. Over a long time, if the fluid were deep, we would observe that the falling mass would approach a constant, terminal velocity v_T. Physically, the terminal velocity occurs when the two forces, the gravitational force and resistive force, balance. Thus $0 = g - (av_T^2/m)$, or

$$v_T = \sqrt{\frac{mg}{a}}.$$

By direct substitution, we note that $v(t) = v_T$ is a constant solution of the differential equation with initial condition $v(0) = v_T$. We call such a constant solution an **equilibrium**, or **steady-state**, solution. It is clear that, regardless of the initial velocity, the system approaches this equilibrium state. This supposition is supported by the observation that $v' > 0$ when $v < v_T$ and $v' < 0$ when $v > v_T$. Figure 1.6 shows what we expect, illustrating several generic solution curves (time series plots) for different initial velocities. To find the position $x(t)$ of the object we would integrate the velocity $v(t)$, once it is determined; that is, $x(t) = \int_0^t v(s)ds$.

Example 1.12

A ball of mass m is tossed upward from a building of height h with initial velocity v_0. If we ignore air resistance, then the only force is that due to gravity, having magnitude mg, directed downward. Taking the positive direction upward with $x = 0$ at the ground, the model that governs the motion (i.e., the height $x = x(t)$ of the ball), is the initial value problem

$$mx'' = -mg, \quad x(0) = h, \quad x'(0) = v_0.$$

Note that the force is negative because the positive direction is upward. Because the right side is a known function (a constant in this case), the differential equation is a pure time equation and can be solved directly by integration (antiderivatives). If $x''(t) = -g$ (i.e., the second derivative is the constant $-g$), then the first derivative must be $x'(t) = -gt + c_1$, where c_1 is some constant (the constant of integration). We can evaluate c_1 using the initial condition $x'(0) = v_0$. We have $x'(0) = -g \times 0 + c_1 = v_0$, giving $c_1 = v_0$. Therefore, at any time the velocity is given by

$$x'(t) = -gt + v_0.$$

Repeating, we take another antiderivative. Then

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + c_2,$$

where c_2 is some constant. Using $x(0) = h$ we find that $c_2 = h$. Therefore the height of the ball at any time t is given by the familiar physics formula

$$x(t) = -\frac{1}{2}gt^2 + v_0 t + h.$$

Example 1.13

Imagine a mass m lying on a table and connected to a spring, which is in turn attached to a rigid wall (figure 1.7). At time $t = 0$ we displace the mass a positive distance x_0 to the right of equilibrium and then release it. If we ignore friction on the table then the mass executes simple harmonic motion; that is, it oscillates back and forth at a fixed frequency. To set up a model for the motion we follow the doctrine of mechanics and write down Newton's second law of motion, $mx'' = F$, where the state function $x = x(t)$ is the position of the mass at time t (we take $x = 0$ to be the equilibrium position and $x > 0$ to the right), and F is the external force. All that is required is to impose the form of the force. Experiments confirm that if the displacement is not too large (which we assume), then the force exerted by the spring is proportional to its displacement from equilibrium. That is,

$$F = -kx. \tag{1.7}$$

The minus sign appears because the force opposes positive motion. The proportionality constant k (having dimensions of force per unit distance) is called the **spring constant**, or **stiffness** of the spring, and equation (1.7) is called **Hooke's law**. Not every spring behaves in this manner, but Hooke's law is used as a model for some springs; it is an example of what in engineering is called a **constitutive relation**. It is an empirical result rather than a law of nature. To give a little more justification for Hooke's law, suppose the force F depends on the displacement x through $F = F(x)$, with $F(0) = 0$. Then by Taylor's theorem,

$$\begin{aligned} F(x) &= F(0) + F'(0)x + \frac{1}{2}F''(0)x^2 + \cdots \\ &= -kx + \frac{1}{2}F''(0)x^2 + \cdots, \end{aligned}$$

where we have defined $F'(0) = -k$. So Hooke's law has a general validity if the displacement is small, allowing the higher-order terms in the series to be neglected. We can measure the stiffness k of a spring by letting it hang from a ceiling without the mass attached; then attach the mass m and measure the elongation L after it comes to rest. The force of gravity mg must balance the restoring force kx of the spring, so $k = mg/L$. Therefore, assuming a Hookean spring, we have the equation of motion

$$mx'' = -kx \tag{1.8}$$

which is the **spring-mass equation**. The initial conditions (released at time zero at position x_0) are

$$x(0) = x_0, \quad x'(0) = 0.$$

We expect oscillatory motion. If we attempt a solution of (1.8) of the form $x(t) = A\cos\omega t$ for some frequency ω and amplitude A, we find upon substitution that $\omega = \sqrt{k/m}$ and $A = x_0$. Therefore the displacement of the mass is given by

$$x(t) = x_0 \cos\sqrt{k/m}t.$$

This solution represents an oscillation of amplitude x_0, frequency $\sqrt{k/m}$, and period $2\pi/\sqrt{k/m}$.

Example 1.14

Continuing with Example 1.13, if there is damping (caused, for example, by friction or submerging the system in a liquid), then the spring-mass equation

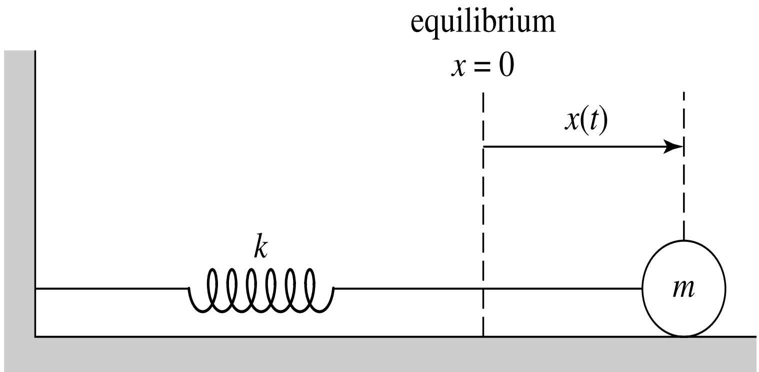

Figure 1.7 Spring-mass oscillator.

must be modified to account for the damping force. The simplest assumption, again a constitutive relation, is to take the resistive force F_{r} to be proportional to the velocity of the mass. Thus, also assuming Hooke's law for the spring force F_{s}, we have the **damped spring-mass equation**

$$mx'' = F_{\mathrm{r}} + F_{\mathrm{s}} = -cx' - kx.$$

The positive constant c is the damping constant. Both forces have negative signs because both oppose positive (to the right) motion. For this case we expect some sort of oscillatory behavior with the amplitude decreasing during each oscillation. In Exercise 1 you will show that solutions representing decaying oscillations do, in fact, occur.

Example 1.15

For conservative mechanical systems, another technique for obtaining the equation of motion is to apply the conservation of energy law: the kinetic energy plus the potential energy remain constant. We illustrate this method by finding the equation governing a frictionless pendulum of length l whose bob has mass m. See figure 1.8. As a state variable we choose the angle θ that it makes with the vertical. As time passes, the bob traces out an arc on a circle of radius l; we let s denote the arclength measured from rest ($\theta = 0$) along the arc. By geometry, $s = l\theta$. As the bob moves, its kinetic energy is one-half its mass times the velocity-squared; its potential energy is mgh, where h is the height above zero-potential energy level, taken where the pendulum is at rest. Therefore $\frac{1}{2}m(s')^2 + mgl(1 - \cos\theta) = E$, where E is the constant energy. In terms of the angle θ,

$$\frac{1}{2}l(\theta')^2 + g(1 - \cos\theta) = C, \tag{1.9}$$

where $C = E/ml$. The initial conditions are $\theta(0) = \theta_0$ and $\theta'(0) = \omega_0$, where θ_0 and ω_0 are the initial angular displacement and angular velocity, respectively. As it stands, the differential equation (1.9) is first-order; the constant C can be determined by evaluating the differential equation at $t = 0$. We get $C = \frac{1}{2}l\omega_0^2 + g(1 - \cos\theta_0)$. By differentiation with respect to t, we can write (1.9) as

$$\theta'' + \frac{g}{l}\sin\theta = 0. \tag{1.10}$$

This is a second-order nonlinear DE in $\theta(t)$ called the **pendulum equation**. It can also be derived directly from Newton's second law by determining the forces, which we leave as an exercise (Exercise 6). We summarize by stating that for a conservative mechanical system the equation of motion can be found either by determining the energies and applying the conservation of energy law, or by finding the forces and using Newton's second law of motion.

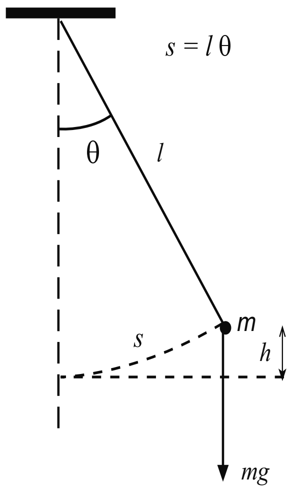

Figure 1.8 A pendulum consisting of a mass m attached to a rigid, weightless, rod of length l. The force of gravity is mg, directed downward. The potential energy is mgh where h is the height of the mass above the equilibrium position.

EXERCISES

1. When a mass of 0.3 kg is placed on a spring hanging from the ceiling, it elongates the spring 15 cm. What is the stiffness k of the spring?

2. Consider a damped spring-mass system whose position $x(t)$ is governed by the equation $mx'' = -cx' - kx$. Show that this equation can have a "decaying-oscillation" solution of the form $x(t) = e^{-\lambda t} \cos \omega t$. (Hint: By substituting into the differential equations, show that the decay constant λ and frequency ω can be determined in terms of the given parameters m, c, and k.)

3. A car of mass m is moving at speed V when it has to brake. The brakes apply a constant force F until the car comes to rest. How long does it take the car to stop? How far does the car go before stopping? Now, with specific data, compare the time and distance it takes to stop if you are going 30 mph vs. 35 mph. Take $m = 1000$ kg and $F = 6500$ N. Write a short paragraph on recommended speed limits in a residential areas.

4. Derive the pendulum equation (1.10) from the conservation of energy law (1.9) by differentiation.

5. A pendulum of length 0.5 meters has a bob of mass 0.1 kg. If the pendulum is released from rest at an angle of 15 degrees, find the total energy in the system.

6. Derive the pendulum equation (1.10) by resolving the gravitational force on the bob in the tangential and normal directions along the arc of motion and then applying Newton's second law. Note that only the tangential component affects the motion.

7. If the amplitude of the oscillations of a pendulum is small, then $\sin \theta$ is nearly equal to θ (why?) and the nonlinear equation (1.10) is approximated by the linear equation $\theta'' + (g/l)\theta = 0$.

   a) Show that the approximate linear equation has a solution of the form $\theta(t) = A \cos \omega t$ for some value of ω, which also satisfies the initial conditions $\theta(0) = A$, $\theta'(0) = 0$. What is the period of the oscillation?

   b) A 650 lb wrecking ball is suspended on a 20 m cord from the top of a crane. The ball, hanging vertically at rest against the building, is pulled back a small distance and then released. How soon does it strike the building?

8. An enemy cannon at distance L from a fort can fire a cannon ball from the top of a hill at height H above the ground level with a muzzle velocity v. How high should the wall of the fort be to guarantee that a cannon ball

will not go over the wall? Observe that the enemy can adjust the angle of its shot. (Hint: Ignoring air resistance, the governing equations follow from resolving Newton's second law for the horizontal and vertical components of the force: $mx'' = 0$ and $my'' = -mg$.)

1.3.2 Autonomous Differential Equations

In this section we introduce some simple qualitative methods to understand the dynamics of an autonomous differential equation

$$u' = f(u).$$

We introduce the methods in the context of population ecology, as well as in some other areas in the life sciences.

Models in biology often have a different character from fundamental laws in the physical sciences, such as Newton's second law of motion in mechanics or Maxwell's equations in electrodynamics. Ecological systems are highly complex and it is often impossible to include every possible factor in a model; the chore of modeling often comes in knowing what effects are important, and what effects are minor. Many models in ecology are often not based on physical law, but rather on observation, experiment, and reasoning.

Ecology is the study of how organisms interact with their environment. A fundamental problem in population ecology is to determine what mechanisms operate to regulate animal populations. Let $p = p(t)$ denote the population of an animal species at time t. For the human population, T. Malthus (in the late 1700s) proposed the model

$$\frac{p'}{p} = r,$$

which states that the "*per capita* growth rate is constant,"where the constant $r > 0$ is the **growth rate** given in dimensions of time^{-1}. We can regard r as the birth rate minus the death rate, or $r = b - d$. This *per capita* law is same as

$$p' = rp,$$

which says that the growth rate is proportional to the population. It is easily verified (check this!) that a one-parameter family of solutions is given by

$$p(t) = Ce^{rt},$$

where C is any constant. If there is an initial condition imposed, that is, $p(0) = p_0$, then $C = p_0$ and we have picked out a particular solution $p(t) = p_0 e^{rt}$ of the DE, that is, the one that satisfies the initial condition. Therefore, the Malthus model predicts exponential population growth (figure 1.9).

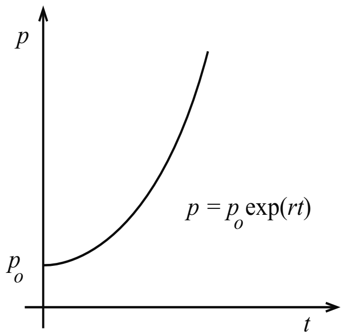

Figure 1.9 The Malthus model for population growth: $p(t) = p_0 e^{rt}$.

The reader should note a difference between the phrases "*per capita* growth rate" and "growth rate." To say that the *per capita* growth rate is 2% (per time) is to say that $p'/p = 0.02$, which gives exponential growth; to say that the growth rate is 2% (animals per time) is to say $p' = 0.02$, which forces $p(t)$ to be of the form $p(t) = 0.02t + K$, (K constant), which is linear growth.

In animal populations, for fairly obvious reasons, we do not expect exponential growth over long times. Environmental factors and competition for resources limit the population when it gets large. Therefore we might expect the *per capita* growth rate r (which is constant in the Malthus model) to decrease as the population increases. The simplest assumption is a linearly decreasing *per capita* growth rate where the rate becomes zero at some maximum carrying capacity K. See figure 1.10. This gives the **logistics model** of population growth (developed by P. Verhulst in the 1800s) by

$$\frac{p'}{p} = r(1 - \frac{p}{K}) \quad \text{or} \quad p' = rp(1 - \frac{p}{K}). \tag{1.11}$$

Clearly we may write this autonomous equation in the form

$$p' = rp - \frac{r}{K}p^2.$$

The first term is a positive **growth term**, which is just the Malthus term. The second term, which is quadratic in p, decreases the population growth rate and is the **competition term**. Note that if there were p animals, then there would be about p^2 encounters among them. So the competition term is proportional to the number of possible encounters, which is a reasonable model. Exercise 11 presents an alternate derivation of the logistics model based on food supply.

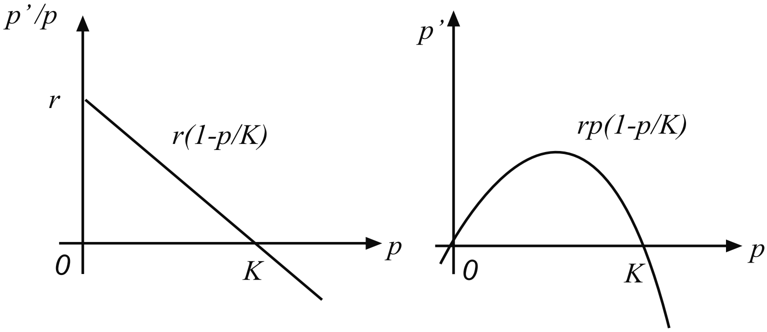

Figure 1.10 Plots of the logistics model of population growth. The left plot shows the *per capita* growth rate vs. population, and the right plot shows the growth rate vs. population. Both plots give important interpretations of the model.

For any initial condition $p(0) = p_0$ we can find the formula for the solution to the logistics equation (1.11). (You will solve the logistics equation in Exercise 8.) But, there are qualitative properties of solutions that can be exposed without actually finding the solution. Often, all we may want are qualitative features of a model. First, we note that there are two constant solutions to (1.11), $p(t) = 0$ and $p(t) = K$, corresponding to no animals (extinction) and to the number of animals represented by the carrying capacity, respectively. These constant solutions are found by setting the right side of the equation equal to zero (because that forces $p' = 0$, or $p =$ constant). The constant solutions are called steady-state, or **equilibrium**, solutions. If the population is between $p = 0$ and $p = K$ the right side of (1.11) is positive, giving $p' > 0$; for these population numbers the population is increasing. If the population is larger than the carrying capacity K, then the right side of (1.11) is negative and the population is decreasing. These facts can also be observed from the growth rate plot in figure 1.10. These observations can be represented conveniently on a **phase line** plot as shown in figure 1.11. We first plot the growth rate p' vs. p, which in this case is a parabola opening downward. The points of intersection on the p axis are the equilibrium solutions 0 and K. We then indicate by a directional arrow on the p axis those values of p where the solution $p(t)$ is increasing (where $p' > 0$) or decreasing ($p' < 0$). Thus the arrow points to the right when the graph of the growth rate is above the axis, and it points to the left when the graph is below the axis. In this context we call the p axis a phase line. We can regard the phase line as a one-dimensional, parametric solution space with the population $p = p(t)$ tracing out points on that line as t increases. In the range $0 < p < K$ the arrow points right because $p' > 0$. So

$p(t)$ increases in this range. For $p > K$ the arrow points left because $p' < 0$. The population $p(t)$ decreases in this range. These qualitative features can be easily transferred to time series plots (figure 1.12) showing $p(t)$ vs. t for different initial conditions.

Both the phase line and the time series plots imply that, regardless of the initial population (if nonzero), the population approaches the carrying capacity K. This equilibrium population $p = K$ is called an **attractor**. The zero population is also an equilibrium population. But, near zero we have $p' > 0$, and so the population diverges away from zero. We say the equilibrium population $p = 0$ is a **repeller**. (We are considering only positive populations, so we ignore the fact that $p = 0$ could be approached on the left side). In summary, our analysis has determined the complete qualitative behavior of the logistics population model.

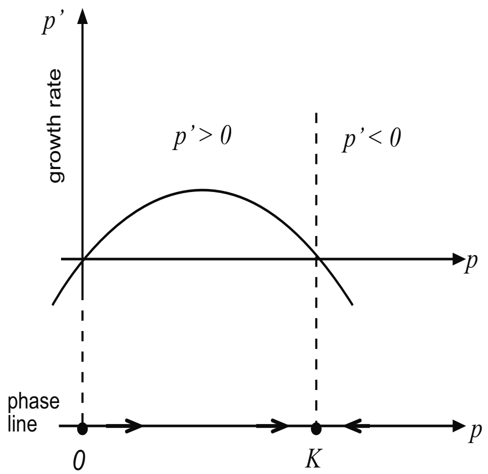

Figure 1.11 The p axis is the phase line, on which arrows indicate an increasing or decreasing population for certain ranges of p.

This qualitative method used to analyze the logistics model is applicable to any autonomous equation

$$u' = f(u). \tag{1.12}$$

The **equilibrium solutions** are the constant solutions, which are roots of the algebraic equation $f(u) = 0$. Thus, if u^* is an equilibrium, then $f(u^*) = 0$.

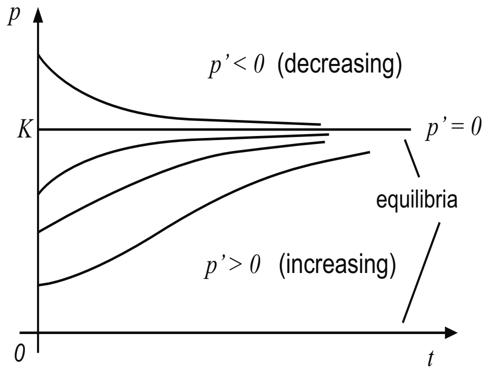

Figure 1.12 Time series plots of solutions to the logistics equation for various initial conditions. For $0 < p < K$ the population increases and approaches K, whereas for $p > K$ the population decreases to K. If $p(0) = K$, then $p(t) = K$ for all times $t > 0$; this is the equilibrium solution.

These are the values where the graph of $f(u)$ vs. u intersects the u-axis. We always assume the equilibria are **isolated**; that is, if u^* is an equilibrium, then there is an open interval containing u^* that contains no other equilibria. Figure 1.13 shows a generic plot where the equilibria are $u^* = a, b, c$. In between the equilibria we can observe the values of u for which the population is increasing ($f(u) > 0$) or decreasing ($f(u) < 0$). We can then place arrows on the phase line, or the u-axis, in between the equilibria showing direction of the movement (increasing or decreasing) as time increases. If desired, the information from the phase line can be translated into time series plots of $u(t)$ vs. t (figure 1.14). In between the constant, equilibrium solutions, the other solution curves increase or decrease; oscillations are not possible. Moreover, assuming f is a well-behaved function ($f'(u)$ is continuous), solution curves actually approach the equilibria, getting closer and closer as time increases. By uniqueness, the curves never intersect the constant equilibrium solutions.

On the phase line, if arrows on both sides of an equilibrium point toward that equilibrium point, then we say the equilibrium point is an **attractor**. If both of the arrows point away, the equilibrium is called a **repeller**. Attractors are called **asymptotically stable** because if the system is in that constant equilibrium state and then it is given a small **perturbation** (i.e., a change or "bump") to a nearby state, then it just returns to that state as $t \to +\infty$. It is clear that real systems will seek out the stable states. Repellers are **unstable** because a small perturbation can cause the system to go to a different equilib-

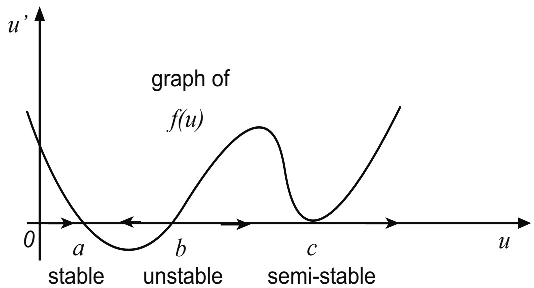

Figure 1.13 A generic plot showing $f(u)$, which is u' vs. u. The points of intersection, a, b, c, on the u-axis are the equilibria. The arrows on the u-axis, or phase line, show how the state u changes with time between the equilibria. The direction of the arrows is read from the plot of $f(u)$. They are to the right when $f(u) > 0$ and to the left when $f(u) < 0$. The phase line can either be drawn as a separate line with arrows, as in figure 1.11, or the arrows can be drawn directly on the u-axis of the plot, as is done here.

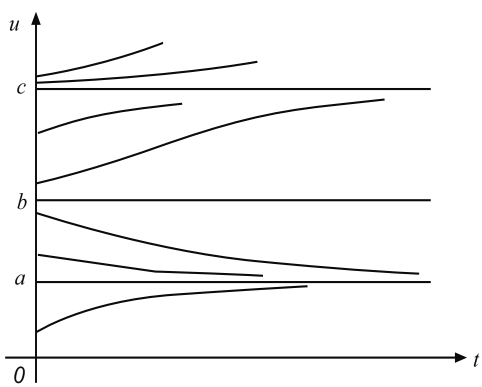

Figure 1.14 Time series plots of (1.12) for different initial conditions. The constant solutions are the equilibria.

rium or even go off to infinity. In the logistics model for population growth we observe (figure 1.11) that the equilibrium $u = K$ is an asymptotically stable attractor, and the zero population $u = 0$ is unstable; all solutions approach the carrying capacity $u = K$ at $t \to +\infty$. Finally, if one of the arrows points toward the equilibrium and one points away, we say the equilibrium is **semi-stable**. Semi-stable equilibria are not stable.

We emphasize that when we say an equilibrium u^* is asymptotically stable, our understanding is that this is with respect to *small* perturbations. To fix the idea, consider a population of fish in a lake that is in an asymptotically stable state u^*. A small death event, say caused by some toxic chemical that is dumped into the lake, will cause the population to drop. Asymptotic stability means that the system will return the original state u^* over time. We call this **local asymptotic stability**. If many fish are killed by the pollution, then the perturbation is not small and there is no guarantee that the fish population will return to the original state u^*. For example, a catastrophe or bonanza could cause the population to jump beyond some other equilibrium. If the population returns to the state u^* for all perturbations, no matter how large, then the state u^* is called **globally asymptotically stable**. A more precise definition of local asymptotic stability can be given as follows. An isolated equilibrium state u^* of (1.12) is locally asymptotically stable if there is an open interval I containing u^* with $\lim_{t\to+\infty} u(t) = u^*$ for any solution $u = u(t)$ of (1.12) with $u(0)$ in I. That is, each solution starting in I converges to u^*.

Note that a semi-stable point is not asymptotically stable; such points are, in fact, not stable.

Example 1.16

(Dimensionless Models) When we formulate a mathematical model we sometimes trade in the dimensioned quantities in our equations for dimensionless ones. In doing so we obtain a *dimensionless* model, often containing fewer parameters than the original model. The idea is simple. If, for example, time t is the independent variable in a model of population growth and a constant r, with dimension time^{-1}, representing the *per capita* growth rate appears in the model, then the variable $\tau = t/r^{-1} = rt$ has no dimensions, that is, it is dimensionless (time divided by time). It can serve as a new independent variable in the model representing "dimensionless time", or time measured relative to the inverse growth rate. We say r^{-1} is a **time scale** in the problem. Every *variable* in a model has a natural **scale** with which we can measure its relative value; these scales are found from the parameters in the problem. The population p of an animal species in a geographical region can be scaled by the carrying capacity K of the region, which is the number of animals the region can support. Then the variable $P = p/K$ is dimensionless (animals divided by animals) and represents the fraction of the region's capacity that is filled. If the carrying capacity is large, the actual population p could be large, requiring us to work with and plot big numbers. However, the dimensionless population P is represented by smaller numbers which are easier to deal with and plot. For some models selecting dimensionless dependent and independent variables can pay off in great benefits—it can help us understand the magnitude of various terms in the equations, and it can reduce the number of parameters in a problem, thus giving simplification. We illustrate this procedure for the initial value problem for the logistics model,

$$p' = rp(1 - \frac{p}{K}), \quad p(0) = p_0. \tag{1.13}$$

There are two variables in the problem, the independent variable t, measured in *time*, and the dependent variable p, measured in *animals*. There are three parameters in the problem: the carrying capacity K and initial population p_0, both measured in animals, and the growth rate r measured in 1/time. Let us define new dimensionless variables $\tau = rt = t/r^{-1}$ and $P = p/K$. These represent a "dimensionless time" and a "dimensionless population"; P is measured relative to the carrying capacity and t is measured relative to the growth rate; the values K and r^{-1} are called scales. Now we transform the DE into the new dimensionless variables. First, we transform the derivative:

$$\frac{dp}{dt} = \frac{d(KP)}{d(\tau/r)} = rK\frac{dP}{d\tau}.$$

Then the logistics DE in (1.13) becomes

$$rK\frac{dP}{d\tau} = r(KP)(1 - \frac{KP}{K}),$$

or

$$\frac{dP}{d\tau} = P(1 - P).$$

In dimensionless variables τ and P, the parameters in the DE disappeared! Next, the initial condition becomes $KP(0) = p_0$, or

$$P(0) = \alpha,$$

where $\alpha = p_0/K$ is a dimensionless parameter (animals divided by animals). In summary, the dimensioned model (1.13), with three parameters, can be replaced by the dimensionless model with only a single dimensionless parameter α:

$$\frac{dP}{d\tau} = P(1 - P), \quad P(0) = \alpha. \tag{1.14}$$

What this tells us is that although three parameters appear in the original problem, only a single combination of those parameters is relevant. We may as well work with the simpler, equivalent, dimensionless model (1.14) where populations are measured relative to the carrying capacity and time is measured relative to how fast the population is growing. For example, if the carrying capacity is $K = 300,000$, and the dimensioned p varies between $0 < p < 300,000$, it is much simpler to have dimensionless populations P with $0 < P < 1$. Furthermore, in the simplified form (1.14) it is easy to see that the equilibria are $P = 0$ and $P = 1$, the latter corresponding to the carrying capacity $p = K$.

We have pointed out that an autonomous model can be easily analyzed qualitatively without ever finding the solution. In this paragraph we introduce a simple method for solving a general autonomous equation

$$u' = f(u). \tag{1.15}$$

The method is called **separation of variables**. If we divide both sides of the equation by $f(u)$, we get

$$\frac{1}{f(u)}u' = 1.$$

Now, remembering that u is a function of t, we integrate both sides with respect to t to obtain

$$\int \frac{1}{f(u)}u'dt = \int 1dt + C = t + C,$$

where C is an arbitrary constant. A substitution $u = u(t)$, $du = u'(t)dt$ reduces the integral on the left and we obtain

$$\int \frac{1}{f(u)}du = t + C. \tag{1.16}$$

This equation, once the integral is calculated, defines the general solution $u = u(t)$ of (1.15) implicitly. We may or may not be able to actually calculate the integral and solve for u in terms of t to determine an explicit solution $u = u(t)$. This method of separating the variables (putting all the terms with u on the left side) is a basic technique in differential equations; it is adapted to more general equations in Chapter 2.

Example 1.17

Consider the **growth–decay** model

$$u' = ru, \tag{1.17}$$

where r is a given constant. If $r < 0$ then the equation models **exponential decay**; if $r > 0$ then the equation models **exponential growth** (e.g., population growth, as in the Malthus model). We apply the separation of variables method. Dividing by u (we could divide by ru, but we choose to leave the constant the right side) and taking antiderivatives gives

$$\int \frac{1}{u} u' dt = \int r dt + C.$$

Because $u' dt = du$, we can write

$$\int \frac{1}{u} du = rt + C.$$

Integrating gives

$$\ln |u| = rt + C \quad \text{or} \quad |u| = e^{rt+C} = e^C e^{rt}.$$

This means either $u = e^C e^{rt}$ or $u = -e^C e^{rt}$. Therefore the general solution of the growth–decay equation can be written compactly as

$$u(t) = C_1 e^{rt},$$

where C_1 has been written for $\pm e^C$, and is an arbitrary constant. If an initial condition

$$u(0) = u_0 \tag{1.18}$$

is prescribed on (1.17), it is straightforward to show that $C_1 = u_0$ and the solution to the initial value problem (1.17)–(1.18) is

$$u(t) = u_0 e^{rt}.$$

The growth–decay equation and its solution given in Example 1.16 occur often enough in applications that they are worthy of memorization. The equation models processes like growth of a population, mortality (death), growth of principal in a money account where the interest is compounded continuously at rate r, and radioactive decay, like the decay of Carbon-14 used in carbon dating.

EXERCISES

1. (The Allee effect) At low population densities it may be difficult for an animal to reproduce because of a limited number of suitable mates. A population model that predicts this behavior is the Allee model (W. C. Allee, 1885–1955)

$$p' = rp\left(\frac{p}{a} - 1\right)\left(1 - \frac{p}{K}\right), \quad 0 < a < K.$$

Find the *per capita* growth rate and plot the *per capita* rate vs. p. Graph p' vs. p, determine the equilibrium populations, and draw the phase line. Which equilibria are attractors and which are repellers? Which are asymptotically stable? From the phase line plot, describe the long time behavior of the system for different initial populations, and sketch generic time series plots for different initial conditions.

2. Modify the logistics model to include harvesting. That is, assume that the animal population grows logistically while, at the same time, animals are being removed (by hunting, fishing, or whatever) at a constant rate of h animals per unit time. What is the governing DE? Determine the equilibria. Which are asymptotically stable? Explain how the system will behave for different initial conditions. Does the population ever become extinct?

3. The **Ricker population law** is

$$p' = rpe^{-ap},$$

where r and a are constants. Determine the dimensions of r and a. At what population is the growth rate maximum? Make a generic sketch of the *per capita* growth rate and write a brief explanation of how a population behaves under this law. Is it possible to use the separation of variables method to find a simple formula for $p(t)$?

4. In this exercise we introduce a simple model of growth of an individual organism over time. For simplicity, we assume it is shaped like a cube having sides equal to $L = L(t)$. Organisms grow because they assimilate nutrients and then use those nutrients in their energy budget for maintenance and to build structure. It is conjectured that the organism's growth rate in volume equals the assimilation rate minus the rate food is used. Food is assimilated at a rate proportional to its surface area because food must ultimately pass across the cell walls; food is used at a rate proportional to its volume because ultimately cells are three-dimensional. Show that the differential equation governing its size $L(t)$ can be written

$$L'(t) = a - bL,$$

where a and b are positive parameters. What is the maximum length the organism can reach? Using separation of variables, show that if the length of the organism at time $t = 0$ is $L(0) = 0$ (it is very small), then the length is given by $L(t) = \frac{a}{b}(1 - e^{-bt})$. Does this function seem like a reasonable model for growth?

5. In a classical ecological study of budworm outbreaks in Canadian fir forests, researchers proposed that the budworm population N was governed by the

law

$$N' = rN\left(1 - \frac{N}{K}\right) - P(N),$$

where the first term on the right represents logistics growth, and where $P(N)$ is a *bird-predation* rate given by

$$P(N) = \frac{aN^2}{N^2 + b^2}.$$

Sketch a graph of the bird-predation rate vs. N and discuss its meaning. What are the dimensions of all the constants and variables in the model? Select new dimensionless independent and dependent variables by

$$\tau = \frac{t}{b/a}, \quad n = \frac{N}{b}$$

and reformulate the model in dimensionless variables and dimensionless constants. Working with the dimensionless model, show that there is at least one and at most three positive equilibrium populations. What can be said about their stability?

6. Use the method of separation of variables to find the general solution to the following autonomous differential equations.

   a) $u' = \sqrt{u}$.

   b) $u' = e^{-2u}$.

   c) $u' = 1 + u^2$.

   d) $u' = 3u - a$, where a is a constant.

   e) $u' = \frac{u}{4+u^2}$.

   f) $u' = e^{u^2}$.

7. In Exercises 6 (a)–(f) find the solution to the resulting IVP when $u(0) = 1$.

8. Find the general solution to the logistics equation $u' = ru(1 - u/K)$ using the separation of variables method. Hint: use the partial fractions decomposition

$$\frac{1}{u(K-u)} = \frac{1/K}{u} + \frac{1/K}{K-u}.$$

Show that a solution curve that crosses the line $u = K/2$ has an inflection point at that position.

9. (Carbon dating) The half-life of Carbon-14 is 5730 years. That is, it takes this many years for half of a sample of Carbon-14 to decay. If the decay of Carbon-14 is modeled by the DE $u' = -ku$, where u is the amount of Carbon-14, find the decay constant k. (Answer: 0.000121 yr^{-1}). In an artifact the percentage of the original Carbon-14 remaining at the present day was measured to be 20 percent. How old is the artifact?

10. In 1950, charcoal from the Lascaux Cave in France gave an average count of 0.09 disintegrations of C^{14} (per minute per gram). Living wood gives 6.68 disintegrations. Estimate the date that individuals lived in the cave.

11. In the usual Malthus growth law $N' = rN$ for a population of size N, assume the growth rate is a linear function of food availability F; that is, $r = bF$, where b is the conversion factor of food into newborns. Assume that F_T is the total, constant food in the system with $F_T = F + cN$, where cN is amount of food already consumed. Write down a differential equation for the population N. What is the carrying capacity? What is the population as t gets large?

12. One model of tumor growth is the Gompertz equation

$$R' = -aR\ln\left(\frac{R}{k}\right),$$

where $R = R(t)$ is the tumor radius, and a and k are positive constants. Find the equilibria and analyze their stability. Can you solve this differential equation for $R(t)$?

13. A population model is given by $p' = rP(P-m)$, where r and m are positive constants. State reasons for calling this the **explosion–extinction** model.

14. In a fixed population of N individuals let I be the number of individuals infected by a certain disease and let S be the number susceptible to the disease with $I+S = N$. Assume that the rate that individuals are becoming infected is proportional to the number of infectives times the number of susceptibles, or $I' = aSI$, where the positive constant a is the transmission coefficient. Assume no individual gets over the disease once it is contracted. If $I(0) = I_0$ is a small number of individuals infected at $t = 0$, find an initial value problem for the number infected at time t. Explain how the disease evolves. Over a long time, how many contract the disease?

15. In Example 1.11 we modeled the velocity of an object falling in a fluid by the equation $mv' = mg - av^2$. If $v(0) = 0$, find an analytic formula for $v(t)$.

1.3.3 Stability and Bifurcation

Differential equations coming from modeling physical phenomena almost always contain one or more parameters. It is of great interest to determine how equilibrium solutions depend upon those parameters. For example, the logistics growth equation

$$p' = rp(1 - \frac{p}{K})$$

has two parameters: the growth rate r and the carrying capacity K. Let us add **harvesting**; that is, we remove animals at a constant rate $H > 0$. We can think of a fish population where fish are caught at a given rate H. Then we have the model

$$p' = rp(1 - \frac{p}{K}) - H. \qquad (1.19)$$

We now ask how possible equilibrium solutions and their stability depend upon the rate of harvesting H. Because there are three parameters in the problem, we can nondimensionalize to simplify it. We introduce new dimensionless variables by

$$u = \frac{p}{K}, \quad \tau = rt.$$

That is, we measure populations relative to the carrying capacity and time relative to the inverse growth rate. In terms of these dimensionless variables, (1.19) simplifies to (check this!)

$$u' = u(1 - u) - h,$$

where $h = H/rK$ is a single dimensionless parameter representing the ratio of the harvesting rate to the product of the growth rate and carrying capacity. We can now study the effects of changing h to see how harvesting influences the steady-state fish populations in the model. In dimensionless form, we think of h as the harvesting parameter; information about changing h will give us information about changing H.

The equilibrium solutions of the dimensionless model are roots of the quadratic equation

$$f(u) = u(1 - u) - h = 0,$$

which are

$$u^* = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4h}.$$

The growth rate $f(u)$ is plotted in figure 1.15 for different values of h. For $h < 1/4$ there are two positive equilibrium populations. The graph of $f(u)$ in this case is concave down and the phase line shows that the smaller one is unstable, and the larger one is asymptotically stable. As h increases these populations begin to come together, and at $h = 1/4$ there is only a single unstable equilibrium. For $h > 1/4$ the equilibrium populations cease to exist. So, when harvesting is small, there are two equilibria, one being stable; as harvesting increases the equilibrium disappears. We say that a **bifurcation** (bifurcation means "dividing") occurs at the value $h = 1/4$. This is the value where there is a significant change in the character of the equilibria. For $h \geq 1/4$ the population will become extinct, regardless of the initial condition (because $f(u) < 0$ for all u). All these facts can be conveniently represented on a **bifurcation diagram**. See figure 1.16. In a bifurcation diagram we plot the equilibrium solutions u^* vs. the parameter h. In this context, h is called the **bifurcation parameter**. The plot is a parabola opening to the left. We observe that the upper branch of the parabola corresponds to the larger equilibrium, and all solutions represented by that branch are asymptotically stable; the lower branch, corresponding to the smaller solution, is unstable.

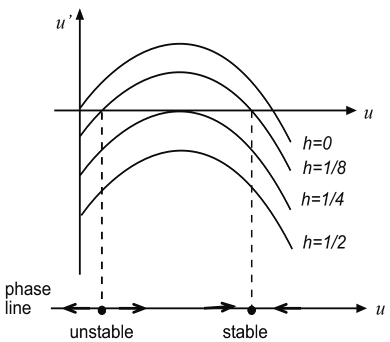

Figure 1.15 Plots of $f(u) = u(1-u) - h$ for different values of h. The phase line is plotted in the case $h = 1/8$.

Finally, we give an analytic criterion that allows us to determine stability of an equilibrium solution by simple calculus. Let

$$u' = f(u) \tag{1.20}$$

be a given autonomous systems and u^* an isolated equilibrium solution, so that $f(u^*) = 0$. We observe from figure 1.13 that when the slope of the graph of $f(u)$ at the equilibrium point is negative, the graph falls from left to right and both arrows on the phase line point toward the equilibrium point. Therefore, a condition that guarantees the equilibrium point u^* is asymptotically stable is $f'(u^*) < 0$. Similarly, if the graph of $f(u)$ strictly increases as it passes through the equilibrium, then $f'(u^*) > 0$ and the equilibrium is unstable. If the slope of $f(u)$ is zero at the equilibrium, then any pattern of arrows is possible and there is no information about stability. If $f'(u^*) = 0$, then u^* is a critical point of

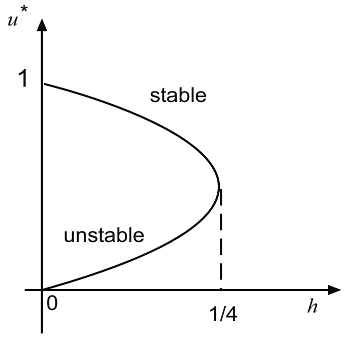

Figure 1.16 Bifurcation diagram: plot of the equilibrium solution as a function of the bifurcation parameter h, $u^* = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4h}$. For $h > \frac{1}{4}$ there are no equilibria and for $h < \frac{1}{4}$ there are two, with the larger one being stable. A bifurcation occurs at $h = \frac{1}{4}$.

f and could be a local maximum, local minimum, or have an inflection point. If there is a local maximum or local minimum, then u^* is semi-stable (which is not stable). If there is an inflection point, then f changes sign at u^* and we obtain either a repeller or an attractor, depending on how the concavity changes, negative to positive, or positive to negative.

Theorem 1.18

Let u^* be an isolated equilibrium for the autonomous system (1.20). If $f'(u^*) < 0$, then u^* is asymptotically stable; if $f'(u^*) > 0$, then u^* is unstable. If $f'(u^*) = 0$, then there is no information about stability.

Example 1.19

Consider the logistics equation $u' = f(u) = ru(1 - u/K)$. The equilibria are $u^* = 0$ and $u^* = K$. The derivative of $f(u)$ is $f'(u) = r - 2ru/K$. Evaluating the derivative at the equilibria gives

$$f'(0) = r > 0, \quad f'(K) = -r < 0.$$

Therefore $u^* = 0$ is unstable and $u^* = K$ is asymptotically stable.

EXERCISES

1. A fish population in a lake is harvested at a constant rate, and it grows logistically. The growth rate is 0.2 per month, the carrying capacity is 40

(thousand), and the harvesting rate is 1.5 (thousand per month). Write down the model equation, find the equilibria, and classify them as stable or unstable. Will the fish population ever become extinct? What is the most likely long-term fish population?

2. For the following equations, find the equilibria and sketch the phase line. Determine the type of stability of all the equilibria. Use Theorem 1.18 to confirm stability or instability.

   a) $u' = u^2(3-u)$.

   b) $u' = 2u(1-u) - \frac{1}{2}u$.

   c) $u' = (4-u)(2-u)^3$.

3. For the following models, which contain a parameter h, find the equilibria in terms of h and determine their stability. Construct a bifurcation diagram showing how the equilibria depend upon h, and label the branches of the curves in the diagram as unstable or stable.

   a) $u' = hu - u^2$.

   b) $u' = (1-u)(u^2-h)$.

4. Consider the model $u' = (\lambda - b)u - au^3$, where a and b are fixed positive constants and λ is a parameter that may vary.

   a) If $\lambda < b$ show that there is a single equilibrium and that it is asymptotically stable.

   b) If $\lambda > b$ find all the equilibria and determine their stability.

   c) Sketch the bifurcation diagram showing how the equilibria vary with λ. Label each branch of the curves shown in the bifurcation diagram as stable or unstable.

5. The biomass P of a plant grows logistically with intrinsic growth rate r and carrying capacity K. At the same time it is consumed by herbivores at a rate

$$\frac{aP}{b+P},$$

per herbivore, where a and b are positive constants. The model is

$$P' = rP(1-\frac{P}{K}) - \frac{aPH}{b+P},$$

where H is the density of herbivores. Assume $aH > br$, and assume r, K, a, and b are fixed. Plot, as a function of P, the growth rate and the consumption rate for several values of H on the same set of axes, and identify the values of P that give equilibria. What happens to the equilibria as the herbivory H is steadily increased from a small value to a large value? Draw a bifurcation diagram showing this effect. That is, plot equilibrium solutions vs. the parameter H. If herbivory is slowly increased so that the plants become extinct, and then it is decreased slowly back to a low level, do the plants return?

6. A deer population grows logistically and is harvested at a rate proportional to its population size. The dynamics of population growth is modeled by

$$P' = rP(1 - \frac{P}{K}) - \lambda P,$$

where λ is the *per capita* harvesting rate. Use a bifurcation diagram to explain the effects on the equilibrium deer population when λ is slowly increased from a small value to a large value.

7. Draw a bifurcation diagram for the model $u' = u^3 - u + h$, where h is the bifurcation parameter. Label branches of the curves as stable or unstable.

8. Consider the model $u' = u(u - e^{\lambda u})$, where λ is a parameter. Draw the bifurcation diagram, plotting the equilibrium solution(s) u^* vs. λ. Label each curve on the diagram as stable or unstable.

1.3.4 Heat Transfer

An object of uniform temperature T_0 (e.g., a potato) is placed in an oven of temperature T_e. It is observed that over time the potato heats up and eventually its temperature becomes that of the oven environment, T_e. We want a model that governs the temperature $T(t)$ of the potato at any time t. **Newton's law of cooling** (heating), a constitutive model inferred from experiment, dictates that the rate of change of the temperature of the object is proportional to the difference between the temperature of the object and the environmental temperature. That is,

$$T' = -h(T - T_e). \tag{1.21}$$

The positive proportionality constant h is the **heat loss coefficient**. There is a fundamental assumption here that the heat is instantly and uniformly distributed throughout the body and there are no temperature gradients, or spatial variations, in the body itself. From the DE we observe that $T = T_e$ is an equilibrium solution. If $T > T_e$ then $T' < 0$, and the temperature decreases; if $T < T_e$ then $T' > 0$, and the temperature increases. Plotting the phase line easily shows that this equilibrium is stable (Exercise!).

We can find a formula for the temperature $T(t)$ satisfying (1.21) using the separation of variables method introduced in the last section. Here, for variety, we illustrate another simple method that uses a **change of variables**. Let $u = T - T_e$. Then $u' = T'$ and (1.21) may be written $u' = -hu$. This is the decay equation and we have memorized its general solution $u = Ce^{-ht}$. Therefore $T - T_e = Ce^{-ht}$, or

$$T(t) = T_e + Ce^{-ht}.$$

This is the general solution of (1.21). If we impose an initial condition $T(0) = T_0$, then one finds $C = T_0 - T_e$, giving

$$T(t) = T_e + (T_0 - T_e)e^{-ht}.$$

We can now see clearly that $T(t) \to T_e$ as $t \to \infty$. A plot of the solution showing how an object heats up is given in figure 1.17.

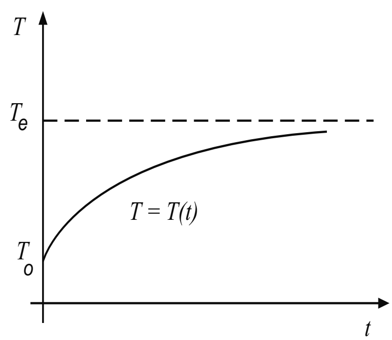

Figure 1.17 Temperature history in Newton's law of cooling.

If the environmental, or ambient, temperature fluctuates, then T_e is not constant but rather a function of time $T_e(t)$. The governing equation becomes

$$T' = -h(T - T_e(t)).$$

In this case there are no constant, or equilibrium, solutions. Writing this model in a different way,

$$T' = -hT + hT_e(t).$$

The first term on the right is internal to the system (the body being heated) and, considered alone with zero ambient temperature, leads to an exponentially decaying temperature (recall that $T' = -hT$ has solution $T = Ce^{-ht}$). Therefore, there is a transient governed by the natural system that decays away. The external, environmental temperature $T_e(t)$ gives rise to time-dependent dynamics and eventually takes over to drive the system; we say the system is "driven", or forced, by the environmental temperature. In Chapter 2 we develop methods to solve this equation with time dependence in the environmental temperature function.

EXERCISES

1. A small solid initially of temperature 22°C is placed in an ice bath of 0°C. It is found experimentally, by measuring temperatures at different times, that the natural logarithm of the temperature $T(t)$ of the solid plots as a linear function of time t; that is,

$$\ln T = -at + b.$$

Show that this equation is consistent with Newton's law of cooling. If the temperature of the object is 8°C degrees after two hours, what is the heat loss coefficient? When will the solid be 2°C?

2. A small turkey at room temperature 70°F is places into an oven at 350°F. If $h = 0.42$ per hour is the heat loss coefficient for turkey meat, how long should you cook the turkey so that it is uniformly 200°F? Comment on the validity of the assumptions being made in this model?

3. The body of a murder victim was discovered at 11:00 A.M. The medical examiner arrived at 11:30 A.M. and found the temperature of the body was 94.6°F. The temperature of the room was 70°F. One hour later, in the same room, he took the body temperature again and found that it was 93.4°F. Estimate the time of death.

4. Suppose the temperature inside your winter home is 68°F at 1:00 P.M. and your furnace then fails. If the outside temperature is 10°F and you notice that by 10:00 P.M. the inside temperature is 57°F, what will be the temperature in your home the next morning at 6:00 A.M.?

5. The temperature $T(t)$ of an exothermic, chemically reacting sample placed in a furnace is governed by the initial value problem

$$T' = -k(T - T_e) + qe^{-\theta/T}, \quad T(0) = T_0,$$

where the term $qe^{-\theta/T}$ is the rate heat is generated by the reaction. What are the dimensions of all the constants (k, T_e, q, T_0, and θ) in the problem? Scale time by k^{-1} and temperature by T_e to obtain the dimensionless model

$$\frac{d\psi}{d\tau} = -(\psi - 1) + ae^{-b/\psi}, \quad \psi(0) = \gamma,$$

for appropriately chosen dimensionless parameters a, b, and c. Fix $a = 1$. How many positive equilibria are possible, depending upon the value of b? (Hint: Graph the heat loss term and the heat generation term vs. ψ on the same set of axes for different values of b).

1.3.5 Chemical Reactors

A **continuously stirred tank reactor** (also called a chemostat, or compartment) is a basic unit of many physical, chemical, and biological processes. A continuously stirred tank reactor is a well-defined geometric volume or entity where substances enter, react, and are then discharged. A chemostat could be an organ in our body, a polluted lake, an industrial chemical reactor, or even an ecosystem. See figure 1.18.

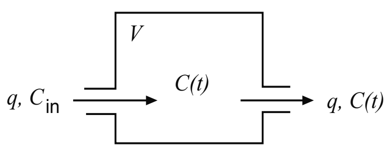

Figure 1.18 A chemostat, or continuously stirred tank reactor.

We illustrate a reactor model with a specific example. Consider an industrial pond with constant volume V cubic meters. Suppose that polluted water containing a toxic chemical of concentration C_{in} grams per cubic meter is dumped into the pond at a constant volumetric flow rate of q cubic meters per day. At the same time the continuously mixed solution in the pond is drained off at the same flow rate q. If the pond is initially at concentration C_0, what is the concentration $C(t)$ of the chemical in the pond at any time t?

The key idea in all chemical mixture problems is to obtain a model by conserving mass: the rate of change of mass in the pond must equal the rate mass flows in minus the rate mass flows out. The total mass in the pond at any time is VC, and the mass flow rate is the volumetric flow rate times the mass concentration; thus mass balance dictates

$$(VC)' = qC_{\text{in}} - qC.$$

Hence, the initial value problem for the chemical concentration is

$$VC' = qC_{\text{in}} - qC, \quad C(0) = C_0, \tag{1.22}$$

where C_0 is the initial concentration in the tank. This initial value problem can be solved by the separation of variables method or the change of variables method (Section 1.3.4). See Exercise 1.

Now suppose we add degradation of the chemical while it is in the pond, assuming that it degrades to inert products at a rate proportional to the amount present. We represent this decay rate as rC gm per cubic meter per day, where r is constant. Then the model equation becomes

$$VC' = qC_{\text{in}} - qC - rVC.$$

Notice that we include a factor V in the last term to make the model dimensionally correct. A similar model holds when the volumetric flow rates in and out are different, which gives a changing volume $V(t)$. Letting q_{in} and q_{out} denote those flow rates, respectively, we have

$$(V(t)C)' = q_{\text{in}}C_{\text{in}} - q_{\text{out}}C - rV(t)C,$$

where $V(t) = V_0 + (q_{\text{in}} - q_{\text{out}})t$, and where V_0 is the initial volume. Methods developed in Chapter 2 show how this equation is solved.

EXERCISES

1. Solve the initial value problem (1.22) and obtain a formula for the concentration in the reactor at time t.

2. An industrial pond having volume 100 m^3 is full of pure water. Contaminated water containing a toxic chemical of concentration 0.0002 kg per m^3 is then is pumped into the pond with a volumetric flow rate of 0.5 m^3 per minute. The contents are well-mixed and pumped out at the same flow rate. Write down an initial value problem for the contaminant concentration $C(t)$ in the pond at any time t. Determine the equilibrium concentration and its stability. Find a formula for the concentration $C(t)$.

3. In the preceding problem, change the flow rate out of the pond to 0.6 m^3 per minute. How long will it take the pond to empty? Write down a revised initial value problem.

4. A vat of volume 1000 gallons initially contains 5 lbs of salt. For $t > 0$ a salt brine of concentration 0.1 lbs per gallon is pumped into the tank at the rate of 2 gallons per minute; the perfectly stirred mixture is pumped out at the same flow rate. Derive a formula for the concentration of salt in the tank at any time t. Check your answer on a computer algebra system, and sketch a graph of the concentration vs. time.

5. Consider a chemostat of constant volume where a chemical $\mathbf{C}$ is pumped into the reactor at constant concentration and constant flow rate. While in the reactor it reacts according to $\mathbf{C} + \mathbf{C} \to$ products. From the law of mass action the rate of the reaction is $r = kC^2$, where k is the rate constant. If the concentration of $\mathbf{C}$ in the reactor is given by $C(t)$, then

mass balance leads the governing equation $(VC)' = qC_{\text{in}} - qC - kVC^2$. Find the equilibrium state(s) and analyze their stability. Redo this problem after nondimensionalizing the equation (pick time scale V/q and concentration scale C_{in}).

6. Work Exercise 5 if the rate of reaction is given by **Michaelis–Menten kinetics**
$$r = \frac{aC}{b+C},$$
where a and b are positive constants.

7. A **batch reactor** is a reactor of volume V where there are no in and out flow rates. Reactants are loaded instantaneously and then allowed to react over a time T, called the residence time. Then the contents are expelled instantaneously. Fermentation reactors and even sacular stomachs of some animals can be modeled as batch reactors. If a chemical is loaded in a batch reactor and it degrades with rate $r(C) = kC$, given in mass per unit time, per unit volume, what is the residence time required for 90 percent of the chemical to degrade?

8. The **Monod equation** for conversion of a chemical substrate of concentration C into its products is
$$\frac{dC}{dt} = -\frac{aC}{b+C},$$
where a and b are positive constants. This equation, with Michaelis–Menten kinetics, describes how the substrate is being used up through chemical reaction. If, in addition to reaction, the substrate is added to the solution at a constant rate R, write down a differential equation for C. Find the equilibrium solution and explain how the substrate concentration evolves for various initial conditions.

9. Consider the chemical reaction $\mathrm{A} + \mathrm{B} \xrightarrow{k} \mathrm{C}$, where one molecule of A reacts with one molecule of B to produce one molecule of C, and the rate of the reaction is k, the rate constant. By the law of mass action in chemistry, the reaction rate is $r = kab$, where a and b represent the time-dependent concentrations of the reactants A and B. Thus, the rates of change of the reactants and product are governed by the three equations
$$a' = -kab, \quad b' = -kab, \quad c' = kab.$$
If, initially, $a(0) = a_0$, $b(0) = b_0$, and $c(0) = 0$, with $a_0 > b_0$, find a single, first-order differential equation that involves only the concentration $a = a(t)$. What is the limiting concentration $\lim_{t\to\infty} a(t)$? What are the other two limiting concentrations?

10. Digestion in the stomach (gut) in some organisms can be modeled as a chemical reactor of volume V, where food enters and is broken down into nutrient products, which are then absorbed across the gut lining; the food–product mixture in the stomach is perfectly stirred and exits at the same rate as it entered. Let S_0 be the concentration of a substrate (food) consumed at rate q (volume per time). In the gut the rate of substrate breakdown into the nutrient product, $S \to P$, is given by kVS, where k is the rate constant and $S = S(t)$ is the substrate concentration. The nutrient product, of concentration $P = P(t)$, is then absorbed across the gut boundary at a rate aVP, where a is the absorption constant. At all times the contents are thoroughly stirred and leave the gut at the flow rate q.

    a) Show that the model equations are

    $$\begin{aligned} VS' &= qS_0 - qS - kVS, \\ VP' &= kVS - aVP - qP. \end{aligned}$$

    b) Suppose the organism eats continuously, in a steady-state mode, where the concentrations become constant. Find the steady-steady, or equilibrium, concentrations S_e and P_e.

    c) Some ecologists believe that animals regulate their consumption rate in order to maximize the absorption rate of nutrients. Show that the maximum nutrient concentration P_e occurs when the consumption rate is $q = \sqrt{ak}V$.

    d) Show that the maximum absorption rate is therefore $\frac{akS_0V}{(\sqrt{a}+\sqrt{k})^2}$.

1.3.6 Electric Circuits

Our modern, technological society is filled with electronic devices of all types. At the basis of these are electrical circuits. The simplest circuit unit is the loop in figure 1.19 that contains an electromotive force (emf) $E(t)$ (a battery or generator that supplies energy), a resistor, an inductor, and a capacitor, all connected in series. A capacitor stores electrical energy on its two plates, a resistor dissipates energy, usually in the form of heat, and an inductor acts as a "choke" that resists changes in current. A basic law in electricity, **Kirchhoff's law,** tells us that the sum of the voltage drops across the circuit elements (as measured, e.g., by a voltmeter) in a loop must equal the applied emf. In symbols,

$$V_L + V_R + V_C = E(t).$$

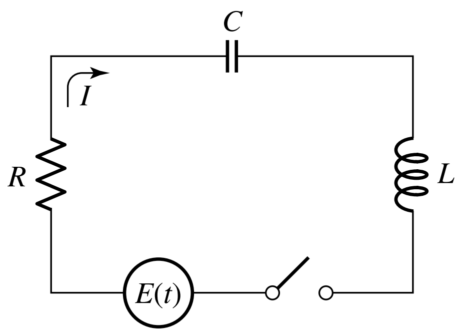

Figure 1.19 An RCL circuit with an electromotive force $E(t)$ supplying the electrical energy.

This law comes from conservation of energy in a current loop, and it is derived in elementary physics texts. A voltage drop across an element is an energy potential that equals the amount of work required to move a charge across that element.

Let $I = I(t)$ denote the current (in amperes, or charge per second) in the circuit, and let $q = q(t)$ denote the charge (in coulombs) on the capacitor. These quantities are related by

$$q' = I.$$

There are several choices of state variables to describe the response of the circuit: charge on the capacitor q, current I, or voltage V_C across the capacitor. Let us write Kirchhoff's law in terms of charge. By Ohm's law the voltage drop across the resistor is proportional to the current, or

$$V_R = RI,$$

where the proportionality constant R is called the resistance (measured in ohms). The voltage drop across a capacitor is proportional to the charge on the capacitor, or

$$V_C = \frac{1}{C}q,$$

where C is the capacitance (measured in farads). Finally, the voltage drop across an inductor is proportional to how fast the current is changing, or

$$V_L = LI',$$

where L is the inductance (measured in henrys). Substituting these voltage drops into Kirchhoff's law gives

$$LI' + RI + \frac{1}{C}q = E(t),$$

or, using $q' = I$,

$$Lq'' + Rq' + \frac{1}{C}q = E(t).$$

This is the **RCL circuit equation**, which is a second-order DE for the charge q. The initial conditions are

$$q(0) = q_0, \quad q'(0) = I(0) = I_0.$$

These express the initial charge on the capacitor and the initial current in the circuit. Here, $E(t)$ may be a given constant (e.g., $E(t) = 12$ for a 12-volt battery) or may be a oscillating function of time t (e.g., $E(t) = A\cos\omega t$ for an alternating voltage potential of amplitude A and frequency ω).

If there is no inductor, then the resulting RC circuit is modeled by the first-order equation

$$Rq' + \frac{1}{C}q = E(t).$$

If $E(t)$ is constant, this equation can be solved using separation of variables or the change of variables method (Exercise 2). We show how to solve second-order differential equations in Chapter 3.

EXERCISES

1. Write down the equation that governs an RC circuit with a 12-volt battery, taking $R = 1$ and $C = \frac{1}{2}$. Determine the equilibrium solution and its stability. If $q(0) = 5$, find a formula for $q(t)$. Find the current $I(t)$. Plot the charge and the current on the same set of axes.

2. In an arbitrary RC circuit with constant emf E, use the method of separation of variables to derive the formula

$$q(t) = Ke^{-t/RC} + EC$$

for the charge on the capacitor, where K is an arbitrary constant. If $q(0) = q_0$, what is K?

3. An RCL circuit with an applied emf given by $E(t)$ has initial charge $q(0) = q_0$ and initial current $I(0) = I_0$. What is $I'(0)$? Write down the circuit equation and the initial conditions in terms of current $I(t)$.

4. Formulate the governing equation of an RCL circuit in terms of the current $I(t)$ when the circuit has an emf given by $E(t) = A\cos\omega t$. What are the appropriate initial conditions?

5. Find the DE model for the charge in an LC circuit with no emf. Show that the response of the circuit may have the form $q(t) = A\cos\omega t$ for some amplitude A and frequency ω.

6. Consider a standard RCL circuit with no emf, but with a voltage drop across the resistor given by a nonlinear function of current,

$$V_R = \frac{1}{2}\left(\frac{1}{3}I^3 - I)\right)$$

(This replaces Ohm's law.) If $C = L = 1$, find a differential equation for the current $I(t)$ in the circuit.

7. Write the RCL circuit equation with the voltage $V_c(t)$ as the unknown state function.

2

Analytic Solutions and Approximations

In the last chapter we studied several first-order DE models and a few elementary techniques to help understand the qualitative behavior of the models. In this chapter we introduce analytic solution techniques for first-order equations and some general methods of approximation, including numerical methods.

2.1 Separation of Variables

In Section 1.3.2 we presented a simple algorithm to obtain an analytic solution to an autonomous equation $u' = f(u)$ called **separation of variables**. Now we show that this method is applicable to a more general class of equations. A **separable equation** is a first-order differential where the right side can be factored into a product of a function of t and a function of u. That is, a separable equation has the form

$$u' = g(t)h(u). \tag{2.1}$$

To solve separable equations we take the expression involving u to the left side and then integrate with respect to t, remembering that $u = u(t)$. Therefore, dividing by $h(u)$ and taking the antiderivatives of both sides with respect to t gives

$$\int \frac{1}{h(u)} u' dt = \int g(t)dt + C,$$

where C is an arbitrary constant of integration. (Both antiderivatives generate an arbitrary constant, but we have combined them into a single constant C). Next we change variables in the integral on the left by letting $u = u(t)$, so that $du = u'(t)dt$. Hence,

$$\int \frac{1}{h(u)} du = \int g(t)dt + C.$$

This equation, once the integrations are performed, yields an equation of the form

$$H(u) = G(t) + C, \tag{2.2}$$

which defines the general solution u implicitly as a function of t. We call (2.2) the **implicit solution**. To obtain an **explicit solution** $u = u(t)$ we must solve (2.2) for u in terms of t; this may or may not be possible. As an aside, we recall that if the antiderivatives have no simple expressions, then we write the antiderivatives with limits on the integrals.

Example 2.1

Solve the initial value problem

$$u' = \frac{t+1}{2u}, \quad u(0) = 1.$$

We recognize the differential equation as separable because the right side is $\frac{1}{2u}(t+1)$. Bringing the $2u$ term to the left side and integrating gives

$$\int 2uu'dt = \int (t+1)dt + C,$$

or

$$\int 2udu = \frac{1}{2}t^2 + t + C.$$

Therefore

$$u^2 = \frac{1}{2}t^2 + t + C.$$

This equation is the general **implicit solution**. We can solve for u to obtain two forms for **explicit solutions**,

$$u = \pm\sqrt{\frac{1}{2}t^2 + t + C}.$$

Which sign do we take? The initial condition requires that u be positive. Thus, we take the plus sign and apply $u(0) = 1$ to get $C = 1$. The solution to the initial value problem is therefore

$$u = \sqrt{\frac{1}{2}t^2 + t + 1}.$$

This solution is valid as long as the expression under the radical is not negative. In the present case the solution is defined for all times $t \in \mathbf{R}$ and so the interval of existence is the entire real line.

Example 2.2

Solve the initial value problem

$$u' = \frac{2\sqrt{u}e^{-t}}{t}, \quad u(1) = 4.$$

Note that we might expect trouble at $t = 0$ because the derivative is undefined there. The equation is separable so we separate variables and integrate with respect to t:

$$\frac{1}{2}\int \frac{u'}{\sqrt{u}}dt = \int \frac{e^{-t}}{t}dt + C.$$

We can integrate the left side exactly, but the integral on the right cannot be resolved in closed form. Hence we write it with variable limits and we have

$$\sqrt{u} = \int_1^t \frac{e^{-t}}{t}dt + C.$$

Judiciously we chose the lower limit as $t = 1$ so that the initial condition would be easy to apply. Clearly we get $C = 2$. Therefore

$$\sqrt{u} = \int_1^t \frac{e^{-t}}{t}dt + 2,$$

or

$$u(t) = \left(\int_1^t \frac{e^{-t}}{t}dt + 2\right)^2.$$

This solution is valid on $0 < t < \infty$. In spite of the apparent complicated form of the solution, which contains an integral, it is not difficult to plot using a computer algebra system. The plot is shown in figure 2.1.

The method of separation of variables is a key technique in differential equations. Many important models turn out to be separable, not the least of which is the autonomous equation.

EXERCISES

1. Find the general solution in explicit form of the following equations.

   a) $u' = \frac{2u}{t+1}$.

   b) $u' = \frac{t\sqrt{t^2+1}}{\cos u}$.

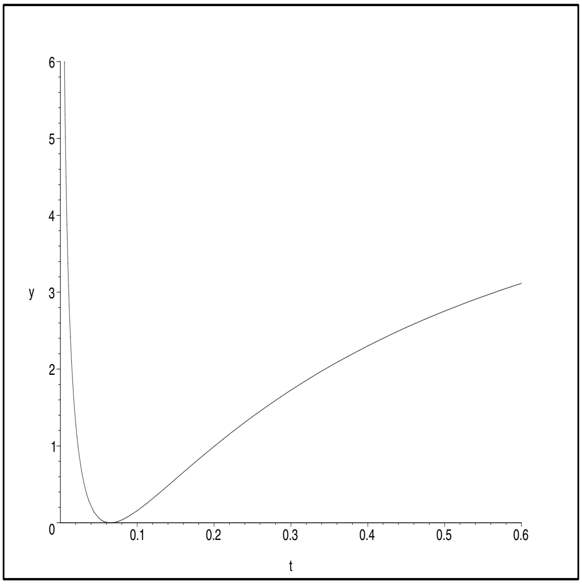

Figure 2.1 This plot is obtained on the interval $(0, 0.6]$ using the Maple command: plot((evalf(2+int(exp(-s)/s, s=1..t)))∧2, t=0..0.6, y=0..6);.

c) $u' = (t+1)(u^2+1)$.

d) $u' + u + \frac{1}{u} = 0$.

2. Find the solution to the initial value problem

$$u' = t^2 e^{-u}, \quad u(0) = \ln 2,$$

and determine the interval of existence.

3. Draw the phase line associated with the DE $u' = u(4-u^2)$ and then solve the DE subject to the initial condition $u(0) = 1$. (Hint: for the integration you will need a partial fractions expansion

$$\frac{1}{u(4-u^2)} = \frac{a}{u} + \frac{b}{2+u} + \frac{c}{2-u},$$

where a, b, and c are to be determined.)

4. Find the general solution in implicit form to the equation

$$u' = \frac{4-2t}{3u^2-5}.$$

Find the solution when $u(1) = 3$ and plot the solution. What is its interval of existence?

5. Solve the initial value problem $u' = \frac{2tu^2}{1+t^2}$, $u(t_0) = u_0$, and find the interval of existence when $u_0 < 0$, when $u_0 > 0$, and when $u_0 = 0$.

6. Find the general solution of the DE
$$u' = 6t(u-1)^{2/3}.$$
Show that there is no value of the arbitrary constant that gives the solution $u = 1$. (A solution to a DE that cannot be obtained from the general solution by fixing a value of the arbitrary constant is called a **singular solution**).

7. Find the general solution of the DE
$$(T^2 - t^2)u' + tu = 0,$$
where T is a fixed, positive parameter. Find the solution to the initial value problem when $u(T/2) = 1$. What is the interval of existence?

8. Allometric growth describes temporal relationships between sizes of different parts of organisms as they grow (e.g., the leaf area and the stem diameter of a plant). We say two sizes u_1 and u_2 are *allometrically* related if their relative growth rates are proportional, or
$$\frac{u_1'}{u_1} = a\frac{u_2'}{u_2}.$$
Show that if u_1 and u_2 are allometrically related, then $u_1 = Cu_2^a$, for some constant C.

9. A differential equation of the form
$$u' = F\left(\frac{u}{t}\right),$$
where the right depends only on the ratio of u and t, is called a **homogeneous**. Show that the substitution $u = ty$ transforms a homogeneous equation into a first-order separable equation for $y = y(t)$. Use this method to solve the equation
$$u' = \frac{4t^2 + 3u^2}{2tu}.$$

10. Solve the initial value problem
$$\frac{d}{dt}\left(u(t)e^{2t}\right) = e^{-t}, \quad u(0) = 3.$$

11. Find the general solution $u = u(r)$ of the DE

$$\frac{1}{r}\frac{d}{dr}(ru'(r)) = -p,$$

where p is a positive constant.

12. A population of u_0 individuals all has HIV, but none has the symptoms of AIDS. Let $u(t)$ denote the number that does not have AIDS at time $t > 0$. If $r(t)$ is the *per capita* rate of individuals showing AIDS symptoms (the conversion rate from HIV to AIDS), then $u'/u = -r(t)$. In the simplest case we can take r to be a linear function of time, or $r(t) = at$. Find $u(t)$ and sketch the solution when $a = 0.2$ and $u_0 = 100$. At what time is the rate of conversion maximum?

13. An arrow of mass m is shot vertically upward with initial velocity 160 ft/sec. It experiences both the deceleration of gravity and a deceleration of magnitude $mv^2/800$ due to air resistance. How high does the arrow go?

14. In very cold weather the thickness of ice on a pond increases at a rate inversely proportional to its thickness. If the ice initially is 0.05 inches thick and 4 hours later it is 0.075 inches thick, how thick will it be in 10 hours?

15. Write the solution to the initial value problem

$$u' = -u^2 e^{-t^2}, \quad u(0) = \frac{1}{2}$$

in terms of the erf function, $\text{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds$.

16. Use separation of variables to solve the following problems. Write the solution explicitly when possible.

    a) $u' = p(t)u$, where $p(t)$ is a given continuous function.

    b) $u' = -2tu, \quad u(1) = 2$. Plot the solution on $0 \le t \le 2$.

    c) $u' = \begin{cases} -2u, & 0 < t < 1 \\ -u^2, & 1 \le t \le 2 \end{cases}, \quad u(0) = 5.$

    Find a continuous solution on the interval $0 \le t \le 3$ and plot the solution.

17. A certain patch is populated with a cohort of newly hatched grasshoppers numbering h_0. As time proceeds they die of natural causes at *per capita* rate m, and they are eaten by spiders at the rate $aH/(1+bH)$ per spider, where

H is the population of grasshoppers, and a and b are positive constants. Thus, the dynamics is given by

$$H' = -mH - \frac{aH}{1+bH}S,$$

where S is the spider population, and time is given in days.

a) Determine the units on the constants m, a, and b.

b) Choose new dimensionless variables $\tau = mt$ and $h = bH$, and reformulate the differential equation and initial condition in a dimensionless problem for $h = h(\tau)$. In your differential equation you should have a single dimensionless constant given by $\lambda = aS/m$.

c) Solve the dimensionless initial value problem to obtain a formula for $h(\tau)$. What is $\lim_{\tau\to\infty} h(\tau)$?

18. Let N_0 be the number of individuals in a cohort at time $t = 0$ and $N = N(t)$ be the number of those individuals alive at time t. If m is the constant *per capita* mortality rate, then $N'/N = -m$, which gives $N(t) = N_0 e^{-mt}$. The **survivorship function** is defined by $S(t) = N(t)/N_0$, and $S(T)$ therefore gives the probability of an individual living to age T. In the case of a constant *per capita* mortality the survivorship curve is a decaying exponential.

a) What fraction die before age T? Calculate the fraction of individuals that die between age a and age b.

b) If the *per capita* death rate depends on time, or $m = m(t)$, find a formula for the survivorship function (your answer will contain an integral).

c) What do you think the human survivorship curve looks like?

2.2 First-Order Linear Equations

A differential equation of the form

$$u' = p(t)u + q(t). \tag{2.3}$$

is called a **first-order linear equation**. The given functions p and q are assumed to be continuous. If $q(t) = 0$, then the equation is called **homogeneous**;

otherwise it is called **nonhomogeneous**. Linear equations have a nice structure to their solution set, and we are able to derive the general solution. The homogeneous equation

$$u' = p(t)u, \tag{2.4}$$

without the nonhomogeneous term $q(t)$, can readily be solved by separation of variables to obtain

$$u_h(t) = Ce^{P(t)}, \quad \text{where} \quad P(t) = \int p(t)dt, \tag{2.5}$$

where C is an arbitrary constant. We have placed a subscript h on this solution to distinguish it from the solution of the nonhomogeneous equation (2.3). (The solution $u_h(t)$ to the homogeneous equation is sometimes called the complementary solution; we just refer to it as the homogeneous solution.) Also, note that we have used the indefinite integral notation for the function $P(t)$; in some cases we have to represent $P(t)$ in the form

$$P(t) = \int_a^t p(s)ds,$$

with limits of integration. We always choose the lower limit a to be the value of time where the initial condition is prescribed.

We now describe a standard technique to solve the nonhomogeneous equation (2.3). The idea is to try a solution of the form (2.5) where we let the constant C in the homogeneous solution vary as a function of t; we then substitute this form into (2.3) to determine the $C = C(t)$. The method is, for obvious reasons, called **variation of parameters**.[1] Thus, assume a solution to (2.3) of the form

$$u(t) = C(t)e^{P(t)}.$$

Then, plugging in,

$$C'(t)e^{P(t)} + C(t)e^{P(t)}P'(t) = p(t)C(t)e^{P(t)} + q(t).$$

But $P' = p$ and therefore two of the terms cancel, giving

$$C'(t)e^{P(t)} = q(t),$$

or

$$C'(t) = e^{-P(t)}q(t).$$

Integration yields

$$C(t) = \int e^{-P(t)}q(t)dt + K,$$

[1] Another method using *integrating factors* is presented in the Exercises.

where K is a constant of integration. So we have

$$\begin{aligned} u(t) &= \left(\int e^{-P(t)} q(t) dt + K \right) e^{P(t)} \\ &= K e^{P(t)} + e^{P(t)} \int e^{-P(t)} q(t) dt, \end{aligned} \tag{2.6}$$

which is the general solution to the general linear, nonhomogeneous equation (2.3). If the antiderivative in the last equation cannot be calculated explicitly, then we write the solution as

$$u(t) = K e^{P(t)} + e^{P(t)} \int_a^t e^{-P(\tau)} q(\tau) d\tau.$$

We urge the reader not to memorize these formulas; rather, remember the *method* and apply it to each problem as you solve it.

Example 2.3

Find the general solution to

$$u' = \frac{1}{t} u + t^3.$$

The homogeneous equation is $u' = \frac{1}{t} u$ and has solution

$$u_h(t) = C e^{\int (1/t) dt} = C e^{\ln t} = Ct.$$

Therefore we vary the parameter C and assume a solution of the original non-homogeneous equation of the form

$$u(t) = C(t) t.$$

Substituting into the equation, we get

$$u' = C(t) + C'(t) t = \frac{1}{t} C(t) t + t^3,$$

or

$$C'(t) = t^2.$$

Therefore $C(t) = \int t^2 dt = \frac{1}{3} t^3 + K$ and the general solution to the original equation is

$$u(t) = \left(\frac{1}{3} t^3 + K \right) t = \frac{1}{3} t^4 + Kt.$$

The arbitrary constant K can be determined by an initial condition.

Example 2.4

Consider the DE

$$u' = 2u + t. \tag{2.7}$$

The associated homogeneous equation is

$$u' = 2u,$$

which has solution $u_h = Ce^{2t}$. Therefore we assume the solution of (2.7) is of the form

$$u(t) = C(t)e^{2t}.$$

Substituting into the original equation gives

$$C(t)2e^{2t} + C'(t)e^{2t} = 2C(t)e^{2t} + t,$$

or

$$C'(t) = te^{-2t}.$$

Integrating,

$$C(t) = \int te^{-2t}dt + K = -\frac{1}{4}e^{-2t}(2t+1) + K.$$

The integral was calculated analytically using integration by parts. Therefore the general solution of (2.7) is

$$\begin{aligned} u(t) &= \left(-\frac{1}{4}e^{-2t}(2t+1) + K\right)e^{2t} \\ &= Ke^{2t} - \frac{1}{4}(2t+1). \end{aligned}$$

Notice that the general solution is composed of two terms, $u_h(t)$ and $u_p(t)$, defined by

$$u_h = Ke^{2t}, \quad u_p = -\frac{1}{4}(2t+1).$$

We know u_h is the general solution to the homogeneous equation, and it is easy to show that u_p is a particular solution to the nonhomogeneous equation (2.7). So, the general solution to the nonhomogeneous equation (2.7) is the sum of the general solution to the associated homogeneous equation and any particular solution to the nonhomogeneous equation (2.7).

Example 2.4 illustrates a general principle that reveals the structure of the solution to a first-order linear DE. The general solution can be written as the sum of the solution to the homogeneous equation and any particular solution of the nonhomogeneous equation. Precisely, the basic structure theorem for first-order linear equations states:

Theorem 2.5

(Structure Theorem) The general solution of the nonhomogeneous equation

$$u' = p(t)u + q(t)$$

is the sum of the general solution u_h of the homogeneous equation $u' = p(t)u$ and a particular solution u_p to the nonhomogeneous equation. In symbols,

$$u(t) = u_h(t) + u_p(t),$$

where $u_h(t) = Ke^{P(t)}$ and $u_p = e^{P(t)} \int e^{-P(t)}q(t)dt$, and where $P(t) = \int p(t)dt$.

Example 2.6

Consider an RC electrical circuit where the resistance is $R = 1$ and the capacitance is $C = 0.5$. Initially the charge on the capacitor is $q(0) = 5$. The current is driven by an emf that generates a variable voltage of $\sin t$. How does the circuit respond? The governing DE for the charge $q(t)$ on the capacitor is

$$Rq' + \frac{1}{C}q = \sin t,$$

or, substituting the given parameters,

$$q' = -2q + \sin t. \tag{2.8}$$

The homogeneous equation $q' = -2q$ has solution $q_h = Ce^{-2t}$. We assume the solution to the nonhomogeneous equation has the form $q = C(t)e^{-2t}$. Substituting into (2.8) gives

$$C(t)(-2q'e^{-2t}) + C'(t)qe^{-2t} = -2C(t)e^{-2t} + \sin t,$$

or

$$C'(t) = e^{2t} \sin t.$$

Integrating,

$$C(t) = \int e^{2t} \sin t dt + K = e^{2t}(\frac{2}{5} \sin t - \frac{1}{5} \cos t) + K,$$

where K is a constant of integration. The integral was calculated using software (or, one can use integration by parts). Therefore the general solution of (2.8) is

$$q(t) = C(t)e^{-2t} = \frac{2}{5} \sin t - \frac{1}{5} \cos t + Ke^{-2t}.$$

Next we apply the initial condition $q(0) = 5$ to obtain $K = 26/5$. Therefore the solution to the initial value problem is

$$q(t) = \frac{2}{5}\sin t - \frac{1}{5}\cos t + \frac{26}{5}e^{-2t}.$$

The solution is consistent with Theorem 2.5. Also, there is an important physical interpretation of the solution. The homogeneous solution is the **transient response** $q_h(t) = \frac{26}{5}e^{-2t}$ that depends upon the initial charge and decays over a time; what remains over a long time is the particular solution, which is regarded as the **steady-state response** $q_p(t) = \frac{26}{5}\sin t - \frac{1}{5}\cos t$. The homogeneous solution ignores the forcing term (the emf), whereas the particular solution arises from the forcing term. After a long time the applied emf drives the response of the system. This behavior is characteristic of forced linear equations coming from circuit theory and mechanics. The solution is a sum of two terms, a contribution due to the internal system and initial data (the decaying transient), and a contribution due to the external forcing term (the steady response). Figure 2.2 shows a plot of the solution.

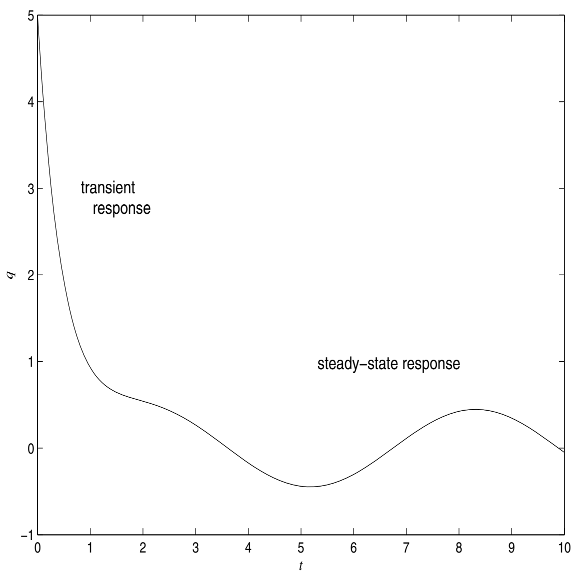

Figure 2.2 Response of the circuit in Example 2.6 showing the initial transient and the long-time steady-state.

Example 2.7

(Sales Response to Advertising) The field of economics has always been a source of interesting phenomena modeled by differential equations. In this example we set up a simple model that allows management to assess the effectiveness of an advertising campaign. Let $S = S(t)$ be the monthly sales of an item. In the absence of advertising it is observed from sales history data that the logarithm of the monthly sales decreases linearly in time, or $\ln S = -at + b$. Thus $S' = -aS$, and sales are modeled by exponential decay. To keep sales up, advertising is required. If there is a lot of advertising, then sales tend to saturate at some maximum value $S = M$; this is because there are only finitely many consumers. The rate of increase in sales due to advertising is jointly proportional to the advertising rate $A(t)$ and to the degree the market is not saturated; that is,

$$rA(t)\left(\frac{M-S}{M}\right).$$

The constant r measures the effectiveness of the advertising campaign. The term $\frac{M-S}{M}$ is a measure of the market share that has still not purchased the product. Then, combining both natural sales decay and advertising, we obtain the model

$$S' = -aS + rA(t)\left(\frac{M-S}{M}\right).$$

The first term on the right is the natural decay rate, and the second term is the rate of sales increase due to advertising, which drives the sales. As it stands, because the advertising rate A is not constant, there are no equilibria (constant solutions). We can rearrange the terms and write the equation in the form

$$S' = -\left(a + \frac{rA(t)}{M}\right)S + rA(t). \tag{2.9}$$

Now we recognize that the sales are governed by a first-order linear DE. The Exercises request some solutions for different advertising strategies.

EXERCISES

1. Find the general solution of $u' = -\frac{1}{t}u + t$.
2. Find the general solution of $u' = -u + e^t$.
3. Show that the general solution to the DE $u' + au = \sqrt{1+t}$ is given by

$$u(t) = Ce^{-at} + \int_0^t e^{-a(t-s)}\sqrt{1+s}ds.$$

4. A decaying battery generating $200e^{-5t}$ volts is connected in series with a 20 ohm resistor, and a 0.01 farad capacitor. Assuming $q = 0$ at $t = 0$, find the charge and current for all $t > 0$. Show that the charge reaches a maximum and find the time it is reached.

5. Solve $u'' + u' = 3t$ by introducing $y = u'$.

6. Solve $u' = (t + u)^2$ by letting $y = t + u$.

7. Express the general solution of the equation $u' = 2tu + 1$ in terms of the erf function.

8. Find the solution to the initial value problem $u' = pu + q, \quad u(0) = u_0$, where p and q are constants.

9. Find a formula for the general solution to the DE $u' = pu + q(t)$, where p is constant. Find the solution satisfying $u(t_0) = u_0$.

10. A differential equation of the form

$$u' = a(t)u + g(t)u^n$$

is called a **Bernoulli equation**, and it arises in many applications. Show that the Bernoulli equation can be reduced to the linear equation

$$y' = (1 - n)a(t)y + (1 - n)g(t)$$

by changing the dependent variable from u to y via $y = u^{1-n}$.

11. Solve the Bernoulli equations (see Exercise 10).

    a) $u' = \frac{2}{3t}u + \frac{2t}{u}$.

    b) $u' = u(1 + ue^t)$.

    c) $u' = -\frac{1}{t}u + \frac{1}{tu^2}$.

12. Initially, a tank contains 60 gal of pure water. Then brine containing 1 lb of salt per gallon enters the tank at 2 gal/min. The perfectly mixed solution is drained off at 3 gal/min. Determine the amount (in lbs) of salt in the tank up until the time it empties.

13. A large industrial retention pond of volume V, initially free of pollutants, was subject to the inflow of a contaminant produced in the factory's processing plant. Over a period of b days the EPA found that the inflow concentration of the contaminant decreased linearly (in time) to zero from its initial value of a (grams per volume), its flow rate q (volume per day) being constant. During the b days the spillage to the local stream was also

q. What is the concentration in the pond after b days? Do a numerical experiment using a computer algebra system where $V = 6000$ cubic meters, $b = 20$ days, $a = 0.03$ grams per cubic meter, and $q = 50$ cubic meters per day. With this data, how long would it take for the concentration in the pond to get below the required EPA level of 0.00001 grams per cubic meter if fresh water is pumped into the pond at the same flow rate, with the same spillover?

14. Determine the dimensions of the various quantities in the sales–advertising model (2.9). If A is constant, what is the equilibrium?

15. (Technology Transfer) Suppose a new innovation is introduced at time $t = 0$ in a community of N possible users (e.g., a new pesticide introduced to a community of farmers). Let $x(t)$ be the number of users who have adopted the innovation at time t. If the rate of adoption of the innovation is jointly proportional to the number of adoptions and the number of those who have not adopted, write down a DE model for $x(t)$. Describe, qualitatively, how $x(t)$ changes in time. Find a formula for $x(t)$.

16. A house is initially at 12 degrees Celsius when its heating–cooling system fails. The outside temperature varies according to $T_e = 9 + 10\cos 2\pi t$, where time is given in days. The heat loss coefficient is $h = 3$ degrees per day. Find a formula for the temperature variation in the house and plot it along with T_e on the same set of axes. What is the time lag between the maximum inside and outside temperature?

17. In the sales response to advertising model (2.9), assume $S(0) = S_0$ and that advertising is constant over a fixed time period T, and is then removed. That is,
$$A(t) = \begin{cases} a, & 0 \le t \le T \\ 0, & t > T \end{cases}$$
Find a formula for the sales $S(t)$. (Hint: solve the problem on two intervals and piece together the solutions in a continuous way).

18. In a community having a fixed population N, the rate that people hear a rumor is proportional to the number of people who have not yet heard the rumor. Write down a DE for the number of people P who have heard the rumor. Over a long time, how many will hear the rumor? Is this a believable model?

19. An object of mass $m = 1$ is dropped from rest at a large height, and as it falls it experiences the force of gravity mg and a time-dependent resistive force of magnitude $F_r = \frac{2}{t+1}v$, where v is its velocity. Write down an initial value problem that governs its velocity and find a formula for the solution.

20. The **MacArthur–Wilson model** of the dynamics of species (e.g., bird species) that inhabit an island located near a mainland was developed in the 1960s. Let P be the number of species in the source pool on the mainland, and let $S = S(t)$ be the number of species on the island. Assume that the rate of change of the number of species is

$$S' = \chi - \mu,$$

where χ is the colonization rate and μ is the extinction rate. In the MacArthur–Wilson model,

$$\chi = I(1 - \frac{S}{P}) \quad \text{and} \quad \mu = \frac{E}{P}S,$$

where I and E are the maximum colonization and extinction rates, respectively.

   a) Over a long time, what is the expected equilibrium for the number of species inhabiting the island? Is this equilibrium stable?

   b) Given $S(0) = S_0$, find an analytic formula for $S(t)$.

   c) Suppose there are two islands, one large and one small, with the larger island having the smaller maximum extinction rate. Both have the same colonization rate. Show that the smaller island will eventually have fewer species.

21. (**Integrating Factor Method**) There is another popular method, called the integrating factor method, for solving first-order linear equations written in the form

$$u' - p(t)u = q(t).$$

If this equation is multiplied by $e^{-\int p(t)dt}$, called an **integrating factor**, show that

$$\left(ue^{-\int p(t)dt}\right)' = q(t)e^{-\int p(t)dt}.$$

(Note that the left side is a total derivative). Next, integrate both sides and show that you obtain (2.6). Use this method to solve Exercises 1 and 2 above.

2.3 Approximation

The fact is that most differential equations cannot be solved with simple analytic formulas. Therefore we are interested in developing methods to approximate solutions to differential equations. Approximations can come in the form of a formula or a data set obtained by computer methods. The latter forms the basis of modern scientific computation.

2.3.1 Picard Iteration

We first introduce an iterative procedure, called Picard iteration (E. Picard, 1856-1941), that is adapted from the classical fixed point method to approximate solutions of nonlinear algebraic equations. In Picard iteration we begin with an assumed first approximation of the solution to an initial value problem and then calculate successively better approximations by an iterative, or recursive, procedure. The result is a set of recursive analytic formulas that approximate the solution. We first review the standard fixed point method for algebraic equations.

Example 2.8

Consider the problem of solving the nonlinear algebraic equation

$$x = \cos x.$$

Graphically, it is clear that there is a unique solution because the curves $y = x$ and $y = \cos x$ cross at a single point. Analytically we can approximate the root by making an initial guess x_0 and then successively calculate better approximations via

$$x_{k+1} = \cos x_k \quad \text{for} \quad k = 0, 1, 2, \ldots$$

For example, if we choose $x_0 = 0.9$, then $x_1 = \cos x_0 = \cos(0.9) = 0.622$, $x_2 = \cos x_1 = \cos(0.622) = 0.813$, $x_3 = \cos x_2 = \cos(0.813) = 0.687$, $x_4 = \cos x_3 = \cos(0.687) = 0.773$, $x_5 = \cos x_4 = \cos(0.773) = 0.716, \ldots$. Thus we have generated a sequence of approximations 0.9, 0.622, 0.813, 0.687, 0.773, $0.716, \ldots$. If we continue the process, the sequence converges to $x^* = 0.739$, which is the solution to $x = \cos x$ (to three decimal places). This method, called **fixed point iteration**, can be applied to general algebraic equations of the form

$$x = g(x).$$

The iterative procedure

$$x_{k+1} = g(x_k), \quad k = 0, 1, 2, \ldots$$

will converge to a root x^* provided $|g'(x^*)| < 1$ and the initial guess x_0 is sufficiently close to x^*. The conditions stipulate that the graph of g is not too steep (its absolute slope at the root must be bounded by one), and the initial guess is close to the root.

We pick up on this iteration idea for algebraic equations to obtain an approximation method for solving the initial value problem

$$\text{(IVP)} \qquad \begin{cases} u' = f(t,u), \\ u(t_0) = u_0. \end{cases} \tag{2.10}$$

First, we turn this initial value problem into an equivalent integral equation by integrating the DE from t_0 to t:

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds.$$

Now we define a type of fixed point iteration, called **Picard iteration**, that is based on this integral equation formulation. We define the iteration scheme

$$u_{k+1}(t) = u_0 + \int_{t_0}^{t} f(s, u_k(s))ds, \quad k = 0, 1, 2, ..., \tag{2.11}$$

where $u_0(t)$ is an initial approximation (we often take the initial approximation to be the constant function $u_0(t) = u_0$). Proceeding in this manner, we generate a sequence $u_1(t)$, $u_2(t)$, $u_3(t)$,... of iterates, called **Picard iterates**, that under certain conditions converge to the solution of the original initial value problem (2.10).

Example 2.9

Consider the linear initial value problem

$$u' = 2t(1+u), \quad u(0) = 0.$$

Then the iteration scheme is

$$u_{k+1}(t) = \int_0^t 2s(1 + u_k(s))ds, \quad k = 0, 1, 2, ...,$$

Take $u_0 = 0$, then

$$u_1(t) = \int_0^t 2s(1+0)ds = t^2.$$

Then

$$u_2(t) = \int_0^t 2s(1 + u_1(s))ds = \int_0^t 2s(1+s^2)ds = t^2 + \frac{1}{2}t^4.$$

Next,

$$u_3(t) = \int_0^t 2s(1+u_2(s))ds = u_{k+1}(t) = \int_0^t 2s(1+s^2+\frac{1}{2}s^4)ds = t^2+\frac{1}{2}t^4+\frac{1}{6}t^6.$$

In this manner we generate a sequence of approximations to the solution to the IVP. In the present case, one can verify that the analytic solution to the IVP is

$$u(t) = e^{t^2} - 1.$$

The Taylor series expansion of this function is

$$u(t) = e^{t^2} - 1 = t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots + \frac{1}{n!}t^{2n} + \cdots,$$

and it converges for all t. Therefore the successive approximations generated by Picard iteration are the partial sums of this series, and they converge to the exact solution.

The Picard procedure (2.11) is especially important from a theoretical viewpoint. The method forms the basis of an existence proof for the solution to a general *nonlinear* initial value problem; the idea is to show that there is a limit to the sequence of approximations, and that limit is the solution to the initial value problem. This topic is discussed in advanced texts on differential equations. Practically, however, Picard iteration is not especially useful for problems in science and engineering. There are other methods, based upon numerical algorithms, that give highly accurate approximations. We discuss these methods in the next section.

Finally, we point out that Picard iteration is guaranteed to converge if the right side of the equation $f(t, u)$ is regular enough; specifically, the first partial derivatives of f must be continuous in an open rectangle of the tu plane containing the initial point. However, convergence is only guaranteed locally, in a small interval about t_0.

EXERCISES

1. Consider the initial value problem

$$u' = 1 + u^2, \quad u(0) = 0.$$

   Apply Picard iteration with $u_0 = 0$ and compute four terms. If the process continues, to what function will the resulting series converge?

2. Apply Picard iteration to the initial value problem

$$u' = t - u, \quad u(0) = 1,$$

   to obtain three Picard iterates, taking $u_0 = 1$. Plot each iterate and the exact solution on the same set of axes.

2.3.2 Numerical Methods

As already emphasized, most differential equations cannot be solved analytically by a simple formula. In this section we develop a class of methods that solve an initial value problem numerically, using a computer algorithm. In industry and science, differential equations are almost always solved numerically because most real-world problems are too complicated to solve analytically. And, even if the problem can be solved analytically, often the solution is in the form of a complicated integral that has to be resolved by a computer calculation anyway. So why not just begin with a computational approach in the first place?

We study numerical approximations by a method belonging to a class called **finite difference methods**. Here is the basic idea. Suppose we want to solve the following initial value problem on the interval $0 \le t \le T$:

$$u' = f(t, u), \quad u(0) = u_0. \tag{2.12}$$

Rather than seek a continuous solution defined at each time t, we develop a strategy of discretizing the problem to determine an approximation at discrete times in the interval of interest. Therefore, the plan is to replace the continuous time model (2.12) with an approximate discrete time model that is amenable to computer solution.

To this end, we divide the interval $0 \le t \le T$ into N segments of constant length h, called the **stepsize**. Thus the stepsize is $h = T/N$. This defines a set of equally spaced discrete times $0 = t_0, t_1, t_2, ..., t_N = T$, where $t_n = nh$, $n = 0, 1, 2, ..., N$. Now, suppose we know the solution $u(t_n)$ of the initial value problem at time t_n. How could we estimate the solution at time t_{n+1}? Let us integrate the DE (2.12) from t_n to t_{n+1} and use the fundamental theorem of calculus. We get the equation

$$u(t_{n+1}) - u(t_n) = \int_{t_n}^{t_{n+1}} f(t, u)dt. \tag{2.13}$$

The integral can be approximated using the left-hand rule, giving

$$u(t_{n+1}) - u(t_n) \approx hf(t_n, u(t_n)).$$

If we denote by u_n the approximation of the solution $u(t_n)$ at $t = t_n$, then this last formula suggests the recursion formula

$$u_{n+1} = u_n + hf(t_n, u_n). \tag{2.14}$$

If $u(0) = u_0$, then (2.14) provides an algorithm for calculating approximations u_1, u_2, u_3, etc., recursively, at times t_1, t_2, t_3,... This method is called the **Euler method**, named after the Swiss mathematician L. Euler (1707–1783). The discrete approximation consisting of the values u_0, u_1, u_2, u_3, etc. is called a **numerical solution** to the initial value problem. The discrete values approximate the graph of the exact solution, and often they are connected with line segments to obtain a continuous curve. It seems evident that the smaller the stepsize h, the better the approximation. One can show that the cumulative error over an interval $0 \leq t \leq T$ is bounded by the stepsize h; thus, the Euler method is said to be of **order** h.

Example 2.10

Consider the initial value problem

$$u' = 1 + tu, \quad u(0) = 0.25.$$

Here $f(t, u) = 1 + tu$ and the Euler difference equation (2.14) with stepsize h is

$$\begin{aligned} u_{n+1} &= u_n + h(1 + t_n u_n) \\ &= u_n + h(1 + nhu_n), \quad n = 0, 1, 2, 3, \ldots \end{aligned}$$

We take $h = 0.1$. Beginning with $u_0 = 0.25$ we have

$$u_1 = u_0 + (0.1)(1 + (0)(0.1)u_0) = 0.25 + (0.1)(1) = 0.350.$$

Then

$$u_2 = u_1 + (0.1)(1 + (1)(0.1)u_1) = 0.35 + (0.1)(1 + (1)(0.1)(0.35)) = 0.454.$$

Next

$$u_3 = u_2 + (0.1)(1 + (2)(0.1)u_2) = 0.454 + (0.1)(1 + (2)(0.1)(0.454)) = 0.563.$$

Continuing in this manner we generate a sequence of numbers at all the discrete time points. We often connect the approximations by straight line segments to generate a continuous curve. In figure 2.3 we compare the discrete solution to the exact solution $u(t) = e^{t^2/2}(0.25 + \int_0^t e^{-s^2/2}ds)$. Because it is tedious to do numerical calculations by hand, one can program a calculator or write a simple set of instructions for a computer algebra system to do the work for us. Most calculators and computer algebra systems have built-in programs that implement the Euler algorithm automatically. Below is a MATLAB m-file to perform the calculations in Example 2.10 and plot the approximate solution on the interval [0,1]. We take 10 steps, so the stepsize is $h = 1/10 = 0.1$.

```
function euler1D
Tmax=1; N=10; h=Tmax/N;
u=0.25; uhistory=0.25;
for n=1:N;
u=u+h*(1+n*h*u);
uhistory=[uhistory, u];
end
T=0:h:Tmax;
plot(T,uhistory)
xlabel('time t'), ylabel('u')
```

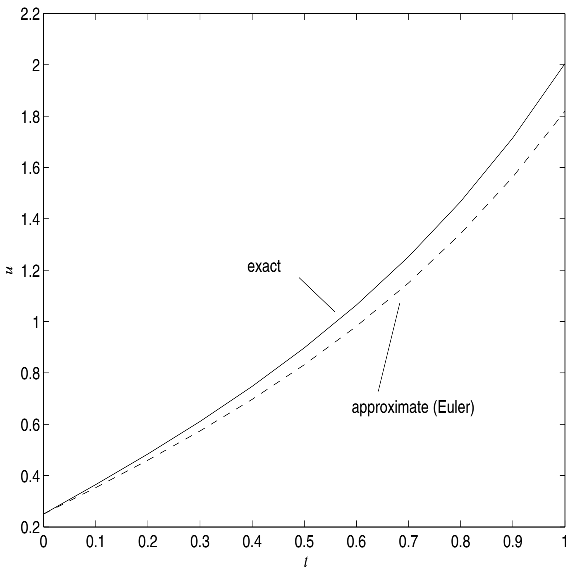

Figure 2.3 The numerical solution (fit with a continuous curve) and exact solution in Example 2.10. Here, $h = 0.1$. A closer approximation can be obtained with a smaller stepsize.

In science and engineering we often write simple programs that implement recursive algorithms; that way we know the skeleton of our calculations, which is often preferred to plugging into an unknown black box containing a canned program.

There is another insightful way to understand the Euler algorithm using the direction field. Beginning at the initial value, we take $u_0 = u(0)$. To find u_1, the approximation at t_1, we march from (t_0, u_0) along the direction field segment with slope $f(t_0, u_0)$ until we reach the point (t_1, u_1) on the vertical line $t = t_1$.

Then, from (t_1, u_1) we march along the direction field segment with slope $f(t_1, u_1)$ until we reach (t_2, u_2). From (t_2, u_2) we march along the direction field segment with slope $f(t_2, u_2)$ until we reach (t_3, u_3). We continue in this manner until we reach $t_N = T$. So how do we calculate the u_n? Inductively, let us assume we are at (t_n, u_n) and want to calculate u_{n+1}. We march along the straight line segment with slope $f(t_n, u_n)$ to (t_{n+1}, u_{n+1}). Thus, writing the slope of this segment in two different ways,

$$\frac{u_{n+1} - u_n}{t_{n+1} - t_n} = f(t_n, u_n).$$

But $t_{n+1} - t_n = h$, and therefore we obtain

$$u_{n+1} = u_n + hf(t_n, u_n),$$

which is again the Euler formula. In summary, the Euler method computes approximate values by moving in the direction of the slope field at each point. This explains why the the numerical solution in Example 2.10 (figure 2.3) lags behind the increasing exact solution.

The Euler algorithm is the simplest method for numerically approximating the solution to a differential equation. To obtain a more accurate method, we can approximate the integral on the right side of (2.13) by the trapezoidal rule, giving

$$u_{n+1} - u_n = \frac{h}{2}[f(t_n, u_n) + f(t_{n+1}, u_{n+1})]. \tag{2.15}$$

This difference equation is not as simple as it may first appear. It does not give the u_{n+1} explicitly in terms of the u_n because the u_{n+1} is tied up in a possibly nonlinear term on the right side. Such a difference equation is called an **implicit equation**. At each step we would have to solve a nonlinear algebraic equation for the u_{n+1}; we can do this numerically, which would be time consuming. Does it pay off in more accuracy? The answer is yes. The Euler algorithm makes a cumulative error over an interval proportional to the stepsize h, whereas the implicit method makes an error of order h^2. Observe that $h^2 < h$ when h is small.

A better approach, which avoids having to solve a nonlinear algebraic equation at each step, is to replace the u_{n+1} on the right side of (2.15) by the u_{n+1} calculated by the simple Euler method in (2.14). That is, we compute a "predictor"

$$\widetilde{u}_{n+1} = u_n + hf(t_n, u_n), \tag{2.16}$$

and then use that to calculate a "corrector"

$$u_{n+1} = u_n + \frac{1}{2}h[f(t_n, u_n) + f(t_{n+1}, \widetilde{u}_{n+1})]. \tag{2.17}$$

This algorithm is an example of a **predictor–corrector method**, and again the cumulative error is proportional to h^2, an improvement to the Euler method. This method is called the **modified Euler method** (also, Heun's method and the second-order Runge–Kutta method).

The Euler and modified Euler methods are two of many numerical constructs to solve differential equations. Because solving differential equations is so important in science and engineering, and because real-world models are usually quite complicated, great efforts have gone into developing accurate, efficient methods. The most popular algorithm is the highly accurate fourth-order **Runge–Kutta method**, where the cumulative error over a bounded interval is proportional to h^4. The Runge–Kutta update formula is

$$u_{n+1} = u_n + \frac{h}{6}(k_1 + k_2 + k_3 + k_4),$$

where

$$\begin{aligned} k_1 &= f(t_n, u_n), \\ k_2 &= f(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_1), \\ k_3 &= f(t_n + \frac{h}{2}, u_n + \frac{h}{2}k_2), \\ k_4 &= f(t_n + h, u_n + hk_3). \end{aligned}$$

We do not derive the formulas here, but they follow by approximating the integral in (2.13) by Simpson's rule, and then averaging. The Runge–Kutta method is built in on computer algebra systems and on scientific calculators.

Note that the order of the error makes a big difference in the accuracy. If $h = 0.1$ then the cumulative errors over an interval for the Euler, modified Euler, and Runge–Kutta methods are proportional to 0.1, 0.01, and 0.0001, respectively.

2.3.3 Error Analysis

Readers who want a detailed account of the errors involved in numerical algorithms should consult a text on numerical analysis or on numerical solution of differential equations. In this section we give only a brief elaboration of the comments made in the last section on the order of the error involved in Euler's method.

Consider again the initial value problem

$$u' = f(t, u), \quad u(0) = u_0 \tag{2.18}$$

on the interval $0 \le t \le T$, with solution $u(t)$. For our argument we assume u has a continuous second derivative on the interval (which implies that the second derivative is bounded). The Euler method, which gives approximations u_n at the discrete points $t_n = nh$, $n = 1, 2, ..., N$, is the recursive algorithm

$$u_{n+1} = u_n + hf(t_n, u_n). \tag{2.19}$$

We want to calculate the error made in performing one step of the Euler algorithm. Suppose at the point t_n the approximation u_n is exact; that is, $u_n = u(t_n)$. Then we calculate the error at the next step. Let $E_{n+1} = u(t_{n+1}) - u_{n+1}$ denote the error at the $(n+1)$st step. Evaluating the DE at $t = t_n$, we get

$$u'(t_n) = f(t_n, u(t_n)).$$

Recall from calculus (Taylor's theorem with remainder) that if u has two continuous derivatives then

$$\begin{aligned} u(t_{n+1}) &= u(t_n + h) = u(t_n) + u'(t_n)h + \frac{1}{2}u''(\tau_n)h^2 \\ &= u(t_n) + hf(t_n, u(t_n)) + \frac{1}{2}u''(\tau_n)h^2 \\ &= u_n + hf(t_n, u_n) + \frac{1}{2}u''(\tau_n)h^2, \end{aligned} \tag{2.20}$$

where the second derivative is evaluated at some point τ_n in the interval (t_n, t_{n+1}). Subtracting (2.19) from (2.20) gives

$$E_{n+1} = \frac{1}{2}u''(\tau_n)h^2.$$

So, if u_n is exact, the Euler algorithm makes an error proportional to h^2 in computing u_{n+1}. So, at each step the Euler algorithm gives an error of order h^2. This is called the **local error**. Notice that the absolute error is $|E_{n+1}| = \frac{1}{2}|u''(\tau_n)|h^2 \le \frac{1}{2}Ch^2$, where C is an absolute bound for the second derivative of u on the entire interval; that is, $|u''(t)| \le C$ for $0 \le t \le T$.

If we apply the Euler method over an entire interval of length T, where $T = Nh$ and N the number of steps, then we expect to make a cumulative error of N times the local error, or an error bounded by a constant times h. This is why we say the cumulative error in Euler's method is order h.

An example will confirm this calculation. Consider the initial value problem for the growth–decay equation:

$$u' = ku, \quad u(0) = u_0,$$

with exact solution $u(t) = u_0 e^{kt}$. The Euler method is

$$u_{n+1} = u_n + hku_n = (1 + hk)u_n.$$

We can iterate to find the exact formula for the sequence of approximations:

$$\begin{aligned} u_1 &= (1+hk)u_0, \\ u_2 &= (1+hk)u_1 = (1+hk)^2u_0, \\ u_3 &= (1+hk)u_2 = (1+hk)^3u_0, \\ &\cdots \\ u_n &= (1+hk)^nu_0. \end{aligned}$$

One can calculate the cumulative error in applying the method over an interval $0 \le t \le T$ with $T = Nh$, where N is the total number of steps. We have

$$E_N = u(T) - u_N = u_0[e^{kT} - (1+hk)^N].$$

The exponential term in the parentheses can be expressed in its Taylor series, $e^{kT} = 1 + kT + \frac{1}{2}(kT)^2 + \cdots$, and the second term can be expanded using the binomial theorem, $(1+hk)^N = 1 + Nhk + \frac{N(N+1)}{2}(hk)^2 + \cdots + (hk)^N$. Using $T = Nh$,

$$\begin{aligned} E_N &= u_0[1 + kT + \frac{1}{2}(kT)^2 + \cdots - 1 - Nhk - \frac{N(N+1)}{2}(hk)^2 - \cdots - (hk)^N] \\ &= -\frac{u_0Tk^2}{2}h + \text{ terms containing at least } h^2. \end{aligned}$$

So the cumulative error is the order of the stepsize h.

EXERCISES

1. Use the Euler method and the modified Euler method to numerically solve the initial value problem $u' = 0.25u - t^2$, $u(0) = 2$, on the interval $0 \le t \le 2$ using a stepsize $h = 0.25$. Find the exact solution and compare it, both graphically and in tabular form, to the numerical solutions. Perform the same calculation with $h = 0.1$, $h = 0.01$, and $h = 0.001$. Confirm that the cumulative error at $t = 2$ is roughly order h for the Euler method and order h^2 for the modified Euler method.

2. Use the Euler method to solve the initial value problem $u' = u\cos t$, $u(0) = 1$ on the interval $0 \le t \le 20$ with 50, 100, 200, and 400 steps. Compare to the exact solution and comment on the accuracy of the numerical algorithm.

3. A population of bacteria, given in millions of organisms, is governed by the law
$$u' = 0.6u\left(1 - \frac{u}{K(t)}\right), \quad u(0) = 0.2,$$
where in a periodically varying environment the carrying capacity is $K(t) = 10 + 0.9\sin t$, and time is given in days. Plot the bacteria population for 40 days.

4. Consider the initial value problem for the decay equation,

$$u' = -ru, \quad u(0) = u_0.$$

Here, r is a given positive decay constant. Find the exact solution to the initial value problem and the exact solution to the sequence of difference approximations $u_{n+1} = u_n - hru_n$ defined by the Euler method. Does the discrete solution give a good approximation to the exact solution for all stepsizes h? What are the constraints on h?

5. Suppose the temperature inside your winter home is 68 degrees at 1:00 P.M. and your furnace then fails. If the outside temperature has an hourly variation over each day given by $15 + 10\cos\frac{\pi t}{12}$ degrees (where $t = 0$ represents 2:00 P.M.), and you notice that by 10:00 P.M. the inside temperature is 57 degrees, what will be the temperature in your home the next morning at 6:00 A.M.? Sketch a plot showing the temperature inside your home and the outside air temperature.

6. Write a program in your computer algebra system that uses the Runge-Kutta method for solving the initial value problem (2.12), and use the program to numerically solve the problem

$$u' = -u^2 + 2t, \quad u(0) = 1.$$

7. Consider the initial value problem $u' = 5u - 6e^{-t}$, $u(0) = 1$. Find the exact solution and plot it on the interval $0 \le t \le 3$. Next use the Euler method with $h = 0.1$ to obtain a numerical solution. Explain the results of this numerical experiment.

8. Consider the initial value problem

$$u' = -u + (15 - u)e^{-a/(u+1)}, \quad u(0) = 1,$$

where a is a parameter. This model arises in the study of a chemically reacting fluid passing through a continuously stirred tank reactor, where the reaction gives off heat. The variable u is related to the temperature in the reactor (Logan 1997, pp. 430–434). Plot the solution for $a = 5.2$ and for $a = 5.3$ to show that the model is sensitive to small changes in the parameter a (this sensitivity is called **structural instability**). Can you explain why this occurs? (Plot the bifurcation diagram with bifurcation parameter a.)

3

Second-Order Differential Equations

Second-order differential equations are one of the most widely studied classes of differential equations in mathematics, physical science, and engineering. One sure reason is that Newton's second law of motion is expressed as a law that involves acceleration of a particle, which is the second derivative of position. Thus, one-dimensional mechanical systems are governed naturally by a second-order equations.

There are really two strategies in dealing with a second-order differential equation. We can always turn a single, second-order differential equation into a system of two simultaneous first-order equations and study the system. Or, we can deal with the equation itself, as it stands. For example, consider the damped spring-mass equation

$$mx'' = -kx - cx'.$$

From Section 1.3 we recall that this equation models the decaying oscillations of a mass m under the action of two forces, a restoring force $-kx$ caused by the spring, and a frictional force $-cx'$ caused by the damping mechanism. This equation is nothing more than a statement of Newton's second law of motion. We can easily transform this equation into a system of two first-order equations with two unknowns by selecting a second unknown state function $y = y(t)$ defined by $y(t) = x'(t)$; thus y is the velocity. Then $my' = -kx - cy$. So the second-order equation is equivalent to

$$\begin{aligned} x' &= y, \\ y' &= -\frac{k}{m}x - \frac{c}{m}y. \end{aligned}$$

This is a simultaneous system of two equations in two unknowns, the position $x(t)$ and the velocity $y(t)$; both equations are first-order. Why do this? Have we gained advantage? Is the system easier to solve than the single equation? The answers to these questions emerge as we study both types of equations in the sequel. Here we just make some general remarks that you may find cryptic. It is probably easier to find the solution formula to the second-order equation directly. But the first-order system embodies a geometrical structure that reveals the underlying dynamics in a far superior way. And, first-order systems arise just as naturally as second-order equations in many other areas of application. Ultimately, it comes down to one's perspective and what information one wants to get from the physical system. Both viewpoints are important.

In this chapter we develop some methods for understanding and solving a single second-order equation. In Chapters 5 and 6 we examine systems of first-order equations.

3.1 Particle Mechanics

Some second-order differential equations can be reduced essentially to a single first-order equation that can be handled by methods from Chapter 2. We place the discussion in the context of particle mechanics to illustrate some of the standard techniques. The general form of Newton's law is

$$mx'' = F(t, x, x'), \tag{3.1}$$

where $x = x(t)$ is the displacement from equilibrium.

(a) If the force does not depend on the position x, then (3.1) is

$$mx'' = F(t, x').$$

We can make the velocity substitution $y = x'$ to obtain

$$my' = F(t, y),$$

which is a first-order differential equation that can be solved with the methods of the preceding chapter. Once the velocity $y = y(t)$ is found, then the position $x(t)$ can be recovered by anti-differentiation, or $x(t) = \int y(t)dt + C$.

(b) If the force does not depend explicitly on time t, then (3.1) becomes

$$mx'' = F(x, x').$$

Again we introduce $y = x'$. Using the chain rule to compute the second derivative (acceleration),

$$x'' = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = y\frac{dy}{dx}.$$

Then

$$my\frac{dy}{dx} = F(x, y),$$

which is a first-order differential equation for the velocity y in terms of the position x. If we solve this equation to obtain $y = y(x)$, then we can recover $x(t)$ by solving the equation $x' = y(x)$ by separation of variables.

(c) In the important, special case where the force F depends only on the position x we say F is a **conservative force**. See also Example 1.5. Then, using the same calculation as in (b), Newton's law becomes

$$my\frac{dy}{dx} = F(x),$$

which is a separable equation. We may integrate both sides with respect to x to get

$$m\int y\frac{dy}{dx}dx = \int F(x)dx + E,$$

or

$$\frac{1}{2}my^2 = \int F(x)dx + E.$$

Note that the left side is the kinetic energy, one-half the mass times the velocity-squared. We use the symbol E for the constant of integration because it must have dimensions of energy. We recall from calculus that the **potential energy** function $V(x)$ is defined by $-dV/dx = F(x)$, or the "force is the negative gradient of the potential." Then $\int F(x)dx = -V(x)$ and we have

$$\frac{1}{2}my^2 + V(x) = E, \tag{3.2}$$

which is the **energy conservation theorem**: the kinetic plus potential energy for a conservative system is constant. The constant E, which represents the total energy in the system, can be computed from knowledge of the initial position $x(0) = x_0$ and initial velocity $y(0) = y_0$, or $E = \frac{1}{2}y_0^2 + V(x_0)$. We regard the conservation of energy law as a reduction of Newton's second law; the latter is a second-order equation, whereas (3.2) is a first-order equation if we replace the position y by dx/dt. It may be recast into

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}\sqrt{E - V(x)}. \tag{3.3}$$

This equation is separable, and its solution would give $x = x(t)$. The appropriate sign is taken depending upon whether the velocity is positive or negative during a certain phase of the motion.

Usually we analyze conservative systems in phase space (xy-space, or the **phase plane**) by plotting y vs. x from equation (3.2) for different values of the parameter E. The result is a one-parameter family of curves, or **orbits**, in the xy plane along which the motion occurs. The set of these curves forms the **phase diagram** for the system. On these orbits we do not know how x and y depend upon time t unless we solve (3.3). But we do know how velocity relates to position.

Example 3.1

Consider a spring-mass system without damping. The governing equation is

$$mx'' = -kx,$$

where k is the spring constant. The force is $-kx$ and the potential energy $V(x)$ is given by

$$V(x) = -\int -kx dx = \frac{k}{2}x^2.$$

We have picked the constant of integration to be zero, which automatically sets the zero level of potential energy at $x = 0$ (i.e., $V(0) = 0$). Conservation of energy is expressed by (3.2), or

$$\frac{1}{2}my^2 + \frac{k}{2}x^2 = E,$$

which plots as a family of concentric ellipses in the xy phase plane, one ellipse for each value of E. These curves represent oscillations, and the mass tracks on one of these orbits in the phase plane, continually cycling as time passes; the position and velocity cycle back and forth. At this point we could attempt to solve (3.3) to determine how x varies in time, but in the next section we find an easier method to solve second-order linear equations for $x(t)$ directly.

EXERCISES

1. Consider a dynamical system governed by the equation $x'' = -x + x^3$. Hence, $m = 1$. Find the potential energy $V(x)$ with $V(0) = 0$. How much total energy E is in the system if $x(0) = 2$ and $x'(0) = 1$? Plot the orbit in the xy phase plane of a particle having this amount of total energy.

2. In a conservative system show that the conservation of energy law (3.2) can be obtained by multiplying the governing equation $mx'' = F(x)$ by x' and noting that $\frac{d}{dt}(x'^2) = 2x'x''$.

3. In a conservative system derive the relation

$$t = \pm \int \frac{dx}{\sqrt{2(E - V(x))}} + C,$$

which gives time as an antiderivative of an expression that is a function of position.

4. A bullet is shot from a gun with muzzle velocity 700 meters per second horizontally at a point target 100 meters away. Neglecting air resistance, by how much does the bullet miss its target?

5. Solve the following differential equations by reducing them to first-order equations.

   a) $x'' = -\frac{2}{t}x'$.

   b) $x'' = xx'$.

   c) $x'' = -4x$.

   d) $x'' = (x')^2$.

   e) $tx'' + x' = 4t$.

6. In a nonlinear spring-mass system the equation governing displacement is $x'' = -2x^3$. Show that conservation of energy for the system can be expressed as $y^2 = C - x^4$, where C is a constant. Plot this set of orbits in the phase plane for different values of C. If $x(0) = x_0 > 0$ and $x'(0) = 0$, show that the period of oscillations is

$$T = \frac{4}{x_0}\int_0^1 \frac{dr}{\sqrt{1-r^4}}.$$

   Sketch a graph of the period T vs. x_0. (Hint: in (3.3) separate variables and integrate over one-fourth of a period.)

3.2 Linear Equations with Constant Coefficients

We recall two models from Chapter 1. For a spring-mass system with damping the displacement $x(t)$ satisfies

$$mx'' + cx' + kx = 0.$$

The current $I(t)$ in an RCL circuit with no emf satisfies

$$LI'' + RI' + \frac{1}{C}I = 0.$$

The similarity between these two models is called the **mechanical-electrical analogy**. The spring constant k is analogous to the inverse capacitance $1/C$; both a spring and a capacitor store energy. The damping constant c is analogous to the resistance R; both friction in a mechanical system and a resistor in an electrical system dissipate energy. The mass m is analogous to the inductance

L; both represent "inertia" in the system. All of the equations we examine in the next few sections can be regarded as either circuit equations or mechanical problems.

After dividing by the leading coefficient, both equations above have the form

$$u'' + pu' + qu = 0, \tag{3.4}$$

where p and q are constants. An equation of the form (3.4) is called a **second-order linear equation with constant coefficients**. Because zero is on the right side (physically, there is no external force or emf), the equation is **homogeneous**. Often the equation is accompanied by initial data of the form

$$u(0) = A, \quad u'(0) = B. \tag{3.5}$$

The problem of solving (3.4) subject to (3.5) is called the **initial value problem** (IVP). Here the initial conditions are given at $t = 0$, but they could be given at any time $t = t_0$. Fundamental to our discussion is the following existence-uniqueness theorem, which we assume to be true. It is proved in advanced texts.

Theorem 3.2

The initial value problem (3.4)–(3.5) has a unique solution that exists on $-\infty < t < \infty$.

The plan is this. We first note that the DE (3.4) always has two independent solutions $u_1(t)$ and $u_2(t)$ (by **independent** we mean one is not a multiple of the other). We prove this fact by actually exhibiting the solutions explicitly. If we multiply each by an arbitrary constant and form the combination

$$u(t) = c_1 u_1(t) + c_2 u_1(t),$$

where c_1 and c_2 are the arbitrary constants, then we can easily check that $u(t)$ is also a solution to (3.4). This combination is called the **general solution** to (3.4). We prove at the end of this section that all solutions to (3.4) are contained in this combination. Finally, to solve the initial value problem we use the initial conditions (3.5) to uniquely determine the constants c_1 and c_2.

We try a solution to (3.4) of the form $u = e^{\lambda t}$, where λ is to be determined. We suspect something like this might work because every term in (3.4) has to be the same type of function in order for cancellation to occur; thus u, u', and u'' must be the same form, which suggests an exponential for u. Substitution of $u = e^{\lambda t}$ into (3.4) instantly leads to

$$\lambda^2 + p\lambda + q = 0, \tag{3.6}$$

which is a quadratic equation for the unknown λ. Equation (3.6) is called the **characteristic equation**. Solving, we obtain roots

$$\lambda = \frac{1}{2}(-p \pm \sqrt{p^2 - 4q}).$$

These roots of the characteristic equation are called the **characteristic values** (or **roots**) corresponding to the differential equation (3.4). There are three cases, depending upon whether the discriminant $p^2 - 4q$ is positive, zero, or negative. The reader should memorize these three cases and the forms of the solution.

Case 1. If $p^2 - 4q > 0$ then there are two real unequal characteristic values λ_1 and λ_2. Hence, there are two independent, exponential-type solutions

$$u_1(t) = e^{\lambda_1 t}, \quad u_2(t) = e^{\lambda_2 t},$$

and the general solution to (3.4) is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \tag{3.7}$$

Case 2. If $p^2 - 4q = 0$ then there is a double root $\lambda = -p/2$. Then one solution is $u_1 = e^{\lambda t}$. A second independent solution in this case is $u_2 = te^{\lambda t}$. Therefore the general solution to (3.4) in this case is

$$u(t) = c_1 e^{\lambda t} + c_2 te^{\lambda t}. \tag{3.8}$$

Case 3. If $p^2 - 4q < 0$ then the roots of the characteristic equation are complex conjugates having the form

$$\lambda = \alpha \pm i\beta.$$

Therefore two *complex* solutions of (3.4) are

$$e^{(\alpha+i\beta)t}, \quad e^{(\alpha-i\beta)t}.$$

To manufacture *real* solutions we use a fundamental result that holds for all linear, homogeneous equations.

Theorem 3.3

If $u = g(t) + ih(t)$ is a complex solution to the differential equation (3.4), then its real and imaginary parts, $g(t)$ and $h(t)$, are real solutions.

The simple proof is requested in the Exercises.

Let us take the first of the complex solutions given above and expand it into its real and imaginary parts using **Euler's formula:** $e^{i\beta t} = \cos\beta t + i\sin\beta t$. We have

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}(\cos\beta t + i\sin\beta t) = e^{\alpha t}\cos\beta t + ie^{\alpha t}\sin\beta t.$$

Therefore, by Theorem 3.3, $u_1 = e^{\alpha t}\cos\beta t$ and $u_2 = e^{\alpha t}\sin\beta t$ are two real, independent solutions to equation (3.4). If we take the second of the complex solutions, $e^{(\alpha-i\beta)t}$ instead of $e^{(\alpha+i\beta)t}$, then we get the same two real solutions. Consequently, in the case that the characteristic values are complex $\lambda = \alpha \pm i\beta$, the general solution to DE (3.4) is

$$u(t) = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t. \tag{3.9}$$

In the case of complex eigenvalues, we recall from trigonometry that (3.9) can be written differently as

$$u(t) = e^{\alpha t}(c_1\cos\beta t + c_2\sin\beta t) = e^{\alpha t}A\cos(\beta t - \varphi),$$

where A is called the **amplitude** and φ is the **phase**. This latter form is called the **phase–amplitude form** of the general solution. Written in this form, A and φ play the role of the two arbitrary constants, instead of c_1 and c_2. One can show that that all these constants are related by

$$A = \sqrt{c_1^2 + c_2^2}, \quad \varphi = \arctan\frac{c_2}{c_1}.$$

This is because the cosine of difference expands to

$$A\cos(\beta t - \varphi) = A\cos(\beta t)\cos\varphi + A\sin(\beta t)\sin\varphi.$$

Comparing this expression to $c_1\cos\beta t + c_2\sin\beta t$, gives

$$A\cos\varphi = c_1, \quad A\sin\varphi = c_2.$$

Squaring and adding this last set of equations determines A, and dividing the set of equations determines φ.

Observe that the solution in the complex case is oscillatory in nature with $e^{\alpha t}$ multiplying the amplitude A. If $\alpha < 0$ then the solution will be a decaying oscillation and if $\alpha > 0$ the solution will be a growing oscillation. If $\alpha = 0$ then the solution is

$$u(t) = c_1\cos\beta t + c_2\sin\beta t = A\cos(\beta t - \varphi),$$

and it oscillates with constant amplitude A and period $2\pi/\beta$. The frequency β is called the **natural frequency** of the system.

There is some useful terminology used in engineering to describe the motion of a spring-mass system with damping, governed by the equation

$$mx'' + cx' + kx = 0.$$

The characteristic equation is

$$m\lambda^2 + c\lambda + k = 0,$$

with roots

$$\lambda = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

If the roots are complex ($c^2 < 4mk$) then the system is **under-damped** (representing a decaying oscillation); if the roots are real and equal ($c^2 = 4mk$) then the system is **critically damped** (decay, no oscillations, and at most one pass through equilibrium $x = 0$); if the roots are real and distinct ($c^2 > 4mk$) then the system is **over-damped** (a strong decay toward $x = 0$). The same terminology can be applied to an RCL circuit.

Example 3.4

The differential equation $u'' - u' - 12u = 0$ has characteristic equation $\lambda^2 - \lambda - 12 = 0$ with roots $\lambda = -3, 4$. These are real and distinct and so the general solution to the DE is $u = c_1 e^{-3t} + c_2 e^{4t}$. Over a long time the contribution e^{-3t} decays and the solution is dominated by the e^{4t} term. Thus, eventually the solution grows exponentially.

Example 3.5

The differential equation $u'' + 4u' + 4u = 0$ has characteristic equation $\lambda^2 + 4\lambda + 4 = 0$, with roots $\lambda = -2, -2$. Thus the eigenvalues are real and equal, and the general solution is $u = c_1 e^{-2t} + c_2 te^{-2t}$. This solution decays as time gets large (recall that a decaying exponential dominates the linear growth term t so that te^{-2t} goes to zero).

Example 3.6

The differential equation $u''+2u'+2u = 0$ models a damped spring-mass system with $m = 1$, $c = 2$, and $k = 2$. It has characteristic equation $\lambda^2 + 2\lambda + 1 = 0$. The quadratic formula gives complex roots $\lambda = -1 \pm 2i$. Therefore the general solution is

$$u = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t,$$

representing a decaying oscillation. Here, the natural frequency of the undamped oscillation is 2. In phase–amplitude form we can write

$$u = Ae^{-t}\cos(2t - \varphi).$$

Let us assume that the mass is given an initial velocity of 3 from an initial position of 1. Then the initial conditions are $u(0) = 1$, $u'(0) = 3$. We can use these conditions directly to determine either c_1 and c_2 in the first form of the solution, or A and φ in the phase-amplitude form. Going the latter route, we apply the first condition to get

$$u(0) = Ae^{-0}\cos(2(0) - \varphi) = A\cos\varphi = 1.$$

To apply the other initial condition we need the derivative. We get

$$u' = -2Ae^{-t}\sin(2t - \varphi).$$

Then

$$u'(0) = -2Ae^{-0}\sin(2(0) - \varphi) = 2A\sin\varphi = 3.$$

Therefore we have

$$A\cos\varphi = 1, \quad A\sin\varphi = \frac{3}{2}.$$

Squaring both equations and summing gives $A^2 = 13/4$, so the amplitude is $A = \sqrt{13/4}$. Note that the cosine is positive and the sine is positive, so the phase angle lies in the first quadrant. The phase is

$$\varphi = \arctan(\frac{3}{2}) \doteq 0.983 \ \text{ radians}.$$

Therefore the solution to the initial value problem is

$$u = \sqrt{\frac{13}{4}}e^{-t}\cos(2t - 0.983).$$

This solution represents a decaying oscillation. The oscillatory part has natural frequency 2 and the period is π. See figure 3.1. The phase has the effect of translating the $\cos 2t$ term by $0.983/2$, which is called the phase shift.

To summarize, we have observed that the differential equation (3.4) always has two independent solutions $u_1(t)$ and $u_2(t)$, and that the combination

$$u(t) = c_1u_1(t) + c_2u_1(t)$$

is also a solution, called the general solution. Now, as promised, we show that the general solution contains all possible solutions to (3.4). To see this let $u_1(t)$ and $u_2(t)$ be special solutions that satisfy the initial conditions

$$u_1(0) = 1, \quad u_1'(0) = 0,$$

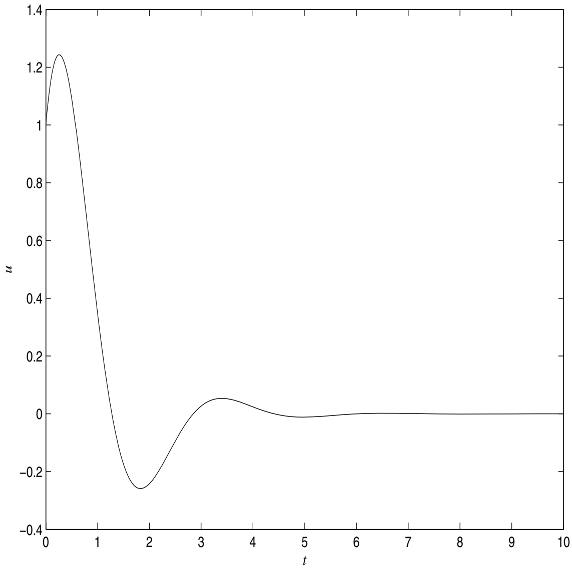

Figure 3.1 Plot of the solution.

and

$$u_2(0) = 0, \quad u_2'(0) = 1,$$

respectively. Theorem 3.2 implies these two solutions exist. Now let $v(t)$ be any solution of (3.4). It will satisfy some conditions at $t = 0$, say, $v(0) = a$ and $v'(0) = b$. But the function

$$u(t) = au_1(t) + bu_1(t)$$

satisfies those same initial conditions, $u(0) = a$ and $u'(0) = b$. Must $u(t)$ therefore equal $v(t)$? Yes, by the uniqueness theorem, Theorem 3.2. Therefore $v(t) = au_1(t)+bu_1(t)$, and the solution $v(t)$ is contained in the general solution.

Two equations occur so frequently that it is worthwhile to memorize them along with their solutions. The pure oscillatory equation

$$u'' + k^2 u = 0$$

has characteristic roots $\lambda = \pm ki$, and the general solution is

$$u = c_1 \cos kt + c_2 \sin kt.$$

On the other hand, the equation

$$u'' - k^2 u = 0$$

has characteristic roots $\lambda = \pm k$, and thus the general solution is

$$u = c_1 e^{kt} + c_2 e^{-kt}.$$

This latter equation can also be written in terms of the the hyperbolic functions cosh and sinh as

$$u = C_1 \cosh kt + C_2 \sinh kt,$$

where

$$\cosh kt = \frac{e^{kt} + e^{-kt}}{2}, \quad \sinh kt = \frac{e^{kt} - e^{-kt}}{2}.$$

Sometimes the hyperbolic form of the general solution is easier to work with.

EXERCISES

1. Find the general solution of the following equations:

   a) $u'' - 4u' + 4u = 0$.

   b) $u'' + u' + 4u = 0$.

   c) $u'' - 5u' + 6u = 0$.

   d) $u'' + 9u = 0$.

   e) $u'' - 2u' = 0$.

   f) $u'' - 12u = 0$.

2. Find the solution to the initial value problem $u'' + u' + u = 0, \quad u(0) = u'(0) = 1$, and write it in phase-amplitude form.

3. A damped spring-mass system is modeled by the initial value problem

   $$u'' + 0.125u' + u = 0, \quad u(0) = 2, \quad u'(0) = 0.$$

   Find the solution and sketch its graph over the time interval $0 \le t \le 50$. If the solution is written in the form $u(t) = Ae^{-t/16} \cos(\omega t - \varphi)$, find A, ω, and φ.

4. For which values of the parameters a and b (if any) will the solutions to $u'' - 2au' + bu = 0$ oscillate with no decay (i.e., be periodic)? Oscillate with decay? Decay without oscillations?

5. An RCL circuit has equation $LI'' + I' + I = 0$. Characterize the types of current responses that are possible, depending upon the value of the inductance L.

6. An oscillator with damping is governed by the equation $x'' + 3ax' + bx = 0$, where a and b are positive parameters. Plot the set of points in the ab plane (i.e., ab parameter space) where the system is critically damped.

7. Find a DE that has general solution $u(t) = c_1 e^{4t} + c_2 e^{-6t}$.
8. Find a DE that has solution $u(t) = e^{-3t} + 2te^{-3t}$. What are the initial conditions?
9. Find a DE that has solution $u(t) = \sin 4t + 3\cos 4t$.
10. Find a DE that has general solution $u(t) = A\cosh 5t + B\sinh 5t$, where A and B are arbitrary constants. Find the arbitrary constants when $u(0) = 2$ and $u'(0) = 0$.
11. Find a DE that has solution $u(t) = e^{-2t}(\sin 4t + 3\cos 4t)$. What are the initial conditions?
12. Describe the current response $I(t)$ of a LC circuit with $L = 5$ henrys, $C = 2$ farads, with $I(0) = 0$, $I'(0) = 1$.
13. Prove Theorem 3.3 by substituting u into the equation and separating real and imaginary parts, using linearity. Then use the fact that a complex quantity is zero if, and only if, its real and imaginary parts are zero.

3.3 The Nonhomogeneous Equation

In the last section we solved the **homogeneous equation**

$$u'' + pu' + qu = 0. \tag{3.10}$$

Now we consider the **nonhomogeneous equation**

$$u'' + pu' + qu = g(t), \tag{3.11}$$

where a known term $g(t)$, called a **source term** or **forcing term**, is included on the right side. In mechanics it represents an applied, time-dependent force; in a circuit it represents an applied voltage (an emf, such as a battery or generator). There is a general structure theorem, analogous to Theorem 1.2 for first-order linear equations, that dictates the form of the solution to the nonhomogeneous equation.

Theorem 3.7

All solutions of the nonhomogeneous equation (3.11) are given by the sum of the general solution to the homogeneous equation (3.10) and any particular solution to the nonhomogeneous equation. That is, the general solution to (3.11) is

$$u(t) = c_1 u_1(t) + c_2 u_1(t) + u_p(t),$$

where u_1 and u_2 are independent solutions to (3.10) and u_p is any solution to (3.11).

This result is very easy to show. If $u(t)$ is any solution whatsoever of (3.11), and $u_p(t)$ is a particular solution, then $u(t)-u_p(t)$ must satisfy the homogeneous equation (3.10). Therefore, by the results in the last section we must have $u(t) - u_p(t) = c_1u_1(t) + c_2u_1(t)$.

3.3.1 Undetermined Coefficients

We already know how to find the solution of the homogeneous equation, so we need techniques to find a particular solution u_p to (3.11). One method that works for many equations is to simply make a judicious guess depending upon the form of the source term $g(t)$. Officially, this method is called the **method of undetermined coefficients** because we eventually have to find numerical coefficients in our guess. This works because all the terms on the left side of (3.11) must add up to give $g(t)$. So the particular solution cannot be too wild if $g(t)$ is not too wild; in fact, it nearly must have the same form as $g(t)$. The method is successful for forcing terms that are exponential functions, sines and cosines, polynomials, and sums and products of these common functions. Here are some basic rules without some caveats, which come later. The capital letters in the list below denote known constants in the source term $g(t)$, and the lowercase letters denote coefficients to be determined in the trial form of the particular solution when it is substituted into the differential equation.

1. If $g(t) = Ae^{\gamma t}$ is an exponential, then we try an exponential $u_p = ae^{\gamma t}$.
2. If $g(t) = A\sin\omega t$ or $g(t) = A\cos\omega t$, then we try a combination $u_p = a\sin\omega t$ $+b\cos\omega t$.
3. If $g(t) = A_nt^n + A_{n-1}t^{n-1} + \cdots + A_0$ is a polynomial of degree n, then we try $u_p = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_0$, a polynomial of degree n.
4. If $g(t) = (A_nt^n + A_{n-1}t^{n-1} + \cdots + A_1t + A_0)e^{\gamma t}$, then we try $u_p = (a_nt^n + a_{n-1}t^{n-1} + \cdots a_1t + a_0)e^{\gamma t}$.
5. If $g(t) = Ae^{\gamma t}\sin\omega t$ or $g(t) = Ae^{\gamma t}\cos\omega t$, then we try $u_p = ae^{\gamma t}\sin\omega t + be^{\gamma t}\cos\omega t$.

If the source term $g(t)$ is a sum of two different types, we take the net guess to be a sum of the two individual guesses. For example, if $g(t) = 3t-1+7e^{-2t}$, a polynomial plus an exponential, then a good guess would be $u_p = at+b+ce^{-2t}$. The following examples show how the method works.

Example 3.8

Find a particular solution to the differential equation

$$u'' - u' + 7u = 5t - 3.$$

The right side, $g(t) = 5t - 3$, is a polynomial of degree 1 so we try $u_p = at + b$. Substituting, $-a + 7(at + b) = 5t - 3$. Equating like terms (constant term and terms involving t) gives $-a + 7b = -3$ and $7a = 5$. Therefore $a = 5/7$ and $b = -16/49$. A particular solution to the equation is therefore

$$u_p(t) = \frac{5}{7} - \frac{16}{49}t.$$

Example 3.9

Consider the equation

$$u'' + 3u' + 3u = 6e^{-2t}.$$

The homogeneous equation has characteristic polynomial $\lambda^2 + 3\lambda + 3 = 0$, which has roots $\lambda = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$. Thus the solution to the homogeneous equation is

$$u_h(t) = c_1 e^{-3t/2} \cos \frac{\sqrt{3}}{2}t + c_2 e^{-3t/2} \sin \frac{\sqrt{3}}{2}t.$$

To find a particular solution to the nonhomogeneous equation note that $g(t) = 6e^{-2t}$. Therefore we guess $u_p = ae^{-2t}$. Substituting this trial function into the nonhomogeneous equation gives, after canceling e^{-2t}, the equation $4a - 6a + 3a = 6$. Thus $a = 1$ and a particular solution to the nonhomogeneous equation is $u_p = e^{-2t}$. The general solution to the original nonhomogeneous equation is

$$u(t) = c_1 e^{-3t/2} \cos \frac{\sqrt{3}}{2}t + c_2 e^{-3t/2} \sin \frac{\sqrt{3}}{2}t + e^{-2t}.$$

Example 3.10

Find a particular solution to the DE

$$u'' + 2u = \sin 3t.$$

Our basic rule above dictates we try a solution of the form $u_p = a \sin 3t + b \cos 3t$. Then, upon substituting,

$$-9a \sin 3t - 9b \cos 3t + 2a \sin 3t + 2b \cos 3t = \sin 3t.$$

Equating like terms gives $-9a + 2a = 1$ and $b = 0$ (there are no cosine terms on the right side). Hence $a = -1/7$ and a particular solution is $u_p = -\frac{1}{7} \sin 3t$.

For this equation, because there is no first derivative, we did not need a cosine term in the guess. If there were a first derivative, a cosine would have been required.

Example 3.11

Next we modify Example 3.10 and consider

$$u'' + 9u = \sin 3t.$$

The rules dictate the trial function $u_p = a \sin 3t + b \cos 3t$. Substituting into (3.15) yields

$$-9a \sin 3t - 9b \cos 3t + 9a \sin 3t + 9b \cos 3t = \sin 3t.$$

But the terms on the left cancel completely and we get $0 = \sin 3t$, an absurdity. The method failed! This is because the homogeneous equation $u'' + 9u = 0$ has eigenvalues $\lambda = \pm 3i$, which lead to independent solutions $u_1 = \sin 3t$ and $u_2 = \cos 3t$. The forcing term $g(t) = \sin 3t$ is not independent from those two basic solutions; it duplicates one of them, and in this case the method as presented above fails. The fact that we get 0 when we substitute our trial function into the equation is no surprise—it is a solution to the homogeneous equation. To remedy this problem, we can modify our original guess by multiplying it by t. That is, we attempt a particular solution of the form

$$u_p = t(a \sin 3t + b \cos 3t).$$

Calculating the second derivative u_p'' and substituting, along with u_p, into the original equation leads to (show this!)

$$6a \cos 3t - 6b \sin 3t = \sin 3t.$$

Hence $a = 0$ and $b = -1/6$. We have found a particular solution

$$u_p = -\frac{1}{6} t \cos 3t.$$

Therefore the general solution of the original nonhomogeneous equation is the homogeneous solution plus the particular solution,

$$u(t) = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{6} t \cos 3t.$$

Notice that the solution to the homogeneous equation is oscillatory and remains bounded; the particular solution oscillates without bound because of the increasing time factor t multiplying that term.

The technique for finding the form of the particular solution that we used in the preceding example works in general; this is the main caveat in the set of rules listed above.

Caveat. *If a term in the initial trial guess for a particular solution u_p duplicates one of the basic solutions for the homogeneous equation, then modify the guess by multiplying by the smallest power of t that eliminates the duplication.*

Example 3.12

Consider the DE

$$u'' - 4u' + u = 5te^{2t}.$$

The initial guess for a particular solution is $u_p = (at + b)e^{2t}$. But, as you can check, e^{2t} and te^{2t} are basic solutions to the homogeneous equation $u'' - 4u' + u = 0$. Multiplying the first guess by t gives $u_p = (at^2 + bt)e^{2t}$, which still does not eliminate the duplication because of the te^{2t} term. So, multiply by another t to get $u_p = (at^3 + bt^2)e^{2t}$. Now no term in the guess duplicates one of the basic homogeneous solutions and so this is the correct form of the particular solution. If desired, we can substitute this form into the differential equation to determine the exact values of the coefficients a and b. But, without actually finding the coefficients, the form of the general solution is

$$u(t) = c_1e^{2t} + c_2te^{2t} + (at^3 + bt^2)e^{2t}.$$

The constants c_1 and c_2 could be determined at this point by initial conditions, if given. Sometimes knowing the form of the solution is enough.

Example 3.13

Consider an RCL circuit where $R = 2$, $L = C = 1$, and the current is driven by an electromotive force of $2\sin 3t$. The circuit equation for the voltage $V(t)$ across the capacitor is

$$V'' + 2V' + V = 2\sin 3t.$$

For initial data we take

$$V(0) = 4, \quad V'(0) = 0.$$

We recognize this as a nonhomogeneous linear equation with constant coefficients. So the general solution will be the sum of the general solution to the homogeneous equation

$$V'' + 2V' + V = 0$$

plus any particular solution to the nonhomogeneous equation. The homogeneous equation has characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ with a double root $\lambda = -1$. Thus the homogeneous solution is

$$V_h = e^{-t}(c_1 + c_2 t).$$

Notice that this solution, regardless of the values of the constants, will decay away in time; this part of the solution is called the **transient response** of the circuit. To find a particular solution we use undetermined coefficients and assume it has the form

$$V_p = a \sin 3t + b \cos 3t.$$

Substituting this into the nonhomogeneous equation gives a pair of linear equations for a and b,

$$-4a - 3b = 1, \quad 7a - 9b = 0.$$

We find $a = -0.158$ and $b = -0.123$. Therefore the general solution is

$$V(t) = e^{-t}(c_1 + c_2 t) - 0.158 \sin 3t - 0.123 \cos 3t.$$

Now we apply the initial conditions. Easily $V(0) = 4$ implies $c_1 = 4.123$. Next we find $V'(t)$ so that we can apply the condition $V'(0) = 0$. Leaving this as an exercise, we find $c_2 = 4.597$. Therefore the voltage on the capacitor is

$$V(t) = e^{-t}(4.123 + 4.597t) - 0.158 \sin 3t - 0.123 \cos 3t.$$

As we observed, the first term always decays as time increases. Therefore we are left with only the particular solution $-0.158 \sin 3t - 0.123 \cos 3t$, which takes over in time. It is called the **steady-state response** of the circuit (figure 3.2).

The method of undetermined coefficients works for nonhomogeneous *first-order* linear equations as well, provided the equation has constant coefficients.

Example 3.14

Consider the equation

$$u' + qu = g(t).$$

The homogeneous solution is $u_h(t) = Ce^{-qt}$. Provided $g(t)$ has the right form, a particular solution $u_p(t)$ can be found by the method of undetermined coefficients exactly as for second-order equations: make a trial guess and substitute into the equation to determine the coefficients in the guess. The general solution to the nonhomogeneous equation is then $u(t) = u_h(t) + u_p(t)$. For example, consider the equation

$$u' - 3u = t - 2.$$

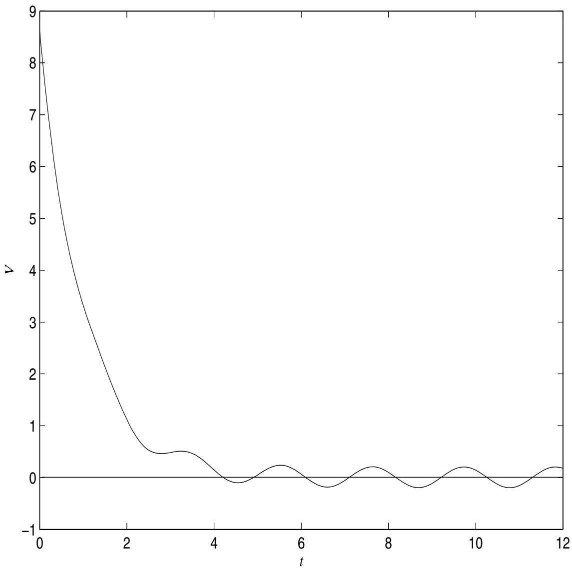

Figure 3.2 A plot of the voltage $V(t)$ in Example 3.13. Initially there is a transient caused by the initial conditions. It decays away and is replaced by a steady-state response, an oscillation, that is caused by the forcing term.

The homogeneous solution is $u_h = Ce^{3t}$. To find a particular solution make the trial guess

$$u_p = at + b.$$

Substituting this into the equation gives $a = -\frac{1}{3}$ and $b = \frac{5}{3}$. Consequently, the general solution is

$$u(t) = Ce^{3t} - \frac{1}{3}t + \frac{5}{3}.$$

EXERCISES

1. Each of the following functions represents $g(t)$, the right side of a nonhomogeneous equation. State the form of an initial trial guess for a particular solution $u_p(t)$.

   a) $3t^3 - 1$.

   b) 12.

   c) $t^2 e^{3t}$.

   d) $5 \sin 7t$.

   e) $e^{2t} \cos t + t^2$.

f) $te^{-t}\sin \pi t$.

2. Find the general solution of the following nonhomogeneous equations:

   a) $u'' + 7u = te^{3t}$.

   b) $u'' - u' = 6 + e^{2t}$.

   c) $u' + u = t^2$.

   d) $u'' - 3u' - 4u = 2t^2$.

   e) $u'' + u = 9e^{-t}$.

   f) $u' + u = 4e^{-t}$.

   g) $u'' - 4u = \cos 2t$.

   h) $u'' + u' + 2u = t\sin 2t$

3. Solve the initial value problem $u'' - 3u' - 40u = 2e^{-t}, \quad u(0) = 0, \quad u'(0) = 1$.

4. Find the solution of $u'' - 2u' = 4$, $u(0) = 1$, $u'(0) = 0$.

5. Find the particular solution to the equation $u'' + u' + 2u = \sin^2 t$? (Hint: use a double angle formula to rewrite the right side.)

6. An RL circuit contains a 2 ohm resistor and a 5 henrys inductor connected in series with a 10 volt battery. If the open circuit is suddenly closed at time zero, find the current for all times $t > 0$. Plot the current vs. time and identify the steady-state response.

7. A circuit contains a 10^{-3} farad capacitor in series with a 20 volt battery and an inductor of 0.4 henrys. At $t = 0$ both $q = 0$ and $I = 0$. Find the charge $q(t)$ on the capacitor and describe the response of the circuit in terms of transients and steady-states.

8. An RCL circuit contains a battery generating 110 volts. The resistance is 16 ohms, the inductance is 2 henrys, and the capacitance is 0.02 farads. If $q(0) = 5$ and $I(0) = 0$, find the charge $q(t)$ current response of the circuit. Identify the transient solution and the steady-state response.

3.3.2 Resonance

The phenomenon of resonance is a key characteristic of oscillating systems. Resonance occurs when the frequency of a forcing term has the same frequency as the natural oscillations in the system; resonance gives rise to large amplitude oscillations. To give an example, consider a pendulum that is oscillating at its natural frequency. What happens when we deliberately force the pendulum (say, by giving it a tap with our finger in the positive angular direction) at a frequency near this natural frequency? So, every time the bob passes through $\theta = 0$ with a positive direction, we give it a positive tap. We will clearly increase its amplitude. This is the phenomenon of resonance. It can occur in circuits where we force (by a generator) the system at its natural frequency, and it can occur in mechanical systems and structures where an external periodic force is applied at the same frequency as the system would naturally oscillate. The results could be disastrous, such as a blown circuit or a fallen building; a building or bridge could have a natural frequency of oscillation, and the wind could provide the forcing function. Another imagined example is a company of soldiers marching in cadence across a suspension bridge at the same frequency as the natural frequency of the structure.

We consider a model problem illustrating this phenomenon, an LC circuit that is forced with a sinusoidal voltage source of frequency β. If $L = 1$ the governing equation for the charge on the capacitor will have the form

$$u'' + \omega^2 u = \sin \beta t, \tag{3.12}$$

where $\omega^2 = 1/C$. Assume first that $\beta \neq \omega$ and take initial conditions

$$u(0) = 0, \quad u'(0) = 1.$$

The homogeneous equation has general solution

$$u_h = c_1 \cos \omega t + c_2 \sin \omega t,$$

which gives natural oscillations of frequency ω. A particular solution has the form $u_p = a \sin \beta t$. Substituting into the DE gives $a = 1/(\omega^2 - \beta^2)$. So the general solution of (3.12) is

$$u = c_1 \cos \omega t + c_2 \sin \omega t + \frac{1}{\omega^2 - \beta^2} \sin \beta t. \tag{3.13}$$

At $t = 0$ we have $u = 0$ and so $c_1 = 0$. Also $u'(0) = 1$ gives $c_2 = -\frac{\beta + \omega(\omega^2 - \beta^2)}{\omega^2 - \beta^2}$. Therefore the solution to the initial value problem is

$$u = -\frac{\beta + \omega(\omega^2 - \beta^2)}{\omega^2 - \beta^2} \sin \omega t + \frac{1}{\omega^2 - \beta^2} \sin \beta t. \tag{3.14}$$

This solution shows that the charge response is a sum of two oscillations of different frequencies. If the forcing frequency β is close to the natural frequency ω, then the amplitude is bounded, but it is obviously large because of the factor $\omega^2 - \beta^2$ in the denominator. Thus the system has large oscillations when β is close to ω.

What happens if $\beta = \omega$? Then the general solution in (3.13) is not valid because there is division by zero, and we have to re-solve the problem. The circuit equation is

$$u'' + \omega^2 u = \sin \omega t, \tag{3.15}$$

where the circuit is forced at the same frequency as its natural frequency. The homogeneous solution is the same as before, but the particular solution will now have the form

$$u_p = t(a \sin \omega t + b \cos \omega t),$$

with a factor of t multiplying the terms. Therefore the general solution of (3.15) has the form

$$u(t) = c_1 \cos \omega t + c_2 \sin \omega t + t(a \sin \omega t + b \cos \omega t).$$

Without actually determining the constants, we can see the nature of the response. Because of the t factor in the particular solution, the amplitude of the oscillatory response $u(t)$ will grow in time. This is the phenomenon of **pure resonance**. It occurs when the frequency of the external force is the same as the natural frequency of the system.

What happens if we include damping in the circuit (i.e., a resistor) and still force it at the natural frequency? Consider

$$u'' + 2\sigma u' + 2u = \sin \sqrt{2}t,$$

where 2σ is a small ($0 < \sigma$) damping coefficient, for example, resistance. The homogeneous equation $u'' + 2\sigma u' + 2u = 0$ has solution $u = e^{-\sigma t}(c_1 \cos \sqrt{2 - \sigma^2}t + c_2 \sin \sqrt{2 - \sigma^2}t)$. Now the particular solution has the form $u_p = a \cos \sqrt{2}t + b \sin \sqrt{2}t$, where a and b are constants (found by substituting into the DE). So, the response of the circuit is

$$u = e^{-\sigma t}(c_1 \cos \sqrt{2 - \sigma^2}t + c_2 \sin \sqrt{2 - \sigma^2}t) + a \cos \sqrt{2}t + b \sin \sqrt{2}t.$$

The transient is a decaying oscillation of frequency $\sqrt{2 - \sigma^2}$, and the steady-state response is periodic of frequency $\sqrt{2}$. The solution will remain bounded, but its amplitude will be large if σ is very small.

EXERCISES

1. Graph the solution (3.14) for several different values of β and ω. Include values where these two frequencies are close.

2. Find the general solution of the equation $u'' + 16u = \cos 4t$.

3. Consider a general LC circuit with input voltage $V_0 \sin \beta t$. If β and the capacitance C are known, what value of the inductance L would cause resonance?

4. Consider the equation

$$u'' + \omega^2 u = \cos \beta t.$$

   a) Find the solution when the initial conditions are $u(0) = u'(0) = 0$ when $\omega \neq \beta$.

   b) Use the trigonometric identity $2\sin A \sin B = \cos(A - B) - \cos(A + B)$ to write the solution as a product of sines.

   c) Take $\omega = 55$ and $\beta = 45$ and plot the solution in part (b).

   d) Show that the solution in (c) can be interpreted as a high-frequency response contained in a low-frequency amplitude envelope. (We say the high frequency is *modulated* by the low frequency.) This is the phenomenon of **beats**.

3.4 Variable Coefficients

Next we consider second-order, linear equations with given variable coefficients $p(t)$ and $q(t)$:

$$u'' + p(t)u' + q(t)u = g(t). \tag{3.16}$$

Except for a few cases, these equations cannot be solved in analytic form using familiar functions. Even the simplest equation of this form,

$$u'' - tu = 0$$

(where $p(t) = g(t) = 0$ and $q(t) = -t$), which is called **Airy's equation**, requires the definition of a new class of functions (Airy functions) to characterize the solutions. Nevertheless, there is a well-developed theory for these equations, and we list some of the main results. We require that the coefficients $p(t)$ and $q(t)$, as well as the forcing term $g(t)$, be continuous functions on the interval I of interest. We list some basic properties of these equations; the reader will observe that these are the same properties shared by second-order, constant coefficient equations studied in Section 3.2.

1. (**Existence-Uniqueness**) If I is an open interval and t_0 belongs to I, then the initial value problem

$$u'' + p(t)u' + q(t)u = g(t), \tag{3.17}$$

$$u(t_0) = a, \quad u'(t_0) = b, \tag{3.18}$$

   has a unique solution on I.

2. (**Superposition of Solutions**) If u_1 and u_2 are independent solutions of the associated homogeneous equation

$$u'' + p(t)u' + q(t)u = 0 \tag{3.19}$$

on an interval I, then $u(t) = c_1u_1 + c_2u_2$ is a solution on the interval I for any constants c_1 and c_2. Moreover, all solutions of the homogeneous equation are contained in the general solution.

3. (**Nonhomogeneous Equation**) All solutions to the nonhomogeneous equation (3.17) can be represented as the sum of the general solution to the homogeneous equation (3.19) and any particular solution to the non-homogeneous equation (3.17). In symbols,

$$u(t) = c_1u_1(t) + c_2u_2(t) + u_p(t),$$

which is called the general solution to (3.17)

The difficulty, of course, is to find two independent solutions u_1 and u_2 to the homogeneous equation, and to find a particular solution. As we remarked, this task is difficult for equations with variable coefficients. The method of writing down the characteristic polynomial, as we did for constant coefficient equations, *does not work*.

3.4.1 Cauchy–Euler Equation

One equation that can be solved analytically is an equation of the form

$$u'' + \frac{b}{t}u' + \frac{c}{t^2}u = 0,$$

or

$$t^2u'' + btu' + cu = 0,$$

which is called a **Cauchy–Euler equation**. In each term the exponent on t coincides with the order of the derivative. Observe that we must avoid $t = 0$ in our interval of solution, because $p(t) = b/t$ and $q(t) = c/t^2$ are not continuous at $t = 0$. We try to find a solution of the form of a power function $u = t^m$. (Think about why this might work). Substituting gives the **characteristic equation**

$$m(m-1) + bm + c = 0,$$

which is a quadratic equation for m. There are three cases. If there are two distinct real roots m_1 and m_2, then we obtain two independent solutions t^{m_1} and t^{m_1}. Therefore the general solution is

$$u = c_1t^{m_1} + c_2t^{m_2}.$$

If the characteristic equation has two equal roots $m_1 = m_2 = m$, then t^m and $t^m \ln t$ are two independent solutions; in this case the general solution is

$$u = c_1 t^m + c_2 t^m \ln t.$$

When the characteristic equation has complex conjugate roots $m = \alpha \pm i\beta$, we note, using the properties of logarithms, exponentials, and Euler's formula, that a complex solution is

$$t^m = t^{\alpha+i\beta} = t^\alpha t^{i\beta} = t^\alpha e^{\ln t^{i\beta}} = t^\alpha e^{i\beta \ln t} = t^\alpha[\cos(\beta \ln t) + i \sin(\beta \ln t)].$$

The real and imaginary parts of this complex function are therefore real solutions (Theorem 3.3). So the general solution in the complex case is

$$u = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t).$$

Figure 3.3 shows a graph of the function $\sin(5 \ln t)$, which is a function of the type that appears in this solution. Note that this function oscillates less and less as t gets large because $\ln t$ grows very slowly. As t nears zero it oscillates infinitely many times. Because of the scale, these oscillations are not apparent on the plot.

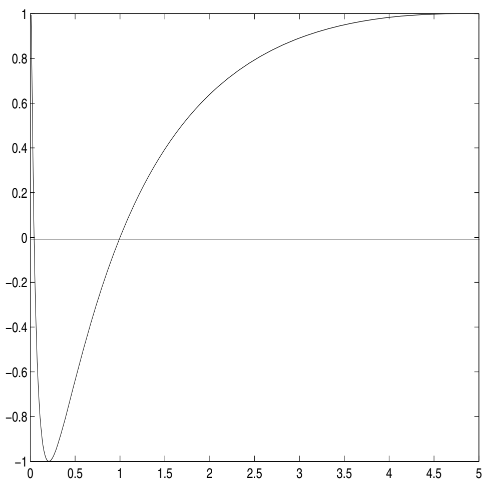

Figure 3.3 Plot of $\sin(5 \ln t)$.

Example 3.15

Consider the equation

$$t^2 u'' + t u' + 9u = 0.$$

The characteristic equation is $m(m-1)+m+9=0$, which has roots $m=\pm 3i$. The general solution is therefore

$$u = c_1 \cos(3\ln t) + c_2 \sin(3\ln t).$$

Example 3.16

Consider the equation

$$u'' = \frac{2}{t}u'.$$

We can write this in Cauchy–Euler form as

$$t^2u'' - 2tu' = 0,$$

which has characteristic equation $m(m-1) - 2m = 0$. The roots are $m = 0$ and $m = 3$. Therefore the general solution is

$$u(t) = c_1 + c_2t^3.$$

Example 3.17

Solve the initial value problem

$$t^2u'' + 3tu' + u = 0, \quad u(1) = 0, \quad u'(1) = 2.$$

The DE is Cauchy–Euler type with characteristic equation $m(m-1)+3m+1 = 0$. This has a double root $m = -1$, and so the general solution is

$$u(t) = \frac{c_1}{t} + \frac{c_2}{t}\ln t.$$

Now, $u(1) = c_1 = 0$ and so $u(t) = \frac{c_2}{t}\ln t$. Taking the derivative, $u'(t) = \frac{c_2}{t^2}(1 - \ln t)$. Then $u'(1) = c_2 = 2$. Hence, the solution to the initial value problem is

$$u(t) = \frac{2}{t}\ln t.$$

A. Cauchy (1789–1857) and L. Euler (1707–1783) were great mathematicians who left an indelible mark on the history of mathematics and science. Their names are encountered often in advanced course in mathematics and engineering.

3.4.2 Power Series Solutions

In general, how are we to solve variable coefficient equations? Some equations can be transformed into the Cauchy–Euler equation, but that is only a small class. If we enter the equation in a computer algebra system such as Maple or Mathematica, the system will often return a general solution that is expressed in terms of so-called special functions (such as Bessel functions, Airy functions, and so on). We could define these special functions by the differential equations that we cannot solve. This is much like defining the natural logarithm function $\ln t$ as the solution to the initial value problem $u' = \frac{1}{t}$, $u(1) = 0$, as in Chapter 1. For example, we could define functions $Ai(t)$ and $Bi(t)$, the Airy functions, as two independent solutions of the DE $u'' - tu = 0$. Many of the properties of these special functions could be derived directly from the differential equation itself. But how could we get a "formula" for those functions? One way to get a representation of solutions to equations with variable coefficients is to use power series.

Let p and q be continuous on an open interval I containing t_0 and also have continuous derivatives of all orders on I. Solutions to the second-order equation with variable coefficients,

$$u'' + p(t)u' + q(t)u = 0, \tag{3.20}$$

can be approximated near $t = t_0$ by assuming a power series solution of the form

$$u(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n = a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + a_3(t - t_0)^3 + \cdots.$$

The idea is to simply substitute the series and its derivatives into the differential equation and collect like terms, thereby determining the coefficients a_n. We recall from calculus that a power series converges only at $t = t_0$, for all t, or in an interval $(t_0 - R, t_0 + R)$, where R is the radius of convergence. Within its radius of convergence the power series represents a function, and the power series may be differentiated term by term to obtain derivatives of the function.

Example 3.18

Consider the DE

$$u'' - (1 + t)u = 0$$

on an interval containing $t_0 = 0$. We have

$$\begin{aligned} u(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots, \\ u'(t) &= a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots, \\ u''(t) &= 2a_2 + 6a_3 t + 12a_4 t^2 + \cdots. \end{aligned}$$

Substituting into the differential equation gives

$$2a_2 + 6a_3t + 12a_4t^2 + \cdots - (1+t)(a_0 + a_1t + a_2t^2 + a_3t^3 + \cdots) = 0.$$

Collecting like terms,

$$(-a_0 + 2a_2) + (-a_0 - a_1 + 6a_3)t + (-a_2 - a_1 + 12a_4)t^2 + \cdots = 0.$$

Therefore

$$\begin{aligned} -a_0 + 2a_2 &= 0, \\ -a_0 - a_1 + 6a_3 &= 0, \\ -a_2 - a_1 + 12a_4 &= 0, \ldots. \end{aligned}$$

Notice that all the coefficients can be determined in terms of a_0 and a_1. We have

$$a_2 = \frac{1}{2}a_0, \quad a_3 = \frac{1}{6}(a_0 + a_1), \quad a_4 = \frac{1}{12}(a_1 + a_2) = \frac{1}{12}(a_1 + \frac{1}{2}a_0), \ldots.$$

Therefore the power series for the solution $u(t)$ can be written

$$\begin{aligned} u(t) &= a_0 + a_1t + \frac{1}{2}a_0t^2 + \frac{1}{6}(a_0 + a_1)t^3 + \frac{1}{12}(a_1 + \frac{1}{2}a_0)t^4 + \cdots \\ &= a_0(1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots) + a_1(t + \frac{1}{6}t^3 + \frac{1}{12}t^4 + \cdots), \end{aligned}$$

which gives the general solution as a linear combination of two independent power series solutions

$$\begin{aligned} u_1(t) &= 1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots, \\ u_2(t) &= t + \frac{1}{6}t^3 + \frac{1}{12}t^4 + \cdots. \end{aligned}$$

The two coefficients a_0 and a_1 can be determined from initial conditions. For example, if

$$u(0) = 1, \quad u'(0) = 3,$$

then $a_0 = 1$ and $a_1 = 3$, which gives the power series solution

$$\begin{aligned} u(t) &= (1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots) + 3(t + \frac{1}{6}t^3 + \frac{1}{12}t^4 + \cdots) \\ &= 1 + 3t + \frac{1}{2}t^2 + \frac{2}{3}t^3 + \frac{7}{24}t^4 + \cdots. \end{aligned}$$

In this example, the power series converges for all t. We have only calculated five terms, and our truncated power series is an approximation to the actual solution to the initial value problem in a neighborhood of $t = 0$. Figure 3.4 shows the polynomial approximations by taking the first term, the first two terms, the first three, and so on.

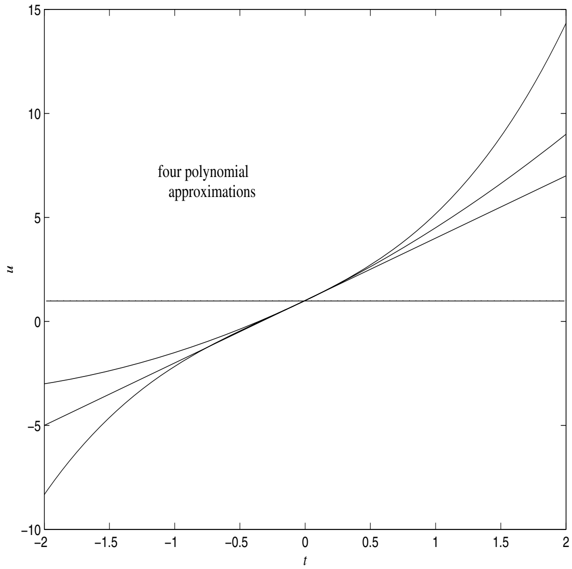

Figure 3.4 Successive polynomial approximations.

There are many important equations of the form (3.20) where the coefficients p and q do not satisfy the regularity properties (having continuous derivatives of all order) mentioned at the beginning of this subsection. However, if p and q are not too ill-behaved at t_0, we can still seek a series solution. In particular, if $(t-t_0)p(t)$ and $(t-t_0)^2q(t)$ have convergent power series expansions in an interval about t_0, then we say t_0 is a **regular singular point** for (3.20), and we attempt a series solution of the form

$$u(t) = t^r \sum_{n=0}^{\infty} a_n (t-t_0)^n,$$

where r is some number. Substitution of this form into (3.20) leads to equations for both r and the coefficients a_n. This technique, which is called the **Frobenius method**, is explored in the Exercises.

3.4.3 Reduction of Order

If one solution $u_1(t)$ of the DE

$$u'' + p(t)u' + q(t)u = 0$$

happens to be known, then a second, linearly independent solution $u_2(t)$ can be found of the form $u_2(t) = v(t)u_1(t)$, for some $v(t)$ to be determined. To find

$v(t)$ we substitute this form for $u_2(t)$ into the differential equation to obtain a first-order equation for $v(t)$. This method is called **reduction of order**, and we illustrate it with an example.

Example 3.19

Consider the DE

$$u'' - \frac{1}{t}u' + \frac{1}{t^2}u = 0.$$

An obvious solution is $u_1(t) = t$. So let $u_2 = v(t)t$. Substituting, we get

$$(2v' + tv'') - \frac{1}{t}(v + tv') + \frac{1}{t^2}vt = 0,$$

which simplifies to

$$tv'' + v' = 0.$$

Letting $w = v'$, we get the first-order equation

$$tw' + w = 0.$$

By separating variables and integrating we get $w = 1/t$. Hence $v = \int \frac{1}{t}dt = \ln t$, and the second independent solution is $u_2(t) = t \ln t$. Consequently, the general solution of the equation is

$$u(t) = c_1 t + c_2 t \ln t.$$

Note that this example is a Cauchy–Euler equation; but the method works on general linear second-order equations.

3.4.4 Variation of Parameters

There is a general formula for the particular solution to the nonhomogeneous equation

$$u'' + p(t)u' + q(t)u = g(t), \tag{3.21}$$

called the variation of parameters formula.

Recall how we attacked the first-order linear equation

$$u' + p(t)u = g(t)$$

in Chapter 1. We first found the solution of the associated homogeneous equation $u' + p(t)u = 0$, as $Ce^{-P(t)}$, where $P(t)$ is an antiderivative of $p(t)$. Then we found the solution to the nonhomogeneous equation by varying the constant

C (i.e., by assuming $u(t) = C(t)e^{-P(t)}$). Substitution of this into the equation yielded $C(t)$ and therefore the solution.

The same method works for second-order equations, but the calculations are more involved. Let u_1 and u_2 be independent solutions to the homogeneous equation

$$u'' + p(t)u' + q(t)u = 0.$$

Then

$$u_h(t) = c_1u_1(t) + c_2u_2(t)$$

is the general solution of the homogeneous equation. To find a particular solution we vary both parameters c_1 and c_2 and take

$$u_p(t) = c_1(t)u_1(t) + c_2(t)u_2(t). \tag{3.22}$$

Now we substitute this expression into the nonhomogeneous equation to get expressions for $c_1(t)$ and $c_2(t)$. This is a tedious task in calculus and algebra, and we leave most of the details to the interested reader. But here is how the argument goes. We calculate u_p' and u_p'' so that we can substitute into the equation. For notational simplicity, we drop the t variable in all of the functions. We have

$$u_p' = c_1u_1' + c_2u_2' + c_1'u_1 + c_2'u_2.$$

There is flexibility in our answer so let us set

$$c_1'u_1 + c_2'u_2 = 0. \tag{3.23}$$

Then

$$\begin{aligned} u_p' &= c_1u_1' + c_2u_2', \\ u_p'' &= c_1u_1'' + c_2u_2'' + c_1'u_1' + c_2'u_2'. \end{aligned}$$

Substituting these into the nonhomogeneous DE gives

$$c_1u_1'' + c_2u_2'' + c_1'u_1' + c_2'u_2' + p(t)[c_1u_1' + c_2u_2'] + q(t)[c_1u_1 + c_2u_2] = g(t).$$

Now we observe that u_1 and u_2 satisfy the homogeneous equation, and this simplifies the last equation to

$$c_1'u_1' + c_2'u_2' = g(t). \tag{3.24}$$

Equations (3.23) and (3.24) form a system of two linear algebraic equations in the two unknowns c_1' and c_2'. If we solve these equations and integrate we finally obtain (readers should fill in the details)

$$c_1(t) = -\int \frac{u_2(t)g(t)}{W(t)}dt, \quad c_2(t) = \int \frac{u_1(t)g(t)}{W(t)}dt, \tag{3.25}$$

where

$$W(t) = u_1(t)u_2'(t) - u_1'(t)u_2(t). \tag{3.26}$$

This expression $W(t)$ is called the **Wronskian**. Combining the previous expressions gives the **variation of parameters formula** for the particular solution of (3.21):

$$u_p(t) = -u_1(t)\int \frac{u_2(t)g(t)}{W(t)}dt + u_2(t)\int \frac{u_1(t)g(t)}{W(t)}dt.$$

The general solution of (3.21) is the homogeneous solution $u_h(t)$ plus this particular solution. If the antiderivatives in (3.25) cannot be computed explicitly, then the integrals should be written with a variable limit of integration.

Example 3.20

Find a particular solution to the DE

$$u'' + 9u = 3\sec 3t.$$

Here the homogeneous equation $u'' + 9u = 0$ has two independent solutions $u_1 = \cos 3t$ and $u_2 = \sin 3t$. The Wronskian is

$$W(t) = 3\cos^2 t + 3\sin^2 3t = 3.$$

Therefore

$$c_1(t) = -\int \frac{\sin 3t \cdot 3\sec 3t}{3}dt, \quad c_2(t) = \int \frac{\cos 3t \cdot 3\sec 3t}{3}dt.$$

Simplifying,

$$c_1(t) = -\int \tan 3t dt = \frac{1}{3}\ln(\cos 3t), \quad c_2(t) = \int 1 dt = t.$$

We do not need constants of integration because we seek only the particular solution. Therefore the particular solution is

$$u_p(t) = \frac{1}{3}\ln(\cos 3t) + t\sin 3t.$$

The general solution is

$$u(t) = c_1\cos 3t + c_2\sin 3t + \frac{1}{3}\ln(\cos 3t) + t\sin 3t.$$

The constants may be determined by initial data, if given.

When the second-order equation has constant coefficients and the forcing term is a polynomial, exponential, sine, or cosine, then the method of undetermined coefficients works more easily than the variation of parameters formula. For other cases we use the formula or Laplace transform methods, which are the subject of Chapter 4. Of course, the easiest method of all is to use a computer algebra system. When you have paid your dues by using analytic methods on several problems, then you have your license and you may use a computer algebra system. The variation of parameters formula is important because it is often used in the theoretical analysis of problems in advanced differential equations.

EXERCISES

1. Solve the following initial value problems:

   a) $t^2u'' + 3tu' - 8u = 0$, $u(1) = 0$, $u'(1) = 2$.

   b) $t^2u'' + tu' = 0$, $u(1) = 0$, $u'(1) = 2$.

   c) $t^2u'' - tu' + 2u = 0$, $u(1) = 0$, $u'(1) = 1$.

2. For what value(s) of β is $u = t^\beta$ a solution to the equation $(1 - t^2)u'' - 2tu' + 2u = 0$?

3. This exercise presents a transformation method for solving a Cauchy–Euler equation. Show that the transformation $x = \ln t$ to a new independent variable x transforms the Cauchy–Euler equation $at^2u'' + btu' + cu = 0$ into an linear equation with constant coefficients. Use this method to solve Exercise 1a.

4. Use the power series method to obtain two independent, power series solutions to $u'' + u = 0$ about $t_0 = 0$ and verify that the series are the expansions of $\cos t$ and $\sin t$ about $t = 0$.

5. Use the power series method to find the first three terms of two independent power series solutions to Airy's equation $u'' - tu = 0$, centered at $t_0 = 0$.

6. Find the first three terms of two independent power series solutions to the equation $(1 + t^2)u'' + u = 0$ near $t_0 = 0$.

7. Solve the first-order nonlinear initial value problem $u' = 1 + u^2$, $u(0) = 1$, using a power series method. Compare the accuracy of the partial sums to the exact solution. (Hint: you will have to square out a power series.)

8. Consider the equation $u'' - 2tu' + 2nu = 0$, which is Hermite's differential equation, an important equation in quantum theory. Show that if n is a nonnegative integer, then there is a polynomial solution $H_n(t)$ of degree n,

which is called a Hermite polynomial of degree n. Find $H_0(t), ..., H_5(t)$ up to a constant multiple.

9. Consider the equation $u'' - 2au' + a^2u = 0$, which has solution $u = e^{at}$. Use reduction of order to find a second independent solution.

10. One solution of
$$u'' - \frac{t+2}{t}u' + \frac{t+2}{t^2}u = 0$$
is $u_1(t) = t$. Find a second independent solution.

11. One solution of
$$t^2u'' + tu' + (t^2 - \frac{1}{4})u = 0$$
is $u_1(t) = \frac{1}{\sqrt{t}} \cos t$. Find a second independent solution.

12. Let $y(t)$ be one solution of the equation $u'' + p(t)u' + q(t)u = 0$. Show that the reduction of order method with $u(t) = v(t)y(t)$ leads to the first-order linear equation
$$yz' + (2y' + py)z = 0, \quad z = v'.$$
Show that
$$z(t) = \frac{Ce^{-\int p(t)dt}}{y(t)^2},$$
and then find a second linear independent solution of the equation in the form of an integral.

13. Use ideas from the last exercise to find a second-order linear equation that has independent solutions e^t and $\cos t$.

14. Let u_1 and u_2 be independent solutions of the linear equation $u'' + p(t)u' + q(t)u = 0$ on an interval I and let $W(t)$ be the Wronskian of u_1 and u_2. Show that
$$W'(t) = -p(t)W(t),$$
and then prove that $W(t) = 0$ for all $t \in I$, or $W(t)$ is never zero on I.

15. Find the general solution of $u'' + tu' + u = 0$ given that $u = e^{-t^2/2}$ is one solution.

16. Use the transformation $u = \exp\left(\int y(t)dt\right)$ to convert the second-order equation $u'' + p(t)u' + q(t)u = 0$ to a **Riccati equation** $y' + y^2 + p(t)y + q(t) = 0$. Conversely, show that the Riccati equation can be reduced to the second-order equation in u using the transformation $y = u'/u$. Solve the first-order nonautonomous equation
$$y' = -y^2 + \frac{3}{t}y.$$

17. Use the variation of parameters formula to find a particular solution to the following equations.

    a) $u'' + \frac{1}{t}u = a$, where a is a constant. Note that 1 and $\ln t$ are two independent solutions to the homogeneous equation.

    b) $u'' + u = \tan t$.

    c) $u'' - u = te^t$.

    d) $u'' - u = \frac{1}{t}$.

    e) $t^2u'' - 2u = t^3$.

18. (Frobenius method) Consider the differential equation (**Bessel's equation** of order k)

$$u'' + \frac{1}{t}u' + \left(1 - \frac{k^2}{t^2}\right)u = 0,$$

where k is a real number.

    a) Show that $t_0 = 0$ is a regular singular point for the equation.

    b) Assuming a solution of the form $u(t) = t^r \sum_{n=0}^{\infty} a_n t^n$, show that $r = \pm k$.

    c) In the case that $k = \frac{1}{3}$, find the first three terms of two independent series solutions to the DE.

    d) Show that if $k = 0$ then the Frobenius method leads to only one series solution, and find the first three terms. (The entire series, which converges for all t, is denoted by $J_0(t)$ and is called a **Bessel function** of the first kind of order zero. Finding a second independent solution is beyond the scope of our treatment.)

3.5 Boundary Value Problems and Heat Flow

Let us consider the following problem in steady-state heat conduction. A cylindrical, uniform, metallic bar of length L and cross-sectional area A is insulated on its lateral side. We assume the left face at $x = 0$ is maintained at T_0 degrees and that the right face at $x = L$ is held at T_L degrees. What is the temperature distribution $u = u(x)$ in the bar after it comes to equilibrium? Here $u(x)$ represents the temperature of the entire cross section of the bar at position x, where $0 < x < L$. We are assuming that heat flows only in the axial direction along the bar, and we are assuming that any transients caused by initial temperatures in the bar have decayed away. In other words, we have waited long enough for the temperature to reach a steady state. One might conjecture that the temperature distribution is a linear function of x along the bar; that is, $u(x) = T_0 + \frac{T_L - T_0}{L}x$. This is indeed the case, which we show below. But also we want to consider a more complicated problems where the bar has both a variable conductivity and an internal heat source along its length. An internal heat source, for example, could be resistive heating produced by a current running through the medium.

The physical law that provides the basic model is conservation of energy. If $[x, x + dx]$ is any small section of the bar, then the rate that heat flows in at x, minus the rate that heat flows out at $x + dx$, plus the rate that heat is generated by sources, must equal zero, because the system is in a steady state. See figure 3.5.

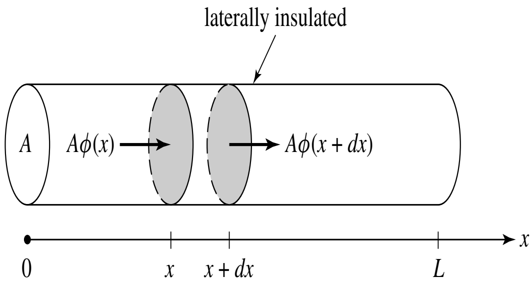

Figure 3.5 Cylindrical bar, laterally insulated, through which heat is flowing in the x-direction. The temperature is uniform in a fixed cross-section.

If we denote by $\phi(x)$ the rate that heat flows to the right at any section x (measured in calories/(area · time), and we let $f(x)$ denote the rate that heat is internally produced at x, measured in calories/(volume · time), then

$$A\phi(x) - A\phi(x + dx) + f(x)A dx = 0.$$

Canceling A, dividing by dx, and rearranging gives

$$\frac{\phi(x + dx) - \phi(x)}{dx} = f(x).$$

Taking the limit as $dx \to 0$ yields

$$\phi'(x) = f(x). \tag{3.27}$$

This is an expression of energy conservation in terms of flux. But what about temperature? Empirically, the flux $\phi(x)$ at a section x is found to be proportional to the negative temperature gradient $-u'(x)$ (which measures the steepness of the temperature distribution, or profile, at that point), or

$$\phi(x) = -K(x)u'(x). \tag{3.28}$$

This is **Fourier's heat conduction law**. The given proportionality factor $K(x)$ is called the **thermal conductivity**, in units of energy/(length · degrees · time), which is a measure of how well the bar conducts heat at location x. For a uniform bar K is constant. The minus sign in (3.28) means that heat flows from higher temperatures to lower temperatures. Fourier's law seems intuitively correct and it conforms with the second law of thermodynamics; the larger the temperature gradient, the faster heat flows from high to low temperatures. Combining (3.27) and (3.28) leads to the equation

$$-(K(x)u'(x))' = f(x), \quad 0 < x < L, \tag{3.29}$$

which is the **steady-state heat conduction equation**. When the **boundary conditions**

$$u(0) = T_0, \quad u(L) = T_1, \tag{3.30}$$

are appended to (3.29), we obtain a **boundary value problem** for the temperature $u(x)$. Boundary conditions are conditions imposed on the unknown state u given at different values of the independent variable x, unlike initial conditions that are imposed at a single value. For boundary value problems we usually use x as the independent variable because boundary conditions usually refer to the boundary of a spatial domain.

Note that we could expand the heat conduction equation to

$$-K(x)u''(x) - K'(x)u'(x) = f(x), \tag{3.31}$$

but there is little advantage in doing so.

Example 3.21

If there are no sources ($f(x) = 0$) and if the thermal conductivity $K(x) = K$ is constant, then the boundary value problem reduces to

$$\begin{aligned} u'' &= 0, \quad 0 < x < L, \\ u(0) &= T_0, \quad u(L) = T_1. \end{aligned}$$

Thus the bar is homogeneous and can be characterized by a constant conductivity. The general solution of $u'' = 0$ is $u(x) = c_1 x + c_2$; applying the boundary conditions determines the constants c_1 and c_2 and gives the linear temperature distribution $u(x) = T_0 + \frac{T_L - T_0}{L} x$, as we previously conjectured.

In nonuniform systems the thermal conductivity K depends upon location x in the system. And, K may depend upon the temperature u as well. Moreover, the heat source term f could depend on location and temperature. In these cases the steady-state heat conduction equation (3.29) takes the more general form

$$-(K(x,u)u')' = f(x,u),$$

which is a nonlinear second-order equation for the steady temperature distribution $u = u(x)$.

Boundary conditions at the ends of the bar may also specify the flux rather than the temperature. For example, in a homogeneous system, if heat is injected at $x = 0$ at a rate of N calories per area per time, then the left boundary condition takes the form $\phi(0) = N$, or

$$-Ku'(0) = N.$$

Thus, a flux condition at an endpoint imposes a condition on the derivative of the temperature at that endpoint. In the case that the end at $x = L$, say, is insulated, so that no heat passes through that end, then the boundary condition is

$$u'(L) = 0,$$

which is called an **insulated boundary condition**. As the reader can see, there are a myriad of interesting boundary value problems associated with heat flow. Similar equations arise in diffusion processes in biology and chemistry, for example, in the diffusion of toxic substances where the unknown is the chemical concentration.

Boundary value problems are much different from initial value problems in that they may have no solution, or they may have infinitely many solutions. Consider the following.

Example 3.22

When $K = 1$ and the heat source term is $f(u) = 9u$ and both ends of a bar of length $L = 2$ are held at $u = 0$ degrees, the boundary value problem becomes

$$\begin{aligned} -u'' &= 9u, \quad 0 < x < 2. \\ u(0) &= 0, \quad u(2) = 0. \end{aligned}$$

The general solution to the DE is $u(x) = c_1 \sin 3x + c_2 \cos 3x$, where c_1 and c_2 are arbitrary constants. Applying the boundary condition at $x = 0$ gives $u(0) = c_1 \sin(3 \cdot 0) + c_2 \cos(3 \cdot 0) = c_2 = 0$. So the solution must have the form $u(x) = c_1 \sin 3x$. Next apply the boundary condition at $x = 2$. Then $u(2) = c_1 \sin(6) = 0$, to obtain $c_1 = 0$. We have shown that the only solution is $u(x) = 0$. There is no nontrivial steady state. But if we make the bar length π, then we obtain the boundary value problem

$$\begin{aligned} -u'' &= 9u, \quad 0 < x < \pi. \\ u(0) &= 0, \quad u(\pi) = 0. \end{aligned}$$

The reader should check that this boundary value problem has infinitely many solutions $u(x) = c_1 \sin 3x$, where c_1 is any number. If we change the right boundary condition, one can check that the boundary value problem

$$\begin{aligned} -u'' &= 9u, \quad 0 < x < \pi. \\ u(0) &= 0, \quad u(\pi) = 1, \end{aligned}$$

has no solution at all.

Example 3.23

Find all real values of λ for which the boundary value problem

$$-u'' = \lambda u, \quad 0 < x < \pi. \tag{3.32}$$

$$u(0) = 0, \quad u'(\pi) = 0, \tag{3.33}$$

has a nontrivial solution. These values are called the **eigenvalues**, and the corresponding nontrivial solutions are called the **eigenfunctions**. Interpreted in the heat flow context, the left boundary is held at zero degrees and the right end is insulated. The heat source is $f(u) = \lambda u$. We are trying to find which linear heat sources lead to nontrivial steady states. To solve this problem we consider different cases because the form of the solution will be different for $\lambda = 0, \lambda < 0, \lambda > 0$. If $\lambda = 0$ then the general solution of $u'' = 0$ is $u(x) = ax+b$. Then $u'(x) = a$. The boundary condition $u(0) = 0$ implies $b = 0$ and the boundary condition $u'(\pi) = 0$ implies $a = 0$. Therefore, when $\lambda = 0$, we get only a trivial solution. Next consider the case $\lambda < 0$ so that the general solution will have the form

$$u(t) = a \sinh \sqrt{-\lambda} x + b \cosh \sqrt{-\lambda} x.$$

The condition $u(0) = 0$ forces $b = 0$. Then $u'(t) = a\sqrt{-\lambda} \cosh \sqrt{-\lambda} x$. The right boundary condition becomes $u'(\pi) = a\sqrt{-\lambda} \cosh(\sqrt{-\lambda} \cdot 0) = 0$, giving $a = 0$. Recall that $\cosh 0 = 1$. Again there is only the trivial solution. Finally assume $\lambda > 0$. Then the general solution takes the form

$$u(t) = a \sin \sqrt{\lambda} x + b \cos \sqrt{\lambda} x.$$

The boundary condition $u(0) = 0$ forces $b = 0$. Then $u(t) = a \sin \sqrt{\lambda} x$ and $u'(x) = a\sqrt{\lambda} \cos \sqrt{\lambda} x$. Applying the right boundary condition gives

$$u'(\pi) = a\sqrt{\lambda} \cos \sqrt{\lambda}\pi = 0.$$

Now we do not have to choose $a = 0$ (which would again give the trivial solution) because we can satisfy this last condition with

$$\cos \sqrt{\lambda}\pi = 0.$$

The cosine function is zero at the values $\pi/2 \pm n\pi$, $n = 0, 1, 2, 3, \ldots$Therefore

$$\sqrt{\lambda}\pi = \pi/2 + n\pi, \qquad n = 0, 1, 2, 3, \ldots$$

Solving for λ yields

$$\lambda = \left(\frac{2n+1}{2}\right)^2, \qquad n = 0, 1, 2, 3, \ldots.$$

Consequently, the values of λ for which the original boundary value problem has a nontrivial solution are $\frac{1}{4}, \frac{9}{4}, \frac{25}{4}, \ldots$. These are the eigenvalues. The corresponding solutions are

$$u(x) = a \sin \left(\frac{2n+1}{2}\right) x, \quad n = 0, 1, 2, 3, \ldots$$

These are the eigenfunctions. Notice that the eigenfunctions are unique only up to a constant multiple. In terms of heat flow, the eigenfunctions represent possible steady-state temperature profiles in the bar. The eigenvalues are those values λ for which the boundary value problem will have steady-state profiles.

Boundary value problems are of great interest in applied mathematics, science, and engineering. They arise in many contexts other than heat flow, including wave motion, quantum mechanics, and the solution of partial differential equations.

EXERCISES

1. A homogeneous bar of length 40 cm has its left and right ends held at 30°C and 10°C, respectively. If the temperature in the bar is in steady-state, what is the temperature in the cross section 12 cm from the left end? If the thermal conductivity is K, what is the rate that heat is leaving the bar at its right face?

2. The thermal conductivity of a bar of length $L = 20$ and cross-sectional area $A = 2$ is $K(x) = 1$, and an internal heat source is given by $f(x) = 0.5x(L - x)$. If both ends of the bar are maintained at zero degrees, what is the steady state temperature distribution in the bar? Sketch a graph of $u(x)$. What is the rate that heat is leaving the bar at $x = 20$?

3. For a metal bar of length L with no heat source and thermal conductivity $K(x)$, show that the steady temperature in the bar has the form

$$u(x) = c_1 \int_0^x \frac{dy}{K(y)} + c_2,$$

where c_1 and c_2 are constants. What is the temperature distribution if both ends of the bar are held at zero degrees? Find an analytic formula and plot the temperature distribution in the case that $K(x) = 1 + x$. If the left end is held at zero degrees and the right end is insulated, find the temperature distribution and plot it.

4. Determine the values of λ for which the boundary value problem

$$\begin{aligned} -u'' &= \lambda u, \quad 0 < x < 1, \\ u(0) &= 0, \quad u(1) = 0, \end{aligned}$$

has a nontrivial solution.

5. Consider the nonlinear heat flow problem

$$\begin{aligned} (uu')' &= 0, \quad 0 < x < \pi, \\ u(0) &= 0, \quad u'(\pi) = 1, \end{aligned}$$

where the thermal conductivity depends on temperature and is given by $K(u) = u$. Find the steady-state temperature distribution.

6. Show that if there is a solution $u = u(x)$ to the boundary value problem (3.29)–(3.30), then the following condition must hold:

$$-K(L)u'(L) + K(0)u'(0) = \int_0^L f(x)dx.$$

Interpret this condition physically.

7. Consider the boundary value problem

$$u'' + \omega^2 u = 0, \quad u(0) = a, \quad u(L) = b.$$

When does a unique solution exist?

8. Find all values of λ for which the boundary value problem

$$\begin{aligned} -u'' - 2u' &= \lambda u, \quad 0 < x < 1, \\ u(0) &= 0, \quad u(1) = 0, \end{aligned}$$

has a nontrivial solution.

9. Show that the eigenvalues of the boundary value problem

$$\begin{aligned} -u'' &= \lambda u, \quad 0 < x < 1, \\ u'(0) &= 0, \quad u(1) + u'(1) = 0, \end{aligned}$$

are given by the numbers $\lambda_n = p_n^2$, $n = 1, 2, 3, ...$, where the p_n are roots of the equation $\tan p = 1/p$. Plot graphs of $\tan p$ and $1/p$ and indicate graphically the locations of the values p_n. Numerically calculate the first four eigenvalues.

10. Find the values of λ (eigenvalues) for which the boundary value problem

$$\begin{aligned} -x^2 u'' - xu' &= \lambda u, \quad 1 < x < e^{\pi}, \\ u(1) &= 0, \quad u(e^{\pi}) = 0, \end{aligned}$$

has a nontrivial solution.

3.6 Higher-Order Equations

So far we have dealt with first- and second-order equations. Higher-order equations occur in some applications. For example, in solid mechanics the vertical deflection $y = y(x)$ of a beam from its equilibrium satisfies a fourth-order equation. However, the applications of higher-order equations are not as extensive as those for their first- and second-order counterparts.

Here, we outline the basic results for a homogeneous, nth-order linear DE with constant coefficients:

$$u^{(n)} + p_{n-1}u^{(n-1)} + \cdots + p_1 u' + p_0 u = 0. \tag{3.34}$$

The p_i, $i = 0, 1, ..., n-1$, are specified constants. The **general solution** of (3.34) has the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + \cdots + c_n u_n(t),$$

where $u_1(t), u_2(t), ..., u_n(t)$ are independent solutions, and where $c_1, c_2, ..., c_n$ are arbitrary constants. In different words, the general solution is a linear combination of n different basic solutions. To find these basic solutions we try the same strategy that worked for a second-order equation, namely assume a solution of the form of an exponential function

$$u(t) = e^{\lambda t},$$

where λ is to be determined. Substituting into the equation gives

$$\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0 = 0, \tag{3.35}$$

which is an nth degree polynomial equation for λ. Equation (3.35) is the **characteristic equation**. From algebra we know that there are n roots $\lambda_1, \lambda_2, ..., \lambda_n$. Here we are counting multiple roots and complex roots (the latter will always occur in complex conjugate pairs $a \pm bi$). A root $\lambda = a$ has **multiplicity** K if $(\lambda - a)^K$ appears in the factorization of the characteristic polynomial.

If the characteristic roots are all *real and distinct*, we will obtain n different basic solutions $u_1(t) = e^{\lambda_1 t}$, $u_2(t) = e^{\lambda_2 t}, ..., u_n(t) = e^{\lambda_n t}$. In this case the general solution of (3.34) will be a linear combination of these,

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}. \tag{3.36}$$

If the roots of (3.35) are not real and distinct then we proceed as might be expected from our study of second-order equations. A complex conjugate pair, $\lambda = a \pm ib$ gives rise to two real solutions $e^{at}\cos bt$ and $e^{at}\sin bt$. A double root λ (multiplicity 2) leads to two solutions $e^{\lambda t}$ and $te^{\lambda t}$. A triple root λ (multiplicity 3) leads to three independent solutions $e^{\lambda t}$, $te^{\lambda t}$, $t^2 e^{\lambda t}$, and so on. In this way we can build up from the factorization of the characteristic polynomial a set of n independent, basic solutions of (3.34). The hardest part of the problem is to find the characteristic roots; computer algebra systems are often useful for this task.

As may be expected from our study of second-order equations, an nth-order nonhomogeneous equation of the form

$$u^{(n)} + p_{n-1}u^{(n-1)} + \cdots + p_1 u' + p_0 u = g(t), \tag{3.37}$$

has a general solution that is the sum of the general solution (3.36) of the homogeneous equation and a particular solution to the equation (3.37). This result is true even if the coefficients p_i are functions of t.

Example 3.24

If the characteristic equation for a 6th-order equation has roots $\lambda = -2 \pm 3i, 4, 4, 4, -1$, the general solution will be

$$u(t) = c_1 e^{-2t}\cos 3t + c_2 e^{-2t}\sin 3t + c_3 e^{4t} + c_4 te^{4t} + c_5 t^2 e^{4t} + c_6 e^{-t}.$$

Initial conditions for an nth order equation (3.34) at $t = 0$ take the form

$$u(0) = \alpha_1, \quad u'(0) = \alpha_2, ..., u^{(n-1)}(0) = \alpha_{n-1},$$

where the α_i are given constants. Thus, for an nth-order initial value problem we specify the value of the function and all of its derivatives up to the $(n-1)$st-order, at the initial time. These initial conditions determine the n arbitrary constants in the general solution and select out a unique solution to the initial value problem.

Example 3.25

Consider

$$u''' - 2u'' - 3u' = 5e^{4t}.$$

The characteristic equation for the homogeneous equation is

$$\lambda^3 - 2\lambda^2 - 3\lambda = 0,$$

or

$$\lambda(\lambda - 3)(\lambda + 1) = 0.$$

The characteristic roots are $\lambda = 0, -1, 3$, and therefore the homogeneous equation has solution

$$u_h(t) = c_1 + c_2e^{-t} + c_3e^{3t}.$$

The particular solution will have the form $u_p(t) = ae^{4t}$. Substituting into the original nonhomogeneous equation gives $a = 1/4$. Therefore the general solution to the equation is

$$u(t) = c_1 + c_2e^{-t} + c_3e^{3t} + \frac{1}{4}e^{4t}.$$

The three constants can be determined from initial conditions. For example, for a third-order equation the initial conditions at time $t = 0$ have the form

$$u(0) = \alpha, \quad u'(0) = \beta, \quad u''(0) = \gamma,$$

for some given constants α, β, γ. Of course, initial conditions can be prescribed at any other time t_0.

EXERCISES

1. Find the general solution of the following differential equations:

   a) $u''' + u' = 0$.

   b) $u'''' + u' = 1$.

c) $u'''' + u'' = 0$.

d) $u''' - u' - 8u = 0$.

e) $u''' + u'' = 2e^t + 3t^2$.

2. Solve the initial value problem $u''' - u'' - 4u' - 4u = 0$, $u(0) = 2$, $u'(0) = -1$, $u''(0) = 5$.

3. Write down a linear, fifth-order differential equation whose general solution is
$$u = c_1 + c_2 t + c_3 e^{-4t} + e^{5t}(c_4 \cos 2t + c_5 \sin 5t).$$

4. Show that the third-order equation $u''' + 2u'' - 5u' - u = 0$ can be written as an equivalent system of three first-order equations in the variables u, v, and w, where $v = u'$ and $w = u''$.

5. What is the general solution of a fourth-order differential equation if the four characteristic roots are $\lambda = 3 \pm i,\ 3 \pm i$? What is the differential equation?

3.7 Summary and Review

One way to think about learning and solving differential equations is in terms of pattern recognition. Although this is a very "compartmentalized" way of thinking, it does help our learning process. When faced with a differential equation, what do we do? The first step is to recognize what type it is. It is like a pianist recognizing a certain set of notes in a complicated musical piece and then playing those notes easily because of long hours of practice. In differential equations we must practice to recognize an equation and learn the solution technique that works for that equation. At this point in your study, what kinds of equations should you surely be able to recognize and solve?

The simplest is the **pure time equation**
$$u' = g(t).$$
Here u is the antiderivative of $g(t)$, and we sometimes have to write the solution as an integral when we cannot find a simple form for the antiderivative. The next simplest equation is the **separable equation**
$$u' = g(t)f(u),$$

where the right side is a product of functions of the dependent and independent variables. These are easy: just separate variables and integrate. **Autonomous equations** have the form

$$u' = f(u),$$

where the right side depends only on u. These equations are separable, should we want to attempt a solution. But often, for autonomous equations, we apply qualitative methods to understand the behavior of solutions. This includes graphing $f(u)$ vs. u, finding the equilibrium solutions, and then drawing arrows on the phase line to determine stability of the equilibrium solutions and whether u is increasing or decreasing. Nearly always these qualitative methods are superior to having an actual solution formula. First-order autonomous equations cannot have oscillatory solutions. Finally, the first-order **linear equation** is

$$u' = p(t)u + g(t).$$

Here we use variation of parameters or integrating factors. Sometimes an equation can be solved by multiple methods; for example, $u' = 2u - 7$ is separable, linear, and autonomous.

There are other first-order nonlinear equations that can be solved, and some of these were introduced in the Exercises. The **Bernoulli equation**

$$u' = p(t)u + g(t)u^n$$

can be transformed into a linear equation for the variable $y = u^{1-n}$, and the **homogeneous equation**

$$u' = f(\frac{u}{t})$$

can be transformed into a separable equation for the variable $y = u/t$. Solutions to special and unusual equations can sometimes be found in mathematical handbooks or in computer algebra systems.

There are really only two second-order linear equations that can be solved simply. These are the **equation with constant coefficients**

$$au'' + bu' + cu = 0,$$

where we have solutions of the form $u = e^{\lambda t}$, with λ satisfying the characteristic equation $a\lambda^2 + b\lambda + c = 0$, and the **Cauchy–Euler equation**

$$at^2u'' + btu' + cu = 0,$$

where we have solutions of the form $u = t^m$, where m satisfies the characteristic equation $am(m-1) + bm + c = 0$. For these two problems we must distinguish when the roots of the characteristic equation are real and unequal, real and equal, or complex. When the right side of either of these equations is nonzero, then the equation is nonhomogeneous. Then we can find particular solutions using the **variation of parameters** method, which works for all linear equations, or use **undetermined coefficients**, which works only for constant coefficient equations with special right sides. Nonhomogeneous linear equations with constant coefficients can also be handled by Laplace transforms, which are discussed in the next chapter. All these methods extend to higher-order equations.

Generally, we cannot easily solve homogeneous second-order linear equations with variable coefficients, or equations having the form

$$u'' + p(t)u' + q(t)u = 0.$$

Many of these equations have solutions that can be written as power series. These power series solutions define **special functions** in mathematics, such as Bessel functions, Hermite polynomials, and so forth. In any case, you can *not* solve these variable coefficient equations using the characteristic polynomial, and nonhomogeneous equations are not amenable to the methods of undetermined coefficients. If you are fortunate enough to find one solution, you can determine a second by reduction of order. If you are lucky enough to find two independent solutions to the homogeneous equation, the method of variation of parameters gives a particular solution.

The basic structure theorem holds for all linear nonhomogeneous equations: the general solution is the sum of the general solution to the homogeneous equation and a particular solution. This result is fundamental.

Second-order equations coming from Newton's second law have the form $x'' = F(t, x, x')$. These can be reduced to first-order equations when t or x is missing from the force F, or when $F = F(x)$, which is the conservative case.

The Exercises give you review and practice in identifying and solving differential equations.

EXERCISES

1. Identify each of the differential equations and find the general solution. Some of the solutions may contain an integral.

   a) $2u'' + 5u' - 3u = 0$.

   b) $u' - Ru = 0$, where R is a parameter.

   c) $u' = \cos t - u \cos t$.

   d) $u' - 6u = e^t$.

   e) $u'' = -\frac{2}{t^2}u$.

   f) $u'' + 6u' + 9u = 5 \sin t$.

g) $u' = -8t + 6.$

h) $u'' + u = t^2 - 2t + 2$

i) $u' + u - tu^3 = 0.$

j) $2u'' + u' + 3u = 0.$

k) $x'' = (x')^3.$

l) $tu' + u = t^2u^2.$

m) $u'' = -3u^2.$

n) $tu' = u - \frac{t}{2}\cos^2\left(\frac{2u}{t}\right).$

o) $u''' + 5u'' - 6u' = 9e^{3t}.$

p) $(6tu - u^3) + (4u + 3t^2 - 3tu^2)u' = 0.$

2. Solve the initial value problem $u' = u^2 \cos t$, $u(0) = 2$, and find the interval of existence.

3. Solve the initial value problem $u' = \frac{2}{t}u + t$, $u(1) = 2$, and find the interval of existence.

4. Use the power series method to find the first three terms of two independent solutions to $u'' + tu' + tu = 0$ valid near $t = 0$.

5. For all cases, find the equilibrium solutions for $u' = (u - a)(u^2 - a)$, where a is a real parameter, and determine their stability. Summarize the information on a bifurcation diagram.

6. A spherical water droplet loses volume by evaporation at a rate proportional to its surface area. Find its radius $r = r(t)$ in terms of the proportionality constant and its initial radius r_0.

7. A population is governed by the law $p' = rp\left(\frac{K-p}{K+ap}\right)$, where r, K, and a are positive constants. Find the equilibria and their stability. Describe, in words, the dynamics of the population.

8. Use the variation of parameters method to find a particular solution to $u'' - u' - 2u = \cosh t$.

9. If e^{-t^2} is one solution to the differential equation $u'' + 4tu' + 2(2t^2 + 1)u = 0$, find the solution satisfying the conditions $u(0) = 3$, $u'(0) = 1$.

10. Sketch the slope field for the differential equation $u' = -t^2 + \sin(u)$ in the window $-3 \le t \le 3$, $-3 \le t \le 3$, and then superimpose on the field the two solution curves that satisfy $u(-2) = 1$ and $u(-1) = 1$, respectively.

11. Solve $u' = 4tu - \frac{2u}{t}\ln u$ by making the substitution $y = \ln u$.

12. Adapt your knowledge about solution methods for Cauchy–Euler equations to solve the third-order initial value problem:

$$t^3u''' - t^2u'' + 2tu' - 2u = 0$$

with $u(1) = 3$, $u'(1) = 2$, $u''(1) = 1$.

4 Laplace Transforms

The Laplace method for solving linear differential equations with constant coefficients is based upon transforming the differential equation into an algebraic equation. It is especially applicable to models containing a nonhomogeneous forcing term (such as the electrical generator in a circuit) that is either discontinuous or is applied only at a single instant of time (an impulse).

This method can be regarded as another tool, in addition to variation of parameters and undetermined coefficients, for solving nonhomogeneous equations. It is often a key topic in engineering where the stability properties of linear systems are addressed.

The material in this chapter is not needed for the remaining chapters, so it may be read at any time.

4.1 Definition and Basic Properties

A successful strategy for many problems is to transform them into simpler ones that can be solved more easily. For example, some problems in rectangular coordinates are better understood and handled in polar coordinates, so we make the usual coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$. After solving the problem in polar coordinates, we can return to rectangular coordinates by the inverse transformation $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x}$. A similar technique holds true for many differential equations using **integral transform methods**. In this chapter we introduce the Laplace transformation which has the effect of turning a differential equation with state function $u(t)$ into an algebra problem for an associated transformed function $U(s)$; we can easily solve the algebra problem for $U(s)$ and then return to $u(t)$ via an inverse transformation. The technique is applicable to both homogeneous and nonhomogeneous linear differential equations with constant coefficients, and it is a standard method for engineers and applied mathematicians. It is particularly useful for differential equations that contain piecewise continuous forcing functions or functions that act as an impulse. The transform goes back to the late 1700s and is named for the great French mathematician and scientist Pierre de Laplace, although the basic integral goes back earlier to L. Euler. The English engineer O. Heaviside developed much of the operational calculus for transform methods in the early 1900s.

Let $u = u(t)$ be a given function defined on $0 \leq t < \infty$. The **Laplace transform** of $u(t)$ is the function $U(s)$ defined by

$$U(s) = \int_0^\infty u(t)e^{-st}dt, \tag{4.1}$$

provided the improper integral exists. The integrand is a function of t and s, and we integrate on t, leaving a function of s. Often we represent the Laplace transform in function notation,

$$\mathcal{L}[u(t)](s) = U(s) \quad \text{or just} \quad \mathcal{L}[u] = U(s).$$

$\mathcal{L}$ represents a function-like operation, called an operator or transform, whose domain and range are sets of functions; $\mathcal{L}$ takes a function $u(t)$ and transforms it into a new function $U(s)$ (see figure 4.1). In the context of Laplace transformations, t and u are called the **time domain** variables, and s and U are called the **transform domain** variables. In summary, the Laplace transform maps functions $u(t)$ to functions $U(s)$ and is somewhat like mappings we consider in calculus, such as $y = f(x) = x^2$, which maps numbers x to numbers y.

We can compute the Laplace transform of many common functions directly from the definition (4.1).

Example 4.1

Let $u(t) = e^{at}$. Then

$$U(s) = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{(a-s)t}dt = \frac{1}{a-s}e^{(a-s)t}|_{t=0}^{t=\infty} = \frac{1}{s-a}, \quad s > a.$$

In different notation, $\mathcal{L}[e^{at}] = \frac{1}{s-a}$. Observe that this transform exists only for $s > a$ (otherwise the integral does not exist). Sometimes we indicate the values of s for which the transformed function $U(s)$ is defined.

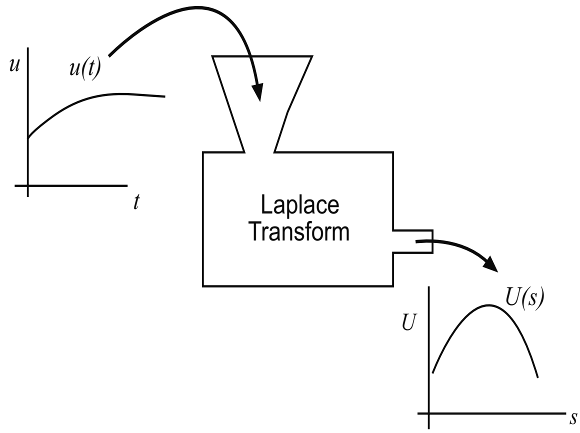

Figure 4.1 The Laplace transform as a machine that transforms functions $u(t)$ to functions $U(s)$.

Example 4.2

Let $u(t) = t$. Then, using integration by parts,

$$U(s) = \int_0^\infty te^{-st}dt = \left[t\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty} - \frac{1}{s}\int_0^\infty 1 \cdot e^{-st}dt = \frac{1}{s^2}, \quad s > 0.$$

Example 4.3

The unit switching function $h_a(t)$ is defined by $h_a(t) = 0$ if $t < a$ and $h_a(t) = 1$ if $t \geq a$. The switch is off if $t < a$, and it is on when $t \geq a$. Therefore the function $h_a(t)$ is a step function where the step from 0 to 1 occurs at $t = a$. The switching function is also called the **Heaviside function**. The Laplace transform of $h_a(t)$ is

$$\begin{aligned}
\mathcal{L}[h_a(t)] &= \int_0^\infty h_a(t)e^{-st}dt \\
&= \int_0^a h_a(t)e^{-st}dt + \int_a^\infty h_a(t)e^{-st}dt \\
&= \int_0^a 0 \cdot e^{-st}dt + \int_a^\infty 1 \cdot e^{-st}dt \\
&= -\frac{1}{s}e^{-st}|_{t=a}^{t=\infty} = \frac{1}{s}e^{-as}, \quad s > 0.
\end{aligned}$$

Example 4.4

The Heaviside function is useful for expressing multi-lined functions in a single formula. For example, let

$$f(t) = \begin{cases} \frac{1}{2}, & 0 \le t < 2 \\ t-1, & 2 \le t \le 3 \\ 5-t^2, & 3 < t \le 6 \\ 0, & t > 6 \end{cases}$$

(The reader should plot this function). This can be written in one line as

$$f(t) = \frac{1}{2}h_0(t) + (t - 1 - \frac{1}{2})h_2(t) + (5 - t^2 - (t-1))h_3(t) - (5-t^2)h_6(t).$$

The first term switches on the function 1/2 at $t = 0$; the second term switches off 1/2 and switches on $t-1$ at time $t = 2$; the third term switches off $t-1$ and switches on $5 - t^2$ at $t = 3$; finally, the last term switches off $5 - t^2$ at $t = 6$. Later we show how to find Laplace transforms of such functions.

As you may have already concluded, calculating Laplace transforms may be tedious business. Fortunately, generations of mathematicians, scientists, and engineers have computed the Laplace transforms of many, many functions, and the results have been catalogued in tables and in software systems. Some of the tables are extensive, but here we require only a short table, which is given at the end of the chapter. The table lists a function $u(t)$ in the first column, and its transform $U(s)$, or $\mathcal{L}u$, in the second. The various functions in the first column are discussed in the sequel. Computer algebra systems also have commands that calculate the Laplace transform (see Appendix B).

Therefore, given $u(t)$, the Laplace transform $U(s)$ can be computed by the definition, given in formula (4.1). We can also think of the opposite problem: given $U(s)$, find a function $u(t)$ whose Laplace transform is $U(s)$. This is the inverse problem. Unfortunately, there is no elementary formula that we can write down that computes $u(t)$ in terms of $U(s)$ (there is a formula, but it involves a contour integration in the complex plane). In elementary treatments we are satisfied with using tables. For example, if $U(s) = \frac{1}{s-2}$, then the table gives $u(t) = e^{2t}$ as the function that has $U(s)$ as its transform. When we think of it this way, we say $u(t) = e^{2t}$ is the "inverse transform" of $U(s) = \frac{1}{s-2}$, and we write

$$e^{2t} = \mathcal{L}^{-1}\left[\frac{1}{s-2}\right].$$

In general we use the notation

$$U = \mathcal{L}(u), \quad u = \mathcal{L}^{-1}[U].$$

We think of $\mathcal{L}$ as an operator (**transform**) and $\mathcal{L}^{-1}$ as the inverse operation (**inverse transform**). The functions $u(t)$ and $U(s)$ form a transform pair, and they are listed together in two columns of a table. Computer algebra systems also supply inverse transforms.

One question that should be addressed concerns the *existence* of the transform. That is, which functions have Laplace transforms? Clearly if a function grows too quickly as t gets large, then the improper integral will not exist and there will be no transform. There are two conditions that guarantee existence, and these are reasonable conditions for most problems in science and engineering. First, we require that $u(t)$ not grow too fast; a way of stating this mathematically is to require that there exist constants $M > 0$ and α for which

$$|u(t)| \le Me^{\alpha t}$$

is valid for all $t > t_0$, where t_0 is some value of time. That is, beyond the value t_0 the function is bounded above and below by an exponential function. Such functions are said to be of **exponential order**. Second, we require that $u(t)$ be **piecewise continuous** on $0 \le t < \infty$. In other words, the interval $0 \le t < \infty$ can be divided into intervals on which u is continuous, and at any point of discontinuity u has finite left and right limits, except possibly at $t = +\infty$. One can prove that if u is piecewise continuous on $0 \le t < \infty$ and of exponential order, then the Laplace transform $U(s)$ exists for all $s > \alpha$.

What makes the Laplace transform so useful for differential equations is that it turns derivative operations in the time domain into multiplication operations in the transform domain. The following theorem gives the crucial operational formulas stating how the derivatives transform.

Theorem 4.5

Let $u(t)$ be a function and $U(s)$ its transform. Then

$$\mathcal{L}[u'] = sU(s) - u(0), \tag{4.2}$$

$$\mathcal{L}[u''] = s^2U(s) - su(0) - u'(0). \tag{4.3}$$

Proof. These facts are easily proved using integration by parts. We have

$$\begin{aligned} \mathcal{L}[u'] &= \int_0^\infty u'(t)e^{-st}dt = \left[u(t)e^{-st}\right]_{t=0}^{t=\infty} - \int_0^\infty -su(t)e^{-st}dt \\ &= -u(0) + sU(s), \quad s > 0. \end{aligned}$$

The second operational formula (4.3) is derived using two successive integration by parts, and we leave the calculation to the reader.

These formulas allow us to transform a differential equation with unknown $u(t)$ into an algebraic problem with unknown $U(s)$. We solve for $U(s)$ and then find $u(t)$ using the inverse transform $u = \mathcal{L}^{-1}[U]$. We elaborate on this method in the next section.

Before tackling the solution of differential equations, we present additional important and useful properties.

(a) (**Linearity**) The Laplace transform is a linear operation; that is, the Laplace transform of a sum of two functions is the sum of the Laplace transforms of each, and the Laplace transform of a constant times a function is the constant times the transform of the function. We can express these rules in symbols by a single formula:

$$\mathcal{L}[c_1 u + c_2 v] = c_1 \mathcal{L}[u] + c_2 \mathcal{L}[v]. \tag{4.4}$$

Here, u and v are functions and c_1 and c_2 are any constants. Similarly, the inverse Laplace transform is a linear operation:

$$\mathcal{L}^{-1}[c_1 u + c_2 v] = c_1 \mathcal{L}^{-1}[u] + c_2 \mathcal{L}^{-1}[v]. \tag{4.5}$$

(b) (**Shift Property**) The Laplace transform of a function times an exponential, $u(t)e^{at}$, shifts the transform of U; that is,

$$\mathcal{L}[u(t)e^{at}] = U(s-a). \tag{4.6}$$

(c) (**Switching Property**) The Laplace transform of a function that switches on at $t = a$ is given by

$$\mathcal{L}[h_a(t)u(t-a)] = U(s)e^{-as}. \tag{4.7}$$

Proofs of some of these relations follow directly from the definition of the Laplace transform, and they are requested in the Exercises.

EXERCISES

1. Use the definition of the Laplace transform to compute the transform of the square pulse function $u(t) = 1$, $1 \le t \le 2$; $u(t) = 0$, otherwise.

2. Derive the operational formula (4.3).

3. Sketch the graphs of $\sin t$, $\sin(t - \pi/2)$, and $h_{\pi/2}(t)\sin(t - \pi/2)$. Find the Laplace transform of each.

4. Find the Laplace transform of $t^2 e^{-3t}$.

5. Find $\mathcal{L}[\sinh kt]$ and $\mathcal{L}[\cosh kt]$ using the fact that $\mathcal{L}\left[e^{kt}\right] = \frac{1}{s-k}$.

6. Find $\mathcal{L}\left[e^{-3t} + 4\sin kt\right]$ using the table. Find $\mathcal{L}\left[e^{-3t}\sin 2t\right]$ using the shift property (4.6).

7. Using the switching property (4.7), find the Laplace transform of the function
$$u(t) = \begin{cases} 0 & t < 2 \\ e^{-t}, & t > 2. \end{cases}$$

8. From the definition (4.1), find $\mathcal{L}\left[1/\sqrt{t}\right]$ using the integral substitution $st = r^2$ and then noting $\int_0^\infty \exp(-r^2)dr = \sqrt{\pi}/2$.

9. Does the function $u(t) = e^{t^2}$ have a Laplace transform? What about $u(t) = 1/t$? Comment.

10. Derive the operational formulas (4.6) and (4.7).

11. Plot the *square-wave* function
$$f(t) = \sum_{n=0}^{\infty} (-1)^n h_n(t)$$
on the interval $t > 0$ and find its transform $F(s)$. (Hint: use the geometric series $1 + x + x^2 + \cdots = \frac{1}{1-x}$.)

12. Show that
$$\mathcal{L}\left[\int_0^t u(r)dr\right] = \frac{U(s)}{s}.$$

13. Derive the formulas
$$\mathcal{L}\left[tu(t)\right] = -U'(s), \quad \mathcal{L}^{-1}[U'(s)] = -tu(t).$$
Use these formulas to find the inverse transform of $\arctan\frac{a}{s}$.

14. Show that
$$\mathcal{L}\left[\frac{u(t)}{t}\right] = \int_s^\infty U(r)dr,$$
and use the result to find
$$\mathcal{L}\left[\frac{\sinh t}{t}\right].$$

15. Show that
$$\mathcal{L}\left[f(t)h_a(t)\right] = e^{-as}\mathcal{L}[f(t+a)],$$
and use this formula to compute $\mathcal{L}[t^2 h_1(t)]$.

16. Find the Laplace transform of the function in Example 4.4.

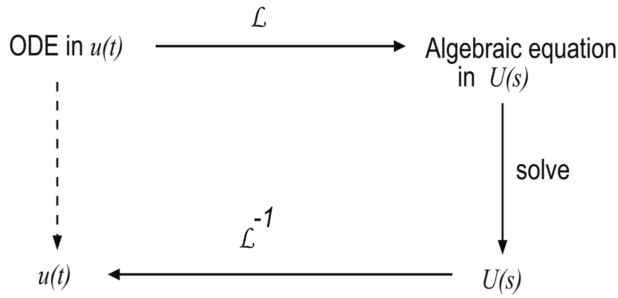

Figure 4.2 A DE for an unknown function $u(t)$ is transformed to an algebraic equation for its transform $U(s)$. The algebraic problem is solved for $U(s)$ in the transform domain, and the solution is returned to the original time domain via the inverse transform.

17. The **gamma function** is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > -1.$$

a) Show that $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$ for nonnegative integers n. Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

b) Show that $\mathcal{L}[t^a] = \frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0$.

4.2 Initial Value Problems

The following examples illustrate how Laplace transforms are used to solve initial value problems for linear differential equations with constant coefficients. The method works on equations of all orders and on systems of several equations in several unknowns. We assume $u(t)$ is the unknown state function. The idea is to take the transform of each term in the equation, using the linearity property. Then, using Theorem 4.5, reduce all of the derivative terms to algebraic expressions and solve for the transformed state function $U(s)$. Finally, invert $U(s)$ to recover the solution $u(t)$. Figure 4.2 illustrates this three-step method.

Example 4.6

Consider the second-order initial value problem

$$u'' + \omega^2 u = 0, \quad u(0) = 0, \quad u'(0) = 1.$$

Taking transforms of both sides and using the linearity property gives

$$\mathcal{L}[u''] + \omega^2 \mathcal{L}[u] = \mathcal{L}[0].$$

Then Theorem 4.5 gives

$$s^2 U(s) - su(0) - u'(0) + \omega^2 U(s) = 0,$$

which is an algebraic equation for the transformed state $U(s)$. Using the initial conditions, we get

$$s^2 U(s) - 1 + \omega^2 U(s) = 0.$$

Solving for the transform function $U(s)$ gives

$$U(s) = \frac{1}{\omega^2 + s^2} = \frac{1}{\omega} \frac{\omega}{\omega^2 + s^2},$$

which is the solution in the transform domain. Therefore, from the table, the inverse transform is

$$u(t) = \frac{1}{\omega} \sin \omega t,$$

which is the solution to the original initial value problem.

Example 4.7

Solve the first-order nonhomogeneous equation

$$u' + 2u = e^{-t}, \quad u(0) = 0.$$

Taking Laplace transforms of each term

$$\mathcal{L}[u'] + \mathcal{L}[2u] = \mathcal{L}[e^{-t}],$$

or

$$sU(s) - u(0) + 2U(s) = \frac{1}{s+1}.$$

Solving for the transformed function $U(s)$ gives

$$U(s) = \frac{1}{(s+1)(s+2)}.$$

Now we can look up the inverse transform in the table. We find

$$u(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}.$$

Example 4.8

(Partial Fractions, I) Sometimes the table may not include an entry for the inverse transform that we seek, and so we may have to algebraically manipulate or simplify our expression so that it can be reduced to table entries. A standard technique is to expand complex fractions into their "partial fraction" expansion. In the last example we had

$$U(s) = \frac{1}{(s+1)(s+2)}.$$

We can expand $U(s)$ as

$$\frac{1}{(s+1)(s+2)} = \frac{a}{(s+1)} + \frac{b}{(s+2)},$$

for some constants a and b to be determined. Combining terms on the right side gives

$$\begin{aligned}\frac{1}{(s+1)(s+2)} &= \frac{a(s+2)+b(s+1)}{(s+1)(s+2)}\\ &= \frac{(a+b)s+2a+b}{(s+1)(s+2)}.\end{aligned}$$

Comparing numerators on the left and right force $a+b=0$ and $2a+b=1$. Hence $a=-b=1$ and we have

$$U(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(s+1)} + \frac{-1}{(s+2)}.$$

We have reduced the complex fraction to the sum of two simple, easily identifiable, fractions that are easily found in the table. Using the linearity property of the inverse transform,

$$\begin{aligned}\mathcal{L}^{-1}[U(s)] &= \mathcal{L}^{-1}\left[\frac{1}{(s+1)}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right]\\ &= e^{-t} - e^{-2t}.\end{aligned}$$

Example 4.9

(Partial Fractions, II) A common expression in the transform domain that requires inversion is a fraction of the form

$$U(s) = \frac{1}{s^2+bs+c}.$$

If the denominator has two distinct real roots, then it factors and we can proceed as in the previous example. If the denominator has complex roots then the following "complete the square" technique may be used. For example, consider

$$U(s) = \frac{1}{s^2 + 3s + 6}.$$

Then, completing the square in the denominator,

$$\begin{aligned} U(s) &= \frac{1}{s^2 + 3s + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 6} \\ &= \frac{1}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2}. \end{aligned}$$

This entry is in the table, up to a factor of $\frac{\sqrt{15}}{2}$. Therefore we multiply and divide by this factor and locate the inverse transform in the table as

$$u(t) = \frac{2}{\sqrt{15}} e^{-3t/2} \sin \frac{\sqrt{15}}{2} t.$$

Example 4.10

In this example we calculate the response of an RC circuit when the emf is a discontinuous function. These types of problems occur frequently in engineering, especially electrical engineering, where discontinuous inputs to circuits are commonplace. Therefore, consider an RC circuit containing a 1 volt battery, and with zero initial charge on the capacitor. Take $R = 1$ and $C = 1/3$. Assume the switch is turned on from $1 \le t \le 2$, and is otherwise switched off, giving a square pulse. The governing equation for the charge on the capacitor is

$$q' + 3q = h_1(t) - h_2(t), \quad q(0) = 0.$$

We apply the basic technique. Taking the Laplace transform gives

$$sQ(s) - q(0) + 3Q(s) = \frac{1}{s}(e^{-s} - e^{-2s}).$$

Solving for $Q(s)$ yields

$$\begin{aligned} Q(s) &= \frac{1}{s(s+3)}(e^{-s} - e^{-2s}) \\ &= \frac{1}{s(s+3)} e^{-s} - \frac{1}{s(s+3)} e^{-2s}. \end{aligned}$$

Now we have to invert, which is usually the hardest part. Each term on the right has the form $U(s)e^{-as}$, and therefore we can apply the switching property (4.7). From the table we have

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3}(1 - e^{-3t}).$$

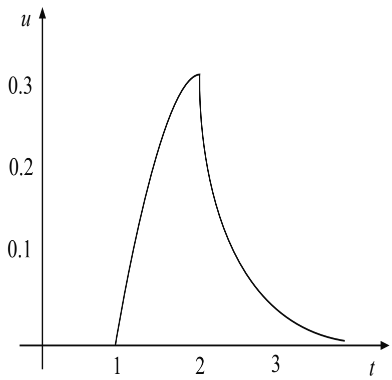

Figure 4.3 The charge response is zero up to time $t = 1$, when the switch is closed. The charge increases until $t = 2$, when the switch is again opened. The charge then decays to zero.

Therefore, by the shift property,

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}e^{-s}\right] = \frac{1}{3}(1 - e^{-3(t-1)})h_1(t).$$

Similarly,

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+3)}e^{-2s}\right] = \frac{1}{3}(1 - e^{-3(t-2)})h_2(t).$$

Putting these two results together gives

$$q(t) = \frac{1}{3}(1 - e^{-3(t-1)})h_1(t) - \frac{1}{3}(1 - e^{-3(t-2)})h_2(t).$$

We can use software to plot the charge response. See figure 4.3.

Because there are extensive tables and computer algebra systems containing large numbers of inverse transforms, the partial fractions technique for inversion is not used as often as in the past.

EXERCISES

1. Find A, B, and C for which

$$\frac{1}{s^2(s-1)} = \frac{As + B}{s^2} + \frac{C}{s-1}.$$

   Then find the inverse Laplace transform of $\frac{1}{s^2(s-1)}$.

2. Find the inverse transform of the following functions.

a) $U(s) = \frac{s}{s^2+7s-8}$.

b) $U(s) = \frac{3-2s}{s^2+2s+10}$.

c) $\frac{2}{(s-5)^4}$.

d) $\frac{7}{s}e^{-4s}$.

3. Solve the following initial value problems using Laplace transforms.

a) $u' + 5u = h_2(t), \quad u(0) = 1.$

b) $u' + u = \sin 2t, \quad u(0) = 0.$

c) $u'' - u' - 6u = 0, \quad u(0) = 2,\ u'(0) = -1$

d) $u'' - 2u' + 2u = 0, \quad u(0) = 0,\ u'(0) = 1.$

e) $u'' - 2u' + 2u = e^{-t}, \quad u(0) = 0,\ u'(0) = 1.$

f) $u'' - u' = 0, \quad u(0) = 1,\ u'(0) = 0.$

g) $u'' + 0.4u' + 2u = 1 - h_5(t), \quad u(0) = 0,\ u'(0) = 0.$

h) $u'' + 9u = \sin 3t, \quad u(0) = 0,\ u'(0) = 0.$

i) $u'' - 2u = 1, \quad u(0) = 1,\ u'(0) = 0.$

4. Use Laplace transforms to solve the two simultaneous differential equations

$$\begin{aligned} x' &= x - 2y - t \\ y' &= 3x + y, \end{aligned}$$

with $x(0) = y(0) = 0$. (Hint: use what you know about solving single equations, letting $\mathcal{L}[x] = X(s)$ and $\mathcal{L}[y] = Y(s)$.)

5. Show that

$$L[t^n u(t)] = (-1)^n U^{(n)}(s)$$

for $n = 1, 2, 3, ...$

4.3 The Convolution Property

The additivity property of Laplace transforms is stated in (4.4): the Laplace transform of a sum is the sum of the transforms. But what can we say about the Laplace transform of a product of two functions? It is *not* the product of the two Laplace transforms. That is, if $u = u(t)$ and $v = v(t)$ with $\mathcal{L}[u] = U(s)$ and $\mathcal{L}[v] = V(s)$, then $\mathcal{L}[uv] \neq U(s)V(s)$. If this is not true, then what is true? We ask it this way. What function has transform $U(s)V(s)$? Or, differently, what is the inverse transform of $U(s)V(s)$. The answer may surprise you because it is nothing one would easily guess. The function whose transform is $U(s)V(s)$ is the convolution of the two functions $u(t)$ and $v(t)$. It is defined as follows. If u and v are two functions defined on $[0,\infty)$, the **convolution** of u and v, denoted by $u * v$, is the function defined by

$$(u * v)(t) = \int_0^t u(\tau)v(t-\tau)d\tau.$$

Sometimes it is convenient to write the convolution as $u(t) * v(t)$. The **convolution property** of Laplace transforms states that

$$\mathcal{L}[u * v] = U(s)V(s).$$

It can be stated in terms of the inverse transform as well:

$$\mathcal{L}^{-1}[U(s)V(s)] = u * v.$$

This property is useful because when solving a DE we often end up with a product of transforms; we may use this last expression to invert the product.

The convolution property is straightforward to verify using a multi-variable calculus technique, interchanging the order of integration. The reader should check the following steps.

$$\begin{aligned}
\mathcal{L}\left(\int_0^t u(\tau)v(t-\tau)d\tau\right) &= \int_0^\infty \left(\int_0^t u(\tau)v(t-\tau)d\tau\right) e^{-st}dt \\
&= \int_0^\infty \left(\int_0^t u(\tau)v(t-\tau)e^{-st}d\tau\right) dt \\
&= \int_0^\infty \left(\int_\tau^\infty u(\tau)v(t-\tau)e^{-st}dt\right) d\tau \\
&= \int_0^\infty \left(\int_\tau^\infty v(t-\tau)e^{-st}dt\right) u(\tau)d\tau \\
&= \int_0^\infty \left(\int_0^\infty v(r)e^{-s(r+\tau)}dr\right) u(\tau)d\tau \\
&= \int_0^\infty \left(\int_0^\infty v(r)e^{-sr}dr\right) e^{-s\tau}u(\tau)d\tau \\
&= \left(\int_0^\infty e^{-s\tau}u(\tau)d\tau\right)\left(\int_0^\infty v(r)e^{-sr}dr\right).
\end{aligned}$$

This last expression is $U(s)V(s)$.

Example 4.11

Find the convolution of 1 and t^2. We have

$$\begin{aligned} 1 * t^2 &= \int_0^t 1 \cdot (t-\tau)^2 d\tau = \int_0^t (t^2 - 2t\tau + \tau^2) d\tau \\ &= t^2 \cdot t - 2t(\frac{t^2}{2}) + \frac{t^3}{3} = \frac{t^3}{3}. \end{aligned}$$

Notice also that the convolution of t^2 and 1 is

$$t^2 * 1 = \int_0^t \tau^2 \cdot 1 d\tau = \frac{t^3}{3}.$$

In the exercises you are asked to show that $u * v = v * u$, so the order of the two functions under convolution does not matter.

Example 4.12

Find the inverse of $U(s) = \frac{3}{s(s^2+9)}$. We can do this by partial fractions, but here we use convolution. We have

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{3}{s(s^2+9)}\right] &= \mathcal{L}^{-1}\left[\frac{1}{s}\frac{3}{(s^2+9)}\right] \\ &= 1 * \sin 3t = \int_0^t \sin 3\tau d\tau \\ &= \frac{1}{3}\left(1 - \cos 3t\right). \end{aligned}$$

Example 4.13

Solve the nonhomogeneous DE

$$u'' + k^2 u = f(t),$$

where f is any given input function, and where $u(0)$ and $u'(0)$ are specified initial conditions. Taking the Laplace transform,

$$s^2 U(s) - su(0) - u'(0) + k^2 U(s) = F(s).$$

Then

$$U(s) = u(0)\frac{s}{s^2+k^2} + u'(0)\frac{1}{s^2+k^2} + \frac{F(s)}{s^2+k^2}.$$

Now we can invert each term, using the convolution property on the last term, to get the solution formula

$$u(s) = u(0)\cos kt + \frac{u'(0)}{k}\sin kt + \frac{1}{k}\int_0^t f(\tau)\sin k(t-\tau)dr.$$

EXERCISES

1. Compute the convolution of $\sin t$ and $\cos t$.
2. Compute the convolution of t and t^2.
3. Use the convolution property to find the general solution of the differential equation $u' = au + q(t)$ using Laplace transforms
4. Use a change of variables to show that the order of the functions used in the definition of the convolution does not matter. That is,

$$(u * v)(t) = (v * u)(t).$$

5. Solve the initial value problem

$$u'' - \omega^2 u = f(t), \quad u(0) = u'(0) = 0.$$

6. Use Exercise 5 to find the solution to

$$u'' - 4u = 1 - h_1(t), \quad u(0) = u'(0) = 0.$$

7. Write an integral expression for the inverse transform of $U(s) = \frac{1}{s}e^{-3s}F(s)$, where $\mathcal{L}[f] = F$.
8. Find a formula for the solution to the initial value problem

$$u'' - u' = f(t), \quad u(0) = u'(0) = 0.$$

9. An integral equation is an equation where the unknown function $u(t)$ appears under an integral sign (see also the exercises in Section 1.2). Consider the integral equation

$$u(t) = f(t) + \int_0^t k(t-\tau)u(\tau)d\tau,$$

where f and k are given functions. Using convolution, find a formula for $U(s)$ in terms of the transforms of F and K of f and k, respectively.

10. Using the idea in the preceding exercise, solve the following integral equations.
    a) $u(t) = t - \int_0^t (t-\tau)u(\tau)d\tau$.
    b) $u(t) = \int_0^t e^{t-\tau}u(\tau)d\tau$.
11. Solve the integral equation for $u(t)$:

$$f(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{u(\tau)}{\sqrt{t-\tau}} d\tau.$$

(Hint: use the gamma function from the Exercise 17 in Section 4.1.)

4.4 Discontinuous Sources

The problems we are solving have the general form

$$\begin{aligned} u'' + bu' + cu &= f(t), \quad t > 0 \\ u(0) &= u_1, \ u'(0) = u_2. \end{aligned}$$

If f is a continuous function, then we can use variation of parameters to find the particular solution; if f has the special form of a polynomial, exponential, sine, or cosine, or sums and products of these forms, we can use the method of undetermined coefficients (judicious guessing) to find the particular solution. If, however, f is a piecewise continuous source with different forms on different intervals, then we would have to find the general solution on each interval and determine the arbitrary constants to match up the solutions at the endpoints of the intervals. This is an algebraically difficult task. However, using Laplace transforms, the task is not so tedious. In this section we present additional examples on how to deal with discontinuous forcing functions.

Example 4.14

As we noted earlier, the Heaviside function is very useful for writing piecewise, or multi-lined, functions in a single line. For example,

$$\begin{aligned} f(t) &= \begin{cases} t, & 0 < t < 1 \\ 2, & 1 \le t \le 3 \\ 0, & t > 3 \end{cases} \\ &= t + (2-t)h_1(t) - 2h_3(t). \end{aligned}$$

The first term switches on the function t at $t = 0$; the second term switches on the function 2 and switches off the function t at $t = 1$; and the last term switches off the function 2 at $t = 3$. By linearity, the Laplace transform of $f(t)$ is given by

$$F(s) = \mathcal{L}[t] + 2\mathcal{L}[h_1(t)] - \mathcal{L}[th_1(t)] - 2\mathcal{L}[h_3(t)].$$

The second and fourth terms are straightforward from Example 4.3, and $\mathcal{L}[t] = 1/s^2$. The third term can be calculated using $\mathcal{L}\left[f(t)h_a(t)\right] = e^{-as}\mathcal{L}[f(t+a)]$. With $f(t) = t$ we have

$$\mathcal{L}\left[th_1(t)\right] = e^{-s}\mathcal{L}[t+1] = \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-s}.$$

Putting all these results together gives

$$F(s) = \frac{1}{s^2} + \frac{2}{s}e^{-s} - \left(\frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-s}\right) - \frac{2}{s}e^{-3s}.$$

Example 4.15

Solve the initial value problem

$$u'' + 9u = e^{-0.5t}h_4(t), \quad u(0) = u'(0) = 0,$$

where the forcing term is an exponential decaying term that switches on at time $t = 4$. The Laplace transform of the forcing term is

$$\mathcal{L}[e^{-0.5t}h_4(t)] = e^{-4s}\mathcal{L}[e^{-0.5(t+4)}] = e^{-2}\frac{1}{s+0.5}e^{-4s}.$$

Then, taking the transform of the the equation,

$$s^2U(s) + 9U(s) = e^{-2}\frac{1}{s+0.5}e^{-4s}.$$

Whence

$$U(s) = e^{-2}\frac{1}{(s+0.5)(s^2+9)}e^{-4s}.$$

Now we need the shift theorem. But first we find the inverse transform of $\frac{1}{(s+0.5)(s^2+9)}$. Here we leave it as an exercise (partial fractions) to show

$$\mathcal{L}^{-1}\left[\frac{1}{(s+0.5)(s^2+9)}\right] = \frac{3e^{-0.5t} - 3\cos 3t + 0.5\sin 3t}{27.75}.$$

Therefore, by the shift property,

$$\begin{aligned} u(t) &= e^{-2}\mathcal{L}^{-1}\left[\frac{e^{-4s}}{(s+0.5)(s^2+9)}\right] \\ &= h_4(t)\frac{3e^{-0.5(t-4)} - 3\cos 3(t-4) + 0.5\sin 3(t-4)}{27.75e^2}, \end{aligned}$$

which is the solution. Notice that the solution does not switch on until $t = 4$. At that time the forcing term turns on, producing a transient; eventually its effects decay away and an oscillating steady-state takes over.

EXERCISES

1. Sketch the function $f(t) = 2h_3(t) - 2h_4(t)$ and find its Laplace transform.
2. Find the Laplace transform of $f(t) = t^2h_3(t)$.
3. Invert $F(s) = (s-2)^{-4}$.
4. Sketch the following function, write it as a single expression, and then find its transform:

$$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 2, & 2 \le t < \pi \\ 6, & \pi \le t \le 7 \\ 0, & t > 7. \end{cases}$$

5. Find the inverse transform of
$$U(s) = \frac{1 - e^{-4s}}{s^2}.$$

6. Solve the initial value problem
$$u'' + 4u = \begin{cases} \cos 2t, & 0 \leq t \leq 2\pi, \\ 0, & t > 2\pi, \end{cases}$$
where $u(0) = u'(0) = 0$. Sketch the solution.

7. Consider the initial value problem $u' = u + f(t)$, $u(0) = 1$, where $f(t)$ is given by
$$f(t) = \begin{cases} 0, & 0 < t \leq 1 \\ -2, & t > 1. \end{cases}$$
Solve this problem in two ways: (a) by solving the problem on two intervals and pasting together the solutions in a continuous way, and (b) by Laplace transforms.

8. An LC circuit with $L = C = 1$ is "ramped-up" with an applied voltage
$$e(t) = \begin{cases} t, & 0 \leq t \leq 9 \\ 9, & t > 9. \end{cases}$$
Initially there is no charge on the capacitor and no current. Find and sketch a graph of the voltage response on the capacitor.

9. Solve $u' = -u + h_1(t) - h_2(t)$, $u(0) = 1$.

10. Solve the initial value problem
$$u'' + \pi^2 u = \begin{cases} \pi^2, & 0 < t < 1, \\ 0, & t > 1, \end{cases}$$
where $u(0) = 1$ and $u'(0) = 0$.

11. Let $f(t)$ be a periodic function with period p. That is, $f(t + p) = f(t)$ for all $t > 0$. Show that the Laplace transform of f is given by
$$F(s) = \frac{1}{1 - e^{-ps}} \int_0^p f(r) e^{-rs} dr.$$
(Hint: break up the interval $(-\infty, +\infty)$ into subintervals $(np, (n+1)p)$, calculate the transform on each subinterval, and finally use the geometric series $1 + x + x^2 + \cdots = \frac{1}{1-x}$.)

12. Show that the Laplace transform of the periodic, square-wave function that takes the value 1 on intervals $[0, a)$, $[2a, 3a)$, $[4a, 5a)$,..., and the value -1 on the intervals $[a, 2a)$, $[3a, 4a)$, $[5a, 6a)$,..., is $\frac{1}{s} \tanh\left(\frac{as}{2}\right)$.

13. Write a single line formula for the function that is 2 between $2n$ and $2n+1$, and 1 between $2n - 1$ and $2n$, where $n = 0, 1, 2, 3, 4, ...$.

4.5 Point Sources

Many physical and biological processes have source terms that act at a single instant of time. For example, we can idealize an injection of medicine (a "shot") into the blood stream as occurring at a single instant; a mechanical system, for example, a damped spring-mass system in a shock absorber on a car, can be given an impulsive force by hitting a bump in the road; an electrical circuit can be closed only for an instant, which leads to an impulsive, applied voltage.

To fix the idea, let us consider an RC circuit with a given emf $e(t)$ and with no initial charge on the capacitor. In terms of the charge $q(t)$ on the capacitor, the governing circuit equation is

$$Rq' + \frac{1}{C}q = e(t), \quad q(0) = 0. \tag{4.8}$$

This is a linear first-order equation, and if the emf is a continuous function, or piecewise continuous function, the problem can be solved by the methods presented in Chapter 2 or by transform methods. We use the latter. Taking Laplace transforms and solving for $Q(s)$, the Laplace transform of $q(t)$, gives

$$Q(s) = \frac{1}{R}\frac{1}{s + 1/RC}E(s),$$

where $E(s)$ is the transform of the emf $e(t)$. Using the convolution property we have the solution

$$q(t) = \frac{1}{R}\int_0^t e^{-(t-\tau)/RC}e(\tau)d\tau. \tag{4.9}$$

But presently we want to consider a special type of electromotive force $e(t)$, one given by an voltage impulse that acts only for a single instant (i.e., a quick surge of voltage). To fix the idea, suppose the source is a 1 volt battery. Imagine that the circuit is open and we just touch the leads together at a single instant at time $t = a$. How does the circuit respond? We denote this unit voltage input by $e(t) = \delta_a(t)$, which is called a **unit impulse** at $t = a$. The question is how to define $\delta_a(t)$, an energy source that acts at a single instant of time. At first it *appears* that we should take $\delta_a(t) = 1$ if $t = a$, and $\delta_a(t) = 0$, otherwise. But this is not correct. To illustrate, we can substitute into (4.9) and write

$$q(t) = \frac{1}{R}\int_0^t e^{-(t-\tau)/RC}\delta_a(\tau)d\tau. \tag{4.10}$$

If $\delta_a(t) = 0$ at all values of t, except $t = a$, the integral must be zero because the integrand is zero except at a single point. Hence, the response of the circuit is $q(t) = 0$, which is incorrect! Something is clearly wrong with this argument and our tentative definition of $\delta_a(t)$.

The difficulty is with the "function" $\delta_a(t)$. We must come to terms with the idea of an impulse. Actually, having the source act at a single instant of time is an idealization. Rather, such a short impulse must occur over a very small interval $[a-\varepsilon/2, a+\varepsilon/2]$, where ε is a small positive number. We do not know the actual form of the applied voltage over this interval, but we want its average value over the interval to be 1 volt. Therefore, let us define an *idealized* applied voltage by

$$\begin{aligned} e_{a,\varepsilon}(t) &= \begin{cases} \frac{1}{\varepsilon}, & a-\varepsilon/2 < t < a+\varepsilon/2 \\ 0, & \text{otherwise}, \end{cases} \\ &= \frac{1}{\varepsilon}(h_{a-\varepsilon/2}(t) - h_{a+\varepsilon/2}(t)). \end{aligned}$$

These idealized impulses are rectangular voltage inputs that get taller and narrower (of height $1/\varepsilon$ and width ε) as ε gets small. But their average value over the small interval $a-\varepsilon/2 < t < a+\varepsilon/2$ is always 1; that is,

$$\int_{a-\varepsilon/2}^{a+\varepsilon/2} e_{a,\varepsilon}(t)dt = 1.$$

This property should hold for all ε, regardless of how small. It seems reasonable therefore to define the unit impulse $\delta_a(t)$ at $t = a$ in a limiting sense, having the property

$$\int_{a-\varepsilon/2}^{a+\varepsilon/2} \delta_a(t)dt = 1, \quad \text{for all } \varepsilon > 0.$$

Engineers and scientists used this condition, along with $\delta_a(t) = 0,\ \ t \neq a$, for decades to define a unit, point source at time $t = a$, called the **delta function**, and they developed a calculus that was successful in obtaining solutions to equations having point sources. But, actually, the unit impulse is not a function at all, and it was shown in the mid-20th century that the unit impulse belongs to a class of so-called **generalized functions** whose actions are not defined pointwise, but rather by how they act when integrated against other functions. Mathematically, the unit impulse is defined by the **sifting property**

$$\int_0^\infty \delta_a(t)\phi(t)dt = \phi(a).$$

That is, when integrated against any nice function $\phi(t)$, the delta function picks out the value of $\phi(t)$ at $t = a$. We check that this works in our problem. If we use this sifting property back in (4.10), then for $t > a$ we have

$$q(t) = \frac{1}{R}\int_0^t e^{-(t-\tau)/RC}\delta_a(\tau)d\tau = \frac{1}{R}e^{-(t-a)/RC}, \quad t > a,$$

which is the correct solution. Note that $q(t) = 0$ up until $t = a$, because there is no source. Furthermore, $q(a) = 1/R$. Therefore the charge is zero up to time a, at which it jumps to the value $1/R$, and then decays away.

To deal with differential equations involving impulses we can use Laplace transforms in a formal way. Using the sifting property, with $\phi(t) = e^{-st}$, we obtain

$$\mathcal{L}[\delta_a(t)] = \int_0^\infty \delta_a(t) e^{-st} dt = e^{-as},$$

which is a formula for the Laplace transform of the unit impulse function. This gives, of course, the inverse formula

$$\mathcal{L}^{-1}[e^{-as}] = \delta_a(t).$$

The previous discussion is highly intuitive and lacks a careful mathematical base. However, the ideas can be made precise and rigorous. We refer to advanced texts for a thorough treatment of generalized functions. Another common notation for the unit impulse $\delta_a(t)$ is $\delta(t-a)$. If an impulse has magnitude f_0, instead of 1, then we denote it by $f_0\delta_a(t)$. For example, an impulse given a circuit by a 12 volt battery at time $t = a$ is $12\delta_a(t)$.

Example 4.16

Solve the initial value problem

$$u'' + u' = \delta_2(t), \quad u(0) = u'(0) = 0,$$

with a unit impulse applied at time $t = 2$. Taking the transform,

$$s^2 U(s) + sU(s) = e^{-2s}.$$

Thus

$$U(s) = \frac{e^{-2s}}{s(s+1)}.$$

Using the table it is simple to find

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = 1 - e^{-t}.$$

Therefore, by the shift property, the solution is

$$u(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s(s+1)}\right] = (1 - e^{-(t-2)})h_2(t).$$

The initial conditions are zero, and so the solution is zero up until time $t = 2$, when the impulse occurs. At that time the solution increases with limit 1 as $t \to \infty$. See figure 4.4.

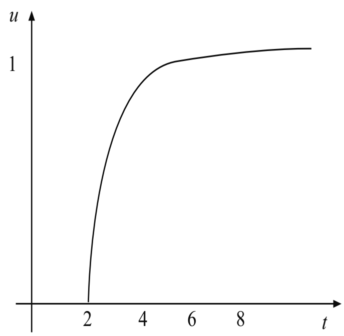

Figure 4.4 Solution in Example 4.16.

EXERCISES

1. Compute $\int_0^\infty e^{-2(t-3)^2}\delta_4(t)dt$.

2. Solve the initial value problem

$$\begin{aligned} u' + 3u &= \delta_1(t) + h_4(t), \\ u(0) &= 1. \end{aligned}$$

Sketch the solution.

3. Solve the initial value problem

$$\begin{aligned} u'' - u &= \delta_5(t), \\ u(0) &= u'(0) = 0. \end{aligned}$$

Sketch the solution.

4. Solve the initial value problem

$$\begin{aligned} u'' + u &= \delta_2(t), \\ u(0) &= u'(0) = 0. \end{aligned}$$

Sketch the solution.

5. Invert the transform $F(s) = \frac{e^{-2s}}{s} + e^{-3s}$.

6. Solve the initial value problem

$$\begin{aligned} u'' + 4u &= \delta_2(t) - \delta_5(t), \\ u(0) &= u'(0) = 0. \end{aligned}$$

7. Consider an LC circuit with $L = C = 1$ and $v(0) = v'(0) = 0$, containing a 1 volt battery, where v is the voltage across the capacitor. At each of the times $t = 0, \pi, 2\pi, 3\pi, ..., n\pi, ...$ the circuit is closed for a single instant. Determine the resulting voltage response $v(t)$ on the capacitor.

8. Compute the Laplace transform of the unit impulse in a different way from that in this section by calculating the transform of $e_{a,\varepsilon}(t)$, and then taking the limit as $\varepsilon \to 0$. Specifically, show

$$\mathcal{L}[e_{a,\varepsilon}(t)] = \mathcal{L}[\frac{1}{\varepsilon}(h_{a-\varepsilon/2}(t) - h_{a+\varepsilon/2}(t))] = \frac{1}{s}e^{-as}\frac{2\sinh\frac{\varepsilon s}{2}}{\varepsilon}.$$

Then use l'Hospital's rule to compute the limit

$$\lim_{\varepsilon \to 0} \frac{2\sinh\frac{\varepsilon s}{2}}{\varepsilon} = s,$$

thereby showing

$$\mathcal{L}[\delta_a(t)] = e^{-as}.$$

4.6 Table of Laplace Transforms

$u(t)$	$U(s)$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, 3, ...$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$
$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$\frac{1}{a-b}(e^{at} - e^{bt})$	$\frac{1}{(s-a)(s-b)}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$u'(t)$	$sU(s) - u(0)$
$u''(t)$	$s^2U(s) - su(0) - u'(0)$
$u^{(n)}(t)$	$s^nU(s) - s^{n-1}u(0) - \cdots - u^{(n-1)}(0)$
$u(at)$	$\frac{1}{a}U(\frac{s}{a})$
$h_a(t)$	$\frac{1}{s}e^{-as}$
$u(t)e^{at}$	$U(s-a)$
$\delta_a(t)$	e^{-as}
$h_a(t)u(t-a)$	$U(s)e^{-as}$
$\int_0^t u(\tau)v(t-\tau)d\tau$	$U(s)V(s)$
$f(t)h_a(t)$	$e^{-as}\mathcal{L}[f(t+a)]$

5
Linear Systems

Up until now we have focused upon a single differential equation with one unknown state function. Yet, most physical systems require several states for their characterization. Therefore, we are naturally led to study several differential equations for several unknowns. Typically, we expect that if there are n unknown states, then there will be n differential equations, and each DE will contain many of the unknown state functions. Thus the equations are coupled together in the same way as simultaneous systems of algebraic equations. If there are n simultaneous differential equations in n unknowns, we call the set of equations an n-dimensional system.

5.1 Introduction

A two-dimensional, linear, homogeneous system of differential equations has the form

$$x' = ax + by, \tag{5.1}$$

$$y' = cx + dy, \tag{5.2}$$

where a, b, c, and d are constants, and where x and y are the unknown states. A solution consists of a pair of functions $x = x(t)$, $y = y(t)$, that, when substituted into the equations, reduce the equations to identities. We can interpret the solution geometrically in two ways. First, we can plot $x = x(t)$ and $y = y(t)$

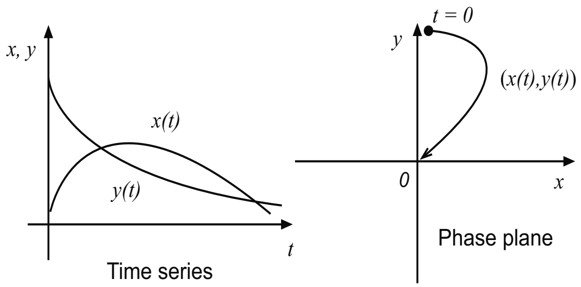

Figure 5.1 Plots showing the two representations of a solution to a system for $t \geq 0$. The plot to the left shows the time series plots $x = x(t)$, $y = y(t)$, and the plot to the right shows the corresponding orbit in the xy-phase plane.

vs. t on the same set of axes as shown in figure 5.1. These are the time series plots and they tell us how the states x and y vary in time. Or, second, we can think of $x = x(t)$, $y = y(t)$ as parametric equations of a curve in an xy plane, with time t as the parameter along the curve. See figure 5.1. In this latter context, the parametric solution representation is called an **orbit**, and the xy plane is called the **phase plane**. Other words used to describe a solution curve in the phase plane, in addition to orbit, are **solution curve**, **path**, and **trajectory**. These words are often used interchangeably. In multi-variable calculus the reader probably used the position vector $\mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ to represent this orbit, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors, but here we use the column vector notation

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

To save vertical space in typesetting, we often write this column vector as $(x(t), y(t))^{\mathrm{T}}$, where "T" denotes transpose; transpose means turn the row into a column. Mostly we use the phase plane representation of a solution rather than the time series representation.

The linear system (5.1)–(5.2) has infinitely many orbits, each defined for all times $-\infty < t < \infty$. When we impose **initial conditions**, which take the form

$$x(t_0) = x_0, \quad y(t_0) = y_0,$$

then a single orbit is selected out. That is, the **initial value problem**, consisting of the system (5.1)–(5.2) and the initial conditions, has a unique solution.

Equations (5.1)–(5.2) also give geometrical information about the direction of the solution curves in the phase plane in much the same way as the slope field of a single differential equation gives information about the slopes of a solution curve (see Section 1.1.2). At any point (x, y) in the xy plane, the right sides of (5.1)–(5.2) define a vector

$$\mathbf{v} = \mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix},$$

which is the tangent vector to the solution curve that goes through that point. Recall from multi-variable calculus that a curve $(x(t), y(t))^{\mathrm{T}}$ has tangent vector $\mathbf{v} = (x'(t), y'(t))^{\mathrm{T}}$. We can plot, or have software plot for us, this vector at a large set of points in the plane to obtain a vector field (a field of vectors) that indicates the "flow", or direction, of the solution curves, as shown in figure 5.2. The orbits fit in so that their tangent vectors coincide with the vector field. A diagram showing several key orbits is called a **phase diagram**, or **phase portrait,** of the system (5.1)–(5.2). The phase portrait may, or may not, include the vector field.

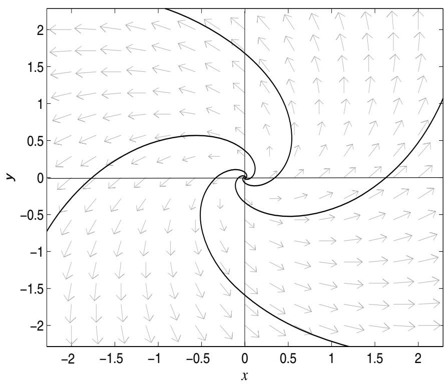

Figure 5.2 In the phase plane, the vector field $\mathbf{v} = (x - y, x + y)^{\mathrm{T}}$ associated with the system $x' = x - y$, $y' = x + y$ and several solution curves $x = x(t)$, $y = y(t)$ which spiral out from the origin. The vector field is tangent to the solution curves. The orbits approach infinity as time goes forward, i.e., $t \to +\infty$, and they approach the origin (but never reach it) as time goes backward, i.e., $t \to -\infty$.

Example 5.1

We observed in Chapter 3 that a second-order differential equation can be reformulated as a system of two first-order equations. For example, the damped, spring-mass equation

$$mx'' = -kx - cx'$$

can be rewritten as

$$\begin{aligned} x' &= y, \\ y' &= -\frac{k}{m}x - \frac{c}{m}y, \end{aligned}$$

where x is position or displacement of the mass from equilibrium and y is its velocity. This system has the form of a two-dimensional linear system. In this manner, mechanical problems can be studied as linear systems. With specific physical parameters $k = m = 1$ and $c = 0.5$, we obtain

$$\begin{aligned} x' &= y, \\ y' &= -x - 0.5y. \end{aligned}$$

The response of this damped spring-mass system is a decaying oscillation. Figure 5.3 shows a phase diagram.

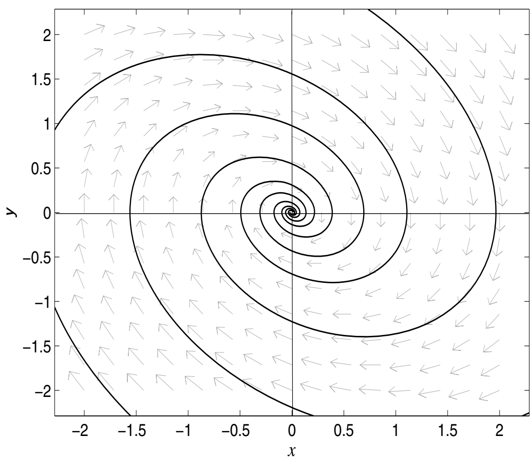

Figure 5.3 Phase plane diagram and vector field for the system $x' = y$, $y' = -x - 0.5y$, showing several orbits, which are spirals approaching the origin. These spirals correspond to time series plots of x and y vs t that oscillate and decay.

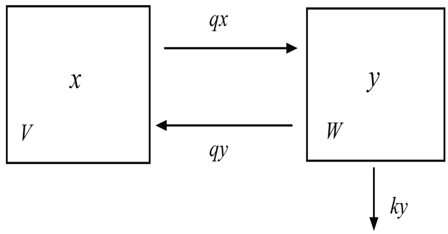

Figure 5.4 Two compartments with arrows indicating the flow rates between them, and the decay rate.

Example 5.2

In this example we generalize ideas presented in Section 1.3.5 on chemical reactors, and the reader should review that material before continuing. The idea here is to consider linked chemical reactors, or several different compartments. Compartmental models play an important role in many areas of science and technology, and they often lead to linear systems. The compartments may be reservoirs, organs, the blood system, industrial chemical reactors, or even classes of individuals. Consider the two compartments in figure 5.4 where a chemical in compartment 1 (having volume V liters) flows into compartment 2 (of volume W liters) at the rate of q liters per minute. In compartment 2 it decays at a rate proportional to its concentration, and it flows back into compartment 1 at the same rate q. At all times both compartments are stirred thoroughly to guarantee perfect mixing. Here we could think of two lakes, or the blood system and an organ. Let x and y denote the concentrations of the chemical in compartments 1 and 2, respectively. We measure concentrations in grams per liter. The technique for finding model equations that govern the concentrations is the same as that noted in Section 1.3.5. Namely, use mass balance. Regardless of the number of compartments, mass must be balanced in each one: the rate of change of mass must equal the rate that mass flows in or is created, minus the rate that mass flows out or is consumed. The mass in compartment 1 is Vx, and the mass in compartment 2 is Wy. A mass flow rate (mass per time) equals the volumetric flow rate (volume per time) times the concentration (mass per volume). So the mass flow rate from compartment 1 to 2 is qx and the mass flow rate from compartment 2 to 1 is qy. Therefore, balancing rates gives

$$\begin{aligned} Vx' &= -qx + qy, \\ Wy' &= qx - qy - Wky, \end{aligned}$$

where k is the decay rate (grams per volume per time) in compartment 2. The volume W must appear as a factor in the decay rate term to make the dimensions correct. We can write this system as

$$\begin{aligned} x' &= -\frac{q}{V}x + \frac{q}{V}y, \\ y' &= \frac{q}{W}x - \left(\frac{q}{W} + k\right)y, \end{aligned}$$

which is the form of a two dimensional linear system. In any compartment model the key idea is to account for all sources and sinks in each compartment. If the volumetric flow rates are not the same, then the problem is complicated by variable volumes, making the problem nonhomogeneous and time dependent. But the technique for obtaining the model equations is the same.

EXERCISES

1. Verify that $\mathbf{x}(t) = (\cos 2t, -2\sin 2t)^{\mathrm{T}}$ is a solution to the system

   $$x' = y, \quad y' = -4x.$$

   Sketch a time series plot of the solution and the corresponding orbit in the phase plane. Indicate the direction of the orbit as time increases. By hand, plot several vectors in the vector field to show the direction of solution curves.

2. Verify that

   $$\mathbf{x}(t) = \begin{pmatrix} 2e^t \\ -3e^t \end{pmatrix}$$

   is a solution to the linear system

   $$x' = 4x + 2y, \quad y' = -3x - y.$$

   Plot this solution in the xy plane for $t \in (-\infty, \infty)$ and find the tangent vectors along the solution curve. Plot the time series on $-\infty < t < +\infty$.

3. Consider the linear system

   $$x' = -x + y, \quad y' = 4x - 4y,$$

   with initial conditions $x(0) = 10$, $y(0) = 0$. Find formulas for the solution $x(t), y(t)$, and plot their time series. (Hint: multiply the first equation by 4 and add the two equations.)

4. In Example 5.2 let $V = 1$ liter, $W = 0.5$ liters, $q = 0.05$ liters per minute, with no decay in compartment 2. If $x(0) = 10$ grams per liter and $y(0) = 0$, find the concentrations $x(t)$ and $y(t)$ in the two compartments. (Hint: add appropriate multiples of the equations.) Plot the time series and the corresponding orbit in the phase plane. What are the concentrations in the compartments after a long time?

5. In Exercise 4 assume there is decay in compartment 2 with $k = 0.2$ grams per liter per minute. Find the concentrations and plot the time series graphs and the phase plot. (Hint: transform the system into a single second-order equation for x.)

6. In the damped spring-mass system in Example 5.1 take $m = 1$, $k = 2$, and $c = \frac{1}{2}$, with initial conditions $x(0) = 4$ and $y(0) = 0$. Find formulas for the position $x(t)$ and velocity $y(t)$. Show the time series and the orbit in the phase plane.

7. Let q and I be the charge and the current in an RCL circuit with no electromotive force. Write down a linear system of first-order equations that govern the two variables q and I. Take $L = 1$, $R = 0$, and $C = \frac{1}{4}$. If $q(0) = 8$ and $I(0) = 0$, find $q(t)$ and $I(t)$. Show a time series plot of the solution and the corresponding orbit in the qI phase plane.

5.2 Matrices

The study of simultaneous differential equations is greatly facilitated by matrices. Matrix theory provides a convenient language and notation to express many of the ideas concisely. Complicated formulas are simplified considerably in this framework, and matrix notation is more or less independent of dimension. In this extended section we present a brief introduction to square matrices. Some of the definitions and properties are given for general n by n matrices, but our focus is on the two- and three-dimensional cases. This section does not represent a thorough treatment of matrix theory, but rather a limited discussion centered on ideas necessary to discuss solutions of differential equations.

A square array A of numbers having n rows and n columns is called a **square matrix** of size n, or an $n \times n$ matrix (we say, "n by n matrix"). The number in the ith row and jth column is denoted by a_{ij}. General 2×2 and 3×3 matrices have the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

The numbers a_{ij} are called the **entries** in the matrix; the first subscript i denotes the row, and the second subscript j denotes the column. The **main diagonal** of a square matrix A is the set of elements $a_{11}, a_{22}, ..., a_{nn}$. We often write matrices using the brief notation $A = (a_{ij})$. An n-**vector x** is a list of n numbers $x_1, x_2, ..., x_n$, written as a *column*; so "vector" means "column list." The numbers $x_1, x_2, ..., x_n$ in the list are called its **components**. For example,

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is a 2-vector. Vectors are denoted by lowercase boldface letters like **x**, **y**, etc., and matrices are denoted by capital letters like A, B, etc. To minimize space in typesetting, we often write, for example, a 2-vector **x** as $(x_1, x_2)^{\mathrm{T}}$, where the T denotes *transpose*, meaning turn the row into a column.

Two square matrices having the same size can be added entry-wise. That is, if $A = (a_{ij})$ and $B = (b_{ij})$ are both $n \times n$ matrices, then the **sum** $A + B$ is an $n \times n$ matrix defined by $A + B = (a_{ij} + b_{ij})$. A square matrix $A = (a_{ij})$ of any size can be multiplied by a constant c by multiplying all the elements of A by the constant; in symbols this **scalar multiplication** is defined by $cA = (ca_{ij})$. Thus $-A = (-a_{ij})$, and it is clear that $A + (-A) = 0$, where 0 is the **zero matrix** having all entries zero. If A and B have the same size, then **subtraction** is defined by $A - B = A + (-B)$. Also, $A + 0 = A$, if 0 has the same size as A. Addition, when defined, is both commutative and associative. Therefore the arithmetic rules of addition for $n \times n$ matrices are the same as the usual rules for addition of numbers.

Similar rules hold for addition of column vectors of the same length and multiplication of column vectors by scalars; these are the definitions you encountered in multi-variable calculus where n-vectors are regarded as elements of $\mathbf{R}^n$. Vectors add component-wise, and multiplication of a vector by a scalar multiplies each component of that vector by that scalar.

Example 5.3

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ 7 & -4 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned} A + B &= \begin{pmatrix} 1 & 0 \\ 10 & -8 \end{pmatrix}, \quad -3B = \begin{pmatrix} 0 & 6 \\ -21 & 12 \end{pmatrix}, \\ 5\mathbf{x} &= \begin{pmatrix} -20 \\ 30 \end{pmatrix}, \quad \mathbf{x}+2\mathbf{y} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}. \end{aligned}$$

The product of two square matrices of the same size is *not* found by multiplying entry-wise. Rather, **matrix multiplication** is defined as follows. Let A and B be two $n \times n$ matrices. Then the matrix AB is defined to be the $n \times n$ matrix $C = (c_{ij})$ where the ij entry (in the ith row and jth column) of the product C is found by taking the product (dot product, as with vectors) of the ith row of A and the jth column of B. In symbols, $AB = C$, where

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj},$$

where $\mathbf{a}_i$ denotes the ith row of A, and $\mathbf{b}_j$ denotes the jth column of B. Generally, matrix multiplication is *not* commutative (i.e., $AB \neq BA$), so the order in which matrices are multiplied is important. However, the associative law $AB(C) = (AB)C$ does hold, so you can regroup products as you wish. The distributive law connecting addition and multiplication, $A(B+C) = AB+AC$, also holds. The powers of a square matrix are defined by $A^2 = AA$, $A^3 = AA^2$, and so on.

Example 5.4

Let

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 5 & 2 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 2\cdot 1 + 3\cdot 5 & 2\cdot 4 + 3\cdot 2 \\ -1\cdot 1 + 0\cdot 5 & -1\cdot 4 + 0\cdot 2 \end{pmatrix} = \begin{pmatrix} 17 & 14 \\ -1 & -4 \end{pmatrix}.$$

Also

$$\begin{aligned} A^2 &= \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2\cdot 2 + 3\cdot(-1) & 2\cdot 3 + 3\cdot 0 \\ -1\cdot 2 + 0\cdot(-1) & -1\cdot 3 + 0\cdot 0 \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ -2 & -3 \end{pmatrix}. \end{aligned}$$

Next we define multiplication of an $n\times n$ matrix A times an n-vector $\mathbf{x}$. The product $A\mathbf{x}$, with the matrix on the left, is defined to be the n-vector whose ith component is $\mathbf{a}_i \cdot \mathbf{x}$. In other words, the ith element in the list $A\mathbf{x}$ is found by taking the product of the ith row of A and the vector $\mathbf{x}$. The product $\mathbf{x}A$ is not defined.

Example 5.5

When $n = 2$ we have

$$A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

For a numerical example take

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

Then

$$A\mathbf{x} = \begin{pmatrix} 2 \cdot 5 + 3 \cdot 7 \\ -1 \cdot 5 + 0 \cdot 7 \end{pmatrix} = \begin{pmatrix} 31 \\ -5 \end{pmatrix}.$$

The special square matrix having ones on the main diagonal and zeros elsewhere else is called the **identity matrix** and is denoted by I. For example, the 2×2 and 3×3 identities are

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that if A is any square matrix and I is the identity matrix of the same size, then $AI = IA = A$. Therefore multiplication by the identity matrix does not change the result, a situation similar to multiplying real numbers by the unit number 1. If A is an $n \times n$ matrix and there exists a matrix B for which $AB = BA = I$, then B is called the **inverse** of A and we denote it by $B = A^{-1}$. If A^{-1} exists, we say A is a **nonsingular** matrix; otherwise it is called **singular**. One can show that the inverse of a matrix, if it exists, is unique. We never write $1/A$ for the inverse of A.

A useful number associated with a square matrix A is its determinant. The **determinant** of a square matrix A, denoted by $\det A$ (also by $|A|$) is a number found by combining the elements of the matrix is a special way. The determinant of a 1×1 matrix is just the single number in the matrix. For a 2×2 matrix we define

$$\det A = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - cb,$$

and for a 3×3 matrix we define

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - bdi - ahf. \tag{5.3}$$

Example 5.6

We have

$$\det \begin{pmatrix} 2 & 6 \\ -2 & 0 \end{pmatrix} = 2 \cdot 0 - (-2) \cdot 6 = 12.$$

There is a general inductive formula that defines the determinant of an $n \times n$ matrix as a sum of $(n-1)\times(n-1)$ matrices. Let $A = (a_{ij})$ be an $n \times n$ matrix, and let M_{ij} denote the $(n-1)\times(n-1)$ matrix found by deleting the ith row and jth column of A; the matrix M_{ij} is called the ij **minor** of A. Then $\det A$ is defined by choosing any fixed column J of A and summing the elements a_{iJ} in that column times the determinants of their minors M_{iJ}, with an associated sign ($\pm$), depending upon location in the column. That is, for any fixed J,

$$\det A = \sum_{i=1}^{n} (-1)^{i+J} a_{iJ} \det(M_{iJ}).$$

This is called the **expansion by minors** formula. One can show that you get the same value regardless of which column J you use. In fact, one can expand on any fixed row I instead of a column and still obtain the same value,

$$\det A = \sum_{j=1}^{n} (-1)^{I+j} a_{Ij} \det(M_{Ij}).$$

So, the determinant is well defined by these equations. The reader should check that these formulas give the values for the 2×2 and 3×3 determinants presented above. A few comments are in order. First, the expansion by minors formulas are useful only for small matrices. For an $n \times n$ matrix, it takes roughly $n!$ arithmetic calculations to compute the determinant using expansion by minors, which is enormous when n is large. Efficient computational algorithms to calculate determinants use row reduction methods. Both computer algebra systems and calculators have routines for calculating determinants.

Using the determinant we can give a simple formula for the inverse of a 2×2 matrix A. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and suppose $\det A \neq 0$. Then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5.4}$$

So the inverse of a 2×2 matrix is found by interchanging the main diagonal elements, putting minus signs on the off-diagonal elements, and dividing by the determinant. There is a similar formula for the inverse of larger matrices; for completeness we will write the formula down, but for the record we comment that there are more efficient ways to calculate the inverse. With that said, the inverse of an $n \times n$ matrix A is the $n \times n$ matrix whose ij entry is $(-1)^{i+j}\det(M_{ji})$, divided by the determinant of A, which is assumed nonzero. In symbols,

$$A^{-1} = \frac{1}{\det A}((-1)^{i+j}\det(M_{ji})). \tag{5.5}$$

Note that the ij entry of A^{-1} is computed from the ji minor, with indices transposed. In the 3×3 case the formula is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} \det M_{11} & -\det M_{21} & \det M_{31} \\ -\det M_{12} & \det M_{22} & -\det M_{32} \\ \det M_{13} & -\det M_{23} & \det M_{33} \end{pmatrix}.$$

Example 5.7

If

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},$$

then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} = \frac{1}{-5} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix}.$$

The reader can easily check that $AA^{-1} = I$.

Equations (5.4) and (5.5) are revealing because they seem to indicate the inverse matrix exists only when the determinant is nonzero (you can't divide by zero). In fact, these two statements are equivalent for any square matrix, regardless of its size: A^{-1} exists if, and only if, $\det A \neq 0$. This is a major theoretical result in matrix theory, and it is a convenient test for invertibility of small matrices. Again, for larger matrices it is more efficient to use row reduction methods to calculate determinants and inverses. The reader should remember the equivalences

$$A^{-1}\text{exists} \Leftrightarrow A \text{ is nonsingular} \Leftrightarrow \det A \neq 0.$$

Matrices were developed to represent and study linear algebraic systems (n linear algebraic equations in n unknowns) in a concise way. For example, consider two equations in two unknowns x_1, x_2 given in standard form by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

Using matrix notation we can write this as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

or just simply as

$$A\mathbf{x} = \mathbf{b}, \tag{5.6}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

A is the **coefficient matrix**, $\mathbf{x}$ is a column vector containing the unknowns, and $\mathbf{b}$ is a column vector representing the right side. If $\mathbf{b} = \mathbf{0}$, the zero vector, then the system (5.6) is called **homogeneous**. Otherwise it is called **nonhomogeneous**. In a two-dimensional system each equation represents a line in the plane. When $\mathbf{b} = \mathbf{0}$ the two lines pass through the origin. A solution vector $\mathbf{x}$ is represented by a point that lies on both lines. There is a unique solution when both lines intersect at a single point; there are infinitely many solutions when both lines coincide; there is no solution if the lines are parallel and different. In the case of three equations in three unknowns, each equation in the system has the form $\alpha x_1 + \beta x_2 + \gamma x_3 = d$ and represents a plane in space. If $d = 0$ then the plane passes through the origin. The three planes represented by the three equations can intersect in many ways, giving no solution (no common intersection points), a unique solution (when they intersect at a single point), a line of solutions (when they intersect in a common line), and a plane of solutions (when all the equations represent the same plane).

The following theorem tells us when a linear system $A\mathbf{x} = \mathbf{b}$ of n equations in n unknowns is solvable. It is a key result that is applied often in the sequel.

Theorem 5.8

Let A be an $n \times n$ matrix. If A is nonsingular, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution given by $\mathbf{x} = A^{-1}\mathbf{b}$; in particular, the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$. If A is singular, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, and the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ may have no solution or infinitely many solutions.

It is easy to show the first part of the theorem, when A is nonsingular, using the machinery of matrix notation. If A is nonsingular then A^{-1} exists. Multiplying both sides of $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1} gives

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b}, \\ I\mathbf{x} &= A^{-1}\mathbf{b}, \\ \mathbf{x} &= A^{-1}\mathbf{b}, \end{aligned}$$

which is the unique solution. If A is singular one can appeal to a geometric argument in two dimensions. That is, if A is singular, then $\det A = 0$, and the two lines represented by the two equations must be parallel (can you show that?). Therefore they either coincide or they do not, giving either infinitely many solutions or no solution. We remark that the method of finding and multiplying by the inverse of the matrix A, as above, is not the most efficient method for solving linear systems. Row reduction methods, introduced in high school algebra (and reviewed below), provide an efficient computational algorithm for solving large systems.

Example 5.9

Consider the homogeneous linear system

$$\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The coefficient matrix has determinant zero, so there will be infinitely many solutions. The two equations represented by the system are

$$4x_1 + x_2 = 0, \quad 8x_1 + 2x_2 = 0,$$

which are clearly not independent; one is a multiple of the other. Therefore we need only consider one of the equations, say $4x_1 + x_2 = 0$. With one equation in two unknowns we are free to pick a value for one of the variables and solve for the other. Let $x_1 = 1$; then $x_2 = -4$ and we get a single solution $\mathbf{x} = (1, -4)^{\mathrm{T}}$. More generally, if we choose $x_1 = \alpha$, where α is any real parameter, then $x_2 = -4\alpha$. Therefore all solutions are given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -4\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \quad a \in \mathbf{R}.$$

Thus all solutions are multiples of $(1, -4)^{\mathrm{T}}$, and the solution set lies along the straight line through the origin defined by this vector. Geometrically, the two equations represent two lines in the plane that coincide.

Next we review the **row reduction method** for solving linear systems when $n = 3$. Consider the algebraic system $A\mathbf{x} = \mathbf{b}$, or

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned} \tag{5.7}$$

At first we assume the coefficient matrix $A = (a_{ij})$ is nonsingular, so that the system has a unique solution. The basic idea is to transform the system into the simpler *triangular form*

$$\begin{aligned} \widetilde{a}_{11}x_1 + \widetilde{a}_{12}x_2 + \widetilde{a}_{13}x_3 &= \widetilde{b}_1, \\ \widetilde{a}_{22}x_2 + \widetilde{a}_{23}x_3 &= \widetilde{b}_2, \\ \widetilde{a}_{33}x_3 &= \widetilde{b}_3. \end{aligned}$$

This triangular system is easily solved by back substitution. That is, the third equation involves only one unknown and we can instantly find x_3. That value is substituted back into the second equation where we can then find x_2, and those two values are substituted back into the first equation and we can find x_1. The process of transforming (5.7) into triangular form is carried out by three admissible operations that do not affect the solution structure.

1. Any equation may be multiplied by a nonzero constant.
2. Any two equations may be interchanged.
3. Any equation may be replaced by that equation plus (or minus) a multiple of any other equation.

We observe that any equation in the system (5.7) is represented by its coefficients and the right side, so we only need work with the numbers, which saves writing. We organize the numbers in an **augmented array**

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{pmatrix}.$$

The admissible operations listed above translate into row operations on the augmented array: any row may be multiplied by a nonzero constant, any two rows may be interchanged, and any row may be replaced by itself plus (or minus) any other row. By performing these row operations we transform the augmented array into a triangular array with zeros in the lower left corner below the main diagonal. The process is carried out one column at a time, beginning from the left.

Example 5.10

Consider the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 0, \\ 2x_1 - 2x_3 &= 2, \\ x_1 - x_2 + x_3 &= 6. \end{aligned}$$

The augmented array is

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{pmatrix}.$$

Begin working on the first column to get zeros in the 2,1 and 3,1 positions by replacing the second and third rows by themselves plus multiples of the first row. So we replace the second row by the second row minus twice the first row and replace the third row by third row minus the first row. This gives

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & 0 & 6 \end{pmatrix}.$$

Next work on the second column to get a zero in the 3,2 position, below the diagonal entry. Specifically, replace the third row by the third row minus the second row:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & 0 & 4 & 4 \end{pmatrix}.$$

This is triangular, as desired. To make the arithmetic easier, multiply the third row by $1/4$ and the second row by $-1/2$ to get

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

with ones on the diagonal. This triangular, augmented array represents the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 0, \\ x_2 + 2x_3 &= -1, \\ x_3 &= 1. \end{aligned}$$

Using back substitution, $x_3 = 1$, $x_2 = -3$, and $x_1 = 2$, which is the unique solution, representing a point $(2, -3, 1)$ in $\mathbf{R}^3$.

If the coefficient matrix A is singular we can end up with different types of triangular forms, for example,

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \quad \begin{pmatrix} 1 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix},$$

where the $*$ denotes an entry. These augmented arrays can be translated back into equations. Depending upon the values of those entries, we will get no solution (the equations are inconsistent) or infinitely many solutions. As examples, suppose there are three systems with triangular forms at the end of the process given by

$$\begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

There would be no solution for the first system (the last row states $0 = 7$), and infinitely many solutions for the second and third systems. Specifically, the second system would have solution $x_3 = 1$ and $x_1 = 0$, with $x_2 = a$, which is arbitrary. Therefore the solution to the second system could be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

with a an arbitrary constant. This represents a line in $\mathbf{R}^3$. A line is a one-dimensional geometrical object described in terms of one parameter. The third system above reduced to $x_1 + 2x_2 = 1$. So we may pick x_3 and x_2 arbitrarily, say $x_2 = a$ and $x_3 = b$, and then $x_1 = 1 - 2a$. The solution to the third system can then be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1-2a \\ a \\ b \end{pmatrix} = a \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

which is a plane in $\mathbf{R}^3$. A plane is a two-dimensional object in $\mathbf{R}^3$ requiring two parameters for its description.

The set of all solutions to a homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the **nullspace** of A. The nullspace may consist of a single point $\mathbf{x} = \mathbf{0}$ when A is nonsingular, or it may be a line or plane passing through the origin in the case where A is singular.

Finally we introduce the notion of independence of column vectors. A set of vectors is said to be a linearly independent set if any one of them cannot be written as a combination of some of the others. We can express this statement mathematically as follows. A set (p of them) of n-vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_p$ is a **linearly independent set** if the equation[1]

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

forces all the constants to be zero; that is, $c_1 = c_2 = \cdots = c_p = 0$. If all the constants are not forced to be zero, then we say the set of vectors is **linearly dependent**. In this case there would be at least one of the constants, say c_r, which is not zero, at which point we could solve for $\mathbf{v}_r$ in terms of the remaining vectors.

Notice that two vectors are independent if one is not a multiple of the other.

[1] A sum of constant multiples of a set of vectors is called a **linear combination** of those vectors.

In the sequel we also need the notion of linear independence for **vector functions**. A vector function in two dimensions has the form of a 2-vector whose entries are functions of time t; for example,

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where t belongs to some interval I of time. The vector function $\mathbf{r}(t)$ is the position vector, and its arrowhead traces out a curve in the plane given by the parametric equations $x = x(t)$, $y = y(t)$, $t \in I$. As observed in Section 5.1, solutions to two-dimensional systems of differential equations are vector functions. Linear independence of a set of n-vector functions $\mathbf{r}_1(t), \mathbf{r}_2(t), ..., \mathbf{r}_p(t)$ on an interval I means that if a linear combination of those vectors is set equal to zero, for all $t \in I$, then the set of constants is forced to be zero. In symbols,

$$c_1\mathbf{r}_1(t)+c_2\mathbf{r}_2(t)+\cdots+c_p\mathbf{r}_p(t) = \mathbf{0}, \;\; t \in I, \quad \text{implies} \quad c_1 = 0, \; c_2 = 0, ..., c_p = 0.$$

Finally, if a matrix has entries that are functions of t, i.e., $A = A(t) = (a_{ij}(t))$, then we define the derivative of the matrix as the matrix of derivatives, or $A'(t) = (a'_{ij}(t))$.

Example 5.11

The two vector functions

$$\mathbf{r}_1(t) = \begin{pmatrix} e^{2t} \\ 7 \end{pmatrix}, \quad \mathbf{r}_2(t) = \begin{pmatrix} 5e^{2t} \\ \sin t \end{pmatrix}$$

form a linearly independent set on the real line because one is not a multiple of the other. Looked at differently, if we set a linear combination of them equal to the zero vector (i.e., $c_1\mathbf{r}_1(t) + c_2\mathbf{r}_2(t) = \mathbf{0}$), and take $t = 0$, then

$$c_1 + 5c_2 = 0, \quad 7c_1 = 0,$$

which forces $c_1 = c_2 = 0$. Because the linear combination is zero for all t, we may take $t = 0$.

Example 5.12

The three vector functions

$$\mathbf{r}_1(t) = \begin{pmatrix} e^{2t} \\ 7 \end{pmatrix}, \quad \mathbf{r}_2(t) = \begin{pmatrix} 5e^{2t} \\ \sin t \end{pmatrix}, \quad \mathbf{r}_3(t) = \begin{pmatrix} 1 \\ 3\sin\frac{t}{2} \end{pmatrix},$$

form a linearly independent set on $\mathbf{R}$ because none can be written as a combination of the others. That is, if we take a linear combination and set it equal to zero; that is, $c_1\mathbf{r}_1(t)+c_2\mathbf{r}_1(t)+c_3\mathbf{r}_1(t)=\mathbf{0}$, for all $t\in\mathbf{R}$, then we are forced into $c_1=c_2=c_3=0$ (see Exercise 15).

EXERCISES

1. Let
$$A=\begin{pmatrix}1&3\\2&4\end{pmatrix},\quad B=\begin{pmatrix}-1&0\\3&7\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}2\\-5\end{pmatrix}.$$
Find $A+B$, $B-4A$, AB, BA, A^2, $B\mathbf{x}$, $AB\mathbf{x}$, A^{-1}, $\det B$, B^3, AI, and $\det(A-\lambda I\ )$, where λ is a parameter.

2. With A given in Exercise 1 and $b=(2,1)^T$, solve the system $A\mathbf{x}=\mathbf{b}$ using A^{-1}. Then solve the system by row reduction.

3. Let
$$A=\begin{pmatrix}0&2&-1\\1&6&-2\\2&0&3\end{pmatrix},\quad B=\begin{pmatrix}1&-1&0\\2&1&4\\-1&-1&1\end{pmatrix},\quad \mathbf{x}=\begin{pmatrix}2\\0\\-1\end{pmatrix}.$$
Find $A+B$, $B-4A$, BA, A^2, $B\mathbf{x}$, $\det A$, AI, $A-3I$, and $\det(B-I)$.

4. Find all values of the parameter λ that satisfy the equation $\det(A-\lambda I)=0$, where A is given in Exercise 1.

5. Let
$$A=\begin{pmatrix}2&-1\\-4&2\end{pmatrix}.$$
Compute $\det A$. Does A^{-1} exist? Find all solutions to $A\mathbf{x}=\mathbf{0}$ and plot the solution set in the plane.

6. Use the row reduction method to determine all values m for which the algebraic system
$$2x+3y=m,\quad -6x-9y=5,$$
has no solution, a unique solution, or infinitely many solutions.

7. Use row reduction to determine the value(s) of m for which the following system has infinitely many solutions.
$$\begin{aligned}x+y&=0,\\2x+y&=0,\\3x+2y+mz&=0.\end{aligned}$$

8. If a square matrix A has all zeros either below its main diagonal or above its main diagonal, show that $\det A$ equals the product of the elements on the main diagonal.

9. Construct simple homogeneous systems $A\mathbf{x} = \mathbf{0}$ of three equations in three unknowns that have: (a) a unique solution, (b) an infinitude of solutions lying on a line in $\mathbf{R}^3$, and (c) an infinitude of solutions lying on a plane in $\mathbf{R}^3$. Is there a case when there is no solution?

10. Let
$$A = \begin{pmatrix} 0 & 2 & -1 \\ 1 & 6 & -2 \\ 2 & 0 & 3 \end{pmatrix}.$$

    a) Find $\det A$ by the expansion by minors formula using the first column, the second column, and the third row. Is A invertible? Is A singular?

    b) Find the inverse of A and use it to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (1, 0, 4)^{\mathrm{T}}$.

    c) Solve $A\mathbf{x} = \mathbf{b}$ in part (b) using row reduction.

11. Find all solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$ if
$$A = \begin{pmatrix} -2 & 0 & 2 \\ 2 & -4 & 0 \\ 0 & 4 & -2 \end{pmatrix}.$$

12. Use the definition of linear independence to show that the 2-vectors $(2, -3)^T$ and $(-4, 8)^T$ are linearly independent.

13. Use the definition to show that the 3-vectors $(0, 1, 0)^{\mathrm{T}}$, $(1, 2, 0)^{\mathrm{T}}$, and $(0, 1, 4)^{\mathrm{T}}$ are linearly independent.

14. Use the definition to show that the 3-vectors $(1, 0, 1)^{\mathrm{T}}$, $(5, -1, 0)^{\mathrm{T}}$, and $(-7, 1, 2)^{\mathrm{T}}$ are linearly dependent.

15. Verify the claim in Example 5.12 by taking two special values of t.

16. Plot each of the following vector functions in the xy plane, where $-\infty < t < +\infty$.
$$\mathbf{r}_1(t) = \begin{pmatrix} 3\cos t \\ 2\sin t \end{pmatrix}, \quad \mathbf{r}_2(t) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} t, \quad \mathbf{r}_3(t) = \begin{pmatrix} t \\ t+1 \end{pmatrix} e^{-t}.$$
    Show that these vector functions form a linearly independent set by setting $c_1\mathbf{r}_1(t) + c_2\mathbf{r}_1(t) + c_3\mathbf{r}_1(t) = \mathbf{0}$ and then choosing special values of t to force the constants to be zero.

17. Show that a 3×3 matrix A is invertible if, and only if, its three columns form an independent set of 3-vectors.

18. Find $A'(t)$ if

$$A(t) = \begin{pmatrix} \cos t & t^2 & 0 \\ 2e^{2t} & 1 & \sin 2t \\ 0 & \sqrt{2t} & \frac{-5}{t^2+1} \end{pmatrix}.$$

5.3 Two-Dimensional Systems

5.3.1 Solutions and Linear Orbits

A two-dimensional linear system of differential equations

$$\begin{aligned} x' &= ax + by, \\ y' &= cx + dy, \end{aligned}$$

where a, b, c, and d are constants, can be written compactly using vectors and matrices. Denoting

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the system can be written

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

or

$$\mathbf{x}'(t) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbf{x}(\mathbf{t}).$$

We often write this simply as

$$\mathbf{x}' = A\mathbf{x}, \tag{5.8}$$

where we have suppressed the understood dependence of $\mathbf{x}$ on t. We briefly reiterate the ideas introduced in the introduction, Section 5.1. A solution to the system (5.8) on an interval is a vector function $\mathbf{x}(t) = (x(t), y(t))^T$, that satisfies the system on the required interval. We can graph $x(t)$ and $y(t)$ vs. t, which gives the state space representation or **time series** plots of the solution. Alternatively, a solution can be graphed as a parametric curve, or vector function, in the xy plane. We call the xy plane the **phase plane**, and we call a solution curve plotted in the xy plane an **orbit**. Observe that a solution is a vector function $\mathbf{x}(t)$ with components $x(t)$ and $y(t)$. In the phase plane, the orbit is represented in parametric form and is traced out as time proceeds. Thus, time is not explicitly displayed in the phase plane representation, but it is a parameter along the orbit. An orbit is traced out in a specific direction as time increases, and we usually denote that direction by an arrow along the curve. Furthermore, time can always be shifted along a solution curve. That is, if $\mathbf{x}(t)$ is a solution, then $\mathbf{x}(t-c)$ is a solution for any real number c and it represents the same solution curve.

Our main objective is to find the phase portrait, or a plot of key orbits of the given system. We are particularly interested in the **equilibrium solutions** of (5.8). These are the *constant* vector solutions $\mathbf{x}^*$ for which $A\mathbf{x}^* = \mathbf{0}$. An equilibrium solution is represented in the phase plane as a point. The vector field vanishes at an equilibrium point. The time series representation of an equilibrium solution is two constant functions. If $\det A \neq 0$ then $\mathbf{x}^* = \mathbf{0}$ is the only equilibrium of (5.8), and it is represented by the origin, $(0,0)$, in the phase plane. We say in this case that the origin is an **isolated equilibrium**. If $\det A = 0$, then there will be an entire line of equilibrium solutions through the origin; each point on the line represents an equilibrium solution, and the equilibria are not isolated. Equilibrium solutions are important because the interesting behavior of the orbits occurs near these solutions. (Equilibrium solutions are also called critical points by some authors.)

Example 5.13

Consider the system

$$\begin{aligned} x' &= -2x - y, \\ y' &= 2x - 5y, \end{aligned}$$

which we write as

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \mathbf{x}.$$

The coefficient determinant is nonzero, so the only equilibrium solution is represented by the origin, $x(t) = 0$, $y(t) = 0$. By substitution, it is straightforward to check that

$$\mathbf{x}_1(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$$

is a solution. Also

$$\mathbf{x}_2(t) = \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-4t}$$

is a solution. Each of these solutions has the form of a constant vector times a scalar exponential function of time t. Why should we expect exponential solutions? The two equations involve both x and y and their derivatives; a solution must make everything cancel out, and so each term must basically have the same form. Exponential functions and their derivatives both have the same form, and therefore exponential functions for both x and y are likely candidates for solutions. We graph these two independent solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ in the phase plane. See figure 5.5. Each solution, or orbit, plots as a ray traced from infinity (as time t approaches $-\infty$) into the origin (as t approaches $+\infty$). The slopes of these ray-like solutions are defined by the constant vectors preceding the scalar exponential factor, the latter of which has the effect of stretching or shrinking the vector. Note that these two orbits approach the origin as time gets large, but they never actually reach it. Another way to look

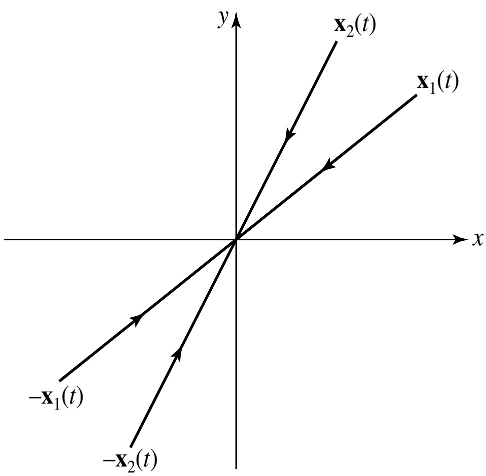

Figure 5.5 $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are shown as linear orbits (rays) entering the origin in the first quadrant. The reflection of those rays in the third quadrant are the solutions $-\mathbf{x}_1(t)$ and $-\mathbf{x}_2(t)$. Note that all four of these linear orbits approach the origin as $t \to +\infty$ because of the decaying exponential factor in the solution. As $t \to -\infty$ (backward in time) all four of these linear orbits go to infinity.

at it is this. If we eliminate the parameter t in the parametric representation $x = e^{-4t}$, $y = 2e^{-4t}$ of $\mathbf{x}_2(t)$, say, then $y = 2x$, which is a straight line in the xy plane. This orbit is on one ray of this straight line, lying in the first quadrant.

Solutions of (5.8) the form $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, where λ is a real constant and $\mathbf{v}$ is a constant, real vector, are called **linear orbits** because they plot as rays in the xy-phase plane.

We are ready to make some observations about the structure of the solution set to the two-dimensional linear system (5.8). All of these properties can be extended to three, or even n, dimensional systems.

1. **(Superposition)** If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are any solutions and c_1 and c_2 are any constants, then the **linear combination** $c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$ is a solution.

2. **(General Solution)** If $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two linear independent solutions (i.e., one is not a multiple of the other), then all solutions are given by $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$, where c_1 and c_2 are arbitrary constants. This combination is called the **general solution** of (5.8).

3. **(Existence-Uniqueness)** The initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(t_0)= \mathbf{x}_0,$$

   where $\mathbf{x}_0$ is a fixed vector, has a unique solution valid for all $-\infty < t < +\infty$.

The existence-uniqueness property actually guarantees that there are two independent solutions to a two-dimensional system. Let $\mathbf{x}_1$ be the unique solution to the initial value problem $\mathbf{x}_1' = A\mathbf{x}_1, \quad \mathbf{x}_1(0)= (1,0)^{\mathrm{T}}$ and $\mathbf{x}_2$ be the unique solution to the initial value problem $\mathbf{x}_2' = A\mathbf{x}_2, \quad \mathbf{x}(0)=(0,1)^{\mathrm{T}}$. These must be independent. Otherwise they would be proportional and we would have

$$\mathbf{x}_1(t) = k\mathbf{x}_2(t),$$

for all t, where k is a nonzero constant. But if we take $t = 0$, we would have

$$(1,0)^T = k(0,1)^T,$$

which is a contradiction.

The question is how to determine two independent solutions so that we can obtain the general solution. This is a central issue we address in the sequel. One method to solve a two-dimensional linear system is to eliminate one of the variables and reduce the problem to a single second-order equation.

Example 5.14

(*Method of Elimination*) Consider

$$\begin{aligned} x' &= 4x - 3y, \\ y' &= 6x - 7y. \end{aligned}$$

Differentiate the first and then use the second to get

$$\begin{aligned} x'' &= 4x' - 3y' = 4(4x-3y) - 3(6x-7y) \\ &= -2x + 9y = -2x + 9(-\frac{1}{3}x' + \frac{4}{3}x) \\ &= -3x' + 10x, \end{aligned}$$

which is a second-order equation. The characteristic equation is $\lambda^2+3\lambda-10=0$ with roots $\lambda = -5, 2$. Thus

$$x(t) = c_1 e^{-5t} + c_2 e^{2t}.$$

Then

$$y(t) = -\frac{1}{3}x' + \frac{4}{3}x = 3c_1 e^{-5t} + \frac{2}{3}c_2 e^{2t}.$$

We can write the solution in vector form as

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} + c_2 \begin{pmatrix} e^{2t} \\ \frac{2}{3}e^{2t} \end{pmatrix}.$$

In this form we can see that two independent vector solutions are

$$\mathbf{x}_1(t) = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} e^{2t} \\ \frac{2}{3}e^{2t} \end{pmatrix},$$

and the general solution is a linear combination of these, $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$. However simple this strategy appears in two dimensions, it does not work as easily in higher dimensions, nor does it expose methods that are easily adaptable to higher-dimensional systems. Therefore we do not often use the elimination method.

But we point out features of the phase plane. Notice that $\mathbf{x}_1$ graphs as a linear orbit in the first quadrant of the xy phase plane, along the ray defined by of the vector $(1,3)^T$. It enters the origin as $t \to \infty$ because of the decaying exponential factor. The other solution, $\mathbf{x}_2$, also represents a linear orbit along the direction defined by the vector $(1, 2/3)^T$. This solution, because of the increasing exponential factor e^{2t}, tends to infinity as $t \to +\infty$. Figure 5.6 shows the linear orbits. Figure 5.7 shows several orbits on the phase diagram obtained by taking different values of the arbitrary constants in the general solution. The structure of the orbital system near the origin, where curves veer away and approach the linear orbits as time goes forward and backward, is called a **saddle point** structure. The linear orbits are sometimes called **separatrices** because they separate different types of orbits. All orbits approach the separatrices as time gets large, either negatively or positively.

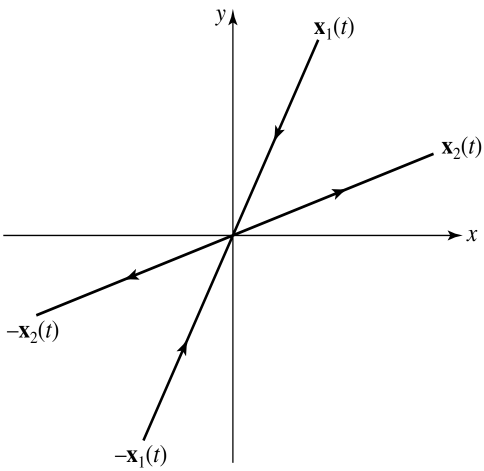

Figure 5.6 Linear orbits in Example 5.14 representing the solutions corresponding to $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, and the companion orbits $-\mathbf{x}_1(t)$ and $-\mathbf{x}_2(t)$. These linear orbits are called separatrices.

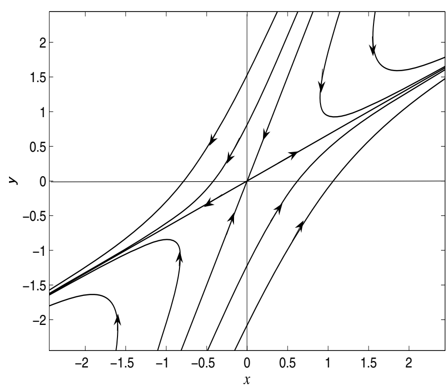

Figure 5.7 Phase portrait for the system showing a saddle point at the origin.

5.3.2 The Eigenvalue Problem

Now we introduce some general methods for the two-dimensional system

$$\mathbf{x}' = A\mathbf{x}. \tag{5.9}$$

We assume that $\det A \neq 0$ so that the only equilibrium solution of the system is at the origin. As examples have shown, we should expect an exponential-type solution. Therefore, we attempt to find a solution of the form

$$\mathbf{x} = \mathbf{v}e^{\lambda t}, \tag{5.10}$$

where λ is a constant and $\mathbf{v}$ is a nonzero constant vector, both to be determined.

Substituting $\mathbf{x} = \mathbf{v}e^{\lambda t}$ and $\mathbf{x}' = \lambda\mathbf{v}e^{\lambda t}$ into the (5.9) gives

$$\lambda\mathbf{v}e^{\lambda t} = A(\mathbf{v}e^{\lambda t}),$$

or

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{5.11}$$

Therefore, if a λ and $\mathbf{v}$ can be found that satisfy (5.11), then we will have determined a solution of the form (5.10). The vector equation (5.11) represents a well-known problem in mathematics called the **algebraic eigenvalue problem**. The eigenvalue problem is to determine values of λ for which (5.11) has a nontrivial solution $\mathbf{v}$. A value of λ for which there is a nontrivial solution $\mathbf{v}$ is called an **eigenvalue**, and a corresponding $\mathbf{v}$ associated with that eigenvalue is called an **eigenvector**. The pair $\lambda, \mathbf{v}$ is called an **eigenpair**. Geometrically we think of the eigenvalue problem like this: A represents a transformation that maps vectors in the plane to vectors in the plane; a vector $\mathbf{x}$ gets transformed to a vector $A\mathbf{x}$. An eigenvector of A is a special vector that is mapped to a multiple (λ) of itself; that is, $Ax = \lambda x$. In summary, we have reduced the problem of finding solutions to a system of differential equations to the problem of finding solutions of an algebra problem—every eigenpair gives a solution.

Geometrically, if λ is real, the linear orbit representing this solution lies along a ray emanating from the origin along the direction defined by the vector $\mathbf{v}$. If $\lambda < 0$ the solution approaches the origin along the ray, and if $\lambda > 0$ the solution goes to infinity along the ray. The situation is similar to that shown in figure 5.6. When there is a solution graphing as a linear orbit, then there is automatically a second, opposite, linear orbit along the ray $-\mathbf{v}$. This is because if $\mathbf{x} = \mathbf{v}e^{\lambda t}$ is a solution, then so is $-\mathbf{x} = -\mathbf{v}e^{\lambda t}$

To solve the eigenvalue problem we rewrite (5.11) as a homogeneous linear system

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{5.12}$$

By Theorem 5.8 this system will have the desired nontrivial solutions if the determinant of the coefficient matrix is zero, or

$$\det(A - \lambda I) = 0. \tag{5.13}$$

Written out explicitly, this system (5.12) has the form

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the coefficient matrix $A - \lambda I$ is the matrix A with λ subtracted from the diagonal elements. Equation (5.13) is, explicitly,

$$\det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = (a-\lambda)(d-\lambda) - cb = 0,$$

or equivalently,

$$\lambda^2 - (a+b)\lambda + (ad - bc) = 0.$$

This last equation can be memorized easily if it is written

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0, \tag{5.14}$$

where $\mathrm{tr}A = a+d$ is called the **trace** of A, defined to be the sum of the diagonal elements of A. Equation (5.14) is called the **characteristic equation** associated with A, and it is a quadratic equation in λ. Its roots, found by factoring or using the quadratic formula, are the two eigenvalues. The eigenvalues may be real and unequal, real and equal, or complex conjugates.

Once the eigenvalues are computed, we can substitute them in turn into the system (5.12) to determine corresponding eigenvectors $\mathbf{v}$. Note that any multiple of an eigenvector is again an eigenvector for that same eigenvalue; this follows from the calculation

$$A(c\mathbf{v}) = cA\mathbf{v} = c(\lambda \mathbf{v}) = \lambda(c\mathbf{v}).$$

Thus, an eigenvector corresponding to a given eigenvalue is not unique; we may multiply them by constants. This is expected from Theorem 5.8. Some calculators display normalized eigenvectors (of length one) found by dividing by their length.

As noted, the eigenvalues may be real and unequal, real and equal, or complex numbers. We now discuss these different cases.

5.3.3 Real Unequal Eigenvalues

If the two eigenvalues are real and unequal, say λ_1 and λ_2, then corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent and we obtain two independent solutions $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$. The general solution of the system is then a linear combination of these two independent solutions,

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t},$$

where c_1 and c_2 are arbitrary constants. Each of the independent solutions represents linear orbits in the phase plane, which helps in plotting the phase diagram. All solutions (orbits) $\mathbf{x}(t)$ are linear combinations of the two independent solutions, with each specific solution obtained by fixing values of the arbitrary constants.

Example 5.15

Consider the linear system

$$\mathbf{x}' = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & -1 \end{pmatrix} \mathbf{x}. \tag{5.15}$$

The characteristic equation (5.14) is

$$\lambda^2 + \frac{5}{2}\lambda + 1 = 0.$$

By the quadratic formula the eigenvalues are

$$\lambda = -\frac{1}{2}, -2.$$

Now we take each eigenvalue successively and substitute it into (5.12) to obtain corresponding eigenvectors. First, for $\lambda = -\frac{1}{2}$, we get

$$\begin{pmatrix} -1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has a solution $(v_1, v_2)^{\mathrm{T}} = (1, 2)^{\mathrm{T}}$. Notice that any multiple of this eigenvector is again an eigenvector, but all we need is one. Therefore an eigenpair is

$$-\frac{1}{2}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Now take $\lambda = -2$. The system (5.12) becomes

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which has solution $(v_1, v_2)^{\mathrm{T}} = (-1, 1)^{\mathrm{T}}$. Thus, another eigenpair is

$$-2, \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The two eigenpairs give two independent solutions

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}. \tag{5.16}$$

Each one plots, along with its negative counterparts, as a linear orbit in the phase plane entering the origin as time increases. The general solution of the system (5.15) is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}.$$

This is a two-parameter family of solution curves, and the totality of all these solution curves, or orbits, represents the phase diagram in the xy plane. These orbits are shown in figure 5.8. Because both terms in the general solution decay as time increases, all orbits enter the origin as $t \to +\infty$. And, as t gets large, the term with $e^{-t/2}$ dominates the term with e^{-2t}. Therefore all orbits approach the origin along the direction $(1, 2)^{\mathrm{T}}$. As $t \to -\infty$ the orbits go to infinity; for large negative times the term e^{-2t} dominates the term $e^{-t/2}$, and the orbits become parallel to the direction $(-1, 1)^{\mathrm{T}}$. Each of the two basic solutions 5.16 represents linear orbits along rays in the directions of the eigenvectors. When both eigenvalues are negative, as in this case, all orbits approach the origin in the direction of one of the eigenvectors. When we obtain this type of phase plane structure, we call the origin an **asymptotically stable node**. When both eigenvalues are positive, then the time direction along the orbits is reversed and we call the origin an **unstable node**. The meaning of the term stable is discussed subsequently.

An initial condition picks out one of the many orbits by fixing values for the two arbitrary constants. For example, if $\mathbf{x}(0) = (1, 4)^{\mathrm{T}}$, or we want an orbit passing through the point $(1, 4)$, then

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix},$$

giving $c_1 = 5/3$ and $c_2 = 2/3$. Therefore the unique solution to the initial value problem is

$$\begin{aligned} \mathbf{x}(t) &= \frac{5}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/2} + \frac{2}{3} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \\ &= \begin{pmatrix} \frac{5}{3} e^{-t/2} - \frac{2}{3} e^{-2t} \\ \frac{10}{3} e^{-t/2} + \frac{2}{3} e^{-2t} \end{pmatrix}. \end{aligned}$$

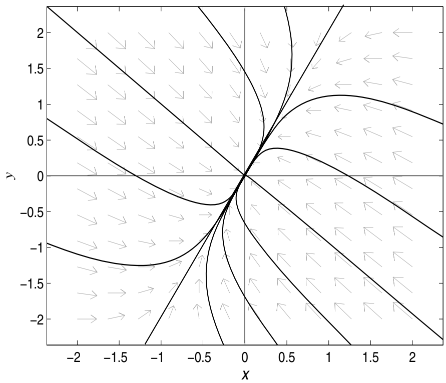

Figure 5.8 A node. All orbits approach the origin, tangent to the direction $(1, 2)$, as $t \to \infty$. Backwards in time, as $t \to -\infty$, the orbits become parallel to the direction $(-1, -1)$. Notice the linear orbits.

Example 5.16

If a system has eigenpairs

$$-2, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad 3, \begin{pmatrix} -1 \\ 5 \end{pmatrix},$$

with real eigenvalues of opposite sign, then the general solution is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{3t}.$$

In this case one of the eigenvalues is positive and one is negative. Now there are two sets of opposite linear orbits, one pair corresponding to -2 approaching the origin from the directions $\pm(3, 2)^{\mathrm{T}}$, and one pair corresponding to $\lambda = 3$ approaching infinity along the directions $\pm(-1, 5)^{\mathrm{T}}$. The orbital structure is that of a **saddle point** (refer to figure 5.7), and we anticipate saddle point structure when the eigenvalues are real and have opposite sign.

5.3.4 Complex Eigenvalues

If the eigenvalues of the matrix A are complex, they must appear as complex conjugates, or $\lambda = a \pm bi$. The eigenvectors will be $\mathbf{v} = \mathbf{w} \pm i\mathbf{z}$. Therefore, taking the eigenpair $a + bi$, $\mathbf{w} + i\mathbf{z}$, we obtain the *complex* solution

$$(\mathbf{w} + i\mathbf{z})e^{(a+bi)t}.$$

Recalling that the real and imaginary parts of a complex solution are real solutions, we expand this complex solution using Euler's formula to get

$$\begin{aligned}(\mathbf{w} + i\mathbf{z})e^{at}e^{ibt} &= e^{at}(\mathbf{w} + i\mathbf{z})(\cos bt + i\sin bt) \\ &= e^{at}(\mathbf{w}\cos bt - \mathbf{z}\sin bt) + ie^{at}(\mathbf{w}\sin bt + \mathbf{z}\cos bt).\end{aligned}$$

Therefore two *real*, independent solutions are

$$\mathbf{x}_1(t) = e^{at}(\mathbf{w}\cos bt - \mathbf{z}\sin bt), \quad \mathbf{x}_2(t) = e^{at}(\mathbf{w}\sin bt + \mathbf{z}\cos bt),$$

and the general solution is a combination of these,

$$\mathbf{x}(t) = c_1 e^{at}(\mathbf{w}\cos bt - \mathbf{z}\sin bt) + c_2 e^{at}(\mathbf{w}\sin bt + \mathbf{z}\cos bt). \tag{5.17}$$

In the case of complex eigenvalues we need not consider both eigenpairs; each eigenpair leads to the same two independent solutions. For complex eigenvalues there are no linear orbits. The terms involving the trigonometric functions are periodic functions with period $2\pi/b$, and they define orbits that rotate around the origin. The factor e^{at} acts as an amplitude factor causing the rotating orbits to expand if $a > 0$, and we obtain spiral orbits going away from the origin. If $a < 0$ the amplitude decays and the spiral orbits go into the origin. In the complex eigenvalue case we say the origin is an **asymptotically stable spiral point** when $a < 0$, and an **unstable spiral point** when $a > 0$.

If the eigenvalues of A are purely imaginary, $\lambda = \pm bi$, then the amplitude factor e^{at} in (5.17) is absent and the solutions are periodic of period $\frac{2\pi}{b}$, given by

$$\mathbf{x}(t) = c_1(\mathbf{w}\cos bt - \mathbf{z}\sin bt) + c_2(\mathbf{w}\sin bt + \mathbf{z}\cos bt).$$

The orbits are closed cycles and plot as either concentric ellipses or concentric circles. In this case we say the origin is a **(neutrally) stable center**.

Example 5.17

Let

$$\mathbf{x}' = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}\mathbf{x}.$$

The matrix A has eigenvalues $\lambda = -2 \pm 3i$. An eigenvector corresponding to $\lambda = -2 + 3i$ is $\mathbf{v}_1 = [-1\ i]^{\mathrm{T}}$. Therefore a complex solution is

$$\begin{aligned}\mathbf{x} &= \begin{pmatrix} -1 \\ i \end{pmatrix} e^{(-2+3i)t} = \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] e^{-2t}(\cos 3t + i\sin 3t) \\ &= \begin{pmatrix} -e^{-2t}\cos 3t \\ -e^{-2t}\sin 3t \end{pmatrix} + i\begin{pmatrix} -e^{-2t}\sin 3t \\ -e^{-2t}\cos 3t \end{pmatrix}.\end{aligned}$$

Therefore two linearly independent solutions are

$$\mathbf{x}_1(t) = \begin{pmatrix} -e^{-2t}\cos 3t \\ -e^{-2t}\sin 3t \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -e^{-2t}\sin 3t \\ -e^{-2t}\cos 3t \end{pmatrix}.$$

The general solution is a linear combination of these two solutions, $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$. In the phase plane the orbits are spirals that approach the origin as $t \to +\infty$ because the real part -2 of the eigenvalues is negative. See figure 5.9. At the point $(1, 1)$ the tangent vector (direction field) is $(-5, 1)$, so the spirals are counterclockwise.

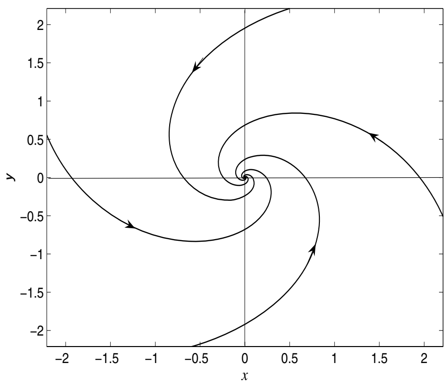

Figure 5.9 A stable spiral from Example 5.17.

5.3.5 Real, Repeated Eigenvalues

One case remains, when A has a repeated real eigenvalue λ with a single eigenvector $\mathbf{v}$. Then $\mathbf{x}_1 = \mathbf{v}e^{\lambda t}$ is one solution (representing a linear orbit), and we need another independent solution. We try a second solution of the form $\mathbf{x}_2 = e^{\lambda t}(t\mathbf{v} + \mathbf{w})$, where $\mathbf{w}$ is to be determined. A more intuitive guess, based on our experience with second-order equations in Chapter 3, would have been $e^{\lambda t}t\mathbf{v}$, but that does not work (try it). Substituting $\mathbf{x}_2$ into the system we get

$$\begin{aligned} \mathbf{x}_2' &= e^{\lambda t}\mathbf{v} + \lambda e^{\lambda t}(t\mathbf{v} + \mathbf{w}), \\ A\mathbf{x}_2 &= e^{\lambda t}A(t\mathbf{v} + \mathbf{w}). \end{aligned}$$

Therefore we obtain an algebraic system for $\mathbf{w}$:

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

This system will always have a solution $\mathbf{w}$, and therefore we will have determined a second linearly independent solution. In fact, this system always has infinitely many solutions, and all we have to do is find one solution. The vector $\mathbf{w}$ is called a **generalized eigenvector**. Therefore, the general solution to the linear system $\mathbf{x}' = A\mathbf{x}$ in the repeated eigenvalue case is

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{w}).$$

If the eigenvalue is negative the orbits enter the origin as $t \to +\infty$, and they go to infinity as $t \to -\infty$. If the eigenvalue is positive, the orbits reverse direction in time.

In the case where the eigenvalues are equal, the origin has a nodal-like structure, as in Example 5.13. When there is a single eigenvector associated with the repeated eigenvalue, we often call the origin a **degenerate node**. It may occur in a special case that a repeated eigenvalue λ has two independent eigenvectors vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ associated with it. When this occurs, the general solution is just $\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda t} + c_2\mathbf{v}_2 e^{\lambda t}$. It happens when the two equations in the system are decoupled, and the matrix is diagonal with equal elements on the diagonal. In this exceptional case all of the orbits are linear orbits entering ($\lambda < 0$) or leaving ($\lambda > 0$) the origin; we refer to the origin in this case as a **star-like** node.

Example 5.18

Consider the system

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{x}.$$

The eigenvalues are $\lambda = 3, 3$ and a corresponding eigenvector is $\mathbf{v} = (1,1)^{\mathrm{T}}$. Therefore one solution is

$$\mathbf{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}.$$

Notice that this solution plots as a linear orbit coming out of the origin and approaching infinity along the direction $(1,1)^{\mathrm{T}}$. There is automatically an opposite orbit coming out of the origin and approaching infinity along the direction $-(1,1)^{\mathrm{T}}$. A second independent solution will have the form $\mathbf{x}_2 = e^{3t}(t\mathbf{v} + \mathbf{w})$ where $\mathbf{w}$ satisfies

$$(A - 3I)\mathbf{w} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This equation has many solutions, and so we choose

$$\mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore a second solution has the form

$$\mathbf{x}_2(t) = e^{3t}(t\mathbf{v} + \mathbf{w}) = e^{3t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} te^{3t} \\ (t+1)e^{3t} \end{pmatrix}.$$

The general solution of the system is the linear combination

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t).$$

If we append an initial condition, for example,

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then we can determine the two constants c_1 and c_2. We have

$$\mathbf{x}(0) = c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence

$$c_1 = 1, \quad c_2 = -1.$$

Therefore the solution to the initial value problem is given by

$$\mathbf{x}(t) = (1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + (-1)\begin{pmatrix} te^{3t} \\ (t+1)e^{3t} \end{pmatrix} = \begin{pmatrix} (1-t)e^{3t} \\ -te^{3t} \end{pmatrix}.$$

As time goes forward ($t \to \infty$), the orbits go to infinity, and as time goes backward ($t \to -\infty$), the orbits enter the origin. The origin is an unstable node.

How to draw a phase diagram. In general, to draw a rough phase diagram for a liner system all you need to know are the eigenvalues and eigenvectors. If the eigenvalues are real then draw straight lines through the origin in the direction of the associated eigenvectors. Label each ray of the line with an arrow that points inward toward the origin if the eigenvalue is negative and outward if the eigenvalue is positive. Then fill in the regions between these linear orbits with consistent solution curves, paying attention to which "eigendirection" dominates as $t \to \infty$ and $t \to -\infty$. Real eigenvalues with the same sign give nodes, and real eigenvalues of opposite signs give saddles.If the eigenvalues are purely imaginary then the orbits are closed loops around the origin, and if they are complex the orbits are spirals. They spiral in if the eigenvalues have negative real part, and they spiral out if the eigenvalues have positive real part. The direction (clockwise or counterclockwise) of the cycles or spirals can be determined directly from the direction field, often by just plotting one vector in the vector field. Another helpful device to get an improved phase diagram is to plot the set of points where the vector field is vertical (the orbits have a vertical tangent) and where the vector field is horizontal (the orbits have a horizontal tangent). These sets of points are found by setting $x^{prime} = ax + by = 0$ and $y^{prime} = cx + dy = 0$, respectively. These straight lines are called the x- and y-**nullclines**.

Example 5.19

The system

$$\mathbf{x}' = \begin{pmatrix} 2 & 5 \\ -2 & 0 \end{pmatrix} \mathbf{x}$$

has eigenvalues $1 \pm 3i$. The orbits spiral outward (because the real part of the eigenvalues, 1, is positive). They are clockwise because the second equation in the system is $y' = -2x$, and so y decreases ($y' < 0$) when $x > 0$. Observe that the orbits are vertical as they cross the nullcline $2x + 5y = 0$, and they are horizontal as they cross the nullcline $x = 0$. With this information the reader should be able to draw a rough phase diagram.

5.3.6 Stability

We mentioned the word stability in the last section. Now we extend the discussion. For the linear system $\mathbf{x}' = A\mathbf{x}$, an **equilibrium solution** is a constant vector solution $\mathbf{x}(t) = \mathbf{x}^*$ representing a point in the phase plane. The zero-vector $\mathbf{x}^* = \mathbf{0}$ (the origin) is always an equilibrium solution to a linear system. Other equilibria will satisfy $A\mathbf{x}^* = \mathbf{0}$, and thus the only time we get a nontrivial equilibrium solution is when $\det A = 0$; in this case there are infinitely many equilibria. If $\det A \neq 0$, then $\mathbf{x}^* = \mathbf{0}$ is the only equilibrium, and it is called an **isolated equilibrium**. For the discussion in the remainder of this section we assume $\det A \neq 0$.

Suppose the system is in its zero equilibrium state. Intuitively, the equilibrium is stable if a small perturbation, or disturbance, does not cause the system to deviate too far from the equilibrium; the equilibrium is unstable if a small disturbance causes the system to deviate far from its original equilibrium state. We have seen in two-dimensional systems that if the eigenvalues of the matrix A are both negative or have negative real parts, then all orbits approach the origin as $t \to +\infty$. In these cases we say that the origin is **asymptotically**

stable node (including degenerate and star-like nodes) or an **asymptotically stable spiral point**. If the eigenvalues are both positive, have positive real parts, or are real of opposite sign, then some or all orbits that begin near the origin do not stay near the origin as $t \to +\infty$, and we say the origin an **unstable node**indexnode, unstable (including degenerate and star-like nodes), an **unstable spiral point**, and a **saddle**, respectively. If the eigenvalues are purely imaginary we obtain periodic solutions, or closed cycles, and the origin is a center. In this case a small perturbation from the origin puts us on one of the elliptical orbits and we cycle near the origin; we say a center is **neutrally stable**, or just **stable**, but not asymptotically stable. Asymptotically stable equilibria are also called **attractors** or **sinks**, and unstable equilibria are called **repellers** or **sources**. Also, we often refer to asymptotically stable spirals and nodes as just stable spirals and nodes—the word asymptotic is understood.

Summary. We make some important summarizing observations that should be remembered for future use (Chapter 6). For two-dimensional systems it is easy to check stability of the origin, and sometimes this is all we want to do. The eigenvalues are roots of the characteristic equation

$$\lambda^2 - (\mathrm{tr}A)\lambda + \det A = 0.$$

By the quadratic formula,

$$\lambda = \frac{1}{2}(\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}).$$

One can easily checks the following facts.

1. If $\det A < 0$, then the eigenvalues are real and have opposite sign, and the origin is a saddle.
2. If $\det A > 0$, then the eigenvalues are real with the same sign (nodes) or complex conjugates (centers and spirals). Nodes have $\mathrm{tr}A)^2 - 4\det A > 0$ and spirals have $\mathrm{tr}A)^2 - 4\det A < 0$. If $\mathrm{tr}A)^2 - 4\det A = 0$ then we obtain degenerate and star-like nodes. If $\mathrm{tr}A < 0$ then the nodes and spirals are stable, and if $\mathrm{tr}A > 0$ they are unstable. If $\mathrm{tr}A = 0$ we obtain centers.
3. If $\det A = 0$, then at least one of the eigenvalues is zero and there is a line of equilibria.

An important result is that *the origin is asymptotically stable if, and only if,* $\mathrm{tr}A < 0$ and $\det A > 0$.

EXERCISES

1. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} -1 & 4 \\ -2 & 5 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & -8 \\ 1 & -2 \end{pmatrix}.$$

2. Write the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ if A has eigenpairs 2, $(1,5)^{\mathrm{T}}$ and -3, $(2,-4)^{\mathrm{T}}$. Sketch the linear orbits in the phase plane corresponding to these eigenpairs. Find the solution curve that satisfies the initial condition $\mathbf{x}(0) = (0,1)^{\mathrm{T}}$ and plot it in the phase plane. Do the same for the initial condition $\mathbf{x}(0) = (-6,12)^{\mathrm{T}}$.

3. Answer the questions in Exercise 2 for a system whose eigenpairs are -6, $(1,2)^{\mathrm{T}}$ and -1, $(1,-5)^{\mathrm{T}}$.

4. For each system find the general solution and sketch the phase portrait. Indicate the linear orbits (if any) and the direction of the solution curves.

a) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\mathbf{x}.$

b) $\mathbf{x}' = \begin{pmatrix} -3 & 4 \\ 0 & -3 \end{pmatrix}\mathbf{x}.$

c) $\mathbf{x}' = \begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix}\mathbf{x}.$

d) $\mathbf{x}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix}\mathbf{x}.$

e) $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x}.$

f) $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\mathbf{x}.$

g) $\mathbf{x}' = \begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$

h) $\mathbf{x}' = \begin{pmatrix} 0 & 9 \\ -1 & 0 \end{pmatrix}\mathbf{x}.$

5. Solve the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

6. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \mathbf{x}.$$

   a) Find the equilibrium solutions and plot them in the phase plane.

   b) Find the eigenvalues and determine if there are linear orbits.

   c) Find the general solution and plot the phase portrait.

7. Determine the behavior of solutions near the origin for the system

$$\mathbf{x}' = \begin{pmatrix} 3 & a \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

   for different values of the parameter a.

8. For the systems in Exercise 4, characterize the origin as to type (node, degenerate node, star-like node, center, spiral, saddle) and stability (unstable, neutrally stable, asymptotically stable).

9. Consider the system

$$\begin{aligned} x' &= -3x + ay, \\ y' &= bx - 2y. \end{aligned}$$

   Are there values of a and b where the solutions are closed cycles (periodic orbits)?

10. In an individual let x be the excess glucose concentration in the blood and y be the excess insulin concentration (positive x and y values measure the concentrations above normal levels, and negative values measure concentrations below normal levels). These quantities are measured in mg per ml and insulin units per ml, respectively, and time is given in hours. One simple model of glucose–insulin dynamics is

$$\begin{aligned} x' &= -ax - by, \\ y' &= cx - dy, \end{aligned}$$

   where $-ax$ is the rate glucose is absorbed in the liver and $-by$ is the rate it is used in the muscle. The rate cx is the rate insulin is produced by the pancreas and $-dy$ is the rate degraded by the liver. A set of values for the constants is $a = 3$, $b = 4.3$, $c = 0.2$, and $d = 0.8$. If $x(0) = 1$ and $y(0) = 0$ find the glucose and insulin concentrations and graph time series plots over a 4 hour period.

11. Find a two-dimensional linear system whose matrix has eigenvalues $\lambda = -2$ and $\lambda = -3$.

12. Rewrite the damped spring-mass equation $mx'' + cx' + kx = 0$ as a system of two first-order equations for x and $y = x'$. Find the characteristic equation of the matrix for the system and show that it coincides with the characteristic equation associated with the second-order DE.

13. Consider an RCL circuit governed by $LCv'' + RCv' + v = 0$, where v is the voltage on the capacitor. Rewrite the equation as a two-dimensional linear system and determine conditions on the constants R, L, and C for which the origin is an asymptotically stable spiral. To what electrical response $v(t)$ does this case correspond?

14. What are the possible behaviors, depending on γ, of the solutions to the linear system

$$\begin{aligned} x' &= -\gamma x - y, \\ y' &= x - \gamma y. \end{aligned}$$

15. Show that A^{-1} exists if, and only if, zero is not an eigenvalue of A.

16. For a 2×2 matrix show that the product of the two eigenvalues equals its determinant, and the sum of the two eigenvalues equals its trace.

17. For a 2×2 matrix A of a linear system, let p equal its trace and q equal its determinant. Sketch the set of points in the pq-plane where the system has an asymptotically stable spiral at the origin. Sketch the region where it has a saddle points.

5.4 Nonhomogeneous Systems

Corresponding to a two-dimensional, linear homogeneous system $\mathbf{x}' = A\mathbf{x}$, we now examine the **nonhomogeneous system**

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t), \tag{5.18}$$

where

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

is a given vector function. We think of this function as the driving force in the system.

To ease the notation in writing the solution we define a **fundamental matrix** $\Phi(t)$ as a 2×2 matrix whose columns are two independent solutions to the associated homogeneous system $\mathbf{x}' = A\mathbf{x}$. So, the fundamental matrix is a square array that holds both vector solutions. It is straightforward to show that $\Phi(t)$ satisfies the *matrix* equation $\Phi'(t) = A\Phi(t)$, and that the general solution to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$ can therefore be written in the form

$$\mathbf{x}_h(t) = \Phi(t)\mathbf{c},$$

where $\mathbf{c} = (c_1, c_2)^{\mathrm{T}}$ is an arbitrary constant vector. (The reader should do Exercise 1 presently, which requires verifying these relations.)

The variation of constants method introduced in Chapter 2 is applicable to a first-order linear system. Therefore we assume a solution to (5.18) of the form

$$\mathbf{x}(t) = \Phi(t)\mathbf{c}(t), \tag{5.19}$$

where we have "varied" the constant vector $\mathbf{c}$. Then, using the product rule for differentiation (which works for matrices),

$$\begin{aligned} \mathbf{x}'(t) &= \Phi(t)\mathbf{c}'(t) + \Phi'(t)\mathbf{c}(t) = \Phi(t)\mathbf{c}'(t) + A\Phi(t)\mathbf{c}(t) \\ &= A\mathbf{x} + \mathbf{f}(t) = A\Phi(t)\mathbf{c}(t) + \mathbf{f}(t). \end{aligned}$$

Comparison gives

$$\Phi(t)\mathbf{c}'(t) = \mathbf{f}(t) \quad \text{or} \quad \mathbf{c}'(t) = \Phi(t)^{-1}\mathbf{f}(t).$$

We can invert the fundamental matrix because its determinant is nonzero, a fact that follows from the independence of its columns. Integrating the last equation from 0 to t then gives

$$\mathbf{c}(t) = \int_0^t \Phi(s)^{-1}\mathbf{f}(s)ds + \mathbf{k},$$

where $\mathbf{k}$ is a arbitrary constant vector. Note that the integral of a vector functions is defined to be the vector consisting of the integrals of the components. Substituting into (5.19) shows that the general solution to the nonhomogeneous equation (5.18) is

$$\mathbf{x}(t) = \Phi(t)\mathbf{k} + \Phi(t)\int_0^t \Phi(s)^{-1}\mathbf{f}(s)ds. \tag{5.20}$$

As for a single first-order linear DE, this formula gives the general solution of (5.18) as a sum of the general solution to the homogeneous equation (first term) and a particular solution to the nonhomogeneous equation (second term). Equation (5.20) is called the **variation of parameters formula** for systems. It is equally valid for systems of any dimension, with appropriate size increase in the vectors and matrices.

It is sometimes a formidable task to calculate the solution (5.20), even in the two-dimensional case. It involves finding the two independent solutions to the homogeneous equation, forming the fundamental matrix, inverting the fundamental matrix, and then integrating.

Example 5.20

Consider the nonhomogeneous system

$$\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

It is a straightforward exercise to find the solution to the homogeneous system

$$\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \mathbf{x}.$$

The eigenpairs are 1, $(1,-1)^{\mathrm{T}}$ and 3, $(-3,1)^{\mathrm{T}}$. Therefore two independent solutions are

$$\begin{pmatrix} e^t \\ -e^t \end{pmatrix}, \begin{pmatrix} -3e^{3t} \\ e^{3t} \end{pmatrix}.$$

A fundamental matrix is

$$\Phi(t) = \begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix},$$

and its inverse is

$$\Phi^{-1}(t) = \frac{1}{\det \Phi} \begin{pmatrix} e^{3t} & 3e^{3t} \\ e^t & e^t \end{pmatrix} = \frac{1}{-2e^{4t}} \begin{pmatrix} e^{3t} & 3e^{3t} \\ e^t & e^t \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} e^{-t} & 3e^{-t} \\ e^{-3t} & e^{-3t} \end{pmatrix}.$$

By the variation of parameters formula (5.20), the general solution is

$$\begin{aligned}
\mathbf{x}(t) &= \Phi(t)\mathbf{k} + \begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \int_0^t -\frac{1}{2} \begin{pmatrix} e^{-s} & 3e^{-s} \\ e^{-3s} & e^{-3s} \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} ds \\
&= \Phi(t)\mathbf{k} - \frac{1}{2} \begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \int_0^t \begin{pmatrix} 3se^{-s} \\ se^{-3s} \end{pmatrix} ds \\
&= \Phi(t)\mathbf{k} - \frac{1}{2} \begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \begin{pmatrix} 3\int_0^t se^{-s} ds \\ \int_0^t se^{-3s} ds \end{pmatrix} \\
&= \Phi(t)\mathbf{k} - \frac{1}{2} \begin{pmatrix} e^t & -3e^{3t} \\ -e^t & e^{3t} \end{pmatrix} \begin{pmatrix} 3 - 3(t+1)e^{-t} \\ \frac{1}{9} - (\frac{t}{3} + \frac{1}{9})e^{-3t} \end{pmatrix} \\
&= \begin{pmatrix} k_1 e^t - 3k_2 e^{3t} \\ -k_1 e^t + k_2 e^{3t} \end{pmatrix} + \begin{pmatrix} t + \frac{4}{3} \\ -\frac{4}{3}t - \frac{13}{9} \end{pmatrix}.
\end{aligned}$$

If the nonhomogeneous term $\mathbf{f}(t)$ is relatively simple, we can use the method of undetermined coefficients (judicious guessing) introduced for second-order equations in Chapter 3 to find the particular solution. In this case we guess a particular solution, depending upon the form of $\mathbf{f}(t)$. For example, if both components are polynomials, then we guess a particular solution with both components being polynomials that have the highest degree that appears. If

$$\mathbf{f}(t) = \begin{pmatrix} 1 \\ t^2 + 2 \end{pmatrix},$$

then a guess for the particular solution would be

$$\mathbf{x}_p(t) = \begin{pmatrix} a_1 t^2 + b_1 t + c_1 \\ a_2 t^2 + b_2 t + c_2 \end{pmatrix}.$$

Substitution into the nonhomogeneous system then determines the six constants. Generally, if a term appears in one component of $\mathbf{f}(t)$, then the guess must have that term appear in all its components. The method is successful on forcing terms with sines, cosines, polynomials, exponentials, and products and sums of those. The rules are the same as for single equations. But the calculations are tedious and a computer algebra system is often preferred.

Example 5.21

We use the method of undetermined coefficients to find a particular solution to the equation in Example 5.20. The forcing function is

$$\begin{pmatrix} 0 \\ t \end{pmatrix},$$

and therefore we guess a particular solution of the form

$$\mathbf{x}_p = \begin{pmatrix} at + b \\ ct + d \end{pmatrix}.$$

Substituting into the original system yields

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} at + b \\ ct + d \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

Simplifying leads to the two equations

$$\begin{aligned} a &= (4a + 3c)t + 4b + 3d, \\ c &= -b + (1 - a)t. \end{aligned}$$

Comparing coefficients gives

$$a = 1, \quad b = -c = \frac{4}{3}, \quad d = -\frac{13}{9}.$$

Therefore a particular solution is

$$\mathbf{x}_p = \begin{pmatrix} t + \frac{4}{3} \\ -\frac{4}{3}t - \frac{13}{9} \end{pmatrix}.$$

EXERCISES

1. Let
$$\mathbf{x}_1 = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$
be independent solutions to the homogeneous equation $\mathbf{x}' = A\mathbf{x}$, and let
$$\Phi(t) = \begin{pmatrix} \phi_1(t) & \psi_1(t) \\ \phi_2(t) & \psi_2(t) \end{pmatrix}$$
be a fundamental matrix. Show, by direct calculation and comparison of entries, that $\Phi'(t) = A\Phi(t)$. Show that the general solution of the homogeneous system can be written equivalently as
$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \Phi(t)\mathbf{c},$$
where $\mathbf{c} = (c_1, c_2)^{\mathrm{T}}$ is an arbitrary constant vector.

2. Two lakes of volume V_1 and V_2 initially have no contamination. A toxic chemical flows into lake 1 at $q + r$ gallons per minute with a concentration c grams per gallon. From lake 1 the mixed solution flows into lake 2 at q gallons per minute, while it simultaneously flows out into a drainage ditch at r gallons per minute. In lake 2 the the chemical mixture flows out at q gallons per minute. If x and y denote the concentrations of the chemical in lake 1 and lake 2, respectively, set up an initial value problem whose solution would give these two concentrations (draw a compartmental diagram). What are the equilibrium concentrations in the lakes, if any? Find $x(t)$ and $y(t)$. Now change the problem by assuming the initial concentration in lake 1 is x_0 and fresh water flows in. Write down the initial value problem and qualitatively, without solving, describe the dynamics of this problem using eigenvalues.

3. Solve the initial value problem
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
if
$$\Phi = \begin{pmatrix} 1+t & -t \\ t & 1-t \end{pmatrix} e^{2t}$$
is a fundamental matrix.

4. Solve the problem in Exercise 3 using undetermined coefficients to find a particular solution.

5. Consider the nonhomogeneous equation

$$\mathbf{x}' = \begin{pmatrix} -5 & 3 \\ 2 & -10 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^{-t} \\ 0 \end{pmatrix}.$$

Find the fundamental matrix and its inverse. Find a particular solution to the system and the general solution.

6. In pharmaceutical studies it is important to model and track concentrations of chemicals and drugs in the blood and in the body tissues. Let x and y denote the amounts (in milligrams) of a certain drug in the blood and in the tissues, respectively. Assume that the drug in the blood is taken up by the tissues at rate r_1x and is returned to the blood from the tissues at rate r_2y. At the same time the drug amount in the blood is continuously degraded by the liver at rate r_3x. Argue that the model equations which govern the drug amounts in the blood and tissues are

$$\begin{aligned} x' &= -r_1x - r_3x + r_2y, \\ y' &= r_1x - r_2y. \end{aligned}$$

Find the eigenvalues of the matrix and determine the response of the system to an initial dosage of $x(0) = x_0$, given intravenously, with $y(0) = 0$. (Hint: show both eigenvalues are negative.)

7. In the preceding problem assume that the drug is administered intravenously and continuously at a constant rate D. What are the governing equations in this case? What is the amount of the drug in the tissues after a long time?

8. An animal species of population $P = P(t)$ has a *per capita* mortality rate m. The animals lay eggs at a rate of b eggs per day, per animal. The eggs hatch at a rate proportional to the number of eggs $E = E(t)$; each hatched egg gives rise to a single new animal.

   a) Write down model equations that govern P and E, and carefully describe the dynamics of the system in the two cases $b > m$ and $b < m$.

   b) Modify the model equations if, at the same time, an egg–eating predator consumes the eggs at a constant rate of r eggs per day.

   c) Solve the model equations in part (b) when $b > m$, and discuss the dynamics.

   d) How would the model change if each hatched egg were multi-yolked and gave rise to y animals?

5.5 Three-Dimensional Systems

In this section we give some examples of solving three linear differential equations in three unknowns. The method is the same as for two-dimensional systems, but now the matrix A for the system is 3×3, and there are three eigenvalues, and so on. We assume $\det A \neq 0$. Eigenvalues λ are found from the characteristic equation $\det(A-\lambda I)=0$, which, when written out, is a cubic equation in λ. For each eigenvalue λ we solve the homogeneous system $(A-\lambda I)\mathbf{v}=0$ to determine the associated eigenvector(s). We will have to worry about real, complex, and equal eigenvalues, as in the two-dimensional case. Each eigenpair $\lambda, \mathbf{v}$ gives a solution $\mathbf{v}e^{\lambda t}$, which, if λ is real, is a linear orbit lying on a ray in $\mathbf{R}^3$ in the direction defined by the eigenvector $\mathbf{v}$. We need three independent solutions $\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)$ to form the general solution, which is the linear combination $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t)$ of those. If all the eigenvalues are real and unequal, then the eigenvectors will be independent and we will have three independent solutions; this is the easy case. Other cases, such as repeated roots and complex roots, are discussed in the examples and in the exercises.

If all the eigenvalues are negative, or have negative real part, then all solution curves approach (0,0,0), and the origin is an asymptotically stable equilibrium. If there is a positive eigenvalue, or complex eigenvalues with positive real part, then the origin is unstable because there is at least one orbit receding from the origin. Three-dimensional orbits can be drawn using computer software, but the plots are often difficult to visualize.

Examples illustrate the key ideas, and we suggest the reader work through the missing details.

Example 5.22

Consider the system

$$\begin{aligned} x_1' &= x_1 + x_2 + x_3 \\ x_2' &= 2x_1 + x_2 - x_3 \\ x_3' &= -8x_1 - 5x_2 - 3x_3 \end{aligned}$$

with matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -8 & -5 & -3 \end{pmatrix}.$$

Eigenpairs of A are given by

$$-1, \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}, \quad -2, \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix}, \quad 2, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

These lead to three independent solutions

$$\mathbf{x}_1 = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t}, \quad \mathbf{x}_2 = \begin{pmatrix} -4 \\ 5 \\ 7 \end{pmatrix} e^{-2t}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t}.$$

Each represents a linear orbit. The general solution is a linear combination of these three; that is, $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t)$. The origin is unstable because of the positive eigenvalue.

Example 5.23

Consider

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix} \mathbf{x}.$$

The eigenvalues, found from $\det(A - \lambda I) = 0$, are $\lambda = -1, 3, 3$. An eigenvector corresponding to $\lambda = -1$ is $(1, 0, -1)^{\mathrm{T}}$, and so

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}$$

is one solution. To find eigenvector(s) corresponding to the other eigenvalue, a double root, we form $(A - 3I)\mathbf{v} = 0$, or

$$\begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This system leads to the single equation

$$v_1 - v_3 = 0,$$

with v_2 arbitrary Letting $v_2 = \beta$ and $v_1 = \alpha$, we can write the solution as as

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

where α and β are arbitrary. Therefore there are two, independent eigenvectors associated with $\lambda = 3$. This gives two independent solutions

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{3t}, \quad \mathbf{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{3t}.$$

Therefore the general solution is a linear combination of the three independent solutions we determined:

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3.$$

We remark that a given eigenvalue with multiplicity two may not yield two independent eigenvectors, as was the case in the last example. Then we must proceed differently to find another independent solution, such as the method given in Section 5.3.3 (see Exercise 2(c) below).

Example 5.24

If the matrix for a three-dimensional system $\mathbf{x}' = A\mathbf{x}$ has one real eigenvalue λ and two complex conjugate eigenvalues $a \pm ib$, with associated eigenvectors $\mathbf{v}$ and $\mathbf{w} \pm i\mathbf{z}$, respectively, then the general solution is, as is expected from Section 5.3.2,

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2e^{at}(\mathbf{w}\cos bt - \mathbf{z}\sin bt) + c_3e^{at}(\mathbf{w}\sin bt + \mathbf{z}\cos bt).$$

EXERCISES

1. Find the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 6 & 2 \\ 0 & 0 & -1 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

2. Find the general solution of the following three-dimensional systems:

a) $\mathbf{x}' = \begin{pmatrix} 3 & 1 & 3 \\ -5 & -3 & -3 \\ 6 & 6 & 4 \end{pmatrix}\mathbf{x}$. (Hint: $\lambda = 4$ is one eigenvalue.)

b) $\mathbf{x}' = \begin{pmatrix} -0.2 & 0 & 0.2 \\ 0.2 & -0.4 & 0 \\ 0 & 0.4 & -0.2 \end{pmatrix}\mathbf{x}$. (Hint: $\lambda = -1$ is one eigenvalue.)

c) $\mathbf{x}' = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 0 & 0 \\ 0 & 2 & -2 \end{pmatrix} \mathbf{x}$. (Hint: see Section 5.3.3.)

d) $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \mathbf{x}$

3. Find the general solution of the system

$$\begin{aligned} x' &= \rho x - y, \\ y' &= x + \rho y, \\ z' &= -2z, \end{aligned}$$

where ρ is a constant.

4. Consider the system

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ -1 & -2 & -3 \end{pmatrix} \mathbf{x}.$$

a) Show that the eigenvalues are $\lambda = -1, -1, -1$.

b) Find an eigenvector $\mathbf{v}_1$ associated with $\lambda = -1$ and obtain a solution to the system.

c) Show that a second independent solution has the form $(\mathbf{v}_2 + t\mathbf{v}_1)e^{-t}$ and find $\mathbf{v}_2$.

d) Show that a third independent solution has the form $(\mathbf{v}_3 + t\mathbf{v}_2 + \frac{1}{2}t^2\mathbf{v}_1)e^{-t}$ and find $\mathbf{v}_3$.

e) Find the general solution and then solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = (0, 1, 0)^{\mathsf{T}}$.

6
Nonlinear Systems

Nonlinear dynamics is common in nature. Unlike linear systems, where we can always find explicit formulas for the solution, nonlinear systems can seldom be solved. For some nonlinear systems we even have to give up on existence and uniqueness. So nonlinear dynamics is much more complicated than linear dynamics, and therefore we rely heavily on qualitative methods to determine their dynamical behavior. As for linear systems, equilibrium solutions and their stability play a fundamental role in the analysis.

6.1 Nonlinear Models

6.1.1 Phase Plane Phenomena

A two-dimensional nonlinear autonomous system has the general form

$$x' = f(x, y) \tag{6.1}$$

$$y' = g(x, y), \tag{6.2}$$

where f and g are given functions of x and y that are assumed to have continuous first partial derivatives in some open region in the plane. This regularity assumption on the first partial derivatives guarantees that the initial value problem associated with (6.1)–(6.2) will have a unique solution through any point in the region. Nonlinear systems arise naturally in mechanics, circuit theory, compartmental analysis, reaction kinetics, mathematical biology, economics, and other areas. In fact, in applications, most systems are nonlinear.

Example 6.1

We have repeatedly noted that a second-order equation can be reformulated as a first-order system. As a reminder, consider Newton's second law of motion for a particle of mass m moving in one dimension,

$$mx'' = F(x, x'),$$

where F is a force depending upon the position and the velocity. Introducing the velocity $y = x'$ as another state variable, we obtain the equivalent first-order system

$$\begin{aligned} x' &= y \\ y' &= \frac{1}{m}F(x, y). \end{aligned}$$

Consequently, we can study mechanical systems in an xy-phase space rather than the traditional position–time space.

In this chapter we are less reliant on vector notation than for linear systems, where vectors and matrices provide a natural language. We review some general terminology of Chapter 5. A **solution** $x = x(t)$, $y = y(t)$ to (6.1)–(6.2) can be represented graphically in two different ways (see figure 5.1 in Chapter 5). We can plot x vs t and y vs t to obtain the time series plots showing how the states x and y vary with time t. Or, we can plot the parametric equations $x = x(t)$, $y = y(t)$ in the xy phase plane. A solution curve in the xy plane is called an **orbit**. On a solution curve in the phase plane, time is a parameter and it may be shifted at will; that is, if $x = x(t)$, $y = y(t)$ is a solution, then $x = x(t - c)$, $y = y(t - c)$ represents the same solution and same orbit for any constant c. This follows because the system is autonomous. The **initial value problem** (IVP) consists of the solving the system (6.1)–(6.2) subject to the **initial conditions**

$$x(t_0) = x_0, \quad y(t_0) = y_0.$$

Geometrically, this means finding the orbit that goes through the point (x_0, y_0) at time t_0. If the functions f and g are continuous and have continuous first partial derivatives on $\mathbf{R}^2$, then the IVP has a unique solution. Therefore, two different solution curves cannot cross in the phase plane. We always assume conditions that guarantee existence and uniqueness.

As is true for their linear counterparts, there is an important geometric interpretation for nonlinear systems in terms of vector fields. For a solution curve $x = x(t)$, $y = y(t)$ we have $(x'(t), y'(t)) = (f(x(t), y(t)), g(x(t), y(t)))$. Therefore, at each point (x, y) in the plane the functions f and g define a vector $(f(x, y), g(x, y))$ that is the tangent vector to the orbit which passes through that point. Thus, the system (6.1)–(6.2) generates a **vector field.** A different way to think about it is this. The totality of all orbits form the *flow* of the vector field. Intuitively, we think of the flow as fluid particle paths with the vector field representing the velocity of the particles at the various points. A plot of several representative or key orbits in the xy-plane is called a **phase diagram** of the system. It is important that f and g do not depend explicitly upon time. Otherwise the vector field would not be stationary and would change, giving a different vector field at each instant of time. This would spoil a simple geometric approach to nonlinear systems. Nonautonomous systems are much harder to deal with than autonomous ones.

Among the most important solutions to (6.1)–(6.2) are the constant solutions, or **equilibrium solutions**. These are solutions $x(t) = x_e$, $y(t) = y_e$, where x_e and y_e are constant. Thus, equilibrium solutions are found as solutions of the algebraic, simultaneous system of equations

$$f(x, y) = 0, \quad g(x, y) = 0.$$

The time series plots of an equilibrium solution are just constant solutions (horizontal lines) in time. In the phase plane an equilibrium solution is represented by a single point (x_e, y_e). We often refer to these as equilibrium points. Nonlinear systems may have several equilibrium points. If an equilibrium point in the phase plane has the property that there is a small neighborhood about the point where there are no other equilibria, then we say the equilibrium point is **isolated**.

Example 6.2

If a particle of mass $m = 1$ moves on an x-axis under the influence of a force $F(x) = 3x^2 - 1$, then the equations of motion in the phase plane take the form

$$\begin{aligned} x' &= y, \\ y' &= 3x^2 - 1, \end{aligned}$$

where the position x and the velocity y are functions of time t. Here we can obtain the actual equation for the orbits in the xy-phase plane, in terms of x and y. Dividing the two equations[1] gives

$$\frac{dy/dt}{dx/dt} = \frac{dy}{dx} = \frac{3x^2 - 1}{y}.$$

[1] Along an orbit $x = x(t)$, $y = y(t)$ we also have y as a function of x, or $y = y(x)$. Then the chain rule dictates $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

Separating variables and integrating yields

$$\int y\,dy = \int (3x^2 - 1)dx,$$

or

$$\frac{1}{2}y^2 = x^3 - x + E, \tag{6.3}$$

where we have chosen the letter E to denote the arbitrary constant of integration (as we soon observe, E stands for total energy). This equation represents a family of orbits in the phase plane giving a relationship between position and velocity. By dividing the equations as we did, time dependence is lost on these orbits. Equation (6.3) has an important physical meaning that is worth reviewing. The term $\frac{1}{2}y^2$ represents the kinetic energy (one-half the mass times the velocity-squared). Secondly, we recall that the potential energy $V(x)$ associated with a conservative force $F(x)$ is $V(x) = -\int F(x)dx$, or $F(x) = -dV/dx$. In the present case $V(x) = -x^3 + x$, where we have taken $V = 0$ at $x = 0$. The orbits (6.3) can be written

$$\frac{1}{2}y^2 + (-x^3 + x) = E,$$

which states that the kinetic energy plus the potential energy is constant. Therefore, the orbits (6.3) represent constant energy curves. The total energy E can be written in terms of the initial position and velocity as $E = \frac{1}{2}y^2(0) + (-x(0)^3 + x(0))$. For each value of E we can plot the locus of points defined by equation (6.3). To carry this out practically, we may solve for y and write

$$y = \sqrt{2}\sqrt{x^3 - x + E}, \quad y = -\sqrt{2}\sqrt{x^3 - x + E}.$$

Then we can plot the curves using a calculator or computer algebra system. (For values of x that make the expression under the radical negative, the curve will not be defined.) Figure 6.1 shows several orbits. Let us discuss their features. There are two points, $x = \sqrt{\frac{1}{3}}$, $y = 0$ and $x = -\sqrt{\frac{1}{3}}$, $y = 0$, where $x' = y' = 0$. These are two equilibrium solutions where the velocity is zero and the force is zero (so the particle cannot be in motion). The equilibrium solution $x = -\sqrt{\frac{1}{3}}$, $y = 0$ has the structure of a center, and for initial values close to this equilibrium the system will oscillate. The other equilibrium $x = \sqrt{\frac{1}{3}}$, $y = 0$ has the structure of an unstable saddle point. Because $x' = y$, for $y > 0$ we have $x' > 0$, and the orbits are directed to the right in the upper half-plane. For $y < 0$ we have $x' < 0$, and the orbits are directed to the left in the lower half-plane. For large initial energies the system does not oscillate but rather goes to $x = +\infty$, $y = +\infty$; that is, the mass moves farther and farther to the right with faster speed.

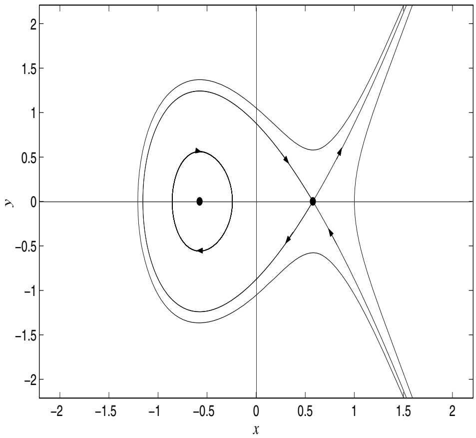

Figure 6.1 Plots of the constant energy curves $\frac{1}{2}y^2 - x^3 + x = E$ in the xy-phase plane. These curves represent the orbits of the system and show how position and velocity relate. Time dependence is lost in this representation of the orbits. Because $x' = y$, the orbits are moving to the right (x is increasing) in the upper half-plane $y > 0$, and to the left (x is decreasing) in the lower half-plane $y < 0$.

Example 6.3

Consider the simple nonlinear system

$$x' = y^2, \tag{6.4}$$

$$y' = -\frac{2}{3}x. \tag{6.5}$$

Clearly, the origin $x = 0$, $y = 0$, is the only equilibrium solution. In this case we can divide the two equations and separate variables to get

$$y^2 y' = -\frac{2}{3}xx'.$$

Integrating with respect to t gives

$$\int y^2 y' dt = -\int \frac{2}{3}xx' dt + C,$$

where C is an arbitrary constant. Changing variables in each integral, $y = y(t)$ in the left integral and $x = x(t)$ in the right, we obtain

$$\int y^2 dy = -\int \frac{2}{3}x dx + C,$$

or

$$y^3 = -x^2 + C.$$

Rearranging,

$$y = (C - x^2)^{1/3}.$$

Consequently, we have obtained the orbits for system (6.4)–(6.5) in terms of x and y. These are easily plotted (e.g., on a calculator, for different values of C), and they are shown in Figure 6.2. This technique illustrates a general

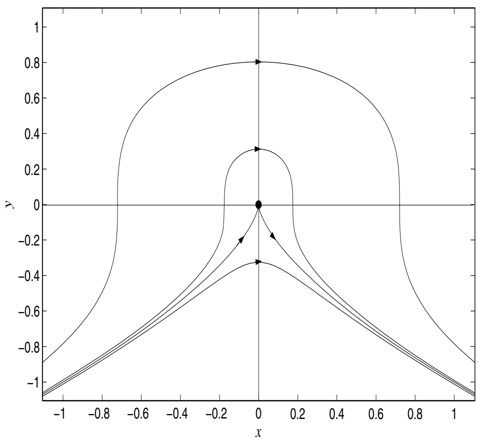

Figure 6.2 Phase diagram for $x' = y^2$, $y' = -\frac{2}{3}x$. Because $x' > 0$, all the orbits are moving to the right as time increases.

method for finding the equation of the orbits for simple equations in terms of the state variables alone: divide the differential equations and integrate, as far as possible. With this technique, however, we lose information about how the states depend on time, or how time varies along the orbits. To find solution curves in terms of time t, we can write (6.4) as

$$x' = y^2 = (C - x^2)^{2/3},$$

which is a single differential equation for $x = x(t)$. We can separate variables, but the result is not very satisfying because we get a complicated integral. This shows that time series solutions are not easily obtained for nonlinear problems. Usually, the qualitative behavior shown in the phase diagram is all we want. If we do need time series plots, we can obtain them using a numerical method, which we discuss later.

We point out an important feature of the phase diagram shown in figure 6.2. The origin does not have the typical type of structure encountered in Chapter 5 for linear systems. There we were able to completely characterize all equilibrium solutions as saddles, spirals, centers, or nodes. The origin for the nonlinear system (6.4)–(6.5) is not one of those. Therefore, nonlinear systems can have an unusual orbital structure near equilibria.

Why are the equilibrium solutions so important? First, much of the "action" in the phase plane takes place near the equilibrium points, so analysis of the flow near those points is insightful. Second, physical systems often seek out and migrate toward equilibria; so equilibrium states can represent persistent states. We think of x and y as representing two competing animal populations. If a system is in an equilibrium state, the two populations coexist. Those populations will remain in the equilibrium states unless the system is perturbed. This means that some event (e.g., a bonanza or catastrophe), would either add or subtract individuals from the populations without changing the underlying processes that govern the population dynamics. If the inflicted population changes are *small*, the populations would be bumped to new values *near* the equilibrium. This brings up the stability issue. Do the populations return to the coexistent state, or do they change to another state? If the populations return to the equilibrium, then it is a persistent state and **asymptotically stable**. If the populations move further away from the equilibrium, then it is not persistent and **unstable**. If the populations remain close to the equilibrium, but do not actually approach it, then the equilibrium is **neutrally stable**. For each model it is important to discover the *locally* stable equilibrium states, or persistent states, in order to understand the dynamics of the model. In Example 6.2 the saddle point is unstable and the center is neutrally stable (figure 6.1), and in Example 6.3 the equilibrium is unstable (figure 6.2). For an unstable equilibrium, orbits that begin near the equilibrium do not remain near. Examples of different types of stability are discussed in the sequel.

The emphasis in the preceding paragraph is on the word *local*. That is, what happens if *small* changes occur near an equilibrium, not large changes. Of course, we really want to know what happens if an equilibrium is disturbed by all possible changes, including an arbitrarily large change. Often the adjectives **local** and **global** are appended to stability statements to indicate what types of perturbations (small or arbitrary) are under investigation. However, we cannot usually solve a nonlinear system, and so we cannot get an explicit resolution of global behavior. Therefore we are content with analyzing local stability properties, and not global stability properties. As it turns out, local stability can be determined because we can approximate the nonlinear system by a tractable linear system near equilibria (Section 6.3).

EXERCISES

1. Consider the uncoupled nonlinear system $x' = x^2$, $y' = -y$.

   a) Find a relation between x and y that describes the orbits. Are all the orbits contained in this relation for different values of the arbitrary constant?

   b) Sketch the vector field at several points near the origin.

   c) Draw the phase diagram. Is the equilibrium stable or unstable?

   d) Find the solutions $x = x(t)$, $y = y(t)$, and plot typical time series. Pick a single time series plot and draw the corresponding orbit in the phase plane.

2. Consider the system $x' = -\frac{1}{y}$, $y' = 2x$.

   a) Are there any equilibrium solutions?

   b) Find a relationship between x and y that must hold on any orbit, and plot several orbits in the phase plane.

   c) From the orbits, sketch the vector field.

   d) Do any orbits touch the x-axis?

3. Consider the nonlinear system $x' = x^2 + y^2 - 4$, $y' = y - 2x$.

   a) Find the two equilibria and plot them in the phase plane.

   b) On the plot in part (a), sketch the set of points where the vector field is vertical (up or down) and the set of points where the vector field is horizontal (left or right).

4. Do parts (a) and (b) of the previous problems for the nonlinear system $x' = y + 1$, $y' = y + x^2$.

5. A nonlinear model of the form

$$\begin{aligned} x' &= y - x \\ y' &= -y + \frac{5x^2}{4 + x^2}, \end{aligned}$$

   has been proposed to describe cell differentiation. Find all equilibrium solutions.

6. Find all equilibria for the system $x' = \sin y$, $y' = 2x$.

7. Consider the nonlinear system $x' = y$, $y' = -x - y^3$. Show that the function $V(x, y) = x^2 + y^2$ decreases along any orbit (i.e., $\frac{d}{dt}V(x(t), y(t)) < 0$), and state why this proves that every orbit approaches the origin as $t \to +\infty$.

8. Consider the nonlinear system $x' = x^2 - y^2$, $y' = x - y$.
   a) Find and plot the equilibrium points in the phase plane. Are they isolated?
   b) Show that, on an orbit, $x + y + 1 = Ce^y$, where C is some constant, and plot several of these curves.
   c) Sketch the vector field.
   d) Describe the fate of the orbit that begins at $(\frac{1}{4}, 0)$ at $t = 0$ as $t \to +\infty$ and as $t \to -\infty$.
   e) Draw a phase plane diagram, being sure to indicate the directions of the orbits.

6.1.2 The Lotka–Volterra Model

Nonlinear equations play an central role in modeling population dynamics in ecology. We formulate and study a model involving predator–prey dynamics. Let $x = x(t)$ be the prey population and $y = y(t)$ be the predator population. We can think of rabbits and foxes, food fish and sharks, or any consumer-resource interaction, including herbivores and plants. If there is no predator we assume the prey dynamics is $x' = rx$, or exponential growth, where r is the *per capita* growth rate. In the absence of prey, we assume that the predator dies via $y' = -my$, where m is the *per capita* mortality rate. When there are interactions, we must include terms that decrease the prey population and increase the predator population. To determine the form of the predation term, we assume that the rate of predation, or the the number of prey consumed per unit of time, per predator, is proportional to the number of prey. That is, the rate of predation is ax. Thus, if there are y predators then the rate that prey is decreased is axy. Note that the interaction term is proportional to xy, the product of the number of predators and the number of prey. For example, if there were 20 prey and 10 predators, there would be 200 possible interactions. Only a fraction of them, a, are assumed to result in a kill. The parameter a depends upon the fraction of encounters and the success of the encounters. The prey consumed cause a rate of increase in predators of εaxy, where ε is the conversion efficiency of the predator population (one prey consumed does not mean one predator born). Therefore, we obtain the simplest model of predator–prey interaction, called the **Lotka–Volterra model**:

$$
\begin{aligned}
x' &= rx - axy \\
y' &= -my + bxy,
\end{aligned}
$$

where $b = \varepsilon a$.

To analyze the Lotka–Volterra model we factor the right sides of the equations to obtain

$$x' = x(r - ay), \quad y' = y(-m + bx). \tag{6.6}$$

Now it is simple to locate the equilibria. Setting the right sides equal to zero gives two solutions, $x = 0$, $y = 0$ and $x = m/b$, $y = r/a$. Thus, in the phase plane, the points $(0, 0)$ and $(m/b, r/a)$ represent equilibria. The origin represents extinction of both species, and the nonzero equilibrium represents a coexistent state. To determine properties of the orbits we usually plot curves in the xy plane where the vector field is vertical (where $x' = 0$) and curves where the vector field is horizontal ($y' = 0$). These curves are called the **nullclines**. They are not (usually) orbits, but rather the curves where the orbits cross vertically or horizontally. The x-nullclines for (6.6) , where $x' = 0$, are $x = 0$ and $y = r/a$. Thus the orbits cross these two lines vertically. The y-nullclines, where $y' = 0$, are $y = 0$ and $x = m/b$. The orbits cross these lines horizontally. Notice that the equilibrium solutions are the intersections of the x- and y-nullclines. The nullclines partition the plane into regions where x' and y' have various signs, and therefore we get a picture of the direction of the flow pattern. See figure 6.3. Next, along each nullcline we can find the direction of

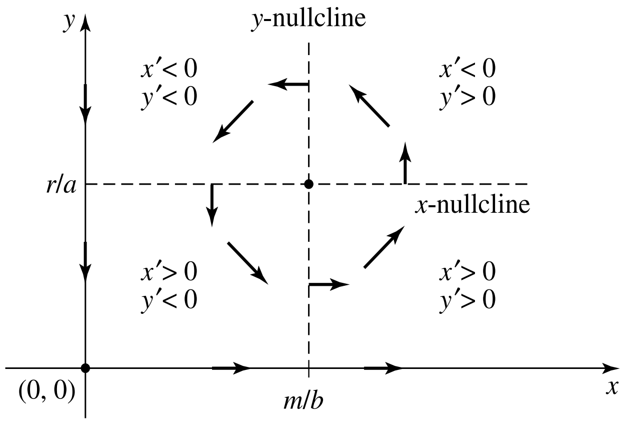

Figure 6.3 Nullclines (dashed) and vector field in regions between nullclines. The x and y axes are nullclines, as well as orbits.

the vector field. For example, on the ray to the right of the equilibrium we have $x > m/b$, $y = r/a$. We know the vector field is vertical so we need only check the sign of y'. We have $y' = y(-m + bx) = (r/a)(-m + bx) > 0$, so the vector field points upward. Similarly we can determine the directions along the other three rays. These are shown in the accompanying figure 6.3. Note that $y = 0$ and $x = 0$, both nullclines, are also orbits. For example, when $x = 0$ we have $y' = -my$, or $y(t) = Ce^{-mt}$; when there are no prey, the foxes die out. Similarly, when $y = 0$ we have $x(t) = Ce^{rt}$, so the rabbits increase in number.

Finally, we can determine the direction of the vector field in the regions between the nullclines either by selecting an arbitrary point in that region and calculating x' and y', or by just noting the sign of x' and y' in that region from information obtained from the system. For example, in the quadrant above and to the right of the nonzero equilibrium, it is easy to see that $x' < 0$ and $y' > 0$; so the vector field points upward and to the left. We can complete this task for each region and obtain the directions shown in figure 6.3. Having the direction of the vector field along the nullclines and in the regions bounded by the nullclines tells us the directions of the solution curves, or orbits. Near $(0, 0)$ the orbits appear to veer away and the equilibrium has a saddle point structure. The equilibrium $(0, 0)$ is unstable. It appears that orbits circle around the nonzero equilibrium in a counterclockwise fashion. But at this time it is not clear if they form closed paths or spirals, so more work is needed.

We attempt to obtain the equation of the orbits by dividing the two equations in (6.6). We get

$$\frac{y'}{x'} = \frac{dy}{dx} = \frac{y(-m + bx)}{x(r - ay)}.$$

Rearranging and integrating gives

$$\int \frac{r - ay}{y} dy = \int \frac{bx - m}{x} dx + C.$$

Carrying out the integration gives

$$r \ln y - ay = bx - m \ln x + C,$$

which is the algebraic equation for the orbits. It is obscure what these curves are because it is not possible to solve for either of the variables. So, cleverness is required. If we exponentiate we get

$$y^r e^{-ay} = e^C e^{bx} x^{-m}.$$

Now consider the y nullcline where x is fixed at a value m/b, and fix a positive C value (i.e., fix an orbit). The right side of the last equation is a positive number A, and so $y^r = Ae^{ay}$. If we plot both sides of this equation (do this!—plot a power function and a growing exponential) we observe that there can be at most two intersections; therefore, this equation can have at most two solutions for y. Hence, along the vertical line $x = m/b$, there can be at most two crossings; this means an orbit cannot spiral into or out from the equilibrium point, because that would mean many values of y would be possible. We conclude

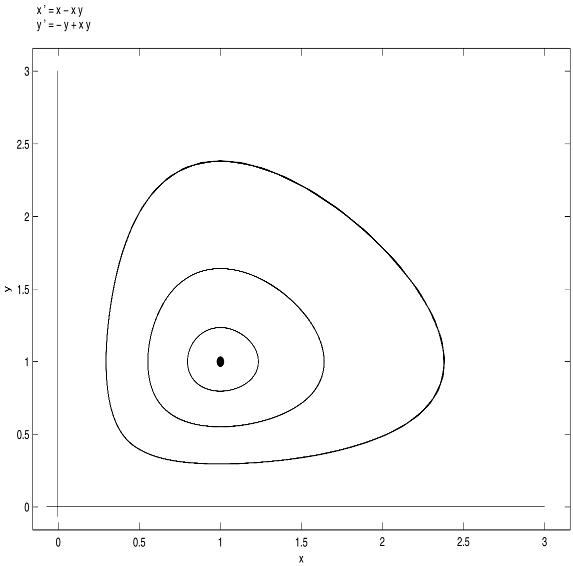

Figure 6.4 Closed, counterclockwise, periodic orbits of the Lotka–Volterra predator–prey model $x' = x - xy$, $y' = -y + xy$. The x-axis is an orbit leaving the origin and the y-axis is an orbit entering the origin.

that the equilibrium is a center with closed, periodic orbits encircling it. A phase diagram is shown in figure 6.4. Time series plots of the prey and predator populations are shown in figure 6.5. When the prey population is high the predators have a high food source and their numbers start to increase, thereby eventually driving down the prey population. Then the prey population gets low, ultimately reducing the number of predators because of lack of food. Then the process repeats, giving cycles.

The Lotka–Volterra model, developed by A. Lotka and V. Volterra in 1925, is the simplest model in ecology showing how populations can cycle, and it was one of the first strategic models to explain qualitative observations in natural systems. Note that the nonzero equilibrium is neutrally stable. A small perturbation from equilibrium puts the populations on a periodic orbit that stays near the equilibrium. But the system does not return to that equilibrium. So the nonzero equilibrium is stable, but not asymptotically stable. The other equilibrium, the origin, which corresponds to extinction of both species, is an unstable saddle point with the two coordinate axes as separatrices.

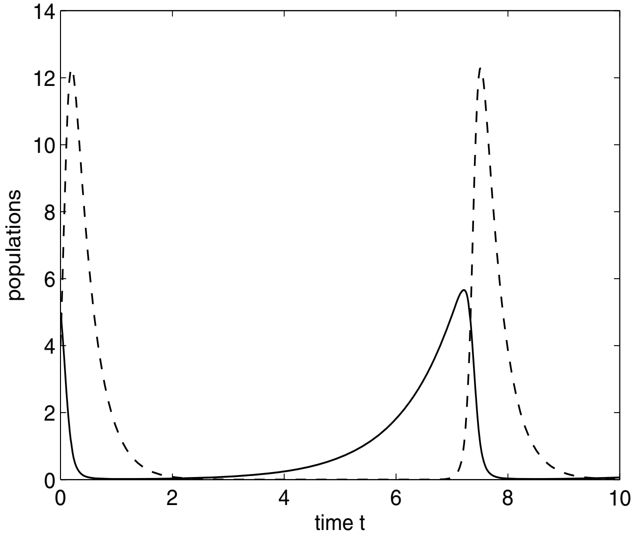

Figure 6.5 Time series solution to the Lotka–Volterra system $x' = x - xy$, $y' = -3y + 3xy$, showing the predator (dashed) and prey (solid) populations.

6.1.3 Holling Functional Responses

Ecology provides a rich source of problems in nonlinear dynamics, and now we take time to introduce another one. In the Lotka–Volterra model the rate of predation (prey per time, per predator) was assumed to be proportional to the number of prey (i.e., ax). Thinking carefully about this leads to concerns. Increasing the prey density indefinitely leads to an extremely high consumption rate, which is clearly impossible for any consumer. It seems more reasonable if the rate of predation would have a limiting value as prey density gets large. In the late 1950s, C. Holling developed a functional form that has this limiting property by partitioning the time budget of the predator. He reasoned that the number N of prey captured by a *single* predator is proportional to the number x of prey and the time T_s allotted for searching.[2] Thus $N = aT_s x$, where the proportionality constant a is the effective encounter rate. But the total time T available to the predator must be partitioned into search time and total handling time T_h, or $T = T_s + T_h$. The total handling time is proportional to the number captured, $T_h = hN$, where h is the time for a predator to handle a

[2] We are thinking of x and y as population numbers, but we can also regard them as *population densities*, or animals per area. There is always an underlying fixed area where the dynamics is occurring.

single prey. Hence $N = a(T - hN)x$. Solving for N/T, which is the predation rate, gives

$$\frac{N}{T} = \frac{ax}{1 + ahx}.$$

This function for the predation rate is called a Holling type II response, or the Holling disk equation. Note that $\lim_{x\to\infty} \frac{ax}{1+ahx} = 1/h$, so the rate of predation approaches a constant value. This quantity N/T is measured in prey per time, per predator, so multiplying by the number of predators y gives the predation rate for y predators.

If the encounter rate a is a function of the prey density (e.g., a linear function $a = bx$), the the predation rate is

$$\frac{N}{T} = \frac{bx^2}{1 + bhx^2},$$

which is called a Holling type III response. Figure 6.6 compares different types of predation rates used by ecologists. For a type III response the predation is turned on once the prey density is high enough; this models, for example, predators that must form a "prey image" before they become aware of the prey, or predators that eat different types of prey. At low densities prey go nearly unnoticed; but once the density reaches an upper threshold the predation rises quickly to its maximum rate.

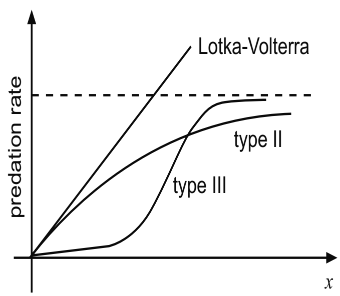

Figure 6.6 Three types of predation rates studied in ecology.

Replacing the linear predation rate ax in the Lotka–Volterra model by the **Holling type II response**, we obtain the model

$$\begin{aligned} x' &= rx - \frac{ax}{1 + ahx}y, \\ y' &= -my + \varepsilon\frac{ax}{1 + ahx}y. \end{aligned}$$

We can even go another step and replace the linear growth rate in the model by a more realistic logistics growth term. Then we obtain the **Rosenzweig–MacArthur** model

$$\begin{aligned} x' &= rx(1 - \frac{x}{K}) - \frac{ax}{1+ahx}y, \\ y &= -my + \varepsilon \frac{ax}{1+ahx}y. \end{aligned}$$

Else, a type III response could be used. All of these models have very interesting dynamics. Questions abound. Do they lead to cycles? Are there persistent states where the predator and prey coexist at constant densities? Does the predator or prey population go to extinction? What happens when a parameter, for example, the carrying capacity K, increases? Some aspects of these models are examined in the Exercises.

Other types of ecological models have been developed for interacting species. A model such as

$$\begin{aligned} x' &= f(x) - axy, \\ y' &= g(y) - bxy \end{aligned}$$

is interpreted as a **competition model** because the interaction terms $-axy$ and $-bxy$ are both negative and lead to a decrease in each population. When both interaction terms are positive, then the model is called a **cooperative model**.

6.1.4 An Epidemic Model

We consider a simple epidemic model where, in a fixed population of size N, the function $I = I(t)$ represents the number of individuals that are infected with a contagious illness and $S = S(t)$ represents the number of individuals that are susceptible to the illness, but not yet infected. We also introduce a removed class where $R = R(t)$ is the number who cannot get the illness because they have recovered permanently, are naturally immune, or have died. We assume $N = S(t) + I(t) + R(t)$, and each individual belongs to only one of the three classes. Observe that N includes the number who may have died. The evolution of the illness in the population can be described as follows. Infectives communicate the disease to susceptibles with a known infection rate; the susceptibles become infectives who have the disease a short time, recover (or die), and enter the removed class. Our goal is to set up a model that describes how the disease progresses with time. These models are called **SIR models**.

In this model we make several assumptions. First, we work in a time frame where we can ignore births and immigration. Next, we assume that the population mixes homogeneously, where all members of the population interact with one another to the same degree and each has the same risk of exposure to the disease. Think of measles, the flu, or chicken pox at an elementary school. We assume that individuals get over the disease reasonably fast. So, we are not modeling tuberculosis, AIDS, or other long-lasting or permanent diseases. Of course, more complicated models can be developed to account for all sorts of factors, such as vaccination, the possibility of reinfection, and so on.

The disease spreads when a susceptible comes in contact with an infective. A reasonable measure of the number of contacts between susceptibles and infectives is $S(t)I(t)$. For example, if there are five infectives and twenty susceptibles, then one hundred contacts are possible. However, not every contact results in an infection. We use the letter a to denote the **transmission coefficient**, or the fraction of those contacts that usually result in infection. For example, a could be 0.02, or 2 percent. The parameter a is the product of two effects, the fraction of the total possible number of encounters that occur, and the fraction of those that result in infection. The constant a has dimensions time^{-1} per individual, aN is a measure of the the average rate that a susceptible individual makes infectious contacts, and $1/(aN)$ is the average time one might expect to get the infection. The quantity $aS(t)I(t)$ is the infection rate, or the rate that members of the susceptible class become infected. Observe that this model is the same as the law of mass action in chemistry where the rate of chemical reaction between two reactants is proportional to the product of their concentrations. Therefore, if no other processes are included, we would have

$$S' = -aSI, \quad I' = aSI.$$

But, as individuals get over the disease, they become part of the removed class R. The **recovery rate** r is the fraction of the infected class that ceases to be infected; thus, the rate of removal is $rI(t)$. The parameter r is measured in time^{-1} and $1/r$ can be interpreted as the average time to recover. Therefore, we modify the last set of equations to get

$$S' = -aSI, \tag{6.7}$$

$$I' = aSI - rI. \tag{6.8}$$

These are our working equations. We do not need an equation for R' because R can be determined directly from $R = N - S - I$. At time $t = 0$ we assume there are I_0 infectives and S_0 susceptibles, but no one yet removed. Thus, initial conditions are given by

$$S(0) = S_0, \quad I(0) = I_0, \tag{6.9}$$

and $S_0 + I_0 = N$. SIR models are commonly diagrammed as in figure 6.7 with S, I, and R compartments and with arrows that indicate the rates that individuals progress from one compartment to the other. An arrow entering a compartment represents a positive rate and an arrow leaving a compartment represents a negative rate.

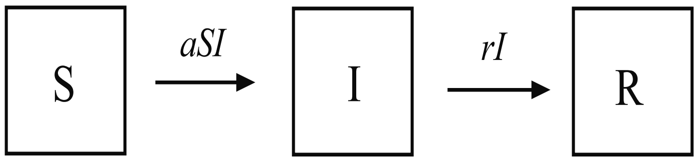

Figure 6.7 Compartments representing the number of susceptibles, the number of infectives, and the number removed, and the flow rates in and out of the compartments.

Qualitative analysis can help us understand how a parametric solution curve $S = S(t)$, $I = I(t)$, or orbit, behaves in the SI-phase plane. First, the initial value must lie on the straight line $I = -S + N$. Where then does the orbit go? Note that S' is always negative so the orbit must always move to the left, decreasing S. Also, because $I' = I(aS-r)$, we see that the number of infectives increases if $S > r/a$, and the number of infectives decreases if $S < r/a$. So, there are two cases to consider: $r/a > N$ and $r/a < N$. That is, it makes a difference if the ratio r/a is greater than the population, or less than the population. The vertical line $S = r/a$ is the I nullcline where the vector field is horizontal. Let us fix the idea and take $r/a < N$. (The other case is requested in the Exercises.) If the initial condition is at point P in figure 6.8, the orbit goes directly down and to the left until it hits $I = 0$ and the disease dies out. If the initial condition is at point Q, then the orbit increases to the left, reaching a maximum at $S = r/a$. Then it decreases to the left and ends on $I = 0$. There are two questions remaining, namely, how steep is the orbit at the initial point, and where on the S axis does the orbit terminate. Figure 6.8 anticipates the answer to the first question. The total number of infectives and susceptibles cannot go above the line $I + S = N$, and therefore the slope of the orbit at $t = 0$ is not as steep as -1, the slope of the line $I + S = N$. To analytically resolve the second issue we can obtain a relationship between S and I along a solution curve as we have done in previous examples. If we divide the equations (6.7)–(6.8) we obtain

$$\frac{I'}{S'} = \frac{dI/dt}{dS/dt} = \frac{dI}{dS} = \frac{aSI - rI}{-aSI} = -1 + \frac{r}{aS}.$$

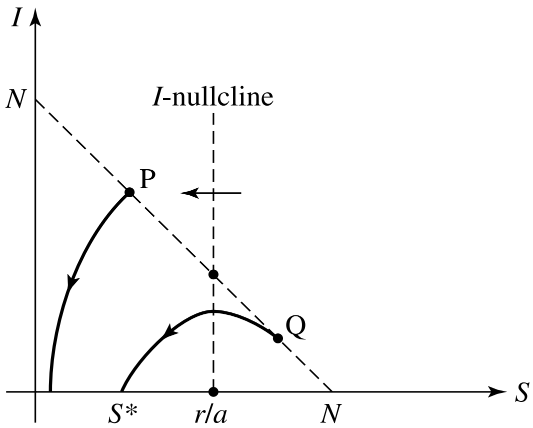

Figure 6.8 The SI phase plane showing two orbits in the case $r/a < N$. One starts at P and and one starts at Q, on the line $I + S = N$. The second shows an epidemic where the number of infectives increases to a maximum value and then decreases to zero; S^* represents the number that does not get the disease.

Thus

$$\frac{dI}{dS} = -1 + \frac{r}{aS}.$$

Integrating both sides with respect to S (or separating variables) yields

$$I = -S + \frac{r}{a} \ln S + C,$$

where C is an arbitrary constant. From the initial conditions, $C = N - (r/a) \ln S_0$. So the solution curve, or orbit, is

$$I = -S + \frac{r}{a} \ln S + N - \frac{r}{a} \ln S_0 = -S + N + \frac{r}{a} \ln \frac{S}{S_0}.$$

This curve can be graphed with a calculator or computer algebra system, once parameter values are specified. Making such plots shows what the general curve looks like, as plotted in figure 6.8. Notice that the solution curve cannot intersect the I axis where $S = 0$, so it must intersect the S axis at $I = 0$, or at the root S^* of the nonlinear equation

$$-S + N + \frac{r}{a} \ln \frac{S}{S_0} = 0.$$

See figure 6.8. This root represents the number of individuals who do not get the disease. Once parameter values are specified, a numerical approximation of S^* can be obtained. In all cases, the disease dies out because of lack of infectives. Observe, again, in this approach we lost time dependence on the orbits. But the qualitative features of the phase plane give good resolution of the disease dynamics. In the next section we show how to obtain accurate time series plots using numerical methods.

Generally, we are interested in the question of whether there will be an epidemic when there are initially a small number of infectives. The number $R_0 = \frac{aS(0)}{r}$ is a threshold quantity called the *reproductive number*, and it determines if there will be an epidemic. If $R_0 > 1$ there will be an epidemic (the number of infectives increase), and if $R_0 < 1$ then the infection dies out.

EXERCISES

1. In the SIR model analyze the case when $r/a > N$. Does an epidemic occur in this case?

2. Referring to figure 6.8, draw the shapes of the times series plots $S(t)$ and $I(t)$ on the same set of axes when the initial point is at point Q.

3. In a population of 200 individuals, 20 were initially infected with an influenza virus. After the flu ran its course, it was found that 100 individuals did not contract the flu. If it took about 3 days to recover, what was the transmission coefficient a? What was the average time that it might have taken for someone to get the flu?

4. In a population of 500 people, 25 have the contagious illness. On the average it takes about 2 days to contract the illness and 4 days to recover. How many in the population will not get the illness? What is the maximum number of infectives at a single time?

5. In a constant population, consider an SIS model (susceptibles become infectives who then become susceptible again after recovery) with infection rate aSI and recovery rate rI. Draw a compartmental diagram as in figure 6.7, and write down the model equations. Reformulate the model as a single DE for the infected class, and describe the dynamics of the disease.

6. If, in the Lotka–Volterra model, we include a constant harvesting rate h of the prey, the model equations become

$$\begin{aligned} x' &= rx - axy - h \\ y' &= -my + bxy. \end{aligned}$$

Explain how the equilibrium is shifted from that in the Lotka–Volterra model. How does the equilibrium shift if both prey and predator are harvested at the same rate?

7. Modify the Lotka–Volterra model to include *refuge*. That is, assume that the environment always provides a constant number of the hiding places

where the prey can avoid predators. Argue that

$$\begin{aligned} x' &= rx - a(x-k)y \\ y' &= -my + b(x-k)y. \end{aligned}$$

How does refuge affect the equilibrium populations compared to no refuge?

8. Formulate a predator–prey model based on Lotka–Volterra, but where the predator migrates out of the region at a constant rate M. Discuss the dynamics of the system.

9. A simple cooperative model where two species depend upon mutual cooperation for their survival is

$$\begin{aligned} x' &= -kx + axy \\ y' &= -my + bxy. \end{aligned}$$

Find the equilibria and identify, insofar as possible, the region in the phase plane where, if the initial populations lie in that region, then both species become extinct. Can the populations ever coexist in a nonzero equilibrium?

10. Beginning with the SIR model, assume that susceptible individuals are vaccinated at a constant rate ν. Formulate the model equations and describe the progress of the disease if, initially, there are a small number of infectives in a large population.

11. Beginning with the SIR model, assume that recovered individuals can lose their immunity and become susceptible again, with rate μR, where r is the recovery rate. Draw a compartmental diagram and formulate a two-dimensional system of model equations. Find the equilibria. Is there a disease-free equilibrium with $I = 0$? Is there an endemic equilibrium with $I > 0$?

12. Two populations X and Y grow logistically and both compete for the same resource. A competition model is given by

$$\frac{dX}{d\tau} = r_1 X\left(1 - \frac{X}{K_1}\right) - b_1 XY, \quad \frac{dY}{d\tau} = r_2 Y\left(1 - \frac{Y}{K_2}\right) - b_2 XY.$$

The competition terms are $b_1 XY$ and $b_2 XY$.

a) Scale time by r_1^{-1} and scale the populations by their respective carrying capacities to derive a dimensionless model

$$x' = x(1-x) - axy, \quad y' = cy(1-y) - bxy,$$

where a, b, and c are appropriately defined dimensionless constants. Give a biological interpretation of the constants.

b) In the case $a > 1$ and $c > b$ determine the equilibria, the nullclines, and the direction of the vector field on and in between the nullclines.

c) Determine the stability of the equilibria by sketching a generic phase diagram. How will an initial state evolve in time?

d) Analyze the population dynamics in the case $a > 1$ and $c < b$.

13. Consider the system

$$x' = \frac{axy}{1+y} - x, \quad y' = -\frac{axy}{1+y} - y + b,$$

where a and b are positive parameters with $a > 1$ and $b > \frac{1}{a-1}$.

a) Find the equilibrium solutions, plot the nullclines, and find the directions of the vector field along the nullclines.

b) Find the direction field in the first quadrant in the regions bounded by the nullclines. Can you determine from this information the stability of any equilibria?

6.2 Numerical Methods

We have formulated a few models that lead to two-dimensional nonlinear systems and have illustrated some elementary methods of analysis. In the next section we advance our technique and show how a more detailed analysis can lead to an overall qualitative picture of the nonlinear dynamics. But first we develop some numerical methods to solve such systems. Unlike two-dimensional linear systems with constant coefficients, nonlinear systems can rarely be resolved analytically by finding solution formulas. So, along with qualitative methods, numerical methods come to the forefront.

We begin with the Euler method, which was formulated in Section 2.4 for a single equation. The idea was to discretize the time interval and replace the derivative in the differential equation by a difference quotient approximation, thereby setting up an iterative method to advance the approximation from time to time. We take the same approach for systems. Consider the nonlinear, autonomous initial value problem

$$\begin{aligned} x' &= f(x,y), \quad y' = g(x,y), \\ x(0) &= x_0, \quad y(0) = y_0, \end{aligned}$$

where a solution is sought on the interval $0 \le t \le T$. First we discretize the time interval by dividing the interval into N equal parts of length $h = T/N$, which is the stepsize; N is the number of steps. The discrete times are $t_n = nh$, $n = 0, 1, 2, ..., N$. We let x_n and y_n denote approximations to the exact solution values $x(t_n)$ and $y(t_n)$ at the discrete points. Then, evaluating the equations at t_n, or $x'(t_n) = f(x(t_n), y(t_n))$, $y'(t_n) = g(x(t_n), y(t_n))$, and then replacing the derivatives by their difference quotient approximations, we obtain, approximately,

$$\begin{aligned} \frac{x(t_{n+1}) - x(t_n)}{h} &= f(x(t_n), x(t_n)), \\ \frac{y(t_{n+1}) - y(t_n)}{h} &= g(x(t_n), x(t_n)). \end{aligned}$$

Therefore, the **Euler method** for computing approximations x_n and y_n is

$$\begin{aligned} x_{n+1} &= x_n + hf(x_n, y_n), \\ y_{n+1} &= y_n + hg(x_n, y_n), \end{aligned}$$

$n = 0, 1, 2, ..., N - 1$. Here, x_0 and y_0 are the prescribed initial conditions that start the recursion process.

The Euler method can be selected on calculators to plot the solution, and it is also available in computer algebra systems. As in Section 2.4, it is easy to write a simple code that calculates the approximate values.

Example 6.4

Consider a mass ($m = 1$) on a nonlinear spring whose oscillations are governed by the second-order equation

$$x'' = -x + 0.1x^3.$$

This is equivalent to the system

$$\begin{aligned} x' &= y, \\ y' &= -x + 0.1x^3. \end{aligned}$$

Euler's formulas are

$$\begin{aligned} x_{n+1} &= x_n + hy_n, \\ y_{n+1} &= y_n + h(-x_n + 0.1x_n^3). \end{aligned}$$

If the initial conditions are $x(0) = 2$ and $y(0) = 0.5$, and if the stepsize is $h = 0.05$, then

$$\begin{aligned} x_1 &= x_0 + hy_0 = 2 + (0.05)(0.5) = 2.025, \\ y_1 &= y_0 + h(-x_0 + 0.1x_0^3) = 0.5 + (0.05)(-2 + (0.1)2^3) = 0.44. \end{aligned}$$

Continuing in this way we can calculate x_2, y_2, and so on, at all the discrete time values. It is clear that calculators and computers are better suited to perform these routine calculations, and Appendix B shows sample computations.

The cumulative error in the Euler method over the interval is proportional to the stepsize h. Just as for a single equation we can increase the order of accuracy with a *modified* Euler method (predictor–corrector), which has a cumulative error of order h^2, or with the classical Runge–Kutta method, which has order h^4. There are other methods of interest, especially those that deal with *stiff* equations where rapid changes in the solution functions occur (such as in chemical reactions or in nerve-firing mechanisms). Runge–Kutta type methods sometimes cannot keep up with rapid changes, so numerical analysts have developed *stiff methods* that adapt to the changes by varying the step size automatically to maintain a small local error. These advanced methods are presented in numerical analysis textbooks. It is clear that the Euler, modified Euler, and Runge–Kutta methods can be extended to three equations in three unknowns, and beyond.

The following exercises require some hand calculation as well as numerical computation. Use a software system or write a program to obtain numerical solutions (see Appendix B for templates).

EXERCISES

1. In Example 6.4 compute x_2, y_2 and x_3, y_3 by hand.

2. Compute, by hand, the first three approximations in Example 6.4 using a modified Euler method.

3. (Trajectory of a baseball) A ball of mass m is hit by a batter. The trajectory is the xy plane. There are two forces on the ball, gravity and air resistance. Gravity acts downward with magnitude mg, and air resistance is directed opposite the velocity vector $\mathbf{v}$ and has magnitude kv^2, where v is the magnitude of $\mathbf{v}$. Use Newton's second law to derive the equations of motion (remember, you have to resolve vertical and horizontal directions). Now take $g = 32$ and $k/m = 0.0025$. Assume the batted ball starts at the origin and the initial velocity is 160 ft per sec at an angle of 30 degrees elevation. Compare a batted ball with air resistance and without air resistance with respect to height, distance, and time to hit the ground.

4. Use a calculator's Runge-Kutta solver, or a computer algebra system, to graph the solution $u = u(t)$ to

$$\begin{aligned} u'' + 9u &= 80\cos 5t, \\ u(0) &= u'(0) = 0, \end{aligned}$$

on the interval $0 \le t \le 6\pi$.

5. Plot several orbits in the phase plane for the system
$$x' = x^2 - 2xy, \quad y' = -y^2 + 2xy.$$

6. Consider a nonlinear mechanical system governed by
$$mx'' = -kx + ax' - b(x')^3,$$
where $m = 2$ and $a = k = b = 1$. Plot the orbit in the phase plane for $t > 0$ and with initial condition $x(0) = 0.01$, $x'(0) = 0$. Plot the time series $x = x(t)$ on the interval $0 \le t \le 60$.

7. The **Van der Pol equation**
$$x'' + a(x^2 - 1)x' + x = 0$$
arises in modeling RCL circuits with nonlinear resistors. For $a = 2$ plot the orbit in the phase plane satisfying $x(0) = 2$, $x'(0) = 0$. Plot the time series graphs, $x = x(t)$ and $y = x'(t)$, on the interval $0 \le t \le 25$. Estimate the period of the oscillation.

8. Consider an influenza outbreak governed by the SIR model (6.7)–(6.8). Let the total population be $N = 500$ and suppose 45 individuals initially have the flu. The data indicate that the likelihood of a healthy individual becoming infected by contact with an individual with the flu is 0.1%. And, once taken ill, an infective is contagious for 5 days. Numerically solve the model equations and draw graphs of S and I vs. time, in days. Draw the orbit in the SI phase plane. How many individuals do not get the flu? What is the maximum number of individuals that have the flu at a single time.

9. Refer to Exercise 8. One way to prevent the spread of a disease is to quarantine some of the infected individuals. Let q be the fraction of infectives that are quarantined. Modify the SIR model to include quarantine, and use the data in Exercise 8 to investigate the behavior of the model for several values of q. Is there a smallest value of q that prevents an epidemic from occurring?

10. The **forced Duffing equation**
$$x'' = x - cx' - x^3 + A\cos t$$
models the damped motion of a mass on a nonlinear spring driven by a periodic forcing function of amplitude A. Take initial conditions $x(0) = 0.5$, $x'(0) = 0$ and plot the phase plane orbit and the time series when $c = 0.25$ and $A = 0.3$. Is the motion periodic? Carry out the same tasks for several other values of the amplitude A and comment on the results.

6.3 Linearization and Stability

For nonlinear systems we have learned how to find equilibrium solutions, nullclines, and the direction of the vector field in regions bounded by the nullclines. What is missing is a detailed analysis of the orbits near the equilibrium points, where much of the action takes place in two-dimensional flows. As mentioned in the last section, we classify equilibrium points as (locally) asymptotically stable, unstable, or neutrally stable, depending upon whether small deviations from equilibrium decay, grow, or remain close. To get an idea of where we are going we consider a simple example.

Example 6.5

Consider

$$x' = x - xy, \quad y' = y - xy. \tag{6.10}$$

This is a simple competition model where two organisms grow with constant *per capita* growth rates, but interaction, represented by the xy terms, has a negative effect on both populations. The origin $(0, 0)$ is an equilibrium point, as is $(1, 1)$. What type are they? Let's try the following strategy. Near the origin both x and y are small. But terms having products of x and y are even smaller, and we suspect we can ignore them. That is, in the first equation x has greater magnitude than xy, and in the second equation y has magnitude greater than xy. Hence, near the origin, the nonlinear system is approximated by

$$x' = x, \quad y' = y.$$

This linearized system has eigenvalues $\lambda = 1, 1$, and therefore $(0, 0)$ is an unstable node. We suspect that the nonlinear system therefore has an unstable node at $(0, 0)$ as well. This turns out to be correct.

Let's apply a similar analysis at the equilibrium $(1, 1)$. We can represent points near $(1, 1)$ as $u = x - 1$, $v = y - 1$ where u and v are small. This is the same as $x = 1 + u$, $y = 1 + v$, so we may regard u and v as small deviations from $x = 1$ and $y = 1$. Rewriting the nonlinear system (6.10) in terms of u and v gives

$$\begin{aligned} u' &= (u + 1)(-v) = -v - uv, \\ v' &= (v + 1)(-u) = -u - uv, \end{aligned}$$

which is a system of differential equations for the small deviations. Again, because the deviations u and v from equilibrium are small we can ignore the products of u and v in favor of the larger linear terms. Then the system can be approximated by

$$u' = -v, \quad v' = -u.$$

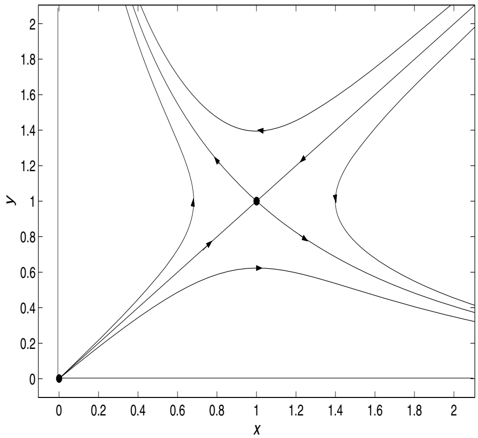

Figure 6.9 Phase portrait for the nonlinear system (6.10) with a saddle at $(1,1)$ and an unstable node at $(0,0)$.

This linear system has eigenvalues $\lambda = -1, 1$, and so $(0,0)$ is a saddle point for the uv-system. This leads us to suspect that $(1,1)$ is a saddle point for the nonlinear system (6.10). We can look at it in this way. If $x = 1+u$ and $y = 1+v$, and changes in u and v have an unstable saddle structure near $(0,0)$, then x and y should have a saddle structure near $(1,1)$. Indeed, the phase portrait for (6.10) is shown in figure 6.9 and it confirms our calculations. Although this is just a toy model of competition with both species having the same dynamics, it leads to an interesting conclusion. Both equilibria are unstable in the sense that small deviations from those equilibria put the populations on orbits that go away from those equilibrium states. There are always perturbations or deviations in a system. So, in this model, there are no persistent states. One of the populations, depending upon where the initial data are, will dominate and the other will approach extinction.

If a nonlinear system has an equilibrium, then the behavior of the orbits near that point is often mirrored by a linear system obtained by discarding the small nonlinear terms. We already know how to analyze linear systems; their behavior is determined by the eigenvalues of the associated matrix for the system. Therefore the general idea is to approximate the nonlinear system by a linear system in a neighborhood of the equilibrium and use the properties of the linear system to deduce the properties of the nonlinear system. This analysis, which is standard fare in differential equations, is called **local stability**

analysis. So, we begin with the system

$$x' = f(x,y) \tag{6.11}$$
$$y' = g(x,y). \tag{6.12}$$

Let $\mathbf{x}^* = (x_e, y_e)$ be an isolated equilibrium and let u and v denote small deviations (often called **small perturbations**) from equilibrium:

$$u = x - x_e, \quad v = y - y_e.$$

To determine if the perturbations grow or decay, we derive differential equations for those perturbations. Substituting into (6.11)–(6.12) we get, in terms of u and v, the system

$$\begin{aligned} u' &= f(x_e + u, y_e + v), \\ v' &= g(x_e + u, y_e + v). \end{aligned}$$

This system of equations for the perturbations has a corresponding equilibrium at $u = v = 0$. Now, in this system, we discard the nonlinear terms in u and v. Formally we can do this by expanding the right sides in Taylor series about point (x_e, y_e) to obtain

$$\begin{aligned} u' &= f(x_e,y_e) + f_x(x_e,y_e)u + f_y(x_e,y_e)v + \text{ higher-order terms in } u \text{ and } v, \\ v' &= g(x_e,y_e) + g_x(x_e,y_e)u + g_y(x_e,y_e)v + \text{ higher-order terms in } u \text{ and } v, \end{aligned}$$

where the higher-order terms are nonlinear terms involving powers of u and v and their products. The first terms on the right sides are zero because (x_e, y_e) is an equilibrium, and the higher-order terms are small in comparison to the linear terms (e.g., if u is small, say 0.1, then u^2 is much smaller, 0.01). Therefore the perturbation equations can be approximated by

$$\begin{aligned} u' &= f_x(x_e,y_e)u + f_y(x_e,y_e)v, \\ v' &= g_x(x_e,y_e)u + g_y(x_e,y_e)v. \end{aligned}$$

This linear system for the small deviations is called the linearized perturbation equations, or simply the **linearization** of (6.11)–(6.12) at the equilibrium (x_e, y_e). It has an equilibrium point at $(0, 0)$ corresponding to (x_e, y_e) for the nonlinear system. In matrix form we can write the linearization as

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} f_x(x_e,y_e) & f_y(x_e,y_e) \\ g_x(x_e,y_e) & g_y(x_e,y_e) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{6.13}$$

The matrix $J = J(x_e, y_e)$ of first partial derivatives of f and g defined by

$$J(x_e,y_e) = \begin{pmatrix} f_x(x_e,y_e) & f_y(x_e,y_e) \\ g_x(x_e,y_e) & g_y(x_e,y_e) \end{pmatrix}$$

is called the **Jacobian matrix** at the equilibrium (x_e, y_e). Note that this matrix is a matrix of numbers because the partial derivatives are evaluated at the equilibrium. We assume that J does not have a zero eigenvalue (i.e., $\det J \neq 0$). If so, we would have to look at the higher-order terms in the Taylor expansions of the right sides of the equations.

We already know that the nature of the equilibrium of (6.13) is determined by the eigenvalues of the matrix J. The question is: does the linearized system for the perturbations u and v near $u = v = 0$ aid in predicting the qualitative behavior in the nonlinear system of the solution curves near an equilibrium point (x_e, y_e)? The answer is yes in all cases except perhaps when the eigenvalues of the Jacobian matrix are purely imaginary (i.e., $\lambda = \pm bi$), or when there are two equal eigenvalues. Stated differently, the phase portrait of a nonlinear system close to an equilibrium point looks essentially the same as that of the linearization provided the eigenvalues have nonzero real part or are equal. Pictorially, near the equilibrium the small nonlinearities in the nonlinear system produce a slightly distorted phase diagram from that of the linearization. We summarize the basic results in the following items.

1. If $(0, 0)$ is asymptotically stable for the linearization (6.13), then the perturbations decay and (x_e, y_e) is asymptotically stable for the nonlinear system (6.11)–(6.12). This will occur when J has negative eigenvalues, or complex eigenvalues with negative real part.

2. If $(0, 0)$ is unstable for the linearization (6.13), then some or all of the perturbations grow and (x_e, y_e) is unstable for the nonlinear system (6.11)–(6.12). This will occur when J has a positive eigenvalue or complex eigenvalues with positive real part.

3. The exceptional case for stability is that of a center. If $(0, 0)$ is a center for the linearization (6.13), then (x_e, y_e) may be asymptotically stable, unstable, or a center for the nonlinear system (6.11)–(6.12). This case occurs when J has purely imaginary eigenvalues.

4. The borderline cases (equal eigenvalues) of degenerate and star-like nodes maintain stability, but the type of equilibria may change. For example, the inclusion of nonlinear terms can change a star-like node into a spiral, but it will not affect stability.

This means if the linearization predicts a regular node, saddle, or spiral at $(0, 0)$, then the nonlinear system will have a regular node, saddle, or spiral at the equilibrium (x_e, y_e). In the case of regular nodes and saddles, the directions of the eigenvectors give the directions of the tangent lines to the special curves that enter or exit the equilibrium point. Such curves are called **separatrices** (singular: **separatrix**). For linear systems the separatrices are the linear orbits entering or leaving the origin in the case of a saddle or node.

Sometimes we are only interested in whether an equilibrium is stable, and not whether it is a node or spiral. Stability can be determined by examining the trace of J and the determinant of J. We recall from Chapter 5:

– The equilibrium (x_e, y_e) is asymptotically stable if and only if

$$\mathrm{tr} J(x_e, y_e) < 0 \quad \text{and} \quad \det J(x_e, y_e) > 0. \tag{6.14}$$

Example 6.6

Consider the decoupled nonlinear system

$$x' = x - x^3, \quad y' = 2y.$$

The equilibria are $(0, 0)$ and $(\pm 1, 0)$. The Jacobian matrix at an arbitrary (x, y) for the linearization is

$$J(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} 1-3x^2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Therefore

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

which has eigenvalues 1 and 2. Thus $(0, 0)$ is an unstable node. Next

$$J(1,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \quad J(-1,0) = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$

and both have eigenvalues -2 and 2. Therefore $(1, 0)$ and $(-1, 0)$ are saddle points. The phase diagram is easy to draw. The vertical nullclines are $x = 0$, $x = 1$, and $x = -1$, and the horizontal nullcline $y = 0$. Along the x axis we have $x' > 0$ if $-1 < x < 1$, and $x' < 0$ if $|x| > 1$. The phase portrait is shown in figure 6.10.

Example 6.7

Consider the Lotka–Volterra model

$$x' = x(r - ay), \quad y' = y(-m + bx). \tag{6.15}$$

The equilibria are (0, 0) and $(m/b, r/a)$. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} r-ay & -ax \\ by & -m+bx \end{pmatrix}.$$

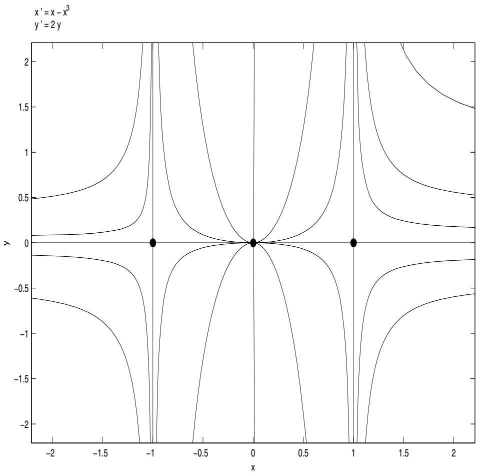

Figure 6.10 Phase diagram for the system $x' = x - x^3, \quad y' = 2y$. In the upper half-plane the orbits are moving upward, and in the lower half-plane they are moving downward.

We have

$$J(0,0) = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix},$$

which has eigenvalues r and $-m$. Thus $(0,0)$ is a saddle. For the other equilibrium,

$$J(m/b, r/a) = \begin{pmatrix} 0 & -am/b \\ rb/a & 0 \end{pmatrix}.$$

The characteristic equation is $\lambda^2 + rm = 0$, and therefore the eigenvalues are purely imaginary: $\lambda = \pm\sqrt{rm}$. This is the exceptional case; we cannot conclude that the equilibrium is a center, and we must work further to determine the nature of the equilibrium. We did this in Section 6.2.1 and found that $(m/b, r/a)$ was indeed a center.

Example 6.8

The nonlinear system

$$\begin{aligned} x' &= \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2), \\ y' &= x + \frac{1}{2}y - \frac{1}{2}(y^3 + yx^2), \end{aligned}$$

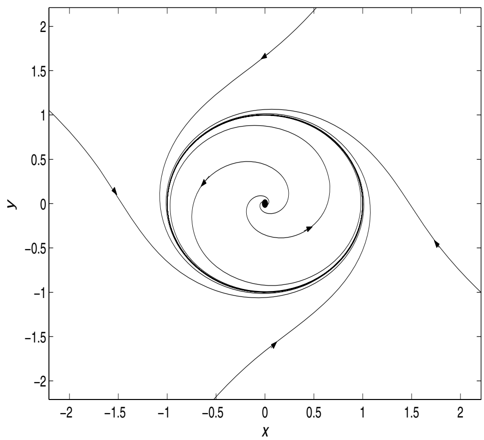

Figure 6.11 Orbits spiral out from the origin and approach the limit cycle $x^2 + y^2 = 1$, which is a closed, periodic orbit. Orbits outside the limit cycle spiral toward it. We say the limit cycle is stable.

has an equilibrium at the origin. The linearized system is

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

with eigenvalues $\frac{1}{2} \pm i$. Therefore the origin is an unstable spiral point. One can check the direction field near the origin to see that the spirals are counterclockwise. Do these spirals go out to infinity? We do not know without further analysis. We have only checked the local behavior, near the equilibrium. What happens beyond that is unknown and is described as the *global behavior* of the system. Using software, in fact, shows that there is cycle at radius one and the spirals coming out of the origin approach that cycle from within. Outside the closed cycle the orbits come in from infinity and approach the cycle. See figure 6.11. A cycle, or periodic solution, that is approached by another orbit as $t \to +\infty$ or as $t \to -\infty$ is called a **limit cycle**.

One can use computer algebra systems, or even a calculator, to draw phase diagrams. With computer algebra systems there are two options. You can write a program to numerically solve and plot the solutions (e.g., a Runge-Kutta routine), or you can use built-in programs that plot solutions automatically. Another option is to use codes developed by others to sketch phase diagrams. One of the best is a MATLAB code, *pplane6,* developed by Professor John Polking at Rice University (see the references for further information).

In summary, we have developed a set of tools to analyze nonlinear systems. We can systematically follow the steps below to obtain a complete phase diagram.

1. Find the equilibrium solutions and check their nature by examining the eigenvalues of the Jacobian J for the linearized system.
2. Draw the nullclines and indicate the direction of the vector field along those lines.
3. Find the direction of the vector field in the regions bounded by the nullclines.
4. Find directions of the separatrices (if any) at equilibria, indicated by the eigenvectors of J.
5. By dividing the equations, find the orbits (this may be impossible in many cases).
6. Use a software package or graphing calculator to get a complete phase diagram.

Example 6.9

A model of vibrations of a nonlinear spring with restoring force $F(x) = -x + x^3$ is

$$x'' = -x + x^3,$$

where the mass is $m = 1$. As a system,

$$x' = y, \quad y' = -x + x^3,$$

where y is the velocity. The equilibria are easily $(0,0)$, $(1,0)$, and $(-1,0)$. Let us check their nature. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} 0 & 1 \\ -1 + 3x^2 & 0 \end{pmatrix}.$$

Then

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J(1,0) = J(-1,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.$$

The eigenvalues of these two matrices are $\pm i$ and $\pm\sqrt{2}$, respectively. Thus $(-1,0)$ and $(1,0)$ are saddles and are unstable; $(0,0)$ is a center for the linearization, which gives us no information about that point for the nonlinear system. It is easy to see that the x-nullcline (vertical vector field) is $y = 0$, or the x-axis, and the y-nullclines (horizontal vector field) are the three lines $x = 0, 1, -1$. The directions of the separatrices coming in and out of the saddle points are given by the eigenvectors of the Jacobian matrix, which are easily found to be $(1, \pm\sqrt{2})^T$. So we have an accurate picture of the phase plane structure except near the origin. To analyze the behavior near the origin we can find formulas for the orbits. Dividing the two differential equations gives

$$\frac{dy}{dx} = \frac{-x + x^3}{y},$$

which, using separation of variables, integrates to

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4 = E,$$

where E is a constant of integration. Again, observe that this expression is just the conservation of energy law because the kinetic energy is $\frac{1}{2}y^2$ and the potential energy is $V(x) = -\int F(x)dx = -\int(-x + x^3)dx = \frac{1}{2}x^2 - \frac{1}{4}x^4$. We can solve for y to obtain

$$y = \pm\sqrt{2}\sqrt{E - \frac{1}{2}x^2 + \frac{1}{4}x^4}.$$

These curves can be plotted for different values of E and we find that they are cycles near the origin. So the origin is a center, which is neutrally stable. A phase diagram is shown in figure 6.12. This type of analysis can be carried out for any conservative mechanical system $x'' = F(x)$. The orbits are always given by $y = \pm\sqrt{2}\sqrt{E - V(x)}$, where $V(x) = -\int F(x)dx$ is the potential energy.

In summary, what we described in this section is **local stability analysis**, that is, how small perturbations from equilibrium evolve in time. Local stability analysis turns a nonlinear problem into a linear one, and it is a procedure that answers the question of what happens when we perturb the states x and y a small amount from their equilibrium values. Local analysis does not give any information about global behavior of the orbits far from equilibria, but it usually does give reliable information about perturbations near equilibria. The local behavior is determined by the eigenvalues of the Jacobian matrix, or the matrix of the linearized system. The only exceptional case is that of a center. One big difference between linear and nonlinear systems is that linear systems, as discussed in Chapter 5, can be solved completely and the global behavior of solutions is known. For nonlinear systems we can often obtain only local behavior near equilibria; it is difficult to tie down the global behavior.

One final remark. In Chapter 1 we investigated a single autonomous equation, and we plotted on a bifurcation diagram how equilibria and their stability change as a function of some parameter in the problem. This same type of behavior is also interesting for systems of equations. As a parameter in a given nonlinear system varies, the equilibria vary and stability can change. Some of the Exercises explore bifurcation phenomena in such systems.

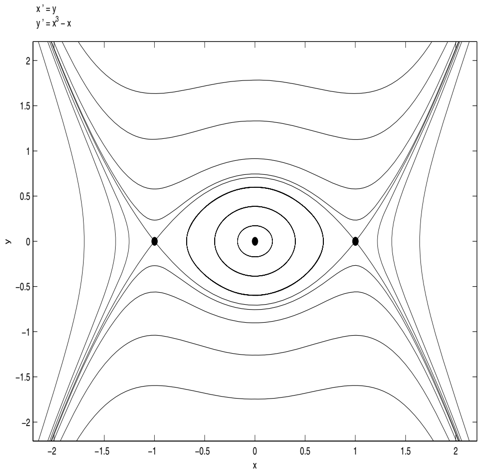

Figure 6.12 Phase portrait of the system $x' = y, \quad y' = -x + x^3$. The orbits are moving to the right in the upper half-plane and to the left in the lower half-plane.

EXERCISES

1. Find the equation of the orbits of the system $x' = e^x - 1, \;\; y' = ye^x$ and plot the the orbits in phase plane.

2. Write down an equation for the orbits of the system $x' = y, \;\; y' = 2y + xy$. Sketch the phase diagram.

3. For the following system find the equilibria, sketch the nullclines and the direction of the flow along the nullclines, and sketch the phase diagram:

$$x' = y - x^2, \;\; y' = 2x - y.$$

   What happens to the orbit beginning at $(1, 3/2)$ as $t \to +\infty$?

4. Determine the nature of each equilibrium of the system $x' = 4x^2 - a, \quad y' = -\frac{y}{4}(x^2 + 4)$, and show how the equilibria change as the parameter a varies.

5. Consider the system

$$\begin{aligned} x' &= 2x(1 - \frac{x}{2}) - xy, \\ y' &= y\left(\frac{9}{4} - y^2\right) - x^2 y. \end{aligned}$$

   Find the equilibria and sketch the nullclines. Use the Jacobian matrix to determine the type and stability of each equilibrium point and sketch the phase portrait.

6. Completely analyze the nonlinear system

$$x' = y, \quad y' = x^2 - 1 - y.$$

7. In some systems there are snails with two types of symmetry. Let R be the number of right curling snails and L be the number of left curling snails. The population dynamics is given by the competition equations

$$\begin{aligned} R' &= R - (R^2 + aRL) \\ L' &= L - (L^2 + aRL), \end{aligned}$$

where a is a positive constant. Analyze the behavior of the system for different values of a. Which snail dominates?

8. Consider the system

$$\begin{aligned} x' &= xy - 2x^2 \\ y' &= x^2 - y. \end{aligned}$$

Find the equilibria and use the Jacobian matrix to determine their types and stability. Draw the nullclines and indicate on those lines the direction of the vector field. Draw a phase diagram.

9. The dynamics of two competing species is governed by the system

$$\begin{aligned} x' &= x(10 - x - y), \\ y' &= y(30 - 2x - y). \end{aligned}$$

Find the equilibria and sketch the nullclines. Use the Jacobian matrix to determine the type and stability of each equilibrium point and sketch the phase diagram.

10. Show that the origin is asymptotically stable for the system

$$\begin{aligned} x' &= y, \\ y' &= 2y(x^2 - 1) - x. \end{aligned}$$

11. Consider the system

$$\begin{aligned} x' &= y, \\ y' &= -x - y^3. \end{aligned}$$

Show that the origin for the linearized system is a center, yet the nonlinear system itself is asymptotically stable. (Hint: show that $\frac{d}{dt}(x^2 + y^2) < 0$.)

12. A particle of mass 1 moves on the x-axis under the influence of a potential $V(x) = x - \frac{1}{3}x^3$. Formulate the dynamics of the particle in x, y coordinates, where y is velocity, and analyze the system in the phase plane. Specifically, find and classify the equilibria, draw the nullclines, determine the xy equation for the orbits, and plot the phase diagram.

13. A system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y), \end{aligned}$$

is called a **Hamiltonian system** if there is a function $H(x,y)$ for which $f = H_y$ and $g = -H_x$. The function H is called the **Hamiltonian**. Prove the following facts about Hamiltonian systems.

a) If $f_x + g_y = 0$, then the system is Hamiltonian. (Recall that $f_x + g_y$ is the divergence of the vector field (f,g).)

b) Prove that along any orbit, $H(x,y) =$ constant, and therefore all the orbits are given by $H(x,y) =$ constant.

c) Show that if a Hamiltonian system has an equilibrium, then it is not a source or sink (node or spiral).

d) Show that any conservative dynamical equation $x'' = f(x)$ leads to a Hamiltonian system, and show that the Hamiltonian coincides with the total energy.

e) Find the Hamiltonian for the system $x' = y, \quad y' = x - x^2$, and plot the orbits.

14. In a Hamiltonian system the Hamiltonian given by $H(x,y) = x^2 + 4y^4$. Write down the system and determine the equilibria. Sketch the orbits.

15. A system

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y), \end{aligned}$$

is called a **gradient system** if there is a function $G(x,y)$ for which $f = G_x$ and $g = G_y$.

a) If $f_y - g_x = 0$, prove that the system is a gradient system. (Recall that $f_y - g_x$ is the curl of the two-dimensional vector field (f,g); a zero curl ensures existence of a potential function on nice domains.)

b) Prove that along any orbit, $\frac{d}{dt}G(x,t) \geq 0$. Show that periodic orbits are impossible in gradient systems.

c) Show that if a gradient system has an equilibrium, then it is not a center or spiral.

d) Show that the system $x' = 9x^2 - 10xy^2, \quad y' = 2y - 10x^2y$ is a gradient system.

e) Show that the system $x' = \sin y, \ \ y' = x\cos y$ has no periodic orbits.

16. The populations of two competing species x and y are modeled by the system

$$\begin{aligned} x' &= (K - x)x - xy, \\ y' &= (1 - 2y)y - xy, \end{aligned}$$

where K is a positive constant. In terms of K, find the equilibria. Explain how the equilibria change, as to type and stability, as the parameter K increases through the interval $0 < K \le 1$, and describe how the phase diagram evolves. Especially describe the nature of the change at $K = 1/2$.

17. Give a thorough description, in terms of equilibria, stability, and phase diagram, of the behavior of the system

$$\begin{aligned} x' &= y + (1 - x)(2 - x), \\ y' &= y - ax^2, \end{aligned}$$

as a function of the parameter $a > 0$.

18. A predator–prey model is given by

$$\begin{aligned} x' &= rx\left(1 - \frac{x}{K}\right) - f(x)y, \\ y' &= -my + cf(x)y, \end{aligned}$$

where r, m, c, and K are positive parameters, and the predation rate $f(x)$ satisfies $f(0) = 0$, $f'(x) > 0$, and $f(x) \to M$ as $x \to \infty$.

a) Show that $(0,0)$ and $(K,0)$ are equilibria.

b) Classify the $(0,0)$ equilibrium. Find conditions that guarantee that $(K,0)$ is unstable and state what type of unstable point it is.

c) Under what conditions will there be an equilibrium in the first quadrant?

19. Consider the dynamical equation $x'' = f(x)$, with $f(x_0) = 0$. Find a condition that guarantees that $(x_0, 0)$ will be a saddle point in the phase plane representation of the problem.

20. The dynamics of two competing species is given by

$$\begin{aligned} x' &= 4x(1 - x/4) - xy, \\ y' &= 2y(1 - ay/2) - bxy. \end{aligned}$$

For which values of a and b can the two species coexist? Physically, what do the parameters a and b represent?

21. A particle of mass $m = 1$ moves on the x-axis under the influence of a force $F = -x + x^3$ as discussed in Example 6.8.

   a) Determine the values of the total energy for which the motion will be periodic.

   b) Find and plot the equation of the orbit in phase space of the particle if its initial position and velocity are $x(0) = 0.5$ and $y(0) = 0$. Do the same if $x(0) = -2$ and $y(0) = 2$.

6.4 Periodic Solutions

We noted the exceptional case in the linearization procedure: if the associated linearization for the perturbations has a center (purely imaginary eigenvalues) at (0,0), then the behavior of the nonlinear system at the equilibrium is undetermined. This fact suggests that the existence of periodic solutions, or (closed) cycles, for nonlinear systems is not always easily decided. In this section we discuss some special cases when we can be assured that periodic solutions do not exist, and when they do exist. The presence of oscillations in physical and biological systems often represent important phenomena, and that is why such solutions are of great interest.

We first state two negative criteria for the nonlinear system

$$x' = f(x, y) \tag{6.16}$$

$$y' = g(x, y). \tag{6.17}$$

1. (**Equilibrium Criterion**) If the nonlinear system (6.16)–(6.17) has a cycle, then the region inside the cycle must contain an equilibrium. Therefore, if there are no equilibria in a given region, then the region can contain no cycles.

2. (**Dulac's Criterion**) Consider the nonlinear system (6.16)–(6.17). If in a given region of the plane there is a function $\beta(x, y)$ for which

$$\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g)$$

is of one sign (strictly positive or strictly negative) entirely in the region, then the system cannot have a cycle in that region.

We omit the proof of the equilibrium criterion (it may be found in the references), but we give the proof of Dulac's criterion because it is a simple application of Green's theorem,[3] which was encountered in multi-variable calculus. The proof is by contradiction, and it assumes that there *is* a cycle of period p given by $x = x(t)$, $y = y(t)$, $\quad 0 \le t \le p$, lying entirely in the region and represented by a simple closed curve C. Assume it encloses a domain R. Without loss of generality suppose that $\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g) > 0$. Then, to obtain a contradiction, we make the following calculation.

$$\begin{aligned} 0 &< \int\int_R \left(\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g)\right) dA = \int_C (-\beta g dx + bf dy) \\ &= \int_0^p (-\beta g x' dt + bf y' dt) = \int_0^p (-\beta g f dt + bf g dt) = 0, \end{aligned}$$

the contradiction being $0 < 0$. Therefore the assumption of a cycle is false, and there can be no periodic solution.

Example 6.10

The system

$$x' = 1 + y^2, \quad y' = x - y + xy$$

does not have any equilibria (note x' can never equal zero), so this system cannot have cycles.

Example 6.11

Consider the system

$$x' = x + x^3 - 2y, \quad y' = -3x + y^3.$$

Then

$$\frac{\partial}{\partial x} f + \frac{\partial}{\partial x} g = \frac{\partial}{\partial x}(x + x^3 - 2y) + \frac{\partial}{\partial x}(-3x + y^3) = 1 + 3x^2 + 3y^2 > 0,$$

which is positive for all x and y. Dulac's criterion implies there are no periodic orbits in the entire plane. Note here that $\beta = 1$.

[3] For a region R enclosed by a simple closed curve C we have $\int_C P dx + Q dy = \int\int_R (Q_x - P_y) dA$, where C is taken counterclockwise.

One must be careful in applying Dulac's criterion. If we find that $\frac{\partial}{\partial x}(\beta f) + \frac{\partial}{\partial y}(\beta g) > 0$ in, say, the first quadrant only, then that means there are no cycles lying entirely in the first quadrant; but there still may be cycles that go out of the first quadrant.

Sometimes cycles can be detected easily in a polar coordinate system. Presence of the expression $x^2 + y^2$ in the system of differential equations often signals that a polar representation might be useful in analyzing the problem.

Example 6.12

Consider the system

$$\begin{aligned} x' &= y + x(1 - x^2 - y^2) \\ y' &= -x + y(1 - x^2 - y^2). \end{aligned}$$

The reader should check, by linearization, that the origin is an unstable spiral point. But what happens beyond that? To transform the problem to polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we note that

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Taking time derivatives and using the chain rule,

$$rr' = xx' + yy', \quad (\sec^2\theta)\theta' = \frac{xy' - yx'}{x^2}.$$

We can solve for r' and θ' to get

$$r' = x'\cos\theta + y'\sin\theta, \quad \theta' = \frac{y'\cos\theta - x'\sin\theta}{r}.$$

Finally we substitute for x' and y' on the right side from the differential equations to get the polar forms of the equations: $r' = F(r, \theta)$, $\theta' = G(r, \theta)$. Leaving the algebra to the reader, we finally get

$$\begin{aligned} r' &= r(1 - r^2), \\ \theta' &= -1. \end{aligned}$$

By direct integration of the second equation, $\theta = -t + C$, so the angle θ rotates clockwise with constant speed. Notice also that $r = 1$ is a solution to the first equation. Thus we have obtained a periodic solution, a circle of radius one, to the system. For $r < 1$ we have $r' > 0$, so r is increasing on orbits, consistent with our remark that the origin is an unstable spiral. For $r > 1$ we have $r' < 0$, so r is decreasing along orbits. Hence, there is a limit cycle that is approached by orbits from its interior and its exterior. Figure 6.13 shows the phase diagram.

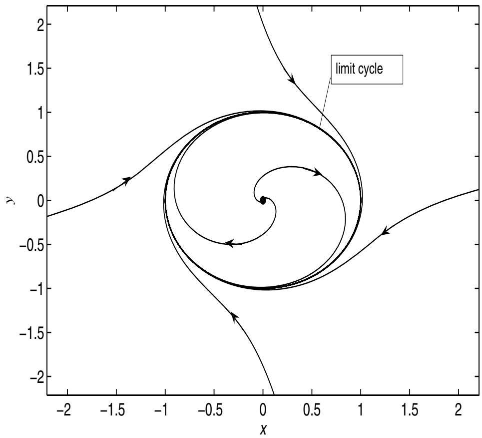

Figure 6.13 Limit cycle. The orbits rotate clockwise.

6.4.1 The Poincaré–Bendixson Theorem

To sum it up, through examples we have observed various nonlinear phenomena in the phase plane, including equilibria, orbits that approach equilibria, orbits that go to infinity, cycles, and orbits that approach cycles. What have we missed? Is there some other complicated orbital structure that is possible? The answer to this question is no; dynamical possibilities in a two-dimensional phase plane are very limited. If an orbit is confined to a closed bounded region in the plane, then as $t \to +\infty$ that orbit must be an equilibrium solution (a point), be a cycle, approach a cycle, or approach an equilibrium. (Recall that a closed region includes its boundary). The same result holds as $t \to -\infty$. This is a famous result called the **Poincaré–Bendixson theorem**, and it is proved in advanced texts. We remark that the theorem is not true in three dimensions or higher where orbits for nonlinear systems can exhibit bizarre behavior, for example, approaching sets of fractal dimension (strange attractors) or showing chaotic behavior. Henri Poincaré (1854–1912) was one of the great contributors to the theory of differential equations and dynamical systems.

Example 6.13

Consider the model

$$\begin{aligned} x' &= \frac{2}{3}x\left(1-\frac{x}{4}\right)-\frac{xy}{1+x}, \\ y' &= ry\left(1-\frac{y}{x}\right), \quad r>0. \end{aligned}$$

In an ecological context, we can think of this system as a predator–prey model. The prey (x) grow logistically and are harvested by the predators (y) with a Holling type II rate. The predator grows logistically, with its carrying capacity depending linearly upon the prey population. The horizontal, y-nullclines, are $y = x$ and $y = 0$, and the vertical, or x-nullcline is the parabola $y = \left(\frac{2}{3} - \frac{1}{6}x\right)(x+1)$. The equilibria are $(1,1)$, and $(4,0)$. The system is not defined when $x = 0$ and we classify the y-axis as a line of *singularities*; no orbits can cross this line. The Jacobian matrix is

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} \frac{2}{3} - \frac{1}{6}x - \frac{y}{(1+x)^2} & \frac{-x}{1+x} \\ \frac{ry^2}{x^2} & r - \frac{2ry}{x} \end{pmatrix}.$$

Evaluating at the equilibria yields

$$J(4,0) = \begin{pmatrix} -\frac{2}{3} & -\frac{4}{5} \\ 0 & r \end{pmatrix}, \quad J(1,1) = \begin{pmatrix} \frac{1}{12} & -\frac{1}{2} \\ r & -r \end{pmatrix}.$$

It is clear that $(4,0)$ is a saddle point with eigenvalues r and $-2/3$. At $(1,1)$ we find $\mathrm{tr}J = \frac{1}{12} - r$ and $\det J = \frac{5}{12}r > 0$. Therefore $(1,1)$ is asymptotically stable if $r > \frac{1}{12}$ and unstable if $r < \frac{1}{12}$. So, there is a bifurcation, or change, at $r = \frac{1}{12}$ because the stability of the equilibrium changes. For a large predator growth rate r there is a nonzero persistent state where predator and prey can coexist. As the growth rate of the predator decreases to a critical value, this persistence goes away. What happens then? Let us imagine that the system is in the stable equilibrium state and other factors, possibly environmental, cause the growth rate of the predator to slowly decrease. How will the populations respond once the critical value of r is reached?

Let us carefully examine the case when $r < \frac{1}{12}$. Consider the direction of the vector field on the boundary of the square with corners $(0,0)$, $(4,0)$, $(4,4)$, $(0,4)$. See figure 6.14. On the left side ($x = 0$) the vector field is undefined, and near that boundary it is nearly vertical; orbits cannot enter or escape along that edge. On the lower side ($y = 0$) the vector field is horizontal ($y' = 0$, $x' > 0$). On the right edge ($x = 4$) we have $x' < 0$ and $y' > 0$, so the vector field points into the square. And, finally, along the upper edge ($y = 4$) we have $x' < 0$ and $y' < 0$, so again the vector field points into the square. The equilibrium at $(1,1)$ is unstable, so orbits go away from equilibrium; but they cannot escape from the

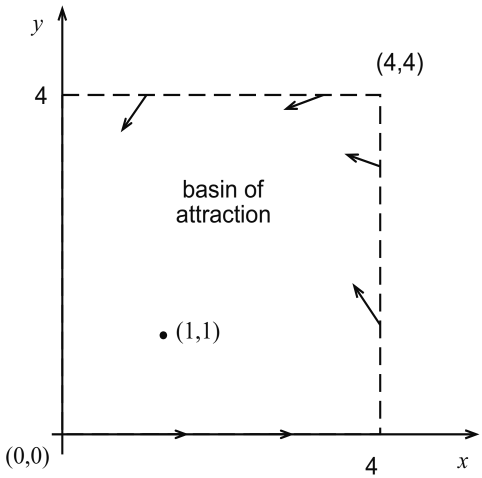

Figure 6.14 A square representing a basin of attraction. Orbits cannot escape the square.

square. On the other hand, orbits along the top and right sides are entering the square. What can happen? They cannot crash into each other! (Uniqueness.) So, there must be a counterclockwise limit cycle in the interior of the square (by the Poincaré–Bendixson theorem). The orbits entering the square approach the cycle from the outside, and the orbits coming out of the unstable equilibrium at $(1, 1)$ approach the cycle from the inside. Now we can state what happens as the predator growth rate r decreases through the critical value. The persistent state becomes unstable and a small perturbation, always present, causes the orbit to approach the limit cycle. Thus, we expect the populations to cycle near the limit cycle. A phase diagram is shown in figure 6.15.

In this example we used a common technique of constructing a region, called a **basin of attraction**, that contains an unstable spiral (or node), yet orbits cannot escape the region. In this case there must be a limit cycle in the region. A similar result holds true for annular type regions (doughnut type regions bounded by concentric simple close curves)—if there are no equilibria in an annular region R and the vector field points inward into the region on both the inner and outer concentric boundaries, then there must be a limit cycle in R.

EXERCISES

1. Does the system

$$\begin{aligned} x' &= x - y - x\sqrt{x^2 + y^2}, \\ y' &= x + y - y\sqrt{x^2 + y^2}, \end{aligned}$$

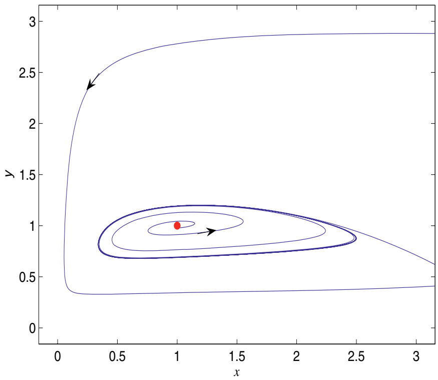

Figure 6.15 Phase diagram showing a counterclockwise limit cycle. Curves approach the limit cycle from the outside and from the inside. The interior equilibrium is an unstable spiral point.

have periodic orbits? Does it have limit cycles?

2. Show that the system

$$x' = 1 + x^2 + y^2,$$
$$y' = (x-1)^2 + 4,$$

has no periodic solutions.

3. Show that the system

$$x' = x + x^3 - 2y,$$
$$y' = y^5 - 3x,$$

has no periodic solutions.

4. Analyze the dynamics of the system

$$x' = y,$$
$$y' = -x(1-x) + cy,$$

for different positive values of c. Draw phase diagrams for each case, illustrating the behavior.

5. An RCL circuit with a nonlinear resistor (the voltage drop across the resistor is a nonlinear function of the current) can be modeled by the Van der Pol equation
$$x'' + a(x^2 - 1)x' + x = 0,$$
where a is a positive constant, and $x = x(t)$ is the current. In the phase plane formulation, show that the origin is unstable. Sketch the nullclines and the vector field. Can you tell if there is a limit cycle? Use a computer algebra system to sketch the phase plane diagram in the case $a = 1$. Draw a time series plot for the current in this case for initial conditions $x(0) = 0.05$, $x'(0) = 0$. Is there a limit cycle?

6. For the system
$$\begin{aligned} x' &= y, \\ y' &= x - y - x^3, \end{aligned}$$
determine the equilibria. Write down the Jacobian matrix at each equilibrium and investigate stability. Sketch the nullclines. Finally, sketch a phase diagram.

7. Let P denote the carbon biomass of plants in an ecosystem and H the carbon biomass of herbivores. Let ϕ denote the constant rate of primary production of carbon in plants due to photosynthesis. Then a model of plant–herbivore dynamics is given by
$$\begin{aligned} P' &= \phi - aP - bHP, \\ H' &= \varepsilon bHP - cH, \end{aligned}$$
where a, b, c, and ε are positive parameters.

   a) Explain the various terms in the model and determine the dimensions of each constant.

   b) Find the equilibrium solutions.

   c) Analyze the dynamics in two cases, that of high primary production ($\phi > ac/\varepsilon b$) and low primary production ($\phi < ac/\varepsilon b$). Determine what happens to the system if the primary production is slowly increased from a low value to a high value.

8. Consider the system
$$x' = ax + y - x(x^2 + y^2), \quad y' = -x + ay - y(x^2 + y^2),$$
where a is a parameter. Discuss the qualitative behavior of the system as a function of the parameter a. In particular, how does the phase plane evolve as a is changed?

9. Show that periodic orbits, or cycles, for the system

$$x' = y, \quad y' = -ky - V'(x)$$

are possible only if $k = 0$.

10. Consider the system

$$x' = x(P - ax + by), \quad y' = y(Q - cy + dx),$$

where a, $c > 0$. Show that there cannot be periodic orbits in the first quadrant of the xy plane. (Hint: take $\beta = (xy)^{-1}$.)

11. Analyze the nonlinear system

$$\begin{aligned} x' &= y - x, \\ y' &= -y + \frac{5x^2}{4 + x^2}. \end{aligned}$$

12. (Project) Consider two competing species where one of the species immigrates or emigrates at constant rate h. The populations are governed by the dynamical equations

$$\begin{aligned} x' &= x(1 - ax) - xy, \\ y' &= y(b - y) - xy + h, \end{aligned}$$

where $a, b > 0$.

a) In the case $h = 0$ (no immigration or emigration) give a complete analysis of the system and indicate in a, b parameter space (i.e., in the ab plane) the different possible behaviors, including where bifurcations occur. Include in your discussion equilibria, stability, and so forth.

b) Repeat part (a) for various fixed values of h, with $h > 0$.

c) Repeat part (a) for various fixed values of h, with $h < 0$.

A
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B
Computer Algebra Systems

There is great diversity in differential equations courses with regard to technology use, and there is equal diversity regarding the choice of technology. MATLAB, Maple, and Mathematica are common computer environments used at many colleges and universities. MATLAB, in particular, has become an important tool in scientific computation; Maple and Mathematica are computer algebra systems that are used for symbolic computation. There is also an add-on symbolic toolbox for the professional version of MATLAB; the student edition comes with the toolbox. In this appendix we present a list of useful commands in Maple and MATLAB. The presentation is only for reference and to present some standard templates for tasks that are commonly faced in differential equations. It is not meant to be an introduction or tutorial to these environments, but only a statement of the syntax of a few basic commands. The reader should realize that these systems are updated regularly, so there is danger that the commands will become obsolete quickly as new versions appear.

Advanced scientific calculators also permit symbolic computation and can perform many of the same tasks. Manuals that accompany these calculators give specific instructions that are not be repeated here.

B.1 Maple

Maple has single, automatic commands that perform most of the calculations and graphics used in differential equations. There are excellent Maple application manuals available, but everything required can be found in the help menu in the program itself. A good strategy is to find what you want in the help menu, copy and paste it into your Maple worksheet, and then modify it to conform to your own problem. Listed below are some useful commands for plotting solutions to differential equations, and for other calculations. The output of these commands is not shown; we suggest the reader type these commands in a worksheet and observe the results. There are packages that must be loaded before making some calculations: `with(plots):` `with(DEtools):` and `with(linalg):` In Maple, a colon suppresses output, and a semicolon presents output.

Define a function $f(t,u) = t^2 - 3u$:

```
f:=(t,u) → t^2-3*u;
```

Draw the slope field for the DE $u' = \sin(t-u)$:

```
DEplot(diff(u(t),t)=sin(t-u(t)),u(t),t=-5..5,u=-5..5);
```

Plot a solution satisfying $u(0) = -0.25$ superimposed upon the slope field:

```
DEplot(diff(u(t),t)=sin(t-u(t)),u(t),t=-5..5,
u=-5..5,[[u(0)=-.25]]);
```

Find the general solution of a differential equation $u' = f(t,u)$ symbolically:

```
dsolve(diff(u(t),t)=f(t,u(t)),u(t));
```

Solve an initial value problem symbolically:

```
dsolve({diff(u(t),t) = f(t,u(t)), u(a)=b}, u(t));
```

Plot solution to: $u'' + \sin u = 0$, $u(0) = 0.5$, $u'(0) = 0.25$.

```
DEplot(diff(u(t),t$2)+sin(u(t)),u(t),t=0..10,
[[u(0)=.5,D(u)(0)=.25]],stepsize=0.05);
```

Euler's method for the IVP $u' = \sin(t-u)$, $u(0) = -0.25$:

```
f:=(t,u) → sin(t-u):
t0:=0: u0:=-0.25: Tfinal:=3:
n:=10: h:=evalf((Tfinal-t0)/n):
t:=t0: u=u0:
for i from 1 to n do
u:=u+h*f(t,u):
t:=t+h:
print(t,u);
od:
```

Set up a matrix and calculate the eigenvalues, eigenvectors, and inverse:

```
with(linalg):
A:=array([[2,2,2],[2,0,-2],[1,-1,1]]);
eigenvectors(A);
eigenvalues(A);
inverse(A);
```

Solve a linear algebraic system:

$A\mathbf{x} = \mathbf{b}$:

```
b:=matrix(3,1,[0,2,3]);
x:=linsolve(A,b);
```

Solve a linear system of DEs with two equations:

```
eq1:=diff(x(t),t)=-y(t):
eq2:=diff(y(t),t)=-x(t)+2*y(t):
dsolve({eq1,eq2},{x(t),y(t)});
dsolve({eq1,eq2,x(0)=2,y(0)=1},{x(t),y(t)});
```

A fundamental matrix associated with the linear system $\mathbf{x}' = A\mathbf{x}$:

```
Phi:=exponential(A,t);
```

Plot a phase diagram in two dimensions:

```
with(DEtools):
eq1:=diff(x(t),t)=y(t):
eq2:=2*diff(y(t),t)=-x(t)+y(t)-y(t)^3:
DEplot([eq1,eq2],[x,y],t=-10..10,x=-5..5,y=-5..5,
{[x(0)=-4,y(0)=-4],[x(0)=-2,y(0)=-2] },
arrows=line, stepsize=0.02);
```

Plot time series:

```
DEplot([eq1,eq2],[x,y],t=0..10,
{[x(0)=1,y(0)=2] },scene=[t,x],arrows=none,stepsize=0.01);
```

Laplace transforms:

```
with(inttrans):
u:=t*sin(t):
U:=laplace(u,t,s):
U:=simplify(expand(U));
u:=invlaplace(U,s,t):
```

Display several plots on same axes:

```
with(plots):
p1:=plot(sin(t), t=0..6): p2:=plot(cos(2*t), t=0..6):
display(p1,p2);
```

Plot a family of curves:

```
eqn:=c*exp(-0.5*t):
curves:={seq(eqn,c=-5..5)}:
plot(curves, t=0..4, y=-6..6);
```

Solve a nonlinear algebraic system: `fsolve({2*x-x*y=0,-y+3*x*y=0},{x,y},
{x=0.1..5,y=0..4});`

Find an antiderivative and definite integral:

```
int(1/(t*(2-t)),t); int(1/(t*(2-t)),t=1..1.5);
```

B.2 MATLAB

There are many references on MATLAB applications in science and engineering. Among the best is Higham & Higham (2000). The MATLAB files *dfield7.m* and *pplane7.m*, developed by J. Polking (2004), are two excellent programs for solving and graphing solutions to differential equations. These programs can be downloaded from his Web site (see references). In the table we list several common MATLAB commands. We do not include commands from the symbolic toolbox.

An m-file for Euler's Method. For scientific computation we often write several lines of code to perform a certain task. In MATLAB, such a code, or program, is written and stored in an **m-file**. The m-file below is a program of the Euler method for solving a pair of DEs, namely, the predator–prey system

$$x' = x - 2 * x^2 - xy, \quad y' = -2y + 6xy,$$

subject to initial conditions $x(0) = 1$, $y(0) = 0.1$. The m-file *euler.m* plots the time series solution on the interval $[0, 15]$.

```
function euler
x=1; y=0.1; xhistory=x; yhistory=y; T=15; N=200; h=T/N;
for n=1:N
u=f(x,y); v=g(x,y);
x=x+h*u; y=y+h*v;
xhistory=[xhistory,x]; yhistory=[yhistory,y];
end
t=0:h:T;
plot(t,xhistory,'-',t,yhistory,'--')
xlabel('time'), ylabel('prey (solid),predator (dashed)')
function U=f(x,y)
U=x-2*x.*x-x.*y;
function V=g(x,y)
V=-2*y+6*x.*y;
```

Direction Fields. The `quiver` command plots a vector field in MATLAB. Consider the system

$$x' = x(8 - 4x - y), \quad y' = y(3 - 3x - y).$$

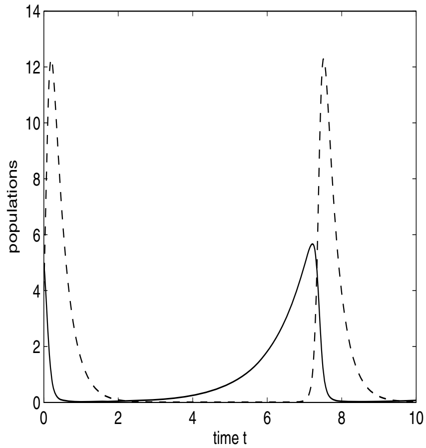

Figure B.1 Predator (dashed) and prey (solid) populations.

To plot the vector field on $0 < x < 3$, $0 < y < 4$ we use:

```
[x,y] = meshgrid(0:0.3:3, 0:0.4:4];
dx = x.*(8-4*x-y);  dy = y.*(3-3*x-y);
quiver(x,y,dx,dy)
```

Using the DE Packages. MATLAB has several differential equations routines that numerically compute the solution to an initial value problem. To use these routines we define the DEs in one m-file and then write a short program in a second m-file that contains the routine and a call to our equations from the first m-file. The files below use the package `ode45`, which is a Runge–Kutta solver with an adaptive stepsize. Consider the initial value problem

$$u' = 2u(1 - 0.3u) + \cos 4t, \quad 0 < t < 3, \quad u(0) = 0.1.$$

We define the differential equation in the m-file:

```
function uprime = f(t,u)
uprime = 2*u.*(1-0.3*u)+cos(4*t);
```

Then we run the m-file:

```
function diffeq
trange = [0  3]; ic=0.1;
[t,u] = ode45(@uprime,trange,ic);
plot(t,u,'*--')
```

Solving a System of DEs. As for a single equation, we first write an m-file that defines the system of DEs. Then we write a second m-file containing a routine that calls the system. Consider the Lotka–Volterra model

$$x' = x - xy, \quad y' = -3y + 3xy,$$

with initial conditions $x(0) = 5$, $y(0) = 4$. Figure B.1 shows the time series plots. The two m-files are:

```
function deriv=lotka(t,z)
deriv=[z(1)-z(1).*z(2); -3*z(2)+3*z(1).*z(2)];

function lotkatimeseries
tspan=[0 10]; ics=[5;4];
[T,X]=ode45(@lotka,tspan,ics);
plot(T,X)
xlabel('time t'), ylabel('populations')
```

Phase Diagrams. To produce phase plane plots we simply plot z(1) versus z(2). We draw two orbits. The main m-file is:

```
function lotkaphase
tspan=[0 10]; ICa=[5;4]; ICb=[4;3];
[ta,ya]=ode45(@lotka,tspan, ICa);
[tb,yb]=ode45(@lotka,tspan, ICb);
plot(ya(:,1),ya(:,2), yb(:,1),yb(:,2))
```

The following table contains several useful MATLAB commands.

MATLAB Command	Instruction
>>	command line prompt
;	seimcolon suppresses output
clc	clear the command screen
Ctrl+C	stop a program
help *topic*	help on MATLAB *topic*
a = 4, A = 5	assigns 4 to a and 5 to A
clear a b	clears the assignments for a and b
clear all	clears all the variable assignments
x=[0, 3,6,9,12,15,18]	vector assignment
x=0:3:18	defines the same vector as above
x=linspace(0,18,7)	defines the same vector as above
+, -, *, /, ∧	operations with numbers
sqrt(a)	square root of a
exp(a), log(a)	e^a and $\ln a$
pi	the number π
.*, ./, .∧	operations on vectors of same length (with dot)
t=0:0.01:5, x=cos(t), plot(t,x)	plots $\cos t$ on $0 \leq t \leq 5$
xlabel('time'), ylabel('state')	labels horizontal and vertical axes
title('Title of Plot')	titles the plot
hold on, hold off	does not plot immediately; releases hold on
for n=1:N,...,end	syntax for a "for-end" loop from 1 to N
bar(x)	plots a bar graph of a vector x
plot(x)	plots a line graph of a vector x
A=[1 2;3 4]	defines a matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
x=A\b	solves Ax=b, where b=[α;β] is a column vector
inv(A)	the inverse matrix
det(A)	determinant of A
[V,D]=eig(A)	computes eigenvalues and eigenvectors of A
q=quad(fun,a,b,tol);	Approximates $\int_a^b \text{fun}(t)dt$, tol = error tolerance
function fun=f(t), fun=t.∧ 2	defines $f(x) = t^2$ in an m-file

C
Sample Examinations

Below are examinations on which students can assess their skills. Solutions are found on the author's Web site (see Preface).

Test 1 (1 hour)

1. Find the general solution to the equation $u'' + 3u' - 10u = 0$.
2. Find the function $u = u(t)$ that solves the initial value problem $u' = \frac{1+t^2}{t}, \quad u(1) = 0$.
3. A mass of 2 kg is hung on a spring with stiffness (spring constant) $k = 3$ N/m. After the system comes to equilibrium, the mass is pulled downward 0.25 m and then given an initial velocity of 1 m/sec. What is the amplitude of the resulting oscillation?
4. A particle of mass 1 moves in one dimension with *acceleration* given by $3 - v(t)$, where $v = v(t)$ is its velocity. If its initial velocity is $v = 1$, when, if ever, is the velocity equal to two?
5. Find $y'(t)$ if
$$y(t) = t^2 \int_1^t \frac{1}{r} e^{-r} \, dr.$$
6. Consider the initial value problem
$$u' = t^2 - u, \quad u(-2) = 0.$$

Use your calculator to draw the graph of the solution on the interval $-2 \leq t \leq 2$. Reproduce the graph on your answer sheet.

7. For the initial value problem in Problem 6, use the Euler method with stepsize $h = 0.25$ to estimate $u(-1)$.

8. For the differential equation in Problem 6, plot in the tu-plane the locus of points where the slope field has value -1.

9. At noon the forensics expert measured the temperature of a corpse and it was 85 degrees F. Two hours later it was 74 degrees. If the ambient temperature of the air was 68 degrees, use Newton's law of cooling to estimate the time of death. (Set up and solve the problem).

Test 2 (1 hour)

1. Consider the system
$$x' = xy, \quad y' = 2y.$$
Find a relation between x and y that must hold on the orbits in the phase plane.

2. Consider the system
$$x' = 2y - x, \quad y' = xy + 2x^2.$$
Find the equilibrium solutions. Find the nullclines and indicate the nullclines and equilibrium solutions on a phase diagram. Draw several interesting orbits.

3. Using a graphing calculator, sketch the solution $u = u(t)$ of the initial value problem
$$u'' + u' - 3\cos 2t = 0, \quad u(0) = 1, \quad u'(0) = 0$$
on the interval $0 < t < 6$.

4. Solve the initial value problem
$$u' - \frac{3}{t}u = t, \quad u(1) = 0.$$

5. Consider the autonomous equation
$$\frac{du}{dt} = -(u-2)(u-4)^2.$$
Find the equilibrium solutions, sketch the phase line, and indicate the type of stability of the equilibrium solutions.

6. Find the general solution to the linear differential equation

$$u'' - \frac{1}{t}u' + \frac{2}{t^2}u = 0.$$

7. Consider the two-dimensional linear system

$$\mathbf{x}' = \begin{pmatrix} 1 & 12 \\ 3 & 1 \end{pmatrix} \mathbf{x}.$$

   a) Find the eigenvalues and corresponding eigenvectors and identify the type of equilibrium at the origin.

   b) Write down the general solution.

   c) Draw a rough phase plane diagram, being sure to indicate the directions of the orbits.

8. A particle of mass $m = 2$ moves on a u-axis under the influence of a force $F(u) = -au$, where a is a positive constant. Write down the differential equation that governs the motion of the particle and then write down the expression for conservation of energy.

Test 3 (1 hour)

1. Find the equation of the orbits in the xy plane for the system $x' = 4y, \quad y' = 2x - 2$.

2. Consider a population model governed by the autonomous equation

$$p' = \sqrt{2}\,p - \frac{4p^2}{1 + p^2}.$$

   a) Sketch a graph of the growth rate p' vs. the population p, and sketch the phase line.

   b) Find the equilibrium populations and determine their stability.

3. For the following system, for which values of the constant b is the origin an unstable spiral?

$$\begin{aligned} x' &= x - (b+1)y \\ y' &= -x + y. \end{aligned}$$

4. Consider the nonlinear system

$$\begin{aligned} x' &= x(1 - xy), \\ y' &= 1 - x^2 + xy. \end{aligned}$$

a) Find all the equilibrium solutions.

b) In the xy plane plot the x and y nullclines.

5. Find a solution representing a linear orbit of the three-dimensional system

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \mathbf{x}.$$

6. Classify the equilibrium as to type and stability for the system

$$x' = x + 13y, \quad y' = -2x - y.$$

7. A two-dimensional system $\mathbf{x}x' = A\mathbf{x}$ has eigenpairs

$$-2, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad 1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

a) If $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, find a formula for $y(t)$ (where $\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

b) Sketch a rough, but accurate, phase diagram.

8. Consider the IVP

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y, \\ x(0) &= 3, \quad y(0) = -3. \end{aligned}$$

a) Use your calculator's graphical DE solver to plot the solution for $t > 0$ in the xy-phase plane.

b) Using your plot in (a), sketch $y(t)$ vs. t for $t > 0$.

Final Examination (2 hrs)

1. Find the general solution of the DE $u'' = u' + \frac{1}{2}u$.
2. Find a particular solution to the DE $u'' + 8u' + 16u = t^2$.
3. Find the (implicit) solution of the DE $u' = \frac{1+t}{3tu^2+t}$ that passes through the point $(1, 1)$.
4. Consider the autonomous system $u' = -u(u-2)^2$. Determine all equilibria and their stability. Draw a rough time series plot (u vs. t) of the solution that satisfies the initial condition $x(0) = 1$.

5. Consider the nonlinear system

$$x' = 4x - 2x^2 - xy, \quad y' = y - y^2 - 2xy.$$

Find all the equilibrium points and determine the type and stability of the equilibrium point $(2, 0)$.

6. An RC circuit has $R = 1$, $C = 2$. Initially the voltage drop across the capacitor is 2 volts. For $t > 0$ the applied voltage (emf) in the circuit is $b(t)$ volts. Write down an IVP for the *voltage* across the capacitor and find a formula for it.

7. Solve the IVP

$$u' + 3u = \delta_2(t) + h_4(t), \quad u(0) = 1.$$

8. Use eigenvalue methods to find the general solution of the linear system

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \mathbf{x}.$$

9. In a recent TV episode of *Miami: CSI*, Horatio took the temperature of a murder victim at the crime scene at 3:20 A.M. and found that it was 85.7 degrees F. At 3:50 A.M. the victim's temperature dropped to 84.8 degrees. If the temperature during the night was 55 degrees, at what time was the murder committed? Note: Body temperature is 98.6 degrees; work in hours.

10. Consider the model $u' = \lambda^2 u - u^3$, where λ is a parameter. Draw the bifurcation diagram (equilibria solutions vs. the parameter) and determine analytically the stability (stable or unstable) of the branch in the first quadrant.

11. Consider the IVP $u'' = \sqrt{u + t}$, $u(0) = 3$, $u'(0) = 1$. Pick step size $h = 0.1$ and use the modified Euler method to find an approximation to $u(0.1)$.

12. A particle of mass $m = 1$ moves on the x-axis under the influence of a potential $V(x) = x^2(1 - x)$.

   a) Write down Newton's second law, which governs the motion of the particle.

   b) In the phase plane, find the equilibrium solutions. If one of the equilibria is a center, find the type and stability of all the other equilibria.

   c) Draw the phase diagram.

D
Solutions and Hints to Selected Exercises

CHAPTER 1

Section 1.1

2. Try a solution of the form $u = at^m$ and determine a and m.

4. Try a solution of the form $u = at^2 + bt + c$.

6. $u' = u/3 + 2te^{3t}$.

8. (a) linear, nonautonomous; (b) nonlinear, nonautonomous; (c) nonlinear, autonomous; (d) linear autonomous.

9. The derivative of $\sqrt{u}$ is $1/(2\sqrt{u})$, which is not continuous when $u = 0$.

12. The slope field is zero on $u = 0$ and $u = 4$.

13. The nullclines are the horizontal lines $u = \pm 1$. The slope field is -3 on the lines $u = \pm 2$.

15. The nullclines are $u = 0$ and $u = 4\sqrt{t}$.

16. Hint: at a positive maximum $u'' < 0$, and so $u'' - u < 0$, a contradiction.

17. Show the derivative of the expression is zero. In the uu' plane the curves plot as a family of hyperbolas.

18. Use the quotient rule to show the time derivative of u_1/u_2 is zero.

Section 1.2

1. $u = \frac{1}{2}\sin(t^2) + C$. And $u(0) = C = 1$.
2. $u = \frac{2}{3}t^{3/2} + 2\sqrt{t} + C$ is the general solution.
5. $u(t) = \int_1^t e^{-s}\sqrt{s}ds$.
6 . $y = -\frac{1}{4}e^{-4t} + C$.
8. $\frac{d}{dt}(\text{erf}(\sin t)) = \text{erf}'(\sin t)\cos t = \frac{2}{\sqrt{\pi}}e^{-\sin^2 t}\cos t$.
9. (a) If the equation is exact, then $f = h_t$ and $g = h_u$. Then $f_u = h_{tu} = h_{ut} = g_t$. (b)(i) $f_u = 3u^2 = g_t$, and so the equation is exact. Then $h_t = u^3$ implies $h = tu^3 + \phi(u)$. Then $h_u = 3tu^2 + \phi'(u) = 3tu^2$. Hence, $\phi'(u) = 0$, or $\phi(u) = C_1$. Therefore $h = tu^3 + C_1 = C_2$, or $tu^3 = C$.
11. Take the derivative and use the fundamental theorem of calculus to get $u' = -2e^{-t} + tu$, $u(0) = 1$.

Section 1.3.1

1. Use $k = mg/L$.
3. The equation is $mv' = -F$, $v(0) = V$ with solution $v = -(F/m)t + V$. Then $x = \int v dt = -(F/2m)t^2 + Vt$.
6. Mass times acceleration equals force, or $ms'' = -mg\sin\theta$. But $s = l\theta$, so $ml\theta'' = -mg\sin\theta$.
7. (a) $\omega = \sqrt{g/l}$. (b) 2.2 sec.
8. The x and y positions of the cannonball are $x = (v\cos\theta)t$, $y = -\frac{1}{2}gt^2 + (v\sin\theta)t + H$, where θ is the angle of elevation of the cannon.

Section 1.3.2

1. The equilibria are $p = 0$ (stable), $p = a$ (unstable), $p = K$ (stable).
2. Find the equilibria by setting $-\frac{r}{K}p^2 + rp - h = 0$ and use the quadratic formula. We get a positive equilibrium only when $r \geq 4h/K$. If $r = 4h/K$ the single equilibrium is semi-stable, and if $r > 4h/K$ the smaller equilibrium is unstable and the larger one is stable.
3. r has dimensions 1/time, and a has dimensions 1/population. The maximum growth rate occurs at $p = 1/a$. There is no simple formula for the antiderivative $\int \frac{e^{ap}}{p}dp$.
4. Maximum length is a/b.

6. (a) $2\sqrt{u} = t + C$; (b) $u = \ln\sqrt{2t + C}$; (c) $u = \tan(t + C)$; (d) $u = Ce^{3t} + a/3$; (e) $4\ln u + 0.5u^2 = t + C$; (f) $\frac{\sqrt{\pi}}{2}\,\mathrm{erf}(u^2) = t + C$.

9. The amount of carbon-14 is $u = u_0 e^{-kt}$; 13,301 years.

10. 35,595.6 years.

11. $N' = bF_T N(1 - cN/F_T)$ (a logistics type equation with carrying capacity F_T/c).

12. $R = k$ is asymptotically stable. The solution is $R(t) = k\exp(Ce^{-at})$.

13. $p = m$ is unstable. If $p(0) < m$ then population becomes extinct, and if $p(0) > m$ then it blows up.

14. $I' = aI(N - I)$, which is logistics type. The asymptotically stable equilibrium is $I = N$.

Section 1.3.3

1. $p' = 0.2p(1 - p/40) - 1.5$ with equilibria $p = 10$ (unstable) and $p = 30$ (stable). If $p(0) \le 10$ then the population becomes extinct.The population will likely approach the stable equilibrium $p = 30$.

2. (a) $u = 0$ (unstable), $u = 3$ (stable). (c) $u = 2$ (stable), $u = 4$ (unstable).

3. (a) Equilibria are $u = 0$ and $u = h$. We have $f_u(u) = h - 2u$, and so $f_u(0) = h$ and $f_u(h) = -h$. If $h > 0$ then $u = 0$ is unstable and $u = h$ is stable; if $h < 0$ then $u = 0$ is stable and $u = h$ is unstable. If $h = 0$ there is no information from the derivative condition. A graph shows $u = 0$ is semi-stable.

7. Hint: Plot h vs. u instead of u vs. h.

Section 1.3.4

1. $h = \ln\sqrt{11/4}$. The solid will be 2° at time $2\ln(1/11)/\ln(4/11)$.

2. About 1.5 hours.

5. k is 1/time, q is degrees/time, and θ, T_e, T_0 are in degrees. The dimensionless equation is $\frac{d\psi}{d\tau} = -(\psi - 1) + ae^{-b/\psi}$, with $b = \theta/T_e$ and $a = q/kT_e$.

Section 1.3.5

1. $C(t) = (C_0 - C_{\mathrm{in}})e^{-qt/V} + C_{\mathrm{in}}$.

2. The equation is $100C' = (0.0002)(0.5) - 0.5C$.

5. The equilibrium $C^* = (-q + \sqrt{q^2 + 4kqVC_{\mathrm{in}}})/2kV$ is stable.

7. $C' = -kVC$ gives $C = C_0 e^{-kVt}$. The residence time is $T = -\ln(0.1)/kV$.
8. $C' = -\frac{aC}{b+C} + R$ has a stable equilibrium $C^* = Rb/(a-R)$, where $a > R$. The concentration approaches C^*.
9. $a' = -ka(a - a_0 + b_0)$; $a \to a_0 - b_0$.
10. (b) Set the equations equal to zero and solve for S and P. (c) With values from part (b), maximize aVP_e.

Section 1.3.6

1. $q(t) = 6 - e^{-2t}$, $I(t) = 2e^{-2t}$.
3. The initial condition is $I'(0) = (-RI(0) + E(0))/L$. The DE is $LI'' + RI' + (1/C)I = E'(t)$.
5. Substitute $q = A\cos\omega t$ into $Lq'' + (1/C)q = 0$ to get $\omega = 1/\sqrt{LC}$, A arbitrary.
6. $I'' + \frac{1}{2}(I^2 - 1)I' + I = 0$.
7. $LCV_c'' + RCV_c' + V_c = E(t)$.

CHAPTER 2

Section 2.1

1. (c) $u = \tan(\frac{1}{2}t^2 + t + C)$. (d) $\ln(1 + u^2) = -2t + C$.
2. $u = \ln(\frac{t^3}{3} + 2)$; interval of existence is $(-6^{1/3}, \infty)$.
4. The interval of existence is $(-\sqrt{5/3}, \sqrt{5/3})$.
6. The general solution is $(u-1)^{1/3} = t^2 + C$. If $u = 1$ for all t, then $t^2 + C = 0$ for all t, which is impossible.
8. Integrate both sides of the equation with respect to t. For example, $\int (u_1'/u_1)dt = \ln u_1 + C$.
9. $y' = \frac{F(y)-y}{t}$. Using $y = u/t$ the given DE can be converted to $y' = \frac{4-y^2}{ty}$, which is separable.
10. $u(t) = -e^{-3t} + Ce^{-2t}$.
11. $u(r) = -\frac{p}{4}r^2 + a\ln r + b$.
12. $u(t) = u_0 \exp(-\frac{at^2}{2})$. The maximum rate of conversion occurs at time $t = 1/\sqrt{a}$.

13. The IVP is $v' = -32 - v^2/800$, $v(0) = 160$.

14. The governing equation is $u' = -k/u$, with solution $u(x) = \sqrt{C - 2kt}$.

16. $u(t) = u_0 \exp(\int_a^t p(s)ds)$.

17. m is 1/time, b is 1/grasshoppers, and a is 1/(time $\cdot$ spiders). The dimensionless equation is $\frac{dh}{d\tau} = -h - \frac{\lambda h}{1+h}$. The population h approaches zero as $t \to \infty$. Separate variables.

18. (a) $1 - e^{-mt}$; $e^{-ma} - e^{-mb}$. (b) $S(t) = \exp(\int_0^t m(s)ds)$.

Section 2.2

1. $u(t) = \frac{1}{3}t^2 + \frac{C}{t}$.

2. $u(t) = Ce^{-t} + \frac{1}{2}e^t$.

4. $q(t) = 10te^{-5t}$. The maximum occurs at $t = 1/5$.

5. The equation becomes $y' + y = 3t$, which is linear.

6. The equation for y is $y' = 1 + y^2$. Then $y = \tan(t + C)$, $u = \tan(t + C) - t$.

7. $u(t) = e^{t^2}(C + \frac{\sqrt{\pi}}{2}\,\mathrm{erf}(t))$.

8. $u(t) = (u_0 + q/p)e^{pt} - q/p$.

11. (b) Let $y = u^{-1}$. Then $y' = -y - e^{-t}$ and $y = Ce^{-t} - \frac{1}{2}e^t$. Then $u = (Ce^{-t} - \frac{1}{2}e^t)^{-1}$. (c) $y' = -3y/t + 3/t$; $y = 1 + C/t^3$; $u = y^{1/3}$.

12. The vat empties at time $t = 60$. The governing equation is $(60 - t)u' = 2 - 3u$.

14. $S^* = raM/(aM + rA)$.

15. $x' = kx(N - x)$, which is similar to the logistics equation.

16. The IVP is $T' = -3(T - 9 - 10\cos(2\pi t))$, $T(0) = 12$.

18. $P' = N - P$. The stable equilibrium is $P = N$, so everyone hears the rumor.

19. $mv' = mg - \frac{2}{t+1}v$, $v(0) = 0$.

20. (a) $S^* = IP/(I + E)$. (b) The equation is linear: $S' = -\frac{1}{P}(I + E)S + I$. The general solution is $S(t) = S^* + Ce^{-\frac{t}{P}(I+E)}$.(c) Use the formula for S^* for each island and compare.

21 . Ex.2: $u' + u = e^t$ has integrating factor e^t. Multiply by the factor to get $(ue^t)' = e^{2t}$; integrate to get $ue^t = \frac{1}{2}e^{2t} + C$.

Section 2.3

1. The Picard iteration scheme is $u_{n+1}(t) = \int_0^t (1 + u_n(s)^2)ds$, $u_0(t) = 0$. It converges to $\tan t = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \cdots$.

2. The Picard iteration scheme is $u_{n+1}(t) = 1 + \int_0^t (s - u_n(s))ds$, $u_0(t) = 1$. We get $u_1(t) = 1 - t + t^2/2 + \cdots$, etc.

Section 2.4

2. $u(t) = e^{\sin t}$.

4. The exact solution is $u_n = u_0(1-hr)^n$. If $h > 1/r$ then the solution oscillates about zero, but the solution to the DE is positive and approaches zero. So we require $h < 1/h$.

7. $u = e^{-t}$. Roundoff error causes the exponentially growing term Ce^{5t} in the general solution to become significant.

CHAPTER 3

Section 3.1

1. $V(x) = x^2/2 - x^4/4$.

2. $mx''x = F(x)x'$. But $\frac{d}{dt}V(x) = \frac{dV}{dx}x' = -F(x)x'$ and $mx''x' = m\frac{d}{dt}(x')^2 = 2mx'x''$. Hence $\frac{1}{2}m\frac{d}{dt}(x')^2 = -\frac{d}{dt}V(x)$. Integrating both sides gives $\frac{1}{2}m(x')^2 = V(x) + C$, which is the conservation of energy law.

5. (a) Make the substitution $v = x'$. The solution is $x(t) = a/t + b$. (b) Make the substitution $v = x'$. The solution is $\int \frac{dx}{a+0.5x^x} = t + b$.

Section 3.2

1. (a) $u = e^{2t}(a + bt)$. (d) $u(t) = a\cos 3t + b\sin 3t$. (e) $u(t) = a + be^{2t}$. (f) $u(t) = a\cosh(\sqrt{12}t) + b\sinh(\sqrt{12}t)$.

4. The solution is periodic if $a = 0$ and $b > 0$; the solution is a decaying oscillation if $a < 0$ and $a^2 < b$; the solution decays without oscillation if $a < 0$ and $a^2 \geq b$.

5. $L = \frac{1}{4}$ (critically damped), $L < \frac{1}{4}$ (over damped), $L > \frac{1}{4}$, (under damped).

6. Critically damped when $9a^2 = 4b$, which plots as a parabola in ab parameter space.

7. $u'' + 2u' - 24u = 0$.

8. $u'' + 6u' + 9u = 0$.

9. $u'' + 16u = 0$.

10. $A = 2$, $B = 0$.

12. $I(t) = \sqrt{10}\sin(t/\sqrt{10})$.

Section 3.3.1

1. (a) $u_p = at3 + bt2 + ct + d$. (b) $u_p = a$. (d) $u_p = a\sin 7t + b\cos 7t$. (f) $u_p = (a + bt)e^{-t}\sin \pi t + (c + dt)e^{-t}\cos \pi t$.

2. (c) $u_p = t^2 - 2t + 2$. (e) $u_p = \frac{9}{2}e^{-t}$.

4. The general solution is $u(t) = c_1 + c_2 e^{2t} - 2t$.

5. Write $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ and then take $u_p = a + b\cos 2t + c\sin 2t$.

7. $u(t) = a\cos 50t + b\sin 50t + 0.02$.

8. The circuit equation is $2q'' + 16q' + 50q = 110$.

Section 3.3.2

2. $u(t) = c_1\cos 4t + c_2\sin 4t + \frac{1}{32}\cos 4t + \frac{1}{8}t\sin 4t$.

3. $L = 1/C\beta^2$.

Section 3.4

1. (b) $u(t) = 2\ln t$.

2. $\beta = 1$.

7. $u(t) = \tan(t + \pi/4)$.

9. The other solution is te^{at}.

11. $\frac{1}{\sqrt{t}}\sin t$.

14. Take the derivative of the Wronskian expression $W = u_1u_2' - u_1'u_2$ and use the fact that u_1 and u_2 are solutions to the differential equation to show $W' = -p(t)W$. Solving gives $W(t) = W(0)\exp(-\int p(t)dt)$, which is always of one sign.

16. The given Riccati equation can be transformed into the Cauchy–Euler equation $u' - \frac{3}{t}u' = 0$.

17. (b) $u_p = -\cos t\ln((1 + \sin t)/\cos t)$. (c) Express the particular solution in terms of integrals. (e) $u_p = t^3/3$.

18. (a) $tp(t) = t\cdot t^{-1} = 1$, and $t^2q(t) = t^2(1 - \frac{k^2}{t^2}) = t^2 - k^2$, which are both power series about $t = 0$.

Section 3.5

2. $u(x) = -\frac{1}{6}x^3 + \frac{1}{240}x^4 + \frac{100}{3}x$. The rate that heat leaves the right end is $-Ku'(20)$ per unit area.

4. There are no nontrivial solutions when $\lambda \leq 0$. There are nontrivial solutions $u_n(x) = \sin n\pi x$ when $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, ...$

5. $u(x) = 2\pi\sqrt{x}$.

6. Integrate the steady-state heat equation from 0 to L and use the fundamental theorem of calculus. This expression states: the rate that heat flows in at $x = 0$ minus the rate it flows out at $x = L$ equals the net rate that heat is generated in the bar.

7 . When ωL is not an integer multiple of π.

8. $\lambda = -1 - n^2\pi^2$, $n = 1, 2, \ldots$.

10. Hint: this is a Cauchy–Euler equation. Consider three cases where the values of λ give characteristic roots that are real and unequal, real and equal, and complex.

Section 3.6

1. (a) $u(t) = c_1 + c_2\cos t + c_3\sin t$. (b) $u(t) = c_1 + e^{t/2}(c_2\cos\frac{\sqrt{3}}{2}t + c_3\sin\frac{\sqrt{3}}{2}t) + c_4e^{-t} + t$. (c) $u(t) = c_1 + c_2t + c_3\cos t + c_4\sin t$.

5. $u(t) = e^{3t}(c_1\cos t + c_2\sin t) + te^{3t}(c_3\cos t + c_4\sin t)$.

Section 3.7

1. (b) $u(t) = Ce^{Rt}$. (c) $u(t) = Ce^{-\sin t} - 1$. (e) $u(t) = \sqrt{t}(c_1\cos(\frac{\sqrt{7}}{2}\ln t) + c_2\sin(\frac{\sqrt{7}}{2}\ln t))$. (g) $u(t) = -4t^2 + 6t + C$. (i) A Bernoulli equation. (k) $x(t) = \pm\int\frac{dt}{\sqrt{A-2t}} + B$. (l) Bernoulli equation. (m) $\pm\int_0^u\frac{dw}{\sqrt{A-2w}} = t + B$. (n) Homogeneous equation. (p) Exact equation.

2. $u(t) = (\frac{1}{2} - \sin t)^{-1}$, $-7\pi/6 < t < \pi/6$.

5. If $a \leq 0$ then $u = a$ is the only equilibrium (unstable). If $0 < a < 1$ then there are three equilibria: $u = \pm\sqrt{a}$ (unstable), and $u = a$ (stable). If $a = 1$ then $u = 1$ is unstable. If $a > 1$ then $u = \sqrt{a}$ (stable), $u = a$, $-\sqrt{a}$ (unstable).

6. $r(t) = -kt + r_0$.

7. $p = 0$ (unstable); $p = K$ (stable).

11. $u(t) = \exp(t^2 + C/t^2)$.

12. $u(t) = t - 3t \ln t + 2t^2$.

CHAPTER 4

Section 4.1

1. $U(s) = \frac{1}{s}(e^{-s} - e^{-2s})$.

2. Integrate by parts twice.

3. $L[\sin t] = \frac{1}{1+s^2}$; $L[\sin(t - \pi/2)] = -\frac{s}{1+s^2}$; $L[h_{\pi/2}(t) \sin(t - \pi/2)] = \frac{s}{1+s^2} e^{-\pi s/2}$.

4. $\frac{2!}{(s+3)^3}$.

5. Use $\sinh kt = \frac{1}{2}(e^{kt} - e^{-kt})$.

7. $\frac{1}{s+1} e^{-2(s+1)}$.

9. e^{t^2} is not of exponential order; the improper integral $\int_0^\infty \frac{1}{t} e^{-st} dt$ does not exist at $t = 0$. Neither transform exists.

11. $\frac{1}{s} \frac{1}{1+e^{-s}}$.

12. Hint: Use the definition of the Laplace transform and integrate by parts using $e^{-st} = -\frac{1}{s} \frac{d}{dt}(e^{-st})$.

13. Hint: $\frac{d}{ds} U(s) = \int_0^\infty u(t) \frac{d}{ds} e^{-st} dt$.

14. $\ln(\sqrt{(s+1)/(s-1)})$, $s > 1$.

15. Integrate by parts.

17. (a) Integrate by parts. Change variables in the integral to write $\Gamma(\frac{1}{2}) = 2 \int_0^\infty e^{-r^2} dr$.

Section 4.2

2. (c) $\frac{1}{3} t^3 e^{5t}$. (d) $7h_4(t)$.

3. (c) $u(t) = \frac{3}{5} e^{3t} + \frac{7}{5} e^{-2t}$. (d) $u(t) = e^t \sin t$. (f) $u(t) = 1$. (i) $\frac{3}{2} \cosh(\sqrt{2}t) - \frac{1}{2}$.

4. Solve for the transforms $X = X(s)$, $Y = Y(s)$ in $sX = X - 2Y - \frac{1}{s^2}$, $sY = 3X + Y$, and then invert.

5. $\frac{d^n}{ds^n} U(s) = \int_0^\infty u(t) \frac{d^n}{ds^n} e^{-st} dt$.

Section 4.3

1. $\frac{1}{2} t \sin t$.

2. $\frac{1}{12}t^4$.

3. $u(t) = u(0)e^{at} + \int_0^t e^{a(t-s)}q(s)ds$.

5. $u(t) = \frac{1}{\omega}\int_0^t \sinh(t-s)f(s)ds$.

7. $u(t) = \int_3^t f(t-s)ds$.

8. $u(t) = \int_0^t (e^{t-s} - 1)f(s)ds$.

9. $U(s) = \frac{F(s)}{1-K(s)}$.

10. (a) $u(t) = \sin(t)$; (b) $u(t) = 0$.

11. The integral is a convolution.

Section 4.4

1. $\frac{2}{s}(e^{-3s} - e^{-4s})$.

2. $e^{-3s}(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s})$.

3. $\frac{1}{6}t^3e^{2t}$.

4. Note $f(t) = 3 - h_2(t) + 4h_\pi(t) - 6h_7(t)$.

5. $t - h_4(t)(t-4)$.

7. $u(t) = e^t$ on $[0,1]$; $u(t) = \frac{e-2}{e}e^t + 2$ on $t > 1$.

8. Solve $q'' + q = t + (9-t)h_9(t)$.

10. $u(t) = 1$, $0 \le t \le 1$; $u(t) = -\cos \pi t$, $t > 1$.

13. $2 - \sum_{n=1}^{\infty}(1)^n h_n(t)$.

Section 4.5

1. $1/e^2$.

3. $u(t) = \sinh(t-5)h_5(t)$.

4. $u(t) = \sin(t-2)h_2(t)$.

5. $u(t) = h_2(t) + \delta_3(t)$.

6. $u(t) = \frac{1}{2}\cos[2(t-2)]h_2(t) - \cos[2(t-5)]h_5(t)$.

7. $v(t) = \sum_{n=0}^{\infty} \sin(t - n\pi)h_{n\pi}(t)$.

CHAPTER 5

Section 5.1

1. The orbit is an ellipse (taken counterclockwise).

2. The tangent vector is $x'(t) = (2, -3)^{\mathrm{T}} e^t$ and it points in the direction $(2, -3)^{\mathrm{T}}$.

3. $x(t) = 8 + 2e^{-5t}$, $y(t) = 8 - 8e^{-5t}$. Over a long time the solution approaches the point (equilibrium) $(8, 8)$.

4. Multiply the first equation by $1/W$, the second by $1/V$, and then add to get $x'/W + y'/V = 0$, or $x/W + y/V = C$. Use $y = V(C - x/W)$ to eliminate y from the first equation to get a single equation in the variable x, namely, $x' = -q(1/V + 1/W)x + qC$. The constant C is determined from the initial condition.

6. Solve the equation $x'' + \frac{1}{2}x' + 2x = 0$ to get a decaying oscillation $x = x(t)$. In the phase plane the solution is a clockwise spiral entering the origin.

7. The system is $q' = I$, $I' = -4q$. We have $q(t) = 8\cos 2t$ and $I(t) = -16\sin 2t$. Both q and I are periodic functions of t with period π, and in the phase plane $q^2/64 + I^2/256 = 1$, which is an ellipse. It is traversed clockwise.

Section 5.2

1. $\det(A - \lambda I) = \lambda^2 - 5\lambda - 2$.

2. $x = 3/2$, $y = 1/6$.

4. $\det(A - \lambda I) = \lambda^2 - 5\lambda - 2 = 0$, so $\lambda = \frac{5}{2} \pm \frac{1}{2}\sqrt{33}$.

5. $\det A = 0$ so A^{-1} does not exist.

6. If $m = -5/3$ then there are infinitely many solutions, and if $m \neq -5/3$, no solution exists.

7. $m = 1$ makes the determinant zero.

8. Use expansion by minors.

10. $\det(A) = -2$, so A is invertible and nonsingular.

11. $\mathbf{x} = a(2, 1, 2)^{\mathrm{T}}$, where a is any real number.

12. Set $c_1(2, -3)^{\mathrm{T}} + c_2(-4, 8)^{\mathrm{T}} = (0, 0)^{\mathrm{T}}$ to get $2c_1 - 4c_2 = 0$ and $-3c_1 + 8c_2 = 0$. This gives $c_1 = c_2 = 0$.

13. Pick $t = 0$ and $t = \pi$.

14. Set a linear combination of the vectors equation to the zero vector and find coefficients c_1, c_2, c_3.

16. $\mathbf{r}_1(t)$ plots as an ellipse; $\mathbf{r}_2(t)$ plots as the straight line $y = 3x$. $\mathbf{r}_2(t)$ plots as a curve approaching the origin along the direction $(1,1)^{\mathrm{T}}$. Choose $t = 0$ to get $c_1 = c_3 = 0$, and then choose $t = 1$ to get $c_2 = 0$.

Section 5.3

1. For A the eigenpairs are 3, $(1,1)^{\mathrm{T}}$ and 1, $(2,1)^{\mathrm{T}}$. For B the eigenpairs are 0, $(3,-2)^{\mathrm{T}}$ and -8, $(1,2)^{\mathrm{T}}$. For C the eigenpairs are $\pm 2i$, $(4, 1 \mp i)^{\mathrm{T}}$.

2. $\mathbf{x} = c_1(1,5)^{\mathrm{T}}e^{2t} + c_2(2,-4)^{\mathrm{T}}e^{-3t}$. The origin has saddle point structure.

3. The origin is a stable node.

4. (a) $\mathbf{x} = c_1(-1,1)^T e^{-t} + c_2(2,3)^T e^{4t}$ (saddle), (c) $\mathbf{x} = c_1(-2,3)^T e^{-t} + c_2(1,2)^T e^{6t}$ (saddle), (d) $\mathbf{x} = c_1(3.1)^T e^{-4t} + c_2(-1,2)^T e^{-11t}$ (stable node), (f) $x(t) = c_1e^t(\cos 2t - \sin 2t) + c_2e^t(\cos 2t + \sin 2t)$, $\quad y(t) = 2c_1e^t\cos 2t + 2c_2e^t \sin 2t$ (unstable spiral), (h) $x(t) = 3c_1\cos 3t + 3c_2 \sin 3t$, $\quad y(t) = -c_1 \sin 3t + c_2 \cos 3t$ (center).

6. (a) Equilibria consist of the entire line $x - 2y = 0$. (b) The eigenvalues are 0 and 5; there is a linear orbit associated with 5, but not 0.

7. The eigenvalues are $\lambda = 2 \pm \sqrt{a+1}$; $a = -1$ (unstable node), $a < -1$ (unstable spiral), $a > -1$ (saddle).

9. The eigenvalues are never purely imaginary, so cycles are impossible.

11. There are many matrices. The simplest is a diagonal matrix with -2 and -3 on the diagonal.

13. The system is $v' = w$, $w' = -(1/LC)v - (R/L)w$. The eigenvalues are $\lambda = -R/2L \pm \sqrt{R^2/4L^2 - 1/LC}$. The eigenvalues are complex when $R^2/4L < 1/C$, giving a stable spiral in the phase plane, representing decaying oscillations in the system.

14. The eigenvalues are $-\gamma \pm i$. When $\gamma = 0$ we get a cycle; when $\gamma > 0$ we get a stable spiral; when $\gamma < 0$ we get an unstable spiral.

Section 5.4

2. The equations are $V_1x' = (q+r)c - qx - rx$, $V_2y' = qx - qy$. The steady-state is $x = y = c$. When freshwater enters the system, $V_1x' = -qx - rx$, $V_2y' = qx - qy$. The eigenvalues are both negative ($-q$ and $-q-r$), and therefore the solution decays to zero. The origin is a stable node.

5. A fundamental matrix is

$$\Phi(t) = \begin{pmatrix} 2e^{-4t} & -e^{-11t} \\ 3e^{-4t} & 2e^{-11t} \end{pmatrix}.$$

The particular solution is $\mathbf{x}_p = -(\frac{9}{42}, \frac{1}{21})^{\mathrm{T}} e^{-t}$.

6. $\det A = r_2 r_3 > 0$ and $\mathrm{tr}(A) = r_1 - r_2 - r_3 < 0$. So the origin is asymptotically stable and both x and y approach zero. The eigenvalues are $\lambda = \frac{1}{2}(\mathrm{tr}(A) \pm \frac{1}{2}\sqrt{\mathrm{tr}(A)2 - 4\det A}$.

7. In the equations in Problem 6, add D to the right side of the first (x') equation. Over a long time the system will approach the equilibrium solution: $x_e = D/(r_1 + r_2 + r_1 r_3/r_2)$, $y_e = (r_1/r_2)x_e$.

Section 5.5

1. The eigenpairs of A are 2, $(1, 0, 0)^{\mathrm{T}}$; 6, $(6, 8, 0)^{\mathrm{T}}$; -1, $(1, -1, 7/2)^{\mathrm{T}}$. The eigenpairs of C are 2, $(1, 0, 1)^{\mathrm{T}}$; 0, $(-1, 0, 1)^{\mathrm{T}}$; 1, $(1, 1, 0)^{\mathrm{T}}$.

2(a). $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix} e^{4t} + c_3 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} e^{2t}.$

2(b). $\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} \cos 0.2t \\ \sin 0.2t \\ -\cos 0.2t - \sin 0.2t \end{pmatrix} + c_3 \begin{pmatrix} -\sin 0.2t \\ \cos 0.2t \\ -\cos 0.2t + \sin 0.2t \end{pmatrix}.$

2(d). $\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{t}.$

4. The eigenvalues are $\lambda = 2, \rho \pm 1$.

CHAPTER 6

Section 6.1.1

1. $y = Ce^{1/x}$, $x(t) = (c_1 - t)^{-1}$, $y(t) = c_2 e^{-t}$.

2. $y = \frac{1}{x^2 + C}$. There are no equilibrium solutions. No solutions can touch the x-axis.

3. Two equilibrium points: $(\sqrt{4/5}, 2\sqrt{4/5}), (-\sqrt{4/5}, -2\sqrt{4/5})$. The vector field is vertical on the circle of radius 2: $x^2 + y^2 = 4$. The vector field is horizontal on the straight line $y = 2x$.

4. $(\pm 1, -1)$. The vector field is vertical on the line $y = -1$ and horizontal on the inverted parabola $y = -x^2$.

5. $(0,0)$, $(1,1)$, and $(4,4)$.

6. $(0, n\pi)$, $n = 0, \pm 1, \pm 2, ...$

7. $dV/dt = 2xx' + 2yy' = -2y^4 < 0$.

8. The equilibria are the entire line $y = x$; they are not isolated.

Section 6.1.2

1. There is no epidemic. The number of infectives decreases from its initial value.

2. $I(t)$ increases to a maximum value, then $S(t)$ decreases to the value S^*.

3. $r = 1/3$ and $a = 0.00196$. The average number of days to get the flu is about 2.5 days.

4. $r = 0.25$ and $a = 0.001$, giving $S^* = 93$. Also $I_{max} = 77$.

5. $I' = aI(N - I) - rI$. The equilibrium is $I = aN - r$.

8. $x' = rx - axy, \quad y' = -my + bxy - M$.

9. The equilibria are $(0,0)$ and $(m/b, k/a)$. The vector field shows that curves veer away from the nonzero equilibrium, so the system could not coexist in that state.

10. $S' = -aSI - \nu, \quad aSI - rI$.

11. $S' = -aSI + \mu(N - S - I), \quad I' = aSI - rI$. The equilibria are $(N, 0)$ and the endemic state $(r/a, I^*)$ where $I^* = \mu(N - r/a)/(r + \mu)$.

Section 6.2

4. Begin by writing the equation as $u' = v, \quad v' = -9u + 80\cos 5t$.

6. The system is $x' = y, \quad y' = -x/2 + y/2 - y^3/2$.

7. The system is $x' = y, \quad y' = -2(x^2 - 1)y - x$.

8. $S(0) = 465$ with $a = 0.001$ and $r = 0.2$.

9. $S' = -aSI, \quad I' = aSI - rI - qI$.

Section 6.3

1. $y = C(e^x - 1)$.

2. $y^2 - x^2 - 4x = C$.

3. Equilibria are $(0,0)$ (a saddle structure) and $(2,4)$ (stable node) and nullclines: $y = x^2$ and $y = 2x$.

4. $a < 0$ (no equilibria); $a = 0$ (origin is equilibrium); $a > 0$ (the equilibria are $(-\sqrt{a}/2, 0)$ and $(\sqrt{a}/2, 0)$, a stable node and a saddle).

6. $(-1,0)$ (stable spiral); $(1,0)$ (saddle).

8. $(2,4)$ (saddle); $(0,0)$ (stable node). The Jacobian matrix at the origin has a zero eigenvalue.

10. $\operatorname{tr}(A) < 0, \quad \det A > 0$. Thus the equilibrium is asymptotically stable.

11. See Exercise 7, Section 6.1.1.

12. The force is $F = -1 + x^2$, and the system is $x' = y, \quad y' = -1 + x^2$. The equilibrium $(1,0)$ is a saddle and $(-1,0)$ is a center. The latter is determined by noting that the orbits are $\frac{1}{2}y^2 + x - \frac{1}{3}x^3 = E$.

13. (a) $\frac{dH}{dt} = H_x x' + H_y y' = H_x H_y + H_y(-H_x) = 0$. (c) The Jacobian matrix at an equilibrium has zero trace. (e) $H = \frac{1}{2}y^2 - \frac{x^2}{2} + \frac{x^3}{3}$.

14. $(0,0)$ is a center.

15. (c) The eigenvalues of the Jacobian matrix are never complex.

16. $(0,0)$, $(0, \frac{1}{2})$, and $(K, 0)$ are always equilibria. If $K \geq 1$ or $K \leq \frac{1}{2}$ then no other positive equilibria occur. If $\frac{1}{2} < K \leq 1$ then there is an additional positive equilibrium.

17. $a = 1/8$ (one equilibrium); $a > 1/8$, (no equilibria); $0 < a < 1/8$ (two equilibria).

19. The characteristic equation is $\lambda^2 = f'(x_0)$. The equilibrium is a saddle if $f'(x_0) > 0$.

Section 6.4

2. There are no equilibrium, and therefore no cycles.

3. $f_x + g_y > 0$ for all x, y, and therefore there are no cycles (by Dulac's criterion).

4. $(1,0)$ is always a saddle, and $(0,0)$ is unstable node if $c > 2$ and an unstable spiral if $c < 2$.

6. $(0,0)$ is a saddle, $(\pm 1, 0)$ are stable spirals.

7. The equilibria are $H = 0$, $P = \phi/a$ and $H = \frac{\varepsilon\phi}{c} - \frac{a}{b}$, $P = \frac{c}{\varepsilon b}$.

8. In polar coordinates, $r' = r(a - r^2), \quad \theta' = 1$. For $a \leq 0$ the origin is a stable spiral. For $a > 0$ the origin is an unstable spiral with the appearance of a limit cycle at $r = \sqrt{a}$.

9. The characteristic equation is $\lambda^2 + k\lambda + V''(x_0) = 0$ and has roots $\lambda = \frac{1}{2}(-k \pm \sqrt{k^2 - 4V''(x_0)})$. These roots are never purely imaginary unless $k = 0$.

10. Use Dulac's criterion.

11. Equilibria at $(0, 0)$, $(1, 1,)$, and $(4, 4)$.

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