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FINANCIAL RISK MANAGER HANDBOOK

FRM® PART I / PART II

Philippe Jorion

GARP | Global Association of Risk Professionals

Financial Risk Manager Handbook Plus Test Bank

Sixth Edition

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Preface

The *Financial Risk Manager Handbook Plus Test Bank* provides the core body of knowledge for financial risk managers. Risk management has rapidly evolved over the past decade and has become an indispensable function in many institutions.

This *Handbook* was originally written to provide support for candidates taking the FRM examination administered by GARP. As such, it reviews a wide variety of practical topics in a consistent and systematic fashion. It covers quantitative methods, major financial products, as well as market, credit, operational, and integrated risk management. It also discusses investment risk management issues essential for risk professionals.

This edition has been thoroughly updated to reflect recent developments in financial markets and changes in the structure of the FRM program. The book is now structured to correspond to the two levels of the FRM exams. All of the chapters have been updated to account for recent developments in financial markets and regulations. In particular, current issues are integrated in the second part of the book. New chapters have been added, including chapters that deal with advanced univariate and multivariate models, as well as advanced option models. Finally, this *Handbook* incorporates the latest questions from the FRM examinations.

Modern risk management systems cut across the entire organization. This breadth is reflected in the subjects covered in this *Handbook*. The *Handbook* was designed to be self-contained, but only for readers who already have some exposure to financial markets. To reap maximum benefit from this book, readers ideally should have taken the equivalent of an MBA-level class on investments.

Finally, I want to acknowledge the help received in writing this *Handbook*. In particular, I would like to thank the numerous readers who shared comments on previous editions. Any comment or suggestion for improvement will be welcome. This feedback will help us to maintain the high quality of the FRM designation.

Philippe Jorion
October 2010

About the Author

Philippe Jorion is a Professor of Finance at the Paul Merage School of Business at the University of California at Irvine. He has also been a professor at Columbia University, Northwestern University, the University of Chicago, and the University of British Columbia. In addition, he taught the risk management class in the Master of Financial Engineering programs at the University of California at Berkeley and University of California at Los Angeles. He holds an M.B.A. and a Ph.D. from the University of Chicago and a degree in engineering from the University of Brussels.

Dr. Jorion is also a managing director at Pacific Alternative Asset Management Company (PAAMCO), a global fund of hedge funds with approximately $10 billion under management. PAAMCO is one of the few funds of funds to require position-level transparency from all invested hedge funds. This information is used to provide various measures of portfolio risk as well as to develop tools that help investors to understand the drivers of the funds' alpha and to detect style drift.

Dr. Jorion is the author of more than 100 publications directed to academics and practitioners on the topics of risk management and international finance. He has also written a number of books, including *Big Bets Gone Bad: Derivatives and Bankruptcy in Orange County*, the first account of the largest municipal failure in U.S. history, and *Value at Risk: The New Benchmark for Managing Financial Risk*, which is aimed at finance practitioners and has become an industry standard.

Philippe Jorion is a frequent speaker at academic and professional conferences. He is on the editorial board of a number of finance journals and was editor in chief of the *Journal of Risk*.

About GARP

Founded in 1996, the Global Association of Risk Professionals (GARP) is the leading not-for-profit association for world-class financial risk certification, education, and training, with close to 100,000 members representing 167 countries. With deep expertise and a strong reputation, GARP sets global standards and creates risk management programs valued worldwide. All GARP programs are developed with input from experts around the world to ensure that concepts and content reflect globally accepted practices.

GARP is dedicated to advancing the risk profession. For more information about GARP, please visit www.garp.com.

FINANCIAL RISK MANAGER (FRM®) CERTIFICATION

The benchmark FRM designation is the globally accepted risk management certification for financial risk professionals. The FRM objectively measures competency in the risk management profession based on globally accepted standards. With a compound annual growth rate of 25% over the past seven years, the FRM program has experienced significant growth in every financial center around the world. Now 16,000+ individuals hold the FRM designation in over 90 countries. In addition, organizations with five or more FRM registrants grew from 105 in 2003 to 424 in 2008, further demonstrating the FRM program's global acceptance.

The FRM Continuing Professional Education (CPE) program, offered exclusively for certified FRM holders, provides the perspective and framework needed to further develop competencies in the ever-evolving field of risk management.

For more information about the FRM program, please visit www.garp.com/frmexam.

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The ICBRR allows individuals to expand their knowledge and understanding of the various risks, regulations, and supervisory requirements banks must face in today's economy, with emphasis on the Basel II Accord. This certificate is ideal for employees who are not professional risk managers but who have a strong need to understand risk concepts. The ICBRR program is designed for employees in nonrisk departments such as internal audit, accounting, information technology

(IT), legal, compliance, and sales, acknowledging that everyone in the organization is a risk manager!

Energy Risk Professional Program

The Energy Risk Professional (ERP®) program is designed to measure a candidate's knowledge of the major energy markets and gauge their ability to manage the physical and financial risks inherent in the complex world of energy. This program is valuable for anyone working in or servicing the energy field, requiring an understanding of the physical and financial markets, how they interrelate, and the risks involved.

GARP DIGITAL LIBRARY

As the world's largest digital library dedicated to financial risk management, the GARP Digital Library (GDL) is the hub for risk management education and research material. The library's unique iReadings™ allow users to download individual chapters of books, saving both time and money. There are over 1,000 readings available from 12 different publishers. The GDL collection offers readings to meet the needs of anyone interested in risk management.

For more information, please visit www.garpdigitallibrary.org.

GARP EVENTS AND NETWORKING

GARP hosts major conventions throughout the world, where risk professionals come together to share knowledge, network, and learn from leading experts in the field. Conventions are bookended with interactive workshops that provide practical insights and case studies presented by the industry's leading practitioners.

GARP regional chapters provide an opportunity for financial risk professionals to network and share new trends and discoveries in risk management. Each of our 52 chapters holds several meetings each year, in some locations more often, focusing on issues of importance to the risk management community, either globally or locally.

Introduction

GARP's formal mission is to be the leading professional association for financial risk managers, managed by and for its members and dedicated to the advancement of the risk profession through education, training, and the promotion of best practices globally. As a part of delivering on that mission, GARP has again teamed with Philippe Jorion to produce the sixth edition of the *Financial Risk Manager Handbook Plus Test Bank*.

The *Handbook* follows GARP's FRM Committee's published FRM Study Guide, which sets forth primary topics and subtopics covered in the FRM exam. The topics are selected by the FRM Committee as being representative of the theories and concepts utilized by risk management professionals as they address current issues.

Over the years the Study Guide has taken on an importance far exceeding its initial intent of providing guidance for FRM candidates. The Study Guide is now being used by universities, educators, and executives around the world to develop graduate-level business and finance courses, as a reference list for purchasing new readings for personal and professional libraries, as an objective outline to assess an employee's or job applicant's risk management qualifications, and as guidance on the important trends currently affecting the financial risk management profession.

Given the expanded and dramatically growing recognition of the financial risk management profession globally, the *Handbook* has similarly assumed a natural and advanced role beyond its original purpose. It has now become the primary reference manual for risk professionals, academicians, and executives around the world. Professional risk managers must be well versed in a wide variety of risk-related concepts and theories, and must also keep themselves up-to-date with a rapidly changing marketplace. The *Handbook* is designed to allow them to do just that. It provides a financial risk management practitioner with the latest thinking and approaches to financial risk-related issues. It also provides coverage of advanced topics with questions and tutorials to enhance the reader's learning experience.

This sixth edition of the *Handbook* includes revised coverage of the primary topic areas covered by the FRM examination. Importantly, this edition also includes the latest lessons from the recent credit crisis, as well as new and more recent sample FRM questions.

The *Handbook* continues to keep pace with the dynamic financial risk profession while simultaneously offering serious risk professionals an excellent and cost-effective tool to keep abreast of the latest issues affecting the global risk management community.

Developing credibility and global acceptance for a professional certification program is a lengthy and complicated process. When GARP first administered its FRM exam in 1997, the concept of a professional risk manager and a global certification relating to that person's skill set was more theory than reality. That has now completely changed, as the number of current FRM holders exceeds 16,000.

The FRM is now the benchmark for a financial risk manager anywhere around the world. Professional risk managers having earned the FRM credential are globally recognized as having achieved a level of professional competency and a demonstrated ability to dynamically measure and manage financial risk in a real-world setting in accordance with global standards.

GARP is proud to continue to make this *Handbook* available to financial risk professionals around the world. Philippe Jorion, a preeminent risk professional, has again compiled an exceptional reference book. Supplemented by an interactive Test Bank, this *Handbook* is a requirement for any risk professional's library.

The Test Bank is a preparatory review for anyone studying for the FRM exam and for risk professionals interested in self-study to review and improve their knowledge of market, credit, and operational risk management. The Test Bank contains hundreds of multiple-choice questions from the 2007, 2008, and 2009 FRM exams, with answers and solutions provided. The Test Bank can be downloaded following the instructions on the FRM® Test Bank Download page at the end of this book.

Global Association of Risk Professionals
October 2010

PART One

Foundations of Risk Management

CHAPTER 1

Risk Management

Financial **risk management** is the process by which financial risks are identified, assessed, measured, and managed in order to create economic value.

Some risks can be measured reasonably well. For those, risk can be quantified using statistical tools to generate a probability distribution of profits and losses. Other risks are not amenable to formal measurement but are nonetheless important. The function of the risk manager is to evaluate financial risks using both quantitative tools and judgment.

As financial markets have expanded over recent decades, the risk management function has become more important. Risk can never be entirely avoided. More generally, the goal is not to minimize risk; it is to take smart risks.

Risk that can be measured can be managed better. Investors assume risk only because they expect to be compensated for it in the form of higher returns. To decide how to balance risk against return, however, requires risk measurement.

Centralized risk management tools such as value at risk (VAR) were developed in the early 1990s. They combine two main ideas. The first is that risk should be measured at the top level of the institution or the portfolio. This idea is not new. It was developed by Harry Markowitz (1952), who emphasized the importance of measuring risk in a total portfolio context.[1] A centralized risk measure properly accounts for hedging and diversification effects. It also reflects the fact that equity is a common capital buffer to absorb all risks. The second idea is that risk should be measured on a forward-looking basis, using the current positions.

This chapter gives an overview of the foundations of risk management. Section 1.1 provides an introduction to the risk measurement process, using an illustration. Next, Section 1.2 discusses how to evaluate the quality of risk management processes. Section 1.3 then turns to the integration of risk measurement with business decisions, which is a portfolio construction problem. These portfolio decisions can be aggregated across investors, leading to asset pricing theories that can be used as yardsticks for performance evaluation and for judging risk management and are covered in Section 1.4. Finally, Section 1.5 discusses how risk management can add economic value.

FRM Exam Part 1 topic. In addition to the topics described in this chapter, FRM candidates should also read the GARP Code of Conduct.

[1] Harry Markowitz, "Portfolio Selection," *Journal of Finance* 7 (1952): 77–91.

1.1 RISK MEASUREMENT

1.1.1 Example

The first step in risk management is the measurement of risk. To illustrate, consider a portfolio with $100 million invested in U.S. equities. Presumably, the investor undertook the position because of an expectation for profit, or investment growth. This portfolio is also risky, however.

The key issue is whether the expected profit for this portfolio warrants the assumed risk. Thus a trade-off is involved, as in most economic problems. To help answer this question, the risk manager should construct the distribution of potential profits and losses on this investment. This shows how much the portfolio can lose, thus enabling the investor to make investment decisions.

Define ΔP as the profit or loss for the portfolio over a fixed horizon, say the coming month. This must be measured in a *risk currency*, such as the dollar. This is also the product of the initial investment value P and the future rate of return R_P. The latter is a random variable, which should be described using its **probability density function**. Using historical data over a long period, for example, the risk manager produces Figure 1.1.

This graph is based on the actual distribution of total returns on the S&P 500 index since 1925. The line is a smoothed histogram and does not assume a simplified model such as the normal distribution.

The vertical axis represents the frequency, or probability, of a gain or loss of a size indicated on the horizontal axis. The entire area under the curve covers all of the possible realizations, so should add up to a total probability of 1.

Most of the weight is in the center of the distribution. This shows that it is most likely that the return will be small, whether positive or negative. The tails have less weight, indicating that large returns are less likely. This is a typical characteristic of returns on financial assets. So far, this pattern resembles the bell-shaped curve for a normal distribution.

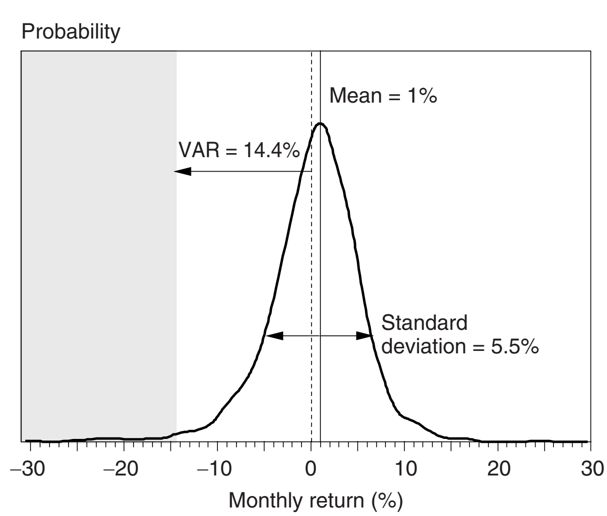

FIGURE 1.1 Distribution of Monthly Returns on U.S. Stocks

On the downside, however, there is a substantial probability of losing 10% or more in a month. This cumulative probability is 3%, meaning that in a repeated sample with 100 months, we should expect to lose 10% or more for a total of three months. This risk is worse than predicted by a normal distribution.

If this risk is too large for the investor, then some money should be allocated to cash. Of course, this comes at the expense of lower expected returns.

The distribution can be characterized in several ways. The entire shape is most informative because it could reveal a greater propensity to large losses than to gains. The distribution could be described by just a few summary statistics, keeping in mind that this is an oversimplification. Other chapters offer formal definitions of these statistics.

- The *mean*, or average return, which is approximately 1% per month. Define this as $\mu(R_P)$, or μ_P in short, or even μ when there is no other asset.
- The *standard deviation*, which is approximately 5.5%. This is often called volatility and is a measure of dispersion around the mean. Define this as σ. This is the square root of the portfolio variance, σ^2.
- The *value at risk* (VAR), which is the cutoff point such that there is a low probability of a greater loss. This is also the percentile of the distribution. Using a 99% confidence level, for example, we find a VAR of 14.4%.

1.1.2 Absolute versus Relative Risk

So far, we have assumed that risk is measured by the dispersion of dollar returns, or in absolute terms. In some cases, however, risk should be measured relative to some **benchmark**. For example, the performance of an active manager is compared to that of an index such as the S&P 500 index for U.S. equities. Alternatively, an investor may have future liabilities, in which case the benchmark is an index of the present value of liabilities. An investor may also want to measure returns after accounting for the effect of inflation. In all of these cases, the investor is concerned with **relative risk**.

- **Absolute risk** is measured in terms of shortfall relative to the initial value of the investment, or perhaps an investment in cash. Using the standard deviation as the risk measure, absolute risk in dollar terms is

$$\sigma(\Delta P) = \sigma(\Delta P/P) \times P = \sigma(R_P) \times P \tag{1.1}$$

- **Relative risk** is measured relative to a benchmark index B. The deviation is $e = R_P - R_B$, which is also known as the **tracking error**. In dollar terms, this is $e \times P$. The risk is

$$\sigma(e)P = [\sigma(R_P - R_B)] \times P = \omega \times P \tag{1.2}$$

where ω is called **tracking error volatility** (TEV).

To compare these two approaches, take the case of an active equity portfolio manager who is given the task of beating a benchmark. In the first year, the active portfolio returns −6% but the benchmark drops by −10%. So, the excess return is positive: $e = -6\% - (-10\%) = 4\%$. In relative terms, the portfolio has done well even though the absolute performance is negative. In the second year, the portfolio returns +6%, which is good using absolute measures, but not so good if the benchmark goes up by +10%.

EXAMPLE 1.1: ABSOLUTE AND RELATIVE RISK

An investment manager is given the task of beating a benchmark. Hence the risk should be measured in terms of

a. Loss relative to the initial investment
b. Loss relative to the expected portfolio value
c. Loss relative to the benchmark
d. Loss attributed to the benchmark

1.2 EVALUATION OF THE RISK MEASUREMENT PROCESS

A major function of the risk measurement process is to estimate the distribution of future profits and losses. The first part of this assignment is easy. The scale of the dollar returns should be proportional to the initial investment. In other words, given the distribution in Figure 1.1, an investment of $100 million should have a standard deviation of $\sigma(\Delta P) = \$100 \times 5.5\% = \$5.5$ million. Scaling the current position by a factor of 2 should increase this risk to $11 million.

The second part of the assignment, which consists of constructing the distribution of future rates of return, is much harder. In Figure 1.1, we have taken the historical distribution and assumed that this provides a good representation of future risks. Because we have a long history of returns over many different cycles, this is a reasonable approach.

This is not always the case, however. The return may have been constant over its recent history. This does not mean that it could not change in the future. For example, the price of gold was fixed to $35 per ounce from 1934 to 1967 by the U.S. government. As a result, using a historical distribution over the 30 years ending in 1967 would have shown no risk. Instead, gold prices started to fluctuate wildly thereafter. By 2008, gold prices had reached $1,000. Thus, the responsibility of the risk manager is to judge whether the history is directly relevant.

How do we evaluate the quality of a risk measurement process? The occurrence of a large loss does not mean that risk management has failed. This could be simply due to bad luck. An investment in stocks would have lost 17% in October 2008. While this is a grievous loss, Figure 1.1 shows that it was not inconceivable. For example, the stock market lost 30% in September 1931 and 22% on October 19, 1987, before recovering. So, the risk manager could have done a perfect job of forecasting the distribution of returns. How can we tell whether this loss is due to bad luck or a flaw in the risk model?

1.2.1 Known Knowns

To help answer this question, it is useful to classify risks into various categories, which we can call (1) known knowns, (2) known unknowns, and (3) unknown unknowns.[2] The first category consists of risks that are properly identified and measured, as in the example of the position in stocks. Losses can still occur due to a combination of bad luck and portfolio decisions.

Such losses, however, should not happen too often. Suppose that VAR at the 99% level of confidence is reported as 14.4%. Under these conditions, a string of consecutive losses of 15% or more several months in a row should be highly unusual. If this were to happen, it would be an indication of a flawed model. A later chapter will show how backtesting can be used to detect flaws in risk measurement systems.

1.2.2 Known Unknowns

The second category, called known unknowns, includes model weaknesses that are known or should be known to exist but are not properly measured by risk managers. For example, the risk manager could have ignored important known risk factors. Second, the distribution of risk factors, including volatilities and correlations, could be measured inaccurately. Third, the mapping process, which consists of replacing positions with exposures on the risk factors, could be incorrect. This is typically called **model risk**. Such risks can be evaluated using stress tests, which shock financial variables or models beyond typical ranges.

As an example, consider the $19 billion loss suffered by UBS in 2007 alone from positions in structured credit securities backed by subprime and Alt-A mortgage-backed loans.[3] UBS had invested in top-rated tranches that the bank thought were perfectly safe (yet yielded high returns). As a result, it had accumulated a position of $90 billion in exposures to these securities, compared to $41 billion in book equity. The bank reported that its risk measurement process relied on simplified models based on a recent period of positive growth in housing prices. As in the example of gold, the recent history gave a biased view of the true risks. In addition, UBS's risk managers overrelied on ratings provided by the credit rating agencies. Because risk management gave little indication of the downside

[2] Philippe Jorion, "Risk Management Lessons from the Credit Crisis," *European Financial Management* 15 (2009): 923–933.

[3] See UBS, *Shareholder Report on UBS's Write-Downs* (Zurich: UBS, 2008). Loans can be classified into prime, Alt-A, and subprime, in order of decreasing credit quality. Subprime loans are loans made to consumers with low credit scores (typically below 640 out of a possible maximum of 850). Alt-A loans, short for Alternative A-paper, are the next category (typically with credit scores below 680 or for loans lacking full documentation). Subprime and Alt-A mortgage loans are expected to have higher credit risk than other (prime) loans.

risk of these investments, these losses can be viewed as a failure of risk management. Even so, the UBS report indicates that the growth strategy undertaken by top management was a "contributing factor to the buildup of UBS's subprime positions which subsequently incurred losses." In other words, top management was largely responsible for the losses.

Another form of known unknown is **liquidity risk**. Most risk models assume that the position can be liquidated over the selected horizon. In practice, this depends on a number of factors. First is the intrinsic liquidity of the asset. Treasury bills, for instance, are much more liquid than high-yield bonds. They trade at a lower spread and with less market impact. Second is the size of the position. This is especially a problem when the position is very large relative to normal trading activity, which would require accepting a large price drop to execute the trade.

1.2.3 Unknown Unknowns

The risks in the last category tend to be the difficult ones. They represent events totally outside the scope of most scenarios. Examples include regulatory risks such as the sudden restrictions on short sales, which can play havoc with hedging strategies, or structural changes such as the conversion of investment banks to commercial banks, which accelerated the deleveraging of the industry. Indeed, a 2010 survey reports that the top concern of risk managers is "government changing the rules."[4]

Similarly, it is difficult to account fully for counterparty risk. It is not enough to know your counterparty; you need to know your counterparty's counterparties, too. In other words, there are network externalities. Understanding the full consequences of Lehman's failure, for example, would have required information on the entire topology of the financial network.[5] Because no individual firm has access to this information, this contagion risk cannot be measured directly.

Similarly, some form of liquidity risk is very difficult to assess. This involves the activity and positions of similar traders, which are generally unknown. In illiquid markets, a forced sale will be much more expensive if a number of similar portfolios are sold at the same time.

This category is sometimes called Knightian **uncertainty**, a form of risk that is immeasurable. Financial institutions cannot possibly carry enough capital to withstand massive counterparty failures, or systemic risk. In such situations, the central bank or the government becomes effectively the risk manager of last resort.

1.2.4 Risk Management Failures

More generally, the role of risk management involves several tasks:

- Identifying all risks faced by the firm
- Assessing and monitoring those risks

[4] *Risk Governance: A Benchmarking Survey* (New York: Capital Markets Risk Advisors, 2010).
[5] A. Haldane, *Why Banks Failed the Stress Test* (London: Bank of England, 2009).

- Managing those risks if given the authority to do so
- Communicating these risks to the decision makers

A large loss is not necessarily an indication of a risk management failure. It could have been within the scope of known knowns and properly communicated to the firm, in which case it reflects bad luck. After all, the objective of risk management is not to prevent losses.

Otherwise, risk management can fail if any of these tasks has not been met. Some risks could go unrecognized. Mismeasurement of risk can occur due to model risk, due to liquidity risk, or if distributions are not adequately measured. Risk limits could not have been enforced. Finally, risk management fails when it does not communicate risks effectively.

EXAMPLE 1.2: FRM EXAM 2009—QUESTION 1-11

Based on the risk assessment of the CRO, Bank United's CEO decided to make a large investment in a levered portfolio of CDOs. The CRO had estimated that the portfolio had a 1% chance of losing $1 billion or more over one year, a loss that would make the bank insolvent. At the end of the first year the portfolio has lost $2 billion and the bank was closed by regulators.

Which of the following statements is correct?

a. The outcome demonstrates a risk management failure because the bank did not eliminate the possibility of financial distress.
b. The outcome demonstrates a risk management failure because the fact that an extremely unlikely outcome occurred means that the probability of the outcome was poorly estimated.
c. The outcome demonstrates a risk management failure because the CRO failed to go to regulators to stop the shutdown.
d. Based on the information provided, one cannot determine whether it was a risk management failure.

1.3 PORTFOLIO CONSTRUCTION

1.3.1 Comparing Multiple Assets

We now turn to the portfolio construction process, which involves combining expected return and risk. Assume that another choice is to invest in long-term U.S. government bonds.

Over the same period, the monthly average return of this other asset class was 0.47%. This is half that of equities. The monthly standard deviation was 2.3%,

TABLE 1.1 Risk and Expected Return on Two Assets

	Average	Volatility	Correlation
Equities	11.2%	19.2%	
Long-term bonds	5.6%	8.1%	0.13

again lower than for equities. To make the numbers more intuitive, monthly returns have been converted to annualized terms, as shown in Table 1.1.

Here, our investor is faced with a typical trade-off, which is to choose between these two alternatives. Neither dominates the other, as shown in Figure 1.2.

This graph describes a simple investment decision. More generally, it also represents more complex business decisions that involve risk. For instance, a bank must decide how much leverage to assume, as defined by the amount of assets divided by the amount of equity on its balance sheet. The horizontal axis could then represent the bank's credit rating. On one hand, higher leverage involves higher risk and accordingly a lower credit rating. In Figure 1.2, this corresponds to a move to the right. On the other hand, higher leverage means that the expected return to equity should be higher. This is because the amount of equity on the balance sheet is lower, implying that profits will be distributed to a smaller equity base. In Figure 1.2, this corresponds to a move up. Again, we observe a trade-off between higher risk and higher return. Without risk measures, deciding where to invest would be difficult.

1.3.2 Risk-Adjusted Performance Measurement

The next question is how the performance can be adjusted for risk in a single measure. The same methods apply to past performance, using historical averages, or prospective performance, using projected numbers.

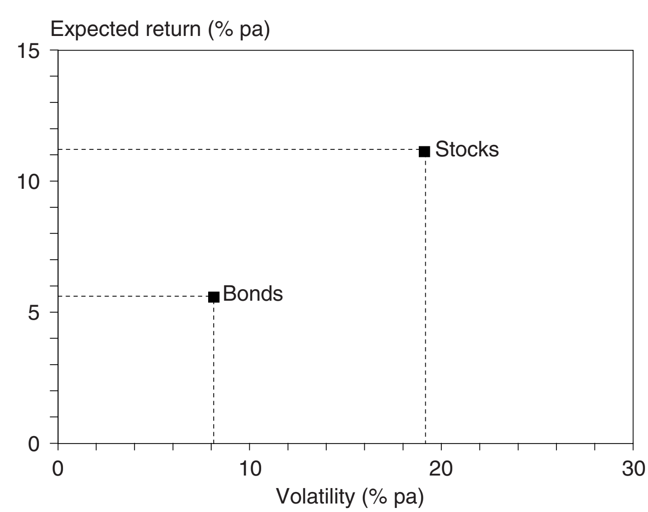

FIGURE 1.2 Comparing Risk and Expected Return

The simplest metric is the **Sharpe ratio** (SR), which is the ratio of the average rate of return, $\mu(R_P)$, in excess of the risk-free rate R_F, to the absolute risk:

$$\mathrm{SR} = \frac{[\mu(R_P) - R_F]}{\sigma(R_P)} \tag{1.3}$$

The Sharpe ratio focuses on total risk measured in absolute terms. This approach can be extended to include VAR in the denominator instead of the volatility of returns.

Figure 1.3 compares the SR for the two investment choices. Assume that we have a risk-free asset, cash, with a return of 3%. The SR is the slope of the line from cash to each asset. This line represents a portfolio mix between cash and each asset. In this case, stocks have a higher SR than bonds. This means that a mix of cash and stocks could be chosen with the same volatility as bonds but with higher returns.

This can be extended to relative risk measures. The **information ratio** (IR) is the ratio of the average rate of return of portfolio P in excess of the benchmark B to the TEV:

$$\mathrm{IR} = \frac{[\mu(R_P) - \mu(R_B)]}{\sigma(R_P - R_B)} \tag{1.4}$$

Table 1.2 presents an illustration. The risk-free interest rate is $R_F = 3\%$ and the portfolio average return is -6%, with volatility of 25%. Hence, the Sharpe ratio of the portfolio is $\mathrm{SR} = [(-6\%) - (3\%)]/25\% = -0.36$. Because this is negative, the absolute performance is poor.

Assume now that the benchmark returned -10% over the same period and that the tracking error volatility was 8%. Hence, the information ratio is $\mathrm{IR} = [(-6\%) - (-10\%)]/8\% = 0.50$, which is positive. The relative performance is good even though the absolute performance is poor.

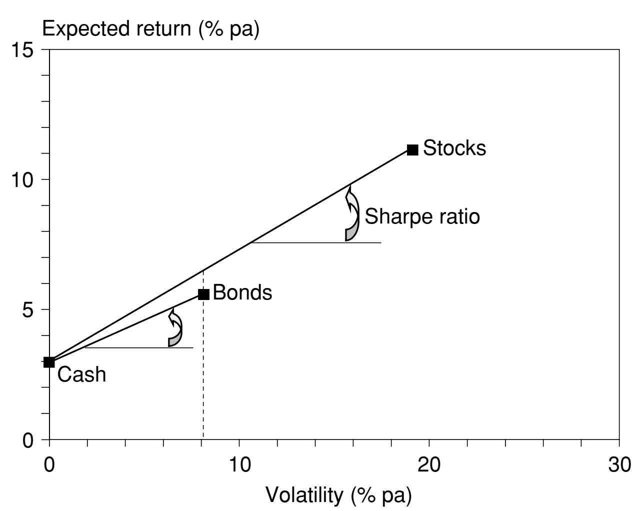

FIGURE 1.3 Comparing Sharpe Ratios

TABLE 1.2 Absolute and Relative Performance

	Average	Volatility	Performance
Cash	3%	0%	
Portfolio P	−6%	25%	$SR = -0.36$
Benchmark B	−10%	20%	$SR = -0.65$
Deviation e	4%	8%	$IR = \quad 0.50$

The tracking error volatility can be derived from the volatilities σ_P and σ_B of the portfolio and the benchmark as well as their correlation ρ. Chapter 2 shows that the variance of a sum of random variables can be expressed in terms of the sum of the individual variances plus twice a covariance term. In terms of difference, the variance is

$$\omega^2 = \sigma_P^2 - 2\rho\sigma_P\sigma_B + \sigma_B^2 \tag{1.5}$$

In this case, if $\sigma_P = 25\%$, $\sigma_B = 20\%$, and $\rho = 0.961$, we have $\omega^2 = 25\%^2 - 2 \times 0.961 \times 25\% \times 20\% + 20\%^2 = 0.0064$, giving $\omega = 8\%$.

The IR has become commonly used to compare active managers in the same peer group. It is a pure measure of active management skill that is scaled for active risk. Consider, for example, two managers. Manager A has TEV of 2% per annum and excess return of 1%. Manager B has TEV of 6% per annum and excess return of 2%. Manager A has lower excess return but a higher information ratio, $1/2 = 0.50$, vs. $2/6 = 0.33$. As a result, it has better management skills. For example, Manager A could be asked to amplify its tracking error by a factor of 3, which would lead to an excess return of 3%, thus beating Manager B with the same level of tracking error of 6%. An information ratio of 0.50 is typical of the performance of the top 25th percentile of money managers and is considered "good."[6]

One of the drawbacks of the information ratio is that the TEV does not adjust for average returns. For instance, a portfolio could be systematically above its benchmark by 0.10% per month. In this case, the tracking error has an average of 0.10% and a standard deviation close to zero. This leads to a very high information ratio, which is not realistic if the active risk cannot be scaled easily.

1.3.3 Mixing Assets

The analysis has so far considered a discrete choice to invest in either asset. More generally, a portfolio can be divided between the two assets. Define w_i as the weight placed on asset i. With full investment, we must have $\sum_{i=1}^{N} w_i = 1$, where N is the total number of assets. In other words, the portfolio weights must sum to 1.

[6] Grinold, Richard and Ronald Kahn, *Active Portfolio Management* (New York: McGraw-Hill, 2000).

Start with a portfolio fully invested in bonds, represented by $w_1 = 1.00, w_2 = 0.00$. As we shift the weight to stocks, we can trace a line that represents the risk and expected return of the portfolio mix. Eventually, this moves to the stock-only position, $w_1 = 0.00, w_2 = 1.00$. Figure 1.4 shows the line that describes all of these portfolios. This is an example of an **asset allocation** problem, where the investor has to decide how much to allocate across asset classes.

The shape of this line depends on the correlation coefficient, ρ, which measures the extent to which the two assets comove with each other. The correlation coefficient is scaled so that it must be between -1 and $+1$. If $\rho = 1$, the two assets move perfectly proportionately to each other and the line becomes a straight line.

More generally, the line is curved. This leads to an interesting observation. The line contains a portfolio with the same level of risk as that of bonds but with a higher return. Thus, a diversified portfolio can dominate one of the assets.

To demonstrate this point, consider the computation of the expected return and volatility for a portfolio. The portfolio variance depends on the weights, the individual asset variances, as well as the correlation:

$$\sigma_P^2 = w_1^2\sigma_1^2 + 2w_1w_2(\rho\sigma_1\sigma_2) + w_2^2\sigma_2^2 \tag{1.6}$$

Hence the portfolio volatility, which is the square root of the variance, is a nonlinear function of the weights. In contrast, the portfolio expected return is a simple linear average:

$$\mu_P = w_1\mu_1 + w_2\mu_2 \tag{1.7}$$

Consider, for instance, the mix of 77% bonds and 23% stocks. The portfolio mean is easy to compute. Using data from Table 1.1, this is

$$\mu_P = 0.77 \times 5.6 + 0.23 \times 11.2 = 6.9\%$$

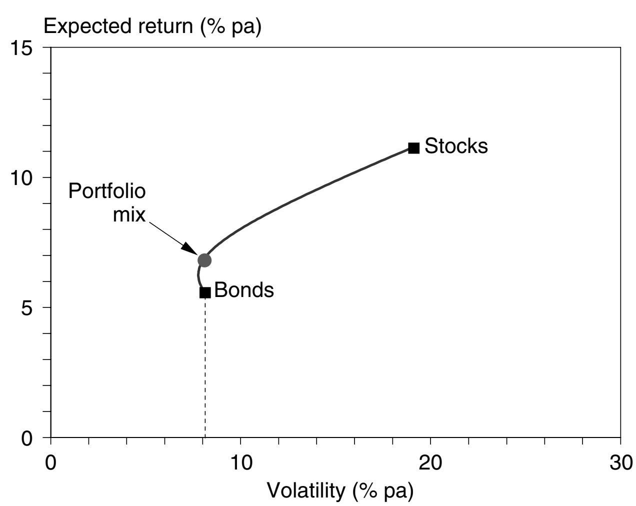

FIGURE 1.4 Mixing Two Assets

When $\rho = 1$, the portfolio variance is, using Equation (1.6):

$$\sigma_P^2 = 0.77^2 8.1^2 + 2 \times 0.77 \times 0.23(1 \times 8.1 \times 19.2) + 0.23^2 19.2^2 = 113.49$$

The portfolio volatility is 10.65. But in this case, this is also a linear average of the two volatilities $\sigma_P = 0.77 \times 8.1 + 0.23 \times 19.2 = 10.65$. Hence this point is on a straight line between the two assets.

Consider next the case where the correlation is the observed value of $\rho = 0.13$, shown in Table 1.1. The portfolio mix then has a volatility of 8.1%. This portfolio has the same volatility as bonds, yet better performance. The mix has expected return of 6.9%, against the 5.6% for bonds, which is an improvement of 1.4%. This demonstrates the power of diversification.

Finally, consider a hypothetical case where the correlation is $\rho = -1$. In this case, the variance drops to 3.32 and the volatility to 1.8%. This illustrates the important point that low correlations help to decrease portfolio risk (at least when the portfolio weights are positive).

1.3.4 Efficient Portfolios

Consider now a more general problem, which is that of diversification across a large number of stocks, for example, $N = 500$ stocks within the S&P 500 index. This seems an insurmountable problem because there are so many different combinations of these stocks. Yet Markowitz showed how the problem can be reduced to a much narrower choice.

The starting point is the assumption that all assets follow a jointly normal distribution. If so, the entire distribution of portfolio returns can be summarized by two parameters only, the mean and the variance.

To solve the diversification problem, all that is needed is the identification of the **efficient set** in this mean-variance space (more precisely, mean–standard deviation). This is the locus of points that represent portfolio mixes with the best risk-return characteristics. More formally, each portfolio, defined by a set of weights $\{w\}$, is such that, for a specified value of expected return μ_p, the risk is minimized:

$$\text{Min}_w \sigma_p^2 \tag{1.8}$$

subject to a number of conditions: (1) the portfolio return is equal to a specified value k, and (2) the portfolio weights sum to 1. Changing this specified value k traces out the efficient set.

When there are no short-sale restrictions on portfolio weights, a closed-form solution exists for the efficient set. Any portfolio is a linear combination of two portfolios. The first is the global **minimum-variance portfolio**, which has the lowest volatility across all fully invested portfolios. The second is the portfolio with the highest Sharpe ratio.

This framework can be generalized to value at risk, which is especially valuable when return distributions have fat tails. In the general case, no closed-form solution exists, however.

1.4 ASSET PRICING THEORIES

1.4.1 Capital Asset Pricing Model (CAPM)

These insights have led to the **capital asset pricing model** (CAPM), developed by William Sharpe (1964).[7] Sharpe's first step was to simplify the covariance structure of stocks with a one-factor model. Define $R_{i,t}$ as the return on stock i during month t, $R_{F,t}$ as the risk-free rate, and $R_{M,t}$ as the market return. Then run a regression of the excess return for i on the market across time:

$$R_{i,t} - R_{F,t} = \alpha_i + \beta_i[R_{M,t} - R_{F,t}] + \epsilon_{i,t}, \quad t = 1, \cdots, T \tag{1.9}$$

The slope coefficient, β_i, measures the exposure of i to the market factor and is also known as **systematic risk**. The intercept is α_i. For an actively managed portfolio α is a measure of management skill. Finally, ϵ_i is the residual, which has mean zero and is uncorrelated to R_M by construction and to all other residuals by assumption.

This is a one-factor model because any interactions between stocks are due to their exposure to the market. To simplify, ignore the risk-free rate and α, which is constant anyway. We now examine the covariance between two stocks, i and j. This is a measure of the comovements between two random variables, and is explained further in Chapter 2.

$$\begin{aligned}\text{Cov}(R_i, R_j) &= \text{Cov}(\beta_i R_M + \epsilon_i, \beta_j R_M + \epsilon_j) \\ &= \beta_i\beta_j\text{Cov}(R_M, R_M) + \beta_i\text{Cov}(R_M,\epsilon_j) + \beta_j\text{Cov}(R_M,\epsilon_i) + \text{Cov}(\epsilon_i,\epsilon_j) \\ &= \beta_i\beta_j\sigma^2(R_M)\end{aligned} \tag{1.10}$$

because all ϵ are uncorrelated with R_M and with each other. Hence, the asset-specific risk, or ϵ, is called **idiosyncratic**.

Such simplified factor structure is extremely useful because it cuts down the number of parameters. With 100 assets, for example, there are in theory $N(N-1)/2 = 4,950$ different pairwise covariances. This is too many to estimate. In contrast, the factor structure in Equation (1.10) involves only 100 parameters, the β_i, plus the variance of the market. This considerably simplifies the analysis.

This type of approximation lies at the heart of **mapping**, a widely used process in risk management. Mapping replaces individual positions by a smaller number of exposures on fundamental risk factors. It will be covered in more detail in a later chapter. Suffice it to say, this requires from the risk manager a good command of quantitative tools in addition to judgment as to whether such simplification is warranted.

Sharpe (1964) then examined the conditions for capital market equilibrium. This requires that the total demand for each asset, as derived from the investors' portfolio optimization, match exactly the outstanding supply of assets (i.e., the

[7] William Sharpe, "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance* 19 (1964): 425–442.

issued shares of equities). The total demand can be aggregated because investors are assumed to have homogeneous expectations about the distribution of rates of return, which is also assumed normal.

In addition, the model assumes the existence of a risk-free asset, which can be used for borrowing or lending, at the same rate. As is usual in most economic models, capital markets are assumed perfect. That is, there are no transaction costs, securities are infinitely divisible, and short sales are allowed.

Under these conditions, Sharpe showed that the market portfolio, defined as the value-weighted average of all stocks in the portfolio, must have the highest Sharpe ratio of any feasible portfolio. Hence, it must be mean-variance efficient. Figure 1.5 shows that the line joining the risk-free asset F to the market portfolio M has the highest Sharpe ratio of any portfolio on the efficient frontier. This line is also known as the **capital market line** (CML). It dominates any combination of cash and stock investment.

Portfolios located between F and M have a positive fraction allocated to each. Investors that are very risk-averse would hold a mix closer to F. Risk-tolerant investors would choose portfolios closer to, or even higher than, the market M. Portfolios above M represent a levered position in the market (i.e., borrowing at the risk-free rate and reinvesting the proceeds in the stock market).

Even so, all investors should hold the same relative proportion of stocks. For instance, Exxon is the largest company in the S&P 500 index, accounting for about 4% of the market. According to the CAPM, all investors should hold 4% of their stock portfolio, whether small or large, in Exxon.

Figure 1.5 demonstrates the concept of **two-fund separation**. Any efficient portfolio must be on the straight line and hence can be separated into two funds, the risk-free asset and the market.

Finally, the mean-variance efficiency of the market implies a linear relationship between expected excess returns and the systematic risk of all stocks in the market portfolio. For any stock i, we must have

$$E(R_i) = R_F + \beta_i[E(R_M) - R_F] \tag{1.11}$$

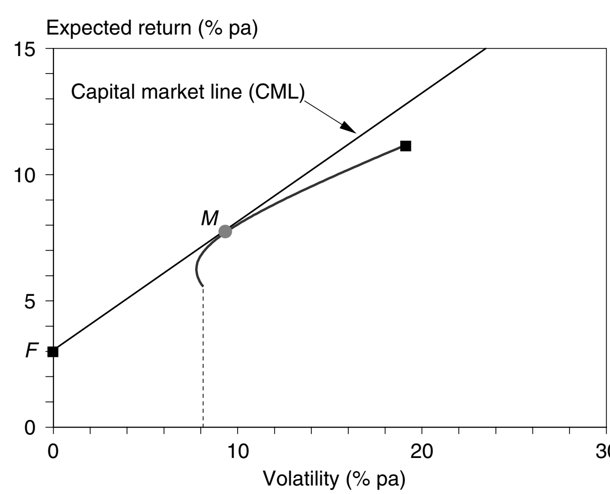

FIGURE 1.5 The Capital Market Line

This required rate of return on equity can also be viewed as the cost of equity capital. In other words, the α_i term in Equation (1.9) should all be zero. This explains why for an actively managed portfolio α is a commonly used term to measure skill.

This theory has profound consequences. It says that investors can easily diversify their portfolios. As a result, most idiosyncratic risk washes away. In contrast, systematic risk cannot be diversified. This explains why Equation (1.11) only contains β_i as a risk factor for stock i. Note that volatility σ_i never appears directly in the pricing of risk.

This leads to an alternative measure of performance, which is the **Treynor ratio** (TR), defined as

$$\mathrm{TR} = \frac{[\mu(R_P) - R_F]}{\beta_P} \tag{1.12}$$

If the CAPM holds, this ratio should be the same for all assets. Equation (1.11) indeed shows that the ratio of $[\mu(R_i) - R_F]/\sigma(R_i)$ should be constant.

The TR penalizes for high β, as opposed to the SR, which penalizes for high σ. For an investor with a portfolio similar to the market, β measures the contribution to the risk of the portfolio. Hence, this is a better performance measure for well-diversified portfolios. In contrast, the SR can be used to adjust the performance of undiversified portfolios. Therefore, a major drawback of the SR and IR is that they do not penalize for systematic risk.

This leads to another performance measure, which directly derives from the CAPM. Suppose an active investment manager claims to add value for portfolio P. The observed average excess return is $\mu(R_P) - R_F$. Some of this, however, may be due to market beta. The proper measure of performance is then **Jensen's alpha**:

$$\alpha_P = \mu(R_P) - R_F - \beta_P[\mu(R_M) - R_F] \tag{1.13}$$

For a fixed portfolio, if all stocks were priced according to the CAPM, the portfolio alpha should be zero (actually it should be negative if one accounts for management fees and other costs).

In summary, the CAPM assumes that (1) investors have homogeneous expectations, (2) distributions are normal, (3) capital markets are perfect, and (4) markets are in equilibrium. The difficulty is that in theory, the market portfolio should contain all investible assets worldwide. As a result, it is not observable.

1.4.2 Arbitrage Pricing Theory (APT)

The CAPM specification starts from a single-factor model. This can be generalized to multiple factors. The first step is to postulate a risk structure where movements in asset returns are due to multiple sources of risk.

For instance, in the stock market, small firms behave differently from large firms. This could be a second factor in addition to the market. Other possible factors are energy prices, interest rates, and so on. With K factors, Equation (1.9) can be generalized to

$$R_i = \alpha_i + \beta_{i1} y_1 + \cdots + \beta_{iK} y_K + \epsilon_i \tag{1.14}$$

Here again, the residuals ϵ are assumed to be uncorrelated with the factors by construction and with each other by assumption. The risk decomposition in Equation (1.10) can be generalized in the same fashion.

The **arbitrage pricing theory** (APT) developed by Stephen Ross (1976) relies on such a factor structure and the assumption that there is no arbitrage in financial markets.[8] Formally, portfolios can be constructed that are well-diversified and have little risk. To prevent arbitrage opportunities, these portfolios should have zero expected returns. These conditions force a linear relationship between expected returns and the factor exposures:

$$E[R_i] = R_F + \sum_{k=1}^{K} \beta_{ik} \lambda_k \tag{1.15}$$

where λ_k is the market price of factor k.

Consider, for example, the simplest case where there is one factor only. We have three stocks, A, B, and C, with betas of 0.5, 1.0, and 1.5, respectively. Assume now that their expected returns are 6%, 8%, and 12%. We can then construct a portfolio long 50% in A, long 50% in C, and short 100% in B. The portfolio beta is $50\% \times 0.5 + 50\% \times 1.5 - 100\% \times 1.0 = 0$. This portfolio has no initial investment but no risk and therefore should have expected return of zero. Computing the expected return gives $50\% \times 6\% + 50\% \times 12\% - 100\% \times 8\% = +1\%$, however. This would create an arbitrage opportunity, which must be ruled out. These three expected returns are inconsistent with the APT. From A and C, we have $R_F = 3\%$ and $\lambda_1 = 6\%$. As a result, the APT expected return on B should be, by Equation (1.15), $E[R_i] = R_F + \beta_{i1}\lambda_1 = 3\% + 1.0 \times 6\% = 9\%$.

Note that the APT expected return is very similar to the CAPM, Equation (1.11), in the case of a single factor. The interpretation, however, is totally different. The APT does not rely on equilibrium but simply on the assumption that there should be no arbitrage opportunities in capital markets, a much weaker requirement. It does not even need the factor model to hold strictly. Instead, it requires only that the residual risk is very small. This must be the case if a sufficient number of common factors is identified and in a well-diversified portfolio, also called **highly granular**.

The APT model does not require the market to be identified, which is an advantage. Unfortunately, tests of this model are ambiguous because the theory provides no guidance as to what the factors should be.

Several approaches to the selection of factors are possible. The first, a structural approach, is to prespecify factors from economic intuition or experience. For

[8] Stephen Ross, "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory* 13 (1976): 341–360.

example, factors such as value, size, and momentum have been widely used to explain expected stock returns. The second is a statistical approach that extracts the factors from the observed data. For example, **principal component analysis** (PCA) is a technique that provides the best fit to the correlation matrix of asset returns. The first PC is a linear combination of asset returns that provides the best approximation to the diagonal of this matrix. The second PC is a linear combination of asset returns that is orthogonal to the first and provides the next best approximation. The analysis can continue until the remaining factors are no longer significant.

Such an approach is very important for risk management when it is a large-scale problem, as for a large portfolio. For example, the risk manager might be able to reduce the dimensionality of a portfolio of 100 stocks using perhaps 10 risk factors. This reduction is particularly important for Monte Carlo simulation, as it cuts down the computing time.

EXAMPLE 1.3: FRM EXAM 2009—QUESTION 1-4

An analyst at CAPM Research Inc. is projecting a return of 21% on Portfolio A. The market risk premium is 11%, the volatility of the market portfolio is 14%, and the risk-free rate is 4.5%. Portfolio A has a beta of 1.5. According to the capital asset pricing model, which of the following statements is true?

a. The expected return of Portfolio A is greater than the expected return of the market portfolio.
b. The expected return of Portfolio A is less than the expected return of the market portfolio.
c. The return of Portfolio A has lower volatility than the market portfolio.
d. The expected return of Portfolio A is equal to the expected return of the market portfolio.

EXAMPLE 1.4: FRM EXAM 2009—QUESTION 1-6

Suppose Portfolio A has an expected return of 8%, volatility of 20%, and beta of 0.5. Suppose the market has an expected return of 10% and volatility of 25%. Finally, suppose the risk-free rate is 5%. What is Jensen's alpha for Portfolio A?

a. 10.0%
b. 1.0%
c. 0.5%
d. 15%

EXAMPLE 1.5: FRM EXAM 2007—QUESTION 132

Which of the following statements about the Sharpe ratio is *false*?

a. The Sharpe ratio considers both the systematic and unsystematic risks of a portfolio.
b. The Sharpe ratio is equal to the excess return of a portfolio over the risk-free rate divided by the total risk of the portfolio.
c. The Sharpe ratio cannot be used to evaluate relative performance of undiversified portfolios.
d. The Sharpe ratio is derived from the capital market line.

EXAMPLE 1.6: SHARPE AND INFORMATION RATIOS

A portfolio manager returns 10% with a volatility of 20%. The benchmark returns 8% with risk of 14%. The correlation between the two is 0.98. The risk-free rate is 3%. Which of the following statements is *correct*?

a. The portfolio has higher SR than the benchmark.
b. The portfolio has negative IR.
c. The IR is 0.35.
d. The IR is 0.29.

1.5 VALUE OF RISK MANAGEMENT

The previous sections have shown that most investment or business decisions require information about the risks of alternative choices. The asset pricing theories also give us a frame of reference to think about how risk management can increase economic value.

1.5.1 Irrelevance of Risk Management

The CAPM emphasizes that investors dislike systematic risk. They can diversify other, nonsystematic risks on their own. Hence, systematic risk is the only risk that needs to be priced.

Companies could use financial derivatives to hedge their volatility. If this changes the volatility but not the market beta, however, the company's cost of capital and hence valuation cannot be affected. Such a result holds only under the perfect capital market assumptions that underlie the CAPM.

As an example, suppose that a company hedges some financial risk that investors could hedge on their own anyway. For example, an oil company could hedge its oil exposure. It would be easy, however, for investors in the company's stock to hedge such oil exposure on their own if they so desired. For instance, they could use oil futures, which can be easily traded. In this situation, the risk management practice by this company should not add economic value.

In this abstract world, risk management is irrelevant. This is an application of the classic Modigliani and Miller (MM) theorem, which says that the value of a firm cannot depend on its financial policies. The intuition for this result is that any financing action undertaken by a corporation that its investors could easily undertake on their own should not add value. Worse, if risk management practices are costly, they could damage the firm's economic value.

1.5.2 Relevance of Risk Management

The MM theorem, however, is based on a number of assumptions: There are no frictions such as financial distress costs, taxes, and access to capital markets; there is no asymmetry of information between financial market participants.

In practice, risk management can add value if some of these assumptions do not hold true.

- Hedging should increase value if it helps avoid a large *cost of financial distress*. Companies that go bankrupt often experience a large drop in value due to forced sales or the legal costs of the bankruptcy process. Consider, for example, the Lehman Brothers bankruptcy. Lehman had around $130 billion of outstanding bonds. The value of these bonds dropped to 9.75 cents on the dollar after September 2008. This implied a huge drop in the firm value due to the bankruptcy event.
- Corporate income *taxes* can be viewed as a form of friction. Assuming no carry-over provisions, the income tax in a very good year is high and is not offset by a tax refund in a year with losses. A tax is paid on income if positive, which is akin to a long position in an option, with a similar convexity effect. Therefore, by stabilizing earnings, corporations reduce the average tax payment over time, which should increase their value.
- Other frictions arise when *external financing* is more costly than internally generated funds. A company could decide not to hedge its financial risks, which leads to more volatile earnings. In good years, projects are financed internally. In bad years, it is always possible to finance projects by borrowing from capital markets. If the external borrowing cost is too high, however, some worthy projects will not be funded in bad years. Hedging helps avoid this **underinvestment problem**, which should increase firm value.
- A form of information asymmetry is due to *agency costs of managerial discretion*. Investors hire managers to serve as their agents, and give them discretion to run the company. Good and bad managers, however, are not always easy to identify. Without hedging, earnings fluctuate due to outside forces. This

makes it difficult to identify the performance of management. With hedging, there is less room for excuses. Bad managers can be identified more easily and fired, which should increase firm value.

- Another form of information asymmetry arises when a large shareholder has *expertise in the firm's business*. This expertise, in addition to the ability to monitor management more effectively than others, may increase firm value. Typically, such investors have a large fraction of their wealth invested in this firm. Because they are not diversified, they may be more willing to invest in the company and hence add value when the company lowers its risk.

In practice, there is empirical evidence that firms that engage in risk management programs tend to have higher valuation than others. This type of analysis is fraught with difficulties, however. Researchers do not have access to identical firms that solely differ by their hedging program. Other confounding effects might be at work. Hedging might be correlated with the quality of management, which does have an effect on firm value. For example, firms with risk management programs are more likely to employ financial risk managers (FRMs).

EXAMPLE 1.7: FRM EXAM 2009—QUESTION 1-8

In perfect markets, risk management expenditures aimed at reducing a firm's diversifiable risk serve to

a. Make the firm more attractive to shareholders as long as costs of risk management are reasonable
b. Increase the firm's value by lowering its cost of equity
c. Decrease the firm's value whenever the costs of such risk management are positive
d. Has no impact on firm value

EXAMPLE 1.8: FRM EXAM 2009—QUESTION 1-2

By reducing the risk of financial distress and bankruptcy, a firm's use of derivatives contracts to hedge its cash flow uncertainty will

a. Lower its value due to the transaction costs of derivatives trading
b. Enhance its value since investors cannot hedge such risks by themselves
c. Have no impact on its value as investors can costlessly diversify this risk
d. Have no impact as only systematic risks can be hedged with derivatives

1.6 IMPORTANT FORMULAS

Absolute risk: $\sigma(\Delta P) = \sigma(\Delta P/P) \times P = \sigma(R_P) \times P$

Relative risk: $\sigma(e)P = [\sigma(R_P - R_B)] \times P = \omega \times P$

Tracking error volatility (TEV): $\omega = \sigma(R_P - R_B)$

Sharpe ratio (SR): $\text{SR} = [\mu(R_P) - R_F]/\sigma(R_P)$

Information ratio (IR): $\text{IR} = [\mu(R_P) - \mu(R_B)]/\omega$

Treynor ratio (TR): $\text{TR} = [\mu(R_P) - R_F]/\beta_P$

Jensen's alpha: $\alpha_P = \mu(R_P) - R_F - \beta_P[\mu(R_M) - R_F]$

Multiple factor model: $R_i = \alpha_i + \beta_{i1}y_1 + \cdots + \beta_{iK}y_K + \epsilon_i$

CAPM expected returns: $E(R_i) = R_F + \beta_i[E(R_M) - R_F]$

APT expected returns: $E[R_i] = R_F + \sum_{k=1}^{K} \beta_{ik}\lambda_k$

1.7 ANSWERS TO CHAPTER EXAMPLES

Example 1.1: Absolute and Relative Risk

c. This is an example of risk measured in terms of deviations of the active portfolio relative to the benchmark. Answers a. and b. are incorrect because they refer to absolute risk. Answer d. is also incorrect because it refers to the absolute risk of the benchmark.

Example 1.2: FRM Exam 2009—Question 1-11

d. It is the role of the CEO to decide on such investments, not the CRO. The CRO had correctly estimated that there was some chance of losing $1 billion or more. In addition, there is no information on the distribution beyond VAR. So, this could have been bad luck. A risk management failure could have occurred if the CRO had stated that this probability was zero.

Example 1.3: FRM Exam 2009—Question 1-4

a. According to the CAPM, the required return on Portfolio A is $R_F + \beta[E(R_M) - R_F] = 4.5 + 1.5[11] = 21\%$ indeed. Because the beta is greater than 1, it must be greater than the expected return on the market, which is 15.5%. Note that the question has a lot of extraneous information.

Example 1.4: FRM Exam 2009—Question 1-6

c. This is the reverse problem. The CAPM return is $R_F + \beta[E(R_M) - R_F] = 5 + 0.5[10 - 5] = 7.5\%$. Hence the alpha is $8 - 7.5 = 0.5\%$.

Example 1.5: FRM Exam 2007—Question 132

c. The SR considers total risk, which includes systematic and unsystematic risks, so a. and b. are correct statements, and incorrect answers. Similarly, the SR is derived from the CML, which states that the market is mean-variance efficient and hence has the highest Sharpe ratio of any feasible portfolio. Finally, the SR can be used to evaluate undiversified portfolios, precisely because it includes idiosyncratic risk.

Example 1.6: Sharpe and Information Ratios

d. The Sharpe ratios of the portfolio and benchmark are $(10\% - 3\%)/20\% = 0.35$ and $(8\% - 3\%)/14\% = 0.36$, respectively. So the SR of the portfolio is lower than that of the benchmark; answer a. is incorrect. The TEV is the square root of $20\%^2 + 14\%^2 - 2 \times 0.98 \times 20\% \times 14\%$, which is $\sqrt{0.00472} = 6.87\%$. So, the IR of the portfolio is $(10\% - 8\%)/6.87\% = 0.29$. This is positive, so answer b. is incorrect. Answer c. is the SR of the portfolio, not the IR, so it is incorrect.

Example 1.7: FRM Exam 2009—Question 1-8

c. In perfect markets, risk management actions that lower the firm's diversifiable risk should not affect its cost of capital, and hence will not increase value. Further, if these activities are costly, the firm value should decrease.

Example 1.8: FRM Exam 2009—Question 1-2

b. The cost of financial distress is a market imperfection, or deadweight cost. By hedging, firms will lower this cost, which should increase the economic value of the firm.

PART Two

Quantitative Analysis

CHAPTER 2

Fundamentals of Probability

The preceding chapter has shown how a risk manager can characterize the risk of a portfolio using a frequency distribution. This process uses the tools of probability, a mathematical abstraction that constructs the distribution of random variables. These random variables are financial risk factors, such as movements in stock prices, in bond prices, in exchange rates, and in commodity prices. These risk factors are then transformed into profits and losses on the portfolio, which can be described by a probability distribution function.

This chapter reviews the fundamental tools of probability theory for risk managers. Section 2.1 lays out the foundations, characterizing random variables by their probability density and distribution functions. These functions can be described by their principal moments, mean, variance, skewness, and kurtosis. Distributions with multiple variables are described in Section 2.2. Section 2.3 then turns to functions of random variables. Section 2.4 presents some examples of important distribution functions for risk management, including the uniform, normal, lognormal, Student's *t*, binomial, and Poisson distributions. Finally, Section 2.5 discusses limit distributions, which can be used to characterize the average of independent random variables.

2.1 CHARACTERIZING RANDOM VARIABLES

The classical approach to probability is based on the concept of the **random variable** (RV). This can be viewed as the outcome from throwing a die, for example. Each realization is generated from a fixed process. If the die is perfectly symmetrical, with six faces, we could say that the probability of observing a face with a specified number in one throw is $p = 1/6$. Although the event itself is random, we can still make a number of useful statements from a fixed data-generating process.

The same approach can be taken to financial markets, where stock prices, exchange rates, yields, and commodity prices can be viewed as random variables. The assumption of a fixed data-generating process for these variables, however, is more tenuous than for the preceding experiment.

FRM Exam Part 1 topic.

2.1.1 Univariate Distribution Functions

A random variable X is characterized by a **distribution function**,

$$F(x) = P(X \leq x) \tag{2.1}$$

which is the probability that the realization of the random variable X ends up less than or equal to the given number x. This is also called a **cumulative distribution function**.

When the variable X takes discrete values, this distribution is obtained by summing the step values less than or equal to x. That is,

$$F(x) = \sum_{x_j \leq x} f(x_j) \tag{2.2}$$

where the function $f(x)$ is called the **frequency function** or the **probability density function** (p.d.f.). Here, $f(x)$ is the probability of observing x. This function is characterized by its shape as well as fixed parameters, θ.

When the variable is continuous, the distribution is given by

$$F(x) = \int_{-\infty}^{x} f(u)du \tag{2.3}$$

The density can be obtained from the distribution using

$$f(x) = \frac{dF(x)}{dx} \tag{2.4}$$

Often, the random variable will be described interchangeably by its distribution or its density.

These functions have notable properties. The density $f(u)$ must be positive for all u. As x tends to infinity, the distribution tends to unity as it represents the total probability of any draw for x:

$$\int_{-\infty}^{\infty} f(u)du = 1 \tag{2.5}$$

Figure 2.1 gives an example of a density function $f(x)$ on the top panel, and of a cumulative distribution function $F(x)$ on the bottom panel. $F(x)$ measures the area under the $f(x)$ curve to the left of x, which is represented by the shaded area. Here, this area is 0.24. For small values of x, $F(x)$ is close to zero. Conversely, for large values of x, $F(x)$ is close to unity.

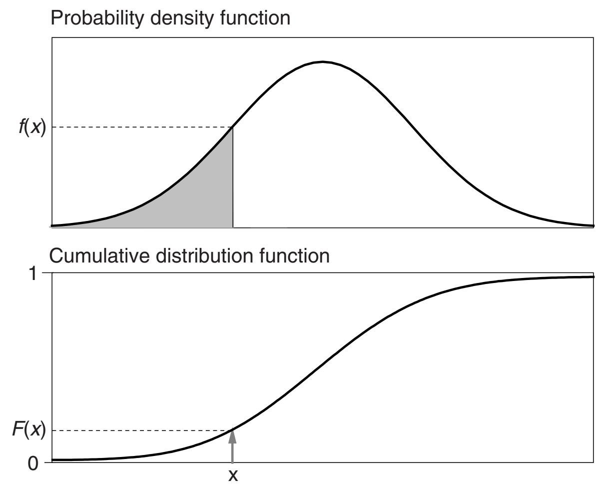

FIGURE 2.1 Density and Distribution Functions

Example: Density Functions

A gambler wants to characterize the probability density function of the outcomes from a pair of dice. Because each has six faces, there are $6^2 = 36$ possible throw combinations. Out of these, there is one occurrence of an outcome of two (each die showing one). So, the frequency of an outcome of two is one. We can have two occurrences of a three (a one and a two and vice versa), and so on.

The gambler compiles the frequency of each value, from 2 to 12, as shown in Table 2.1. From this, the gambler can compute the probability of each outcome. For instance, the probability of observing three is equal to 2, the frequency $n(x)$, divided by the total number of outcomes, of 36, which gives 0.0556. We can verify that all the probabilities indeed add up to 1, since all occurrences must be accounted for. From the table, we see that the probability of an outcome of three or less is 8.33%.

TABLE 2.1 Probability Density Function

Outcome x_i	Frequency $n(x)$	Probability $f(x)$		Cumulative Probability $F(x)$
2	1	1/36	0.0278	0.0278
3	2	2/36	0.0556	0.0833
4	3	3/36	0.0833	0.1667
5	4	4/36	0.1111	0.2778
6	5	5/36	0.1389	0.4167
7	6	6/36	0.1667	0.5833
8	5	5/36	0.1389	0.7222
9	4	4/36	0.1111	0.8333
10	3	3/36	0.0833	0.9167
11	2	2/36	0.0556	0.9722
12	1	1/36	0.0278	1.0000
Sum	36	1	1.0000	

2.1.2 Moments

A random variable is characterized by its distribution function. Instead of having to report the whole function, it is convenient to summarize it by a few parameters, or **moments**.

For instance, the expected value for x, or **mean**, is given by the integral

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx \tag{2.6}$$

which measures the *central tendency*, or *center of gravity* of the population.

The distribution can also be described by its **quantile**, which is the cutoff point x with an associated probability c:

$$F(x) = \int_{-\infty}^{x} f(u) du = c \tag{2.7}$$

So, there is a probability of c that the random variable will fall *below* x. Because the total probability adds up to 1, there is a probability of $p = 1 - c$ that the random variable will fall *above* x. Define this quantile as $Q(X, c)$. The 50% quantile is known as the **median**.

In fact, value at risk (VAR) can be interpreted as the cutoff point such that a loss will not happen with probability greater than $p = 95\%$, say. If $f(u)$ is the distribution of profit and losses on the portfolio, VAR is defined from

$$F(x) = \int_{-\infty}^{x} f(u) du = (1 - p) \tag{2.8}$$

where p is the right-tail probability, and c the usual left-tail probability. VAR can be defined as minus the quantile itself, or alternatively, the deviation between the expected value and the quantile,

$$\text{VAR}(c) = E(X) - Q(X, c) \tag{2.9}$$

Note that VAR is typically reported as a loss (i.e., a positive number), which explains the negative sign. Figure 2.2 shows an example with $c = 5\%$.

Another useful moment is the squared dispersion around the mean, or **variance**:

$$\sigma^2 = V(X) = \int_{-\infty}^{+\infty} [x - E(X)]^2 f(x) dx \tag{2.10}$$

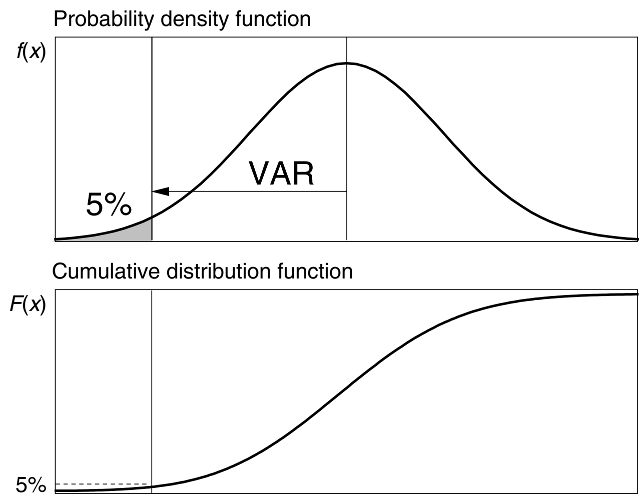

FIGURE 2.2 VAR as a Quantile

The **standard deviation** is more convenient to use as it has the same units as the original variable X:

$$\mathrm{SD}(X) = \sigma = \sqrt{V(X)} \tag{2.11}$$

Next, the scaled third moment is the **skewness**, which describes departures from symmetry. It is defined as

$$\gamma = \left(\int_{-\infty}^{+\infty} [x - E(X)]^3 f(x)dx \right) \Big/ \sigma^3 \tag{2.12}$$

Negative skewness indicates that the distribution has a long left tail, which indicates a high probability of observing large negative values. If this represents the distribution of profits and losses for a portfolio, this is a dangerous situation. Figure 2.3 displays distributions with various signs for the skewness.

The scaled fourth moment is the **kurtosis**, which describes the degree of flatness of a distribution, or width of its tails. It is defined as

$$\delta = \left(\int_{-\infty}^{+\infty} [x - E(X)]^4 f(x)dx \right) \Big/ \sigma^4 \tag{2.13}$$

Because of the fourth power, large observations in the tail will have a large weight and hence create large kurtosis. Such a distribution is called **leptokurtic**, or **fat-tailed**. This parameter is very important for risk measurement. A kurtosis of 3 is considered average, and represents a normal distribution. High kurtosis indicates a higher probability of extreme movements. A distribution with kurtosis lower than 3 is called **platykurtic**. Figure 2.4 displays distributions with various values for the kurtosis.

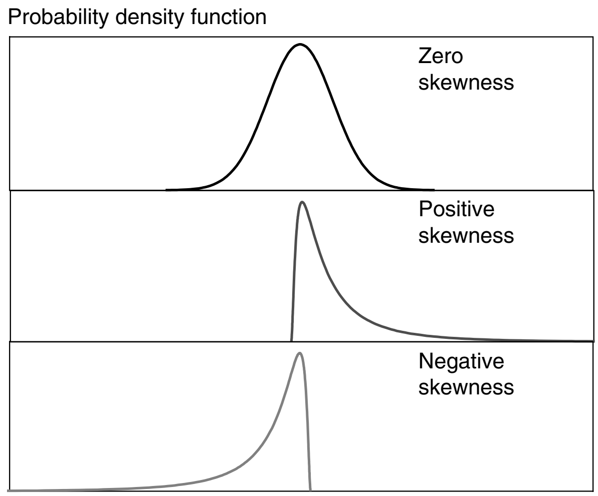

FIGURE 2.3 Effect of Skewness

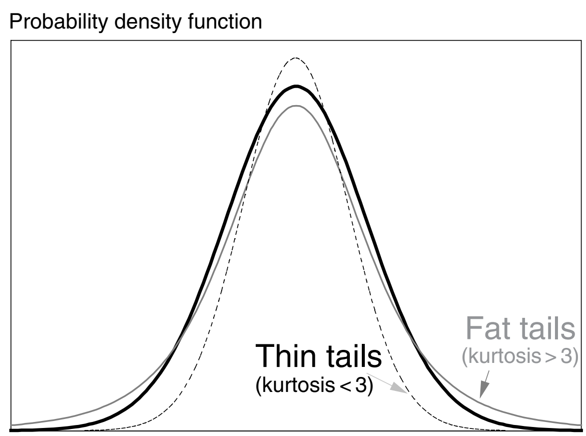

FIGURE 2.4 Effect of Kurtosis

Example: Computing Moments

Our gambler wants to know the expected value of the outcome of throwing two dice. He computes the product of each outcome and associated probability, as shown in Table 2.2. For instance, the first entry is $xf(x) = 2 \times 0.0278 = 0.0556$, and so on. Summing across all events, the mean is $\mu = 7.000$. This is also the median, since the distribution is perfectly symmetrical.

Next, we can use Equation (2.10) to compute the variance. The first term is $(x - \mu)^2 \ f(x) = (2 - 7)^2 0.0278 = 0.6944$. These terms add up to 5.8333, or, taking the square root, $\sigma = 2.4152$. The skewness terms sum to zero, because for each entry with a positive deviation $(x - \mu)^3$, there is an identical one with a negative sign and with the same probability. Finally, the kurtosis terms $(x - \mu)^4 f(x)$ sum to 80.5. Dividing by $\sigma^4 = 34.0278$, this gives a kurtosis of $\delta = 2.3657$.

TABLE 2.2 Computing Moments of a Distribution

Outcome x_i	Prob. $f(x)$	Mean $xf(x)$	Variance $(x-\mu)^2 f(x)$	Skewness $(x-\mu)^3 f(x)$	Kurtosis $(x-\mu)^4 f(x)$
2	0.0278	0.0556	0.6944	−3.4722	17.3611
3	0.0556	0.1667	0.8889	−3.5556	14.2222
4	0.0833	0.3333	0.7500	−2.2500	6.7500
5	0.1111	0.5556	0.4444	−0.8889	1.7778
6	0.1389	0.8333	0.1389	−0.1389	0.1389
7	0.1667	1.1667	0.0000	0.0000	0.0000
8	0.1389	1.1111	0.1389	0.1389	0.1389
9	0.1111	1.0000	0.4444	0.8889	1.7778
10	0.0833	0.8333	0.7500	2.2500	6.7500
11	0.0556	0.6111	0.8889	3.5556	14.2222
12	0.0278	0.3333	0.6944	3.4722	17.3611
Sum	1.0000	7.0000	$\sigma^2 = 5.8333$	0.0000	80.5000
Denominator				$\sigma^3 = 14.0888$	$\sigma^4 = 34.0278$
		Mean	StdDev	Skewness	Kurtosis
		$\mu = 7.00$	$\sigma = 2.4152$	$\gamma = 0.0000$	$\delta = 2.3657$

EXAMPLE 2.1: FRM EXAM 2009—QUESTION 2-3

An analyst gathered the following information about the return distributions for two portfolios during the same time period:

Portfolio	Skewness	Kurtosis
A	−1.6	1.9
B	0.8	3.2

The analyst states that the distribution for Portfolio A is more peaked than a normal distribution and that the distribution for Portfolio B has a long tail on the left side of the distribution. Which of the following is correct?

a. The analyst's assessment is correct.
b. The analyst's assessment is correct for Portfolio A and incorrect for Portfolio B.
c. The analyst's assessment is not correct for Portfolio A but is correct for Portfolio B.
d. The analyst's assessment is incorrect for both portfolios.

2.2 MULTIVARIATE DISTRIBUTION FUNCTIONS

In practice, portfolio payoffs depend on numerous random variables. To simplify, start with two random variables. This could represent two currencies, or two interest rate factors, or default and credit exposure, to give just a few examples.

2.2.1 Joint Distributions

We can extend Equation (2.1) to

$$F_{12}(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) \tag{2.14}$$

which defines a joint bivariate distribution function. In the continuous case, this is also

$$F_{12}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{12}(u_1, u_2) du_1 du_2 \tag{2.15}$$

where $f(u_1, u_2)$ is now the **joint density**. In general, adding random variables considerably complicates the characterization of the density or distribution functions.

The analysis simplifies considerably if the variables are **independent**. In this case, the joint density separates out into the product of the densities:

$$f_{12}(u_1, u_2) = f_1(u_1) \times f_2(u_2) \tag{2.16}$$

and the integral reduces to

$$F_{12}(x_1, x_2) = F_1(x_1) \times F_2(x_2) \tag{2.17}$$

This is very convenient because we only need to know the individual densities to reconstruct the joint density. For example, a credit loss can be viewed as a combination of (1) default, which is a random variable with a value of one for default and zero otherwise, and (2) the exposure, which is a random variable representing the amount at risk, for instance the positive market value of a swap. If the two variables are independent, we can construct the distribution of the credit loss easily. In the case of the two dice, the events are indeed independent. As a result, the probability of a joint event is simply the product of probabilities. For instance, the probability of throwing two ones is equal to $1/6 \times 1/6 = 1/36$.

It is also useful to characterize the distribution of x_1 abstracting from x_2. By integrating over all values of x_2, we obtain the **marginal density**:

$$f_1(x_1) = \int_{-\infty}^{\infty} f_{12}(x_1, u_2) du_2 \tag{2.18}$$

and similarly for x_2. We can then define the **conditional density** as

$$f_{1\cdot 2}(x_1 \mid x_2) = \frac{f_{12}(x_1, x_2)}{f_2(x_2)} \tag{2.19}$$

Here, we keep x_2 fixed and divide the joint density by the marginal probability of x_2. This normalization is necessary to ensure that the conditional density is a proper density function that integrates to one. This relationship is also known as **Bayes' rule**.

2.2.2 Covariances and Correlations

When dealing with two random variables, the comovement can be described by the **covariance**

$$\mathrm{Cov}(X_1, X_2) = \sigma_{12} = \int_1 \int_2 [x_1 - E(X_1)][x_2 - E(X_2)] f_{12}(x_1, x_2) dx_1 dx_2 \quad (2.20)$$

It is often useful to scale the covariance into a unitless number, called the **correlation coefficient**, obtained as

$$\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} \quad (2.21)$$

The correlation coefficient is a measure of linear dependence. One can show that the correlation coefficient always lies in the $[-1, +1]$ interval. A correlation of one means that the two variables always move in the same direction. A correlation of minus one means that the two variables always move in the opposite direction.

Equation (2.21) defines what is also called the **Pearson correlation**. Another measure is the **Spearman correlation**, which replaces the value of the variables by their rank. This nonparametric measure is less sensitive to outliers, and hence more robust than the usual correlation when there might be errors in the data.

If the variables are independent, the joint density separates out and this becomes

$$\mathrm{Cov}(X_1, X_2) = \left\{ \int_1 [x_1 - E(X_1)] f_1(x_1) dx_1 \right\} \left\{ \int_2 [x_2 - E(X_2)] f_2(x_2) dx_2 \right\} = 0$$

by Equation (2.6), since the average deviation from the mean is zero. In this case, the two variables are said to be **uncorrelated**. Hence independence implies zero correlation (the reverse is not true, however).

Example: Multivariate Functions

Consider two variables, such as the exchange rates for the Canadian dollar and the euro. Table 2.3a describes the joint density function $f_{12}(x_1, x_2)$, assuming two payoffs only for each variable. Note first that the density indeed sums to $0.30 + 0.20 + 0.15 + 0.35 = 1.00$.

TABLE 2.3a Joint Density Function

x_1 / x_2	-5	$+5$
-10	0.30	0.15
$+10$	0.20	0.35

From this, we can compute the marginal density for each variable, along with its mean and standard deviation. For instance, the marginal probability of $x_1 = -5$

is given by $f_1(x_1) = f_{12}(x_1, x_2 = -10) + f_{12}(x_1, x_2 = +10) = 0.30 + 0.20 = 0.50$. The marginal probability of $x_1 = +5$ must be 0.50 as well. Table 2.3b shows that the means and standard deviations are, respectively, $\bar{x}_1 = 0.0, \sigma_1 = 5.0$, and $\bar{x}_2 = 1.0, \sigma_2 = 9.95$.

TABLE 2.3b Marginal Density Functions

Variable 1				Variable 2			
x_1	Prob. $f_1(x_1)$	Mean $x_1 f_1(x_1)$	Variance $(x_1 - \bar{x}_1)^2 f_1(x_1)$	x_2	Prob. $f_2(x_2)$	Mean $x_2 f_2(x_2)$	Variance $(x_2 - \bar{x}_2)^2 f_2(x_2)$
−5	0.50	−2.5	12.5	−10	0.45	−4.5	54.45
+5	0.50	+2.5	12.5	+10	0.55	+5.5	44.55
Sum	1.00	0.0	25.0	Sum	1.00	1.0	99.0
		$\bar{x}_1 = 0.0$	$\sigma_1 = 5.0$			$\bar{x}_2 = 1.0$	$\sigma_2 = 9.95$

Finally, Table 2.3c details the computation of the covariance, which gives Cov = 15.00. Dividing by the product of the standard deviations, we get $\rho = \text{Cov}/(\sigma_1\sigma_2) = 15.00/(5.00 \times 9.95) = 0.30$. The positive correlation indicates that when one variable goes up, the other is more likely to go up than down.

TABLE 2.3c Covariance and Correlation

	$(x_1 - \bar{x}_1)(x_2 - \bar{x}_2) f_{12}(x_1, x_2)$	
	$x_1 = -5$	$x_1 = +5$
$x_2 = -10$	$(-5 - 0)(-10 - 1)0.30 = 16.50$	$(+5 - 0)(-10 - 1)0.15 = -8.25$
$x_2 = +10$	$(-5 - 0)(+10 - 1)0.20 = -9.00$	$(+5 - 0)(+10 - 1)0.35 = 15.75$
Sum	Cov = 15.00	

EXAMPLE 2.2: FRM EXAM 2000—QUESTION 81

Which one of the following statements about the correlation coefficient is *false*?

- **a.** It always ranges from −1 to +1.
- **b.** A correlation coefficient of zero means that two random variables are independent.
- **c.** It is a measure of linear relationship between two random variables.
- **d.** It can be calculated by scaling the covariance between two random variables.

EXAMPLE 2.3: FRM EXAM 2007—QUESTION 93

The joint probability distribution of random variables X and Y is given by $f(x, y) = k \times x \times y$ for $x = 1, 2, 3$, $y = 1, 2, 3$, and k is a positive constant. What is the probability that $X + Y$ will exceed 5?

a. 1/9
b. 1/4
c. 1/36
d. Cannot be determined

2.3 FUNCTIONS OF RANDOM VARIABLES

Risk management is about uncovering the distribution of portfolio values. Consider a security that depends on a unique source of risk, such as a bond. The risk manager could model the change in the bond price as a random variable (RV) directly. The problem with this choice is that the distribution of the bond price is not stationary, because the price converges to the face value at expiration.

Instead, the practice is to model the change in yields as a random variable because its distribution is better behaved. The next step is to use the relationship between the bond price and the yield to uncover the distribution of the bond price.

This illustrates a general principle of risk management, which is to model the risk factor first, then to derive the distribution of the instrument from information about the function that links the instrument value to the risk factor. This may not be easy to do, unfortunately, if the relationship is highly nonlinear. In what follows, we first focus on the mean and variance of simple transformations of random variables.

2.3.1 Linear Transformation of Random Variables

Consider a transformation that multiplies the original random variable by a constant and add a fixed amount, $Y = a + bX$. The expectation of Y is

$$E(a + bX) = a + bE(X) \tag{2.22}$$

and its variance is

$$V(a + bX) = b^2 V(X) \tag{2.23}$$

Note that adding a constant never affects the variance since the computation involves the *difference* between the variable and its mean. The standard deviation is

$$\text{SD}(a + bX) = b\text{SD}(X) \tag{2.24}$$

Example: Currency Position Plus Cash

A dollar-based investor has a portfolio consisting of \$1 million in cash plus a position in 1,000 million Japanese yen. The distribution of the dollar/yen exchange rate X has a mean of $E(X) = 1/100 = 0.01$ and volatility of $\mathrm{SD}(X) = 0.10/100 = 0.001$.

The portfolio value can be written as $Y = a + bX$, with fixed parameters (in millions) $a = \$1$ and $b = Y1{,}000$. Therefore, the portfolio expected value is $E(Y) = \$1 + Y1{,}000 \times 1/100 = \$11$ million, and the standard deviation is $\mathrm{SD}(Y) = Y1{,}000 \times 0.001 = \$1$ million.

2.3.2 Sum of Random Variables

Another useful transformation is the summation of two random variables. A portfolio, for instance, could contain one share of Intel plus one share of Microsoft. The rate of return on each stock behaves as a random variable.

The expectation of the sum $Y = X_1 + X_2$ can be written as

$$E(X_1 + X_2) = E(X_1) + E(X_2) \tag{2.25}$$

and its variance is

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2\mathrm{Cov}(X_1, X_2) \tag{2.26}$$

When the variables are uncorrelated, the variance of the sum reduces to the sum of variances. Otherwise, we have to account for the cross-product term.

KEY CONCEPT

The expectation of a sum is the sum of expectations. The variance of a sum, however, is the sum of variances only if the variables are uncorrelated.

2.3.3 Portfolios of Random Variables

More generally, consider a linear combination of a number of random variables. This could be a portfolio with fixed weights, for which the rate of return is

$$Y = \sum_{i=1}^{N} w_i X_i \tag{2.27}$$

where N is the number of assets, X_i is the rate of return on asset i, and w_i its weight.

To shorten notation, this can be written in matrix notation, replacing a string of numbers by a single vector:

$$Y = w_1 X_1 + w_2 X_2 + \cdots + w_N X_N = \left[w_1 w_2 \ldots w_N\right] \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = w' X \tag{2.28}$$

where w' represents the transposed vector (i.e., horizontal) of weights and X is the vertical vector containing individual asset returns. Appendix A provides a brief review of matrix multiplication.

The portfolio expected return is now

$$E(Y) = \mu_p = \sum_{i=1}^{N} w_i \mu_i \tag{2.29}$$

which is a weighted average of the expected returns $\mu_i = E(X_i)$. The variance is

$$V(Y) = \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \sigma_{ij} \tag{2.30}$$

Using matrix notation, the variance can be written as

$$\sigma_p^2 = [w_1 \ldots w_N] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \ldots & \sigma_{1N} \\ \vdots & & & & \vdots \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \ldots & \sigma_{N} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

Defining Σ as the covariance matrix, the variance of the portfolio rate of return can be written more compactly as

$$\sigma_p^2 = w' \Sigma w \tag{2.31}$$

This is a useful expression to describe the risk of the total portfolio.

Example: Computing the Risk of a Portfolio

Consider a portfolio invested in Canadian dollars and euros. The joint density function is given by Table 2.3a. Here, x_1 describes the payoff on the Canadian dollar, with $\mu_1 = 0.00$, $\sigma_1 = 5.00$, and $\sigma_1^2 = 25$. For the euro, $\mu_2 = 1.00$, $\sigma_2 = 9.95$, and $\sigma_2^2 = 99$. The covariance was computed as $\sigma_{12} = 15.00$, with the correlation $\rho = 0.30$. If we have 60% invested in Canadian dollars and 40% in euros, what is the portfolio volatility?

Following Equation (2.31), we write

$$\sigma_p^2 = [0.60 \quad 0.40] \begin{bmatrix} 25 & 15 \\ 15 & 99 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix} = [0.60 \quad 0.40] \begin{bmatrix} 25 \times 0.60 + 15 \times 0.40 \\ 15 \times 0.60 + 99 \times 0.40 \end{bmatrix}$$

$$\sigma_p^2 = [0.60 \quad 0.40] \begin{bmatrix} 21.00 \\ 48.60 \end{bmatrix} = 0.60 \times 21.00 + 0.40 \times 48.60 = 32.04$$

Therefore, the portfolio volatility is $\sigma_p = \sqrt{32.04} = 5.66$. Note that this is hardly higher than the volatility of the Canadian dollar alone, even though the risk of the euro is much higher. The portfolio risk has been kept low due to a diversification effect, or low correlation between the two assets.

2.3.4 Product of Random Variables

Some risks result from the product of two random variables. A credit loss, for instance, arises from the product of the occurrence of default and the loss given default.

Using Equation (2.20), the expectation of the product $Y = X_1 X_2$ can be written as

$$E(X_1 \; X_2) = E(X_1)E(X_2) + \text{Cov}(X_1, X_2) \tag{2.32}$$

When the variables are independent, this reduces to the product of the means.

The variance is more complex to evaluate. With independence, it reduces to:

$$V(X_1 \; X_2) = E(X_1)^2 V(X_2) + V(X_1)E(X_2)^2 + V(X_1)V(X_2) \tag{2.33}$$

2.3.5 Distributions of Transformations of Random Variables

The preceding results focus on the mean and variance of simple transformations only. They do not fully describe the distribution of the transformed variable $Y = g(X)$. This, unfortunately, is usually complicated for all but the simplest transformations $g(\cdot)$ and densities $f(X)$.

Even if there is no closed-form solution for the density, we can describe the cumulative distribution function of Y when $g(X)$ is a one-to-one transformation from X into Y. This implies that the function can be inverted, or that for a given y, we can find x such that $x = g^{-1}(y)$. We can then write

$$P[Y \leq y] = P[g(X) \leq y] = P[X \leq g^{-1}(y)] = F_X(g^{-1}(y)) \tag{2.34}$$

where $F(\cdot)$ is the cumulative distribution function of X. Here, we assumed the relationship is positive. Otherwise, the right-hand term is changed to $1 - F_X(g^{-1}(y))$.

This allows us to derive the quantile of, say, the bond price from information about the probability distribution of the yield. Suppose we consider a zero-coupon bond, for which the market value V is

$$V = \frac{100}{(1+r)^T} \tag{2.35}$$

where r is the yield. This equation describes V as a function of r, or $Y = g(X)$. Using $r = 6\%$ and $T = 30$ years, the current price is $V = \$17.41$. The inverse function $X = g^{-1}(Y)$ is

$$r = (100/V)^{1/T} - 1 \tag{2.36}$$

We wish to estimate the probability that the bond price could fall below a cut-off price $V = \$15$. We invert the price-yield function and compute the associated yield level, $g^{-1}(y) = (100/\$15)^{1/30} - 1 = 6.528\%$. Lower prices are associated with higher yield levels. Using Equation (2.34), the probability is given by

$$P[V \leq \$15] = P[r \geq 6.528\%]$$

Assuming the yield change is normal with volatility 0.8%, this gives a probability of 25.5%.[1] Even though we do not know the density of the bond price, this method allows us to trace out its cumulative distribution by changing the cutoff price of \$15. Taking the derivative, we can recover the density function of the bond price. Figure 2.5 shows that this p.d.f. is skewed to the right.

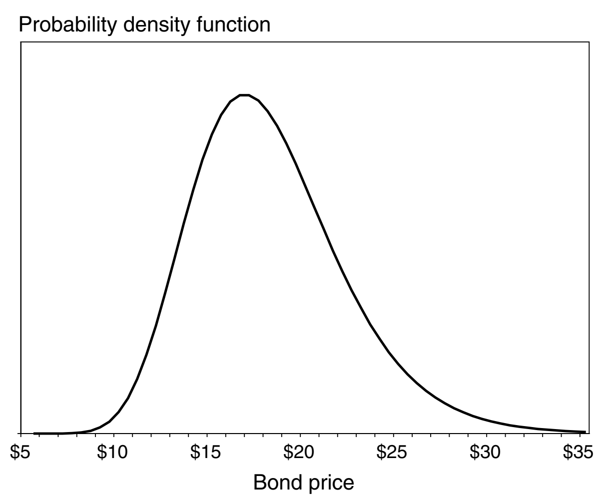

FIGURE 2.5 Density Function for the Bond Price

[1] We will see later that this is obtained from the standard normal variable $z = (6.528 - 6.000)/0.80 = 0.660$. Using standard normal tables, or the NORMSDIST(-0.660) Excel function, this gives 25.5%.

On the extreme right, if the yield falls to zero, the bond price will go to \$100. On the extreme left, if the yield goes to infinity, the bond price will fall to, but not go below, zero. Relative to the current value of \$17.41, there is a greater likelihood of large movements up than down.

This method, unfortunately, cannot be easily extended. For general density functions and transformations, risk managers turn to numerical methods, especially when the number of random variables is large. This is why credit risk models, for instance, all describe the distribution of credit losses through simulations.

EXAMPLE 2.4: FRM EXAM 2007—QUESTION 127

Suppose that A and B are random variables, each follows a standard normal distribution, and the covariance between A and B is 0.35. What is the variance of $(3A + 2B)$?

a. 14.47
b. 17.20
c. 9.20
d. 15.10

EXAMPLE 2.5: FRM EXAM 2002—QUESTION 70

Given that x and y are random variables and a, b, c, and d are constants, which one of the following definitions is *wrong*?

a. $E(ax + by + c) = aE(x) + bE(y) + c$, if x and y are correlated.
b. $V(ax + by + c) = V(ax + by) + c$, if x and y are correlated.
c. $\text{Cov}(ax + by, cx + dy) = acV(x) + bdV(y) + (ad + bc)\text{Cov}(x, y)$, if x and y are correlated.
d. $V(x - y) = V(x + y) = V(x) + V(y)$, if x and y are uncorrelated.

2.4 IMPORTANT DISTRIBUTION FUNCTIONS

2.4.1 Uniform Distribution

The simplest continuous distribution function is the **uniform distribution**. This is defined over a range of values for x, $a \leq x \leq b$. The density function is

$$f(x) = \frac{1}{(b - a)}, \quad a \leq x \leq b \tag{2.37}$$

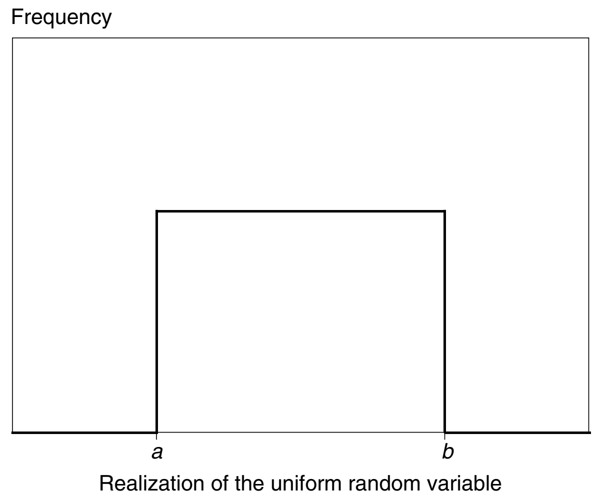

FIGURE 2.6 Uniform Density Function

which is constant and indeed integrates to unity. This distribution puts the same weight on each observation within the allowable range, as shown in Figure 2.6. We denote this distribution as $U(a, b)$.

Its mean and variance are given by

$$E(X) = \frac{a+b}{2} \tag{2.38}$$

$$V(X) = \frac{(b-a)^2}{12} \tag{2.39}$$

The uniform distribution $U(0, 1)$ is widely used as a starting distribution for generating random variables from any distribution $F(Y)$ in simulations. We need to have analytical formulas for the p.d.f. $f(Y)$ and its cumulative distribution $F(Y)$. As any cumulative distribution function ranges from zero to unity, we first draw X from $U(0, 1)$ and then compute $y = F^{-1}(x)$. The random variable Y will then have the desired distribution $f(Y)$.

EXAMPLE 2.6: FRM EXAM 2002—QUESTION 119

The random variable X with density function $f(x) = 1/(b-a)$ for $a < x < b$, and 0 otherwise, is said to have a uniform distribution over (a, b). Calculate its mean.

a. $(a+b)/2$
b. $a - b/2$
c. $a + b/4$
d. $a - b/4$

2.4.2 Normal Distribution

Perhaps the most important continuous distribution is the **normal distribution**, which represents adequately many random processes. This has a bell-like shape with more weight in the center and tails tapering off to zero. The daily rate of return in a stock price, for instance, has a distribution similar to the normal p.d.f.

The normal distribution can be characterized by its first two moments only, the mean μ and variance σ^2. The first parameter represents the location; the second, the dispersion. The normal density function has the following expression:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2] \tag{2.40}$$

Its mean is $E[X] = \mu$ and variance $V[X] = \sigma^2$. We denote this distribution as $N(\mu, \sigma^2)$. Because the function can be fully specified by these two parameters, it is called a **parametric function**.

Instead of having to deal with different parameters, it is often more convenient to use a **standard normal variable** as ϵ, which has been standardized, or normalized, so that $E(\epsilon) = 0$, $V(\epsilon) = \sigma(\epsilon) = 1$. Figure 2.7 plots the **standard normal density**. Appendix B describes the standard normal distribution.

First, note that the function is symmetrical around the mean. Its mean of zero is the same as its **mode** (which is also the most likely, or highest, point on this curve) and **median** (which is such that the area to the left is a 50% probability). The skewness of a normal distribution is 0, which indicates that it is symmetrical around the mean. The kurtosis of a normal distribution is 3. Distributions with fatter tails have a greater kurtosis coefficient.

About 95 percent of the distribution is contained between values of $\epsilon_1 = -2$ and $\epsilon_2 = +2$, and 68 percent of the distribution falls between values of $\epsilon_1 = -1$ and $\epsilon_2 = +1$. Table 2.4 gives the values that correspond to right-tail probabilities, such that

$$\int_{-\alpha}^{\infty} f(\epsilon)d\epsilon = c \tag{2.41}$$

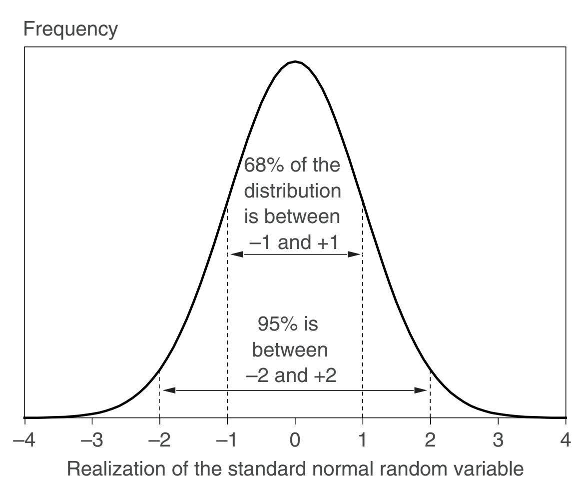

FIGURE 2.7 Normal Density Function

TABLE 2.4 Lower Quantiles of the Standardized Normal Distribution

	Confidence Level (%)								
c	99.99	99.9	99	97.72	97.5	95	90	84.13	50
$(-\alpha)$	−3.715	−3.090	−2.326	−2.000	−1.960	−1.645	−1.282	−1.000	−0.000

For instance, $-\alpha = -1.645$ is the quantile that corresponds to a 95% probability (see Table 2.4).[2]

This distribution plays a central role in finance because it represents adequately the behavior of many financial variables. It enters, for instance, the Black-Scholes option pricing formula where the function $N(\cdot)$ represents the cumulative standardized normal distribution function.

The distribution of any normal variable can then be recovered from that of the standard normal, by defining

$$X = \mu + \epsilon\sigma \tag{2.42}$$

Using Equations (2.22) and (2.23), we can show that X has indeed the desired moments, as $E(X) = \mu + E(\epsilon)\sigma = \mu$ and $V(X) = V(\epsilon)\sigma^2 = \sigma^2$.

Define, for instance, the random variable as the change in the dollar value of a portfolio. The expected value is $E(X) = \mu$. To find the quantile of X at the specified confidence level c, we replace ϵ by $-\alpha$ in Equation (2.42). This gives $Q(X, c) = \mu - \alpha\sigma$. Using Equation (2.9), we can compute VAR as

$$\text{VAR} = E(X) - Q(X, c) = \mu - (\mu - \alpha\sigma) = \alpha\sigma \tag{2.43}$$

For example, a portfolio with a standard deviation of \$10 million would have a VAR, or potential downside loss, of \$16.45 million at the 95% confidence level.

KEY CONCEPT

With normal distributions, the VAR of a portfolio is obtained from the product of the portfolio standard deviation and a standard normal deviate factor that reflects the confidence level, for instance 1.645 at the 95% level.

An important property of the normal distribution is that it is one of the few distributions that is *stable* under addition. In other words, a linear combination of jointly normally distributed random variables has a normal distribution.[3] This

[2] More generally, the cumulative distribution can be found from the Excel function NORMDIST($\cdot$). For example, we can verify that NORMSDIST(−1.645) yields 0.04999, or a 5% left-tail probability.

[3] Strictly speaking, this is true only under either of the following conditions: (1) the univariate variables are independently distributed, or (2) the variables are multivariate normally distributed (this invariance property also holds for jointly elliptically distributed variables).

is extremely useful because we need to know only the mean and variance of the portfolio to reconstruct its whole distribution.

KEY CONCEPT

A linear combination of jointly normal variables has a normal distribution.

When we have N random variables, the joint normal density can be written as a function of the vector x, of the means μ, and the covariance matrix Σ:

$$f(x_1, \ldots, x_N) = \frac{1}{(2\pi)^{N/2} \mid \Sigma \mid^{1/2}} \exp[-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)] \tag{2.44}$$

EXAMPLE 2.7: FRM EXAM 2009—QUESTION 2-18

Assume that a random variable follows a normal distribution with a mean of 80 and a standard deviation of 24. What percentage of this distribution is *not* between 32 and 116?

a. 4.56%
b. 8.96%
c. 13.36%
d. 18.15%

EXAMPLE 2.8: FRM EXAM 2003—QUESTION 21

Which of the following statements about the normal distribution is *not* accurate?

a. Kurtosis equals 3.
b. Skewness equals 1.
c. The entire distribution can be characterized by two moments, mean and variance.
d. The normal density function has the following expression:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2]$$

EXAMPLE 2.9: FRM EXAM 2006—QUESTION 11

Which type of distribution produces the lowest probability for a variable to exceed a specified extreme value that is greater than the mean, assuming the distributions all have the same mean and variance?

a. A leptokurtic distribution with a kurtosis of 4
b. A leptokurtic distribution with a kurtosis of 8
c. A normal distribution
d. A platykurtic distribution

2.4.3 Lognormal Distribution

The normal distribution is a good approximation for many financial variables, such as the rate of return on a stock, $r = (P_1 - P_0)/P_0$, where P_0 and P_1 are the stock prices at time 0 and 1.

Strictly speaking, this is inconsistent with reality since a normal variable has infinite tails on both sides. In theory, r could end up below -1, which implies $P_1 < 0$. In reality, due to the limited liability of corporations, stock prices cannot turn negative. In many situations, however, this is an excellent approximation. For instance, with short horizons or small price moves, the probability of having a negative price is so small that it is negligible. If this is not the case, we need to resort to other distributions that prevent prices from going negative. One such distribution is the lognormal.

A random variable X is said to have a **lognormal distribution** if its logarithm $Y = \ln(X)$ is normally distributed. Define here $X = (P_1/P_0)$. Because the argument X in the logarithm function must be positive, the price P_1 can never go below zero.

The lognormal density function has the following expression:

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\ln(x) - \mu)^2\right], \quad x > 0 \tag{2.45}$$

Note that this is more complex than simply plugging $\ln(x)$ into Equation (2.40), because x also appears in the denominator. Its mean is

$$E[X] = \exp\left[\mu + \frac{1}{2}\sigma^2\right] \tag{2.46}$$

and variance $V[X] = \exp[2\mu + 2\sigma^2] - \exp[2\mu + \sigma^2]$. The parameters were chosen to correspond to those of the normal variable, $E[Y] = E[\ln(X)] = \mu$ and $V[Y] = V[\ln(X)] = \sigma^2$.

Conversely, if we set $E[X] = \exp[r]$, the mean of the associated normal variable is $E[Y] = E[\ln(X)] = (r - \sigma^2/2)$. We will see later that this adjustment is also

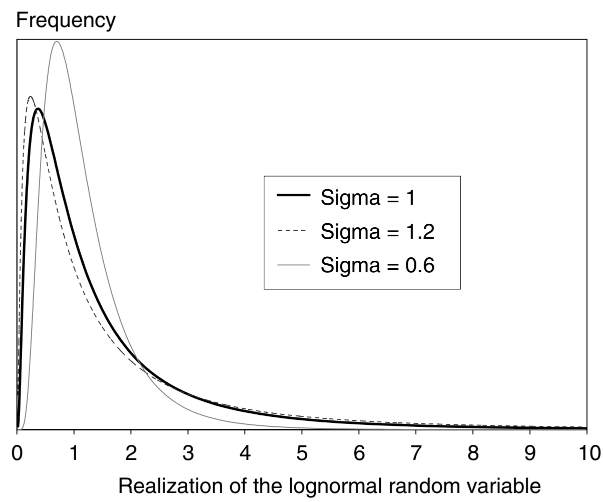

FIGURE 2.8 Lognormal Density Function

used in the Black-Scholes option valuation model, where the formula involves a trend in $(r - \sigma^2/2)$ for the log-price ratio.

Figure 2.8 depicts the lognormal density function with $\mu = 0$, and various values $\sigma = 1.0, 1.2, 0.6$. Note that the distribution is skewed to the right. The tail increases for greater values of σ. This explains why as the variance increases, the mean is pulled up in Equation (2.46).

We also note that the distribution of the bond price in our previous example, Equation (2.35), resembles a lognormal distribution. Using continuous compounding instead of annual compounding, the price function is

$$V = 100 \exp(-rT) \tag{2.47}$$

which implies $\ln(V/100) = -rT$. Thus if r is normally distributed, V has a lognormal distribution.

EXAMPLE 2.10: FRM EXAM 1999—QUESTION 5

Which of the following statements best characterizes the relationship between the normal and lognormal distributions?

a. The lognormal distribution is the logarithm of the normal distribution.
b. If the natural log of the random variable X is lognormally distributed, then X is normally distributed.
c. If X is lognormally distributed, then the natural log of X is normally distributed.
d. The two distributions have nothing to do with one another.

EXAMPLE 2.11: FRM EXAM 2007—QUESTION 21

The skew of a lognormal distribution is always

a. Positive
b. Negative
c. 0
d. 3

EXAMPLE 2.12: FRM EXAM 2002—QUESTION 125

Consider a stock with an initial price of \$100. Its price one year from now is given by $S = 100 \times \exp(r)$, where the rate of return r is normally distributed with a mean of 0.1 and a standard deviation of 0.2. With 95% confidence, after rounding, S will be between

a. \$67.57 and \$147.99
b. \$70.80 and \$149.20
c. \$74.68 and \$163.56
d. \$102.18 and \$119.53

EXAMPLE 2.13: FRM EXAM 2000—QUESTION 128

For a lognormal variable X, we know that $\ln(X)$ has a normal distribution with a mean of zero and a standard deviation of 0.5. What are the expected value and the variance of X?

a. 1.025 and 0.187
b. 1.126 and 0.217
c. 1.133 and 0.365
d. 1.203 and 0.399

2.4.4 Student's *t* Distribution

Another important distribution is the **Student's *t* distribution**. This arises in hypothesis testing, because it describes the distribution of the ratio of the estimated coefficient to its standard error.

This distribution is characterized by a parameter k known as the **degrees of freedom.** Its density is

$$f(x) = \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} \frac{1}{\sqrt{k\pi}} \frac{1}{(1 + x^2/k)^{(k+1)/2}} \tag{2.48}$$

where Γ is the gamma function, defined as $\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx$. As k increases, this function converges to the normal p.d.f.

The distribution is symmetrical with mean zero and variance

$$V[X] = \frac{k}{k-2} \tag{2.49}$$

provided $k > 2$. Its kurtosis is

$$\delta = 3 + \frac{6}{k-4} \tag{2.50}$$

provided $k > 4$. It has fatter tails than the normal distribution, which often provides a better representation of typical financial variables. Typical estimated values of k are around four to six for stock returns. Figure 2.9 displays the density for $k = 4$ and $k = 50$. The latter is close to the normal. With $k = 4$, however, the p.d.f. has fatter tails. As was done for the normal density, we can also use the Student's t to compute VAR as a function of the volatility

$$\text{VAR} = \alpha_k \sigma \tag{2.51}$$

where the multiplier now depends on the degrees of freedom k.

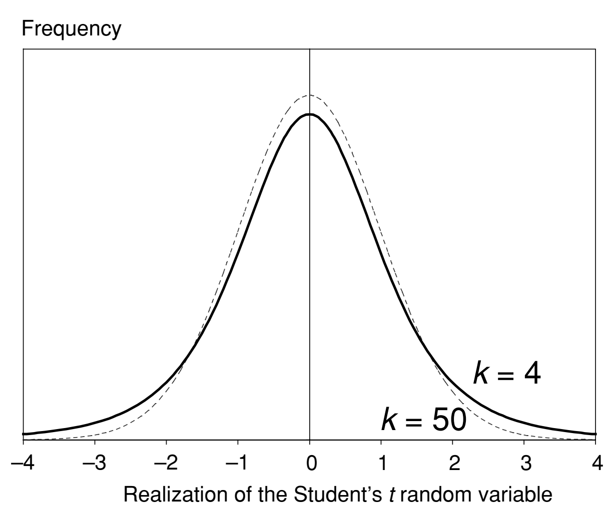

FIGURE 2.9 Student's t Density Function

Another distribution derived from the normal is the **chi-square distribution**, which can be viewed as the sum of independent squared standard normal variables

$$x = \sum_{j=1}^{k} z_j^2 \tag{2.52}$$

where k is also called the degrees of freedom. Its mean is $E[X] = k$ and variance $V[X] = 2k$. For values of k sufficiently large, $\chi^2(k)$ converges to a normal distribution $N(k, 2k)$. This distribution describes the sample variance.

Finally, another associated distribution is the ***F* distribution**, which can be viewed as the ratio of independent chi-square variables divided by their degrees of freedom

$$F(a, b) = \frac{\chi^2(a)/a}{\chi^2(b)/b} \tag{2.53}$$

This distribution appears in joint tests of regression coefficients.

EXAMPLE 2.14: FRM EXAM 2003—QUESTION 18

Which of the following statements is the most accurate about the relationship between a normal distribution and a Student's t distribution that have the same mean and standard deviation?

a. They have the same skewness and the same kurtosis.
b. The Student's t distribution has larger skewness and larger kurtosis.
c. The kurtosis of a Student's t distribution converges to that of the normal distribution as the number of degrees of freedom increases.
d. The normal distribution is a good approximation for the Student's t distribution when the number of degrees of freedom is small.

2.4.5 Binomial Distribution

Consider now a random variable that can take discrete values between zero and n. This could be, for instance, the number of times VAR is exceeded over the last year, also called the number of **exceptions**. Thus, the binomial distribution plays an important role for the backtesting of VAR models.

A binomial variable can be viewed as the result of n independent **Bernoulli trials**, where each trial results in an outcome of $y = 0$ or $y = 1$. This applies, for example, to credit risk. In case of default, we have $y = 1$, otherwise $y = 0$. Each Bernoulli variable has expected value of $E[Y] = p$ and variance $V[Y] = p(1 - p)$.

A random variable is defined to have a **binomial distribution** if the discrete density function is given by

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n \tag{2.54}$$

where $\binom{n}{x}$ is the number of combinations of n things taken x at a time, or

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \tag{2.55}$$

and the parameter p is between zero and one. This distribution also represents the total number of successes in n repeated experiments where each success has a probability of p.

The binomial variable has mean and variance

$$E[X] = pn \tag{2.56}$$

$$V[X] = p(1-p)n \tag{2.57}$$

It is described in Figure 2.10 in the case where $p = 0.25$ and $n = 10$. The probability of observing $X = 0, 1, 2 \ldots$ is 5.6%, 18.8%, 28.1%, and so on.

For instance, we want to know what is the probability of observing $x = 0$ exceptions out of a sample of $n = 250$ observations when the true probability is 1%. We should expect to observe 2.5 exceptions on average across many such

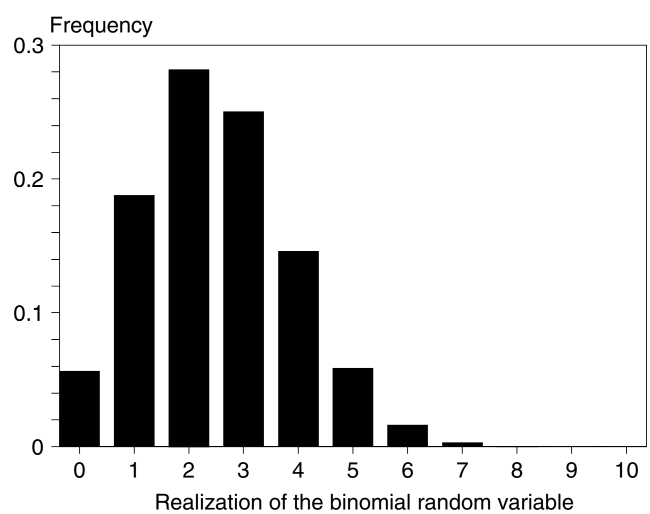

FIGURE 2.10 Binomial Density Function with $p = 0.25$, $n = 10$

samples. There will be, however, some samples with no exceptions at all. This probability is

$$f(X=0) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} = \frac{250!}{1 \times 250!} 0.01^0 0.99^{250} = 0.081$$

So, we would expect to observe 8.1% of samples with zero exceptions, under the null hypothesis. We can repeat this calculation with different values for x. For example, the probability of observing eight exceptions is $f(X=8) = 0.20\%$ only. We can use this information to test the null hypothesis. Because this probability is so low, observing eight exceptions would make us question whether the true probability is 1%.

A related distribution is the **negative binomial distribution**. Instead of a fixed number of trials, this involves repeating the experiment until a fixed number of failures r has occurred. The density is

$$f(x) = \binom{x+r-1}{r-1} p^x (1-p)^r, \quad x = 0, 1, \ldots \tag{2.58}$$

In other words, this is the distribution of the number of successes before the rth failure of a Bernoulli process. Its mean is $E[X] = r\frac{p}{(1-p)}$. Also note that x does not have an upper limit, unlike the variable in the usual binomial density.

EXAMPLE 2.15: FRM EXAM 2006—QUESTION 84

On a multiple-choice exam with four choices for each of six questions, what is the probability that a student gets fewer than two questions correct simply by guessing?

a. 0.46%
b. 23.73%
c. 35.60%
d. 53.39%

2.4.6 Poisson Distribution

The Poisson distribution is a discrete distribution, which typically is used to describe the number of events occurring over a fixed period of time, assuming events are independent of each other. It is defined as

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \ldots \tag{2.59}$$

where λ is a positive number representing the average arrival rate during the period. This distribution, for example, is widely used to represent the frequency, or number of occurrences, of operational losses over a year.

The parameter λ represents the expected value of X and also its variance

$$E[X] = \lambda \tag{2.60}$$

$$V[X] = \lambda \tag{2.61}$$

The Poisson distribution is the limiting case of the binomial distribution, as n goes to infinity and p goes to zero, while $np = \lambda$ remains fixed. In addition, when λ is large the Poisson distribution is well approximated by the normal distribution with mean and variance of λ, through the central limit theorem.

If the number of arrivals follows a Poisson distribution, then the time period between arrivals follows an **exponential distribution** with mean $1/\lambda$. The latter has density taking the form $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$. For example, if we expect $\lambda = 12$ losses per year, the average time interval between losses should be one year divided by 12, or one month.

EXAMPLE 2.16: FRM EXAM 2004—QUESTION 60

When can you use the normal distribution to approximate the Poisson distribution, assuming you have n independent trials, each with a probability of success of p?

a. When the mean of the Poisson distribution is very small
b. When the variance of the Poisson distribution is very small
c. When the number of observations is very large and the success rate is close to 1
d. When the number of observations is very large and the success rate is close to 0

2.5 DISTRIBUTION OF AVERAGES

The normal distribution is extremely important because of the **central limit theorem** (CLT), which states that the mean of n independent and identically distributed (i.i.d.) variables converges to a normal distribution as the number of observations n increases. This very powerful result is valid for any underlying distribution, as long as the realizations are independent. For instance, the distribution of total credit losses converges to a normal distribution as the number of loans increases to a large value, assuming defaults are always independent of each other.

Define $\bar{X}$ as the mean $\frac{1}{n}\sum_{i=1}^{n} X_i$, where each variable has mean μ and standard deviation σ. We have

$$\bar{X} \rightarrow N\left(\mu, \frac{\sigma^2}{n}\right) \tag{2.62}$$

Standardizing the variable, we can write

$$\frac{\bar{X} - \mu}{(\sigma/\sqrt{n})} \rightarrow N(0, 1) \tag{2.63}$$

Thus, the normal distribution is the limiting distribution of the average, which explains why it has such a prominent place in statistics.

As an example, consider the binomial variable, which is the sum of independent Bernoulli trials. When n is large, we can use the CLT and approximate the binomial distribution by the normal distribution. Using Equation (2.63) for the sum, we have

$$z = \frac{x - pn}{\sqrt{p(1-p)n}} \rightarrow N(0, 1) \tag{2.64}$$

which is much easier to evaluate than the binomial distribution.

Consider, for example, the issue of whether the number of exceptions x we observe is compatible with a 99% VAR. For our example, the mean and variance of x are $E[X] = 0.01 \times 250 = 2.5$ and $V[X] = 0.01(1 - 0.01) \times 250 = 2.475$. We observe $x = 8$, which gives $z = (8 - 2.5)/\sqrt{2.475} = 3.50$. We can now compare this number to the standard normal distribution. Say, for instance, that we decide to reject the hypothesis that VAR is correct if the statistic falls outside a 95% two-tailed confidence band.[4] This interval is $(-1.96, +1.96)$ for the standardized normal distribution. Here, the value of 3.50 is much higher than the cutoff point of $+1.96$. As a result, we would reject the null hypothesis that the true probability of observing an exception is 1% only. In other words, there are simply too many exceptions to be explained by bad luck. It is more likely that the VAR model underestimates risk.

2.6 IMPORTANT FORMULAS

Probability density function: $f(x) = \text{Prob}(X = x)$

(Cumulative) distribution function: $F(x) = \int_{-\infty}^{x} f(u)du$

Mean: $E(X) = \mu = \int x f(x)dx$

Variance: $V(X) = \sigma^2 = \int [x - \mu]^2 f(x)dx$

[4] Note that the choice of this confidence level has nothing to do with the VAR confidence level. Here, the 95% level represents the rate at which the decision rule will commit the error of falsely rejecting a correct model.

Skewness: $\gamma = \left(\int [x-\mu]^3 f(x)dx\right)/\sigma^3$
Kurtosis: $\delta = \left(\int [x-\mu]^4 f(x)dx\right)/\sigma^4$
Quantile, VAR: $\text{VAR} = E(X) - Q(X,c) = \alpha\sigma$
Independent joint densities: $f_{12}(x_1 x_2) = f_1(x_1) \times f_2(x_2)$
Marginal densities: $f_1(x_1) = \int f_{12}(x_1, u_2)du_2$,
Conditional densities: $f_{1\cdot 2}(x_1 \mid x_2) = \frac{f_{12}(x_1,x_2)}{f_2(x_2)}$
Covariance: $\sigma_{12} = \int_1 \int_2 [x_1-\mu_1][x_2-\mu_2] f_{12} dx_1 dx_2$
Correlation: $\rho_{12} = \sigma_{12}/(\sigma_1\sigma_2)$
Linear transformation of random variables: $E(a+bX) = a + bE(X)$,
$V(a+bX) = b^2 V(X)$, $\sigma(a+bX) = b\sigma(X)$
Sum of random variables: $E(X_1 + X_2) = \mu_1 + \mu_2$,
$V(X_1 + X_2) = \sigma_1^2 + \sigma_2^2 + 2\sigma_{12}$
Portfolios of random variables: $Y = w'X$, $E(Y) = \mu_p = w'\mu$, $\sigma_p^2 = w'\Sigma w$
Product of random variables: $E(X_1\ X_2) = \mu_1\mu_2 + \sigma_{12}$,
$V(X_1\ X_2) = \mu_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \sigma_1^2\sigma_2^2$
Uniform distribution: $E(X) = \frac{a+b}{2}$, $V(X) = \frac{(b-a)^2}{12}$
Normal distribution: $E(X) = \mu$, $V(X) = \sigma^2$, $\gamma = 0$, $\delta = 3$
Lognormal distribution: for X if $Y = \ln(X)$ is normal, $E[X] = \exp\left[\mu + \frac{1}{2}\sigma^2\right]$,
$V[X] = \exp[2\mu + 2\sigma^2] - \exp[2\mu + \sigma^2]$
Student's t distribution: $V[X] = \frac{k}{k-2}$, $\gamma = 0$, $\delta = 3 + \frac{6}{k-4}$
Binomial distribution: $E[X] = pn$, $V[X] = p(1-p)n$
Poisson distribution: $E[X] = \lambda$, $V[X] = \lambda$
Distribution of averages, central limit theorem (CLT): $\bar{X} \rightarrow N\left(\mu, \frac{\sigma^2}{n}\right)$

2.7 ANSWERS TO CHAPTER EXAMPLES

Example 2.1: FRM Exam 2009—Question 2-3

b. Portfolio A has a longer left tail, due to negative skewness. In addition, it has less kurtosis (1.9) than for a normal distribution, which implies that it is more peaked.

Example 2.2: FRM Exam 2000—Question 81

b. Correlation is a measure of linear association. Independence implies zero correlation, but the reverse is not always true.

Example 2.3: FRM Exam 2007—Question 93

b. The function $x \times y$ is described in the following table. The sum of the entries is 36. The scaling factor k must be such that the total probability is 1. Therefore, we have $k = 1/36$. The table shows one instance where $X + Y > 5$, which is $x = 3$, $y = 3$. The probability is $p = 9/36 = 1/4$.

$x \times y$	$x = 1$	2	3
$y = 1$	1	2	3
2	2	4	6
3	3	6	9

Example 2.4: FRM Exam 2007—Question 127

b. The variance is $V(3A + 2B) = 3^2 V(A) + 2^2 V(B) + 2 \times 3\ 2\ \text{Cov}(A, B) = 9\ 1 + 4\ 1 + 12 \times 0.35 = 17.2$.

Example 2.5: FRM Exam 2002—Question 70

b. Statement a. is correct, as it is a linear operation. Statement c. is correct, as in Equation (2.30). Statement d. is correct, as the covariance term is zero if the variables are uncorrelated. Statement b. is false, as adding a constant c to a variable cannot change the variance. The constant drops out because it is also in the expectation.

Example 2.6: FRM Exam 2002—Question 119

a. The mean is the center of the distribution, which is the average of a and b.

Example 2.7: FRM Exam 2009—Question 2-18

b. First convert the cutoff points of 32 and 116 into standard normal deviates. The first is $z_1 = (32 - 80)/24 = -48/24 = -2$, and the second is $z_2 = (116 - 80)/24 = 36/24 = 1.5$. From normal tables, $P(Z > +1.5) = N(-1.5) = 0.0668$ and $P(Z < -2.0) = N(-2.0) = 0.0228$. Summing gives 8.96%.

Example 2.8: FRM Exam 2003—Question 21

b. Skewness is 0, kurtosis 3, the entire distribution is described by μ and σ, and the p.d.f. is correct.

Example 2.9: FRM Exam 2006—Question 11

d. A platykurtic distribution has kurtosis less than 3, less than the normal p.d.f. Because all other answers have higher kurtosis, this produces the lowest extreme values.

Example 2.10: FRM Exam 1999—Question 5

c. X is said to be lognormally distributed if its logarithm $Y = \ln(X)$ is normally distributed.

Example 2.11: FRM Exam 2007—Question 21

a. A lognormal distribution is skewed to the right. Intuitively, if this represents the distribution of prices, prices can fall at most by 100% but can increase by more than that.

Example 2.12: FRM Exam 2002—Question 125

c. Note that this is a two-tailed confidence band, so that $\alpha = 1.96$. We find the extreme values from $\$100\exp(\mu \pm \alpha\sigma)$. The lower limit is then $V_1 = \$100\exp(0.10 - 1.96 \times 0.2) = \$100\exp(-0.292) = \$74.68$. The upper limit is $V_2 = \$100\exp(0.10 + 1.96 \times 0.2) = \$100\exp(0.492) = \$163.56$.

Example 2.13: FRM Exam 2000—Question 128

c. Using Equation (2.46), we have $E[X] = \exp[\mu + 0.5\sigma^2] = \exp[0 + 0.5 * 0.5^2] = 1.1331$. Assuming there is no error in the answers listed for the variance, it is sufficient to find the correct answer for the expected value.

Example 2.14: FRM Exam 2003—Question 18

c. The two distributions have the same skewness of zero but the Student's t has higher kurtosis. As the number of degrees of freedom increases, the Student's t distribution converges to the normal distribution, so c. is the correct answer.

Example 2.15: FRM Exam 2006—Question 84

d. We use the density given by Equation (2.54). The number of trials is $n = 6$. The probability of guessing correctly just by chance is $p = 1/4 = 0.25$. The probability of zero lucky guesses is $\binom{6}{0}0.25^0 0.75^6 = 0.75^6 = 0.17798$. The probability of one lucky guess is $\binom{6}{1}0.25^1 0.75^5 = 6\ 0.25\ 0.75^5 = 0.35596$. The sum is 0.5339.

Note that the same analysis can be applied to the distribution of scores on an FRM examination with 100 questions. It would be virtually impossible to have a score of zero, assuming random guesses; this probability is $0.75^{100} = 3.2E - 13$. Also, the expected percentage score under random guesses is $p = 25\%$.

Example 2.16: FRM Exam 2004—Question 60

c. The normal approximation to the Poisson improves when the success rate, λ, is very high. Because this is also the mean and variance, answers a. and b. are wrong. In turn, the binomial density is well approximated by the Poisson density when $np = \lambda$ is large.

APPENDIX A: REVIEW OF MATRIX MULTIPLICATION

This appendix briefly reviews the mathematics of matrix multiplication. Say that we have two matrices, A and B, that we wish to multiply to obtain the new matrix $C = AB$. The respective dimensions are $(n \times m)$ for A, that is, n rows and m columns, and $(m \times p)$ for B. The number of columns for A must exactly match (or conform to) the number of rows for B. If so, this will result in a matrix C of dimensions $(n \times p)$.

We can write the matrix A in terms of its individual components a_{ij}, where i denotes the row and j denotes the column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

As an illustration, take a simple example where the matrices are of dimension (2×3) and (3×2).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$C = AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

To multiply the matrices, each row of A is multiplied element-by-element by each column of B. For instance, c_{12} is obtained by taking

$$c_{12} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

The matrix C is then:

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Matrix multiplication can be easily implemented in Excel using the function MMULT. First, we highlight the cells representing the output matrix C, say f1:g2. Then we enter the function, for instance MMULT(a1:c2; d1:e3), where the first range represents the first matrix, A, here 2 by 3, and the second range represents the second matrix, B, here 3 by 2. The final step is to hit the three keys Control-Shift-Return simultaneously.

APPENDIX B: NORMAL DISTRIBUTION

Cumulative Distribution, from Minus Infinity to z, $P(-\infty < Z < z)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Far Right Tail Probability, Above z

z	$P(Z \geq z)$	z	$P(Z \geq z)$	z	$P(Z \geq z)$	z	$P(Z \geq z)$
2.0	0.022750	3.0	0.0013499	4.0	3.167E-05	5.0	2.867E-07
2.1	0.017864	3.1	0.0009676	4.1	2.066E-05	5.5	1.899E-08
2.2	0.013903	3.2	0.0006871	4.2	1.335E-05	6.0	9.866E-10
2.3	0.010724	3.3	0.0004834	4.3	8.540E-06	6.5	4.016E-11
2.4	0.008198	3.4	0.0003369	4.4	5.413E-06	7.0	1.280E-12
2.5	0.006210	3.5	0.0002326	4.5	3.398E-06	7.5	3.191E-14
2.6	0.004661	3.6	0.0001591	4.6	2.112E-06	8.0	6.221E-16
2.7	0.003467	3.7	0.0001078	4.7	1.301E-06	8.5	9.480E-18
2.8	0.002555	3.8	0.0000723	4.8	7.933E-07	9.0	1.129E-19
2.9	0.001866	3.9	0.0000481	4.9	4.792E-07	9.5	1.049E-21

CHAPTER 3

Fundamentals of Statistics

The preceding chapter was mainly concerned with the theory of probability, including distribution theory. In practice, researchers have to find methods to choose among distributions and to estimate distribution parameters from real data. The subject of sampling brings us now to the theory of statistics. Whereas probability assumes the distributions are known, statistics attempts to make inferences from actual data.

Here, we sample from the distribution of a population, say the return on a stock market index, to make inferences about the population. Issues of interest are the choices of the best distribution and of the best parameters.

In addition, risk measurement deals with large numbers of random variables. As a result, we also need to characterize the relationships between risk factors. For instance, what is the correlation between U.S. and UK stock indices? This leads to the need to develop decision rules to test hypotheses, for instance whether the volatility for a risk factor remains stable over time, or whether the relationship between these stock indices is significant.

These examples illustrate two important problems in statistical inference: **estimation** and **tests of hypotheses**. With estimation, we wish to estimate the value of an unknown parameter from sample data. With tests of hypotheses, we wish to verify a conjecture about the data.

This chapter reviews the fundamental tools of statistics theory for risk managers. Parameter estimation and hypothesis testing are presented in Section 3.1. Section 3.2 then turns to regression analysis, summarizing important results as well as common pitfalls in their interpretation.

3.1 PARAMETER ESTIMATION

3.1.1 Parameters

The first step in risk measurement is to define the risk factors. These can be movements in stock prices, interest rates, exchange rates, or commodity prices.

The next step is to measure their distribution. This usually involves choosing a particular distribution function and then estimating parameters. For instance,

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define X as the random variable of interest. We observe a sequence of T realized values for x, $x_1, x_2, \ldots, x_T$.

As an example, we could assume that the observed values for x are drawn from a normal distribution

$$x \sim \Phi(\mu, \sigma) \tag{3.1}$$

with mean μ and standard deviation σ. Generally, we also need to assume that the random variables are independent and identically distributed (i.i.d.). As we will see later, estimation is still possible if this is not the case but requires additional steps, fitting a model to r until the residuals are i.i.d.

Even in simple cases, the i.i.d. assumption requires a basic transformation of the data. For example, r should be the rate of change in the stock index, not its level P. We know that the level tomorrow cannot be far from the level today. What is random is whether the level will go up or down. So, the random variable should be the rate of change in the level.

Armed with our i.i.d. sample of T observations, we can start estimating the parameters of interest, such as the sample mean, the variance, and other moments. Say that the random variable X has a normal distribution with **parameters** μ and σ^2. These are unknown values that must be estimated. This approach can also be used to check whether the parametric distribution is appropriate. For instance, the normal distribution implies a value of three for the kurtosis coefficient. We can estimate an estimate for the kurtosis for the sample at hand and test whether it is equal to three. If not, the assumption that the distribution is normal must be rejected and the risk manager must search for another distribution that fits the data better.

3.1.2 Parameter Estimators

The expected return, or mean, $\mu = E(X)$ can be estimated by the sample mean,

$$m = \widehat{\mu} = \frac{1}{T}\sum_{i=1}^{T} x_i \tag{3.2}$$

The sample mean m is an **estimator**, which is a function of the data. The particular value of this estimator for this sample is a **point estimate**.

Note that we assign the same weight of $1/T$ to all observations because they all have the same probability due to the i.i.d. property. Other estimators are possible, however. For instance, the pattern of weight could be different, as long as they sum to 1. A good estimator should have several properties.

- It should be **unbiased**, meaning that its expectation is equal to the parameter of interest; for example, $E[m] = \mu$. Otherwise, the estimator is biased.
- It should be **efficient**, which implies that it has the smallest standard deviation of all possible estimators; for example, $V[m - \mu]$ is lowest.

The sample mean, for example, satisfies all of these conditions. An estimator that is unbiased and efficient among all linear combinations of the data is said to be **best linear unbiased estimator** (BLUE).

A weaker condition is for an estimator to be **consistent**. This means that it converges to the true parameter as the sample size T increases, or asymptotically. Take for instance, Equation (3.2), but multiply it instead by $T/(T+1)$. This new estimator is biased but as T goes to infinity will tend to μ as well; hence it is consistent.

Next, the variance, $\sigma^2 = E[(X-\mu)^2]$, can be estimated by the sample variance,

$$s^2 = \hat{\sigma}^2 = \frac{1}{(T-1)}\sum_{i=1}^{T}(x_i - \hat{\mu})^2 \tag{3.3}$$

Note that we divide by $T-1$ instead of T. This is because we estimate the variance around an unknown parameter, the mean. So, we have fewer degrees of freedom than otherwise. As a result, we need to adjust s^2 to ensure that its expectation equals the true value, or that it is unbiased. In most situations, however, T is large so this adjustment is minor.

It is essential to note that these estimated values depend on the particular sample and, hence, have some inherent variability. The sample mean itself is distributed as

$$m = \hat{\mu} \sim N(\mu, \sigma^2/T) \tag{3.4}$$

If the population distribution is normal, this exactly describes the distribution of the sample mean. Otherwise, the central limit theorem states that this distribution is only valid asymptotically (i.e., for large samples).

$$\text{se}(m) = \sigma\sqrt{\frac{1}{T}} \tag{3.5}$$

KEY CONCEPT

With independent draws, the standard deviation of the sample average is given by the population volatility divided by the square root of number of observations T.

For the distribution of the sample variance $\hat{\sigma}^2$, one can show that, when X is normal, the following ratio is distributed as a chi-square with $(T-1)$ degrees of freedom:

$$\frac{(T-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(T-1) \tag{3.6}$$

If the sample size T is large enough, the chi-square distribution converges to a normal distribution:

$$\widehat{\sigma}^2 \sim N\left(\sigma^2, \sigma^4 \frac{2}{(T-1)}\right) \tag{3.7}$$

Using the same approximation, the sample standard deviation has a normal distribution with a standard error of

$$\text{se}(\widehat{\sigma}) = \sigma\sqrt{\frac{1}{2T}} \tag{3.8}$$

Note also that the precision of these estimators, or standard error, is proportional to $1/\sqrt{T}$ for both Equation (3.5) and (3.8). This is a typical result, which is due to the fact that the observations are independent of each other.

We can used this information for **hypothesis testing**. For instance, we would like to detect a constant trend in X. Here, the **null hypothesis** is that $\mu = 0$. To answer the question, we use the distributional assumption in Equation (3.4) and compute a standard normal variable as the ratio of the estimated mean to its standard error, or

$$z = \frac{(m-0)}{\sigma/\sqrt{T}} \tag{3.9}$$

Because this is now a standard normal variable, we would not expect to observe values far away from zero. We need to decide on a **significance level** for the test. This is also one minus the confidence level. Call this c. Typically, we would set $c = 95\%$, which translates into a two-tailed interval for z_c of $[-1.96, +1.96]$. The significance level here is 5%.

Roughly, this means that, if the absolute value of z is greater than two, we would reject the hypothesis that m came from a distribution with a mean of zero. We can have some confidence that the true μ is indeed different from zero.

In fact, we do not know the true σ and use the estimated s instead. The distribution then becomes a Student's t with T degrees of freedom:

$$t = \frac{(m-0)}{s/\sqrt{T}} \tag{3.10}$$

for which the cutoff values can be found from Student's tables. The quantile values for the interval are then t_c. For large values of T, however, this distribution is close to the normal.

These test statistics can be transformed into **confidence intervals**. These are random intervals that contain the true parameter with a fixed level of confidence

$$c = P[m - z_c \times \text{se}(m) \leq \mu \leq m + z_c \times \text{se}(m)] \tag{3.11}$$

Say for instance that we want to determine a 95% confidence interval that contains μ. If T is large, we can use the normal distribution, and the multiplier z_c is 1.96. The confidence interval for the mean is then

$$m \pm z_c \text{se}(m) = [m - 1.96 \times \text{se}(m), m + 1.96 \times \text{se}(m)] \quad (3.12)$$

In this case, the confidence interval is symmetric because the distribution is normal. More generally, this interval could be asymmetric. For instance, the distribution of the sample variance is chi-square, which is asymmetric. A confidence interval can be constructed using the 2.5 percent lower and 2.5 percent higher cutoff values from the chi-square distribution. Define, for example, the first cutoff value as $q_{2.5\%} = q[2.5\%, \chi^2(T-1)]$. Setting this equal to the chi-square value on the right-hand side of Equation (3.6) gives the lower bound for the variance of $q_{2.5\%} \times s^2/(T-1)$.

3.1.3 Example: Yen Exchange Rate

We want to characterize movements in the monthly yen/dollar exchange rate from historical data, taken over 1990 to 2009. Returns are defined using continuously compounded changes. The sample size is $T = 240$, and estimated parameters are $m = -0.18\%$ and $s = 3.24\%$ (per month).

Using Equation (3.4), the standard error of the mean is approximately $\text{se}(m) = s/\sqrt{T} = 3.24\%/\sqrt{240} = 0.21\%$. For the null of $\mu = 0$, this gives a t-ratio of $t = m/\text{se}(m) = -0.18\%/0.21\% = -0.86$. Because this number is less than 2 in absolute value, we cannot reject the hypothesis that the mean is zero at the 95% confidence level. This is a typical result for financial series. Over short horizons, the mean is not precisely estimated.

Next, we turn to the precision in the sample standard deviation. By Equation (3.8), its standard error is $\text{se}(s) = \sigma\sqrt{1/(2T)} = 3.24\%\sqrt{1/480} = 0.15\%$. For the null of $\sigma = 0$, this gives a ratio of $z = s/\text{se}(s) = 3.24\%/0.15\% = 21.9$, which is very high. So, the volatility is not zero. Therefore, there is much more precision in the measurement of s than in that of m.

Furthermore, we can construct 95% confidence intervals around the estimated values. These are:

$$\mu: [-0.18\% - 1.96 \times 0.21\%, -0.18\% + 1.96 \times 0.21\%] = [-0.59\%, +0.23\%]$$

$$\sigma: [3.24\% - 1.96 \times 0.15\%, 3.24\% + 1.96 \times 0.15\%] = [2.948\%, 3.527\%]$$

So, we could be reasonably confident that the volatility is between 3% and 3.5%, but we cannot even be sure that the mean is different from zero.

To be more precise, we could use the chi-square distribution for the variance with $k = T - 1 = 239$. For the $\chi^2(239)$, the 2.5% lower and 2.5% higher quantiles are $q_{2.5\%} = 198.1$ and $q_{97.5\%} = 283.7$, respectively. The exact confidence band is then $\sqrt{198.1/239} \times 3.24\%$ to $\sqrt{283.7/239} \times 3.24\%$, or $[2.949\%, 3.530\%]$, which is virtually identical to the one using the normal approximation. This is because T is large, in which case the normal distribution is a very good approximation for the χ^2 distribution.

3.1.4 Choosing Significance Levels for Tests

Hypothesis testing requires the choice of a significance level, which needs careful consideration. Two types of errors can arise, as described in Table 3.1. A *type 1* error involves rejecting a correct model. A *type 2* error involves accepting an incorrect model. For a given test, increasing the significance level will decrease the probability of a type 1 error but increase the probability of a type 2 error.

Hence there is a trade-off between the two errors for a given test methodology. The choice of the significance level should reflect the cost of each of these errors. At a minimum, however, the test methodology should be designed so that it is *powerful*. This means that, for a fixed type 1 error, the probability of making a type 2 error should be as low as possible.

This type of situation arises, for example, when a risk manager or regulator must decide whether to accept a VAR model. Assume that the bank reports a daily VAR of \$100 million at the 99% level of confidence. The first step is to record the number of exceptions, or losses worse than VAR forecasts over the past 250 days, for example. Under the null hypothesis that the VAR model is correctly calibrated, the number of exceptions should follow a binomial distribution with expected value of $E[X] = np = 250(1 - 0.99) = 2.5$. The risk manager then has to pick a cutoff number of exceptions above which the model would be rejected.

The type 1 error rate is the probability of observing higher numbers than the cutoff point. Say the risk manager chooses $n = 4$, which corresponds to a type 1 error rate or significance level of 10.8%. Above 4, the risk model is rejected.

Suppose, for instance, the regulator for Bank A observes six exceptions. The ***p*-value** of observing six or more exceptions is 4.1%. Because this is below the selected significance level, the regulator would conclude that the VAR model is incorrect. This is the wrong decision, however, due to bad luck.

Once this cutoff point is selected, however, it could lead the regulator to incorrectly accept a VAR model that produces numbers that are too low. So, there is always the possibility of making a type 2 error. For instance, suppose that Bank B knows that its true VAR is \$100 million but reports a number of \$90 million to save regulatory capital. It has exactly the same positions as Bank A. Having a

TABLE 3.1 Decision Errors

	Model	
Decision	**Correct**	**Incorrect**
Accept	Okay	Type 2 error
Reject	Type 1 error	Okay

lower VAR must lead to more exceptions than if VAR = \$100 million. However, the bank could be lucky and report only three exceptions. This leads to the wrong decision, which is not to reject.

The trade-off between type 1 and type 2 errors can be illustrated by changing the cutoff point from 4 to 9. This would decrease the probability of a type 1 error, but increase that of a type 2 error. On one hand, Bank A will not be penalized, which is the correct decision. On the other hand, it will be more difficult to catch Bank B.

The choice of the cutoff point depends on the cost of the two errors. Which is worse? Penalizing a bank that has too many exceptions due to bad luck, or failing to identify banks that understate their risk? A regulator who places more weight on the second type of error would choose a low cutoff point.

3.1.5 Precision of Estimates

Equation (3.4) shows that, when the sample size increases, the standard error of $\widehat{\mu}$ shrinks at a rate proportional to $1/\sqrt{T}$. The precision of the estimate increases as the number of observations increases.

This result will prove useful to assess the precision of estimates generated from **numerical simulations**, which are widely used in risk management. Numerical simulations create independent random variables over a fixed number of replications T. If T is too small, the final estimates will be imprecisely measured. If T is very large, the estimates will be accurate. The precision of the estimates increases at a rate proportional to $1/\sqrt{T}$.

KEY CONCEPT

With independent draws, the standard deviation of most statistics is inversely related to the square root of the number of observations T. Thus, more observations make for more precise estimates.

EXAMPLE 3.1: FRM EXAM 2007—QUESTION 137

What does a hypothesis test at the 5% significance level mean?

a. P(not reject H0 | H0 is true) = 0.05
b. P(not reject H0 | H0 is false) = 0.05
c. P(reject H0 | H0 is true) = 0.05
d. P(reject H0 | H0 is false) = 0.05

EXAMPLE 3.2: FRM EXAM 2009—QUESTION 9

When testing a hypothesis, which of the following statements is *correct* when the level of significance of the test is decreased?

a. The likelihood of rejecting the null hypothesis when it is true decreases.
b. The likelihood of making a type 1 error increases.
c. The null hypothesis is rejected more frequently, even when it is actually false.
d. The likelihood of making a type 2 error decreases.

EXAMPLE 3.3: FRM EXAM 2009—QUESTION 6

A population has a known mean of 1,000. Suppose 1,600 samples are randomly drawn (with replacement) from this population. The mean of the observed samples is 998.7, and the standard deviation of the observed samples is 100. What is the standard error of the sample mean?

a. 0.025
b. 0.25
c. 2.5
d. 25

3.1.6 Hypothesis Testing for Distributions

The analysis so far has focused on hypothesis testing for specific parameters. Another application is to test the hypothesis that the sample comes from a specific distribution such as the normal distribution. Such a hypothesis can be tested using a variety of tools. A widely used test focuses on the moments. Define $\hat{\gamma}$ and $\hat{\delta}$ as the estimated skewness and kurtosis. With a normal distribution, the true values are $\gamma = 0$ and $\delta = 3$.

The **Jarque-Bera** (JB) statistic measures the deviations from the expected values

$$\mathrm{JB} = T[\frac{\hat{\gamma}^2}{6} + \frac{(\hat{\delta} - 3)^2}{24}] \tag{3.13}$$

which under the null hypothesis has a chi-square distribution with two degrees of freedom. The cutoff point at the 95% level of confidence is 5.99. Hence if the observed value of the JB statistic is above 5.99, we would have to reject the hypothesis that the observations come from a normal distribution.

As an illustration, continue examining the dollar-yen distribution over the period 1990 to 2009. The sample skewness and kurtosis are $\widehat{\gamma} = -0.47$ and $\widehat{\delta} = 5.82$. This skewness is negative but small. The kurtosis, however, is much greater than three, which indicates that the distribution has fat tails. The JB statistic is 88.4, which is much greater than the cutoff value of 5.99. Hence the distribution of this exchange rate is definitely not normal.

3.2 REGRESSION ANALYSIS

Regression analysis has particular importance for risk management, because it can be used to model relationships between financial variables. It can also be used for the mapping process. For example, positions in individual stocks can be replaced by exposures on a smaller set of stock indices. This considerably reduces the dimensions of the risk space, using exposures estimated via regressions.

3.2.1 Bivariate Regression: Ordinary Least Squares (OLS) Estimation

In a **linear regression**, the **dependent variable** y is projected on a set of N predetermined **independent variables**, x. In the simplest bivariate case, with two variables, we write

$$y_t = \alpha + \beta x_t + \epsilon_t, \qquad t = 1, \ldots, T \tag{3.14}$$

where α is called the **intercept**, or constant, β is called the **slope**, and ϵ is called the **error term**. Here, y is called the **regressand**, and x the **regressor** (as in predictor).

This regression could represent a time series or a cross section of variables. It is **linear** in the coefficients but not necessarily in the variables. The variables themselves could have been transformed. For example, when size or total market value is used, it is commonly transformed by taking logs, in which case $x = \ln(X)$.

Next comes the question of how to estimate these parameters, which are unobservable usually. The estimated regression is

$$y_t = \widehat{\alpha} + \widehat{\beta} x_t + \widehat{\epsilon}_t, \qquad t = 1, \ldots, T \tag{3.15}$$

where the estimated error $\widehat{\epsilon}$ is called the **residual**, or deviation between the observed and fitted values. Figure 3.1 displays the decomposition from a linear regression. The value y_t is split into a forecast of y conditional on x, $E[y_t|x_t] = \alpha + \beta x_t$, and an error term e_t.

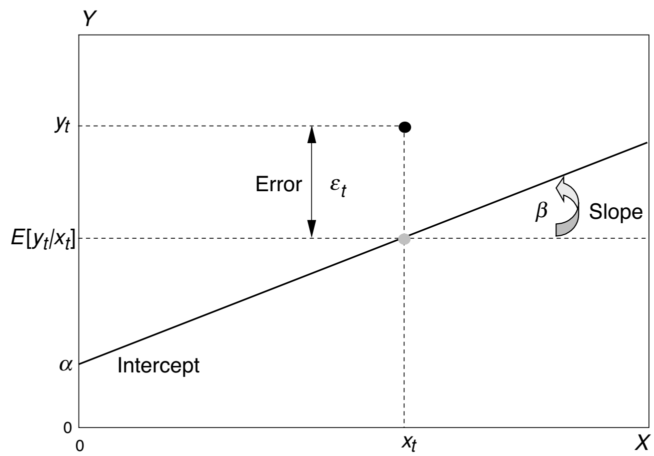

FIGURE 3.1 Regression Decomposition

A particularly convenient regression model is based on **ordinary least squares** (OLS), using these assumptions:

- The errors are independent of the variables x.
- The errors have constant variance.
- The errors are uncorrelated across observations t.
- The errors are normally distributed.

As seen in the previous chapter, independence between two random variables u and v is a strong assumption; it implies that the covariance $\text{Cov}(u, v)$ and the correlation $\rho(u, v)$ are zero. Otherwise, the assumption of normality leads to exact statistical results. More generally, the conditions of the central limit theorem hold as T grows large, which leads to similar results.

OLS provides estimators of the coefficients by minimizing the sum of squared errors, $\sum \widehat{\epsilon}_t^2$. Thus, the estimators provide the least sum of squared errors.

Under the model assumptions, the Gauss-Markov theorem states that OLS provides the best linear unbiased estimators (BLUEs). The estimator $\widehat{\beta}$ is unbiased, or has expectation that is the true value β. Best means that the OLS estimator is efficient, or has the lowest variance compared to others. When the errors are normal, OLS is a maximum likelihood estimator. However, it also has good properties for broader distributions.

The OLS beta is

$$\widehat{\beta} = \frac{[1/(T-1)] \sum_t (x_t - \bar{x})(y_t - \bar{y})}{[1/(T-1)] \sum_t (x_t - \bar{x})^2} \tag{3.16}$$

where $\bar{x}$ and $\bar{y}$ are the means of x_t and y_t. Alpha is estimated by

$$\widehat{\alpha} = \bar{y} - \widehat{\beta}\bar{x} \tag{3.17}$$

When the regression includes an intercept, it can be shown that the estimated residuals must average to zero by construction

$$(1/T)\sum_{t=1}^{T}\widehat{\epsilon} = 0 \tag{3.18}$$

Note that the numerator in Equation (3.16) is also the sample covariance between two series x_i and x_j, which can be written as

$$\widehat{\sigma_{ij}} = \frac{1}{(T-1)}\sum_{t=1}^{T}(x_{t,i} - \widehat{\mu_i})(x_{t,j} - \widehat{\mu_j}) \tag{3.19}$$

To interpret β, we can take the covariance between y and x, which is

$$\text{Cov}(y, x) = \text{Cov}(\alpha + \beta x + \epsilon, x) = \beta\text{Cov}(x, x) = \beta V(x)$$

because ϵ is uncorrelated with x. This shows that the population β is also

$$\beta(y, x) = \frac{\text{Cov}(y, x)}{V(x)} = \frac{\rho(y, x)\sigma(y)\sigma(x)}{\sigma^2(x)} = \rho(y, x)\frac{\sigma(y)}{\sigma(x)} \tag{3.20}$$

Once the parameters have been estimated, we can construct a forecast for y conditional on observations on x:

$$\widehat{y_t} = E[y_t|x_t] = \widehat{\alpha} + \widehat{\beta}x_t \tag{3.21}$$

This is a very convenient representation for risk measurement. It shows that the variable y can be mapped on variable x when the regression provides a good fit, meaning that the error terms are relatively small.

3.2.2 Bivariate Regression: Quality of Fit

The **regression fit** can be assessed by examining the size of the residuals, obtained by subtracting the fitted values $\widehat{y_t}$ from y_t,

$$\widehat{\epsilon_t} = y_t - \widehat{y_t} = y_t - \widehat{\alpha} - \widehat{\beta}x_t \tag{3.22}$$

and taking the estimated variance as

$$V(\hat{\epsilon}) = \frac{1}{(T-2)} \sum_{t=1}^{T} \hat{\epsilon}_t^2 \tag{3.23}$$

We divide by $T - 2$ because the estimator uses two unknown quantities, $\hat{\alpha}$ and $\hat{\beta}$. Without this adjustment, it would be too small, or biased downward. Also note that, because the regression includes an intercept, the average value of $\hat{\epsilon}$ has to be exactly zero.

The quality of the fit can be assessed using a unitless measure called the **regression *R*-squared**, also called the **coefficient of determination**. This is defined as

$$R^2 = 1 - \frac{\text{SSE}}{\text{SSY}} = 1 - \frac{\sum_t \hat{\epsilon}_t^2}{\sum_t (y_t - \bar{y})^2} \tag{3.24}$$

where SSE is the sum of squared errors (residuals, to be precise), and SSY is the sum of squared deviations of y around its mean. If the regression includes a constant, we always have $0 \leq R^2 \leq 1$. In this case, R-squared is also the square of the usual correlation coefficient,

$$R^2 = \rho(y, x)^2 \tag{3.25}$$

The R^2 measures the degree to which the size of the errors is smaller than that of the original dependent variable y. To interpret R^2, consider two extreme cases. On one hand, if the fit is excellent, all the errors will be zero and the numerator in Equation (3.24) will be zero, which gives $R^2 = 1$. On the other hand, if the fit is poor, SSE will be as large as SSY and the ratio will be one, giving $R^2 = 0$.

Alternatively, we can interpret the R-squared by decomposing the variance of $y_t = \alpha + \beta x_t + \epsilon_t$. Because ϵ and x are uncorrelated, this yields

$$V(y) = \beta^2 V(x) + V(\epsilon) \tag{3.26}$$

Dividing by $V(y)$,

$$1 = \frac{\beta^2 V(x)}{V(y)} + \frac{V(\epsilon)}{V(y)} \tag{3.27}$$

Because the R-squared is also $R^2 = 1 - V(\epsilon)/V(y)$, it is equal to $= \beta^2 V(x)/V(y)$, which is the contribution in the variation of y due to β and x.

Assessing the quality of fit also involves checking whether the OLS assumptions are satisfied. For instance, the errors should have a distribution with constant variance: $V(\epsilon_t|x_t) = \sigma^2$. The absence of subscript in σ^2 means that it is constant. This can be checked by plotting the squared residual against x_t.

3.2.3 Bivariate Regression: Hypothesis Testing

Finally, we can derive the distribution of the estimated coefficients, which is normal and centered around the true values. For the slope coefficient, $\widehat{\beta} \sim N(\beta, V(\widehat{\beta}))$, with variance given by

$$V(\widehat{\beta}) = V(\widehat{\epsilon})\frac{1}{\sum_t (x_t - \bar{x})^2} \tag{3.28}$$

This can be used to test whether the slope coefficient is significantly different from zero. The associated test statistic

$$t = \widehat{\beta}/\sigma(\widehat{\beta}) \tag{3.29}$$

has a Student's t distribution.

The usual practice is to check whether the absolute value of the statistic is above 2. If so, we would reject the hypothesis that there is no relationship between y and x. This corresponds to a two-tailed significance level of 5%. Equation (3.29) can also be used to construct confidence intervals for the population coefficients.

3.2.4 Autoregression

A particularly useful application is a regression of a variable on a lagged value of itself, called **autoregression**:

$$y_t = \alpha + \beta_k y_{t-k} + \epsilon_t, \qquad t = 1, \ldots, T \tag{3.30}$$

If the β coefficient is significant, previous movements in the variable can be used to predict future movements. Here, the coefficient β_k is known as the kth-order **autocorrelation coefficient**.

Consider, for instance, a first-order autoregression, where the daily change in the stock market is regressed on the previous day's value. A positive coefficient $\widehat{\beta_1}$ indicates a trend, or momentum. A negative coefficient indicates mean reversion. As an example, assume that we find that $\widehat{\beta_1} = 0.10$, with zero intercept. One day, the market goes up by 2%. Our best forecast for the next day is another up move of

$$E[y_{t+1}] = \beta_1 y_t = 0.10 \times 2\% = 0.2\%$$

In practice, such slope coefficient for most financial variables is not statistically significantly different from zero.

Autocorrelation changes normal patterns in risk across horizons. When there is no autocorrelation, risk increases with the square root of time. With positive autocorrelation, shocks have a longer-lasting effect and risk increases faster than the square root of time.

3.2.5 Multivariate Regression

More generally, the regression in Equation (3.14) can be written, with N independent variables:

$$y_t = \alpha + \sum_{i=1}^{N} \beta_i x_{t,i} + \epsilon_t \tag{3.31}$$

Stacking all the y variables, we have

$$\begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} x_{11} \; x_{12} \; x_{13} \; \ldots \; x_{1N} \\ \vdots \\ x_{T1} \; x_{T2} \; x_{T3} \; \ldots \; x_{TN} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_T \end{bmatrix} \tag{3.32}$$

This can include the case of a constant when the first column of X is a vector of ones, in which case β_1 is the usual α. In matrix notation,

$$y = X\beta + \epsilon \tag{3.33}$$

The estimated coefficients can be written succinctly as

$$\widehat{\beta} = (X'X)^{-1}X'y \tag{3.34}$$

which requires another assumption for OLS estimation:

- The matrix X must have full rank, meaning that a variable cannot be a linear combination of others.

If X were not of full rank, the matrix $(X'X)$ would not be invertible. This assumption rules out perfect correlations between variables. Sometimes, however, two variables can be highly correlated, which is described as **multicollinearity**. When this is the case, the regression output is unstable. Small changes in the data can produce large changes in the estimates. Indeed, coefficients will have very high standard errors.

Generalizing Equation (3.28), the covariance matrix of coefficients is

$$V(\widehat{\beta}) = V(\widehat{\epsilon})(X'X)^{-1} \tag{3.35}$$

using

$$V(\widehat{\epsilon}) = \frac{1}{(T-N)} \sum_{t=1}^{T} \widehat{\epsilon_t}^2 \tag{3.36}$$

where the denominator is adjusted for the number of estimated coefficients N.

We can extend the t-statistic to a multivariate environment. Say we want to test whether the last m coefficients are jointly zero. Define $\widehat{\beta_m}$ as these grouped coefficients and $V_m(\widehat{\beta})$ as their covariance matrix. We set up a statistic

$$F = \frac{\widehat{\beta}_m' V_m(\widehat{\beta})^{-1} \widehat{\beta}_m / m}{\mathrm{SSE}/(T-N)} \tag{3.37}$$

which has an F-distribution with m and $T - N$ degrees of freedom. As before, we would reject the hypothesis if the value of F is too large compared to critical values from tables.

Finally, we can also report a regression R-squared, as in Equation (3.24). We should note, however, that the R-squared mechanically increases as variables are added to the regression. With more variables, the variance of the residual term must be small, because this is in-sample fitting with more variables. Sometimes an **adjusted *R*-squared** is used:

$$\bar{R}^2 = 1 - \frac{\mathrm{SSE}/(T-N)}{\mathrm{SSY}/(T-1)} = 1 - (1 - R^2)\frac{T-1}{T-N} \tag{3.38}$$

This more properly penalizes for the number of independent variables.

3.2.6 Example

This section gives the example of a regression of a stock return on the market. Such analysis is commonly used to assess whether movements in the stock can be hedged using stock-market index futures. Alternatively, the stock can be mapped on the index (i.e., its position replaced by the fitted index exposure).

We consider 10 years of data for Intel and the S&P 500 index, using total rates of return over a month. Figure 3.2 plots the 120 combination of returns, or (y_t, x_t). Apparently, there is a positive relationship between the two variables, as shown by the straight line that represents the regression fit $(\widehat{y_t}, x_t)$.

Table 3.2 displays the regression results. The regression shows a positive relationship between the two variables, with $\widehat{\beta} = 1.349$. This is significantly positive, with a standard error of 0.229 and t-statistic of 5.90. The t-statistic is very high, with an associated probability value (p-value) close to zero. Thus we can be fairly confident of a positive association between the two variables.

This beta coefficient is also called **systematic risk**, or exposure to general market movements. Typically, technology stocks have greater systematic risk than

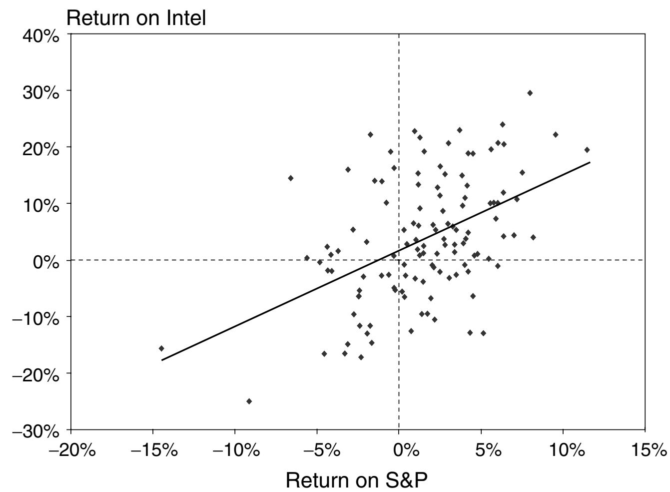

FIGURE 3.2 Intel Return vs. S&P Return

the average. Indeed, the slope in Intel's regression is greater than unity. To test whether β is significantly different from one, we can compute a z-score as

$$z = \frac{(\widehat{\beta} - 1)}{s(\widehat{\beta})} = \frac{(1.349 - 1)}{0.229} = 1.53$$

This is less than the usual cutoff value of two, so we cannot say for certain that Intel's systematic risk is greater than one.

The R-squared of 22.8% can be also interpreted by examining the reduction in dispersion from y to $\widehat{\epsilon}$, which is from 10.94% to 9.62%. The R-squared can be written as

$$R^2 = 1 - \frac{9.62\%^2}{10.94\%^2} = 22.8\%$$

Thus about 23% of the variance of Intel's returns can be attributed to the market.

TABLE 3.2 Regression Results

$y = \alpha + \beta x$, y = Intel return, x = S&P return				
R-squared		0.228		
Standard error of y		10.94%		
Standard error of $\widehat{\epsilon}$		9.62%		
Coefficient	**Estimate**	**Standard Error**	***T*-Statistic**	***P*-Value**
Intercept $\widehat{\alpha}$	0.0168	0.0094	1.78	0.77
Slope $\widehat{\beta}$	1.349	0.229	5.90	0.00

EXAMPLE 3.4: FRM EXAM 2004—QUESTION 4

Consider the following linear regression model: $Y = a + b\ X + e$. Suppose $a = 0.05$, $b = 1.2$, $SD(Y) = 0.26$, and $SD(e) = 0.1$. What is the correlation between X and Y?

a. 0.923
b. 0.852
c. 0.701
d. 0.462

EXAMPLE 3.5: FRM EXAM 2007—QUESTION 22

Consider two stocks, A and B. Assume their annual returns are jointly normally distributed, the marginal distribution of each stock has mean 2% and standard deviation 10%, and the correlation is 0.9. What is the expected annual return of stock A if the annual return of stock B is 3%?

a. 2%
b. 2.9%
c. 4.7%
d. 1.1%

EXAMPLE 3.6: FRM EXAM 2009—QUESTION 8

A portfolio manager is interested in the systematic risk of a stock portfolio, so he estimates the linear regression: $R_{Pt} - R_F = \alpha_P + \beta_P[R_{Mt} - R_F] + \epsilon_{Pt}$ where R_{Pt} is the return of the portfolio at time t, R_{Mt} is the return of the market portfolio at time t, and R_F is the risk-free rate, which is constant over time. Suppose that $\alpha = 0.008$, $\beta = 0.977$, $\sigma(R_P) = 0.167$, and $\sigma(R_M) = 0.156$.

What is the approximate coefficient of determination in this regression?

a. 0.913
b. 0.834
c. 0.977
d. 0.955

EXAMPLE 3.7: FRM EXAM 2004—QUESTION 23

Which of the following statements about the linear regression of the return of a portfolio over the return of its benchmark presented below are *correct*?

Portfolio Parameter	Value
Beta	1.25
Alpha	0.26
Coefficient of determination	0.66
Standard deviation of error	2.42

I. The correlation is 0.71.
II. About 34% of the variation in the portfolio return is explained by variation in the benchmark return.
III. The portfolio is the dependent variable.
IV. For an estimated portfolio return of 12%, the confidence interval at 95% is (7.16% − 16.84%).

a. II and IV
b. III and IV
c. I, II, and III
d. II, III, and IV

3.2.7 Pitfalls with Regressions

As with any quantitative method, the usefulness of regression analysis depends on the underlying assumptions being fulfilled for the problem at hand. We now briefly mention potential problems of interpretation.

The original OLS setup assumes that the X variables are predetermined (i.e., exogenous or fixed), as in a controlled experiment. In practice, regressions are performed on actual, existing data that do not satisfy these strict conditions. In the previous regression, returns on the S&P 500 index are certainly not predetermined.

If the X variables are stochastic, however, most of the OLS results are still valid as long as the X variables are distributed independently of the errors and their distribution does not involve β and σ^2.

Violations of this assumption are serious because they create biases in the slope coefficients. Biases could lead the researcher to come to the wrong conclusion. For instance, we could have measurement errors in the X variables, which causes the measured X to be correlated with ϵ. This so-called **errors in the variables** problem causes a downward bias in the estimated coefficients. The slope coefficients are systematically lower than their true value, which is a serious error. Note that errors in the y variables are not an issue, because they are captured by the error component ϵ.

A related problem is that of **specification error**. Suppose the true model has N variables but we use only a subset N_1. If the omitted variables are correlated with the included variables, the estimated coefficients will be biased. This is a very serious problem because it is difficult to identify. Biases in the coefficients cause problems with **estimation**.

Another class of problems has to do with potential biases in the standard errors of the coefficients. These errors are especially serious if standard errors are underestimated, creating a sense of false precision in the regression results and perhaps leading to the wrong conclusions. The OLS approach assumes that the errors are independent across observations. This is generally the case for financial time series, but often not in cross-sectional setups. For instance, consider a cross section of mutual fund returns on some attribute. Mutual fund families often have identical funds, except for the fee structure (e.g., called *A* for a front load, *B* for a deferred load). These funds, however, are invested in the same securities and have the same manager. Thus, their returns are certainly not independent. If we run a standard OLS regression with all funds, the standard errors will be too small because we overestimate the number of independent observations. More generally, one has to check that there is no systematic correlation pattern in the residuals. Even with time series, problems can arise with **autocorrelation** in the errors. Biases in the standard errors cause problems with **inference**, as one could conclude erroneously that a coefficient is statistically significant.

Problems with **efficiency** arise when the estimation does not use all available information. For instance, the residuals can have different variances across observations, in which case we have **heteroskedasticity**. This is the opposite of the constant variance case, or **homoskedasticity**. Conditional heteroskedasticity occurs when the variance is systematically related to the independent variables. For instance, large values of X could be associated with high error variances. These problems can be identified by diagnostic checks on the residuals. If heteroskedasticity is present, one could construct better standard errors or try an alternative specification. This is much less a problem than problems with estimation or inference, however. Inefficient estimates do not necessarily create biases.

Also, regressions may be subject to **multicollinearity**. This arises when the X variables are highly correlated. Some of the variables may be superfluous, for example using two currencies that are fixed to each other. As a result, the matrix $(X'X)$ in Equation (3.34) will be unstable, and the estimated β will be unreliable. This problem will show up in large standard errors, however. It can be fixed by discarding some of the variables that are highly correlated with others.

Last, even if all the OLS conditions are satisfied, one has to be extremely careful about using a regression for forecasting. Unlike physical systems, which are inherently stable, financial markets are dynamic and relationships can change quickly. Indeed, financial anomalies, which show up as strongly significant coefficients in historical regressions, have an uncanny ability to disappear as soon as one tries to exploit them.

EXAMPLE 3.8: FRM EXAM 2009—QUESTION 7

You built a linear regression model to analyze annual salaries for a developed country. You incorporated two independent variables, age and experience, into your model. Upon reading the regression results, you notice that the coefficient of experience is negative, which appears to be counterintuitive. In addition, you discover that the coefficients have low *t*-statistics but the regression model has a high R^2. What is the most likely cause of these results?

a. Incorrect standard errors
b. Heteroskedasticity
c. Serial correlation
d. Multicollinearity

EXAMPLE 3.9: FRM EXAM 2004—QUESTION 59

Which of the following statements regarding linear regression is *false*?

a. Heteroskedasticity occurs when the variance of residuals is not the same across all observations in the sample.
b. Unconditional heteroskedasticity leads to inefficient estimates, whereas conditional heteroskedasticity can lead to problems with both inference and estimation.
c. Serial correlation occurs when the residual terms are correlated with each other.
d. Multicollinearity occurs when a high correlation exists between or among two or more of the independent variables in a multiple regression.

EXAMPLE 3.10: FRM EXAM 1999—QUESTION 2

Under what circumstances could the explanatory power of regression analysis be overstated?

a. The explanatory variables are not correlated with one another.
b. The variance of the error term decreases as the value of the dependent variable increases.
c. The error term is normally distributed.
d. An important explanatory variable is omitted that influences the explanatory variables included and the dependent variable.

3.3 IMPORTANT FORMULAS

Discrete returns, log returns: $r_t = (P_t - P_{t-1})/P_{t-1},\ \ R_t = \ln[P_t/P_{t-1}]$

Time aggregation: $\mathrm{E}(R_T) = \mathrm{E}(R_1)T,\ \ V(R_T) = V(R_1)T,\ \ \mathrm{SD}(R_T) = \mathrm{SD}(R_1)\sqrt{T}$

Portfolio rate of return: $r_{p,t+1} = \sum_{i=1}^{N} w_{i,t} r_{i,t+1} = w'R$

Portfolio variance: $V[r_{p,t+1}] = w'\Sigma\, w$

Estimated mean: $m = \widehat{\mu} = \frac{1}{T}\sum_{i=1}^{T} x_i$

Estimated variance: $s^2 = \widehat{\sigma}^2 = \frac{1}{(T-1)}\sum_{i=1}^{T}(x_i - \widehat{\mu})^2$

Distribution of estimated mean: $m = \widehat{\mu} \sim N(\mu, \sigma^2/T)$

Distribution of estimated variance: $\frac{(T-1)\widehat{\sigma}^2}{\sigma^2} \sim \chi^2(T-1),\ \widehat{\sigma}^2 \rightarrow N\left(\sigma^2, \sigma^4\frac{2}{(T-1)}\right)$

Standard error of estimated standard deviation: $\mathrm{se}(\widehat{\sigma}) = \sigma\sqrt{1/(2T)}$

Bivariate regression: $y_t = \alpha + \beta x_t + \epsilon_t$

Multivariate regression: $y_t = \alpha + \sum_{i=1}^{N} \beta_i x_{t,i} + \epsilon_t = X_t\beta + \epsilon_t$

Estimated beta: $\widehat{\beta} = (X'X)^{-1}X'y$

Population beta: $\beta(y, x) = \frac{\mathrm{Cov}(y,x)}{V(x)} = \frac{\rho(y,x)\sigma(y)\sigma(x)}{\sigma^2(x)} = \rho(y, x)\frac{\sigma(y)}{\sigma(x)}$

Regression R-squared: $R^2 = 1 - \frac{\mathrm{SSE}}{\mathrm{SSY}} = 1 - \frac{\sum_t \widehat{\epsilon}_t^2}{\sum_t (y_t - \bar{y})^2}$

Variance decomposition: $V(y) = \beta^2 V(x) + V(\epsilon)$

T-statistic for hypothesis of zero coefficient: $t = \widehat{\beta}/\sigma(\widehat{\beta})$

3.4 ANSWERS TO CHAPTER EXAMPLES

Example 3.1: FRM Exam 2007—Question 137

c. The significance level is the probability of committing a type 1 error, or rejecting a correct model. This is also P(reject H0 | H0 is true). By contrast, the type 2 error rate is P(not reject H0 | H0 is false).

Example 3.2: FRM Exam 2009—Question 9

a. The significance level is also the probability of making a type 1 error, or to reject the null hypothesis when true, which decreases. This is the opposite of answers b. and c., which are false. This leads to an increase in the likelihood of making a type 2 error, which is to accept a false hypothesis, so answer d. is false.

Example 3.3: FRM Exam 2009—Question 6

c. This is $\sigma/\sqrt{T}$, or $1{,}000/sqrt1{,}600 = 1{,}000/40 = 2.5$. Other numbers are irrelevant.

Example 3.4: FRM Exam 2004—Question 4

a. We can find the volatility of X from the variance decomposition, Equation (3.26). This gives $V(x) = [V(y) - V(e)]/\beta^2 = [0.26^2 - 0.10^2)/1.2^2 = 0.04$. Then $\text{SD}(X) = 0.2$, and $\rho = \beta\text{SD}(X)/\text{SD}(Y) = 1.20.2/0.26 = 0.923$.

Example 3.5: FRM Exam 2007—Question 22

b. The information in this question can be used to construct a regression model of A on B. We have $R_A = 2\% + 0.9(10\%/10\%)(R_B - 2\%) + \epsilon$. Next, replacing R_B by 3% gives $\hat{R}_A = 2\% + 0.9(3\% - 2\%) = 2.9\%$.

Example 3.6: FRM Exam 2009—Question 8

b. Using Equation (3.27), the R-squared is given by $\beta^2\sigma^2_M/\sigma^2_P = 0.977^2 \times 0.156^2/0.167^2 = 0.83$.

Example 3.7: FRM Exam 2004—Question 23

b. The correlation is given by $\sqrt{0.66} = 0.81$, so answer I. is incorrect. Next, 66% of the variation in Y is explained by the benchmark, so answer II. is incorrect. The portfolio return is indeed the dependent variable Y, so answer III. is correct. Finally, to find the 95% two-tailed confidence interval, we use α from a normal distribution, which covers 95% within plus or minus 1.96, close to 2.00. The interval is then $y - 2\text{SD}(e)$, $y + 2\text{SD}(e)$, or $(7.16 - 16.84)$. So answers III. and IV. are correct.

Example 3.8: FRM Exam 2009—Question 7

d. Age and experience are likely to be highly correlated. Generally, multicollinearity manifests itself when standard errors for coefficients are high, even when the R^2 is high.

Example 3.9: FRM Exam 2004—Question 59

b. Heteroskedasticity indeed occurs when the variance of the residuals is not constant, so a. is correct. This leads to inefficient estimates but otherwise does not cause problems with inference and estimation. Statements c. and d. are correct.

Example 3.10: FRM Exam 1999—Question 2

d. If the true regression includes a third variable z that influences both y and x, the error term will not be conditionally independent of x, which violates one of the assumptions of the OLS model. This will artificially increase the explanatory power of the regression. Intuitively, the variable x will appear to explain more of the variation in y simply because it is correlated with z.

CHAPTER 4

Monte Carlo Methods

The two preceding chapters dealt with probability and statistics. The former involves the generation of random variables from known distributions. The second deals with estimation of distribution parameters from actual data. With estimated distributions in hand, we can proceed to the next step, which is the simulation of random variables for the purpose of risk management.

Such simulations, called **Monte Carlo** (MC) simulations, are central to financial engineering and risk management. They allow financial engineers to price complex financial instruments. They allow risk managers to build the distribution of portfolios that are too complex to model analytically.

Simulation methods are quite flexible and are becoming easier to implement with technological advances in computing. Their drawbacks should not be underestimated, however. For all their elegance, simulation results depend heavily on the model's assumptions: the shape of the distribution, the parameters, and the pricing functions. Risk managers need to be keenly aware of the effect that errors in these assumptions can have on the results.

This chapter shows how Monte Carlo methods can be used for risk management. Section 4.1 introduces a simple case with just one source of risk. Section 4.2 shows how to apply these methods to construct value at risk (VAR) measures, as well as to price derivatives. Multiple sources of risk are then considered in Section 4.3.

4.1 SIMULATIONS WITH ONE RANDOM VARIABLE

Simulations involve creating artificial random variables with properties similar to those of the risk factors in the portfolio. These include stock prices, exchange rates, bond yields or prices, and commodity prices.

4.1.1 Simulating Markov Processes

In efficient markets, financial prices should display a random walk pattern. More precisely, prices are assumed to follow a **Markov process**, which is a particular stochastic process independent of its past history; the entire distribution of the

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future price relies on the current price only. The past history is irrelevant. These processes are built from the following components, described in order of increasing complexity.

- **The Wiener process.** This describes a variable Δz, whose change is measured over the interval Δt such that its mean change is zero and variance proportional to Δt:

$$\Delta z \sim N(0, \Delta t) \tag{4.1}$$

  If ϵ is a standard normal variable $N(0, 1)$, this can be written as $\Delta z = \epsilon\sqrt{\Delta t}$. In addition, the increments Δz are independent across time.
- **The generalized Wiener process.** This describes a variable Δx built up from a Wiener process, with, in addition, a constant trend a per unit of time and volatility b:

$$\Delta x = a\Delta t + b\Delta z \tag{4.2}$$

  A particular case is the **martingale**, which is a zero-drift stochastic process, $a = 0$, which leads to $E(\Delta x) = 0$. This has the convenient property that the expectation of a future value is the current value

$$E(x_T) = x_0 \tag{4.3}$$

- **The Ito process.** This describes a generalized Wiener process, whose trend and volatility depend on the *current* value of the underlying variable and time:

$$\Delta x = a(x, t)\Delta t + b(x, t)\Delta z \tag{4.4}$$

  This is a Markov process because the distribution depends only on the current value of the random variable x, as well as time. In addition, the innovation in this process has a normal distribution.

4.1.2 The Geometric Brownian Motion

A particular example of Ito process is the **geometric Brownian motion** (GBM), which is described for the variable S as

$$\Delta S = \mu S\Delta t + \sigma S\Delta z \tag{4.5}$$

The process is geometric because the trend and volatility terms are proportional to the current value of S. This is typically the case for stock prices, for which *rates of return* appear to be more stationary than raw dollar returns, ΔS. It is also used for currencies. Because $\Delta S/S$ represents the capital appreciation only, abstracting from dividend payments, μ represents the expected total rate of return on the asset minus the rate of income payment, or dividend yield in the case of stocks.

Example: A Stock Price Process

Consider a stock that pays no dividends, has an expected return of 10% per annum, and has volatility of 20% per annum. If the current price is \$100, what is the process for the change in the stock price over the next week? What if the current price is \$10?

The process for the stock price is

$$\Delta S = S(\mu \Delta t + \sigma \sqrt{\Delta t} \times \epsilon)$$

where ϵ is a random draw from a standard normal distribution. If the interval is one week, or $\Delta t = 1/52 = 0.01923$, the mean is $\mu \Delta t = 0.10 \times 0.01923 = 0.001923$ and $\sigma\sqrt{\Delta t} = 0.20 \times \sqrt{0.01923} = 0.027735$. The process is $\Delta S = \$100(0.001923 + 0.027735 \times \epsilon)$. With an initial stock price at \$100, this gives $\Delta S = 0.1923 + 2.7735\epsilon$. With an initial stock price at \$10, this gives $\Delta S = 0.01923 + 0.27735\epsilon$. The trend and volatility are scaled down by a factor of 10.

This model is particularly important because it is the underlying process for the Black-Scholes formula. The key feature of this distribution is the fact that the volatility is proportional to S. This ensures that the stock price will stay positive. Indeed, as the stock price falls, its variance decreases, which makes it unlikely to experience a large down move that would push the price into negative values. As the limit of this model is a normal distribution for $dS/S = d\ln(S)$, S follows a **lognormal distribution**.

This process implies that, over an interval $T - t = \tau$, the logarithm of the ending price is distributed as

$$\ln(S_T) = \ln(S_t) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \tag{4.6}$$

where ϵ is a standardized normal variable.

Example: A Stock Price Process (*Continued*)

Assume the price in one week is given by $S = \$100\exp(R)$, where R has annual expected value of 10% and volatility of 20%. Construct a two-tailed 95% confidence interval for S.

The standard normal deviates that corresponds to a 95% confidence interval are $\alpha_{MIN} = -1.96$ and $\alpha_{MAX} = 1.96$. In other words, we have 2.5% in each tail. The 95% confidence band for R is then $R_{MIN} = \mu\Delta t - 1.96\sigma\sqrt{\Delta t} = 0.001923 - 1.96 \times 0.027735 = -0.0524$ and $R_{MAX} = \mu\Delta t + 1.96\sigma\sqrt{\Delta t} = 0.001923 + 1.96 \times 0.027735 = 0.0563$. This gives $S_{MIN} = \$100\exp(-0.0524) = \$94.89$, and $S_{MAX} = \$100\exp(0.0563) = \$105.79$.

Whether a lognormal distribution is much better than the normal distribution depends on the horizon considered. If the horizon is one day only, the choice of the lognormal versus normal assumption does not really matter. It is highly unlikely that the stock price would drop below zero in one day, given typical volatilities. However, if the horizon is measured in years, the two assumptions do lead to different results. The lognormal distribution is more realistic as it prevents prices from turning negative.

In simulations, this process is approximated by small steps with a normal distribution with mean and variance given by

$$\frac{\Delta S}{S} \sim N(\mu\Delta t, \sigma^2\Delta t) \tag{4.7}$$

To simulate the future price path for S, we start from the current price S_t and generate a sequence of independent standard normal variables ϵ, for $i = 1, 2, \ldots, n$. The next price, S_{t+1}, is built as $S_{t+1} = S_t + S_t(\mu\Delta t + \sigma\epsilon_1\sqrt{\Delta t})$. The following price, S_{t+2}, is taken as $S_{t+1} + S_{t+1}(\mu\Delta t + \sigma\epsilon_2\sqrt{\Delta t})$, and so on until we reach the target horizon, at which point the price $S_{t+n} = S_T$ should have a distribution close to the lognormal.

Table 4.1 illustrates a simulation of a process with a drift (μ) of 0% and volatility (σ) of 20% over the total interval, which is divided into 100 steps.

The initial price is \$100. The local expected return is $\mu\Delta t = 0.0/100 = 0.0$, and the volatility is $0.20 \times \sqrt{1/100} = 0.02$. The second column shows the realization of a uniform $U(0, 1)$ variable. The value for the first step is $u_1 = 0.0430$. The next column transforms this variable into a normal variable with mean 0.0 and volatility of 0.02, which gives -0.0343. The price increment is then obtained by multiplying the random variable by the previous price, which gives $-\$3.433$. This generates a new value of $S_1 = \$100 - \$3.43 = \$96.57$. The process is repeated until the final price of \$125.31 is reached at the 100th step.

This experiment can be repeated as often as needed. Define K as the number of **replications**, or random trials. Figure 4.1 displays the first three trials. Each leads

TABLE 4.1 Simulating a Price Path

	Random Variable		Price	
Step i	Uniform u_i	Normal $\mu\Delta t + \sigma\Delta z$	Increment ΔS_i	Price S_{t+i}
0				100.00
1	0.0430	−0.0343	−3.433	96.57
2	0.8338	0.0194	1.872	98.44
3	0.6522	0.0078	0.771	99.21
4	0.9219	0.0284	2.813	102.02
...				
99				124.95
100	0.5563	0.0028	0.354	125.31

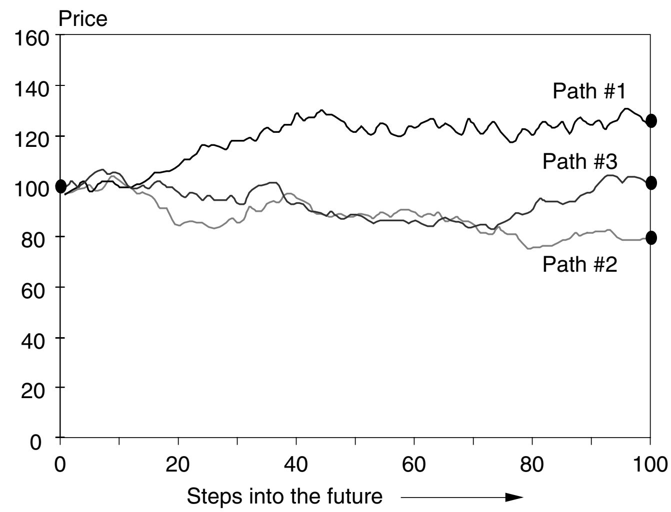

FIGURE 4.1 Simulating Price Paths

to a simulated final value S_T^k. This generates a distribution of simulated prices S_T. With just one step $n = 1$, the distribution must be normal. As the number of steps n grows large, the distribution tends to a lognormal distribution.

While very useful for modeling stock prices, this model has shortcomings. Price increments are assumed to have a normal distribution. In practice, we observe that price changes for most financial assets typically have fatter tails than the normal distribution. Returns may also experience changing variances.

In addition, as the time interval Δt shrinks, the volatility shrinks as well. This implies that large discontinuities cannot occur over short intervals. In reality, some assets experience discrete jumps, such as commodities, or securities issued by firms that go bankrupt. In such cases, the stochastic process should be changed to accommodate these observations.

EXAMPLE 4.1: FRM EXAM 2009—QUESTION 14

Suppose you simulate the price path of stock HHF using a geometric Brownian motion model with drift $\mu = 0$, volatility $\sigma = 0.14$, and time step $\Delta t = 0.01$. Let S_t be the price of the stock at time t. If $S_0 = 100$, and the first two simulated (randomly selected) standard normal variables are $\epsilon_1 = 0.263$ and $\epsilon_2 = -0.475$, what is the simulated stock price after the second step?

a. 96.79
b. 99.79
c. 99.97
d. 99.70

EXAMPLE 4.2: FRM EXAM 2003—QUESTION 40

In the geometric Brown motion process for a variable S,

I. S is normally distributed.
II. $d\ln(S)$ is normally distributed.
III. dS/S is normally distributed.
IV. S is lognormally distributed.

a. I only
b. II, III, and IV
c. IV only
d. III and IV

EXAMPLE 4.3: FRM EXAM 2002—QUESTION 126

Consider that a stock price S that follows a geometric Brownian motion $dS = aSdt + bSdz$, with b strictly positive. Which of the following statements is *false*?

a. If the drift a is positive, the price one year from now will be above today's price.
b. The instantaneous rate of return on the stock follows a normal distribution.
c. The stock price S follows a lognormal distribution.
d. This model does not impose mean reversion.

4.1.3 Drawing Random Variables

Most spreadsheets or statistical packages have functions that can generate uniform or standard normal random variables. This can be easily extended to distributions that better reflect the data (e.g., with fatter tails or nonzero skewness).

The methodology involves the inverse cumulative probability distribution function (p.d.f.). Take the normal distribution as an example. By definition, the cumulative p.d.f. $N(x)$ is always between 0 and 1. Because we have an analytical formula for this function, it can be easily inverted.

First, we generate a uniform random variable u drawn from $U(0, 1)$. Next, we compute x such that $u = N(x)$, or $x = N^{-1}(u)$. For example, set $u = 0.0430$, as in the first line of Table 4.1. This gives $x = -1.717$.[1] Because u is less than 0.5, we

[1] In Excel, a uniform random variable can be generated with the function u_i = RAND(). From this, a standard normal random variable can be computed with NORMSINV(u_i).

verify that x is negative. The variable can be transformed into any normal variable by multiplying by the standard deviation and adding the mean. More generally, any distribution function can be generated as long as the cumulative distribution function can be inverted.

4.1.4 Simulating Yields

The GBM process is widely used for stock prices and currencies. Fixed-income products are another matter, however.

Bond prices display long-term reversion to the face value, which represents the repayment of principal at maturity (assuming there is no default). Such a process is inconsistent with the GBM process, which displays no such mean reversion. The volatility of bond prices also changes in a predictable fashion, as duration shrinks to zero. Similarly, commodities often display mean reversion.

These features can be taken into account by modeling bond yields directly in a first step. In the next step, bond prices are constructed from the value of yields and a pricing function. The dynamics of interest rates r_t can be modeled by

$$\Delta r_t = \kappa(\theta - r_t)\Delta t + \sigma {r_t}^{\gamma} \Delta z_t \tag{4.8}$$

where Δz_t is the usual Wiener process. Here, we assume that $0 \leq \kappa < 1, \theta \geq 0$, $\sigma \geq 0$. Because there is only one stochastic variable for yields, the model is called a **one-factor model**.

This Markov process has a number of interesting features. First, it displays mean reversion to a long-run value of θ. The parameter κ governs the speed of mean reversion. When the current interest rate is high (i.e., $r_t > \theta$), the model creates a negative drift $\kappa(\theta - r_t)$ toward θ. Conversely, low current rates create a positive drift toward θ.

The second feature is the volatility process. This model includes the **Vasicek model** when $\gamma = 0$. Changes in yields are normally distributed because Δr is then a linear function of Δz, which is itself normal. The Vasicek model is particularly convenient because it leads to closed-form solutions for many fixed-income products. The problem, however, is that it could potentially lead to negative interest rates when the initial rate starts from a low value. This is because the volatility of the change in rates does not depend on the level, unlike that in the geometric Brownian motion.

Equation (4.8) is more general, however, because it includes a power of the yield in the variance function. With $\gamma = 1$, this is the **lognormal model**. Ignoring the trend, this gives $\Delta r_t = \sigma r_t \Delta z_t$, or $\Delta r_t / r_t = \sigma \Delta z_t$. This implies that the *rate of change* in the yield dr/r has a fixed variance. Thus, as with the GBM model, smaller yields lead to smaller movements, which makes it unlikely the yield will drop below zero. This model is more appropriate than the normal model when the initial yield is close to zero.

With $\gamma = 0.5$, this is the **Cox, Ingersoll, and Ross (CIR) model**. Ultimately, the choice of the exponent γ is an empirical issue. Recent research has shown that $\gamma = 0.5$ provides a good fit to the data.

This class of models is known as **equilibrium models**. They start with some assumptions about economic variables and imply a process for the short-term interest rate r. These models generate a predicted term structure, whose shape depends on the model parameters and the initial short rate. The problem with these models, however, is that they are not flexible enough to provide a good fit to today's term structure. This can be viewed as unsatisfactory, especially by practitioners who argue they cannot rely on a model that cannot be trusted to price today's bonds.

In contrast, **no-arbitrage models** are designed to be consistent with today's term structure. In this class of models, the term structure is an input into the parameter estimation. The earliest model of this type was the **Ho and Lee model**:

$$\Delta r_t = \theta(t)\Delta t + \sigma \Delta z_t \tag{4.9}$$

where $\theta(t)$ is a function of time chosen so that the model fits the initial term structure. This was extended to incorporate mean reversion in the **Hull and White model**:

$$\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma \Delta z_t \tag{4.10}$$

Finally, the **Heath, Jarrow, and Morton model** goes one step further and assumes that the volatility is a function of time.

The downside of these no-arbitrage models is that they do not impose any consistency between parameters estimated over different dates. The function $\theta(t)$ could be totally different from one day to the next, which is illogical. No-arbitrage models are also more sensitive to outliers, or data errors in bond prices used to fit the term structure.

4.1.5 Binomial Trees

Simulations are very useful to mimic the uncertainty in risk factors, especially with numerous risk factors. In some situations, however, it is also useful to describe the uncertainty in prices with discrete trees. When the price can take one of two steps, the tree is said to be **binomial**.

The binomial model can be viewed as a discrete equivalent to the geometric Brownian motion. As before, we subdivide the horizon T into n intervals $\Delta t = T/n$. At each node, the price is assumed to go either up with probability p or down with probability $1 - p$.

The parameters u, d, p are chosen so that, for a small time interval, the expected return and variance equal those of the continuous process. One could choose

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu \Delta t} - d}{u - d} \tag{4.11}$$

TABLE 4.2 Binomial Tree

Step						
0		1		2		3
						u^3S
					↗	
				u^2S		
			↗		↘	
		uS				u^2dS
	↗		↘		↗	
S				udS		
	↘		↗		↘	
		dS				d^2uS
			↘		↗	
				d^2S		
					↘	
						d^3S

This matches the mean, for example,

$$E[\frac{S_1}{S_0}] = pu + (1-p)d = \frac{e^{\mu\Delta t} - d}{u-d}u + \frac{u - e^{\mu\Delta t}}{u-d}d = \frac{e^{\mu\Delta t}(u-d) - du + ud}{u-d} = e^{\mu\Delta t}$$

Table 4.2 shows how a binomial tree is constructed. As the number of steps increases, the discrete distribution of S_T converges to the lognormal distribution. This model will be used in a later chapter to price options.

EXAMPLE 4.4: INTEREST RATE MODEL

The Vasicek model defines a risk-neutral process for r that is $dr = a(b - r)dt + \sigma dz$, where a, b, and σ are constant, and r represents the rate of interest. From this equation we can conclude that the model is a

a. Monte Carlo type model
b. Single-factor term-structure model
c. Two-factor term-structure model
d. Decision tree model

EXAMPLE 4.5: INTEREST RATE MODEL INTERPRETATION

The term $a(b - r)$ in the previous question represents which term?

a. Gamma
b. Stochastic
c. Reversion
d. Vega

EXAMPLE 4.6: FRM EXAM 2000—QUESTION 118

Which group of term-structure models do the Ho-Lee, Hull-White, and Heath-Jarrow-Morton models belong to?

a. No-arbitrage models
b. Two-factor models
c. Lognormal models
d. Deterministic models

EXAMPLE 4.7: FRM EXAM 2000—QUESTION 119

A plausible stochastic process for the short-term rate is often considered to be one where the rate is pulled back to some long-run average level. Which one of the following term-structure models does *not* include this characteristic?

a. The Vasicek model
b. The Ho-Lee model
c. The Hull-White model
d. The Cox-Ingersoll-Ross model

4.2 IMPLEMENTING SIMULATIONS

4.2.1 Simulation for VAR

Implementing Monte Carlo (MC) methods for risk management follows these steps:

1. Choose a stochastic process for the risk factor price S (i.e., its distribution and parameters, starting from the current value S_t).
2. Generate pseudo-random variables representing the risk factor at the target horizon, S_T.
3. Calculate the value of the portfolio at the horizon, $F_T(S_T)$.
4. Repeat steps 2 and 3 as many times as necessary. Call K the number of replications.

These steps create a distribution of values, $F_T^1, \ldots, F_T^K$, which can be sorted to derive the VAR. We measure the cth quantile $Q(F_T, c)$ and the average value $\text{Ave}(F_T)$. If VAR is defined as the deviation from the expected value on the target date, we have

$$\text{VAR}(c) = \text{Ave}(F_T) - Q(F_T, c) \tag{4.12}$$

4.2.2 Simulation for Derivatives

Readers familiar with derivatives pricing will have recognized that this method is similar to the Monte Carlo method for valuing derivatives. In that case, we simply focus on the expected value on the target date discounted into the present:

$$F_t = e^{-r(T-t)} \text{Ave}(F_T) \tag{4.13}$$

Thus derivatives valuation focuses on the discounted center of the distribution, while VAR focuses on the quantile on the target date.

Monte Carlo simulations have been long used to price derivatives. As will be seen in a later chapter, pricing derivatives can be done by assuming that the underlying asset grows at the risk-free rate r (assuming no income payment). For instance, pricing an option on a stock with expected return of 20% is done assuming that (1) the stock grows at the risk-free rate of 10% and (2) we discount at the same risk-free rate. This is called the **risk-neutral approach**.

In contrast, risk measurement deals with actual distributions, sometimes called **physical distributions**. For measuring VAR, the risk manager must simulate asset growth using the actual expected return μ of 20%. Therefore, risk management uses physical distributions, whereas pricing methods use risk-neutral distributions.

It should be noted that simulation methods are not applicable to all types of options. These methods assume that the value of the derivative instrument at expiration can be priced solely as a function of the end-of-period price S_T, and perhaps of its sample path. This is the case, for instance, with an Asian option, where the payoff is a function of the price *averaged* over the sample path. Such an option is said to be **path-dependent**.

Simulation methods, however, are inadequate to price American options, because such options can be exercised early. The optimal exercise decision, however, is complex to model because it should take into account *future* values of the option. This cannot be done with regular simulation methods, which only consider present and past information. Instead, valuing American options requires a **backward recursion**, for example with binomial trees. This method examines whether the option should be exercised, starting from the end and working backward in time until the starting time.

4.2.3 Accuracy

Finally, we should mention the effect of **sampling variability**. Unless K is extremely large, the empirical distribution of S_T will only be an approximation of the true distribution. There will be some natural variation in statistics measured from Monte Carlo simulations. Since Monte Carlo simulations involve *independent* draws, one can show that the standard error of statistics is inversely related to the square root of K. Thus more simulations will increase precision, but at a slow rate. For example, accuracy is increased by a factor of 10 going from $K = 10$ to $K = 1{,}000$, but then requires going from $K = 1{,}000$ to $K = 100{,}000$ for the same factor of 10.

This accuracy issue is worse for risk management than for pricing, because the quantiles are estimated less precisely than the average. For VAR measures, the precision is also a function of the selected confidence level. Higher confidence levels generate fewer observations in the left tail and hence less precise VAR measures. A 99% VAR using 1,000 replications should be expected to have only 10 observations in the left tail, which is not a large number. The VAR estimate is derived from the 10th and 11th sorted numbers. In contrast, a 95% VAR is measured from the 15th and 51st sorted numbers, which is more precise. In addition, the precision of the estimated quantile depends on the shape of the distribution. Relative to a symmetric distribution, a short option position has negative skewness, or a long left tail. The observations in the left tail therefore will be more dispersed, making is more difficult to estimate VAR precisely.

Various methods are available to speed up convergence:

- **Antithetic variable technique.** This technique uses twice the same sequence of random draws from t to T. It takes the original sequence and changes the sign of all their values. This creates twice the number of points in the final distribution of F_T without running twice the number of simulations.
- **Control variate technique.** This technique is used to price options with trees when a similar option has an analytical solution. Say that f_E is a European option with an analytical solution. Going through the tree yields the values of an American option and a European option, F_A and F_E. We then assume that the error in F_A is the same as that in F_E, which is known. The adjusted value is $F_A - (F_E - f_E)$.
- **Quasi-random sequences.** These techniques, also called quasi Monte Carlo (QMC), create draws that are not independent but instead are designed to fill the sample space more uniformly. Simulations have shown that QMC methods converge faster than Monte Carlo methods. In other words, for a fixed number of replications K, QMC values will be on average closer to the true value.

  The advantage of traditional MC, however, is that it also provides a standard error, which is on the order of $1/\sqrt{K}$ because draws are independent. So, we have an idea of how far the estimate might be from the true value, which is useful in deciding on the number of replications. In contrast, QMC methods give no measure of precision.

EXAMPLE 4.8: FRM EXAM 2005—QUESTION 67

Which one of the following statements about Monte Carlo simulation is *false*?

a. Monte Carlo simulation can be used with a lognormal distribution.
b. Monte Carlo simulation can generate distributions for portfolios that contain only linear positions.
c. One drawback of Monte Carlo simulation is that it is computationally very intensive.
d. Assuming the underlying process is normal, the standard error resulting from Monte Carlo simulation is inversely related to the square root of the number of trials.

EXAMPLE 4.9: FRM EXAM 2007—QUESTION 66

A risk manager has been requested to provide some indication of the accuracy of a Monte Carlo simulation. Using 1,000 replications of a normally distributed variable S, the relative error in the one-day 99% VAR is 5%. Under these conditions,

a. Using 1,000 replications of a long option position on S should create a larger relative error.
b. Using 10,000 replications should create a larger relative error.
c. Using another set of 1,000 replications will create an exact measure of 5.0% for relative error.
d. Using 1,000 replications of a short option position on S should create a larger relative error.

EXAMPLE 4.10: SAMPLING VARIATION

The measurement error in VAR, due to sampling variation, should be greater with

a. More observations and a high confidence level (e.g., 99%)
b. Fewer observations and a high confidence level
c. More observations and a low confidence level (e.g., 95%)
d. Fewer observations and a low confidence level

4.3 MULTIPLE SOURCES OF RISK

We now turn to the more general case of simulations with many sources of financial risk. Define N as the number of risk factors. If the factors S_j are independent, the randomization can be performed independently for each variable. For the GBM model,

$$\Delta S_{j,t} = S_{j,t-1}\mu_j \Delta t + S_{j,t-1}\sigma_j \epsilon_{j,t}\sqrt{\Delta t} \tag{4.14}$$

where the standard normal variables ϵ are independent across time and factor $j = 1, \ldots, N$.

In general, however, risk factors are correlated. The simulation can be adapted by, first, drawing a set of independent variables η, and, second, transforming them into correlated variables ϵ. As an example, with two factors only, we write

$$\begin{aligned} \epsilon_1 &= \eta_1 \\ \epsilon_2 &= \rho\eta_1 + (1-\rho^2)^{1/2}\eta_2 \end{aligned} \tag{4.15}$$

Here, ρ is the correlation coefficient between the variables ϵ. Because the η's have unit variance and are uncorrelated, we verify that the variance of ϵ_2 is one, as required

$$V(\epsilon_2) = \rho^2 V(\eta_1) + [(1-\rho^2)^{1/2}]^2 V(\eta_2) = \rho^2 + (1-\rho^2) = 1$$

Furthermore, the correlation between ϵ_1 and ϵ_2 is given by

$$\text{Cov}(\epsilon_1, \epsilon_2) = \text{Cov}(\eta_1, \rho\eta_1 + (1-\rho^2)^{1/2}\eta_2) = \rho\,\text{Cov}(\eta_1, \eta_1) = \rho$$

Defining ϵ as the *vector* of values, we verified that the covariance matrix of ϵ is

$$V(\epsilon) = \begin{bmatrix} \sigma^2(\epsilon_1) & \text{Cov}(\epsilon_1, \epsilon_2) \\ \text{Cov}(\epsilon_1, \epsilon_2) & \sigma^2(\epsilon_2) \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = R$$

Note that this covariance matrix, which is the expectation of squared deviations from the mean, can also be written as

$$V(\epsilon) = E[(\epsilon - E(\epsilon)) \times (\epsilon - E(\epsilon))'] = E(\epsilon \times \epsilon')$$

because the expectation of ϵ is zero. To generalize this approach to many more risk factors, however, we need a systematic way to derive the transformation in Equation (4.15).

4.3.1 The Cholesky Factorization

We would like to generate N joint values of ϵ that display the correlation structure $V(\epsilon) = E(\epsilon\epsilon') = R$. Because the matrix R is a symmetric real matrix, it can be decomposed into its so-called Cholesky factors:

$$R = TT' \tag{4.16}$$

where T is a lower triangular matrix with zeros on the upper right corners (above the diagonal). This is known as the **Cholesky factorization**.

As in the previous section, we first generate a vector of independent η. Thus, their covariance matrix is $V(\eta) = I$, where I is the identity matrix with zeros everywhere except for the diagonal.

We then construct the transformed variable $\epsilon = T\eta$. The covariance matrix is now $V(\epsilon) = E(\epsilon\epsilon') = E((T\eta)(T\eta)') = E(T\eta\eta'T') = TE(\eta\eta')T' = TV(\eta)T' = TIT' = TT' = R$. This transformation therefore generates ϵ variables with the desired correlations.

To illustrate, let us go back to our two-variable case. The correlation matrix can be decomposed into its Cholesky factors:

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} \\ a_{21}a_{11} & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

To find the entries a_{11}, a_{21}, a_{22}, we solve each of the three equations

$$a_{11}^2 = 1$$
$$a_{11}a_{21} = \rho$$
$$a_{21}^2 + a_{22}^2 = 1$$

This gives $a_{11} = 1, a_{21} = \rho$, and $a_{22} = (1 - \rho^2)^{1/2}$. The Cholesky factorization is then

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & (1-\rho^2)^{1/2} \end{bmatrix}$$

Note that this conforms to Equation (4.15):

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

In practice, this decomposition yields a number of useful insights. The decomposition will fail if the number of independent factors implied in the correlation matrix is less than N. For instance, if $\rho = 1$, the two assets are perfectly correlated, meaning that we have twice the same factor. This could be, for instance, the case of two currencies fixed to each other. The decomposition gives $a_{11} = 1, a_{21} = 1, a_{22} = 0$. The new variables are then $\epsilon_1 = \eta_1$ and $\epsilon_2 = \eta_1$. In this case, the second variable η_2 is totally superfluous and simulations can be performed with one variable only.

4.3.2 The Curse of Dimensionality

Modern risk management is about measuring the risk of large portfolios, typically exposed to a large number of risk factors. The problem is that the number of computations increases geometrically with the number of factors N. The covariance matrix, for instance, has dimensions $N(N+1)/2$. A portfolio with 500 variables requires a matrix with 125,250 entries.

In practice, the risk manager should simplify the number of risk factors, discarding those that do not contribute significantly to the risk of the portfolio. Simulations based on the full set of variables would be inordinately time-consuming. The art of simulation is to design parsimonious experiments that represent the breadth of movements in risk factors.

This can be done by an economic analysis of the risk factors and portfolio strategies, as done in Part Three of this handbook. Alternatively, the risk manager can perform a statistical decomposition of the covariance matrix. A widely used method for this is the **principal component analysis** (PCA), which finds linear combinations of the risk factors that have maximal explanatory power. This type of analysis, which is as much an art as it is a science, can be used to reduce the dimensionality of the risk space.

EXAMPLE 4.11: FRM EXAM 2007—QUESTION 28

Let N be a $1 \times n$ vector of independent draws from a standard normal distribution, and let V be a covariance matrix of market time-series data. Then, if L is a diagonal matrix of the eigenvalues of V, E is a matrix of the eigenvectors of V, and $C'C$ is the Cholesky factorization of V, which of the following would generate a normally distributed random vector with mean zero and covariance matrix V to be used in a Monte Carlo simulation?

a. $NC'CN'$
b. NC'
c. $E'LE$
d. Cannot be determined from data given

EXAMPLE 4.12: FRM EXAM 2006—QUESTION 82

Consider a stock that pays no dividends, has a volatility of 25% pa, and has an expected return of 13% pa. The current stock price is $S_0 = \$30$. This implies the model $S_{t+1} = S_t(1 + 0.13\Delta t + 0.25\sqrt{\Delta t}\,\epsilon)$, where ϵ is a standard normal random variable. To implement this simulation, you generate a path of the stock price by starting at $t = 0$, generating a sample for ϵ, updating the stock price according to the model, incrementing t by 1, and repeating this process until the end of the horizon is reached. Which of the following strategies for generating a sample for ϵ will implement this simulation properly?

a. Generate a sample for ϵ by using the inverse of the standard normal cumulative distribution of a sample value drawn from a uniform distribution between 0 and 1.
b. Generate a sample for ϵ by sampling from a normal distribution with mean 0.13 and standard deviation 0.25.
c. Generate a sample for ϵ by using the inverse of the standard normal cumulative distribution of a sample value drawn from a uniform distribution between 0 and 1. Use Cholesky decomposition to correlate this sample with the sample from the previous time interval.
d. Generate a sample for ϵ by sampling from a normal distribution with mean 0.13 and standard deviation 0.25. Use Cholesky decomposition to correlate this sample with the sample from the previous time interval.

EXAMPLE 4.13: FRM EXAM 2006—QUESTION 83

Continuing with the previous question, you have implemented the simulation process discussed earlier using a time interval $\Delta t = 0.001$, and you are analyzing the following stock price path generated by your implementation.

t	S_{t-1}	ϵ	ΔS
0	30.00	0.0930	0.03
1	30.03	0.8493	0.21
2	30.23	0.9617	0.23
3	30.47	0.2460	0.06
4	30.53	0.4769	0.12
5	30.65	0.7141	0.18

Given this sample, which of the following simulation steps most likely contains an error?

a. Calculation to update the stock price
b. Generation of random sample value for ϵ
c. Calculation of the change in stock price during each period
d. None of the above

4.4 IMPORTANT FORMULAS

Wiener process: $\Delta z \sim N(0, \Delta t)$

Generalized Wiener process: $\Delta x = a\Delta t + b\Delta z$

Ito process: $\Delta x = a(x, t)\Delta t + b(x, t)\Delta z$

Geometric Brownian motion: $\Delta S = \mu S\Delta t + \sigma S\Delta z$

One-factor equilibrium model for yields: $\Delta r_t = \kappa(\theta - r_t)\Delta t + \sigma {r_t}^{\gamma}\Delta z_t$

Vasicek model, $\gamma = 0$

Lognormal model, $\gamma = 1$

CIR model, $\gamma = 0.5$

No-arbitrage models:

Ho and Lee model, $\Delta r_t = \theta(t)\Delta t + \sigma\Delta z_t$

Hull and White model, $\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma\Delta z_t$

Heath, Jarrow, and Morton model, $\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma(t)\Delta z_t$

Binomial trees: $u = e^{\sigma\sqrt{\Delta t}}$, $\quad d = (1/u)$, $\; p = \frac{e^{\mu\Delta t} - d}{u-d}$

Cholesky factorization: $R = TT'$, $\epsilon = T\eta$

4.5 ANSWERS TO CHAPTER EXAMPLES

Example 4.1: FRM Exam 2009—Question 14

d. The process for the stock prices has mean of zero and volatility of $\sigma\sqrt{\Delta t} = 0.14\sqrt{0.01} = 0.014$. Hence the first step is $S_1 = S_0(1 + 0.014 \times 0.263) = 100.37$. The second step is $S_2 = S_1(1 + 0.014 \times -0.475) = 99.70$.

Example 4.2: FRM Exam 2003—Question 40

b. Both dS/S or $d\ln(S)$ are normally distributed. As a result, S is lognormally distributed. The only incorrect answer is I.

Example 4.3: FRM Exam 2002—Question 126

a. All the statements are correct except a., which is too strong. The expected price is higher than today's price but certainly not the price in all states of the world.

Example 4.4: Interest Rate Model

b. This model postulates only one source of risk in the fixed-income market. This is a single-factor term-structure model.

Example 4.5: Interest Rate Model Interpretation

c. This represents the expected return with mean reversion.

Example 4.6: FRM Exam 2000—Question 118

a. These are no-arbitrage models of the term structure, implemented as either one-factor or two-factor models.

Example 4.7: FRM Exam 2000—Question 119

b. Both the Vasicek and CIR models are one-factor equilibrium models with mean reversion. The Hull-White model is a no-arbitrage model with mean reversion. The Ho-Lee model is an early no-arbitrage model without mean reversion.

Example 4.8: FRM Exam 2005—Question 67

b. MC simulations do account for options. The first step is to simulate the process of the risk factor. The second step prices the option, which properly accounts for nonlinearity.

Example 4.9: FRM Exam 2007—Question 66

d. Short option positions have long left tails, which makes it more difficult to estimate a left-tailed quantile precisely. Accuracy with independent draws increases with the square root of K. Thus increasing the number of replications should shrink the standard error, so answer b. is incorrect.

Example 4.10: Sampling Variation

b. Sampling variability (or imprecision) increases with (1) fewer observations and (2) greater confidence levels. To show (1), we can refer to the formula for the precision of the sample mean, which varies inversely with the square root of the number of data points. A similar reasoning applies to (2). A greater confidence level involves fewer observations in the left tails, from which VAR is computed.

Example 4.11: FRM Exam 2007—Question 28

b. In the notation of the text, N is the vector of i.i.d. random variables η and $C'C = TT'$. The transformed variable is $T\eta$, or $C'N$, or its transpose.

Example 4.12: FRM Exam 2006—Question 82

a. The variable ϵ should have a standard normal distribution (i.e., with mean zero and unit standard deviation). Answer b. is incorrect because ϵ is transformed afterward to the desired mean and standard deviation. The Cholesky decomposition is not applied here because the sequence of random variables has no serial correlation.

Example 4.13: FRM Exam 2006—Question 83

b. The random variable ϵ should have a standard normal distribution, which means that it should have negative as well as positive values, which should average close to zero. This is not the case here. This is probably a uniform variable instead.

CHAPTER 5

Modeling Risk Factors

We now turn to an analysis of the distribution of risk factors used in financial risk management. A common practice is to use the volatility as a single measure of dispersion. More generally, risk managers need to consider the entire shape of the distribution as well as potential variation in time of this distribution.

The normal distribution is a useful starting point due to its attractive properties. Unfortunately, most financial time series are characterized by fatter tails than the normal distribution. In addition, there is ample empirical evidence that risk changes in a predictable fashion. This phenomenon, called **volatility clustering**, could also explain the appearance of fat tails. Extreme observations could be drawn from periods with high volatility. This could cause the appearance of fat tails when combining periods of low and high volatility.

Section 5.1 discusses the sampling of real data and the construction of returns. It shows how returns can be aggregated across time or, for a portfolio, across assets. Section 5.2 then describes the normal and lognormal distributions and explains why these choices are so popular, whereas Section 5.3 discusses distributions that have fatter tails than the normal distribution.

Section 5.4 then turns to time variation in risk. We describe the generalized autoregressive conditional heteroskedastic (GARCH) model and a special case, which is RiskMetrics' exponentially weighted moving average (EWMA). These models place more weight on more recent data and have proved successful in explaining volatility clustering. They should be part of the tool kit of risk managers.

5.1 REAL DATA

5.1.1 Measuring Returns

To start with an example, let us say that we observe movements in the daily yen/dollar exchange rate and wish to characterize the distribution of tomorrow's exchange rate. The risk manager's job is to assess the range of potential gains and losses on a trader's position. He or she observes a sequence of past prices $P_0, P_1, \ldots, P_t$, from which the distribution of tomorrow's price, P_{t+1}, should be inferred.

FRM Exam Part 1 topic.

The truly random component in tomorrow's price is not its level, but rather its change relative to today's price. We measure the *relative rate of change* in the spot price:

$$r_t = (P_t - P_{t-1})/P_{t-1} \tag{5.1}$$

Alternatively, we could construct the logarithm of the price ratio:

$$R_t = \ln[P_t/P_{t-1}] \tag{5.2}$$

which is equivalent to using continuous instead of discrete compounding. This is also

$$R_t = \ln[1 + (P_t - P_{t-1})/P_{t-1}] = \ln[1 + r_t]$$

Because $\ln(1 + x)$ is close to x if x is small, R_t should be close to r_t provided the return is small. For daily data, there is typically little difference between R_t and r_t.

The next question is whether the sequence of variables r_t can be viewed as independent observations. Independent observations have the very nice property that their joint distribution is the product of their marginal distribution, which considerably simplifies the analysis. The obvious question is whether this assumption is a workable approximation. In fact, there are good economic reasons to believe that rates of change on financial prices are close to independent.

The hypothesis of **efficient markets** postulates that current prices convey all relevant information about the asset. If so, any change in the asset price must be due to news, or events that are by definition impossible to forecast (otherwise, the event would not be news). This implies that changes in prices are unpredictable and, hence, satisfy our definition of independent random variables.

This hypothesis, also known as the **random walk** theory, implies that the conditional distribution of returns depends on only current prices, and not on the previous history of prices. If so, technical analysis must be a fruitless exercise. Technical analysts try to forecast price movements from past price patterns. If in addition the distribution of returns is constant over time, the variables are said to be **independent and identically distributed** (i.i.d.).

5.1.2 Time Aggregation

It is often necessary to translate parameters over a given horizon to another horizon. For example, we have data for daily returns, from which we compute a daily volatility that we want to extend to a monthly volatility. This is a **time aggregation** problem.

Returns can be easily aggregated when we use the log of the price ratio, because the log of a product is the sum of the logs of the individual terms. Over two periods, for instance, the price movement can be described as the sum of the price movements over each day:

$$R_{t,2} = \ln(P_t/P_{t-2}) = \ln(P_t/P_{t-1}) + \ln(P_{t-1}/P_{t-2}) = R_{t-1} + R_t \tag{5.3}$$

The expected return and variance are then $E(R_{t,2}) = E(R_{t-1}) + E(R_t)$ and $V(R_{t,2}) = V(R_{t-1}) + V(R_t) + 2\text{Cov}(R_{t-1}, R_t)$. Assuming returns are uncorrelated (i.e., that the covariance term is zero) and have identical distributions across days, we have $E(R_{t,2}) = 2E(R_t)$ and $V(R_{t,2}) = 2V(R_t)$.

More generally, define T as the number of steps. The multiple-period expected return and volatility are

$$\mu_T = \mu T \tag{5.4}$$

$$\sigma_T = \sigma\sqrt{T} \tag{5.5}$$

KEY CONCEPT

When successive returns are uncorrelated, the volatility increases as the horizon extends following the square root of time.

Assume now that the distribution is stable under addition, which means that it stays the same whether over one period or over multiple periods. This is the case for the normal distribution. If so, we can use the same multiplier α that corresponds to a selected confidence level for a one-period and T-period return. The multiple-period VAR is

$$\text{VAR}_T = \alpha(\sigma\sqrt{T})W = \text{VAR}_1\sqrt{T} \tag{5.6}$$

In other words, extension to a multiple period follows a square root of time rule. Figure 5.1 shows how VAR grows with the length of the horizon for various confidence levels. The figure shows that VAR increases more slowly than time. The one-month 99% VAR is 0.67, but increases to only 2.33 at a one-year horizon.

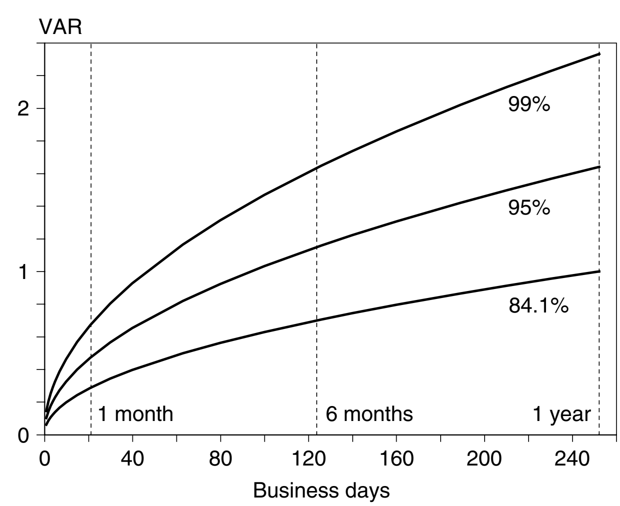

FIGURE 5.1 VAR at Increasing Horizons

In summary, the square root of time rule applies to parametric VAR under the following conditions:

- The distribution is the same at each period (i.e., there is no predictable time variation in expected return nor in risk).
- Returns are uncorrelated across each period.
- The distribution is the same for one- or T-period, or is stable under addition, such as the normal distribution.

If returns are not independent, we may be able to characterize longer-term risks. For instance, when returns follow a first-order autoregressive process,

$$R_t = \rho R_{t-1} + u_t \tag{5.7}$$

we can write the variance of two-day returns as

$$V[R_t + R_{t-1}] = V[R_t] + V[R_{t-1}] + 2\text{Cov}[R_t, R_{t-1}] = \sigma^2 + \sigma^2 + 2\rho\sigma^2 \tag{5.8}$$

or

$$V[R_t + R_{t-1}] = \sigma^2 \times 2[1 + \rho] \tag{5.9}$$

In this case,

$$\text{VAR}_2 = \alpha(\sigma\sqrt{2(1+\rho)})W = [\text{VAR}_1\sqrt{2}]\sqrt{(1+\rho)} \tag{5.10}$$

Because we are considering correlations in the time series of the same variable, ρ is called the **autocorrelation coefficient**, or the **serial autocorrelation coefficient**. A positive value for ρ describes a situation where a movement in one direction is likely to be followed by another in the same direction. This implies that markets display **trends**, or **momentum**. In this case, the longer-term volatility increases faster than with the usual square root of time rule.

A negative value for ρ, by contrast, describes a situation where a movement in one direction is likely to be reversed later. This is an example of **mean reversion**. In this case, the longer-term volatility increases more slowly than with the usual square root of time rule.

EXAMPLE 5.1: TIME SCALING

Consider a portfolio with a one-day VAR of $1 million. Assume that the market is trending with an autocorrelation of 0.1. Under this scenario, what would you expect the two-day VAR to be?

a. $2 million
b. $1.414 million
c. $1.483 million
d. $1.449 million

EXAMPLE 5.2: INDEPENDENCE

A fundamental assumption of the random walk hypothesis of market returns is that returns from one time period to the next are statistically independent. This assumption implies

a. Returns from one time period to the next can never be equal.
b. Returns from one time period to the next are uncorrelated.
c. Knowledge of the returns from one time period does not help in predicting returns from the next time period.
d. Both b. and c. are true.

EXAMPLE 5.3: FRM EXAM 2002—QUESTION 3

Consider a stock with daily returns that follow a random walk. The annualized volatility is 34%. Estimate the weekly volatility of this stock assuming that the year has 52 weeks.

a. 6.80%
b. 5.83%
c. 4.85%
d. 4.71%

EXAMPLE 5.4: FRM EXAM 2002—QUESTION 2

Assume we calculate a one-week VAR for a natural gas position by rescaling the daily VAR using the square root of time rule. Let us now assume that we determine the *true* gas price process to be mean reverting and recalculate the VAR. Which of the following statements is true?

a. The recalculated VAR will be less than the original VAR.
b. The recalculated VAR will be equal to the original VAR.
c. The recalculated VAR will be greater than the original VAR.
d. There is no necessary relationship between the recalculated VAR and the original VAR.

5.1.3 Portfolio Aggregation

Let us now turn to aggregation of returns across assets. Consider, for example, an equity portfolio consisting of investments in N shares. Define the number of each share held as q_i with unit price S_i. The portfolio value at time t is then

$$W_t = \sum_{i=1}^{N} q_i S_{i,t} \tag{5.11}$$

We can write the weight assigned to asset i as

$$w_{i,t} = \frac{q_i S_{i,t}}{W_t} \tag{5.12}$$

which by construction sum to unity. Using weights, however, rules out situations with zero net investment, $W_t = 0$, such as some derivatives positions. But we could have positive and negative weights if short selling is allowed, or weights greater than one if the portfolio can be leveraged.

The next period, the portfolio value is

$$W_{t+1} = \sum_{i=1}^{N} q_i S_{i,t+1} \tag{5.13}$$

assuming that the unit price incorporates any income payment. The gross, or dollar, return is then

$$W_{t+1} - W_t = \sum_{i=1}^{N} q_i (S_{i,t+1} - S_{i,t}) \tag{5.14}$$

and the *rate* of return is

$$\frac{W_{t+1} - W_t}{W_t} = \sum_{i=1}^{N} \frac{q_i S_{i,t}}{W_t} \frac{(S_{i,t+1} - S_{i,t})}{S_{i,t}} = \sum_{i=1}^{N} w_{i,t} \frac{(S_{i,t+1} - S_{i,t})}{S_{i,t}} \tag{5.15}$$

So, the portfolio rate of return is a linear combination of the asset returns

$$r_{p,t+1} = \sum_{i=1}^{N} w_{i,t}\, r_{i,t+1} \tag{5.16}$$

The dollar return is then

$$W_{t+1} - W_t = \left[\sum_{i=1}^{N} w_{i,t} r_{i,t+1} \right] W_t \tag{5.17}$$

and has a normal distribution if the individual returns are also normally distributed.

Alternatively, we could express the individual positions in dollar terms,

$$x_{i,t} = w_{i,t} W_t = q_i S_{i,t} \tag{5.18}$$

The dollar return is also, using dollar amounts,

$$W_{t+1} - W_t = \left[\sum_{i=1}^{N} x_{i,t} r_{i,t+1} \right] \tag{5.19}$$

As we have seen in the previous chapter, the variance of the portfolio dollar return is

$$V[W_{t+1} - W_t] = x' \Sigma x \tag{5.20}$$

Because the portfolio follows a normal distribution, it is fully characterized by its expected return and variance. The portfolio VAR is then

$$\text{VAR} = \alpha \sqrt{x' \Sigma x} \tag{5.21}$$

where α depends on the confidence level and the selected density function.

EXAMPLE 5.5: FRM EXAM 2004—QUESTION 39

Consider a portfolio with 40% invested in asset X and 60% invested in asset Y. The mean and variance of return on X are 0 and 25 respectively. The mean and variance of return on Y are 1 and 121 respectively. The correlation coefficient between X and Y is 0.3. What is the nearest value for portfolio volatility?

a. 9.51
b. 8.60
c. 13.38
d. 7.45

5.2 NORMAL AND LOGNORMAL DISTRIBUTIONS

5.2.1 Why the Normal?

The normal, or Gaussian, distribution is usually the first choice when modeling asset returns. This distribution plays a special role in statistics, as it is easy to handle and is stable under addition, meaning that a combination of jointly normal variables is itself normal. It also provides the limiting distribution of the average of *independent* random variables (through the central limit theorem).

Empirically, the normal distribution provides a rough, first-order approximation to the distribution of many random variables: rates of changes in currency prices, rates of changes in stock prices, rates of changes in bond prices, changes in yields, and rates of changes in commodity prices. All of these are characterized by many occurrences of small moves and fewer occurrences of large moves. This provides a rationale for a distribution with more weight in the center, such as the bell-shaped normal distribution. For many applications, this is a sufficient approximation. This may not be appropriate for measuring tail risk, however.

5.2.2 Computing Returns

In what follows, the random variable is the new price P_1, given the current price P_0. Defining $r = (P_1 - P_0)/P_0$ as the rate of return in the price, we can start with the assumption that this random variable is drawn from a normal distribution,

$$r \sim \Phi(\mu, \sigma) \tag{5.22}$$

with some mean μ and standard deviation σ. Turning to prices, we have $P_1 = P_0(1 + r)$ and

$$P_1 \sim P_0 + \Phi(P_0\mu, P_0\sigma) \tag{5.23}$$

For instance, starting from a stock price of \$100, if $\mu = 0\%$ and $\sigma = 15\%$, we have $P_1 \sim \$100 + \Phi(\$0, \$15)$.

In this case, however, the normal distribution cannot be even theoretically correct. Because of limited liability, stock prices cannot go below zero. Similarly, commodity prices and yields cannot turn negative. This is why another popular distribution is the **lognormal distribution**, which is such that

$$R = \ln(P_1/P_0) \sim \Phi(\mu, \sigma) \tag{5.24}$$

By taking the logarithm, the price is given by $P_1 = P_0 \exp(R)$, which precludes prices from turning negative, as the exponential function is always positive. Figure 5.2 compares the normal and lognormal distributions over a one-year horizon with $\sigma = 15\%$ annually. The distributions are very similar, except for the tails. The lognormal is skewed to the right.

The difference between the two distributions is driven by the size of the volatility parameter over the horizon. Small values of this parameter imply that the distributions are virtually identical. This can happen either when the asset is not very risky, that is, when the annual volatility is small, or when the horizon is very short. In this situation, there is very little chance of prices turning negative. The limited liability constraint is not important.

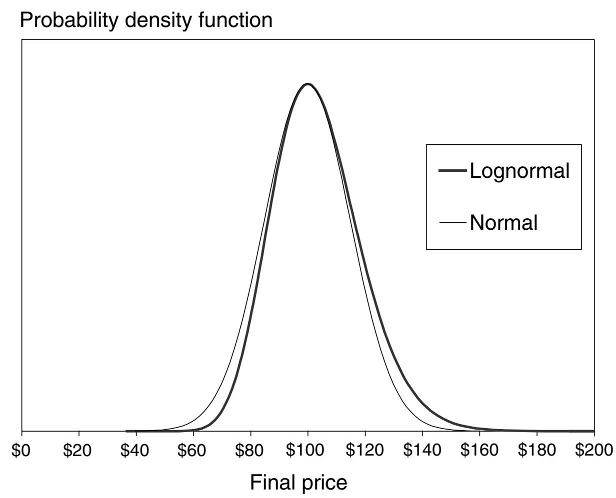

FIGURE 5.2 Normal and Lognormal Distributions—Annual Horizon

TABLE 5.1 Comparison between Discrete and Log Returns

	Daily	Annual
Initial price	100	100
Ending price	101	115
Discrete return	1.0000	15.0000
Log return	0.9950	13.9762
Relative difference	0.50%	7.33%

KEY CONCEPT

The normal and lognormal distributions are very similar for short horizons or low volatilities.

As an example, Table 5.1 compares the computation of returns over a one-day horizon and a one-year horizon. The one-day returns are 1.000% and 0.995% for discrete and log returns, respectively, which translates into a relative difference of 0.5%, which is minor. In contrast, the difference is more significant over longer horizons.

5.3 DISTRIBUTIONS WITH FAT TAILS

Perhaps the most serious problem with the normal distribution is the fact that its tails disappear too fast, at least faster than what is empirically observed in financial data. We typically observe that every market experiences one or more daily moves of four standard deviations or more per year. Such frequency is incompatible with a normal distribution. With a normal distribution, the probability of this happening is 0.0032% for one day, which implies a frequency of once every 125 years.

KEY CONCEPT

Every financial market experiences one or more daily price moves of four standard deviations or more each year. And in any year, there is usually at least one market that has a daily move greater than 10 standard deviations.

This empirical observation can be explained in a number of ways: (1) The true distribution has fatter tails (e.g., the Student's *t*), or (2) the observations are drawn from a mix of distributions (e.g., a mix of two normals, one with low risk, the other with high risk), or (3) the distribution is nonstationary.

The first explanation is certainly a possibility. Figure 5.3 displays the density function of the normal and Student's *t* distribution, with 4 and 6 degrees of freedom (df). The Student's *t* density has fatter tails, which better reflect the occurrences of extreme observations in empirical financial data.

The distributions are further compared in Table 5.2. The left-side panel reports the tail probability of an observation lower than the deviate. For instance, the probability of observing a draw less than −3 is 0.001, or 0.1% for the normal, 0.012 for the Student's *t* with 6 degrees of freedom, and 0.020 for the Student's *t* with 4 degrees of freedom. There is a greater probability of observing an extreme move when the data is drawn from a Student's *t* rather than from a normal distribution.

We can transform these into an expected number of occurrences in one year, or 250 business days. The right-side panel shows that the corresponding numbers are 0.34, 3.00, and 4.99 for the respective distributions. In other words, with a

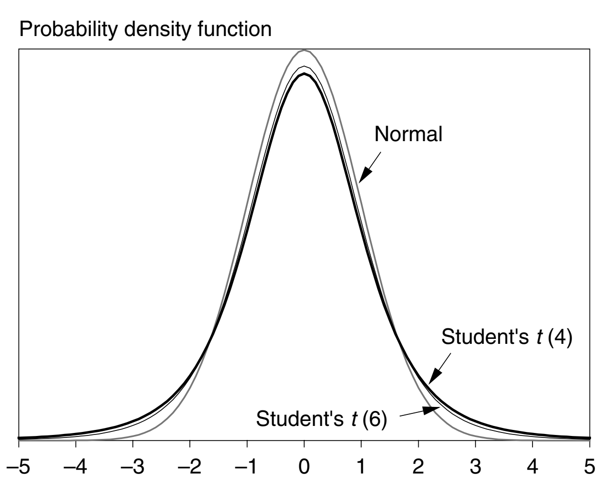

FIGURE 5.3 Normal and Student's *t* Distributions

TABLE 5.2 Comparison of the Normal and Student's t Distributions

	Tail Probability			Expected Number in 250 Days		
Deviate	Normal	t df = 6	t df = 4	Normal	t df = 6	t df = 4
−5	0.00000	0.00123	0.00375	0.00	0.31	0.94
−4	0.00003	0.00356	0.00807	0.01	0.89	2.02
−3	0.00135	0.01200	0.01997	0.34	3.00	4.99
−2	0.02275	0.04621	0.05806	5.69	11.55	14.51
−1	0.15866	0.17796	0.18695	39.66	44.49	46.74
				Deviate (Alpha)		
Probability = 1%				2.33	3.14	3.75
Ratio to normal				1.00	1.35	1.61

normal distribution, we should expect that this extreme movement below $z = -3$ will occur one day or less on average. With a Student's t with df = 4, the expected number is five in a year, which is closer to reality.

The bottom panel reports the deviate that corresponds to a 99% right-tail confidence level, or 1% left tail. For the normal distribution, this is the usual 2.33. For the Student's t with df = 4, α is 3.75, much higher. The ratio of the two is 1.61. Thus a rule of thumb would be to correct the VAR measure from a normal distribution by a ratio of 1.61 to achieve the desired coverage in the presence of fat tails. More generally, this explains why safety factors are used to multiply VAR measures, such as the Basel multiplicative factor of 3.

5.4 TIME VARIATION IN RISK

Fat tails can also occur when risk factors are drawn from a distribution with time-varying volatility. To be practical, this time variation must have some predictability.

5.4.1 Moving Average

Consider a traditional problem where a risk manager observes a sequence of T returns r_t, from which the variance must be estimated. To simplify, ignore the mean return. At time t, the traditional variance estimate is

$$\sigma_t^2 = (1/T)\sum_{i=1}^{T} r_{t-i}^2 \tag{5.25}$$

This is a simple average where the weight on each past observation is $w_i = 1/T$. This may not be the best use of the data, however, especially if more recent observations are more relevant for the next day.

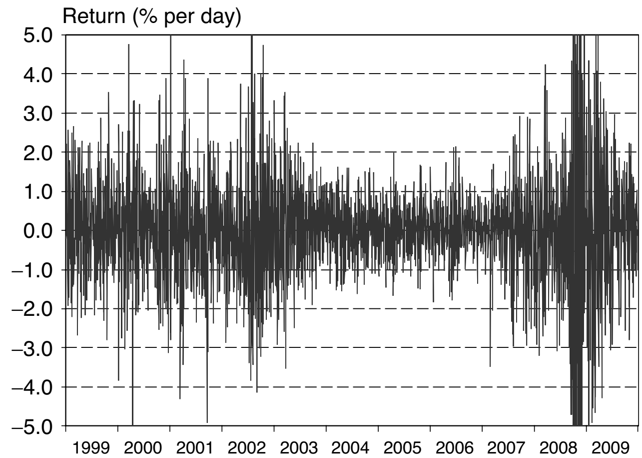

FIGURE 5.4 Daily Return for U.S. Equities

This is illustrated in Figure 5.4, which plots daily returns on the S&P 500 index. We observe **clustering** in volatility. Some periods are particularly hectic. After the Lehman bankruptcy in September 2008, there was a marked increase in the number of large returns, both positive and negative. Other periods, such as 2004 to 2006, were much more quiet. Simply taking the average over the entire period will underestimate risk during 2008 and overestimate risk during 2004 to 2006.

5.4.2 GARCH

A practical model for volatility clustering is the **generalized autoregressive conditional heteroskedastic (GARCH)** model developed by Engle (1982) and Bollerslev (1986). This class of models assumes that the return at time t has a particular distribution such as the normal, conditional on parameters μ_t and σ_t:

$$r_t \sim \Phi(\mu_t, \sigma_t) \tag{5.26}$$

The important point is that σ is indexed by time. We define the **conditional variance** as that conditional on current information. This may differ from the **unconditional variance**, which is the same for the whole sample. Thus the average variance is unconditional, whereas a time-varying variance is conditional.

There is substantial empirical evidence that conditional volatility models successfully forecast risk. The general assumption is that the conditional returns have a normal distribution, although this could be extended to other distributions such as the Student's t.

The GARCH model assumes that the conditional variance depends on the latest innovation, and on the previous conditional variance. Define $h_t = \sigma_t^2$ as the conditional variance, using information up to time $t-1$, and r_{t-1} as the previous day's return, also called innovation. The simplest such model is the GARCH(1,1) process,

$$h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1} \tag{5.27}$$

which involves one lag of the innovation and one lag of the previous forecast. The β term is important because it allows persistence in the variance, which is a realistic feature of the data. Here, we ignored the mean μ_t, which is generally small if the horizon is short. More generally, the GARCH(p, q) model has p lagged terms on historical returns and q lagged terms on previous variances.

The average unconditional variance is found by setting $E[r_{t-1}^2] = h_t = h_{t-1} = h$. Solving for h, we find

$$h = \frac{\alpha_0}{1 - \alpha_1 - \beta} \tag{5.28}$$

This model is stationary when the sum of parameters $\gamma = \alpha_1 + \beta$ are less than unity. This sum is also called the **persistence**, as it defines the speed at which shocks to the variance revert to their long-run values.

Figure 5.5 displays the one-day GARCH forecast for the S&P 500 index. The GARCH long-run volatility has been around 1.1% per day. Volatility peaked in September 2008, at the time of the Lehman Brothers bankruptcy, when it reached 5%. This reverted slowly to the long-run average later, which is a typical pattern of this forecast. Also note that the GARCH model identifies extended periods of low volatility, from 2004 to 2006.

To understand how the process works, consider Table 5.3. The parameters are $\alpha_0 = 0.01$, $\alpha_1 = 0.03$, $\beta = 0.95$. The unconditional variance is $0.01/(1 - 0.03 - 0.95) = 0.5$, or 0.7 daily volatility, which is typical of a currency series, as it translates into an annualized volatility of 11%. The process is stationary because $\alpha_1 + \beta = 0.98 < 1$.

At time 0, we start with the variance at $h_0 = 1.1$ (expressed in percent squared). The conditional volatility is $\sqrt{h_0} = 1.05\%$. The next day, there is a large return of 3%. The new variance forecast is then $h_1 = 0.01 + 0.03 \times 3^2 + 0.95 \times 1.1 = 1.32$. The conditional volatility just went up to 1.15%. If nothing happens the following days, the next variance forecast is $h_2 = 0.01 + 0.03 \times 0^2 + 0.95 \times 1.32 = 1.27$. And so on.

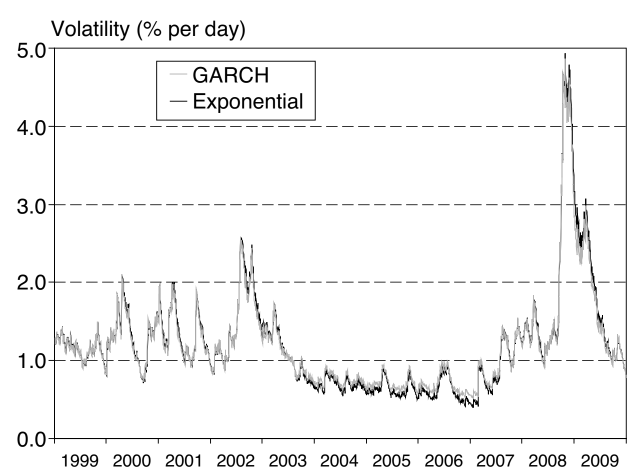

FIGURE 5.5 GARCH and EWMA Volatility Forecast for U.S. Equities

TABLE 5.3 Building a GARCH Forecast

Time	Return	Conditional Variance	Conditional Risk	Conditional 95% Limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.10
1	3.0	1.32	1.15	±2.30
2	0.0	1.27	1.13	±2.25
3	0.0	1.22	1.10	±2.20

How are the GARCH parameters derived? The parameters are estimated by a **maximum likelihood** method. This involves a numerical optimization of the likelihood of the observations. Typically, the scaled residuals $\epsilon_t = r_t/\sqrt{h_t}$ are assumed to have a normal distribution and to be independent. For each observation, the density is

$$f(r_t|\alpha_0, \alpha_1, \beta) = \frac{1}{\sqrt{2\pi h_t}} \exp[-\frac{1}{2h_t}(r_t - \mu_t)^2]$$

If we have T observations, their joint density is the product of the densities for each time period t. The likelihood function then depends on the three GARCH parameters, again ignoring the mean.

The optimization maximizes the logarithm of the likelihood function

$$\max\ F(\alpha_0, \alpha_1, \beta \mid r) = \sum_{t=1}^{T} \ln f(r_t \mid h_t) = \sum_{t=1}^{T} [\ln \frac{1}{\sqrt{2\pi h_t}} - \frac{r_t^2}{2h_t}] \tag{5.29}$$

where f is the normal density function. The optimization must be performed recursively. We fix some values for the three parameters, then start with h_0 and recursively compute h_1 to h_T. We then compute the likelihood function and let the optimizer converge to a maximum.

Finally, the GARCH process can be extrapolated to later days. This creates a **volatility term structure**. For the next-day forecast,

$$E_{t-1}(r_{t+1}^2) = \alpha_0 + \alpha_1 E_{t-1}(r_t^2) + \beta h_t = \alpha_0 + \alpha_1 h_t + \beta h_t = \alpha_0 + \gamma h_t$$

For the following day,

$$E_{t-1}(r_{t+2}^2) = \alpha_0 + \alpha_1 E_{t-1}(r_{t+1}^2) + \beta E_{t-1}(h_{t+1}) = \alpha_0 + (\alpha_1 + \beta) E_{t-1}(r_{t+1}^2)$$
$$E_{t-1}(r_{t+2}^2) = \alpha_0 + \gamma(\alpha_0 + \gamma h_t)$$

Generally,

$$E_{t-1}(r_{t+n}^2) = \alpha_0(1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1}) + \gamma^n h_t$$

Figure 5.6 illustrates the dynamics of shocks to a GARCH process for various values of the persistence parameter. As the conditional variance deviates from the

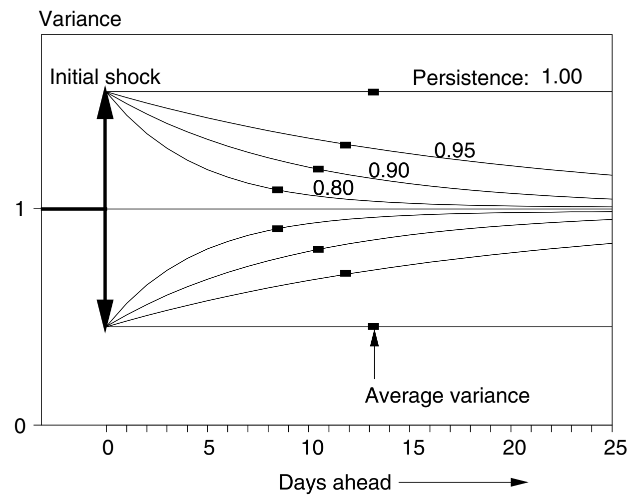

FIGURE 5.6 Shocks to a GARCH Process

starting value, it slowly reverts to the long-run value at a speed determined by $\alpha_1 + \beta$.

Note that these are forecasts of one-day variances at forward points in time. The total variance over the horizon is the sum of one-day variances. The *average* variance is marked with a black rectangle on the graph.

The graph also shows why the square root of time rule for extrapolating returns does not apply when risk is time-varying. If the initial value of the variance is greater than the long-run average, simply extrapolating the one-day variance to a longer horizon will overstate the average variance. Conversely, starting from a lower value and applying the square root of time rule will understate risk.

KEY CONCEPT

The square root of time rule used to scale one-day returns into longer horizons is generally inappropriate when risk is time-varying.

EXAMPLE 5.6: FRM EXAM 2009—QUESTION 2-13

Suppose σ_t^2 is the estimated variance at time t and u_t is the realized return at t. Which of the following GARCH(1,1) models will take the longest time to revert to its mean?

a. $\sigma_t^2 = 0.04 + 0.02u_{t-1}^2 + 0.92\sigma_{t-1}^2$
b. $\sigma_t^2 = 0.02 + 0.04u_{t-1}^2 + 0.94\sigma_{t-1}^2$
c. $\sigma_t^2 = 0.03 + 0.02u_{t-1}^2 + 0.95\sigma_{t-1}^2$
d. $\sigma_t^2 = 0.03 + 0.03u_{t-1}^2 + 0.93\sigma_{t-1}^2$

EXAMPLE 5.7: FRM EXAM 2006—QUESTION 132

Assume you are using a GARCH model to forecast volatility that you use to calculate the one-day VAR. If volatility is mean reverting, what can you say about the T-day VAR?

a. It is less than the $\sqrt{T}\times$ one-day VAR.
b. It is equal to $\sqrt{T}\times$ one-day VAR.
c. It is greater than the $\sqrt{T}\times$ one-day VAR.
d. It could be greater or less than the $\sqrt{T}\times$ one-day VAR.

EXAMPLE 5.8: FRM EXAM 2007—QUESTION 34

A risk manager estimates daily variance h_t using a GARCH model on daily returns r_t: $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1}$, with $\alpha_0 = 0.005$, $\alpha_1 = 0.04$, $\beta = 0.94$. The long-run *annualized* volatility is approximately

a. 13.54%
b. 7.94%
c. 72.72%
d. 25.00%

EXAMPLE 5.9: FRM EXAM 2009—QUESTION 2-17

Which of the following statements is *incorrect* regarding the volatility term structure predicted by a GARCH(1,1) model: $\sigma_t^2 = \omega + \alpha u_{t-1}^2 + \beta\sigma_{t-1}^2$, where $\alpha + \beta < 1$?

a. When the current volatility estimate is below the long-run average volatility, this GARCH model estimates an upward-sloping volatility term structure.
b. When the current volatility estimate is above the long-run average volatility, this GARCH model estimates a downward-sloping volatility term structure.
c. Assuming the long-run estimated variance remains unchanged as the GARCH parameters α and β increase, the volatility term structure predicted by this GARCH model reverts to the long-run estimated variance more slowly.
d. Assuming the long-run estimated variance remains unchanged as the GARCH parameters α and β increase, the volatility term structure predicted by this GARCH model reverts to the long-run estimated variance faster.

5.4.3 EWMA

The RiskMetrics approach is a specific case of the GARCH process and is particularly simple and convenient to use. Variances are modeled using an **exponentially weighted moving average (EWMA)** forecast. The forecast is a weighted average of the previous forecast, with weight λ, and of the latest squared innovation, with weight $(1 - \lambda)$:

$$h_t = \lambda h_{t-1} + (1 - \lambda) r_{t-1}^2 \tag{5.30}$$

The λ parameter, with $0 < \lambda < 1$, is also called the **decay factor**. It determines the relative weights placed on previous observations. The EWMA model places geometrically declining weights on past observations, assigning greater importance to recent observations. By recursively replacing h_{t-1} in Equation (5.30), we have

$$h_t = (1 - \lambda)[r_{t-1}^2 + \lambda r_{t-2}^2 + \lambda^2 r_{t-3}^2 + \cdots] \tag{5.31}$$

The weights therefore decrease at a geometric, or exponential, rate. The lower λ, the more quickly older observations are forgotten. RiskMetrics has chosen $\lambda = 0.94$ for daily data and $\lambda = 0.97$ for monthly data.

Table 5.4 shows how to build the EWMA forecast using a parameter of $\lambda = 0.95$, which is consistent with the previous GARCH example. At time 0, we start with the variance at $h_0 = 1.1$, as before. The next day, we have a return of 3%. The new variance forecast is then $h_1 = 0.05 \times 3^2 + 0.95 \times 1.1 = 1.50$. The next day, this moves to $h_2 = 0.05 \times 0^2 + 0.95 \times 1.50 = 1.42$. And so on.

This model is a special case of the GARCH process, where α_0 is set to 0, and α_1 and β sum to unity. The model therefore has permanent persistence. Shocks to the volatility do not decay, as shown in Figure 5.6 when the persistence is 1.00. Thus longer-term extrapolation from the GARCH and EWMA models may give quite different forecasts. Indeed, Equation (5.28) shows that the unconditional variance is not defined. Over a one-day horizon, however, the two models are quite similar and often indistinguishable from each other.

TABLE 5.4 Building a EWMA Forecast

Time	Return	Conditional Variance	Conditional Risk	Conditional 95% Limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.1
1	3.0	1.50	1.22	±2.4
2	0.0	1.42	1.19	±2.4
3	0.0	1.35	1.16	±2.3

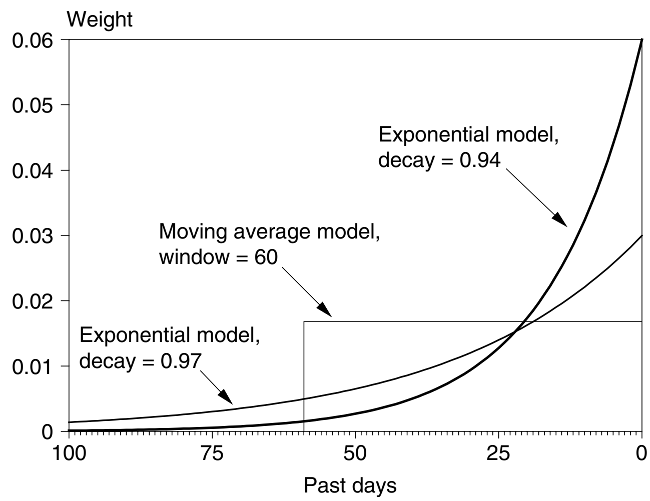

FIGURE 5.7 Weights on Past Observations

Figure 5.5 also displays the EWMA forecast, which follows a similar pattern to the GARCH forecast, reflecting the fact that it is a special case of the other. The EWMA forecast, however, has less mean reversion. For instance, it stays lower in 2004 to 2006.

Figure 5.7 displays the pattern of weights for previous observations. With $\lambda = 0.94$, the weights decay quickly. The weight on the last day is $(1 - \lambda) = (1 - 0.94) = 0.06$. The weight on the previous day is $(1 - \lambda)\lambda = 0.0564$, and so on. The weight drops below 0.00012 for data more than 100 days old. With $\lambda = 0.97$, the weights start at a lower level but decay more slowly. In comparison, moving average (MA) models have a fixed window, with equal weights within the window but otherwise zero. MA models with shorter windows give a greater weight to recent observations. As a result, they are more responsive to current events and more volatile.

EXAMPLE 5.10: FRM EXAM 2007—QUESTION 46

A bank uses the exponentially weighted moving average (EWMA) technique with λ of 0.9 to model the daily volatility of a security. The current estimate of the daily volatility is 1.5%. The closing price of the security is USD 20 yesterday and USD 18 today. Using continuously compounded returns, what is the updated estimate of the volatility?

a. 3.62%
b. 1.31%
c. 2.96%
d. 5.44%

EXAMPLE 5.11: FRM EXAM 2006—QUESTION 40

Using a daily RiskMetrics EWMA model with a decay factor $\lambda = 0.95$ to develop a forecast of the conditional variance, which weight will be applied to the return that is four days old?

a. 0.000
b. 0.043
c. 0.048
d. 0.950

EXAMPLE 5.12: EFFECT OF WEIGHTS ON OBSERVATIONS

Until January 1999 the historical volatility for the Brazilian real versus the U.S. dollar had been very small for several years. On January 13, Brazil abandoned the defense of the currency peg. Using the data from the close of business on January 13, which of the following methods for calculating volatility would have shown the greatest jump in measured historical volatility?

a. 250-day equal weight
b. Exponentially weighted with a daily decay factor of 0.94
c. 60-day equal weight
d. All of the above

EXAMPLE 5.13: FRM EXAM 2008—QUESTION 1-8

Which of the following four statements on models for estimating volatility is *incorrect*?

a. In the EWMA model, some positive weight is assigned to the long-run average variance rate.
b. In the EWMA model, the weights assigned to observations decrease exponentially as the observations become older.
c. In the GARCH(1,1) model, a positive weight is estimated for the long-run average variance rate.
d. In the GARCH(1,1) model, the weights estimated for observations decrease exponentially as the observations become older.

EXAMPLE 5.14: FRM EXAM 2009—QUESTION 2-16

Assume that an asset's daily return is normally distributed with zero mean. Suppose you have historical return data, $u_1, u_2, \ldots, u_m$ and that you want to use the maximum likelihood method to estimate the parameters of a EWMA volatility model. To do this, you define $v_i = \sigma_i^2$ as the variance estimated by the EWMA model on day i, so that the likelihood that these m observations occurred is given by: $\Pi_{i=1}^{m}[\frac{1}{\sqrt{2\pi v_i}}\exp[-u_i^2/(2v_i)]$. To maximize the likelihood that these m observations occurred, you must:

a. Find the value of λ that minimizes: $\sum_{i=1}^{m}[-\ln(v_i) - u_i^2/(2v_i)]$
b. Find the value of λ that maximizes: $\sum_{i=1}^{m}[-\ln(v_i) - u_i^2/(2v_i)]$
c. Find the value of λ that minimizes: $-m\ln(v_i) - \sum_{i=1}^{m}[u_i^2/(2v_i)]$
d. Find the value of λ that maximizes: $-m\ln(v_i) - \sum_{i=1}^{m}[u_i^2/(2v_i)]$

5.5 IMPORTANT FORMULAS

Multiperiod expected return and volatility for i.i.d. returns: $\mu_T = \mu T$, $\sigma_T = \sigma\sqrt{T}$

Two-period variance with nonzero autocorrelation: $V[R_t + R_{t-1}] = \sigma^2 \times 2[1+\rho]$

VAR assuming i.i.d. returns: $\text{VAR}_T = \alpha(\sigma\sqrt{T})W = \text{VAR}_1\sqrt{T}$

Portfolio VAR, from covariance matrix Σ and dollar exposures x: $\text{VAR} = \alpha\sqrt{x'\Sigma x}$

GARCH process: $h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1}$

GARCH long-run mean: $h = \alpha_0/(1-\alpha_1-\beta)$

EWMA process: $h_t = \lambda h_{t-1} + (1-\lambda)r_{t-1}^2$

5.6 ANSWERS TO CHAPTER EXAMPLES

Example 5.1: Time Scaling

c. Knowing that the variance is V(2-day) $= V$(1-day) $[2+2\rho]$, we find VAR(2-day) = VAR(1-day) $\sqrt{2+2\rho} = \$1\sqrt{2+0.2} = \$1.483$, assuming the same distribution for the different horizons.

Example 5.2: Independence

d. The term *efficient markets* implies that the distribution of future returns does not depend on past returns. Hence, returns cannot be correlated. It could happen, however, that return distributions are independent but that, just by chance, two successive returns are equal.

Example 5.3: FRM Exam 2002—Question 3

d. Assuming a random walk, we can use the square root of time rule. The weekly volatility is then $34\% \times 1/\sqrt{52} = 4.71\%$.

Example 5.4: FRM Exam 2002—Question 2

a. With mean reversion, the volatility grows more slowly than the square root of time.

Example 5.5: FRM Exam 2004—Question 39

d. The variance of the portfolio is given by $\sigma_p^2 = (0.4)^2 25 + (0.6)^2 121 + 2(0.4)(0.6)0.3\sqrt{25 \times 121} = 55.48$. Hence the volatility is 7.45.

Example 5.6: FRM Exam 2009—Question 2-13

b. The persistence $(\alpha_1 + \beta)$ is, respectively, 0.94, 0.98, 0.97, and 0.96. Hence the model with the highest persistence will take the longest time to revert to the mean.

Example 5.7: FRM Exam 2006—Question 132

d. If the initial volatility were equal to the long-run volatility, then the T-day VAR could be computed using the square root of time rule, assuming normal distributions. If the starting volatility were higher, then the T-day VAR should be less than the $\sqrt{T}\times$ one-day VAR. Conversely, if the starting volatility were lower, then the T-day VAR should be more than the long-run value. However, the question does not indicate the starting point. Hence, answer d. is correct.

Example 5.8: FRM Exam 2007—Question 34

b. The long-run mean variance is $h = \alpha_0/(1 - \alpha_1 - \beta) = 0.005/(1 - 0.04 - 0.94) = 0.25$. Taking the square root, this gives 0.5 for daily volatility. Multiplying by $\sqrt{252}$, we have an annualized volatility of 7.937%.

Example 5.9: FRM Exam 2009—Question 2-17

d. The GARCH model has mean reversion in the conditional volatility, so statements a. and b. are correct. When σ_t is lower than the long-run average, the volatility structure goes up. Higher persistence $\alpha + \beta$ means that mean reversion is slower, so statement c. is correct.

Example 5.10: FRM Exam 2007—Question 46

a. The log return is $\ln(18/20) = -10.54\%$. The new variance forecasts is $h = 0.90 \times (1.5^2) + (1 - 0.90) \times 10.54^2 = 0.001313$, or taking the square root, 3.62%.

Example 5.11: FRM Exam 2006—Question 40

b. The weight of the last day is $(1 - 0.95) = 0.050$. For the day before, this is 0.05×0.95, and for four days ago, $0.05 \times 0.95^3 = 0.04287$.

Example 5.12: Effect of Weights on Observations

b. The EWMA model puts a weight of 0.06 on the latest observation, which is higher than the weight of $(1/60) = 0.0167$ for the 60-day MA and $(1/250) = 0.004$ for the 250-day MA.

Example 5.13: FRM Exam 2008—Question 1-8

a. The GARCH model has a finite unconditional variance, so statement c. is correct. In contrast, because $\alpha_1 + \beta$ sum to 1, the EWMA model has undefined long-run average variance. In both models weights decline exponentially with time.

Example 5.14: FRM Exam 2009—Question 2-16

b. The optimal parameter must maximize (not minimize) the likelihood function. Otherwise, the log-likelihood function is the log of the product, which is the sum of the logs. This gives, up to a constant, $\sum_{i=1}^{m}[-\ln(v_i) - u_i^2/(2v_i)]$, and there is no way to take the first term outside the summation because it depends on i. So, answers c. and d. are incorrect.

PART Three

Financial Markets and Products

CHAPTER 6

Bond Fundamentals

Risk management starts with the pricing of assets. The simplest assets to study are regular, fixed-coupon bonds. Because their cash flows are predetermined, we can translate their stream of cash flows into a present value by discounting at a fixed interest rate. Thus the valuation of bonds involves understanding compounded interest and discounting, as well as the relationship between present values and interest rates.

Risk management goes one step further than pricing, however. It examines potential changes in the price of assets as the interest rate changes. In this chapter, we assume that there is a single interest rate, or yield, that is used to price the bond. This will be our fundamental risk factor. This chapter describes the relationship between bond prices and yields and presents indispensable tools for the management of fixed-income portfolios.

This chapter starts our coverage of financial markets by discussing bond fundamentals. Section 6.1 reviews the concepts of discounting, present values, and future values. These concepts are fundamental to understand the valuation of financial assets. Section 6.2 then plunges into the price-yield relationship. It shows how the Taylor expansion rule can be used to relate movements in bond prices to those in yields. This Taylor expansion rule, however, covers much more than bonds. It is a building block of risk measurement methods based on local valuation, as we shall see later. Section 6.3 applies this expansion rule to the computation of partial derivatives for bonds. Section 6.4 then presents an economic interpretation of duration and convexity. Fixed-income managers routinely use these measures to help navigate their portfolios.

6.1 DISCOUNTING, PRESENT VALUE, AND FUTURE VALUE

An investor buys a zero-coupon bond that pays \$100 in 10 years. Because the investment is guaranteed by the U.S. government, we assume that there is no credit risk. So, this is a default-free bond, which is exposed to market risk only. Market risk arises because of possible fluctuations in the market price of this bond.

FRM Exam Part 1 topic. This chapter also covers basic risk models for bonds.

To value this \$100 payment, we need a **discounting factor**. This is also the **interest rate**, or more simply the **yield**. Define C_t as the cash flow at time t and the discounting factor as y. We define T as the number of periods until maturity (e.g., number of years), also known as **tenor**. The **present value** (PV) of the bond can be computed as

$$\text{PV} = \frac{C_T}{(1+y)^T} \tag{6.1}$$

For instance, a payment of $C_T = \$100$ in 10 years discounted at 6% is worth only \$55.84 now. So, all else fixed, the market value of zero-coupon bonds decreases with longer maturities. Also, keeping T fixed, the value of the bond decreases as the yield increases.

Conversely, we can compute the **future value** (FV) of the bond as

$$\text{FV} = \text{PV} \times (1+y)^T \tag{6.2}$$

For instance, an investment now worth $\text{PV} = \$100$ growing at 6% will have a future value of $\text{FV} = \$179.08$ in 10 years.

Here, the yield has a useful interpretation, which is that of an **internal rate of return** on the bond, or annual growth rate. It is easier to deal with rates of return than with dollar values. Rates of return, when expressed in percentage terms and on an annual basis, are directly comparable across assets. An annualized yield is sometimes defined as the **effective annual rate (EAR)**.

It is important to note that the interest rate should be stated along with the method used for compounding. Annual compounding is very common. Other conventions exist, however. For instance, the U.S. Treasury market uses semiannual compounding. Define in this case y^S as the rate based on semiannual compounding. To maintain comparability, it is expressed in annualized form (i.e., after multiplication by 2). The number of periods, or semesters, is now $2T$. The formula for finding y^S is

$$\text{PV} = \frac{C_T}{(1+y^S/2)^{2T}} \tag{6.3}$$

For instance, a Treasury zero-coupon bond with a maturity of $T = 10$ years would have $2T = 20$ semiannual compounding periods. Comparing with Equation (6.1), we see that

$$(1+y) = (1+y^S/2)^2 \tag{6.4}$$

Continuous compounding is often used when modeling derivatives. It is the limit of the case where the number of compounding periods per year increases to infinity. The continuously compounded interest rate y^C is derived from

$$PV = C_T \times e^{-y^C T} \tag{6.5}$$

where $e^{(\cdot)}$, sometimes noted as $\exp(\cdot)$, represents the exponential function.

Note that in all of these Equations (6.1), (6.3), and (6.5), the present value and future cash flows are identical. Because of different compounding periods, however, the yields will differ. Hence, the compounding period should always be stated.

Example: Using Different Discounting Methods

Consider a bond that pays \$100 in 10 years and has a present value of \$55.8395. This corresponds to an annually compounded rate of 6% using $PV = C_T/(1+y)^{10}$, or $(1+y) = (C_T/PV)^{1/10}$.

This rate can be transformed into a semiannual compounded rate, using $(1+y^S/2)^2 = (1+y)$, or $y^S/2 = (1+y)^{1/2} - 1$, or $y^S = [(1+0.06)^{(1/2)} - 1] \times 2 = 0.0591 = 5.91\%$. It can be also transformed into a continuously compounded rate, using $\exp(y^C) = (1+y)$, or $y^C = \ln(1+0.06) = 0.0583 = 5.83\%$.

Note that as we increase the frequency of the compounding, the resulting rate decreases. Intuitively, because our money works harder with more frequent compounding, a lower investment rate will achieve the same payoff at the end.

KEY CONCEPT

For fixed present value and cash flows, increasing the frequency of the compounding will decrease the associated yield.

EXAMPLE 6.1: FRM EXAM 2002—QUESTION 48

An investor buys a Treasury bill maturing in one month for \$987. On the maturity date the investor collects \$1,000. Calculate effective annual rate (EAR).

a. 17.0%
b. 15.8%
c. 13.0%
d. 11.6%

EXAMPLE 6.2: FRM EXAM 2009—QUESTION 4-9

Lisa Smith, the treasurer of Bank AAA, has \$100 million to invest for one year. She has identified three alternative one-year certificates of deposit (CDs), with different compounding periods and annual rates. CD1: monthly, 7.82%; CD2: quarterly, 8.00%; CD3: semiannually, 8.05%; and CD4: continuous, 7.95%. Which CD has the highest effective annual rate (EAR)?

a. CD1
b. CD2
c. CD3
d. CD4

EXAMPLE 6.3: FRM EXAM 2002—QUESTION 51

Consider a savings account that pays an annual interest rate of 8%. Calculate the amount of time it would take to double your money. Round to the nearest year.

a. 7 years
b. 8 years
c. 9 years
d. 10 years

6.2 PRICE-YIELD RELATIONSHIP

6.2.1 Valuation

The fundamental discounting relationship from Equation (6.1) can be extended to any bond with a fixed cash-flow pattern. We can write the present value of a bond P as the discounted value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{6.6}$$

where: C_t = the cash flow (coupon or principal) in period t
t = the number of periods (e.g., half-years) to each payment
T = the number of periods to final maturity
y = the discounting factor per period (e.g., $y^S/2$)

A typical cash-flow pattern consists of a fixed coupon payment plus the repayment of the principal, or **face value** at expiration. Define c as the coupon *rate* and F as the face value. We have $C_t = cF$ prior to expiration, and at expiration, we have $C_T = cF + F$. The appendix reviews useful formulas that provide closed-form solutions for such bonds.

When the coupon rate c precisely matches the yield y, using the same compounding frequency, the present value of the bond must be equal to the face value. The bond is said to be a **par bond**. If the coupon is greater than the yield, the price must be greater than the face value, which means that this is a **premium bond**. Conversely, if the coupon is lower, or even zero for a zero-coupon bond, the price must be less than the face value, which means that this is a **discount bond**.

Equation (6.6) describes the relationship between the yield y and the value of the bond P, given its cash-flow characteristics. In other words, the value P can also be written as a nonlinear function of the yield y:

$$P = f(y) \tag{6.7}$$

Conversely, we can set P to the current market price of the bond, including any accrued interest. From this, we can compute the implied yield that will solve this equation.

Figure 6.1 describes the price-yield function for a 10-year bond with a 6% annual coupon. In risk management terms, this is also the relationship between the payoff on the asset and the risk factor. At a yield of 6%, the price is at par, $P = \$100$. Higher yields imply lower prices. This is an example of a **payoff function**, which links the price to the underlying risk factor.

Over a wide range of yield values, this is a highly nonlinear relationship. For instance, when the yield is zero, the value of the bond is simply the sum of cash flows, or \$160 in this case. When the yield tends to very large values, the bond price tends to zero. For small movements around the initial yield of 6%, however, the relationship is quasilinear.

There is a particularly simple relationship for **consols**, or **perpetual bonds**, which are bonds making regular coupon payments but with no redemption date. For a consol, the maturity is infinite and the cash flows are all equal to a fixed

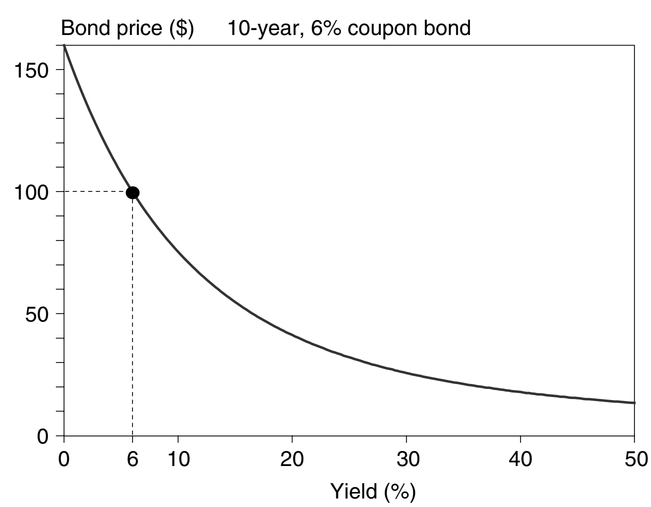

FIGURE 6.1 Price-Yield Relationship

percentage of the face value, $C_t = C = cF$. As a result, the price can be simplified from Equation (6.6) to

$$P = cF\left[\frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] = \frac{c}{y}F \tag{6.8}$$

as shown in the appendix. In this case, the price is simply proportional to the inverse of the yield. Higher yields lead to lower bond prices, and vice versa.

Example: Valuing a Bond

Consider a bond that pays \$100 in 10 years and a 6% annual coupon. Assume that the next coupon payment is in exactly one year. What is the market value if the yield is 6%? If it falls to 5%?

The bond cash flows are $C_1 = \$6, C_2 = \$6, \ldots, C_{10} = \$106$. Using Equation (6.6) and discounting at 6%, this gives the present value of cash flows of \$5.66, \$5.34, ..., \$59.19, for a total of \$100.00. The bond is selling at par. This is logical because the coupon is equal to the yield, which is also annually compounded. Alternatively, discounting at 5% leads to a price of \$107.72.

6.2.2 Taylor Expansion

Let us say that we want to see what happens to the price if the yield changes from its initial value, called y_0, to a new value, $y_1 = y_0 + \Delta y$. Risk management is all about assessing the effect of changes in risk factors such as yields on asset values. Are there shortcuts to help us with this?

We could recompute the new value of the bond as $P_1 = f(y_1)$. If the change is not too large, however, we can apply a very useful shortcut. The nonlinear relationship can be approximated by a **Taylor expansion** around its initial value[1]

$$P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2}f''(y_0)(\Delta y)^2 + \cdots \tag{6.9}$$

where $f'(\cdot) = \frac{dP}{dy}$ is the first derivative and $f''(\cdot) = \frac{d^2P}{dy^2}$ is the second derivative of the function $f(\cdot)$ valued at the starting point.[2] This expansion can be generalized to situations where the function depends on two or more variables. For bonds, the first derivative is related to the *duration* measure, and the second to *convexity*.

Equation (6.9) represents an infinite expansion with increasing powers of Δy. Only the first two terms (linear and quadratic) are ever used by finance

[1] This is named after the English mathematician Brook Taylor (1685–1731), who published this result in 1715. The full recognition of the importance of this result came only in 1755 when Euler applied it to differential calculus.

[2] This first assumes that the function can be written in polynomial form as $P(y + \Delta y) = a_0 + a_1\Delta y + a_2(\Delta y)^2 + \cdots$, with unknown coefficients a_0, a_1, a_2. To solve for the first, we set $\Delta y = 0$. This gives $a_0 = P_0$. Next, we take the derivative of both sides and set $\Delta y = 0$. This gives $a_1 = f'(y_0)$. The next step gives $2a_2 = f''(y_0)$. Here, the term *derivatives* takes the usual mathematical interpretation, and has nothing to do with *derivatives products* such as options.

practitioners. They provide a good approximation to changes in prices relative to other assumptions we have to make about pricing assets. If the increment is very small, even the quadratic term will be negligible.

Equation (6.9) is fundamental for risk management. It is used, sometimes in different guises, across a variety of financial markets. We will see later that this Taylor expansion is also used to approximate the movement in the value of a derivatives contract, such as an option on a stock. In this case, Equation (6.9) is

$$\Delta P = f'(S)\Delta S + \frac{1}{2} f''(S)(\Delta S)^2 + \cdots \tag{6.10}$$

where S is now the price of the underlying asset, such as the stock. Here, the first derivative $f'(S)$ is called *delta*, and the second $f''(S)$, *gamma*.

The Taylor expansion allows easy aggregation across financial instruments. If we have x_i units (numbers) of bond i and a total of N different bonds in the portfolio, the portfolio derivatives are given by

$$f'(y) = \sum_{i=1}^{N} x_i f_i'(y) \tag{6.11}$$

EXAMPLE 6.4: FRM EXAM 2009—QUESTION 4-8

A five-year corporate bond paying an annual coupon of 8% is sold at a price reflecting a yield to maturity of 6%. One year passes and the interest rates remain unchanged. Assuming a flat term structure and holding all other factors constant, the bond's price during this period will have

a. Increased
b. Decreased
c. Remained constant
d. Cannot be determined with the data given

6.3 BOND PRICE DERIVATIVES

For fixed-income instruments, the derivatives are so important that they have been given a special name.[3] The negative of the first derivative is the **dollar duration (DD):**

$$f'(y_0) = \frac{dP}{dy} = -D^* \times P_0 \tag{6.12}$$

[3] Note that this chapter does not present duration in the traditional textbook order. In line with the advanced focus on risk management, we first analyze the properties of duration as a sensitivity measure. This applies to any type of fixed-income instrument. Later, we will illustrate the usual definition of duration as a weighted average maturity, which applies for fixed-coupon bonds only.

where D^* is called the **modified duration**. Thus, dollar duration is

$$\text{DD} = D^* \times P_0 \tag{6.13}$$

where the price P_0 represent the *market* price, including any accrued interest. Sometimes, risk is measured as the **dollar value of a basis point (DVBP)**,

$$\text{DVBP} = \text{DD} \times \Delta y = [D^* \times P_0] \times 0.0001 \tag{6.14}$$

with 0.0001 representing an interest rate change of one basis point (bp) or one-hundredth of a percent. The **DVBP**, sometimes called the **DV01**, measures can be easily added up across the portfolio.

KEY CONCEPT

The dollar value of a basis point is the dollar exposure of a bond price for a change in yield of 0.01%. It is also the duration times the value of the bond and is additive across the entire portfolio.

The second derivative is the **dollar convexity (DC)**:

$$f''(y_0) = \frac{d^2P}{dy^2} = C \times P_0 \tag{6.15}$$

where C is called the **convexity**.

For fixed-income instruments with known cash flows, the price-yield function is known, and we can compute analytical first and second derivatives. Consider, for example, our simple zero-coupon bond in Equation (6.1) where the only payment is the face value, $C_T = F$. We take the first derivative, which is

$$\frac{dP}{dy} = \frac{d}{dy}\left[\frac{F}{(1+y)^T}\right] = (-T)\frac{F}{(1+y)^{T+1}} = -\frac{T}{(1+y)}P \tag{6.16}$$

Comparing with Equation (6.12), we see that the modified duration must be given by $D^* = T/(1+y)$. The conventional measure of **duration** is $D = T$, which does not include division by $(1+y)$ in the denominator. This is also called **Macaulay duration**. Note that duration is expressed in periods, like T. With annual compounding, duration is in years. With semiannual compounding, duration is in semesters. It then has to be divided by 2 for conversion to years. Modified duration D^* is related to Macaulay duration D:

$$D^* = \frac{D}{(1+y)} \tag{6.17}$$

Modified duration is the appropriate measure of interest rate exposure. The quantity $(1 + y)$ appears in the denominator because we took the derivative of the present value term with discrete compounding. If we use continuous compounding, modified duration is identical to the conventional duration measure. In practice, the difference between Macaulay and modified duration is usually small.

Let us now go back to Equation (6.16) and consider the second derivative, or

$$\frac{d^2 P}{dy^2} = -(T+1)(-T)\frac{F}{(1+y)^{T+2}} = \frac{(T+1)T}{(1+y)^2} \times P \tag{6.18}$$

Comparing with Equation (6.15), we see that the convexity is $C = (T+1)T/(1+y)^2$. Note that its dimension is expressed in period squared. With semiannual compounding, convexity is measured in semesters squared. It then has to be divided by 4 for conversion to years squared.[4] So, convexity must be positive for bonds with fixed coupons.

Putting together all these equations, we get the Taylor expansion for the change in the price of a bond, which is

$$\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2 + \cdots \tag{6.19}$$

Therefore duration measures the first-order (linear) effect of changes in yield, and convexity measures the second-order (quadratic) term.

Example: Computing the Price Approximation[5]

Consider a 10-year zero-coupon Treasury bond trading at a yield of 6%. The present value is obtained as $P = 100/(1 + 6/200)^{20} = 55.368$. As is the practice in the Treasury market, yields are semiannually compounded. Thus all computations should be carried out using semesters, after which final results can be converted into annual units.

Here, Macaulay duration is exactly 10 years, as $D = T$ for a zero-coupon bond. Its modified duration is $D^* = 20/(1 + 6/200) = 19.42$ semesters, which is 9.71 years. Its convexity is $C = 21 \times 20/(1 + 6/200)^2 = 395.89$ semesters squared, which is 98.97 in years squared. Dollar duration is $\text{DD} = D^* \times P = 9.71 \times \$55.37 = \$537.55$. The DVBP is $\text{DVBP} = \text{DD} \times 0.0001 = \$0.0538$.

[4] This is because the conversion to annual terms is obtained by multiplying the semiannual yield Δy by 2. As a result, the duration term must be divided by 2 and the convexity term by 2^2, or 4, for conversion to annual units.

[5] For such examples in this handbook, please note that intermediate numbers are reported with fewer significant digits than actually used in the computations. As a result, using rounded-off numbers may give results that differ slightly from the final numbers shown here.

We want to approximate the change in the value of the bond if the yield goes to 7%. Using Equation (6.19), we have $\Delta P = -[9.71 \times \$55.37](0.01) + 0.5[98.97 \times \$55.37](0.01)^2 = -\$5.375 + \$0.274 = -\$5.101$. Using the linear term only, the new price is $\$55.368 - \$5.375 = \$49.992$. Using the two terms in the expansion, the predicted price is slightly higher, at $\$55.368 - \$5.375 + \$0.274 = \$50.266$.

These numbers can be compared with the exact value, which is \$50.257. The linear approximation has a relative pricing error of −0.53%, which is not bad. Adding a quadratic term reduces this to an error of only 0.02%, which is very small, given typical bid-ask spreads.

More generally, Figure 6.2 compares the quality of the Taylor series approximation. We consider a 10-year bond paying a 6% coupon semiannually. Initially, the yield is also at 6% and, as a result, the price of the bond is at par, at \$100. The graph compares three lines representing

1. The actual, exact price $\quad P = f(y_0 + \Delta y)$
2. The duration estimate $\quad P = P_0 - D^* P_0 \Delta y$
3. The duration and convexity estimate $\quad P = P_0 - D^* P_0 \Delta y + (1/2) C P_0 (\Delta y)^2$

The actual price curve shows an increase in the bond price if the yield falls and, conversely, a depreciation if the yield increases. This effect is captured by the tangent to the true price curve, which represents the linear approximation based on duration. For small movements in the yield, this linear approximation provides a reasonable fit to the exact price. For large movements in price, however, the price-yield function becomes more curved and the linear fit deteriorates. Under these conditions, the quadratic approximation is noticeably better.

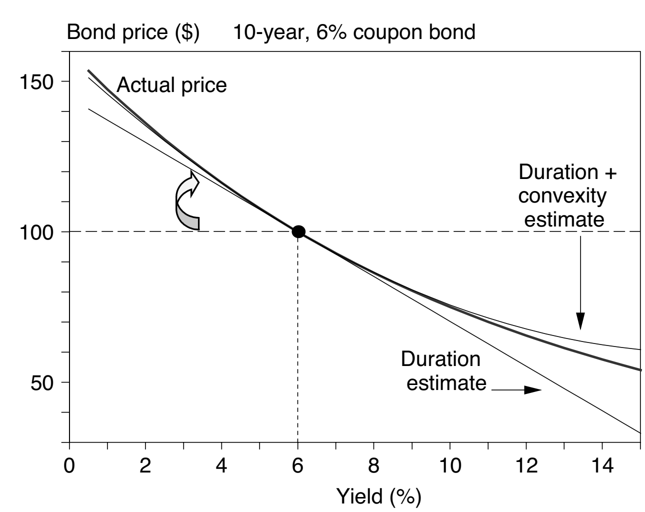

FIGURE 6.2 Price Approximation

KEY CONCEPT

Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point.

We should also note that the curvature is away from the origin, which explains the term *convexity* (as opposed to concavity). This curvature is beneficial since the second-order effect $0.5[C \times P](\Delta y)^2$ *must* be positive when convexity is positive. Some types of bonds, which involve the granting of an option to the investor, have negative convexity instead. In this case, the quadratic approximation is below the straight line instead of above as with positive convexity in Figure 6.2.

Figure 6.3 compares curves with different values for convexity. As the figure shows, when the yield rises, the price drops but less than predicted by the tangent. Conversely, if the yield falls, the price increases faster than along the tangent. In other words, the quadratic term is always beneficial.

KEY CONCEPT

Convexity is always positive for regular coupon-paying bonds. Greater convexity is beneficial for both falling and rising yields.

The bond's modified duration and convexity can also be computed directly from numerical derivatives. Duration and convexity cannot be computed directly for some bonds, such as mortgage-backed securities, because their cash flows are

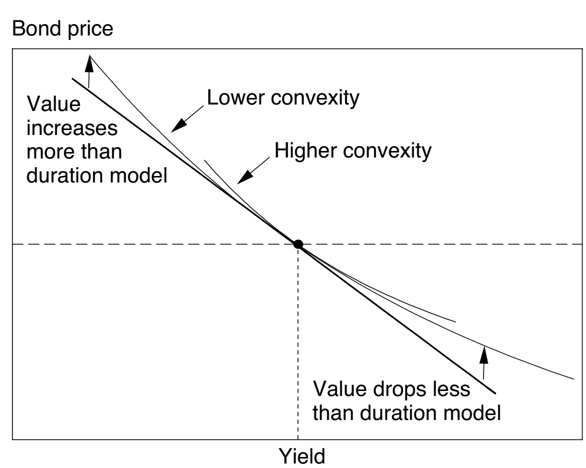

FIGURE 6.3 Effect of Convexity

uncertain. Instead, the portfolio manager has access to pricing models that can be used to reprice the securities under various yield environments.

As shown in Figure 6.4, we choose a change in the yield, Δy, and reprice the bond under an up move scenario, $P_+ = P(y_0 + \Delta y)$, and down move scenario, $P_- = P(y_0 - \Delta y)$. **Effective duration** is measured by the numerical derivative. Using $D^* = -(1/P)dP/dy$, it is estimated as

$$D^E = \frac{[P_- - P_+]}{(2P_0\Delta y)} = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2\Delta y)P_0} \tag{6.20}$$

Using $C = (1/P)d^2P/dy^2$, **effective convexity** is estimated as

$$C^E = [D_- - D_+]/\Delta y = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0\Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0\Delta y)}\right]/\Delta y \tag{6.21}$$

To illustrate, consider a 30-year zero-coupon bond with a yield of 6%, semi-annually compounded. The initial price is \$16.9733. We revalue the bond at 5% and 7%, with prices shown in Table 6.1. The effective duration in Equation (6.20) uses the two extreme points. The effective convexity in Equation (6.21) uses the difference between the dollar durations for the up move and down move. Note that convexity is positive if duration increases as yields fall, or if $D_- > D_+$.

The computations are detailed in Table 6.1, which shows an effective duration of 29.56. This is very close to the true value of 29.13, and would be even closer if the step Δy was smaller. Similarly, the effective convexity is 869.11, which is close to the true value of 862.48.

Finally, this numerical approach can be applied to get an estimate of the duration of a bond by considering bonds with the same maturity but different coupons. If interest rates decrease by 1%, the market price of a 6% bond should go up to a value close to that of a 7% bond. Thus we replace a drop in yield of

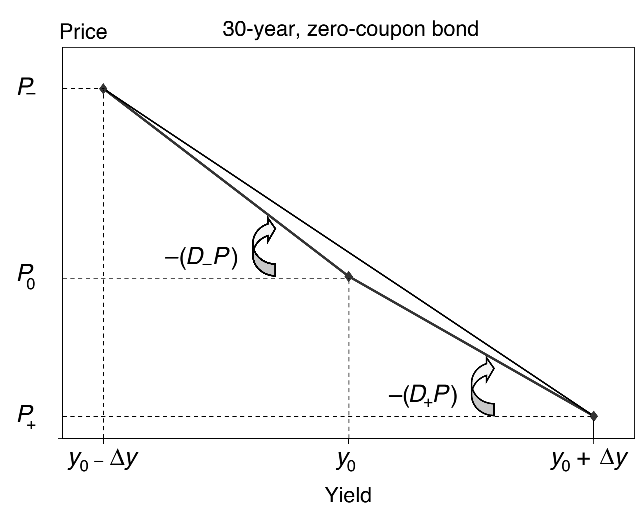

FIGURE 6.4 Effective Duration and Convexity

TABLE 6.1 Effective Duration and Convexity

State	Yield (%)	Bond Value	Duration Computation	Convexity Computation
Initial y_0	6.00	16.9733		
Up $y_0 + \Delta y$	7.00	12.6934		Duration up: 25.22
Down $y_0 - \Delta y$	5.00	22.7284		Duration down: 33.91
Difference in values			−10.0349	8.69
Difference in yields			0.02	0.01
Effective measure			29.56	869.11
Exact measure			29.13	862.48

Δy with an increase in coupon Δc and use the effective duration method to find the **coupon curve duration:**[6]

$$D^{CC} = \frac{[P_+ - P_-]}{(2P_0 \Delta c)} = \frac{P(y_0; c + \Delta c) - P(y_0; c - \Delta c)}{(2\Delta c)P_0} \tag{6.22}$$

This approach is useful for securities that are difficult to price under various yield scenarios. It only requires the market prices of securities with different coupons.

Example: Computation of Coupon Curve Duration

Consider a 10-year bond that pays a 7% coupon semiannually. In a 7% yield environment, the bond is selling at par and has modified duration of 7.11 years. The prices of 6% and 8% coupon bonds are \$92.89 and \$107.11, respectively. This gives a coupon curve duration of (107.11 – 92.89)/(0.02 × 100) = 7.11, which in this case is the same as modified duration.

EXAMPLE 6.5: FRM EXAM 2006—QUESTION 75

A zero-coupon bond with a maturity of 10 years has an annual effective yield of 10%. What is the closest value for its modified duration?

a. 9 years
b. 10 years
c. 99 years
d. 100 years

[6] For a more formal proof, we could take the pricing formula for a consol at par and compute the derivatives with respect to y and c. Apart from the sign, these derivatives are identical when $y = c$.

EXAMPLE 6.6: FRM EXAM 2007—QUESTION 115

A portfolio manager has a bond position worth USD 100 million. The position has a modified duration of eight years and a convexity of 150 years. Assume that the term structure is flat. By how much does the value of the position change if interest rates increase by 25 basis points?

a. USD −2,046,875
b. USD −2,187,500
c. USD −1,953,125
d. USD −1,906,250

EXAMPLE 6.7: FRM EXAM 2009—QUESTION 4-15

A portfolio manager uses her valuation model to estimate the value of a bond portfolio at USD 125.482 million. The term structure is flat. Using the same model, she estimates that the value of the portfolio would increase to USD 127.723 million if all interest rates fell by 30bp and would decrease to USD 122.164 million if all interest rates rose by 30bp. Using these estimates, the effective duration of the bond portfolio is closest to:

a. 8.38
b. 16.76
c. 7.38
d. 14.77

6.4 DURATION AND CONVEXITY

6.4.1 Economic Interpretation

The preceding section has shown how to compute analytical formulas for duration and convexity in the case of a simple zero-coupon bond. We can use the same approach for coupon-paying bonds. Going back to Equation (6.6), we have

$$\frac{dP}{dy} = \sum_{t=1}^{T} \frac{-tC_t}{(1+y)^{t+1}} = -\left[\sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}\right] / P \times \frac{P}{(1+y)} = -\frac{D}{(1+y)} P \qquad (6.23)$$

which defines duration as

$$D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} / P \tag{6.24}$$

The economic interpretation of duration is that it represents the average time to wait for each payment, weighted by the present value of the associated cash flow. Indeed, replacing P, we can write

$$D = \sum_{t=1}^{T} t \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} t \times w_t \tag{6.25}$$

where the weights w_t represent the ratio of the present value of each cash flow C_t relative to the total, and sum to unity. This explains why the duration of a zero-coupon bond is equal to the maturity. There is only one cash flow and its weight is one.

KEY CONCEPT

(Macaulay) duration represents an average of the time to wait for all cash flows.

Figure 6.5 lays out the present value of the cash flows of a 6% coupon, 10-year bond. Given a duration of 7.80 years, this coupon-paying bond is equivalent to a zero-coupon bond maturing in exactly 7.80 years.

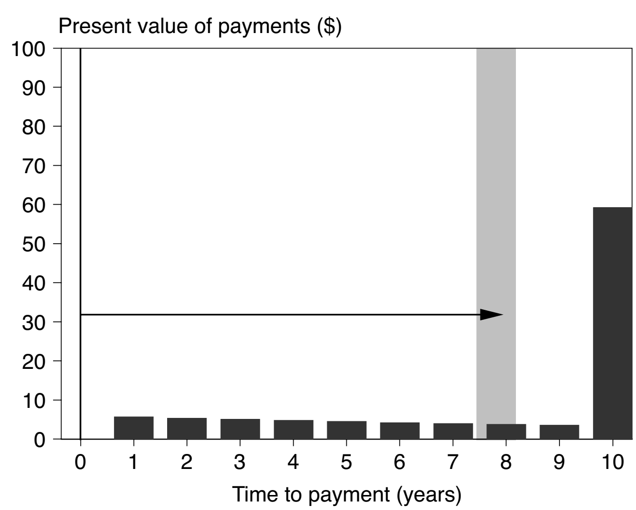

FIGURE 6.5 Duration as the Maturity of a Zero-Coupon Bond

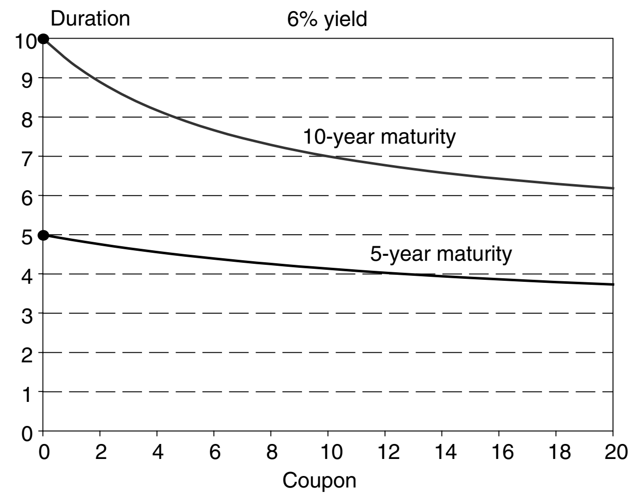

FIGURE 6.6 Duration and Coupon

For bonds with fixed coupons, duration is less than maturity. For instance, Figure 6.6 shows how the duration of a 10-year bond varies with its coupon, in a fixed 6% yield environment. With a zero coupon, Macaulay duration is equal to maturity. So, the curve starts at $D = 10$ years. Higher coupons place more weight on prior payments and therefore reduce duration.

Next, duration also varies with the current level of yield. Figure 6.7 shows that the duration of a 10-year bond decreases as the yield increases. This is because higher yields reduce the weight of distant cash flows in the current bond price, thus reducing its duration. The relationship is stronger for bonds with longer maturities. In addition, yield has even more effect on modified duration because it also appears in the denominator.

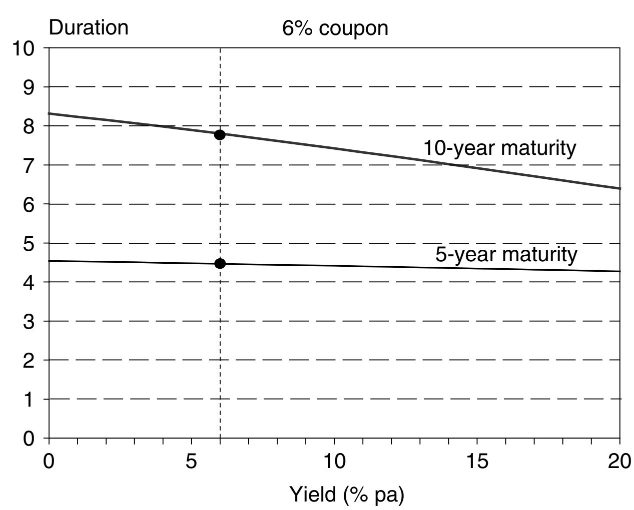

FIGURE 6.7 Duration-Yield Relationship

Duration can be expressed in a simple form for **consols**. From Equation (6.8), we have $P = (c/y)F$. Taking the derivative, we find

$$\frac{dP}{dy} = cF\frac{(-1)}{y^2} = (-1)\frac{1}{y}\left[\frac{c}{y}F\right] = (-1)\frac{1}{y}P = -\frac{D_C}{(1+y)}P \tag{6.26}$$

Hence the Macaulay duration for the consol D_C is

$$D_C = \frac{(1+y)}{y} \tag{6.27}$$

This shows that the duration of a consol is finite even if its maturity is infinite. Also, this duration does not depend on the coupon.

This formula provides a useful rule of thumb. For a long-term coupon-paying bond, duration should be lower than $(1+y)/y$ in most cases. For instance, when $y = 6\%$, the upper limit on duration is $D_C = 1.06/0.06$, or 17.7 years. In this environment, the duration of a par 30-year bond is 14.25, which is indeed lower than 17.7 years.

KEY CONCEPT

The duration of a long-term bond can be approximated by an upper bound, which is that of a consol with the same yield, $D_C = (1+y)/y$.

Figure 6.8 describes the relationship between duration, maturity, and coupon for regular bonds in a 6% yield environment. For the zero-coupon bond, $D = T$, which is a straight line going through the origin. For the par 6% bond, duration

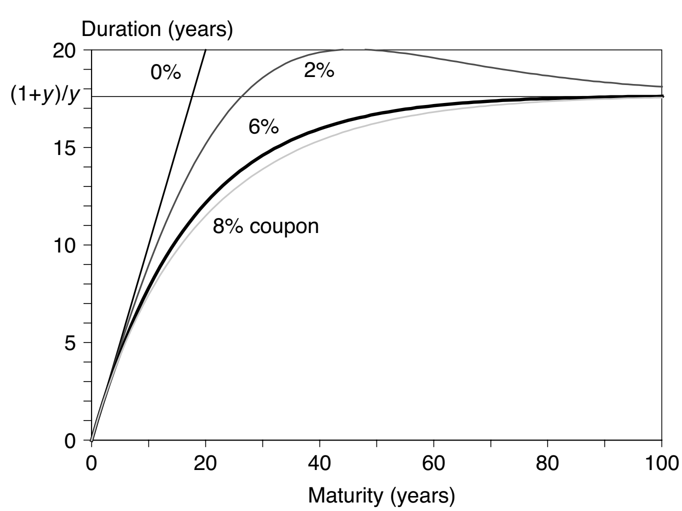

FIGURE 6.8 Duration and Maturity

increases monotonically with maturity until it reaches the asymptote of D_C. The 8% bond has lower duration than the 6% bond for fixed T. Greater coupons, for a fixed maturity, decrease duration, as more of the payments come early.

Finally, the 2% bond displays a pattern intermediate between the zero-coupon and 6% bonds. It initially behaves like the zero, exceeding D_C, and then falls back to the asymptote, which is the same for all coupon-paying bonds.

Taking now the second derivative in Equation (6.23), we have

$$\frac{d^2 P}{dy^2} = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} = \left[\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} / P \right] \times P \tag{6.28}$$

which defines convexity as

$$C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} / P \tag{6.29}$$

Convexity can also be written as

$$C = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times w_t \tag{6.30}$$

Because the squared t term dominates in the fraction, this basically involves a weighted average of the square of time. Therefore, convexity is much greater for long-maturity bonds because they have payoffs associated with large values of t. The formula also shows that convexity is always positive for such bonds, implying that the curvature effect is beneficial. As we will see later, convexity can be negative for bonds that have uncertain cash flows, such as **mortgage-backed securities** (MBSs) or callable bonds.

Figure 6.9 displays the behavior of convexity, comparing a zero-coupon bond and a 6% coupon bond with identical maturities. The zero-coupon bond always has greater convexity, because there is only one cash flow at maturity. Its convexity is roughly the square of maturity, for example about 900 for the 30-year zero. In contrast, the 30-year coupon bond has a convexity of about 300 only.

KEY CONCEPT

All else equal, duration and convexity both increase for longer maturities, lower coupons, and lower yields.

As an illustration, Table 6.2 details the steps of the computation of duration and convexity for a two-year, 6% semiannual coupon-paying bond. We first

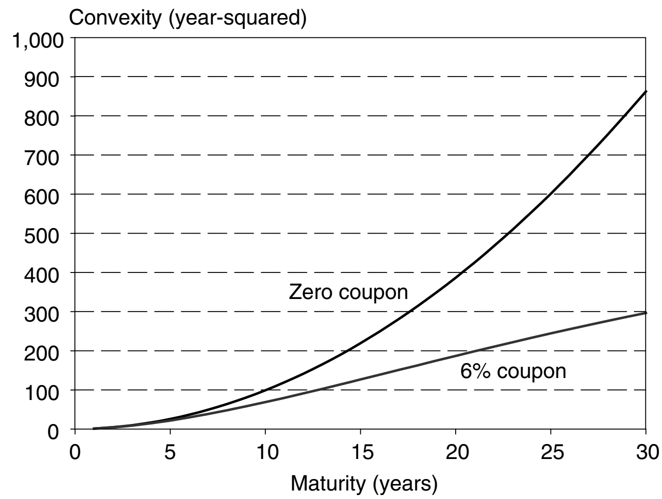

FIGURE 6.9 Convexity and Maturity

convert the annual coupon and yield into semiannual equivalent, \$3 and 3% each. The PV column then reports the present value of each cash flow. We verify that these add up to \$100, since the bond must be selling at par.

Next, the duration term column multiplies each PV term by time, or more precisely the number of half years until payment. This adds up to \$382.86, which divided by the price gives $D = 3.83$. This number is measured in half years, and we need to divide by 2 to convert to years. Macaulay duration is 1.91 years, and modified duration $D^* = 1.91/1.03 = 1.86$ years. Note that, to be consistent, the adjustment in the denominator involves the semiannual yield of 3%.

Finally, the rightmost column shows how to compute the bond's convexity. Each term involves PV_t times $t(t+1)/(1+y)^2$. These terms sum to 1,777.755, or divided by the price, 17.78. This number is expressed in units of time squared

TABLE 6.2 Computing Duration and Convexity

Period (Half Year) t	Payment C_t	Yield (%) (6 Mo.)	PV of Payment $C_t/(1+y)^t$	Duration Term tPV_t	Convexity Term $t(t+1)PV_t$ $\times[1/(1+y)^2]$
1	3	3.00	2.913	2.913	5.491
2	3	3.00	2.828	5.656	15.993
3	3	3.00	2.745	8.236	31.054
4	103	3.00	91.514	366.057	1,725.218
Sum:			100.00	382.861	1,777.755
(Half years)				3.83	17.78
(Years)				1.91	
Modified duration				1.86	
Convexity					4.44

and must be divided by 4 to be converted in annual terms. We find a convexity of $C = 4.44$, in year-squared.

EXAMPLE 6.8: FRM EXAM 2003—QUESTION 13

Suppose the face value of a three-year option-free bond is USD 1,000 and the annual coupon is 10%. The current yield to maturity is 5%. What is the modified duration of this bond?

a. 2.62
b. 2.85
c. 3.00
d. 2.75

EXAMPLE 6.9: FRM EXAM 2002—QUESTION 118

A Treasury bond has a coupon rate of 6% per annum (the coupons are paid semiannually) and a semiannually compounded yield of 4% per annum. The bond matures in 18 months and the next coupon will be paid 6 months from now. Which number of years is closest to the bond's Macaulay duration?

a. 1.023 years
b. 1.457 years
c. 1.500 years
d. 2.915 years

EXAMPLE 6.10: DURATION AND COUPON

A and B are two perpetual bonds; that is, their maturities are infinite. A has a coupon of 4% and B has a coupon of 8%. Assuming that both are trading at the same yield, what can be said about the duration of these bonds?

a. The duration of A is greater than the duration of B.
b. The duration of A is less than the duration of B.
c. A and B both have the same duration.
d. None of the above.

EXAMPLE 6.11: FRM EXAM 2004—QUESTION 16

A manager wants to swap a bond for a bond with the same price but higher duration. Which of the following bond characteristics would be associated with a higher duration?

I. A higher coupon rate
II. More frequent coupon payments
III. A longer term to maturity
IV. A lower yield

a. I, II, and III
b. II, III, and IV
c. III and IV
d. I and II

EXAMPLE 6.12: FRM EXAM 2001—QUESTION 104

When the maturity of a plain coupon bond increases, its duration increases

a. Indefinitely and regularly
b. Up to a certain level
c. Indefinitely and progressively
d. In a way dependent on the bond being priced above or below par

EXAMPLE 6.13: FRM EXAM 2000—QUESTION 106

Consider the following bonds:

Bond Number	Maturity (Years)	Coupon Rate	Frequency	Yield (Annual)
1	10	6%	1	6%
2	10	6%	2	6%
3	10	0%	1	6%
4	10	6%	1	5%
5	9	6%	1	6%

How would you rank the bonds from the shortest to longest duration?

a. 5-2-1-4-3
b. 6-2-3-4-5
c. 5-4-3-1-2
d. 2-4-5-1-3

EXAMPLE 6.14: FRM EXAM 2000—QUESTION 110

Which of the following statements is/are *true*?

I. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 10-year 6% bond.
II. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 6% bond with a duration of 10 years.
III. Convexity grows proportionately with the maturity of the bond.
IV. Convexity is always positive for all types of bonds.
V. Convexity is always positive for straight bonds.

a. I only
b. I and II only
c. I and V only
d. II, III, and V only

6.4.2 Portfolio Duration and Convexity

Fixed-income portfolios often involve very large numbers of securities. It would be impractical to consider the movements of each security individually. Instead, portfolio managers aggregate the duration and convexity across the portfolio. A manager who believes that rates will increase should shorten the portfolio duration relative to that of the benchmark. Say, for instance, that the benchmark has a duration of five years. The manager shortens the portfolio duration to one year only. If rates increase by 2%, the benchmark will lose approximately $5y \times 2\% = 10\%$. The portfolio, however, will lose only $1y \times 2\% = 2\%$, hence beating the benchmark by 8%.

Because the Taylor expansion involves a summation, the portfolio duration is easily obtained from the individual components. Say we have N components indexed by i. Defining D_p^* and P_p as the portfolio modified duration and value, the portfolio dollar duration (DD) is

$$D_p^* P_p = \sum_{i=1}^{N} D_i^* x_i P_i \tag{6.31}$$

where x_i is the number of units of bond i in the portfolio. A similar relationship holds for the portfolio dollar convexity (DC). If yields are the same for all components, this equation also holds for the Macaulay duration.

Because the portfolio's total market value is simply the summation of the component market values,

$$P_p = \sum_{i=1}^{N} x_i P_i \tag{6.32}$$

we can define the **portfolio weight** w_i as $w_i = x_i P_i / P_p$, provided that the portfolio market value is nonzero. We can then write the portfolio duration as a weighted average of individual durations:

$$D_p^* = \sum_{i=1}^{N} D_i^* w_i \tag{6.33}$$

Similarly, the portfolio convexity is a weighted average of convexity numbers:

$$C_p = \sum_{i=1}^{N} C_i w_i \tag{6.34}$$

As an example, consider a portfolio invested in three bonds, described in Table 6.3. The portfolio is long a 10-year and 1-year bond, and short a 30-year zero-coupon bond. Its market value is \$1,301,600. Summing the duration for each component, the portfolio dollar duration is \$2,953,800, which translates into a duration of 2.27 years. The portfolio convexity is $-76{,}918{,}323/1{,}301{,}600 = -59.10$, which is negative due to the short position in the 30-year zero, which has very high convexity.

Alternatively, assume the portfolio manager is given a benchmark, which is the first bond. The manager wants to invest in bonds 2 and 3, keeping the portfolio duration equal to that of the target, or 7.44 years. To achieve the target value and dollar duration, the manager needs to solve a system of two equations in the numbers x_1 and x_2:

$$\text{Value:} \quad \$100 = x_1\$94.26 + x_2\$16.97$$

$$\text{Dollar duration:} \quad 7.44 \times \$100 = 0.97 \times x_1\$94.26 + 29.13 \times x_2\$16.97$$

TABLE 6.3 Portfolio Dollar Duration and Convexity

	Bond 1	Bond 2	Bond 3	Portfolio
Maturity (years)	10	1	30	
Coupon	6%	0%	0%	
Yield	6%	6%	6%	
Price P_i	\$100.00	\$94.26	\$16.97	
Modified duration D_i^*	7.44	0.97	29.13	
Convexity C_i	68.78	1.41	862.48	
Number of bonds x_i	10,000	5,000	−10,000	
Dollar amounts $x_i P_i$	\$1,000,000	\$471,300	−\$169,700	\$1,301,600
Weight w_i	76.83%	36.21%	−13.04%	100.00%
Dollar duration $D_i^* P_i$	\$744.00	\$91.43	\$494.34	
Portfolio DD: $x_i D_i^* P_i$	\$7,440,000	\$457,161	−\$4,943,361	\$2,953,800
Portfolio DC: $x_i C_i P_i$	68,780,000	664,533	−146,362,856	−76,918,323

The solution is $x_1 = 0.817$ and $x_2 = 1.354$, which gives a portfolio value of \$100 and modified duration of 7.44 years.[7] The portfolio convexity is 199.25, higher than the index. Such a portfolio consisting of very short and very long maturities is called a **barbell portfolio**. In contrast, a portfolio with maturities in

EXAMPLE 6.15: FRM EXAM 2002—QUESTION 57

A bond portfolio has the following composition:

1. Portfolio A: price \$90,000, modified duration 2.5, long position in 8 bonds
2. Portfolio B: price \$110,000, modified duration 3, short position in 6 bonds
3. Portfolio C: price \$120,000, modified duration 3.3, long position in 12 bonds

All interest rates are 10%. If the rates rise by 25 basis points, then the bond portfolio value will decrease by

a. \$11,430
b. \$21,330
c. \$12,573
d. \$23,463

EXAMPLE 6.16: FRM EXAM 2006—QUESTION 61

Consider the following portfolio of bonds (par amounts are in millions of USD).

Bond	Price	Par Amount Held	Modified Duration
A	101.43	3	2.36
B	84.89	5	4.13
C	121.87	8	6.27

What is the value of the portfolio's DV01 (dollar value of 1 basis point)?

a. \$8,019
b. \$8,294
c. \$8,584
d. \$8,813

[7] This can be obtained by first expressing x_2 in the first equation as a function of x_1 and then substituting back into the second equation. This gives $x_2 = (100 - 94.26x_1)/16.97$, and $744 = 91.43x_1 + 494.34x_2 = 91.43x_1 + 494.34(100 - 94.26x_1)/16.97 = 91.43x_1 + 2{,}913.00 - 2{,}745.79x_1$. Solving, we find $x_1 = (-2{,}169.00)/(-2{,}654.36) = 0.817$ and $x_2 = (100 - 94.26 \times 0.817)/16.97 = 1.354$.

> **EXAMPLE 6.17: FRM EXAM 2008—QUESTION 2-33**
>
> Which of the following statements is *correct* regarding the effects of interest rate shift on fixed-income portfolios with similar durations?
>
> **a.** A barbell portfolio has greater convexity than a bullet portfolio because convexity increases linearly with maturity.
> **b.** A barbell portfolio has greater convexity than a bullet portfolio because convexity increases with the square of maturity.
> **c.** A barbell portfolio has lower convexity than a bullet portfolio because convexity increases linearly with maturity.
> **d.** A barbell portfolio has lower convexity than a bullet portfolio because convexity increases with the square of maturity.

the same range is called a **bullet portfolio**. Note that the barbell portfolio has a much greater convexity than the bullet portfolio because of the payment in 30 years. Such a portfolio would be expected to outperform the bullet portfolio if yields moved by a large amount.

In sum, duration and convexity are key measures of fixed-income portfolios. They summarize the linear and quadratic exposure to movements in yields. This explains why they are essential tools for fixed-income portfolio managers.

6.5 IMPORTANT FORMULAS

Compounding: $(1 + y)^T = (1 + y^S/2)^{2T} = e^{y^C T}$

Fixed-coupon bond valuation: $P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$

Taylor expansion: $P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2} f''(y_0)(\Delta y)^2 + \cdots$

Duration as exposure: $\frac{dP}{dy} = -D^* \times P$, $\text{DD} = D^* \times P$, $\text{DVBP} = \text{DD} \times 0.0001$

Modified and conventional duration: $D^* = \frac{D}{(1+y)}$, $D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} / P$

Convexity: $\frac{d^2 P}{dy^2} = C \times P$, $C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} / P$

Price change: $\Delta P = -[D^* \times P](\Delta y) + 0.5[C \times P](\Delta y)^2 + \cdots$

Consol: $P = \frac{c}{y} F$, $D = \frac{(1+y)}{y}$

Portfolio duration and convexity: $D_p^* = \sum_{i=1}^{N} D_i^* w_i$, $C_p = \sum_{i=1}^{N} C_i w_i$

6.6 ANSWERS TO CHAPTER EXAMPLES

Example 6.1: FRM Exam 2002—Question 48

a. The EAR is defined by $\text{FV/PV} = (1 + \text{EAR})^T$. So $\text{EAR} = (\text{FV/PV})^{1/T} - 1$. Here, $T = 1/12$. So, $\text{EAR} = (1{,}000/987)^{12} - 1 = 17.0\%$.

Example 6.2: FRM Exam 2009—Question 4-9

d. A dollar initially invested will grow to (CD1) $(1 + 7.82\%/12)^{12} = 1.08107$, (CD2) $(1 + 8.00\%/4)^4 = 1.08243$, (CD3) $(1 + 8.05\%/2)^2 = 1.08212$, (CD4) $\exp(7.95\%) = 1.08275$. Hence, CD4 gives the highest final amount and EAR.

Example 6.3: FRM Exam 2002—Question 51

c. The time T relates the current and future values such that $\mathrm{FV}/\mathrm{PV} = 2 = (1 + 8\%)^T$. Taking logs of both sides, this gives $T = \ln(2)/\ln(1.08) = 9.006$.

Example 6.4: FRM Exam 2009—Question 4-8

b. Because the coupon is greater than the yield, the bond must be selling at a premium, or current price greater than the face value. If yields do not change, the bond price will converge to the face value. Given that it starts higher, it must decrease.

Example 6.5: FRM Exam 2006—Question 75

a. Without doing any computation, the Macaulay duration must be 10 years because this is a zero-coupon bond. With annual compounding, modified duration is $D^* = 10/(1 + 10\%)$, or close to 9 years.

Example 6.6: FRM Exam 2007—Question 115

c. The change in price is given by $\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2$ $= -[8 \times 100](0.0025) + 0.5[150 \times 100](0.0025)^2 = -2.000000 + 0.046875 = -1.953125$.

Example 6.7: FRM Exam 2009—Question 4-15

c. By Equation (6.20), effective duration is $D^E = \frac{[P_- - P_+]}{(2P_0\Delta y)} = \frac{[127.723 - 122.164]}{(125.482 \times 0.6\%)} = 7.38$.

Example 6.8: FRM Exam 2003—Question 13

a. As in Table 6.2, we lay out the cash flows and find

Period t	Payment C_t	Yield y	$\mathrm{PV}_t = C_t/(1+y)^t$	$t\mathrm{PV}_t$
1	100	5.00	95.24	95.24
2	100	5.00	90.71	181.41
3	1,100	5.00	950.22	2,850.66
Sum:			1,136.16	3,127.31

Duration is then 2.75, and modified duration 2.62.

Example 6.9: FRM Exam 2002—Question 118

b. For coupon-paying bonds, Macaulay duration is slightly less than the maturity, which is 1.5 years here. So, b. would be a good guess. Otherwise, we can compute duration exactly.

Example 6.10: Duration and Coupon

c. Going back to the duration equation for the consol, Equation (6.27), we see that it does not depend on the coupon but only on the yield. Hence, the durations must be the same. The price of bond A, however, must be half that of bond B.

Example 6.11: FRM Exam 2004—Question 16

c. Higher duration is associated with physical characteristics that push payments into the future, that is, longer term, lower coupons, and less frequent coupon payments, as well as lower yields, which increase the relative weight of payments in the future.

Example 6.12: FRM Exam 2001—Question 104

b. With a fixed coupon, the duration goes up to the level of a consol with the same coupon. See Figure 6.8.

Example 6.13: FRM Exam 2000—Question 106

a. The nine-year bond (number 5) has shorter duration because the maturity is shortest, at nine years, among comparable bonds. Next, we have to decide between bonds 1 and 2, which differ only in the payment frequency. The semiannual bond (number 2) has a first payment in six months and has shorter duration than the annual bond. Next, we have to decide between bonds 1 and 4, which differ only in the yield. With lower yield, the cash flows further in the future have a higher weight, so that bond 4 has greater duration. Finally, the zero-coupon bond has the longest duration. So, the order is 5-2-1-4-3.

Example 6.14: FRM Exam 2000—Question 110

c. Because convexity is proportional to the square of time to payment, the convexity of a bond is mainly driven by the cash flows far into the future. Answer I. is correct because the 10-year zero has only one cash flow, whereas the coupon bond has several others that reduce convexity. Answer II. is false because the 6% bond with 10-year duration must have cash flows much further into the future, say in 30 years, which will create greater convexity. Answer III. is false because convexity grows with the square of time. Answer IV. is false because some bonds, for example MBSs or callable bonds, can have negative convexity. Answer V. is correct because convexity must be positive for coupon-paying bonds.

Example 6.15: FRM Exam 2002—Question 57

a. The portfolio dollar duration is $D^* P = \sum x_i D_i^* P_i = +8 \times 2.5 \times \$90{,}000 - 6 \times 3.0 \times \$110{,}000 + 12 \times 3.3 \times \$120{,}000 = \$4{,}572{,}000$. The change in portfolio value is then $-(D * P)(\Delta y) = -\$4{,}572{,}000 \times 0.0025 = -\$11{,}430$.

Example 6.16: FRM Exam 2006—Question 61

c. First, the market value of each bond is obtained by multiplying the par amount by the ratio of the market price divided by 100. Next, this is multiplied by D^* to get the dollar duration DD. Summing, this gives $85.841 million. We multiply by 1,000,000 to get dollar amounts and by 0.0001 to get the DV01, which gives $8,584.

Bond	Price	Par	Market Value	D^*	DD
A	101.43	3	3.043	2.36	7.181
B	84.89	5	4.245	4.13	15.530
C	121.87	8	9.750	6.27	61.130
Sum					85.841

Example 6.17: FRM Exam 2008—Question 2-33

b. The statement compares two portfolios with the same duration. A barbell portfolio consists of a combination of short-term and long-term bonds. A bullet portfolio has only medium-term bonds. Because convexity is a quadratic function of time to wait for the payments, the long-term bonds create a large contribution to the convexity of the barbell portfolio, which must be higher than that of the bullet portfolio.

APPENDIX: APPLICATIONS OF INFINITE SERIES

When bonds have fixed coupons, the bond valuation problem often can be interpreted in terms of combinations of infinite series. The most important infinite series result is for a sum of terms that increase at a geometric rate:

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{6.a} \tag{6.35}$$

This can be proved, for instance, by multiplying both sides by $(1 - a)$ and canceling out terms.

Equally important, consider a geometric series with a finite number of terms, say N. We can write this as the difference between two infinite series:

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = (1 + a + a^2 + a^3 + \cdots) - a^N(1 + a + a^2 + a^3 + \cdots) \tag{6.36}$$

such that all terms with order N or higher will cancel each other.

We can then write

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = \frac{1}{6.a} - a^N \frac{1}{6.a} \tag{6.37}$$

These formulas are essential to value bonds. Consider first a consol with an infinite number of coupon payments with a fixed coupon rate c. If the yield is y and the face value F, the value of the bond is

$$\begin{aligned} P &= cF\left[\tfrac{1}{(1+y)} + \tfrac{1}{(1+y)^2} + \tfrac{1}{(1+y)^3} + \cdots\right] \\ &= cF\tfrac{1}{(1+y)}[1 + a^2 + a^3 + \cdots] \\ &= cF\tfrac{1}{(1+y)}\left[\tfrac{1}{6.a}\right] \\ &= cF\tfrac{1}{(1+y)}\left[\tfrac{1}{(1-1/(1+y))}\right] \\ &= cF\tfrac{1}{(1+y)}\left[\tfrac{(1+y)}{y}\right] \\ &= \tfrac{c}{y}F \end{aligned}$$

Similarly, we can value a bond with a *finite* number of coupons over T periods at which time the principal is repaid. This is really a portfolio with three parts:

(1) a long position in a consol with coupon rate c
(2) a short position in a consol with coupon rate c that starts in T periods
(3) a long position in a zero-coupon bond that pays F in T periods

Note that the combination of (1) and (2) ensures that we have a finite number of coupons. Hence, the bond price should be:

$$P = \frac{c}{y}F - \frac{1}{(1+y)^T}\frac{c}{y}F + \frac{1}{(1+y)^T}F = \frac{c}{y}F\left[1 - \frac{1}{(1+y)^T}\right] + \frac{1}{(1+y)^T}F \tag{6.38}$$

where again the formula can be adjusted for different compounding methods.

This is useful for a number of purposes. For instance, when $c = y$, it is immediately obvious that the price must be at par, $P = F$. This formula also can be used to find closed-form solutions for duration and convexity.

CHAPTER 7

Introduction to Derivatives

This chapter provides an overview of derivatives markets. Derivatives are financial contracts traded in private **over-the-counter** (OTC) markets or on **organized exchanges**. As the term implies, derivatives derive their value from some underlying index, typically the price of an asset. Depending on the type of relationship, they can be broadly classified into two categories: linear and nonlinear instruments.

To the first category belong forward rate agreements (FRAs), futures, and swaps. Their value is a linear function of the underlying index. These are *obligations* to exchange payments according to a specified schedule. Forward contracts involve one payment and are relatively simple to evaluate. So are futures, which are traded on exchanges. Swaps involve a series of payments and generally can be reduced to portfolios of forward contracts. To the second category belong options, whose value is a nonlinear function of the underlying index. These are much more complex to evaluate and will be covered in the next chapter.

This chapter describes the general characteristics as well as the pricing of linear derivatives. Pricing is the first step toward risk measurement. The second step consists of combining the valuation formula with the distribution of underlying risk factors to derive the distribution of contract values. This will be done later, in the market risk section (Part Five of this book).

Section 7.1 provides an overview of the size of the derivatives markets as well as trading mechanisms. Section 7.2 then presents the valuation and pricing of forwards. Sections 7.3 and 7.4 introduce futures and swap contracts, respectively.

7.1 DERIVATIVES MARKETS

7.1.1 Definitions

A **derivative instrument** can be generally defined as a private contract whose value derives from some underlying asset price, reference rate, or index—such as a stock, bond, currency, or commodity. In addition, the contract must also specify a principal, or **notional** amount, which is defined in terms of currency, shares,

FRM Exam Part 1 topic.

bushels, or some other unit. Movements in the value of the derivative depend on the notional and on the underlying price or index.

In contrast with **securities**, such as stocks and bonds, which are issued to raise capital, derivatives are **contracts**, or private agreements between two parties. Thus the sum of gains and losses on derivatives contracts must be zero. For any gain made by one party, the other party must have suffered a loss of equal magnitude.

7.1.2 Size of Derivatives Markets

At the broadest level, derivatives markets can be classified by the underlying instrument, as well as by the type of trading. Table 7.1 describes the size and growth of the global derivatives markets. As of 2009, the total notional amounts add up to almost $688 trillion, of which $615 trillion is on OTC markets and $73 trillion on organized exchanges. These markets have grown exponentially, from $56 trillion in 1995.

TABLE 7.1 Global Derivatives Markets, 1995–2009 (Billions of U.S. Dollars)

	Notional Amounts	
	March 1995	**Dec. 2009**
OTC Instruments	**47,530**	**614,674**
Interest rate contracts	**26,645**	**449,793**
Forwards (FRAs)	4,597	51,749
Swaps	18,283	349,236
Options	3,548	48,808
Foreign exchange contracts	**13,095**	**49,196**
Forwards and forex swaps	8,699	23,129
Swaps	1,957	16,509
Options	2,379	9,558
Equity-linked contracts	**579**	**6,591**
Forwards and swaps	52	1,830
Options	527	4,762
Commodity contracts	**318**	**2,944**
Credit default swaps	**0**	**32,693**
Others	**6,893**	**73,456**
Exchange-Traded Instruments	**8,838**	**73,140**
Interest rate contracts	**8,380**	**67,057**
Futures	5,757	20,628
Options	2,623	46,429
Foreign exchange contracts	**88**	**311**
Futures	33	164
Options	55	147
Stock index contracts	**370**	**5,772**
Futures	128	965
Options	242	4,807
Total	**55,910**	**687,814**

Source: Bank for International Settlements.

The table shows that interest rate contracts, especially swaps, are the most widespread type of derivatives. On the OTC market, currency contracts are also widely used, especially outright forwards and **forex swaps**, which are a combination of spot and short-term forward transactions. Among exchange-traded instruments, interest rate futures and options are the most common.

The magnitude of the notional amount of $688 trillion is difficult to grasp. This number is several times the world **gross domestic product (GDP)**, which amounted to approximately $61 trillion in 2008. It is also greater than the total outstanding value of stocks ($34 trillion) and of debt securities ($83 trillion) at that time.

Notional amounts give an indication of equivalent positions in cash markets. For example, a long futures contract on a stock index with a notional of $1 million is equivalent to a cash position in the stock market of the same magnitude. They are also important because they drive the fees to the financial industry, which are typically set as a fraction of the notionals.

Notional amounts, however, do not give much information about the risks of the positions. The current (positive) market value of OTC derivatives contracts, for instance, is estimated at $22 trillion. This is only 3% of the notional. More generally, the risk of these derivatives is best measured by the potential change in mark-to-market values over the horizon—in other words, by a value at risk measure.

7.1.3 Trading Mechanisms

Derivatives can be traded in private decentralized markets, called **over-the-counter** (OTC) markets, or on **organized exchanges**. Trading OTC is generally done with a **derivatives dealer**, which is a specially organized firm, usually associated with a major financial institution, that buys and sells derivatives.

A major issue when dealing with derivatives is **counterparty risk**. A counterparty is defined as the opposite side of a financial transaction. Suppose that Bank A entered a derivative contract with Hedge Fund B, where A agrees to purchase the euro by selling dollars at a fixed price of $1.3. If the euro goes up, the contract moves in-the-money for A. This means, however, that B suffers a loss. If this loss is large enough, B could default. Hence, this contract creates credit risk. Suffice to say, this topic will be developed at length in another section of this book.

Derivatives dealers manage their counterparty risk through a variety of means. Even so, these potential exposures are a major concern. This explains why American International Group (AIG) was rescued by the U.S. government. AIG had sold credit default swaps (CDSs) to many other institutions, and its failure could have caused domino effects for other banks.

Counterparty risk can be addressed with a **clearinghouse**. A clearinghouse is a financial institution that provides settlement and clearing services for financial transactions that may have taken place on an organized exchange or OTC.

The purpose of a clearinghouse is to reduce counterparty risk by interposing itself between the buyer and the seller, thereby ensuring payments on the contract. This is why clearinghouses are said to provide **central counterparty** (CCP) clearing. This is done through a process called **novation**, which refers to the replacement of a contract between two counterparties with a contract between the remaining party and a third party.[1] Clearinghouses, however, may also offer other services than CCP clearing.

Clearinghouses reduce counterparty risk by a variety of means. First, they allow netting, or offsetting, transactions between counterparties due to the fungibility of the contracts.[2] Suppose, for instance, that Bank A bought $100 million worth of euros from B. At the same time, it sold an otherwise identical contract to Bank C in the amount of $80 million. If the two trades had gone through the same clearinghouse, the net exposure of Bank A would be $20 million only.

Second, clearinghouses have in place procedures to manage their credit risk. They require **collateral deposits**, also called margins, which must be adjusted on a daily basis or more frequently. They mark contracts to market based on independent valuation of trades. Should the **clearing member** fail to add to the collateral when needed, the clearinghouse has the right to liquidate the position. In addition, clearinghouses generally rely on a guarantee fund provided by clearing members that can be used to cover losses in a default event.

Turning now to the trading aspects, an **organized exchange**, or **bourse**, is a highly organized market where financial instruments are bought and sold. Trading can occur either in a physical location or electronically. The most active stock market in the world is the **New York Stock Exchange** (NYSE). Another example is the **CME Group**, which is the largest derivatives market in the world. Trading at the CME is conducted in open outcry format, or electronically through Globex. In 2009, Globex accounted for 80% of total trading at the CME.

Most exchanges are **public**, trade public securities, and require registration with a local regulator, such as the Securities and Exchange Commission (SEC). **Private** exchanges, in contrast, are not registered and allow for the trading of unregistered securities, such as private placements.

All exchanges have a clearinghouse. A clearinghouse may be dedicated to one or several exchanges. As an example of the first case, which is also called **vertical integration**, CME Clearing is the in-house clearer for the CME Group. In contrast, the **Options Clearing Corporation** (OCC) supports 14 exchanges and platforms, including the Chicago Board Options Exchange and NYSE Amex. LCH.Clearnet Group handles many of the trades in major European financial exchanges. This model is called **horizontal integration**.

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[1] Novation requires the consent of all parties. This is in contrast to an assignment, which is valid as long as the other party is given notice.

[2] An important example, covered in a credit risk chapter in Part Six, is the netting of currency payments created by the CLS Bank.

OTC trading does not require a clearinghouse. This is changing slowly, however. About 25% of interest rate swaps are now routed through clearinghouses. Similarly, ICE Trust, the U.S.-based electronic futures exchange group, has started to clear CDS index contracts. Legislation will surely push more trading toward CCPs. OTC derivatives dealers, however, are generally opposed to CCPs because these would cut into their business and profits.

CCPs have other advantages. They create **transparency** in financial markets. Because the trading information is now stored centrally, CCPs can disseminate information about transactions, prices, and positions. This was a major issue with AIG because apparently regulators were not aware of the extent of AIG's short CDS positions. Thus CCPs could be used for trade **reporting**. Currently, most electronic trades of credit derivatives are reported to the **Depository Trust & Clearing Corporation** (DTCC), which is a financial institution that settles the vast majority of securities transactions in the United States.

Clearinghouses, however, have disadvantages as well. They do not eliminate counterparty risk in the financial system but rather concentrate it among themselves. Even if CCPs are more creditworthy than most other parties, a CCP failure could create systemic risk. CCPs generally have very high credit ratings, for example AAA for the OCC. CCP failures have been extremely rare, but could happen again.[3]

Regulators are contemplating whether CCPs should have access to the lending facilities of a central bank. Of course, this creates a problem of moral hazard, which is a situation where clearers would have less incentive to create a robust structure because they know that central banks would ride to their rescue anyway.

Second, the risk management aspect of positions is very important. CCPs have been able to handle simple, standardized contracts, which are easy to price. Forcing more complex instruments to clearinghouses increases the probability of operational problems. Even standardized instruments may not be very liquid, which could cause large losses in case of a forced liquidation.

Third, CCPs' netting benefits would be defeated by having too many existing CCPs. In our example of Bank A with clients B and C, there is no netting if the two trades are routed to two different CCPs. Further, having CCPs for different products would reduce the effectiveness of International Swaps and Derivatives Association (ISDA) agreements because these allow netting across many different products. Such agreements are called **bilateral** because they cover a variety of trades between only two parties.

The establishment of CCPs creates a global coordination problem. The United States, United Kingdom, and continental Europe all want their own CCP because their CCP might need to be backstopped by the local central bank. Regulation might also be needed to avoid CCPs competing against each other by setting low margins, which would make them less safe.

[3] Recorded failures include one in France in 1974, in Malaysia in 1983, and in Hong Kong in 1987. There have been near failures as well. In the wake of the October 1987 crash, both the CME and OCC encountered difficulty in receiving margins.

EXAMPLE 7.1: NOVATION

Novation is the process of:

a. Creating a new trade between two counterparties
b. Terminating an existing trade between two counterparties
c. Discharging a contract between the original counterparties and creating two new contracts, each with a central counterparty
d. Assigning a trade to another party

EXAMPLE 7.2: CENTRAL COUNTERPARTIES

Which of the following is *not* an advantage of establishing CCPs?

a. CCPs allow netting of contracts.
b. CCPs can be applied to some types of OTC trades.
c. CCPs can create more transparency in trading.
d. CCPs eliminate all counterparty risk in the financial system.

7.2 FORWARD CONTRACTS

7.2.1 Overview

The most common transactions in financial instruments are **spot transactions**, that is, for physical delivery as soon as practical (perhaps in two business days or in a week). Historically, grain farmers went to a centralized location to meet buyers for their product. As markets developed, the farmers realized that it would be beneficial to trade for delivery at some future date. This allowed them to hedge out price fluctuations for the sale of their anticipated production.

This gave rise to **forward contracts**, which are private agreements to exchange a given asset against cash (or sometimes another asset) at a fixed point in the future. The terms of the contract are the quantity (number of units or shares), date, and price at which the exchange will be done.

A position that implies buying the asset is said to be **long**. A position to sell is said to be **short**. Any gain to one party must be a loss to the other.

These instruments represent contractual obligations, as the exchange must occur whatever happens to the intervening price, unless default occurs. Unlike an option contract, there is no choice to take delivery or not.

To avoid the possibility of losses, the farmer could enter a forward sale of grain for dollars. By so doing, the farmer locks up a price now for delivery in the future. We then say that the farmer is **hedged** against movements in the price.

We use the notations

$$
\begin{aligned}
t &= \text{current time} \\
T &= \text{time of delivery} \\
\tau = T - t &= \text{time to maturity} \\
S_t &= \text{current spot price of the asset in dollars} \\
F_t(T) &= \text{current forward price of the asset for delivery at } T \\
&\quad \text{(also written as } F_t \text{ or } F \text{ to avoid clutter)} \\
V_t &= \text{current value of contract} \\
r &= \text{current domestic risk-free rate for delivery at } T \\
n &= \text{quantity, or number of units in contract}
\end{aligned}
$$

The **face amount**, or **principal value**, of the contract is defined as the amount nF to pay at maturity, like a bond. This is also called the **notional amount**. We will assume that interest rates are continuously compounded so that the present value of a dollar paid at expiration is $\text{PV}(\$1) = e^{-r\tau}$.

Say that the initial forward price is $F_t = \$100$. A speculator agrees to buy $n = 500$ units for F_t at T. At expiration, the payoff on the forward contract is determined in two steps as follows:

1. The speculator pays $nF = \$50{,}000$ in cash and receives 500 units of the underlying.
2. The speculator could then sell the underlying at the prevailing spot price S_T, for a profit of $n(S_T - F)$. For example, if the spot price is at $S_T = \$120$, the profit is $500 \times (\$120 - \$100) = \$10{,}000$. This is also the mark-to-market value of the contract at expiration.

In summary, the value of the forward contract at expiration, for one unit of the underlying asset, is:

$$V_T = S_T - F \tag{7.1}$$

Here, the value of the contract at expiration is derived from the purchase and **physical delivery** of the underlying asset. There is a payment of cash in exchange for the actual asset.

Another mode of settlement is **cash settlement**. This involves simply measuring the market value of the asset upon maturity, S_T, and agreeing for the long side to receive $nV_T = n(S_T - F)$. This amount can be positive or negative, involving a profit or loss.

Figures 7.1 and 7.2 present the payoff patterns on long and short positions in a forward contract, respectively. It is important to note that the payoffs are *linear* in the underlying spot price. Also, the positions in the two figures are symmetrical around the horizontal axis. For a given spot price, the sum of the profit or loss for the long and the short is zero, because these are private contracts.

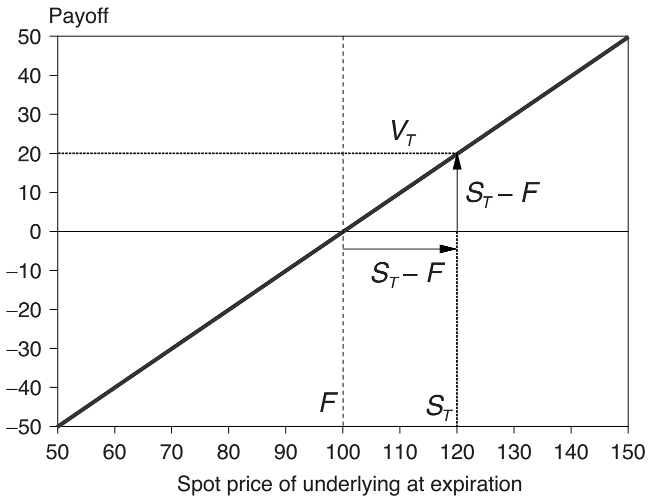

FIGURE 7.1 Payoff of Profits on Long Forward Contract

7.2.2 Valuing Forward Contracts

When evaluating forward contracts, two important questions arise. First, how is the current forward price F_t determined? Second, what is the current value V_t of an outstanding forward contract?

Initially, we assume that the underlying asset pays no income. This will be generalized in the next section. We also assume no transaction costs, that is, zero bid-ask spread on spot and forward quotations, as well as the ability to lend and borrow at the same risk-free rate.

Generally, forward contracts are established so that their initial value is zero. This is achieved by setting the forward price F_t appropriately by a **no-arbitrage relationship** between the cash and forward markets. No-arbitrage is a situation where positions with the same payoffs have the same price. This rules out situations

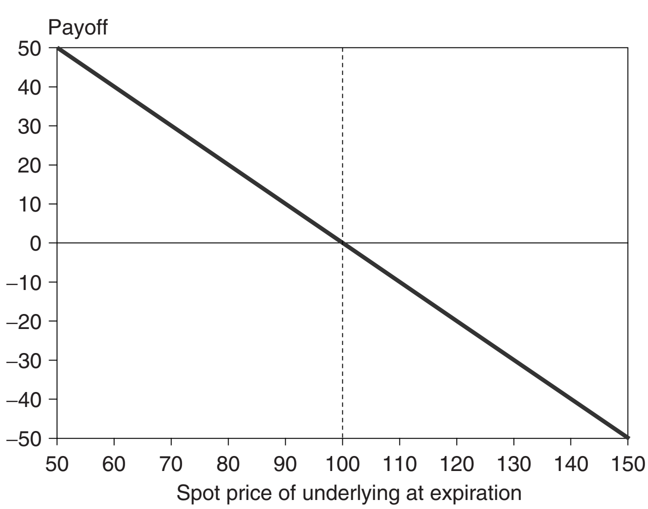

FIGURE 7.2 Payoff of Profits on Short Forward Contract

where **arbitrage profits** can exist. Arbitrage is a zero-risk, zero-net investment strategy that still generates profits.

Consider these strategies:

- Buy one share/unit of the underlying asset at the spot price S_t and hold to time T.
- Enter a forward contract to buy one share/unit of same underlying asset at the forward price F_t. In order to have sufficient funds at maturity to pay F_t, we invest the present value of F_t in an interest-bearing account. This is the present value $F_t e^{-r\tau}$. The forward price F_t is set so that the initial cost of the forward contract, V_t, is zero.

The two portfolios are economically equivalent because they will be identical at maturity. Each will contain one share of the asset. Hence their up-front cost must be the same. To avoid arbitrage, we must have:

$$S_t = F_t e^{-r\tau} \tag{7.2}$$

This equation defines the fair forward price F_t such that the initial value of the contract is zero. More generally, the term multiplying F_t is the present value factor for maturity τ, or PV($1). For instance, assuming $S_t = \$100$, $r = 5\%$, $\tau = 1$, we have $F_t = S_t e^{r\tau} = \$100 \times \exp(0.05 \times 1) = \$105.13$.

We see that the forward rate is higher than the spot rate. This reflects the fact that there is no down payment to enter the forward contract, unlike for the cash position. As a result, the forward price must be higher than the spot price to reflect the time value of money.

Abstracting from transaction costs, any deviation creates an arbitrage opportunity. This can be taken advantage of by buying the cheap asset and selling the expensive one. Assume, for instance, that $F = \$110$. We determined that the fair value is $S_t e^{r\tau} = \$105.13$, based on the cash price. We apply the principle of buying low at $105.13 and selling high at $110. We can lock in a sure profit in two steps by:

1. Buying now the asset spot at $100.
2. Selling now the asset forward at $110.

This can be done by borrowing the $100 to buy the asset now. At expiration, we will owe principal plus interest, or $105.13, but receive $110, for a profit of $4.87. This would be a blatant arbitrage opportunity, or "money machine."

Now consider a mispricing where $F = \$102$. We apply the principle of buying low at $102 and selling high at $105.13. We can lock in a sure profit in two steps by:

1. Short-selling now the asset spot at $100.
2. Buying now the asset forward at $102.

From the short sale, we invest the cash, which will grow to \$105.13. At expiration, we will have to deliver the stock, but this will be acquired through the forward purchase. We pay \$102 for this and are left with a profit of \$3.13.

This transaction involves the **short sale** of the asset, which is more involved than an outright purchase. When purchasing, we pay \$100 and receive one share of the asset. When short-selling, we borrow one share of the asset and promise to give it back at a future date; in the meantime, we sell it at \$100.[4]

7.2.3 Valuing an Off-Market Forward Contract

We can use the same reasoning to evaluate an outstanding forward contract with a locked-in delivery price of K. In general, such a contract will have nonzero value because K differs from the prevailing forward rate. Such a contract is said to be **off-market**.

Consider these strategies:

- Buy one share/unit of the underlying asset at the spot price S_t and hold it until time T.
- Enter a forward contract to buy one share/unit of same underlying asset at the price K; in order to have sufficient funds at maturity to pay K, we invest the present value of K in an interest-bearing account. This present value is also $Ke^{-r\tau}$. In addition, we have to pay the market value of the forward contract, or V_t.

The up-front cost of the two portfolios must be identical. Hence, we must have $V_t + Ke^{-r\tau} = S_t$, or

$$V_t = S_t - Ke^{-r\tau} \tag{7.3}$$

which defines the market value of an outstanding long position.[5] This gains value when the underlying S increases in value. A short position would have the reverse sign. Later, we will extend this relationship to the measurement of risk by considering the distribution of the underlying risk factors, S_t and r.

For instance, assume we still hold the previous forward contract with $F_t = \$105.13$ and after one month the spot price moves to $S_t = \$110$. The fixed rate is $K = \$105.13$ throughout the life of the contract. The interest has not changed at $r = 5\%$, but the maturity is now shorter by one month, $\tau = 11/12$. The new value of the contract is $V_t = S_t - Ke^{-r\tau} = \$110 - \$105.13\exp(-0.05 \times 11/12) = \$110 - \$100.42 = \$9.58$. The contract is now more valuable than before because the spot price has moved up.

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[4] In practice, we may not get full access to the proceeds of the sale when it involves individual stocks. The broker will typically allow us to withdraw only 50% of the cash. The rest is kept as a performance bond should the transaction lose money.

[5] Note that V_t is not the same as the forward price F_t. The former is the value of the contract; the latter refers to a specification of the contract.

7.2.4 Valuing Forward Contracts with Income Payments

We previously considered a situation where the asset produces no income payment. In practice, the asset may be

- A stock that pays a regular dividend
- A bond that pays a regular coupon
- A stock index that pays a dividend stream approximated by a continuous yield
- A foreign currency that pays a foreign-currency-denominated interest rate

Whichever income is paid on the asset, we can usefully classify the payment into **discrete**, that is, fixed dollar amounts at regular points in time, or on a **continuous** basis, that is, accrued in proportion to the time the asset is held. We must assume that the income payment is fixed or is certain. More generally, a storage cost is equivalent to a negative dividend.

We use these definitions:

$D =$ discrete (dollar) dividend or coupon payment
$r_t^*(T) =$ foreign risk-free rate for delivery at T
$q_t(T) =$ dividend yield

Whether the payment is a dividend or a foreign interest rate, the principle is the same. We can afford to invest less in the asset up front to get one unit at expiration. This is because the income payment can be reinvested into the asset. Alternatively, we can borrow against the value of the income payment to increase our holding of the asset.

It is also important to note that all prices (S, F) are measured in the domestic currency. For example, S could be expressed in terms of the U.S. dollar price of the euro, in which case r is the U.S. interest rate and r^* is the euro interest rate. Conversely, if S is the Japanese yen price of the U.S. dollar, r will represent the Japanese interest rate, and r^* the U.S. interest rate.

Continuing our example, consider a stock priced at \$100 that pays a dividend of $D = \$1$ in three months. The present value of this payment discounted over three months is $De^{-r\tau} = \$1 \exp(-0.05 \times 3/12) = \$0.99$. We only need to put up $S_t - \text{PV}(D) = \$100.00 - 0.99 = \$99.01$ to get one share in one year. Put differently, we buy 0.9901 fractional shares now and borrow against the (sure) dividend payment of \$1 to buy an additional 0.0099 fractional share, for a total of 1 share.

The pricing formula in Equation (7.2) is extended to

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{7.4}$$

where PV(D) is the present value of the dividend/coupon payments. If there is more than one payment, PV(D) represents the sum of the present values of each individual payment, discounted at the appropriate risk-free rate. With storage costs, we need to *add* the present value of storage costs PV(C) to the right side of Equation (7.4).

The approach is similar for an asset that pays a continuous income, defined per unit of time instead of discrete amounts. Holding a foreign currency, for instance, should be done through an interest-bearing account paying interest that accrues with time. Over the horizon τ, we can afford to invest less up front, $S_t e^{-r^*\tau}$, in order to receive one unit at maturity. The right-hand side of Equation (7.4) is now

$$F_t e^{-r\tau} = S_t e^{-r^*\tau} \tag{7.5}$$

Hence the forward price should be

$$F_t = S_t e^{-r^*\tau} / e^{-r\tau} \tag{7.6}$$

If instead interest rates are annually compounded, this gives

$$F_t = S_t(1 + r)^\tau / (1 + r^*)^\tau \tag{7.7}$$

Equation (7.6) can be also written in terms of the forward premium or discount, which is

$$\frac{(F_t - S_t)}{S_t} = e^{-r^*\tau} / e^{-r\tau} - 1 = \exp(r - r^*)\tau \approx (r - r^*)\tau \tag{7.8}$$

If $r^* < r$, we have $F_t > S_t$ and the asset trades at a **forward premium**. Conversely, if $r^* > r$, $F_t < S_t$ and the asset trades at a **forward discount**. Thus the forward price is higher or lower than the spot price, depending on whether the yield on the asset is lower than or higher than the domestic risk-free interest rate.

Equation (7.6) is also known as **interest rate parity** when dealing with currencies. Also note that both the spot and forward prices must be expressed in dollars per unit of the foreign currency when the domestic currency interest rate is r. This is the case, for example, for the dollar/euro or dollar/pound exchange rate. If, however, the exchange rate is expressed in foreign currency per dollar, then r must be the rate on the foreign currency. For the yen/dollar rate, for example, S is in yen per dollar, r is the yen interest rate, and r^* is the dollar interest rate.

KEY CONCEPT

The forward price differs from the spot price to reflect the time value of money and the income yield on the underlying asset. It is higher than the spot price if the yield on the asset is lower than the domestic risk-free interest rate, and vice versa.

With income payments, the value of an outstanding forward contract is

$$V_t = S_t e^{-r^*\tau} - K e^{-r\tau} \tag{7.9}$$

If F_t is the new, current forward price, we can also write

$$V_t = F_t e^{-r\tau} - K e^{-r\tau} = (F_t - K) e^{-r\tau} \tag{7.10}$$

This provides a useful alternative formula for the valuation of a forward contract. The intuition here is that we could liquidate the outstanding forward contract by entering a reverse position at the current forward rate. The payoff at expiration is $(F - K)$, which, discounted back to the present, gives Equation (7.10).

KEY CONCEPT

The current value of an outstanding forward contract can be found by entering an offsetting forward position and discounting the net cash flow at expiration.

EXAMPLE 7.3: FRM EXAM 2008—QUESTION 2-15

The one-year U.S. dollar interest rate is 2.75% and one-year Canadian dollar interest rate is 4.25%. The current USD/CAD spot exchange rate is 1.0221–1.0225. Calculate the one-year USD/CAD forward rate. Assume annual compounding.

a. 1.0076
b. 1.0074
c. 1.0075
d. 1.03722

EXAMPLE 7.4: FRM EXAM 2005—QUESTION 16

Suppose that U.S. interest rates rise from 3% to 4% this year. The spot exchange rate quotes at 112.5 JPY/USD and the forward rate for a one-year contract is at 110.5. What is the Japanese interest rate?

a. 1.81%
b. 2.15%
c. 3.84%
d. 5.88%

EXAMPLE 7.5: FRM EXAM 2002—QUESTION 56

Consider a forward contract on a stock market index. Identify the *false* statement. Everything else being constant,

a. The forward price depends directly on the level of the stock market index.
b. The forward price will fall if underlying stocks increase the level of dividend payments over the life of the contract.
c. The forward price will rise if time to maturity is increased.
d. The forward price will fall if the interest rate is raised.

EXAMPLE 7.6: FRM EXAM 2007—QUESTION 119

A three-month futures contract on an equity index is currently priced at USD 1,000. The underlying index stocks are valued at USD 990 and pay dividends at a continuously compounded rate of 2%. The current continuously compounded risk-free rate is 4%. The potential arbitrage profit per contract, given this set of data, is closest to

a. USD 10.00
b. USD 7.50
c. USD 5.00
d. USD 1.50

7.3 FUTURES CONTRACTS

7.3.1 Overview

Forward contracts allow users to take positions that are economically equivalent to those in the underlying cash markets. Unlike cash markets, however, they do not involve substantial up-front payments. Thus, forward contracts can be interpreted as having *leverage*. Leverage is efficient, as it makes our money work harder.

Leverage creates credit risk for the counterparty, however. For a cash trade, there is no leverage. When a speculator buys a stock at the price of $100, the counterparty receives the cash and has no credit risk. Instead, when a speculator enters a forward contract to buy an asset at the price of $105, there is no up-front payment. In effect, the speculator borrows from the counterparty to invest in the asset. There is a risk of default should the value of the contract to the speculator fall sufficiently. In response, futures contracts have been structured so as to minimize credit risk for all counterparties. Otherwise, from a market risk standpoint, futures contracts are basically identical to forward contracts.

Futures contracts are standardized, negotiable, and exchange-traded contracts to buy or sell an underlying asset. They differ from forward contracts as follows.

- **Trading on organized exchanges.** In contrast to forwards, which are OTC contracts tailored to customers' needs, futures are traded on organized exchanges.
- **Standardization.** Futures contracts are offered with a limited choice of expiration dates. They trade in fixed contract sizes. This standardization ensures an active secondary market for many futures contracts, which can be easily traded, purchased, or resold. In other words, most futures contracts have good liquidity. The trade-off is that futures are less precisely suited to the needs of some hedgers, which creates basis risk (to be defined later).
- **Clearinghouse.** Futures contracts are also standardized in terms of the counterparty. After each transaction is confirmed, the clearinghouse basically interposes itself between the buyer and the seller, ensuring the performance of the contract. Thus, unlike forward contracts, counterparties do not have to worry about the credit risk of the other side of the trade.
- **Marking to market.** As the clearinghouse now has to deal with the credit risk of the two original counterparties, it has to monitor credit risk closely. This is achieved by daily marking to market, which involves settlement of the gains and losses on the contract every day. This will avoid the accumulation of large losses over time, potentially leading to an expensive default.
- **Margins.** Although daily settlement accounts for past losses, it does not provide a buffer against future losses. This is the goal of **margins**, which represent posting of collateral that can be seized should the other party default. The **initial margin** must be posted when initiating the position. If the equity in the account falls below the **maintenance margin**, the customer is required to provide additional funds to cover the initial margin. Note that the amount to fill up is not to the maintenance margin, but to the initial margin. The level of margin depends on the instrument and the type of position; in general, less volatile instruments or hedged positions require lower margins.

Example: Margins for a Futures Contract

Consider a futures contract on 1,000 units of an asset worth \$100. A long futures position is economically equivalent to holding \$100,000 worth of the asset directly. To enter the futures position, a speculator has to post only \$5,000 in margin, for example. This amount is placed in an equity account with the broker.

The next day, the futures price moves down by \$3, leading to a loss of \$3,000 for the speculator. The loss is subtracted from the equity account, bringing it down to \$5,000 − \$3,000 = \$2,000. The speculator would then receive a **margin call** from the broker, asking to have an additional \$3,000 of capital posted to the account. If he or she fails to meet the margin call, the broker has the right to liquidate the position.

Since futures trading is centralized on an exchange, it is easy to collect and report aggregate trading data. **Volume** is the number of contracts traded during the day, which is a flow item. **Open interest** represents the outstanding number of contracts at the close of the day, which is a stock item.

7.3.2 Valuing Futures Contracts

Valuation principles for futures contracts are very similar to those for forward contracts. The main difference between the two types of contracts is that any profit or loss accrues *during* the life of the futures contract instead of all at once, at expiration.

When interest rates are assumed constant or deterministic, forward and futures prices must be equal. With stochastic interest rates, there may be a small difference, depending on the correlation between the value of the asset and interest rates.

If the correlation is zero, then it makes no difference whether payments are received earlier or later. The futures price must be the same as the forward price. In contrast, consider a contract whose price is positively correlated with the interest rate. If the value of the contract goes up, it is more likely that interest rates will go up as well. This implies that profits can be withdrawn and reinvested at a higher rate. Relative to forward contracts, this marking-to-market feature is beneficial to a long futures position. As a result, the futures price must be higher in equilibrium.

In practice, this effect is observable only for interest-rate futures contracts, whose value is *negatively* correlated with interest rates. Because this feature is unattractive for the long position, the futures price must be *lower* than the forward price. Chapter 8 explains how to compute the adjustment, called the **convexity effect**.

EXAMPLE 7.7: FRM EXAM 2004—QUESTION 38

An investor enters into a short position in a gold futures contract at USD 294.20. Each futures contract controls 100 troy ounces. The initial margin is USD 3,200, and the maintenance margin is USD 2,900. At the end of the first day, the futures price drops to USD 286.6. Which of the following is the amount of the variation margin at the end of the first day?

a. 0
b. USD 34
c. USD 334
d. USD 760

EXAMPLE 7.8: FRM EXAM 2004—QUESTION 66

Which one of the following statements is *incorrect* regarding the margining of exchange-traded futures contracts?

a. Day trades and spread transactions require lower margin levels.
b. If an investor fails to deposit variation margin in a timely manner, the positions may be liquidated by the carrying broker.
c. Initial margin is the amount of money that must be deposited when a futures contract is opened.
d. A margin call will be issued only if the investor's margin account balance becomes negative.

7.4 SWAP CONTRACTS

Swap contracts are OTC agreements to exchange a *series* of cash flows according to prespecified terms. The underlying asset can be an interest rate, an exchange rate, an equity, a commodity price, or any other index. Typically, swaps are established for longer periods than forwards and futures.

For example, a 10-year currency swap could involve an agreement to exchange every year 5 million dollars against 3 million pounds over the next 10 years, in addition to a principal amount of 100 million dollars against 50 million pounds at expiration. The principal is also called **notional principal**.

Another example is that of a five-year interest rate swap in which one party pays 8% of the principal amount of 100 million dollars in exchange for receiving an interest payment indexed to a floating interest rate. In this case, since both payments are the same amount in the same currency, there is no need to exchange principal at maturity.

Swaps can be viewed as a portfolio of forward contracts. They can be priced using valuation formulas for forwards. Our currency swap, for instance, can be viewed as a combination of 10 forward contracts with various face values, maturity dates, and rates of exchange. We will give detailed examples in later chapters.

7.5 IMPORTANT FORMULAS

Forward price, no income on the asset: $F_t e^{-r\tau} = S_t$

Forward price, income on the asset:

Discrete dividend, $F_t e^{-r\tau} = S_t - \text{PV}(D)$

Continuous dividend, $F_t e^{-r\tau} = S_t e^{-r^*\tau}$

Forward premium or discount: $\frac{(F_t - S_t)}{S_t} \approx (r - r^*)\tau$

Valuation of outstanding forward contract: $V_t = S_t e^{-r^*\tau} - Ke^{-r\tau} = F_t e^{-r\tau} - Ke^{-r\tau} = (F_t - K)e^{-r\tau}$

7.6 ANSWERS TO CHAPTER EXAMPLES

Example 7.1: Novation

c. Novation involves the substitution of counterparties. Clearinghouses use this process to interpose themselves between buyers and sellers. This requires consent from all parties, unlike an assignment.

Example 7.2: Central Counterparties

d. CCPs generally reduce counterparty risk but can be a source of systemic risk if they fail.

Example 7.3: FRM Exam 2008—Question 2-15

a. The spot price is the middle rate of \$1.0223. Using annual (not continuous) compounding, the forward price is $F = S(1 + r)/(1 + R^*) = 1.0223(1.0275)/(1.0425) = 1.0076$.

Example 7.4: FRM Exam 2005—Question 16

b. As is the convention in the currency markets, the exchange rate is defined as the yen price of the dollar, which is the foreign currency. The foreign currency interest rate is the latest U.S. dollar rate, or 4%. Assuming discrete compounding, the pricing formula for forward contracts is $F(JPY/USD)/(1 + rT) = S(JPY/USD)/(1 + r^*T)$. Therefore, $(1 + rT) = (F/S)(1 + r^*T) = (110.5/112.5)(1.04) = 1.0215$, and $r = 2.15\%$. Using continuous compounding gives a similar result. Another approach would consider the forward discount on the dollar, which is $(F - S)/S = -1.8\%$. Thus the dollar is 1.8% cheaper forward than spot, which must mean that the Japanese interest rate must be approximately 1.8% lower than the U.S. interest rate.

Example 7.5: FRM Exam 2002—Question 56

d. Defining the dividend yield as q, the forward price depends on the cash price according to $F\ \exp(-rT) = S\ \exp(-qT)$. This can also be written as $F = S\ \exp[+(r - q)T]$. Generally, $r > q$. Statement a. is correct: F depends directly on S. Statement b. is also correct, as higher q decreases the term between brackets and hence F. Statement c. is correct because the term $r - q$ is positive, leading to a larger term in brackets as the time to maturity T increases. Statement d. is false, as increasing r makes the forward contract more attractive, or increases F.

Example 7.6: FRM Exam 2007—Question 119

c. The fair value of the futures contract is given by $F = S\exp(-r^*T)/\exp(-rT) = 990\exp(-0.02 \times 3/12)/\exp(-0.04 \times 3/12) = 994.96$. Hence the actual futures price is too high by $(1{,}000 - 995) = 5$.

Example 7.7: FRM Exam 2004—Question 38

a. This is a tricky question. Because the investor is short and the price fell, the position creates a profit and there is no variation margin. However, for the long the loss is \$760, which would bring the equity to \$3,200 – \$760 = \$2,440. Because this is below the maintenance margin of \$2,900, an additional payment of \$760 is required to bring back the equity to the initial margin.

Example 7.8: FRM Exam 2004—Question 66

d. All the statements are correct, except d. If the margin account balance falls below the maintenance margin (not zero), a margin call will be issued.

CHAPTER 8

Option Markets

This chapter now turns to nonlinear derivatives, or options. As described in the previous chapter, options account for a large part of the derivatives markets. On organized exchanges, options represent more than $50 trillion in derivatives outstanding. Over-the-counter options add up to more than $60 trillion in notional amounts.

Although the concept behind these instruments is not new, option markets have blossomed since the early 1970s, because of a breakthrough in pricing options (the Black-Scholes formula) and advances in computing power. We start with **plain-vanilla** options: calls and puts. These are the basic building blocks of many financial instruments. They are also more common than complicated, **exotic** options.

The purpose of this chapter is to present a compact overview of important concepts for options, including their pricing. We will cover option sensitivities (the "Greeks") in a future chapter. Section 8.1 presents the payoff functions on basic options and combinations thereof. We then discuss option prices, or premiums, in Section 8.2. The Black-Scholes pricing approach is presented in Section 8.3. Next, Section 8.4 briefly summarizes more complex options. Finally, Section 8.5 shows how to value options using a numerical, binomial tree model.

8.1 OPTION PAYOFFS

8.1.1 Basic Options

Options are instruments that give their holder the *right* to buy or sell an asset at a specified price until a specified expiration date. The specified delivery price is known as the **delivery price**, or **exercise price**, or **strike price**, and is denoted by K.

Options to buy are **call options**. Options to sell are **put options**. As options confer a right to the purchaser of the option, but not an obligation, they will be exercised only if they generate profits. In contrast, forwards involve an obligation to either buy or sell and can generate profits or losses. Like forward contracts, options can be purchased or sold. In the latter case, the seller is said to **write** the option.

Depending on the timing of exercise, options can be classified into European or American options. **European options** can be exercised at maturity only. **American**

FRM Exam Part 1 topic. This chapter also covers the valuation of options.

options can be exercised at any time, before or at maturity. Because American options include the right to exercise at maturity, they must be at least as valuable as European options. In practice, however, the value of this early exercise feature is small, as an investor can generally receive better value by reselling the option on the open market instead of exercising it.

We use these notations, in addition to those in the previous chapter:

K = exercise price
c = value of European call option
C = value of American call option
p = value of European put option
P = value of American put option

To illustrate, take an option on an asset that currently trades at \$85 with a delivery price of \$100 in one year. If the spot price stays at \$85 at expiration, the holder of the call will not **exercise** the option, because the option is not profitable with a stock price less than \$100. In contrast, if the price goes to \$120, the holder will exercise the right to buy at \$100, will acquire the stock now worth \$120, and will enjoy a paper profit of \$20. This profit can be realized by selling the stock. For put options, a profit accrues if the spot price ends up below the exercise price $K = \$100$.

Thus the payoff profile of a long position in a call option at expiration is

$$C_T = \text{Max}(S_T - K, 0) \tag{8.1}$$

The payoff profile of a long position in a put option is

$$P_T = \text{Max}(K - S_T, 0) \tag{8.2}$$

If the current asset price S_t is close to the strike price K, the option is said to be **at-the-money**. If the current asset price S_t is such that the option could be exercised now at a profit, the option is said to be **in-the-money**. If the remaining situation is present, the option is said to be **out-of-the-money**. A call will be in-the-money if $S_t > K$. A put will be in-the-money if $S_t < K$.

As in the case of forward contracts, the payoff at expiration can be cash settled. Instead of actually buying the asset, the contract could simply pay \$20 if the price of the asset is \$120.

Because buying options can generate only profits (at worst zero) at expiration, an option contract must be a valuable asset (or at worst have zero value). This means that a payment is needed to acquire the contract. This up-front payment, which is much like an insurance premium, is called the option **premium**. This premium cannot be negative. An option becomes more expensive as it moves in-the-money.

Thus the payoffs on options must take into account this cost (for long positions) or benefit (for short positions). To compute the total payoff, we should

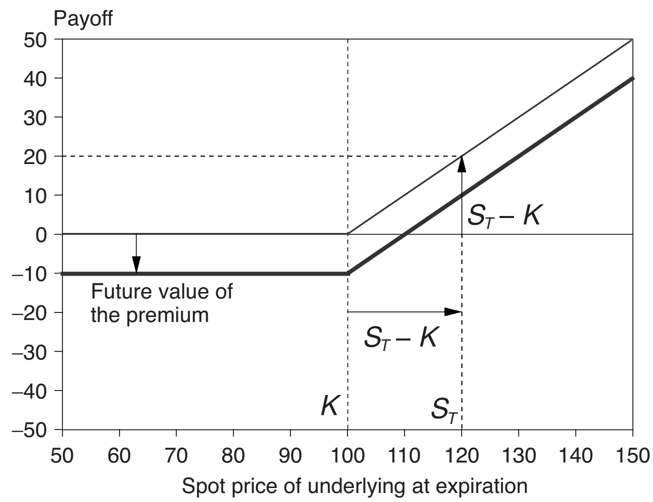

FIGURE 8.1 Profit Payoffs on Long Call

translate all option payoffs by the *future* value of the premium, that is, $ce^{r\tau}$, for European call options.

Figure 8.1 displays the total profit payoff on a call option as a function of the asset price at expiration. Assuming that $S_T = \$120$, the proceeds from exercise are $\$120 - \$100 = \$20$, from which we have to subtract the future value of the premium, say \$10. In the graphs that follow, we always take into account the cost of the option.

Figure 8.2 summarizes the payoff patterns on long and short positions in a call and a put contract. Unlike those of forwards, these payoffs are **nonlinear** in the underlying spot price. Sometimes they are referred to as the "hockey stick" diagrams. This is because forwards are obligations, whereas options are rights. Note that the positions for the same contract are symmetrical around the horizontal axis. For a given spot price, the sum of the profit or loss for the long and for the short is zero.

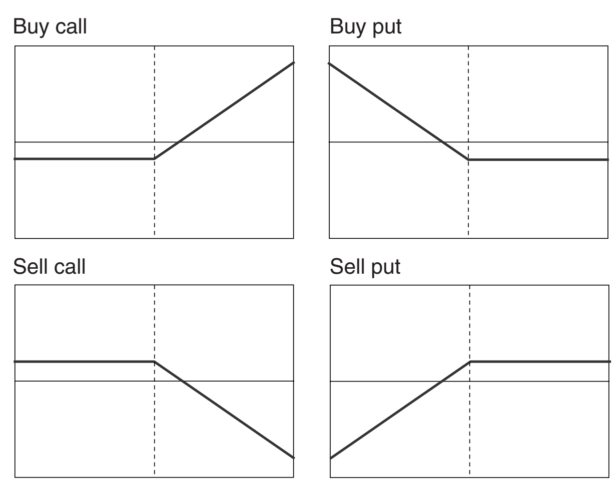

FIGURE 8.2 Profit Payoffs on Long and Short Calls and Puts

In the market risk section of this handbook (Part Five), we combine these payoffs with the distribution of the risk factors. Even so, it is immediately obvious that long option positions have limited downside risk, which is the loss of the premium. Short call option positions have unlimited downside risk because there is no upper limit on S. The worst loss on short put positions occurs if S goes to zero.

So far, we have covered options on cash instruments. Options can also be struck on futures. When exercising a call, the investor becomes long the futures contract. Conversely, exercising a put creates a short position in the futures contract. Because positions in futures are equivalent to leveraged positions in the underlying cash instrument, options on cash instruments and on futures are economically equivalent.

8.1.2 Put-Call Parity

These option payoffs can be used as the basic building blocks for more complex positions. A long position in the underlying asset can be decomposed into a long call plus a short put with the same strike prices and maturities, as shown in Figure 8.3.

The figure shows that the long call provides the equivalent of the upside while the short put generates the same downside risk as holding the asset. This link creates a relationship between the value of the call and that of the put, also known as **put-call parity**. The relationship is illustrated in Table 8.1, which examines the payoff at initiation and at expiration under the two possible states of the world. We consider only European options with the same maturity and exercise price. Also, we assume that there is no income payment on the underlying asset.

The portfolio consists of a long position in the call, a short position in the put, and an investment to ensure that we will be able to pay the exercise price at maturity. Long positions are represented by negative values, as they represent outflows, or costs.

The table shows that the final payoffs to portfolio (1) add up to S_T in the two states of the world, which is the same as a long position in the asset itself. Hence, to avoid arbitrage, the initial payoff must be equal to the current cost of the asset,

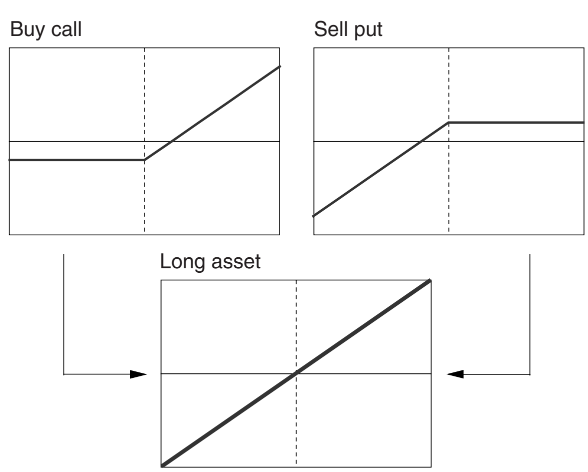

FIGURE 8.3 Decomposing a Long Position in the Asset

TABLE 8.1 Put-Call Parity

Portfolio	Position	Initial Payoff	Final Payoff $S_T < K$	Final Payoff $S_T \geq K$
(1)	Buy call	$-c$	0	$S_T - K$
	Sell put	$+p$	$-(K - S_T)$	0
	Invest	$-Ke^{-r\tau}$	K	K
	Total	$-c + p - Ke^{-r\tau}$	S_T	S_T
(2)	Buy asset	$-S$	S_T	S_T

which is $S_t = S$. So, we must have $-c + p - Ke^{-r\tau} = -S$. More generally, with income paid at the rate of r^*, put-call parity can be written as

$$c - p = Se^{-r^*\tau} - Ke^{-r\tau} = (F - K)e^{-r\tau} \tag{8.3}$$

Because $c \geq 0$ and $p \geq 0$, this relationship can also be used to determine lower bounds for European calls and puts. Note that the relationship does not hold exactly for American options since there is a likelihood of early exercise, which could lead to mismatched payoffs.

Finally, this relationship can be used to determine the **implied dividend yield** from market prices. We observe c, p, S, and r and can solve for y or r^*. This yield is used for determining the forward rate in **dividend swaps**, which are contracts where the payoff is indexed to the actual dividends paid over the horizon, minus the implied dividends.

KEY CONCEPT

A long position in an asset is equivalent to a long position in a European call with a short position in an otherwise identical put, combined with a risk-free position.

EXAMPLE 8.1: FRM EXAM 2007—QUESTION 84

According to put-call parity, buying a put option on a stock is equivalent to

a. Buying a call option and buying the stock with funds borrowed at the risk-free rate
b. Selling a call option and buying the stock with funds borrowed at the risk-free rate
c. Buying a call option, selling the stock, and investing the proceeds at the risk-free rate
d. Selling a call option, selling the stock, and investing the proceeds at the risk-free rate

EXAMPLE 8.2: FRM EXAM 2005—QUESTION 72

A one-year European put option on a non-dividend-paying stock with strike at EUR 25 currently trades at EUR 3.19. The current stock price is EUR 23 and its annual volatility is 30%. The annual risk-free interest rate is 5%. What is the price of a European call option on the same stock with the same parameters as those of this put option? Assume continuous compounding.

a. EUR 1.19
b. EUR 3.97
c. EUR 2.41
d. Cannot be determined with the data provided

EXAMPLE 8.3: FRM EXAM 2008—QUESTION 2-10

The current price of stock ABC is $42 and the call option with a strike at $44 is trading at $3. Expiration is in one year. The corresponding put is priced at $2. Which of the following trading strategies will result in arbitrage profits? Assume that the risk-free rate is 10% and that the risk-free bond can be shorted costlessly. There are no transaction costs.

a. Long position in both the call option and the stock, and short position in the put option and risk-free bond
b. Long position in both the call option and the put option, and short position in the stock and risk-free bond
c. Long position in both the call option and the risk-free bond, and short position in the stock and the put option
d. Long position in both the put option and the risk-free bond, and short position in the stock and the call option

EXAMPLE 8.4: FRM EXAM 2006—QUESTION 74

Jeff is an arbitrage trader, who wants to calculate the implied dividend yield on a stock while looking at the over-the-counter price of a five-year European put and call on that stock. He has the following data: $S = \$85$, $K = \$90$, $r = 5\%$, $c = \$10$, $p = \$15$. What is the continuous implied dividend yield of that stock?

a. 2.48%
b. 4.69%
c. 5.34%
d. 7.71%

8.1.3 Combination of Options

Options can be combined in different ways, either with each other or with the underlying asset. Consider first combinations of the underlying asset and an option. A long position in the stock can be accompanied by a short sale of a call to collect the option premium. This operation, called a **covered call**, is described in Figure 8.4. Likewise, a long position in the stock can be accompanied by a purchase of a put to protect the downside. This operation is called a **protective put**.

Options can also be combined with an underlying position to limit the range of potential gains and losses. Suppose an investor is long a stock, currently trading at \$10. The investor can buy a put with a low strike price (e.g., \$7), partially financed by the sale of a call with high strike (e.g., \$12). Ignoring the net premium, the highest potential gain is \$2 and the worst loss is \$3. Such a strategy is called a **collar**. If the strike prices were the same, this would be equivalent to a short stock position, which creates a net payoff of exactly zero.

We can also combine a call and a put with the same or different strike prices and maturities. When the strike prices of the call and the put and their maturities are the same, the combination is referred to as a **straddle**. Figure 8.5 shows how to construct a long straddle (i.e., buying a call and a put with the same maturity and strike price). This position is expected to benefit from a large price move, whether up or down. The reverse position is a short straddle. When the strike prices are different, the combination is referred to as a **strangle**. Since strangles are out-of-the-money, they are cheaper to buy than straddles.

Thus far, we have concentrated on positions involving two classes of options. One can, however, establish positions with one class of options, called **spreads**. Calendar or **horizontal spreads** correspond to different maturities. **Vertical spreads** correspond to different strike prices. The names of the spreads are derived from the manner in which they are listed in newspapers: time is listed horizontally and strike prices are listed vertically. **Diagonal spreads** move across maturities and strike prices.

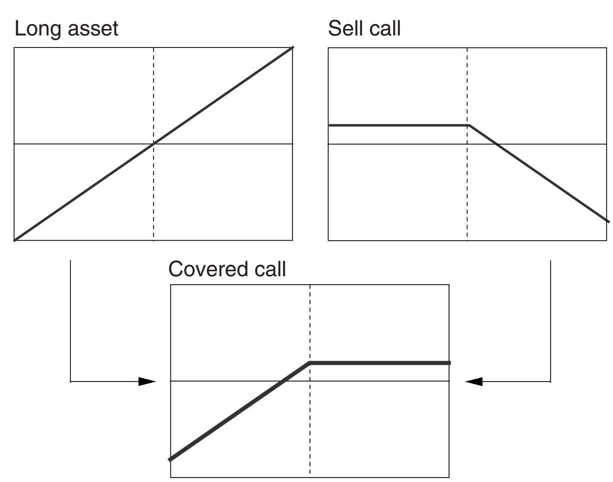

FIGURE 8.4 Creating a Covered Call

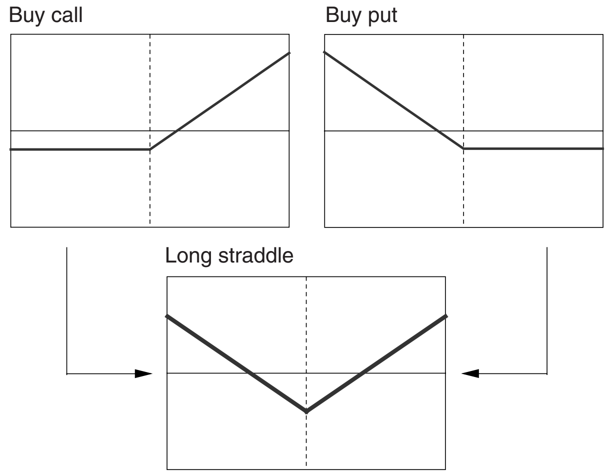

FIGURE 8.5 Creating a Long Straddle

For instance, a **bull spread** is positioned to take advantage of an increase in the price of the underlying asset. Conversely, a **bear spread** represents a bet on a falling price. Figure 8.6 shows how to construct a bull(ish) vertical spread with two calls with the same maturity. This could also be constructed with puts, however. Here, the spread is formed by buying a call option with a low exercise price K_1 and selling another call with a higher exercise price K_2. Note that the cost of the first call $c(S, K_1)$ must exceed the cost of the second call $c(S, K_2)$, because the first option is more in-the-money than the second. Hence, the sum of the two premiums represents a net cost. At expiration, when $S_T > K_2$, the payoff is $\text{Max}(S_T - K_1, 0) - \text{Max}(S_T - K_2, 0) = (S_T - K_1) - (S_T - K_2) = K_2 - K_1$, which is positive. Thus this position is expected to benefit from an up move, while incurring only limited downside risk.

Spreads involving more than two positions are referred to as butterfly or sandwich spreads. A **butterfly spread** involves three types of options with the same maturity: for example, a long call at a strike price K_1, two short calls at a higher strike price K_2, and a long call position at a higher strike price K_3, with the

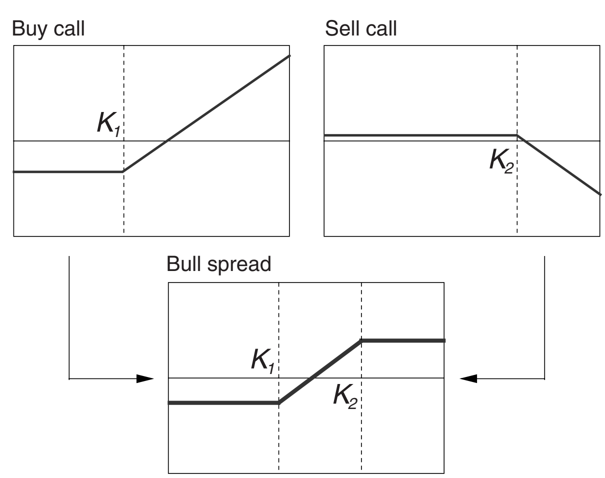

FIGURE 8.6 Creating a Bull Spread

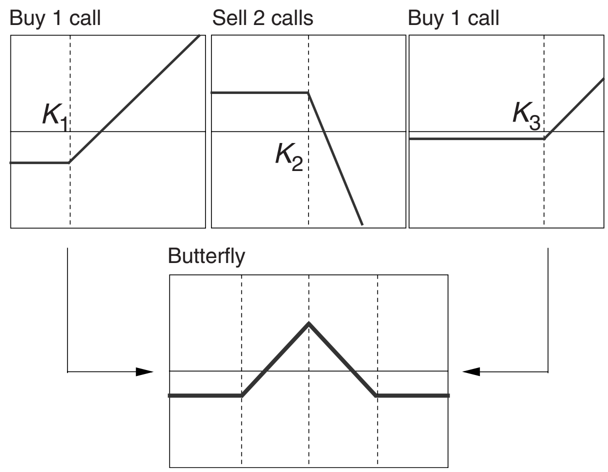

FIGURE 8.7 Creating a Butterfly Spread

same spacing. Figure 8.7 shows that this position is expected to benefit when the underlying asset price stays stable, close to K_2. The double position in the middle is called the body, and the others the wings. A sandwich spread is the opposite of a butterfly spread.

EXAMPLE 8.5: RISK OF OPTION CONTRACTS

Which of the following is the riskiest form of speculation using option contracts?

a. Setting up a spread using call options
b. Buying put options
c. Writing naked call options
d. Writing naked put options

EXAMPLE 8.6: FRM EXAM 2007—QUESTION 103

An investor sells a June 2008 call of ABC Limited with a strike price of USD 45 for USD 3 and buys a June 2008 call of ABC Limited with a strike price of USD 40 for USD 5. What is the name of this strategy and the maximum profit and loss the investor could incur?

a. Bear spread, maximum loss USD 2, maximum profit USD 3
b. Bull spread, maximum loss unlimited, maximum profit USD 3
c. Bear spread, maximum loss USD 2, maximum profit unlimited
d. Bull spread, maximum loss USD 2, maximum profit USD 3

EXAMPLE 8.7: FRM EXAM 2006—QUESTION 45

A portfolio manager wants to hedge his bond portfolio against changes in interest rates. He intends to buy a put option with a strike price below the portfolio's current price in order to protect against rising interest rates. He also wants to sell a call option with a strike price above the portfolio's current price in order to reduce the cost of buying the put option. What strategy is the manager using?

a. Bear spread
b. Strangle
c. Collar
d. Straddle

EXAMPLE 8.8: FRM EXAM 2002—QUESTION 42

Consider a bearish option strategy of buying one $50 strike put for $7, selling two $42 strike puts for $4 each, and buying one $37 put for $2. All options have the same maturity. Calculate the final profit per share of the strategy if the underlying is trading at $33 at expiration.

a. $1 per share
b. $2 per share
c. $3 per share
d. $4 per share

EXAMPLE 8.9: FRM EXAM 2009—QUESTION 3-8

According to an in-house research report, it is expected that USDJPY (quoted as JPY/USD) will trade near 97 at the end of March. Frankie Shiller, the investment director of a house fund, decides to use an option strategy to capture this opportunity. The current level of the USDJPY exchange rate is 97 on February 28. Accordingly, which of the following strategies would be the most appropriate for the largest profit while the potential loss is limited?

a. Long a call option on USDJPY and long a put option on USDJPY with the same strike price of USDJPY 97 and expiration date
b. Long a call option on USDJPY with strike price of USDJPY 97 and short a call option on USDJPY with strike price of USDJPY 99 and the same expiration date
c. Short a call option on USDJPY and long a put option on USDJPY with the same strike price of USDJPY 97 and expiration date
d. Long a call option with strike price of USDJPY 96, long a call option with strike price of USDJPY 98, and sell two call options with strike price of USDJPY 97, all of them with the same expiration date

8.2 OPTION PREMIUMS

8.2.1 General Relationships

So far, we have examined the payoffs at expiration only. Also important is the instantaneous relationship between the option value and the current price S, which is displayed in Figures 8.8 and 8.9.

For a call, a higher price S increases the current value of the option, but in a nonlinear, convex fashion. For a put, lower values for S increase the value of the option, also in a convex fashion. As time goes by, the curved line approaches the hockey stick line.

Figures 8.8 and 8.9 decompose the current premium into

- An **intrinsic value**, which basically consists of the value of the option if exercised today, or $\text{Max}(S_t - K, 0)$ for a call and $\text{Max}(K - S_t, 0)$ for a put
- A **time value**, which consists of the remainder, reflecting the possibility that the option will create further gains in the future

Consider for example a one-year call with strike $K = \$100$. The current price is $S = \$120$ and interest rate $r = 5\%$. The asset pays no dividend. Say the call premium is \$26.17. This can be decomposed into an intrinsic value of \$120 – \$100 = \$20 and time value of \$6.17. The time value increases with the volatility of the underlying asset. It also generally increases with the maturity of the option.

As shown in the figures, options can be classified into:

- **At-the-money**, when the current spot price is close to the strike price
- **In-the-money**, when the intrinsic value is large

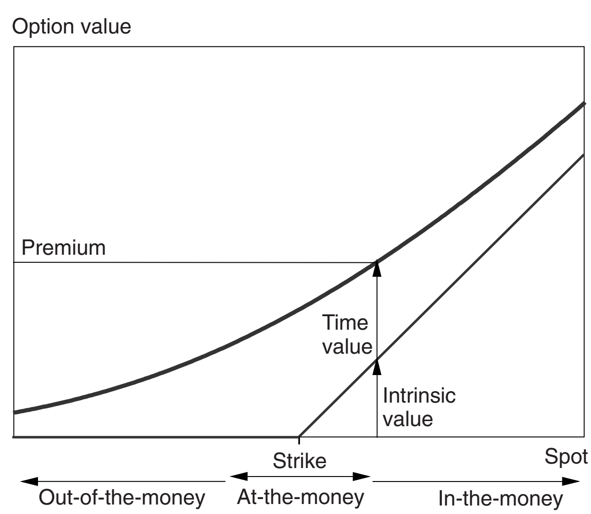

FIGURE 8.8 Relationship between Call Value and Spot Price

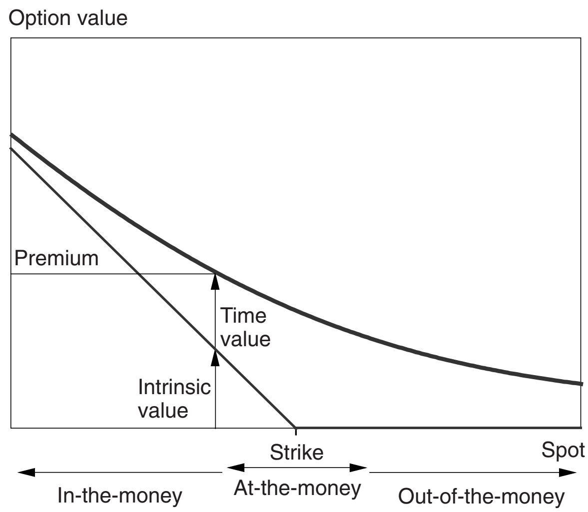

FIGURE 8.9 Relationship between Put Value and Spot Price

- **Out-of-the-money**, when the spot price is much below the strike price for calls and conversely for puts (out-of-the-money options have zero intrinsic value)

We can also identify some general bounds for European options that should always be satisfied; otherwise there would be an arbitrage opportunity (i.e., a money machine). For simplicity, assume there is no dividend. We know that a European option is worth less than an American option. First, the current value of a call must be less than, or equal to, the asset price:

$$c_t \leq C_t \leq S_t \tag{8.4}$$

This is because, in the limit, an option with zero exercise price is equivalent to holding the stock in this case. We are sure to exercise the option.

Second, the value of a European call must be greater than, or equal to, the price of the asset minus the present value of the strike price:

$$c_t \geq S_t - Ke^{-r\tau} \tag{8.5}$$

To prove this, we could simply use put-call parity, or Equation (8.3) with $r^* = 0$, imposing the condition that $p \geq 0$. Note that, since $e^{-r\tau} < 1$, we must have $S_t - Ke^{-r\tau} > S_t - K$ before expiration. Thus $S_t - Ke^{-r\tau}$ is a more informative lower bound than $S_t - K$. As an example, continue with our call option. The lower bound is $S_t - Ke^{-r\tau} = \$120 - \$100 \exp(-5\% \times 1) = \$24.88$. This is more informative than $S - K = \$20$.

We can also describe upper and lower bounds for put options. The value of a put cannot be worth more than K:

$$p_t \leq P_t \leq K \tag{8.6}$$

which is the upper bound if the price falls to zero. Using put-call parity, we can show that the value of a European put must satisfy the following lower bound:

$$p_t \geq Ke^{-r\tau} - S_t \tag{8.7}$$

8.2.2 Early Exercise of Options

These relationships can be used to assess the value of early exercise for American options. The basic trade-off arises between the value of the American option **dead**, that is, exercised, or **alive**, that is, nonexercised. Thus, the choice is between exercising the option and selling it on the open market.

Consider an American call on a non-dividend-paying stock. By exercising early, the holder gets exactly $S_t - K$. The value of the option alive, however, must be worth more than that of the equivalent European call. From Equation (8.5), this must satisfy $c_t \geq S_t - Ke^{-r\tau}$, which is strictly greater than $S_t - K$ because $e^{-r\tau} < 1$ with positive interest rates. Hence, an American call on a non-dividend-paying stock *should never* be exercised early.

In our example, the lower bound on the European call is \$24.88. If we exercise the American call, we get only $S - K = \$120 - \$100 = \$20$. Because this is less than the minimum value of the European call, the American call should not be exercised. As a result, the value of the American feature is zero and we always have $c_t = C_t$.

The only reason to exercise a call early is to capture a dividend payment. Intuitively, a high income payment makes holding the asset more attractive than holding the option. Thus American options on income-paying assets may be exercised early. Note that this applies also to options on futures, since the implied income stream on the underlying is the risk-free rate.

KEY CONCEPT

An American call option on a non-dividend-paying stock (or asset with no income) should never be exercised early. If the asset pays income, early exercise may occur, with a probability that increases with the size of the income payment.

For an American put, we must have

$$P_t \geq K - S_t \tag{8.8}$$

because it could be exercised now. Unlike the relationship for calls, this lower bound $K - S_t$ is strictly greater than the lower bound for European puts $Ke^{-r\tau} - S_t$. So, we could have early exercise.

To decide whether to exercise early, the holder of the option has to balance the benefit of exercising, which is to receive K now instead of later, against the loss of killing the time value of the option. Because it is better to receive money now than later, it may be worth exercising the put option early.

Thus, American puts on non-income-paying assets *could* be exercised early, unlike calls. The probability of early exercise decreases for lower interest rates and with higher income payments on the asset. In each case, it becomes less attractive to sell the asset.

KEY CONCEPT

An American put option on a non-dividend-paying stock (or asset with no income) may be exercised early. If the asset pays income, the possibility of early exercise decreases with the size of the income payments.

EXAMPLE 8.10: FRM EXAM 2002—QUESTION 50

Given strictly positive interest rates, the best way to close out a long American call option position early (on a stock that pays no dividends) would be to

a. Exercise the call
b. Sell the call
c. Deliver the call
d. Do none of the above

EXAMPLE 8.11: FRM EXAM 2005—QUESTION 15

You have been asked to verify the pricing of a two-year European call option with a strike price of USD 45. You know that the initial stock price is USD 50, and the continuous risk-free rate is 3%. To verify the possible price range of this call, you consider using price bounds. What is the difference between the upper and lower bounds for that European call?

a. 0.00
b. 7.62
c. 42.38
d. 45.00

EXAMPLE 8.12: FRM EXAM 2008—QUESTION 2-6

Which two of the following four statements are *correct* about the early exercise of American options on non-dividend-paying stocks?

I. It is never optimal to exercise an American call option early.
II. It can be optimal to exercise an American put option early.
III. It can be optimal to exercise an American call option early.
IV. It is never optimal to exercise an American put option early.

a. I and II
b. I and IV
c. II and III
d. III and IV

8.3 VALUING OPTIONS

8.3.1 Pricing by Replication

We now turn to the pricing of options. The philosophy of pricing models consists of replicating the payoff on the instrument by a portfolio of assets. To avoid arbitrage, the price of the instrument must then equal the price of the replicating portfolio.

Consider a call option on a stock whose price is represented by a binomial process. The initial price of $S_0 = \$100$ can only move up or down to two values (hence the name *binomial*), $S_1 = \$150$ or $S_2 = \$50$. The option is a call with $K = \$100$, and therefore can only take values of $c_1 = \$50$ or $c_2 = \$0$. We assume that the rate of interest is $r = 25\%$, so that a dollar invested now grows to \$1.25 at maturity.

$$
\begin{array}{lcll}
 & & S_1 = \$150 & c_1 = \$50 \\
 & \nearrow & & \\
S_0 = \$100 & & & \\
 & \searrow & & \\
 & & S_2 = \$50 & c_2 = \$0
\end{array}
$$

The key idea of derivatives pricing is that of **replication**. In other words, we replicate the payoff on the option by a suitable portfolio of the underlying asset plus a position, long or short, in a risk-free bill. This is feasible in this simple setup because we have two states of the world and two instruments, the stock and the bond. To prevent arbitrage, the current value of the derivative must be the same as that of the portfolio.

The portfolio consists of n shares and a risk-free investment currently valued at B (a negative value implies borrowing). We set $c_1 = nS_1 + B$, or $\$50 = n\$150 + B$ and $c_2 = nS_2 + B$, or $\$0 = n\$50 + B$ and solve the 2 by 2 system, which gives

$n = 0.5$ and $B = -\$25$. At time $t = 0$, the value of the loan is $B_0 = \$25/1.25 = \$20$. The current value of the portfolio is $nS_0 + B_0 = 0.5 \times \$100 - \$20 = \$30$. Hence the current value of the option must be $c_0 = \$30$. This derivation shows the essence of option pricing methods.

Note that we did not need the actual probability of an up move. Define this as p. To see how this can be derived, we write the current value of the stock as the discounted expected payoff assuming investors were risk-neutral:

$$S_0 = [p \times S_1 + (1 - p) \times S_2]/(1 + r) \tag{8.9}$$

where the term between brackets is the expectation of the future spot price, given by the probability times its value for each state. Solving for $100 = [p \times 150 + (1 - p) \times 50]/1.25$, we find a risk-neutral probability of $p = 0.75$. We now value the option in the same fashion:

$$c_0 = [p \times c_1 + (1 - p) \times c_2]/(1 + r) \tag{8.10}$$

which gives

$$c_0 = [0.75 \times \$50 + 0.25 \times \$0]/1.25 = \$30$$

This simple example illustrates a very important concept, which is that of **risk-neutral pricing**.

8.3.2 Black-Scholes Valuation

The Black-Scholes (BS) model is an application of these ideas that provides an elegant closed-form solution to the pricing of European calls. The derivation of the model is based on four assumptions:

Black-Scholes Model Assumptions

1. *The price of the underlying asset moves in a continuous fashion.*
2. *Interest rates are known and constant.*
3. *The variance of underlying asset returns is constant.*
4. *Capital markets are perfect* (i.e., short sales are allowed, there are no transaction costs or taxes, and markets operate continuously).

The most important assumption behind the model is that prices are continuous. This rules out discontinuities in the sample path, such as jumps, which cannot be hedged in this model.

The statistical process for the asset price is modeled by a geometric Brownian motion: Over a very short time interval, dt, the logarithmic return has a normal distribution with mean = μdt and variance = $\sigma^2 dt$. The total return can be modeled as

$$dS/S = \mu dt + \sigma dz \tag{8.11}$$

where the first term represents the drift component, and the second is the stochastic component, with dz distributed normally with mean zero and variance dt.

This process implies that the logarithm of the ending price is distributed as

$$\ln(S_T) = \ln(S_0) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \tag{8.12}$$

where ϵ is a $N(0, 1)$ random variable. Hence, the price is lognormally distributed.

Based on these assumptions, Black and Scholes (1972) derived a closed-form formula for European options on a non-dividend-paying stock, called the **Black-Scholes model**. The key point of the analysis is that a position in the option can be replicated by a delta position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is locally risk-free, that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate.

As a result, we can directly compute the present value of the derivative as the discounted expected payoff

$$f_t = E_{\text{RN}}[e^{-r\tau} F(S_T)] \tag{8.13}$$

where the underlying asset is assumed to grow at the risk-free rate, and the discounting is also done at the risk-free rate. Here, the subscript RN refers to the fact that the analysis assumes **risk neutrality**. In a risk-neutral world, the expected return on all securities must be the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a risk premium to induce them to take risks. The BS model value can be computed assuming that all payoffs grow at the risk-free rate and are discounted at the same risk-free rate.

This risk-neutral valuation approach is without a doubt the most important tool in derivatives pricing. Before the Black-Scholes breakthrough, Samuelson had derived a very similar model in 1965, but with the asset growing at the rate μ and discounting as some other rate μ^*.[1] Because μ and μ^* are unknown, the Samuelson model was not practical. The risk-neutral valuation is merely an artificial method to obtain the correct solution, however. It does not imply that investors are risk-neutral.

Furthermore, this approach has limited uses for risk management. The BS model can be used to derive the **risk-neutral probability** of exercising the option. For risk management, however, what matters is the actual probability of exercise, also called **physical probability**. This can differ from the RN probability.

We now turn to the formulation of the BS model. In the case of a European call, the final payoff is $F(S_T) = \text{Max}(S_T - K, 0)$. Initially, we assume no dividend payment on the asset. The current value of the call is given by:

$$c = SN(d_1) - Ke^{-r\tau}N(d_2) \tag{8.14}$$

[1] Paul Samuelson, "Rational Theory of Warrant Price," *Industrial Management Review* 6 (1965): 13–39.

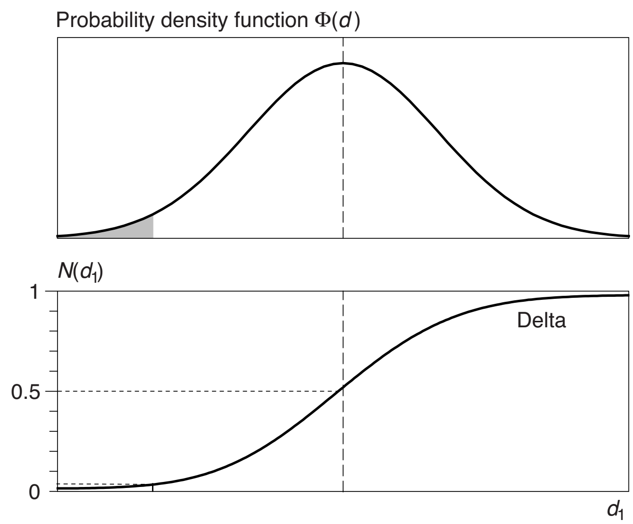

FIGURE 8.10 Cumulative Distribution Function

where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx$$

with Φ defined as the standard normal density function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d, as shown in Figure 8.10. Note that, since the normal density is symmetrical, $N(d) = 1 - N(-d)$, or the area to the left of d is the same as the area to the right of $-d$.

The values of d_1 and d_2 are:

$$d_1 = \frac{ln\,(S/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put-call parity, the European put option value is:

$$p = S[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{8.15}$$

Example: Computing the Black-Scholes Value

Consider an at-the-money call on a stock worth $S = \$100$, with a strike price of $K = \$100$ and maturity of six months. The stock has annual volatility of $\sigma = 20\%$ and pays no dividend. The risk-free rate is $r = 5\%$.

First, we compute the present value factor, which is $e^{-r\tau} = \exp(-0.05 \times 6/12) = 0.9753$. We then compute the value of $d_1 = \ln[S/Ke^{-r\tau}]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = 0.2475$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.1061$. Using standard normal tables or the NORMSDIST Excel function, we find $N(d_1) = 0.5977$ and $N(d_2) = 0.5422$. Note that both values are greater than 0.5 since d_1 and d_2 are both positive. The option is at-the-money. As S is close to K, d_1 is close to zero and $N(d_1)$ close to 0.5.

The value of the call is $c = SN(d_1) - Ke^{-r\tau}N(d_2) = \$6.89$.

The value of the call can also be viewed as an equivalent position of $N(d_1) =$ 59.77% in the stock and some borrowing: $c = \$59.77 - \$52.88 = \$6.89$. Thus this is a leveraged position in the stock.

The value of the put is \$4.42. Buying the call and selling the put costs \$6.89 − \$4.42 = \$2.47. This indeed equals $S - Ke^{-r\tau} = \$100 - \$97.53 = \$2.47$, which confirms put-call parity.

We should note that Equation (8.14) can be reinterpreted in view of the discounting formula in a risk-neutral world, Equation (8.13):

$$c = E_{\mathrm{RN}}[e^{-r\tau}\mathrm{Max}(S_T - K, 0)] = e^{-r\tau}[\int_K^{\infty} Sf(S)dS] - Ke^{-r\tau}[\int_K^{\infty} f(S)dS] \tag{8.16}$$

We see that the integral term multiplying K is the risk-neutral probability of exercising the call, or that the option will end up in-the-money $S > K$. Matching this up with Equation (8.14), this gives

$$\text{Risk} - \text{Neutral Probability of Exercise} = [\int_K^{\infty} f(S)dS] = N(d_2) \tag{8.17}$$

8.3.3 Extensions

Merton (1973) expanded the BS model to the case of a stock paying a continuous dividend yield q. Garman and Kohlhagen (1983) extended the formula to foreign currencies, reinterpreting the yield as the foreign rate of interest $q = r^*$, in what is called the **Garman-Kohlhagen model**.

The Merton model then replaces all occurrences of S by $Se^{-r^*\tau}$. The call is worth

$$c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2) \tag{8.18}$$

It is interesting to take the limit of Equation (8.14) as the option moves more in-the-money, that is, when the spot price S is much greater than K. In this case, d_1 and d_2 become very large and the functions $N(d_1)$ and $N(d_2)$ tend to unity. The value of the call then tends to

$$c(S \gg K) = Se^{-r^*\tau} - Ke^{-r\tau} \tag{8.19}$$

which is the valuation formula for a forward contract. A call that is deep in-the-money is equivalent to a long forward contract, because we are almost certain to exercise.

The **Black model** (1976) applies the same formula to options on futures. The only conceptual difference lies in the income payment to the underlying instrument. With an option on cash, the income is the dividend or interest on the cash instrument. In contrast, with a futures contract, the economically equivalent stream of income is the riskless interest rate. The intuition is that a futures contract can be viewed as equivalent to a position in the underlying asset with the investor setting aside an amount of cash equivalent to the present value of F.

KEY CONCEPT

With an option on futures, the implicit income is the risk-free rate of interest.

For the Black model, we simply replace S by F, the current futures quote, and replace r^* by r, the domestic risk-free rate. The Black model for the valuation of options on futures is:

$$c = [FN(d_1) - KN(d_2)]e^{-r\tau} \tag{8.20}$$

EXAMPLE 8.13: FRM EXAM 2001—QUESTION 91

Using the Black-Scholes model, calculate the value of a European call option given the following information: spot rate = 100; strike price = 110; risk-free rate = 10%; time to expiry = 0.5 years; $N(d_1) = 0.457185$; $N(d_2) = 0.374163$.

a. \$10.90
b. \$9.51
c. \$6.57
d. \$4.92

EXAMPLE 8.14: PROBABILITY OF EXERCISE

In the Black-Scholes expression for a European call option, the term used to compute option probability of exercise is

a. d_1
b. d_2
c. $N(d_1)$
d. $N(d_2)$

8.4 OTHER OPTION CONTRACTS

The options described so far are standard, plain-vanilla options. Many other types of options, however, have been developed.

Binary options, also called **digital options**, pay a fixed amount, say Q, if the asset price ends up above the strike price:

$$c_T = Q \times I(S_T - K) \tag{8.21}$$

where $I(x)$ is an indicator variable that takes the value of 1 if $x \geq 0$ and 0 otherwise. The payoff function is illustrated in Figure 8.11 when $K = \$100$.

Because the probability of ending in-the-money in a risk-neutral world is $N(d_2)$, the initial value of this option is simply

$$c = Qe^{-r\tau} N(d_2) \tag{8.22}$$

These options involve a sharp discontinuity around the strike price. Just below K, their value is zero. Just above, the value is the notional Q. Due to this discontinuity, these options are very difficult to hedge.

Another important class of options is barrier options. **Barrier options** are options where the payoff depends on the value of the asset hitting a barrier during a certain period of time. A **knock-out option** disappears if the price hits a certain barrier. A **knock-in option** comes into existence when the price hits a certain barrier.

An example of a knock-out option is the **down-and-out call**. This disappears if S hits a specified level H during its life. In this case, the knock-out price H must be lower than the initial price S_0. The option that appears at H is the **down-and-in call**. With identical parameters, the two options are perfectly complementary. When one disappears, the other appears. As a result, these two options must add

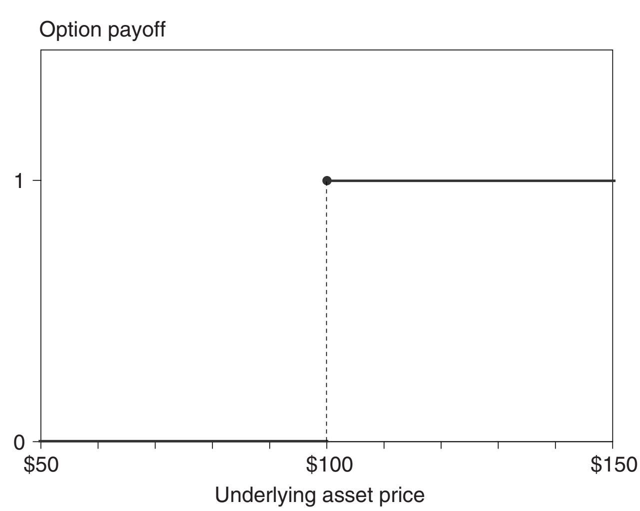

FIGURE 8.11 Payoff on a Binary Option

up to a regular call option. Similarly, an **up-and-out call** ceases to exist when S reaches $H > S_0$. The complementary option is the **up-and-in call**.

Figure 8.12 compares price paths for the four possible combinations of calls. In all figures, the dark line describes the relevant price path during which the option is alive; the gray line describes the remaining path.

The graphs illustrate that the down-and-out call and down-and-in call add up to the regular price path of a regular European call option. Thus, at initiation, the value of these two options must add up to

$$c = c_{\mathrm{DO}} + c_{\mathrm{DI}} \tag{8.23}$$

Because all these values are positive (or at worst zero), the value of each premium c_{DO} and c_{DI} must be no greater than that of c. A similar reasoning applies to the two options in the right panels. Sometimes the option offers a **rebate** if it is knocked out.

Similar combinations exist for put options. An **up-and-out put** ceases to exist when S reaches $H > S_0$. A **down-and-out put** ceases to exist when S reaches $H < S_0$. The only difference with Figure 8.12 is that the option is exercised at maturity if $S < K$.

Barrier options are attractive because they are cheaper than the equivalent European option. This, of course, reflects the fact that they are less likely to be exercised than other options.

In addition, these options are difficult to hedge because a discontinuity arises as the spot price get closer to the barrier. Just above the barrier, the option has positive value. For a very small movement in the asset price, going below the barrier, this value disappears.

Another widely used class of options is Asian options. **Asian options**, or **average rate options**, generate payoffs that depend on the average value of the

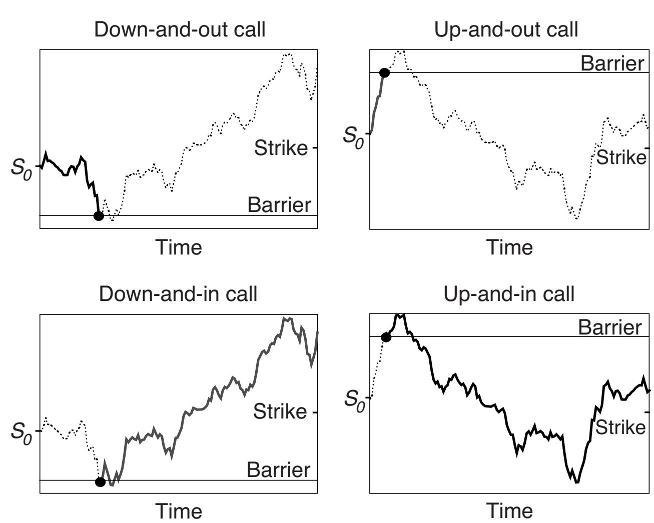

FIGURE 8.12 Paths for Knock-Out and Knock-In Call Options

underlying spot price during the life of the option, instead of the ending value. Define this as $S_{\text{AVE}}(t, T)$. The final payoff for a call is

$$c_T = \text{Max}(S_{\text{AVE}}(t, T) - K, 0) \quad (8.24)$$

Because an average is less variable than the final value at the end of the same period, such options are cheaper than regular options due to lower volatility. In fact, the price of the option can be treated like that of an ordinary option with the volatility set equal to $\sigma/\sqrt{3}$ and an adjustment to the dividend yield.[2] As a result of the averaging process, such options are easier to hedge than ordinary options.

Chooser options allow the holder to choose whether the option is a call or a put. At that point in time, the value of the option is

$$f_t = \text{Max}(c_t, p_t) \quad (8.25)$$

Thus it is a package of two options, a regular call plus an option to convert to a put. As a result, these options are more expensive than plain-vanilla options.

Compound options are options on options. A call on a call (cacall), for example, allows the holder to pay a fixed strike K_1 on the first exercise date T_1 to receive a call, which itself gives the right to buy the asset at a fixed strike K_2 on the second exercise date T_2. The first option will be exercised only if the value of the second option on that date $c(S, K_2, T_2)$ is greater than the strike price K_1.

These options are used, for example, to hedge bids for business projects that may or may not be accepted and that involve foreign currency exposure. If the project is accepted at date T_1, the option is more likely to be exercised. The compound option is cheaper than a regular call option at inception, at the expense of a higher total cost if both options are exercised. Compound options also include calls on puts, puts on puts, and puts on calls.

Finally, **lookback options** have payoffs that depend on the extreme values of S over the option's life. Define S_{MAX} as the maximum and S_{MIN} as the minimum. A fixed-strike lookback call option pays $\text{Max}(S_{MAX} - K, 0)$. A floating-strike lookback call option pays $\text{Max}(S_T - S_{MIN}, 0)$. These options are even more expensive than regular options.

EXAMPLE 8.15: FRM EXAM 2003—QUESTION 34

Which of the following options is strongly path-dependent?

a. An Asian option
b. A binary option
c. An American option
d. A European call option

[2] This is strictly true only when the averaging is a geometric average. In practice, average options involve an arithmetic average, for which there is no analytic solution; the lower volatility adjustment is just an approximation.

EXAMPLE 8.16: FRM EXAM 2006—QUESTION 59

All else being equal, which of the following options would cost more than plain-vanilla options that are currently at-the-money?

I. Lookback options
II. Barrier options
III. Asian options
IV. Chooser option

a. I only
b. I and IV
c. II and III
d. I, III, and IV

EXAMPLE 8.17: FRM EXAM 2002—QUESTION 19

Of the following options, which one does *not* benefit from an increase in the stock price when the current stock price is \$100 and the barrier has not yet been crossed?

a. A down-and-out call with barrier at \$90 and strike at \$110
b. A down-and-in call with barrier at \$90 and strike at \$110
c. An up-and-in put with barrier at \$110 and strike at \$100
d. An up-and-in call with barrier at \$110 and strike at \$100

8.5 VALUING OPTIONS BY NUMERICAL METHODS

Some options have analytical solutions, such as the Black-Scholes models for European vanilla options. For more general options, however, we need to use numerical methods.

The basic valuation formula for derivatives is Equation (8.13), which states that the current value is the discounted present value of expected cash flows, where all assets grow at the risk-free rate and are discounted at the same risk-free rate.

We can use the Monte Carlo simulation methods presented in Chapter 4 to generate sample paths, determine final option values, and discount them into the present. Such simulation methods can be used for European or even path-dependent options, such as Asian options.

Table 8.2 gives an example. Suppose we need to price a European call with parameters $S = 100$, $K = 100$, $T = 1$, $r = 5\%$, $r^* = 0$, $\sigma = 20\%$. We set up the simulation with, for instance, $n = 100$ steps over the horizon of one year.

TABLE 8.2 Example of Simulation for a European Call Option

Replication	Final Payoff S_T	c_T	Discounted Value
1	114.06	14.06	13.37
2	75.83	0.00	0.00
3	108.76	8.76	8.33
...			
Average			10.33

For each step, the trend is $r/n = 0.05/100$; the volatility is $\sigma/\sqrt{(n)} = 0.20/\sqrt{100}$. Each replication starts from a price of \$100 until the horizon. For instance, the first replication gives a final price of $S_T = \$114.06$. The option is in-the-money and is worth $c_T = \$14.06$. We then discount this number into the present and get \$13.37. In the second replication, $S_T = \$75.83$ and the option expires worthless: $c_T = 0$. Averaging across the K replications gives an average of \$10.33 in this case. The result is close to the actual Black-Scholes model price of \$10.45, obtained with Equation (8.14). The simulation, however, is much more general. The payoff at expiration could be a complicated function of the final price or even its intermediate values.

Simulation methods, however, cannot account for the possibility of early exercise, because they do not consider intermediate exercise choices. Instead, binomial trees must be used to value American options. As explained previously, the method consists of chopping up the time horizon into n intervals Δt and setting up the tree so that the characteristics of price movements fit the lognormal distribution.

At each node, the initial price S can go up to uS with probability p or down to dS with probability $(1 - p)$. The parameters u, d, p are chosen so that, for a small time interval, the expected return and variance equal those of the continuous process. One could choose, for instance,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu\Delta t} - d}{u - d} \tag{8.26}$$

Since this a risk-neutral process, the total expected return must be equal to the risk-free rate r. Allowing for an income payment of r^*, this gives $\mu = r - r^*$.

The tree is built starting from the current time to maturity, from the left to the right. Next, the derivative is valued by starting at the end of the tree and working backward to the initial time, from the right to the left.

Consider first a European call option. At time T (maturity) and node j, the call option is worth $\text{Max}(S_{Tj} - K, 0)$. At time $T - 1$ and node j, the call option is the discounted expected value of the option at time T and nodes j and $j + 1$:

$$c_{T-1,j} = e^{-r\Delta t}[pc_{T,j+1} + (1 - p)c_{T,j}] \tag{8.27}$$

We then work backward through the tree until the current time.

For American options, the procedure is slightly different. At each point in time, the holder compares the value of the option *alive* (i.e., unexercised) and *dead* (i.e., exercised). The American call option value at node $T-1,\ j$ is

$$C_{T-1,j} = \text{Max}[(S_{T-1,j} - K), c_{T-1,j}] \tag{8.28}$$

Example: Computing an American Option Value

Consider an at-the-money call on a foreign currency with a spot price of \$100, a strike price of $K = \$100$, and a maturity of six months. The annualized volatility is $\sigma = 20\%$. The domestic interest rate is $r = 5\%$; the foreign rate is $r^* = 8\%$. Note that we require an income payment for the American feature to be valuable. If $r^* = 0$, we know that the American option is worth the same as a European option, which can be priced with the Black-Scholes model. There would be no point in using a numerical method.

First, we divide the period into four intervals, for instance, so that $\Delta t = 0.50/4 = 0.125$. The discounting factor over one interval is $e^{-r\Delta t} = 0.9938$. We then compute:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.20\sqrt{0.125}} = 1.0733$$

$$d = (1/u) = 0.9317$$

$$a = e^{(r-r^*)\Delta t} = e^{(-0.03)0.125} = 0.9963$$

$$p = \frac{a-d}{u-d} = (0.9963 - 0.9317)/(1.0733 - 0.9317) = 0.4559$$

The procedure for pricing the option is detailed in Table 8.3. First, we lay out the tree for the spot price, starting with $S = 100$ at time $t = 0$, then $uS = 107.33$ and $dS = 93.17$ at time $t = 1$, and so on.

This allows us to value the European call. We start from the end, at time $t = 4$, and set the call price to $c = S - K = 132.69 - 100.00 = 32.69$ for the highest spot price, 15.19 for the next price, and so on, down to $c = 0$ if the spot price is below $K = 100.00$. At the previous step and highest node, the value of the call is

$$c = 0.9938[0.4559 \times 32.69 + (1 - 0.4559) \times 15.19] = 23.02$$

Continuing through the tree to time 0 yields a European call value of \$4.43. The Black-Scholes formula gives an exact value of \$4.76. Note how close the binomial approximation is, with just four steps. A finer partition would quickly improve the approximation.

Next, we examine the American call. At time $t = 4$, the values are the same as for the European call since the call expires. At time $t = 3$ and node $j = 4$, the option holder can either keep the call, in which case the value is still \$23.02, or exercise. When exercised, the option payoff is $S - K = 123.63 - 100.00 = 23.63$. Since this is greater than the value of the option alive, the holder should optimally

TABLE 8.3 Computation of American Option Value

	0	1	2	3	4
Spot price S_t	→	→	→	→	→
					132.69
				123.63	115.19
			115.19	107.33	100.00
		107.33	100.00	93.17	86.81
	100.00	93.17	86.81	80.89	75.36
European call c_t	←	←	←	←	←
					32.69
				23.02	15.19
			14.15	6.88	0.00
		8.10	3.12	0.00	0.00
	4.43	1.41	0.00	0.00	0.00
Exercised call $S_t - K$					
					32.69
				23.63	15.19
			15.19	7.33	0.00
		7.33	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
American call C_t	←	←	←	←	←
					32.69
				23.63	15.19
			15.19	7.33	0.00
		8.68	3.32	0.00	0.00
	4.74	1.50	0.00	0.00	0.00

exercise the option. We replace the European option value by \$23.63 at that node. Continuing through the tree in the same fashion, we find a starting value of \$4.74. The value of the American call is slightly greater than the European call price, as expected.

EXAMPLE 8.18: FRM EXAM 2006—QUESTION 86

Which one of the following statements about American options is *incorrect*?

a. American options can be exercised at any time until maturity.
b. American options are always worth at least as much as European options.
c. American options can easily be valued with Monte Carlo simulation.
d. American options can be valued with binomial trees.

8.6 IMPORTANT FORMULAS

Payoff on a long call and put: $C_T = \text{Max}(S_T - K, 0)$, $P_T = \text{Max}(K - S_T, 0)$

Put-call parity: $c - p = Se^{-r^*\tau} - Ke^{-r\tau} = (F - K)e^{-r\tau}$

Bounds on call value (no dividends): $c_t \leq C_t \leq S_t,\ c_t \geq S_t - Ke^{-r\tau}$

Bounds on put value (no dividends): $p_t \leq P_t \leq K,\ p_t \geq Ke^{-r\tau} - S_t$

Geometric Brownian motion: $\ln(S_T) = \ln(S_0) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\ \epsilon$

Risk-neutral discounting formula: $f_t = E_{\text{RN}}[e^{-r\tau} F(S_T)]$

Black-Scholes call option pricing: $c = SN(d_1) - Ke^{-r\tau} N(d_2)$, $d_1 = \frac{ln\ (S/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2},\ d_2 = d_1 - \sigma\sqrt{\tau}$

Black-Scholes put option pricing: $p = S[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1]$

Black-Scholes pricing with dividend, Garman-Kohlhagen model:
$c = Se^{-r^*\tau} N(d_1) - Ke^{-r\tau} N(d_2)$

Black model, option on futures: $c = [FN(d_1) - KN(d_2)]e^{-r\tau}$

Pricing of binary option: $c = Qe^{-r\tau} N(d_2)$

Asian option: $c_T = \text{Max}(S_{\text{AVE}}(t, T) - K, 0)$

Binomial process: $u = e^{\sigma\sqrt{\Delta t}},\ d = (1/u),\ p = \frac{e^{\mu\Delta t} - d}{u - d}$

8.7 ANSWERS TO CHAPTER EXAMPLES

Example 8.1: FRM Exam 2007—Question 84

c. Buying a put creates a gain if the stock price falls, which is similar to selling the stock on the downside. On the upside, the loss is capped by buying a call.

Example 8.2: FRM Exam 2005—Question 72

c. By put-call parity, $c = p + Se^{-r^*\tau} - Ke^{-r\tau} = 3.19 + 23 - 25\exp(-0.05 \times 1) = 3.19 - 0.78 = 2.409$. Note that the volatility information is not useful.

Example 8.3: FRM Exam 2008—Question 2-10

c. Answers a. and b. have payoffs that depend on the stock price and therefore cannot create arbitrage profits. Put-call parity says that $c - p = 3 - 2 = \$1$ should equal $S - Ke^{-r\tau} = 42 - 44 \times 0.9048 = \$2.19$. The call option is cheap. Therefore buy the call and hedge it by selling the stock, for the upside. The benefit from selling the stock if S goes down is offset by selling a put.

Example 8.4: FRM Exam 2006—Question 74

c. By put-call parity, $c - p = Se^{-r^*\tau} - Ke^{-r\tau}$. Therefore, $Se^{-r^*\tau} = (c - p + Ke^{-r\tau}) = (10 - 15 + 90\exp(0.05 \times 5) = 65.09$. The dividend yield is then $y = -(1/T)\ln(65.09/85) = 5.337\%$.

Example 8.5: Risk of Option Contracts

c. Long positions in options can lose at worst the premium, so b. is wrong. Spreads involve long and short positions in options and have limited downside loss, so

a. is wrong. Writing options exposes the seller to very large losses. In the case of puts, the worst loss is the strike price K, if the asset price goes to zero. In the case of calls, however, the worst loss is in theory unlimited because there is a small probability of a huge increase in S. Between c. and d., c. is the better answer.

Example 8.6: FRM Exam 2007—Question 103

d. This position is graphed in Figure 8.6. It benefits from an increase in the price between 40 and 45, so is a bull spread. The worst loss occurs below $K_1 = 40$, when none of the options is exercised and the net lost premium is $5 - 3 = 2$. The maximum profit occurs above $K_2 = 45$, when the two options are exercised, for a net profit of \$5 minus the lost premium, which gives \$3.

Example 8.7: FRM Exam 2006—Question 45

c. The manager is long a portfolio, which is protected by buying a put with a low strike price and selling a call with a higher strike price. This locks in a range of profits and losses and is a collar. If the strike prices were the same, the hedge would be perfect.

Example 8.8: FRM Exam 2002—Question 42

b. Because the final price is below the lowest of the three strike prices, all the puts will be exercised. The final payoff is $(\$50 - \$33) - 2(\$42 - \$33) + (\$37 - \$33) = \$17 - \$18 + \$4 = \$3$. From this, we have to deduct the up-front cost, which is $-\$7 + 2(\$4) - \$2 = -\$1$. The total profit is then, ignoring the time value of money, $\$3 - \$1 = \$2$ per share.

Example 8.9: FRM Exam 2009—Question 3-8

d. The best strategy among these is a long butterfly, which benefits if the spot stays at the current level. Answer a. is a long straddle, which is incorrect because this will lose money if the spot rate does not move. Answer b. is a bull spread, which is incorrect because it assumes the spot price will go up. Answer c. is the same as a short spot position, which is also incorrect.

Example 8.10: FRM Exam 2002—Question 50

b. When there is no dividend, there is never any reason to exercise an American call early. Instead, the option should be sold to another party.

Example 8.11: FRM Exam 2005—Question 15

c. The upper bound is $S = 50$. The lower bound is $c \geq S - Ke^{-r\tau} = 50 - 45\exp(-0.03 \times 2) = 50 - 42.38 = 7.62$. Hence, the difference is 42.38.

Example 8.12: FRM Exam 2008—Question 2-6

a. If the stock does not pay a dividend, the value of the American call option alive is always higher than if exercised (basically because there is no dividend to capture).

Hence, it never pays to exercise a call early. On the other hand, exercising an American put early may be rational because it is better to receive the strike price now than later, with positive interest rates.

Example 8.13: FRM Exam 2001—Question 91

c. We use Equation (8.14), assuming there is no income payment on the asset. This gives $c = SN(d_1) - K \exp(-r\tau)N(d_2) = 100 \times 0.457185 - 110 \exp(-0.1 \times 0.5) \times 0.374163 = \$6.568$.

Example 8.14: Probability of Exercise

d. This is the term multiplying the present value of the strike price, by Equation (8.17).

Example 8.15: FRM Exam 2003—Question 34

a. The payoff of an Asian option depends on the average value of S and therefore is path-dependent.

Example 8.16: FRM Exam 2006—Question 59

b. Lookback options use the maximum stock price over the period, which must be more than the value at the end. Hence they must be more valuable than regular European options. Chooser options involve an additional choice during the life of the option, and as a result are more valuable than regular options. Asian options involve the average, which is less volatile than the final price, so must be less expensive than regular options. Finally, barrier options can be structured so that the sum of two barrier options is equal to a regular option. Because each premium is positive, a barrier option must be less valuable than regular options.

Example 8.17: FRM Exam 2002—Question 19

b. A down-and-in call comes alive only when the barrier is touched; so an increase in S brings it away from the barrier. This is not favorable, so b. is the correct answer. A down-and-out call (answer a.) where the barrier has not been touched is still alive and hence benefits from an increase in S. An up-and-in put (c.) would benefit from an increase in S as this would bring it closer to the barrier of \$110. Finally, an up-and-in call (d.) would also benefit if S gets closer to the barrier.

Example 8.18: FRM Exam 2006—Question 86

c. This statement is incorrect because Monte Carlo simulations are strictly backward-looking, and cannot take into account optimal future exercise, which a binomial tree can do.

CHAPTER 9

Fixed-Income Securities

The next two chapters provide an overview of fixed-income markets, securities, and their derivatives. At the most basic level, **fixed-income securities** refer to bonds that promise to make fixed coupon payments. Over time, this narrow definition has evolved to include any security that obligates the borrower to make specific payments to the bondholder on specified dates. Thus, a **bond** is a security that is issued in connection with a borrowing arrangement. In exchange for receiving cash, the borrower becomes obligated to make a series of payments to the bondholder.

Section 9.1 provides an overview of the different segments of the bond market. Section 9.2 then introduces the various types of fixed-income securities. Section 9.3 reviews the basic tools for pricing fixed-income securities, including the determination of cash flows, discounting, and the term structure of interest rates, including yields, spot rates, and forward rates. Finally, Section 9.4 describes movements in risk factors in fixed-income markets.

Fixed-income derivatives are instruments whose value derives from some bond price, interest rate, or other bond market variable. Due to their complexity, these instruments are analyzed in the next chapter. Because of their importance, mortgage-backed securities (MBSs) and other securitized products will be covered in a later chapter.

9.1 OVERVIEW OF DEBT MARKETS

Fixed-income markets are truly global. They include domestic bonds, foreign bonds, and Eurobonds. Bonds issued by resident entities and denominated in the domestic currency are called **domestic bonds**. In contrast, **foreign bonds** are those floated by a foreign issuer in the domestic currency and subject to domestic country regulations (e.g., by the government of Sweden in dollars in the United States). **Eurobonds** are mainly placed outside the country of the currency in which they are denominated and are sold by an international syndicate of financial institutions (e.g., a dollar-denominated bond issued by IBM and marketed in London). The latter bond should not be confused with bonds denominated in the euro currency, which can be of any type.

FRM Exam Part 1 topic. This chapter also contains material on valuation.

The domestic bond market can be further decomposed into these categories:

- **Government bonds**, issued by central governments, or also called **sovereign bonds** (e.g., by the United States federal government in dollars)
- **Government agency and guaranteed bonds**, issued by agencies or guaranteed by the central government (e.g., by Fannie Mae, a U.S. government agency), which are public financial institutions
- **State and local bonds**, issued by local governments rather than the central government, also known as **municipal bonds** (e.g., by the state of California)
- Bonds issued by private **financial institutions**, including banks, insurance companies, or issuers of asset-backed securities (e.g., by Citibank in the U.S. market)
- **Corporate bonds**, issued by private nonfinancial corporations, including industrials and utilities (e.g., by IBM in the U.S. market)

Table 9.1 breaks down the world debt securities market, which was worth $90 trillion at the end of 2009. This includes the **bond markets**, traditionally defined as fixed-income securities with remaining maturities beyond one year, and the shorter-term **money markets**, with maturities below one year. The table includes all publicly tradable debt securities sorted by country of issuer and issuer type.

The table shows that U.S. entities have issued a total of $25.1 trillion in domestic debt and $6.6 trillion in international debt, for a total amount of $31.7 trillion, by far the biggest debt securities market. Next comes the Eurozone market, with a size of $24.9 trillion, and the Japanese market, with $11.9 trillion.

Focusing now on the type of borrower, domestic government debt is the largest sector. The domestic financial market is also important, especially for mortgage-backed securities. **Mortgage-backed securities** (MBSs) are securities issued in conjunction with mortgage loans, which are loans secured by the collateral of a specific real estate property. Payments on MBSs are repackaged cash flows supported by mortgage payments made by property owners. MBSs can be issued by government agencies as well as by private financial corporations. More generally, **asset-backed securities** (ABSs) are securities whose cash flows are supported by assets such as credit card receivables or car loan payments.

TABLE 9.1 Global Debt Securities Markets, 2009 (Billions of U.S. Dollars)

		Type				
Country of Issuer	Domestic	Gov't	Financial	Corporate	Int'l	Total
United States	25,065	9,475	12,805	2,785	6,646	31,711
Japan	11,522	9,654	1,085	783	380	11,902
Eurozone	14,043	6,872	5,032	2,139	10,879	24,922
United Kingdom	1,560	1,189	349	22	3,045	4,605
Others	12,032	6,914	3,792	1,326	5,128	17,160
Total	64,222	34,104	23,063	7,055	26,078	90,300

Source: Bank for International Settlements.

Finally, the remainder of the domestic market represents bonds raised by private, nonfinancial corporations. This sector is very large in the United States. In contrast, Japan and continental Europe rely more on bank debt to raise funds.

9.2 FIXED-INCOME SECURITIES

9.2.1 Instrument Types

Bonds pay interest on a regular basis, semiannually for U.S. Treasury and corporate bonds, annually for others such as Eurobonds, or quarterly for others. The most common types of bonds are

- **Fixed-coupon bonds**, which pay a fixed percentage of the principal every period and the principal as a **balloon** (one-time) payment at maturity.
- **Zero-coupon bonds**, which pay no coupons but only the principal; their return is derived from price appreciation only.
- **Annuities**, which pay a constant amount over time, which includes interest plus amortization, or gradual repayment, of the principal.
- **Perpetual bonds** or **consols**, which have no set redemption date and whose value derives from interest payments only.
- **Floating-coupon bonds**, which pay interest equal to a reference rate plus a margin, reset on a regular basis; these are usually called **floating-rate notes** (FRNs).
- **Structured notes**, which have more complex coupon patterns to satisfy the investor's needs.
- **Inflation-protected notes**, whose principal is indexed to the **Consumer Price Index** (CPI), hence providing protection against an increasing rate of inflation.[1]

There are many variations on these themes. For instance, **step-up bonds** have fixed coupons that start at a low rate and increase over time.

It is useful to consider floating-rate notes in more detail. Take, for instance, a 10-year \$100 million FRN paying semiannually six-month LIBOR in arrears.[2] **LIBOR**, the London Interbank Offered Rate, is a benchmark cost of borrowing for highly rated (AA) credits. Every semester, on the **reset date**, the value of six-month LIBOR is recorded. Say LIBOR is initially at 6%. At the next coupon date, the payment will be $(1/2) \times \$100 \times 6\% = \$3$ million. Simultaneously, we record a new value for LIBOR, say 8%. The next payment will then increase to

[1] In the United States, these government bonds are called **Treasury inflation-protected securities** (TIPS). The coupon payment is fixed in real terms, say 3%. If after six months, the cumulative inflation is 2%, the principal value of the bond increases from \$100 to $\$100 \times (1 + 2\%) = \$102$. The first semiannual coupon payment is then $(3\%/2) \times \$102 = \$1.53$.

[2] Note that the index could be defined differently. The floating payment could be tied to a Treasury rate, or LIBOR with a different maturity—say three-month LIBOR. The pricing of the FRN will depend on the index. Also, the coupon will typically be set to LIBOR plus some spread that depends on the creditworthiness of the issuer.

\$4 million, and so on. At maturity, the issuer pays the last coupon plus the principal. Therefore, the coupon payment floats with the current interest rate (like a cork at the end of a fishing line).

Application: LIBOR and Other Benchmark Interest Rates

LIBOR, the London Interbank Offered Rate, is a reference rate based on interest rates at which banks borrow unsecured funds from each other in the London interbank market.

LIBOR is published daily by the **British Bankers' Association** (BBA) around 11:45 A.M. London time, and is computed from an average of the distribution of rates provided by reporting banks. LIBID, the London Interbank Bid Rate, represents the average deposit rate.

LIBOR is calculated for 10 different currencies and various maturities, from overnight to one year. LIBOR rates are the most widely used reference rates for short-term futures contracts, such as the Eurodollar futures.

For the euro, however, **EURIBOR**, or Euro Interbank Offered Rate, is most often used. It is sponsored by the European Banking Federation (EBF) and published by Reuters at 11 A.M. Central European Time (CET). In addition, **EONIA** (Euro Overnight Index Average) is an overnight unsecured lending rate that is published every day before 7 P.M. CET. The same panel of banks contributes to EURIBOR and EONIA. The equivalent for sterling is **SONIA** (Sterling Overnight Index Average).

Among structured notes, we should mention **inverse floaters**, also known as reverse floaters, which have coupon payments that vary inversely with the level of interest rates. A typical formula for the coupon is $c = 12\% - \text{LIBOR}$, if positive, payable semiannually. Assume the principal is \$100 million. If LIBOR starts at 6%, the first coupon will be $(1/2) \times \$100 \times (12\% - 6\%) = \$3$ million. If after six months LIBOR moves to 8%, the second coupon will be $(1/2) \times \$100 \times (12\% - 8\%) = \$2$ million. The coupon will go to zero if LIBOR moves above 12%. Conversely, the coupon will increase if LIBOR drops. Hence, inverse floaters do best in a falling interest rate environment.

Bonds can also be issued with option features. The most important are:

- **Callable bonds**, where the issuer has the right to call back the bond at fixed prices on fixed dates, the purpose being to call back the bond when the cost of issuing new debt is lower than the current coupon paid on the bond.
- **Puttable bonds**, where the investor has the right to put the bond back to the issuer at fixed prices on fixed dates, the purpose being to dispose of the bond should its price deteriorate.
- **Convertible bonds**, where the bond can be converted into the common stock of the issuing company at a fixed price on a fixed date, the purpose being to partake in the good fortunes of the company (these will be covered in a separate chapter).

The key to analyzing these bonds is to identify and price the option feature. For instance, a callable bond can be decomposed into a long position in a straight bond minus a call option on the bond price. The call feature is unfavorable for investors who require a lower price to purchase the bond, thereby increasing its yield. Conversely, a put feature will make the bond more attractive, increasing its price and lowering its yield. Similarly, the convertible feature allows companies to issue bonds at a lower yield than otherwise.

EXAMPLE 9.1: FRM EXAM 2003—QUESTION 95

With any other factors remaining unchanged, which of the following statements regarding bonds is *not* valid?

a. The price of a callable bond increases when interest rates increase.
b. Issuance of a callable bond is equivalent to a short position in a straight bond plus a long call option on the bond price.
c. The put feature in a puttable bond lowers its yield compared with the yield of an equivalent straight bond.
d. The price of an inverse floater decreases as interest rates increase.

9.2.2 Methods of Quotation

Most bonds are quoted on a **clean price** basis, that is, without accounting for the accrued income from the last coupon. For U.S. bonds, this clean price is expressed as a percentage of the face value of the bond with fractions in 32nds, for instance as 104-12, which means $104 + 12/32$, for the 6.25% May 2030 Treasury bond. Transactions are expressed in number of units (e.g., \$20 million face value).

Actual payments, however, must account for the accrual of interest. This is factored into the **gross price**, also known as the **dirty price**, which is equal to the clean price plus accrued interest. In the U.S. Treasury market, accrued interest (AI) is computed on an *actual/actual* basis:

$$\text{AI} = \text{Coupon} \times \frac{\text{Actual number of days since last coupon}}{\text{Actual number of days between last and next coupon}} \tag{9.1}$$

The fraction involves the actual number of days in both the numerator and denominator. For instance, say the 6.25% May 2030 Treasury bond paid the last coupon on November 15 and will pay the next coupon on May 15. The denominator is, counting the number of days in each month, $15 + 31 + 31 + 29 + 31 + 30 + 15 = 182$. If the trade settles on April 26, there are $15 + 31 + 31 + 29 + 31 + 26 = 163$ days into the period. The accrued interest is computed from the \$3.125 coupon as

$$\$3.125 \times \frac{163}{182} = \$2.798763$$

The total, gross price for this transaction is:

$$(\$20{,}000{,}000/100) \times [(104 + 12/32) + 2.798763] = \$21{,}434{,}753$$

Different markets have different day count conventions. A 30/360 convention, for example, assumes that all months count for 30 days exactly.

We should note that the accrued interest in the LIBOR market is based on *actual/360*. For instance, the interest accrued on a 6% \$1 million loan over 92 days is

$$\$1{,}000{,}000 \times 0.06 \times \frac{92}{360} = \$15{,}333.33$$

Another notable pricing convention is the discount basis for Treasury bills. These bills are quoted in terms of an annualized discount rate (DR) to the face value, defined as

$$\text{DR} = (\text{Face} - P)/\text{Face} \times (360/t) \tag{9.2}$$

where P is the price and t is the actual number of days. The dollar price can be recovered from

$$P = \text{Face} \times [1 - \text{DR} \times (t/360)] \tag{9.3}$$

For instance, a bill quoted at 5.19% discount with 91 days to maturity could be purchased for

$$\$100 \times [1 - 5.19\% \times (91/360)] = \$98.6881$$

This price can be transformed into a conventional yield to maturity, using

$$F/P = (1 + y \times t/365) \tag{9.4}$$

which gives 5.33% in this case. Note that the yield is greater than the discount rate because it is a rate of return based on the initial price. Because the price is lower than the face value, the yield must be greater than the discount rate.

9.2.3 Duration and Convexity

A previous chapter has explained the concept of duration, which is perhaps the most important risk measure in fixed-income markets. **Duration** represents the linear sensitivity of the bond rate of return to movements in yields. **Convexity** is the quadratic effect.

When cash flows are fixed, duration can be computed exactly as the weighted maturity of each payment, where the weights are proportional to the present value of the cash flows. In many other cases, we can infer duration from an economic analysis of the security. Consider a **floating-rate note** (FRN) with no credit risk. Just before the reset date, we know that the coupon will be set to the prevailing interest rate. The FRN is then similar to cash, or a money market instrument, which has no interest rate risk and hence is selling at par with zero duration. Just after the reset date, the investor is locked into a fixed coupon over the accrual period. The FRN is then economically equivalent to a zero-coupon bond with maturity equal to the time to the next reset date.

We have also seen that bonds with fixed or zero coupons have positive convexity. However, an investor in a callable bond has given the company the right to repurchase the bond at a fixed price. Suppose that rates fall, in which case a noncallable bond would see its price increase from \$100 to \$110. A company that has the right to call the bond at \$105 would then exercise this right, because it can buy the bond cheaply. This will create negative convexity in the bond. Conversely, bonds where the investor is long an option have greater positive convexity.

EXAMPLE 9.2: CALLABLE BOND DURATION

A 10-year zero-coupon bond is callable annually at par (its face value) starting at the beginning of year 6. Assume a flat yield curve of 10%. What is the bond duration?

a. 5 years
b. 7.5 years
c. 10 years
d. Cannot be determined based on the data given

EXAMPLE 9.3: DURATION OF FLOATERS

A money markets desk holds a floating-rate note with an eight-year maturity. The interest rate is floating at the three-month LIBOR rate, reset quarterly. The next reset is in one week. What is the approximate duration of the floating-rate note?

a. 8 years
b. 4 years
c. 3 months
d. 1 week

EXAMPLE 9.4: FRM EXAM 2009—QUESTION 4-16

From the time of issuance until the bond matures, which of the following bonds is most likely to exhibit negative convexity?

a. A puttable bond
b. A callable bond
c. An option-free bond selling at a discount
d. A zero-coupon bond

9.3 PRICING OF FIXED-INCOME SECURITIES

9.3.1 The NPV Approach

Fixed-income securities can be valued by, first, laying out their cash flows and, second, computing their net present value (NPV) using the appropriate discount rate. Let us write the market value of a bond P as the present value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{9.5}$$

where: C_t = cash flow (coupon and/or principal repayment) in period t
t = number of periods (e.g., half-years) to each payment
T = number of periods to final maturity
y = yield to maturity for this particular bond
P = price of the bond, including accrued interest

For a fixed-rate bond with face value F, the cash flow C_t is cF at each period, where c is the coupon rate, plus F upon maturity. Other cash flow patterns are possible, however. Figure 9.1 illustrates the time profile of the cash flows C_t for three bonds with initial market value of \$100, 10-year maturity, and 6% annual interest. The figure describes a straight coupon-paying bond, an annuity, and a zero-coupon bond. As long as the cash flows are predetermined, the valuation is straightforward.

Given the market price, solving for y gives the yield to maturity. This yield is another way to express the price of the bond and is more convenient when comparing various bonds. The yield is also the *expected* rate of return on the bond, provided all coupons are reinvested at the same rate. This interpretation fails, however, when the cash flows are random or when the life of the bond can change due to option-like features. Movements in interest rates also create **reinvestment risk**. This risk can be avoided only by investing in zero-coupon bonds, which do not make intermediate payments.

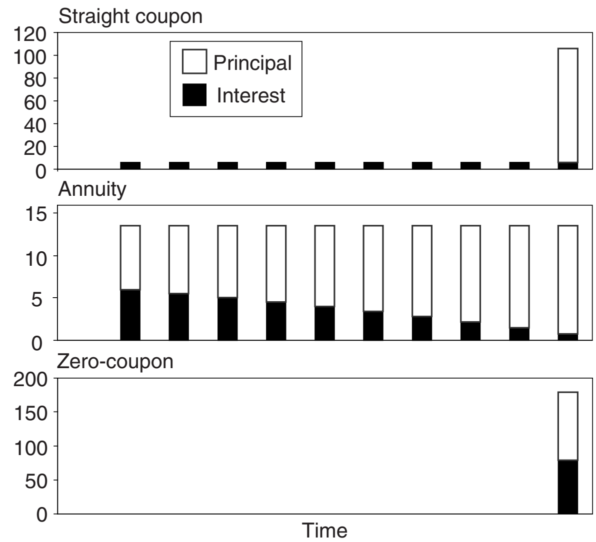

FIGURE 9.1 Time Profile of Cash Flows

EXAMPLE 9.5: FRM EXAM 2009—QUESTION 4-12

Your boss wants to devise a fixed-income strategy such that there is no reinvestment risk over five years. Reinvestment risk will not occur if:

I. Interest rates remain constant over the time period the bonds are held.
II. The bonds purchased are callable.
III. The bonds purchased are issued at par.
IV. Only zero-coupon bonds with a five-year maturity are purchased.

a. I only
b. I and II only
c. III only
d. I and IV

9.3.2 Pricing

We can also use information from the fixed-income market to assess the fair value of the bond. Say we observe that the yield to maturity for comparable bonds is y_T. We can then discount the cash flows using the same, market-determined yield. This gives a fair value for the bond:

$$\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1 + y_T)^t} \tag{9.6}$$

Note that the discount rate y_T does not depend on t, but is fixed for all payments for this bond.

This approach, however, ignores the shape of the term structure of interest rates. Short maturities, for example, could have much lower rates, in which case it is inappropriate to use the same yield. We should really be discounting each cash flow at the zero-coupon rate that corresponds to each time period. This rate R_t is called the **spot interest rate** for maturity t. The fair value of the bond is then

$$\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t)^t} \tag{9.7}$$

We can then check whether the market price is higher or lower. If the term structure is flat, the two approaches will be identical.

Note that the spot rates should be used to discount cash flows in the same risk class (i.e., for the same currency and credit risk). For instance, Treasury bonds should be priced using rates that have the same risk as the U.S. government. For high-quality corporate credits, the **swap curve** is often used. Swap rates correspond to the credit risk of AA-rated counterparties.

Another approach to assess whether a bond is rich or cheap is to add a fixed amount, called the **static spread** (SS), to the spot rates so that the NPV equals the current price:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t + \mathrm{SS})^t} \tag{9.8}$$

All else being equal, a bond with a large static spread is preferable to another with a lower spread. It means the bond is cheaper, or has a higher expected rate of return.

It is simpler, but less accurate, to compute a **yield spread** (YS), using yield to maturity, such that

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + y_T + \mathrm{YS})^t} \tag{9.9}$$

Table 9.2 gives an example of a 7% coupon, two-year bond. The term structure environment, consisting of spot rates and par yields, is described on the left side. The right side lays out the present value of the cash flows (PVCF) using different approaches. Let us first use Equation (9.7). Discounting the two cash flows at the spot rates gives a fair value of $\hat{P} = \$101.9604$. In fact, the bond is selling at a price of $P = \$101.5000$, so the bond is slightly cheap.

Next, we can use Equation (9.9), starting from the par yield of 5.9412%. The yield to maturity on this bond is 6.1798%, which implies a yield spread of YS = 6.1798 − 5.9412 = 0.2386%. Finally, we can use Equation (9.8). Using the static spread approach, we find that adding SS = 0.2482% to the spot rates gives the current price. The last two approaches provide a plug-in so that the NPV exactly matches the observed market price.

TABLE 9.2 Bond Price and Term Structure

Maturity (Year) i	Term Structure Spot Rate R_i	Par Yield y_i	7% Bond PVCF Discounted at Spot SS = 0	Yield + YS $\Delta y = 0.2386$	Spot + SS SS = 0.2482
1	4.0000	4.0000	6.7308	6.5926	6.7147
2	6.0000	5.9412	95.2296	94.9074	94.7853
Sum			101.9604	101.5000	101.5000
Price			101.5000	101.5000	101.5000

For risk management purposes, it is better to consider the spot rates as the risk factors rather than the rate of return on price. This is more intuitive and uses variables that are more likely to be stationary. In contrast, using the history of bond prices is less useful because the maturity of a bond changes with the passage of time, which systematically alters its characteristics.

EXAMPLE 9.6: FRM EXAM 2009—QUESTION 4-11

Consider a bond with par value of EUR 1,000 and maturity in three years, and that pays a coupon of 5% annually. The spot rate curve is as follows: 1-year, 6%; 2-year, 7%; and 3-year, 8%. The value of the bond is closest to:

a. 904
b. 924
c. 930
d. 950

9.3.3 Spot and Forward Rates

Thus, fixed-income pricing relies heavily on **spot rates**, which are zero-coupon investment rates that start at the current time. From spot rates, we can infer **forward rates**, which are rates that start at a future date. Both are essential building blocks for the pricing of bonds. In addition, forward rates can be viewed as market-implied forecasts of future spot rates. Interest rate forecasting can add value only if the portfolio manager's forecast differs from the forward rate.

Consider, for instance, a one-year rate that starts in one year. This forward rate is defined as $F_{1,2}$ and can be inferred from the one-year and two-year spot rates, R_1 and R_2. The forward rate is the break-even future rate that equalizes the return on investments of different maturities.

To demonstrate this important concept, consider an investor who has two choices. The first is to lock in a two-year investment at the two-year rate. The second is to invest for a term of one year and roll over at the one-to-two-year

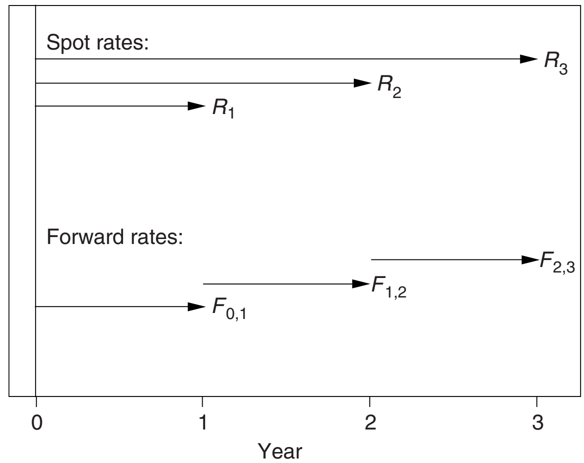

FIGURE 9.2 Spot and Forward Rates

forward rate. The two portfolios will have the same payoff when the future rate $F_{1,2}$ is such that

$$(1 + R_2)^2 = (1 + R_1)(1 + F_{1,2}) \tag{9.10}$$

For instance, if $R_1 = 4.00\%$ and $R_2 = 4.62\%$, we must have $F_{1,2} = 5.24\%$.

More generally, the T-period spot rate can be written as a geometric average of the spot and consecutive one-year forward rates:

$$(1 + R_T)^T = (1 + R_1)(1 + F_{1,2}) \ldots (1 + F_{T-1,T}) \tag{9.11}$$

where $F_{i,i+1}$ is the forward rate of interest prevailing now (at time t) over a horizon of i to $i + 1$. This sequence is shown in Figure 9.2. Table 9.3 displays a sequence of spot rates, forward rates, and par yields, using annual compounding. The last

TABLE 9.3 Spot Rates, Forward Rates, and Par Yields

Maturity (Year) i	Spot Rate R_i	Forward Rate $F_{i-1,i}$	Par Yield y_i	Discount Function $D(t_i)$
1	4.000	4.000	4.000	0.9615
2	4.618	5.240	4.604	0.9136
3	5.192	6.350	5.153	0.8591
4	5.716	7.303	5.640	0.8006
5	6.112	7.712	6.000	0.7433
6	6.396	7.830	6.254	0.6893
7	6.621	7.980	6.451	0.6383
8	6.808	8.130	6.611	0.5903
9	6.970	8.270	6.745	0.5452
10	7.112	8.400	6.860	0.5030

column is the **discount function**, which is simply the current price of a dollar paid at t.

Alternatively, one could infer a series of forward rates for various maturities, all starting in one year:

$$(1 + R_3)^3 = (1 + R_1)(1 + F_{1,3})^2, \ldots, \; (1 + R_T)^T = (1 + R_1)(1 + F_{1,T})^{T-1} \quad (9.12)$$

This defines a term structure in one year, $F_{1,2}, F_{1,3}, \ldots, F_{1,T}$.

The forward rate can be interpreted as a measure of the slope of the term structure. To illustrate this point, expand both sides of Equation (9.10). After neglecting cross-product terms, we have

$$F_{1,2} \approx R_2 + (R_2 - R_1) \quad (9.13)$$

Thus, with an upward-sloping term structure, R_2 is above R_1, and $F_{1,2}$ will also be above R_2. In the preceding example, this gives $R_2 + (R_2 - R_1) = 4.62\% + (4.62\% - 4.00\%) = 4.62\% + 0.62\% = 5.24\%$, which is the correct number for $F_{1,2}$.

With an upward-sloping term structure, the spot rate curve is above the par yield curve. Consider a bond with two payments. The two-year par yield y_2 is implicitly defined from:

$$P = \frac{cF}{(1 + y_2)} + \frac{(cF + F)}{(1 + y_2)^2} = \frac{cF}{(1 + R_1)} + \frac{(cF + F)}{(1 + R_2)^2}$$

where P is set to par $P = F$. The par yield can be viewed as a weighted average of spot rates. In an upward-sloping environment, par yield curves involve coupons that are discounted at shorter and thus lower rates than the final payment. As a result, the par yield curve lies below the spot rate curve.[3] When the spot rate curve is flat, the spot curve is identical to the par yield curve and to the forward curve. In general, the curves differ. Figure 9.3 displays the case of an upward-sloping term structure. It shows that the yield curve is below the spot curve, while the forward curve is above the spot curve. With a downward-sloping term structure, as shown in Figure 9.4, the yield curve is above the spot curve, which is above the forward curve.

Note that, because interest rates have to be positive, forward rates have to be positive; otherwise there would be an arbitrage opportunity.[4]

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[3] For a formal proof, consider a two-period par bond with a face value of \$1 and coupon of y_2. We can write the price of this bond as $1 = y_2/(1 + R_1) + (1 + y_2)/(1 + R_2)^2$. After simplification, this gives $y_2 = R_2(2 + R_2)/(2 + F_{1,2})$. In an upward-sloping environment, $F_{1,2} > R_2$ and thus $y_2 < R_2$.

[4] We abstract from transaction costs and assume we can invest and borrow at the same rate. For instance, $R_1 = 11.00\%$ and $R_2 = 4.62\%$ gives $F_{1,2} = -1.4\%$. This means that $(1 + R_1) = 1.11$ is greater than $(1 + R_2)^2 = 1.094534$. To take advantage of this discrepancy, we borrow \$1 million for two years and invest it for one year. After the first year, the proceeds are kept in cash, or under the proverbial mattress, for the second period. The investment gives \$1,110,000 and we have to pay back \$1,094,534 only. This would create an arbitrage profit of \$15,466 out of thin air, which is highly unlikely.

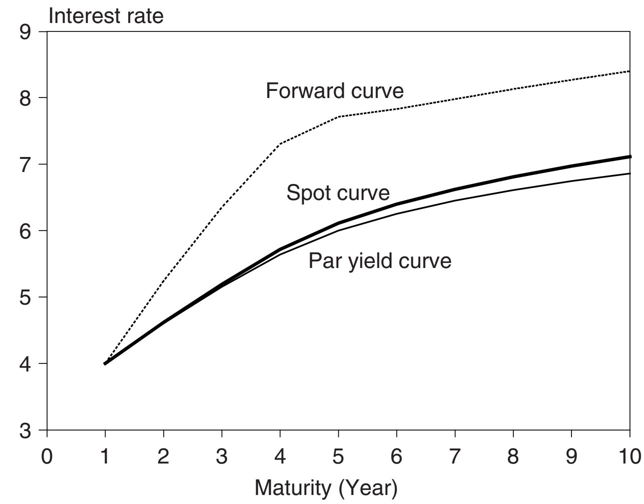

FIGURE 9.3 Upward-Sloping Term Structure

Forward rates allow us to project future cash flows that depend on future rates. The $F_{1,2}$ forward rate, for example, can be taken as the market's expectation of the second coupon payment on an FRN with annual payments and resets. We will also show later that positions in forward rates can be taken easily with derivative instruments.

As a result, the forward rate can be viewed as an expectation of the future spot rate. According to the **expectations hypothesis,**

$$F^t_{1,2} = E(R^{t+1}_1) \tag{9.14}$$

This assumes that there is no risk premium embedded in forward rates. An upward-sloping term structure implies that short-term rates will rise in the future.

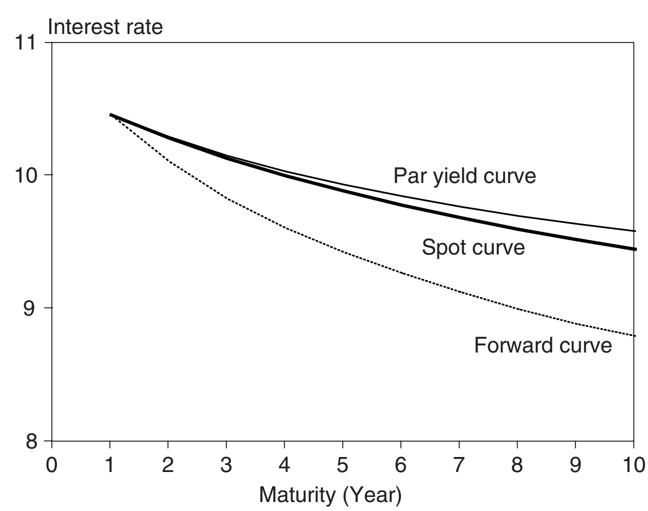

FIGURE 9.4 Downward-Sloping Term Structure

In Figure 9.3, the forward curve traces out the path of future one-year spot rates. Conversely, a downward-sloping curve would imply that future spot rates are expected to fall.

If this hypothesis is correct, then it does not matter which maturity should be selected for investment purposes. Longer maturities benefit from higher coupons but will suffer a capital loss, due to the increase in rates, that will offset this benefit exactly.

In practice, forward rates may contain a risk premium. Generally, investors prefer the safety of short-term instruments. They can be coaxed into buying longer maturities if the latter provide a positive premium. This explains why the spot or yield curves are upward-sloping most of the time.

KEY CONCEPT

In an upward-sloping term structure environment, the forward curve is above the spot curve, which is above the par yield curve. According to the expectations hypothesis, this implies a forecast for rising interest rates.

EXAMPLE 9.7: FRM EXAM 2007—QUESTION 32

The price of a three-year zero-coupon government bond is $85.16. The price of a similar four-year bond is $79.81. What is the one-year implied forward rate from year 3 to year 4?

a. 5.4%
b. 5.5%
c. 5.8%
d. 6.7%

EXAMPLE 9.8: FRM EXAM 2009—QUESTION 3-24

The term structure of swap rates is: 1-year, 2.50%; 2-year, 3.00%; 3-year, 3.50%; 4-year, 4.00%; 5-year, 4.50%. The two-year forward swap rate starting in three years is closest to

a. 3.50%
b. 4.50%
c. 5.51%
d. 6.02%

EXAMPLE 9.9: SHAPE OF TERM STRUCTURE

Suppose that the yield curve is upward sloping. Which of the following statements is *true*?

- **a.** The forward rate yield curve is above the zero-coupon yield curve, which is above the coupon-bearing bond yield curve.
- **b.** The forward rate yield curve is above the coupon-bearing bond yield curve, which is above the zero-coupon yield curve.
- **c.** The coupon-bearing bond yield curve is above the zero-coupon yield curve, which is above the forward rate yield curve.
- **d.** The coupon-bearing bond yield curve is above the forward rate yield curve, which is above the zero-coupon yield curve.

EXAMPLE 9.10: FRM EXAM 2004—QUESTION 61

According to the pure expectations hypothesis, which of the following statements is *correct* concerning the expectations of market participants in an upward-sloping yield curve environment?

- **a.** Interest rates will increase and the yield curve will flatten.
- **b.** Interest rates will increase and the yield curve will steepen.
- **c.** Interest rates will decrease and the yield curve will flatten.
- **d.** Interest rates will decrease and the yield curve will steepen.

9.4 FIXED-INCOME RISK

9.4.1 Price and Yield Volatility

Fixed-income risk arises from potential movements in the level and volatility of the risk factors, usually taken as bond yields.

Using the duration approximation, the volatility of the rate of return in the bond price can be related to the volatility of yield changes

$$\sigma\left(\frac{dP}{P}\right) \approx |\ D^*\ | \times \sigma(dy) \tag{9.15}$$

We now illustrate yield risk for a variety of markets.

9.4.2 Factors Affecting Yields

Movements in yields reflect economic fundamentals. The primary determinant of movements in the levels of interest rates is **inflationary expectations**. This is because investors care about the real (after-inflation) return on their investments. They are specially sensitive to inflation when they hold long-term bonds that pay fixed coupons. As a result, any perceived increase in the rate of inflation will make bonds with fixed nominal coupons less attractive, thereby increasing their yield.

Figure 9.5 compares the level of short-term U.S. interest rates with the concurrent level of inflation. This rate is the yield on U.S. government bills, so is risk-free. The graphs show that most of the long-term movements in nominal rates can be explained by inflation. In more recent years, however, inflation has been subdued. Rates have fallen accordingly.

The **real interest rate** is defined as the nominal rate minus the rate of inflation over the same period. This is generally positive but in recent years has been negative as the Federal Reserve has kept nominal rates to very low levels in order to stimulate economic activity.

Next, we consider a second effect, which is the shape of the term structure. Even though yields move largely in parallel across different maturities, the slope of the term structure changes as well. This can be measured by considering the difference in yields for two maturities. In practice, market observers focus on a long-term rate (e.g., from the 10-year Treasury note) and a short-term rate (e.g., from the three-month Treasury bill), as shown in Figure 9.6.

Generally, the two rates move in tandem, although the short-term rate displays more variability. The **term spread** is defined as the difference between the long rate and the short rate. Figure 9.7 relates the term spread to economic activity. Shaded areas indicate periods of U.S. economic recessions, as officially recorded by the National Bureau of Economic Research (NBER). As the graph shows, periods of recessions usually witness an increase in the term spread. Slow economic activity decreases the demand for capital, which in turn decreases short-term rates and

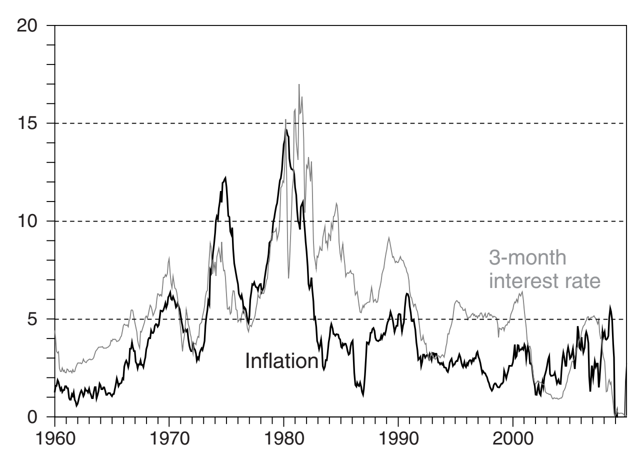

FIGURE 9.5 Inflation and Interest Rates

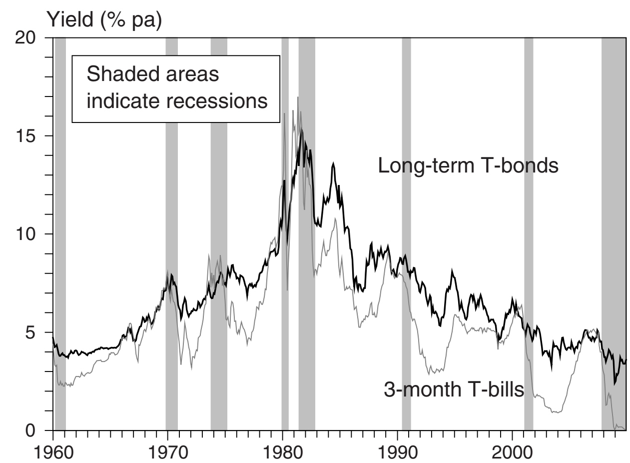

FIGURE 9.6 Movements in the Yields

increases the term spread. Equivalently, the central bank lowers short-term rates to stimulate the economy.

9.4.3 Bond Price and Yield Volatility

Table 9.4 compares the RiskMetrics volatility forecasts for U.S. bond prices as of December 2006. These numbers are exponentially weighted moving average (EWMA) daily and monthly forecasts. Monthly numbers are also shown as annualized, after multiplying by the square root of 12. The table includes Eurodeposits and zero-coupon Treasury rates for maturities ranging from 30 days to 30 years.

The table shows that short-term investments have very little price risk, as expected, due to their short maturities and durations. The price risk of 10-year bonds is around 6%, which is similar to that of floating currencies. The risk of 30-year bonds is higher, around 20%, which is similar to that of equities.

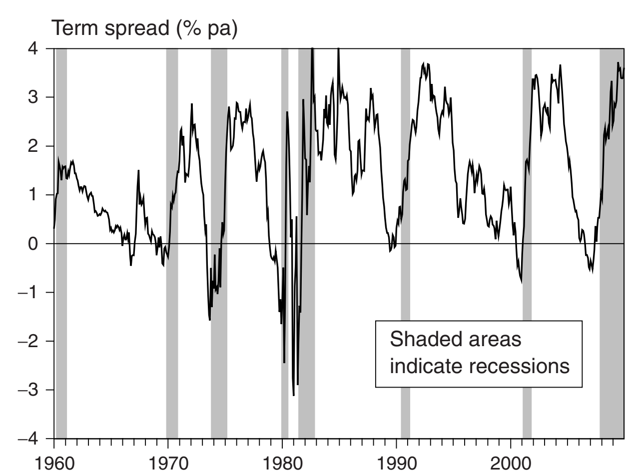

FIGURE 9.7 Term Structure Spread

TABLE 9.4 U.S. Fixed-Income Volatility (Percent), 2006

Type/Maturity	Code	Yield Level	$\sigma(dP/P)$ Daily	$\sigma(dP/P)$ Monthly	$\sigma(dP/P)$ Annual	$\sigma(dy)$ Annual
Euro-30d	R030	5.325	0.001	0.004	0.01	0.17
Euro-360d	R360	5.338	0.028	0.148	0.51	0.54
Zero-2Y	Z02	4.811	0.088	0.444	1.54	0.22
Zero-5Y	Z05	4.688	0.216	1.084	3.76	0.44
Zero-10Y	Z10	4.698	0.375	1.874	6.49	0.48
Zero-20Y	Z20	4.810	0.690	3.441	11.92	0.49
Zero-30Y	Z30	4.847	1.014	5.049	17.49	0.63

The table shows yield volatilities in the last column. Yield volatility is around 0.50% for most maturities. This is also the case for bond markets in other currencies. There were, however, periods of higher inflation during which bond yields were much more volatile, as can be seen from Figure 9.6. For example, the early 1980s experienced wide swings in yields.

EXAMPLE 9.11: FRM EXAM 2007—QUESTION 50

A portfolio consists of two zero-coupon bonds, each with a current value of \$10. The first bond has a modified duration of one year and the second has a modified duration of nine years. The yield curve is flat, and all yields are 5%. Assume all moves of the yield curve are parallel shifts. Given that the daily volatility of the yield is 1%, which of the following is the best estimate of the portfolio's daily value at risk (VAR) at the 95% confidence level?

a. USD 1.65
b. USD 2.33
c. USD 1.16
d. USD 0.82

9.4.4 Real Yield Risk

So far, the analysis has considered only **nominal interest rate risk**, as most bonds represent obligations in nominal terms (e.g., in dollars for the coupon and principal payment). Recently, however, many countries have issued inflation-protected bonds, which make payments that are fixed in real terms but indexed to the rate of inflation.

In this case, the source of risk is **real interest rate risk**. This real yield can be viewed as the internal rate of return that will make the discounted value of promised real bond payments equal to the current real price. This is a new source of risk, as movements in real interest rates may not correlate perfectly with movements in nominal yields. In addition, these two markets can be used to infer inflationary expectations. The implied rate of inflation can be measured as the nominal yield minus the real yield.

Example: Real and Nominal Yields

Consider, for example, the 10-year Treasury Inflation-Protected Security (TIPS) paying a 3% coupon in real terms. The actual coupon and principal payments are indexed to the increase in the Consumer Price Index (CPI).

The TIPS is now trading at a clean real price of 108-23+. Discounting the coupon payments and the principal gives a real yield of $r = 1.98\%$. Note that since the security is trading at a premium, the real yield must be lower than the coupon.

Projecting the rate of inflation at $\pi = 2\%$, semiannually compounded, we infer the projected nominal yield as $(1 + y/200) = (1 + r/200)(1 + \pi/200)$, which gives 4.00%. This is the same order of magnitude as the current nominal yield on the 10-year Treasury note, which is 3.95%. The two securities have very different risk profiles, however. If the rate of inflation moves to 5%, payments on the TIPS will grow at 5% plus 2%, while the coupon on the regular note will stay fixed.

9.4.5 Credit Spread Risk

Credit spread risk is the risk that yields on duration-matched credit-sensitive bonds and risk-free bonds could move differently. The topic of credit risk will be analyzed in more detail in Part Six of this book. Suffice to say that the credit spread represents a compensation for the loss due to default, plus perhaps a risk premium that reflects investor risk aversion.

A position in a credit spread can be established by investing in credit-sensitive bonds, such as corporates, agencies, and mortgage-backed securities (MBSs), and shorting Treasuries with the appropriate duration. This type of position benefits from a stable or shrinking credit spread, but loses from a widening of spreads. Because credit spreads cannot turn negative, their distribution is asymmetric, however. When spreads are tight, large moves imply increases in spreads rather than decreases. Thus positions in credit spreads can be exposed to large losses.

Figure 9.8 displays the time series of credit spreads since 1960, taken from the Baa-Treasury spread. The graph shows that credit spreads display cyclical patterns, increasing during recessions and decreasing during economic expansions. Greater spreads during recessions reflect the greater number of defaults during difficult times. At the time of the 2008 credit crisis, the spread exceeded 5%, which was a record high. Investors were worried that many companies were going to default, which would have decreased the value of their bonds.

As with the term structure of risk-free rates, there is a term structure of credit spreads. For good credit firms, this slopes upward, reflecting the low probability of default in the near term but the inevitable greater uncertainty at longer horizons. In 2009, for example, there were only four U.S. nonfinancials with a perfect credit

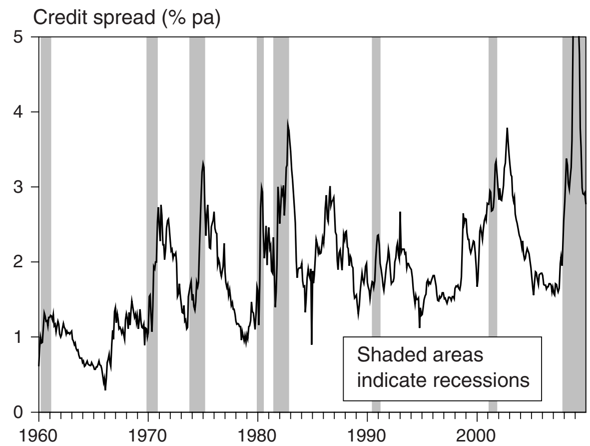

FIGURE 9.8 Credit Spreads

rating of AAA. There were 14 such companies in 1994. Over long horizons, credit tends to deteriorate, which explains why credit spreads widen for longer maturities.

EXAMPLE 9.12: FRM EXAM 2002—QUESTION 128

During 2002, an Argentinean pension fund with 80% of its assets in dollar-denominated debt lost more than 40% of its value. Which of the following reasons could explain all of the 40% loss?

a. The assets were invested in a diversified portfolio of AAA firms in the United States.
b. The assets invested in local currency in Argentina lost all of their value, while the value of the dollar-denominated assets stayed constant.
c. The dollar-denominated assets were invested in U.S. Treasury debt, but the fund had bought credit protection on sovereign debt from Argentina.
d. The fund had invested 80% of its funds in dollar-denominated sovereign debt from Argentina.

EXAMPLE 9.13: FRM EXAM 2008—QUESTION 2-41

Which of the following would *not* cause an upward-sloping yield curve?

a. An investor preference for short-term instruments
b. An expected decline in interest rates
c. An improving credit risk outlook
d. An expected increase in the inflation rate

9.5 ANSWERS TO CHAPTER EXAMPLES

Example 9.1: FRM Exam 2003—Question 95

a. Answer b. is valid because a short position in a callable bond is the same as a short position in a straight bond plus a long position in a call (the issuer can call the bond back). Answer c. is valid because a put is favorable for the investor, so it lowers the yield. Answer d. is valid because an inverse floater has high duration.

Example 9.2: Callable Bond Duration

c. Because this is a zero-coupon bond, it will always trade below par, and the call should never be exercised. Hence its duration is the maturity, 10 years.

Example 9.3: Duration of Floaters

d. Duration is not related to maturity when coupons are not fixed over the life of the investment. We know that at the next reset, the coupon on the FRN will be set at the prevailing rate. Hence, the market value of the note will be equal to par at that time. The duration or price risk is only related to the time to the next reset, which is one week here.

Example 9.4: FRM Exam 2009—Question 4-16

b. A callable bond is short an option, which creates negative convexity for some levels of interest rates. Regular bonds, as in answers c. and d., have positive convexity, as well as puttable bonds.

Example 9.5: FRM Exam 2009—Question 4-12

d. Reinvestment risk occurs when the intermediate coupon payments have to be reinvested at a rate that differs from the initial rate. This does not happen if interest rates stay constant, or with zero-coupon bonds. Callable bonds can be called early, which creates even more reinvestment risk for the principal.

Example 9.6: FRM Exam 2009—Question 4-11

b. The price is $50/(1 + 6\%) + 50/(1 + 7\%)^2 + 1{,}050/(1 + 8\%)^3 = 924.36$.

Example 9.7: FRM Exam 2007—Question 32

d. The forward rate can be inferred from $P_4 = P_3/(1 + F_{3,4})$, or $(1 + R_4)^4 = (1 + R_3)^3(1 + F_{3,4})$. Solving, this gives $F_{3,4} = (85.16/79.81) - 1 = 0.067$.

Example 9.8: FRM Exam 2009—Question 3-24

d. We compute first the accrual of a dollar over three and five years. For $T = 3$, this is $(1 + 3.50\%)^3 = 1.10872$. For $T = 5$, this is $(1 + 4.50\%)^5 = 1.24618$. This gives $1.24618 = (1 + F_{3,5})^2 \times 1.10872$. Solving, we find 6.018%. Note that we can use Equation (9.13) for an approximation. Here, this is $5R_5 = 3R_3 + 2F_{3,5}$, or $F_{3,5} = R_5 + (3/2)(R_5 - R_3) = 4.50\% + 1.5(4.50\% - 3.50\%) = 6\%$.

Example 9.9: Shape of Term Structure

a. See Figures 9.3 and 9.4. The coupon yield curve is an average of the spot, zero-coupon curve; hence it has to lie below the spot curve when it is upward sloping. The forward curve can be interpreted as the spot curve plus the slope of the spot curve. If the latter is upward sloping, the forward curve has to be above the spot curve.

Example 9.10: FRM Exam 2004—Question 61

a. An upward-sloping term structure implies forward rates higher than spot rates, or that short-term rates will increase. Because short-term rates increase more than long-term rates, this implies a flattening of the yield curve.

Example 9.11: FRM Exam 2007—Question 50

a. The dollar duration of the portfolio is $1 \times \$10 + 9 \times \$10 = \$100$. Multiplied by 0.01 and 1.65, this gives \$1.65.

Example 9.12: FRM Exam 2002—Question 128

d. In 2001, Argentina defaulted on its debt, both in the local currency and in dollars. Answer a. is wrong because a diversified portfolio could not have lost so much. The funds were invested at 80% in dollar-denominated assets, so b. is wrong; even a total wipeout of the local-currency portion could not explain a loss of 40% on the portfolio. If the fund had bought credit protection, it would not have lost as much, so c. is wrong. The fund must have had credit exposure to Argentina, so answer d. is correct.

Example 9.13: FRM Exam 2008—Question 2-41

b. An upward-sloping yield curve could be explained by a preference for short-term maturities (answer a.), which requires a higher long-term yield, so answer a. is not the correct choice. An upward-sloping yield curve could also be explained by expectations of higher interest rates or higher inflation (d.). Finally, improving credit conditions (c.) would reduce the cumulative probability of default and thus flatten the term structure. Only an expected decline in interest rates (b.) would *not* cause an upward-sloping yield curve.

CHAPTER 10

Fixed-Income Derivatives

This chapter turns to the analysis of fixed-income derivatives. These are instruments whose value derives from a bond price, interest rate, or other bond market variable. As shown in Table 7.1, fixed-income derivatives account for the largest proportion of the global derivatives markets. Understanding fixed-income derivatives is also important because many fixed-income securities have derivative-like characteristics.

This chapter focuses on the use of fixed-income derivatives, as well as their valuation. Pricing involves finding the fair market value of the contract. For risk management purposes, however, we also need to assess the range of possible movements in contract values. This will be further examined in the chapters on market risk.

This chapter presents the most important interest rate derivatives and discusses fundamentals of pricing. Section 10.1 discusses interest rate forward contracts, also known as forward rate agreements (FRAs). Section 10.2 then turns to the discussion of interest rate futures, covering Eurodollar and Treasury bond futures. Although these products are dollar-based, similar products exist on other capital markets. Interest rate swaps are analyzed in Section 10.3. Swaps are very important instruments due to their widespread use. Finally, interest rate options are covered in Section 10.4, including caps and floors, swaptions, and exchange-traded options.

10.1 FORWARD CONTRACTS

Forward rate agreements (FRAs) are over-the-counter (OTC) financial contracts that allow counterparties to lock in an interest rate starting at a future time. The buyer of an FRA locks in a borrowing rate, and the seller locks in a lending rate. In other words, the long benefits from an increase in rates, and the short benefits from a fall in rates.

As an example, consider an FRA that settles in one month on three-month LIBOR. Such FRA is called 1×4. The first number corresponds to the first settlement date, the second to the time to final maturity. Call τ the period to which LIBOR applies, three months in this case. On the settlement date, in one month,

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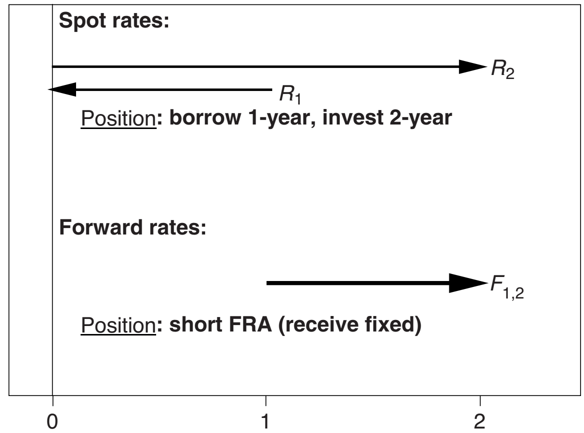

FIGURE 10.1 Decomposition of a Short FRA Position

the payment to the long involves the net value of the difference between the spot rate S_T (the prevailing three-month LIBOR rate) and the locked-in forward rate F. The payoff is $S_T - F$, as with other forward contracts, present valued to the first settlement date. This gives

$$V_T = (S_T - F) \times \tau \times \text{Notional} \times \text{PV}(\$1) \tag{10.1}$$

where PV($1) $= \$1/(1 + S_T\tau)$. The amount is settled in cash.

Figure 10.1 shows that a *short* position in an FRA is equivalent to borrowing short-term to finance a long-term investment. In both cases, there is no up-front investment. The duration is equal to the difference between the durations of the two legs, and can be inferred from the derivative of Equation (10.1). The duration of a short FRA is τ. Its dollar duration is $\text{DD} = \tau \times \text{Notional} \times \text{PV}(\$1)$.

Example: Using an FRA

A company will receive $100 million in six months to be invested for a six-month period. The Treasurer is afraid rates will fall, in which case the investment return will be lower. The company needs to take a position that will offset this loss by generating a gain when rates fall. Because a short FRA gains when rates fall, the Treasurer needs to *sell* a 6 × 12 FRA on $100 million at the rate of, say, $F = 5\%$. This locks in an investment rate of 5% starting in six months.

When the FRA expires in six months, assume that the prevailing six-month spot rate is $S_T = 3\%$. This will lower the investment return on the cash received, which is the scenario the Treasurer feared. Using Equation (10.1), the FRA has a payoff of $V_T = -(3\% - 5\%) \times (6/12) \times \$100$ million = $1,000,000, which multiplied by the 3% present value factor gives $985,222. In effect, this payment offsets the lower return that the company received on a floating investment, guaranteeing a return equal to the forward rate. This contract is also equivalent to borrowing the present value of $100 million for six months and investing the proceeds for 12 months.

KEY CONCEPT

A long FRA position benefits from an increase in rates. A short FRA position is similar to a long position in a bond. Its duration is positive and equal to the difference between the two maturities.

EXAMPLE 10.1: FRM EXAM 2002—QUESTION 27

A long position in a 2×5 FRA is equivalent to the following positions in the spot market:

a. Borrowing in two months to finance a five-month investment
b. Borrowing in five months to finance a two-month investment
c. Borrowing half a loan amount at two months and the remainder at five months
d. Borrowing in two months to finance a three-month investment

EXAMPLE 10.2: FRM EXAM 2005—QUESTION 57

ABC, Inc., entered a forward rate agreement (FRA) to receive a rate of 3.75% with continuous compounding on a principal of USD 1 million between the end of year 1 and the end of year 2. The zero rates are 3.25% and 3.50% for one and two years. What is the value of the FRA when the deal is just entered?

a. USD 35,629
b. USD 34,965
c. USD 664
d. USD 0

EXAMPLE 10.3: FRM EXAM 2001—QUESTION 70

Consider the buyer of a 6×9 FRA. The contract rate is 6.35% on a notional amount of $10 million. Calculate the settlement amount of the *seller* if the settlement rate is 6.85%. Assume a 30/360-day count basis.

a. −12,500
b. −12,290
c. +12,500
d. +12,290

10.2 FUTURES

Whereas FRAs are over-the-counter contracts, futures are traded on organized exchanges. We will cover the most important types of futures contracts, Eurodollar and T-bond futures.

10.2.1 Eurodollar Futures

Eurodollar futures are futures contracts tied to a forward LIBOR rate. Since their creation on the Chicago Mercantile Exchange, Eurodollar futures have spread to equivalent contracts such as EURIBOR futures (denominated in euros),[1] Euroyen futures (denominated in Japanese yen), and so on. These contracts are akin to FRAs involving three-month forward rates starting on a wide range of dates, up to 10 years into the future.

The formula for calculating the value of one contract is

$$P_t = 10{,}000 \times [100 - 0.25(100 - \mathrm{FQ}_t)] = 10{,}000 \times [100 - 0.25F_t] \tag{10.2}$$

where FQ_t is the quoted Eurodollar futures price. This is quoted as 100.00 minus the interest rate F_t, expressed in percent, that is, $\mathrm{FQ}_t = 100 - F_t$. The 0.25 factor represents the three-month maturity, or 0.25 years. For instance, if the market quotes $\mathrm{FQ}_t = 94.47$, we have $F_t = 100 - 94.47 = 5.53$, and the contract value is $P = 10{,}000[100 - 0.25 \times 5.53] = \$986{,}175$. At expiration, the contract value settles to

$$P_T = 10{,}000 \times [100 - 0.25S_T] \tag{10.3}$$

where S_T is the three-month Eurodollar spot rate prevailing at T. Payments are cash settled.

As a result, F_t can be viewed as a three-month forward rate that starts at the maturity of the futures contract. The formula for the contract price may look complicated but in fact is structured so that an increase in the interest rate leads to a decrease in the price of the contract, as is usual for fixed-income instruments. Also, because the change in the price is related to the interest rate by a factor of 0.25, this contract has a constant duration of three months. It is useful to remember that the DV01 is $\$10{,}000 \times 0.25 \times 0.01 = \$25$. This is so even though the notional amount is \$1 million. In this case, the notional amount is nowhere close to what could be lost on the contract even in the worst-case scenario. Even if the rate changes by 100bp, the loss would be only \$2,500.

Example: Using Eurodollar Futures

As in the previous section, the Treasurer wants to hedge a future investment of \$100 million in six months for a six-month period. The company needs to take a position that will offset the earnings loss by generating a gain when rates fall.

[1] EURIBOR futures are based on the European Banking Federation's Euro Interbank Offered Rate (EBF EURIBOR).

Because a long Eurodollar futures position gains when rates fall, the Treasurer should *buy* Eurodollar futures.

If the futures contract trades at $FQ_t = 95.00$, the dollar value of one contract is $P = 10{,}000 \times [100 - 0.25(100 - 95)] = \$987{,}500$. The Treasurer needs to buy a suitable number of contracts that will provide the best hedge against the loss of earnings. The computation of this number will be detailed in a future chapter.

A previous chapter has explained that the pricing of forwards is similar to that of futures, except when the value of the futures contract is strongly correlated with the reinvestment rate. This is the case with Eurodollar futures.

Interest rate futures contracts are designed to move like a bond, that is, to lose value when interest rates increase. The correlation is negative. This implies that when interest rates rise, the futures contract loses value and in addition funds have to be provided precisely when the borrowing cost or reinvestment rate is higher. Conversely, when rates drop, the contract gains value and the profits can be withdrawn but are now reinvested at a lower rate. Relative to forward contracts, this marking-to-market feature is *disadvantageous* to long futures positions. This has to be offset by a *lower* futures contract value. Given that the value is negatively related to the futures rate, by $P_t = 10{,}000 \times [100 - 0.25 \times F_t]$, this implies a *higher* Eurodollar futures rate F_t.

The difference is called the **convexity adjustment** and can be described as[2]

$$\textbf{Futures Rate} = \textbf{Forward Rate} + (1/2)\sigma^2 t_1 t_2 \tag{10.4}$$

where σ is the volatility of the change in the short-term rate, t_1 is the time to maturity of the futures contract, and t_2 is the maturity of the rate underlying the futures contract.

Example: Convexity Adjustment

Consider a 10-year Eurodollar contract, for which $t_1 = 10$, $t_2 = 10.25$. The maturity of the futures contract itself is 10 years and that of the underlying rate is 10 years plus three months.

Typically, $\sigma = 1\%$, so that the adjustment is $(1/2)0.01^2 \times 10 \times 10.25 = 0.51\%$. So, if the forward price is 6%, the equivalent futures rate would be 6.51%. Note that the effect is significant for long maturities only. Changing t_1 to one year and t_2 to 1.25, for instance, reduces the adjustment to 0.006%, which is negligible.

10.2.2 T-Bond Futures

T-bond futures are futures contracts tied to a pool of Treasury bonds that consists of all bonds with a remaining maturity greater than 15 years (and noncallable

[2] This formula is derived from the Ho-Lee model. See, for instance, John C. Hull, *Options, Futures, and Other Derivatives*, 7th ed. (Upper Saddle River, NJ: Prentice Hall, 2008).

within 15 years). Similar contracts exist on shorter rates, including 2-, 5-, and 10-year Treasury notes. Government bond futures also exist in other markets, including Canada, the United Kingdom, the Eurozone, and Japan.

Futures contracts are quoted like T-bonds, for example 97-02, in percent plus 32nds, with a notional of \$100,000. Thus the price of the contract is $P =$ \$100,000 $\times$ (97 + 2/32)/100 = \$97,062.50. The next day, if yields go up and the quoted price falls to 96-0, the new value is \$96,000, and the loss on the long position is $P_2 - P_1 = -\$1,062.50$.

It is important to note that the T-bond futures contract is settled by physical delivery. To ensure interchangeability between the deliverable bonds, the futures contract uses a **conversion factor** (CF) for delivery. This factor multiplies the futures price for payment to the short. The goal of the CF is to attempt to equalize the net cost of delivering the various eligible bonds.

The conversion factor is needed because bonds trade at widely different prices. High-coupon bonds trade at a premium, low-coupon bonds at a discount. Without this adjustment, the party with the short position (the short) would always deliver the same cheap bond and there would be little exchangeability between bonds. Exchangeability is an important feature, however, as it minimizes the possibility of market squeezes. A **squeeze** occurs when holders of the short position cannot acquire or borrow the securities required for delivery under the terms of the contract.

So, the short buys a bond, delivers it, and receives the quoted futures price times a conversion factor that is specific to the delivered bond (plus accrued interest). The short should rationally pick the bond that minimizes the net cost:

$$\text{Cost} = \text{Price} - \text{Futures Quote} \times \text{CF} \tag{10.5}$$

The bond with the lowest net cost is called **cheapest to deliver** (CTD).

In practice, the CF is set by the exchange at initiation of the contract for each bond. It is computed by discounting the bond's cash flows at a notional 6% rate, assuming a flat term structure. Take, for instance, the $7\frac{5}{8}\%$ of 2025. The CF is computed as

$$\text{CF} = \frac{(7.625\%/2)}{(1+6\%/2)^1} + \cdots + \frac{(1+7.625\%/2)}{(1+6\%/2)^T} \tag{10.6}$$

which gives CF = 1.1717. High-coupon bonds have higher CFs. Also, because the coupon is greater than 6%, the CF is greater than one.

The net cost calculations are illustrated in Table 10.1 for three bonds. The net cost for the first bond in the table is \$104.375 − 110.8438 × 0.9116 = \$3.330. For the 6% coupon bond, the CF is exactly unity. The net cost for the third bond in the table is \$1.874. Because this is the lowest entry, this bond is the CTD for this group. Note how the CF adjustment brings the cost of all bonds much closer to each other than their original prices.

The adjustment is not perfect when current yields are far from 6%, or when the term structure is not flat, or when bonds do not trade at their theoretical prices. Assume, for instance, that we operate in an environment where yields are flat at

TABLE 10.1 Calculation of CTD

Bond	Price	Futures	CF	Cost
$5\frac{1}{4}$% Nov. 2028	104.3750	110.8438	0.9116	3.330
6% Feb. 2026	112.9063	110.8438	1.0000	2.063
$7\frac{5}{8}$% Feb. 2025	131.7500	110.8438	1.1717	1.874

5% and all bonds are priced at par. Discounting at 6% will create CF factors that are lower than one; the longer the maturity of the bond, the greater the difference. The net cost $P - F \times$ CF will then be greater for longer-term bonds. This tends to favor short-term bonds for delivery. When the term structure is upward sloping, the opposite occurs, and there is a tendency for long-term bonds to be delivered.

As a first approximation, this CTD bond drives the characteristics of the futures contract. As before, the equilibrium futures price is given by

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{10.7}$$

where S_t is the gross price of the CTD and PV(D) is the present value of the coupon payments. This has to be further divided by the conversion factor for this bond. The duration of the futures contract is also given by that of the CTD.

In fact, this relationship is only approximate, because the short has an *option* to deliver the cheapest of a group of bonds. The value of this delivery option should depress the futures price because the party who is long the futures is also short the option. As a result, this requires a lower acquisition price. In addition, the short has a **timing option**, which allows delivery on any day of the delivery month. Unfortunately, these complex options are not easy to evaluate.

EXAMPLE 10.4: FRM EXAM 2009—QUESTION 3-11

The yield curve is upward sloping. You have a short T-bond futures position. The following bonds are eligible for delivery:

Bond	A	B	C
Spot price	102-14/32	106-19/32	98-12/32
Coupon	4%	5%	3%
Conversion factor	0.98	1.03	0.952

The futures price is 103-17/32 and the maturity date of the contract is September 1. The bonds pay their coupon semiannually on June 30 and December 31. The cheapest to deliver bond is:

a. Bond A
b. Bond B
c. Bond C
d. Insufficient information

EXAMPLE 10.5: FRM EXAM 2009—QUESTION 3-23

Which of the following statements related to forward and futures prices is *true*?

a. If the forward price does not equal the futures price, arbitragers will exploit this arbitrage opportunity.
b. The level of interest rates determines whether the forward price is higher or lower than the futures price.
c. The volatility of interest rates determines whether the forward price is higher or lower than the futures price.
d. Whether the forward price will be higher or lower than the futures price depends on correlation between interest rate and futures price.

EXAMPLE 10.6: FRM EXAM 2007—QUESTION 80

Consider a forward rate agreement (FRA) with the same maturity and compounding frequency as a Eurodollar futures contract. The FRA has a LIBOR underlying. Which of the following statements is *true* about the relationship between the forward rate and the futures rate?

a. The forward rate is normally higher than the futures rate.
b. They have no fixed relationship.
c. The forward rate is normally lower than the futures rate.
d. They should be exactly the same.

10.3 SWAPS

Swaps are agreements by two parties to exchange cash flows in the future according to a prearranged formula. Interest rate swaps have payments tied to an interest rate. The most common type of swap is the **fixed-for-floating** swap, where one party commits to pay a fixed percentage of notional against a receipt that is indexed to a floating rate, typically LIBOR. The risk is that of a change in the level of rates.

Other types of swaps are **basis swaps**, where both payments are indexed to a floating rate. For instance, the swap can involve exchanging payments tied to three-month LIBOR against a three-month Treasury bill rate. The risk is that of a change in the spread between the reference rates.

10.3.1 Instruments

Consider two counterparties, A and B, that can raise funds at either fixed or floating rates, $100 million over 10 years. A wants to raise floating, and B wants to raise fixed.

Table 10.2a displays capital costs. Company A has an **absolute advantage** in the two markets, as it can raise funds at rates systematically lower than B can. Company A, however, has a **comparative advantage** in raising fixed, as the cost is 1.2% lower than for B. In contrast, the cost of raising floating is only 0.70% lower than for B. Conversely, company B must have a comparative advantage in raising floating.

This provides a rationale for a swap that will be to the mutual advantage of both parties. If both companies directly issue funds in their final desired markets, the total cost will be LIBOR + 0.30% (for A) and 11.20% (for B), for a total of LIBOR + 11.50%. In contrast, the total cost of raising capital where each has a comparative advantage is 10.00% (for A) and LIBOR + 1.00% (for B), for a total of LIBOR + 11.00%. The gain to both parties from entering a swap is

TABLE 10.2a Cost of Capital Comparison

Company	Fixed	Floating
A	10.00%	LIBOR + 0.30%
B	11.20%	LIBOR + 1.00%

TABLE 10.2b Swap to Company A

Operation	Fixed	Floating
Issue debt	Pay 10.00%	
Enter swap	Receive 10.00%	Pay LIBOR + 0.05%
Net		Pay LIBOR + 0.05%
Direct cost		Pay LIBOR + 0.30%
Savings		0.25%

TABLE 10.2c Swap to Company B

Operation	Floating	Fixed
Issue debt	Pay LIBOR + 1.00%	
Enter swap	Receive LIBOR + 0.05%	Pay 10.00%
Net		Pay 10.95%
Direct cost		Pay 11.20%
Savings		0.25%

11.50% − 11.00% = 0.50%. For instance, the swap described in Tables 10.2b and 10.2c splits the benefit equally between the two parties.

Company A issues fixed debt at 10.00%, and then enters a swap whereby it promises to pay LIBOR + 0.05% in exchange for receiving 10.00% fixed payments. Its net, effective funding cost is therefore LIBOR + 0.05%, which is less than the direct cost by 25bp.

Similarly, company B issues floating debt at LIBOR + 1.00%, and then enters a swap whereby it receives LIBOR + 0.05% in exchange for paying 10.00% fixed. Its net, effective funding cost is therefore 11.00% − 0.05% = 10.95%, which is less than the direct cost by 25bp. Both parties benefit from the swap.

In terms of actual cash flows, swap payments are typically *netted* against each other. For instance, if the first LIBOR rate is at 9% assuming annual payments, company A would be owed 10% × \$100 = \$1 million, and have to pay LIBOR + 0.05%, or 9.05% × \$100 = \$0.905 million. This gives a net receipt of \$95,000. There is no need to exchange principals since both involve the same amount.

10.3.2 Quotations

Swaps can be quoted in terms of spreads relative to the yield of similar-maturity Treasury notes. For instance, a dealer may quote 10-year swap spreads as 31/34bp against LIBOR. If the current note yield is 6.72, this means that the dealer is willing to pay 6.72 + 0.31 = 7.03% against receiving LIBOR, or that the dealer is willing to receive 6.72 + 0.34 = 7.06% against paying LIBOR. Of course, the dealer makes a profit from the spread, which is rather small, at 3bp only. Equivalently, the outright quote is 7.03/7.06 for the swap.

Note that the swap should trade at a positive credit spread to Treasuries. This is because the other leg is quoted in relation to LIBOR, which also has credit risk. More precisely, swap rates correspond to the credit risk of AA-rated counterparties.

Table 7.1 has shown that the interest rate swap market is by far the largest derivative market in terms of notional. Because the market is very liquid, market quotations for the fixed-rate leg have become benchmark interest rates. Thus, swap rates form the basis for the **swap curve**, which is also called the par curve, because it is equivalent to yields on bonds selling at par. Because the floating-rate leg is indexed to LIBOR, which carries credit risk, the swap curve is normally higher than the par curve for government bonds in the same currency.

10.3.3 Pricing

We now discuss the pricing of interest rate swaps. Consider, for instance, a three-year \$100 million swap, where we receive a fixed coupon of 5.50% against LIBOR. Payments are annual and we ignore credit spreads. We can price the swap using either of two approaches: taking the difference between two bond prices or valuing a sequence of forward contracts. This is illustrated in Figure 10.2.

The top part of the figure shows that this swap is equivalent to a long position in a three-year fixed-rate 5.5% bond and a short position in a three-year

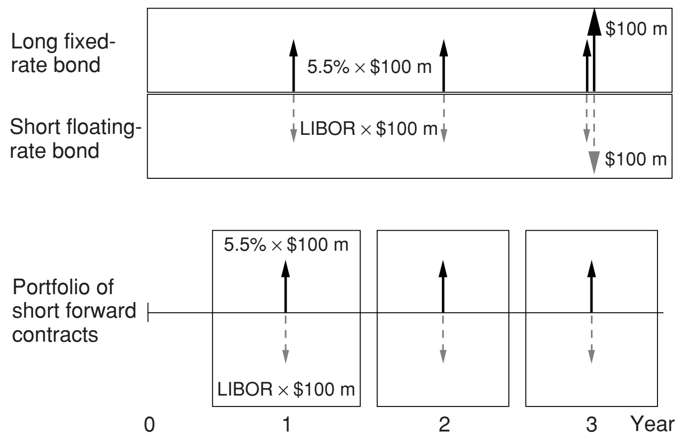

FIGURE 10.2 Alternative Decompositions for Swap Cash Flows

floating-rate note (FRN). If B_F is the value of the fixed-rate bond and B_f is the value of the FRN, the value of the swap is $V = B_F - B_f$.

The value of the FRN should be close to par. Just before a reset, B_f will behave exactly like a cash investment, as the coupon for the next period will be set to the prevailing interest rate. Therefore, its market value should be close to the face value. Just after a reset, the FRN will behave like a bond with a one-year maturity. But overall, fluctuations in the market value of B_f should be small.

Consider now the swap value. If at initiation the swap coupon is set to the prevailing par yield, B_F is equal to the face value, $B_F = 100$. Because $B_f = 100$ just before the reset on the floating leg, the value of the swap is zero: $V = B_F - B_f = 0$. This is like a forward contract at initiation.

After the swap is consummated, its value will be affected by interest rates. If rates fall, the swap will move in-the-money, since it receives higher coupons than prevailing market yields. B_F will increase, whereas B_f will barely change.

Thus the duration of a receive-fixed swap is similar to that of a fixed-rate bond, including the fixed coupons and principal at maturity. This is because the duration of the floating leg is close to zero. The fact that the principals are not exchanged does not mean that the duration computation should not include the principal. Duration should be viewed as an interest rate sensitivity.

KEY CONCEPT

A position in a receive-fixed swap is equivalent to a long position in a bond with similar coupon characteristics and maturity offset by a short position in a floating-rate note. Its duration is close to that of the fixed-rate note.

We now value the three-year swap using term-structure data from the preceding chapter. The time is just before a reset, so $B_f = \$100$ million. We compute

B_F (in millions) as

$$B_F = \frac{\$5.5}{(1 + 4.000\%)} + \frac{\$5.5}{(1 + 4.618\%)^2} + \frac{\$105.5}{(1 + 5.192\%)^3} = \$100.95$$

The outstanding value of the swap is therefore $V = \$100.95 - \$100 = \$0.95$ million.

Alternatively, the swap can be valued as a sequence of forward contracts, as shown in the bottom part of Figure 10.2. Recall from Chapter 7 that the value of a unit position in a long forward contract is given by

$$V_i = (F_i - K)\exp(-r_i \tau_i) \tag{10.8}$$

where F_i is the current forward rate, K the prespecified rate, and r_i the spot rate for time τ_i. Extending this to multiple maturities, and to discrete compounding using R_i, the swap can be valued as

$$V = \sum_i n_i (F_i - K)/(1 + R_i)^{\tau_i} \tag{10.9}$$

where n_i is the notional amount for maturity i.

A long forward rate agreement benefits if rates go up. Indeed, Equation (10.8) shows that the value increases if F_i goes up. In the case of our swap, we *receive* a fixed rate K. So, the position loses money if rates go up, as we could have received a higher rate. Hence, the sign on Equation (10.9) must be reversed.

Using the forward rates listed in Table 9.3, we find

$$V = -\frac{\$100(4.000\% - 5.50\%)}{(1 + 4.000\%)} - \frac{\$100(5.240\% - 5.50\%)}{(1 + 4.618\%)^2} - \frac{\$100(6.350\% - 5.50\%)}{(1 + 5.192\%)^3}$$

$$V = +1.4423 + 0.2376 - 0.7302 = \$0.95 \text{ million}$$

This is identical to the previous result, as it should be. The swap is in-the-money primarily because of the first payment, which pays a rate of 5.5% whereas the forward rate is only 4.00%.

Thus, interest rate swaps can be priced and hedged using a sequence of forward rates, such as those implicit in Eurodollar contracts. The practice of daily marking to market of futures induces a slight convexity bias in futures rates, which have to be adjusted downward to get forward rates.

Figure 10.3 compares a sequence of quarterly forward rates with the five-year swap rate prevailing at the same time. Because short-term forward rates are less than the swap rate, the near payments are in-the-money. In contrast, the more distant payments are out-of-the-money. The current market value of this swap is zero, which implies that all the near-term positive values must be offset by distant negative values.

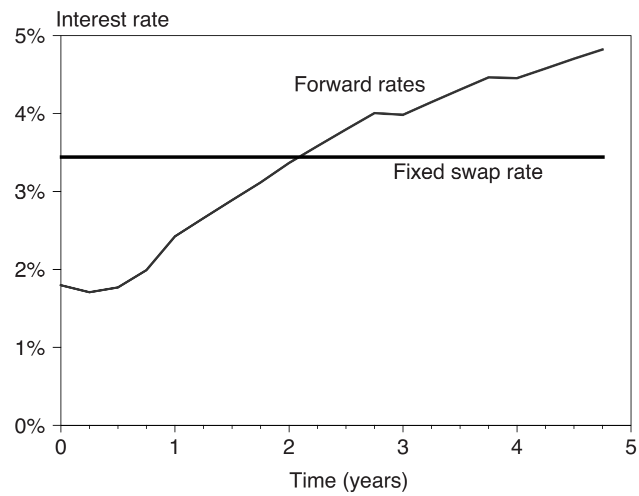

FIGURE 10.3 Sequence of Forward Rates and Swap Rate

EXAMPLE 10.7: FRM EXAM 2005—QUESTION 51

Consider the following information about an interest rate swap: two-year term, semiannual payment, fixed rate = 6%, floating rate = LIBOR + 50 basis points, notional USD 10 million. Calculate the net coupon exchange for the first period if LIBOR is 5% at the beginning of the period and 5.5% at the end of the period.

a. Fixed-rate payer pays USD 0.
b. Fixed-rate payer pays USD 25,000.
c. Fixed-rate payer pays USD 50,000.
d. Fixed-rate payer receives USD 25,000.

EXAMPLE 10.8: FRM EXAM 2000—QUESTION 55

Bank XYZ enters into a five-year swap contract with ABC Co. to pay LIBOR in return for a fixed 8% rate on a principal of $100 million. Two years from now, the market rate on three-year swaps at LIBOR is 7%. At this time ABC Co. declares bankruptcy and defaults on its swap obligation. Assume that the net payment is made only at the end of each year for the swap contract period. What is the market value of the loss incurred by Bank XYZ as a result of the default?

a. $1.927 million
b. $2.245 million
c. $2.624 million
d. $3.011 million

EXAMPLE 10.9: FRM EXAM 2009—QUESTION 3-4

A bank entered into a three-year interest rate swap for a notional amount of USD 250 million, paying a fixed rate of 7.5% and receiving LIBOR annually. Just after the payment was made at the end of the first year, the continuously compounded spot one-year and two-year LIBOR rates are 8% and 8.5%, respectively. The value of the swap at that time is closest to

a. USD 14 million
b. USD −6 million
c. USD −14 million
d. USD 6 million

10.4 OPTIONS

There is a large variety of fixed-income options. We briefly describe here caps and floors, swaptions, and exchange-traded options. In addition to these stand-alone instruments, fixed-income options are embedded in many securities. For instance, a callable bond can be viewed as a regular bond plus a short position in an option.

When considering fixed-income options, the underlying can be a yield or a price. Due to the negative price-yield relationship, a call option on a bond can also be viewed as a put option on the underlying yield.

10.4.1 Caps and Floors

A **cap** is a call option on interest rates with unit value

$$C_T = \text{Max}[i_T - K, 0] \tag{10.10}$$

where $K = i_C$ is the cap rate and i_T is the rate prevailing at maturity.

In practice, caps are purchased jointly with the issuance of floating-rate notes (FRNs) that pay LIBOR plus a spread on a periodic basis for the term of the note. By purchasing the cap, the issuer ensures that the cost of capital will not exceed the capped rate. Such caps are really a combination of individual options, called **caplets**.

The payment on each caplet is determined by C_T, the notional, and an accrual factor. Payments are made in **arrears**, that is, at the end of the period. For instance, take a one-year cap on a notional of \$1 million and a six-month LIBOR cap rate of 5%. The agreement period is from January 15 to the next January with a reset on July 15. Suppose that on July 15, LIBOR is at 5.5%. On the following January, the payment is

$$\$1 \text{ Million} \times (0.055 - 0.05)(184/360) = \$2{,}555.56$$

using *Actual*/360 interest accrual. If the cap is used to hedge an FRN, this would help to offset the higher coupon payment, which is now 5.5%.

A **floor** is a put option on interest rates with value

$$P_T = \text{Max}[K - i_T, 0] \tag{10.11}$$

where $K = i_F$ is the floor rate. A **collar** is a combination of buying a cap and selling a floor. This combination decreases the net cost of purchasing the cap protection. Figure 10.4 shows an example of a price path, with a cap rate of 3.5% and a floor rate of 2%. There are three instances where the cap is exercised, leading to a receipt of payment. There is one instance where the rate is below the floor, requiring a payment.

When the cap and floor rates converge to the same value $K = i_C = i_F$, the overall debt cost becomes fixed instead of floating. The collar is then the same as a pay-fixed swap, which is the equivalent of put-call parity,

$$\text{Long Cap}(i_C = K) + \text{Short Floor}(i_F = K) = \text{Long Pay-Fixed Swap} \tag{10.12}$$

Caps are typically priced using a variant of the Black model, assuming that interest rate changes are lognormal. The value of the cap is set equal to a portfolio of N caplets, which are European-style individual options on different interest rates with regularly spaced maturities

$$c = \sum_{j=1}^{N} c_j \tag{10.13}$$

For each caplet, the unit price is

$$c_j = [F\,N(d_1) - K\,N(d_2)]\text{PV}(\$1) \tag{10.14}$$

where F is the current forward rate for the period t_j to t_{j+1}, K is the cap rate, and PV($1) is the discount factor to time t_{j+1}. To obtain a dollar amount, we must adjust for the notional amount as well as the length of the accrual period.

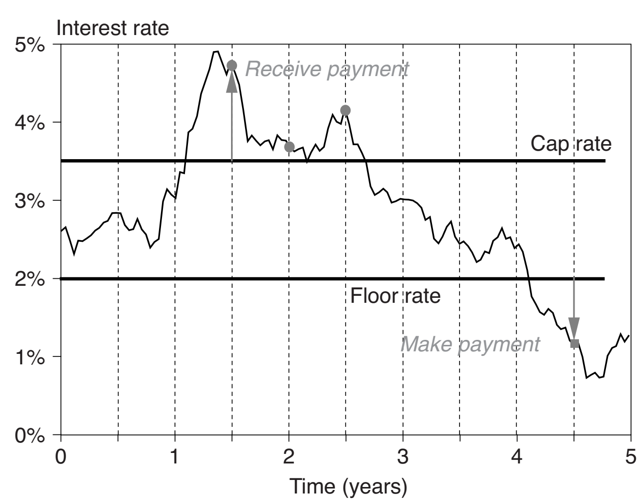

FIGURE 10.4 Exercise of Cap and Floor

The volatility entering the pricing model, σ, is that of the forward rate between now and the expiration of the option contract, that is, at t_j. Generally, volatilities are quoted as one number for all caplets within a cap; this is called **flat volatilities**.

$$\sigma_j = \sigma$$

Alternatively, volatilities can be quoted separately for each forward rate in the caplet; this is called **spot volatilities**.

Example: Computing the Value of a Cap

Consider the previous cap on \$1 million at the capped rate of 5%. Assume a flat term structure at 5.5% and a volatility of 20% pa. The reset is on July 15, in 181 days. The accrual period is 184 days.

Since the term structure is flat, the six-month forward rate starting in six months is also 5.5%. First, we compute the present value factor, which is $\text{PV}(\$1) = 1/(1 + 0.055 \times 365/360) = 0.9472$, and the volatility, which is $\sigma\sqrt{\tau} = 0.20\sqrt{181/360} = 0.1418$.

We then compute the value of $d_1 = \ln[F/K]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = \ln[0.055/0.05]/0.1418 + 0.1418/2 = 0.7430$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.7430 - 0.1418 = 0.6012$. We find $N(d_1) = 0.7713$ and $N(d_2) = 0.7261$. The unit value of the call is $c = [F\,N(d_1) - K\,N(d_2)]\text{PV}(\$1) = 0.5789\%$. Finally, the total price of the call is \$1 million $\times$ 0.5789% $\times$ (184/360) = \$2,959.

Figure 10.3 can be taken as an illustration of the sequence of forward rates. If the cap rate is the same as the prevailing swap rate, the cap is said to be *at-the-money*. In the figure, the near caplets are out-of-the-money because $F_i < K$. The distant caplets, however, are in-the-money.

EXAMPLE 10.10: FRM EXAM 2002—QUESTION 22

An interest rate cap runs for 12 months based on three-month LIBOR with a strike price of 4%. Which of the following is generally *true*?

a. The cap consists of three caplet options with maturities of three months, the first one starting today based on three-month LIBOR set in advance and paid in arrears.
b. The cap consists of four caplets starting today, based on LIBOR set in advance and paid in arrears.
c. The implied volatility of each caplet will be identical no matter how the yield curve moves.
d. Rate caps have only a single option based on the maturity of the structure.

EXAMPLE 10.11: FRM EXAM 2004—QUESTION 10

The payoff to a swap where the investor receives fixed and pays floating can be replicated by all of the following *except*

a. A short position in a portfolio of FRAs
b. A long position in a fixed-rate bond and a short position in a floating-rate bond
c. A short position in an interest rate cap and a long position in a floor
d. A long position in a floating-rate note and a short position in a floor

EXAMPLE 10.12: FRM EXAM 2003—QUESTION 27

A portfolio management firm manages the fixed-rate corporate bond portfolio owned by a defined-benefit pension fund. The duration of the bond portfolio is five years; the duration of the pension fund's liabilities is seven years. Assume that the fund sponsor strongly believes that rates will decline over the next six months and is concerned about the duration mismatch between portfolio assets and pension liabilities. Which of the following strategies would be the best way to eliminate the duration mismatch?

a. Enter into a swap transaction in which the firm pays fixed and receives floating.
b. Enter into a swap transaction in which the firm receives fixed and pays floating.
c. Purchase an interest rate cap expiring in six months.
d. Sell Eurodollar futures contracts.

10.4.2 Swaptions

Swaptions are over-the-counter (OTC) options that give the buyer the right to enter a swap at a fixed point in time at specified terms, including a fixed coupon rate.

These contracts take many forms. A **European swaption** is exercisable on a single date at some point in the future. On that date, the owner has the right to enter a swap with a specific rate and term. Consider, for example, a 1Y × 5Y swaption. This gives the owner the right to enter in one year a long or short position in a five-year swap.

A fixed-term **American swaption** is exercisable on any date during the exercise period. In our example, this would be during the next year. If, for instance, exercise occurs after six months, the swap would terminate in five years and six months from now. So, the termination date of the swap depends on the exercise date. In contrast, a **contingent American swaption** has a prespecified termination date, for instance exactly six years from now. Finally, a **Bermudan option** gives the holder the right to exercise on a specific set of dates during the life of the option.

As an example, consider a company that, in one year, will issue five-year floating-rate debt. The company wishes to have the option to swap the floating payments into fixed payments. The company can purchase a swaption that will give it the right to create a five-year pay-fixed swap at the rate of 8%. If the prevailing swap rate in one year is higher than 8%, the company will exercise the swaption, otherwise not. The value of the option at expiration will be

$$P_T = \mathrm{Max}[V(i_T) - V(K), 0] \tag{10.15}$$

where $V(i)$ is the value of a swap to pay a fixed rate i, i_T is the prevailing swap rate for the swap maturity, and K is the locked-in swap rate. This contract is called a European 6/1 put swaption, or one into five-year payer option.

Such a swap is equivalent to an option on a bond. As this swaption creates a profit if rates rise, it is akin to a one-year put option on a six-year bond. A put option benefits when the bond value falls, which happens when rates rise. Conversely, a swaption that gives the right to receive fixed is akin to a call option on a bond. Table 10.3 summarizes the terminology for swaps, caps and floors, and swaptions.

Swaptions can be used for a variety of purposes. Consider an investor in a mortgage-backed security (MBS). If long-term rates fall, prepayment will increase, leading to a shortfall in the price appreciation in the bond. This risk can be hedged by buying receiver swaptions. If rates fall, the buyer will exercise the option, which creates a profit to offset the loss on the MBS. Alternatively, this risk can also be hedged by issuing callable debt. This creates a long position in an option that generates a profit if rates fall. As an example, Fannie Mae, a government-sponsored enterprise that invests heavily in mortgages, uses these techniques to hedge its prepayment risk.

TABLE 10.3 Summary of Terminology for OTC Swaps and Options

Product	Buy (Long)	Sell (Short)
Fixed/floating swap	Pay fixed Receive floating	Pay floating Receive fixed
Cap	Pay premium Receive $\mathrm{Max}(i - i_C, 0)$	Receive premium Pay $\mathrm{Max}(i - i_C, 0)$
Floor	Pay premium Receive $\mathrm{Max}(i_F - i, 0)$	Receive premium Pay $\mathrm{Max}(i_F - i, 0)$
Put swaption (payer option)	Pay premium Option to pay fixed and receive floating	Receive premium If exercised, receive fixed and pay floating
Call swaption (receiver option)	Pay premium Option to pay floating and receive fixed	Receive premium If exercised, receive floating and pay fixed

Finally, swaptions are typically priced using a variant of the Black model, assuming that interest rates are lognormal. The value of the swaption is then equal to a portfolio of options on different interest rates, all with the same maturity. In practice, swaptions are traded in terms of volatilities instead of option premiums. The applicable forward rate starts at the same time as the option, with a term equal to that of the option.

EXAMPLE 10.13: FRM EXAM 2003—QUESTION 56

As your company's risk manager, you are looking for protection against adverse interest rate changes in five years. Using Black's model for options on futures to price a European swap option (swaption) that gives the option holder the right to cancel a seven-year swap after five years, which of the following would you use in the model?

a. The two-year forward par swap rate starting in five years' time
b. The five-year forward par swap rate starting in two years' time
c. The two-year par swap rate
d. The five-year par swap rate

10.4.3 Exchange-Traded Options

Among exchange-traded fixed-income options, we describe options on Eurodollar futures and on T-bond futures.

Options on Eurodollar futures give the owner the right to enter a long or short position in Eurodollar futures at a fixed price. The payoff on a put option, for example, is

$$P_T = \text{Notional} \times \text{Max}[K - \text{FQ}_T, 0] \times (90/360) \tag{10.16}$$

where K is the strike price and FQ_T the prevailing futures price quote at maturity. In addition to the cash payoff, the option holder enters a position in the underlying futures. Since this is a put, it creates a short position after exercise, with the counterparty taking the opposing position. Note that, since futures are settled daily, the value of the contract is zero.

Since the futures price can also be written as $\text{FQ}_T = 100 - i_T$ and the strike price as $K = 100 - i_C$, the payoff is also

$$P_T = \text{Notional} \times \text{Max}[i_T - i_C, 0] \times (90/360) \tag{10.17}$$

which is equivalent to that of a cap on rates. Thus a put on Eurodollar futures is equivalent to a caplet on LIBOR.

In practice, there are minor differences in the contracts. Options on Eurodollar futures are American style instead of European style. Also, payments are made at the expiration date of Eurodollar futures options instead of in arrears.

Options on T-bond futures give the owner the right to enter a long or short position in futures at a fixed price. The payoff on a call option, for example, is

$$C_T = \text{Notional} \times \text{Max}[F_T - K, 0] \tag{10.18}$$

An investor who thinks that rates will fall, or that the bond market will rally, could buy a call on T-bond futures. In this manner, he or she will participate in the upside, without downside risk.

EXAMPLE 10.14: FRM EXAM 2007—QUESTION 95

To hedge against future, unanticipated, and significant increases in borrowing rates, which of the following alternatives offers the greatest flexibility for the borrower?

a. Interest rate collar
b. Fixed for floating swap
c. Call swaption
d. Interest rate floor

EXAMPLE 10.15: FRM EXAM 2009—QUESTION 2-24

The yield curve is upward sloping and a portfolio manager has a long position in 10-year Treasury notes funded through overnight repurchase agreements. The risk manager is concerned with the risk that market rates may increase further and reduce the market value of the position. What hedge could be put on to reduce the position's exposure to rising rates?

a. Enter into a 10-year pay-fixed and receive-floating interest rate swap.
b. Enter into a 10-year receive-fixed and pay-floating interest rate swap.
c. Establish a long position in 10-year Treasury note futures.
d. Buy a call option on 10-year Treasury note futures.

10.5 IMPORTANT FORMULAS

Long 1 $\times$ 4 FRA = Invest for one period, borrow for four

Payment on FRA: $V_T = (S_T - F) \times \tau \times \text{Notional} \times \text{PV}(\$1)$

Valuation of Eurodollar contract: $P_t = 10{,}000 \times [100 - 0.25(100 - \text{FQ}_t)] = 10{,}000 \times [100 - 0.25F_t]$

Eurodollar contract risk: DV01 = \$25

Futures convexity adjustment: Futures Rate = Forward Rate + $(1/2)\sigma^2 t_1 t_2$ (negative relationship between contract value and rates)

T-bond futures net delivery cost: Cost = Price − Futures Quote × CF

T-bond futures conversion factor: CF = NPV of bond at 6%

Valuation of interest rate swap: $V = B_F(\text{Fixed-Rate}) - B_f(\text{Floating-Rate})$ Long Receive-Fixed = Long Fixed-Coupon Bond + Short FRN

Valuation of interest rate swap as forward contracts: $V = \sum_i n_i(F_i - K)/(1 + R_i)^{\tau_i}$

Interest-rate cap: $C_T = \text{Max}[i_T - K, 0]$

Interest-rate floor: $P_T = \text{Max}[K - i_T, 0]$

Collar: Long cap plus short floor

Cap valuation: $c = \sum_{j=1}^{N} c_j, \quad c_j = [FN(d_1) - KN(d_2)]\text{PV}(\$1)$

Put swaption (1Y × 5Y) (right to pay fixed, starting in one year for five years): $P_T = \text{Max}[V(i_T) - V(K), 0]$

Put option on Eurodollar futures = cap on rates

10.6 ANSWERS TO CHAPTER EXAMPLES

Example 10.1: FRM Exam 2002—Question 27

b. An FRA defined as $t_1 \times t_2$ involves a forward rate starting at time t_1 and ending at time t_2. The buyer of this FRA locks in a borrowing rate for months 3 to 5. This is equivalent to borrowing for five months and reinvesting the funds for the first two months.

Example 10.2: FRM Exam 2005—Question 57

d. The market-implied forward rate is given by $\exp(-R_2 \times 2) = \exp(-R_1 \times 1 - F_{1,2} \times 1)$, or $F_{1,2} = 2 \times 3.50 - 1 \times 3.25 = 3.75\%$. Given that this is exactly equal to the quoted rate, the value must be zero. If instead this rate was 3.50%, for example, the value would be $V = \$1{,}000{,}000 \times (3.75\% - 3.50\%) \times (2 - 1) \exp(-3.50\% \times 2) = 2{,}331$.

Example 10.3: FRM Exam 2001—Question 70

b. The seller of an FRA agrees to receive fixed. Since rates are now higher than the contract rate, this contract must show a loss for the seller. The loss is \$10,000,000 × (6.85% − 6.35%) × (90/360) = \$12,500 when paid in arrears (i.e., in nine months). On the settlement date (i.e., brought forward by three months), the loss is \$12,500/(1 + 6.85% × 0.25) = \$12,290.

Example 10.4: FRM Exam 2009—Question 3-11

b. The cost of delivering each bond is the price divided by the conversion factor. This gives, respectively, $(102 + 14/32)/0.98 = 104.53$, 103.49, and 103.55. Hence the CTD is bond B. All other information is superfluous.

Example 10.5: FRM Exam 2009—Question 3-23

d. Forward rates may not equal futures rates due to the correlation between the interest rate, or reinvestment rate, and the futures contract profit. As seen in Equation (10.4), the volatility determines the size of the bias but not the direction.

Example 10.6: FRM Exam 2007—Question 80

c. Equation (10.4) shows that the futures rate exceeds the forward rate.

Example 10.7: FRM Exam 2005—Question 51

b. The floating leg uses LIBOR at the beginning of the period, plus 50bp, or 5.5%. The payment is given by $\$10,000,000 \times (0.06 - 0.055) \times 0.5 = 25,000$.

Example 10.8: FRM Exam 2000—Question 55

c. Using Equation (10.9) for three remaining periods, we have the discounted value of the net interest payment, or $(8\% - 7\%)\$100\text{m} = \$1\text{m}$, discounted at 7%, which is $\$934,579 + \$873,439 + \$816,298 = \$2,624,316$.

Example 10.9: FRM Exam 2009—Question 3-4

d. This question differs from the previous one, which gave the swap rate. Here, we have the spot rates for maturities of one and two years. The coupon is 7.5. The net present value (NPV) of the payments is then $V = \$18.75\exp(-1 \times 8\%) + (\$250 + \$18.75)\exp(-2 \times 8.5\%) = \$244$ million. Right after the reset, the value of the FRN is \$250 million, leading to a gain of \$6 million. This is a gain because the bank must pay a fixed rate but current rates are higher.

Example 10.10: FRM Exam 2002—Question 22

a. Interest rate caps involve multiple options, or caplets. The first one has terms that are set in three months. It locks in $\text{Max}[R(t+3) - 4\%, 0]$. Payment occurs in arrears in six months. The second one is a function of $\text{Max}[R(t+6) - 4\%, 0]$. The third is a function of $\text{Max}[R(t+9) - 4\%, 0]$ and is paid at $t + 12$. The sequence then stops because the cap has a term of 12 months only. This means there are three caplets.

Example 10.11: FRM Exam 2004—Question 10

d. A receive-fixed swap position is equivalent to being long a fixed-rate bond, or being short a portfolio of FRAs (which gain if rates go down), or selling a cap and buying a floor with the same strike price (which gains if rates go up). A short position in a floor does not generate a gain if rates drop. It is asymmetric anyway.

Example 10.12: FRM Exam 2003—Question 27

b. The manager should increase the duration of assets, or buy coupon-paying bonds. This can be achieved by entering a receive-fixed swap, so b. is correct and a. is wrong. Buying a cap will not provide protection if rates drop. Selling Eurodollar futures will lose money if rates drop.

Example 10.13: FRM Exam 2003—Question 56

a. The forward rate should start at the beginning of the option in five years, with a maturity equal to the duration of the option, or two years.

Example 10.14: FRM Exam 2007—Question 95

c. A swaption gives the borrower the flexibility to lock in a low rate. A regular swap does not offer flexibility as an option. A collar fixes a range of rates, but not much flexibility. A floor involves protection if rates go down, not up. (Note that buying a cap would have been another good choice.)

Example 10.15: FRM Exam 2009—Question 2-24

a. The bond position has positive duration. Entering a pay-fixed swap gains if rates go up; this negative duration can provide a hedge against the original position. Answer b. is thus incorrect. Answer c. is the same as the original position and is not a hedge. In answer d., a call on futures would not create a profit if rates go up, in which case the futures would go down. Buying a put would be a correct answer.

CHAPTER 11

Equity, Currency, and Commodity Markets

Having covered fixed-income instruments, we now turn to equity, currency, and commodity markets. Equities, or common stocks, represent ownership shares in a corporation. Due to the uncertainty in their cash flows, as well as in the appropriate discount rate, equities are much more difficult to value than fixed-income securities. They are also less amenable to the quantitative analysis that is used in fixed-income markets. Equity derivatives, however, can be priced reasonably precisely in relation to underlying stock prices.

Next, the foreign currency markets include spot, forward, options, futures, and swaps markets. The foreign exchange markets are by far the largest financial markets in the world, with daily turnover above $3 trillion.

Commodity markets consist of agricultural products, metals, energy, and other products. Commodities differ from financial assets, as their holding provides an implied benefit known as convenience yield but also incurs storage costs.

Section 11.1 introduces equity markets and presents valuation methods as well as some evidence on equity risk. Section 11.2 then provides an overview of important equity derivatives, including stock index contracts such as futures, options, and swaps as well as derivatives on single stocks. For most of these contracts, pricing methods have been developed in the previous chapters and do not require special treatment. Convertible bonds and warrants will be covered in a separate chapter.

Section 11.3 presents a brief introduction to currency markets. Currency derivatives are discussed in Section 11.4. We analyze currency swaps in some detail because of their unique features and importance. Finally, Sections 11.5 and 11.6 discuss commodity markets and commodity derivatives.

11.1 EQUITIES

11.1.1 Overview

Common stocks, also called **equities**, are securities that represent ownership in a corporation. Bonds are *senior* to equities; that is, they have a prior claim on

FRM Exam Part 1 topic.

the firm's assets in case of bankruptcy. Hence equities represent **residual claims** to what is left of the value of the firm after bonds, loans, and other contractual obligations have been paid off.

Another important feature of common stocks is their **limited liability**, which means that the most shareholders can lose is their original investment. This is unlike owners of unincorporated businesses, whose creditors have a claim on the personal assets of the owner should the business turn bad.

Table 11.1 describes the global equity markets. The total market value of common stocks was worth approximately \$48 trillion at the end of 2009. The United States accounts for the largest share, followed by Japan, the Eurozone, and the United Kingdom. In 2008, global stocks fell by 42%, which implies a loss of market value of about \$26 trillion. About \$10 trillion of this was recovered in 2009. Thus, investing in equities involves substantial risks.

Preferred stocks differ from common stock because they promise to pay a specific stream of dividends. So, they behave like a perpetual bond, or consol. Unlike bonds, however, failure to pay these dividends does not result in default. Instead, the corporation must withhold dividends to holders of common stock until the preferred dividends have been paid out. In other words, preferred stocks are junior to bonds, but senior to common stocks.

With **cumulative preferred dividends**, all current and previously postponed dividends must be paid before any dividends on common stock shares can be paid. Preferred stocks usually have no voting rights.

Unlike interest payments, preferred stocks' dividends are not tax-deductible expenses. Preferred stocks, however, have an offsetting tax advantage. Corporations that receive preferred dividends pay taxes on only 30% of the amount received, which lowers their income tax burden. As a result, most preferred stocks are held by corporations. The market capitalization of preferred stocks is much lower than that of common stocks, as seen from the IBM example. Trading volumes are also much lower.

TABLE 11.1 Global Equity Markets, 2009 (Billions of U.S. Dollars)

United States	15,077
Japan	3,444
Eurozone	7,271
United Kingdom	2,796
Other Europe	2,109
Other Pacific	4,084
Canada	1,677
Developed	36,459
Emerging	7,239
World	47,783

Source: World Federation of Exchanges.

Example: IBM Preferred Stock

IBM issued 11.25 million preferred shares in June 1993. These are traded as 45 million depositary shares, each representing one-fourth of the preferred, under the ticker IBM-A on the New York Stock Exchange (NYSE). Dividends accrue at the rate of \$7.50 per annum, or \$1.875 per depositary share.

As of April 2001, the depositary shares were trading at \$25.40, within a narrow 52-week trading range between \$25.00 and \$26.25. Using the valuation formula for a consol, the shares were trading at an implied yield of 7.38%. The total market capitalization of the IBM-A shares amounts to approximately \$1,143 million. In comparison, the market value of the common stock is \$214,602 million, which is much larger.

11.1.2 Valuation

Common stocks are difficult to value. Like any other asset, their value derives from their future benefits, that is, from their stream of future cash flows (i.e., dividend payments) or future stock price.

We have seen that valuing Treasury bonds is relatively straightforward, as the stream of cash flows, coupon, and principal payments can be easily laid out and discounted into the present. It is an entirely different affair for common stocks. Consider for illustration a simple case where a firm pays out a dividend D over the next year that grows at the constant rate of g. We ignore the final stock value and discount at the constant rate of r, such that $r > g$. The firm's value, P, can be assessed using the net present value formula, like a bond:

$$\begin{aligned} P &= \textstyle\sum_{t=1}^{\infty} C_t/(1+r)^t \\ &= \textstyle\sum_{t=1}^{\infty} D(1+g)^{(t-1)}/(1+r)^t \\ &= [D/(1+r)] \textstyle\sum_{t=0}^{\infty} [(1+g)/(1+r)]^t \\ &= [D/(1+r)] \times \left[\frac{1}{1-(1+g)/(1+r)}\right] \\ &= [D/(1+r)] \times [(1+r)/(r-g)] \end{aligned}$$

This is also the so-called Gordon growth model,

$$P = \frac{D}{r-g} \tag{11.1}$$

The problem with equities is that the growth rate of dividends is uncertain and, in addition, it is not clear what the required discount rate should be. To make things even harder, some companies simply do not pay any dividend and instead create value from the appreciation of their share price.

Still, this valuation formula indicates that large variations in equity prices can arise from small changes in the discount rate or in the growth rate of dividends, thus explaining the large volatility of equities. More generally, the value of the equity depends on the underlying business fundamentals as well as on the amount of leverage, or debt, in the capital structure.

11.1.3 Equity Risk

Equity risk arises from potential movements in the value of stock prices. We can usefully decompose the total risk of an equity portfolio into a marketwide risk and stock-specific risk. Focusing on volatility as a single measure of risk, stock index volatility typically ranges from 12% to 20% per annum.

Markets that are less diversified are typically more volatile. **Concentration** refers to the proportion of the index due to the biggest stocks. In Finland, for instance, half of the index represents one firm only, Nokia, which makes the index more volatile than it otherwise would be.

11.2 EQUITY DERIVATIVES

Equity derivatives can be traded in over-the-counter (OTC) markets as well as on organized exchanges. We consider only the most popular instruments.

11.2.1 Stock Index Contracts

Derivative contracts on stock indices are widely used due to their ability to hedge general stock market risks. Active contracts include stock index futures and their options as well as index swaps.

Stock index futures are the most active derivative contracts on stock indices and are traded all over the world. In fact, the turnover corresponding to the notional amount is sometimes greater than the total amount of trading in physical stocks in the same market. The success of these contracts can be explained by their versatility for risk management. Stock index futures allow investors to manage efficiently their exposure to broad stock market movements. Speculators can take easily directional bets with futures, on the upside or downside. Hedgers also find that futures provide a cost-efficient method to protect against price risk.

Perhaps the most active contract is the S&P 500 futures contract on the Chicago Mercantile Exchange (now CME Group). The contract notional is defined as $250 times the index level. Table 11.2 displays quotations as of December 31, 1999.

The table shows that most of the volume was concentrated in the near contract, that is, March in this case. Translating the trading volume in number of contracts into a dollar equivalent, we find $\$250 \times 1{,}484.2 \times 34{,}897$, which gives

TABLE 11.2 Sample S&P Futures Quotations

Maturity	Open	Settle	Change	Volume	Open Interest
March	1,480.80	1,484.20	+3.40	34,897	356,791
June	1,498.00	1,503.10	+3.60	410	8,431

\$13 billion. More recently, in 2009, the average daily volume was \$33 billion. This is nearly half the trading volume of \$71 billion for stocks on the New York Stock Exchange (NYSE). So, these markets are very liquid.

We can also compute the daily profit on a long position, which would have been \$250 × (+3.40), or \$850 on that day. In relative terms, this daily move was +3.4/1,480.8, which is only 0.23%. The typical daily standard deviation is about 1%, which gives a typical profit or loss of \$3,710.50.

These contracts are cash settled. They do not involve delivery of the underlying stocks at expiration. In terms of valuation, the futures contract is priced according to the usual cash-and-carry relationship,

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{11.2}$$

where y is the dividend yield defined per unit time. For instance, the yield on the S&P was $y = 0.94\%$ per annum on that day.

Here, we assume that the dividend yield is known in advance and paid on a continuous basis, which is a good approximation. With a large number of firms in the index, dividends will be spread reasonably evenly over the quarter.

To check whether the futures contract was fairly valued, we need the spot price, $S = 1{,}469.25$, the short-term interest rate, $r = 5.3\%$, and the number of days to maturity, which was 76 (to March 16). Note that rates are not continuously compounded. The present value factor is $\mathrm{PV}(\$1) = 1/(1 + r\tau) = 1/(1 + 5.3\%(76/365)) = 0.9891$. Similarly, the present value of the dividend stream is $1/(1 + y\tau) = 1/(1 + 0.94\%(76/365)) = 0.9980$. The fair price is then

$$F = [S/(1 + y\tau)]\,(1 + r\tau) = [1{,}469.25 \times 0.9980]/0.9891 = 1{,}482.6$$

This is rather close to the settlement value of $F = 1{,}484.2$. The discrepancy is probably because the quotes were not measured simultaneously. Because the yield is less than the interest rate, the forward price is greater than the spot price.

Figure 11.1 displays the convergence of futures and cash prices for the S&P 500 stock index futures contract traded on the CME. Note two major features. First, the futures price is always above the spot price as predicted. The difference, however, shrinks to zero as the contract goes to maturity. Second, the correlation between the two prices is very high, reflecting the cash-and-carry relationship in Equation (11.2).

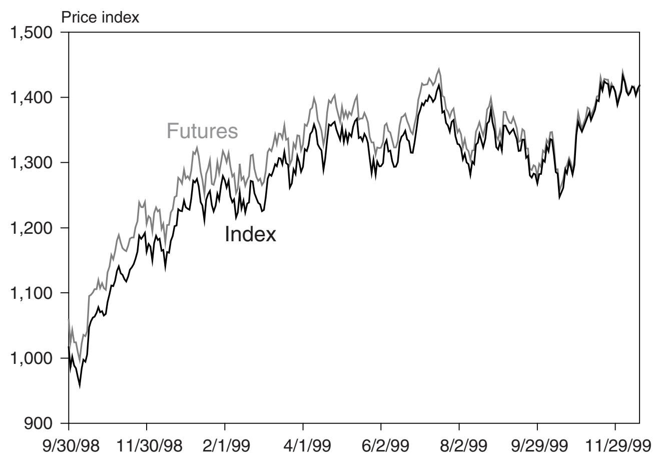

FIGURE 11.1 Futures and Cash Prices for S&P 500 Futures

Because financial institutions engage in stock index arbitrage, we would expect the cash-and-carry relationship to hold very well. One notable exception was during the market crash of October 19, 1987. The market lost more than 20% in a single day. Throughout the day, however, futures prices were more up-to-date than cash prices because of execution delays in cash markets. As a result, the S&P stock index futures value was very cheap compared with the underlying cash market. Arbitrage, however, was made difficult due to chaotic market conditions.

Next, **equity swaps** are agreements to exchange cash flows tied to the return on a stock market index in exchange for a fixed or floating rate of interest. An example is a swap that provides the return on the S&P 500 index every six months in exchange for payment of LIBOR plus a spread. The swap will be typically priced so as to have zero value at initiation. Equity swaps can be valued as portfolios of forward contracts, as in the case of interest rate swaps. These swaps are used by investment managers to acquire exposure to, for example, an emerging stock market without having to invest in the market itself. In some cases, these swaps can also be used to skirt restrictions on foreign investments.

EXAMPLE 11.1: FRM EXAM 2000—QUESTION 12

Suppose the price for a six-month S&P index futures contract is 552.3. If the risk-free interest rate is 7.5% per year and the dividend yield on the stock index is 4.2% per year, and the market is complete and there is no arbitrage, what is the price of the index today?

a. 543.26
b. 552.11
c. 555.78
d. 560.02

EXAMPLE 11.2: FRM EXAM 2009—QUESTION 3-1

A stock index is valued at USD 750 and pays a continuous dividend at the rate of 2% per annum. The six-month futures contract on that index is trading at USD 757. The risk-free rate is 3.50% continuously compounded. There are no transaction costs or taxes. Is the futures contract priced so that there is an arbitrage opportunity? If yes, which of the following numbers comes closest to the arbitrage profit you could realize by taking a position in one futures contract?

a. $4.18
b. $1.35
c. $12.60
d. There is no arbitrage opportunity.

11.2.2 Single Stock Contracts

Derivative contracts tied to single stocks are also widely used. These include futures and options as well as contracts for differences.

In late 2000, the United States passed legislation authorizing trading in **single stock futures**, which are futures contracts on individual stocks. Such contracts were already trading in Europe and elsewhere. In the United States, electronic trading started in November 2002 and now takes place on OneChicago, a joint venture of Chicago exchanges.

Each contract gives the obligation to buy or sell 100 shares of the underlying stock. Settlement usually involves physical delivery, that is, the exchange of the underlying stock. Relative to trading in the underlying stocks, single stock futures have many advantages. Positions can be established more efficiently due to their low margin requirements, which are generally 20% of the cash value. In contrast, margins for stocks are higher. Also, short selling eliminates the costs and inefficiencies associated with the stock loan process. Other than physical settlement, these contracts trade like stock index futures.

Contracts for differences (CFDs) are contracts whose payoff is tied to the value of the underlying stock. CFDs were originally developed in the early 1990s in London, in large part to avoid an expensive stamp duty, which is a UK tax on trades involving the physical trading of stocks. Like futures, CFDs are subject to margin requirements. Payoffs are tied to the change in the price of the stock and a financing charge. Dividends are passed on to the long CFD position. CFDs have no expiration and can be rolled over as needed, provided the margin requirements are met. CFDs are traded over-the-counter with a broker or market maker.

11.2.3 Equity Options

Options can be traded on individual stocks, on stock indices, or on stock index futures. In the United States, stock options trade, for example, on the Chicago

Board Options Exchange (CBOE). Each option gives the right to buy or sell a round lot of 100 shares. Settlement involves physical delivery.

Traded options are typically American-style, so their valuation should include the possibility of early exercise. In practice, however, their values do not differ much from those of European options, which can be priced by the Black-Scholes model. When the stock pays no dividend, the values are the same. For more precision, we can use numerical models such as binomial trees to take into account dividend payments.

The most active *index* options in the United States are options on the S&P 100 and S&P 500 index traded on the CBOE. The former are American-style, while the latter are European-style. These options are cash settled, as it would be too complicated to deliver a basket of 100 or 500 underlying stocks. Each contract is for $100 times the value of the index. European options on stock indices can be priced using the Black-Scholes formula. Finally, options on S&P 500 stock index futures are also popular. These give the right to enter a long or short futures position at a fixed price.

11.3 CURRENCIES

11.3.1 Overview

The foreign exchange (**forex**) or currency markets have enormous trading activity, with daily turnover estimated at $3,210 billion in 2007. Their size and growth are described in Table 11.3. This trading activity dwarfs that of bond or stock markets. In comparison, the daily trading volume on the New York Stock Exchange (NYSE) is approximately $80 billion. Even though the largest share of these transactions is between dealers or with other financial institutions, the volume of trading with other, nonfinancial institutions is still quite large, at $549 billion daily.

TABLE 11.3 Average Daily Trading Volume in Currency Markets (Billions of U.S. Dollars)

Year	Spot	Forwards, Forex Swaps	Total
1989	350	240	590
1992	416	404	820
1995	517	673	1,190
1998	592	898	1,490
2001	399	811	1,210
2004	656	1,224	1,880
2007	1,005	2,076	3,210
Of which, between:			
Dealers			1,374
Financials			1,287
Others			549

Source: Bank for International Settlements surveys.

Spot transactions are exchanges of two currencies for settlement as soon as it is practical, typically in two business days. They account for about 35% of trading volume. Other transactions are outright forward contracts and forex swaps. **Outright forward contracts** are agreements to exchange two currencies at a future date, and account for about 12% of the total market. **Forex swaps** involve two transactions, an exchange of currencies on a given date and a reversal at a later date, and account for 53% of the total market. Note that forex swaps are typically of a short-term nature and should not be confused with long-term currency swaps, which involve a stream of payments over longer horizons.

In addition to these contracts, the market also includes OTC forex options ($212 billion daily) and exchange-traded derivatives ($72 billion daily). The most active currency futures are traded on the Chicago Mercantile Exchange (now CME Group) and settled by physical delivery. The CME also trades options on currency futures.

Currencies are generally quoted in **European terms**, that is, in units of the foreign currency per dollar. The yen, for example, is quoted as 120 yen per U.S. dollar. Two notable exceptions are the British pound (sterling) and the euro, which are quoted in **American terms**, that is in dollars per unit of the foreign currency. The pound, for example, is quoted as 1.6 dollars per pound.

EXAMPLE 11.3: FRM EXAM 2003—QUESTION 2

The current spot CHF/USD rate is 1.3680 CHF. The three-month USD interest rate is 1.05%, and the three-month Swiss interest rate is 0.35%, both continuously compounded and per annum. A currency trader notices that the three-month forward price is USD 0.7350. In order to arbitrage, the trader should

a. Borrow CHF, buy USD spot, go long Swiss franc forward
b. Borrow CHF, sell Swiss franc spot, go short Swiss franc forward
c. Borrow USD, buy Swiss franc spot, go short Swiss franc forward
d. Borrow USD, sell USD spot, go long Swiss franc forward

11.3.2 Currency Risk

Currency risk arises from potential movements in the value of foreign exchange rates. Currency risk arises in the following environments.

In a *pure currency float*, the external value of a currency is free to move, to depreciate or appreciate, as pushed by market forces. An example is the dollar/euro exchange rate. In these cases, currency volatility typically ranges from 6% to 10% per annum. This is considerably less than the volatility of equities.

In a *fixed currency system*, a currency's external value is fixed (or pegged) to another currency. An example is the Hong Kong dollar, which is fixed against the U.S. dollar. This does not mean there is no risk, however, due to possible readjustments in the **parity value**, called devaluations or revaluations. Thus, they are subject to **devaluation risk**.

In a *change in currency regime*, a currency that was previously fixed becomes flexible, or vice versa. For instance, the Argentinian peso was fixed against the dollar until 2001, and floated thereafter. Changes in regime can also lower currency risk, as in the case of the euro.[1]

EXAMPLE 11.4: FRM EXAM 2009—QUESTION 3-19

Bonumeur SA is a French company that produces strollers for children and is specialized in strollers for twins and triplets for the EU market. The company buys the wheels of the strollers on the U.S. market. Invoices are paid in USD. What is Bonumeur's currency risk and how can the company hedge its exposure?

a. EUR depreciating against USD; selling EUR against buying USD forward
b. EUR depreciating against USD; selling USD against buying EUR forward
c. EUR appreciating against USD; selling EUR against buying USD forward
d. EUR appreciating against USD; selling USD against buying EUR forward

11.4 CURRENCY DERIVATIVES

Currency markets offer the full range of financial instruments, including futures, forwards, and options. These derivatives can be priced according to standard valuation models, specifying the income payment to be a continuous flow defined by the foreign interest rate. For currency forwards, for example, the relationship between forward and spot prices is very similar to that in Figure 11.1. The two prices are highly correlated and converge to each other at maturity.

Because of their importance, currency swaps are examined in more detail. **Currency swaps** are agreements by two parties to exchange a stream of cash flows in different currencies according to a prearranged formula.

11.4.1 Currency Swaps

Consider two counterparties, company A and company B, which can raise funds either in dollars or in yen, \$100 million or ¥10 billion at the current rate of

[1] As of 2009, the Eurozone includes a block of 16 countries. Early adopters in 1999 include Austria, Belgium, Luxembourg, Finland, France, Germany, Ireland, Italy, Netherlands, Portugal, and Spain. Greece joined on January 1, 2001. Slovenia joined on January 1, 2007. Cyprus and Malta joined on January 1, 2008. Slovakia joined on January 1, 2009. Currency risk is not totally eliminated, however, as there is always a possibility that the currency union could dissolve.

TABLE 11.4a Cost of Capital Comparison

Company	Yen	Dollar
A	5.00%	9.50%
B	6.50%	10.00%

TABLE 11.4b Swap to Company A

Operation	Yen	Dollar
Issue debt	Pay yen 5.00%	
Enter swap	Receive yen 5.00%	Pay dollar 9.00%
Net		Pay dollar 9.00%
Direct cost		Pay dollar 9.50%
Savings		0.50%

TABLE 11.4c Swap to Company B

Operation	Dollar	Yen
Issue debt	Pay dollar 10.00%	
Enter swap	Receive dollar 9.00%	Pay yen 5.00%
Net		Pay yen 6.00%
Direct cost		Pay yen 6.50%
Savings		0.50%

100¥/$, over 10 years. Company A wants to raise dollars, and company B wants to raise yen. Table 11.4a displays borrowing costs. This example is similar to that of interest rate swaps, except that rates are now in different currencies.

Company A has an **absolute advantage** in the two markets as it can raise funds at rates systematically lower than company B. Company B, however, has a **comparative advantage** in raising dollars, as the cost is only 0.50% higher than for company A, compared to the cost difference of 1.50% in yen. Conversely, company A must have a comparative advantage in raising yen.

This provides the basis for a swap that will be to the mutual advantage of both parties. If both institutions directly issue funds in their final desired market, the total cost will be 9.50% (for A) and 6.50% (for B), for a total of 16.00%. In contrast, the total cost of raising capital where each has a comparative advantage is 5.00% (for A) and 10.00% (for B), for a total of 15.00%. The gain to both parties from entering a swap is $16.00 - 15.00 = 1.00\%$. For instance, the swap described in Tables 11.4b and 11.4c splits the benefit equally between the two parties.

Company A issues yen debt at 5.00%, then enters a swap whereby it promises to pay 9.00% in dollars in exchange for receiving 5.00% yen payments. Its effective funding cost is therefore 9.00%, which is less than the direct cost by 50bp.

Similarly, company B issues dollar debt at 10.00%, then enters a swap whereby it receives 9.00% dollars in exchange for paying 5.00% yen. If we add up the difference in dollar funding cost of 1.00% to the 5.00% yen funding costs, the effective funding cost is therefore 6.00%, which is less than the direct cost by 50bp.[2] Both parties benefit from the swap.

While payments are typically netted for an interest rate swap, because they are in the same currency, this is not the case for currency swaps. Full interest payments are made in different currencies. In addition, at initiation and termination, there is exchange of principal in different currencies. For instance, assuming annual payments, company A will receive 5.00% on a notional of ¥10 billion, which is ¥500 million in exchange for paying 9.00% on a notional of $100 million, or $9 million every year.

11.4.2 Swap Pricing

Consider now the pricing of the swap to company A. This involves receiving 5.00% yen in exchange for paying 9.00% dollars. As with interest rate swaps, we can price the swap using either of two approaches, taking the difference between two bond prices or valuing a sequence of forward contracts.

This swap is equivalent to a long position in a fixed-rate 5% 10-year yen-denominated bond and a short position in a 9% 10-year dollar-denominated bond. The value of the swap is that of a long yen bond minus a dollar bond. Defining S as the dollar price of the yen and P and P^* as the dollar and yen bond, respectively, we have:

$$V = S(\$/¥)P^*(¥) - P(\$) \tag{11.3}$$

Here, we indicate the value of the yen bond by an asterisk, P^*.

In general, the bond value can be written as $P(c, r, F)$ where the coupon is c, the yield is r, and the face value is F. Our swap is initially worth (in millions):

$$V = \frac{1}{100}P^*(5\%, 5\%, ¥10{,}000) - P(9\%, 9\%, \$100)$$
$$= \frac{\$1}{¥100}¥10{,}000 - \$100 = \$0$$

Thus, the initial value of the swap is zero, assuming a flat term structure for both countries and no credit risk.

We can identify three conditions under which the swap will be in-the-money. This will happen (1) if the value of the yen S appreciates, or (2) if the yen interest

[2] Note that B is somewhat exposed to currency risk, as funding costs cannot be simply added when they are denominated in different currencies. The error, however, is of a second-order magnitude.

rate r^* falls, or (3) if the dollar interest rate r goes up. Thus the swap is exposed to three risk factors: the spot rate and two interest rates. The latter exposures are given by the duration of the equivalent bond.

KEY CONCEPT

A position in a receive foreign currency swap is equivalent to a long position in a foreign currency bond offset by a short position in a dollar bond.

The swap can be alternatively valued as a sequence of forward contracts. Recall that the valuation of a forward contract on one yen is given by

$$V_i = (F_i - K)\exp(-r_i\tau_i) \tag{11.4}$$

using continuous compounding. Here, r_i is the dollar interest rate, F_i is the prevailing forward rate (in \$/yen), and K is the locked-in rate of exchange, defined as the ratio of the dollar-to-yen payment on this maturity. Extending this to multiple maturities, the swap is valued as

$$V = \sum_i n_i(F_i - K)\exp(-r_i\tau_i) \tag{11.5}$$

where $n_i F_i$ is the dollar value of the yen payments translated at the forward rate and the other term, $n_i K$, is the dollar payment in exchange.

Table 11.5 compares the two approaches for a three-year swap with annual payments. Market rates have now changed and are $r = 8\%$ for U.S. yields and $r^* = 4\%$ for yen yields. We assume annual compounding. The spot exchange rate has moved from 100¥/\$ to 95¥/\$, reflecting a depreciation of the dollar (or appreciation of the yen).

The middle panel shows the valuation using the difference between the two bonds. First, we discount the cash flows in each currency at the newly prevailing yield. This gives $P = \$102.58$ for the dollar bond and ¥10,277.51 for the yen bond. Translating the latter at the new spot rate of ¥95, we get \$108.18. The swap is now valued at \$108.18 − \$102.58, which is a positive value of $V = \$5.61$ million. The appreciation of the swap is principally driven by the appreciation of the yen.

The bottom panel shows how the swap can be valued by a sequence of forward contracts. First, we compute the forward rates for the three maturities. For example, the one-year rate is $95 \times (1 + 4\%)/(1 + 8\%) = 91.48$ ¥/\$, by interest rate parity. Next, we convert each yen receipt into dollars at the forward rate, for example ¥500 million in one year, which is \$5.47 million. This is offset against a payment of \$9 million, for a net planned cash outflow of −\$3.53 million. Discounting and adding up the planned cash flows, we get $V = \$5.61$ million, which must be exactly equal to the value found using the alternative approach.

TABLE 11.5 Pricing a Currency Swap

	Specifications		Market Data
	Notional Amount (Millions)	Contract Rates	Market Rates
Dollar	$100	9%	8%
Yen	¥10,000	5%	4%
Exchange rate		100¥/$	95¥/$

Valuation Using Bond Approach (Millions)

	Dollar Bond			Yen Bond		
Time (Year)	Dollar Payment	PV($1)	PV(CF)	Yen Payment	PV(¥1)	PV(CF)
1	9	0.9259	8.333	500	0.9615	480.769
2	9	0.8573	7.716	500	0.9246	462.278
3	109	0.7938	86.528	10,500	0.8890	9,334.462
Total			$102.58			¥10,277.51
Swap ($)			−$102.58			$108.18
Value						$5.61

Valuation Using Forward Contract Approach (Millions)

Time (Year)	Forward Rate (¥/$)	Yen Receipt (¥)	Yen Receipt ($)	Dollar Payment ($)	Difference CF ($)	PV(CF) ($)
1	91.48	500	5.47	−9.00	−3.534	−3.273
2	88.09	500	5.68	−9.00	−3.324	−2.850
3	84.83	10,500	123.78	−109.00	14.776	11.730
Value						$5.61

EXAMPLE 11.5: FRM EXAM 2008—QUESTION 2-27

Which of the following statements is correct when comparing the differences between an interest rate swap and a currency swap?

a. At maturity, the counterparties to interest rate swaps and the counterparties to currency swaps both exchange the principal of the swap.
b. At maturity, the counterparties to interest rate swaps do not exchange the principal, but the counterparties to currency swaps exchange the value difference in principal determined by prevailing exchange rates.
c. At maturity, the counterparties to interest rate swaps do not exchange the principal, and counterparties to currency swaps do exchange the principal.
d. Counterparties to interest rate swaps are exposed to more counterparty credit risk due to the magnifying effect of currency, interest rate, and settlement risk embedded within the transaction.

EXAMPLE 11.6: FRM EXAM 2006—QUESTION 88

You have entered into a currency swap in which you receive 4%pa in yen and pay 6%pa in dollars once a year. The principals are 1,000 million yen and 10 million dollars. The swap will last for another two years, and the current exchange rate is 115 yen/\$. The annualized spot rates (with continuous compounding) are 2.00% and 2.50% in yen for one- and two-year maturities, and 4.50% and 4.75% in dollars. What is the value of the swap to you in million dollars?

a. −1.270
b. −0.447
c. 0.447
d. 1.270

EXAMPLE 11.7: FRM EXAM 2007—QUESTION 87

Your company is expecting a major export order from a London-based client. The receivables under the contract are to be billed in GBP, while your reporting currency is USD. Since the order is a large sum, your company does not want to bear the exchange risk and wishes to hedge it using derivatives. To minimize the cost of hedging, which of the following is the most suitable contract?

a. A chooser option for GBP/USD pair
b. A currency swap where you pay fixed in USD and receive floating in GBP
c. A barrier put option to sell GBP against USD
d. An Asian call option on GBP against USD

11.5 COMMODITIES

11.5.1 Overview

Commodities are typically traded on exchanges. Contracts include spot, futures, and options on futures. There is also an OTC market for long-term commodity swaps, where payments are tied to the price of a commodity against a fixed or floating rate.

Commodity contracts can be classified into:

- **Agricultural products**, including grains and oilseeds (corn, wheat, soybeans), food and fiber (cocoa, coffee, sugar, orange juice)
- **Livestock and meat** (cattle, hogs)

- **Base metals** (aluminum, copper, nickel, zinc)
- **Precious metals** (gold, silver, platinum)
- **Energy products** (natural gas, heating oil, unleaded gasoline, crude oil)
- **Weather derivatives** (temperature, hurricanes, snow)
- **Environmental products** (CO_2 allowances)

The **S&P GSCI**, formerly the Goldman Sachs Commodity Index, is a broad production-weighted index of commodity price performance, which is composed of 24 liquid exchange-traded futures contracts. As of December 2009, the index contains 70% energy products, 8% industrial metals, 3% precious metals, 14% agricultural products, and 4% livestock products. The CME Group trades futures and options contracts on the S&P GSCI.

In the past few years, active markets have developed for **electricity products**, electricity futures for delivery at specific locations, for instance California/Oregon border (COB), Palo Verde, and so on. These markets have mushroomed following the deregulation of electricity prices, which has led to more variability in electricity prices.

More recently, OTC markets and exchanges have introduced **weather derivatives**, where the payout is indexed to temperature or precipitation. On the CME, for instance, contract payouts are based on the Degree Day Index over a calendar month. This index measures the extent to which the daily temperature deviates from the average. These contracts allow users to hedge situations where their income is negatively affected by extreme weather. Markets are constantly developing new products.

Such commodity markets allow participants to exchange risks. Farmers, for instance, can sell their crops at a fixed price on a future date, insuring themselves against variations in crop prices. Likewise, consumers can buy these crops at a fixed price.

11.6 COMMODITY DERIVATIVES

11.6.1 Valuation

Commodities differ from financial assets in two notable dimensions: they may be expensive, even impossible, to store and they may generate a flow of benefits that are not directly measurable.

The first dimension involves the cost of carrying a physical inventory of commodities. For most financial instruments, this cost is negligible. For bulky commodities, this cost may be high. Other commodities, like electricity, cannot be stored easily.

The second dimension involves the benefit from holding the physical commodity. For instance, a company that manufactures copper pipes benefits from an inventory of copper, which is used up in its production process. This flow is also called **convenience yield** for the holder. For a financial asset, this flow would be a monetary income payment for the investor. When an asset such as gold can be lent out for a profit, the yield represents the **lease rate**, which is the return to lending gold short-term.

Consider the first factor, storage cost only. The cash-and-carry relationship should be modified as follows. We compare two positions. In the first, we buy the commodity spot plus pay up front the present value of storage costs PV(C). In the second, we enter a forward contract and invest the present value of the forward price. Since the two positions are identical at expiration, they must have the same initial value:

$$F_t e^{-r\tau} = S_t + \text{PV}(C) \tag{11.6}$$

where $e^{-r\tau}$ is the present value factor. Alternatively, if storage costs are incurred per unit time and defined as c, we can restate this relationship as

$$F_t e^{-r\tau} = S_t e^{c\tau} \tag{11.7}$$

Due to these costs, the forward rate should be much greater than the spot rate, as the holder of a forward contract benefits not only from the time value of money but also from avoiding storage costs.

Example: Computing the Forward Price of Gold

Let us use data from December 1999. The spot price of gold is $S = \$288$, the one-year interest rate is $r = 5.73\%$ (continuously compounded), and storage costs are \$2 per ounce per year, paid up front. The fair price for a one-year forward contract should be $F = [S + \text{PV}(C)]e^{r\tau} = [\$288 + \$2]e^{5.73\%} = \$307.1$.

Let us now turn to the convenience yield, which can be expressed as y per unit time. In fact, y represents the net benefit from holding the commodity, after storage costs. Following the same reasoning as before, the forward price on a commodity should be given by

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{11.8}$$

where $e^{-y\tau}$ is an actualization factor. This factor may have an economically identifiable meaning, reflecting demand and supply conditions in the cash and futures markets. Alternatively, it can be viewed as a *plug-in* that, given F, S, and $e^{-r\tau}$, will make Equation (11.8) balance.

Let us focus, for example, on the one-year contract. Using $S = \$25.60$, $F = \$20.47$, $r = 5.73\%$ and solving for y,

$$y = r - \frac{1}{\tau}\ln(F/S) \tag{11.9}$$

we find $y = 28.10\%$, which is quite large. In fact, variations in y can be substantial. Just one year before, a similar calculation would have given $y = -9\%$, which

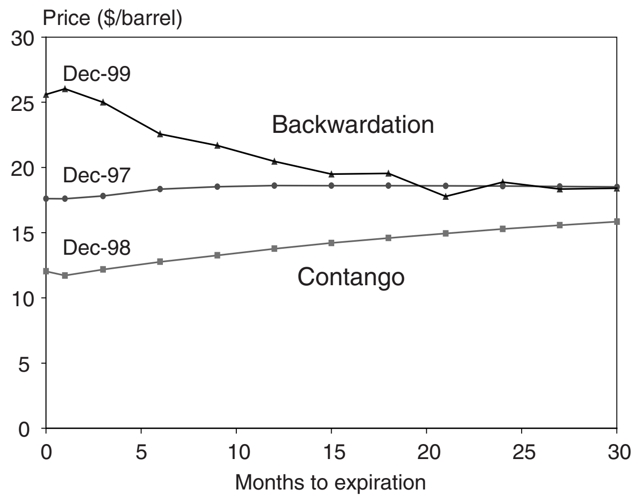

FIGURE 11.2 Term Structure of Futures Prices for Crude Oil

implies a negative convenience yield, or a storage cost. This yield depends on the maturity of the contract.

Figure 11.2, for example, displays the shape of the term structure of spot and futures prices for the New York Mercantile Exchange (NYMEX) crude oil contract. In December 1997, the term structure is relatively flat. In December 1998, the term structure becomes strongly upward sloping. The market is said to be in a **contango** when the futures price trades at a premium relative to the spot price. Using Equation (11.9), this implies that the convenience yield is smaller than the interest rate $y < r$.

In contrast, the term structure is downward sloping in December 1999. A market is said to be in **backwardation** (or inverted) when forward prices trade at a discount relative to spot prices. This implies that the convenience yield is greater than the interest rate $y > r$. In other words, a high convenience yields puts a higher price on the cash market, as there is great demand for immediate consumption of the commodity.

Table 11.6 displays futures prices for selected contracts. Futures prices are generally increasing with maturity, reflecting the time value of money, storage cost, and low convenience yields. Corn, for example, is in contango. There are some irregularities, however, reflecting anticipated imbalances between demand

TABLE 11.6 Futures Prices as of December 31, 2009

Maturity	Corn	Sugar	Copper	Gold	Natural Gas	Gasoline	Heating Oil
Jan.			333.8	1,095.2		205.3	211.9
March	414.5	26.95	334.7	1,096.9	5.532	207.2	212.2
July	433.0	23.02	337.1	1,099.5	5.695	219.9	215.3
Sept.	437.5	22.20	337.6	1,101.0	5.795	218.9	219.4
Dec.	440.8	21.50	338.2	1,104.1	6.548	209.4	226.6
Mar. 11	449.8	21.05	338.5	1,108.4	6.560	216.0	230.7
...							
Dec. 11	447.8	17.50	339.4	1,126.9	6.820	217.4	239.0

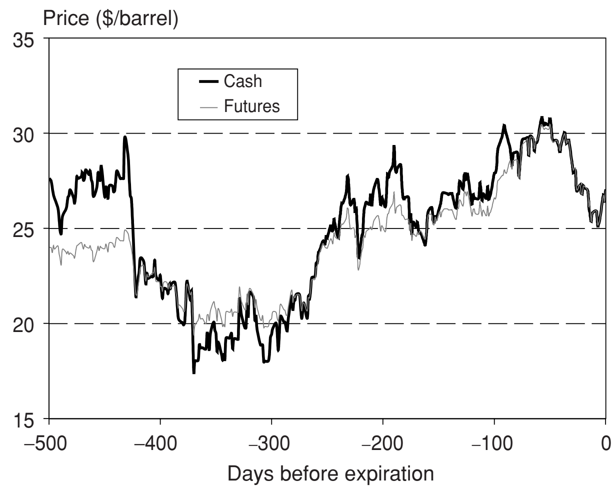

FIGURE 11.3 Futures and Spot Prices for Crude Oil

and supply. For instance, sugar is in backwardation. Also, gasoline futures prices tend to increase in the summer due to increased automobile driving. Heating oil displays the opposite pattern, where prices increase during the winter due to the demand for heating. Agricultural products can also be highly seasonal. In contrast, futures prices for gold are going up monotonically with time, since this is a perfectly storable good.

Finally, Figure 11.3 compares the spot and futures prices for crude oil. There is substantial variation in the basis between the spot and futures prices for crude oil. The market switches from backwardation ($S > F$) to contango ($S < F$).

KEY CONCEPT

Markets are in contango if spot prices are lower than forward prices. This occurs when the convenience yield is lower than the interest rate. Markets are in backwardation if spot prices are higher than forward prices. This occurs when there is high current demand for the commodity, which implies high convenience yields.

EXAMPLE 11.8: FRM EXAM 2008—QUESTION 2-30

If the lease rate of commodity A is less than the risk-free rate, what is the market structure of commodity A?

a. Backwardation
b. Contango
c. Flat
d. Inversion

11.6.2 Futures and Expected Spot Prices

An interesting issue is whether today's futures price gives the best forecast of the future spot price. If so, it satisfies the **expectations hypothesis**, which can be written as:

$$F_t = E_t[S_T] \tag{11.10}$$

The reason this relationship may hold is as follows. Say that the one-year oil futures price is $F = \$20.47$. If the market forecasts that oil prices in one year will be at \$25, one could make a profit by going long a futures contract at the cheap futures price of $F = \$20.47$, waiting a year, then buying oil at \$20.47, and reselling it at the higher price of \$25. In other words, deviations from this relationship imply **speculative profits**.

To be sure, these profits are not risk-free. Hence, they may represent some compensation for risk. For instance, if the market is dominated by producers that want to hedge by selling oil futures, F will be abnormally low compared with expectations. Thus the relationship between futures prices and expected spot prices can be complex.

For financial assets for which the arbitrage between cash and futures is easy, the futures or forward rate is solely determined by the cash-and-carry relationship (i.e., the interest rate and income on the asset). For commodities, however, the arbitrage may not be so easy. As a result, the futures price may deviate from the cash-and-carry relationship through this convenience yield factor. Such prices may reflect expectations of future spot prices, as well as speculative and hedging pressures.

A market trades in **contango** when the futures price trades at a premium relative to the spot price. Normally, the size of the premium should be limited by arbitrage opportunities. If this became too large, traders could buy the commodity spot, put it in storage, and simultaneously sell it for future delivery at the higher forward price. In December 2008, however, the premium for one-year oil contracts reached an all-time high of \$13 per barrel. This was explained by the credit crunch, which prevented oil traders from securing loans to finance oil storage.

With backwardation, the futures price tends to increase as the contract nears maturity. In such a situation, a **roll-over strategy** should be profitable, provided that prices do not move too much. This involves buying a long-maturity contract, waiting, and then selling it at a higher price in exchange for buying a cheaper, longer-term contract.

This strategy is comparable to **riding the yield curve** when upward sloping. This involves buying long maturities and waiting to have yields fall due to the passage of time. If the shape of the yield curve does not change too much, this will generate a capital gain from bond price appreciation. Because of the negative price-yield relationship, a positively sloped yield curve is equivalent to backwardation in bond prices.

This was basically the strategy followed by Metallgesellschaft Refining & Marketing (MGRM), the U.S. subsidiary of **Metallgesellschaft**, which had made large sales of long-term oil to clients on the OTC market. These were hedged by rolling over long positions in West Texas Intermediate (WTI) crude oil futures. This made money as long as the market was in backwardation. When the market turned to contango, however, the long positions started to lose money as they got closer to maturity. In addition, the positions were so large that they moved markets against MGRM. These losses caused cash-flow or liquidity problems. MGRM ended up liquidating the positions, which led to a realized loss of $1.3 billion.

A similar problem afflicted **Amaranth**, a hedge fund that lost $6.6 billion as a result of bad bets against natural gas futures. In September 2006, the price of natural gas fell sharply. In addition, the spread between prices in winter and summer months collapsed. As the size of the positions was huge, this led to large losses that worsened when the fund attempted to liquidate the contracts.

EXAMPLE 11.9: FRM EXAM 2007—QUESTION 29

On January 1, a risk manager observes that the one-year continuously compounded interest rate is 5% and the storage cost of a commodity product A is USD 0.05 per quarter (payable at each quarter end). The manager further observes the following forward prices for product A: March, 5.35; June, 5.90; September, 5.30; December, 5.22. Given the following explanation of supply and demand for this product, how would you best describe its forward price curve from June to December?

a. Backwardation as the supply of product A is expected to decline after summer
b. Contango as the supply of product A is expected to decline after summer
c. Contango as there is excess demand for product A in early summer
d. Backwardation as there is excess demand for product A in early summer

EXAMPLE 11.10: FRM EXAM 2007—QUESTION 30

Continuing with the previous question, what is the annualized rate of return earned on a cash-and-carry trade entered into in March and closed out in June?

a. 9.8%
b. 8.9%
c. 39.1%
d. 35.7%

EXAMPLE 11.11: FRM EXAM 2008—QUESTION 4-16

In late 1993, MGRM reported losses of about $1.3 billion in connection with the implementation of a hedging strategy in the oil futures market. In 1992, the company had begun a new strategy to sell petroleum to independent retailers at fixed prices above the prevailing market price for periods of up to 10 years. At the same time, MGRM implemented a hedging strategy using a large number of short-term derivative contracts such as swaps and futures on crude oil. This led to a timing (maturity) mismatch between the short-term hedges and the long-term liability. Unfortunately, the company suffered significant losses with its hedging strategy when oil market conditions abruptly changed to:

a. Contango, which occurs when the futures price is above the spot price
b. Contango, which occurs when the futures price is below the spot price
c. Normal backwardation, which occurs when the futures price is above the spot price
d. Normal backwardation, which occurs when the futures price is below the spot price

11.6.3 Commodity Risk

Commodity risk arises from potential movements in the value of commodity prices. Table 11.7 describes the risks of a sample of commodity contracts.[3] These can be grouped into *precious metals* (gold, platinum, silver); *base metals* (aluminum, copper, nickel, zinc); and *energy products* (natural gas, heating oil, unleaded gasoline, crude oil—West Texas Intermediate). The table reports the annualized volatility for spot or short-term contracts as well as for longer-term (typically 12- to 15-month) futures.

Precious and base metals have an annual volatility ranging from 20% to 30%, higher than for stock markets. Energy products, in contrast, are much more volatile with numbers ranging from 20% to 70%. This is because energy products are less storable than metals and, as a result, are much more affected by variations in demand and supply.

As Table 11.7 shows, futures prices are less volatile for longer maturities. This decreasing term structure of volatility is more marked for energy products and less so for base metals.

In terms of correlations, Figure 11.3 has shown that movements in futures prices are much less tightly related to spot prices than for financial contracts. Thus, the futures contract represents a separate risk factor. In addition, correlations

[3] These data are provided by RiskMetrics as of December 2006. Volatilities are derived from an exponentially weighted moving average (EWMA) model with a forecast horizon of one month, and annualized.

TABLE 11.7 Commodity Volatility, 2006 (Percent per Annum)

Commodity	Spot	Futures
Gold	17	
Platinum	29	
Silver	33	
Aluminium	28	20
Copper	30	27
Nickel	41	45
Zinc	36	28
Natural gas	72	41
Heating oil	33	20
Unleaded gas	36	25
Crude oil	28	19

across maturities are lower for energy products than for metals. This explains why trading energy requires risk measurement systems with numerous risk factors, across maturities, grades, and locations.

EXAMPLE 11.12: FRM EXAM 2006—QUESTION 115

Assume the risk-free rate is 5% per annum, the cost of storing oil for a year is 1% per annum, the convenience yield for owning oil for a year is 2% per annum, and the current price of oil is USD 50 per barrel. All rates are continuously compounded. What is the forward price of oil in a year?

a. USD 49.01
b. USD 52.04
c. USD 47.56
d. USD 49.50

EXAMPLE 11.13: FRM EXAM 2006—QUESTION 138

Imagine a stack-and-roll hedge of monthly commodity deliveries that you continue for the next five years. Assume the hedge ratio is adjusted to take into account the mistiming of cash flows but is not adjusted for the basis risk of the hedge. In which of the following situations is your calendar basis risk likely to be greatest?

a. Stack-and-roll in the front month in oil futures
b. Stack-and-roll in the 12-month contract in natural gas futures
c. Stack-and-roll in the three-year contract in gold futures
d. All three situations will have the same basis risk.

11.7 IMPORTANT FORMULAS

Gordon growth model for valuation of stocks: $P = \frac{D}{r-g}$

Stock index futures: $F_t e^{-r\tau} = S_t e^{-y\tau}$

Pricing a currency swap as two bond positions: $V = S(\$/Y)P^*(Y) - P(\$)$

Pricing a currency swap as a sequence of forwards: $V = \sum_i n_i(F_i - K)\exp(-r_i\tau_i)$

Pricing of commodity futures with storage costs: $F_t e^{-r\tau} = S_t + \text{PV(C)}$, or $F_t e^{-r\tau} = S_t e^{c\tau}$

Expectations hypothesis: $F_t = E_t[S_T]$

Contango: $F_t > S_t,\ y < r$

Backwardation: $F_t < S_t,\ y > r$

11.8 ANSWERS TO CHAPTER EXAMPLES

Example 11.1: FRM Exam 2000—Question 12

a. This is the cash-and-carry relationship, solved for S. We have $Se^{-y\tau} = Fe^{-r\tau}$, or $S = 552.3 \times \exp(-7.5/200)/\exp(-4.2/200) = 543.26$. We verify that the forward price is greater than the spot price since the dividend yield is less than the risk-free rate.

Example 11.2: FRM Exam 2009—Question 3-1

b. The fair forward price is $F = Se^{-y\tau}/e^{-r\tau} = 750\exp(-0.02 \times 6/12)/\exp(-0.035 \times 6/12) = 750 \times 0.9905/0.9827 = 755.65$. The actual price is 757.00. Hence buying at the cheap price and selling at the forward price gives a profit of \$1.35.

Example 11.3: FRM Exam 2003—Question 2

c. For consistency, translate the spot rate into dollars, $S = 0.7310$. The CHF interest rate is lower than the USD rate, so the CHF must be selling at a forward premium. The fair forward price is $F = S\exp((r - r^*)\tau) = 0.7310\exp((0.0105 - 0.0035)\,0.25) = 0.7323$. Because this is less than the observed price of 0.7350, we sell at the expensive forward price and borrow USD, buy CHF spot, and invest in CHF. At maturity, we liquidate the CHF investment to satisfy the forward sale into dollars, repay the loan, and make a tidy profit.

Example 11.4: FRM Exam 2009—Question 3-19

a. Because the company has revenues fixed in EUR and some costs in USD, it would be hurt if the USD appreciated. So, the risk is that of a depreciation of the

EUR against the USD. This can be hedged by buying the USD forward, which will lock in the EUR payment even if the USD appreciates.

Example 11.5: FRM Exam 2008—Question 2-27

c. Because principals on currency swaps are in different currencies, they need to be exchanged. In contrast, the principal amounts for interest rate swaps are in the same currency and are not exchanged.

Example 11.6: FRM Exam 2006—Question 88

a. The net present values of the payoffs in two currencies are described in the following table. As a result, the value of the currency swap is given by the dollar value of a long position in the yen bond minus a position in the dollar bond, or $(1/115)1{,}000(102.85/100) - 10(102.13/100) = \$8.943 - \$10.213 = -\$1.270$.

	Yen			USD		
T	Rate	CF	NPV	Rate	CF	NPV
1	2.00%	4	3.92	4.50%	6	5.74
2	2.50%	104	98.93	4.75%	106	96.39
Sum			102.85			102.13

Example 11.7: FRM Exam 2007—Question 87

c. A cross-currency swap is inappropriate because there is no stream of payment but just one. Also, one would want to pay GBP, not receive it. An Asian option is generally cheap, but this should be a put option, not a call. Among the two remaining choices, the chooser option is more expensive because it involves a call and a put.

Example 11.8: FRM Exam 2008—Question 2-30

b. If the lease rate is, for example, zero, the futures price must be greater than the spot price, which describes a contango.

Example 11.9: FRM Exam 2007—Question 29

d. From June to December, prices go down, which is backwardation. June prices are abnormally high because of excess demand, which pushes prices up.

Example 11.10: FRM Exam 2007—Question 30

d. The trade involves now going long a March contract and short a June contract. In practice, this means taking delivery of the commodity and holding it for three months until resale in June. The final payout is $5.90 - 0.05$ on a base of 5.35. This gives an annualized rate of return of $r = 4\ln(5.85/5.35) = 35.7\%$.

Example 11.11: FRM Exam 2008—Question 4-16

a. MGRM had purchased oil in short-term futures market as a hedge against the long-term sales. The long futures positions lost money due to the move into contango, which involves the spot price falling below longer-term prices.

Example 11.12: FRM Exam 2006—Question 115

b. Using $F_t e^{-r\tau} = S_t e^{-y\tau}$, we have $F = S\exp(-(y - c)\tau + r\tau) = 50\exp(-(0.02 - 0.01) + 0.05) = 52.04$.

Example 11.13: FRM Exam 2006—Question 138

a. For gold, forward rates closely follow spot rates, so there is little basis risk. For oil and natural gas, there is most movement at the short end of the term structure of futures prices. So using short maturities, or the front month, has the greatest basis risk.

PART Four

Valuation and Risk Models

CHAPTER 12

Introduction to Risk Models

This chapter provides an introduction to risk models. Modern risk management is **position-based**. This is more forward-looking than **return-based** information. Position-based risk measures are more informative because they can be used to manage the portfolio, which involves changing the positions.

Part Four of this book focuses primarily on market risk models. Ideally, risk should be measured at the top level of the portfolio or institution. This has led to a push toward risk measures that are comparable across different types of risk. One such summary measure is **value at risk** (VAR). VAR is a statistical measure of *total* portfolio risk, taken as the worst loss at a specified confidence level over the horizon. More generally, risk managers should evaluate the entire distribution of profits and losses. In addition, the analysis should be complemented by **stress-testing**, which identifies potential losses under extreme market conditions that may not show up in the recent history.

Section 12.1 gives a brief overview of financial market risks. Section 12.2 describes the broad components of a VAR system. Section 12.3 then shows how to compute VAR for a simple portfolio exposed to one risk factor only. It also discusses caveats, or pitfalls to be aware of when interpreting VAR numbers. Section 12.4 turns to the choice of VAR parameters, that is, the confidence level and horizon. Next, Section 12.5 shows how to implement stress tests. Finally, Section 12.6 describes how risk models can be classified into local valuation and full valuation methods.

12.1 INTRODUCTION TO FINANCIAL MARKET RISKS

12.1.1 Types of Financial Risks

Financial risks include market risk, credit risk, and operational risk. **Market risk** is the risk of losses due to movements in financial market prices or volatilities. This usually includes **liquidity risk**, which is the risk of losses due to the need to liquidate positions to meet funding requirements. Liquidity risk, unfortunately, is not amenable to formal quantification. Because of its importance, it will be covered in Chapter 26. **Credit risk** is the risk of losses due to the fact that counterparties

FRM Exam Part 1 topic.

may be unwilling or unable to fulfill their contractual obligations. Credit risk will be covered in Part Six. **Operational risk** is the risk of loss resulting from failed or inadequate internal processes, systems, and people, or from external events. This subject will be covered in Chapter 25. Oftentimes, however, these three categories interact with each other, so that any classification is, to some extent, arbitrary.

For example, credit risk can interact with other types of risks. At the most basic level, it involves the risk of default on the asset, such as a loan or bond. When the asset is traded, however, market risk also reflects credit risk. Take a corporate bond, for example. Some of the price movement may be due to movements in risk-free interest rates, which is pure market risk. The remainder will reflect the market's changing perception of the likelihood of default. Thus, for traded assets, there is no clear-cut delineation of market and credit risk. Some arbitrary classification must take place. Furthermore, operational risk is often involved as well.

Consider a simple transaction whereby a trader purchases 1 million worth of British pound (GBP) spot from Bank A. The current rate is $1.5/GBP, for settlement in two business days. So, our bank will have to deliver $1.5 million in two days in exchange for receiving GBP 1 million. This simple transaction involves a series of risks.

- *Market risk:* During the day, the spot rate could change. Say that after a few hours the rate moves to $1.4/GBP. The trader cuts the position and enters a spot sale with another bank, Bank B. The million pounds is now worth only $1.4 million. The loss of $100,000 is the change in the market value of the investment.
- *Credit risk:* The next day, Bank B goes bankrupt. The trader must now enter a new, replacement trade with Bank C. If the spot rate has dropped further from $1.4/GBP to $1.35/GBP, the gain of $50,000 on the spot sale with Bank B is now at risk. The loss is the change in the market value of the investment, if positive. Thus there is interaction between market and credit risk.
- *Settlement risk:* The next day, our bank wires the $1.5 million to Bank A in the morning, which defaults at noon and does not deliver the promised GBP 1 million. This is also known as **Herstatt risk** because this German bank defaulted on such obligations in 1974, potentially destabilizing the whole financial system. The loss is now potentially the whole principal in dollars.
- *Operational risk:* Suppose that our bank wired the $1.5 million to a wrong bank, Bank D. After two days, our back office gets the money back, which is then wired to Bank A plus compensatory interest. The loss is the interest on the amount due.

12.1.2 Risk Management Tools

In the past, risks were measured using a variety of ad hoc tools, which were not comparable across types of risk. These included **notional amounts** and **sensitivity measures**. While these measures provide a useful intuition of risk, they do not provide consistent estimates of the potential for downside loss across the portfolio. They fail to take into account differences in volatilities across markets, correlations across risk factors, as well as the probability of adverse moves in the risk factors.

Consider, for instance, a five-year **inverse floater**, which pays a coupon equal to 16% minus twice current LIBOR, if positive, on a notional principal of $100 million. The initial market value of the note is $100 million. This type of investment is extremely sensitive to movements in interest rates. If rates go up, the coupon payments will drop sharply. In addition, the discount rate also increases. As with all bonds, this investment can be priced by discounting the future cash flows into the present. This combination of lower cash flows and higher discount rate will push the bond price down sharply.

The question is, how much could an investor lose on this investment over a specified horizon? The *notional amount* is only indirectly informative. The worst-case scenario is one where interest rates rise above 8%. In this situation, the coupon will drop to $16 - 2 \times 8 =$ zero. The bond becomes a zero-coupon bond, whose value is $68 million, discounted at 8%. This gives a loss of $100 − $68 = $32 million. While sizable, this is still less than the notional.

A *sensitivity measure* such as duration is more helpful. In this case, the bond has three times the modified duration of a similar five-year note, which gives $D = 3 \times 4.5 = 13.5$ years. So, if interest rates go up by 1%, the bond would lose 13.5% of its value. This duration measure reveals the extreme sensitivity of the bond to interest rates but does not answer the question of whether such a disastrous movement in interest rates is likely. It also ignores the nonlinearity between the note price and yields.

Another general problem is that these sensitivity measures do not allow the investor to aggregate risk across different markets. Let us say that this investor also holds a position in a bond denominated in another currency, the euro. Do the risks add up, or diversify each other?

VAR provides a uniform answer to all these questions. One number aggregates the risks across the whole portfolio, taking into account leverage and diversification, and providing a risk measure with an associated probability.

If the worst increase in yield at the 95% level is 1.65% over the next year, we can compute VAR as

$$\text{VAR} = (\text{Market Value} \times \text{Modified Duration}) \times \text{Worst Yield Increase} \tag{12.1}$$

In this case, VAR = $100 × 13.5 × 0.0165 = $22 million. The investor can now make a statement such as: The worst loss at the 95% confidence level is approximately $22 million. The risk manager can now explain the risk of the investment in a simple, intuitive fashion.

Measures such as notional amounts and exposures have been, and are still, used to set limits, in an attempt to control risk before it occurs, or *ex ante*. These measures should be supplemented by VAR, which is an *ex ante* measure of the potential dollar loss. Other risk management tools include **stop losses**, which are rules enforcing position cuts after losses occur, that is, *ex post*. While stop losses are useful, especially in trending markets, they provide only partial protection because they are applied *after* a loss. In other words, this is too late, except for preventing further losses.

12.1.3 Sources of Loss

Our example can help isolate the sources of a market loss. The bond's value change can be described as

$$dP = -(D^* P) \times dy \tag{12.2}$$

where $D^* P$ is the *dollar duration* and dy the change in the yield.

This illustrates the general principle that losses can occur because of a combination of two components:

1. The exposure to the factor, or dollar duration. This is a choice variable that represents the positions.
2. The movement in the risk factor itself. This is external to the portfolio.

This is a general decomposition. It also applies to *systematic risk*, or exposure to the stock market. We can generally decompose the return on stock i, R_i into a component due to the market R_M and some residual risk

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \approx \beta_i \times R_M \tag{12.3}$$

We ignore the constant α_i because it does not contribute to risk, as well as the residual ϵ_i, which is diversified. Note that R_i is expressed here in terms of **rate of return** and, hence, has no dimension. To get a change in a dollar price, we write

$$dP_i = R_i P_i \approx (\beta P_i) \times R_M \tag{12.4}$$

The term between parentheses is the exposure, a choice variable.

This concept of linear exposure also applies to an option *delta*, defined as Δ.[1] The change in the value of a derivative f can be expressed in terms of the change in the price of the underlying asset S:

$$df = (\Delta) \times dS \tag{12.5}$$

Equations (12.2), (12.4), and (12.5) all reveal that the change in value is linked to an **exposure** coefficient and a change in a market variable:

Market Loss = Exposure × Adverse Movement in Financial Variable

To have a loss, we need to have some exposure *and* an unfavorable move in the risk factor. Thus we can manage the portfolio risk by changing its exposure.

[1] To avoid confusion, we use the conventional notation of Δ for the first partial derivative of the option. Changes are expressed in infinitesimal amounts df and dS.

For instance, moving a bond portfolio into cash creates a dollar duration of zero, in which case interest rate movements have no effect on the value of the portfolio. More generally, the relationship between the portfolio value and the risk factor need not be linear.

EXAMPLE 12.1: FRM EXAM 2005—QUESTION 32

Which of the following statements about trader limits are *correct*?

I. Stop loss limits are useful if markets are trending.
II. Exposure limits do not allow for diversification.
III. VAR limits are not susceptible to arbitrage.
IV. Stop loss limits are effective in preventing losses.

a. I and II
b. III and IV
c. I and III
d. II and IV

12.2 COMPONENTS OF RISK MEASUREMENT SYSTEMS

As described in Figure 12.1, a risk measurement system combines the following three steps:

1. Collect the **portfolio positions** and map them onto the risk factors.
2. From market data, construct the distribution of **risk factors** (e.g., normal, empirical, or other).
3. Construct the **distribution of portfolio returns** using one of the three methods (parametric, historical, Monte Carlo), and summarize the downside risk with VAR.

Consider for instance a position of $4 billion short the yen, long the dollar. This position corresponds to a well-known hedge fund that took a bet that the yen would fall in value against the dollar. The portfolio manager rightfully asks: How much could this position lose over a day?

12.2.1 Portfolio Positions

We start with portfolio positions. In this example, the current position is short the Japanese yen in the dollar amount of $4 billion.

The assumption is that all positions are constant over the horizon. This, of course, cannot be true in an environment where traders turn over their portfolios actively. Rather, it is a simplification.

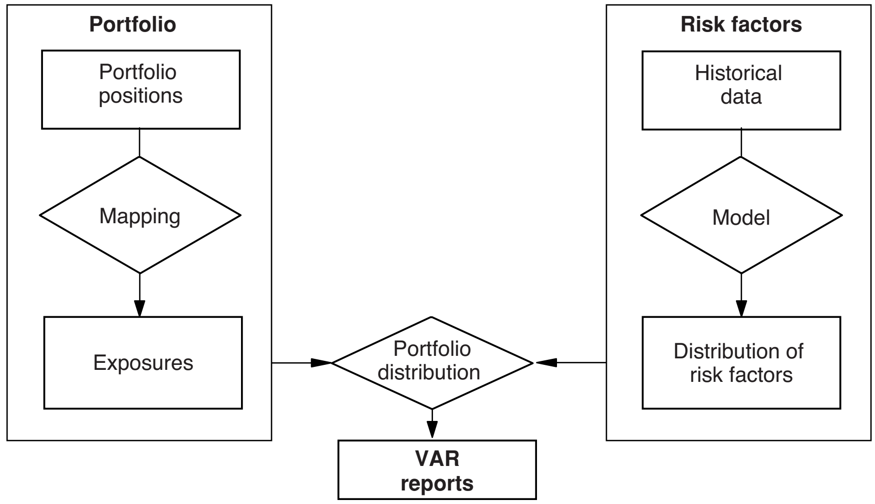

FIGURE 12.1 Components of a Risk System

The true risk can be greater or lower than the VAR measure. It can be greater if VAR is based on close-to-close positions that reflect lower trader limits and if traders take more risks during the day. Conversely, the true risk can be lower if management enforces loss limits, in other words, cuts down the risk that traders can take if losses develop.

12.2.2 Risk Factors

Next comes the choice of the risk factors. In this example of a single position, the main risk factor is obviously the change in the yen/dollar exchange rate. We start by collecting a relevant history of the exchange rate. This is an example where traditional risk models will give useful results because the historical data reveals a lot of movements in the risk factor, which are representative of future risks.

The **risk factors** represent a subset of all market variables that adequately span the risks of the current, or allowed, portfolio. For large portfolios, there are literally tens of thousands of securities available, but a much more restricted set of useful risk factors.

The key is to choose market factors that are adequate for the portfolio. For a simple fixed-income portfolio, one bond market risk factor may be enough. In contrast, for a highly leveraged portfolio, multiple risk factors are needed. For an option portfolio, volatilities should be added as risk factors. In general, the more complex the strategies, the greater the number of risk factors that should be used.

12.2.3 Portfolio Distribution

Finally, information about the portfolio positions and the movements in the risk factors should be combined to build the distribution of portfolio returns.

The next section illustrates different methods. The choice depends on the nature of the portfolio. A simple method may be sufficient for simple portfolios. For a fixed-income portfolio, a linear method may be adequate. In contrast, if the portfolio contains options, we need to include nonlinear effects. For simple, plain-vanilla options, we may be able to approximate their price behavior with a first and second derivative (delta and gamma). For more complex options, such as digital or barrier options, this may not be sufficient.

This is why risk management is as much an art as it is a science. Risk managers need to make reasonable approximations to come up with a cost-efficient measure of risk. They also need to be aware of the fact that traders could be induced to find holes in the risk management system.

Once this risk measurement system is in place, it can also be used to perform stress tests. The risk manager can easily submit the current portfolio to various scenarios, which are simply predefined movements in the risk factors. Therefore, stress tests are simple extensions of VAR systems.

EXAMPLE 12.2: POSITION-BASED RISK MEASURES

The standard VAR calculation for extension to multiple periods also assumes that positions are fixed. If risk management enforces loss limits, the true VAR will be

a. The same
b. Greater than calculated
c. Less than calculated
d. Unable to be determined

12.3 DOWNSIDE RISK MEASURES

12.3.1 VAR: Definition

VAR appeared as a risk measure in 1993, after its endorsement by the Group of Thirty (G-30).[2] The methodology behind VAR, however, is not new.

VAR is a summary measure of downside risk expressed in dollars, or in the reference currency. A general definition is:

> *VAR is the maximum loss over a target horizon such that there is a low, prespecified probability that the actual loss will be larger.*

12.3.2 VAR: Historical Simulation

Let us go back to our position of $4 billion short the yen. To measure its risk, we could use 10 years of historical daily data on the yen/dollar rate, say from 2000

[2] The G-30 is a private, nonprofit association, consisting of senior representatives of the private and public sector and of academia. In the wake of the derivatives disasters of the early 1990s, the G-30 issued a report that has become a milestone document for risk management. Group of Thirty, *Derivatives: Practices and Principles* (New York: Group of Thirty, 1993).

through 2009. We then simulate a daily return in dollars as

$$R_t(\$) = Q_0(\$)[S_t - S_{t-1}]/S_{t-1} \tag{12.6}$$

where Q_0 is the current dollar value of the position and S is the spot rate in yen per dollar measured over two consecutive days.

For instance, for two hypothetical days $S_1 = 112.0$ and $S_2 = 111.8$. The simulated return is

$$R_2(\$) = \$4{,}000 \text{ million} \times [111.8 - 112.0]/112.0 = -\$7.2 \text{ million}$$

Repeating this operation over the entire sample, or 2,527 trading days, creates a time series of fictitious returns, which is plotted in Figure 12.2. The method is called **historical simulation** because it simulates the current portfolio using the recent history.

We can now construct a frequency distribution of daily returns. This is based on ordered losses from worst to best. For instance, there are two losses below \$150 million, eight losses between \$150 million and \$100 million, and so on. The histogram, or frequency distribution, is graphed in Figure 12.2.

We now wish to summarize the distribution by one number. We could describe the quantile, that is, the level of loss that will not be exceeded at some high **confidence level**. Select, for instance, this confidence level as $c = 95\%$. This corresponds to a **right-tail probability**. We could as well define VAR in terms of a **left-tail probability**, which we write as $p = 1 - c$.

Define x as the dollar profit or loss. VAR is typically reported as a positive number, even if it is a loss. It is defined implicitly by

$$c = \int_{-\text{VAR}}^{\infty} f(x)dx \tag{12.7}$$

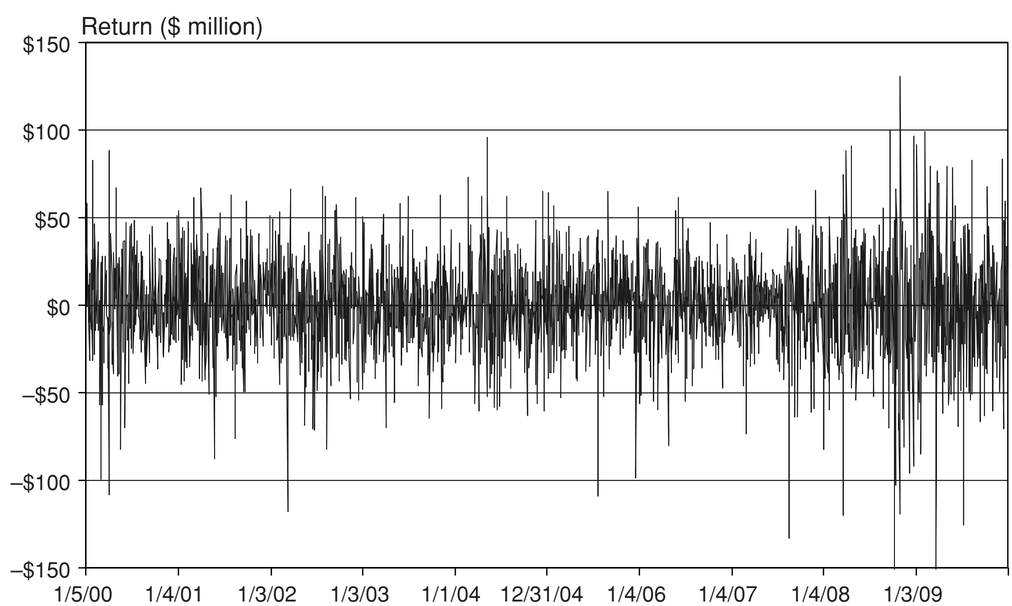

FIGURE 12.2 Simulated Daily Returns

When the outcomes are discrete, VAR is the smallest loss such that the right-tail probability is at least c.

Sometimes, VAR is reported as the deviation between the mean and the quantile. This second definition is more consistent than the usual one. Because it considers the deviation between two values on the target date, it takes into account the time value of money. In most applications, however, the time horizon is very short, in which case the average return on financial series is close to zero. As a result, the two definitions usually give similar values.

In this hedge fund example, we want to find the cutoff value $R^* > 0$ such that the probability of a loss worse than $-R^*$ is $p = 1 - c = 5\%$. With a total of $T = 2{,}527$ observations, this corresponds to a total of $pT = 0.05 \times 2{,}527 = 126$ observations in the left tail. We pick from the ordered distribution the cutoff value, which is $R^* = \$42$ million. We can now make a statement such as: The maximum loss over one day is about \$42 million at the 95% confidence level. This describes risk in a way that notional amounts or exposures cannot convey.

12.3.3 VAR: Parametric

Another approach to VAR measurement is to assume that the distribution of returns belongs to a particular density function, such as the normal distribution. Other distributions are possible, however. The dispersion parameter is measured by the usual standard deviation (SD), defined as

$$\mathrm{SD}(X) = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} [x_i - E(X)]^2} \tag{12.8}$$

The advantage of this measure is that it takes into account all observations, not just the few around the quantile. Any large negative value, for example, will affect the computation of the variance, increasing $\mathrm{SD}(X)$. If we are willing to take a stand on the shape of the distribution, say normal or Student's t, we do know that the standard deviation is the most efficient measure of dispersion. For example, for our yen position, this value is SD = \$26.8 million.

We can translate this standard deviation into a VAR measure, using a multiplier $\alpha(c)$ that depends on the distribution and the selected confidence level c:

$$\mathrm{VAR} = \alpha\sigma W \tag{12.9}$$

where σ is the volatility of the rate of return, which is unitless, and W is the amount invested, measured in the reference currency. Here $\mathrm{SD} = \sigma W$, which is in dollars.

With a normal distribution and $c = 95\%$, we have $\alpha = 1.645$. This gives a VAR estimate of $1.645 \times 26.8 = \$44$ million, which is not far from the empirical quantile of \$42 million.

Note that Equation (12.9) measures VAR relative to the mean, because the standard deviation is a measure of dispersion around the mean. If it is important to measure the loss relative to the initial value, VAR is then

$$VAR = (\alpha\sigma - \mu)W \tag{12.10}$$

where μ is the expected rate of return over the horizon. In this case, the mean is very small, at $-\$0.1$ million, which hardly affects VAR.

The disadvantage of the standard deviation is that it is symmetrical and cannot distinguish between large losses or gains. Also, computing VAR from SD requires a distributional assumption, which may not be valid.

Using the standard deviation to compute VAR is an example of the **parametric approach** (because it relies on a distribution with parameters). In the previous section, VAR was computed from the empirical distribution, which is an example of a **nonparametric approach**.

12.3.4 VAR: Monte Carlo

Finally, a third approach to risk measurement is to simulate returns using Monte Carlo simulations. This involves assuming a particular density for the distribution of risk factors and then drawing random samples from these distributions to generate returns on the portfolio.

EXAMPLE 12.3: FRM EXAM 2005—QUESTION 43

The 10-Q report of ABC Bank states that the monthly VAR of ABC Bank is USD 10 million at the 95% confidence level. What is the proper interpretation of this statement?

a. If we collect 100 monthly gain/loss data of ABC Bank, we will always see five months with losses larger than \$10 million.
b. There is a 95% probability that the bank will lose less than \$10 million over a month.
c. There is a 5% probability that the bank will gain less than \$10 million each month.
d. There is a 5% probability that the bank will lose less than \$10 million over a month.

12.3.5 VAR: Caveats

VAR is a useful summary measure of risk but is subject to caveats:

- *VAR does not describe the worst possible loss.* This is not what VAR is designed to measure. Indeed, we would expect the VAR number to be exceeded

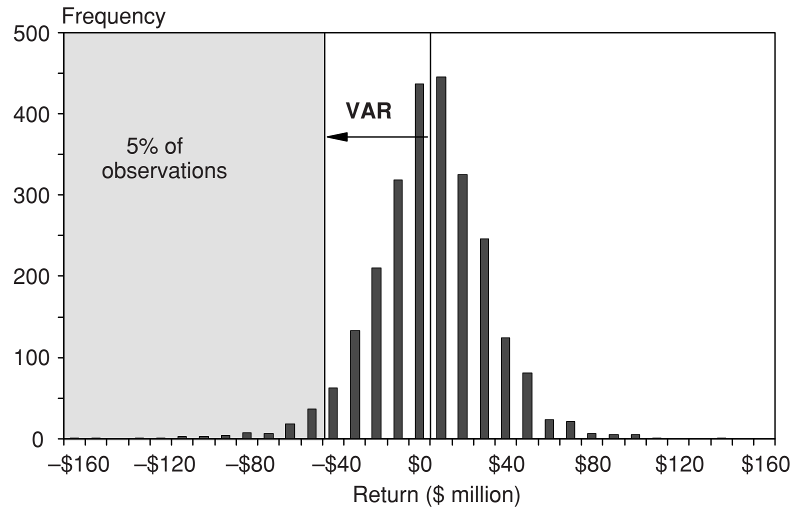

FIGURE 12.3 Distribution of Daily Returns

with a frequency of p, that is five days out of a hundred for a 95% confidence level. This is perfectly normal. In fact, backtesting procedures are designed to check whether the frequency of exceedences is in line with p. Backtesting will be covered in Chapter 16.

- *VAR does not describe the losses in the left tail.* VAR does not say anything about the distribution of losses in its left tail. It just indicates the probability of such a value occurring. For the same VAR number, however, we can have very different distribution shapes. In the case of Figure 12.3, the average value of the losses worse than $42 million is around $63 million, which is 50% worse than the VAR. So, it would be unusual to sustain many losses beyond $200 million.

  Other distributions are possible, however, while maintaining the same VAR. Figure 12.4 illustrates a distribution with 125 occurrences of large losses of $160 million. Because there is still one observation left just below $42 million, VAR is unchanged at $42 million. Yet this distribution implies a high probability of sustaining very large losses, unlike the original one.

  This can create other strange results. For instance, one can construct examples, albeit stretched, where the VAR of a portfolio is greater than the sum of the VARs for its components. In this case, the risk measure is said to fail the *subadditivity* property. The standard deviation, however, is subadditive: The SD of a portfolio must be smaller than, or at worst equal to, the sum of the SDs of subportfolios. As a result, VAR computed from the standard deviation is subadditive as well.
- *VAR is measured with some error.* The VAR number itself is subject to normal sampling variation. In our example, we used 10 years of daily data. Another sample period, or a period of different length, will lead to a different VAR number. Different statistical methodologies or simplifications can also lead to different VAR numbers. One can experiment with sample periods and methodologies to get a sense of the precision in VAR. Hence, it is

useful to remember that there is limited precision in VAR numbers. What matters is the first-order magnitude. It would not make sense to report VAR as $41.989 million, for example. Only the first two digits are meaningful in this case.

An advantage of the parametric approach is that VAR is more precisely estimated than with historical simulation. This is because the standard deviation estimator uses all the observations in the sample, in contrast with the sample quantile, which uses just one or two observations, in addition to the count in the left tail.

In addition, VAR measures are subject to the same problems that affect all risk measures based on a window of recent historical data. Ideally, the past window should reflect the range of future outcomes. If not, all risk measures based on recent historical data may be misleading.

12.3.6 Alternative Measures of Risk

The conventional VAR measure is the *quantile* of the distribution measured in dollars. This single number is a convenient summary, but its very simplicity can be dangerous. We see in Figure 12.4 that the same VAR can hide very different distribution patterns. Chapter 15 reviews desirable properties for risk measures and shows that VAR can display undesirable properties under some conditions. In particular, the VAR of a portfolio can be greater than the sum of subportfolio VARs. If so, merging portfolios can increase risk, which is an unexpected result. Alternative measures of risk are described next.

The Conditional VAR A related concept is the expected value of the loss when it exceeds VAR. This measures the average of the loss conditional on the fact that

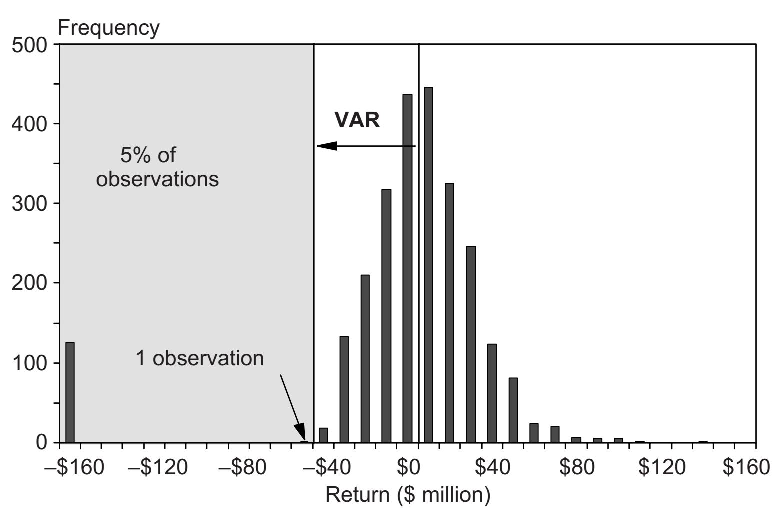

FIGURE 12.4 Altered Distribution with Same VAR

it is greater than VAR. Define the VAR number as $-q$. Formally, the **conditional VAR** (CVAR) is the negative of

$$E[X \mid X < q] = \int_{-\infty}^{q} x f(x) dx \Big/ \int_{-\infty}^{q} f(x) dx \qquad (12.11)$$

Note that the denominator represents the probability of a loss exceeding VAR, which is also $p = 1 - c$. This ratio is also called **expected shortfall, tail conditional expectation**, **conditional loss**, or **expected tail loss**. CVAR indicates the potential loss if the portfolio is "hit" beyond VAR. Because CVAR is an average of the tail loss, one can show that it qualifies as a *subadditive* risk measure. For our yen position, the average loss beyond the \$42 million VAR is CVAR = \$63 million.

The Semistandard Deviation This is a simple extension of the usual standard deviation that considers only data points that represent a loss. Define N_L as the number of such points. The measure is

$$\mathrm{SD}_L(X) = \sqrt{\frac{1}{(N_L)} \sum_{i=1}^{N} [\mathrm{Min}(x_i, 0)]^2} \qquad (12.12)$$

The advantage of this measure is that it accounts for asymmetries in the distribution (e.g., negative skewness, which is especially dangerous). The semistandard deviation is sometimes used to report downside risk, but is much less intuitive and less popular than VAR.

The Drawdown **Drawdown** is the decline from peak over a fixed time interval. Define x^{MAX} as the local maximum over this period $[0, T]$, which occurs at time $t_{MAX} \in [0, T]$. Relative to this value, the drawdown at time t is

$$\mathrm{DD}(X) = \frac{(x^{MAX} - x_t)}{x^{MAX}} \qquad (12.13)$$

The maximum drawdown is the largest such value over the period, or decline from peak to trough (local maximum to local minimum).

This measure is useful if returns are not independent from period to period. When a market trends, for example, the cumulative loss over a longer period is greater than the loss extrapolated from a shorter period. Alternatively, drawdowns are useful measures of risk if the portfolio is actively managed. A portfolio insurance program, for example, should have lower drawdowns relative to a fixed position in the risky asset because it cuts the position as losses accumulate.

The disadvantage of this measure is that it is backward-looking. It cannot be constructed from the current position, as in the case of VAR. In addition, the maximum drawdown corresponds to different time intervals (i.e., $t_{MAX} - t_{MIN}$). As a result, maximum drawdown measures are not directly comparable across portfolios, in contrast with VAR or the standard deviation, which are defined over a fixed horizon or in annual terms.

EXAMPLE 12.4: FRM EXAM 2003—QUESTION 5

Given the following 30 ordered percentage returns of an asset, calculate the VAR and expected shortfall at a 90% confidence level: −16, −14, −10, −7, −7, −5, −4, −4, −4, −3, −1, −1, 0, 0, 0, 1, 2, 2, 4, 6, 7, 8, 9, 11, 12, 12, 14, 18, 21, 23.

a. VAR (90%) = 10, expected shortfall = 14
b. VAR (90%) = 10, expected shortfall = 15
c. VAR (90%) = 14, expected shortfall = 15
d. VAR (90%) = 18, expected shortfall = 22

EXAMPLE 12.5: FRM EXAM 2009—QUESTION 4-4

Worse-than-VAR scenarios are defined as scenarios that lead to losses in the extreme left tail of the return distribution equal to or exceeding VAR at a given level of confidence. Which of the following statements is an accurate description of VAR?

a. VAR is the average of the worse-than-VAR scenario returns.
b. VAR is the standard deviation of the worse-than-VAR scenario returns.
c. VAR is the most pessimistic scenario return (maximum loss) from the worse-than-VAR scenarios.
d. VAR is the most optimistic scenario return (minimum loss) from the worse-than-VAR scenarios.

12.4 VAR PARAMETERS

To measure VAR, we first need to define two quantitative parameters: the confidence level and the horizon.

12.4.1 Confidence Level

The higher the confidence level c, the greater the VAR measure. Varying the confidence level provides useful information about the return distribution and potential extreme losses. It is not clear, however, whether one should stop at 99%, 99.9%, 99.99%, or higher. Each of these values will create an increasingly larger loss, but a loss that is increasingly less likely.

Another problem is that as c increases, the number of occurrences below VAR shrinks, leading to poor measures of high quantiles. With 1,000 observations, for example, VAR can be taken as the 10th lowest observation for a 99% confidence level. If the confidence level increases to 99.9%, VAR is taken from the lowest observation only. Finally, there is no simple way to estimate a 99.99% VAR from this sample because it has too few observations.

The choice of the confidence level depends on the use of VAR. For most applications, VAR is simply a benchmark measure of downside risk. If so, what really matters is *consistency* of the VAR confidence level across trading desks or time.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a high confidence level is advisable. Obviously, institutions would prefer to go bankrupt very infrequently. This **capital adequacy** use, however, applies to the overall institution and not to trading desks.

Another important point is that VAR models are useful only insofar as they can be verified. This is the purpose of backtesting, which systematically checks whether the frequency of losses exceeding VAR is in line with $p = 1 - c$. For this purpose, the risk manager should choose a value of c that is not too high. Picking, for instance, $c = 99.99\%$ should lead, on average, to one exceedence out of 10,000 trading days, or 40 years. In other words, it is going to be impossible to verify if the true probability associated with VAR is indeed 99.99%. For all these reasons, the usual recommendation is to pick a confidence level that is not too high, such as 95% to 99%.

12.4.2 Horizon

The longer the horizon T, the greater the VAR measure. This extrapolation is driven by two factors: the behavior of the risk factors and the portfolio positions.

To extrapolate from a one-day horizon to a longer horizon, we need to assume that returns are independent and identically distributed (i.i.d.). If so, the daily volatility can be transformed into a multiple-day volatility by multiplication by the square root of time. We also need to assume that the distribution of daily returns is unchanged for longer horizons, which restricts the class of distribution to the so-called stable family, of which the normal is a member. If so, we have

$$\text{VAR}(T \text{ days}) = \text{VAR}(1 \text{ day}) \times \sqrt{T} \tag{12.14}$$

This requires (1) the distribution to be invariant to the horizon (i.e., the same α as for the normal), (2) the distribution to be the same for various horizons (i.e., no time decay in variances), and (3) innovations to be independent across days.

KEY CONCEPT

VAR can be extended from a one-day horizon to T days by multiplication by the square root of time. This adjustment is valid with independent and identically distributed (i.i.d.) returns that have a normal distribution.

The choice of the horizon also depends on the characteristics of the portfolio. If the positions change quickly, or if exposures (e.g., option deltas) change as underlying prices change, increasing the horizon will create slippage in the VAR measure.

Again, the choice of the horizon depends on the use of VAR. If the purpose is to provide an accurate benchmark measure of downside risk, the horizon should be relatively short, ideally less than the average period for major portfolio rebalancing.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a long horizon is advisable. For **capital adequacy** purposes, institutions will want to have enough time for corrective action as problems start to develop. The VAR horizon should also be long enough to allow for orderly liquidation of the positions. In other words, less liquid assets should be evaluated with a longer horizon.

In practice, the horizon cannot be less than the frequency of reporting of profits and losses (P&L). Typically, banks measure P&L on a daily basis, and corporates on a longer interval (ranging from daily to monthly). This interval is the minimum horizon for VAR.

Another criterion relates to the need for backtesting. Shorter time intervals create more data points that can be used to match VAR with the subsequent P&L. For statistical tests, having more data points means that the tests will be more powerful, or more likely to identify problems in the VAR model. So, for the purpose of backtesting, it is advisable to have a horizon as short as possible.

For all these reasons, the usual recommendation is to pick a horizon that is as short as feasible for trading desks, for instance one day. For institutions such as pension funds, for instance, a one-month horizon may be more appropriate.

In summary, the choice of the confidence level and horizon depend on the intended use for the risk measures. For backtesting purposes, we should select a low confidence level and a short horizon. For capital adequacy purposes, a high confidence level and a long horizon are required. In practice, these conflicting objectives can be accommodated by a more complex rule, as is the case for the Basel market risk charge.

12.4.3 Application: The Basel Rules

An important use of risk models is for capital adequacy purposes. The Basel Committee on Banking Supervision has laid out minimum capital requirements for commercial banks to cover the market risk of their trading portfolios. The rules define a **Market Risk Charge** (MRC) that is based on the bank's internal VAR measures. The original rules, as laid out in 1996, require the following parameters:

- A horizon of 10 trading days, or two calendar weeks
- A 99% confidence interval

- An observation period based on at least a year of historical data and updated at least once a quarter

Under the **Internal Models Approach** (IMA) as defined in 1996, the MRC includes a **general market risk charge** (GMRC) plus other components:

$$\mathrm{GMRC}_t = \mathrm{Max}\left(k\frac{1}{60}\sum_{i=1}^{60}\mathrm{VAR}_{t-i},\ \mathrm{VAR}_{t-1}\right) \tag{12.15}$$

The GMRC involves the average of the trading VAR over the last 60 days, times a supervisor-determined multiplier k (with a minimum value of 3), as well as yesterday's VAR. The Basel Committee allows the 10-day VAR to be obtained from an extrapolation of one-day VAR figures. Thus VAR is really

$$\mathrm{VAR}_t(10, 99\%) = \sqrt{10} \times \mathrm{VAR}_t(1, 99\%)$$

Presumably, the 10-day period corresponds to the time required for corrective action by bank regulators, should an institution start to run into trouble. Presumably as well, the 99% confidence level corresponds to a low probability of bank failure due to market risk. Even so, one occurrence every 100 periods implies a high frequency of failure. There are $52/2 = 26$ two-week periods in one year. Thus, one failure should be expected to happen every $100/26 = 3.8$ years, which is still much too frequent. This explains why the Basel Committee has applied a multiplier factor, $k \geq 3$, to guarantee further safety. In addition, this factor is supposed to protect against fat tails, unstable parameters, changing positions, and, more generally, model risk.

In 2009, the rules were revised to require updating at least every month. In addition, the GMRC was expanded to include a stressed VAR measure, which is explained in Chapter 28.

EXAMPLE 12.6: FRM EXAM 2008—QUESTION 2-2

Assume that the P&L distribution of a liquid asset is i.i.d. normally distributed. The position has a one-day VAR at the 95% confidence level of \$100,000. Estimate the 10-day VAR of the same position at the 99% confidence level.

a. \$1,000,000
b. \$450,000
c. \$320,000
d. \$220,000

EXAMPLE 12.7: FRM EXAM 2009—QUESTION 4-3

Assume that portfolio daily returns are independent and identically normally distributed. Sam Neil, a new quantitative analyst, has been asked by the portfolio manager to calculate portfolio VARs over 10, 15, 20, and 25 days. The portfolio manager notices something amiss with Sam's calculations, displayed here. Which one of the following VARs on this portfolio is inconsistent with the others?

a. VAR(10-day) = USD 316M
b. VAR(15-day) = USD 465M
c. VAR(20-day) = USD 537M
d. VAR(25-day) = USD 600M

EXAMPLE 12.8: MARKET RISK CHARGE

The 95%, one-day RiskMetrics VAR for a bank trading portfolio is \$1,000,000. What is the approximate general market risk charge, as defined in 1996?

a. \$3,000,000
b. \$9,500,000
c. \$4,200,000
d. \$13,400,000

12.5 STRESS-TESTING

12.5.1 Limitations of VAR Measures

We have seen in a previous section that VAR measures have inherent limitations. In addition, the traditional application of historical simulation creates special problems due to the choice of the **moving window**. This typically uses one to three years of historical data.

During the credit crisis that started in 2007, risk management systems failed at many banks. Some banks suffered losses that were much more frequent and much worse than they had anticipated. In 2007 alone, for example, UBS suffered 29 exceptions, or losses worse than VAR, instead of the expected number of two or three (i.e., 1% of 250 days).

This was in part due to the fact that 2007 followed an extended period of stability. Figure 12.5, for example, plots the daily volatility forecast for the S&P 500 stock index using an exponentially weighted moving average (EWMA) with

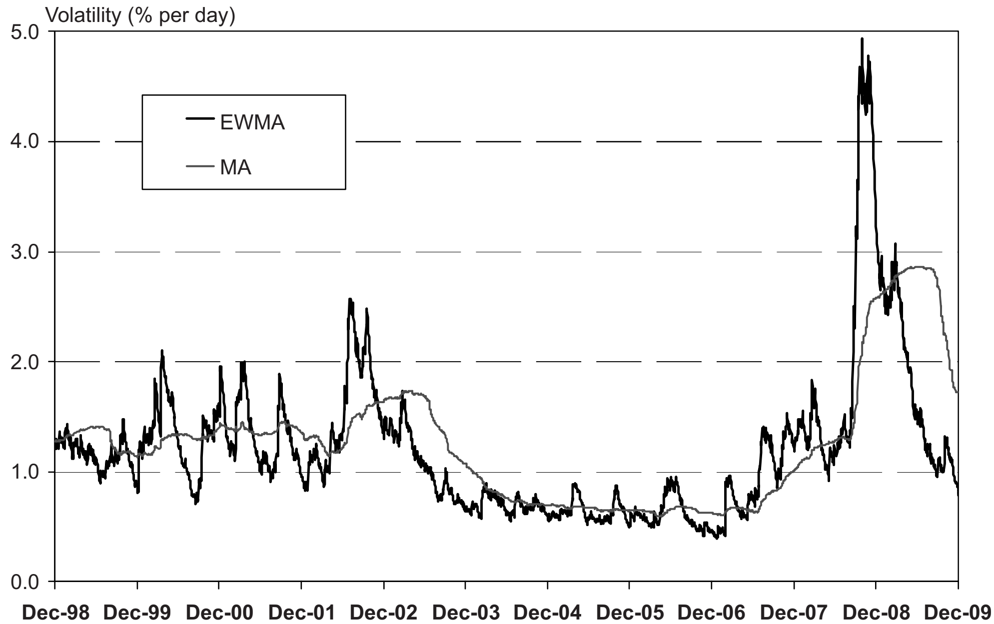

FIGURE 12.5 Volatility of S&P 500 Stock Index

decay of 0.94. This model shows that during 2004 to 2006, the volatility was very low, averaging 0.7% daily. As a result, many financial institutions entered 2007 with high levels of leverage.

Banks, however, do not model their risk using this EWMA forecast. Typically, they use a moving window with equal weight on each day, which is essentially a moving average (MA) model. Therefore, the graph also shows a volatility forecast coming from an MA model. The figure shows that the MA model systematically underestimated the EWMA volatility starting in mid-2007, which explains the high number of exceptions. The lesson from this episode is that relying on recent data may not be sufficient to assess risks. This is why traditional VAR models must be complemented by stress tests.

12.5.2 Principles of Stress Tests

VAR should be complemented by **stress-testing**, which aims at identifying situations that could create extraordinary losses but plausible losses. One drawback of stress tests is that they are more subjective than VAR measures. A VAR number reflects realized risk. An extreme scenario, by contrast, may be more difficult to accept by senior management if it does not reflect an actual observation and if it looks too excessive.

In the case of our hedge fund with a \$4 billion position short the yen, we have seen that the daily VAR at the 95 percent level of confidence is on the order of \$42 million. In addition, it would be informative to see how worse the loss could be. Going back over the last twenty years, for example, the worst movement in the exchange rate was a loss of −5.4% on October 7, 1998. This leads to a stress loss of \$215 million. Such a loss is plausible.

Stress-testing is a key risk management process, which includes (1) scenario analysis; (2) stressing models, volatilities, and correlations; and (3) developing policy responses. **Scenario analysis** submits the portfolio to large movements in financial market variables. These scenarios can be created using a number of methods.

- *Moving key variables one at a time*, which is a simple and intuitive method. Unfortunately, it is difficult to assess realistic comovements in financial variables. It is unlikely that all variables will move in the worst possible direction at the same time.
- *Using historical scenarios*, for instance the 1987 stock market crash, the devaluation of the British pound in 1992, the bond market debacle of 1984, the Lehman bankruptcy, and so on.
- *Creating prospective scenarios*, for instance working through the effects, direct and indirect, of a U.S. stock market crash. Ideally, the scenario should be tailored to the portfolio at hand, assessing the worst thing that could happen to current positions.
- *Reverse stress tests* start from assuming a large loss and then explore the conditions that would lead to this loss. This type of analysis forces institutions to think of other scenarios and to address issues not normally covered in regular stress tests, such as financial contagion.

Stress-testing is useful to guard against **event risk**, which is the risk of loss due to an observable political or economic event. The problem (from the viewpoint of stress-testing) is that such events are relatively rare and may be difficult to anticipate. These include:

- *Changes in governments* leading to changes in economic policies
- *Changes in economic policies*, such as default, capital controls, inconvertibility, changes in tax laws, expropriations, and so on
- *Coups, civil wars, invasions*, or other signs of political instability
- *Currency devaluations*, which are usually accompanied by other drastic changes in market variables

Even so, designing stress tests is not an easy matter. Recent years have demonstrated that markets seem to be systematically taken by surprise. Few people seem to have anticipated the Russian default, for instance. The Argentinian default of 2001 was also unique in many respects.

Example: Turmoil in Argentina

Argentina is a good example of political risk in emerging markets. Up to 2001, the Argentine peso was fixed to the U.S. dollar at a one-to-one exchange rate. The government had promised it would defend the currency at all costs. Argentina, however, suffered from the worst economic crisis in decades, compounded by the cost of excessive borrowing.

In December 2001, Argentina announced it would stop paying interest on its $135 billion foreign debt. This was the largest sovereign default recorded so far. Economy Minister Cavallo also announced sweeping restrictions on withdrawals from bank deposits to avoid capital flight. On December 20, President Fernando de la Rua resigned after 25 people died in street protests and rioting. President Duhalde took office on January 2 and devalued the currency on January 6. The exchange rate promptly moved from 1 peso/dollar to more than 3 pesos.

Such moves could have been factored into risk management systems by scenario analysis. What was totally unexpected, however, was the government's announcement that it would treat bank loans and deposits differentially. Dollar-denominated bank deposits were converted into devalued pesos, but dollar-denominated bank loans were converted into pesos at a one-to-one rate. This mismatch rendered much of the banking system technically insolvent, because loans (bank assets) overnight became less valuable than deposits (bank liabilities). Whereas risk managers had contemplated the market risk effect of a devaluation, few had considered this possibility of such political actions.

By 2005, the Argentinian government proposed to pay back about 30% of the face value of its debt. This recovery rate was very low by historical standards.

The goal of stress-testing is to identify areas of potential vulnerability. This is not to say that the institution should be totally protected against every possible contingency, as this would make it impossible to take any risk. Rather, the objective of stress-testing and management response should be to ensure that the institution can withstand likely scenarios without going bankrupt. Stress-testing can be easily implemented once the VAR structure is in place. In Figure 12.1, all that is needed is to enter the scenario values into the risk factor inputs.

EXAMPLE 12.9: FRM EXAM 2008—QUESTION 2-29

Which of the following statements about stress testing are *true*?

I. Stress testing can complement VAR estimation in helping risk managers identify crucial vulnerabilities in a portfolio.
II. Stress testing allows users to include scenarios that did not occur in the lookback horizon of the VAR data but are nonetheless possible.
III. A drawback of stress testing is that it is highly subjective.
IV. The inclusion of a large number of scenarios helps management better understand the risk exposure of a portfolio.

a. I and II only.
b. III and IV only.
c. I, II, and III only.
d. I, II, III, and IV.

EXAMPLE 12.10: FRM EXAM 2006—QUESTION 87

Which of the following is true about stress testing?

a. It is used to evaluate the potential impact on portfolio values of unlikely, although plausible, events or movements in a set of financial variables.
b. It is a risk management tool that directly compares predicted results to observed actual results. Predicted values are also compared with historical data.
c. Both a. and b. are true.
d. None of the above are true.

EXAMPLE 12.11: FRM EXAM 2008—QUESTION 2-18

John Flag, the manager of a \$150 million distressed bond portfolio, conducts stress tests on the portfolio. The portfolio's annualized return is 12%, with an annualized return volatility of 25%. In the past two years, the portfolio encountered several days when the daily value change of the portfolio was more than 3 standard deviations. If the portfolio would suffer a 4-sigma daily event, estimate the change in the value of this portfolio.

a. \$9.48 million
b. \$23.70 million
c. \$37.50 million
d. \$150 million

12.6 VAR: LOCAL VERSUS FULL VALUATION

This section turns to a general classification of risk models into local valuation and full valuation methods. **Local valuation methods** make use of the valuation of the instruments at the current point, along with the first and perhaps the second partial derivatives. **Full valuation methods**, in contrast, reprice the instruments over a broad range of values for the risk factors.

The various approaches to VAR are described in Figure 12.6. The left branch describes local valuation methods, also known as **analytical methods**. These include linear models and nonlinear models. Linear models are based on the covariance matrix approach. The covariance matrix can be simplified using factor models, or even a diagonal model, which is a one-factor model.

Nonlinear models take into account the first and second partial derivatives. The latter are called gamma or convexity. Next, the right branch describes full valuation methods and includes historical or Monte Carlo simulations.

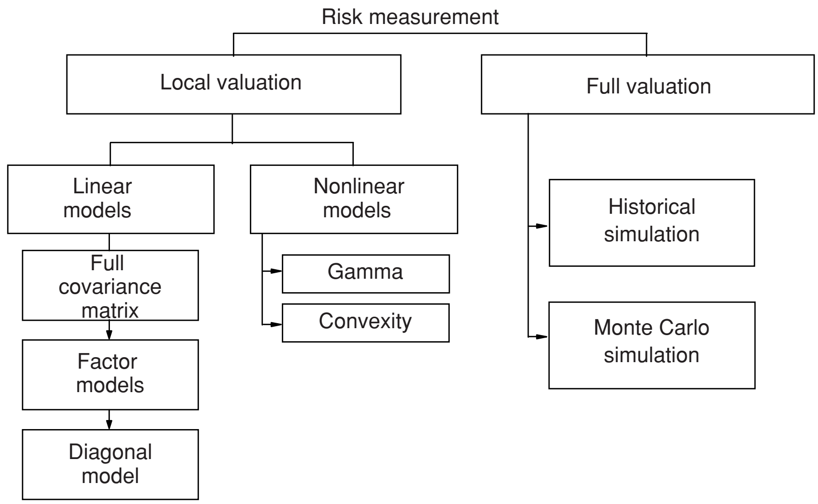

FIGURE 12.6 VAR Methods

12.6.1 Local Valuation

VAR was born from the recognition that we need an estimate that accounts for various sources of risk and expresses loss in terms of probability. Extending the duration equation to the worst change in yield at some confidence level dy, we have

$$(\text{Worst } dP) = (-D^* P) \times (\text{Worst } dy) \tag{12.16}$$

where D^* is modified duration. For a long position in the bond, the worst movement in yield is an increase at, say, the 95% confidence level. This will lead to a fall in the bond value at the same confidence level. We call this approach **local valuation**, because it uses information about the initial price and the exposure at the initial point. As a result, the VAR for the bond is given by

$$\text{VAR}(dP) = \mid -D^* P \mid \times \text{VAR}(dy) \tag{12.17}$$

More generally, the **delta-normal** method replaces all positions by their delta exposures and assumes that risk factors have multivariate normal distributions. In this case, VAR is

$$\text{VAR}(df) = \mid \Delta \mid \text{VAR}(dS) \tag{12.18}$$

The main advantage of this approach is its simplicity: The distribution of the price is the same as that of the change in yield. This is particularly convenient for portfolios with numerous sources of risks, because linear combinations of normal distributions are normally distributed. Figure 12.7, for example, shows how the linear exposure combined with the normal density (in the right panel) creates a normal density. This linear model can be extended to an approximation that accounts for quadratic terms, called delta-gamma, which are detailed in Chapter 14.

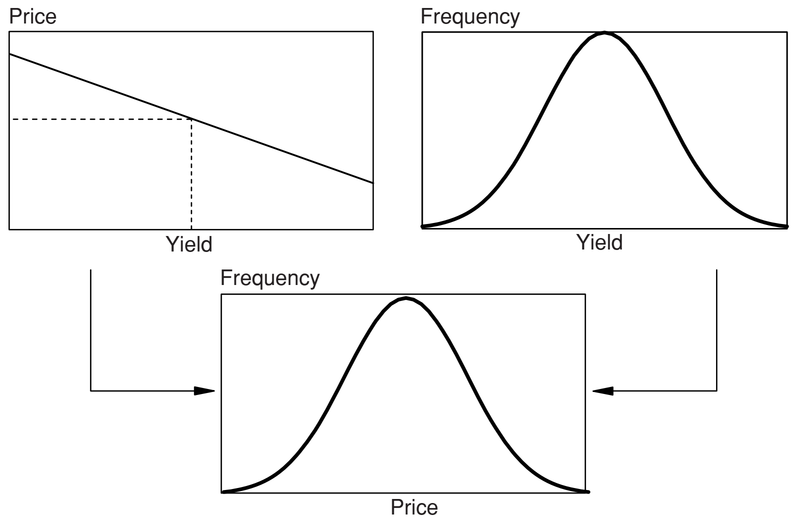

FIGURE 12.7 Distribution with Linear Exposures

12.6.2 Full Valuation

More generally, to take into account nonlinear relationships, one would have to reprice the bond under different scenarios for the yield. Defining y_0 as the initial yield,

$$(\text{Worst } dP) = P[y_0 + (\text{Worst } dy)] - P[y_0] \tag{12.19}$$

We call this approach **full valuation**, because it requires repricing the asset.

This approach is illustrated in Figure 12.8, where the nonlinear exposure combined with the normal density creates a distribution that is no longer symmetrical,

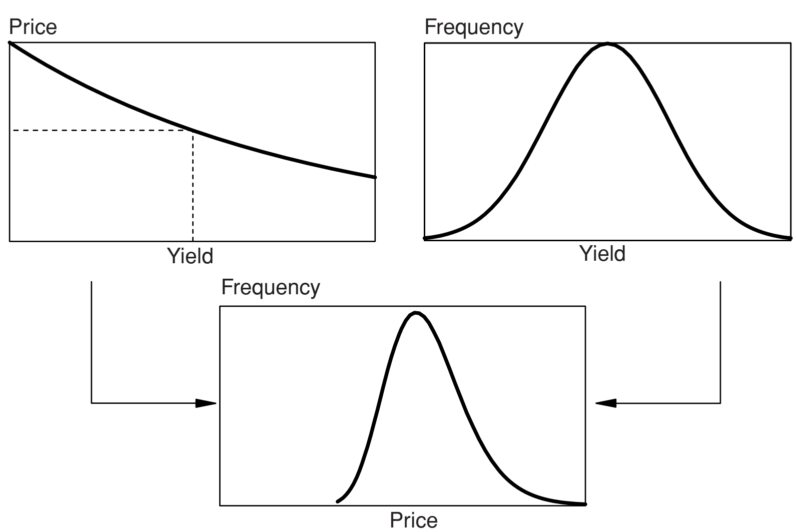

FIGURE 12.8 Distribution with Nonlinear Exposures

but skewed to the right. This is more precise but, unfortunately, more complex than a simple, linear valuation method.

Full valuation methods are needed when the portfolio has options, especially in situations where movements in risk factors are large. This explains why stress tests require full valuation.

EXAMPLE 12.12: FRM EXAM 2004—QUESTION II-60

Which of the following methodologies would be most appropriate for stress-testing your portfolio?

a. Delta-gamma valuation
b. Full revaluation
c. Marking to market
d. Delta-normal VAR

12.7 IMPORTANT FORMULAS

VAR: $c = \int_{-\text{VAR}}^{\infty} f(x)dx$

CVAR: $E[X \mid X < q] = \int_{-\infty}^{q} xf(x)dx \Big/ \int_{-\infty}^{q} f(x)dx$

Drawdown: $\text{DD}(X) = \frac{(x^{MAX} - x_t)}{x^{MAX}}$

Square root of time adjustment: $\text{VAR}(T \text{ days}) = \text{VAR}(1 \text{ day}) \times \sqrt{T}$

Market Risk Charge: $\text{GMRC}_t^{IMA} = \text{Max}\left(k\frac{1}{60}\sum_{i=1}^{60}\text{VAR}_{t-i},\ \text{VAR}_{t-1}\right)$

Linear VAR, fixed-income: $\text{VAR}(dP) = \mid -D^* P \mid \times \text{VAR}(dy)$

Full-valuation VAR, fixed-income: $(\text{Worst } dP) = P[y_0 + (\text{Worst } dy)] - P[y_0]$

Delta VAR: $\text{VAR}(df) = \mid \Delta \mid \text{VAR}(dS)$

12.8 ANSWERS TO CHAPTER EXAMPLES

Example 12.1: FRM Exam 2005—Question 32

a. Stop loss limits cut down the positions after a loss is incurred, which is useful if markets are trending. Exposure limits do not allow for diversification, because correlations are not considered. VAR limits can be arbitraged, especially with weak VAR models. Finally, stop loss limits are put in place after losses are incurred, so cannot prevent all losses. As a result, statements I. and II. are correct.

Example 12.2: Position-Based Risk Measures

c. The true VAR will be less than calculated. Loss limits cut down the positions as losses accumulate. This is similar to a long position in an option, where the delta increases as the price increases, and vice versa. Long positions in options have shortened left tails, and hence involve less risk than unprotected positions.

Example 12.3: FRM Exam 2005—Question 43

b. VAR is the worst loss such that there is a 95% probability that the losses will be less severe. Alternatively, there is a 5% probability that the losses will be worse. So b. is correct. Answer d. says “lose less” and therefore is incorrect.

Example 12.4: FRM Exam 2003—Question 5

b. The 10% lower cutoff point is the third lowest observation, which is VAR = 10. The expected shortfall is then the average of the observations in the tails, which is 15.

Example 12.5: FRM Exam 2009—Question 4-4

d. CVAR is the average of losses worse than VAR, so answer a. is incorrect. Expressed in absolute value, VAR is lower than any other losses used for CVAR, so VAR must be the most optimistic loss.

Example 12.6: FRM Exam 2008—Question 2-2

b. We need to scale the VAR to a 99% level using $\$100,000 \times 2.326/1.645 = \$141,398$. Multiplying by $\sqrt{10}$ then gives \$447,140.

Example 12.7: FRM Exam 2009—Question 4-3

a. We compute the daily VAR by dividing each VAR by the square root of time. This gives $316/\sqrt{10} = 100$, then 120, 120, and 120. So, answer a. is out of line.

Example 12.8: Market Risk Charge

d. First, we have to convert the 95% VAR to a 99% measure, assuming a normal distribution in the absence of other information. The GMRC is then $3 \times VAR \times \sqrt{10} = 3 \times \$1,000,000(2.33)/1.65) \times \sqrt{10} = \$13,396,000$.

Example 12.9: FRM Exam 2008—Question 2-29

c. All the statements are correct except IV., because too many scenarios will make it more difficult to interpret the risk exposure.

Example 12.10: FRM Exam 2006—Question 87

a. Stress testing is indeed used to evaluate the effect of extreme events. Answer b. is about backtesting, not stress-testing.

Example 12.11: FRM Exam 2008—Question 2-18

a. First, we transform the volatility into a daily measure, which is $25\%/\sqrt{252} = 1.57\%$. Multiplying, we get $150 \times 1.57\% \times 4 = \$9.45$.

Example 12.12: FRM Exam 2004—Question II-60

b. By definition, stress-testing involves large movements in the risk factors. This requires a full revaluation of the portfolio.

CHAPTER 13

Managing Linear Risk

Risk that has been measured can be managed. This chapter turns to the active management of market risks. An important aspect of managing risk is **hedging**, which consists of taking positions that lower the risk profile of the portfolio.

Techniques for hedging have been developed in the futures markets, where farmers, for instance, use financial instruments to hedge the price risk of their products. In this case, the objective is to find the optimal position in a futures contract that minimizes the volatility of the total portfolio. This portfolio consists of two positions, a fixed inventory exposed to a risk factor and a hedging instrument.

In this chapter, we examine hedging in cases where the value of the hedging instrument is linearly related to the underlying risk factor. This involves futures, forwards, and swaps. The next chapter examines risk management using nonlinear instruments (i.e., options).

In general, hedging can create **hedge slippage**, or **basis risk**. Basis risk arises when changes in payoffs on the hedging instrument do not perfectly offset changes in the value of the inventory position. Hedging is effective when basis risk is much less than outright price risk.

This chapter discusses the management of risk with linear instruments. Section 13.1 presents unitary hedging, where the quantity hedged is the same as the quantity protected. Section 13.2 then turns to a general method for finding the optimal hedge ratio. This method is applied in Section 13.3 for hedging bonds and equities.

13.1 UNITARY HEDGING

13.1.1 Futures Hedging

Consider the situation of a U.S. exporter who has been promised a payment of 125 million Japanese yen in seven months. This defines the underlying position, which can be viewed as an anticipated inventory. The perfect hedge would be to enter a seven-month forward contract over-the-counter (OTC). Assume for this illustration that this OTC contract is not convenient. Instead, the exporter decides to turn to an exchange-traded futures contract, which can be bought or sold easily.

FRM Exam Part 1 topic.

TABLE 13.1 Example of a Futures Hedge

Item	Initial Time	Exit Time	Gain or Loss
Market Data:			
Maturity (months)	9	2	
U.S. interest rate	6%	6%	
Yen interest rate	5%	2%	
Spot (¥/$)	¥125.00	¥150.00	
Futures (¥/$)	¥124.07	¥149.00	
Contract Data:			
Spot ($/¥)	0.008000	0.006667	−$166,667
Futures ($/¥)	0.008060	0.006711	$168,621
Basis ($/¥)	−0.000060	−0.000045	$1,954

The Chicago Mercantile Exchange (now CME Group) lists yen contracts with face amount of ¥12,500,000 that expire in nine months. The exporter places an order to sell 10 contracts, with the intention of reversing the position in seven months, when the contract will still have two months to maturity.[1] Because the amount sold is the same as the underlying, this is called a **unitary hedge**.

In this case, the hedge stays in place until the end of the hedging horizon. More generally, we can distinguish between **static hedging**, which consists of putting on, and leaving, a position until the hedging horizon, and **dynamic hedging**, which consists of continuously rebalancing the portfolio to the horizon. Dynamic hedging is associated with options, which we examine in the next chapter.

Table 13.1 describes the initial and final conditions for the contract. At each date, the futures price is determined by interest parity. Suppose that the yen depreciates sharply, or that the dollar goes up from ¥125 to ¥150. This leads to a loss on the anticipated cash position of ¥125,000,000 × (0.006667 − 0.00800) = −$166,667. This loss, however, is offset by a gain on the futures, which is (−10) × ¥12,500,000 × (0.006711 − 0.00806) = $168,621. The net is a small gain of $1,954. Overall, the exporter has been hedged.

This example shows that futures hedging can be quite effective, removing the effect of fluctuations in the risk factor. Define Q as the amount of yen transacted and S and F as the spot and futures rates, indexed by 1 at the initial time and by 2 at the exit time. The P&L on the unhedged transaction is

$$Q[S_2 - S_1] \tag{13.1}$$

[1] In practice, if the liquidity of long-dated contracts is not adequate, the exporter could use nearby contracts and roll them over prior to expiration into the next contracts. When there are multiple exposures, this practice is known as a **stack hedge**. Another type of hedge is the **strip hedge**, which involves hedging the exposures with a number of different contracts. While a stack hedge has superior liquidity, it also entails greater basis risk than a strip hedge. Hedgers must decide whether the greater liquidity of a stack hedge is worth the additional basis risk.

With unit hedging, the total profit is

$$Q[(S_2 - S_1)] - Q[(F_2 - F_1)] = Q[(S_2 - F_2) - (S_1 - F_1)] = Q[b_2 - b_1] \quad (13.2)$$

where $b = S - F$ is the **basis**. The profit depends on only the movement in the basis. Hence the effect of hedging is to transform price risk into basis risk. A short hedge position is said to be *long the basis*, since it benefits from an increase in the basis.

KEY CONCEPT

A short hedge position is long the basis; that is, it benefits when the basis widens or strengthens. This is because the position is short the hedging instrument, which falls in value (relative to the spot price) when the basis widens.

In this case, the basis risk is minimal for a number of reasons. First, the cash and futures correspond to the same asset. Second, the cash-and-carry relationship holds very well for currencies. Third, the remaining maturity at exit is rather short. This is not always the case, however.

13.1.2 Basis Risk

Basis risk arises when the characteristics of the futures contract differ from those of the underlying position. Futures contracts are standardized to a particular grade, say West Texas Intermediate (WTI) for oil futures traded on the New York Mercantile Exchange (NYMEX). This defines the grade of crude oil deliverable against the contract. A hedger, however, may have a position in a different grade, which may not be perfectly correlated with WTI. Thus basis risk is the uncertainty whether the cash-futures spread will widen or narrow during the hedging period. Hedging can be effective, however, if movements in the basis are dominated by movements in cash markets.

For most commodities, basis risk is inevitable. Organized exchanges strive to create enough trading and liquidity in their listed contracts, which requires standardization. Speculators also help to increase trading volumes and provide market liquidity. Thus there is a trade-off between liquidity and basis risk.

Basis risk is higher with **cross-hedging**, which involves using a futures on a totally different asset or commodity than the cash position. For instance, a U.S. exporter who is due to receive a payment in Norwegian kroner (NK) could hedge using a futures contract on the $/euro exchange rate. Relative to the dollar, the euro and the NK should behave similarly, but there is still some basis risk.

Basis risk is lowest when the underlying position and the futures correspond to the same asset. Even so, some basis risk remains because of differing maturities. As we have seen in the yen hedging example, the maturity of the futures contract is nine months instead of seven months. As a result, the liquidation price of the futures is uncertain.

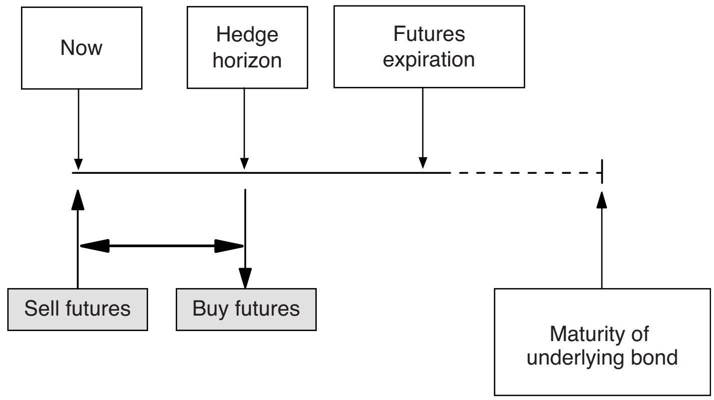

FIGURE 13.1 Hedging Horizon and Contract Maturity

Figure 13.1 describes the various time components for a hedge using T-bond futures. The first component is the *maturity of the underlying bond*, say 20 years. The second component is the *time to futures expiration*, say nine months. The third component is the *hedge horizon*, say seven months. Basis risk occurs when the hedge horizon does not match the time to futures expiration.

EXAMPLE 13.1: FRM EXAM 2000—QUESTION 79

Under which scenario is basis risk likely to exist?

a. A hedge (which was initially matched to the maturity of the underlying) is lifted before expiration.
b. The correlation of the underlying and the hedge vehicle is less than one and their volatilities are unequal.
c. The underlying instrument and the hedge vehicle are dissimilar.
d. All of the above are correct.

EXAMPLE 13.2: FRM EXAM 2009—QUESTION 3-14

Mary has IBM stock and will sell it two months from now at a specified date in the middle of the month. Mary would like to hedge the price of risk of IBM stock. How could she best hedge the IBM stock without incurring basis risk?

a. Short a two-month forward contract on IBM stock
b. Short a three-month futures contract on IBM stock
c. Short a two-month forward contract on the S&P 500 index
d. Answers a. and b. are correct.

EXAMPLE 13.3: FRM EXAM 2009—QUESTION 3-15

Which of the following statements is/are *true* with respect to basis risk?

I. Basis risk arises in cross-hedging strategies, but there is no basis risk when the underlying asset and hedge asset are identical.
II. A short hedge position benefits from unexpected strengthening of basis.
III. A long hedge position benefits from unexpected strengthening of basis.

a. I and II
b. I and III
c. II only
d. III only

EXAMPLE 13.4: FRM EXAM 2007—QUESTION 99

Which of the following trades contain mainly basis risk?

I. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Nov 07 NYMEX WTI Crude Oil contracts
II. Long 1,000 lots Nov 07 ICE Brent Oil contracts and long 2,000 lots Nov 07 ICE Brent Oil at-the-money put
III. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Dec 07 ICE Brent Oil contracts
IV. Long 1,000 lots Nov 07 ICE Brent Oil contracts and short 1,000 lots Dec 07 NYMEX WTI Crude Oil contracts

a. II and IV only
b. I and III only
c. I, III, and IV only
d. III and IV only

13.2 OPTIMAL HEDGING

The previous section gave an example of a unit hedge, where the quantities transacted are identical in the two markets. In general, this is not appropriate. We have to decide how much of the hedging instrument to transact.

Consider a situation where a portfolio manager has an inventory of carefully selected corporate bonds that should do better than their benchmark. The manager wants to guard against interest rate increases, however, over the next three months.

In this situation, it would be too costly to sell the entire portfolio only to buy it back later. Instead, the manager can implement a temporary hedge using derivative contracts, for instance T-bond futures.

Here, we note that the only risk is **price risk**, as the quantity of the inventory is known. This may not always be the case, however. Farmers, for instance, have uncertainty over both the price and size of their crop. If so, the hedging problem is substantially more complex as it involves hedging *revenues*, which involves analyzing demand and supply conditions.

13.2.1 The Optimal Hedge Ratio

Define ΔS as the change in the dollar value of the inventory and ΔF as the change in the dollar value of one futures contract. The inventory, or position to be hedged, can be existing or **anticipatory**, that is, to be received in the future with a great degree of certainty. The manager is worried about potential movements in the value of the inventory ΔS.

If the manager goes long N futures contracts, the total change in the value of the portfolio is

$$\Delta V = \Delta S + N\Delta F \tag{13.3}$$

One should try to find the hedge that reduces risk to the minimum level. The variance of total profits is equal to

$$\sigma^2_{\Delta V} = \sigma^2_{\Delta S} + N^2\sigma^2_{\Delta F} + 2N\sigma_{\Delta S,\Delta F} \tag{13.4}$$

Note that volatilities are initially expressed in dollars, not in rates of return, as we attempt to stabilize dollar values.

Taking the derivative with respect to N

$$\frac{\partial\sigma^2_{\Delta V}}{\partial N} = 2N\sigma^2_{\Delta F} + 2\sigma_{\Delta S,\Delta F} \tag{13.5}$$

For simplicity, drop the Δ in the subscripts. Setting Equation (13.5) equal to zero and solving for N, we get

$$N^* = -\frac{\sigma_{\Delta S,\Delta F}}{\sigma^2_{\Delta F}} = -\frac{\sigma_{SF}}{\sigma^2_F} = -\rho_{SF}\frac{\sigma_S}{\sigma_F} \tag{13.6}$$

where σ_{SF} is the covariance between futures and spot price changes. Here, N^* is the **minimum variance hedge ratio**.

In practice, there is often confusion about the definition of the portfolio value and unit prices. Here S consists of the number of units (shares, bonds, bushels, gallons) times the unit price (stock price, bond price, wheat price, fuel price).

It is sometimes easier to deal with unit prices and to express volatilities in terms of *rates of changes in unit prices*, which are unitless. Defining quantities Q

and unit prices s, we have $S = Qs$. Similarly, the notional amount of one futures contract is $F = Q_f f$. We can then write

$$\sigma_{\Delta S} = Q\sigma(\Delta s) = Qs\sigma(\Delta s/s)$$
$$\sigma_{\Delta F} = Q_f\sigma(\Delta f) = Q_f f\sigma(\Delta f/f)$$
$$\sigma_{\Delta S,\Delta F} = \rho_{sf}[Qs\sigma(\Delta s/s)][Q_f f\sigma(\Delta f/f)]$$

Using Equation (13.6), the optimal hedge ratio N^* can also be expressed as

$$N^* = -\rho_{SF}\frac{Qs\sigma(\Delta s/s)}{Q_f f\sigma(\Delta f/f)} = -\rho_{SF}\frac{\sigma(\Delta s/s)}{\sigma(\Delta f/f)}\frac{Qs}{Q_f f} = -\beta_{sf}\frac{Q\times s}{Q_f\times f} \tag{13.7}$$

where β_{sf} is the coefficient in the regression of $\Delta s/s$ over $\Delta f/f$. The second term represents an adjustment factor for the size of the cash position and of the futures contract.

The optimal amount N^* can be derived from the slope coefficient of a regression of $\Delta s/s$ on $\Delta f/f$:

$$\frac{\Delta s}{s} = \alpha + \beta_{sf}\frac{\Delta f}{f} + \epsilon \tag{13.8}$$

As seen in Chapter 3, standard regression theory shows that

$$\beta_{sf} = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf}\frac{\sigma_s}{\sigma_f} \tag{13.9}$$

Thus the **best hedge** is obtained from a regression of (change in) the value of the inventory on the value of the hedge instrument.

KEY CONCEPT

The optimal hedge is given by the negative of the beta coefficient of a regression of changes in the cash value on changes in the payoff on the hedging instrument.

We can do more than this, though. At the optimum, we can find the variance of profits by replacing N in Equation (13.4) by N^*, which gives

$$\sigma_V^{*2} = \sigma_S^2 + \left(\frac{\sigma_{SF}}{\sigma_F^2}\right)^2\sigma_F^2 + 2\left(\frac{-\sigma_{SF}}{\sigma_F^2}\right)\sigma_{SF} = \sigma_S^2 + \frac{\sigma_{SF}^2}{\sigma_F^2} + 2\frac{-\sigma_{SF}^2}{\sigma_F^2} = \sigma_S^2 - \frac{\sigma_{SF}^2}{\sigma_F^2} \tag{13.10}$$

We can measure the quality of the optimal hedge ratio in terms of the amount by which we decreased the variance of the original portfolio:

$$R^2 = \frac{(\sigma_S^2 - \sigma_V^{*2})}{\sigma_S^2} \tag{13.11}$$

After substitution of Equation (13.10), we find that $R^2 = (\sigma_S^2 - \sigma_S^2 + \sigma_{SF}^2/\sigma_F^2)/\sigma_S^2 = \sigma_{SF}^2/(\sigma_F^2\sigma_S^2) = \rho_{SF}^2$. This unitless number is also the coefficient of determination, or the percentage of variance in $\Delta s/s$ explained by the independent variable $\Delta f/f$ in Equation (13.8). Thus this regression also gives us the **effectiveness** of the hedge, which is measured by the proportion of variance eliminated.

We can also express the volatility of the hedged position from Equation (13.10) using the R^2 as

$$\sigma_V^* = \sigma_S\sqrt{(1 - R^2)} \tag{13.12}$$

This shows that if $R^2 = 1$, the regression fit is perfect, and the resulting portfolio has zero risk. In this situation, the portfolio has no basis risk. However, if the R^2 is very low, the hedge is not effective.

13.2.2 Example

An airline knows that it will need to purchase 10,000 metric tons of jet fuel in three months. It wants some protection against an upturn in prices using futures contracts.

The company can hedge using heating oil futures contracts traded on NYMEX. The notional for one contract is 42,000 gallons. As there is no futures contract on jet fuel, the risk manager wants to check if heating oil could provide an efficient hedge instead. The current price of jet fuel is \$277/metric ton. The futures price of heating oil is \$0.6903/gallon. The standard deviation of the rate of change in jet fuel prices over three months is 21.17%, that of futures is 18.59%, and the correlation is 0.8243.

Compute

1. The notional and standard deviation of the unhedged fuel cost in dollars
2. The optimal number of futures contract to buy/sell, rounded to the closest integer
3. The standard deviation of the hedged fuel cost in dollars

Answer

1. The position notional is Qs = \$2,770,000. The standard deviation in dollars is

$$\sigma(\Delta s/s)sQ = 0.2117 \times \$277 \times 10{,}000 = \$586{,}409$$

For reference, that of one futures contract is

$$\sigma(\Delta f/f) f Q_f = 0.1859 \times \$0.6903 \times 42{,}000 = \$5{,}389.72$$

with a futures notional of $f Q_f = \$0.6903 \times 42{,}000 = \$28{,}992.60$.

2. The cash position corresponds to a payment, or liability. Hence, the company will have to *buy* futures as protection. First, we compute beta, which is $\beta_{sf} = 0.8243(0.2117/0.1859) = 0.9387$. The corresponding covariance term is $\sigma_{sf} = 0.8243 \times 0.2117 \times 0.1859 = 0.03244$. Adjusting for the notionals, this is $\sigma_{SF} = 0.03244 \times \$2{,}770{,}000 \times \$28{,}993 = 2{,}605{,}268{,}452$. The optimal hedge ratio is, using Equation (13.7),

$$N^* = \beta_{sf} \frac{Q \times s}{Q_f \times f} = 0.9387 \frac{10{,}000 \times \$277}{42{,}000 \times \$0.69} = 89.7$$

or 90 contracts after rounding (which we ignore in what follows).

3. To find the risk of the hedged position, we use Equation (13.10). The volatility of the unhedged position is $\sigma_S = \$586{,}409$. The variance of the hedged position is

$$\begin{array}{lll} \sigma_S^2 = (\$586{,}409)^2 & & = +343{,}875{,}515{,}281 \\ -\sigma_{SF}^2/\sigma_F^2 = -(2{,}605{,}268{,}452/5{,}390)^2 & & = -233{,}653{,}264{,}867 \\ \hline \text{V(hedged)} & & = +110{,}222{,}250{,}414 \end{array}$$

Taking the square root, the volatility of the hedged position is $\sigma_V^* = \$331{,}997$. Thus the hedge has reduced the risk from \$586,409 to \$331,997. Computing the R^2, we find that one minus the ratio of the hedged and unhedged variances is $(1 - 110{,}222{,}250{,}414/343{,}875{,}515{,}281) = 67.95\%$. This is exactly the square of the correlation coefficient, $0.8243^2 = 0.6795$, or effectiveness of the hedge.

Figure 13.2 displays the relationship between the risk of the hedged position and the number of contracts. With no hedging, the volatility is \$586,409. As N increases, the risk decreases, reaching a minimum for $N^* = 90$ contracts. The graph also shows that the quadratic relationship is relatively flat for a range of values around the minimum. Choosing anywhere between 80 and 100 contracts will have little effect on the total risk. Given the substantial reduction in risk, the risk manager could choose to implement the hedge.

13.2.3 Liquidity Issues

Although futures hedging can be successful at mitigating market risk, it can create other risks. Futures contracts are marked to market daily. Hence they can involve large cash inflows or outflows. Cash outflows, in particular, can create liquidity problems, especially when they are not offset by cash inflows from the underlying position.

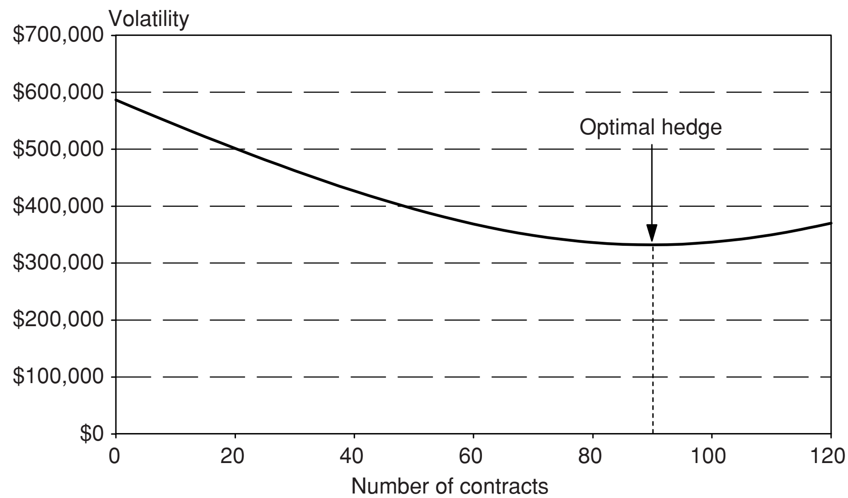

FIGURE 13.2 Risk of Hedged Position and Number of Contracts

EXAMPLE 13.5: FRM EXAM 2001—QUESTION 86

If two securities have the same volatility and a correlation equal to −0.5, their minimum variance hedge ratio is

a. 1:1
b. 2:1
c. 4:1
d. 16:1

EXAMPLE 13.6: FRM EXAM 2007—QUESTION 125

A firm is going to buy 10,000 barrels of West Texas Intermediate Crude Oil. It plans to hedge the purchase using the Brent Crude Oil futures contract. The correlation between the spot and futures prices is 0.72. The volatility of the spot price is 0.35 per year. The volatility of the Brent Crude Oil futures price is 0.27 per year. What is the hedge ratio for the firm?

a. 0.9333
b. 0.5554
c. 0.8198
d. 1.2099

EXAMPLE 13.7: FRM EXAM 2009—QUESTION 3-26

XYZ Co. is a gold producer and will sell 10,000 ounces of gold in three months at the prevailing market price at that time. The standard deviation of the change in the price of gold over a three-month period is 3.6%. In order to hedge its price exposure, XYZ Co. decides to use gold futures to hedge. The contract size of each gold futures contract is 10 ounces. The standard deviation of the gold futures price is 4.2%. The correlation between quarterly changes in the futures price and the spot price of gold is 0.86. To hedge its price exposure, how many futures contracts should XYZ Co. go long or short?

a. Short 632 contracts
b. Short 737 contracts
c. Long 632 contracts
d. Long 737 contracts

13.3 APPLICATIONS OF OPTIMAL HEDGING

The linear framework presented here is completely general. We now specialize it to two important cases, duration and beta hedging. The first applies to the bond market, the second to the stock market.

13.3.1 Duration Hedging

Modified duration can be viewed as a measure of the exposure of relative changes in prices to movements in yields. Using the definitions in Chapter 6, we can write

$$\Delta P = (-D^* P)\Delta y \tag{13.13}$$

where D^* is the modified duration. The **dollar duration** is defined as $(D^* P)$.

Assuming the duration model holds, which implies that the change in yield Δy does not depend on maturity, we can rewrite this expression for the cash and futures positions

$$\Delta S = (-D_S^* S)\Delta y \qquad \Delta F = (-D_F^* F)\Delta y$$

where D_S^* and D_F^* are the modified durations of S and F, respectively. Note that these relationships are supposed to be perfect, without an error term. The variances and covariance are then

$$\sigma_S^2 = (D_S^* S)^2 \sigma^2(\Delta y) \qquad \sigma_F^2 = (D_F^* F)^2 \sigma^2(\Delta y) \qquad \sigma_{SF} = (D_F^* F)(D_S^* S)\sigma^2(\Delta y)$$

We can replace these in Equation (13.6):

$$N^* = -\frac{\sigma_{SF}}{\sigma_F^2} = -\frac{(D_F^* F)(D_S^* S)}{(D_F^* F)^2} = -\frac{(D_S^* S)}{(D_F^* F)} \tag{13.14}$$

Alternatively, this can be derived as follows. Write the total portfolio payoff as

$$\begin{aligned}\Delta V &= \Delta S + N \Delta F \\ &= (-D_S^* S)\Delta y + N(-D_F^* F)\Delta y \\ &= -[(D_S^* S) + N(D_F^* F)] \times \Delta y\end{aligned}$$

which is zero when the net exposure, represented by the term between brackets, is zero. In other words, the optimal hedge ratio is simply minus the ratio of the dollar duration of cash relative to the dollar duration of the hedge. This ratio can also be expressed in dollar value of a basis point (DVBP). This gives

$$N^* = -\frac{\text{DVBP}_S}{\text{DVBP}_F} \tag{13.15}$$

More generally, we can use N as a tool to modify the total duration of the portfolio. If we have a target duration of D_V, this can be achieved by setting $[(D_S^* S) + N(D_F^* F)] = D_V^* V$, or

$$N = \frac{(D_V^* V - D_S^* S)}{(D_F^* F)} \tag{13.16}$$

of which Equation (13.14) is a special case.

KEY CONCEPT

The optimal duration hedge is given by the ratio of the dollar duration of the position to that of the hedging instrument.

Example: T-Bond Futures Hedging

A portfolio manager holds a bond portfolio worth \$10 million with a modified duration of 6.8 years, to be hedged for three months. The current futures price is 93-02, with a notional of \$100,000. We assume that its duration can be derived from that of the cheapest-to-deliver bond, which is 9.2 years.[2]

To be consistent, all of these values should be measured as of the hedge horizon date. So, the portfolio and futures durations are forecast to be 6.8 and 9.2 years

[2] T-bond futures are described in Chapter 10. Note that this hedging method ignores the effect of options held by the short, which include a delivery and timing option. Also, the duration of the futures should be the duration of the cheapest-to-deliver bond divided by the conversion factor.

in three months. The \$10 million should be the forward value of the portfolio, although in practice it is often taken from the current value.

Compute

1. The dollar value of the futures contract notional
2. The number of contracts to buy/sell for optimal protection

Answer

1. The dollar notional is $[93 + (2/32)]/100 \times \$100{,}000 = \$93{,}062.5$.
2. The optimal number to *sell* is from Equation (13.14):

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{6.8 \times \$10{,}000{,}000}{9.2 \times \$93{,}062.5} = -79.4$$

or 79 contracts after rounding. Note that the DVBP of the futures is about $9.2 \times \$93{,}000 \times 0.01\% = \$85$.

Example: Eurodollar Futures Hedging

On February 2, a corporate Treasurer wants to hedge a July 17 issue of \$5 million of commercial paper with a maturity of 180 days, leading to anticipated proceeds of \$4.52 million. The September Eurodollar futures contract trades at 92, and has a notional amount of \$1 million.[3]

Compute

1. The current dollar value of the futures contract
2. The number of contracts to buy/sell for optimal protection

Answer

1. The current dollar price is given by $\$10{,}000[100 - 0.25(100 - 92)] = \$980{,}000$. Note that the duration of the futures is always three months (90 days), since the contract refers to three-month LIBOR.
2. If rates increase, the cost of borrowing will be higher. We need to offset this by a gain, or a short position in the futures. The optimal number is from Equation (13.14):

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{180 \times \$4{,}520{,}000}{90 \times \$980{,}000} = -9.2$$

or 9 contracts after rounding. Note that the DVBP of the futures is about $0.25 \times \$1{,}000{,}000 \times 0.01\% = \$25$. In this case, the DVBP of the position to be hedged is \$226. Dividing by \$25 gives a hedge ratio of 9.

[3] Eurodollar futures are described in Chapter 10.

EXAMPLE 13.8: DURATION HEDGING

What assumptions does a duration-based hedging scheme make about the way in which interest rates move?

- **a.** All interest rates change by the same amount.
- **b.** A small parallel shift occurs in the yield curve.
- **c.** Any parallel shift occurs in the term structure.
- **d.** Interest rates' movements are highly correlated.

EXAMPLE 13.9: HEDGING WITH EURODOLLAR FUTURES

If all spot interest rates are increased by one basis point, a value of a portfolio of swaps will increase by $1,100. How many Eurodollar futures contracts are needed to hedge the portfolio?

- **a.** 44
- **b.** 22
- **c.** 11
- **d.** 1,100

EXAMPLE 13.10: FRM EXAM 2007—QUESTION 17

On June 2, a fund manager with USD 10 million invested in government bonds is concerned that interest rates will be highly volatile over the next three months. The manager decides to use the September Treasury bond futures contract to hedge the portfolio. The current futures price is USD 95.0625. Each contract is for the delivery of USD 100,000 face value of bonds. The duration of the manager's bond portfolio in three months will be 7.8 years. The cheapest-to-deliver (CTD) bond in the Treasury bond futures contract is expected to have a duration of 8.4 years at maturity of the contract. At the maturity of the Treasury bond futures contract, the duration of the underlying benchmark Treasury bond is nine years. What position should the fund manager undertake to mitigate his interest rate risk exposure?

- **a.** Short 94 contracts
- **b.** Short 98 contracts
- **c.** Short 105 contracts
- **d.** Short 113 contracts

EXAMPLE 13.11: FRM EXAM 2004—QUESTION 4

Albert Henri is the fixed income manager of a large Canadian pension fund. The present value of the pension fund's portfolio of assets is CAD 4 billion while the expected present value of the fund's liabilities is CAD 5 billion. The respective modified durations are 8.254 and 6.825 years. The fund currently has an actuarial deficit (assets < liabilities) and Albert must avoid widening this gap. There are currently two scenarios for the yield curve: The first scenario is an upward shift of 25bp, with the second scenario a downward shift of 25bp. The most liquid interest rate futures contract has a present value of CAD 68,336 and a duration of 2.1468 years. Analyzing both scenarios separately, what should Albert Henri do to avoid widening the pension fund gap? Choose the best option.

First Scenario	Second Scenario
a. Do nothing.	Buy 7,559 contracts
b. Do nothing.	Sell 7,559 contracts.
c. Buy 7,559 contracts.	Do nothing.
d. Do nothing.	Do nothing.

13.3.2 Beta Hedging

We now turn to equity hedging using stock index futures. **Beta**, or **systematic risk**, can be viewed as a measure of the exposure of the rate of return on a portfolio i to movements in the market m:

$$R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it} \tag{13.17}$$

where β represents the systematic risk, α the intercept (which is not a source of risk and therefore is ignored for risk management purposes), and ϵ the residual component, which is uncorrelated with the market. We can also write, in line with the previous sections and ignoring the residual and intercept,

$$(\Delta S/S) \approx \beta(\Delta M/M) \tag{13.18}$$

Now, assume that we have at our disposal a stock index futures contract, which has a beta of unity $(\Delta F/F) = 1(\Delta M/M)$. For options, the beta is replaced by the net delta, $(\Delta C) = \delta(\Delta M)$.

As in the case of bond duration, we can write the total portfolio payoff as

$$\begin{aligned}\Delta V &= \Delta S + N\Delta F \\ &= (\beta S)(\Delta M/M) + NF(\Delta M/M) \\ &= [(\beta S) + NF] \times (\Delta M/M)\end{aligned}$$

which is set to zero when the net exposure, represented by the term between brackets, is zero. The optimal number of contracts to short is

$$N^* = -\frac{\beta S}{F} \tag{13.19}$$

KEY CONCEPT

The optimal hedge with stock index futures is given by the beta of the cash position times its value divided by the notional of the futures contract.

Example

A portfolio manager holds a stock portfolio worth $10 million with a beta of 1.5 relative to the S&P 500. The current futures price is 1,400, with a multiplier of $250.

Compute

1. The notional of the futures contract
2. The number of contracts to sell short for optimal protection

Answer

1. The notional amount of the futures contract is $\$250 \times 1{,}400 = \$350{,}000$.
2. The optimal number of contracts to short is, from Equation (13.19),

$$N^* = -\frac{\beta S}{F} = -\frac{1.5 \times \$10{,}000{,}000}{1 \times \$350{,}000} = -42.9$$

or 43 contracts after rounding.

13.3.3 General Considerations

The quality of the hedge depends on the size of the residual risk in the market model of Equation (13.17). For large portfolios, the approximation may be good because residual risk diversifies away. In contrast, hedging an individual stock with stock index futures may give poor results.

For instance, the correlation of a typical U.S. stock with the S&P 500 is 0.50. For an industry index, it is typically 0.75. Using the regression effectiveness in Equation (13.12), we find that the volatility of the hedged portfolio is still about $\sqrt{1 - 0.5^2} = 87\%$ of the unhedged volatility for a typical stock and about 66% of the unhedged volatility for a typical industry. Thus hedging a portfolio of stocks (an industry index) with a general market hedge is more effective.

A final note on hedging is in order. If the objective of hedging is to lower volatility, hedging will eliminate downside risk but also any upside potential. The objective of hedging is to lower risk, not to make profits, so this is a double-edged sword. Whether hedging is beneficial should be examined in the context of the trade-off between risk and return.

EXAMPLE 13.12: FRM EXAM 2009—QUESTION 3-10

You have a portfolio of USD 5 million to be hedged using index futures. The correlation coefficient between the portfolio and futures being used is 0.65. The standard deviation of the portfolio is 7% and that of the hedging instrument is 6%. The futures price of the index futures is USD 1,500 and one contract size is 100 futures. Among the following positions, which one reduces risk the most?

a. Long 33 futures contracts
b. Short 33 futures contracts
c. Long 25 futures contracts
d. Short 25 futures contracts

EXAMPLE 13.13: FRM EXAM 2007—QUESTION 107

The current value of the S&P 500 index is 1,457, and each S&P futures contract is for delivery of 250 times the index. A long-only equity portfolio with market value of USD 300,100,000 has a beta of 1.1. To reduce the portfolio beta to 0.75, how many S&P futures contracts should you sell?

a. 288 contracts
b. 618 contracts
c. 906 contracts
d. 574 contracts

13.4 IMPORTANT FORMULAS

Profit on position with unit hedge: $Q[(S_2 - S_1) - (F_2 - F_1)] = Q[b_2 - b_1]$

Short hedge position = long the basis, or benefits when the basis widens/strengthens

Optimal hedge ratio: $N^* = -\beta_{sf} \frac{Q \times s}{Q_f \times f}$

Optimal hedge ratio (unitless): $\beta_{sf} = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf} \frac{\sigma_s}{\sigma_f}$

Volatility of the hedged position: $\sigma_V^* = \sigma_S\sqrt{(1 - R^2)}$
Duration hedge: $N^* = -\frac{(D_S^* S)}{(D_F^* F)}$
Beta hedge: $N^* = -\beta\frac{S}{F}$

13.5 ANSWERS TO CHAPTER EXAMPLES

Example 13.1: FRM Exam 2000—Question 79

d. Basis risk occurs if movements in the value of the cash and hedged positions do not offset each other perfectly. This can happen if the instruments are dissimilar or if the correlation is not unity. Even with similar instruments, if the hedge is lifted before the maturity of the underlying, there is some basis risk.

Example 13.2: FRM Exam 2009—Question 3-14

a. Basis risk is minimized when the maturity of the hedging instrument coincides with the horizon of the hedge (i.e., two months) and when the hedging instrument is exposed to the same risk factor (i.e., IBM).

Example 13.3: FRM Exam 2009—Question 3-15

c. Basis risk can arise if the maturities are different, so answer I. is incorrect. A short hedge position is long the basis, which means that it benefits when the basis strengthens, because this means that the futures price drops relative to the spot price, which generates a profit.

Example 13.4: FRM Exam 2007—Question 99

c. There is mainly basis risk for positions that are both long and short either different months or contracts. Position II. is long twice the same contract and thus has no basis risk (but a lot of directional risk).

Example 13.5: FRM Exam 2001—Question 86

b. Set x as the amount to invest in the second security, relative to that in the first (or the hedge ratio). The variance is then proportional to $1 + x^2 + 2x\rho$. Taking the derivative and setting to zero, we have $x = -\rho = 0.5$. Thus, one security must have twice the amount in the other. Alternatively, the hedge ratio is given by $N^* = -\rho\frac{\sigma_S}{\sigma_F}$, which gives 0.5. Answer b. is the only one that is consistent with this number or its inverse.

Example 13.6: FRM Exam 2007—Question 125

a. The optimal hedge ratio is $\beta_{sf} = \rho_{sf}\frac{\sigma_s}{\sigma_f} = 0.72\ 0.35/0.27 = 0.933$.

Example 13.7: FRM Exam 2009—Question 3-26

b. XYZ will incur a loss if the price of gold falls, so should short futures as a hedge. The optimal hedge ratio is $\rho\sigma_s/\sigma_f = 0.86 \times 3.6/4.2 = 0.737$. Taking into account the size of the position, the number of contracts to sell is $0.737 \times 10{,}000/10 = 737$.

Example 13.8: Duration Hedging

b. The assumption is that of (1) parallel and (2) small moves in the yield curve. Answers a. and c. are the same, and omit the size of the move. Answer d. would require perfect, not high, correlation plus small moves.

Example 13.9: Hedging with Eurodollar Futures

a. The DVBP of the portfolio is \$1,100. That of the futures is \$25. Hence the ratio is $1{,}100/25 = 44$.

Example 13.10: FRM Exam 2007—Question 17

b. The number of contracts to short is $N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -(7.8 \times 10{,}000{,}000)/(8.4 \times (95.0625) \times 1{,}000) = -97.7$, or 98 contracts. Note that the relevant duration for the futures is that of the CTD; other numbers are irrelevant.

Example 13.11: FRM Exam 2004—Question 4

a. We first have to compute the dollar duration of assets and liabilities, which gives, in millions, $4{,}000 \times 8.254 = 33{,}016$ and $5{,}000 \times 6.825 = 34{,}125$, respectively. Because the DD of liabilities exceeds that of assets, a decrease in rates will increase the liabilities more than the assets, leading to a worsening deficit. Albert needs to buy interest rate futures as an offset. The number of contracts is $(34{,}125 - 33{,}016)/(68{,}336 \times 2.1468/1{,}000{,}000) = 7{,}559$.

Example 13.12: FRM Exam 2009—Question 3-10

d. To hedge, the portfolio manager should sell index futures, to create a profit if the portfolio loses value. The portfolio beta is $0.65 \times (7\%/6\%) = 0.758$. The number of contracts is $N^* = -\beta S/F = -(0.758 \times 5{,}000{,}000)/(1{,}500 \times 100) = -25.3$, or 25 contracts.

Example 13.13: FRM Exam 2007—Question 107

a. This is as in the previous question, but the hedge is partial (i.e., for a change of 1.10 to 0.75). So, $N^* = -\beta S/F = -(1.10 - 0.75)300{,}100{,}000/(1{,}457 \times 250) = -288.3$ contracts.

CHAPTER 14

Nonlinear (Option) Risk Models

The previous chapter focused on linear risk models (e.g., hedging using contracts such as forwards and futures whose values are linearly related to the underlying risk factors). Because linear combinations of normal random variables are also normally distributed, linear hedging maintains normal distributions, which considerably simplifies the risk analysis.

Nonlinear risk models, however, are much more complex. In particular, option values can have sharply asymmetrical distributions. Even so, it is essential for risk managers to develop an ability to evaluate this type of risk, because options are so widespread in financial markets. Since options can be replicated by dynamic trading, this also provides insights into the risks of active trading strategies.

In a previous chapter, we have seen that market losses can be ascribed to the combination of two factors: exposure and adverse movements in the risk factor. Thus a large loss could occur because of the risk factor, which is bad luck. Too often, however, losses occur because the exposure profile is similar to a short option position. This is less forgivable, because exposure is under the control of the portfolio manager.

The challenge is to develop measures that provide an intuitive understanding of the exposure profile. Section 14.1 introduces option pricing and the Taylor approximation. It starts from the Black-Scholes formula that was presented in Chapter 8. Partial derivatives, also known as Greeks, are analyzed in Section 14.2. Section 14.3 then turns to the distribution profile of option positions and the measurement of value at risk (VAR) using delta and gamma.

14.1 OPTION MODELS

14.1.1 Definitions

We consider a **derivative** instrument whose value depends on an underlying asset, which can be a price, an index, or a rate. As an example, consider a call option

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FRM Exam Part 1 topic. Also note that FRM Exam Part 1 includes credit ratings, which are covered in Chapter 20.

where the underlying asset is a foreign currency. We use these definitions:

S_t = current spot price of the asset in dollars
F_t = current forward price of the asset
K = exercise price of option contract
f_t = current value of derivative instrument
r_t = domestic risk-free rate
r_t^* = foreign risk-free rate (also written as y)
σ_t = annual volatility of the rate of change in S
τ = time to maturity

More generally, r^* represents the income payment on the asset, which represents the *annual rate* of dividend or coupon payments on a stock index or bond.

For most options, we can write the value of the derivative as the function

$$f_t = f(S_t, r_t, r_t^*, \sigma_t, \; K, \tau) \tag{14.1}$$

The contract specifications are represented by K and the time to maturity by τ. The other factors are affected by market movements, creating volatility in the value of the derivative. For simplicity, we drop the time subscripts in what follows.

Derivatives pricing is all about finding the value of f, given the characteristics of the option at expiration and some assumptions about the behavior of markets. For a forward contract, for instance, the expression is very simple. It reduces to

$$f = Se^{-r^*\tau} - Ke^{-r\tau} \tag{14.2}$$

More generally, we may not be able to derive an analytical expression for the function f, requiring numerical methods.

14.1.2 Taylor Expansion

We are interested in describing the movements in f. The exposure profile of the derivative can be described *locally* by taking a Taylor expansion,

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \cdots \tag{14.3}$$

Because the value depends on S in a nonlinear fashion, we added a quadratic term for S. The terms in Equation (14.3) approximate a nonlinear function by linear and quadratic polynomials.

Option pricing is about finding f. **Option hedging** uses the partial derivatives. **Risk management** is about combining those with the movements in the risk factors.

Figure 14.1 describes the relationship between the value of a European call and the underlying asset. The actual price is the solid thick line. The straight thin line is the linear (delta) estimate, which is the tangent at the initial point. The other line is the quadratic (delta plus gamma) estimates; it gives a much better fit because it has more parameters.

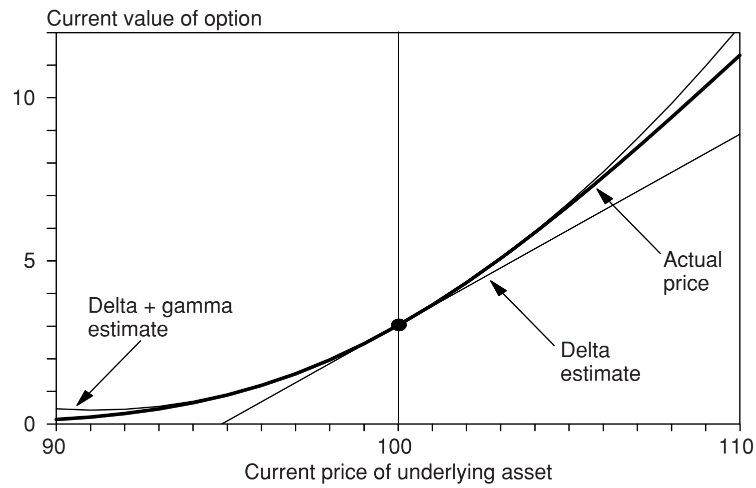

FIGURE 14.1 Delta-Gamma Approximation for a Long Call

Note that, because we are dealing with sums of local price movements, we can aggregate the sensitivities at the portfolio level. This is similar to computing the portfolio duration from the sum of durations of individual securities, appropriately weighted.

Defining $\Delta = \frac{\partial f}{\partial S}$, for example, we can summarize the portfolio or book Δ_P in terms of the total sensitivity,

$$\Delta_P = \sum_{i=1}^{N} x_i \Delta_i \tag{14.4}$$

where x_i is the number of options of type i in the portfolio. To hedge against first-order price risk, it is sufficient to hedge the *net* portfolio delta. This is more efficient than trying to hedge every single instrument individually.

The Taylor expansion will provide a bad approximation in a number of cases:

- *Large movements in the underlying risk factor*
- *Highly nonlinear exposures*, such as options near expiry or exotic options
- *Cross-partial effects*, such as σ changing in relation with S

If this is the case, we need to turn to a **full revaluation** of the instrument. Using the subscripts 0 and 1 as the initial and final values, the change in the option value is

$$f_1 - f_0 = f(S_1, r_1, r_1^*, \sigma_1,\ K, \tau_1) - f(S_0, r_0, r_0^*, \sigma_0,\ K, \tau_0) \tag{14.5}$$

14.1.3 Option Pricing

We now present the various partial derivatives for conventional European call and put options. As we have seen in Chapter 8, the **Black-Scholes** (BS) model provides a closed-form solution, from which these derivatives can be computed analytically.

The key point of the BS derivation is that a position in the option can be replicated by a delta position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is risk-free locally, that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate. The option value is the discounted expected payoff:

$$f_t = E_{\mathrm{RN}}[e^{-r\tau} F(S_T)] \tag{14.6}$$

where E_{RN} represents the expectation of the future payoff in a risk-neutral world, that is, assuming the underlying asset grows at the risk-free rate and the discounting also employs the risk-free rate.

In the case of a European call, the final payoff is $F(S_T) = \mathrm{Max}(S_T - K, 0)$, and the current value of the call is given by

$$c = Se^{-r^*\tau} N(d_1) - Ke^{-r\tau} N(d_2) \tag{14.7}$$

where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx$$

with Φ defined as the standard normal density function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d. The values of d_1 and d_2 are

$$d_1 = \frac{ln(Se^{-r^*\tau}/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put-call parity, the European put option value is

$$p = Se^{-r^*\tau}[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{14.8}$$

14.2 OPTION GREEKS

14.2.1 Option Sensitivities: Delta and Gamma

Given these closed-form solutions for European options, we can derive all partial derivatives. The most important sensitivity is the **delta**, which is the first partial derivative with respect to the price. For a call option, this can be written explicitly as:

$$\Delta_c = \frac{\partial c}{\partial S} = e^{-r^*\tau} N(d_1) \tag{14.9}$$

which is always positive and below unity.

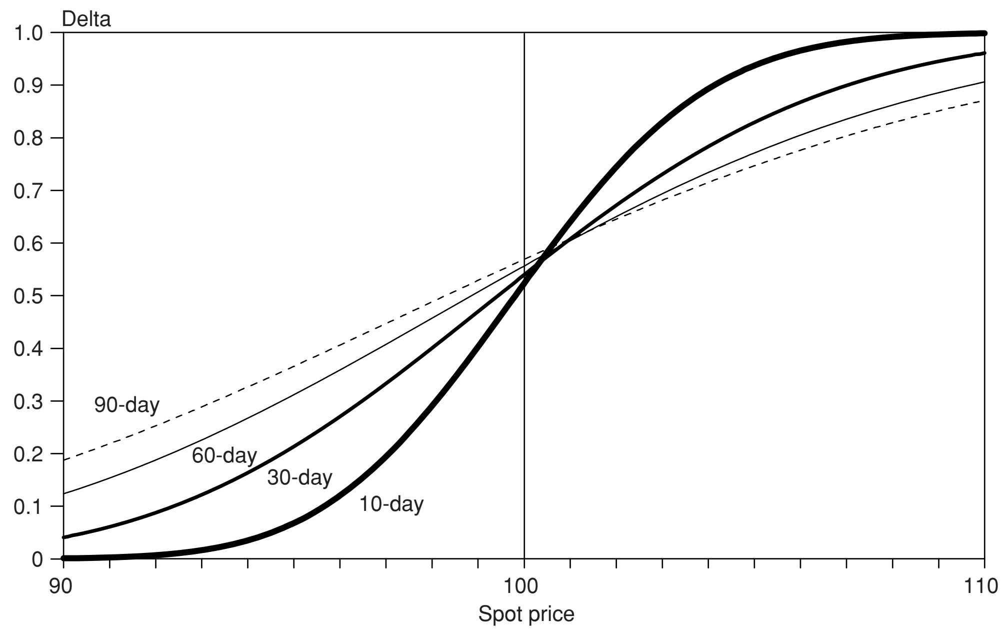

FIGURE 14.2 Option Delta

Figure 14.2 relates delta to the current value of S, for various maturities. The essential feature of this figure is that Δ varies substantially with the spot price and with time. As the spot price increases, d_1 and d_2 become very large, and Δ tends toward $e^{-r^*\tau}$, close to 1 for short maturities. In this situation, the option behaves like an outright position in the asset. Indeed, the limit of Equation (14.7) is $c = Se^{-r^*\tau} - Ke^{-r\tau}$, which is exactly the value of our forward contract, Equation (14.2).

At the other extreme, if S is very low, Δ is close to zero and the option is not very sensitive to S. When S is close to the strike price K, Δ is close to 0.5, and the option behaves like a position of 0.5 in the underlying asset.

KEY CONCEPT

The delta of an at-the-money call option is close to 0.5. Delta moves to 1 as the call goes deep in-the-money (ITM). It moves to zero as the call goes deep out-of-the-money (OTM).

The delta of a put option is

$$\Delta_p = \frac{\partial p}{\partial S} = e^{-r^*\tau}[N(d_1) - 1] \tag{14.10}$$

which is always negative. It behaves similarly to the call Δ, except for the sign. The delta of an at-the-money (ATM) put is about -0.5.

KEY CONCEPT

The delta of an at-the-money put option is close to −0.5. Delta moves to −1 as the put goes deep in-the-money. It moves to zero as the put goes deep out-of-the-money.

The figure also shows that, as the option nears maturity, the Δ function becomes more curved. The function converges to a step function (i.e., 0 when $S < K$, and 1 otherwise). Close-to-maturity options have unstable deltas.

For a European call or put, gamma (Γ) is the second-order term,

$$\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r^*\tau}\Phi(d_1)}{S\sigma\sqrt{\tau}} \tag{14.11}$$

which has the bell shape of the normal density function Φ. This is also the derivative of Δ with respect to S. Thus Γ measures the instability in Δ. Note that gamma is identical for a call and put with identical characteristics.

Figure 14.3 plots the call option gamma. At-the-money options have the highest gamma, which indicates that Δ changes very fast as S changes. In contrast, both in-the-money options and out-of-the-money options have low gammas because their deltas are constant, close to one or zero, respectively. The figure also shows that as the maturity nears, the option gamma increases. This leads to a useful rule (see box).

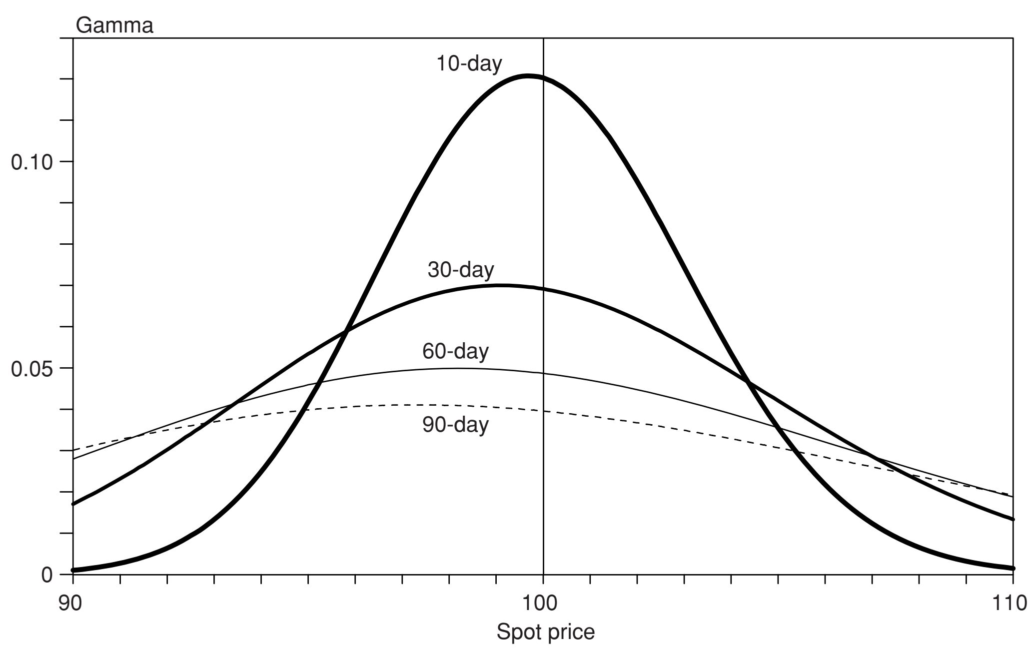

FIGURE 14.3 Option Gamma

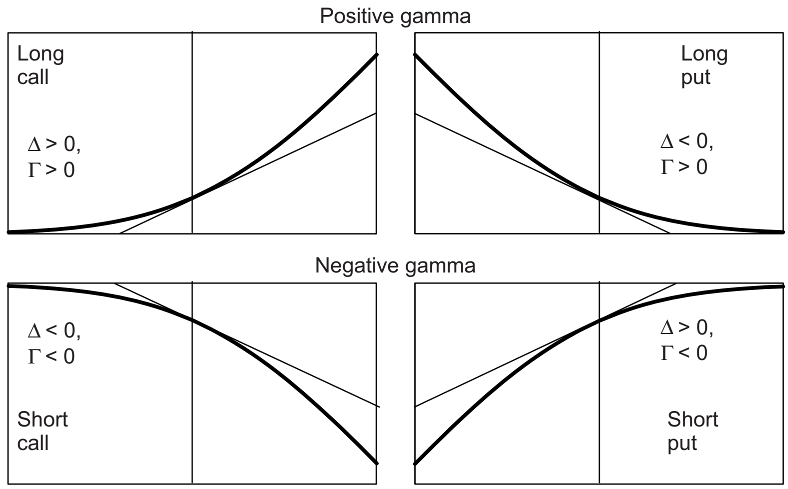

FIGURE 14.4 Delta and Gamma of Option Positions

> **KEY CONCEPT**
>
> For vanilla options, gamma is the highest, or nonlinearities are most pronounced, for short-term at-the-money options.

Thus, gamma is similar to the concept of convexity developed for bonds. Fixed-coupon bonds, however, always have positive convexity, whereas options can create positive or negative convexity. Positive convexity or gamma is beneficial, as it implies that the value of the asset drops more slowly and increases more quickly than otherwise. In contrast, negative convexity can be dangerous because it implies faster price falls and slower price increases.

Figure 14.4 summarizes the delta and gamma exposures of positions in options. Long positions in options, whether calls or puts, create positive convexity. Short positions create negative convexity. In exchange for assuming the harmful effect of this negative convexity, option sellers receive the premium.

> **EXAMPLE 14.1: FRM EXAM 2006—QUESTION 91**
>
> The dividend yield of an asset is 10% per annum. What is the delta of a long forward contract on the asset with six months to maturity?
>
> **a.** 0.95
> **b.** 1.00
> **c.** 1.05
> **d.** Cannot determine without additional information

EXAMPLE 14.2: FRM EXAM 2004—QUESTION 21

A 90-day European put option on Microsoft has an exercise price of \$30. The current market price for Microsoft is \$30. The delta for this option is close to

a. −1
b. −0.5
c. 0.5
d. 1

EXAMPLE 14.3: FRM EXAM 2006—QUESTION 80

You are given the following information about a European call option: Time to maturity = 2 years; continuous risk-free rate = 4%; continuous dividend yield = 1%; $N(d_1) = 0.64$. Calculate the delta of this option.

a. −0.64
b. 0.36
c. 0.63
d. 0.64

EXAMPLE 14.4: FRM EXAM 2009—QUESTION 4-27

An analyst is doing a study on the effect on option prices of changes in the price of the underlying asset. The analyst wants to find out when the deltas of calls and puts are most sensitive to changes in the price of the underlying. Assume that the options are European and that the Black-Scholes formula holds. An increase in the price of the underlying has the largest absolute value impact on delta for:

a. Calls deep in-the-money and puts deep out-of-the-money
b. Deep in-the-money puts and calls
c. Deep out-of-the-money puts and calls
d. At-the-money puts and calls

EXAMPLE 14.5: FRM EXAM 2001—QUESTION 79

A bank has sold USD 300,000 of call options on 100,000 equities. The equities trade at 50, the option strike price is 49, the maturity is in three months, volatility is 20%, and the interest rate is 5%. How does the bank delta-hedge?

a. Buy 65,000 shares
b. Buy 100,000 shares
c. Buy 21,000 shares
d. Sell 100,000 shares

EXAMPLE 14.6: FRM EXAM 2006—QUESTION 106

Suppose an existing short option position is delta-neutral, but has a gamma of −600. Also assume that there exists a traded option with a delta of 0.75 and a gamma of 1.50. In order to maintain the position gamma-neutral and delta-neutral, which of the following is the appropriate strategy to implement?

a. Buy 400 options and sell 300 shares of the underlying asset.
b. Buy 300 options and sell 400 shares of the underlying asset.
c. Sell 400 options and buy 300 shares of the underlying asset.
d. Sell 300 options and buy 400 shares of the underlying asset.

14.2.2 Option Sensitivities: Vega

Unlike linear contracts, options are exposed to movements not only in the direction of the spot price, but also in its volatility. Options therefore can be viewed as volatility bets.

The sensitivity of an option to volatility is called the option **vega** (sometimes also called lambda, or kappa). For European calls and puts, this is

$$\Lambda = \frac{\partial c}{\partial \sigma} = Se^{-r^*\tau}\sqrt{\tau}\,\Phi(d_1) \tag{14.12}$$

which also has the bell shape of the normal density function Φ. As with gamma, vega is identical for similar call and put positions. Vega must be positive for long option positions.

Figure 14.5 plots the call option vega. The graph shows that at-the-money options are the most sensitive to volatility. The time effect, however, is different from that for gamma, because the term $\sqrt{\tau}$ appears in the numerator instead of denominator. Thus, vega decreases with maturity, unlike gamma, which increases with maturity.

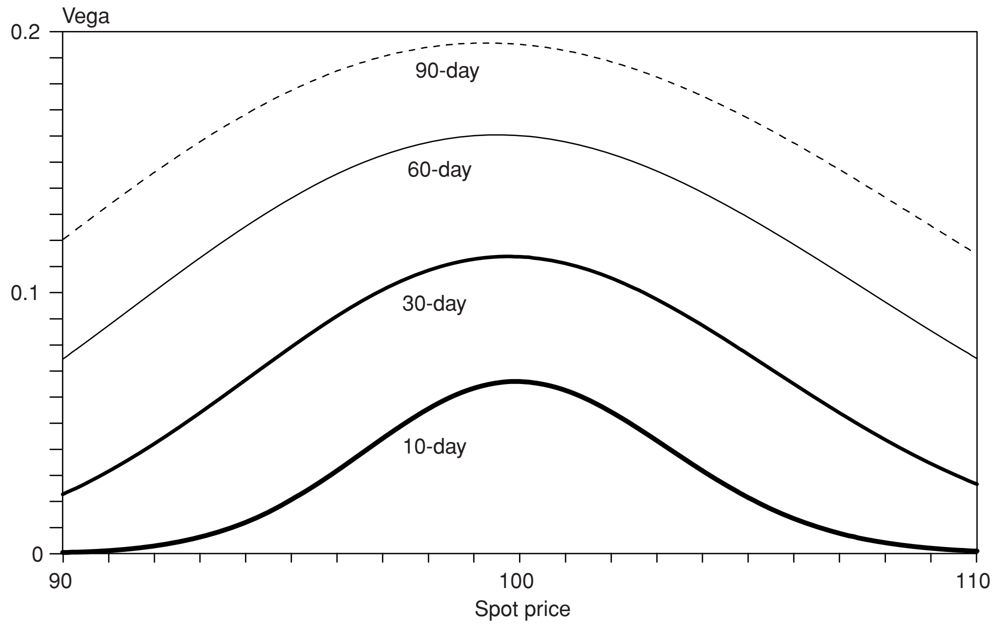

FIGURE 14.5 Option Vega

> **KEY CONCEPT**
>
> Vega is highest for long-term at-the-money options.

14.2.3 Option Sensitivities: Rho

The sensitivity to the domestic interest rate, also called **rho**, is, for a call,

$$\rho_c = \frac{\partial c}{\partial r} = Ke^{-r\tau}\tau N(d_2) \tag{14.13}$$

For a put,

$$\rho_p = \frac{\partial p}{\partial r} = -Ke^{-r\tau}\tau N(-d_2) \tag{14.14}$$

An increase in the rate of interest increases the value of the call, as the underlying asset grows at a higher rate, which increases the probability of exercising the call, with a fixed strike price K. In the limit, for an infinite interest rate, the probability of exercise is 1 and the call option is equivalent to the stock itself.

As shown in Figure 14.6, rho is positive for a call option and higher when the call is in-the-money. The reasoning is opposite for a put option, for which rho is negative. In each case, the sensitivity is roughly proportional to the remaining time to maturity.

The exposure to the yield on the asset is, for calls and puts, respectively,

$$\rho^*_C = \frac{\partial c}{\partial r^*} = -Se^{-r^*\tau}\tau N(d_1) \tag{14.15}$$

$$\rho^*_P = \frac{\partial p}{\partial r^*} = Se^{-r^*\tau}\tau N(-d_1) \tag{14.16}$$

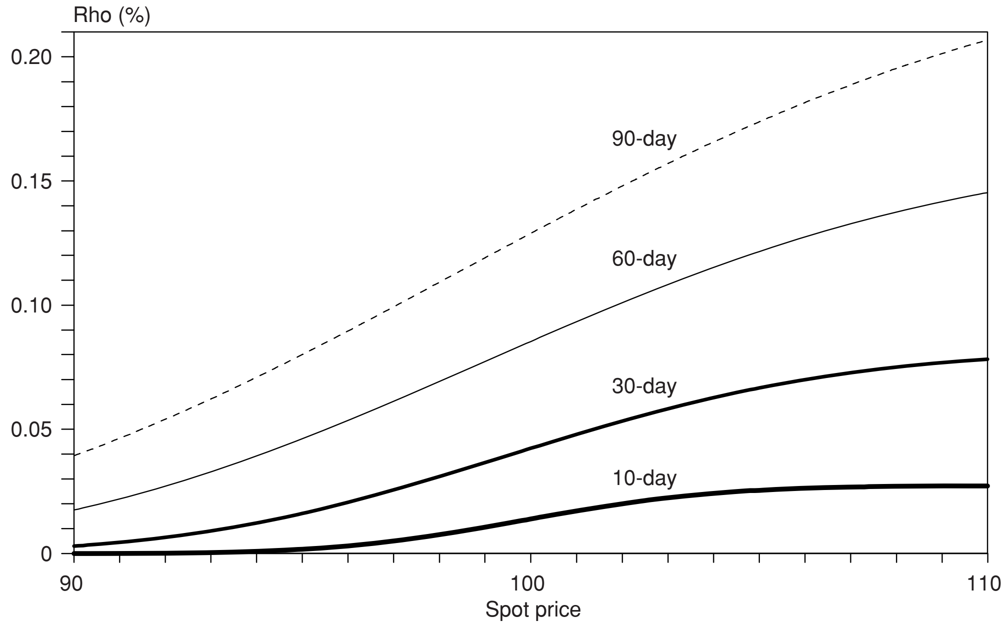

FIGURE 14.6 Call Option Rho

An increase in the dividend yield decreases the growth rate of the underlying asset, which is harmful to the value of the call but helpful to the value of a put.

As shown in Figure 14.7, rho is negative and also greater in absolute value when the call is in-the-money. Again, the reasoning is opposite for a put option. In each case, the sensitivity is proportional to the remaining time to maturity.

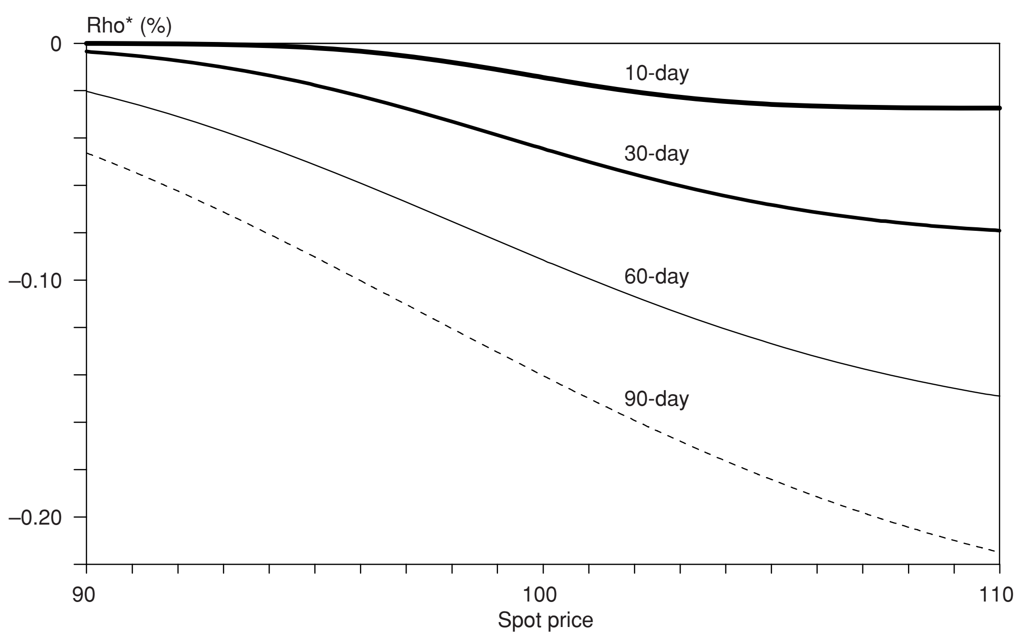

FIGURE 14.7 Call Option Rho*

EXAMPLE 14.7: FRM EXAM 2009—QUESTION 4-26

Ms. Zheng is responsible for the options desk in a London bank. She is concerned about the impact of dividends on the options held by the options desk. She asks you to assess which options are the most sensitive to dividend payments. What would be your answer if the value of the options is found by using the Black-Scholes model adjusted for dividends?

a. Everything else equal, out-of-the-money call options experience a larger decrease in value than in-the-money call options as expected dividends increase.
b. The increase in the value of in-the-money put options caused by an increase in expected dividends is always larger than the decrease in value of in-the-money call options.
c. Keeping the type of option constant, in-the-money options experience the largest absolute change in value and out-of-the-money options the smallest absolute change in value as expected dividends increase.
d. Keeping the type of option constant, at-the-money options experience the largest absolute change in value and out-of-the-money options the smallest absolute change in value as a result of dividend payment.

14.2.4 Option Sensitivities: Theta

Finally, the variation in option value due to the passage of time is called **theta**. This is also the **time decay**. Unlike other factors, however, the movement in remaining maturity is perfectly predictable. Time is not a risk factor.

For a European call, this is

$$\Theta_c = \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} + r^* Se^{-r^*\tau}N(d_1) - rKe^{-r\tau}N(d_2) \qquad (14.17)$$

For a European put, this is

$$\Theta_p = \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} = -\frac{Se^{-r^*\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} - r^* Se^{-r^*\tau}N(-d_1) + rKe^{-r\tau}N(-d_2) \qquad (14.18)$$

Theta is generally negative for long positions in both calls and puts. This means that the option loses value as time goes by.

For American options, however, Θ is *always* negative. Because they give their holder the choice to exercise early, shorter-term American options are unambiguously less valuable than longer-term options. For European options, the positive

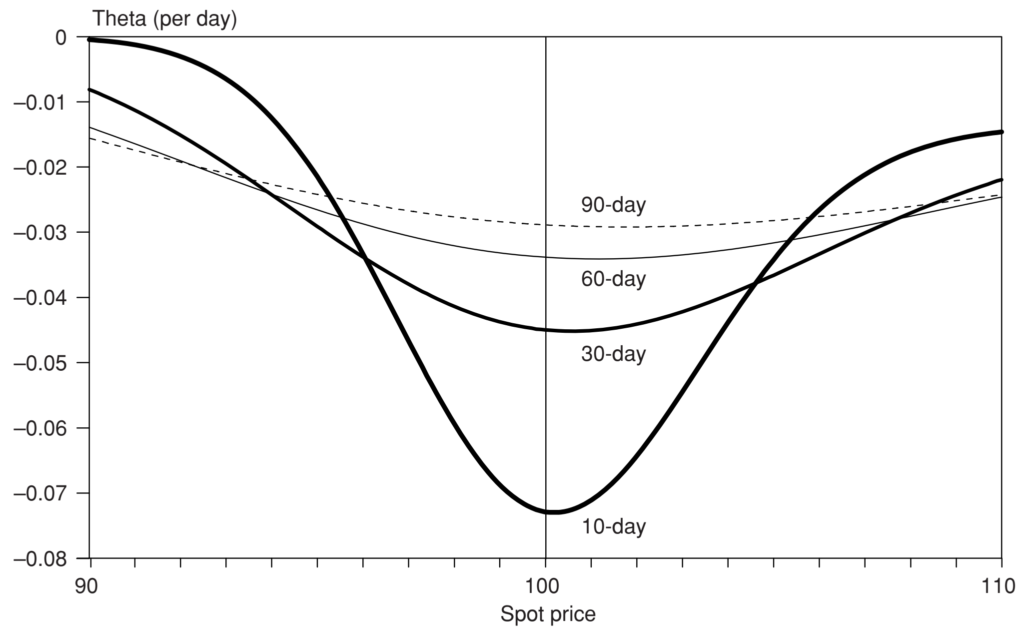

FIGURE 14.8 Option Theta

terms in Equations (14.17) and (14.18) indicate that theta could be positive for some parameter values, albeit this would be unusual.

Figure 14.8 displays the behavior of a call option theta for various prices of the underlying asset and maturities. For long positions in options, theta is negative, which reflects the fact that the option is a wasting asset. Like gamma, theta is greatest for short-term at-the-money options, when measured in absolute value. At-the-money options lose a great proportion of their value when the maturity is near.

14.2.5 Option Pricing and the Greeks

Having defined the option sensitivities, we can illustrate an alternative approach to the derivation of the Black-Scholes formula. Recall that the underlying process for the asset follows a stochastic process known as a **geometric Brownian motion** (GBM),

$$dS = \mu S dt + \sigma S dz \tag{14.19}$$

where dz has a normal distribution with mean zero and variance dt.

Considering only this *single* source of risk, we can return to the Taylor expansion in Equation (14.3). The value of the derivative is a function of S and time, which we can write as $f(S, t)$. The question is, how does f evolve over time?

We can relate the stochastic process of f to that of S using **Ito's lemma,** named after its creator. This can be viewed as an extension of the Taylor approximation to a stochastic environment. Applied to the GBM, this gives

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial \tau}\right) dt + \left(\frac{\partial f}{\partial S}\sigma S\right) dz \tag{14.20}$$

This is also

$$df = (\Delta\,\mu S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\,\sigma S)dz \tag{14.21}$$

The first term, including dt, is the trend. The second, including dz, is the stochastic component.

Next, we construct a portfolio delicately balanced between S and f that has no exposure to dz. Define this portfolio as

$$\Pi = f - \Delta\, S \tag{14.22}$$

Using Equations (14.19) and (14.21), its stochastic process is

$$\begin{aligned} d\Pi &= [(\Delta\,\mu S + \tfrac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\,\sigma S)dz] - \Delta\,[\mu S dt + \sigma S dz] \\ &= (\Delta\,\mu S)dt + (\tfrac{1}{2}\Gamma\sigma^2 S^2)dt + \Theta dt + (\Delta\,\sigma S)dz - (\Delta\,\mu S)dt - (\Delta\,\sigma S)dz \\ &= (\tfrac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt \end{aligned} \tag{14.23}$$

This simplification is extremely important. Note how the terms involving dz cancel each other out. The portfolio has been immunized against this source of risk. At the same time, the terms in μS also happened to cancel each other out. The fact that μ disappears from the trend in the portfolio is important because it explains why the trend of the underlying asset does not appear in the Black-Scholes formula.

Continuing, we note that the portfolio Π has no risk. To avoid arbitrage, it must return the risk-free rate:

$$d\Pi = [r\Pi]dt = r(f - \Delta\, S)dt \tag{14.24}$$

If the underlying asset has a dividend yield of y, this must be adjusted to

$$d\Pi = (r\Pi)dt + y\Delta\, Sdt = r(f - \Delta\, S)dt + y\Delta\, Sdt = [rf - (r - y)\Delta\, S]dt \tag{14.25}$$

Setting the trends in Equations (14.23) and (14.25) equal to each other, we must have

$$(r - y)\Delta\, S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf \tag{14.26}$$

This is the Black-Scholes **partial differential equation (PDE)**, which applies to any contract, or portfolio, that derives its value from S. The solution of this equation, with appropriate boundary conditions, leads to the BS formula for a European call, Equation (14.7).

We can use this relationship to understand how the sensitivities relate to each other. Consider a portfolio of derivatives, all on the same underlying asset, that is delta-hedged. Setting $\Delta = 0$ in Equation (14.26), we have

$$\frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf \tag{14.27}$$

This shows that, for such portfolio, when Γ is large and positive, Θ must be negative if rf is small. In other words, a delta-hedged position with positive gamma, which is beneficial in terms of price risk, must have negative theta, or time decay. An example is the long straddle examined in Chapter 8. Such a position is delta-neutral and has large gamma or convexity. It would benefit from a large move in S, whether up or down. This portfolio, however, involves buying options whose values decay very quickly with time. Thus, there is an intrinsic trade-off between Γ and Θ.

KEY CONCEPT

For delta-hedged portfolios, Γ and Θ must have opposite signs. Portfolios with positive convexity, for example, must experience time decay.

14.2.6 Option Sensitivities: Summary

We now summarize the sensitivities of option positions with some illustrative data in Table 14.1. Three strike prices are considered, $K = 90$, 100, and 110. We verify that the $\Gamma,\ \Lambda,\ \Theta$ measures are all highest when the option is at-the-money ($K = 100$). Such options have the most nonlinear patterns.

The table also shows the loss for the worst daily movement in each risk factor at the 95% confidence level. For S, this is $dS = -1.645 \times 20\% \times \$100/\sqrt{252} = -\$2.08$. We combine this with delta, which gives a potential loss of $\Delta \times dS = -\$1.114$, or about a fourth of the option value.

Next, we examine the second-order term, S^2. The worst squared daily movement is $dS^2 = 2.08^2 = 4.33$ in the risk factor at the 95% confidence level. We combine this with gamma, which gives a potential gain of $\frac{1}{2}\Gamma \times dS^2 = 0.5 \times 0.039 \times 4.33 = \$0.084$. Note that this is a gain because gamma is positive, but much smaller than the first-order effect. Thus the worst loss due to S would be $-\$1.114 + \$0.084 = -\$1.030$ using the linear and quadratic effects.

For σ, we observe a volatility of daily changes in σ on the order of 1.5%. The worst daily move is therefore $-1.645 \times 1.5 = -2.5$, expressed in percent, which gives a worst loss of $-\$0.495$. Finally, for r, we assume an annual volatility of changes in rates of 1%. The worst daily move is then $-1.645 \times 1/\sqrt{252} = -0.10$, in percent, which gives a worst loss of $-\$0.013$. So, most of the risk originates

TABLE 14.1 Derivatives for a European Call
Parameters: $S = \$100, \sigma = 20\%, r = 5\%, y = 3\%, \tau = 3$ months

			Strike			Worst Loss	
	Variable	Unit	$K = 90$	$K = 100$	$K = 110$	Variable	Loss
c		Dollars	\$11.02	\$4.22	\$1.05		
		Change per:					
Δ	Spot price	Dollar	0.868	0.536	0.197	−\$2.08	−\$1.114
Γ	Spot price	Dollar	0.020	0.039	0.028	4.33	\$0.084
Λ	Volatility	(% pa)	0.103	0.198	0.139	−2.5	−\$0.495
ρ	Interest rate	(% pa)	0.191	0.124	0.047	−0.10	−\$0.013
ρ^*	Asset yield	(% pa)	−0.220	−0.135	−0.049	0.10	−\$0.014
Θ	Time	Day	−0.014	−0.024	−0.016		

from S. In this case, a linear approximation using Δ only would capture most of the downside risk. For near-term at-the-money options, however, the quadratic effect is more important.

EXAMPLE 14.8: FRM EXAM 2004—QUESTION 65

Which of the following statements is *true* regarding options Greeks?

a. Theta tends to be large and positive when buying at-the-money options.
b. Gamma is greatest for in-the-money options with long maturities.
c. Vega is greatest for at-the-money options with long maturities.
d. Delta of deep in-the-money put options tends toward +1.

EXAMPLE 14.9: FRM EXAM 2006—QUESTION 33

Steve, a market risk manager at Marcat Securities, is analyzing the risk of its S&P 500 index options trading desk. His risk report shows the desk is net long gamma and short vega. Which of the following portfolios of options shows exposures consistent with this report?

a. The desk has substantial long-expiry long call positions and substantial short-expiry short put positions.
b. The desk has substantial long-expiry long put positions and substantial long-expiry short call positions.
c. The desk has substantial long-expiry long call positions and substantial short-expiry short call positions.
d. The desk has substantial short-expiry long call positions and substantial long-expiry short call positions.

EXAMPLE 14.10: FRM EXAM 2006—QUESTION 54

Which of the following statements is *incorrect*?

a. The vega of a European-style call option is highest when the option is at-the-money.
b. The delta of a European-style put option moves toward zero as the price of the underlying stock rises.
c. The gamma of an at-the-money European-style option tends to increase as the remaining maturity of the option decreases.
d. Compared to an at-the-money European-style call option, an out-of-the-money European-style option with the same strike price and remaining maturity has a greater negative value for theta.

EXAMPLE 14.11: VEGA AND GAMMA

How can a trader produce a short vega, long gamma position?

a. Buy short-maturity options, sell long-maturity options.
b. Buy long-maturity options, sell short-maturity options.
c. Buy and sell options of long maturity.
d. Buy and sell options of short maturity.

EXAMPLE 14.12: VEGA AND THETA

An option portfolio exhibits high unfavorable sensitivity to increases in implied volatility and while experiencing significant daily losses with the passage of time. Which strategy would the trader most likely employ to hedge the portfolio?

a. Sell short-dated options and buy long-dated options.
b. Buy short-dated options and sell long-dated options.
c. Sell short-dated options and sell long-dated options.
d. Buy short-dated options and buy long-dated options.

14.3 OPTION RISKS

14.3.1 Distribution of Option Payoffs

Unlike linear derivatives such as forwards and futures, payoffs on options are intrinsically asymmetric. This is not necessarily because of the distribution of the underlying factor, which is often symmetrical, but rather is due to the exposure

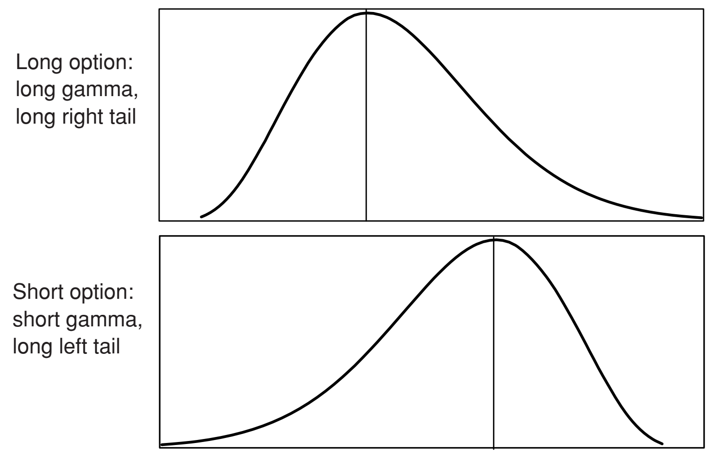

FIGURE 14.9 Distributions of Payoffs on Long and Short Options

profile. Long positions in options, whether calls or puts, have positive gamma, positive skewness, or long right tails. In contrast, short positions in options are short gamma and hence have negative skewness or long left tails. This is illustrated in Figure 14.9.

14.3.2 Option VAR: Linear

We now summarize VAR formulas for simple option positions. Assuming a normal distribution, the VAR of the underlying asset is

$$\text{VAR}(dS) = \alpha S \sigma(dS/S) \tag{14.28}$$

where α corresponds to the desired confidence level (e.g., $\alpha = 1.645$ for a 95% confidence level).

Next, we relate the movement in the option to that in the asset value. Consider a long position in a call for which Δ is positive:

$$dc = \Delta \times dS \tag{14.29}$$

The option VAR is the positive number $\text{VAR}_1(dc) = -dc = \Delta \times -dS = \Delta \times \text{VAR}(dS)$. Generally, the linear VAR for an option is

$$\text{VAR}_1(dc) = |\Delta| \times \text{VAR}(dS) \tag{14.30}$$

14.3.3 Option VAR: Quadratic

Next, we want to take into account nonlinear effects using the Taylor approximation.

$$df \approx \frac{\partial f}{\partial S} dS + (1/2) \frac{\partial^2 f}{\partial S^2} dS^2 = \Delta\, dS + (1/2) \Gamma\, dS^2 \tag{14.31}$$

When the value of the instrument is a monotonic function of the underlying risk factor, we can use the Taylor expansion to find the worst move in the value f from the worst move in the risk factor S.

For a long call option, for example, the worst value is achieved as the underlying price moves down by VAR(dS). The quadratic VAR for this option is

$$\mathrm{VAR}_2(dc) = |\Delta| \times \mathrm{VAR}(dS) - (1/2)\Gamma \times \mathrm{VAR}(dS)^2 \tag{14.32}$$

This method is called **delta-gamma** because it provides an analytical, second-order correction to the delta-normal VAR. This simple adjustment, unfortunately, works only when the payoff function is monotonic, that is, involves a one-to-one relationship between the option value f and S.

Equation (14.32) is fundamental. It explains why long positions in options with positive gamma have less risk than with a linear model. Conversely, short positions in options have negative gamma and thus greater risk than implied by a linear model.

This also applies to fixed-income positions, where the Taylor expansion uses modified duration D^* and convexity C:

$$dP \approx \frac{\partial P}{\partial y} dy + (1/2)\frac{\partial^2 P}{\partial y^2}(dy)^2 = (-D^* P)dy + (1/2)CP(dy)^2 \tag{14.33}$$

Because the price is a monotonic function of the underlying yield, we can use the Taylor expansion to find the worst down move in the bond price from the worst move in the yield. Calling this $dy^* = \mathrm{VAR}(dy)$, we have

$$(\text{Worst}\, dP) = P(y_0 + dy^*) - P(y_0) \approx (-D^* P)(dy^*) + (1/2)(C\ P)(dy^*)^2 \tag{14.34}$$

Similar to Equation (14.32), this gives

$$\mathrm{VAR}(dP) = |-D^* P| \times \mathrm{VAR}(dy) - (1/2)(C\ P) \times \mathrm{VAR}(dy)^2 \tag{14.35}$$

Lest we think that such options require sophisticated risk management methods, what matters is the *extent* of nonlinearity. Figure 14.10 illustrates the risk of a call option with a maturity of three months. It shows that the degree of nonlinearity also depends on the horizon. With a VAR horizon of two weeks, the range of possible values for S is quite narrow. If S follows a normal distribution, the option value will be approximately normal. However, if the VAR horizon is set at two months, the nonlinearities in the exposure combine with the greater range of price movements to create a heavily skewed distribution.

So, for plain-vanilla options, the linear approximation may be adequate as long as the VAR horizon is kept short. For more exotic options, or longer VAR horizons, risk managers must account for nonlinearities.

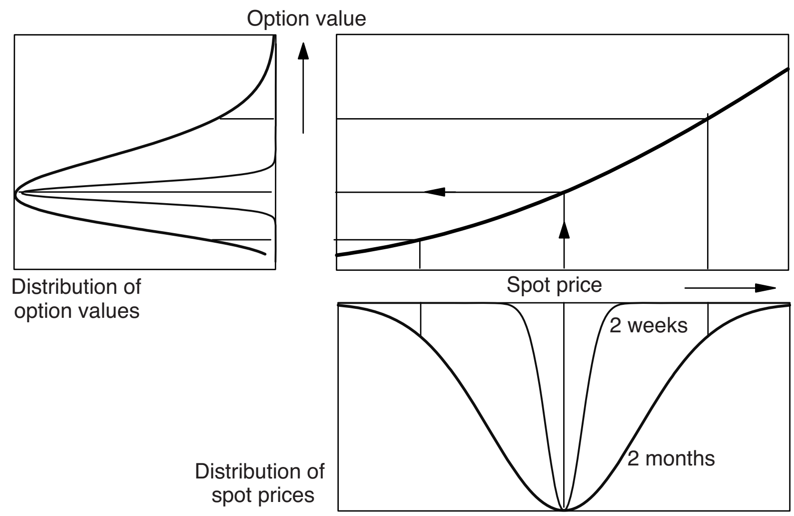

FIGURE 14.10 Skewness and VAR Horizon

EXAMPLE 14.13: FRM EXAM 2005—QUESTION 130

An option on the Bovespa stock index is struck on 3,000 Brazilian reais (BRL). The delta of the option is 0.6, and the annual volatility of the index is 24%. Using delta-normal assumptions, what is the 10-day VAR at the 95% confidence level? Assume 260 days per year.

a. 44 BRL
b. 139 BRL
c. 2,240 BRL
d. 278 BRL

EXAMPLE 14.14: FRM EXAM 2009—QUESTION 4-6

An investor is long a short-term at-the-money put option on an underlying portfolio of equities with a notional value of USD 100,000. If the 95% VAR of the underlying portfolio is 10.4%, which of the following statements about the VAR of the option position is *correct* when second-order terms are considered?

a. The VAR of the option position is slightly more than USD 5,200.
b. The VAR of the option position is slightly more than USD 10,400.
c. The VAR of the option position is slightly less than USD 5,200.
d. The VAR of the option position is slightly less than USD 10,400.

14.4 IMPORTANT FORMULAS

Black-Scholes option pricing model: $c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2)$

Taylor series expansion:

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \cdots$$

$$df = \Delta\, dS + \frac{1}{2}\Gamma\, dS^2 + \rho\, dr + \rho^* dr^* + \Lambda\, d\sigma + \Theta d\tau + \cdots$$

Delta: $\Delta_c = \frac{\partial c}{\partial S} = e^{-r^*\tau}N(d_1),\ \ \Delta_p = \frac{\partial p}{\partial S} = e^{-r^*\tau}[N(d_1) - 1]$

Gamma (for calls and puts): $\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r^*\tau}}{S\sigma\sqrt{\tau}}\Phi(d_1)$

Vega (for calls and puts): $\Lambda = \frac{\partial c}{\partial \sigma} = Se^{-r^*\tau}\sqrt{\tau}\ \Phi(d_1)$

	Long Call			Long Put		
	OTM	ATM	ITM	ITM	ATM	OTM
Δ	$\rightarrow 0$	0.5	$\rightarrow 1$	$\rightarrow -1$	-0.5	$\rightarrow 0$
Γ	Low	High, > 0 esp. short-term	Low	Low	High, > 0 esp. short-term	Low
Λ	Low	High, > 0 esp. long-term	Low	Low	High, > 0 esp. long-term	Low
Θ	Low	High, < 0 esp. short-term	Low	Low	High, < 0 esp. short-term	Low

Black-Scholes PDE: $(r - y)\Delta S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf$

Linear VAR for an option: $\text{VAR}_1(dc) = |\Delta| \times \text{VAR}(dS)$

Quadratic VAR for an option: $\text{VAR}_2(dc) = |\Delta| \times \text{VAR}(dS) - \frac{1}{2}\Gamma \times \text{VAR}(dS)^2$

14.5 ANSWERS TO CHAPTER EXAMPLES

Example 14.1: FRM Exam 2006—Question 91

a. The delta of a long forward contract is $e^{-r^*\tau} = \exp(-0.10 \times 0.5) = 0.95$.

Example 14.2: FRM Exam 2004—Question 21

b. The option is ATM because the strike price is close to the spot price. This is a put, so the delta must be close to -0.5.

Example 14.3: FRM Exam 2006—Question 80

c. This is a call option, so delta must be positive. This is given by $\Delta = \exp(-r^*\tau)$ $N(d_1) = \exp(-0.01 \times 2) \times 0.64 = 0.63$.

Example 14.4: FRM Exam 2009—Question 4-27

d. From Figure 14.3, the delta is most sensitive, or gamma the highest, for ATM short-term options. Under the BS model, gamma is the same for calls and puts.

Example 14.5: FRM Exam 2001—Question 79

a. This is an at-the-money option with a delta of about 0.5. Since the bank sold calls, it needs to delta-hedge by buying the shares. With a delta of 0.54, it would need to buy approximately 50,000 shares. Answer a. is the closest. Note that most other information is superfluous.

Example 14.6: FRM Exam 2006—Question 106

a. Because gamma is negative, we need to buy a call to increase the portfolio gamma back to zero. The number is $600/1.5 = 400$ calls. This, however, will increase the delta from zero to $400 \times 0.75 = 300$. Hence, we must sell 300 shares to bring the delta back to zero. Note that positions in shares have zero gamma.

Example 14.7: FRM Exam 2009—Question 4-26

c. OTM call options are not very sensitive to dividends, as indicated in Figure 14.7, so answer a. is incorrect. This also shows that ITM options have the highest ρ^* in absolute value.

Example 14.8: FRM Exam 2004—Question 65

c. Theta is negative for long positions in ATM options, so a. is incorrect. Gamma is small for ITM options, so b. is incorrect. Delta of ITM puts tends to -1, so d. is incorrect.

Example 14.9: FRM Exam 2006—Question 33

d. Long gamma means that the portfolio is long options with high gamma, typically short-term (short-expiry) ATM options. Short vega means that the portfolio is short options with high vega, typically long-term (long-expiry) ATM options.

Example 14.10: FRM Exam 2006—Question 54

d. Vega is highest for ATM European options, so statement a. is correct. Delta is negative and moves to zero as S increases, so statement b. is correct. Gamma increases as the maturity of an ATM option decreases, so statement c. is correct. Theta is greater (in absolute value) for short-term ATM options, so statement d. is incorrect.

Example 14.11: Vega and Gamma

a. Long positions in options have positive gamma and vega. Gamma (or instability in delta) increases near maturity; vega decreases near maturity. So, to obtain positive gamma and negative vega, we need to buy short-maturity options and sell long-maturity options.

Example 14.12: Vega and Theta

a. Such a portfolio is short vega (volatility) and short theta (time). We need to implement a hedge that is delta-neutral and involves buying and selling options with different maturities. Long positions in short-dated options have high negative theta and low positive vega. Hedging can be achieved by selling short-term options and buying long-term options.

Example 14.13: FRM Exam 2005—Question 130

b. The linear VAR is derived from the worst move in the index value, which is $\alpha S\sigma\sqrt{T} = 1.645 \times 3{,}000(24\%/\sqrt{260})\sqrt{10} = 232.3$. Multiplying by the delta of 0.6 gives 139.

Example 14.14: FRM Exam 2009—Question 4-6

c. The delta must be around 0.5, which implies a linear VAR of $\$100{,}000 \times 10.4\% \times 0.5 = \$5{,}200$. The position is long an option and has positive gamma. As a result, the quadratic VAR must be lower than \$5,200.

PART Five

Market Risk Management

CHAPTER 15

Advanced Risk Models: Univariate

We now turn to more advanced risk models. First, we consider univariate risk models. Multivariate models are presented in the next chapter.

This chapter covers improvements to traditional risk models. In practice, the implementation of risk models for large institutions involves many shortcuts, simplifications, and judgment calls. The role of the risk manager is to design a system that provides a reasonable approximation to the risk of the portfolio with acceptable speed and cost. The question is how to judge whether accuracy is reasonable.

This is why risk models must always be complemented by a backtesting procedure. This involves systematically comparing the risk forecast with the subsequent outcome. The framework for backtesting is presented in Section 15.1.

Next, we examine a method to improve the estimation of the tail quantile beyond the traditional historical-simulation and delta-normal methods, which can be defined as nonparametric and parametric, respectively. Section 15.2 turns to **extreme value theory** (EVT), which can be used to fit an analytical distribution to the left tail. This method, which can be described as nonparametric, gives more precise value at risk (VAR) estimates. In addition, the analytical function can be used to extrapolate VAR to other confidence levels.

Finally, Section 15.3 considers properties for risk measure. A risk measure that satisfies the selected properties is called coherent. It shows that, in some cases, VAR is not coherent. Expected shortfall, however, satisfies this property.

15.1 BACKTESTING

Any risk model should be checked for consistency with reality. **Backtesting** is a process to compare systematically the VAR forecasts with actual returns. This process is uniquely informative for risk managers. It should detect weaknesses in the models and point to areas for improvement.

Backtesting is also important because it is one of the reasons that bank regulators allow banks to use their internal risk measures to determine the amount of regulatory capital required to support their trading portfolios. Thus this section

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FRM Exam Part 2 topic.

also covers the backtesting framework imposed by the Basel Committee in the Market Risk Amendment to the Basel I Capital Accord.[1]

Backtesting compares the daily VAR forecast with the realized profit and loss (P&L) the next day. If the actual loss is worse than the VAR, the event is recorded as an **exception**. The risk manager then counts the number of exceptions x over a window with T observations.

15.1.1 Measuring Exceptions

But first, we have to define the **trading outcome**. One definition is the profit or loss from the **actual portfolio** over the next day. This return, however, does not exactly correspond to the previous day's VAR. All VAR measures assume a *frozen* portfolio from the close of a trading day to the next, and ignore fee income. In practice, trading portfolios do change. Intraday trading will generally increase risk. Fee income is more stable and decreases risk. Although these effects may offset each other, the actual portfolio may have more or less volatility than predicted by VAR.

This is why it is recommended that **hypothetical portfolios** be constructed so as to match the VAR measure exactly. Their returns are obtained from fixed positions applied to the actual price changes on all securities, measured from close to close.

The Basel framework recommends using both hypothetical and actual trading outcomes in backtests. The two approaches are likely to provide complementary information on the quality of the risk management system. For instance, suppose the backtest fails using the actual but not the hypothetical portfolio. This indicates that the model is sound but that actual trading increases volatility. Conversely, if the backtest fails using the hypothetical model, the conclusion should be that the risk model is flawed.

15.1.2 Binomial Distribution

Consider a VAR measure over a daily horizon defined at the 99% level of confidence c. This implies a probability of $p = 1 - c = 1\%$ for daily exception. The window for backtesting is $T = 250$ days.

The number of exceptions is a random variable X, which is the result of T independent **Bernoulli trials**, where each trial results in an outcome of $y = 0$ or $y = 1$, pass or fail. As a result, X has a **binomial distribution**, with density

$$f(x) = \binom{T}{x} p^x (1-p)^{T-x}, \quad x = 0, 1, \ldots, n \tag{15.1}$$

where $\binom{T}{x}$ is the number of combinations of T things taken x at a time, or

$$\binom{T}{x} = \frac{T!}{x!(T-x)!} \tag{15.2}$$

[1] Basel Committee on Banking Supervision, *Supervisory Framework for the Use of "Backtesting" in Conjunction with the Internal Models Approach to Market Risk Capital Requirements* (Basel: Bank for International Settlements, 1996).

and the parameter p is the probability of an exception, and is between zero and one. The binomial variable has mean and variance $E[X] = pT$ and $V[X] = p(1-p)T$.

For instance, we want to know what is the probability of observing $x = 0$ exceptions out of a sample of $T = 250$ observations when the true probability is 1%. We should expect to observe $p \times T = 2.5$ exceptions on average across many such samples. There will be, however, some samples with no exceptions at all simply due to luck. This probability is

$$f(X=0) = \frac{T!}{x!(T-x)!} p^x (1-p)^{T-x} = \frac{250!}{1 \times 250!} 0.01^0 0.99^{250} = 0.081$$

So, we would expect to observe 8.1% of samples with zero exceptions under the null hypothesis. We can repeat this calculation with different values for x. For example, the probability of observing eight exceptions is $f(X=8) = \binom{250}{8} 0.01^8 (0.99)^{242} = 0.02\%$. Because this probability is so low, this outcome should raise questions as to whether the true probability is 1%.

EXAMPLE 15.1: FRM EXAM 2003—QUESTION 11

Based on a 90% confidence level, how many exceptions in backtesting a VAR would be expected over a 250-day trading year?

a. 10
b. 15
c. 25
d. 50

EXAMPLE 15.2: FRM EXAM 2007—QUESTION 101

A large, international bank has a trading book whose size depends on the opportunities perceived by its traders. The market risk manager estimates the one-day VAR, at the 95% confidence level, to be USD 50 million. You are asked to evaluate how good a job the manager is doing in estimating the one-day VAR. Which of the following would be the most convincing evidence that the manager is doing a poor job, assuming that losses are identical and independently distributed (i.i.d.)?

a. Over the past 250 days, there are eight exceptions.
b. Over the past 250 days, the largest loss is USD 500 million.
c. Over the past 250 days, the mean loss is USD 60 million.
d. Over the past 250 days, there is no exception.

15.1.3 Normal Approximation

When T is large, we can use the central limit theorem and approximate the binomial distribution by the normal distribution

$$z = \frac{x - pT}{\sqrt{p(1-p)T}} \approx N(0, 1) \tag{15.3}$$

which provides a convenient shortcut. If the decision rule is defined at the two-tailed 95 percent test confidence level, then the cutoff value of $|z|$ is 1.96.

For instance, the z-value for observing eight exceptions is $z = (8 - 2.5)/1.573 = 3.50$, which is very high. So, it is unlikely that a well-calibrated model with a confidence level of 99% would produce eight exceptions.

15.1.4 Decision Rule for Backtests

On average, we would expect 1% of 250, or 2.5 instances of exceptions over the past year. Too many exceptions indicate that either the model is understating VAR or the trader is unlucky. How do we decide which explanation is more likely?

Such statistical testing framework must account for two types of errors:

- **Type 1 errors**, which describe the probability of rejecting a correct model, due to bad luck
- **Type 2 errors**, which describe the probability of not rejecting a model that is false

Ideally, one would want to create a decision rule that has low type 1 and type 2 error rates. In practice, one has to trade off one type of error against the other. Most statistical tests fix the type 1 error rate, say at 5%, and structure the test so as to minimize the type 2 error rate, or to maximize the test's power.[2]

Table 15.1 shows how the cumulative distribution can be used to compute a cutoff value for the number of exceptions as a function of the type 1 error rate. For example, the cumulative probability of observing five exceptions or more is 10.78%. This is one minus the cumulative probability of observing up to four observations, which is 0.8922 from the middle column. Using this cutoff point will penalize VAR models that are correct in about 11% of the cases. A higher cutoff point would lower this type 1 error rate. Say, for example, that the decision rule is changed to reject after 10 or more exceptions. This will reduce the type 1 error rate from 10.78% to 0.03%. So, we will almost never reject models that are correct.

However, this will make it more likely that we will miss VAR models that are misspecified. Suppose, for example, that the trader tries to willfully understate

[2] The **power of a test** is also one minus the type 2 error rate.

TABLE 15.1 Distribution for Number of Exceptions ($T = 250,\ p = 0.01$)

Number of Exceptions	Probability	Cumulative Probability	Type 1 Error Rate
0	0.0811	0.0811	100.00%
1	0.2047	0.2858	91.89%
2	0.2574	0.5432	71.42%
3	0.2150	0.7581	45.68%
4	0.1341	0.8922	24.19%
5	0.0666	0.9588	10.78%
6	0.0275	0.9863	04.12%
7	0.0097	0.9960	01.37%
8	0.0030	0.9989	00.40%
9	0.0008	0.9998	00.10%
10	0.0002	0.9999	00.03%

VAR, using a confidence level of 96% instead of 99%. This leads to an expected number of exceptions of $4\% \times 250 = 10$. As a result, the type 2 error rate is very high, around 50%. So, this cheating trader will not be easy to catch.

15.1.5 The Basel Rules for Backtests

The Basel Committee put in place a framework based on the daily backtesting of VAR. Having up to four exceptions is acceptable, which defines a green zone. If the number of exceptions is five or more, the bank falls into a yellow or red zone and incurs a progressive penalty, which is enforced with a higher capital charge. Roughly, the capital charge is expressed as a multiplier of the 10-day VAR at the 99% level of confidence. The normal multiplier k is 3. After an incursion into the yellow zone, the multiplicative factor, k, is increased from 3 to 4, or by a **plus factor** described in Table 15.2.

An incursion into the red zone generates an *automatic*, nondiscretionary penalty. This is because it would be extremely unlikely to observe 10 or more exceptions if the model was indeed correct.

TABLE 15.2 The Basel Penalty Zones

Zone	Number of Exceptions	Potential Increase in k
Green	0 to 4	0.00
Yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
Red	≥ 10	1.00

If the number of exceptions falls within the yellow zone, the supervisor has discretion to apply a penalty, depending on the causes for the exceptions. The Basel Committee uses these categories:

- *Basic integrity of the model:* The deviation occurred because the positions were incorrectly reported or because of an error in the program code. This is a very serious flaw. In this case, a penalty should apply and corrective action should be taken.
- *Deficient model accuracy:* The deviation occurred because the model does not measure risk with enough precision (e.g., does not have enough risk factors). This is a serious flaw, too. A penalty should apply and the model should be reviewed.
- *Intraday trading:* Positions changed during the day. Here, a penalty "should be considered." If the exception disappears with the hypothetical return, the problem is not in the bank's VAR model.
- *Bad luck:* Markets were particularly volatile or correlations changed. These exceptions "should be expected to occur at least some of the time" and may not suggest a deficiency of the model but simply bad luck.

15.1.6 Evaluation of Backtesting

Finally, we should note that exception tests focus only on the frequency of occurrences of exceptions. This is consistent with the idea that VAR is simply a quantile. The counting approach, however, totally ignores the size of losses. This is a general weakness of VAR-based risk measures, which can be remedied with conditional value at risk (CVAR), which is discussed in a later section.

Another issue with the traditional backtesting approach is that it ignores the time pattern of losses. Ideally, exceptions should occur uniformly over the period. In contrast, a pattern where exceptions tend to *bunch* over a short period indicates a weakness of the risk measure. In response, the risk model should take into account time variation in risk, using, for example, the generalized autoregressive conditional heteroskedastic (GARCH) model explained in Chapter 5. For example, even if a backtest produces only four exceptions over the past year, which passes the Basel requirements, the fact that these four exceptions occurred in the last month should cause concern, because it is more likely that the portfolio will suffer large losses over the coming days.

EXAMPLE 15.3: FRM EXAM 2002—QUESTION 20

Which of the following procedures is essential in validating the VAR estimates?

a. Stress-testing
b. Scenario analysis
c. Backtesting
d. Once approved by regulators, no further validation is required.

EXAMPLE 15.4: PENALTY ZONES

The Market Risk Amendment to the Basel Capital Accord defines the yellow zone as the following range of exceptions out of 250 observations:

a. 3 to 7
b. 5 to 9
c. 6 to 9
d. 6 to 10

EXAMPLE 15.5: FRM EXAM 2002—QUESTION 23

Backtesting routinely compares daily profits and losses with model-generated risk measures to gauge the quality and accuracy of their risk measurement systems. The 1996 Market Risk Amendment describes the backtesting framework that is to accompany the internal models capital requirement. This backtesting framework involves

I. The size of outliers
II. The use of risk measure calibrated to a one-day holding period
III. The size of outliers for a risk measure calibrated to a 10-day holding period
IV. Number of outliers

a. II and III
b. II only
c. I and II
d. II and IV

EXAMPLE 15.6: FRM EXAM 2009—QUESTION 5-6

Tycoon Bank announced that there were eight days in the previous year for which losses exceeded the daily 99% VAR. As a result, concerns emerged about the accuracy of the VAR implementation. Assuming that there are 250 days in the year, which of the following statements is/are correct?

I. Using a two-tailed 99% confidence level z-score test, the current VAR implementation understates the actual risk in the bank's portfolio.
II. Using a two-tailed 99% confidence level z-score test, the current VAR implementation overstates the actual risk in the bank's portfolio.
III. The bank's exception rates for VAR may be inaccurate if the bank's portfolio changes incorporate the returns from low-risk but highly profitable intraday market making activities.
IV. If these eight exceptions all happened in the previous month, the model should be reexamined for faulty assumptions and invalid parameters.

a. I and III
b. I, III, and IV
c. III only
d. I, II, and IV

15.2 EXTREME VALUE THEORY

As we have seen in Chapter 12, VAR measures can be computed using either of two approaches. The first is *nonparametric* and relies on a simulation of recent history of returns. The second is *parametric* because it imposes an analytical density function for returns, such as the normal, which is summarized in a standard deviation (the parameter), from which VAR is computed. The nonparametric method is more general but not very powerful, which is usually the case when few assumptions are made. As a result, the VAR estimates can be very imprecise, due to the effect of sampling variation, especially at high confidence levels.

Instead, this section turns to a third method, which can be described as *semi-parametric*. **Extreme value theory** (EVT) can be used to fit an analytical distribution, but just to the left tail. This leads to more precise VAR estimates. In addition, the analytical function can be used to extrapolate VAR to other confidence levels.

15.2.1 EVT Distribution

EVT can be viewed as an extension of the central limit theorem, which states that the average of independent random variables tends to the normal distribution, irrespective of the original distribution. This deals with the mean, or center, of the distribution.

For risk management purposes, the tails of the distribution are of interest. The EVT theorem says that the limit distribution for values x beyond a cutoff point u belongs to the following family

$$\begin{aligned} F(y) &= 1 - (1 + \xi y)^{-1/\xi}, \ \xi \neq 0 \\ F(y) &= 1 - exp(-y), \ \xi = 0 \end{aligned} \tag{15.4}$$

where $y = (x - u)/\beta$. To simplify, we defined the loss x as a positive number so that y is also positive. The distribution is characterized by $\beta > 0$, a *scale* parameter, and by ξ, a *shape* parameter that determines the speed at which the tail disappears.

This approach is called **peaks over threshold** (POT), where u is the fixed threshold. Note that this is a limit theorem, which means that the EVT distribution is only asymptotically valid (i.e., as u grows large).

This distribution is called the **generalized Pareto** (GP) distribution because it subsumes other distributions as special cases. For instance, the normal distribution corresponds to $\xi = 0$, in which case the tails disappear at an exponential speed. Typical financial data have $\xi > 0$, which implies *fat tails*. This class of distribution includes the Gumbel, Fréchet, and Weibull families, as $\xi \to 0$, $\xi > 0$, and $\xi < 0$, respectively. Among these, the Fréchet distribution is most relevant for financial risk management because most financial risk factors have fatter tails than the normal. This family includes the Student's t distribution and the Pareto distribution.

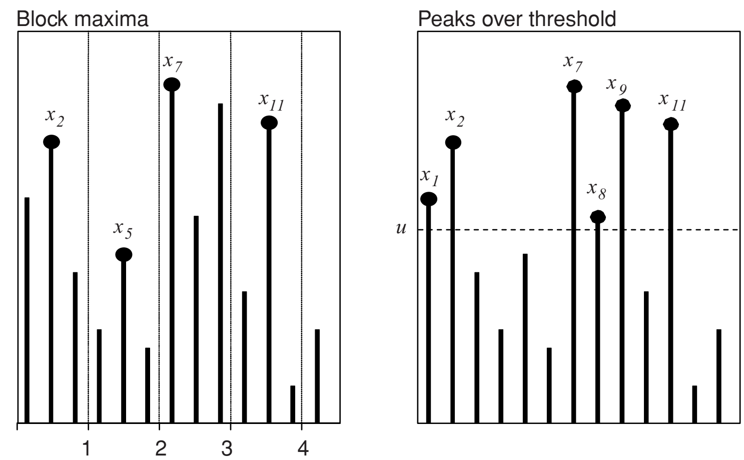

FIGURE 15.1 EVT Approaches

Another approach is the **block maxima**, where the sample is grouped into successive blocks, from which each maximum is identified. In this case, the limiting distribution of normalized maxima is the **generalized extreme value** (GEV) distribution.

The two approaches are compared in Figure 15.1. In the left panel, the selected observations are the maxima of groups of three. In the right panel, observations are selected whenever they are greater than u. The block maxima approach ignores extreme values x_9, x_1, and x_8, because they appear in a block with already one outlier.

In practice, the POT method is more widely used because it uses data more efficiently, even though it requires the choice of the threshold. It is better adapted to the risk measurement of tail losses because it focuses on the distribution of exceedances over a threshold.

Figure 15.2 illustrates the shape of the density function for U.S. stock market data using the GP distribution. The normal density falls off fairly quickly. With $\xi = 0.2$, the EVT density has a fatter tail than the normal density, implying a higher probability of experiencing large losses. This is an important observation for risk management purposes.

Note that the EVT density is only defined for the tail (i.e., when the loss x exceeds an arbitrary cutoff point, which is taken as 2 in this case). It says nothing about the rest of the distribution.

15.2.2 VAR and EVT

VAR, as well as CVAR, can be derived in closed-form solution from the analytical distribution in Equation (15.4). This requires estimation of the tail parameter ξ and of the dispersion parameter β.

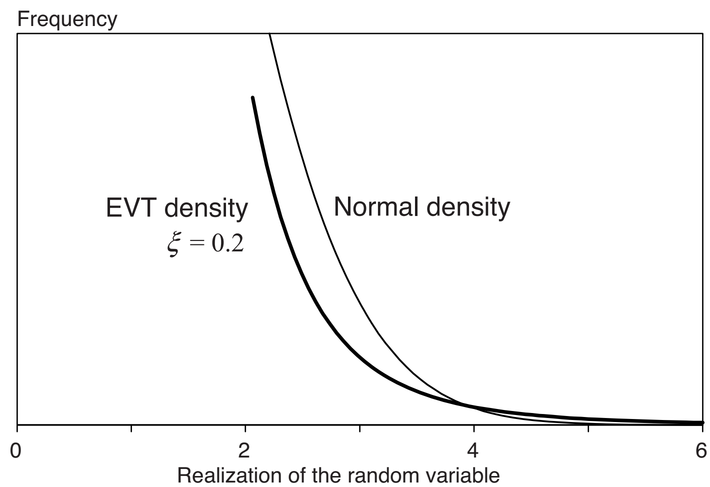

FIGURE 15.2 EVT and Normal Densities

This can be performed using a variety of statistical approaches. One method is **maximum likelihood**. First, we define a cutoff point u. This needs to be chosen so that there are a sufficient number of observations in the tail. However, the theory is most valid far into the tail. A good, ad hoc, choice is to choose u so as to include 5% of the data in the tail. For example, if we have $T = 1{,}000$ observations, we would consider only the 50 in the left tail. Second, we consider only losses beyond u and then maximize the likelihood of the observations over the two parameters ξ and β.

Another estimation method is the **method of moments**. This consists of fitting the parameters so that the GP moments equal the observed moments. This method is easier to implement but less efficient.

A third method, which is widely used, is **Hill's estimator**. The first step is to sort all observations from highest to lowest. The tail index is then estimated from

$$\widehat{\xi} = \frac{1}{k}\sum_{i=1}^{k} \ln X_i - \ln X_{k+1} \tag{15.5}$$

In other words, this is the average of the logarithm of observations from 1 to k minus the logarithm of the next observation. Unfortunately, there is no theory to help us choose k. In practice, one can plot ξ against k and choose the value in a flat area, where the estimator is not too sensitive to the choice of the cutoff point.

One issue with EVT is that it still relies on a small number of observations in the tail. Hence, estimates are sensitive to changes in the sample, albeit less so than with nonparametric VAR. More generally, EVT results also depend on the assumptions and estimation method. And in the end, it still relies on historical data, which may not give a complete picture of all financial risks.

EXAMPLE 15.7: FRM EXAM 2009—QUESTION 5-12

Extreme value theory (EVT) provides valuable insight about the tails of return distributions. Which of the following statements about EVT and its applications is *incorrect*?

a. The peaks over threshold (POT) approach requires the selection of a reasonable threshold, which then determines the number of observed exceedances; the threshold must be sufficiently high to apply the theory, but sufficiently low so that the number of observed exceedances is a reliable estimate.
b. EVT highlights that distributions justified by the central limit theorem (e.g., normal) can be used for extreme value estimation.
c. EVT estimates are subject to considerable model risk, and EVT results are often very sensitive to the precise assumptions made.
d. Because observed data in the tails of distribution is limited, EV estimates can be very sensitive to small sample effects and other biases.

EXAMPLE 15.8: FRM EXAM 2007—QUESTION 110

Which of the following statements regarding extreme value theory (EVT) is *incorrect*?

a. In contrast to conventional approaches for estimating VAR, EVT considers only the tail behavior of the distribution.
b. Conventional approaches for estimating VAR that assume that the distribution of returns follows a unique distribution for the entire range of values may fail to properly account for the fat tails of the distribution of returns.
c. EVT attempts to find the optimal point beyond which all values belong to the tail and then models the distribution of the tail separately.
d. By smoothing the tail of the distribution, EVT effectively ignores extreme events and losses that can generally be labeled outliers.

15.3 COHERENT RISK MEASURES

15.3.1 Desirable Properties for Risk Measures

The purpose of a risk measure is to summarize the entire distribution of dollar returns X by one number, $\rho(X)$. Artzner et al. (1999) list four desirable properties of risk measures for capital adequacy purposes:[3]

[3] See P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, "Coherent Measures of Risk," *Mathematical Finance* 9 (1999): 203–228.

1. **Monotonicity:** if $X_1 \leq X_2$, $\rho(X_1) \geq \rho(X_2)$.
   In other words, if a portfolio has systematically lower values than another, that is, in each state of the world, it must have greater risk.
2. **Translation invariance:** $\rho(X + k) = \rho(X) - k$.
   In other words, adding cash k to a portfolio should reduce its risk by k. This reduces the lowest portfolio value. As with X, k is measured in dollars.
3. **Homogeneity:** $\rho(bX) = b\rho(X)$.
   In other words, increasing the size of a portfolio by a factor b should scale its risk measure by the same factor b. This property applies to the standard deviation.
4. **Subadditivity:** $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.
   In other words, the risk of a portfolio must be less than, or at worst equal to, the sum of separate risks. If so, merging portfolios cannot increase risk.

The usefulness of these criteria is that they force us to think about ideal properties of risk measures and, more importantly, potential problems with simplified risk measures.

For instance, homogeneity seems reasonable in most cases. It is, however, questionable in the case of huge portfolios that could not be liquidated without substantial market impact. In this case, risk increases more than proportionately with the size of the portfolio. Thus, this property ignores liquidity risk. In practice, nearly all risk measures have this problem, which is very difficult to deal with.

Next, subadditivity implies that the risk of a portfolio must be less than the sum of risks for portfolio components. As we show in the next section, the quantile-based VAR measure fails to satisfy this property.

Assuming a normal distribution, however, the standard deviation–based VAR satisfies the subadditivity property. This is because the volatility of a portfolio is less than, or at worst equal to, the sum of volatilities: $\sigma(X_1 + X_2) \leq \sigma(X_1) + \sigma(X_2)$. More generally, subadditivity holds for **elliptical distributions**, for which contours of equal density are ellipsoids, such as the Student's t.

15.3.2 Example: VAR and Subadditivity

We now give an example where VAR fails to satisfy subadditivity. Consider a trader with an investment in a corporate bond with face value of $100,000 and default probability of 0.5%. Over the next period, we can either have no default with a return of zero or default with a loss of $100,000. The payoffs are thus −$100,000 with probability of 0.5% and +$0 with probability 99.5%. Since the probability of getting $0 is greater than 99%, the VAR at the 99% confidence level is $0, without taking the mean into account. This is consistent with the definition that VAR is the smallest loss such that the right-tail probability is at least 99%.

Now, consider a portfolio invested in three bonds (A, B, and C) with the same characteristics and independent payoffs. The VAR numbers add up to $\text{VAR}_S = \sum_i \text{VAR}_i = \$0$. We report the payoffs and probabilities in Table 15.3.

TABLE 15.3 Subadditivity and VAR: Example

State	Bonds	Probability	Payoff
No default		$0.995 \times 0.995 \times 0.995 = 0.9850749$	\$0
1 default	A, B, or C	$3 \times 0.005 \times 0.995 \times 0.995 = 0.0148504$	−\$100,000
2 defaults	AB, AC, or BC	$3 \times 0.005 \times 0.005 \times 0.995 = 0.0000746$	−\$200,000
3 defaults	A, B, and C	$0.005 \times 0.005 \times 0.005 = 0.0000001$	−\$300,000

Here, the probability of zero default is 0.9851, which is less than 99%. The portfolio VAR is therefore \$100,000, which is the lowest number such that the probability exceeds 99%. Thus the portfolio VAR is greater than the sum of individual VARs, which is zero. In this example, VAR is not subadditive. This is an undesirable property because it creates disincentives to aggregate the portfolio, since it appears to have higher risk.

Admittedly, this example is a bit contrived. Nevertheless, it illustrates the danger of focusing on VAR as a sole measure of risk.

15.3.3 Expected Shortfall

In contrast, **conditional VAR** (CVAR), also called **expected shortfall** or **expected tail loss**, does satisfy the subadditivity property. CVAR is the average of losses beyond VAR; $\text{CVAR} = E[-X \mid X < -\text{VAR}]$.

For each individual bond, there is only one observation in the tail, which leads to $\text{CVAR}_i = \$100{,}000$. The sum is $\text{CVAR}_S = \$300{,}000$. We now compute the CVAR for the portfolio. This is the probability-weighted average of losses worse than \$100,000, or $(0.0000746 \times \$200{,}000 + 0.0000001 \times \$300{,}000)/0.0000747 = \$200{,}167$. This is less than CVAR_S, hence showing that CVAR is a subadditive risk measure.

In addition, CVAR is better justified than VAR in terms of decision theory. Suppose an investor has to choose between two portfolios A and B with different distributions. Decision rules can be based on various definitions of stochastic dominance. One example is **first-order stochastic dominance** (FSD). This requires that the cumulative distribution function for A be systematically lower than that for B. So, B has a higher probability of a bad outcome. This is a very strict rule, however. Another example is **second-order stochastic dominance** (SSD). Portfolio A would dominate B if it has higher mean and lower risk. This is a more realistic rule than the first. Using CVAR as a risk measure is consistent with SSD, whereas VAR requires FSD, which is less realistic.

In practice, CVAR is rarely reported in the financial industry. It is more commonly used in the insurance industry, which has had traditionally greater focus on tail losses. In addition, statistical distributions for mortality rates and natural catastrophe events have long histories and are more amenable to expected shortfall analysis.

Even so, risk managers need to be aware of the shortcoming of summarizing an entire distribution with one number such as VAR. Traders might decide to create a portfolio with low VAR but very high CVAR, creating infrequent but very large losses. This is an issue with asymmetrical positions, such as short positions in options or undiversified portfolios exposed to credit risk. The next chapter gives an example of billions of losses suffered from senior tranches of collateralized debt obligations backed by subprime mortgages. Such senior tranches are similar to short positions in out-of-the-money options, which involve rare but catastrophic losses.

EXAMPLE 15.9: FRM EXAM 2008—QUESTION 2-25

A market risk manager uses historical information on 1,000 days of profit/loss information to calculate a daily VAR at the 99th percentile, or \$8 million. Loss observations beyond the 99th percentile are then used to estimate the conditional VAR. If the losses beyond the VAR level, in millions, are \$9, \$10, \$11, \$13, \$15, \$18, \$21, \$24, and \$32, then what is the CVAR?

a. \$9 million
b. \$32 million
c. \$15 million
d. \$17 million

EXAMPLE 15.10: FRM EXAM 2009—QUESTION 5-8

Greg Lawrence is a risk analyst at ES Bank. After estimating the 99%, one-day VAR of the bank's portfolio using historical simulation with 1,200 past days, he is concerned that the VAR measure is not providing enough information about tail losses. He decides to reexamine the simulation results. Sorting the simulated daily P&L from worst to best gives the following results:

Rank	1	2	3	4	5	6
P&L	−2,833	−2,333	−2,228	−2,084	−1,960	−1,751
Rank	7	8	9	10	11	12
P&L	−1,679	−1,558	−1,542	−1,484	−1,450	−1,428
Rank	13	14	15			
P&L	−1,368	−1,347	−1,319			

What is the 99%, one-day expected shortfall (ES) of the portfolio?

a. USD 433
b. USD 1,428
c. USD 1,861
d. USD 2,259

EXAMPLE 15.11: FRM EXAM 2009—QUESTION 5-14

Which of the following statements about expected shortfall (ES) is *incorrect*?

a. ES provides a consistent risk measure across different positions and takes account of correlations.
b. ES tells what to expect in bad states: It gives an idea of how bad the portfolio payoff can be expected to be if the portfolio has a bad outcome.
c. ES-based rule is consistent with expected utility maximization if risks are ranked by a second-order stochastic dominance rule.
d. Like VAR, ES does not always satisfy subadditivity (i.e., the risk of a portfolio must be less than or equal to the sum of the risks of its individual positions).

15.4 IMPORTANT FORMULAS

Expected number of exceptions in a sample of size T with VAR at confidence level $c = 1 - p$: $E[X] = p \times T$

Distribution of exceptions: $f(x) = \binom{T}{x} p^x (1-p)^{T-x}, \quad x = 0, 1, \ldots, n$

Basel rules for number n exceptions with $T = 252$, and $c = 99\%$:

Green zone: $0 \leq n \leq 4$

Yellow zone: $5 \leq n \leq 9$

Red zone: $10 \leq n$

Distribution of tails (EVT): $y = (x - u)/\beta \rightarrow$ generalized Pareto distribution

15.5 ANSWERS TO CHAPTER EXAMPLES

Example 15.1: FRM Exam 2003—Question 11

c. This is $p \times T = 10\% \times 250 = 25$.

Example 15.2: FRM Exam 2007—Question 101

d. We should expect $(1 - 95\%)250 = 12.5$ exceptions on average. Having eight exceptions is too few, but the difference could be due to luck. Having zero exceptions, however, would be very unusual, with a probability of $1 - (1 - 5\%)^{250}$, which is very low. This means that the risk manager is providing VAR estimates that are much too high. Otherwise, the largest or mean losses are not directly useful without more information on the distribution of profits.

Example 15.3: FRM Exam 2002—Question 20

c. VAR estimates need to be compared to actual P&L results to be validated, which is called backtesting.

Example 15.4: Penalty Zones

b. See Table 15.2.

Example 15.5: FRM Exam 2002—Question 23

d. The backtesting framework in the IMA only counts the number of times a daily exception occurs (i.e., a loss worse than VAR). So, this involves the number of outliers and the daily VAR measure.

Example 15.6: FRM Exam 2009—Question 5-6

b. The z-score gives $(8 - 2.5)/\sqrt{250 \times 0.01 \times 0.99} = 3.5$. This is too high (greater than 2), which leads to rejection of the null that the VAR model is well calibrated. Hence, VAR is too low and statement I. is correct. Statement II. is incorrect. However, this may be due to intraday trading, so III. is correct, too. Finally, if all eight exceptions occurred in the last month, there is bunching, and the model should be reexamined, so IV. is correct.

Example 15.7: FRM Exam 2009—Question 5-12

b. EVT estimates are subject to estimation risk, so statement c. and d. are correct. However, EVT does not apply the central limit theorem (CLT), which states that the average (as opposed to the tail) of i.i.d. random variables is normal.

Example 15.8: FRM Exam 2007—Question 110

d. EVT uses only information in the tail, so statement a. is correct. Conventional approaches such as delta-normal VAR assume a fixed probability density function (p.d.f.) for the entire distribution, which may understate the extent of fat tails, so statement b. is correct. The first step in EVT is to choose a cutoff point for the tail, and then to estimate the parameters of the tail distribution, so statement c. is correct. Finally, EVT does not ignore extreme events (as long as they are in the sample).

Example 15.9: FRM Exam 2008—Question 2-25

d. CVAR is the average of observations beyond VAR. This gives \$17 million. Answers a. and b. can be dismissed out of hand because they are too low and too high, respectively.

Example 15.10: FRM Exam 2009—Question 5-8

c. This looks like a computationally intensive question, but it can be answered using judgment. The 1% left tail for $T = 1{,}200$ is 12 observations, so VAR $= 1{,}428$. This rules out answers a. and b. The ES is then the average from observations 1 to 11. Using simple rank, the point in the middle is for observation 6, which is $-1{,}751$. The closest is 1,861, or answer c.

Example 15.11: FRM Exam 2009—Question 5-14

d. ES, like VAR, does provide a consistent measure of risk that takes diversification into account, so statement a. is correct. Unlike VAR, however, CVAR is a subadditive risk measure.

CHAPTER 16

Advanced Risk Models: Multivariate

We now turn to the measurement of risk across large portfolios. A risk system has three components: (1) a portfolio position system, (2) a risk factor modeling system, and (3) an aggregation system. The first component is described in Section 16.1. Portfolio positions must be collected and then processed through **mapping**, which consists of replacing each instrument by its exposures on selected risk factors. Mapping considerably simplifies the risk measurement process. It would not be feasible to model all instruments individually, because there are too many. The art of risk management consists of choosing a set of limited risk factors that will adequately cover the spectrum of risks for the portfolio at hand.

The second step is to describe the joint movements in the risk factors. Two approaches are possible. The first specifies an analytical distribution, for example, a normal joint distribution. More generally, the joint movements can be characterized by copulas, which are described in Section 16.2. In a second approach, the joint distribution could be simply taken from empirical observations, without making any additional assumptions.

The third step brings together positions and risk factors. Section 16.3 describes the three main value at risk (VAR) methods, which include the delta-normal method, the historical simulation (HS) method, and the Monte Carlo simulation method. The methods are illustrated with an example in Section 16.5.

A question of particular interest is the performance of VAR models during the recent credit crisis. This is discussed in Section 16.4, which also expands on general drawbacks of risk models.

16.1 RISK MAPPING

16.1.1 Risk Simplification

The fundamental idea behind modern risk measurement methods is to aggregate the portfolio risk at the highest level. In practice, it is often too complex to model

FRM Exam Part 2 topic.

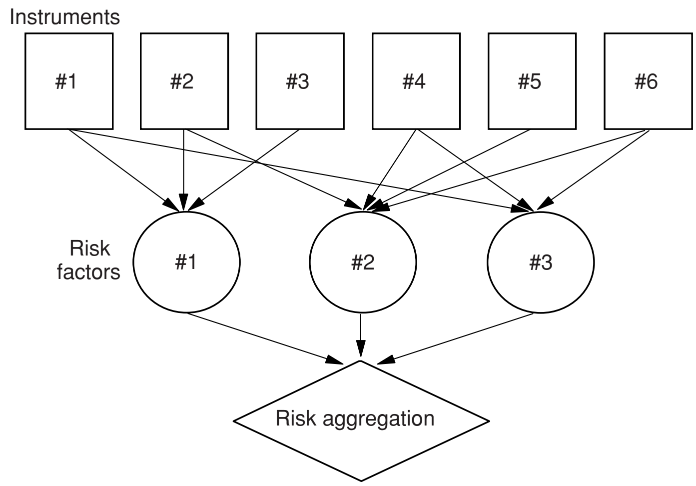

FIGURE 16.1 Mapping Approach

each position individually. Instead, some simplification is required. This is the role of **risk mapping**, which replaces exposures to individual positions by aggregate exposures to major risk factors.

Of course, this idea is not new. Consider, for example, a portfolio with many stock positions. William Sharpe devised a method to simplify the measurement of risk for these portfolios. His **diagonal model** decomposes individual stock return movements into a common index component and an idiosyncratic component. In large, well-diversified portfolios, the idiosyncratic component washes out, leaving the common component as the main driver of risk. This justifies mapping individual stocks to positions on the same index.

More generally, this methodology can be used in any market. Figure 16.1 illustrates the mapping process in a case with six instruments, say different forward contracts on the same currency but with different maturities. The risk manager judges that these positions can be replaced by exposures on three risk factors only. We give a fully worked-out example later in this chapter.

There are three steps in the process:

1. Replace each of the $N = 6$ positions with a $K = 3$ exposure on the risk factors. Define $x_{i,k}$ as the exposure of instrument i to risk factor k.
2. Aggregate the K exposures across the positions in the portfolio, $x_k = \sum_{i=1}^{N} x_{i,k}$. This generates $K = 3$ values for dollar exposures.
3. Derive the distribution of the portfolio return $R_{p,t+1}$ from the exposures and movements in risk factors, Δf, using one of the three VAR methods.

16.1.2 Mapping with Factor Models

The diagonal model is a simplified form of a factor model with one factor only, selected as a passive market index. This model starts with a statistical decomposition of the return on stock i into a marketwide return and a residual term, sometimes called idiosyncratic or specific. We decompose the return on stock i, R_i, into (1) a constant, (2) a component due to the market, R_M, and (3) a residual term:

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \tag{16.1}$$

where β_i is called systematic risk of stock i. Note that the residual is uncorrelated with R_M by assumption. The diagonal model adds the assumption that all specific risks are uncorrelated. Hence, any correlation across two stocks must come from the joint effect of the market.

The contribution of William Sharpe was to show that equilibrium in capital markets imposes restrictions on the α_i. For risk managers, however, the intercept is not the primary focus and is neglected in what follows. Instead, this model simplifies the risk measurement process.

Consider a portfolio that consists of simple, linear positions w_i on the various assets. We have

$$R_p = \sum_{i=1}^{N} w_i R_i \tag{16.2}$$

Using Equation (16.1), the portfolio return is also

$$R_p = \sum_{i=1}^{N} (w_i \beta_i R_M + w_i \epsilon_i) = \beta_p R_M + \sum_{i=1}^{N} (w_i \epsilon_i) \tag{16.3}$$

where the weighted average beta is

$$\beta_p = \sum_{i=1}^{N} w_i \beta_i \tag{16.4}$$

The portfolio variance is

$$V[R_p] = \beta_p^2 V[R_M] + \sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i] \right) \tag{16.5}$$

since all the residual terms are uncorrelated. Suppose that, for simplicity, the portfolio is equally weighted, $w_i = w = 1/N$, and that the residual variances are all the same, $V[\epsilon_i] = V$. As the number of assets increases, the second term tends to

$$\sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i] \right) \rightarrow N \times [(1/N)^2 V] = (V/N)$$

which should vanish as N increases. In this situation, the only remaining risk is the general market risk, consisting of the beta squared times the variance of the market:

$$V[R_p] \rightarrow \beta_p^2 V[R_M]$$

So, this justifies ignoring specific risk in large, well-diversified portfolios. The mapping approach replaces a dollar amount of x_i in stock i by a dollar amount of $x_i \beta_i$ on the index:

$$x_i \text{ on stock } i \rightarrow (x_i \beta_i) \text{ on index} \tag{16.6}$$

More generally, this approach can be expanded to multiple factors. The appendix at the end of this chapter shows how this approach can be used to build a covariance matrix from general market factors. Each security is first mapped on the selected risk factors. Exposures are then added up across the entire portfolio, for which risk is aggregated at the top level.

This mapping approach is particularly useful when there is no history of returns for some positions. Instead, the positions can be mapped on selected risk factors. Consider, for example, a stock that just went through an **initial public offering** (IPO). This stock, has no history. Such stocks can be quite risky. A practical solution is to map this position on a stock index with similar characteristics (e.g., an index of small-cap, high-tech stocks).

16.1.3 Mapping Fixed-Income Portfolios

As another important example of portfolio simplification, we turn to the analysis of a risk-free bond portfolio. This portfolio will have different payments coming due at different points in time, ranging from the next day to 30 years from now. It would be impractical to model all these maturities individually. Instead, simplifications are used for the risk-free term structure.

One simplification is **maturity mapping**, which replaces the current value of each bond by a position on a risk factor with the same maturity. This ignores intervening cash flows, however. A better approach is **duration mapping**, which maps the bond on a zero-coupon risk factor with a maturity equal to the duration of the bond. A third approach, which is even more precise but more complex, is **cash flow** (CF) **mapping**, which maps the current value of each bond payment on a zero-coupon risk factor with maturity equal to the time to wait for each cash flow.

Consider now another example, which is a corporate bond portfolio. This increases the dimensionality of risk factors into both risk-free term structure factors and credit factors. Again, it would be impractical to try to model all securities individually. There may not be sufficient price history on each bond. In addition, the history may not be relevant if it does not account for the probability of default. More generally, the history may not represent the current credit rating nor the duration of this bond.

In what follows, we assume that the risk manager uses duration mapping and takes movements in yield curves rather than prices as risk factors. The risk manager judges that risk factors can be restricted to a set of J Treasury zero-coupon rates, z_j, and of K credit spreads, s_k, sorted by credit rating. Ideally, these should be sufficient to provide a good approximation of the risk of the portfolio.

We then model the movement in each corporate bond yield y_i by a movement in the Treasury factor z_j at the closest maturity and in the credit rating s_k class to which it belongs. Alternatively, we can interpolate. The remaining component is ϵ_i, which is assumed to be independent across i. We have $y_i = z_j + s_k + \epsilon_i$. This decomposition is illustrated in Figure 16.2 for a corporate bond rated BBB with a 20-year maturity.

The movement in the value of bond price i is

$$\Delta P_i = -\text{DVBP}_i \Delta y_i = -\text{DVBP}_i\, \Delta z_j - \text{DVBP}_i\, \Delta s_k - \text{DVBP}_i\, \Delta \epsilon_i \tag{16.7}$$

where DVBP is the total dollar value of a basis point for the associated risk factor, which is directly derived from modified duration times the bond value.

We now aggregate the exposures across N positions. We hold n_i units of each bond,

$$P = \sum_{i=1}^{N} n_i P_i \tag{16.8}$$

which gives a price change of

$$\Delta P = \sum_{i=1}^{N} n_i \Delta P_i = -\sum_{i=1}^{N} n_i \text{DVBP}_i \Delta y_i \tag{16.9}$$

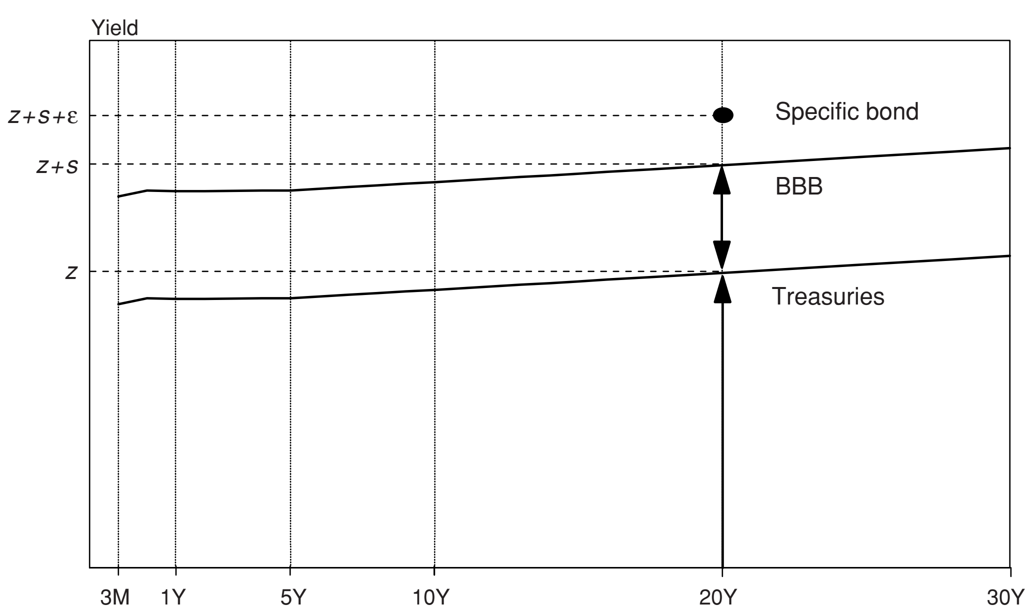

FIGURE 16.2 Yield Decomposition

Using the risk factor decomposition, the portfolio price movement is

$$\Delta P = -\sum_{j=1}^{J} \mathrm{DVBP}_j^z \, \Delta z_j - \sum_{k=1}^{K} \mathrm{DVBP}_k^s \, \Delta s_k - \sum_{i=1}^{N} n_i \mathrm{DVBP}_i \, \Delta \epsilon_i \qquad (16.10)$$

where DVBP_j^z results from the summation of $n_i\mathrm{DVBP}_i$ for all bonds that are exposed to the jth maturity, and likewise for DVBP_k^s. As in Equation (16.5) for equity portfolios, the total variance can be decomposed into

$$V[\Delta P] = \text{General Risk} + \sum_{i=1}^{N} n_i^2 \mathrm{DVBP}_i^2 \, V[\Delta \epsilon_i] \qquad (16.11)$$

If the portfolio is well diversified, the general risk term should dominate. So, we could simply ignore the second term. Ignoring specific risk, a portfolio composed of thousands of securities can be characterized by its exposure to just a few government maturities and credit spreads. This is a considerable simplification.

The mapping approach replaces a dollar amount of x_i in bond i by dollar exposures on two risk factors:

$$x_i \text{ on bond } i \rightarrow (n_i \mathrm{DVBP}_i) \text{ on yield}_j + (n_i \mathrm{DVBP}_i) \text{ on spread}_k \qquad (16.12)$$

16.1.4 Mapping: Choice of Risk Factors

The choice of risk factors should be driven by the nature of the portfolio. A diagonal risk model may be sufficient for portfolios of stocks that have many small positions well dispersed across sectors. For a portfolio with a small number of stocks concentrated in one sector, however, this approach will underestimate risk. Similarly, an equity market-neutral portfolio, which consists of long and short equity positions with essentially zero beta, will appear as having no risk, which is not the case.

A simple mapping approach on one interest rate risk factor may be perfectly adequate for a long-only portfolio. Essentially, all the risk exposures are summarized in one number, which is the dollar duration.

Consider next a trading portfolio where the portfolio manager is both long and short various bonds and has a net duration of zero. In this case, the duration model gives a total risk of zero, which is misleading. More factors are needed. Another example is a portfolio that has long and short positions in various options, across strike prices and maturities. In this case, simply considering the linear or even quadratic exposure to the underlying risk factor may not be sufficient. The risk manager should add movements in implied volatilities. As markets evolve toward more complex financial products, the risk manager has to make sure that the risk models do not lag behind and miss major risks.

KEY CONCEPT

The number of risk factors and the complexity of risk models depend on the depth of the trading strategies. In general, complex portfolios require more complex risk models.

EXAMPLE 16.1: FRM EXAM 2009—QUESTION 2-7

Which of these statements regarding risk factor mapping approaches is/are correct?

I. Under the cash flow (CF) mapping approach, only the risk associated with the average maturity of a fixed-income portfolio is mapped.
II. Cash flow mapping is the least precise method of risk mapping for a fixed-income portfolio.
III. Under the duration mapping approach, the risk of a bond is mapped to a zero-coupon bond of the same duration.
IV. Using more risk factors generally leads to better risk measurement but also requires more time to be devoted to the modeling process and risk computation.

a. I and II
b. I, III, and IV
c. III and IV
d. IV only

EXAMPLE 16.2: FRM EXAM 2002—QUESTION 44

The historical simulation (HS) approach is based on the empirical distributions and a large number of risk factors. The RiskMetrics approach assumes normal distributions and uses mapping on equity indices. The HS approach is more likely to provide an accurate estimate of VAR than the RiskMetrics approach for a portfolio that consists of

a. A small number of emerging market securities
b. A small number of broad market indices
c. A large number of emerging market securities
d. A large number of broad market indices

EXAMPLE 16.3: FRM EXAM 2007—QUESTION 11

A hedge fund manager has to choose a risk model for a large equity market-neutral portfolio, which has zero beta. Many of the stocks held are recent IPOs. Among the following alternatives, the best is

a. A single index model with no specific risk, estimated over the last year
b. A diagonal index model with idiosyncratic risk, estimated over the last year
c. A model that maps positions on industry and style factors
d. A full covariance matrix model using a very short window

16.2 JOINT DISTRIBUTIONS OF RISK FACTORS

The second component in risk systems is setting up the joint distribution of risk factors. One approach is nonparametric and consists of using recent observations. The other approach is parametric and requires specifying an analytical function for the joint distribution as well as its parameters. This section introduces the concept of the copula, which is central to the joint distribution of risk factors.

Copulas are used extensively for modeling financial instruments such as **collateralized debt obligations** (CDOs). CDOs are pools of N debt obligations, the value of each of which depends on the creditworthiness of the borrower. The risk of each credit taken one at a time can be described by a marginal distribution. Ultimately, however, what matters is the distribution of losses on the total portfolio. Hence, the financial engineering of CDOs requires modeling the joint movements in the individual credits. For portfolios of traded assets, the joint distributions can be assessed from historical data. Credit portfolios, however, do not have this luxury because the current credits in the portfolio do not have a history of defaults. Hence constructing the distribution of losses on a CDO must rely on a parametric approach, which requires the specification of a copula. Copulas are also used for enterprise-wide risk measurement, to aggregate market risk, credit risk, and operational risk.

16.2.1 Marginal Densities and Distributions

Marginal distributions consider each risk factor in isolation. This reduced dimensionality makes it easier to model the risk factor. In contrast, joint distributions are much more complex because of the higher dimensionality, which requires many more parameters to estimate. This creates serious difficulties, which in practice have caused major losses in financial markets, as we will see in a later section.

From Chapter 2, recall that a probability density function $f(u)$ describes the probability of observing a value around u. A normal density, for example, has the familiar bell-shaped curve. In contrast, a **distribution function** is the cumulative density $F(x) = \int_{-\infty}^{x} f(u)du$, and is denoted as an upper case. The value of this function is always between zero and one: $0 \leq F(x) \leq 1$.

Consider now a simple case with two risk factors. The question is how to link marginal densities, or distributions, across these risk factors.

16.2.2 Copulas

When the two variables are independent, the joint density is simply the product of the marginal densities. It is rarely the case, however, that financial variables are independent. Dependencies can be modeled by a function called the **copula**, which links, or attaches, marginal distributions into a joint distribution.

Formally, the copula is a function of the values of the marginal distributions $F(x)$ plus some parameters, θ, that are specific to this function (and not to the marginals). In the bivariate case, it has two arguments:

$$c_{12}[F_1(x_1), F_2(x_2); \theta] \tag{16.13}$$

The link between the joint and marginal distribution is made explicit by *Sklar's theorem*, which states that for any joint density there exists a copula that links the marginal densities:

$$f_{12}(x_1, x_2) = f_1(x_1) \times f_2(x_2) \times c_{12}[F_1(x_1), F_2(x_2); \theta] \tag{16.14}$$

With independence, the copula function is a constant always equal to one.

Thus the copula contains all the information on the nature of the dependence between the random variables but gives no information on the marginal distributions.

As an example, consider a normal multivariate density with N random variables. The joint normal density can be written as a function of the vector x, of the means μ, and of the covariance matrix Σ:

$$f^N(x_1, \ldots, x_N) = \frac{1}{(2\pi)^{N/2} \mid \Sigma \mid^{1/2}} \exp[-\frac{1}{2}(x-\mu)'\Sigma(x-\mu)] \tag{16.15}$$

Using the concept of copulas, this can be separated into N different marginal normal densities and a joint normal copula. With two variables,

$$f_{12}^N(x_1, x_2) = f_1^N(x_1) \times f_2^N(x_2) \times c_{12}^N[F_1(x_1), F_2(x_2); \rho] \tag{16.16}$$

Here, both f_1^N and f_2^N are normal marginals. They have parameters μ_1 and σ_1, and μ_2 and σ_2. In addition, c_{12}^N is the normal copula. Note that its sole parameter is the correlation coefficient ρ_{12}.

In particular, with standard normal variables, where $\mu_i = 0$ and $\sigma_i = 1$, the bivariate normal density reduces to

$$f_{12}^N(x_1, x_2) = \frac{1}{2\pi\sqrt{(1-\rho^2)}}\exp\left\{-\frac{(x_1^2 + 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)}\right\} \quad (16.17)$$

When the correlation is zero, this gives

$$f_{12}^N(x_1, x_2) = \frac{1}{2\pi}\exp\left\{-\frac{(x_1^2 + x_2^2)}{2}\right\} = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x_1^2)}{2}\right\} \times \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(x_2^2)}{2}\right\} \quad (16.18)$$

which is indeed the product of the marginal normal densities for x_1 and x_2, and of the copula, which is unity because the variables are independent.

Thus the copula can be separated from the marginals. In the case of a joint normal distribution, both the marginals and the copula are normal. This need not be the case, however. One could mix marginals from one family with a copula from another family.

The normal copula assumes a linear relationship between the risk factors, which is measured by the usual Pearson correlation coefficient. In general, however, the relationship does not need to be linear, which can be accommodated by other types of copulas.

The risk manager should choose the copula that provides the best fit to the data. This involves a choice of, first, an analytical function and, second, the best parameters using standard statistical tools such as maximum likelihood estimation.

An important feature of copulas is their **tail dependence**. This derives from the conditional probabilities of an extreme move in one variable given an extreme move in another variable. Formally, the upper and lower conditional probabilities are, for a given confidence level c,

$$P_U(c) = P[X_2 > F_{X_2}^{-1}(c) | X_1 > F_{X_1}^{-1}(c)] \quad (16.19)$$

$$P_L(c) = P[X_2 \leq F_{X_2}^{-1}(c) | X_1 \leq F_{X_1}^{-1}(c)] \quad (16.20)$$

Upper tail dependence is the limit when c goes to one; lower tail dependence when c goes to zero.

When the tail dependence parameters are zero, a copula is said to exhibit **tail independence**. This is the case for the normal copula. Consider for example a lower cutoff probability of 16%, which corresponds to $F_X^{-1} = -1$. Next, compute the conditional probability that $X_2 \leq -1$ given that $X_1 \leq -1$. Assuming a correlation of 0.50, this is $P_L(16\%) = P[X_2 \leq -1 | X_1 \leq -1] = 42\%$. Next, change the cutoff point from -1 to -1.645. The conditional probability then goes down from 42% to $P_L(5\%) = 22\%$. Continue with a lower cutoff point of -2.326. The conditional probability continues to decrease to $P_L(1\%) = 9\%$. Eventually, this converges to a tail dependence of zero. Because the normal distribution is symmetrical, both upper and lower tail dependences are zero.

Thus the normal copula is unable to reproduce tail dependencies. In practice, markets typically drop simultaneously during financial crises. Consider, for example, a portfolio with a long position in U.S. and U.K. stocks. When the U.S. market falls sharply, the U.K. market tends to fall in unison, which aggravates the portfolio loss.[1] This makes the normal copula unrealistic for tail events. Other copulas can be used, however. One example is the Student's *t* copula, which does exhibit tail dependence and is thus more realistic. Thus, the statistical modeling of risk factors involves delicate choices.

EXAMPLE 16.4: FRM EXAM 2009—QUESTION 2-9

Brenda Williams is a risk analyst who wants to model the dependence between asset returns using copulas and must convince her manager that this is the best approach. Which of the following statements are correct?

I. The dependence between the return distributions of portfolio assets is critical for risk measurement.
II. Correlation estimates often appear stable in periods of low market volatility and then become volatile in stressed market conditions. Risk measures calculated using correlations estimated over long horizons will therefore underestimate risk in stressed periods.
III. Pearson correlation is a linear measure of dependence between the return of two assets equal to the ratio of the covariance of the asset returns to the product of their volatilities.
IV. Using copulas, one can construct joint return distribution functions from marginal distribution functions in a way that allows for more general types of dependence structure of the asset returns.

a. I, II, and III
b. II and IV
c. I, II, III, and IV
d. I, III, and IV

16.3 VAR METHODS

We now turn to the next step. Once exposures have been aggregated across the portfolio, these need to be combined with movements in risk factors to create a

[1] F. Longin and B. Solnik, "Extreme Correlations of International Equity Markets," *Journal of Finance* 56 (2001): 649–676.

profile of the portfolio return distribution, which can be summarized with VAR. Three methods are commonly used.

16.3.1 Delta-Normal Method

The **delta-normal method** is the simplest VAR approach. It assumes that the portfolio exposures are linear and that the risk factors are jointly normally distributed. As such, it is a local valuation method.

Because the portfolio return is a linear combination of normal variables, it is itself normally distributed. Using matrix notations, the portfolio variance is given by

$$\sigma^2(R_{p,t+1}) = x_t' \Sigma_{t+1} x_t \tag{16.21}$$

where Σ_{t+1} is the forecast of the covariance matrix over the horizon. Because the method relies on the covariance matrix, it is sometimes called the **variance-covariance matrix** method.

If the portfolio volatility is measured in dollars, VAR is directly obtained from the standard normal deviate α that corresponds to the confidence level c:

$$\text{VAR} = \alpha\sigma(R_{p,t+1}) \tag{16.22}$$

This is called the **diversified VAR**, because it accounts for diversification effects. In contrast, the **undiversified VAR** is simply the sum of the individual VARs for each risk factor. It assumes that all prices will move in the worst direction simultaneously, which is unrealistic.

The main benefit of this approach is its appealing simplicity. The distribution is described by the variance, which is obtained quickly as a closed-form solution. This also allows analytical expressions for marginal and component VAR. The VAR measure is also more precise, or has less sampling variability, than nonparametric measures.

This simplicity, however, is also its drawback. The delta-normal method cannot account for nonlinear effects inherent in option positions or embedded options. It may also underestimate the occurrence of large observations because of its reliance on a normal distribution.

16.3.2 Historical Simulation Method

These drawbacks explain why the most commonly used method is **historical simulation** (HS). HS consists of going back in time (e.g., over the past 250 days), and using the factor movements over this period to project hypothetical factor values, from which portfolio values can be derived, usually with full revaluation. In other words, this method replays a “tape” of history to current positions.

Define the current time as t; we observe data from 1 to t. The current portfolio value is P_t, which is a function of the current risk factors

$$P_t = P[f_{1,t}, f_{2,t}, \ldots, f_{N,t}] \tag{16.23}$$

We sample the factor movements from the historical distribution, without replacement:

$$\Delta f_i^k = \{\Delta f_{i,1}, \Delta f_{i,2}, \ldots, \Delta f_{i,t}\} \tag{16.24}$$

From this we can construct hypothetical factor values, starting from the current one,

$$f_i^k = f_{i,t} + \Delta f_i^k \tag{16.25}$$

which are used to construct a hypothetical value of the current portfolio under the new scenario, using Equation (16.23):

$$P^k = P\left[f_1^k, f_2^k, \ldots, f_N^k\right] \tag{16.26}$$

We can now compute changes in portfolio values from the current position $R^k = (P^k - P_t)/P_t$.

We then sort the t returns and pick the one that corresponds to the cth quantile, $R_p(c)$. VAR is obtained from the difference between the average and the quantile,

$$\text{VAR} = \text{Ave}[R_p] - R_p(c) \tag{16.27}$$

when VAR is measured relative to the mean.

The advantage of this method is that it makes no specific distributional assumption about return distributions, other than relying on historical data. This is an improvement over the normal distribution, as historical data typically contain fat tails. HS is also more intuitive because a loss can be traced to a particular episode, where movements in risk factors give insights into the drivers of a large loss.

The main drawback of the method is its reliance on a short historical moving window to infer movements in market prices. If this window does not contain some market moves that are likely, it may miss some risks.

16.3.3 Monte Carlo Simulation Method

The **Monte Carlo simulation method** is similar to the historical simulation method, except that the movements in risk factors are generated by drawings from some prespecified distribution

$$\Delta f^k \sim g(\theta), \quad k = 1, \ldots K \tag{16.28}$$

where g is the joint distribution and θ its parameters. Here, the risk manager needs to specify the marginal distribution of risk factors as well as their copula. This can include a normal or Student's t, for example.

The method consists of sampling **pseudo-random numbers** from this distribution, repricing the portfolio using full valuation, and generating pseudo-dollar returns, which are then sorted to produce the desired VAR.

This method is the most flexible, but also carries an enormous computational burden. It requires users to make assumptions about the stochastic process and to understand the sensitivity of the results to these assumptions. Thus, it is subject to **model risk**. A good example of this has been the practice of credit rating agencies to use a normal copula for measuring the risk of CDOs. As explained in the previous section, the choice of this copula understates the true risk.

Monte Carlo methods also create inherent sampling variability because of the randomization. Different random numbers will lead to different results. It may take a large number of iterations to converge to a stable VAR measure. Unlike other methods, Monte Carlo simulation makes explicit the sampling variability in the risk numbers.

Finally, it should be noted that when all risk factors have a normal distribution and exposures are linear, the method should converge to the VAR produced by the delta-normal VAR. More generally, it is good practice to compare the outputs of the three VAR methods and to check that the numbers relate to each other as expected.

16.3.4 Comparison of Methods

Table 16.1 provides a summary comparison of the three mainstream VAR methods. Among these methods, the delta-normal is by far the easiest and fastest to

TABLE 16.1 Comparison of Approaches to VAR

Features	Delta-Normal	Historical Simulation	Monte Carlo Simulation
Valuation	Linear	Full	Full
Distribution			
Shape	Normal	Actual	General
Extreme events	Low probability	In recent data	Possible
Implementation			
Ease of computation	Yes	Average	No
Communicability	Average	Easy	Difficult
VAR precision	Excellent	Poor with short window	Good with many iterations
Major pitfalls	Nonlinearities, fat tails	Time variation in risk, unusual events	Model risk

implement. For simple portfolios with little optionality, this may be perfectly appropriate.

In contrast, the presence of options requires a full valuation method. In addition, most financial series display fat tails, which makes the normal approximation unreliable at high confidence levels. This explains why historical simulation is the most widely used VAR method.

EXAMPLE 16.5: FRM EXAM 2004—QUESTION 51

In early 2000, a risk manager calculates the VAR for a technology stock fund based on the past three years of data. The strategy of the fund is to buy stocks and write out-of-the-money puts. The manager needs to compute VAR. Which of the following methods would yield results that are *least* representative of the risks inherent in the portfolio?

a. Historical simulation with full repricing
b. Delta-normal VAR assuming zero drift
c. Monte Carlo style VAR assuming zero drift with full repricing
d. Historical simulation using delta equivalents for all positions

EXAMPLE 16.6: FRM EXAM 2006—QUESTION 114

Which of the following is most accurate with respect to delta-normal VAR?

a. The delta-normal method provides accurate estimates of VAR for assets that can be expressed as a linear or nonlinear combination of normally distributed risk factors.
b. The delta-normal method provides accurate estimates of VAR for options that are near or at-the-money and close to expiration.
c. The delta-normal method provides estimates of VAR by generating a covariance matrix and measuring VAR using relatively simple matrix multiplication.
d. The delta-normal method provides accurate estimates of VAR for options and other derivatives over ranges even if deltas are unstable.

EXAMPLE 16.7: FRM EXAM 2005—QUESTION 94

Which of the following statements about VAR estimation methods is *wrong*?

a. The delta-normal VAR method is more reliable for portfolios that implement portfolio insurance through dynamic hedging than for portfolios that implement portfolio insurance through the purchase of put options.
b. The full-valuation VAR method based on historical data is more reliable for large portfolios that contain significant option-like investments than the delta-normal VAR method.
c. The delta-normal VAR method can understate the true VAR for stock portfolios when the distribution of the return of the stocks has high kurtosis.
d. Full-valuation VAR methods based on historical data take into account nonlinear relationships between risk factors and security prices.

EXAMPLE 16.8: FRM EXAM 2005—QUESTION 128

Natural gas prices exhibit seasonal volatility. Specifically, the entire forward curve is more volatile during the wintertime. Which of the following statements concerning VAR is correct if the VAR is estimated using unweighted historical simulation and a three-year sample period?

a. We will overstate VAR in the summer and understate VAR in the winter.
b. We will overstate VAR in the summer and overstate VAR in the winter.
c. We will understate VAR in the summer and understate VAR in the winter.
d. We will understate VAR in the summer and overstate VAR in the winter.

EXAMPLE 16.9: FRM EXAM 2004—QUESTION 30

You are given the following information about the returns of stock P and stock Q: variance of return of stock P = 100; variance of return of stock Q = 225; covariance between the return of stock P and the return of stock Q = 53.2. At the end of 1999, you are holding USD 4 million in stock P. You are considering a strategy of shifting USD 1 million into stock Q and keeping USD 3 million in stock P. What percentage of risk, as measured by standard deviation of return, can be reduced by this strategy?

a. 0.5%
b. 5.0%
c. 7.4%
d. 9.7%

16.4 LIMITATIONS OF RISK SYSTEMS

The goal of risk measurement systems is to describe the distribution of potential losses on the portfolio. VAR is a single summary measure of dispersion in portfolio returns and consequently has limitations that should be obvious.

16.4.1 Illiquid Assets

The first and most obvious limitation of all risk measures based on a history of prices is the effect of **illiquidity**. Chapter 26 is entirely devoted to liquidity risk. Illiquid assets do not trade often, which means that observed prices may not represent recent transactions, in which case the prices are **stale**. Some bonds, for instance, hardly trade within a month. In this case, daily reported prices will be flat, with sporadic adjustments. As a result, volatility measures will be biased downward.

Stale prices manifest themselves with particular patterns of autocorrelations, which can be used to try to improve traditional risk measures. Even so, if prices are themselves not very meaningful, traditional risk measures must inherit the same qualification.

16.4.2 Losses Beyond VAR

As explained in Chapter 12, a VAR number cannot be viewed as a worst-loss measure. Instead, it should be viewed as a measure of dispersion that should be exceeded with some regularity (e.g., in 1% of the cases with the usual 99% confidence level). In addition, VAR does not describe the extent of losses in the left tail. Instruments such as short positions in options could generate infrequent but extreme losses when they occur. To detect such vulnerabilities, the distribution of losses beyond VAR should be examined as well.

16.4.3 Issues with Mapping

In addition, as we have seen in this chapter, the implementation of VAR systems often requires simplifications, obtained by mapping the positions on the selected risk factors. Thus, risk managers should be cognizant of weaknesses in their risk systems.

During the credit crisis that started in 2007, risk management systems failed at many banks. Some banks suffered losses that were much more frequent and much worse than they had anticipated. In 2007 alone, for example, UBS suffered losses of $19 billion from positions in mortgage-backed securities. Instead of experiencing the expected number of two or three exceptions (i.e., 1% of 250 days), this bank suffered 29 exceptions, or losses worse than VAR.[2] Note that the same limitations apply to stress tests.

[2] For a lucid explanation of risk management weaknesses, see the *Shareholder Report on UBS's Write-Downs* (April 2008).

In addition to the effects of heightened volatility in all financial markets, many banks experienced large losses on super-senior, AAA-rated tranches of securities backed by subprime mortgages. As will be seen in Chapter 23, these structures are fairly complex to model due to the need to estimate joint default probabilities. Investing in super-senior tranches can be viewed as selling out-of-the-money put options, which, as we have seen, involve nonlinear payoffs. As long as the real estate market continued to go up, the default rate on subprime debt was relatively low and the super-senior debt was safe, experiencing no price volatility. However, as the real estate market corrected sharply, the put options moved in-the-money, which led to large losses on the super-senior debt. Of course, none of these movements showed up in the recent historical data because this reflected only a sustained appreciation in the housing market but also because of the inherent nonlinearity in these securities.

Instead of modeling these complexities, some banks simply mapped the super-senior debt on AAA-rated corporate bond curves. This ignored the nonlinearities in the securities and was an act of blind faith in the credit rating. In this case, the mapping process was flawed and gave no warning sign of the impending risks.

16.4.4 Reliance on Recent Historical Data

The traditional application of VAR models, such as historical simulation, involves **moving windows**. The choice of the window involves a trade-off between using longer windows, which give more precise and more stable estimates, and using shorter windows, which may be more appropriate when markets are changing or when older data simply do not exist.

The effect of this moving window muddies the interpretation of changes in VAR measures, however. Changes can be due to either changes in the positions or changes in the moving window, or both.[3]

The shorter the window, the more volatile the VAR number. Typically, most banks use short windows, with one to three years of historical data at most. After periods of stability in financial markets, such windows may underestimate future risks, as was explained with the example of Figure 12.5.

This is why stress tests are needed as a complement to traditional VAR models. Indeed, as will be explained in Chapter 28, the Basel Committee has added a **stressed VAR** to the capital requirements against market risks. This is still based on current positions but uses fixed shocks in the risk factors.

16.4.5 Procyclicality

A related issue is the combination of leverage and portfolio management using risk-sensitive measures. Risk tends to move in cycles. Periods of economic expansion

[3] This can be resolved as follows. We start with two VAR measures. The first is the previous VAR, using previous positions and risk settings. The second is the current VAR, with current positions and risk settings. The effect can be disentangled by adding a new VAR measure using the current positions and the risk settings from the previous window. The difference from the previous VAR then represents solely the effect of changing positions.

tend to be accompanied with low credit spreads and low volatility of risk factors. This is because the volatility of fundamentals is lower during expansions but also because of lower default rates and lower risk aversion. In contrast, periods of recession are marked by wider spreads and higher volatility. Inevitably, we will experience business cycles and attendant cycles in volatility.

A good risk-sensitive measurement system will reflect this volatility and give proper warnings of periods of lower and higher risk. In some cases, however, this information can be used to decide on the amount of **leverage** in a portfolio. Periods of low volatility can induce traders to increase leverage if they have a fixed volatility target.

Suppose, for instance, that a fund manager has a fixed volatility target of $\overline{\sigma} = 10\%$. The manager has a single position in an asset with time-varying volatility. Currently, this is $\sigma_t = 5\%$. The fund manager can then leverage the asset by a factor of $l = 2$ to reach the target volatility. If, however, the asset volatility goes up to $\sigma_t = 10\%$, the portfolio manager will be forced to deleverage from $l = 2$ to $l = 1$, which involves selling half of the portfolio.

Thus, a sustained increase in volatility can force traders, hedge funds, and banks to deleverage their books. The process is aggravated if they have sustained losses. The combination of risk-sensitive measures and leverage could create a **procyclical effect**. More generally, this describes a situation that could magnify economic or financial fluctuations. This is a major issue for bank regulators, as will be seen in Chapter 28. The dilemma is that rules that are more risk sensitive are automatically more procyclical.

Another example is the margins set by the exchanges when trading, for example, futures contracts. These margins are designed to protect the clearinghouse in the event of a customer default. They are typically based on a daily VAR measure at a high confidence level. For instance, the typical initial margin on a position with a notional of \$100,000 is $m = 5\%$. This allows a leverage $l = 1/m = 20$. In times of increased volatility, exchanges can increase their margins. This is only prudent. However, it could force customers to liquidate their positions if they cannot contribute to the higher margin.

Similarly, banks or prime brokers provide financing to other banks and asset managers using **reverse repurchase agreements**. Under this agreement, the client sells a security to the bank in exchange for cash in the amount of the security value minus a **haircut**. In effect, this haircut functions as a margin, or buffer, to protect the bank against a loss due to client default. Table 16.2 shows that haircuts went up sharply during the credit crisis. Even for Treasuries, the most creditworthy and liquid securities on the list, haircuts went from 0.25% to 3%. For less liquid securities, haircuts went to 100%, meaning that the banks did not want these as collateral.

This was a prudent risk management practice by banks, which did not want to inherit securities that they could not value and for which there was no market. In the meantime, however, these practices exacerbated deleveraging pressures, adding to the panic of 2007.[4]

[4] G. Gorton, "The Panic of 2007" (working paper, Yale School of Management, 2008).

TABLE 16.2 Typical Haircuts

Security	April 2007	August 2008
U.S. Treasuries	0.25%	3%
Investment-grade bonds	0–3%	8–12%
Speculative-grade bonds	10–15%	25–40%
Senior leveraged loans	10–12%	15–20%
Prime MBSs	2–4%	10–20%
Asset-backed CDOs, AAA	2–4%	95%
Asset-backed CDOs, equity	50%	100%

Source: Joint FSF-CGFS Working Group, *The Role of Valuation and Leverage in Procyclicality* (Basel: Bank for International Settlements, 2009).

16.4.6 Crowded Trades

Another major limitation of risk measurement models is that they basically assume that the firm is a price taker. Risk measures seldom account for the price impact of liquidating the firm's portfolio. Even worse, they cannot possibly account for the likelihood of forced sales of the same (long) positions by other traders at the same time, because these other positions are not disclosed. Simultaneous sales of the same positions by traders or asset managers are sometimes described as **herding**. Trades that are susceptible to this effect are called **crowded trades**.[5]

16.5 EXAMPLE

16.5.1 Mark-to-Market (MTM)

We now illustrate the computation of VAR for a simple example. The problem at hand is to evaluate the one-day downside risk of a currency forward contract. We show that to compute VAR we need first to value the portfolio, mapping the value of the portfolio on fundamental risk factors, then to generate movements in these risk factors, and finally, to combine the risk factors with the valuation model to simulate movements in the contract value.

Assume that on December 31, 1998, we have a forward contract to buy £10 million in exchange for delivering \$16.5 million in three months. We use these definitions:

S_t = current spot price of the pound in dollars
F_t = current forward price
K = purchase price set in contract
f_t = current value of contract
r_t = domestic risk-free rate
r_t^* = foreign risk-free rate
τ = time to maturity

[5] Note that crowded trades require a particular coincidence of conditions. A group of traders or funds must be (1) leveraged and (2) long or short very similar positions. In addition, it is difficult to identify the drivers of herding because actions are often triggered by the release of new information instead of sales by other funds.

To be consistent with conventions in the foreign exchange market, we define the present value factors using discrete compounding:

$$P_t = \mathrm{PV}(\$1) = \frac{1}{1 + r_t\tau} \qquad P_t^* = \mathrm{PV}(\$1) = \frac{1}{1 + r_t^*\tau} \tag{16.29}$$

The current market value of a forward contract to buy one pound is given by

$$f_t = S_t \frac{1}{1 + r_t^*\tau} - K \frac{1}{1 + r_t\tau} = S_t P_t^* - K P_t \tag{16.30}$$

which is exposed to three risk factors: the spot rate and the two interest rates. In addition, we can use this equation to derive the exposures on the risk factors. After differentiation, we have

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial P^*} dP^* + \frac{\partial f}{\partial P} dP = P^* dS + S dP^* - K dP \tag{16.31}$$

Alternatively,

$$df = (SP^*) \frac{dS}{S} + (SP^*) \frac{dP^*}{P^*} - (KP) \frac{dP}{P} \tag{16.32}$$

Intuitively, the forward contract is equivalent to

- A long position of (SP^*) on the spot rate
- A long position of (SP^*) in the foreign bill
- A short position of (KP) in the domestic bill (borrowing)

We can now mark to market our contract. If Q represents our quantity, £10 million, the current market value of our contract is

$$V_t = Qf_t = \$10{,}000{,}000\ S_t \frac{1}{1 + r_t^*\tau} - \$16{,}500{,}000 \frac{1}{1 + r_t\tau} \tag{16.33}$$

On the valuation date, we have $S_t = 1.6637$, $r_t = 4.9375\%$, and $r_t^* = 5.9688\%$. Hence

$$P_t = \frac{1}{1 + r_t\tau} = \frac{1}{(1 + 4.9375\% \times 90/360)} = 0.9879$$

and similarly, $P_t^* = 0.9854$. The current market value of our contract is

$$V_t = \$10{,}000{,}000 \times 1.6637 \times 0.9854 - \$16{,}500{,}000 \times 0.9879 = \$93{,}581$$

which is slightly in-the-money. We are going to use this formula to derive the distribution of contract values under different scenarios for the risk factors.

16.5.2 Risk Factors

Assume now that we consider only the last 100 days to be representative of movements in market prices. Table 16.3 displays quotations on the spot and three-month rates for the past 100 business days, starting on August 10.

We first need to convert these quotes into true random variables, that is, with zero mean and constant dispersion. Table 16.4 displays the one-day changes in interest rates dr, as well as the relative changes in the associated present value factors dP/P and in spot rates dS/S. For instance, for the first day,

$$dr_1 = 5.5625 - 5.5938 = -0.0313$$

and

$$dS/S_1 = (1.6315 - 1.6341)/1.6341 = -0.0016$$

This information is now used to construct the distribution of risk factors.

16.5.3 VAR: Historical Simulation

The **historical simulation method** takes historical movements in the risk factors to simulate potential future movements. For instance, one possible scenario for

TABLE 16.3 Historical Market Factors

	Market Factors			
Date	\$ Eurorate (3mo-%pa)	£ Eurorate (3mo-%pa)	Spot Rate S(\$/£)	Number
8/10/98	5.5938	7.4375	1.6341	
8/11/98	5.5625	7.5938	1.6315	1
8/12/98	6.0000	7.5625	1.6287	2
8/13/98	5.5625	7.4688	1.6267	3
8/14/98	5.5625	7.6562	1.6191	4
8/17/98	5.5625	7.6562	1.6177	5
8/18/98	5.5625	7.6562	1.6165	6
8/19/98	5.5625	7.5625	1.6239	7
8/20/98	5.5625	7.6562	1.6277	8
8/21/98	5.5625	7.6562	1.6387	9
8/24/98	5.5625	7.6562	1.6407	10
...				
12/15/98	5.1875	6.3125	1.6849	90
12/16/98	5.1250	6.2188	1.6759	91
12/17/98	5.0938	6.3438	1.6755	92
12/18/98	5.1250	6.1250	1.6801	93
12/21/98	5.1250	6.2812	1.6807	94
12/22/98	5.2500	6.1875	1.6789	95
12/23/98	5.2500	6.1875	1.6769	96
12/24/98	5.1562	6.1875	1.6737	97
12/29/98	5.1875	6.1250	1.6835	98
12/30/98	4.9688	6.0000	1.6667	99
12/31/98	**4.9375**	**5.9688**	**1.6637**	**100**

TABLE 16.4 Movements in Market Factors

	Movements in Market Factors				
Number	*dr* ($1)	*dr* (£1)	*dP/P* ($1)	*dP/P* (£1)	*dS* ($/£) /S
1	−0.0313	0.1563	0.00000	−0.00046	−0.0016
2	0.4375	−0.0313	−0.00116	0.00000	−0.0017
3	−0.4375	−0.0937	0.00100	0.00015	−0.0012
4	0.0000	0.1874	−0.00008	−0.00054	−0.0047
5	0.0000	0.0000	−0.00008	−0.00008	−0.0009
6	0.0000	0.0000	−0.00008	−0.00008	−0.0007
7	0.0000	−0.0937	−0.00008	0.00015	0.0046
8	0.0000	0.0937	−0.00008	−0.00031	0.0023
9	0.0000	0.0000	−0.00008	−0.00008	0.0068
10	0.0000	0.0000	−0.00008	−0.00008	0.0012
...					
90	0.0937	0.0625	−0.00031	−0.00023	−0.0044
91	−0.0625	−0.0937	0.00008	0.00015	−0.0053
92	−0.0312	0.1250	0.00000	−0.00038	−0.0002
93	0.0312	−0.2188	−0.00015	0.00046	0.0027
94	0.0000	0.1562	−0.00008	−0.00046	0.0004
95	0.1250	−0.0937	−0.00039	0.00015	−0.0011
96	0.0000	0.0000	−0.00008	−0.00008	−0.0012
97	−0.0938	0.0000	0.00015	−0.00008	−0.0019
98	0.0313	−0.0625	−0.00015	0.00008	0.0059
99	−0.2187	−0.1250	0.00046	0.00023	−0.0100
100	−0.0313	−0.0312	0.00000	0.00000	−0.0018

the U.S. interest rate is that, starting from the current value $r_0 = 4.9375$, the movement the next day could be similar to that observed on August 11, which is a decrease of $dr_1 = -0.0313$. The new value is $r(1) = 4.9062$.

We compute the simulated values of other variables as $r^*(1) = 5.9688 + 0.1563 = 6.1251$ and $S(1) = 1.6637 \times (1 - 0.0016) = 1.6611$. Armed with these new values, we can reprice the forward contract, now worth $V_t = \$10{,}000{,}000 \times 1.6611 \times 0.9849 - \$16{,}500{,}000 \times 0.9879 = \$59{,}941$.

Note that, because the contract is long the pound that fell in value, the current value of the contract has decreased relative to the initial value of $93,581.

We record the new contract value and repeat this process for all the movements from day 1 to day 100. This creates a distribution of contract values, which is reported in the last column of Table 16.5.

The final step consists of sorting the contract values, as shown in Table 16.6. Suppose we want to report VAR relative to the initial value (instead of relative to the average on the target date). The last column in the table reports the *change* in the portfolio value, $V(k) - V_0$. These range from a loss of $200,752 to a gain of $280,074.

We can now characterize the risk of the forward contract by its entire distribution, which is shown in Figure 16.3. The purpose of VAR is to report a single number as a downside risk measure. Let us take, for instance, the 95% lower quantile. From Table 16.6, we identify the fifth lowest value out of a

TABLE 16.5 Simulated Market Factors

	Simulated Market Factors					
Number	r (\$1)	r (£1)	S (\$/£)	PV (\$1)	PV (£1)	Hypothetical MTM Contract
1	4.9062	6.1251	1.6611	0.9879	0.9849	\$59,941
2	5.3750	5.9375	1.6608	0.9867	0.9854	\$84,301
3	4.5000	5.8751	1.6617	0.9889	0.9855	\$59,603
4	4.9375	6.1562	1.6559	0.9878	0.9848	\$9,467
5	4.9375	5.9688	1.6623	0.9878	0.9853	\$79,407
6	4.9375	5.9688	1.6625	0.9878	0.9853	\$81,421
7	4.9375	5.8751	1.6713	0.9878	0.9855	\$172,424
8	4.9375	6.0625	1.6676	0.9878	0.9851	\$128,149
9	4.9375	5.9688	1.6749	0.9878	0.9853	\$204,361
10	4.9375	5.9688	1.6657	0.9878	0.9853	\$113,588
...						
90	5.0312	6.0313	1.6564	0.9876	0.9851	\$23,160
91	4.8750	5.8751	1.6548	0.9880	0.9855	\$7,268
92	4.9063	6.0938	1.6633	0.9879	0.9850	\$83,368
93	4.9687	5.7500	1.6683	0.9877	0.9858	\$148,705
94	4.9375	6.1250	1.6643	0.9878	0.9849	\$93,128
95	5.0625	5.8751	1.6619	0.9875	0.9855	\$84,835
96	4.9375	5.9688	1.6617	0.9878	0.9853	\$74,054
97	4.8437	5.9688	1.6605	0.9880	0.9853	\$58,524
98	4.9688	5.9063	1.6734	0.9877	0.9854	\$193,362
99	4.7188	5.8438	1.6471	0.9883	0.9856	−\$73,811
100	4.9062	5.9376	1.6607	0.9879	0.9854	\$64,073
	4.9375	**5.9688**	**1.6637**	**0.9879**	**0.9854**	**\$93.581**

hundred, which is minus \$127,232. Ignoring the mean, the 95% VAR is $\text{VAR}_{\text{HS}} = \$127{,}232$.

16.5.4 VAR: Delta-Normal Method

The **delta-normal** method takes a different approach to constructing the distribution of the portfolio value. We assume that the three risk factors (dS/S), (dP/P), (dP^*/P^*) are jointly normally distributed.

We can write Equation (16.32) as

$$df = (SP^*)\frac{dS}{S} + (SP^*)\frac{dP^*}{P^*} - (KP)\frac{dP}{P} = x_1 dz_1 + x_2 dz_2 + x_3 dz_3 \qquad (16.34)$$

where the dz are normal variables and x are exposures.

Define Σ as the (3 by 3) covariance matrix of the dz, and x as the vector of exposures. We compute VAR from $\sigma^2(df) = x'\Sigma x$. Table 16.7 details the steps. First, we compute the covariance matrix of the three risk factors. The top of the table shows the standard deviation of daily returns as well as correlations. From these, we construct the covariance matrix.

Next, the table shows the vector of exposures, x'. The matrix multiplication Σx is shown on the following lines. After that, we compute $x'(\Sigma x)$, which yields the variance. Taking the square root, we have $\sigma(df) = \$77{,}306$. Finally, we

TABLE 16.6 Distribution of Portfolio Values

	Sorted Values	
Number	Hypothetical MTM	Change in MTM
1	−$107,171	−$200,752
2	−$73,811	−$167,392
3	−$46,294	−$139,875
4	−$37,357	−$130,938
5	−$33,651	**−$127,232**
6	−$22,304	−$115,885
7	−$11,694	−$105,275
8	$7,268	−$86,313
9	$9,467	−$84,114
10	$13,744	−$79,837
...		
90	$193,362	$99,781
91	$194,405	$100,824
92	$204,361	$110,780
93	$221,097	$127,515
94	$225,101	$131,520
95	$228,272	$134,691
96	$233,479	$139,897
97	$241,007	$147,426
98	$279,672	$186,091
99	$297,028	$203,447
100	$373,655	$280,074

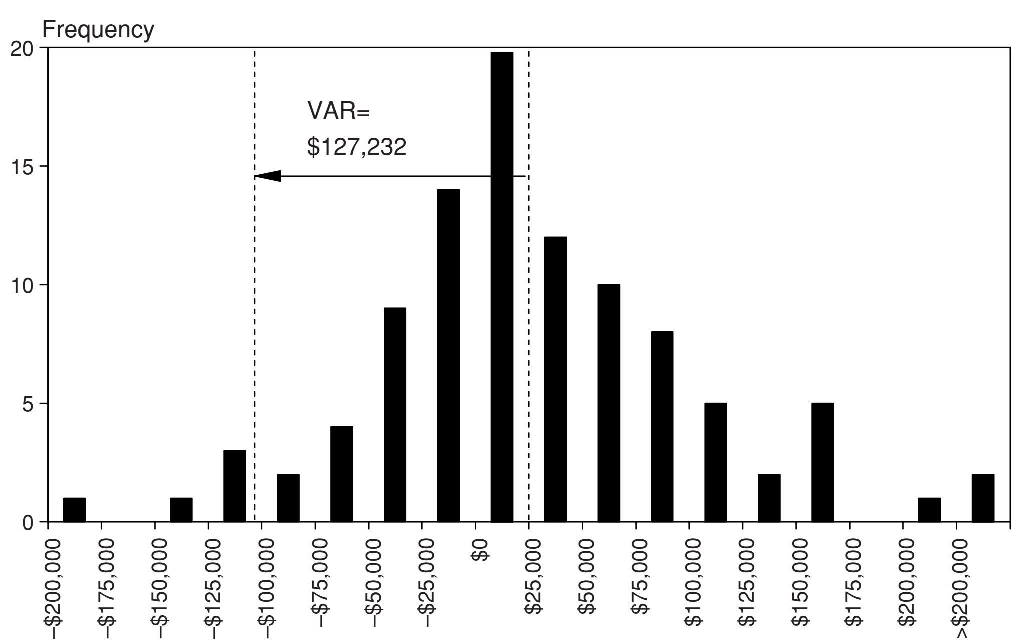

FIGURE 16.3 Empirical Distribution of Value Changes

TABLE 16.7 Covariance Matrix Approach

		$dP/P(\$1)$	$dP/P(£1)$	$dS(\$/£)/S$				
Standard Deviation:		0.022%	0.026%	0.473%				
Correlation Matrix:		$dP/P(\$1)$	$dP/P(£1)$	$dS(\$/£)/S$				
$\sum$	$dP/P(\$1)$	1.000	0.137	0.040				
	$dP/P(£1)$	0.137	1.000	−0.063				
	$dS(\$/£)/S$	0.040	−0.063	1.000				
Covariance Matrix:		$dP/P(\$1)$	$dP/P(£1)$	$dS(\$/£)/S$				
$\sum$	$dP/P(\$1)$	4.839E-08	7.809E-09	4.155E-08				
	$dP/P(£1)$	7.809E-09	6.720E-08	−7.688E-08				
	$dS(\$/£)/S$	4.155E-08	−7.688E-08	2.237E-05				
Exposures:								
$\mathbf{x}'$		−\$16,300,071	\$16,393,653	\$16,393,653				
$\sum \mathbf{x}$		4.839E-08	7.809E-09	4.155E-08		−\$16,300,071		\$0.020
		7.809E-09	6.720E-08	−7.688E-08	×	\$16,393,653	=	−\$0.286
		4.155E-08	−7.688E-08	2.237E-05		\$16,393,653		\$364.852
$\sigma^2 = x'(\sum x)$ **Variance:**						**\$0.020**		
		−\$16,300,071	\$16,393,653	\$16,393,653	×	−\$0.286	=	\$5,976,242,188
						\$364.852		
σ		Standard deviation						**\$77,306**

transform into a 95% quantile by multiplying by 1.645, which gives VAR_{DN} = \$127,169.

Note how close this number is to the VAR_{HS} of \$127,232 we found previously. This suggests that the distribution of these variables is close to a normal distribution. Indeed, the empirical distribution in Figure 16.3 roughly looks like a normal distribution. The fitted distribution is shown in Figure 16.4.

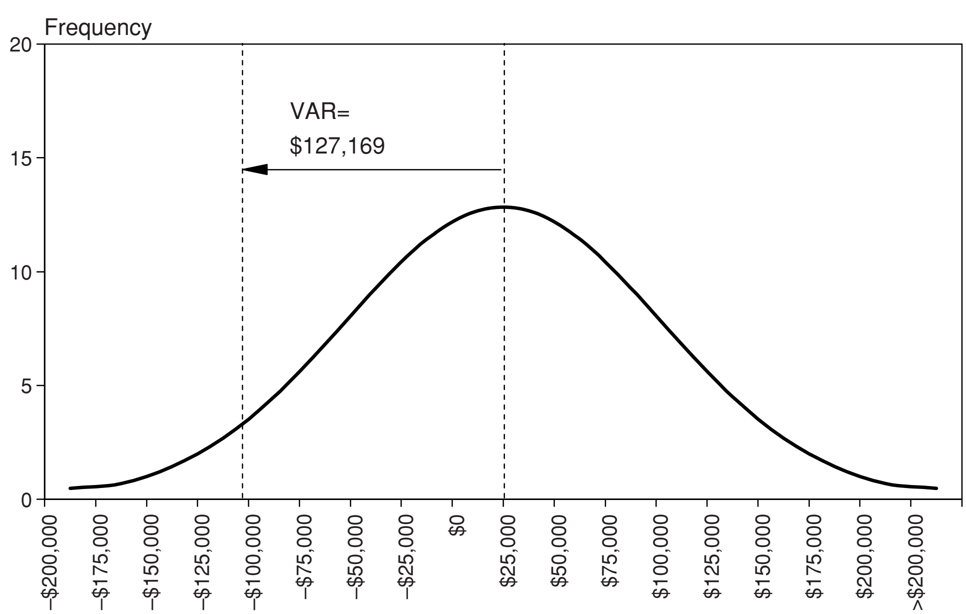

FIGURE 16.4 Normal Distribution of Value Changes

EXAMPLE 16.10: FRM EXAM 2008—QUESTION 2-35

A trading book consists of the following two assets, with correlation of 0.2.

Asset	Expected Return	Annual Volatility	Value
A	10%	25%	\$100
B	20%	20%	\$50

How would the daily VAR at the 99% level change if the bank sells \$50 worth of A and buys \$50 worth of B? Assume a normal distribution and 250 trading days.

a. 0.2286
b. 0.4571
c. 0.7705
d. 0.7798

16.6 IMPORTANT FORMULAS

Decomposition of a joint distribution into marginals and its copula (Sklar's theorem): $f_{12}(x_1, x_2) = f_1(x_1) \times f_2(x_2) \times c_{12}[F_1(x_1), F_2(x_2); \theta]$

Delta-normal VAR: $\text{VAR} = \alpha\sigma(R_{p,t+1}),\ \sigma^2(R_{p,t+1}) = x_t'\Sigma_{t+1}x_t$

Historical simulation VAR: Quantile of simulated portfolio using $\Delta f_i^k = \{\Delta f_{i,1}, \Delta f_{i,2}, \ldots, \Delta f_{i,t}\}$

Monte Carlo simulation VAR: Quantile of simulated portfolio using $\Delta f^k \sim g(\theta)$

16.7 ANSWERS TO CHAPTER EXAMPLES

Example 16.1: FRM Exam 2009—Question 2-7

c. Under the cash flow (CF) mapping approach, each payment (and not only the last one) is associated with a different risk factor, so statement I. is incorrect. Statement II. is incorrect because the CF mapping approach is more correct than duration or maturity mapping.

Example 16.2: FRM Exam 2002—Question 44

a. The question deals with the distribution of the assets and the effect of diversification. Emerging market securities are more volatile and less likely to be normally distributed than broad market indices. In addition, a small portfolio is less likely to be well represented by a mapping approach, and is less likely to be normal. The RiskMetrics approach assumes that the conditional distribution is normal and simplifies risk by mapping. This will be acceptable with a large number of securities with distributions close to the normal, which is answer d. Answer a. describes the least diversified portfolio, for which the HS method is best.

Example 16.3: FRM Exam 2007—Question 11

c. Answer a. is incorrect because it considers only the portfolio beta, which is zero by construction. So, it would erroneously conclude that there is no risk. Answer b. is better but would miss the risk of the IPO positions because they have no history. Answer d. will produce unreliable numbers because of the short window. The best solution is to replace the IPO positions by exposures on industry and style factors.

Example 16.4: FRM Exam 2009—Question 2-9

d. The dependence is critical, so statement I. is correct. The usual Pearson correlation is a linear measure of dependence, so statement III. is correct. Statement IV. is also correct. For statement II., correlations indeed change over stressed periods, but it is not clear whether this biases long-term correlations upward or downward. Also, the effect on the portfolio risk depends on the positioning. Hence, there is not enough information to support statement II.

Example 16.5: FRM Exam 2004—Question 51

d. Because the portfolio has options, methods a. or c. based on full repricing would be appropriate. Next, recall that technology stocks had a big increase in price until March 2000. From 1996 to 1999, the NASDAQ index went from 1,300 to 4,000. This creates a positive drift in the series of returns. So, historical simulation without an adjustment for this drift would bias the simulated returns upward, thereby underestimating VAR.

Example 16.6: FRM Exam 2006—Question 114

c. The delta-normal approach will perform poorly with nonlinear payoffs, so answer a. is false. Similarly, the approach will fail to measure risk properly for options if the delta changes, which is the case for at-the-money options, so answers b. and d. are false.

Example 16.7: FRM Exam 2005—Question 94

a. Full-valuation methods are more precise for portfolios with options, so answers b. and d. are correct. The delta-normal VAR understates the risk when distributions have fat tails, so answer c. is correct. Answer a. is indeed wrong. The delta-normal method will be poor for outright positions in options, or their dynamic replication.

Example 16.8: FRM Exam 2005—Question 128

a. This method essentially estimates the average volatility over a three-year window, ignoring seasonality. As a result, if the conditional volatility is higher during the winter, the method will understate the true risk, and conversely for the summer.

Example 16.9: FRM Exam 2004—Question 30

b. The variance of the original portfolio is 1,600, implying a volatility of 40. The new portfolio has variance of $3^2 \times 100 + 1^2 \times 225 + 2 \times 53.2 \times 3 \times 1 = 1{,}444$. This gives a volatility of 38, which is a reduction of 5%.

Example 16.10: FRM Exam 2008—Question 2-35

b. We compute first the variance of the current portfolio. This is $(100 \times 0.25)^2 + (50 \times 0.20)^2 + 2 \times 0.2(100 \times 0.25)(50 \times 0.20) = 825$. VAR is then $\sqrt{825} \times 2.33/\sqrt{250} = 4.226$. The new portfolio has positions of \$50 and \$100, respectively. The variance is $(50 \times 0.25)^2 + (100 \times 0.20)^2 + 2 \times 0.2(50 \times 0.25)(100 \times 0.20) = 656.25$. VAR is then 3.769 and the difference is -0.457. The new VAR is lower because of the greater weight on asset B, which has lower volatility. Also note that the expected return is irrelevant.

APPENDIX: SIMPLIFICATION OF THE COVARIANCE MATRIX

This appendix shows how the diagonal model can be used to construct a simplified covariance matrix, which is useful for analytical risk models. Say that we have $N = 100$ assets. This implies a covariance matrix with $N(N+1)/2 = 5{,}050$ different entries. We start from a one-factor model, as in Equation (16.1):

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \tag{16.35}$$

As a result, the covariance between any two stocks is

$$\text{Cov}[R_i, R_j] = \text{Cov}[\beta_i R_M + \epsilon_i, \beta_j R_M + \epsilon_j] = \beta_i \beta_j \sigma_M^2 \tag{16.36}$$

using the assumption that the residual components are uncorrelated with each other and with the market. The variance of a stock is

$$\text{Cov}[R_i, R_i] = \beta_i^2 \sigma_M^2 + \sigma_{\epsilon,i}^2 \tag{16.37}$$

Therefore, the covariance matrix is

$$\Sigma = \begin{bmatrix} \beta_1^2\sigma_M^2 + \sigma_{\epsilon,1}^2 & \beta_1\beta_2\sigma_M^2 & \ldots & \beta_1\beta_N\sigma_M^2 \\ \vdots & & & \\ \beta_N\beta_1\sigma_M^2 & \beta_N\beta_2\sigma_M^2 & \ldots & \beta_N^2\sigma_M^2 + \sigma_{\epsilon,N}^2 \end{bmatrix}$$

which can also be written as

$$\Sigma = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} [\beta_1 \dots \beta_N] \sigma_M^2 + \begin{bmatrix} \sigma_{\epsilon,1}^2 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sigma_{\epsilon,N}^2 \end{bmatrix}$$

Using matrix notation, we have

$$\Sigma = \beta\beta'\sigma_M^2 + D_\epsilon \tag{16.38}$$

This consists of N elements in the vector β, of N elements on the diagonal of the matrix D_ϵ, plus the variance of the market itself. The diagonal model reduces the number of parameters from $N \times (N+1)/2$ to $2N+1$, a considerable improvement. For example, with 100 assets the number is reduced from 5,050 to 201.

The variance of a portfolio with weights w is

$$\sigma_p^2 = w'\Sigma w = (w'\beta)^2\sigma_M^2 + w' D_\epsilon w \tag{16.39}$$

This depends on the portfolio beta, $\beta_p = w'\beta$, and the variance of the residuals, which should be small if the portfolio is well diversified.

In summary, this diagonal model substantially simplifies the risk structure of an equity portfolio. Risk managers can focus on managing the overall market risk of the portfolio, while controlling the concentration risk of individual securities.

Still, this one-factor model could miss common effects among groups of stocks, such as industry effects. To account for these, Equation (16.35) can be generalized to K factors

$$R_i = \alpha_i + \beta_{i1} y_1 + \dots + \beta_{iK} y_K + \epsilon_i \tag{16.40}$$

where $y_1, \dots, y_K$ are the factors. When these are independent of each other, the covariance matrix generalizes Equation (16.38) to

$$\Sigma = \beta_1\beta_1'\sigma_1^2 + \dots + \beta_K\beta_K'\sigma_K^2 + D_\epsilon \tag{16.41}$$

The number of parameters is now $(N \times K + K + N)$. For example, with 100 assets and five factors, this number is 605, which is still much lower than 5,050 for the unrestricted model. Again, the portfolio risk can be controlled by managing the portfolio exposures to the risk factors $\beta_{p,1}, \dots, \beta_{p,K}$.

CHAPTER 17

Managing Volatility Risk

Options are nonlinear instruments whose value depends on, among other things, a volatility parameter. Thus trading options involves taking volatility bets as well as directional bets on the underlying asset prices. Previous chapters have covered the Black-Scholes pricing formula as well as partial derivatives. This chapter examines more advanced models involving volatility trading.

Section 17.1 explains how an implied volatility can be recovered from option market prices. A notable example is the Volatility Index (VIX), which is the implied volatility measure for U.S. stocks that has become widely watched as a broad measure of risk. More generally, VIX should be extended to various maturities and strike prices, which leads to the concept of an implied volatility surface. Option portfolios present special challenges for risk measurement. Complicated portfolios require modeling the entire volatility surface for all the underlying risk factors, which is a complex undertaking.

At the portfolio level, comparisons of volatilities lead to the concept of average correlations, which are discussed in Section 17.2. Next, Section 17.3 presents derivative contracts whose values are directly tied to the realized variance or correlation. Section 17.4 then turns to the interpretation of dynamic hedging. Risk managers need to have a good understanding of these strategies because they give important insights into many active trading strategies.

Finally, Section 17.5 analyzes convertible bonds and warrants. These differ from the usual equity options in that exercising them creates new shares. Convertible bond trading is a widespread activity for hedge funds.

17.1 IMPLIED VOLATILITY

17.1.1 Concept

We can generally express the value of a derivative instrument as a function of the value of the underlying asset, plus other parameters

$$f_t = f(S_t, r_t, r_t^*, \sigma_t,\ K, \tau) \tag{17.1}$$

FRM Exam Part 2 topic. Readers should review Chapter 8, which introduced options and their pricing, in particular exotic options. In addition, Chapter 14 covered the partial derivatives, or Greeks.

where S_t is the spot price of the underlying asset, and r_t and r_t^* are the domestic interest rate and asset yield, respectively. The contract defines the strike price K and the time to maturity τ. For example, the Black-Scholes (BS) model for a European call is

$$c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2) \tag{17.2}$$

All of the parameters are observable, except for σ_t, the volatility of the asset value over the life of the option. Knowing this parameter yields the theoretical value of the option.

Alternatively, we can invert the model. Suppose that we observe a market price for the option premium, c_t. We can then recover the **implied standard deviation** (ISD), also called **implied volatility**, from the market price and other parameters, using the valuation function

$$\mathrm{ISD}_t = f^{-1}(c_t, S_t, r_t, r_t^*, \ K, \tau) \tag{17.3}$$

Because the function is nonlinear, solving for the ISD requires a numerical procedure, such as a **Newton-Raphson method**. This optimization process uses the partial derivative with respect to the ISD to converge to the solution and is fairly quick.

In fact, some markets quote the ISD directly instead of premiums. Premiums, which are required for actual trading, are then inferred from a standard pricing model.[1] Quoting volatility is often more intuitive for traders than are quoting premiums. This is similar to the convention of quoting bonds in terms of yields, which are more intuitive and easier to compare across securities and across time than are prices.

This ISD reflects the market's view about the asset volatility. Remember, however, that the BS formula assumes a risk-neutral world. Hence the ISD is a **risk-neutral** volatility, which is not necessarily the same as the actual or objective volatility. Thus, for predicting the risk of the asset price, the ISD may need some adjustment.

Similarly, the term $N(d_2)$ reflects the risk-neutral probability that the call will be exercised. This is useful in pricing binary options, for example, which pay Q if S ends up above K. Their price is simply $c = Qe^{-r\tau}N(d_2)$. The actual probability of exercise may be different, though.

Even so, for risk management purposes, the ISD is crucial. To project the future value of the option, we need to forecast this implied volatility as well. Thus the ISD is a major risk factor when measuring option risk.

17.1.2 VIX

Changes in the volatility parameter can be a substantial source of risk. Figure 17.1 illustrates the time variation in the **implied volatility** for options on the S&P

[1] For European-style options the BS model is standard. For American-style options, exchanges use the Whaley pricing model, which provides a quadratic and efficient approximation to the premium.

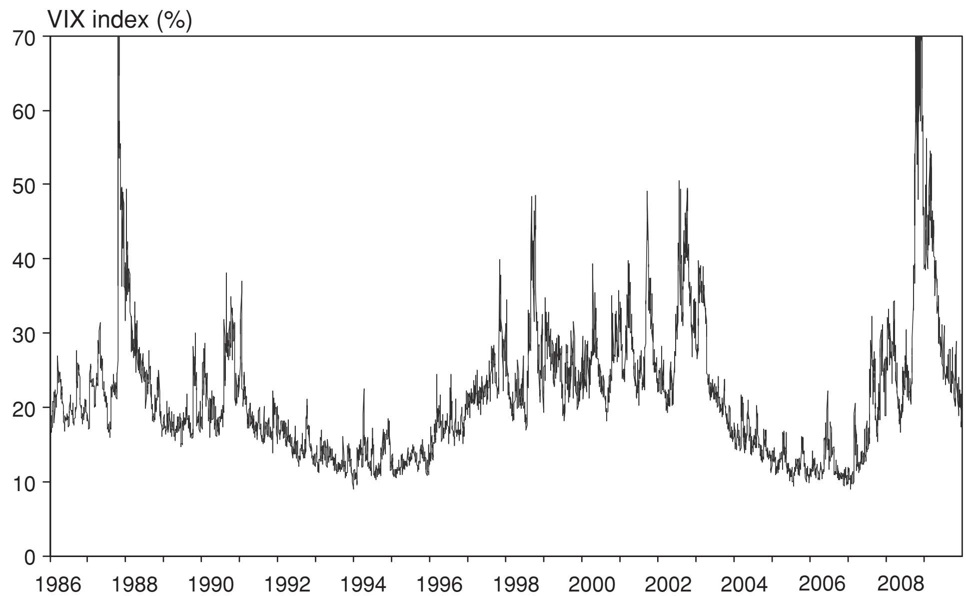

FIGURE 17.1 VIX: Implied Volatility

stock index, also known as the **Volatility Index** (**VIX**). The implied volatility, formerly derived from the market prices of at-the-money near-term options on the S&P 100 index, is calculated by the Chicago Board Options Exchange.[2] The VIX is sometimes called the **fear index** because it represents an aggregate measure of risk aversion.

Over this period, the average value of the VIX index was on the order of 21%. The volatility in the daily change in the VIX was about 2.4%.[3] Assuming a normal distribution to simplify, this implies that most of the daily movements should be within plus or minus 5%, as shown in Figure 17.2.

The distribution, however, is far from normal and instead displays sharp spikes, reflecting increased uncertainty. In particular, the VIX exceeded 40% during the crash of October 1987, during the Long-Term Capital Management (LTCM) crisis of September 1998, after the World Trade Center attack of September 2001, at the bottom of the 2000–2002 bear market in July 2002, and during the credit crisis that suddenly worsened in September 2008.

17.1.3 Implied Volatility Surface

In theory, the BS model relies on a single volatility parameter. If the model were correct, the ISDs should be constant across strike prices and maturities. In fact, this is not what we observe. ISDs vary systematically with the strike price and with the maturity.

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[2] In 2003, the methodology was changed; the new VIX index is derived from the prices of S&P 500 index options across a wide range of strike prices. The graph here shows VXO, which uses the old methodology but goes back to 1986.

[3] Note that, because of mean reversion, daily volatilities cannot be extrapolated to annual data.

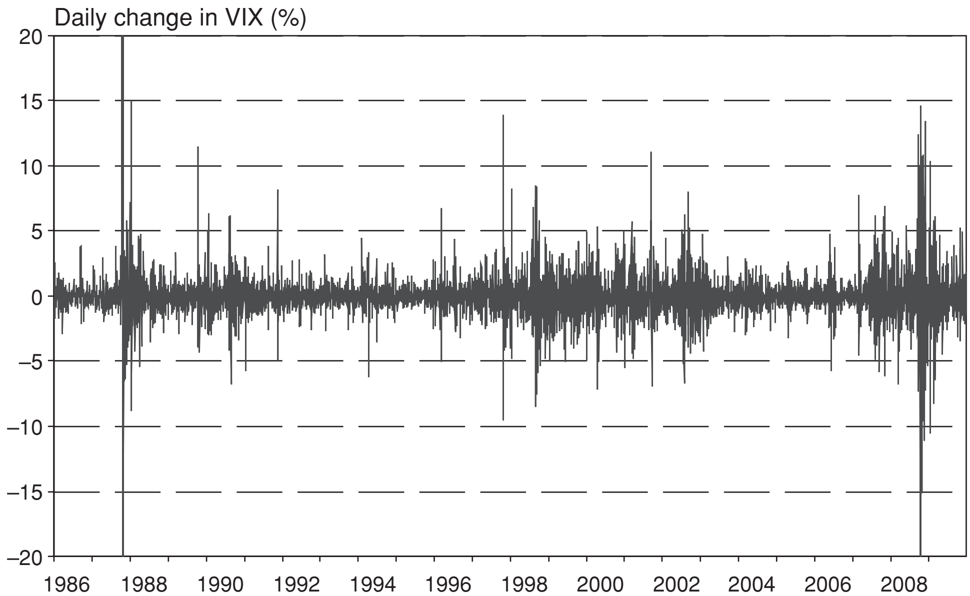

FIGURE 17.2 Daily Changes in Implied Volatility

The ISDs can be represented by an **implied volatility surface**, which is a three-dimensional plot of ISD across maturities and strike prices. Option traders typically scrutinize the shape of this volatility surface to identify sectors where the ISDs seem out of line. This is illustrated in Figure 17.3, which plots the surface for options on the S&P 500 stock market index. At that time, the spot price was around 1,130.

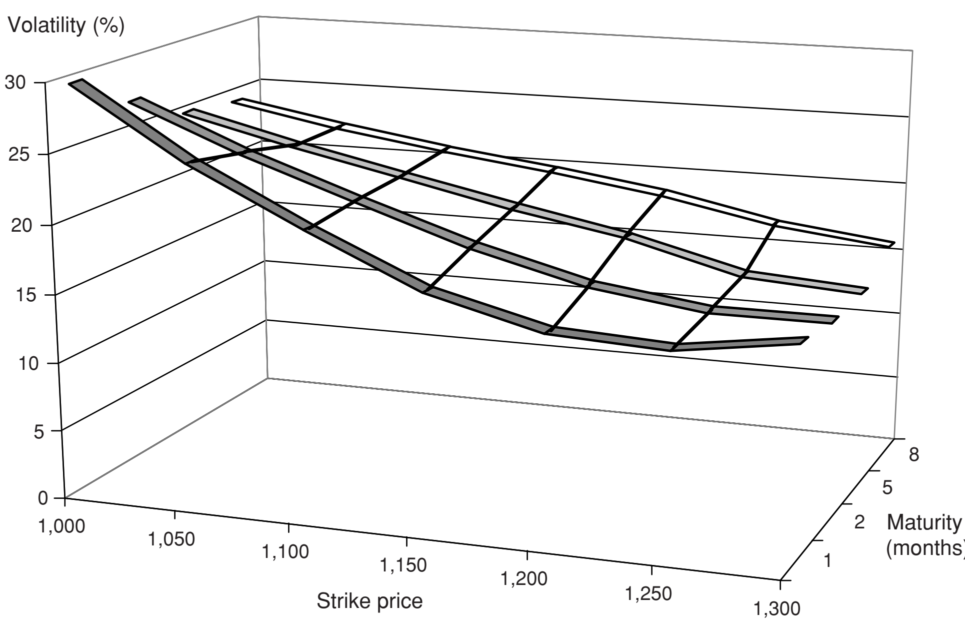

FIGURE 17.3 Implied Volatility Surface

Plots of the ISD against the strike price generally display what is called a **volatility smile** pattern, meaning that ISDs increase for low and high values of K relative to the current price. This effect has been observed in a variety of markets, in particular foreign currency options.

For equity index options, the effect is more asymmetrical, with very high ISDs for low strike prices. Because of the negative slope, this is called a **volatility skew**. A skewed smile is sometimes called a **smirk**. In other words, out-of-the-money (OTM) put options are priced with a higher ISD than at-the-money (ATM) or even in-the-money (ITM) put options. Before the stock market crash of October 1987, this effect was minor. Since then, it has become more pronounced.

Another feature is the **term structure of volatility**, which refers to the observation that the ISD differs across maturities. This arises when uncertainty is not uniformly distributed over various horizons. For instance, for individual stocks, the realized volatility tends to increase on the day of an earnings announcement. In Figure 17.3, the term structure is decreasing for $K = 1{,}000$ and increasing for $K = 1{,}200$.

Like interest rates, volatilities can be measured on a spot and forward basis. Define $\sigma_\tau^2\tau$ as the total variance between now and time τ into the future. This is a spot measure. Assuming serially uncorrelated movements in the underlying risk factor, variances simply add up across time, without annualization. Combinations of spot measures can be used to define a forward volatility $\sigma_{1,2}$:

$$\sigma_2^2 \times \tau_2 = \sigma_1^2 \times \tau_1 + \sigma_{1,2}^2 \times (\tau_2 - \tau_1) \tag{17.4}$$

For example, in Figure 17.3, the one-month and two-month ATM volatilities are 16.5% and 17.8%, respectively. The respective variances are $16.5\%^2 \times (1/12) = 0.002269$ and $17.8\%^2 \times (2/12) = 0.005287$. The one- to two-month forward variance is then 0.003018. Annualizing and taking the square root gives 19.0%. Thus the market expects an annualized volatility of 16.5% over the first month and of 19.0% over the second month. Perhaps some major uncertainty is expected to be resolved during the second month, which could reflect the earnings season, for example.

17.1.4 Prediction of the Volatility Surface

To predict returns on options, traders also need to forecast the evolution of the implied volatility surface. Figure 17.4 gives an example of the evolution of a volatility skew. The initial curve has an ISD of 18% for ATM options, with a strike price of $K = 100$. The question is what the curve will look like if the spot price moves from $S = 100$ to $S = 110$. Traders typically use heuristic approaches to the extrapolation of the curve over the investment horizon.

In a first scenario, called **sticky strike**, the curve does not change and the ISD drops from 18 to 17.5. This assumes that there is no structural change in the

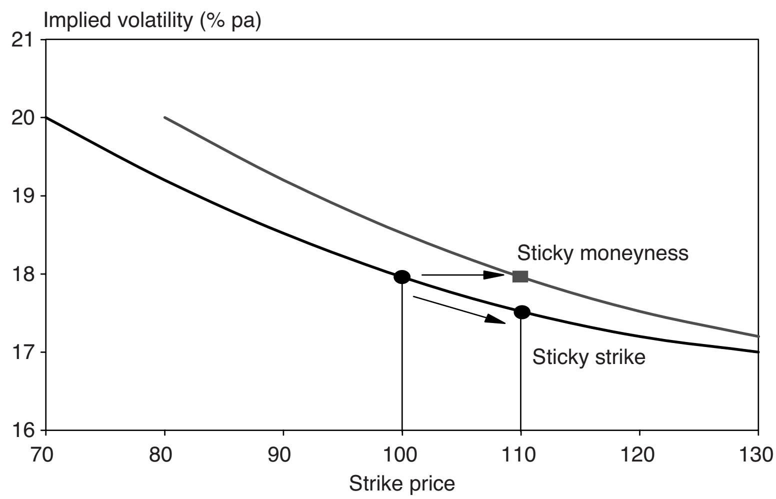

FIGURE 17.4 Evolution of Volatility Skew

volatility curve and that the price movement is largely temporary. In a second scenario, called **sticky moneyness**, the curve shifts to the right and the ISD stays at 18 (the moneyness is not changed and is still 100%). This assumes a permanent shift in the volatility curve. Based on these assumptions, the trader can then examine the return on different option trading strategies and choose the most appropriate one. More generally, this demonstrates that the implied volatility is a major risk factor when trading options.

EXAMPLE 17.1: FRM EXAM 2008—QUESTION 2-11

You are asked to mark to market a book of plain-vanilla stock options. The trader is short deep out-of-the-money options and long at-the-money options. There is a pronounced smile for these options. The trader's bonus increases as the value of his book increases. Which approach should you use to mark the book?

a. Use the implied volatility of at-the-money options because the estimation of the volatility is more reliable.
b. Use the average of the implied volatilities for the traded options for which you have data because all options should have the same implied volatility with Black-Scholes and you don't know which one is the right one.
c. For each option, use the implied volatility of the most similar option traded on the market.
d. Use the historical volatility because doing so corrects for the pricing mistakes in the option market.

EXAMPLE 17.2: FRM EXAM 2009—QUESTION 5-1

Assume that implied volatilities from equity option prices display a volatility skew and that implied vols from currency option prices display a volatility smile. Which of the following statements about option price implied volatility curves are *true*?

I. The implied volatility of a deep out-of-the-money equity put option is higher than that of a deep-in-the-money equity put.
II. The implied volatility of a deep out-of-the-money equity call option is higher than that of an at-the-money equity call option.
III. The implied volatility of a deep in-the-money currency call option cannot be the same as that of a deep in-the-money currency put option.
IV. The implied volatility of a deep out-of-the-money currency call option is higher than that of an at-the-money currency call option.

a. I and III only
b. I and IV only
c. II and III only
d. II and IV only

17.2 IMPLIED CORRELATION

17.2.1 Concept

The concept of implied volatility can be extended to implied correlation, provided we have enough options. First, we should note that standard options involve a choice to exchange cash for the asset. This is a special case of an **exchange option**, which involves the surrender of an asset (call it B) in exchange for acquiring another (call it A). The payoff on such a call is

$$c_T = \text{Max}(S_T^A - S_T^B, 0) \tag{17.5}$$

where S^A and S^B are the respective spot prices. Some financial instruments involve the maximum of the value of two assets, which is equivalent to a position in one asset plus an exchange option:

$$\text{Max}(S_t^A, S_t^B) = S_T^B + \text{Max}(S_T^A - S_T^B, 0) \tag{17.6}$$

Margrabe (1978) has shown that the valuation formula is similar to the usual model, except that K is replaced by the price of asset B (S_B), and the risk-free rate by the yield on asset B (q_B).[4] The volatility σ is now that of the difference between the two assets, which is

$$\sigma^2_{AB} = \sigma^2_A + \sigma^2_B - 2\rho_{AB}\sigma_A\sigma_B \tag{17.7}$$

These options also involve the correlation coefficient. So, if we have a triplet of options, involving A, B, and the option to exchange B into A, we can compute σ_A, σ_B, and σ_{AB}. This allows us to infer the implied correlation coefficient. The pricing formula is called the **Margrabe model**.

More generally, this is an example of a **rainbow option**, which is an option exposed to two or more sources of uncertainty. An **option on a basket**, for example, derives its value from the basket with the two assets, $S^A_T + S^B_T$, whose volatility also depends on the implied correlation $\sigma^2_{AB} = \sigma^2_A + \sigma^2_B + 2\rho_{AB}\sigma_A\sigma_B$.

17.2.2 Currency Implied Correlation

A direct application involves quotes on a triplet of currency options. Exchange rates are expressed relative to a base currency, usually the dollar. The **cross rate** is the exchange rate between two currencies other than the reference currency. For instance, say that S_1 represents the dollar price of the British pound (GBP) and that S_2 represents the dollar/euro (EUR) rate. Then the euro/pound rate is given by the ratio

$$S_3(\text{EUR/GBP}) = \frac{S_1(\$/\text{GBP})}{S_2(\$/\text{EUR})} \tag{17.8}$$

Using logs, we can write

$$\ln[S_3] = \ln[S_1] - \ln[S_2] \tag{17.9}$$

The volatility of the cross rate is

$$\sigma^2_3 = \sigma^2_1 + \sigma^2_2 - 2\rho_{12}\sigma_1\sigma_2 \tag{17.10}$$

This shows again that we can infer the correlation coefficient ρ_{12} from the triplet of variances.

In practice, some care should be used when setting up this equation. Equation (17.8) assumes that both the numerator and denominator are in the same currency, in which case the log of the cross rate is the difference. Otherwise, the log of the cross rate is the sum of the logs, and the negative sign in Equation (17.10) must be changed to a positive sign.

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[4] W. Margrabe, "The Value of an Option to Exchange One Asset for Another," *Journal of Finance* 33 (1978): 177–186. See also R. Stulz, "Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications," *Journal of Financial Economics* 10 (1982): 161–185.

17.2.3 Portfolio Average Correlation

The ISD of a portfolio of assets can be related to the ISD of its components through the **average correlation**. Normally, the portfolio variance is related to the individual volatilities using

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \rho_{ij} \sigma_i \sigma_j \tag{17.11}$$

Assume now that there is a constant correlation ρ across all pairs of assets that maintains the portfolio variance

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j (\rho) \sigma_i \sigma_j \tag{17.12}$$

This correlation is a weighted average of the pairwise correlations ρ_{ij}. With option ISDs measured for the portfolio and all the constituents, we can use Equation (17.12) to infer the portfolio average correlation. This can be applied to either realized volatility or to implied volatility.

This average correlation is a summary measure of diversification benefits across the portfolio. All else equal, an increasing correlation increases the total portfolio risk.

Trading strategies can be designed to take advantage of changes in the average correlation. For example a **dispersion trade** takes a short position in index volatility, which is offset by long positions in the volatility of the index components.

EXAMPLE 17.3: IMPLIED CORRELATION

What is the implied correlation between JPY/EUR and EUR/USD when given the following volatilities for foreign exchange rates? JPY/USD at 8%; JPY/EUR at 10%; EUR/USD at 6%.

a. 60%
b. 30%
c. −30%
d. −60%

17.3 VARIANCE SWAPS

A **variance swap** is a forward contract on the realized variance. The payoff is computed as

$$V_T = (\sigma_{t_0,T}^2 - K_V)N \tag{17.13}$$

where N is the notional amount, and σ^2 is the realized variance over the life of the contract, usually measured as

$$\sigma^2 = \frac{252}{\tau}\sum_{i=1}^{\tau}[\ln(S_i/S_{i-1})]^2 \tag{17.14}$$

and K_V is the strike price, or forward price. Variance swaps can be written on any underlying asset, but are most common for equities or equity indices. They allow trades based on direct views on variance. Long positions are bets on high volatility. A similar contract is the **correlation swap**, where the payoff is tied to the realized average correlation in a portfolio over the selected period.

For example, suppose a dealer quotes a one-year contract on the S&P 500 index, with $K_V = (15\%)^2$ and notional of $N = \$100{,}000/(\text{one volatility point})^2$. If at expiration the realized volatility is 17%, the payoff to the long position is $[\$100{,}000/(1^2)][(17)^2 - (15)^2] = \$100{,}000(289 - 225) = \$6{,}400{,}000$. Therefore, the payoff is a quadratic function of the volatility. In theory, it is unlimited.[5] Like any forward contract, K_V is determined so that the initial value of the contract is zero. In fact, the new version of the **VIX index** is the fair strike price for a variance swap on the S&P 500 index, quoted as volatility.

The market value of an outstanding variance swap with $\tau = T - t$ days remaining to maturity is

$$V_t = Ne^{-r\tau}[w(\sigma^2_{t_0,t} - K_V) + (1 - w)(K_t - K_V)] \tag{17.15}$$

where $\sigma^2_{t_0,t}$ is the elapsed variance between the initial time t_0 and the current time, t, w is the fraction of days elapsed since t_0, and K_t is the current forward price. We can verify that at the initial time, $w = 0$, and V_0 is simply proportional to $K_t - K_V$, which is zero if the contract starts at-the-market. At expiration, this converges to Equation (17.13).

Such contracts also allow **correlation trading**. Consider, for example, an index of two stocks. A variance swap is available for each constituent stock as well as for the index. The realized variance of the index depends on the two variances as well as the correlation coefficient. All else equal, a higher correlation translates into a higher portfolio variance. A **long correlation** trade would buy a variance swap on the index and short variance swaps on the components.[6] If the correlation increases, the long position should gain more than the short positions, thereby generating a gain.

17.4 DYNAMIC TRADING

The BS derivation taught us how to price and hedge options. Perhaps even more importantly, it showed that holding a call option is equivalent to holding a fraction of the underlying asset, where the fraction dynamically changes over time.

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[5] In practice, most contracts are capped to a maximum value for the variance equal to m^2K_V. Volatility swaps are also available but are much less common. This is because variance swaps can be hedged relatively easily, using a combination of options. This is not the case for volatility swaps.

[6] Note that keeping the position variance-neutral requires a greater notional amount for the index swaps than for the sum of component swaps.

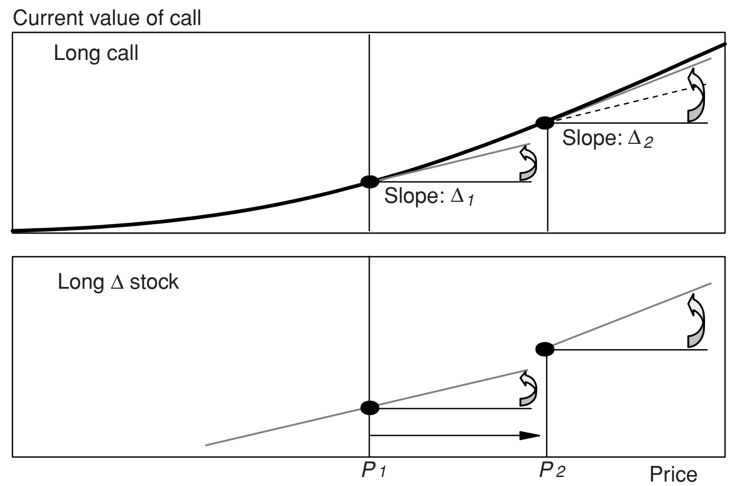


FIGURE 17.5 Dynamic Replication of a Call Option

17.4.1 Dynamic Option Replication

This equivalence is illustrated in Figure 17.5, which displays the current value of a call as a function of the current spot price. The long position in a call is replicated by a partial position in the underlying asset. For an at-the-money position, the initial delta is about 0.5.

As the stock price increases from P_1 to P_2, the slope of the option curve, or delta, increases from Δ_1 to Δ_2. As a result, the option can be replicated by a larger position in the underlying asset. Conversely, when the stock price decreases, the size of the position is cut, as in a graduated stop-loss order. Thus the dynamic adjustment buys more of the asset as its price goes up, and conversely, sells it after a fall.

Figure 17.6 shows the dynamic replication of a put. We start at-the-money with Δ close to -0.5. As the price S goes up, Δ increases toward 0. Note that this is an increase since the initial delta was negative. As with the long call position,

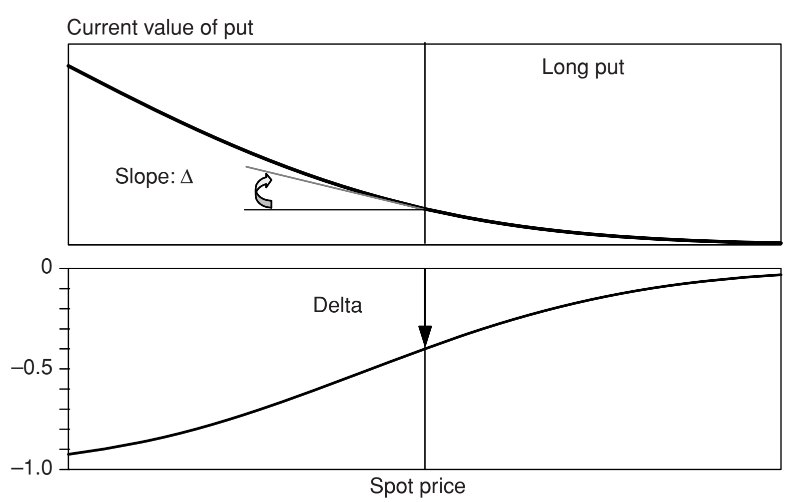


FIGURE 17.6 Dynamic Replication of a Put Option

we *buy* more of the asset *after* its price has gone up. In contrast, short positions in calls and puts imply opposite patterns. Dynamic replication of a short option position implies buying more of the asset after its price has gone down.

Dynamic hedging of traditional options is relatively straightforward because the hedge ratio changes smoothly. In other words, the gamma, or change in delta, is not too high. Exotic options can be easier or harder to hedge.

For Asian options, for example, hedging is easier because the payoff depends on an average price, which is more stable than the price at expiration. For options with discontinuities, in contrast, hedging is more difficult. One example is a binary option, which pays at expiration zero just below the boundary and one just above. This payoff is identical to the delta of a vanilla call option. As a result, their delta has the same shape as the gamma from the BS call option model. A highly variable delta makes dynamic hedging more difficult. Other examples involve barrier options, where the delta is highly unstable just around the barrier.

Even with traditional options, the success of this option replication strategy critically hinges on the assumption of a continuous geometric Brownian motion (GBM) price process. With continuous price movements, it is theoretically possible to closely approximate the option payoff by frequent rebalancing, as often as needed. In practice, the replication may fail if prices experience drastic jumps. If the price gaps down by a large amount within a short interval, the loss incurred on the long delta position will be large before delta can be cut.

17.4.2 Static Option Replication

The previous section has shown how option portfolio can be replicated by dynamic trading in the underlying assets. This is convenient when the option is mispriced, or when there is no traded option with the desired characteristics. Suppose, for instance, that the investor wants to replicate the pattern of a long position in a put option with a maturity of two years. Such options are not actively traded on organized exchanges. Or the desired option may be a more exotic option.

Another approach to replication is to use a portfolio of options that is rebalanced infrequently. **Static replication** is achieved by matching the value of the target option with the portfolio of options at selected boundaries and dates.

Consider, for example, an up-and-out call option that expires in a year with a strike price of 100 and barrier of 120. The current stock price is at 100. If it hits 120 at any time before expiration, the option dies. For boundaries, we choose $c_T = \text{Max}(S_T - K, 0)$, if $S_T < 120$, and $c_t = 0$ when $S_t \geq 120$. To replicate the payoff at maturity, we could choose one long call option with $K = 100$ plus two short calls with $K = 120$, which represent the loss in value if $S > 120$. With more options, the value of the replicating portfolio converges to the desired pattern.

17.4.3 Implications for Trading

For risk managers, these patterns are extremely important for a number of reasons. First, as strange as it sounds, dynamic replication of a long option position is bound to lose money. This is because it buys the asset *after* the price has gone up—in other words, too late. Each transaction loses a small amount of money, which will accumulate precisely to an option premium. Note, however, that this premium is driven by the volatility realized over the horizon instead of the implied volatility.

KEY CONCEPT

The dynamic replication of a long position in an option systematically buys after the price has gone up and sells after the price has gone down. This process is bound to lose money, in an amount that represents an option premium based on the realized volatility (instead of the implied volatility).

Consider next a strategy of selling an option and dynamically hedging it using the underlying instrument, or its futures. Assume that a trader sells a call. This has negative delta, which needs to be hedged dynamically with a positive delta position in the underlying. As a whole, the portfolio is delta-neutral. The question is, should this strategy create systematic gains or losses?

The answer depends on the price of the option. Selling the call locks in a revenue that is a function of the implied volatility. Implementing the dynamic hedging will create a cost that is a function of the realized volatility. Hence, if the implied volatility tends to be greater than the realized volatility, this strategy should on average generate a profit.

Indeed, this is what we observe in practice. In equity markets, implied volatilities tend to be greater than realized vols. This explains why selling implied volatility is a common strategy employed by hedge funds and other traders. The reason for the difference is because the implied vol is a risk-neutral measure, which may systematically differ from the realized volatility because of a risk premium.

Indeed, we can find evidence that could explain this risk premium. The VIX is negatively correlated with stock market movements. Figure 17.7 displays monthly returns over the period 1990 to 2009. The graph has a pronounced and significant negative slope, indicating that the VIX tends to go up when the market falls. In these situations, selling delta-hedged options should generate a loss.

Hence, our delta-neutral portfolio (short the option, which is dynamically hedged) has negative beta. It tends to lose money precisely when the stock market falls. This could explain why volatility-selling trading strategies tend to generate profits over time. Such profits could simply reflect a portion of the equity premium.

17.4.4 General Implications

Dynamic hedging has more general implications. These automatic trading systems, if applied on a large scale, have the potential to be destabilizing. Selling on a downturn in price can exacerbate the downside move. Some have argued that the crash of 1987 was due to the large-scale selling of portfolio insurers in a falling

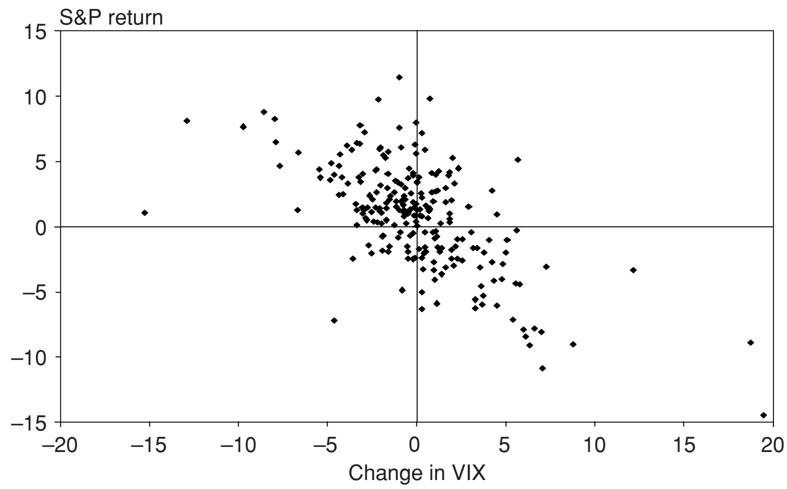


FIGURE 17.7 Relationship between S&P Return and Change in VIX

market. These portfolio insurers were, in effect, replicating long positions in puts, blindly selling when the market was falling.[7]

Another point is that this pattern of selling an asset after its price has gone down is similar to prudent risk-management practices. Typically, traders must cut down their positions after they incur large losses. This is similar to decreasing Δ when S drops. Thus, loss-limit policies bear some resemblance to a long position in an option. They have positive gamma, less potential for a large downside, and positive skewness.

EXAMPLE 17.4: FRM EXAM 2009—QUESTION 4-25

You are the risk manager of your bank responsible for the derivatives desk. A trader has sold 300 call option contracts each on 100 shares of Nissan Motors with time to maturity of 90 days at USD 1.80. The delta of the option on one share is 0.60. You have hedged the option exposure by buying 18,000 shares of the underlying. The next day, the stock price falls and the delta of the options falls to 0.54. In order to keep the options hedged, you have to

a. Buy 1,800 shares of Nissan Motors
b. Sell 1,800 shares of Nissan Motors
c. Buy 1,080 shares of Nissan Motors
d. Sell 1,080 shares of Nissan Motors

[7] The exact role of portfolio insurance, however, is still hotly debated. Others have argued that the crash was aggravated by a breakdown in market structures (i.e., the additional uncertainty due to the inability of the stock exchanges to handle abnormal trading volumes).

EXAMPLE 17.5: PROFITS FROM DYNAMIC HEDGING

A trader buys an at-the-money call option with the intention of delta-hedging it to maturity. Which one of the following is likely to be the most profitable over the life of the option?

a. An increase in implied volatility
b. The underlying price steadily rising over the life of the option
c. The underlying price steadily decreasing over the life of the option
d. The underlying price drifting back and forth around the strike over the life of the option

EXAMPLE 17.6: FRM EXAM 2004—QUESTION 26

A non-dividend-paying stock has a current price of \$100 per share. You have just sold a six-month European call option contract on 100 shares of this stock at a strike price of \$101 per share. You want to implement a dynamic delta-hedging scheme to hedge the risk of having sold the option. The option has a delta of 0.50. You believe that delta would fall to 0.44 if the stock price falls to \$99 per share. Identify what action you should take **now** (i.e., when you have just written the option contract) to make your position delta-neutral. After the option is written, if the stock price falls to \$99 per share, identify what action should be taken at that time (i.e., **later**) to rebalance your delta-hedged position.

a. **Now**: buy 50 shares of stock; **later**: buy 6 shares of stock.
b. **Now**: buy 50 shares of stock; **later**: sell 6 shares of stock.
c. **Now**: sell 50 shares of stock; **later**: buy 6 shares of stock.
d. **Now**: sell 50 shares of stock; **later**: sell 6 shares of stock.

EXAMPLE 17.7: FRM EXAM 2009—QUESTION 5-3

Trader A purchases a down-and-out call with a strike price of USD 100 and a barrier at USD 96 from Trader B. Both traders need to unwind their delta hedge at the barrier. Which trader is more at risk if there is a price gap (discontinuity) that prevents them from exiting the trade at the barrier?

a. Trader A has the bigger risk.
b. Trader B has the bigger risk.
c. They both have the same risk.
d. Neither trader has any risk because both are hedged.

17.5 CONVERTIBLE BONDS AND WARRANTS

17.5.1 Concepts

We now turn to convertible bonds and warrants. While these instruments have option-like features, they differ from regular options. When a call option is exercised, for instance, the long purchases an outstanding share from the short. There is no net creation of shares. In contrast, the exercise of convertible bonds, of warrants, and of executive stock options entails the creation of new shares, as the option is sold by the corporation itself. Because the number of shares goes up, the existing shares are said to be **diluted** by the creation of new shares.

Warrants are long-term call options issued by a corporation on its own stock. They are typically created at the time of a bond issue, but they trade separately from the bond to which they were originally attached. When a warrant is exercised, it results in a cash inflow to the firm that issues more shares.

Convertible bonds are bonds issued by a corporation that can be converted into equity at certain times using a predetermined exchange ratio. They are equivalent to a regular bond plus a warrant. This allows the company to issue debt with a lower coupon than otherwise.

For example, a bond with a **conversion ratio** of 10 allows its holder to convert one bond with par value of \$1,000 into 10 shares of the common stock. The **conversion price**, which is really the strike price of the option, is \$1,000/10 = \$100. The corporation will typically issue the convertible deep out-of-the-money, for example when the stock price is at \$50. When the stock price moves, for instance to \$120, the bond can be converted into stock for an immediate option profit of (\$120 − \$100) × 10 = \$200.

Figure 17.8 describes the relationship between the value of the convertible bond and the **conversion value**, defined as the current stock price times the conversion ratio. The convertible bond value must be greater than the price of an otherwise identical straight bond and the conversion value.

For high values of the stock price, the firm is unlikely to default and the straight bond price is constant, reflecting the discounting of cash flows at the

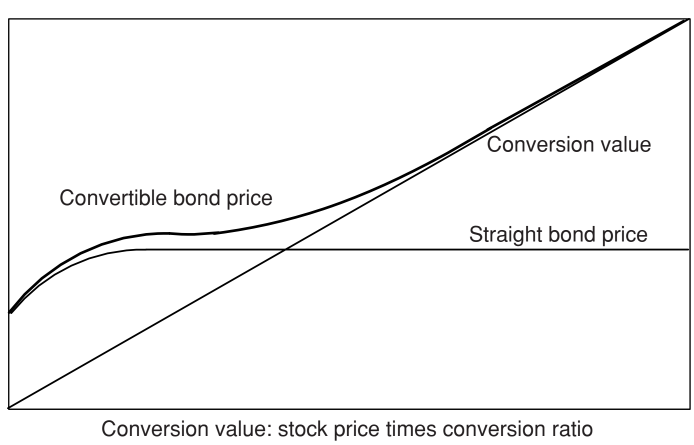


FIGURE 17.8 Convertible Bond Price and Conversion Value

risk-free rate. In this situation, it is almost certain the option will be exercised and the convertible value is close to the conversion value. For low values of the stock price, the firm is likely to default and the straight bond price drops, reflecting the likely loss upon default. In this case, the convertible is said to be **busted**. In this situation, it is almost certain the option will not be exercised, and the convertible value is close to the straight bond value. In the intermediate region, the convertible value depends on both the conversion and straight bond values.

Example: A Convertible Bond

Consider an 8% annual coupon, 10-year convertible bond with face value of $1,000. The yield on similar-maturity straight debt issued by the company is currently 8.5%, which gives a current value of straight debt of $967. The bond can be converted into common stock at a ratio of 10 to 1.

Assume first that the stock price is $50. The conversion value is then $500, much less than the straight debt value of $967. This corresponds to the left area of Figure 17.8. If the convertible trades at $972, its promised yield is 8.42%. This is close to the yield of straight debt, as the option has little value.

Assume next that the stock price is $150. The conversion value is then $1,500, much higher than the straight debt value of $967. This corresponds to the right area of Figure 17.8. If the convertible trades at $1,505, its promised yield is 2.29%. In this case, the conversion option is in-the-money, which explains why the yield is so low.

One complication is that most convertibles are also callable at the discretion of the firm. Convertible securities can be called for several reasons. First, an issue can be called to force conversion into common stock when the stock price is high enough. Bondholders have typically a month during which they can still convert, in which case this is a **forced conversion**. This call feature gives the corporation more control over conversion. It also allows the company to raise equity capital by forcing the bondholders to pay the exercise price.

Second, the call may be exercised when the option value is worthless and the firm can refinance its debt at a lower coupon. This is similar to the call of a nonconvertible bond, except that the convertible must be busted, which occurs when the stock price is much lower than the conversion price.

17.5.2 Valuation

Warrants and convertibles can be valued by adapting standard option pricing models to the dilution effect of new shares. Consider a company with N outstanding shares and M outstanding warrants, each allowing the holder to purchase γ shares at the fixed price of K. At origination, the value of the firm equity includes the warrant, or

$$V_0 = NS_0 + MW_0 \tag{17.16}$$

where S_0 is the initial stock price just before issuing the warrant, and W_0 is the up-front value of the warrant.

After dilution, the total value of the firm equity includes the value of the equity before exercise (including the original value of the warrants) plus the proceeds from exercise (i.e., $V_T + M\gamma K$). The number of shares then increases to $N + \gamma M$. The total payoff to the warrant holder is

$$W_T = \gamma \text{Max}(S_T - K, 0) = \gamma(S_T - K) = \gamma\left(\frac{V_T + M\gamma K}{N + \gamma M} - K\right) \qquad (17.17)$$

which must be positive. After simplification, this is also

$$W_T = \gamma\left(\frac{V_T - NK}{N + \gamma M}\right) = \frac{\gamma}{N + \gamma M}(V_T - NK) = \frac{\gamma N}{N + \gamma M}\left(\frac{V_T}{N} - K\right) \qquad (17.18)$$

which is equivalent to $n = \gamma N/(N + \gamma M)$ options on the stock price. The warrant can be valued by standard option models with the asset value equal to the stock price plus the warrant proceeds, multiplied by the factor n,

$$W_0 = n \times c\left(S_0 + \frac{M}{N} W_0, K, \tau, \sigma, r, d\right) \qquad (17.19)$$

with the usual parameters. Here, the unit asset value is $\frac{V_0}{N} = S_0 + \frac{M}{N} W_0$. This must be solved iteratively since W_0 appears on both sides of the equation. If, however, M is small relative to the current float, or number of outstanding shares N, the formula reduces to a simple call option in the amount γ:

$$W_0 = \gamma \; c(S_0, K, \tau, \sigma, r, d) \qquad (17.20)$$

Example: Pricing a Convertible Bond

Consider a zero-coupon, 10-year convertible bond with face value of \$1,000. The yield on similar-maturity straight debt issued by the company is currently 8.158%, using continuous compounding, which gives a straight debt value of \$442.29.

The bond can be converted into common stock at a ratio of 10 to 1 at expiration only. This gives a strike price of $K = \$100$. The current stock price is \$60. The stock pays no dividend and has annual volatility of 30%. The risk-free rate is 5%, also continuously compounded.

Ignoring dilution effects, the Black-Scholes model gives an option value of \$216.79. So, the theoretical value for the convertible bond is $P = \$442.29 + \$216.79 = \$659.08$. If the market price is lower than \$659, the convertible is said to be cheap. This, of course, assumes that the pricing model and input assumptions are correct.

17.5.3 Risk Management

We have seen that convertibles involve a long position in a call option on the firm's equity. As such, the option value should increase with the usual factors that affect options, in particular increasing stock prices and increasing implied volatility. The option is more complex when the convertible is also callable by the firm, which is a short option position for the bond investor. In addition, the value of a convertible bond is also exposed to increasing interest rates and credit spreads, like regular corporate bonds.

Some hedge fund strategies are based on **convertible bond arbitrage**, which will be covered in more detail in Chapter 30. These hedge fund managers typically buy convertible bonds because they are cheap but then hedge the resulting delta by shorting the underlying stock. The short is actively managed by dynamic trading, also called **gamma trading**. Recall that replicating a long position in an option is bound to create a loss. Conversely, the convertible arb manager replicates a short position in a call option, which is bound to create a gain that depends on the realized volatility. Provided the implied volatility is low enough, the sum of these activities should generate a gain.

It is important to note that the implied volatility derived from convertible bonds is not the same as the VIX, which is a short-term volatility derived from exchange-listed options. The convertible's implied volatility reflects the convertible bond price, which may also be affected by liquidity factors. Indeed, in 2008, the convertible market lost value even though the VIX went to all-time highs. This reflected severe liquidity problems in bond markets, as prime brokers forced hedge funds to deleverage, which in turn forced selling of convertible bonds in a market with few buyers. The problem was made even worse when the Securities and Exchange Commission (SEC) imposed restrictions on the short selling of shares, which wreaked havoc on convertible hedging strategies. This largely explains why the Credit Suisse First Boston (CSFB) convertible arbitrage hedge fund index lost 32% in 2008. The following year, however, the sector gained 47%.

EXAMPLE 17.8: CONVERTIBLE BOND DECISIONS

A corporate bond with face value of $100 is convertible at $40 and the corporation has called it for redemption at $106. The bond is currently selling at $115 and the stock's current market price is $45. Which of the following would a bondholder most likely do?

a. Sell the bond
b. Convert the bond into common stock
c. Allow the corporation to call the bond at 106
d. None of the above

EXAMPLE 17.9: CALLS IN CONVERTIBLE BONDS

What is the main reason why convertible bonds are generally issued with a call?

a. To make their analysis less easy for investors
b. To protect against unwanted takeover bids
c. To reduce duration
d. To force conversion if in-the-money

EXAMPLE 17.10: FRM EXAM 2009—QUESTION 3-20

What is the effect on the value of a callable convertible bond of a decrease in interest rate volatility and stock price volatility?

a. An increase in value due to both interest rate volatility and stock price volatility
b. An increase and decrease in value, respectively
c. A decrease and increase in value, respectively
d. A decrease in value due to both

17.6 IMPORTANT FORMULAS

Implied standard deviation, or volatility ISD $= f^{-1}(c_t, S_t, r_t, r_t^*, K, \tau)$

Margrabe model: Replace S by S_A, K by S_B, set $\sigma_{AB}^2 = \sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B$

Cross-exchange rate: $S_3(\text{EUR/GBP}) = \frac{S_1(\$/\text{GBP})}{S_2(\$/\text{EUR})}$, $\sigma_3^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$

Portfolio average correlation: ρ from $\sigma_p^2 = \sum_{i=1}^N w_i^2\sigma_i^2 + 2\sum_{i=1}^N \sum_{j<i}^N w_i w_j(\rho)\sigma_i\sigma_j$

Payoff on a variance swap: $V_T = (\sigma^2 - K_V)N$

Valuation of an outstanding variance swap: $V_t = Ne^{-r\tau}[w(\sigma_{t_0,t}^2 - K_V) + (1-w)(K_t - K_V)]$

Warrant valuation: $W_0 = n \times c\left(S_0 + \frac{M}{N}W_0, K, \tau, \sigma, r, d\right)$

17.7 ANSWERS TO CHAPTER EXAMPLES

Example 17.1: FRM Exam 2008—Question 2-11

c. The book should be marked using volatilities that give prices that are closest to market prices. This means using the ISDs of the most similar options. Also, using

ATM ISDs, as suggested in answer a., will understate the value of the short OTM options, which artificially inflates the trader's profit.

Example 17.2: FRM Exam 2009—Question 5-1

b. A volatility skew means that, for equities, the ISD of out-of-the-money (OTM) puts is greater than that of ITM puts, so answer I. is true. Conversely, the ISD of ITM puts, or equivalently that of OTM calls, is similar to that of ATM options, so answer II. is false. A volatility skew means that, for currencies, the ISD of out-of-the-money options is greater than that of ATM options, so answer IV. is true. On the other hand, OTM and ITM options might have similar vols (for currency options), so answer III. is false.

Example 17.3: Implied Correlation

d. The logs of JPY/EUR and EUR/USD add up to that of JPY/USD: ln[JPY/USD] = ln[JPY/EUR] + ln[EUR/USD]. So, $\sigma^2(\text{JPY/USD}) = \sigma^2(\text{JPY/EUR}) + \sigma^2(\text{EUR/USD}) + 2\rho\sigma(\text{JPY/EUR})\sigma(\text{EUR/USD})$, or $8^2 = 10^2 + 6^2 + 2\rho 10 \times 6$, or $2\rho 10 \times 6 = -72$, or $\rho = -0.60$.

Example 17.4: FRM Exam 2009—Question 4-25

b. First, we verify that the initial amount purchased is correct. This is $0.60 \times 300 \times 100 = 18{,}000$ shares. If the delta falls to 0.54, or by 0.06, the risk manager will have to sell $0.06 \times 300 \times 100 = 1{,}800$ shares.

Example 17.5: Profits from Dynamic Hedging

d. An important aspect of the question is the fact that the option is held to maturity. Answer a. is incorrect because changes in the implied volatility would change the value of the option, but this has no effect when holding to maturity. The profit from the dynamic portfolio will depend on whether the actual volatility differs from the initial implied volatility. It does not depend on whether the option ends up in-the-money, so answers b. and c. are incorrect. The portfolio will be profitable if the actual volatility is small, which implies small moves around the strike price (answer d.).

Example 17.6: FRM Exam 2004—Question 26

b. The dynamic hedge should replicate a long position in the call. Due to the positive delta, this implies a long position of $\Delta \times 100 = 50$ shares. If the delta falls, the position needs to be adjusted by selling $(0.5 - 0.44) \times 100 = 6$ shares.

Example 17.7: FRM Exam 2009—Question 5-3

b. Each trader replicates dynamically the down-and-out call as a hedge. Trader B sold the option, so needs to replicate a long position in this call. The hedge ratio for a down-and-out call resembles the usual one except that it has an abrupt discontinuity, dropping to zero below the barrier. Just above the barrier, Trader B is long the asset in the amount of the hedge ratio (e.g., 0.4). When the price jumps down below the barrier, Trader B will be stuck with a large loss. Intuitively, this loss is the gain to Trader A, who has the opposite position.

Example 17.8: Convertible Bond Decisions

a. The conversion rate is expressed here in terms of the conversion price. The conversion rate for this bond is \$100 into \$40, or 1 bond into 2.5 shares. Immediate conversion will yield $2.5 \times \$45 = \$112.5$. The call price is \$106. Since the market price is higher than the call price and the conversion value, and the bond is being called, the best value is achieved by selling the bond.

Example 17.9: Calls in Convertible Bonds

d. Companies issue convertible bonds because the coupon is lower than for regular bonds. In addition, these bonds are callable in order to force conversion into the stock at a favorable ratio. In the previous question, for instance, conversion would provide equity capital to the firm at the price of \$40, while the market price is at \$45.

Example 17.10: FRM Exam 2009—Question 3-20

b. A decrease in stock price volatility decreases the value of the equity conversion option and thus the convertible bond price. A decrease in interest rate volatility decreases the value of the interest rate call option. Because the bond investor is short the interest rate option, this increases the value of the convertible.

CHAPTER 18

Mortgage-Backed Securities Risk

This chapter turns to the pricing and risk management of mortgage-backed securities (MBSs). These are securities whose cash flows are backed by pools of mortgage loans, residential or commercial. The MBS market is an example of **securitization**, which is the process by which assets are pooled, and securities representing interests in the pool are issued.

Payments on mortgage loans are typically structured as annuities, which involve a regular payment covering interest and principal amortization. A key feature of a mortgage loan, however, is that the homeowner has the ability to repay the principal early at any time. Thus, the homeowner is long a prepayment option.

This makes the risk analysis of MBSs considerably more complex than for regular bonds. In addition to the usual interest rate risk and credit risk, MBSs involve prepayment risk. This is covered in Section 18.1. Due to this embedded short option, MBSs are exposed to volatility risk. This explains why yields on MBS securities are higher than otherwise, by an amount known as the yield spread. This spread can be decomposed into an option component and an option-adjusted spread (OAS).

Section 18.2 then discusses MBSs and the process of securitization. It discusses drawbacks to this securitization process that have become apparent in the latest credit crisis. Finally, Section 18.3 describes the tranching of MBS products into collateralized mortgage obligations (CMOs).

18.1 PREPAYMENT RISK

18.1.1 Mortgages as Annuities

As usual with fixed-income securities, the first step in the pricing of mortgages is to lay out the projected cash flows. Mortgages can be structured to have fixed or floating payments. To simplify, we assume a fixed payment of C_t every month.

FRM Exam Part 2 topic.

The price-yield relationship is

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{18.1}$$

Consider a typical 30-year, fixed-rate mortgage on a balance of \$100,000 with an annual yield of 6%. The monthly payment is then $C = \$600$. Initially, \$500 of this goes to interest, which is $I_t = (6\%/12) \times \$100{,}000$, and the remainder, $P_t = C - I_t = \$100$, to principal payment. At the beginning of the following month, the principal is reduced to \$99,900, which lowers the next interest payment and increases the regularly scheduled principal payment. Toward the end, most of the annuity payment goes to repayment of principal.

Even though the maturity of this bond is $T = 30$ years, a better measure is the **average life** (AL), which takes into account the time profile of principal repayment,

$$\mathrm{AL} = \sum_{t=1}^{T} t P_t / P \tag{18.2}$$

In this case, AL = 19.3 years. We can also compute a duration measure, which accounts for both principal and interest payments, and is $D = 10.8$ years in this case. This also represents the sensitivity of the bond price to movements in yields. Note that these measures are considerably lower than for an otherwise identical bullet bond, for which AL = 30 and $D = 14.0$.

When dealing with a pool of mortgages, its characteristics are described by the weighted average maturity (WAM), the weighted average coupon (WAC), and the weighted average life (WAL).

18.1.2 Prepayment Speed

Even with fixed payments, however, the cash flows are uncertain because the homeowner has the possibility of making early payments of principal. For the borrower, this represents a long position in an option. For the lender, this is a short position.

In some cases, these prepayments are random, for instance when the homeowner sells the home due to changing job or family conditions. In other cases, these prepayments are more predictable. When interest rates fall, prepayments increase as homeowners can refinance at a lower cost. Thus, this is akin to callable bonds, which are common in the corporate bond markets, and where the borrower has the option to call back its bonds at fixed prices at fixed points in time.[1]

Generally, these factors affect mortgage refinancing patterns:

- *Spread between the mortgage rate and current rates:* If current mortgage rates fall relative to what the homeowner is paying, the borrower has an incentive

[1] Some bonds are also puttable, which means that investors have the option to put back the bond to the borrower. In this case, the investor is long an option instead.

to refinance the loan at a lower rate. Thus, a widening spread increases prepayments. Like a callable bond, there is a greater benefit to refinancing if it achieves a significant cost saving.

- *Age of the loan:* Prepayment rates are generally low just after the mortgage loan has been issued, presumably because borrowing involves fixed costs. Prepayments then gradually increase over time until they reach a stable, or seasoned, level. This effect is known as **seasoning**.
- *Refinancing incentives:* The smaller the costs of refinancing, the more likely homeowners will refinance often.
- *Previous path of interest rates:* Refinancing is more likely to occur if rates have been high in the past but recently dropped. In this scenario, past prepayments have been low but should rise sharply. In contrast, if rates are low but have been so for a while, most of the principal will already have been prepaid. This path dependence is usually referred to as **burnout**.
- *Level of mortgage rates:* Lower rates increase affordability and turnover.
- *Economic activity:* An economic environment where more workers change job locations creates greater job turnover, which is more likely to lead to prepayments.
- *Seasonal effects:* There is typically more home-buying in the spring, leading to increased prepayments in early fall.

The prepayment rate is summarized into what is called the **conditional prepayment rate (CPR)**, which is expressed in annual terms. This number can be translated into a monthly number, known as the **single monthly mortality (SMM) rate**, using the adjustment:

$$(1 - \text{SMM})^{12} = (1 - \text{CPR}) \tag{18.3}$$

For instance, if CPR = 6% annually, the monthly proportion of principal paid early will be $\text{SMM} = 1 - (1 - 0.06)^{1/12} = 0.005143$, or 0.514% monthly. For a loan with a beginning monthly balance (BMB) of BMB = \$50,525 and a scheduled principal (SP) payment of SP = \$67, the prepayment will be $0.005143 \times (\$50{,}525 - \$67) = \$260$.

To price the mortgage, the portfolio manager should describe the schedule of projected prepayments during the remaining life of the bond. This depends on all the factors described earlier.

Prepayments can be described relative to an industry standard, known as the **Public Securities Association (PSA)** prepayment model. The PSA model assumes a CPR of 0.2% for the first month, going up by 0.2% per month for the next 30 months, until 6% thereafter. Formally, this is:

$$\text{CPR} = \text{Min}[6\% \times (t/30), 6\%] \tag{18.4}$$

This pattern is described in Figure 18.1 as the 100% PSA speed. By convention, prepayment patterns are expressed as a percentage of the PSA speed. For example, 165% PSA would represent a faster pattern and 70% PSA a slower pattern.

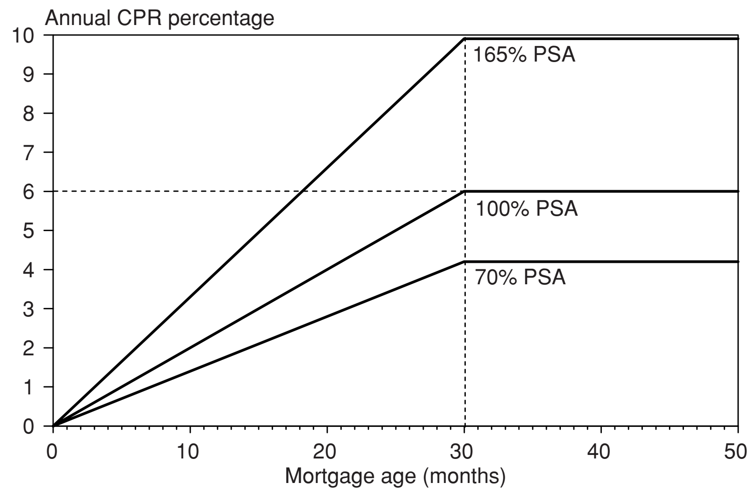


FIGURE 18.1 Prepayment Pattern

Example: Computing the CPR

Consider a mortgage issued 20 months ago with a speed of 150% PSA. What are the CPR and SMM?

The PSA speed is $\text{Min}[6\% \times (20/30), 6\%] = 0.04$. Applying the 150 factor, we have $\text{CPR} = 150\% \times 0.04 = 0.06$. This implies $\text{SMM} = 0.514\%$.

The next step is to project cash flows based on the prepayment speed pattern. Figure 18.2 displays cash-flow patterns for a 30-year loan with a face amount of

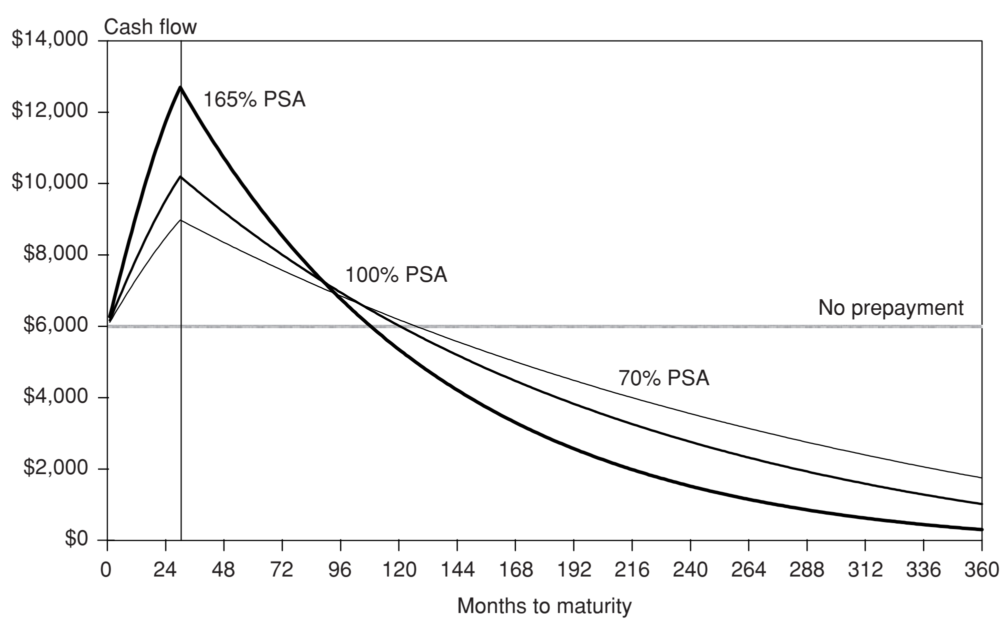


FIGURE 18.2 Cash Flows for Various PSA Speeds

$1 million and 6% interest rate. The horizontal "No prepayment" line describes the fixed annuity payment of $6,000 without any prepayment. The "100% PSA" line describes an increasing pattern of cash flows, peaking in 30 months and decreasing thereafter. This point corresponds to the stabilization of the CPR at 6%. This pattern is more marked for the "165% PSA" line, which assumes a faster prepayment speed.

Early prepayments create lower payments later, which explains why the 100% PSA line is initially higher than the 70% line, and then lower as the principal has been paid off more quickly.

EXAMPLE 18.1: FRM EXAM 2008—QUESTION 2-37

MBSs are a class of securities where the underlying is a pool of mortgages. In addition to the credit risk of a borrower defaulting on the loan, mortgages also have prepayment risk because the borrower has the option to repay the loan early (at any time) usually due to favorable interest rate changes. From an investor's point of view, a mortgage-backed security is equivalent to holding a long position in a nonrepayable mortgage pool and which of the following?

a. A long American call option on the underlying
b. A short American call option on the underlying
c. A short European put option on the underlying
d. A long American put option on the underlying

EXAMPLE 18.2: COMPUTING THE SMM

Suppose the annual conditional prepayment rate (CPR) for a mortgage-backed security is 6%. What is the corresponding single monthly mortality (SMM) rate?

a. 0.514%
b. 0.334%
c. 0.5%
d. 1.355%

18.1.3 Measuring Prepayment Risk

Armed with a forecast of cash flows, we can now discount these into a current value. The next step is the measurement of risk, which is obtained from changes in these values. Like other fixed-income instruments, mortgages are subject to

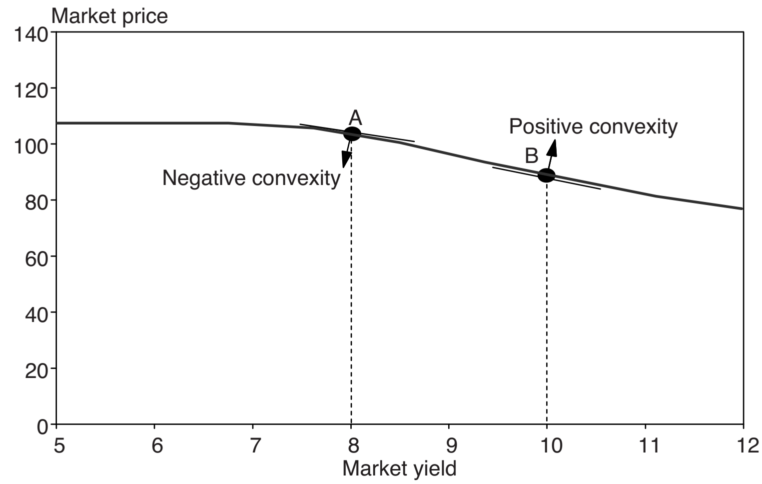


FIGURE 18.3 Negative Convexity of Mortgages

market risk, due to fluctuations in interest rates, and to credit risk, due to homeowner default. The previous section has shown that mortgages are also subject to **prepayment risk**, which is the risk that the principal will be repaid early.

Consider, for instance, an 8% mortgage, which is illustrated in Figure 18.3. If rates drop to 6%, homeowners will rationally prepay early to refinance the loan, as it is cheaper to pay 6% than 8%. Because the average life of the loan is shortened, this is called **contraction risk**. Conversely, if rates increase to 10%, homeowners will be less likely to refinance early, and prepayments will slow down. Because the average life of the loan is extended, this is called **extension risk**.

As shown in Figure 18.3, this feature creates negative convexity at point A. This reflects the fact that the investor in a mortgage is short an option. At point B, in contrast, interest rates are very high and it is unlikely that the homeowner will refinance early. Hence, the option is nearly worthless and the mortgage behaves like a regular bond, with the usual positive convexity.

This changing cash-flow pattern makes standard duration measures unreliable. Instead, sensitivity measures are computed using **effective duration** and **effective convexity**, as explained in Chapter 1. The measures are based on the estimated price of the mortgage for three yield values, making suitable assumptions about how changes in rates should affect prepayments.

Table 18.1 shows an example. In each case, we consider an up move and a down move of 25bp. In the first, "unchanged" panel, the PSA speed is assumed to be constant at 165 PSA. In the second, "changed" panel, we assume a higher PSA speed if rates drop and lower speed if rates increase. When rates drop, the mortgage value goes up but slightly less than with a constant PSA speed. This reflects contraction risk. When rates increase, the mortgage value drops by more than if the prepayment speed had not changed. This reflects extension risk.

As we have seen in Chapter 6, **effective duration** is measured as

$$D^E = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2P_0 \Delta y)} \tag{18.5}$$

TABLE 18.1 Computing Effective Duration and Convexity

	Initial	Unchanged PSA		Changed PSA	
Yield	7.50%	+25bp	−25bp	+25bp	−25bp
PSA		165 PSA	165 PSA	150 PSA	200 PSA
Price	100.125	98.75	101.50	98.7188	101.3438
Duration		5.49 years		5.24 years	
Convexity		0		−299	

Effective convexity is measured as

$$C^E = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0 \Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0 \Delta y)} \right] \Big/ \Delta y \tag{18.6}$$

In the first, "unchanged" panel, the effective duration is 5.49 years and the convexity is close to zero. In the second, "changed" panel, the effective duration is 5.24 years and the convexity is negative, as expected, and quite large.

KEY CONCEPT

Mortgage investments have negative convexity, which reflects the short position in an option granted to the homeowner to repay early. This creates extension risk when rates increase or contraction risk when rates fall.

The option feature in mortgages increases their yield. To ascertain whether the securities represent good value, portfolio managers need to model the option component. The approach most commonly used is the **option-adjusted spread** (OAS).

We start from the **static spread** (SS), which is the difference between the yield of the MBS and that of a Treasury note with the same weighted average life. A better measure is the **zero spread** (ZS), which is a fixed spread added to zero-coupon rates so that the discounted value of the projected cash flows equals the current price

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t + ZS)^t} \tag{18.7}$$

The OAS method involves running simulations of various interest rate scenarios and prepayments to establish the option cost. The OAS is then measured as the plug-in

$$\text{OAS} = \text{Zero Spread} - \text{Option Cost} \tag{18.8}$$

which represents the net richness or cheapness of the instrument. Within the same risk class, a security trading at a high OAS is preferable to others.

The OAS is more stable over time than the spread, because the latter is affected by the option component. This explains why during market rallies (i.e., when long-term Treasury yields fall) yield spreads on current coupon mortgages often widen. These mortgages are more likely to be prepaid early, which makes them less attractive. Their option cost increases, pushing up the yield spread. In addition, the option cost tends to increase when the volatility of interest rate increases, pushing up the zero spread as well.

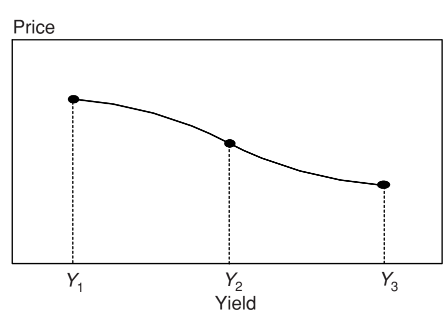
EXAMPLE 18.3: FRM EXAM 2000—QUESTION 3

How would you describe the typical price behavior of a low-premium mortgage pass-through security?

- **a.** It is similar to a U.S. Treasury bond.
- **b.** It is similar to a plain-vanilla corporate bond.
- **c.** When interest rates fall, its price increase would exceed that of a comparable-duration U.S. Treasury bond.
- **d.** When interest rates fall, its price increase would lag that of a comparable-duration U.S. Treasury bond.

EXAMPLE 18.4: FRM EXAM 2003—QUESTION 52

What bond type does the following price-yield curve represent and at which yield level is convexity equal to zero?



- **a.** Puttable bond with convexity close to zero at Y_2.
- **b.** Puttable bond with convexity close to zero at Y_1 and Y_3.
- **c.** Callable bond with convexity close to zero at Y_2.
- **d.** Callable bond with convexity close to zero at Y_1 and Y_3.

EXAMPLE 18.5: FRM EXAM 2006—QUESTION 93

You are analyzing two comparable (same credit rating, maturity, liquidity, rate) U.S. callable corporate bonds. The following data is available for the nominal spread over the U.S. Treasury yield curve and *Z* spread and option-adjusted spread (OAS) relative to the U.S. Treasury spot curve:

	X	Y
Nominal spread	145	130
Z spread	120	115
OAS	100	105

The nominal spread on the comparable option-free bonds in the market is 100 basis points. Which of the following statements is correct?

a. X only is undervalued.
b. Y only is undervalued.
c. X and Y both are undervalued.
d. Neither X nor Y is undervalued.

18.2 SECURITIZATION

18.2.1 Principles of Securitization

One problem with mortgage loans is that they are not tradable. In the past, they were originated and held by financial institutions such as savings and loans. This arrangement, however, concentrates risk in an industry that may not be able to hedge it efficiently. Also, it limits the amount of capital that can flow into mortgages. **Mortgage-backed securities** (MBSs) were created to solve this problem. MBSs are tradable securities that represent claims on pools of mortgage loans.[2]

This is an example of **securitization**, which is the process by which assets are pooled and securities representing interests in the pool are issued. These assets are created by an **originator**, or **issuer**.

The first step of the process is to create a new legal entity, called a **special-purpose vehicle** (**SPV**), or **special-purpose entity** (**SPE**). The originator then pools a group of assets and sells them to the SPV. In the next step, the SPV issues tradable claims, or securities, that are backed by the financial assets. Figure 18.4 describes a basic securitization structure.

[2] The MBS market was developed largely by Salomon Brothers in the early 1980s. This is described in a very entertaining book by Michael Lewis, *Liar's Poker* (New York: Norton, 1989).

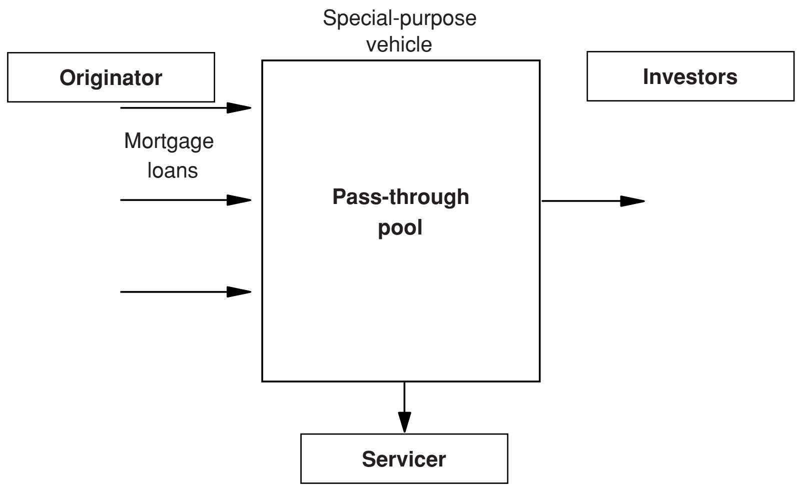


FIGURE 18.4 Securitization

A major advantage of this structure is that it shields the asset-backed security (ABS) investor from the credit risk of the originator. This requires, however, a clean sale of the assets to the SPV. Otherwise, the creditors of the originators might try to seize the SPV's assets in a bankruptcy proceeding. Other advantages are that pooling offers ready-made diversification across many assets.

The growth of securitization is being fueled by the **disintermediation** of banks as main providers of capital to everyone. When banks act as financial intermediaries, they raise funds (recorded as liabilities on the balance sheet) that are used for making loans (recorded as assets). With securitization, both assets and liabilities are removed from the balance sheet, requiring less equity capital to operate. Securitization provides regulatory capital relief if it enables the originator to hold proportionately less equity capital than otherwise. For instance, if the capital requirements for mortgages are too high, the bank will benefit from spinning off mortgage loans into securities, because its required capital will drop sharply.

For the originator, securitization creates an additional source of funding. Securitization can also be used to manage the bank's risk profile. If the securitized assets have the same risk as the rest of the bank's assets, the relative risk of the bank is not changed, even though its size shrinks. In contrast, if the collateral is much riskier than the rest of the assets, the bank will have lowered its risk profile with securitization.

All sorts of assets can be included in ABSs, including mortgage loans, auto loans, student loans, credit card receivables, accounts receivables, and debt obligations. These assets are called **collateral**. In general, collecting payments on the collateral requires ongoing servicing activities. This is done by the **servicing agent**. Usually, the originator also performs the servicing, in exchange for a servicing fee.

The cash flows from the assets, minus the servicing fees, flow through the SPV to securities holders. When the securitization is structured as a **pass-through**, there is one class of bonds, and all investors receive the same proportional interests in the cash flows. When the SPV issues different classes of securities, the bonds are called **tranches**.[3] In addition, derivative instruments can be created to exchange claims on the ABS tranches, as we will see in Chapter 23.

So far, we have examined off-balance-sheet securitizations. Another group is on-balance-sheet securitizations, called **covered bonds** or ***Pfandbriefe*** in Germany. In these structures, the bank originates the loans and issues securities secured by these loans, which are kept on its books. Such structures are similar to secured corporate bonds, but have stronger legal protection in many European civil-law countries. Another difference is that investors have recourse against the bank in the case of defaults on the mortgages. Effectively, the bank provides a guarantee against credit risk.

In the case of MBS securitizations, the collateral consists of residential or commercial mortgage loans. These are called RMBSs and CMBSs, respectively. Their basic cash-flow patterns start from an annuity, where the homeowner makes a monthly fixed payment that covers principal and interest. As a result, the net present value of these cash flows is subject to interest rate risk, prepayment risk, and default risk.

In practice, however, most MBSs have third-party guarantees against credit risk. For instance, MBSs issued by Fannie Mae, an agency that is sponsored by the U.S. government, carry a guarantee of full interest and principal payment, even if the original borrower defaults. In this case, the **government-sponsored enterprise** (GSE) is the **mortgage insurer**. Such mortgage pass-throughs are sometimes called **participation certificates**.

In contrast, private-label MBSs are exposed to credit risk, and may receive a credit rating. The credit risk of the underlying mortgage loan can be assessed using a measure of borrower creditworthiness such as the **FICO credit score**.[4] Other factors include the ratio of the loan to the house value, the ratio of borrower debt to income or assets, and documentation provided in the loan. Loans that are in the lowest credit category, typically with FICO scores below 640, are called **subprime**. Between prime and subprime loans, Alt-A loans contain nonstandard features but have borrowers of A-rated creditworthiness.

18.2.2 Issues with Securitization

The financial crisis that started in 2007 has highlighted serious deficiencies with the securitization process. Securitization allows banks to move assets off their balance sheets, freeing up capital and spreading the risk among many different investors. In theory, this provides real benefits.

In practice, failures in the **originate-to-distribute** model contributed to the credit crisis. The model failed in a number of key places, including underwriting, credit rating, and investor due diligence.

[3] This is the French word for *slice*, as in a cake.

[4] Fair Isaac Corporation (FICO) is a publicly traded corporation (founded by Bill Fair and Earl Isaac) that provides widely used measures of consumer credit ratings. The FICO score is between 300 and 850, with a median of 723.

In the traditional banking model, banks or savings institutions underwrite mortgage loans and keep them on their balance sheets. This creates an incentive to screen loans carefully and to monitor their quality closely. In contrast, when a loan is securitized, the originator has less incentive to worry about the quality of the loan, because its revenues depend on the volume of issuance. After all, another party bears the losses. This has led to a large increase in poor-quality loans, in particular within the subprime category, which were packaged in securities and went bad rather quickly. This is a **moral hazard** problem, where an institution behaves differently than if it were fully exposed to the risk of the activity.

An additional problem arises when a bank decides to securitize a loan because of negative information about the borrower, rather than for diversification or capital relief reasons. This creates a systematic bias toward lower-quality loans among securitized loans. To protect themselves, investors in asset-backed securities can require the issuing bank to retain some of the risk of the loans, but this is not always sufficient. This is an **adverse selection** problem, due to **asymmetric information**. Indeed, Berndt and Gupta (2008) find that borrowers whose loans are sold in the secondary market have stock prices that underperform other firms.[5]

In addition, most of these loans ended up in complex securitization structures. Investors had relied blindly on credit ratings that turned out to be inaccurate because the credit rating agencies had not performed sufficient in-depth analysis of the underlying credits.

Also, because it was so easy to securitize assets, the total amount of lending in the residential sector went up, creating additional demand for housing and perhaps pushing housing prices even farther away from their fundamental values.

The securitization process also came back to haunt originating banks, many of which had created separate entities that invested in asset-backed securities. One such example is **structured investment vehicles** (SIVs), which are basically virtual banks, investing in asset-backed securities and funding themselves using short-term debt. During the credit crisis, investors in this debt became worried about the solvability of these SIVs and refused to roll over their investments. Thus SIVs were badly exposed to liquidity risk.

This forced many banks to absorb the assets of the failing SIVs they had sponsored back on their balance sheets, requiring them to raise additional equity capital in a particularly unfavorable environment. As a result, securitization failed to provide the expected capital relief.

Finally, when the securitization market froze, many banks and loan originators were stuck with loans that were **warehoused**, or held in a pipeline that was supposed to be temporary. As the real estate market plummeted and many of these loans went bad, these banks realized sizable losses on what they thought should have been temporary holdings.

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[5] An alternative explanation is that borrowers whose loans are sold are not subject to the discipline of bank monitoring, and undertake suboptimal investment. See A. Berndt and A. Gupta, "Moral Hazard and Adverse Selection in the Originate-to-Distribute Model of Bank Credit" (working paper, Carnegie Mellon University, 2008).

EXAMPLE 18.6: FRM EXAM 2009—QUESTION 9-2

Many observers have blamed the originate-to-distribute model of banks for having aggravated or even caused the credit crisis. Which of the following statements about the performance of the originate-to-distribute model of banks during the credit crisis is *incorrect*?

a. The originate-to-distribute model failed across all loan types equally.
b. Mortgage originators made large losses during the crisis and many went bankrupt.
c. Warehouse risk was a major cause of losses for originators of subprime mortgages.
d. The originate-to-distribute model can reduce systemic risk by having fewer risks located in banks, but a surprise during the crisis was the extent to which banks had kept risks.

18.3 TRANCHING

18.3.1 Concept

As we have seen, mortgage-backed securities are exposed to prepayment risk, which induces negative convexity. This feature is unattractive to investors who want fixed-income securities with predictable payments so they can match their liabilities.

In response, the industry developed classes of securities based on MBSs with more appealing characteristics. These are the **collateralized mortgage obligations (CMOs)**, which are new securities that redirect the cash flows of an MBS pool to various segments.

Figure 18.5 illustrates this tranching process. The cash flows from the MBS pool go into the SPV, which issues different claims with various characteristics, called tranches. These are structured so that the cash flow from the first tranche, for instance, is more predictable than the original cash flows. The uncertainty is then pushed into the other tranches.

Starting from an MBS pool, financial engineering creates securities that are better tailored to investors' needs. It is important to realize, however, that the cash flows and risks are fully preserved. They are only redistributed across tranches. Whatever transformation is brought about, the resulting package must obey basic laws of conservation for the underlying securities and package of resulting securities.[6]

[6] As Antoine Lavoisier, the French chemist who was executed during the French Revolution, said, *Rien ne se perd, rien ne se crée* (Nothing is lost, nothing is created).

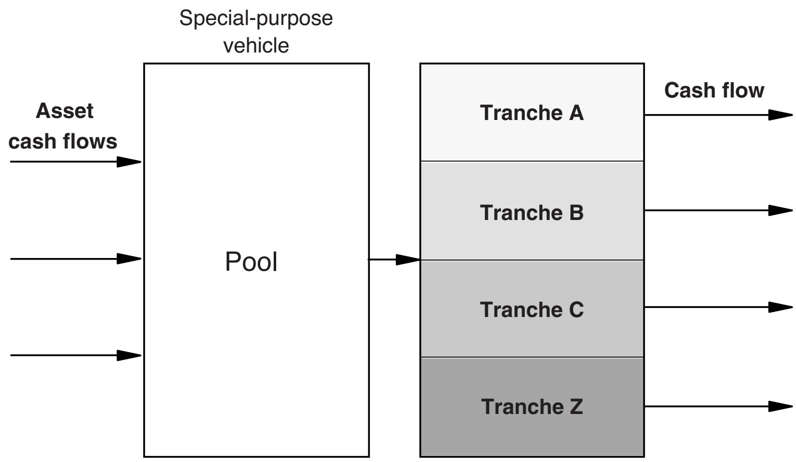

FIGURE 18.5 Tranching

At every single point in time, we must have the same cash flows going into and coming out of the SPV. As a result, we must have the same market value and the same risk profile. In particular, the weighted duration and convexity of the portfolio of tranches must add up to the original duration and convexity. If Tranche A has less convexity than the underlying securities, the other tranches must have more convexity.

Similar structures apply to **collateralized bond obligations** (CBOs), **collateralized loan obligations** (CLOs), **collateralized debt obligations** (CDOs), which are sets of tradable bonds backed by bonds, loans, or debt (bonds and loans), respectively. These structures rearrange credit risk and will be explained in more detail in a later chapter.

KEY CONCEPT

Tranching rearranges the total cash flows, total value, and total risk of the underlying securities. At all times, the total cash flows, value, and risk of the tranches must equal those of the collateral. If some tranches are less risky than the collateral, others must be more risky.

EXAMPLE 18.7: FRM EXAM 2000—QUESTION 13

A CLO is generally

a. A set of loans that can be traded individually in the market
b. A pass-through
c. A set of bonds backed by a loan portfolio
d. None of the above

EXAMPLE 18.8: FRM EXAM 2004—QUESTION 57

When evaluating asset-backed securitization issues, which of following would be *least* important during the investor's analysis process?

a. The liability concentration levels of the asset originator
b. The structure of the underlying securitization transaction
c. The quality of the loan servicer for the underlying assets in the transaction
d. The quality of the underlying assets within the securitization structure

18.3.2 Inverse Floaters

To illustrate the concept of tranching, we consider a simple example with a two-tranche structure that splits up interest rate risk. The collateral consists of a regular five-year, 6% coupon \$100 million note. This can be split up into a floating-rate note (FRN) that pays LIBOR on a notional of \$50 million, and an **inverse floater** that pays 12% − LIBOR on a notional of \$50 million. Because the coupon C_{IF} on the inverse floater cannot go below zero, this imposes another condition on the floater coupon C_F. The exact formulas are:

$$\text{Coupon}_F = \text{Min}(\text{LIBOR}, 12\%) \quad \text{Coupon}_{\text{IF}} = \text{Max}(12\% - \text{LIBOR}, 0)$$

We verify that the outgoing cash flows exactly add up to the incoming flows. For each coupon payment, we have, in millions

$$\$50 \times \text{LIBOR} + \$50 \times (12\% - \text{LIBOR}) = \$100 \times 6\% = \$6$$

so this is a perfect match. At maturity, the total payments of twice \$50 million add up to \$100 million, so this matches as well.

We can also decompose the risk of the original structure into that of the two components. Assume a flat term structure and say the duration of the original five-year note is $D = 4.5$ years. The portfolio dollar duration is:

$$\$50{,}000{,}000 \times D_F + \$50{,}000{,}000 \times D_{\text{IF}} = \$100{,}000{,}000 \times D$$

Just before a reset, the duration of the floater is close to zero $D_F = 0$. Hence, the duration of the inverse floater must be $D_{\text{IF}} = (\$100{,}000{,}000/\$50{,}000{,}000) \times D = 2 \times D$, or nine years, which is twice that of the original note. Note that the duration of the inverse floater is much greater than its maturity. This illustrates the point that duration is an interestrate sensitivity measure. When cash flows are uncertain, duration is not necessarily related to maturity. Intuitively, the first tranche, the floater, has zero risk so that all of the risk must be absorbed into the second tranche to conserve the total risk of the portfolio.

This analysis can be easily extended to inverse floaters with greater leverage. Suppose the coupon is tied to twice LIBOR, for example $18\% - 2 \times \text{LIBOR}$. The principal must be allocated in the amount x for the floater and $100 - x$ for the inverse floater so that the coupon payment is preserved. We set

$$x \times \text{LIBOR} + (100 - x) \times (18\% - 2 \times \text{LIBOR}) = \$6$$

$$[x - 2(100 - x)] \times \text{LIBOR} + (100 - x) \times 18\% = \$6$$

Because LIBOR will change over time, this can be satisfied only if the term between brackets is always zero. This implies $3x - 200 = 0$, or $x = \$66.67$ million. Thus, two-thirds of the notional must be allocated to the floater, and one-third to the inverse floater. The inverse floater now has three times the duration of the original note.

EXAMPLE 18.9: DURATION OF INVERSE FLOATERS

Suppose that the coupon and the modified duration of a 10-year bond priced to par are 6.0% and 7.5, respectively. What is the approximate modified duration of a 10-year inverse floater priced to par with a coupon of $18\% - 2 \times \text{LIBOR}$?

a. 7.5
b. 15.0
c. 22.5
d. 0.0

EXAMPLE 18.10: FRM EXAM 2004—QUESTION 69

With LIBOR at 4%, a manager wants to increase the duration of his portfolio. Which of the following securities should he acquire to increase the duration of his portfolio the most?

a. A 10-year reverse floater that pays $8\% - \text{LIBOR}$, payable annually
b. A 10-year reverse floater that pays $12\% - 2 \times \text{LIBOR}$, payable annually
c. A 10-year floater that pays LIBOR, payable annually
d. A 10-year fixed-rate bond carrying a coupon of 4% payable annually

EXAMPLE 18.11: FRM EXAM 2003—QUESTION 91

Which of the following statements most accurately reflects characteristics of a reverse floater (with no options attached)?

a. A portfolio of reverse floaters carries a marginally higher duration risk than a portfolio of similar-maturity normal floaters.
b. A holder of a reverse floater can synthetically convert the position into a fixed-rate bond by receiving floating and paying fixed on an interest rate swap.
c. A reverse floater hedges against rising benchmark yields.
d. A reverse floater's price changes by as much as that in a similar-maturity fixed-rate bond for a given change in yield.

18.3.3 CMOs

When the collateral consists of mortgages, CMOs can be defined by prioritizing the payment of principal into different tranches. This is defined as **sequential-pay tranches**. Tranche A, for instance, will receive the principal payment on the whole underlying mortgages first. This creates more certainty in the cash flows accruing to Tranche A, which makes it more appealing to some investors. Of course, this is to the detriment of others. After principal payments to Tranche A are exhausted, Tranche B then receives all principal payments on the underlying MBS, and so on for other tranches.

Prepayment risk can be minimized further by **planned amortization class** (PAC). All prepayment risk is then transferred to other bonds in the CMO structure, called **support** bonds. PAC bonds offer a fixed redemption schedule as long as prepayments on the collateral stay within a specific PSA range, say 100 to 250 PSA, called the **PAC collar**. When the structure is set up, the principal payment is set at the minimum payment of these two extreme values for every month of its life. Over time, this ensures a more stable pattern of payments.

Another popular construction is the IO/PO structure. This strips the MBS into two components. The **interest-only (IO)** tranche receives only the interest payments on the underlying MBS. The **principal-only (PO)** tranche then receives only the principal payments. As before, the market value of the IO and PO must exactly add to that of the MBS. Figure 18.6 describes the price behavior of the IO and PO. Note that the vertical addition of the two components always equals the value of the MBS.

To analyze the PO, it is useful to note that the sum of all principal payments is constant (because we have no default risk). Only the timing is uncertain. In contrast, the sum of all interest payments depends on the timing of principal payments. Later principal payments create greater total interest payments.

If interest rates fall, principal payments will come early, which reflects contraction risk. Because the principal is paid earlier and the discount rate decreases, the PO should appreciate sharply in value. In contrast, the faster prepayments

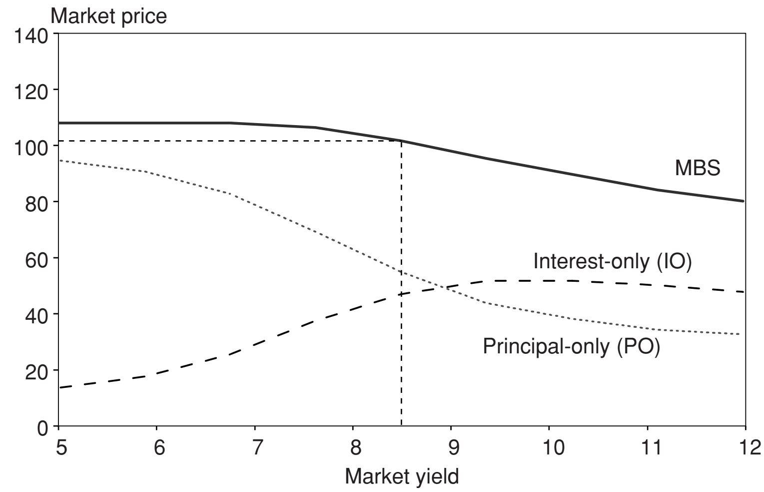

FIGURE 18.6 Creating an IO and PO from an MBS

mean less interest payments over the life of the MBS, which is unfavorable to the IO. The IO should depreciate.

Conversely, if interest rates rise, prepayments will slow down, which reflects extension risk. Because the principal is paid later and the discount rate increases, the PO should lose value. However, the slower prepayments mean more interest payments over the life of the MBS, which is favorable to the IO. The IO appreciates in value, up to the point where the higher discount rate effect dominates. Thus, IOs are bullish securities with negative duration. Figure 18.6 indeed shows that the IO line has a positive slope, which signifies negative duration.

EXAMPLE 18.12: FRM EXAM 2006—QUESTION 43

Which of the following mortgage-backed securities has a negative duration?

a. Interest-only (IO) strips
b. Inverse floaters
c. Mortgage pass-throughs
d. Principal-only (PO) strips

EXAMPLE 18.13: FRM EXAM 2004—QUESTION 45

As the chief risk officer (CRO) of a firm specializing in MBSs, you have been asked to explain how interest-only (IO) strips and principal-only (PO) strips would react if interest rates change. Which of the following is *true*?

a. When interest rates fall, both PO and IO strips will increase in value.
b. When interest rates fall, POs will increase in value, IOs decrease in value.
c. When interest rates rise, POs will increase in value, IOs decrease in value.
d. When interest rates rise, both PO and IO strips will increase in value.

EXAMPLE 18.14: FRM EXAM 2009—QUESTION 5-10

Which of following statements about mortgage-backed securities is *incorrect*?

a. An MBS price is more sensitive to yield curve twists than are zero-coupon bonds.
b. When the yield is higher than the coupon rate of an MBS, the MBS behaves similarly to corporate bonds as interest rates change.
c. As yield volatility increases, the value of an MBS grows, too.
d. Due to changes in prepayment rates, mortgages and MBSs exhibit negative convexity; that is, when interest rates decrease, prepayments increase.

EXAMPLE 18.15: FRM EXAM 2009—QUESTION 5-11

George Smith is an analyst in the risk management department and he is reviewing a pool of mortgages. Prepayment risk introduces complexity to the valuation of mortgages. Which of the following factors are generally considered to affect prepayment risk for a mortgage?

I. Changes to interest rates
II. Age of the mortgage
III. Season of the year
IV. Age of the home
V. Amount of principal outstanding

a. I, II, and V
b. I, II, III, and V
c. I, II, IV, and V
d. III and IV

18.4 IMPORTANT FORMULAS

Average life: $\text{AL} = \sum_{t=1}^{T} t P_t / P$

Conditional prepayment rate (CPR), single monthly mortality (SMM) rate: $(1 - \text{SMM})^{12} = (1 - \text{CPR})$

Public Securities Association (PSA) model: $\text{CPR} = \text{Min}[6\% \times (t/30), 6\%]$

Option-adjusted spread: OAS = Zero Spread − Option Cost

18.5 ANSWERS TO CHAPTER EXAMPLES

Example 18.1: FRM Exam 2008—Question 2-37

b. This is similar to an American call option because the borrower can repay at any time (American vs. European) if the yield drops or price goes up.

Example 18.2: Computing the SMM

a. Using $(1 - 6\%) = (1 - \text{SMM})^{12}$, we find SMM = 0.51%.

Example 18.3: FRM Exam 2000—Question 3

d. MBSs are unlike regular bonds, Treasuries, or corporates, because of their negative convexity. When rates fall, homeowners prepay early, which means that the price appreciation is less than that of comparable-duration regular bonds.

Example 18.4: FRM Exam 2003—Question 52

c. This has to be a callable bond because the price is capped if rates fall, reflecting the fact that the borrower would call back the bond. At Y_1, convexity is negative; at Y_2, close to zero.

Example 18.5: FRM Exam 2006—Question 93

b. The nominal spreads and Z spreads do not take into account the call option. Instead, comparisons should focus on the OAS, which is higher for Y, and also higher than the 100bp for option-free bonds.

Example 18.6: FRM Exam 2009—Question 9-2

a. The problems with mortgage loans, in particular subprime loans, were much larger than for other types of debt. Otherwise, statements b. and c. are correct. Originators lost money due to warehouse risk, which involves keeping the loans on the balance sheet before they can be securitized. This led to losses when these loans fell in value.

Example 18.7: FRM Exam 2000—Question 13

c. Like a CMO, a CLO represents a set of tradable securities backed by some collateral, in this case a loan portfolio.

Example 18.8: FRM Exam 2004—Question 57

a. Bankruptcy by the originator would not affect the SPV, so the financial condition of the originator is the least important factor. All of the other factors would be important in evaluating the securitization.

Example 18.9: Duration of Inverse Floaters

c. Following the same reasoning as earlier, we must divide the fixed-rate bonds into two-thirds FRN and one-third inverse floater. This will ensure that the inverse floater payment is related to twice LIBOR. As a result, the duration of the inverse floater must be three times that of the bond.

Example 18.10: FRM Exam 2004—Question 69

b. The duration of a floater is about zero. The duration of a 10-year regular bond is about nine years. The first reverse floater (answer a.) has a duration of about $2 \times 9 = 18$ years, the second (answer b.), $3 \times 9 = 27$ years.

Example 18.11: FRM Exam 2003—Question 91

b. The duration of a reverse floater is higher than that of an FRN, which is close to zero, or even than that of a fixed-date bond with the same maturity. So, answers a. and d. are wrong. It loses money when yields rise, so c. is wrong. A reverse floater is equivalent as a long position in a fixed-rate bond plus a receive-fixed/pay-floating swap. Hence b. is correct.

Example 18.12: FRM Exam 2006—Question 43

a. IOs increase in value as interest rates increase because in this scenario, there will be less prepayment of mortgages. Less early payment means more total interest payments, which increases the value of the IO.

Example 18.13: FRM Exam 2004—Question 45

b. POs have positive duration, IOs negative. Hence they react in opposite directions to falls in interest rates.

Example 18.14: FRM Exam 2009—Question 5-10

c. As yield volatility grows, the option cost goes up and so does the yield, which implies a drop in the price. Statement d. is correct, as MBSs have negative convexity. Statement b. is correct, as a high yield implies a lower prepayment rate, which makes the MBS similar to a regular bond.

Example 18.15: FRM Exam 2009—Question 5-11

b. All factors affect prepayment risk, except the age of the home.

PART Six

Credit Risk Management

CHAPTER **19**

Introduction to Credit Risk

Credit risk is the risk of an economic loss from the failure of a counterparty to fulfill its contractual obligations. Its effect is measured by the cost of replacing cash flows if the other party defaults.

This chapter provides an introduction to the measurement of credit risk. The study of credit risk has undergone vast developments in the past few years. Fueled by advances in the measurement of market risk, institutions are now, for the first time, attempting to quantify credit risk on a portfolio basis.

Credit risk, however, offers unique challenges. It requires constructing the distribution of default probabilities, of loss given default (LGD), and of credit exposures, all of which contribute to credit losses and should be measured in a portfolio context. In comparison, the measurement of market risk using value at risk (VAR) is a simple affair. These challenges explain why many of these models performed poorly during the credit crisis that started in 2007.

For most institutions, however, market risk pales in significance compared with credit risk. Indeed, the amount of risk-based capital for the banking system reserved for credit risk is vastly greater than that for market risk. The history of financial institutions has also shown that the biggest banking failures were due to credit risk.

Credit risk involves the possibility of nonpayment, either on a future obligation or during a transaction. Section 19.1 introduces **settlement risk**, which arises from the exchange of principals in different currencies during a short window, typically a day. We discuss exposure to settlement risk and methods to deal with it.

Traditionally, however, credit risk is viewed as **presettlement risk**, which arises during the life of the obligation. Section 19.2 analyzes the components of a credit risk system and the evolution of credit risk measurement systems. Section 19.3 then shows how to construct the distribution of credit losses for a portfolio given default probabilities for the various credits in the portfolio.

The key drivers of portfolio credit risk are the correlations between defaults. Section 19.4 takes a fixed $100 million portfolio with an increasing number of obligors and shows how the distribution of losses is dramatically affected by correlations.

FRM Exam Part 2 topic.

19.1 SETTLEMENT RISK

19.1.1 Presettlement versus Settlement Risk

Counterparty credit risk consists of both presettlement and settlement risk. **Presettlement risk** is the risk of loss due to the counterparty's failure to perform on an obligation during the life of the transaction. This includes default on a loan or bond or failure to make the required payment on a derivative transaction. Presettlement risk exists over long periods—years—starting from the time it is contracted until settlement.

In contrast, **settlement risk** is due to the *exchange* of cash flows and is of a much shorter-term nature. This risk arises as soon as an institution makes the required payment and exists until the offsetting payment is received. This risk is greatest when payments occur in different time zones, especially for foreign exchange transactions where notionals are exchanged in different currencies. Failure to perform on settlement can be caused by counterparty default, liquidity constraints, or operational problems.

Most of the time, settlement failure due to operational problems leads to minor economic losses, such as additional interest payments. In some cases, however, the loss can be quite large, extending to the full amount of the transferred payment. An example of major settlement risk is the 1974 failure of Herstatt Bank. The day the bank went bankrupt, it had received payments from a number of counterparties but defaulted before payments were made on the other legs of the transactions.

19.1.2 Managing Settlement Risk

In March 1996, the Bank for International Settlements (BIS) issued a report warning that the private sector should find ways to reduce settlement risk in the \$1.2-trillion-a-day global foreign exchange market.[1] The report noted that central banks had "significant concerns regarding the risk stemming from the current arrangements for settling FX trades." It explained that "the amount at risk to even a single counterparty could exceed a bank's capital," which creates **systemic risk**. The threat of regulatory action led to a reexamination of settlement risk.

Examining the various stages of a trade, its status can be classified into five categories:

1. *Revocable:* When the institution can still cancel the transfer without the consent of the counterparty
2. *Irrevocable:* After the payment has been sent and before payment from the other party is due

[1] Committee on Payment and Settlement Systems, *Settlement Risk in Foreign Exchange Transactions*, BIS (1996) (online), available at www.bis.org/publ/cpss17.pdf.

3. *Uncertain:* After the payment from the other party is due but before it is actually received
4. *Settled:* After the counterparty payment has been received
5. *Failed:* After it has been established that the counterparty has not made the payment

Settlement risk occurs during the periods of irrevocable and uncertain status, which can take from one to three days.

While this type of credit risk can lead to substantial economic losses, the short-term nature of settlement risk makes it fundamentally different from presettlement risk. Managing settlement risk requires unique tools, such as **real-time gross settlement** (RTGS) systems. These systems aim at reducing the time interval between the time an institution can no longer stop a payment and the receipt of the funds from the counterparty.

Settlement risk can be further managed with netting agreements. One such form is **bilateral netting**, which involves two banks. Instead of making payments of gross amounts to each other, the banks total up the balance and settle only the net balance outstanding in each currency. At the level of instruments, netting also occurs with **contracts for differences** (CFDs). Instead of exchanging principals in different currencies, the contracts are settled in dollars at the end of the contract term.[2]

The next step up is a **multilateral netting system**, also called **continuous-linked settlements**, where payments are netted for a group of banks that belong to the system. This idea became reality when the **CLS Bank**, established in 1998 with 60 bank participants, became operational on September 9, 2002. Every evening, CLS Bank provides a schedule of payments for the member banks to follow during the next day. Payments are not released until funds are received and all transactions confirmed. The risk now has been reduced to that of the netting institution. In addition to reducing settlement risk, the netting system has the advantage of reducing the number of trades between participants, by up to 90%, which lowers transaction costs.

EXAMPLE 19.1: SETTLEMENT RISK

Settlement risk in foreign exchange is generally due to

a. Notionals being exchanged
b. Net value being exchanged
c. Multiple currencies and countries involved
d. High volatility of exchange rates

[2] These are similar to **nondeliverable forwards**, which are used to trade emerging-market currencies outside the jurisdiction of the emerging-market regime and are also settled in dollars.

EXAMPLE 19.2: MULTILATERAL NETTING SYSTEMS

Which one of the following statements about multilateral netting systems is *not* accurate?

a. Systemic risks can actually increase because they concentrate risks on the central counterparty, the failure of which exposes all participants to risk.
b. The concentration of risks on the central counterparty eliminates risk because of the high quality of the central counterparty.
c. By altering settlement costs and credit exposures, multilateral netting systems for foreign exchange contracts could alter the structure of credit relations and affect competition in the foreign exchange markets.
d. In payment netting systems, participants with net debit positions will be obligated to make a net settlement payment to the central counterparty that, in turn, is obligated to pay those participants with net credit positions.

19.2 OVERVIEW OF CREDIT RISK

19.2.1 Drivers of Credit Risk

We now examine the drivers of credit risk, traditionally defined as presettlement risk. Credit risk measurement systems attempt to quantify the risk of losses due to counterparty default. The distribution of credit risk can be viewed as a compound process driven by these variables:

- **Default**, which is a discrete state for the counterparty—either the counterparty is in default or not. This occurs with some **probability of default** (PD).
- **Credit exposure** (CE), which is the economic or market value of the claim on the counterparty. It is also called **exposure at default** (EAD) at the time of default.
- **Loss given default** (LGD), which represents the fractional loss due to default. As an example, take a situation where default results in a **fractional recovery rate** of 30% only. LGD is then 70% of the exposure.

It is important to realize that a long credit position involves an embedded short option position. This represents the fact that the borrower has the option to default. Consider, for example, a regular credit-sensitive bond. This pays a higher yield than a risk-free bond but at the expense of a capital loss should the borrower default. This payment pattern, receiving a regular premium in exchange for a potential future capital loss, is exactly equivalent to a short position in an option.

Traditionally, credit risk has been measured in the context of loans or bonds for which the exposure, or economic value, of the asset is close to its notional, or face value. This is an acceptable approximation for bonds but certainly not for derivatives, which can have positive or negative value. Credit exposure is defined as the positive value of the asset:

$$\text{CE}_t = \text{Max}(V_t, 0) \tag{19.1}$$

This is so because if the counterparty defaults with money owed to it, the full amount has to be paid. In contrast, if the counterparty owes money, only a fraction may be recovered. This asymmetry can be explained in view of the fact that most contracts now have *no walk-away clauses*, which prevent the party that owes money to a bankrupt entity to walk away from the contract. Thus, presettlement risk arises only when the contract's replacement cost has a positive value to the institution (i.e., is in-the-money).

19.2.2 Measurement of Credit Risk

The evolution of credit risk measurement tools has gone through these four steps:

1. **Notional amounts**, adding up simple exposures
2. **Risk-weighted amounts**, adding up exposures with a rough adjustment for risk
3. **Notional amounts combined with credit ratings**, adding up exposures adjusted for default probabilities
4. **Internal portfolio credit models**, integrating all dimensions of credit risk

Initially, risk measures were based on the total notional amount. A multiplier, say 8%, was applied to this amount to establish the amount of required capital to hold as a reserve against credit risk.

The problem with this approach is that it ignores variations in the probability of default. In 1988, the Basel Committee instituted a rough categorization of credit risk by *risk class*, providing risk weights to multiply notional amounts. This was the first attempt to force banks to carry capital in relation to the risks they were taking. The Basel rules are explained in more detail in a later chapter.

These risk weights proved to be too simplistic, however, creating incentives for banks to alter their portfolios in order to maximize their shareholder returns subject to the Basel capital requirements. This had the perverse effect of introducing more risk into the balance sheets of commercial banks, which was certainly not the intended purpose of the 1988 rules. As an example, there was no differentiation between AAA-rated and C-rated corporate credits. Because loans to lower-quality credits are generally more profitable than those to high-quality credits but require the same amount of regulatory capital, some banks responded by shifting the loan mix toward lower-rated credits. This led to the Basel II rules, which allow banks to use their own internal or external credit ratings. These credit ratings provide more refined representation of credit risk.

Even with these improvements, the credit risk charges are computed separately for all exposures and then added up. This approach may not properly account for diversification effects, unlike internal portfolio credit models. Such models,

TABLE 19.1 Comparison of Market Risk and Credit Risk

Item	Market Risk	Credit Risk
Sources of risk	Market risk only	Default risk, recovery risk, market risk
Distributions	Mainly symmetrical, perhaps fat tails	Skewed to the left
Time horizon	Short-term (days)	Long-term (years)
Aggregation	Business/trading unit	Whole firm vs. counterparty
Legal issues	Not applicable	Very important

however, create special challenges and as a result are not accepted by the Basel Committee for capital adequacy requirements.

19.2.3 Credit Risk versus Market Risk

The tools recently developed to measure market risk have proved invaluable in assessing credit risk. Even so, there are a number of major differences between market and credit risks, which are listed in Table 19.1.

As mentioned previously, credit risk results from a compound process with three types of risk. The nature of this risk creates a distribution that is strongly skewed to the left, unlike most market risk factors. This is because credit risk is akin to short positions in options. At best, the counterparty makes the required payment and there is no loss. At worst, the entire amount due is lost.

The time horizon is also different. Whereas the time required for corrective action is relatively short in the case of market risk, it is much longer for credit risk. Positions also turn over much more slowly for credit risk than for market risk, although the advent of credit derivatives has made it easier to hedge credit risk.

Finally, the level of aggregation is different. Limits on market risk may apply at the level of a trading desk, a business unit, and eventually the whole firm. In contrast, limits on credit risk must be defined at the counterparty level for all positions taken by the institution.

Legal issues are also very important for credit risk. Exposures can be controlled by netting contracts, for example. Finally, an event of default invariably puts into motion a legal process.

19.3 MEASURING CREDIT RISK

19.3.1 Credit Losses

To develop the intuition of credit models, let us start with a simple case where losses are due to the effect of defaults only. This is what is called **default mode**. In other words, there is no intermediate marking to market. This example would be typical of loans held in a banking book as opposed to bonds held in a trading account.

The distribution of credit losses (CLs) from a portfolio of N instruments issued by different obligors can be described as

$$\mathrm{CL} = \sum_{i=1}^{N} b_i \times \mathrm{CE_i} \times (1 - f_i) \tag{19.2}$$

where

b_i = a (Bernoulli) random variable that takes the value of 1 if default occurs and 0 otherwise, with probability p_i, such that $E[b_i] = p_i$
$\mathrm{CE_i}$ = the credit exposure at time of default
f_i = the recovery rate, or $\mathrm{LGD}_i = (1 - f_i)$ the loss given default

In theory, all of these could be random variables. If they are independent, the expected credit loss is

$$E[\mathrm{CL}] = \sum_{i=1}^{N} E[b_i] \times E[\mathrm{CE_i}] \times E[\mathrm{LGD}_i] = \sum_{i=1}^{N} p_i \times E[\mathrm{CE_i}] \times E[\mathrm{LGD}_i] \tag{19.3}$$

To understand the drivers of the credit distribution, consider a simple case where all creditors have the same distribution of default and loss given default. Exposures are the same and fixed at \$1.

The actual credit loss is then

$$\mathrm{CL} = \sum_{i=1}^{n} \mathrm{LGD}_i \tag{19.4}$$

where n is the actual number of losses, between 0 and N. This is a random sum of random numbers, each consisting of the actual LGD.

As before, the expected value of the credit loss is

$$E[\mathrm{CL}] = E[n]E[\mathrm{LGD}] = NpE[\mathrm{LGD}] \tag{19.5}$$

All else equal, the expected credit loss is linear in the default probability p.

Consider now the dispersion of CL, measured by the standard deviation. By the law of total variance, we can write

$$V[\mathrm{CL}] = E\{V[\mathrm{CL}|n]\} + V\{E[\mathrm{CL}|n]\} \tag{19.6}$$

This gives

$$V[\mathrm{CL}] = E\{nV(\mathrm{LGD}]\} + V\{nE[\mathrm{LGD}]\}$$

where the first term is due to the fact that the variance of a sum of independent variables is the sum of variances. Next,

$$V[\mathrm{CL}] = E[n]V[\mathrm{LGD}] + V[n]\{E[\mathrm{LGD}]\}^2 = NpV[\mathrm{LGD}] + Np(1-p)\{E[\mathrm{LGD}]\}^2$$

When $N = 1$, this gives

$$\mathrm{SD[CL]} = \sqrt{pV[\mathrm{LGD}] + p(1-p)\{E[\mathrm{LGD}]\}^2} \tag{19.7}$$

We see here that the volatility of the credit loss is driven by both the variance of LGD and that of the default indicator, $V[b] = p(1-p)$.

The relationship between SD[CL] and p, however is nonlinear, unlike for the expected credit loss. When p is small, it is approximately proportional to $\sqrt{p}$. As a result, SD[CL] increases more quickly than E[CL] with p. For instance, when p goes from 0 to 0.01, its square root goes from 0 to 0.10. Thus, the dispersion of credit losses tends to be greater than the average loss for higher default rates.

EXAMPLE 19.3: FRM EXAM 2002—QUESTION 130

You have granted an unsecured loan to a company. This loan will be paid off by a single payment of \$50 million. The company has a 3% chance of defaulting over the life of the transaction and your calculations indicate that if it defaults you would recover 70% of your loan from the bankruptcy courts. If you are required to hold a credit reserve equal to your expected credit loss, how great a reserve should you hold?

a. \$450,000
b. \$750,000
c. \$1,050,000
d. \$1,500,000

EXAMPLE 19.4: FRM EXAM 2009—QUESTION 6-7

A bank has booked a loan with total commitment of \$50,000 of which 80% is currently outstanding. The default probability of the loan is assumed to be 2% for the next year and loss given default (LGD) is estimated at 50%. The standard deviation of LGD is 40%. Drawdown on default (i.e., the fraction of the undrawn loan) is assumed to be 60%. The expected and unexpected losses (standard deviation) for the bank are

a. Expected loss = \$500, unexpected loss = \$4,140
b. Expected loss = \$500, unexpected loss = \$3,220
c. Expected loss = \$460, unexpected loss = \$3,220
d. Expected loss = \$460, unexpected loss = \$4,140

EXAMPLE 19.5: FRM EXAM 2008—QUESTION 3-5

Define unexpected loss (UL) as the standard deviation of losses and expected loss (EL) as the average loss. Further define LGD as loss given default, and EDF as the expected default frequency. Which of the following statements hold(s) *true*?

I. EL increases linearly with increasing EDF.
II. EL is often higher than UL.
III. With increasing EDF, UL increases at a much faster rate than EL.
IV. The lower the LGD, the higher the percentage loss for both the EL and UL.

a. I only
b. I and II
c. I and III
d. II and IV

19.3.2 Joint Events

More generally, with a large number of credits, the dispersion in credit losses critically depends on the correlations between the default events. Closed-form solutions can be easily derived when the default events are statistically independent. This simplifies the analysis considerably, as the probability of any joint event is simply the product of the individual event probabilities:

$$p(A \text{ and } B) = p(A) \times p(B) \tag{19.8}$$

At the other extreme, if the two events are perfectly correlated, that is, if B always defaults when A defaults, we have

$$p(A \text{ and } B) = p(B \mid A) \times p(A) = 1 \times p(A) = p(A) \tag{19.9}$$

when the marginal probabilities are equal, $p(A) = p(B)$.

Suppose, for instance, that the marginal probabilities are $p(A) = p(B) = 1\%$. Then the probability of the joint event is 0.01% in the independence case and 1% in the perfect-correlation case.

More generally, one can show that the probability of a joint default depends on the marginal probabilities and the correlations. As we have seen in Chapter 2, the expectation of the product can be related to the covariance

$$E[b_A \times b_B] = \text{Cov}[b_A, b_B] + E[b_A]E[b_B] = \rho\sigma_A\sigma_B + p(A)p(B) \tag{19.10}$$

TABLE 19.2 Joint Probabilities (Default Correlation = 0.5)

	B		
A	**Default**	**No Default**	**Marginal**
Default	0.00505	0.00495	0.01
No default	0.00495	0.98505	0.99
Marginal	0.01	0.99	

Given that b_A is a Bernoulli variable, Chapter 2 has shown that its standard deviation is $\sigma_A = \sqrt{p(A)[1 - p(A)]}$ and similarly for b_B. We then have

$$p(A \text{ and } B) = \text{Corr}(A, B)\sqrt{p(A)[1 - p(A)]}\sqrt{p(B)[1 - p(B)]} + p(A)p(B) \tag{19.11}$$

For example, if the correlation is unity and $p(A) = p(B) = p$, we have

$$p(A \text{ and } B) = 1 \times [p(1 - p)]^{1/2} \times [p(1 - p)]^{1/2} + p^2 = [p(1 - p)] + p^2 = p$$

as shown in Equation (19.9).

If the correlation is 0.5 and $p(A) = p(B) = 0.01$, however, then we have $p(A \text{ and } B) = 0.00505$, which is only half of the marginal probabilities. This example is illustrated in Table 19.2, which lays out the full joint distribution. Note how the probabilities in each row and column sum to the marginal probability. From this information, we can infer all remaining probabilities. For example, the probability of B not defaulting when A is in default is $0.01 - 0.00505 = 0.00495$. This allows us to fill the entire table with joint probabilities.

19.3.3 An Example

As an example of credit loss distribution, consider a portfolio of \$100 million with three bonds, A, B, and C, with various probabilities of default. To simplify, we assume (1) that the exposures are constant, (2) that the recovery in case of default is zero, and (3) that default events are independent across the three issuers.

Table 19.3 displays the exposures and default probabilities. We can easily compute the expected loss as $E[\text{CL}] = \sum p_i \times \text{CE}_i = 0.05 \times 25 + 0.10 \times 30 + 0.20 \times 45$, or

$$E[\text{CL}] = \sum p_i \times \text{CE}_i = \$13.25 \text{ million}$$

This is the average credit loss over many repeated, hypothetical samples. This computation is very easy and does not require any information about the distribution other than default probabilities. Also note that we define the loss as a positive number, which is the usual convention.

TABLE 19.3 Portfolio Exposures, Default Risk, and Credit Losses (Dollar Amounts in Millions)

Issuer	Exposure	Probability
A	\$25	0.05
B	\$30	0.10
C	\$45	0.20

Default i	Loss L_i	Probability $p(L_i)$	Cumulative Probability	Expected $L_i p(L_i)$	Variance $(L_i - EL_i)^2 p(L_i)$
None	\$0	0.6840	0.6840	0.000	120.08
A	\$25	0.0360	0.7200	0.900	4.97
B	\$30	0.0760	0.7960	2.280	21.32
C	\$45	0.1710	0.9670	7.695	172.38
A, B	\$55	0.0040	0.9710	0.220	6.97
A, C	\$70	0.0090	0.9800	0.630	28.99
B, C	\$75	0.0190	0.9990	1.425	72.45
A, B, C	\$100	0.0010	1.0000	0.100	7.53
Sum				\$13.25	434.7

In contrast, we do need to describe the full distribution to derive a worst-loss measure. This is done in the second panel, which lists all possible states. In state 1, there is no default, which has a probability of $(1 - p_1)(1 - p_2)(1 - p_3) = (1 - 0.05)(1 - 0.10)(1 - 0.20) = 0.684$, given independence. In state 2, bond A defaults and the others do not, with probability $p_1(1 - p_2)\,(1 - p_3) = 0.05(1 - 0.10)\,(1 - 0.20) = 0.036$ (and so on for the other states).

Figure 19.1 plots the frequency distribution of credit losses. The table also shows how to compute the variance as

$$V[\mathrm{CL}] = \sum_{i=1}^{N}(L_i - E[\mathrm{CL}])^2 p(L_i) = 434.7$$

which gives a standard deviation of $\sigma(\mathrm{CL}) = \sqrt{434.7} = \$20.9$ million.

Alternatively, we can express the range of losses with a 95% quantile, which is the lowest number CL_i such that

$$P(\mathrm{CL} \ \leq \mathrm{CL}_i) \geq 95\% \qquad (19.12)$$

Because losses are recorded as positive numbers, the quantile is the lowest loss such that the cumulative probability of a lower loss is at or just above 95%.

From Table 19.3, we see that the fourth row has a cumulative probability of 0.9670, just above 0.95, and corresponds to a loss of \$45 million. Figure 19.2 plots the cumulative distribution function and shows that the 95% quantile is \$45 million.

In terms of deviations from the mean, this gives an unexpected loss of 45 − 13.2 = \$32 million. This is a measure of **credit VAR**. Sometimes, however, credit VAR is measured as the total loss, or \$45 million in this case.

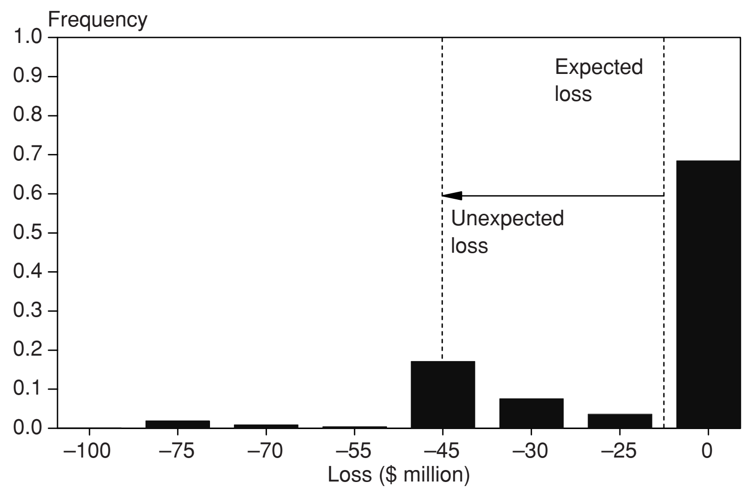

FIGURE 19.1 Distribution of Credit Losses

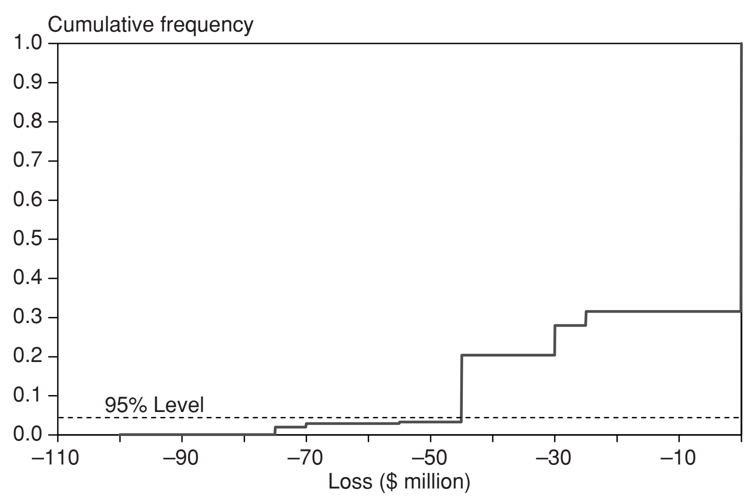

FIGURE 19.2 Cumulative Distribution of Credit Losses

This very simple three-bond portfolio provides a useful example of the measurement of the distribution of credit risk. It shows that the distribution is skewed to the left. In addition, the distribution has irregular bumps that correspond to the default events. Chapter 24 on managing credit risk further elaborates on this point.

KEY CONCEPT

The expected credit loss depends on default probabilities but not on default correlation. In contrast, the higher the default correlation, the higher the unexpected credit loss.

EXAMPLE 19.6: FRM EXAM 2003—QUESTION 17

An investor holds a portfolio of $100 million. This portfolio consists of A-rated bonds ($40 million) and BBB-rated bonds ($60 million). Assume that the one-year probabilities of default for A-rated and BBB-rated bonds are 3% and 5%, respectively, and that they are independent. If the recovery value for A-rated bonds in the event of default is 70% and the recovery value for BBB-rated bonds is 45%, what is the one-year expected credit loss from this portfolio?

a. $1,672,000
b. $1,842,000
c. $2,010,000
d. $2,218,000

EXAMPLE 19.7: FRM EXAM 2007—QUESTION 73

A portfolio consists of two bonds. The credit VAR is defined as the maximum loss due to defaults at a confidence level of 98% over a one-year horizon. The probability of joint default of the two bonds is 1.27%, and the default correlation is 30%. The bond value, default probability, and recovery rate are USD 1,000,000, 3%, and 60% for one bond, and USD 600,000, 5%, and 40% for the other. What is the expected credit loss of the portfolio?

a. USD 0
b. USD 9,652
c. USD 20,348
d. USD 30,000

EXAMPLE 19.8: FRM EXAM 2007—QUESTION 74

Continuing with the previous question, what is the best estimate of the unexpected credit loss (away from the ECL), or credit VAR, for this portfolio?

a. USD 570,000
b. USD 400,000
c. USD 360,000
d. USD 370,000

EXAMPLE 19.9: FRM EXAM 2007—QUESTION 102

Suppose Bank Z lends EUR 1 million to X and EUR 5 million to Y. Over the next year, the PD for X is 0.2 and for Y is 0.3. The PD of joint default is 0.1. The loss given default is 40% for X and 60% for Y. What is the expected loss of default in one year for the bank?

a. EUR 0.72 million
b. EUR 0.98 million
c. EUR 0.46 million
d. EUR 0.64 million

EXAMPLE 19.10: FRM EXAM 2004—QUESTION 46

Consider an A-rated bond and a BBB-rated bond. Assume that the one-year probabilities of default for the A- and BBB-rated bonds are 2% and 4%, respectively, and that the joint probability of default of the two bonds is 0.15%. What is the default correlation between the two bonds?

a. 0.07%
b. 2.6%
c. 93.0%
d. The default correlation cannot be calculated with the information provided.

19.4 CREDIT RISK DIVERSIFICATION

Modern banking was built on the sensible notion that a portfolio of loans is less risky than are single loans. As with market risk, the most important feature of credit risk management is the ability to diversify across defaults.

To illustrate this point, Figure 19.3 presents the distribution of losses for a \$100 million loan portfolio. The probability of default is fixed at 1%. If default occurs, recovery is zero.

In the first panel, we have one loan only. We can have either no default with probability 99% or a loss of \$100 million with probability 1%. The expected loss is

$$\text{EL} = 0.01 \times \$100 + 0.99 \times 0 = \$1 \text{ million}$$

The problem, of course, is that if default occurs it will be a big hit to the bottom line, possibly bankrupting the lending bank.

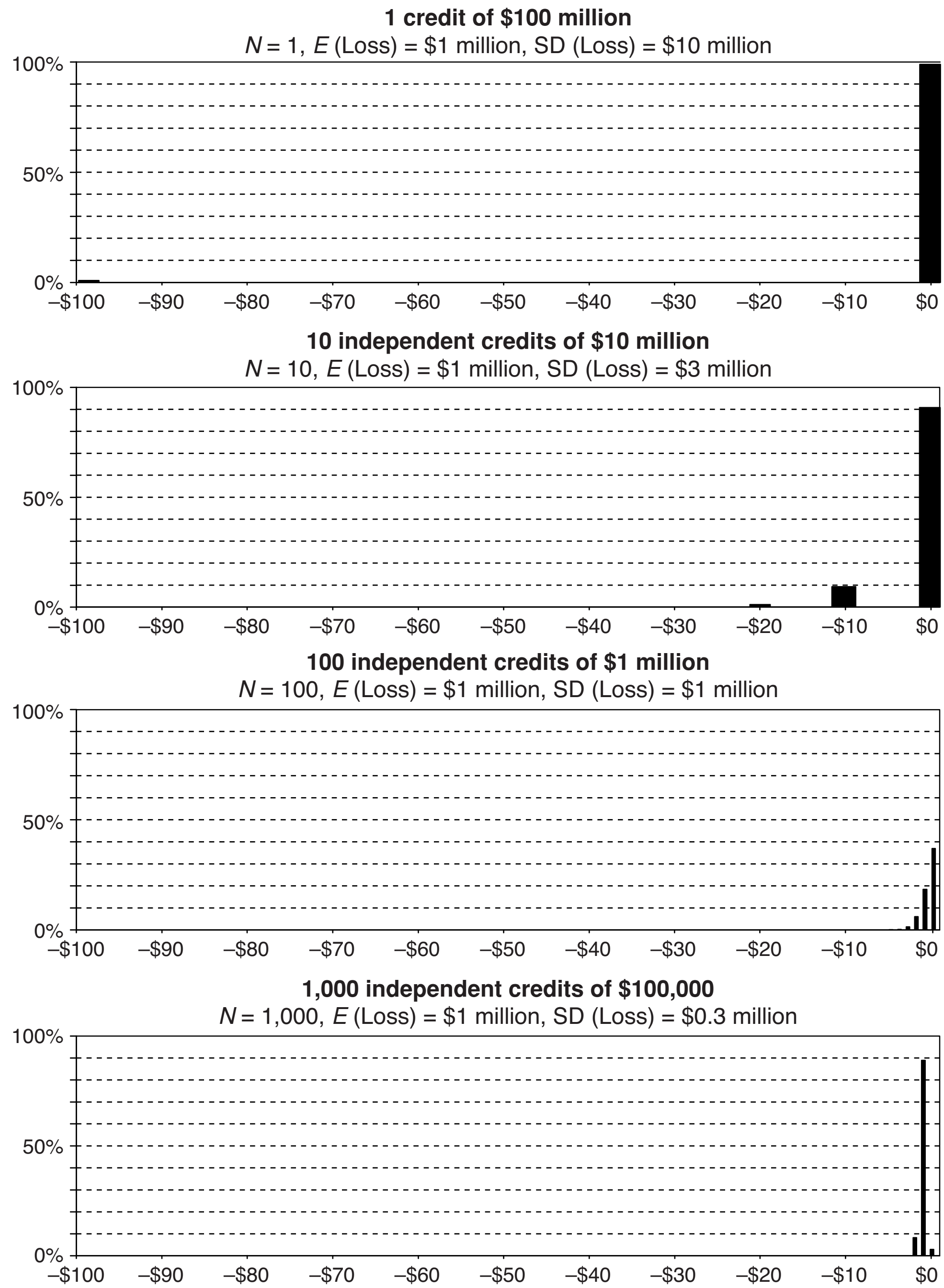

FIGURE 19.3 Distribution of Credit Losses

Basically, this is what happened to Peregrine Investments Holdings, one of Hong Kong's leading investment banks which shut down due to the Asian crisis of 1997. The bank failed in large part from a single loan to PT Steady Safe, an Indonesian taxicab operator, that amounted to $235 million, a quarter of the bank's equity capital.

In the case of our single loan, the spread of the distribution is quite large, with a variance of 99, which implies a standard deviation (SD) of about $10 million.

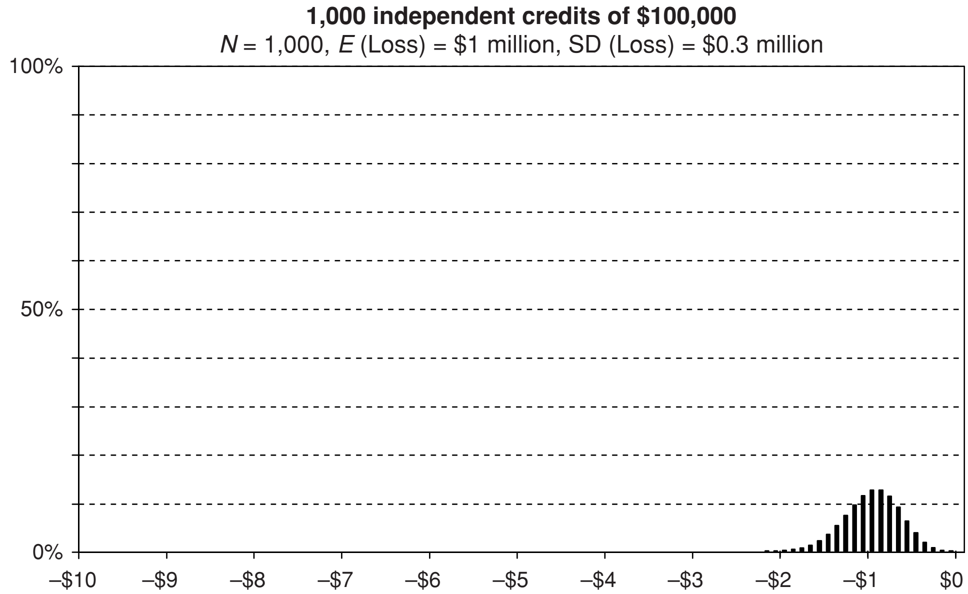

FIGURE 19.3 (*Continued*)

Simply focusing on the standard deviation, however, ignores the severe skewness in the distribution.

In the second panel, we consider 10 loans, each for $10 million. The total notional is the same as before. We assume that defaults are independent. The expected loss is still $1 million, or 10 × 0.01 × $10 million. The SD, however, is now $3 million, much less than before.

Next, the third panel considers 100 loans of $1 million each. The expected loss is still $1 million, but the SD is now $1 million, even lower. Finally, the fourth panel considers a thousand loans of $100,000, which create an SD of $0.3 million.

For comparability, all of these graphs use the same vertical and horizontal scale. This, however, does not reveal the distributions fully. This is why the fifth panel expands the distribution with 1,000 counterparties, which looks similar to a normal distribution. This reflects the **central limit theorem** (CLT), which states that the distribution of the sum of *independent* variables tends to a normal distribution. Remarkably, even starting from a highly skewed distribution, we end up with a normal distribution due to diversification effects.

In addition, the spread of the distribution becomes very small. This explains why portfolios of consumer loans, which are spread over a large number of credits, are less risky than typical portfolios of corporate loans.

With N events that occur with the same probability p, define the variable $X = \sum_{i=1}^{N} b_i$ as the number of defaults (where $b_i = 1$ when default occurs). The expected credit loss on our portfolio is then

$$E[\text{CL}] = E[X] \times \$100/N = pN \times \$100/N = p \times \$100 \qquad (19.13)$$

which depends not on N but rather on the average probability of default and total exposure, \$100 million. When the events are independent, the variance of this variable is, using the results from a binomial distribution,

$$V[\text{CL}] = V[X] \times (\$100/N)^2 = p(1-p)N \times (\$100/N)^2 \tag{19.14}$$

which gives a standard deviation of

$$\text{SD}[\text{CL}] = \sqrt{p(1-p)} \times \$100/\sqrt{N} \tag{19.15}$$

For a constant total notional, this shrinks to zero as N increases.

We should note the crucial assumption that the credits are independent. When this is not the case, the distribution will lose its asymmetry more slowly. Even with a very large number of consumer loans, the dispersion will not tend to zero because the general state of the economy is a common factor behind consumer credits. Indeed, many more defaults occur in a recession than in an expansion. This is one of the reasons for being suspicious of credit risk models that have been calibrated over periods of expansion.

Institutions loosely attempt to achieve diversification by **concentration limits**. In other words, they limit the extent of exposure, say loans, to a particular industrial or geographical sector. The rationale behind this is that defaults are more highly correlated within sectors than across sectors. Conversely, **concentration risk** is the risk that too many defaults could occur at the same time.

The distributions in this section were derived using closed-form solutions, which assume *homogeneity*, or the same probability of default p, and independence. In this situation, all the possible states of the world can be described by using the **binomial expansion**. This general theorem states that

$$(x+y)^N = a_0 x^N + a_1 x^{N-1} y^1 + a_2 x^{N-2} y^2 + \cdots + a_{N-1} x^1 y^{N-1} + a_N y^N \tag{19.16}$$

where the coefficients a_i in this expansion are the number of combinations of N things taken i at a time, or

$$a_i = \binom{N}{i} = \frac{N!}{i!(N-i)!} \tag{19.17}$$

If we define $x = p$ and $y = 1 - p$, the expansion must sum to unity, and each term gives the probability of this particular combination of events.

$$1 = p^N + Np^{N-1}(1-p)^1 + \frac{N(N-1)}{2} p^{N-2}(1-p)^2 + \cdots + Np^1(1-p)^{N-1} + (1-p)^N \tag{19.18}$$

As an example, with $N = 3$, we have

$$1 = p^3 + 3p^2(1 - p) + 3p(1 - p)^2 + (1 - p)^3 \tag{19.19}$$

The first term gives the probability of three defaults. The second is the probability of exactly two defaults, which involves three events.[3] The last term gives the probability of no defaults. This is a convenient decomposition but is valid only when defaults are independent.

EXAMPLE 19.11: FRM EXAM 2002—QUESTION 92

A portfolio of bonds consists of five bonds whose default correlation is zero. The one-year probabilities of default of the bonds are: 1%, 2%, 5%, 10%, and 15%. What is the one-year probability of no default within the portfolio?

a. 71%
b. 67%
c. 85%
d. 99%

EXAMPLE 19.12: FRM EXAM 2004—QUESTION 15

There are 10 bonds in a credit default swap basket. The probability of default for each of the bonds is 5%. The probability of any one bond defaulting is completely independent of what happens to the other bonds in the basket. What is the probability that exactly one bond defaults?

a. 5%
b. 50%
c. 32%
d. 3%

19.5 IMPORTANT FORMULAS

Credit loss variable: $CL = \sum_{i=1}^{N} b_i \times CE_i \times (1 - f_i)$

Credit loss with identical events, independent b and LGD, fixed CE = 1.

Expected value: $E[CL] = E[n]E[LGD] = NpE[LGD]$

Standard deviation: $SD[CL] = \sqrt{NpV[LGD] + Np(1 - p)\{E[LGD]\}^2}$

[3] The events are (1) bond 1 in no default with bonds 2 and 3 in default, (2) bond 2 in no default with others in default, and (3) bond 3 in no default with others in default.

Joint probability with independence: $p(A \text{ and } B) = p(A) \times p(B)$

Joint probability:

$$p(A \text{ and } B) = \text{Corr}(A, B)\sqrt{p(A)[1 - p(A)]}\sqrt{p(B)[1 - p(B)]} + p(A)p(B),$$
using $E[b_A \times b_B] = \text{Cov}[b_A, b_B] + E[b_A]E[b_B]$

Binomial expansion:

$$1 = p^N + Np^{N-1}(1 - p)^1 + \frac{N(N-1)}{2} p^{N-2}(1 - p)^2 + \cdots + Np^1(1 - p)^{N-1} + (1 - p)^N$$

19.6 ANSWERS TO CHAPTER EXAMPLES

Example 19.1: Settlement Risk

a. Settlement risk is due to the exchange of notional principal in different currencies at different points in time, which exposes one counterparty to default after it has made payment. There would be less risk with netted payments.

Example 19.2: Multilateral Netting Systems

b. Answers c. and d. are both correct. Answers a. and b. are contradictory. A multilateral netting system concentrates the credit risk into one institution. This could potentially create much damage if this institution fails.

Example 19.3: FRM Exam 2002—Question 130

a. The expected credit loss (ECL) is the notional amount times the probability of default times the loss given default. This is $\$50,000,000 \times 0.03 \times (1 - 70\%) =$ $450,000.

Example 19.4: FRM Exam 2009—Question 6-7

d. First, we compute the exposure at default. This is the drawn amount, or $80\% \times \$50,000 = \$40,000$ plus the drawdown on default, which is $60\% \times \$10,000 = \$6,000$, for a total of CE = $46,000. The expected loss is this amount times $p \times E[\text{LGD}] = 0.02 \times 50\% = 1\%$, or EL = $460. Next, we compute the standard deviation of losses using Equation (19.7). The variance is $pV[\text{LGD}] + p(1 - p)\{E[\text{LGD}]\}^2 = 0.02(0.40)^2 + 0.02(1 - 0.02)(0.50)^2 = 0.00810$. Taking the square root gives 0.090. Multiplying by $46,000 gives $4,140. Ignoring V[LGD] gives the incorrect answer of $3,220. Note that the unexpected loss is much greater than the expected loss.

Example 19.5: FRM Exam 2008—Question 3-5

c. Equation (19.5) shows that EL increases linearly with p, so answer I. is correct. Answer II. is not correct, certainly for concentrated portfolios. Equation (19.7) shows that UL increases faster than EL linearly with p, so answer III. is correct. Finally, Answer II. is incorrect, as higher (not lower) LGD would lead to higher credit losses.

Example 19.6: FRM Exam 2003—Question 17

c. The expected loss is $\sum_i p_i \times \text{CE}_i \times (1 - f_i) = \$40{,}000{,}000 \times 0.03(1 - 0.70) + \$60{,}000{,}000 \times 0.05(1 - 0.45) = \$2{,}010{,}000$.

Example 19.7: FRM Exam 2007—Question 73

d. The ECL is for the first bond $1{,}000{,}000 \times 3\% \times (1 - 60\%) = 12{,}000$, and for the second $600{,}000 \times 5\% \times (1 - 40\%) = 18{,}000$. This adds up to \$30,000. Note that this number does not depend on the default correlation.

Example 19.8: FRM Exam 2007—Question 74

d. Here, the joint default probability matters. If the two bonds default, the loss is $\$1{,}000{,}000 \times (1 - 40\%) + \$600{,}000 \times (1 - 60\%) = \$400{,}000 + \$360{,}000 = \$760{,}000$. This will happen with probability 1.27%. The next biggest loss is \$400,000, which has probability of $3.00 - 1.27 = 1.73\%$. Its cumulative probability must be $100.00 - 1.17 = 98.73\%$. This is slightly above 98%, so \$400,000 is the quantile at the 98% level of confidence or higher. Subtracting the mean gives \$370,000.

Example 19.9: FRM Exam 2007—Question 102

b. The joint PD does not matter for the ECL. This is $\text{ECL} = 1 \times 20\% \times 40\% + 5 \times 30\% \times 60\%, = 0.08 + 0.90$, or EUR 0.98 million.

Example 19.10: FRM Exam 2004—Question 46

b. From Equation (19.11), the default correlation is $\text{Corr}(A, B) = [p(A \text{ and } B) - p(A)p(B)]/\{\sqrt{p(A)[1 - p(A)]}\sqrt{p(B)[1 - p(B)]}\} = [0.0015 - 0.02 \times 0.04]/\{\sqrt{0.02[1 - 0.02]}\sqrt{0.04[1 - 0.04]}\} = 0.025516$.

Example 19.11: FRM Exam 2002—Question 92

a. Because the events are independent, the joint probability is given by the product $(1 - p_1)(1 - p_2)(1 - p_3)(1 - p_4)(1 - p_5) = (1 - 1\%)(1 - 2\%)(1 - 5\%)(1 - 10\%)(1 - 20\%) = 70.51\%$.

Example 19.12: FRM Exam 2004—Question 15

c. Using the second term in Equation (19.18), we have $a_1 = (10/1) = 10$, and the probability is $10p^1(1 - p)^9 = 10 \times 0.05 \times (1 - 0.05)^9 = 0.315$.

CHAPTER 20

Measuring Actuarial Default Risk

Default risk is the primary driver of credit risk. It is represented by the **probability of default** (PD). When default occurs, the actual loss is the combination of **exposure at default** (EAD) and **loss given default** (LGD).

Default risk can be measured using two approaches: (1) **actuarial methods**, which provide objective measures of default rates, usually based on historical default data; and (2) **market-price methods**, which infer from traded prices of debt, equity, or credit derivatives risk-neutral measures of default risk.

Risk-neutral measures were introduced earlier in relation to option prices. They are required for proper asset *pricing*. In addition, a major benefit of risk-neutral measures is that they are forward-looking and responsive to the latest news because they are based on current market prices.

For *risk management* purposes, however, they are contaminated by the effect of risk premiums and contain measures of loss given default. As a result, they do not directly measure default probabilities. In contrast, objective measures describe the actual or natural probability of default.

Actuarial measures of default probabilities are provided by **credit rating agencies** (CRAs), which classify borrowers by credit ratings that are supposed to quantify default risk. Such ratings are **external** to the institution using them. Similar techniques can be used to develop **internal** ratings.

Ratings usually start with **accounting variables models**, which relate the occurrence of default to a list of accounting variables, or more generally firm characteristics. Statistical techniques such as discriminant analysis can be used to examine how these variables are related to the occurrence or nonoccurrence of default. Accounting variables are augmented by information from financial markets and about the economic environment. Rating agencies also have access to management and private information about the firm.

This chapter focuses on actuarial measures of default risk. Market-based measures of default risk are examined in the next chapter. Section 20.1 examines first the definition of a credit event. Section 20.2 then examines credit ratings, describing how historical default rates can be used to infer default probabilities. Recovery

FRM Exam Parts 1 and 2 topic.

rates are analyzed in Section 20.3. Section 20.4 assesses corporate and sovereign credit ratings. Finally, Section 20.5 discusses the role of credit rating agencies.

20.1 CREDIT EVENT

A credit event is a discrete state of the world. Either it happens or it does not, depending on the definition of the event, which must be framed as precisely as possible.

The definition of default for a bond obligation is very narrow. Default on a bond occurs when payment on that bond is missed. The state of **default** is defined by **Standard & Poor's** (S&P), a credit rating agency, as

> *The first occurrence of a payment default on any financial obligation, rated or unrated, other than a financial obligation subject to a bona fide commercial dispute; an exception occurs when an interest payment missed on the due date is made within the grace period.*

Default on a particular bond, however, generally reflects the creditor's financial distress and is typically accompanied by default on other obligations. This is why rating agencies give a credit rating for the *issuer* in addition to a rating for specific bonds. The rating for specific bonds can be higher or lower than this issuer rating, depending on their relative priority.

Default, however, needs to be defined more precisely for credit derivatives, whose payoffs are directly related to credit events. We cover credit derivatives in Chapter 23. The definition of a **credit event** has been formalized by the **International Swaps and Derivatives Association** (ISDA), an industry group, which lists these events:

- **Bankruptcy**, which is defined as a situation involving either of the following:
  - The *dissolution* of the obligor (other than merger)
  - The *insolvency*, or inability to pay its debt
  - The *assignment* of claims
  - The institution of *bankruptcy* proceeding
  - The appointment of *receivership*
  - The *attachment* of substantially all assets by a third party
- **Failure to pay**, which means failure of the creditor to make due payment; this is usually triggered after an agreed-upon grace period and when the payment due is above a certain amount.
- **Obligation/cross default**, which means the occurrence of a default (other than failure to make a payment) on any other similar obligation.
- **Obligation/cross acceleration**, which means the occurrence of a default (other than failure to make a payment) on any other similar obligation, resulting in that obligation becoming due immediately.
- **Repudiation/moratorium**, which means that the counterparty is rejecting, or challenging, the validity of the obligation.
- **Restructuring**, which means a waiver, deferral, or rescheduling of the obligation with the effect that the terms are less favorable than before.

Notably, credit events occurred in 2008 for Fannie Mae and Freddie Mac (receivership), Lehman Brothers Holdings and Washington Mutual (bankruptcy), and Ecuador (failure to pay). In addition, other events sometimes included are:

- **Downgrading**, which means the credit rating is lower than previously, or is withdrawn.
- **Currency inconvertibility**, which means the imposition of exchange controls or other currency restrictions by a governmental or associated authority.
- **Governmental action**, which means either (1) declarations or actions by a government or regulatory authority that impair the validity of the obligation, or (2) the occurrence of war or other armed conflict that impairs the functioning of the government or banking activities.

Ideally, the industry should agree on a common set of factors defining a credit event, to minimize the possibility of disagreements and costly legal battles that create uncertainty for everybody. The ISDA definitions are designed to minimize **legal risk** by precisely wording the definition of credit event.

Even so, unforeseen situations sometimes develop. For example, there have been differences of opinion as to whether a bank debt restructuring constitutes a credit event, as in cases involving Conseco, Xerox, and Marconi.

Another notable situation is that of Argentina, which represents the largest sovereign default recorded so far, in terms of external debt. Argentina announced in November 2001 a restructuring of its local debt that was more favorable to itself. Some holders of credit default swaps argued that this was a credit event, since the exchange was coerced, and that they were entitled to payment. Swap sellers disagreed. This became an unambiguous default, however, when Argentina announced in December it would stop paying interest on its $135 billion foreign debt. Nonetheless, the situation was unresolved for holders of credit swaps that expired just before the official default.

EXAMPLE 20.1: DEFINITION OF A CREDIT EVENT

Which of the following events is *not* a credit event?

a. Bankruptcy
b. Calling back a bond
c. Downgrading
d. Default on payments

20.2 DEFAULT RATES

20.2.1 Credit Ratings

A **credit rating** is an “evaluation of creditworthiness” issued by a credit rating agency (CRA). The major U.S. bond rating agencies are Moody’s Investors Service, Standard & Poor's (S&P), and Fitch Ratings. More technically, a credit rating is defined by Moody's as an

> *opinion of the future ability, legal obligation, and willingness of a bond issuer or other obligor to make full and timely payments on principal and interest due to investors.*

Table 20.1 presents the interpretation of various credit ratings issued by Moody's and S&P. These ratings correspond to long-term debt; other ratings apply to short-term debt. Generally, the two agencies provide similar ratings for the same issuer.

Ratings are divided into the following:

- **Investment grade**, that is, at and above BBB for S&P and Baa for Moody's
- **Speculative grade**, or **below investment grade**, for the rest

Each letter is known as a class. In addition, the CRAs use modifiers, also called **notches**. For instance, S&P subdivides the BBB category into BBB+, BBB, and BBB−. For Moody's, the equivalent ratings are Baa1, Baa2, and Baa3.

These ratings represent objective (or actuarial) probabilities of default.[1] Indeed, the agencies have published studies that track the frequency of bond defaults, classified by initial ratings for different horizons. These frequencies can be used to convert ratings to default probabilities.

The agencies use a number of criteria to decide on the credit rating, including various accounting ratios. Table 20.2 presents median values for selected accounting ratios for U.S. industrial corporations. The first column (under "leverage") shows that the ratio of total debt to total capital (debt plus book equity, or assets) varies systematically across ratings. Highly rated companies have low leverage

TABLE 20.1 Classification by Credit Ratings

Explanation	**Standard & Poor's**	**Moody's Investors Service**
Investment grade:		
Highest grade	AAA	Aaa
High grade	AA	Aa
Upper medium grade	A	A
Medium grade	BBB	Baa
Speculative grade:		
Lower medium grade	BB	Ba
Speculative	B	B
Poor standing	CCC	Caa
Highly speculative	CC	Ca
Lowest quality, no interest	C	C
In default	D	

Modifiers: A+, A, A−, and A1, A2, A3

[1] In fact, the ratings measure the probability of default (PD) for S&P and the joint effect of PD × LGD for Moody's, where LGD is the proportional loss given default.

TABLE 20.2 S&P's Industrial Financial Ratios across Ratings (2005 to 2007 Averages)

	Leverage: (Percent)	Cash Flow Coverage: (Multiplier)	
Rating	Total Debt/Capital	EBITDA/Interest	EBIT/Interest
AAA	12	32.0	26.2
AA	35	19.5	16.4
A	37	13.5	11.2
BBB	45	7.8	5.8
BB	53	4.8	3.4
B	73	2.3	1.4
CCC	99	1.1	0.4

ratios, 12% for AAA firms. In contrast, BB-rated (just below investment grade) companies have a leverage ratio of 53%. This implies a capital-to-equity leverage ratio of 2.1 to 1.[2]

The right-hand panel (under "cash flow coverage") also shows systematic variations in a measure of free cash flow divided by interest payments. This represents the number of times the cash flow can cover interest payments. Focusing on earnings before interest and taxes (EBIT), AAA-rated companies have a safe cushion of 26.2, whereas BB-rated companies have coverage of only 3.4.

A related model for bankruptcy prediction is the **multiple discriminant analysis** (MDA), such as the ***z*-score** model developed by Altman.[3] MDA constructs a linear combination of accounting data that provides the best fit with the observed states of default and nondefault for the sample firms.

The variables used in the z-score are (1) working capital over total assets, (2) retained earnings over total assets, (3) EBIT over total assets, (4) market value of equity over total liabilities, and (5) net sales over total assets. Lower scores indicate a higher likelihood of default. Each variable enters with a positive sign, meaning that an increase in each of these variables decreases the probability of bankruptcy.

EXAMPLE 20.2: FRM EXAM 2003—QUESTION 100

What is the lowest tier of an investment-grade credit rating by Moody's?

a. Baa1
b. Ba1
c. Baa3
d. Ba3

[2] Defining D and E as debt and equity, the debt-to-asset ratio is $D/(D+E) = 53\%$. We then have an asset-to-equity ratio of $(D+E)/E = [D/E]+1 = [D/(D+E)]/[E/(D+E)]+1 = 53\%/(1-53\%)+1 = 2.1$

[3] E. Altman, "Financial Ratios, Discriminant Analysis and the Prediction of Corporate Bankruptcy," *Journal of Finance* 23 (1968): 589–609.

EXAMPLE 20.3: FRM EXAM 2005—QUESTION 86

You are considering an investment in one of three different bonds. Your investment guidelines require that any bond you invest in carry an investment-grade rating from at least two recognized bond rating agencies. Which, if any, of the bonds listed would meet your investment guidelines?

a. Bond A carries an S&P rating of BB and a Moody's rating of Baa
b. Bond B carries an S&P rating of BBB and a Moody's rating of Ba
c. Bond C carries an S&P rating of BBB and a Moody's rating of Baa
d. None of the above

EXAMPLE 20.4: FRM EXAM 2002—QUESTION 110

If Moody's and S&P are equally good at rating bonds, the average default rate on BB bonds by S&P will be lower than the average default rate on bonds rated by Moody's as

a. Baa3
b. Ba1
c. Ba
d. Ba3

20.2.2 Historical Default Rates

Tables 20.3 and 20.4 display historical default rates as reported by Moody's and Standard & Poor's, respectively. These describe the proportion of firms that default, $\bar{X}$, which is a statistical estimate of the true default probability:

$$E(\bar{X}) = p \tag{20.1}$$

For example, borrowers with an initial Moody's rating of Baa experienced an average default rate of 0.29% over the next year. Similar rates are obtained for S&P's BBB-rated credits, who experienced an average 0.23% default rate over the next year. A-rated firms experience a default rate around 0.07% over the next year. Firms at or below Caa have a default rate of 13.73%. Higher ratings are associated with lower default rates. As a result, this information could be used to derive estimates of default probability for an initial rating class.

In addition, the tables show that the cumulative default rate increases sharply with the horizon, for a given initial credit rating. The default rate for Baa firms increases from 0.29% over one year to 7.06% over the following 10 years. Because these are cumulative default rates, the number must necessarily increase with the

TABLE 20.3 Moody's Cumulative Default Rates (Percent), 1920–2007

	Year																			
Rating	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Aaa	0.00	0.00	0.02	0.08	0.16	0.26	0.37	0.53	0.70	0.90	1.07	1.21	1.36	1.41	1.45	1.53	1.62	1.68	1.77	1.83
Aa	0.06	0.18	0.29	0.45	0.70	1.01	1.34	1.65	1.95	2.29	2.68	3.10	3.51	3.93	4.25	4.49	4.68	4.88	5.09	5.27
A	0.07	0.24	0.50	0.81	1.12	1.45	1.80	2.13	2.50	2.90	3.34	3.77	4.15	4.50	4.92	5.28	5.56	5.81	6.08	6.33
Baa	0.29	0.85	1.56	2.34	3.14	3.94	4.71	5.48	6.28	7.06	7.80	8.54	9.24	9.89	10.44	11.01	11.53	12.00	12.44	12.91
Ba	1.34	3.20	5.32	7.49	9.59	11.56	13.36	15.11	16.73	18.44	20.00	21.52	23.04	24.34	25.51	26.64	27.81	28.91	29.85	30.78
B	4.05	8.79	13.49	17.72	21.43	24.66	27.59	30.04	32.15	33.93	35.64	37.26	38.69	40.08	41.40	42.68	43.73	44.52	45.07	45.38
Caa–C	13.73	22.46	29.03	33.92	37.64	40.58	42.87	44.92	47.00	48.98	50.99	53.07	55.05	57.11	59.12	60.98	62.63	64.20	65.68	67.13
Inv.	0.14	0.43	0.81	1.23	1.69	2.16	2.63	3.09	3.58	4.08	4.58	5.09	5.57	6.00	6.42	6.79	7.12	7.41	7.71	8.00
Spec.	3.59	7.24	10.75	13.92	16.71	19.18	21.37	23.34	25.11	26.83	28.44	30.00	31.50	32.87	34.13	35.35	36.52	37.57	38.46	39.28
All	1.41	2.88	4.32	5.63	6.80	7.85	8.80	9.67	10.48	11.28	12.05	12.80	13.50	14.15	14.74	15.29	15.79	16.25	16.66	17.06

TABLE 20.4 S&P's Cumulative Global Default Rates (Percent), 1981–2007

	Year														
Rating	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
AAA	0.00	0.00	0.09	0.18	0.28	0.41	0.48	0.59	0.63	0.67	0.67	0.67	0.67	0.73	0.79
AA	0.01	0.05	0.09	0.19	0.29	0.40	0.52	0.62	0.71	0.81	0.91	0.99	1.09	1.17	1.21
A	0.06	0.16	0.29	0.45	0.64	0.85	1.11	1.32	1.53	1.76	1.95	2.11	2.26	2.39	2.61
BBB	0.23	0.65	1.13	1.75	2.38	2.98	3.47	3.96	4.42	4.89	5.37	5.75	6.22	6.68	7.20
BB	1.00	2.93	5.19	7.36	9.30	11.19	12.72	14.05	15.27	16.24	17.13	17.87	18.51	18.96	19.43
B	4.57	10.06	14.72	18.39	21.08	23.19	24.94	26.37	27.55	28.74	29.80	30.70	31.61	32.47	33.26
CCC	25.59	34.06	39.04	41.86	44.50	45.62	46.67	47.25	48.86	49.76	50.50	51.26	51.87	52.50	52.50
Inv.	0.10	0.30	0.52	0.81	1.11	1.42	1.69	1.95	2.19	2.44	2.66	2.85	3.05	3.24	3.47
Spec.	4.11	8.11	11.66	14.57	16.90	18.84	20.45	21.79	23.01	24.08	25.05	25.87	26.64	27.30	27.90
All	1.45	2.91	4.21	5.33	6.26	7.06	7.73	8.30	8.81	9.29	9.72	10.08	10.44	10.76	11.09

horizon. For investment-grade credits, however, the increase is more than proportional with the horizon. The ratio is $7.06/0.29 = 24$ for Baa-rated credits, which is more than 10. In contrast, the ratio for B-rated credits is $33.93/4.05 = 8$. For speculative-grade credits, the increase is less than proportional with the horizon.

One problem with such historical information, however, is the relative paucity of data in some cells. There are few defaults for highly rated firms over a one-year horizon. For example, the one-year default rate for Aa firms is 0.06%. This corresponds to very few defaults. From 1939 to 2007, only two companies initially rated Aa defaulted during the following year, which happened in 1989. Changing the sample period or having another number of defaults could have a substantial effect on this average default rate.

In addition, the sample size decreases as the horizon lengthens. Consider, for instance, default rates reported by S&P over the period 1981 to 2007. The one-year default rate is an average using 27 periods—that is, 1981, 1982, and so on to 2007. For 15-year horizons, however, the first period is 1981 to 1995, the second is 1982 to 1996, and so on until the last period of 1993 to 2007. In this case, the average therefore uses only 13 periods, which is a much shorter sample. The data are also overlapping and therefore not independent. So, omitting or adding a few borrowers can drastically alter the reported default rate.

This can lead to inconsistencies in the tables. For instance, the default rate for CCC obligors is the same, at 52.50%, from year 14 to 15. This implies that there is no further risk of default after 14 years, which is an unrealistic implication. Also, when the categories are further broken down into modifiers, default rates sometimes do not decrease monotonically as the rating increases, which is probably a small-sample effect.

We can try to assess the accuracy of these default rates by computing their standard errors. Consider, for instance, the default rate over the first year for AA-rated credits, which averaged out to $\bar{X} = 0.01\%$ in this S&P sample. Assume that this was taken out of a total of about $N = 10{,}000$ independent observations. The variance of the average is, from the distribution of a binomial process,

$$V(\bar{X}) = \frac{p(1-p)}{N} \tag{20.2}$$

which gives a standard error of about 0.01%. This is on the same order as the average of 0.01%, indicating that there is substantial imprecision in this average default rate. So we could not really distinguish an AA credit from an AAA credit.

The problem is made worse with lower sample sizes, which is the case in non-U.S. markets or when the true p is changing over time. For instance, if we observe a 5% default rate among 100 observations, the standard error becomes 2.2%, which is very large. Therefore, a major issue with credit risk is that estimation of default rates for low-probability events can be very imprecise.

20.2.3 Cumulative and Marginal Default Rates

The default rates reported in Tables 20.3 and 20.4 are **cumulative default rates** for an initial credit rating; that is, they measure the total frequency of default *at any*

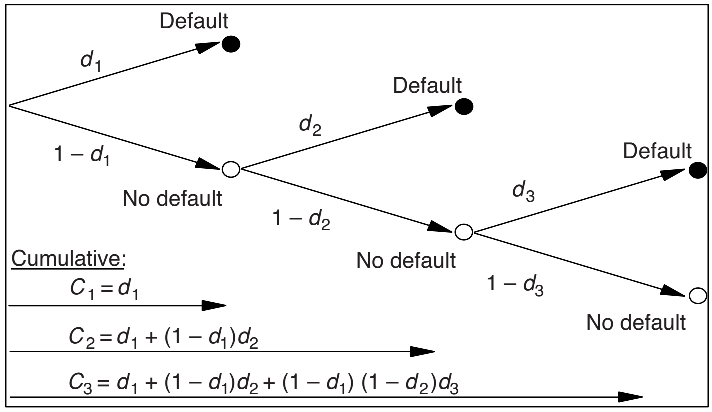

FIGURE 20.1 Sequential Default Process

time between the starting date and year T. It is also informative to measure the **marginal default rate**, which is the frequency of default *during* year T.

The default process is illustrated in Figure 20.1. Here, d_1 is the marginal default rate during year 1, and d_2 is the marginal default rate during year 2. To default during the second year, the firm must have survived the first year and defaulted in the second. Thus, the probability of defaulting in year 2, is given by $(1 - d_1)d_2$. The cumulative probability of defaulting up to year 2 is then $C_2 = d_1 + (1 - d_1)d_2$. Subtracting and adding 1, this is also $C_2 = 1 - (1 - d_1)(1 - d_2)$, which perhaps has a more intuitive interpretation, as this is 1 minus the probability of surviving the entire period.

More formally,

- $m[t + N \mid R(t)]$ is the number of issuers rated R at the end of year t that default in year $T = t + N$.
- $n[t + N \mid R(t)]$ is the number of issuers rated R at the end of year t that have not defaulted by the beginning of year $t + N$.

Marginal Default Rate during Year *T* This is the proportion of issuers initially rated R at initial time t that default in year T, relative to the remaining number at the beginning of the same year T:

$$d_N(R) = \frac{m[t + N \mid R(t)]}{n[t + N \mid R(t)]}$$

Survival Rate This is the proportion of issuers initially rated R that will not have defaulted by T:

$$S_N(R) = \Pi_{i=1}^{N}(1 - d_i(R)) \tag{20.3}$$

Marginal Default Rate from Start to Year *T* This is the proportion of issuers initially rated R that defaulted in year T, relative to the initial number in year t.

For this to happen, the issuer will have survived until year $t + N - 1$, and then default the next year. Hence, this is:

$$k_N(R) = S_{N-1}(R)d_N(R) \tag{20.4}$$

Cumulative Default Rate This is the proportion of issuers rated R that defaulted at any point until year T:

$$C_N(R) = k_1(R) + k_2(R) + \cdots + k_N(R) = 1 - S_N(R) \tag{20.5}$$

Average Default Rate We can express the total cumulative default rate as an average per-period default rate d, by setting

$$C_N = 1 - \Pi_{i=1}^{N}(1 - d_i) = 1 - (1 - d)^N \tag{20.6}$$

As we move from annual to semiannual and ultimately continuous compounding, the average default rate becomes

$$C_N = 1 - (1 - d^a)^N = 1 - (1 - d^s/2)^{2N} \to 1 - e^{-d^c N} \tag{20.7}$$

where d^a, d^s, and d^c are default rates using annual, semiannual, and continuous compounding. This is equivalent to the various definitions for the compounding of interest.

Example: Computing Cumulative Default Probabilities

Consider a B-rated firm that has default rates of $d_1 = 5\%$ and $d_2 = 7\%$. Compute the cumulative default probabilities.

Answer

In the first year, $k_1 = d_1 = 5\%$. After one year, the survival rate is $S_1 = 0.95$. The probability of defaulting in year 2 is then $k_2 = S_1 \times d_2 = 0.95 \times 0.07 = 6.65\%$. After two years, the survival rate is $(1 - d_1)(1 - d_2) = 0.95 \times 0.93 = 0.8835$. Thus, the cumulative probability of defaulting in years 1 and 2 is $5\% + 6.65\% = 11.65\%$.

Based on this information, we can map these *forward*, or marginal, default rates from cumulative default rates for various credit ratings. Figure 20.2, for instance, displays cumulative default rates reported by Moody's in Table 20.3. The corresponding marginal default rates are plotted in Figure 20.3.

It is interesting to see that the marginal probability of default increases with maturity for initial high credit ratings, but decreases for initial low credit ratings. The increase is due to a mean-reversion effect. The fortunes of an Aaa-rated firm can only stay the same at best, and often will deteriorate over time. In contrast, a B-rated firm that has survived the first few years must have a decreasing probability of defaulting as time goes by. This is a survival effect.

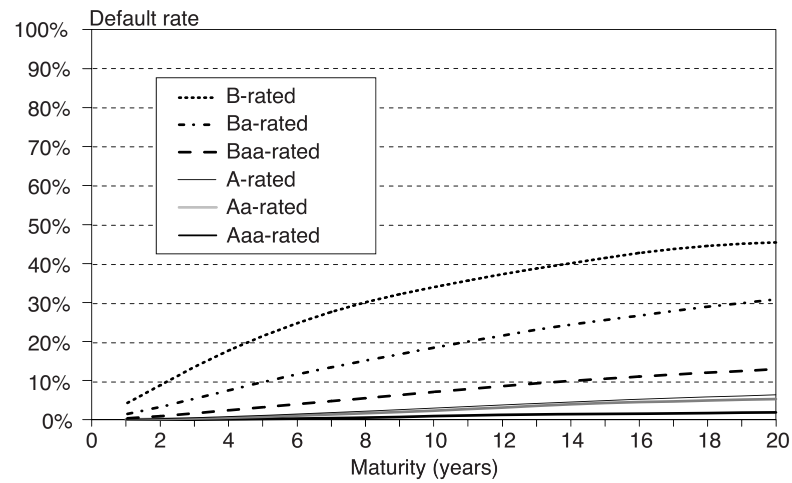

FIGURE 20.2 Moody's Cumulative Default Rates, 1920–2007

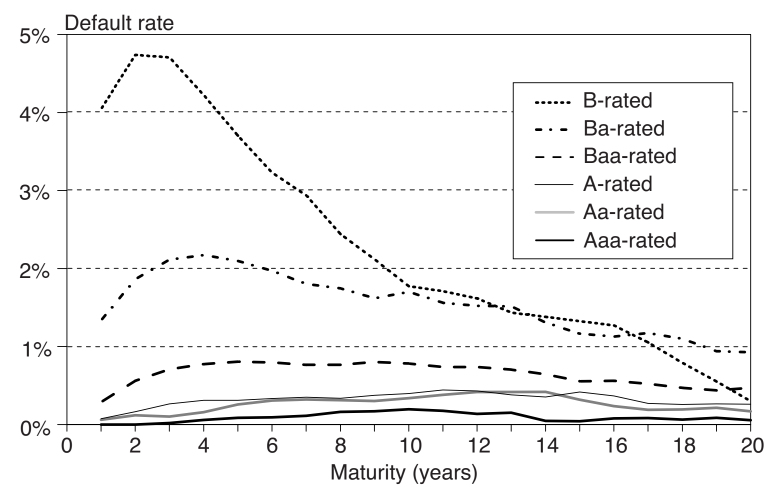

FIGURE 20.3 Moody's Marginal Default Rates, 1920–2007

EXAMPLE 20.5: FRM EXAM 2004—QUESTION 1

Company ABC was incorporated on January 1, 2004. It has an expected annual default rate of 10%. Assuming a constant quarterly default rate, what is the probability that company ABC will *not* have defaulted by April 1, 2004?

a. 2.40%
b. 2.50%
c. 97.40%
d. 97.50%

EXAMPLE 20.6: FRM EXAM 2002—QUESTION 77

If the default probability for an A-rated company over a three-year period is 0.3%, the most likely probability of default for this company over a six-year period is:

a. 0.30%
b. Between 0.30% and 0.60%
c. 0.60%
d. Greater than 0.60%

EXAMPLE 20.7: FRM EXAM 2006—QUESTION 21

What is the survival rate at the end of three years if the annual default probabilities are 8%, 12%, and 15% in the first, second, and third years, respectively?

a. 68.8%
b. 39.1%
c. 99.9%
d. 65.0%

EXAMPLE 20.8: FRM EXAM 2008—QUESTION 3-1

The marginal default probabilities for an A-rated issue are, respectively, for years 1, 2, and 3: 0.300%, 0.450%, and 0.550%. Assume that defaults, if they take place, happen only at the end of the year. Calculate the cumulative default rate at the end of each of the next three years.

a. 0.300%, 0.750%, 1.300%
b. 0.300%, 0.150%, 0.250%
c. 0.300%, 0.749%, 1.295%
d. 0.300%, 0.449%, 0.548%

EXAMPLE 20.9: DEFAULT PROBABILITY DEFINITIONS

What is the difference between the marginal default probability and the cumulative default probability?

a. Marginal default probability is the probability that a borrower will default in any given year, whereas the cumulative default probability is over a specified multiyear period.
b. Marginal default probability is the probability that a borrower will default due to a particular credit event, whereas the cumulative default probability is for all possible credit events.
c. Marginal default probability is the minimum probability that a borrower will default, whereas the cumulative default probability is the maximum probability.
d. Both a. and c. are correct.

20.2.4 Transition Probabilities

As we have seen, the measurement of long-term default rates can be problematic with small sample sizes. The computation of these default rates can be simplified by assuming a Markov process for the ratings migration, described by a transition matrix. **Migration** is a discrete process that consists of credit ratings changing from one period to the next.

The **transition matrix** gives the probability of moving to one rating conditional on the rating at the beginning of the period. The usual assumption is that these moves follow a **Markov process**, or that migrations across states are independent from one period to the next.[4] This type of process exhibits *no carry-over effect*. More formally, a **Markov chain** describes a stochastic process where the conditional distribution, given today's value, is constant over time. Only present values are relevant.

Table 20.5 gives an example of a simplified transition matrix for four states, A, B, C, D, where the last represents default. Consider a company in year 0 in the B category. The company could default

- In year 1, with probability $D[t_1 \mid B(t_0)] = P(D_1 \mid B_0) = 3\%$
- In year 2, after going from B to A in the first year, then A to D in the second; or from B to B, then to D; or from B to C, then to D. The default probability is $P(D_2 \mid A_1)P(A_1) + P(D_2 \mid B_1)P(B_1) + P(D_2 \mid C_1)P(C_1) = 0.00 \times 0.02 + 0.03 \times 0.93 + 0.23 \times 0.02 = 3.25\%$.

[4] There is some empirical evidence, however, that credit downgrades are not independent over time but instead display a momentum effect.

TABLE 20.5 Credit Ratings Transition Probabilities

	Ending State				
Starting State	**A**	**B**	**C**	**D**	**Total Probability**
A	0.97	0.03	0.00	0.00	1.00
B	0.02	0.93	0.02	0.03	1.00
C	0.01	0.12	0.64	0.23	1.00
D	0	0	0	1.00	1.00

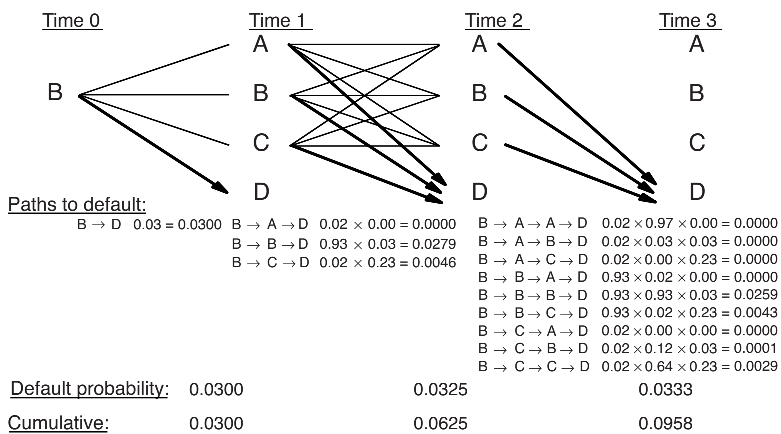

FIGURE 20.4 Paths to Default

The cumulative probability of default over the two years is then 3%+3.25%= 6.25%. Figure 20.4 illustrates the various paths to default in years 1, 2, and 3.

The advantage of using this approach is that the resulting data are more robust and consistent. For instance, the 15-year cumulative default rate obtained this way will always be greater than the 14-year default rate.

EXAMPLE 20.10: FRM EXAM 2005—QUESTION 105

A rating transition table includes sufficient information to find all but which of the following items?

a. The likelihood that an AA-rated firm will fall to a BB rating over five years
b. The price of a bond that has been downgraded to BB from BBB
c. The probability of default on a B-rated bond
d. The probability that a high-yield bond will be upgraded to investment grade

EXAMPLE 20.11: FRM EXAM 2007—QUESTION 51

Fitch Ratings provides a table indicating that the number of A-rated issuers migrating to AAA is 2, to AA is 5, staying at A is 40, migrating to BBB is 2, and going into default is 3. Based on this information, what is the probability that an issue with a rating of A at the beginning of the year will be downgraded by the end of the year?

a. 13.46%
b. 13.44%
c. 9.62%
d. 3.85%

EXAMPLE 20.12: FRM EXAM 2009—QUESTION 4-18

A two-year zero-coupon bond issued by ABC Co. is currently rated A. The market expects that one year from now the probability that the rating of ABC remains at A, is downgraded to BBB, or is upgraded to AA are, respectively, 80%, 15%, and 5%. Suppose that the risk-free rate is flat at 1% and that credit spreads for AA-, A-, and BBB-rated debt are flat at 80, 150, and 280 basis points, respectively. All rates are compounded annually. What is the best approximation of the expected value of the zero-coupon bond one year from now?

a. 97.41
b. 97.37
c. 94.89
d. 92.44

EXAMPLE 20.13: FRM EXAM 2008—QUESTION 3-20

As a result of the credit crunch, a small retail bank wants to better predict and model the likelihood that its larger commercial loans might default. It is developing an internal ratings-based approach to assess its commercial customers. Given this one-year transition matrix, what is the probability that a loan currently rated at B will default over a two-year period?

Rating at Beginning of Period	Rating at End of Period			
	A	B	C	Default
A	0.90	0.10	0.00	0.00
B	0.00	0.75	0.15	0.10
C	0.00	0.05	0.55	0.40

a. 17.5%
b. 20.0%
c. 21.1%
d. 23.5%

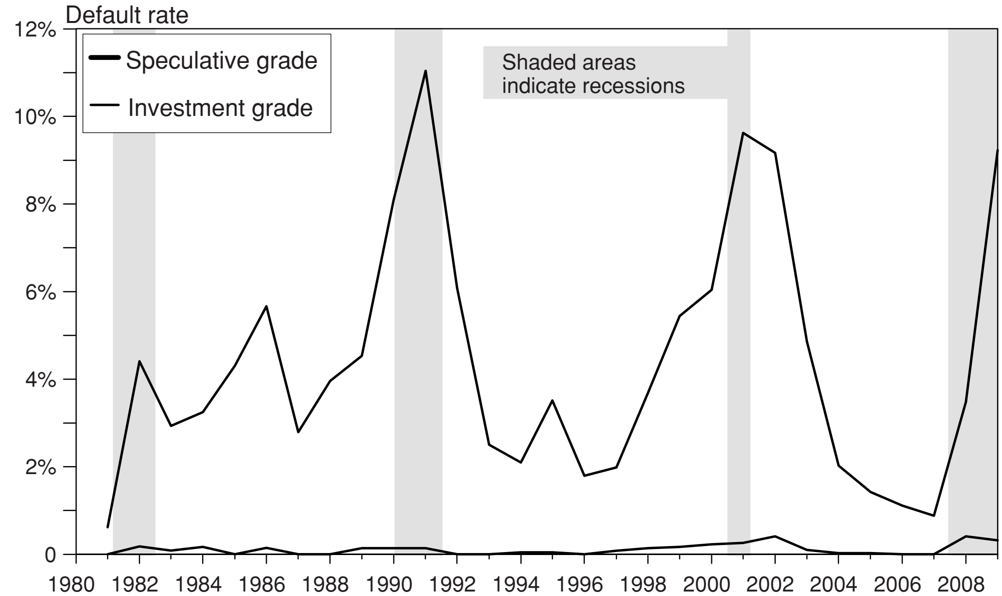

FIGURE 20.5 Time Variation in Defaults (from S&P)

20.2.5 Time Variation in Default Probabilities

Defaults are also correlated with economic activity. Moody's, for example, has compared the annual default rate to the level of industrial production since 1920. Moody's reports a marked increase in the default rate in the 1930s at the time of the Great Depression and during the recent recessions. These default rates, however, do not control for structural shifts in the credit quality. In recent years, many issuers came to the market with a lower initial credit rating than in the past. This should lead to more defaults even with a stable economic environment.

To control for this effect, Figure 20.5 plots the default rate for investment-grade and speculative credits over the years 1981 to 2009. As expected, the default rate of investment-grade bonds is very low. More interestingly, however, it displays minimal variation through time. We do observe, however, significant variation in the default rate of speculative-grade credits, which peaks during the recessions that started in 1981, 1990, 2001, and 2007. Thus, economic activity significantly affects the frequency of defaults. This effect is most marked for speculative-grade bonds.

20.3 RECOVERY RATES

Credit risk also depends on the **loss given default** (LGD). This can be measured as 1 minus the **recovery rate**, or fraction recovered after default.

20.3.1 The Bankruptcy Process

Normally, default is a state that affects all obligations of an issuer equally, especially when accompanied by a bankruptcy filing. In most countries, a formal

TABLE 20.6 Pecking Order in U.S. Federal Bankruptcy Law

Seniority	Type of Creditor
Highest (paid first)	1. Secured creditors (up to the extent of secured collateral)
	2. Priority creditors
	• Firms that lend money during bankruptcy period
	• Providers of goods and services during bankruptcy period (e.g., employees, lawyers, vendors)
	• Taxes
	3. General creditors
	• Unsecured creditors before bankruptcy
Lowest (paid last)	• Shareholders

bankruptcy process provides a centralized forum for resolving all the claims against the corporation. The bankruptcy process creates a **pecking order** for a company's creditors. This spells out the sequence in which creditors are paid, thereby creating differences in the recovery rate across creditors. Within each class, however, creditors should be treated equally.

In the United States, firms that are unable to make required payments can file for either **Chapter 7** bankruptcy, which leads to the liquidation of the firm's assets, or **Chapter 11** bankruptcy, which leads to a reorganization of the firm during which the firm continues to operate under court supervision.

Under Chapter 7, the proceeds from liquidation should be divided according to the **absolute priority rule**, which states that payments should be made first to claimants with the highest priority. Table 20.6 describes the pecking order in bankruptcy proceedings. At the top of the list are **secured creditors**, who, because of their property right, are paid to the fullest extent of the value of their collateral. Then come **priority creditors**, which consist mainly of post-bankruptcy creditors. Finally, **general creditors** can be paid if funds remain after distribution to others.

Similar rules apply under Chapter 11. In this situation, the firm must submit a **reorganization plan**, which specifies new financial claims to the firm's assets. The absolute priority rule is often violated in Chapter 11 settlements, however. Junior debt holders and stockholders often receive some proceeds even though senior shareholders are not paid in full. This is allowed to facilitate timely resolution of the bankruptcy and to avoid future lawsuits. Even so, there remain sharp differences in recovery rates across seniorities.

20.3.2 Estimates of Recovery Rates

Recovery rates are commonly estimated from the market prices of defaulted debt shortly after default. This is viewed as the best estimate of the future recovery and takes into account the value of the firm's assets, the estimated cost of the bankruptcy process, and various means of payment (e.g., using equity to pay bondholders), discounted into the present.

The recovery rate has been shown to depend on a number of factors:

- *The status or seniority of the debtor*. Claims with higher seniority have higher recovery rates. More generally, a greater **debt cushion**, or the percentage of total company debt below the instrument, also leads to higher recovery rates.

- *The state of the economy*. Recovery rates tend to be higher when the economy is in an expansion, and lower when in a recession.
- *The obligor's characteristics*. Recovery rates tend to be higher when the borrower's assets are tangible and when the previous rating was high. Utilities have more **tangible assets**, such as power-generating plants, than other industries and consequently have higher recovery rates. Also, companies with greater interest coverage and higher credit ratings typically have higher recovery rates.
- *The type of default*. Distressed exchanges, as opposed to bankruptcy proceedings, usually lead to higher recovery rates. Unlike a bankruptcy proceeding, which causes all debts to go into default, a distressed exchange involves only the instruments that have defaulted.

Ratings can also include the loss given default. The same borrower may have various classes of debt, which may have different credit ratings due to the different levels of protection. If so, debt with lower seniority should carry a lower rating.

Table 20.7 displays recovery rates for corporate debt, from Moody's. The average recovery rate for senior unsecured debt is around $f = 37\%$. Derivative instruments rank as senior unsecured creditors and should have the same recovery rates as senior unsecured debt.

Bank loans are usually secured and therefore have higher recovery rates, typically around 60%. As expected, subordinated bonds have the lowest recovery rates, typically around 20% to 30%.

There is, however, much variation around the average recovery rates. The table reports not only the average value but also the standard deviation, minimum, maximum, and 10th and 90th percentiles. Recovery rates vary widely. In addition, recovery rates are negatively related to default rates. During years with more bond defaults, prices after default are more depressed than usual. This correlation creates bigger losses, which extends the tail of the credit loss distribution. In practice, the distribution of recovery rates is often modeled with a beta distribution, which has an argument ranging from 0 to 1.

TABLE 20.7 Moody's Recovery Rates for Global Corporate Debt (Percent)

Priority	Count	Mean	S.D.	Min.	10th	Median	90th	Max.
All bank loans	310	61.6	23.4	5.0	25.0	67.0	90.0	98.0
Equipment trust	86	40.2	29.9	1.5	10.6	31.0	90.0	103.0
Senior secured	238	53.1	26.9	2.5	10.0	34.0	82.0	125.0
Senior unsecured	1,095	37.4	27.2	0.3	7.0	30.0	82.2	122.6
Senior subordinated	450	32.0	24.0	0.5	5.0	27.0	66.5	123.0
Subordinated	477	30.4	21.3	0.5	5.0	27.1	60.0	102.5
Junior subordinated	22	23.6	19.0	1.5	3.8	16.4	48.5	74.0
All bonds	2,368	36.8	26.3	0.3	7.5	30.0	80.0	125.0

Source: Adapted from Moody's, based on 1982–2002 defaulted bond prices.

TABLE 20.8 Moody's Mean Recovery Rates (Percent): Europe and North America

Instrument	Europe	North America
Bank loans	47.6	61.7
Bonds		
Senior secured	52.2	52.7
Senior unsecured	25.6	37.5
Senior subordinated	24.3	32.1
Subordinated	13.9	31.3
Junior subordinated	NA	24.5
All bonds	28.4	35.3
Preferred stock	3.4	10.9
All instruments	27.6	35.9

Source: Adapted from Moody's, from 1982–2002 defaulted bond prices.

The legal environment is also a main driver of recovery rates. Differences across national jurisdictions cause differences among recovery rates. Table 20.8 compares mean recovery rates across Europe and North America. Recovery rates are significantly higher in the United States than in Europe.

Using trading prices of debt shortly after default as estimates of recovery is convenient because the bankruptcy process can be slow, often taking years. Computing the total value of payments to debt holders can also be complicated, and should take into account the time value of money.

The evidence, however, is that trading prices are on average lower than the discounted recovery rate, as shown in Table 20.9. The average discounted recovery rate is systematically higher than the indication given by trading prices. This could be due to different clienteles for the two markets, or to a risk premium in trading prices. In other words, trading prices may be artificially depressed because investors want to get rid of defaulted securities in their portfolios. If so, this creates an interesting trading opportunity. Buying the defaulted debt and working through the recovery process should create value. Indeed, this largely explains the existence of the hedge fund category called **distressed securities funds**. Such funds invest in selected distressed securities and benefit from their subsequent increase in value.

TABLE 20.9 S&P's Recovery Rates for Corporate Debt (Percent)

Instrument	Trading Prices 15–45 Days	Discounted Recovery
Bank loans	58.0	81.6
Senior secured bonds	48.6	67.0
Senior unsecured bonds	34.5	46.0
Senior subordinated bonds	28.4	32.4
Subordinated bonds	28.9	31.2

Source: Adapted from S&P, from 1988–2002 defaulted debt.

EXAMPLE 20.14: FRM EXAM 2005—QUESTION 74

Moody's estimates the average recovery rate for senior unsecured debt to be nearest to

a. 20%
b. 40%
c. 60%
d. 80%

EXAMPLE 20.15: FRM EXAM 2002—QUESTION 123

The recovery rate on credit instruments is defined as 1 minus the loss rate. The loss rate can be significantly influenced by the volatility of the value of a firm's assets before default. All other things being equal, in the event of a default, which type of company would we expect to have the highest recovery rate?

a. A trading company active in volatile markets
b. An Internet merchant of trendy consumer products
c. An asset-intensive manufacturing company
d. A highly leveraged hedge fund

20.4 ASSESSING CORPORATE AND SOVEREIGN RATINGS

20.4.1 Corporate Ratings

Rating agencies expend considerable effort and financial resources in coming up with publicly available credit ratings. As explained in Table 20.2, the primary inputs for the credit rating process are accounting variables such as balance sheet leverage and debt coverage. The weight assigned to these variables may change if their informativeness changes over time (i.e., if earnings management is suspected).

By nature, however, accounting information is backward looking. The economic prospects of a company are also crucial for assessing credit risk. These include growth potential, market competition, and exposure to financial risk factors. Rating agencies also have access to private information, including meetings with management during which they might be provided with confidential information.

Rating agencies also need to account for structural differences across countries. These could arise because of a number of factors:

- *Differences in financial stability across countries.* Countries differ in terms of financial market structures and government policies. The mishandling of economic policy, for instance, can turn what should be a minor devaluation into a major problem, leading to a recession.

- *Differences in legal systems.* The protection accorded to creditors can vary widely across countries, some of which have not yet established a bankruptcy process.

In theory, ratings provided by credit rating agencies are supposed to be *consistent* across countries and industrial sectors. In other words, they should take into account such variations and represent the same probability of default.

Finally, credit ratings are supposed to look **through the cycle**, where a business cycle typically covers several years. This means that the rating should not depend on the current position in the business cycle. There is no point in assigning a high rating to a company enjoying peak prosperity if that performance is expected to be temporary, for instance due to high consumer demand that will revert soon to a long-run average. Figure 20.6 illustrates how corporate performance depends on the cycle. In theory, ratings should be constant through time.

In practice, a down cycle can have a lasting impact on credit quality, in extreme cases leading to default. As a result, actual ratings may be affected by the business cycle, in particular for speculative-grade firms.

Ratings less sensitive to cyclical factors should be more stable over time. This should reduce the **procyclical effect** of capital charges based on external credit ratings. Lower credit ratings during a recession would lead to higher capital charges, precisely after banks have suffered credit losses, which would force them to raise additional equity and to reduce lending, which would aggravate the recession. This effect is further discussed in Chapter 28. In contrast, the practice of rating through the cycle implies that credit ratings may underestimate the probabilities of default during a recession, and conversely overestimate such probabilities during an expansion.

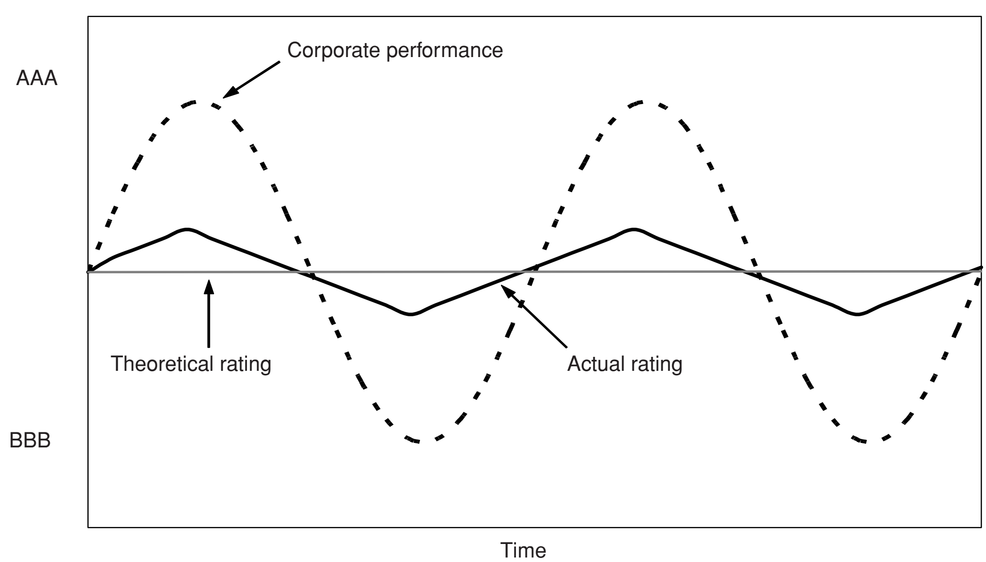

FIGURE 20.6 Business Cycle and Credit Ratings

20.4.2 Sovereign Ratings

Rating agencies have only recently started to rate sovereign bonds. In 1975, S&P rated only seven countries, all of which were investment grade. By 1990, the pool had expanded to 31 countries, of which only nine were from emerging markets. Now, S&P rates approximately 120 countries. The history of default, however, is even more sparse. As a result, it is difficult to generalize from a very small sample.

Assessing credit risk for sovereign nations is significantly more complex than for corporates. When a corporate borrower defaults, legal action can be taken by the creditors. For instance, an unsecured creditor can file an action against a debtor and have the defendant's assets seized under a writ of attachment. This creates a **lien** on its assets, or a claim on the assets as security for the payment of the debt. In contrast, it is impossible to attach the domestic assets of a sovereign nation. As a result, recovery rates on sovereign debt are usually lower than recovery rates on corporate debt. Thus, sovereign credit evaluation involves not only **economic risk** (the ability to repay debts when due), but also **political risk** (the willingness to pay). Some countries have unilaterally repudiated their debt (i.e., refused to make payments even when they had the ability to do so).

Sovereign credit ratings also differ depending on whether the debt is **local currency debt** or **foreign currency debt**. Table 20.10 displays the factors involved in local and foreign currency ratings.

Political risk factors (e.g., degree of political consensus, integration in global trade and financial system, and internal or external security risk) play an important part in sovereign credit risk. Countries with greater political stability have higher credit ratings.

The second group of factors includes income and economic structure, and economic growth prospects. Countries that are richer or growing faster tend to have higher credit ratings.

The third group of factors includes fiscal flexibility, the public debt burden, and contingent liabilities. Countries with lower budget deficits and lower debt amounts in relation to the size of their economy tend to have higher credit ratings. The evaluation of debt, however, must also include public-sector enterprises where a default would require government funding, as well as the financial sector, where the government may have to inject funds to ensure stability.

TABLE 20.10 Credit Ratings Factors

Category	Local Currency	Foreign Currency
Political risk	x	x
Income and economic structure	x	x
Economic growth prospects	x	x
Fiscal flexibility	x	x
Public debt burden	x	x
Contingent liabilities	x	x
Monetary flexibility	x	x
External liquidity		x
External debt burden		x

The fourth group includes monetary flexibility. High rates of inflation typically reflect economic mismanagement and are associated with political instability. Countries with high inflation rates tend to have low credit ratings.

All of the previous factors affect both *local currency debt* and *foreign currency debt*. Foreign currency debt is also affected by external liquidity and the external debt burden. External liquidity is assessed by balance of payment flows. Countries with large current account deficits, reflecting excess imports, tend to have low credit ratings, especially when financed by volatile capital inflows. In particular, the ratio of external interest payments to exports is closely watched. The maturity profile of flows is also important. In the case of the 1997 Asian crisis, rating agencies seem to have overlooked other important aspects of creditworthiness, such as the currency and maturity structure of national debt. Too many Asian creditors had borrowed short-term in dollars to invest in the local currency, which created a severe liquidity problem when their currency devalued. Finally, the external debt burden is assessed by the international investment position of a country (that is, public and private external debt) as well as the stock of foreign currency reserves.

Because local currency debt is backed by the taxation power of the government, local currency debt is considered to have less credit risk than foreign currency debt. Table 20.11 displays local and foreign currency debt ratings for a sample of

TABLE 20.11 S&P's Sovereign Credit Ratings, September 2010

Issuer	Local Currency	Foreign Currency
Argentina	B	B
Australia	AAA	AAA
Belgium	AA+	AA+
Brazil	BBB+	BBB−
Canada	AAA	AAA
China	A+	A+
France	AAA	AAA
Germany	AAA	AAA
Hong Kong	AA+	AA+
India	BBB−	BBB−
Italy	A+	A+
Japan	AA	AA
Mexico	A	BBB
Netherlands	AAA	AAA
Russia	BBB+	BBB
South Africa	A+	BBB+
South Korea	A+	A
Spain	AA	AA
Switzerland	AAA	AAA
Taiwan	AA−	AA−
Thailand	A−	BBB+
Turkey	BB+	BB
United Kingdom	AAA	AAA
United States	AAA	AAA

countries. Ratings for foreign currency debt are the same as, or generally only one notch below, those of local currency debt. Similarly, sovereign debt is typically rated higher than corporate debt in the same country. Governments can repay foreign currency debt, for instance, by controlling capital flows or seizing foreign currency reserves.

Overall, sovereign debt ratings are considered less reliable than corporate ratings. Indeed, bond spreads are generally greater for sovereigns than for corporate issuers. There are also greater differences in sovereign ratings across agencies than for corporates. Thus, the evaluation of sovereign credit risk is a much more subjective process than for corporates.

EXAMPLE 20.16: FRM EXAM 2005—QUESTION 79

In the context of evaluating sovereign risk, which of the following statements is *incorrect*?

a. Bankruptcy law does not typically protect investors from sovereign risk.
b. Debt repudiation is a postponement of all current and future foreign debt obligations of a borrower.
c. Debt rescheduling occurs when a group of creditors declares a moratorium on debt obligations and seeks to reschedule terms.
d. Sovereign risk can be a cause of default in a nongovernmental borrower of high credit quality.

EXAMPLE 20.17: FRM EXAM 2009—QUESTION 4-19

Rating agencies typically assign two ratings to debt-issuing countries. The first is the local currency debt rating and the second is the foreign currency debt rating. Historically, defaults have been more frequent on foreign-currency-denominated debt than on local-currency-denominated debt. What is the main reason behind this difference?

a. This is a statistical anomaly, as the default rate theoretically should be the same in both cases.
b. Foreign-currency-denominated debt is usually less collateralized than local-currency-denominated debt.
c. Local-currency-denominated debt obligations could be met through monetary expansion.
d. In distressed situations, governments tend to default on their foreign-currency-denominated debt first for political reasons.

20.5 REGULATION OF CREDIT RATING AGENCIES

20.5.1 Role of Credit Rating Agencies

Credit rating agencies play an important role in financial markets. They provide widely available summary information about the credit risk of a wide range of instruments and obligors. They help mitigate the asymmetry of information between borrowers and investors. This is especially useful for small investors, who do not have the time or the resources for detailed credit analyses.

Investors use these credit ratings to assess credit risk and to comply with investment guidelines and regulations. Sell-side firms such as broker-dealers use them to determine the amount of collateral to hold against credit exposure.

The role of credit ratings is also officially recognized. Indeed, credit rating agencies registered with the Securities and Exchange Commission (SEC) are known as **Nationally Recognized Statistical Rating Organizations** (NRSROs). Several regulations at the federal and state levels explicitly use ratings from NRSROs. For example, the Basel II rules for commercial banks include capital charges that depend on external credit ratings.

20.5.2 Conflicts of Interest

Credit rating agencies, however, are beset by conflicts of interest. Even though their ratings are viewed as opinions to the reader, they are paid directly by the firms that they rate. This is because it would not be economically feasible to rely on a model where investors pay.[5]

The model where the debt issuer pays, however, creates conflicting objectives. On the one hand, rating agencies may have an incentive to *maximize profits* by providing easy ratings to many client firms. On the other hand, they have a countervailing incentive to *protect their reputations* as neutral third parties that provide independent and objective advice. If their ratings became worthless, they would quickly lose their business franchise.

Credit rating agencies argue that there is no strong incentive to accommodate the preferences of bond issuers, because each single issuer represents a small fraction of revenues. For a long time, these countervailing incentives were delicately balanced and the system seemed to work well.

20.5.3 Structured Products

In recent years, however, credit rating agencies have expanded into rating the different tranches of structured credit products. This was a booming market with complex products that will be described later. The business of rating structured products was highly profitable, much more so than before, and less dispersed across issuers. Credit rating agencies were also involved in the design of the

[5] Only one CRA, Egan-Jones Ratings Company, derives its revenues from charging investors. Its market share, however, has remained low.

products, for instance offering advice as to the width of various tranches to achieve the desired rating. This was quite a step from giving an opinion on the default risk of a bond issued by a corporate entity. The banks that designed these structured products could also shop around the three major credit rating agencies, giving their business to the most accommodating one. These pressures seem to have led to a marked relaxation of credit standards. Credit rating agencies gave very high grades to securities that quickly went bad when subprime-backed debt started to default in 2007.

Investors, burned by these losses, lost faith in the quality of credit ratings and withdrew from credit markets, sometimes abruptly. This put in motion the sequence of events that led to the recent credit crisis.

20.5.4 Regulatory Response

Because the credit rating agencies are widely viewed as having played a role in this crisis, their regulation has been tightened.

In 2010, the **Dodd-Frank Wall Street Reform and Consumer Protection Act** strengthened the regulation of CRAs by giving more supervisory authority to the SEC. Now CRAs may not provide both ratings and advice on how to structure securities. Agencies are now required to provide more information on their methodologies and track records. The Act also expanded their potential liability, by making litigation easier against CRAs. The Act removes their exemption from **Regulation Fair Disclosure** (FD), which allowed them to have access to company information not released to the public. Finally, the Act requires CRAs to improve their internal controls and governance. This leaves their core model, where debt issuers pay for ratings, unchanged, but hopefully with better management of conflicts of interest.

In addition, EU regulators will create a pan-European supervisory body that will handle the registration and oversight of credit rating agencies in Europe.

20.6 IMPORTANT FORMULAS

Credit ratings by Standard & Poor's (modifiers, +, −): AAA, AA, A, BBB (investment grade); BB, B, CCC, and below (speculative grade)

Credit ratings by Moody's (modifiers, 1, 2, 3): Aaa, Aa, A, Baa (investment grade); Ba, B, Caa, and below (speculative grade)

Default rate $\bar{X}$ mean and variance: $E(\bar{X}) = p,\ V(\bar{X}) = p(1-p)/N$

Marginal default rate for firm initially rated R during year $T = t + N$: $d_N(R) = \frac{m[t+N]}{n[t+N]}$

Survival rate for N years: $S_N(R) = \Pi_{i=1}^{N}(1 - d_i(R))$

Marginal default rate from start to year T: $k_N(R) = S_{N-1}(R)d_N(R)$

Cumulative default rate: $C_N(R) = k_1(R) + k_2(R) + \cdots + k_N(R) = 1 - S_N(R)$

Average default rate, d: $C_N = 1 - \Pi_{i=1}^{N}(1 - d_i) = 1 - (1 - d)^N$

20.7 ANSWERS TO CHAPTER EXAMPLES

Example 20.1: Definition of a Credit Event

b. Calling back a bond occurs when the borrower wants to refinance its debt at a lower cost, which is not a credit event.

Example 20.2: FRM Exam 2003—Question 100

c. Baa3 is the lowest investment-grade rating for Moody's.

Example 20.3: FRM Exam 2005—Question 86

c. The lowest investment-grade ratings are BBB and Baa.

Example 20.4: FRM Exam 2002—Question 110

d. The BB rating by S&P is similar to a Ba rating by Moody's. A BB bond will have lower default rate than a bond rated lower. Hence, the answer is the next lower rating category by Moody's.

Example 20.5: FRM Exam 2004—Question 1

c. The probability of survival for one year is $S_1 = (1 - d) = (1 - d^Q)^4$. This gives a probability of surviving the first quarter of $(1 - d^Q) = (1 - 0.10)^{1/4} = 0.974$.

Example 20.6: FRM Exam 2002—Question 77

d. The marginal default rate increases with maturity. So, this could be, for example, 0.50% over the last three years of the six-year period. This gives a cumulative default probability greater than 0.60%.

Example 20.7: FRM Exam 2006—Question 21

a. The survival rate is $S_3 = (1 - d_1)(1 - d_2)(1 - d_3) = (1 - 0.08)(1 - 0.12)(1 - 0.15) = 68.8\%$.

Example 20.8: FRM Exam 2008—Question 3-1

c. The default rate to the end of year 2 is the survival rate for year 1 times the year 2 default rate, $(1 - d_1)d_2 = (1 - 0.003)0.0045 = 0.449\%$. Hence the year 2 cumulative default rate is $0.300 + 0.449 = 0.749\%$. The default rate to the end of year 3 is $(1 - d_1)(1 - d_2)d_3 = (1 - 0.003)(1 - 0.0045)0.0055 = 0.546\%$. Hence the year 3 cumulative default rate is $0.749 + 0.546 = 1.295\%$.

Example 20.9: Default Probability Definitions

a. The marginal default rate is the probability of defaulting over the next year, conditional on having survived to the beginning of the year.

Example 20.10: FRM Exam 2005—Question 105

b. Transition matrices have no information about prices, so answer b. is the correct choice. A rating transition table has probabilities of changing from one rating to another over one year, which can be extrapolated over several years (statement a.). This also includes default (statement c.). The probabilities can be used to group ratings (statement d.).

Example 20.11: FRM Exam 2007—Question 51

c. This is given by the ratio of entries to BBB and D, which is $2 + 3$ over the total of 52, which is 0.096.

Example 20.12: FRM Exam 2009—Question 4-18

a. After one year, the bond becomes a one-year zero-coupon bond. The respective values are, for AA, A, and BBB, $P_{AA} = 100/(1 + 0.0180) = 98.23$, 97.56, and 96.34. Note that prices are lower for lower ratings. The expected value is given by $P = \sum \pi_i P_i = 5\%98.23 + 80\%97.56 + 15\%96.34 = 97.41$.

Example 20.13: FRM Exam 2008—Question 3-20

d. B can go into default the first year, with probability of 0.10. Or it could go to A, then D, with probability of $0.00 \times 0.00 = 0$. Or, it could go to B, then D, with probability of $0.75 \times 0.10 = 0.075$. Or it could go to C, then D, with probability of $0.15 \times 0.40 = 0.060$. The total is 0.235.

Example 20.14: FRM Exam 2005—Question 74

b. From Table 20.7, the typical recovery rate for senior unsecured debt is nearest to 40%.

Example 20.15: FRM Exam 2002—Question 123

c. The recovery rate is higher when the assets of the firm in default consist of tangible assets that can be resold easily. More volatile assets mean that there is a greater probability of a fall in market value upon liquidation. So, the tangible assets of a manufacturing company (c.) is the best answer.

Example 20.16: FRM Exam 2005—Question 79

b. Statements a., c., and d. are all correct. Debt repudiation is a cancellation, not a postponement, so b. is incorrect.

Example 20.17: FRM Exam 2009—Question 4-19

c. The higher default rate on foreign-currency-denominated debt is consistent with the observation that credit ratings are lower. So, this is not a statistical anomaly (statement a.). The main reason is that governments could force the central bank to print more money, creating inflation that reduces the real value of local currencies. This option is not possible with foreign-currency-denominated debt.

CHAPTER 21

Measuring Default Risk from Market Prices

The previous chapter discussed how to quantify credit risk into credit ratings from actuarial methods. Credit risk can also be assessed from market prices of securities whose values are affected by default. These include corporate bonds, equities, and credit derivatives. In principle, these should provide more up-to-date and accurate measures of credit risk because financial markets have access to a very large amount of information and are forward-looking. Agents also have very strong financial incentives to impound this information in trading prices. This chapter shows how to infer default risk from market prices.

Section 21.1 will show how to use information about the market prices of credit-sensitive bonds to infer default risk. In this chapter, we call defaultable debt interchangeably "credit-sensitive," "corporate," and "risky" debt. Here *risky* refers to credit risk and not market risk. We show how to break down the yield on a corporate bond into a default probability, a recovery rate, and a risk-free yield.

Section 21.2 turns to equity prices. The advantage of using equity prices is that they are much more widely available and of much better quality than corporate bond prices. We show how equity can be viewed as a call option on the value of the firm and how a default probability can be inferred from the value of this option. This approach also explains why credit positions are akin to short positions in options and are characterized by distributions that are skewed to the left. Chapter 23 discusses credit derivatives, which can also be used to infer default risk.

21.1 CORPORATE BOND PRICES

To assess the credit risk of a transaction with a counterparty, consider **credit-sensitive** bonds issued by the same counterparty. We assume that default is a state that affects all obligations equally.

FRM Exam Part 2 topic.

21.1.1 Spreads and Default Risk

Assume for simplicity that the bond makes only one payment of \$100 in one period. We can compute a market-determined yield y^* from the price P^* as

$$P^* = \frac{\$100}{(1 + y^*)} \tag{21.1}$$

This can be compared with the risk-free yield over the same period y.

The payoffs on the bond can be described by a simplified default process, which is illustrated in Figure 21.1. At maturity, the bond can be in default or not. Its value is \$100 if there is no default and $f \times \$100$ if default occurs, where f is the fractional recovery. We define π as the default rate over the period. How can we value this bond?

Using **risk-neutral pricing**, the current price must be the mathematical expectation of the values in the two states, discounting the payoffs at the risk-free rate. Hence,

$$P^* = \frac{\$100}{(1 + y^*)} = \left[\frac{\$100}{(1 + y)}\right] \times (1 - \pi) + \left[\frac{f \times \$100}{(1 + y)}\right] \times \pi \tag{21.2}$$

Note that the discounting uses the risk-free rate y because there is no risk premium with risk-neutral valuation. After rearranging terms,

$$(1 + y) = (1 + y^*)[1 - \pi(1 - f)] \tag{21.3}$$

which implies a default probability of

$$\pi = \frac{1}{(1 - f)}\left[1 - \frac{(1 + y)}{(1 + y^*)}\right] \tag{21.4}$$

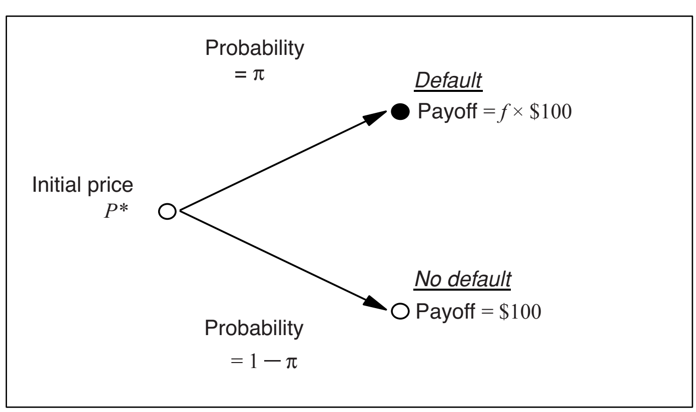

FIGURE 21.1 Simplified Bond Default Process

Assuming that yields and default probabilities are small, and dropping second-order terms, this simplifies to

$$y^* \approx y + \pi(1 - f) \tag{21.5}$$

This equation shows that the credit spread $y^* - y$ measures credit risk. More specifically, it includes the probability of default (PD), π, times the loss given default (LGD), $(1 - f)$. This makes sense because there is no potential credit loss either if the default probability is zero or if the loss given default is zero.

Let us now consider multiple periods, which number T. We compound interest rates and default rates over each period. In other words, π^a is now the *average* annual default rate. Assuming one payment only, the present value is

$$P^* = \frac{\$100}{(1+y^*)^T} = \left[\frac{\$100}{(1+y)^T}\right] \times (1 - \pi^a)^T + \left[\frac{f \times \$100}{(1+y)^T}\right] \times [1 - (1 - \pi^a)^T] \tag{21.6}$$

which can be written as

$$(1+y)^T = (1+y^*)^T\{(1-\pi^a)^T + f[1-(1-\pi^a)^T]\} \tag{21.7}$$

Unfortunately, this does not simplify easily. Alternatively, we can use the cumulative default probability,

$$\frac{1}{(1+y^*)^T} = \left[\frac{1}{(1+y)^T}\right] \times (1-\pi) + \left[\frac{f \times 1}{(1+y)^T}\right] \times [1-(1-\pi)] \tag{21.8}$$

or

$$\frac{1}{(1+y^*)^T} = \frac{1}{(1+y)^T} \times [1 - \pi(1 - f)] \tag{21.9}$$

for which a very rough approximation is

$$y^* \approx y + (\pi/T)(1 - f) \tag{21.10}$$

When we have risky bonds of various maturities, they can be used to compute default probabilities for different horizons. If we have two periods, for example, we could use Equation (21.3) to find the probability of defaulting over the first period, π_1, and Equation (21.7) to find the annualized, or average, probability of defaulting over the first two periods, π_2. As we saw in the previous chapter, the marginal probability of defaulting in the second period, d_2, is given by solving

$$(1 - \pi_2)^2 = (1 - \pi_1)(1 - d_2) \tag{21.11}$$

This enables us to recover a term structure of forward default probabilities from a sequence of zero-coupon bonds. In practice, if we have access to only coupon-paying bonds, the computation becomes more complicated because we need to consider the payments in each period with and without default.

21.1.2 Risk Premium

It is worth emphasizing that the preceding approach assumed risk neutrality. As in the methodology for pricing options, we assumed that the value of any asset grows at the risk-free rate and can be discounted at the same risk-free rate. Thus the probability measure π is a risk-neutral measure, which is not necessarily equal to the objective, physical probability of default.

Defining this objective probability as π' and the discount rate as y', the current price can also be expressed in terms of the true expected value discounted at the risky rate y':

$$P^* = \frac{\$100}{(1 + y^*)} = \left[\frac{\$100}{(1 + y')}\right] \times (1 - \pi') + \left[\frac{f \times \$100}{(1 + y')}\right] \times \pi' \tag{21.12}$$

Equation (21.4) allows us to recover a risk-neutral default probability only. More generally, if investors require some compensation for bearing credit risk, the credit spread will include a risk premium, rp:

$$y^* \approx y + \pi'(1 - f) + \text{rp} \tag{21.13}$$

To be meaningful, this risk premium must be tied to some measure of bond riskiness as well as investor risk aversion. In addition, this premium may incorporate a **liquidity premium** and tax effects.[1]

KEY CONCEPT

The yield spread between a corporate bond and an otherwise identical bond with no credit risk reflects the expected actuarial loss, or annual default rate times the loss given default, plus a risk premium.

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[1] For a decomposition of the yield spread into risk premium effects, see E. Elton, M. Gruber, D. Agrawal, and C. Mann, "Explaining the Rate Spread on Corporate Bonds," *Journal of Finance*, 2001. The authors find a high risk premium, which is related to common risk factors from the stock market. Part of the risk premium is also due to tax effects. Because Treasury coupon payments are not taxable at the state level, investors are willing to accept a lower yield on Treasury bonds, which increases the corporate yield spread.

Example: Deriving Default Probabilities

We wish to compare a 10-year U.S. Treasury strip and a 10-year zero issued by International Business Machines (IBM), which is rated A by S&P and Moody's. The respective yields are 6% and 7%, using semiannual compounding. Assuming that the recovery rate is 45% of the face value, what does the credit spread imply for the probability of default?

Equation (21.9) shows that $\pi(1 - f) = 1 - (1 + y/200)^{20}/(1 + y^*/200)^{20} = 0.0923$. Hence, $\pi = 9.23\%/(1 - 45\%) = 16.8\%$. Therefore, the cumulative (risk-neutral) probability of defaulting during the next 10 years is 16.8%. This number is rather high compared with the historical record for this risk class. Table 20.3 shows that Moody's reports a historical 10-year default rate for A credits around 3% only.

If these historical default rates are used as the future probability of default, the implication is that a large part of the credit spread reflects a risk premium. For instance, assume that 80 basis points out of the 100-basis-point credit spread reflects a risk premium. We change the 7% yield to 6.2% and find a probability of default of 3.5%. This is more in line with the actual default experience of such issuers.

EXAMPLE 21.1: FRM EXAM 2007—QUESTION 77

The risk-free rate is 5% per year and a corporate bond yields 6% per year. Assuming a recovery rate of 75% on the corporate bond, what is the approximate market-implied one-year probability of default of the corporate bond?

a. 1.33%
b. 4.00%
c. 8.00%
d. 1.60%

EXAMPLE 21.2: FRM EXAM 2007—QUESTION 48

The spread on a one-year BBB-rated bond relative to the risk-free Treasury of similar maturity is 2%. It is estimated that the contribution to this spread by all noncredit factors (e.g., liquidity risk, taxes) is 0.8%. Assuming the loss given default rate for the underlying credit is 60%, what is, approximately, the implied default probability for this bond?

a. 3.33%
b. 5.00%
c. 3.00%
d. 2.00%

EXAMPLE 21.3: FRM EXAM 2008—QUESTION 3-12

A risk analyst seeks to find out the credit-linked yield spread on a BB-rated one-year coupon bond issued by a multinational petroleum company. If the prevailing annual risk-free rate is 3%, the default rate for BB-rated bonds is 7%, and the loss given default is 60%, then the yield to maturity of the bond is

a. 2.57%
b. 5.90%
c. 7.45%
d. 7.52%

21.1.3 Cross-Section of Yield Spreads

We now turn to actual market data. Figure 21.2 illustrates a set of par yield curves for various credits as of December 1998. For reference, the spreads are listed in Table 21.1. The curves are sorted by credit rating, from AAA to B, using S&P's ratings.

These curves bear a striking resemblance to the cumulative default rate curves reported in the previous chapter. They increase with maturity and with lower credit quality.

The lowest curve is the Treasuries curve, which represents risk-free bonds. Spreads for AAA credits are low, starting at 46bp at short maturities and increasing to 60bp at longer maturities. Spreads for B credits are much wider; they also increase faster, from 275 to 450. Finally, note how close together the AAA

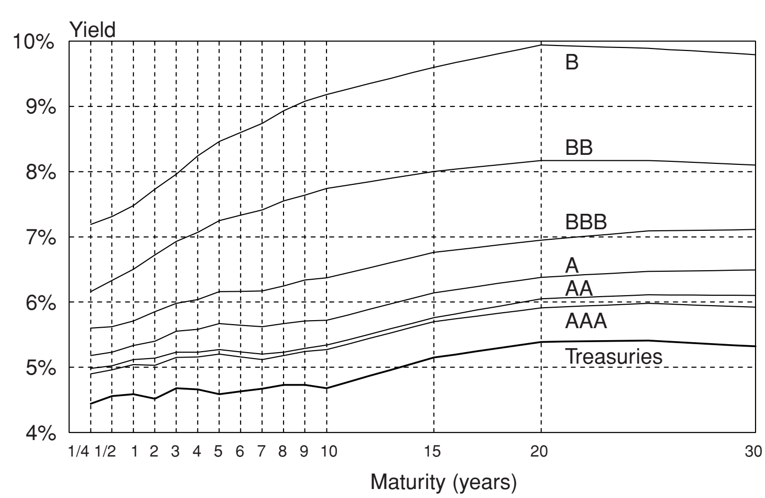

FIGURE 21.2 Yield Curves for Different Credits

TABLE 21.1 Credit Spreads

Maturity (Years)	Credit Rating					
	AAA	AA	A	BBB	BB	B
3M	46	54	74	116	172	275
6M	40	46	67	106	177	275
1	45	53	74	112	191	289
2	51	62	88	133	220	321
3	47	55	87	130	225	328
4	50	57	92	138	241	358
5	61	68	108	157	266	387
6	53	61	102	154	270	397
7	45	53	95	150	274	407
8	45	50	94	152	282	420
9	51	56	98	161	291	435
10	59	66	104	169	306	450
15	55	61	99	161	285	445
20	52	66	99	156	278	455
30	60	78	117	179	278	447

and AA spreads are, in spite of the fact that default probabilities approximately double from AAA to AA. The transition from Treasuries to AAA credits most likely reflects other factors, such as liquidity and tax effects, rather than actuarial credit risk.

The previous sections showed that we could use information in corporate bond yields to make inferences about credit risk. Indeed, bond prices represent the best assessment of traders, or real bets on credit risk. Thus we would expect bond prices to be the best predictors of credit risk and to outperform credit ratings. To the extent that agencies use public information to form their credit ratings, this information should be subsumed into market prices. Bond prices are also revised more frequently than credit ratings. As a result, movements in corporate bond prices tend to *lead* changes in credit ratings.

EXAMPLE 21.4: FRM EXAM 2002—QUESTION 81

Which of the following is true?

- **a.** Changes in bond spreads tend to lead changes in credit ratings.
- **b.** Changes in bond spreads tend to lag changes in credit ratings.
- **c.** Changes in bond spreads tend to occur at the exact same time as changes in credit ratings.
- **d.** There is absolutely no perceived general relationship in the timing of changes in bond spreads and changes in credit ratings.

EXAMPLE 21.5: TERM STRUCTURE OF CREDIT SPREADS

Suppose XYZ Corp. has two bonds paying semiannually according to the following table:

Remaining Maturity	Coupon (sa 30/360)	Price	T-Bill Rate (Bank Discount)
Six months	8.0%	99	5.5%
One year	9.0%	100	6.0%

The recovery rate for each in the event of default is 50%. For simplicity, assume that each bond will default only at the end of a coupon period. The market-implied risk-neutral probability of default for XYZ Corp. is

- **a.** Greater in the first six-month period than in the second
- **b.** Equal between the two coupon periods
- **c.** Greater in the second six-month period than in the first
- **d.** Cannot be determined from the information provided

21.1.4 Time Variation in Credit Spreads

Credit spreads reflect potential losses caused by default risk, and perhaps a risk premium. Some of this default risk is specific to the issuer and requires a detailed analysis of its prospective financial condition. Part of this risk, however, can be attributed to common credit risk factors. These common factors are particularly important, as they cannot be diversified away in a large portfolio of credit-sensitive bonds.

First among these factors are general economic conditions. Economic growth is negatively correlated with credit spreads. When the economy slows down, more companies are likely to have cash-flow problems and to default on their bonds.

Figure 21.3 compares the speculative-grade default rate from Moody's and the Baa-Treasury credit spread. Shaded areas indicate periods of recession. The graph shows that both default rates and credit spreads tend to increase around recessions. Because spreads are forward-looking, however, they tend to lead default rates, which peak after recessions. Also, the effect of the 2007–2008 credit crisis is apparent from the unprecedented widening of credit spreads. These reflect a combination of higher risk aversion and anticipation of very high default rates.

Volatility is also a factor. In a more volatile environment, investors may require larger risk premiums, thus increasing credit spreads. When this happens, liquidity may also dry up. Investors may then require a greater credit spread in order to hold increasingly illiquid securities.

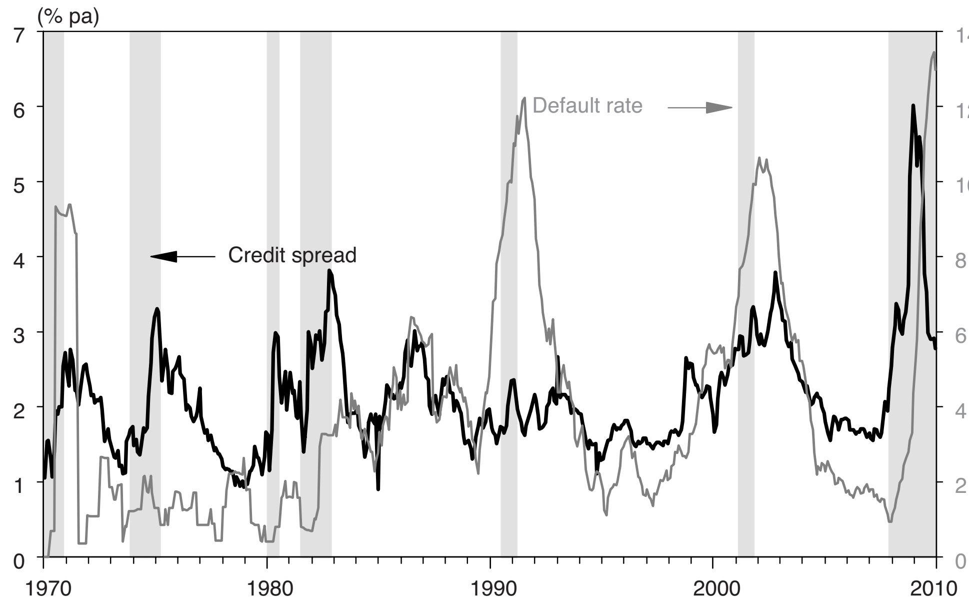

FIGURE 21.3 Default Rates and High Yield Spreads

Finally, volatility has another effect through an option channel. Corporate bond indices include many callable bonds, unlike Treasury indices. The buyer of a callable bond requires a higher yield in exchange for granting the call option. Higher volatility should increase the value of this option and therefore this yield, all else equal. Thus, credit spreads directly increase with volatility.

21.2 EQUITY PRICES

The credit spread approach, unfortunately, is useful only when there is good bond market data. The problem is that this is rarely the case, for a number of reasons.

- Many countries do not have a well-developed corporate bond market. As Table 7.2 has shown, the United States has by far the largest corporate bond market in the world. This means that other countries have fewer outstanding bonds and a much less active market.
- The counterparty may not have an outstanding publicly traded bond or if so, the bond may contain other features such as a call that make it more difficult to interpret the yield.
- The bond may not trade actively, and instead reported prices may simply be **matrix prices**, that is, interpolated from yields on other issuers.

An alternative is to turn to default risk models based on stock prices, because equity prices are available for a larger number of companies and because equities are more actively traded than corporate bonds. The Merton (1974) model views equity as akin to a call option on the assets of the firm, with an exercise price given by the face value of debt.

21.2.1 The Merton Model

To simplify to the extreme, consider a firm with total value V that has one bond due in one period with face value K. If the value of the firm exceeds the promised payment, the bond is repaid in full and stockholders receive the remainder. However, if V is less than K, the firm is in default and the bondholders receive V only. The value of equity goes to zero. We assume that there are no transaction costs and that the absolute-priority rule is followed. Hence, the value of the stock at expiration is

$$S_T = \text{Max}(V_T - K, 0) \tag{21.14}$$

Because the bond and equity add up to the firm value, the value of the bond must be

$$B_T = V_T - S_T = V_T - \text{Max}(V_T - K, 0) = \text{Min}(V_T, K) \tag{21.15}$$

The current stock price, therefore, embodies a forecast of default probability in the same way that an option embodies a forecast of being exercised. Figures 21.4 and 21.5 describe how the value of the firm can be split up into the bond and stock values.

Note that the bond value can also be described as

$$B_T = K - \text{Max}(K - V_T, 0) \tag{21.16}$$

In other words, a long position in a risky bond is equivalent to a long position in a risk-free bond plus a short put option, which is really a credit derivative.

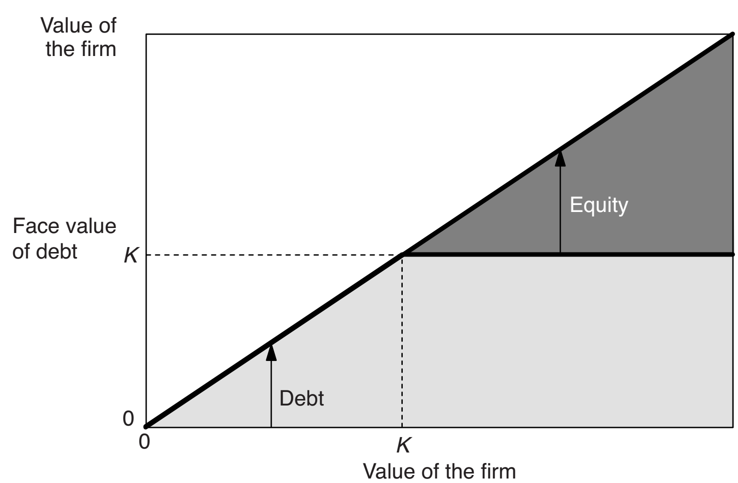

FIGURE 21.4 Equity as an Option on the Value of the Firm

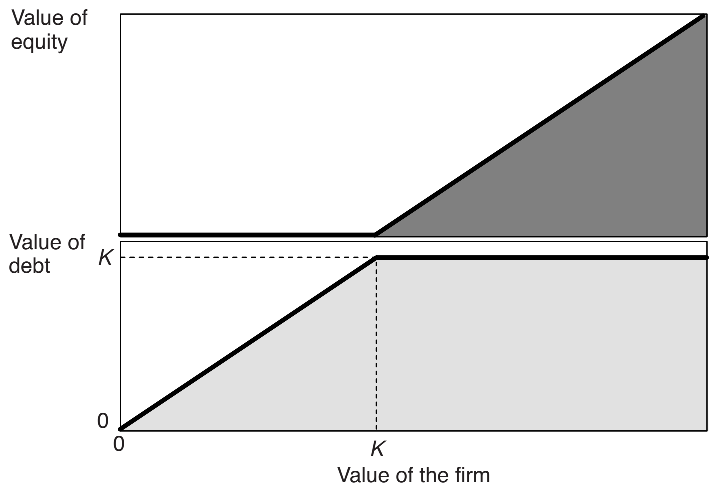

FIGURE 21.5 Components of the Value of the Firm

KEY CONCEPT

Equity can be viewed as a call option on the firm value with strike price equal to the face value of debt. Corporate debt can be viewed as risk-free debt minus a put option on the firm value.

This approach is particularly illuminating because it demonstrates that corporate debt has a payoff akin to a short position in an option, explaining the left skewness that is characteristic of credit losses. In contrast, equity is equivalent to a long position in an option due to its **limited-liability feature**. In other words, investors can lose no more than their equity investment.

21.2.2 Pricing Equity and Debt

To illustrate, we proceed along the lines of the usual Black-Scholes (BS) framework, assuming that the firm value follows the geometric Brownian motion process:

$$dV = \mu V\, dt + \sigma V\, dz \tag{21.17}$$

If we assume that markets are frictionless and that there are no bankruptcy costs, the value of the firm is simply the sum of the firm's equity (S) and debt (B): $V = B + S$.

To price a claim on the value of the firm, we need to solve a partial differential equation with appropriate boundary conditions. The corporate bond price is obtained as

$$B = F(V, t), \quad F(V, T) = \text{Min}[V, B_F] \tag{21.18}$$

where $B_F = K$ is the face value of the bond to be repaid at expiration, or the strike price.

Similarly, the equity value is

$$S = f(V, t), \qquad f(V, T) = \mathrm{Max}[V - B_F, 0] \tag{21.19}$$

Stock Valuation With no dividend, the value of the stock is given by the BS formula,

$$S = \mathrm{Call} = VN(d_1) - Ke^{-r\tau}N(d_2) \tag{21.20}$$

where $N(d)$ is the cumulative distribution function for the standard normal distribution, and

$$d_1 = \frac{\ln(V/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}$$

where $\tau = T - t$ is the time to expiration, r is the risk-free interest rate, and σ is the volatility of *asset* value. The option value depends on two factors, $x = Ke^{-r\tau}/V$ and $\sigma\sqrt{\tau}$. The first factor is the debt-value ratio and is inversely related to leverage, which can be written as $l = V/(V - Ke^{-r\tau})$. The value of the stock increases as the volatility σ increases and as x decreases, or when leverage increases.

Firm Volatility Note that this application is different from the BS model, where we plug in the value of V and of its volatility, $\sigma = \sigma_V$, and solve for the value of the call. Here we observe the market value of the firm S and the equity volatility σ_S and must infer the values of V and its volatility so that Equation (21.20) is satisfied. This can only be done iteratively. Defining $\Delta = N(d_1)$ as the hedge ratio, we have

$$dS = \frac{\partial S}{\partial V}dV = \Delta\, dV \tag{21.21}$$

Defining σ_S as the volatility of (dS/S), we have $(\sigma_S S) = \Delta(\sigma_V V)$ and

$$\sigma_V = (1/\Delta)\,\sigma_S(S/V) \tag{21.22}$$

Bond Valuation Next, the value of the bond is given by $B = V - S$, or

$$B = Ke^{-r\tau}N(d_2) + V[1 - N(d_1)] \tag{21.23}$$

$$B/Ke^{-r\tau} = [N(d_2) + (V/Ke^{-r\tau})N(-d_1)] \tag{21.24}$$

Hence, the value of the bond is related to the firm by

$$dB = \frac{\partial B}{\partial V}dV = N(-d_1)dV \tag{21.25}$$

Equations (21.21) and (21.25) are sometimes used for **capital arbitrage** trades, which involve buying and selling different types of claims on the firm, using these hedge ratios to try to minimize risk—that is, assuming the model is correct.

The bond price can be expressed in terms of the annualized credit spread s

$$B = Ke^{-(r+s)\tau} \tag{21.26}$$

which gives

$$s = -(1/\tau)[N(d_2) + (1/x)N(-d_1)] \tag{21.27}$$

The value of the bond decreases when volatility increases and when leverage increases. The spread moves conversely. Equation (21.27) can create rich patterns in the term structure of credit spreads. For firms with low leverage and volatility, spreads are low for near maturities and increase with maturity. Firms with high leverage have high spreads, with a term structure that can have a negative slope.

Risk-Neutral Dynamics of Default In the Black-Scholes model, $N(d_2)$ is also the probability of exercising the call, or that the bond will not default. Conversely, $1 - N(d_2) = N(-d_2)$ is the risk-neutral probability of default. Defining $z = -d_2$, we use

$$z = \frac{\ln(K/V) - r\tau + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{21.28}$$

To get a physical default probability, the risk-free rate in this equation can be replaced by δ, which is the expected rate of growth in the firm's assets.

$$\text{PD} = N[z] = N[\frac{\ln(K/V) - \delta\tau + 0.5\sigma^2\tau}{\sigma\sqrt{\tau}}] \tag{21.29}$$

Pricing Credit Risk At maturity, the credit loss is the value of the risk-free bond minus the corporate bond, $\text{CL} = B_F - B_T$. At initiation, the expected credit loss (ECL) is

$$\begin{aligned} B_F e^{-r\tau} - B &= Ke^{-r\tau} - \{Ke^{-r\tau}N(d_2) + V[1 - N(d_1)]\} \\ &= Ke^{-r\tau}[1 - N(d_2)] - V[1 - N(d_1)] \\ &= Ke^{-r\tau}N(-d_2) - VN(-d_1) \\ &= N(-d_2)[Ke^{-r\tau} - VN(-d_1)/N(-d_2)] \end{aligned}$$

This decomposition is quite informative. Multiplying by the future value factor $e^{r\tau}$ shows that the ECL at maturity is

$$\text{ECL}_T = N(-d_2)[K - Ve^{r\tau}N(-d_1)/N(-d_2)] = p \times [\text{Exposure} \times \text{LGD}] \tag{21.30}$$

This involves two terms. The first is the probability of default, $N(-d_2)$. The second, between brackets, is the loss when there is default. This is obtained as the face value of the bond K minus the recovery value of the loan when in default, $Ve^{r\tau}N(-d_1)/N(-d_2)$, which is also the expected value of the firm in the state of default. Note that the recovery rate is endogenous here, as it depends on the value of the firm, time, and debt ratio.

Credit Option Valuation This approach can also be used to value the put option component of the credit-sensitive bond. This option pays $K - B_T$ in case of default. A portfolio with the bond plus the put is equivalent to a risk-free bond $Ke^{-r\tau} = B + \text{Put}$. Hence, using Equation (21.23), the credit put should be worth

$$\text{Put} = Ke^{-r\tau} - \{Ke^{-r\tau}N(d_2) + V[1 - N(d_1)]\} = -V[N(-d_1)] + Ke^{-r\tau}[N(-d_2)] \tag{21.31}$$

This will be used later in Chapter 23 on credit derivatives.

EXAMPLE 21.6: FRM EXAM 2001—QUESTION 14

To what sort of option on the counterparty's assets can the current exposure of a credit-risky position better be compared?

a. A short call
b. A short put
c. A short knock-in call
d. A binary option

21.2.3 Applying the Merton Model

These valuation formulas can be used to recover, given the current value of equity and of nominal liabilities, the value of the firm and its probability of default. Figure 21.6 illustrates the evolution of the value of the firm. The firm defaults if this value falls below the liabilities at the horizon. We measure this risk-neutral probability by $N(-d_2)$.

In practice, default is much more complex than depicted here. We would have to collect information about all the liabilities of the company, as well as their maturities. Default can also occur with coupon payments. So instead of default on a target date, we could measure default probability as a function of the distance relative to a moving floor that represents liabilities. This is essentially the approach undertaken by **KMV Corporation**, now part of Moody's, which sells **expected default frequencies** (EDFs) for global firms. The approach is explained in Chapter 24.

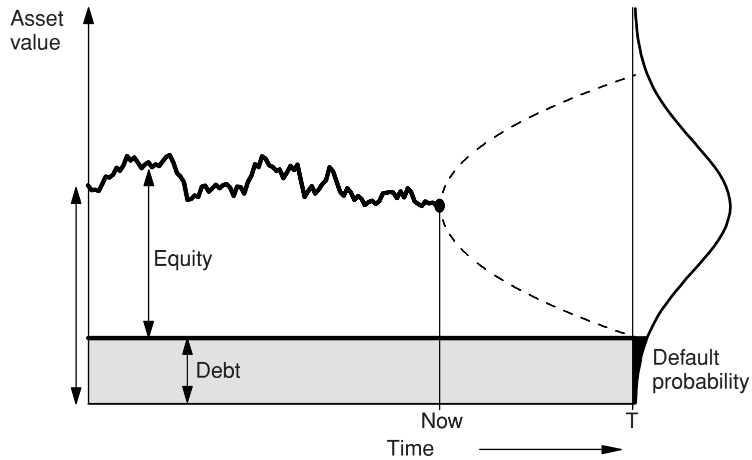

FIGURE 21.6 Default in the Merton Model

The Merton approach has many advantages. First, it relies on the prices of equities, which are more actively traded than bonds. Second, correlations between equity prices can generate correlations between defaults, which would be otherwise difficult to measure. Perhaps the most important advantage of this model is that it generates movements in EDFs that seem to *lead* changes in credit ratings.

Figure 21.7 displays movements in EDFs and credit rating for WorldCom, using the same vertical scale. WorldCom went bankrupt on July 21, 2002. With $104 billion in assets, this was the largest U.S. bankruptcy until the Lehman Brothers $639 billion bankruptcy on September 15, 2008. The agency rating was BBB until April 2002. It gave no warning of the impending default. In contrast,

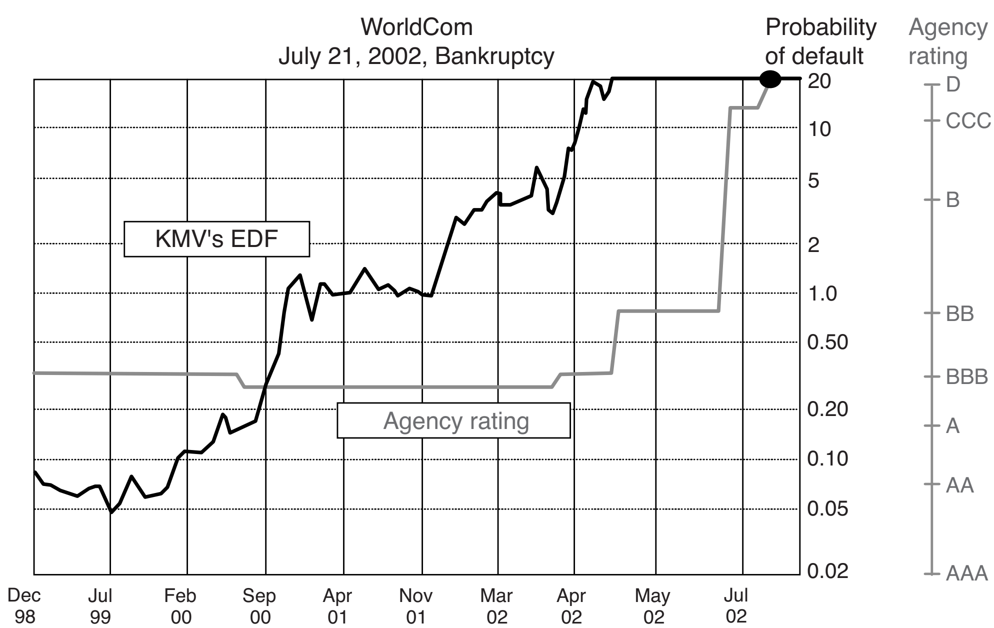

FIGURE 21.7 KMV's EDF and Credit Rating

starting one year before the default, the EDF began to move up. In April, it reached 20%, presaging bankruptcy.

The Merton model has disadvantages as well, however. The first limitation of the model is that it cannot be used to price sovereign credit risk, as countries obviously do not have a stock price. This is a problem for credit derivatives, where a large share of the market consists of sovereign risks.

A more fundamental drawback is that it relies on a static model of the firm's capital and risk structure. The debt level is assumed to be constant over the horizon. Similarly, the model cannot handle new injections of equity, which protect existing debt holders. Also, the model needs to be expanded to a more realistic setting where debt matures at various points in time, which is not an obvious extension.

Another problem is that management could undertake new projects that increase not only the value of equity but also its volatility, thereby increasing the credit spread. This runs counter to the fundamental intuition of the Merton model, which is that, all else equal, a higher stock price reflects a lower probability of default and hence should be associated with a smaller credit spread.

Finally, this class of models fails to explain the magnitude of credit spreads we observe on credit-sensitive bonds. Recent work has attempted to add other sources of risk, such as interest rate risk, but still falls short of explaining these spreads. Thus these models are most useful in tracking *changes* in EDFs over time. Indeed, KMV calibrates the risk-neutral default probabilities to actual default data.

21.2.4 Example

It is instructive to work through a simplified example. Consider a firm with assets worth $V = \$100$ and with volatility $\sigma_V = 20\%$. In practice, one would have to start from the observed stock price and volatility and iterate to find σ_V.

The horizon is $\tau = 1$ year. The risk-free rate is $r = 10\%$ using continuous compounding. The debt face value is $K = \$99.46$, which implies a risk-free current value of $Ke^{-r\tau} = \$90$ and a leverage factor of $x = 0.9$.

Working through the Merton analysis, one finds that the current stock price should be $S = \$13.59$. Hence the current bond price is

$$B = V - S = \$100 - \$13.59 = \$86.41$$

which implies a yield of $\ln(K/B)/\tau = \ln(99.46/86.41) = 14.07\%$, or a yield spread of 4.07%. The current value of the credit put is then

$$P = Ke^{-r\tau} - B = \$90 - \$86.41 = \$3.59$$

The analysis also generates values for $N(d_2) = 0.6653$ and $N(d_1) = 0.7347$. Thus the *risk-neutral* probability of default is $\text{EDF} = N(-d_2) = 1 - N(d_2) = 33.47\%$. Note that this could differ from the *actual* or *objective* probability of default since the stock could very well grow at a rate that is greater than the risk-free rate of 10%.

Finally, let us decompose the expected loss at expiration from Equation (21.30), which gives

$$N(-d_2)[K - Ve^{r\tau} N(-d_1)/N(-d_2)] = 0.3347 \times [\$99.46 - \$110.56 \times 0.2653/0.3347]$$
$$= 0.3347 \times [\$11.85] = \$3.96$$

This combines the probability of default with the expected loss upon default, which is \$11.85. This future expected credit loss of \$3.96 must also be the future value of the credit put, or $\$3.59e^{r\tau} = \$3.96$.

Note that the model needs very high leverage, here $x = 90\%$, to generate a reasonable credit spread of 4.07%. This implies a debt-to-equity ratio of 0.9/0.1 = 900%, which is unrealistically high for this type of spread.

With lower leverage, say $x = 0.7$, the credit spread shrinks rapidly, to 0.36%. At $x = 50\%$ or below, the predicted spread goes to zero. As this leverage would be considered normal, the model fails to reproduce the size of observed credit spreads. Perhaps it is most useful for tracking time variation in estimated default frequencies.

EXAMPLE 21.7: FRM EXAM 2002—QUESTION 97

Among the following variables, which one is the main driver of the probability of default in the KMV model?

a. Stock prices
b. Bond prices
c. Bond yield
d. Loan prices

EXAMPLE 21.8: FRM EXAM 2008—QUESTION 3-9

The capital structure of HighGear Corporation consists of two parts: one five-year zero-coupon bond with a face value of \$100 million and the rest is equity. The current market value of the firm's assets is \$130 million and the expected rate of change of the firm's value is 25%. The firm's assets have an annual volatility of 30%. Assume that firm value is lognormally distributed, with constant volatility. The firm's risk management division estimates the distance to default using the Merton model, or $[\ln(K/V) - \delta\tau + 0.5\sigma^2\tau]/\sigma\sqrt{\tau}$. Given the distance to default, the estimated default probability is

a. 2.74%
b. 12.78%
c. 12.79%
d. 30.56%

EXAMPLE 21.9: FRM EXAM 2005—QUESTION 108

The KMV model produces a measure called expected default frequency. Which of the following statements about this variable is correct?

- **a.** It decreases when the leverage of the firm falls.
- **b.** It increases when the stock price of the firm has been rising.
- **c.** It is the risk-neutral probability of default from Merton's model.
- **d.** It tells investors how the default risk of a bond is correlated with the default risk of other bonds in the portfolio.

EXAMPLE 21.10: FRM EXAM 2007—QUESTION 82

Using the Merton model, the value of the debt increases if all other parameters are fixed and

- **I.** The value of the firm decreases.
- **II.** The riskless interest rate decreases.
- **III.** Time to maturity increases.
- **IV.** The volatility of the firm value decreases.

- **a.** I and II only
- **b.** I and IV only
- **c.** II and III only
- **d.** II and IV only

EXAMPLE 21.11: FRM EXAM 2005—QUESTION 134

You have a large position of bonds of firm XYZ. You hedge these bonds with equity using Merton's debt valuation model. The value of the debt falls unexpectedly, but the value of equity does not fall, so you make a loss. Consider the following statements:

- **I.** Interest rates increased.
- **II.** Volatility fell.
- **III.** Volatility increased.
- **IV.** A liquidity crisis increased the liquidity component of the credit spreads.

Which statements are possible explanations for why your hedge did not work out?

- **a.** I and II only
- **b.** I and III only
- **c.** I, III, and IV only
- **d.** III and IV only

EXAMPLE 21.12: FRM EXAM 2008—QUESTION 3-24

The Merton model is used to predict default. It builds on several very strong assumptions and its applicability is hampered by practical difficulties. Which of the following statements does *not* correctly identify limiting assumptions or practical difficulties of using the model?

a. The model relies on a simplistic capital structure with only one debt issue.
b. The asset value volatility cannot be estimated because firm value does not trade.
c. The model assumes that debt does not pay a coupon while most publicly traded debt is coupon debt.
d. The model assumes a constant riskless interest rate.

21.3 IMPORTANT FORMULAS

Implied default probability, 1 period: $(1 + y) = (1 + y^*)[1 - \pi(1 - f)]$

Approximation of implied default probability: $y^* \approx y + \pi(1 - f)$

Implied default probability, T period: $(1 + y)^T = (1 + y^*)^T\{(1 - \pi)^T + f[1 - (1 - \pi)^T]\}$

Approximation of physical default probability: $y^* \approx y + \pi'(1 - f) + \text{rp}$

Merton model for stock price: $S_T = \text{Max}(V_T - K, 0)$

Merton model for bond price: $B_T = V_T - S_T = \text{Min}(V_T, K)$

Stock valuation: $S = \text{Call} = VN(d_1) - Ke^{-r\tau}N(d_2)$

Firm value and stock volatility: $\sigma_V = (1/\Delta)\sigma_S(S/V)$

Bond valuation: $B =$ Risk-free bond $-$ Put, $B/Ke^{-r\tau} = [N(d_2) + (V/Ke^{-r\tau})N(-d_1)]$

Risk-neutral PD: $1 - N(d_2) = N(-d_2)$

Physical PD: $\text{PD} = N[z] = N\{[\ln(K/V) - \delta\tau + 0.5\sigma^2\tau]/[\sigma\sqrt{\tau}]\}$

Credit default swap, or put option: $\text{Put} = Ke^{-r\tau} - \{Ke^{-r\tau}N(d_2) + V[1 - N(d_1)]\} = -V[N(-d_1)] + Ke^{-r\tau}[N(-d_2)]$

21.4 ANSWERS TO CHAPTER EXAMPLES

Example 21.1: FRM Exam 2007—Question 77

b. The spread is $7 - 6 = 1\%$. Dividing by the loss given default of $(1 - f) = 0.25$, we get $\pi = (y^* - y)/(1 - f) = 4\%$.

Example 21.2: FRM Exam 2007—Question 48

d. The part of the spread due to expected credit losses is $2.00 - 0.80 = 1.20\%$. Dividing by the LGD of $(1 - f) = 0.65$, we get 2%.

Example 21.3: FRM Exam 2008—Question 3-12

d. From Equation (21.3), $(1 + y) = (1 + y^*)[1 - \pi \times \text{LGD}]$. This gives $(1 + y^*) = (1 + y)/[1 - \pi \times \text{LGD}] = 1.03/[1 - 0.07 \times 60\%] = 1.0752$, or $y^* = 7.52\%$.

Example 21.4: FRM Exam 2002—Question 81

a. Changes in market prices, including bond spreads, tend to lead changes in credit ratings. This is because market prices reflect all publicly available information about a company.

Example 21.5: Term Structure of Credit Spreads

a. First, we compute the current yield on the six-month bond, which is selling at a discount. We solve for y^* such that $99 = 104/(1 + y^*/200)$ and find $y^* = 10.10\%$. Thus the yield spread for the first bond is $10.1 - 5.5 = 4.6\%$. The second bond is at par, so the yield is $y^* = 9\%$. The spread for the second bond is $9 - 6 = 3\%$. The default rate for the first period must be greater. The recovery rate is the same for the two periods, so it does not matter for this problem.

Example 21.6: FRM Exam 2001—Question 14

b. The lender is short a put option, since exposure exists only if the value of assets falls below the amount lent.

Example 21.7: FRM Exam 2002—Question 97

a. Stock prices are the main driver of KMV's estimated default frequency (EDF), because they drive the value of equity. These models also use the volatility of asset values and the value of liabilities.

Example 21.8: FRM Exam 2008—Question 3-9

a. We compute $z = [\ln(K/V) - \delta\tau + 0.5\sigma^2\tau]/\sigma\sqrt{\tau} = [\ln(100/130) - (25\%)5 + 0.5(30\%^2)5]/[30\%\sqrt{5}] = -1.919$. The PD is then $N(z) = N(-1.919) = 2.749\%$.

Example 21.9: FRM Exam 2005—Question 108

a. The EDF, similarly to the risk-neutral PD, decreases when the stock price goes up, when the leverage goes down, or when the volatility goes down. It is a transformation of the PD from a Merton-type model. The KMV framework can be extended to finding correlations, but the EDF is not sufficient.

Example 21.10: FRM Exam 2007—Question 82

d. The value of credit-sensitive debt is $B = Ke^{-(r+s)\tau}$. This increases (1) if the risk-free interest rate decreases, or (2) if the credit spread decreases, or (3) if the maturity decreases. The credit spread decreases if the value of the firm goes up, or if the leverage goes down, or if the volatility goes down. Hence, the value of debt increases if the riskless rate decreases or if the volatility decreases.

Example 21.11: FRM Exam 2005—Question 134

b. We need to identify shocks that decrease the value of debt but not that of equity. An increase in the risk-free rate will decrease the value of the debt but not the equity (because this decreases leverage). An increase in volatility will have the opposite effect on debt and equity. Finally, a liquidity crisis cannot explain the divergent behavior, because, as we have seen during 2008, it would affect both corporate bonds and equity adversely. Answers I and III are correct.

Example 21.12: FRM Exam 2008—Question 3-24

b. This statement is incorrect because the asset volatility can be recovered iteratively from the equity volatility and model prices. Other statements are correct weaknesses of this model.

CHAPTER 22

Credit Exposure

Credit exposure is the amount at risk during the life of the financial instrument. Upon default, it is called **exposure at default** (EAD). When banking simply consisted of making loans, exposure was essentially the face value of the loan. In this case, the exposure is the notional amount and is fixed.

Since the development of the swap markets, the measurement of credit exposure has become more complicated. This is because swaps, like most derivatives, have an up-front value that is much smaller than the notional amount. Indeed, the initial value of a swap is typically zero, which means that at the outset, there is no credit risk because there is nothing to lose.

As the swap contract matures, however, it can turn into a positive or negative value. The asymmetry of bankruptcy treatment is such that a credit loss can occur only if the instrument has positive value, or is a claim against the defaulted counterparty. Thus, the credit exposure is the value of the asset if it is positive, like an option.

This chapter turns to the quantitative measurement of credit exposure. Section 22.1 describes the general features of credit exposure for various types of financial instruments, including loans or bonds, guarantees, credit commitments, repos, and derivatives. Section 22.2 shows how to compute the distribution of credit exposure and gives detailed examples of exposures of interest rate and currency swaps. Section 22.3 discusses exposure modifiers, or techniques that have been developed to reduce credit exposure. It shows how credit risk can be controlled by marking to market, margins, position limits, recouponing, and netting agreements. For completeness, Section 22.4 includes credit risk modifiers such as credit triggers and time puts, which also control default risk.

22.1 CREDIT EXPOSURE BY INSTRUMENT

Credit exposure is the positive part of the value of the asset at various points during its life. In particular, the **current exposure** is the value of the asset at the current time V_t if positive:

$$CE_t = \text{Max}(V_t, 0) \qquad (22.1)$$

The **potential exposure** represents the exposure on some future date, or sets of dates. Based on this definition, we can characterize the exposure of a variety of financial instruments. The measurement of current and potential exposure also motivates regulatory capital charges for credit risk, which are explained in Chapter 28.

22.1.1 Loans or Bonds

Loans or **bonds** are balance sheet assets whose current and potential exposure basically is the notional, or amount lent or invested. To be more precise, this should be the market value of the asset given current interest rates, but, as we will show, this is not very far from the notional. The exposure is also the notional for **receivables** and **trade credits**, as the potential loss is the amount due.

22.1.2 Guarantees

Guarantees are off-balance-sheet contracts whereby the bank has underwritten, or agrees to assume, the obligations of a third party. The exposure is the notional amount, because this will be fully drawn when default occurs. By nature, guarantees are **irrevocable**, that is, unconditional and binding, whatever happens.

An example of a **guarantee** is a contract whereby bank A makes a loan to client C only if it is guaranteed by bank B. Should C default, B is exposed to the full amount of the loan. Another example is an **acceptance**, whereby a bank agrees to pay the face value of a bill at maturity. Alternatively, **standby facilities**, or **financial letters of credit**, provide a guarantee to a third party of the making of a payment should the obligor default.

22.1.3 Commitments

Commitments are off-balance-sheet contracts whereby the bank commits to a future transaction that may result in creating a credit exposure at a *future* date. For instance, a bank may provide a **note issuance facility** whereby it promises a minimum price for notes regularly issued by a borrower. If the notes cannot be placed at the market at the minimum price, the bank commits to buy them at a fixed price. Such commitments have less risk than guarantees because it is not certain that the bank will have to provide backup support.

It is also useful to distinguish between **irrevocable commitments**, which are unconditional and binding on the bank, and **revocable commitments**, where the bank has the option to revoke the contract should the counterparty's credit quality deteriorate. This option substantially decreases the credit exposure.

22.1.4 Swaps or Forwards

Swaps or **forwards** contracts are off-balance-sheet items that can be viewed as irrevocable commitments to purchase or sell some asset on prearranged terms. The current and potential exposure will vary from zero to a large amount depending on movements in the driving risk factors. Similar arrangements are **sale-repurchase agreements** (repos), whereby an institution sells an asset to another in exchange for a promise to buy it back later.

22.1.5 Long Options

Options are off-balance-sheet items that may create credit exposure. The current and potential exposure also depends on movements in the driving risk factors. Here there is no possibility of negative values because options always have positive value, or zero value at worst: $V_t \geq 0$.

22.1.6 Short Options

Unlike long options, the current and potential exposure for short options is zero because the bank writing the option can incur only a negative cash flow, assuming the option premium has been fully paid.

Exposure also depends on the features of any embedded option. With an American option, for instance, the holder of an in-the-money swap may want to exercise early if the credit rating of its counterparty starts to deteriorate. This decreases the exposure relative to an equivalent European option.

EXAMPLE 22.1: FRM EXAM 2006—QUESTION 95

A credit loss on market-driven instruments such as swaps and forwards arises if:

a. Market rates move in your favor.
b. Market rates move against you.
c. Market rates move against you and the counterparty defaults.
d. Market rates move in your favor and the counterparty defaults.

EXAMPLE 22.2: FRM EXAM 2009—QUESTION 6-2

Capital Bank is concerned about its counterparty credit exposure to City Bank. Which of the following trades by Capital Bank would increase its credit exposure to City Bank?

I. Buying a put option from City Bank
II. Selling a call option to City Bank
III. Selling a forward contract to City Bank
IV. Buying a secondary loan granted to Sunny Inc. from City Bank

a. I and II only
b. I and III only
c. II and III only
d. I, III, and IV only

EXAMPLE 22.3: FRM EXAM 2006—QUESTION 117

Which of the following will have the greatest potential credit exposure?

- **a.** Long 3,000 ounces of gold for delivery in one year
- **b.** Long 3,000 ounces of gold for delivery in two years
- **c.** Short 3,000 ounces of gold for delivery in two years
- **d.** Selling an at-the-money call option on 10,000 ounces of gold for delivery in two years

EXAMPLE 22.4: FRM EXAM 2004—QUESTION 8

Your company has reached its credit limit to Ford but Ford is insisting that your firm provide some increased protection in the event a major project Ford is undertaking results in some unforeseen liability. Ignoring settlement risk and assuming option premiums are paid immediately at the time of the transaction, which of these strategies will *not* give rise to increased credit exposure to Ford?

- **a.** Selling a costless collar to Ford
- **b.** Buying an option from Ford
- **c.** Selling an option to Ford
- **d.** None of the above

EXAMPLE 22.5: FRM EXAM 2001—QUESTION 84

If a counterparty defaults before maturity, which of the following situations will cause a credit loss?

- **a.** You are short euros in a one-year euro/USD forward foreign exchange (FX) contract, and the euro has appreciated.
- **b.** You are short euros in a one-year euro/USD forward FX contract, and the euro has depreciated.
- **c.** You sold a one-year OTC euro call option, and the euro has appreciated.
- **d.** You sold a one-year OTC euro call option, and the euro has depreciated.

22.2 DISTRIBUTION OF CREDIT EXPOSURE

Credit exposure consists of the **current exposure**, which is readily observable, and the **potential exposure**, or future exposure, which is random. Define x as the potential value of the asset on the target date. We describe this variable by its probability density function $f(x)$. This is where market risk mingles with credit risk.

22.2.1 Expected and Worst Exposure

The **expected credit exposure** (ECE) is the expected value of the asset replacement value x, if positive, on a target date:

$$\text{ECE} = \int_{-\infty}^{+\infty} \text{Max}(x, 0)\, f(x) dx \tag{22.2}$$

The **worst credit exposure** (WCE) is the largest (worst) credit exposure at some level of confidence. It is implicitly defined as the value that is not exceeded at the given confidence level p:

$$1 - p = \int_{\text{WCE}}^{\infty} f(x) dx \tag{22.3}$$

To model the potential credit exposure, we need to (1) model the distribution of risk factors, and (2) evaluate the instrument given these risk factors. This process is identical to a market value at risk (VAR) computation except that the aggregation takes place at the counterparty level if contracts can be netted.

To simplify to the extreme, suppose that the payoff x, or net claim against a particular counterparty, is normally distributed with mean zero and volatility σ. The expected credit exposure is then

$$\text{ECE} = \frac{1}{2} E(x \mid x > 0) = \frac{1}{2}\sigma\sqrt{\frac{2}{\pi}} = \frac{\sigma}{\sqrt{2\pi}} \tag{22.4}$$

Note that we divided by 2 because there is a 50% probability that the value will be positive. The worst credit exposure at the 95% level is given by

$$\text{WCE} = 1.645\sigma \tag{22.5}$$

Figure 22.1 illustrates the measurement of ECE and WCE for a normal distribution. Note that negative values of x are not considered.

22.2.2 Time Profile

The distribution can be summarized by the expected and worst credit exposures at each point in time. To summarize even further, we can express the average credit exposure by taking a simple arithmetic average over the life of the instrument.

The **average expected credit exposure** (AECE) is the average of the expected credit exposure over time, from now to maturity T:

$$\text{AECE} = (1/T) \int_{t=0}^{T} \text{ECE}_t \, dt \tag{22.6}$$

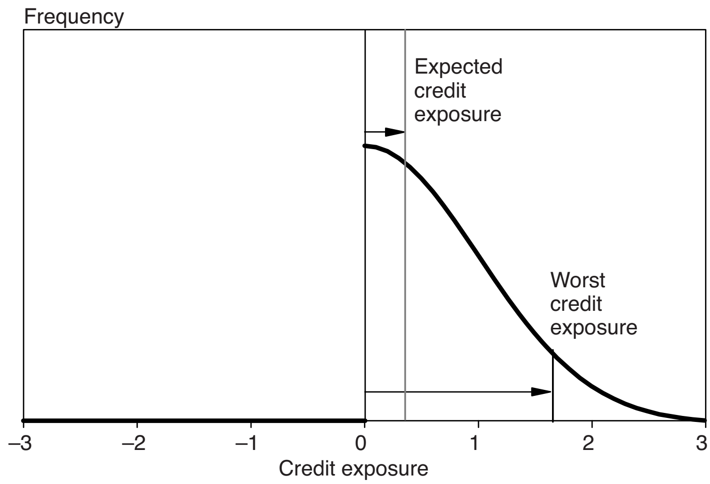

FIGURE 22.1 Expected and Worst Credit Exposures—Normal Distribution

The **average worst credit exposure** (AWCE) is defined similarly:

$$\text{AWCE} = (1/T)\int_{t=0}^{T} \text{WCE}_t \, dt \tag{22.7}$$

22.2.3 Exposure Profile for Interest Rate Swaps

We now consider the computation of the exposure profile for an interest rate swap. In general, we need to define (1) the market risk factors, (2) the function and parameters for the joint stochastic processes, and (3) the pricing model for the swap. This is a good illustrative example for an instrument that is widely employed.

We start with a one-factor stochastic process for the interest rate, defining the movement in the rate r_t at time t as

$$dr_t = \kappa(\theta - r_t)dt + \sigma r_t^{\gamma} dz_t \tag{22.8}$$

as given in Chapter 4. The first term imposes **mean reversion**. When the current value of r_t is higher than the long-run value, the term in parentheses is negative, which creates a downward trend. More generally, the mean term could reflect the path implied in forward interest rates.

The second term defines the innovation, which can be given a normal distribution. An important issue is whether the volatility of the innovation should be constant or proportional to some power γ of the current value of the interest rate r_t. If the horizon is short, this issue is not so important because the current rate will be close to the initial rate.

When $\gamma = 0$, changes in yields are normally distributed, which is the Vasicek model (1977). As seen in a previous chapter, a typical volatility for *absolute* changes in yields is 1% per annum. A potential problem with this is that the volatility is the same whether the yield starts at 20% or 1%. As a result, the yield could turn negative, depending on the initial starting point and the strength of the mean reversion.

Another class of models is the lognormal model, which takes $\gamma = 1$. The model can then be rewritten in terms of $dr_t/r_t = d\ln(r_t)$. This specification ensures that the volatility shrinks as r gets close to zero, avoiding negative values. A typical volatility of *relative* changes in yields is 15% per annum, which is also the 1% value for changes in the level of rates divided by an initial rate of 6.7%.

For illustration purposes, we choose the normal process $\gamma = 0$ with mean reversion $\kappa = 0.02$ and volatility $\sigma = 0.25\%$ per month, which are realistic parameters based on recent U.S. data. The initial and long-run values of r are both 6%. Typical simulation values are shown in Figure 22.2. Note how rates can deviate from their initial value but are pulled back to the long-term value of 6%.

This model is convenient because it leads to closed-form solutions. The distribution of future values for r is summarized in Figure 22.3 by its mean and two-tailed 90% confidence bands (called maximum and minimum values). The graph shows that the mean is 6%, which is also the long-run value. The confidence bands initially widen due to the increasing horizon, and then converge to a fixed value due to the mean-reversion effect.

The next step is to value the swap. At each point in time, the current market value of the receive-fixed swap is the difference between the value of a fixed-coupon bond and a floating-rate note:

$$V_t = B(F, t, T, c, r_t) - B(F, \mathrm{FRN}) \tag{22.9}$$

Here, F is the notional amount or face value, c is the annualized *fixed* coupon rate, and T is the maturity date. The risk to the swap comes from the fact that the fixed leg has a coupon c that could differ from prevailing market rates. The principals are not exchanged.

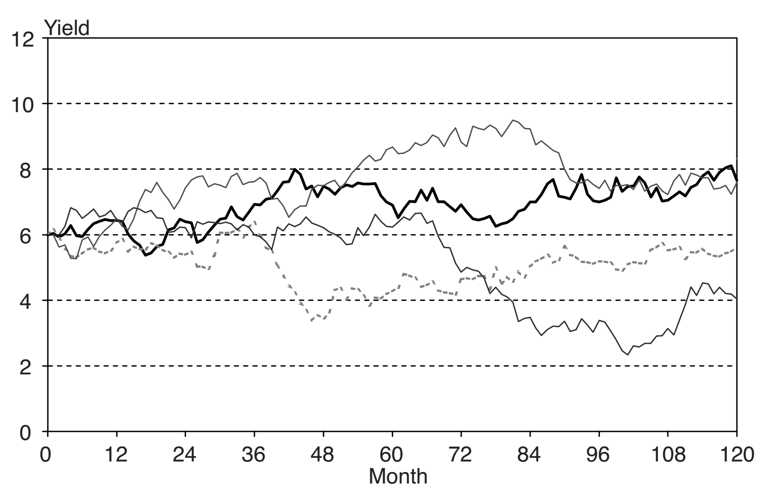

FIGURE 22.2 Simulation Paths for the Interest Rate

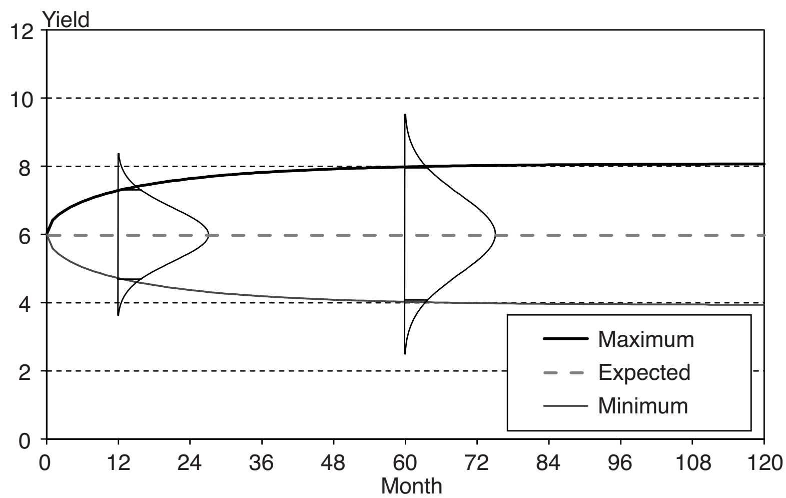

FIGURE 22.3 Distribution Profile for the Interest Rate

Figure 22.4 illustrates the changes in cash flows that could arise from a drop in rates from 6% to 4% after five years. Assume the swap has notional of $N = \$100$ million and pays semiannually. Every six months, the receive-fixed party is owed $\$100 \times (6 - 4)\% \times 0.5 = \$1$ million until the maturity of the swap. With 10 payments remaining, this adds up to a positive credit exposure of approximately \$10 million. More precisely, discounting over the life of the remaining payments gives \$8.1 million as of the valuation date.

In what follows, we assume that the swap receives fixed payments that are paid at a continuous rate instead of semiannually, which simplifies the example. Otherwise, there would be discontinuities in cash-flow patterns, and we would have to consider the risk of the floating leg as well. We also use continuous

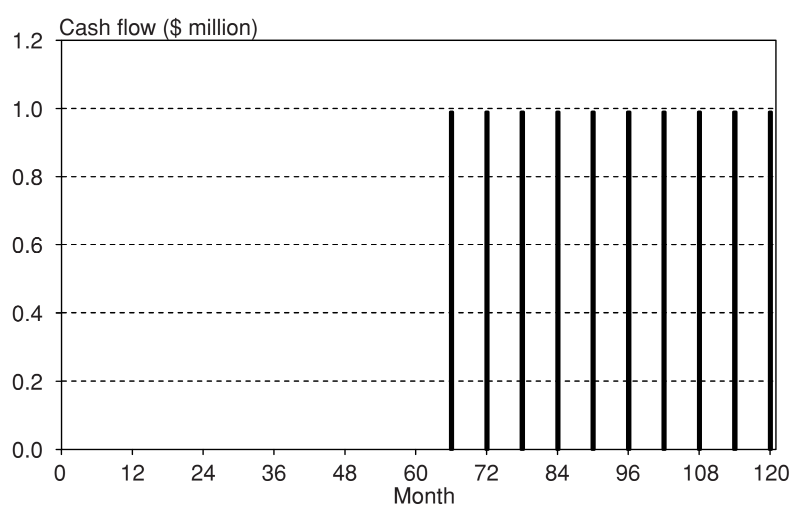

FIGURE 22.4 Net Cash Flows When Rates Fall to 4% after Five Years

compounding. Defining N as the number of remaining years, the coupon bond value is

$$B(\$100, N, c, r) = \$100\frac{c}{r}[1 - e^{-rN}] + \$100e^{-rN} \tag{22.10}$$

as seen in the appendix to Chapter 6. The first term is the present value of the fixed-coupon cash flows discounted at the current rate r. The second term is the repayment of principal. For the special case where the coupon rate is equal to the current market rate ($c = r$), the market value is indeed \$100 for this par bond. The floating-rate note can be priced in the same way, but with a coupon rate that is always equal to the current rate. Hence, its value is always at par.

To understand the exposure profile of the coupon bond, we need to consider two opposing effects as time goes by:

1. The **diffusion effect**, which increases the uncertainty in the interest rate
2. The **amortization effect**, which decreases the bond's duration toward zero

The latter effect is described in Figure 22.5, which shows the bond's duration converging to zero. This explains why the bond's market value converges to the face value upon maturity, whatever happens to the current interest rate.

Because the bond is a strictly monotonic function of the current yield, we can compute the 90% confidence bands by valuing the bond using the extreme interest rates range at each point in time. We use Equation (22.10) at each point in time in Figure 22.3. This exposure profile is shown in Figure 22.6.

Initially, the market value of the bond is \$100. After two or three years, the range of values is the greatest, from \$87 to \$115. Thereafter, the range converges to the face value of \$100. But overall, the fluctuations as a *proportion* of the face value are relatively small. Considering other approximations in the measurement of credit risk, such as the imprecision in default probability and recovery rate, assuming a constant exposure for the bond is not a bad approximation.

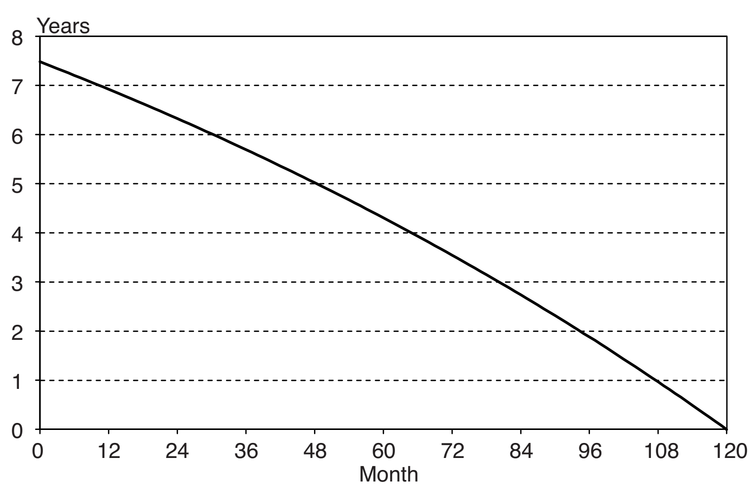

FIGURE 22.5 Duration Profile for a 10-Year Bond

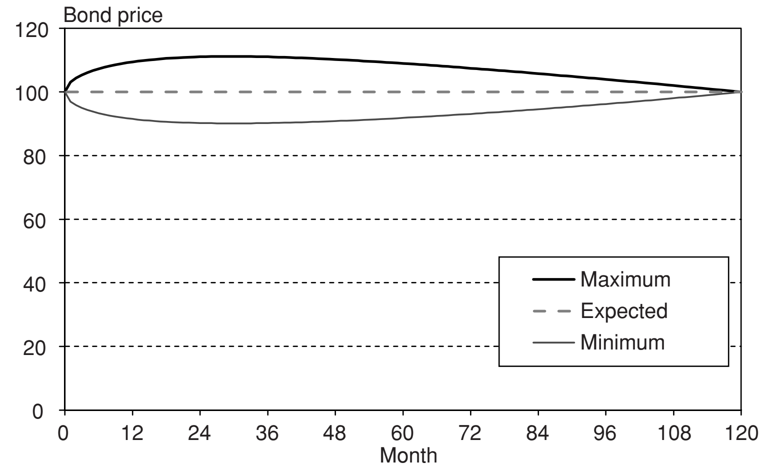

FIGURE 22.6 Exposure Profile for a 10-Year Bond

This is not the case, however, for the interest rate swap. Its value can be found by subtracting \$100 (the value of the floating-rate note) from the value of the coupon bond. Initially, its value is zero. Thereafter, it can take on positive or negative values. Credit exposure is the positive value only. Figure 22.7 presents the profile of the expected exposure and of the maximum (worst) exposure at the one-sided 95% level. Numbers are scaled relative to a notional of \$100. The chart also shows the average maximum exposure over the whole life of the swap.

Intuitively, the value of the swap is derived from the difference between the fixed and floating cash flows. Consider a swap with two remaining payments and a notional amount of \$100. Its value is

$$V_t = \$100\left[\frac{c}{(1+r)} + \frac{c}{(1+r)^2} + \frac{1}{(1+r)^2}\right] - \$100\left[\frac{r}{(1+r)} + \frac{r}{(1+r)^2} + \frac{1}{(1+r)^2}\right]$$
$$= \$100\left[\frac{(c-r)}{(1+r)} + \frac{(c-r)}{(1+r)^2}\right] \qquad (22.11)$$

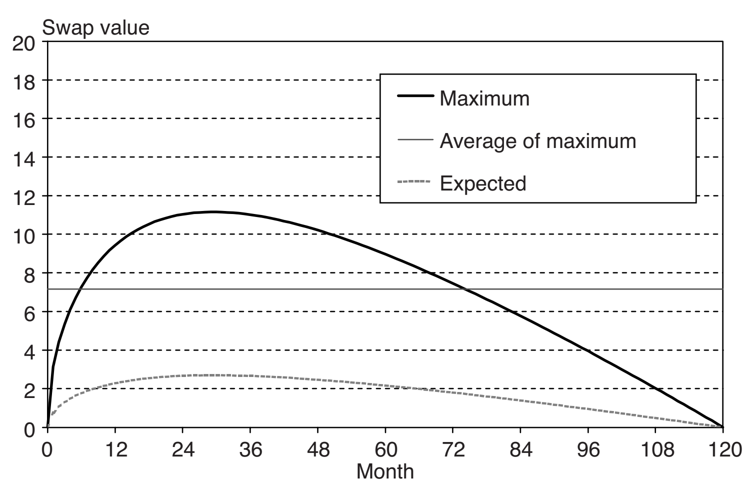

FIGURE 22.7 Exposure Profile for a 10-Year Interest Rate Swap

Note how the principal payments cancel out and we are left with the discounted *net* difference between the fixed coupon and the prevailing rate $(c - r)$.

This information can be used to assess the expected exposure and worst exposure on a target date. The peak exposure occurs around the second year into the swap, or at about one-fourth of the swap's life. At that point, the expected exposure is about 3% to 4% of the notional, which is much less than that of the bond. The worst exposure peaks at about 10% to 15% of notional. In practice, these values depend on the particular stochastic process used, but the exposure profiles will be qualitatively similar.

To assess the potential variation in swap values, we can make some approximations based on duration. Consider first the very short-term exposure, for which mean reversion and changes in durations are not important. The volatility of changes in rates then simply increases with the square root of time. Given a 0.25%-per-month volatility and 7.5-year initial duration, we can approximate the volatility of the swap value over the next year as

$$\sigma(V) = \$100 \times 7.5 \times 0.25\%\sqrt{12} = \$6.5 \text{ million}$$

Multiplying by 1.645, we get \$10.7 million, which is close to the actual \$9.4 million 95% worst exposure in a year reported in Figure 22.7.

The trade-off between declining duration and increasing risk can be formalized with a simple example. Assume that the bond's (modified) duration is proportional to the remaining life, $D = k(T - t)$ at any date t. The volatility from 0 to time t can be written as $\sigma(r_t - r_0) = \sigma\sqrt{t}$. Hence, the swap volatility is

$$\sigma(V) = [k(T - t)] \times \sigma\sqrt{t} \tag{22.12}$$

To see where it reaches a maximum, we differentiate with respect to t:

$$\frac{d\sigma(V)}{dt} = [k(-1)]\sigma\sqrt{t} + [k(T - t)]\sigma\frac{1}{2\sqrt{t}}$$

Setting this to zero, we have

$$\sqrt{t} = (T - t)\frac{1}{2\sqrt{t}}, \quad 2t = (T - t)$$

which gives

$$t_{\text{MAX}} = (1/3)T \tag{22.13}$$

The maximum exposure occurs at one-third of the life of the swap. This occurs later than the one-fourth point reported previously because we assumed no mean reversion.

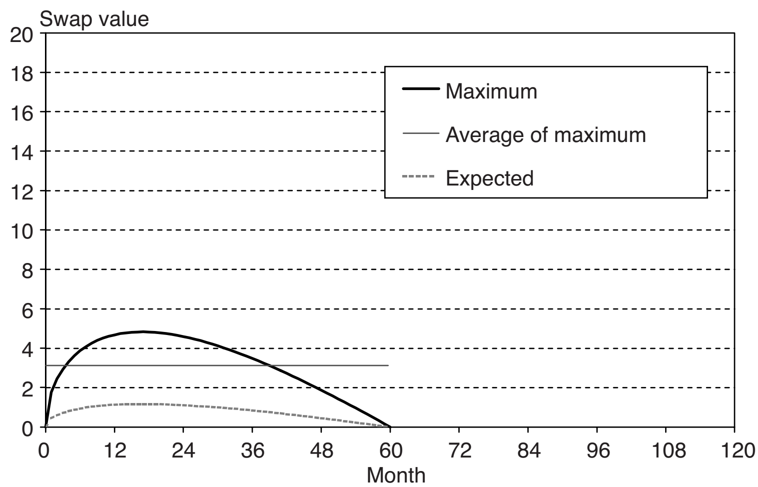

FIGURE 22.8 Exposure Profile for a Five-Year Interest Rate Swap

Further, we can check how this evolves with the maturity of the contract. At that point, the worst credit exposure will be

$$1.645\ \sigma(V_{\mathrm{MAX}}) = 1.645\left[k(2/3)T\sigma\sqrt{T/3}\right] = \left[1.645k(2/3)\sigma\sqrt{1/3}\right]T^{3/2} \tag{22.14}$$

which shows that the WCE increases as $T^{3/2}$, which is faster than the maturity.

Figure 22.8 shows the exposure profile of a five-year swap. Here again, the peak exposure occurs at one-third of the swap's life. As expected, the magnitude is lower, with the peak expected exposure only about 1% of the notional.

Finally, Figure 22.9 displays the exposure profile when the initial interest rate is at 5% with a coupon of 6%. The swap starts in-the-money, with a current value of \$7.9 million. With a long-run rate of 6%, the total exposure profile starts from a positive value, reaches a maximum after about two years, then converges to zero.

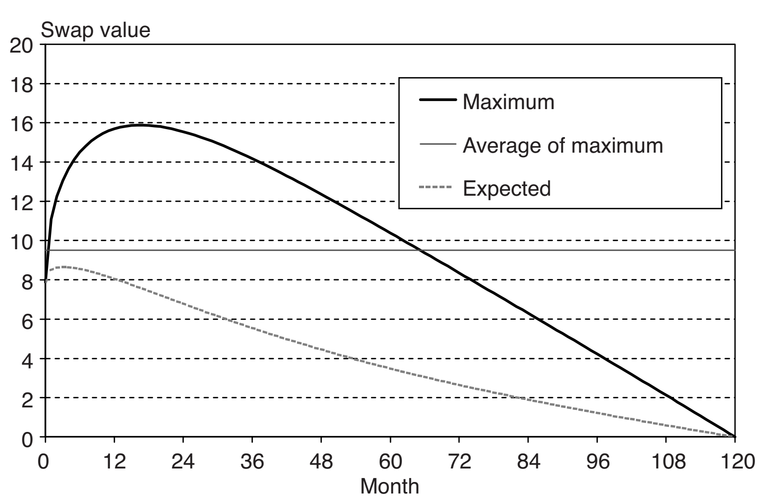

FIGURE 22.9 Exposure Profile for a 10-Year In-the-Money Swap

EXAMPLE 22.6: FRM EXAM 2004—QUESTION 43

In determining the amount of credit risk in a derivatives transaction, which of the following factors are used?

I. Notional principal amount of the underlying transaction
II. Current exposure
III. Potential exposure
IV. Peak exposure—the replacement cost in a worst-case scenario

a. I and II
b. I, III, and IV
c. III and IV
d. II, III, and IV

EXAMPLE 22.7: FRM EXAM 2005—QUESTION 61

Assume that a bank enters into a USD 100 million, four-year annual-pay interest rate swap, where the bank receives 6% fixed against 12-month LIBOR. Which of the following numbers best approximates the current exposure at the end of year 1 if the swap rate declines 125 basis points over the year?

a. USD 3,420,069
b. USD 4,458,300
c. USD 3,341,265
d. USD 4,331,382

EXAMPLE 22.8: PEAK EXPOSURE

Assume that the DV01 of an interest rate swap is proportional to its time to maturity (which at the initiation is equal to T). Assume that interest rate curve moves are parallel, stochastic with constant volatility, normally distributed, and independent. At what time will the maximum potential exposure be reached?

a. $T/4$
b. $T/3$
c. $T/2$
d. $3T/4$

EXAMPLE 22.9: FRM EXAM 2002—QUESTION 83

Assume that you have entered into a fixed-for-floating interest rate swap that starts today and ends in six years. Assume that the duration of your position is proportional to the time to maturity. Also assume that all changes in the yield curve are parallel shifts, and that the volatility of interest rates is proportional to the square root of time. When would the maximum potential exposure be reached?

a. In two months
b. In two years
c. In six years
d. In four years and five months

22.2.4 Exposure Profile for Currency Swaps

Exposure profiles are substantially different for other swaps. Consider, for instance, a currency swap where the notionals are 100 million U.S. dollars against 50 million British pounds (GBP), set at an initial exchange rate of $S(\$/\text{GBP}) = 2$.

The market value of a currency swap that receives foreign currency is

$$V_t = S_t(\$/\text{GBP})B^*(\text{GBP}50, t, T, c^*, r^*) - B(\$100, t, T, c, r) \tag{22.15}$$

Following the usual conventions, asterisks refer to foreign-currency values.

In general, this swap is exposed to domestic as well as foreign interest rate risk. When we just have two remaining coupons, the value of the swap evolves according to

$$\begin{aligned} V = S \times 50 &\left[\frac{c^*}{(1+r^*)} + \frac{c^*}{(1+r^*)^2} + \frac{1}{(1+r^*)^2}\right] \\ -\$100 &\left[\frac{c}{(1+r)} + \frac{c}{(1+r)^2} + \frac{1}{(1+r)^2}\right] \end{aligned} \tag{22.16}$$

Note that the principals do not cancel each other, unlike an interest rate swap. Instead, they are paid at maturity in different currencies, which is a major source of credit exposure.

In what follows, we will assume for simplicity that there is no interest rate risk, or that the value of the swap is dominated by currency risk. Further, we assume that the coupons are the same in the two currencies; otherwise there would be an asymmetrical accumulation of payments. As before, we have to choose a stochastic process for the spot rate. Say this is a lognormal process with constant variance and no trend:

$$dS_t = \sigma S_t \, dz_t \tag{22.17}$$

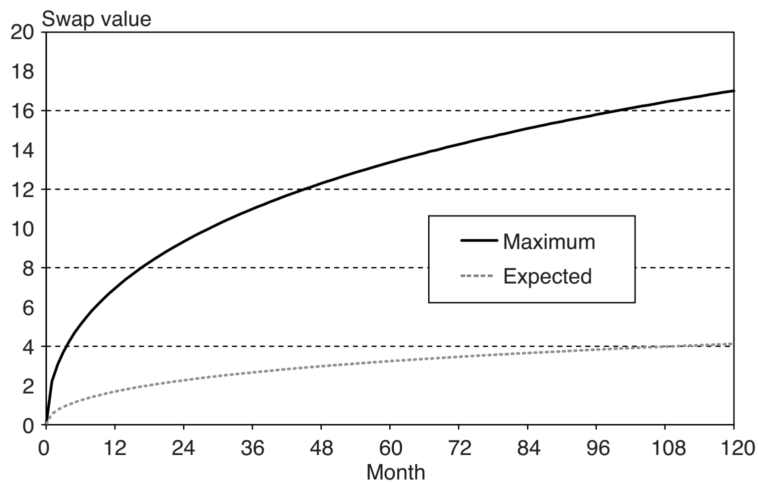

FIGURE 22.10 Exposure Profile for a 10-Year Currency Swap

We choose $\sigma = 12\%$ annually, which is realistic, as seen in the chapter on market risk factors. This process ensures that the rate never becomes negative.

Figure 22.10 presents the exposure profile of a 10-year currency swap. Here there is no amortization effect, and exposure increases continuously over time. The peak exposure occurs at the end of the life of the swap. At that point, the expected exposure is about 10% of the notional, which is much higher than for the interest rate swap. The worst exposure is commensurately high, at about 45% of notional.

Although these values depend on the particular stochastic process and parameters used, this example demonstrates that credit exposures for currency swaps are far greater than for interest rate swaps, even with identical maturities.

22.2.5 Exposure Profile for Different Coupons

So far, we have assumed a flat term structure and equal coupon payments in different currencies, which creates a symmetric situation for the exposure for the long and short parties. In reality, these conditions will not hold, and the exposure patterns will be asymmetric.

Consider, for instance, the interest rate swap in Equation (22.11). If the term structure slopes upward, the coupon rate is greater than the floating rate, $c > r$, in which case the net payment to the party receiving fixed is initially positive. The value of the two-period swap can be analyzed by projecting floating payments at the forward rate:

$$V_t = \frac{(c - s_1)}{(1 + s_1)} + \frac{(c - f_{12})}{(1 + s_2)^2}$$

where s_1 and s_2 are the one- and two-year spot rates, and f_{12} is the one- to two-year forward rate.

Example

Consider a \$100 million interest rate swap with two remaining payments. We have $s_1 = 5\%$, $s_2 = 6.03\%$; hence, using $(1 + s_2)^2 = (1 + s_1)(1 + f_{12})$, we have $f_{12} = 7.07\%$. The coupon yield of $c = 6\%$ is such that the swap has zero initial value. The following table shows that the present value of the first payment (to the party receiving fixed) is positive and equal to \$0.9524. The second payment then must be negative, and is equal to −\$0.9524. The two payments exactly offset each other because the swap has zero value.

Time	Expected Spot	Expected Payment	Discounted
1	5%	6.00 − 5.00 = +1.00	+0.9524
2	7.07%	6.00 − 7.07 = −1.07	−0.9524
Total			0.0000

This pattern of payments, however, creates more credit exposure to the fixed payer because it involves a payment in the first period offset by a receipt in the second. If the counterparty defaults shortly after the first payment is made, there could be a credit loss even if interest rates have not changed.

KEY CONCEPT

With a positively sloped term structure, the receiver of the floating rate (payer of the fixed rate) has a greater credit exposure than the counterparty.

A similar issue arises with currency swaps when the two coupon rates differ. Low nominal interest rates imply a higher forward exchange rate. The party that receives payments in a low-coupon currency is expected to receive greater payments later during the exchange of principal. If the counterparty defaults, there could be a credit loss even if rates have not changed.

KEY CONCEPT

The receiver of a low-coupon currency has a greater credit exposure than the counterparty.

EXAMPLE 22.10: FRM EXAM 2000—QUESTION 47

Which one of the following deals would have the greatest credit exposure for a $1,000,000 deal size (assume the counterparty in each deal is an AAA-rated bank and has no settlement risk)?

a. Pay fixed in an Australian dollar (AUD) interest rate swap for one year.
b. Sell USD against AUD in a one-year forward foreign exchange contract.
c. Sell a one-year AUD cap.
d. Purchase a one-year certificate of deposit (CD).

EXAMPLE 22.11: FRM EXAM 2001—QUESTION 8

Which of the following 10-year swaps has the highest potential credit exposure?

a. A cross-currency swap *after* two years
b. A cross-currency swap *after* nine years
c. An interest rate swap *after* two years
d. An interest rate swap *after* nine years

EXAMPLE 22.12: FRM EXAM 2004—QUESTION 14

BNP Paribas has just entered into a plain-vanilla interest-rate swap as a pay-fixed counterparty. Credit Agricole is the receive-fixed counterparty in the same swap. The forward spot curve is upward-sloping. If LIBOR starts trending down and the forward spot curve flattens, the credit risk from the swap will:

a. Increase only for BNP Paribas
b. Increase only for Credit Agricole
c. Decrease for both BNP Paribas and Credit Agricole
d. Increase for both BNP Paribas and Credit Agricole

22.3 EXPOSURE MODIFIERS

In a continuing attempt to decrease credit exposures, the industry has developed a number of methods to limit exposures. This section analyzes marking to market, margins and collateral, exposure limits, recouponing, and netting arrangements.

Other modifiers include **third-party guarantees** and purchasing **credit derivatives**. The former involve receiving, typically from a bank, a guarantee of payment should the counterparty fail. Credit derivatives are covered in the next chapter.

22.3.1 Marking to Market

The ultimate form of reducing credit exposure is marking to market (MTM). **Marking to market** involves settling the variation in the contract value on a regular basis (e.g., daily). For over-the-counter (OTC) contracts, counterparties can agree to longer periods (e.g., monthly or quarterly). If the MTM treatment is symmetrical across the two counterparties, it is called **two-way marking to market**. Otherwise, if one party settles losses only, it is called **one-way marking to market**.

Marking to market has long been used by organized exchanges to deal with credit risk. The reason is that exchanges are accessible to a wide variety of investors, including retail speculators, who are more likely to default than others. On OTC markets, in contrast, institutions interacting with each other typically have an ongoing relationship. As one observer put it,

> *Futures markets are designed to permit trading among strangers, as against other markets which permit only trading among friends.*

With daily marking to market, the *current* exposure is reduced to zero. There is still, however, *potential* exposure because the value of the contract could change before the next settlement. Potential exposure arises from (1) the time interval between MTM periods and (2) the time required for liquidating the contract when the counterparty defaults.

In the case of a retail client, the broker can generally liquidate the position fairly quickly, within a day. When positions are very large, as in the case of brokers dealing with Long-Term Capital Management (LTCM), however, the liquidation period could be much longer. Indeed, LTCM's bailout in 1998 was motivated by the potential disruption to financial markets had brokers attempted to liquidate their contracts with LTCM at the same time.

Marking to market introduces other types of risks, however:

- **Operational risk**, which is due to the need to keep track of contract values and to make or receive payments daily
- **Liquidity risk**, because the institution now needs to keep enough cash to absorb variations in contract values

22.3.2 Margins

Potential exposure is covered by margin requirements. **Margins** represent cash or securities that must be advanced in order to open a position. The purpose of these funds is to provide a buffer against potential exposure.

Exchanges, for instance, require a customer to post an **initial margin** when establishing a new position. This margin serves as a performance bond to offset possible future losses should the customer default. Contract gains and losses are then added to the posted margin in the **equity account**. Whenever the value of this equity account falls below a threshold, set at a **maintenance margin**, new funds must be provided.

Margins are set in relation to price volatility and to the type of position, speculative or hedging. Margins increase for more volatile contracts. Margins are typically lower for hedgers because a loss on the futures position can be offset by a gain on the physical, assuming no basis risk. Some exchanges set margins at a level that covers the 99th percentile of worst daily price changes, which is a daily VAR system for credit risk.

22.3.3 Collateral

Over-the-counter markets may allow posting securities as **collateral** instead of cash. This collateral protects against current and potential exposure. Typically, the amount of the collateral will exceed the funds owed by an amount known as the **haircut**. Collateral is typically managed within the International Swaps and Derivatives Association (ISDA) **credit support annex** (CSA).

The haircut reflects both default risk and market risk. Safe counterparties will in general have lower haircuts. This also depends, however, on the downside risk of the asset. For instance, cash can have a haircut of zero, which means that there is full protection against current exposure. Government securities can require a haircut of 1%, 3%, and 8% for short-term, medium-term, and longer-term maturities, respectively. With greater price volatility, there is an increasing chance of losses if the counterparty defaults and the collateral loses value, which explains the increasing haircuts.

As an example, assume that hedge fund A enters a swap with bank B. To mitigate A's credit risk, the two parties enter a collateral agreement that specifies the conditions under which B can ask for collateral. Now suppose the contract moves in-the-money for B, which requests $1 million in collateral from A. The funds are legally still the property of A but under the administration of B. If A defaults, B is entitled to sell the collateral and terminate the contract. Any positive excess value is returned to A. Conversely, if the collateral is not sufficient, B will have a claim against A.

22.3.4 Exposure Limits

Credit exposure can also be managed by setting **position limits** on the exposure to a counterparty. Ideally, these should be evaluated in a portfolio context, taking into account all the contracts between an institution and a counterparty.

To enforce limits, information on transactions must be centralized in middle-office systems. This generates an *exposure profile* for each counterparty, which can be used to manage credit line usage for several maturity buckets. Proposed new trades with the same counterparty should then be examined for their incremental effect.

These limits can also be set at the instrument level. In the case of a swap, for instance, an **exposure cap** requires a payment to be made whenever the value of the contract exceeds some amount. Figure 22.11 shows the effect of a $5 million

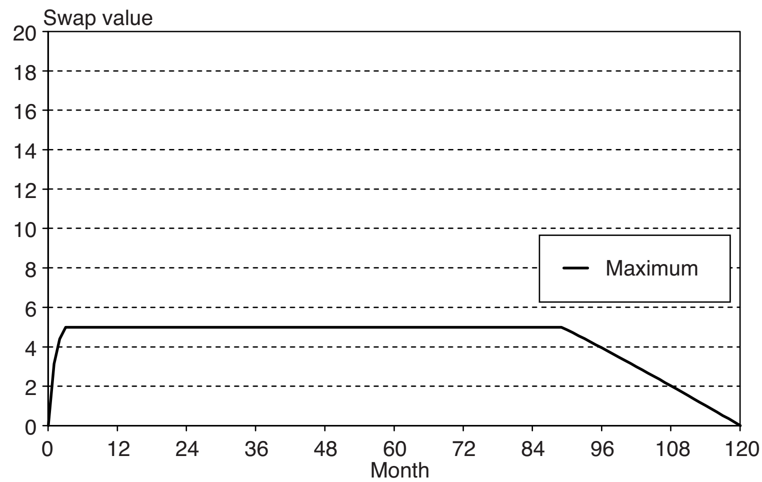

FIGURE 22.11 Effect of Exposure Cap

cap on our 10-year swap. If, after two years, say, the contract suddenly moves into a positive value of $11 million, the counterparty would be required to make a payment of $6 million to bring the swap's outstanding value back to $5 million. This limits the worst exposure to $5 million and also lowers the average exposure.

22.3.5 Recouponing

Another method for controlling exposure at the instrument level is recouponing. **Recouponing** refers to a clause in the contract requiring the contract to be marked to market at some fixed date. This involves (1) exchanging cash to bring the MTM value to zero and (2) resetting the coupon or the exchange rate to prevailing market values.

In the case of our 10-year swap, for instance, this reduces the exposure to zero after five years. Thereafter, the exposure profile is that of a swap with a remaining five-year maturity.

22.3.6 Netting Arrangements

Perhaps the most powerful mechanism for controlling exposures is **netting agreements**. These are now a common feature of standardized **master swap agreements** such as the one established in 1992 by the **International Swaps and Derivatives Association** (ISDA). ISDA's master agreement is explained in more detail in the appendix of this chapter.

The purpose of these agreements is to provide for the **netting** of payments across a set of contracts. In case of default, a counterparty cannot stop payments on contracts that have negative value while demanding payment on positive-value contracts. As a result, this system reduces the exposure to the net value for all the contracts covered by the netting agreement. This prevents *cherry-picking* by the administrator of the defaulted counterparty.

Netting can be classified into three types:

1. **Payment netting** involves the daily offsetting of several claims in the same currency. An example is an interest rate swap, where only the net payment, floating against fixed, is exchanged.
2. **Novation netting** involves the cancellation of several contracts between the two parties, resulting in a replacement contract with new, net payments. As an example, consider a forward trade where A must pay 10 million euros in exchange for receiving 15 million dollars from B. In another trade, A must receive 5 million euros from B and pay 7 million dollars in exchange. Under novation, the two contracts are reduced to a payment of 5 million euros from A in exchange for a receipt of 8 million dollars from B.
3. **Close-out netting** involves the cancellation of all transactions under the master agreement in the event of bankruptcy or any other specified default event. The trades are then netted at market value. The ability to **terminate** financial contracts upon an event of default is central to the effective management of financial risk. Without a close-out or termination clause, counterparties would helplessly watch their contracts fluctuate in value during the bankruptcy process, which could take years.

Table 22.1 gives an example with four contracts. Without a netting agreement, the exposure of the first two contracts is the sum of the positive part of each, or \$100 million. In contrast, if the first two fall under a netting agreement, their values would offset each other, resulting in a net exposure of \$100 − \$60 = \$40 million. If contracts 3 and 4 do not fall under the netting agreement, the exposure is then increased to \$40 + \$25 = \$65 million.

To summarize, the **net exposure** with netting is

$$\text{Net Exposure} = \text{Max}(V, 0) = \text{Max}\left(\sum_{i=1}^{N} V_i, 0\right) \tag{22.18}$$

TABLE 22.1 Comparison of Exposure with and without Netting

Contract	Contract Value	Exposure: No Netting	Exposure: With Netting for 1 and 2
	Under netting agreement		
1	+\$100	+\$100	
2	−\$60	+\$0	
Total, 1 and 2	+\$40	+\$100	+\$40
	No netting agreement		
3	+\$25	+\$25	
4	−\$15	+\$0	
Grand total, 1 to 4	+\$50	+\$125	+\$65

Without netting agreement, the **gross exposure** is the sum of all positive-value contracts:

$$\text{Gross Exposure} = \sum_{i=1}^{N} \text{Max}(V_i, 0) \tag{22.19}$$

This is always greater than or equal to the exposure under the netting agreement.

The benefit from netting depends on the number of contracts N and the extent to which contract values covary. The larger the value of N and the lower the correlation, the greater the benefit from netting. It is easy to verify from Table 22.1 that if all contracts move into positive value at the same time, or have high correlation, there will be no benefit from netting.

Figures 22.12 and 22.13 illustrate the effect of netting on a portfolio of two swaps with the same counterparty. In each case, interest rates could increase or decrease with the same probability.

In Figure 22.12, the bank is long both a receive-fixed 10-year swap and a receive-fixed five-year swap. The top panel describes the worst exposure when rates fall. In this case there is positive exposure for both contracts, which we add to get the total portfolio exposure. Whether there is netting or not does not matter, because the two positions are positive at the same time. The bottom panel describes the worst exposure when rates increase. Both positions, as well as the portfolio, have zero exposure.

In Figure 22.13, the bank is long the 10-year swap and short the five-year swap. When rates fall, the first swap has positive value and the second has negative value. With netting, the worst exposure profile is reduced. In contrast, with

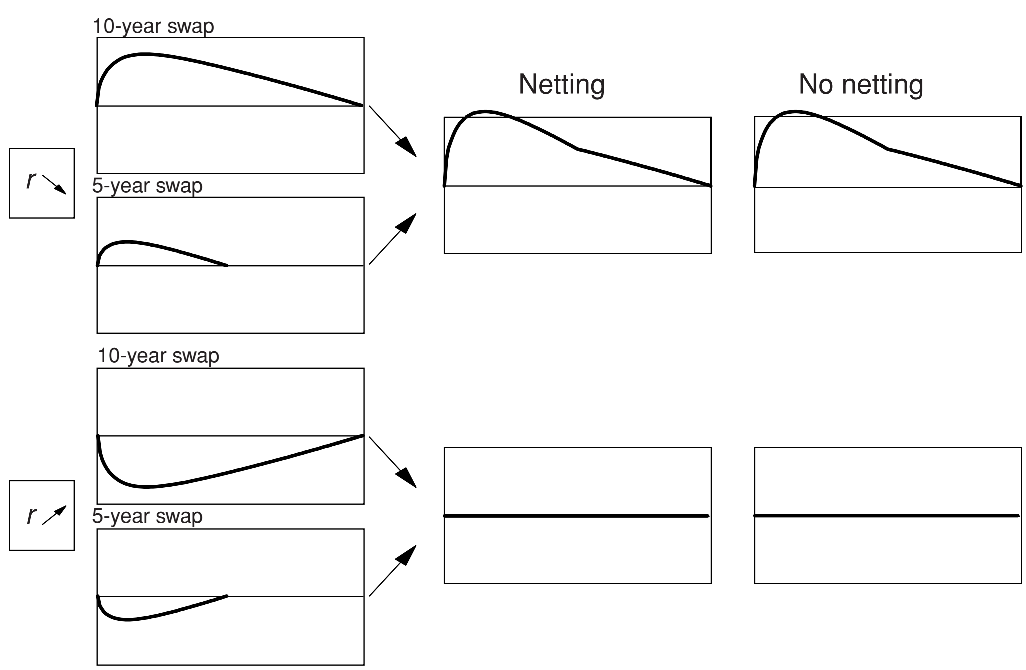

FIGURE 22.12 Netting with Two Long Positions

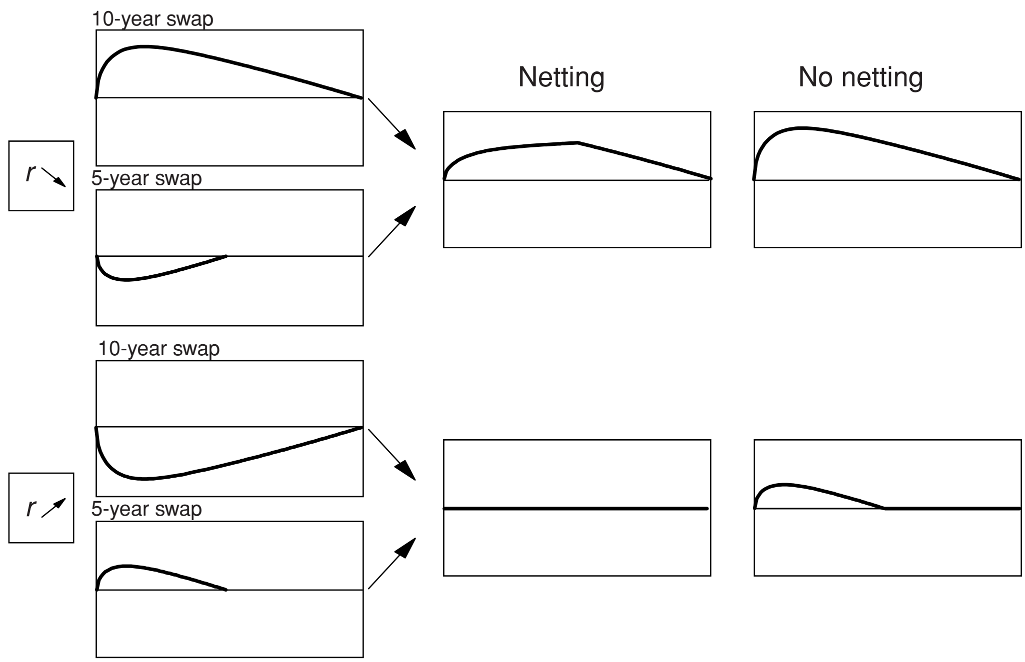

FIGURE 22.13 Netting with a Long and a Short Position

no netting the exposure is that of the 10-year swap. Conversely, when rates increase, the swap value is negative for the first and positive for the second. With netting, the exposure profile is zero, whereas without netting it is the same as that of the five-year swap. This shows that netting is more effective with diversified positions.

Banks provide some information in their annual reports about the benefit of netting for their current exposure. Without netting agreements or collateral, the **gross replacement value** (GRV) is reported as the sum of the worst-case exposures if all counterparties K default at the same time:

$$\mathrm{GRV} = \sum_{k=1}^{K} (\text{Gross exposure})_k = \sum_{k=1}^{K} \left[\sum_{i=1}^{N_k} \mathrm{Max}(V_i, 0) \right] \tag{22.20}$$

With netting agreements and collateral, the resulting exposure is defined as the **net replacement value** (NRV). This is the sum, over all counterparties, of the net positive exposure:

$$\mathrm{NRV} = \sum_{k=1}^{K} (\text{Net exposure})_k = \sum_{k=1}^{K} \left[\mathrm{Max} \left(\sum_{i=1}^{N_k} V_i, 0 \right) \right] \tag{22.21}$$

If collateral is held, this should be subtracted from the net exposure.

The effectiveness of netting can be assessed from Bank for International Settlements (BIS) statistics for the OTC derivatives markets. As described in Chapter 7, the total notional amounts added up to $688 trillion as of December 2009. The gross market value, defined as the summation of the positive part of all contracts, was estimated at $22 trillion. The net credit exposure was reduced to $3.5 trillion. Thus, netting reduces the exposure by 80%.

EXAMPLE 22.13: FRM EXAM 2002—QUESTION 89

If we assume that the value at risk (VAR) for the portfolio of trades with a given counterparty can be viewed as a measure of potential credit exposure, which of the following could *not* be used to decrease this credit exposure?

a. A netting agreement
b. Collateral
c. A credit derivative that pays out if the counterparty defaults
d. An offsetting trade with a different counterparty

EXAMPLE 22.14: FRM EXAM 2005—QUESTION 96

Which of the following statements correctly describes the impact of signing a netting agreement with a counterparty?

a. It will increase or have no effect on the total credit exposure.
b. It will decrease or have no effect on the total credit exposure.
c. It will increase exposure if exposure is net long and decrease exposure if it is net short.
d. Its impact is impossible to determine based on the available information.

EXAMPLE 22.15: FRM EXAM 2006—QUESTION 39

What are the benefits of novation?

a. Both parties are allowed to walk away from the contract in the event of default.
b. In a bilateral contract, it is specified that on default, the nondefaulting party nets gains and losses with the defaulting counterparty to a single payment for all covered transactions.
c. Financial market contracts can be terminated upon an event of default prior to the bankruptcy process.
d. Obligations are amalgamated with others.

EXAMPLE 22.16: FRM EXAM 2003—QUESTION 24

Bank A, which is AAA rated, trades a 10-year interest rate swap (semiannual payments) with Bank B, rated A−. Because of Bank B's poor credit rating, Bank A is concerned about its 10-year exposure. Which of the following measures would help mitigate Bank A's credit exposure to Bank B?

I. Negotiate a CSA with Bank B and efficiently manage the collateral management system.
II. Execute the swap deal as a reset swap wherein the swap will be marked to market every six months.
III. Execute the swap deal with a break clause in the fifth year.
IV. Decrease the frequency of coupon payments from semiannual to annual.

a. I only
b. IV only
c. I, II, III, and IV
d. I, II, and III

EXAMPLE 22.17: FRM EXAM 2002—QUESTION 73

Consider the following information. You have purchased 10,000 barrels of oil for delivery in one year at a price of $25/barrel. The rate of change of the price of oil is assumed to be normally distributed with zero mean and annual volatility of 30%. Margin is to be paid within two days if the credit exposure becomes greater than $50,000. There are 252 business days in the year. Assuming enforceability of the margin agreement, which of the following is the closest number to the 95% one-year credit risk of this deal governed under the margining agreement?

a. $50,000
b. $58,000
c. $61,000
d. $123,000

EXAMPLE 22.18: FRM EXAM 2008—QUESTION 3-35

Suppose BSM, a large derivative market maker, has six contracts with a counterparty, all transacted in New York (i.e., the same legal jurisdiction). The current market values (PV) for these contracts are: 125, 75, 25, −10, −65, and −140. Suppose BSM does not currently have a legally enforceable netting agreement with the counterparty. By how much would BSM's current credit exposure to this counterparty improve if it did have a legally enforceable netting agreement with the counterparty?

a. 0
b. 10
c. 215
d. 225

22.4 CREDIT RISK MODIFIERS

Credit risk modifiers operate on credit exposure, default risk, or a combination of the two. For completeness, this section discusses modifiers that affect default risk.

22.4.1 Credit Triggers

Credit triggers specify that if either counterparty's credit rating falls below a specified level, the other party has the right to have the swap cash settled. These are not exposure modifiers, but rather attempt to reduce the probability of default on that contract. For instance, if all outstanding swaps can be terminated when the counterparty rating falls below A, the probability of default is lowered to the probability that a counterparty will default when rated A or higher.

These triggers are useful when the credit rating of a firm deteriorates slowly, because few firms jump directly from investment grade into bankruptcy. The increased protection can be estimated by analyzing transition probabilities, as discussed in a previous chapter. For example, say a transaction with an AA-rated borrower has a cumulative probability of default of 0.81% over 10 years. If the contract can be terminated whenever the rating falls to BB or below, this probability falls to 0.23%.

22.4.2 Time Puts

Time puts, or **mutual termination options**, permit either counterparty to terminate the transaction unconditionally on one or more dates in the contract. This feature decreases both the default risk and the exposure. It allows one counterparty to terminate the contract if the exposure is large and the other party's rating starts to slip.

Triggers and puts, which are types of **contingent requirements**, can cause serious trouble, however. They create calls on liquidity precisely in states of the world where the company is faring badly, putting additional pressure on the company's liquidity. Indeed, triggers in some of Enron's securities forced the company to make large cash payments and propelled it into bankruptcy. Rather than offering protection, these clauses can trigger bankruptcy, affecting all creditors adversely.

EXAMPLE 22.19: FRM EXAM 2009—QUESTION 6-1

Which of the following statements about counterparty exposure is correct?

a. Potential future exposure is the minimum amount of exposure expected to occur on a future date with a high degree of statistical confidence.
b. Netting rights, collateral agreements, and early settlement provisions are all examples of credit risk mitigants.
c. Current exposure refers to the current value of the exposure to a subsidiary.
d. Wrong-way exposures are exposures that are positively correlated with the credit quality of the counterparty.

22.5 IMPORTANT FORMULAS

Credit exposure: $CE_t = \text{Max}(V_t, 0)$

Long options: $CE_t = V_t$; short options, $CE_t = 0$

Expected credit exposure (ECE): $ECE = \int_{-\infty}^{+\infty} \text{Max}(x, 0) f(x)dx$

Worst credit exposure (WCE): $1 - p = \int_{WCE}^{\infty} f(x)dx$

Credit exposure for an interest rate swap, from: $V_t = B(F, t, T, c, r_t) - B(F, \text{FRN})$

Volatility of credit exposure for an interest rate swap: $\sigma(V_t) = [k(T - t)] \times \sigma\sqrt{t}$

Credit exposure for a currency swap, from: $V_t = S_t(\$/\text{FC})B^*(F^*, t, T, c^*, r^*) - B(F, t, T, c, r)$

Gross credit exposure: $\sum_{i=1}^{N} \text{Max}(V_i, 0)$

Net credit exposure with netting: $\text{Max}(V, 0) = \text{Max}\left(\sum_{i=1}^{N} V_i, 0\right)$

Gross replacement value (GRV): $GRV = \sum_{k=1}^{K}(\text{Gross exposure})_k = \sum_{k=1}^{K}\left[\sum_{i=1}^{N_k} \text{Max}(V_i, 0)\right]$

Net replacement value (NRV): $NRV = \sum_{k=1}^{K}(\text{Net exposure})_k = \sum_{k=1}^{K}\left[\text{Max}\left(\sum_{i=1}^{N_k} V_i, 0\right)\right]$ minus collateral held

22.6 ANSWERS TO CHAPTER EXAMPLES

Example 22.1: FRM Exam 2006—Question 95

d. For a loss to occur, the exposure must be positive, meaning rates move in your favor, and the counterparty must default.

Example 22.2: FRM Exam 2009—Question 6-2

b. Buying an option or entering a forward contract creates credit exposure because both contracts could move in-the-money. Selling an option, in contrast, does not create exposure. Buying a loan creates exposure to Sunny, not City Bank.

Example 22.3: FRM Exam 2006—Question 117

b. Selling an option does not create exposure, so answer d. is wrong. Longer horizons create a potential for larger price movements, so answer a. is wrong. The potential gain from being long is greater than being short. Prices can go up several times from the initial price for a long position. For a short position, the maximum gain is if the price goes to zero.

Example 22.4: FRM Exam 2004—Question 8

c. This is the only answer that involves truly selling an option, which has no credit exposure. A collar involves the sale and purchase of an option.

Example 22.5: FRM Exam 2001—Question 84

b. Being short an option creates no credit exposure, so answers c. and d. are false. With the short forward contract, a gain will be realized if the euro has depreciated.

Example 22.6: FRM Exam 2004—Question 43

d. All measures of exposure are important: current, potential, and peak. The notional amount, however, is not at risk, because it is subsumed in exposures.

Example 22.7: FRM Exam 2005—Question 61

a. The value of the fixed-rate bond is $6/(1+4.75\%)^1 + 6/(1+4.75\%)^2 + 106/(1+4.75\%)^3 = 103.420$. Subtracting \$100 for the floating leg gives an exposure of \$3.4 million. More intuitively, the sum of the coupon difference is 3 times $(6\% - 4.75\%)\$100 = \$1.25$, or around \$3.75 million without discounting.

Example 22.8: Peak Exposure

b. See Equation (22.14).

Example 22.9: FRM Exam 2002—Question 83

b. Exposure is a function of duration, which decreases with time, and interest rate volatility, which increases with the square root of time. Define T as the original maturity and k as a constant. This gives $\sigma(V_t) = k(T - t)\sqrt{t}$. Taking the derivative with respect to t gives a maximum at $t = (T/3)$. This gives $t = (6/3) = 2$ years.

Example 22.10: FRM Exam 2000—Question 47

d. The CD has the whole notional at risk. Otherwise, the next greatest exposures are for the forward currency contract and the interest rate swap. The short cap position has no exposure if the premium has been collected. Note that the question eliminates settlement risk for the forward contract.

Example 22.11: FRM Exam 2001—Question 8

a. The question asks about potential exposure for various swaps during their lives. Interest rate swaps generally have lower exposure than currency swaps because there is no market risk on the principals. Currency swaps with longer remaining maturities have greater potential exposure. This is the case for the 10-year currency swap, which after two years has eight years remaining to maturity.

Example 22.12: FRM Exam 2004—Question 14

b. With an upward-sloping term structure, the fixed payer has greater credit exposure. It receives less initially, but receives more later. This backloading of payments increases credit exposure. Conversely, if the forward curve flattens, the fixed payer (i.e., BNP Paribas) has less credit exposure. Credit Agricole must have greater credit exposure. Alternatively, if LIBOR drifts down, BNP will have to pay more, and its counterparty will have greater credit exposure.

Example 22.13: FRM Exam 2002—Question 89

d. An offsetting trade with a different party will provide no credit protection. If the first party defaults while the contract is in-the-money, there will be a credit loss.

Example 22.14: FRM Exam 2005—Question 96

b. Netting should decrease the credit exposure if contracts with the same counterparty have positive and negative values. In the worst case, that of one contract with positive value, there is no effect.

Example 22.15: FRM Exam 2006—Question 39

d. Answer a. is incorrect because this is a walk-away clause. Answer b. is incorrect because this is close-out netting. Answer c. is incorrect because this is a termination clause.

Example 22.16: FRM Exam 2003—Question 24

d. Collateral management will lower credit exposure, so answer I. is correct. Resetting, or recouponing the swap, also will lower exposure. A break clause in five years will allow marking to market, which also lowers exposure. In contrast, decreasing the frequency of coupons will not change the exposure much. In fact, extending the period will increase exposure because there is a longer time to wait for the next payment, increasing the chance that the market will move in favor of one counterparty.

Example 22.17: FRM Exam 2002—Question 73

c. The worst credit exposure is the \$50,000 plus the worst move over two days at the 95% level. The worst potential move is $\alpha\sigma\sqrt{T} = 1.645 \times 30\% \times \sqrt{(2/252)} =$ 4.40%. Applied to the position worth \$250,000, this gives a worst move of \$10,991. Adding this to \$50,000 gives \$60,991.

Example 22.18: FRM Exam 2008—Question 3-35

c. The sum of positive exposures is 225. This is the credit exposure without netting. The sum of negative exposures is 215. With netting, the exposure goes to 10, or a drop of 215.

Example 22.19: FRM Exam 2009—Question 6-1

b. Statement a. is incorrect because exposure is the maximum amount, not the minimum amount, which is zero. Statement c. is incorrect because exposure occurs with a counterparty, not subsidiary. Statement d. is incorrect because wrong-way exposures are negatively correlated with the credit quality. The problem is when exposures are high and the credit quality goes down.

APPENDIX: ISDA MASTER NETTING AGREEMENT

At the beginning of the 1980s, swaps were tailor-made financial contracts that required documentation to be drafted on a case-by-case basis. This was very time-consuming and costly, and it introduced a time lag between the commercial agreement and the signing of the legally binding contract.

In response, the industry developed standardized terms for swaps. As with futures, this made it easier to offset the contracts, increasing liquidity and decreasing legal uncertainty. Out of this effort came the **master netting agreement** established by the ISDA in 1987 and revised in 1992 and in 2002. This form establishes (1) a template for a standardized contract, which is supplemented by (2) a **schedule to the master agreement** and (3) the actual **confirmation of contract**. Parties have the flexibility to select parts of the agreement or to amend the base document through the schedule. The more specific clauses (e.g., confirmation) override more general clauses. Thus, the order of precedence in the case of conflict is first the confirmation, then the schedule, and finally the master agreement. In addition, the **credit support annex** (CSA) manages the exchange of collateral between parties.

The ISDA master agreement contains the following provisions:

- A list of *obligations*, detailing the mechanics of payment conditions (section 2 in the ISDA agreement), including the netting of obligations.
- A list of *credit provisions*, which describe events of default and termination (section 5), early termination (section 6), and credit support provisions (e.g., the system of collateral payments). The event of default includes:
  - Failure to pay
  - Breach of agreement
  - Credit support default (e.g., failure to provide collateral when due)
  - Misrepresentation
  - Default under a specified transaction
  - Cross-default, which is optional
  - Acts pertaining to bankruptcy or liquidation
  - Mergers without the successor assuming the obligation to perform under the swap

  Termination includes:
  - An illegality in which a party is unable to perform due to a change in law or regulation
  - A tax event such as a change in tax law that causes a party to make an additional payment (called gross-up)
  - A tax event upon merger
  - A credit event upon merger where the creditworthiness of the successor is materially weaker than the original entity
- A list of contractual *boilerplate statements*, including representations (section 3), agreements (section 4), transfer provisions (section 7), governing law (section 13), and so on.

Although the ISDA forms attempt to provide comprehensive and standardized coverage of swap events, they cannot anticipate every eventuality. When Russia defaulted on its domestic-currency debt on August 17, 1998, it imposed a moratorium on foreign-currency debt payments as well as a 90-day freeze on forward foreign exchange contracts. It has maintained payment on its foreign debt, however. Whether this constitutes a credit event on the foreign debt was not clearly defined by the swap agreements in place. This has created considerable disagreement over the interpretation of standard contracts. By 1999, the ISDA had published a revised set of definitions for credit derivatives that considers both sovereign and nonsovereign entities. Finally, on April 8, 2009, ISDA introduced modifications, called the **Big Bang Protocol**, which apply to CDS contracts and are explained next in the credit derivatives chapter.

CHAPTER 23

Credit Derivatives and Structured Products

Credit derivatives are the latest tool in the management of portfolio credit risk. **Credit derivatives** are contracts whose value derives from the credit risk of an underlying obligor, corporate, sovereign, or multiname. They allow the exchange of credit risk from one counterparty to another. Credit derivatives initially grew from the need of banks to modify their credit exposure but since then have become essential portfolio management tools.

Like other derivatives, they can be traded on a stand-alone basis or embedded in some other instrument, such as a credit-linked note (CLN). This market has led to the expansion of **structured credit products**, through which portfolios of credit exposures are repackaged to better suit the needs of investors. This transformation relies on the securitization process, which was first applied to mortgage pools and was described in Chapter 18.

Section 23.1 presents an introduction to the size and rationale of these markets. Section 23.2 describes credit default swaps and their pricing. Other contracts such as total return swaps, credit spread forwards, and option contracts are covered in Section 23.3. Section 23.4 then presents credit structured products, including credit-linked notes and collateralized debt obligations (CDOs). Because of its importance, Section 23.5 describes the CDO market in more detail. Finally, Section 23.6 discusses the pros and cons of credit derivatives and structured products as well as recent regulatory developments.

23.1 INTRODUCTION

23.1.1 Market Size

From 1996 to 2007, the market for credit derivatives is estimated to have grown from about \$40 billion in gross notional to more than \$62 trillion, all of which is currently traded in **over-the-counter** (OTC) markets. As a reference, Chapter 9 has shown that the size of the global domestic corporate bond markets is now approximately \$7 trillion. Including governments, financials, and international bonds, this adds up to a total market size of \$90 trillion.

As is usual with OTC markets, however, only a fraction of this growth represents a net economic exchange of credit risk. A recent Fitch Ratings survey estimates that the ratio of gross-to-net exposures is 50, implying net exposures around $1.2 trillion.

Gross exposures are very high because dealers had a practice of not canceling existing trades but instead simply added new ones with offsetting characteristics. This practice, however, creates a backlog of paperwork, and increases operational risk as well as counterparty risk.

As a result, the industry is starting to implement **portfolio compression**, which is a process that reduces the overall size and number of items in credit portfolios without changing the risk parameters of the portfolio. In 2008 alone, $30 trillion worth of contracts were canceled, leading for the first time to a decline in notional amount during the year. By the end of 2009, the credit derivatives market had gone back to $30 trillion.

23.1.2 Markets for Exchanges of Risks

Credit derivatives have grown so quickly because they provide an efficient mechanism to exchange credit risk. While modern banking is built on the sensible notion that a portfolio of loans is less risky than single ones, banks still tend to be too concentrated in geographic or industrial sectors. This is because their comparative advantage is "relationship banking," which is usually limited to a clientele that banks know best. So far, it has been difficult to lay off this credit exposure, as there is only a limited market for secondary loans. In addition, borrowers may not like to see their bank selling their loans to another party, even for diversification reasons. Credit derivatives solve this dilemma by allowing banks to keep the loans on their books and to buy protection with credit derivatives.

In fact, credit derivatives are not totally new. **Bond insurance** is a contract between a bond issuer and a guarantor (a bank or insurer) to provide additional payment should the issuer fail to make full and timely payment. A **letter of credit** is a guarantee by a bank to provide a payment to a third party should the primary credit fail on its obligations. The **call feature** in corporate bonds involves an option on the risk-free interest rate as well as the credit spread. Indeed, the borrower can also call back the bond should its credit rating improve. At an even more basic level, a long position in a **corporate bond** is equivalent to a long position in a risk-free (meaning default-free) bond plus a short position in a credit default swap (CDS).

Thus, many existing instruments embed some form of credit derivative. What is new is the transparency and trading made possible by credit derivatives. Corporate bonds, notably, are difficult to short. In contrast, this position can be replicated easily by the purchase of a CDS contract. Thus, credit derivatives open new possibilities for investors, hedgers, and speculators.

23.1.3 Types of Credit Derivatives

Credit derivatives are over-the-counter contracts that allow credit risk to be exchanged across counterparties. They can be classified in terms of the following:

- *The underlying credit*, which can be either a single entity (single name) or a group of entities (multiname)

- *The exercise conditions*, which can be a credit event (such as default or a rating downgrade, or an increase in credit spreads)
- *The payoff function*, which can be a fixed amount or a variable amount with a linear or nonlinear payoff

The credit derivatives market includes plain-vanilla credit default swaps, total return swaps, credit spread forwards, and options. These instruments are bilateral OTC contracts. They also appear in credit structured products, which will be discussed later in this chapter. A recent survey breaks down the market into 33% for single-name CDSs, 30% for index CDSs, 16% for synthetic CDOs, 8% for tranche index trades, 3% for credit-linked notes, and 1% for credit spread options.[1] Thus the most common instruments are CDS contracts.

23.2 CREDIT DEFAULT SWAPS

23.2.1 Definition

In a **credit default swap** contract, a protection buyer (say A) pays a premium to the protection seller (say B), in exchange for payment if a credit event occurs. The **premium payment** can be a lump sum or periodic. The **contingent payment** is triggered by a credit event (CE) on the underlying credit, say a bond issued by company Y. The structure of this swap is described in Figure 23.1. Thus, these contracts represent the purest form of credit derivatives, as settlement payment occurs only in default mode (DM).

Note that these contracts are really options, not swaps. The main difference from a regular option is that the cost of the option is paid in installments instead of up front. When the premium is paid up front, these contracts are called *default put options*.[2] The annual payment is referred to as the **CDS spread**.[3]

Example

The protection buyer, call it A, enters a one-year credit default swap on a notional of $100 million worth of 10-year bonds issued by XYZ. The swap entails an annual payment of 50bp. The bond is called the *reference credit asset*.

At the beginning of the year, A pays $500,000 to the protection seller. Say that at the end of the year, company XYZ defaults on this bond, which now trades at 40 cents on the dollar. The counterparty then has to pay $60 million to A. If A holds this bond in its portfolio, the credit default swap provides protection against credit loss due to default.

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[1] British Bankers' Association, *BBA Credit Derivatives Report 2006* (London: BBA, 2006).

[2] Default swaps and default options are not totally identical instruments, however, because a default swap requires premium payments only until a triggering credit event occurs.

[3] This should not be confused with the bid-ask spread, which is the difference between the buying rate and the selling rate. For instance, the bid rate may be 45bp, and the ask rate 55bp. So, the buyer would pay 0.55% annually to acquire protection. A protection seller would receive 0.45% only.

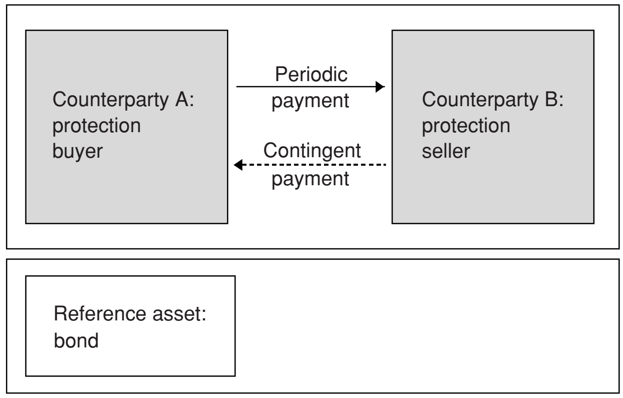

FIGURE 23.1 Credit Default Swap

Most CDS contracts are quoted in terms of an annual spread, with the payment made on a quarterly basis. Distressed names, however, can trade *up front*. For instance, on September 17, 2008, Washington Mutual was quoted at 44 points up front. This meant that the buyer of protection on $100 million would have to pay $44 million up front plus the usual spread of 500bp per year. WaMu incurred a credit event on September 27, triggering payments on CDS contracts.

Default swaps are embedded in many financial products: Investing in a risky (credit-sensitive) bond is equivalent to investing in a risk-free bond plus selling a credit default swap. Say, for instance, that the risky bond sells at $90 and promises to pay $100 in one year. The risk-free bond sells at $95. Buying the risky bond is equivalent to buying the risk-free bond at $95 and selling a credit default swap worth $5 now. The up-front cost is the same, $90. If the company defaults, the final payoff will be the same.

KEY CONCEPT

A long position in a defaultable bond is economically equivalent to a long position in a default-free bond plus a short position in a CDS on the same underlying credit.

23.2.2 Settlement

Credit events must be subject to precise definitions. Chapter 20 provided such a list, drawn from the ISDA's master netting agreement. Ideally, there should be no uncertainty about the interpretation of a credit event. Otherwise, credit derivative transactions can create legal risks.

The payment on default reflects the loss to the holders of the reference asset when the credit event occurs. Define Q as the value of this payment per unit of notional. This can take a number of forms.

- **Cash settlement**, or a payment equal to the strike minus the prevailing market value of the underlying bond.

- **Physical delivery** of the defaulted obligation in exchange for a fixed payment.
- A **lump sum**, or a fixed amount based on some pre-agreed recovery rate. For instance, if the CE occurs, the recovery rate is set at 40%, leading to a payment of 60% of the notional.

Thus, the payoff on a credit default swap is

$$\text{Payment} = \text{Notional} \times Q \times I(\text{CE}) \tag{23.1}$$

where the indicator function $I(\text{CE})$ is 1 if the credit event has occurred and zero otherwise.

The swap spread reflects both the probability of default and the loss given default, both of which are unknown. A slight variant on the usual CDS contract is the **binary credit default swap**, which pays a fixed amount $Q = 1$ if the credit event occurs. The two contracts can be combined to extract a market-implied estimate of the recovery rate.

With physical settlement, the contract usually defines a list of bonds that can be delivered. The bonds can trade at different prices but must all be exchanged for their face value. Naturally, the buyer of protection will select the cheapest bond, which creates a **delivery option**.

The growth of the CDS markets, however, has led to cases where the outstanding CDS notional far exceeds the available supply of bonds issued by a particular obligor. Cash settlement can be conducted through an **auction**, which defines the recovery rate. For instance, the final price for CDS contracts tied to Lehman Brothers Holdings was fixed on October 10, 2008, at a recovery rate of 8.625 cents on the dollar.[4] This meant that sellers of credit protection had to pay 91.375% of face value, an unexpectedly high fraction. Given an estimated $400 billion in outstanding CDS contracts, some feared that the settlement process would cause major disruptions to financial markets. In fact, the **Depository Trust & Clearing Corporation** (DTCC), which processes a large fraction of the trades, reported that only $5.2 billion had to change hands after the auction. This was because netting sharply reduced the gross exposures of $400 billion. Indeed, dividing by the estimated gross-to-net ratio of 50 gives a net exposure of only $8 billion. In addition, sellers of credit protection already had to post collateral. Thus, the CDS market has successfully handled a major default.

23.2.3 Big Bang Protocol

The **International Swaps and Derivatives Association** (ISDA), a trade organization, introduced major changes to the Standard North American Corporate CDS contract, nicknamed SNAC, effective April 8, 2009. These changes have been called the **Big Bang Protocol**.[5]

The goal was to make the new contracts more standardized, and hence more fungible (i.e., easier to resell), transparent, and liquid. Greater standardization

[4] Other notable auctions were Fannie Mae (91.51 cents for senior debt) and Washington Mutual (57 cents).

[5] For more detail, see Markit, "The CDS Big Bang," 2009, available at www.markit.com.

can be achieved by fixing the annual coupon payment. This allows a perfect offset of long and short positions with the same underlying credit and maturity date, making netting easier. Going forward, all contracts trade at fixed spreads of 100bp and 500bp, respectively, for investment-grade and speculative-grade contracts. Concurrently, contracts now trade with up-front points. The combination of these up-front points and coupons can always be converted into an equivalent CDS spread that makes the initial contract value equal to zero. In addition, SNAC contracts trade without restructuring as an applicable credit event.

The Big Bang Protocol also introduces the following changes:

- It adopts the **auction model** as the default settlement mechanism.
- It establishes **Credit Derivatives Determination Committees** for the purpose of deciding whether the credit event has taken place.
- It applies a uniform lookback period for credit and succession events for contracts with the same maturity.

In June 2009, ISDA announced changes in market practices for European and emerging markets CDSs, which now must have four coupons (25, 100, 500, and 1,000 basis points). European CDSs continue to include restructuring as a potential credit event.

23.2.4 Pricing

CDS contracts can be priced by considering the present value of the cash flows on each side of the contract. Define PV_t as the present value of a dollar paid at time t. For simplicity, assume that default occurs at the end of the year. As seen in Chapter 20, the marginal default rate from now to year t is $k_t = S_{t-1}d_t$, where S_t is the survival probability until the end of year t and d_t is the marginal probability of defaulting in year t. The survival probability is linked to the cumulative default probability $C_t = k_1 + k_2 + \cdots + k_t = 1 - S_t$.

Table 23.1 describes the annual default probabilities in the left panel. These represent typical market quotes for a credit initially rated BBB. The market-implied five-year cumulative default rate is 15.43%. Using $C_T = 1 - (1 - d)^T$, this gives

TABLE 23.1 Payoffs on a Credit Default Swap

	Probability (%)				Discount	Payoff Payments		Spread Payments	
Year t	Cumul. C_t	Annual d_t	Marg. k_t	Survival S_t	Factor PV_t	Expected $k_t(1-f)$	PV	Expected sS_{t-1}	PV
1	2.64	2.640	2.640	0.9736	0.9434	1.584	1.494	s1.000	s0.943
2	5.48	2.917	2.840	0.9452	0.8900	1.704	1.517	s0.974	s0.867
3	8.57	3.269	3.090	0.9143	0.8396	1.854	1.557	s0.945	s0.794
4	11.89	3.631	3.320	0.8811	0.7921	1.992	1.578	s0.914	s0.724
5	15.43	4.018	3.540	0.8457	0.7473	2.124	1.587	s0.881	s0.658
Total			15.430		4.2124		7.733		s3.986

an annual average default rate d of 3.30%. The second panel gives the discount factor assuming a risk-free interest rate of 6%.

Let us examine first the payoff payments. Upon default, the protection buyer receives the face value minus the recovery rate f, here assumed to be 40%. This occurs with probability k_t every year. The third panel in Table 23.1 illustrates the computations. On a notional of \$100, the PV of the expected payment in the first year is $k_1(1-f) \times \$100 \times PV_1 = 2.640\%(1-0.40) \times \$100 \times 0.9434 = 1.494$. Adding across the five years of the contract gives \$7.733.

In exchange, the protection buyer must make annual payments tied to a spread of s, defined in percent. In case of default, payments have to be made in arrears until the time of default, then stop. For the first year, the PV of the expected payment is $s\$100 \times S_0 \times PV_1 = (s/100)\$100 \times 1.000 \times 0.9434 = s0.943$. Here, $S_0 = 1.000$ because we are sure to make this first payment given that default happens at the end of the year. For the second year, this is $s\$100 \times S_1 \times PV_2 = (s/100)\$100 \times 0.9736 \times 0.8900 = s0.867$. Summing across the five years gives $s3.986$.

The fair value of the spread is the number that sets the initial value of the CDS contract to zero. This solves

$$V = (\text{PV Payoff}) - s(\text{PV Spread}) = \left(\sum_{t=1}^{T} k_t(1-f)\text{PV}_t\right) - s\left(\sum_{t=1}^{T} S_{t-1}\text{PV}_t\right) \tag{23.2}$$

In this case, the fair CDS spread solves $0 = 7.733 - s3.986$, which gives $s = 1.94\%$ because the spread was defined in percent. Note that this is very close to an approximation based on the annual average default rate times the loss given default, which is $3.30\% \times (1-0.40) = 1.98\%$.

Equation (23.2) can also be used to price an outstanding CDS contract. Assume, for instance, that the contract was entered with a spread of 1.50% and that the probabilities and interest rates in Table 23.1 represent current market conditions. The value of the CDS is then $V = 7.733 - 1.50 \times 3.986 = \$1.753$. This is a profit to the CDS buyer because the current spread is now greater than the locked-in value.

Additionally, Equation (23.2) can also be used to compute the spread duration of the contract. As an approximation, we can shock the market spreads $k_t(1-f)$ upward by 1bp, which gives a gain of $\sum \text{PV}_t = 4.21$bp. This represents a (negative) spread duration of 4.21 years, slightly less than the maturity of the contract.

Note that the default probabilities used to price the CDS contract must be *risk-neutral* (RN) probabilities, not real-world probabilities. These RN probabilities π can be inferred from bond prices and CDS prices. For instance, assume that we observe a five-year CDS spread quote of 1.50%. Using the simplified approach, this gives $\pi = 1.50\%/(1-0.40) = 2.50\%$. For more precision, we could reverse the process in Table 23.1, using market quotes for one-, two-, three-, four-, and five-year CDS contracts to derive RN default probabilities for all the maturities.

Abstracting from counterparty risk, the CDS spread should be approximately equal to the difference between the yield on a corporate bond issued by the same obligor and the risk-free yield for the same maturity. If the CDS spread were markedly lower than this difference, an investor could make an arbitrage profit by buying the corporate bond, hedging pure interest rate risk by shorting a Treasury bond, and buying the CDS contract.

KEY CONCEPT

The CDS swap spread should approximately equal the yield on a corporate bond issued by the same obligor minus the risk-free yield.

In general, however, the **basis** between the CDS spread and the cash yield spread is slightly positive. To some extent, this is influenced by demand and supply considerations, including the availability of arbitrage capital. Because investors can only short credit by buying CDS contracts, this pushes up the spread. In addition, all else equal, the basis should be wider because of the delivery option, which makes buying credit protection more attractive because the buyer is long the delivery option. In contrast, counterparty credit risk should decrease the basis because there is a risk the payoff may not be made if the credit event is triggered.

23.2.5 Counterparty Risk

It is important to realize that entering a credit swap does not eliminate credit risk entirely. Instead, the protection buyer decreases exposure to the reference credit Y but assumes new credit exposure to the CDS seller. Protection will be effective with a low correlation between the default risk of the underlying credit and of the counterparty. Just to be sure, the contracts may involve the posting of collateral from the protection seller.

Like options, these instruments are **unfunded**, meaning that each party is responsible for making payments (i.e., premiums and settlement amount) without recourse to other assets. In contrast, in a **funded** instrument, the protection seller makes a payment that could be used to settle any potential credit event. In the latter case, the protection buyer is not exposed to counterparty risk.

Table 23.2 illustrates the effect of the counterparty for the pricing of the CDS on a BBB credit. If the counterparty is default free, the CDS spread should be

TABLE 23.2 CDS Spreads for Different Counterparties

	Counterparty Credit Rating			
Correlation	AAA	AA	A	BBB
0.0	194	194	194	194
0.2	191	190	189	186
0.4	187	185	181	175
0.6	182	178	171	159
0.8	177	171	157	134

Source: Adapted from J. Hull and A. White, "Valuing Credit Default Swaps II: Modeling Default Correlations," *Journal of Derivatives* 8 (2001): 12–21.

194bp. The spread depends on the default risk for the counterparty as well as the correlation with the reference credit. In the worst case in the table, with a BBB rating for the counterparty and correlation of 0.8, protection is less effective, and the CDS spread is only 134bp.

EXAMPLE 23.1: FRM EXAM 2004—QUESTION 9

If an investor holds a five-year IBM bond, it will give a return very close to the return of the following position:

a. A five-year IBM credit default swap on which the investor pays fixed and receives a payment in the event of default
b. A five-year IBM credit default swap on which the investor receives fixed and makes a payment in the event of default
c. A five-year U.S. Treasury bond plus a five-year IBM credit default swap on which the investor pays fixed and receives a payment in the event of default
d. A five-year U.S. Treasury bond plus a five-year IBM credit default swap on which the investor receives fixed and makes a payment in the event of default

EXAMPLE 23.2: FRM EXAM 2009—QUESTION 6-3

A six-year CDS on a AA-rated issuer is offered at 150bp with semiannual payments while the yield on a six-year annual coupon bond of this issuer is 8%. There is no counterparty risk on the CDS. The annualized LIBOR rate paid every six months is 4.6% for all maturities. Which strategy would exploit the arbitrage opportunity? How much would your return exceed LIBOR?

a. Buy the bond and the CDS with a risk-free gain of 1.9%.
b. Buy the bond and the CDS with a risk-free gain of 0.32%.
c. Short the bond and sell CDS protection with a risk-free gain of 4.97%.
d. There is no arbitrage opportunity as any apparent risk-free profit is necessarily compensation for being exposed to the credit risk of the issuer.

EXAMPLE 23.3: FRM EXAM 2007—QUESTION 120

Bank A makes a USD 10 million five-year loan and wants to offset the credit exposure to the obligor. A five-year credit default swap (CDS) with the loan as the reference asset trades on the market at a swap premium of 50 basis points paid quarterly. In order to hedge its credit exposure, bank A

a. Sells the five-year CDS and receives a quarterly payment of USD 50,000.
b. Buys the five-year CDS and makes a quarterly payment of USD 12,500.
c. Buys the five-year CDS and receives a quarterly payment of USD 12,500.
d. Sells the five-year CDS and makes a quarterly payment of USD 50,000.

EXAMPLE 23.4: FRM EXAM 2004—QUESTION 50

The table shows the bid-ask quotes by UBS for CDS spreads for companies A, B, and C. CSFB has excessive credit exposure to company C and wants to reduce it through the CDS market.

	1 Year	3 Years	5 Years
A	15/25	21/32	27/36
B	43/60	72/101	112/152
C	71/84	93/113	141/170

Since the farthest maturity of its exposure to C is three years, CSFB buys a USD 200 million three-year protection on C from UBS. In order to make its purchase of this protection cheaper, based on its views on companies A and B, CSFB decides to sell USD 300 million five-year protection on company A and to sell USD 100 million one-year protection on company B to UBS. What is the net annual premium payment made by CSFB to UBS in the first year?

a. USD 1.02 million
b. USD 0.18 million
c. USD 0.58 million
d. USD 0.62 million

EXAMPLE 23.5: FRM EXAM 2004—QUESTION 65

When an institution has sold exposure to another institution (i.e., purchased protection) in a CDS, it has exchanged the risk of default on the underlying asset for which of the following?

a. Default risk of the counterparty
b. Default risk of a credit exposure identified by the counterparty
c. Joint risk of default by the counterparty and of the credit exposure identified by the counterparty
d. Joint risk of default by the counterparty and the underlying asset

EXAMPLE 23.6: FRM EXAM 2007—QUESTION 85

Bank A has exposure to USD 100 million of debt issued by company R. Bank A enters into a credit default swap transaction with bank B to hedge its debt exposure to company R. Bank B would fully compensate bank A if company R defaults in exchange for a premium. Assume that the defaults of bank A, bank B, and company R are independent and that their default probabilities are 0.3%, 0.5%, and 3.6%, respectively. What is the probability that bank A will suffer a credit loss in its exposure to company R?

a. 4.1%
b. 3.6%
c. 0.0108%
d. 0.0180%

EXAMPLE 23.7: FRM EXAM 2005—QUESTION 111

You enter into a credit default swap with bank B that settles based on the performance of company C. Assuming that bank B and company C have the same initial credit rating and everything else remains the same, what is the impact on the value of your credit default swap if bank B buys company C?

a. The credit default swap value increases.
b. The credit default swap value remains the same.
c. The credit default swap value decreases.
d. It is impossible to determine based in the information provided.

23.3 OTHER CONTRACTS

23.3.1 CDS Variants

Credit default swaps can also be written on multiple names. For instance, the **first-of-basket-to-default swap** gives the protection buyer the right to deliver *one and only one* defaulted security out of a basket of selected securities. Because the protection buyer has more choices—that is, default can occur across a basket instead of just one reference credit—this type of protection will be more expensive than a single credit swap, all else kept equal. The price of protection also depends on the correlation between credit events. The lower the correlation, the more expensive the swap. Conversely, the higher the correlation, the lower the swap rate. To illustrate this point, consider the extreme case of perfect correlation. In such a case, all underlying credits default at the same time, and this basket swap is equivalent to a regular single-name CDS.

With an **Nth-to-default swap**, payment is triggered after N defaults in the underlying portfolio, but not before. When N is large, the cost of protection will be high when the default correlation is high, making it more likely that N names will default simultaneously.

CDS indices are widely used to track the performance of this market. The iTraxx indices cover the most liquid names in European and Asian credit markets. The North American and emerging markets are covered by the CDX indices. For example, the CDX.NA.IG index is composed of 125 investment-grade entities domiciled in North America. The CDX.NA.HY index covers 100 non-investment-grade (high-yield) borrowers. The CDX.EM index covers borrowers from emerging markets. The indices are rebalanced every six months. Because these contracts are very liquid and trade at tight bid-ask spreads, they provide an easy way to buy and sell market-wide or sectoral credit risk. These indices also have tradable tranches, using the CDO methodology described later in this chapter.

CDS indices trade at tight spreads, even more narrow than for single-name contracts. Assume, for instance, that a dealer quotes 201/203 for five-year CDX.NA.IG. A trader wants \$80,000 of protection on each of the 125 companies in the index, which add up to a notional of \$10 million. The total cost is $0.0203 \times \$10{,}000{,}000 = \$203{,}000$.[6] If one company defaults, the buyer receives the usual CDS payment, and the notional is then reduced by \$80,000.

23.3.2 Total Return Swaps

A **total return swap** (TRS) is a contract where one party, called the protection buyer, makes a series of payments linked to the total return on a reference asset. In exchange, the protection seller makes a series of payments tied to a reference rate, such as the yield on an equivalent Treasury issue (or LIBOR) plus a spread.

[6] In practice, the pattern of payment is more complicated, with a fixed coupon and an initial price that depends on the quoted spread.

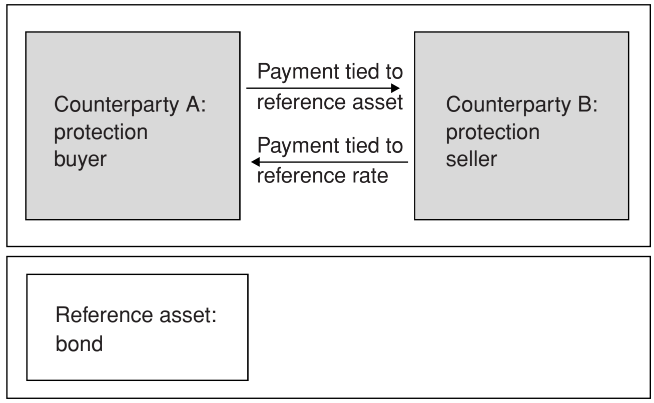

FIGURE 23.2 Total Return Swap

If the price of the asset goes down, the protection buyer receives a payment from the counterparty; if the price goes up, a payment is due in the other direction. The structure of this swap is described in Figure 23.2.

This type of swap is tied to changes in the market value of the underlying asset and provides protection against credit risk in a mark-to-market (MTM) framework. For the protection buyer, the TRS removes all the economic risk of the underlying asset without selling it. Unlike a CDS, the TRS involves both credit risk and market risk, the latter reflecting pure interest rate risk.

Example

Suppose that a bank (call it bank A) has made a \$100 million loan to company XYZ at a fixed rate of 10%. The bank can hedge its exposure by entering a TRS with counterparty B, whereby it promises to pay the interest on the loan plus the change in the market value of the loan in exchange for LIBOR plus 50bp. If the market value of the loan decreases, the payment tied to the reference asset will become negative, providing a hedge for the bank.

Say that LIBOR is currently at 9% and that after one year, the value of the loan drops from \$100 million to \$95 million. The *net* obligation from bank A is the sum of

- Outflow of $10\% \times \$100 = \$10$ million, for the loan's interest payment
- Inflow of $9.5\% \times \$100 = \$9.5$ million, for the reference payment
- Outflow of $\frac{(95-100)}{100}\% \times \$100 = -\$5$ million, for the movement in the loan's value

This sums to a net receipt of $-10 + 9.5 - (-5) = \$4.5$ million. Bank A has been able to offset the change in the economic value of this loan by a gain on the TRS.

23.3.3 Credit Spread Forwards and Options

These instruments are derivatives whose value is tied to an underlying credit spread between a risky and risk-free bond.

In a **credit spread forward contract**, the buyer receives the difference between the credit spread at maturity and an agreed-upon spread, if positive. Conversely, a payment is made if the difference is negative. An example of the formula for the cash payment is

$$\text{Payment} = (S - F) \times \text{MD} \times \text{Notional} \tag{23.3}$$

where MD is the modified duration, S is the prevailing spread, and F is the agreed-upon spread. Alternatively, this could be expressed in terms of prices:

$$\text{Payment} = [P(y + F, \tau) - P(y + S, \tau)] \times \text{Notional} \tag{23.4}$$

where y is the yield to maturity of an equivalent Treasury, and $P(y + S, \tau)$ is the present value of the security with τ years to expiration, discounted at y plus a spread, in percent. Note that if $S > F$, the payment will be positive as in the previous expression.

In a **credit spread option contract**, the buyer pays a premium in exchange for the right to put any increase in the spread to the option seller at a predefined maturity:

$$\text{Payment} = \text{Max}(S - K, 0) \times \text{MD} \times \text{Notional} \tag{23.5}$$

where K is the predefined spread. The purchaser of the option buys credit protection, or the right to put the bond to the seller if it falls in value. The payout formula could also be expressed directly in terms of prices, as in Equation (23.4).

Example

A credit spread option has a notional of \$100 million with a maturity of one year. The underlying security is an 8% 10-year bond issued by the corporation XYZ. The current spread is 150bp against 10-year Treasuries. The option is European type with a strike of 160bp.

Assume that, at expiration, Treasury yields have moved from 6.5% to 6% and the credit spread has widened to 180bp. The price of an 8% coupon, nine-year semiannual bond discounted at $y + S = 6 + 1.8 = 7.8\%$ is \$101.276. The price of the same bond discounted at $y + K = 6 + 1.6 = 7.6\%$ is \$102.574. Using the notional amount, the payout is $(102.574 - 101.276)/100 \times \$100{,}000{,}000 = \$1{,}297{,}237$.

EXAMPLE 23.8: FRM EXAM 2005—QUESTION 14

Sylvia, a portfolio manager, established a Yankee bond portfolio. However, she wants to hedge the credit and interest rate risk of her portfolio. Which of the following derivatives will best fit Sylvia's need?

a. A total return swap
b. A credit default swap
c. A credit spread option
d. A currency swap

EXAMPLE 23.9: FRM EXAM 2008—QUESTION 3-31

Helman Bank has made a loan of USD 300 million at 6.5% per annum. Helman enters into a total return swap under which it will pay the interest on the loan plus the change in the marked-to-market value of the loan, and in exchange Helman will receive LIBOR + 50 basis points. Settlement payments are made *semiannually*. What is the cash flow for Helman on the first settlement date if the mark-to-market value of the loan falls by 2% and LIBOR is 4%?

a. Net inflow of USD 9.0 million
b. Net inflow of USD 12.0 million
c. Net outflow of USD 9.0 million
d. Net outflow of USD 12.0 million

23.4 STRUCTURED PRODUCTS

23.4.1 Creating Structured Products

Structured products generally can be defined as instruments created to meet specific needs of investors or borrowers that cannot be met with conventional financial instruments. A typical example is retail demand for investments that participate in the appreciation of stock markets but also preserve capital. The payoff profile of the product can be replicated from a combination of existing or sometimes new instruments. In this case, for example, the payoff can be replicated by an investment in a risk-free bond with notional equal to the guaranteed capital, plus long positions in a call option, either through direct investment in a portfolio of options or indirectly replicated through dynamic trading. This instrument is a **principal-protected note**, and can be indexed to a variety of markets, including equities, currencies, and commodities.

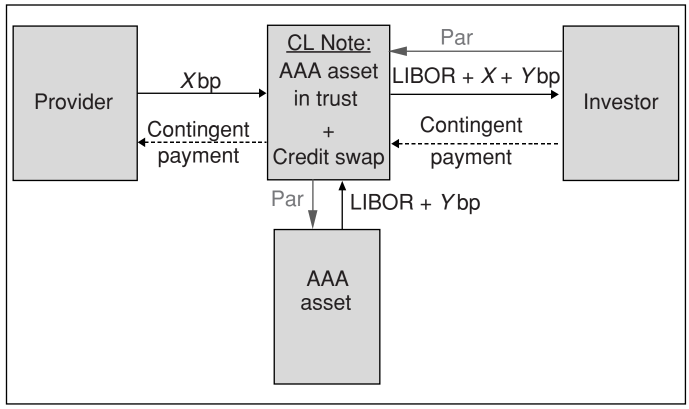

FIGURE 23.3 Credit-Linked Note

In recent years, the market for credit structured products has expanded enormously. The advent of credit derivatives has made possible a flurry of innovative products where payoffs are linked to credit events.

23.4.2 Credit-Linked Notes

Credit-linked notes (CLNs) are structured securities that combine a credit derivative with a regular bond. In a CLN, the buyer of protection transfers credit risk to an investor via an intermediary bond-issuing entity. This entity can be the buyer itself or a special-purpose vehicle (SPV).

An example of the first case is a bank with exposure to an emerging country, say Mexico. The bank issues a note with an embedded short position in a credit default swap on Mexico. The note is a liability on the bank's balance sheet. Investors receive a high coupon but will lose some of the principal if Mexico defaults on its debt. This structure achieves its goal of reducing the bank's exposure if Mexico defaults. In this case, because the note is a liability of the bank, the investor is exposed to a default of either Mexico or the bank.

An example of the SPV structure is provided in Figure 23.3. In this case, the investor's initial funds are placed in a top-rated investment that pays LIBOR plus a spread of Ybp. The SPV takes a short position in a credit default swap, for an additional annual receipt of Xbp. The annual payment to the investor is then LIBOR $+ Y + X$. In return for this higher yield, the investor must be willing to lose some of the principal should a default event occur.

Relative to a regular investment in, say, a note issued by the government of Mexico, this structure may carry a higher yield if the CDS spread is greater than the bond yield spread. This structure may also be attractive to investors who are precluded from investing directly in derivatives.

23.4.3 Collateralized Debt Obligations

Much of financial engineering is about repackaging financial instruments to make them more palatable to investors, creating value in the process. In the 1980s,

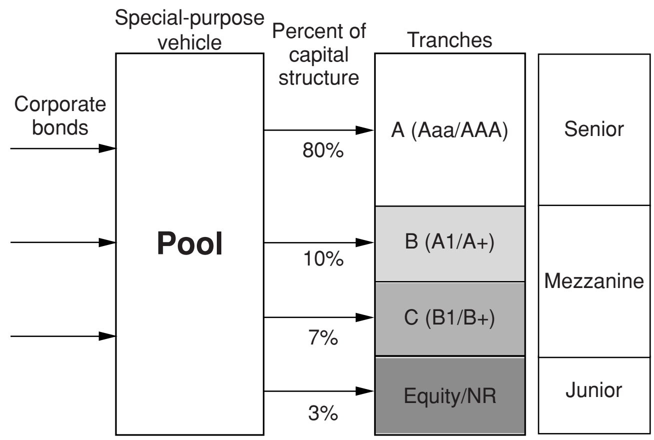

FIGURE 23.4 Collateralized Debt Obligation Structure

collateralized mortgage obligations (CMOs) brought mortgage-backed securities to the masses by repackaging their cash flows into **tranches** with different characteristics.

The same magic is performed with **collateralized debt obligations** (CDOs), which are securities backed by a pool of debt. **Collateralized bond obligations** (CBOs) and **collateralized loan obligations** (CLOs) are backed by bonds and loans, respectively. Figure 23.4 illustrates a typical CDO structure.[7]

The first step is to place a package of corporate bonds in a **special-purpose vehicle** (SPV). Assume that we have a total of \$1 billion, representing exposures of \$10 million to 100 entities. Multiple tranches are then issued by the SPV, with a specified **waterfall** structure, or priority of payments to the various tranches. Tranches are categorized as senior, mezzanine, and subordinated or equity. In the simplest structure, the SPV is ideally a passive entity. It redistributes cash flows according to well-defined rules. There is no need for other management action.

In this example, 80% of the capital structure is apportioned to tranche A, which has the highest credit rating of Aaa, using Moody's rating, or AAA. It pays LIBOR + 45bp, for example. Other tranches have lower priorities and ratings. These intermediate, *mezzanine*, tranches are typically rated A, Baa, Ba, or B (A, BBB, BB, B, using S&P's ratings). For instance, tranche C would absorb losses from 3% to 10%. These numbers are called, respectively, the **attachment point** and the **detachment point**.

At the bottom comes the equity tranche, which is not rated. Due to leverage, the return on the equity tranche can be very high if there is no default. In exchange, this is exposed to the first dollar loss in the portfolio. Special conventions apply to trading in equity tranches. The investor first pays the notional amount, which is \$30 million in this case. In exchange, this protection seller receives a spread, called running spread, and an up-front fee. This fee is quoted in percent and is typically

[7] This structure has similarities with the CLN structure. The differences are that CDOs are always issued by an SPV, involve a pool with a large number of underlying assets, and are usually tranched.

around 40% for an investment-grade CDO. In this case, the investor would get $40\% \times \$30 = \$12$ million up front, and the running spread, say 500bp.

Cumulative losses of \$20 million would reduce the notional of the equity tranche to $\$30 - \$20 = \$10$ million. The running spread then applies to the new notional of \$10 million.

For losses amounting to \$45 million, the first tranche is wiped out, with an excess loss of $\$45 - \$30 = \$15$ million. Investors in tranche C receive only $\$70 - \$15 = \$55$ million back. Thus, the rating enhancement for the senior classes is achieved through prioritizing the cash flows. Rating agencies have developed internal models to rate the senior tranches based on the probability of shortfalls due to defaults.

Essentially, these are portfolio credit risk models similar to those discussed in Chapters 19 and 24. The risk manager builds the distribution of total losses in the portfolio based on the default probability of the underlying credits, their losses given default, as well as their default correlations. The width of the lower tranches then defines the default probability of the senior tranche. Among these parameters, the (average) default correlation is very important. Lower default correlations lead to more diversified or tighter distributions. For a fixed width for the junior tranches, this makes it less likely that the portfolio loss will wipe out the junior tranche, hence making the senior tranche safer. Conversely, if all underlying bonds default at the same time, the loss can be very large and more likely to affect the senior tranche. Indeed, during the credit crisis, many senior tranches of structures based on mortgage debt suffered unexpected losses due to the higher incidence of simultaneous defaults on mortgages.

KEY CONCEPT

A long position in the senior tranche of a CDO is *short* the (average) correlation across defaults. Lower correlation implies a more diversified portfolio, which makes it less likely that many bonds will default at the same time and reach the senior tranche.

Whatever transformation is brought about, the resulting package must obey some basic laws of conservation. For the underlying and resulting securities, we must have the same cash flows at each point in time, apart from transaction costs. As a result, this implies (1) the same total market value, and (2) the same risk profile, for both interest rate and default risk. The weighted duration of the final package must equal that of the underlying securities. The expected default rate, averaged by market values, must be the same. So, if some tranches are less risky, others must bear more risk. Like CMOs, CDOs are structured so that most of the tranches have less risk than the collateral. Inevitably, the remaining **residual** tranche is more risky. This is sometimes called "toxic waste." If this residual is cheap enough, however, some investors should be willing to buy it. Oftentimes, the institution sponsoring the CDO will retain the most subordinate equity tranche to convince investors of the quality of the pool. Credit investors have developed sophisticated trading strategies that involve going long and short different tranches of these capital structures.

Example: Correlation Trading

Take a synthetic $1 billion CDO with 100 names worth $10 million each. Say that the equity tranche is $30 million, which represents the first 3% of losses. For simplicity, suppose that all premiums are measured in net present value terms and that there is no recovery. The investor gets paid $15 million up front for assuming the equity risk, so the worst net loss on the tranche is $15 million. The investor then hedges by buying CDSs on the same 100 names, with notional of $3 million each. The present value of the spread is 2%, which gives a payment of $\$300 \times 2\% = \$6$ million. If there is no default, the principal is returned, and the net gain is $\$15 - \$6 = \$9$ million. If all 100 names default, the position loses the principal of the equity and gains the CDS payments, which gives $(\$15 - \$30) - \$6 + \$300 = \$279$ million. Of course, this is very unlikely.

We need to explore other scenarios that could generate losses. If only three names default, the equity tranche is wiped out. The investor then exercises three CDS contracts and unwinds the 97 remaining CDS hedges, which are no longer necessary. As a worst-case situation, suppose the CDS spreads have tightened and that the contracts are sold for $4.8 million. This translates into a net loss of $(\$15 - \$30) - \$6 + (3 \times \$3) + \$4.8 = -\$7.2$ million.

EXAMPLE 23.10: FRM EXAM 2004—QUESTION 63

A CDO consisting of three tranches has an underlying portfolio of n corporate bonds with a total principal of USD N million. Tranche 1 has 10% of N and absorbs the first 10% of the default losses. Tranche 2 has 20% of N and absorbs the next 20% of default losses. The final Tranche 3 has 70% of N and absorbs the residual default loss. Which of the following statements is/are *true*?

I. Tranche 2 has the highest yield.
II. Tranche 1 is usually called "toxic waste."
III. Tranche 3 would typically be rated as AAA by S&P.
IV. Tranche 3 has the lowest yield.

a. I only
b. IV only
c. II, III, and IV only
d. II and IV only

EXAMPLE 23.11: FRM EXAM 2002—QUESTION 32

A collateralized bond obligation (CBO) consists of several tranches of notes from a repackaging of corporate bonds, ranging from equity to super-senior. Which of the following is generally *true* of these structures?

a. The total yield of all the CBO tranches is slightly less than the underlying repackaged bonds to allow the issuer to recover their fees/costs/profits.
b. The super-senior tranche has expected loss rate higher than the junior tranche.
c. The super-senior tranche is typically rated below AAA and sold to bond investors.
d. The equity tranche does not absorb the first losses of the structure.

EXAMPLE 23.12: FRM EXAM 2009—QUESTION 6-6

An investor has sold default protection on the most senior tranche of a CDO. If the default correlation decreases unexpectedly, assuming everything else is unchanged, the investor's position will

a. Gain value since the probability of exercising the protection falls.
b. Lose value, since the investor's protection will gain value.
c. Neither gain nor lose value since only expected default losses matter and correlation does not affect expected default losses.
d. It depends on the pricing model used and the market conditions.

EXAMPLE 23.13: FRM EXAM 2007—QUESTION 130

A three-year credit-linked note (CLN) with underlying company Z has a LIBOR + 60bp semiannual coupon. The face value of the CLN is USD 100. LIBOR is 5% for all maturities. The current three-year CDS spread for company Z is 90bp. The fair value of the CLN is closest to

a. USD 100.00
b. USD 111.05
c. USD 101.65
d. USD 99.19

EXAMPLE 23.14: FRM EXAM 2009—QUESTION 6-5

A fixed-income investor is considering investing in an asset-backed security (ABS) that has the following structure.

Senior tranche	USD 250 million
Junior tranche	USD 100 million
Subordinated tranche A	USD 60 million
Subordinated tranche B	USD 30 million
Total	USD 440 million

If the assets in the pool are worth USD 450 million, what amount of losses will cause the investor to begin to lose money if he invested in the senior tranche?

a. USD 200 million
b. USD 190 million
c. USD 100 million
d. USD 90 million

23.5 CDO MARKET

23.5.1 Balance Sheet and Arbitrage CDOs

Table 23.3 describes the evolution of the CDO market from 2004 to 2008. From 2004 to 2006, the market doubled every year, reaching more than $520 billion in new issues during 2006. As a result of the credit crisis, however, issuances have spiraled down.

CDO transactions are typically classified by purpose, as balance sheet or arbitrage. The primary goal of **balance sheet CDOs** is to move loans off the balance sheet of commercial banks to lower regulatory capital requirements.

TABLE 23.3 Evolution of the CDO Market Annual Issues ($ Million)

Explanation	2004	2006	2009
By type:			
Cash flow	119,531	410,504	43,458
Synthetic	37,237	66,503	1,346
Market value	650	43,638	16,392
By purpose:			
Arbitrage	146,998	454,971	47,938
Balance sheet	10,420	65,674	12,949
Total	157,418	520,645	61,197

Source: Bond Market Association

In contrast, **arbitrage CDOs** are designed to capture the spread between the portfolio of underlying securities and that of highly rated overlying tranches. Because CDO senior tranches should be relatively safe due to diversification effects, they pay a tight spread over LIBOR. The arbitrage profit then goes into the equity tranche (but also into management and investment banking fees).

Senior tranches also seem attractive for investors. Generally, AA-rated corporate borrowers pay LIBOR. Higher credit pay rates below LIBOR. A typical AAA-rated senior CDO tranche, however, pays a higher rate than LIBOR. This explains why investors were attracted to this market, putting blind faith into the credit ratings. Some commentators wondered whether this was a true arbitrage or involved some type of model risk. By now, it is clear that the models used by the credit rating agencies were flawed (see also Chapter 20).

23.5.2 Cash-Flow and Synthetic CDOs

Credit risk transfer can be achieved by cash-flow or synthetic structures. The example in Figure 23.4 is typical of traditional, or *funded*, **cash-flow CDOs**. The physical assets are sold to an SPV and the underlying cash flows are used to back payments to the issued notes.

In contrast, the credit risk exposure of **synthetic CDOs** is achieved with credit default swaps. We know that a long position in a defaultable bond is equivalent to a long position in a default-free bond plus a short position in a CDS. Synthetic CDOs create higher yields by first, funding or placing the initial investment in default-free, or Treasury, securities, and second, selling a group of CDSs to replicate a cash-flow CDO.

Synthetic CDOs offer several advantages. First, they are easier to manage than cash-flow CDOs. In case of bankruptcy of one of the underlying credits, the management of a cash-flow SPV has to take part in the bankruptcy process. With a short CDS position that is cash settled, there is no need for the SPV to get involved with the bankruptcy process. Second, the issue does not need to be fully funded. In a **full capital structure CDO**, the total notional amount of notes issued is equal to the total notional amount of the underlying portfolio. So, it is fully funded. In contrast, a **single-tranche CDO** is a **bespoke** transaction where the bank and the investor agree on the terms of a deal, including size, credit rating, and underlying credits.[8] Effectively, the bank holds the rest of the capital structure and does not place it.

Take as an example a $10 million tranche rated A+ paying a coupon of six-month LIBOR plus 111bp, on a reference portfolio of $1,000 million of 100 North American investment-grade entities. The attachment point is 5%. The detachment point is 6%. The investor will receive the promised payments as long as cumulative losses in the reference portfolio remain below 5%. Above that, the investor will have to take a loss. For example, if cumulative dollar losses on the portfolio

[8] The term *bespoke* was originally used to describe clothing made to a customer's specification. The term comes from the word *bespeak*, meaning to ask for or order something.

amount to \$52 million, or 5.2% of total, the investor will lose $(5.2 - 5.0)/(6.0 - 5.0) = 20\%$ of the capital, which is \$2 million in this case. The coupon then applies to the reduced notional amount. A bank can sell such structure to an investor without fully funding it.

23.5.3 Cash-Flow and Market-Value CDOs

In the case of cash-flow CDOs, payments to investors solely come from collateral cash flows. In contrast, with **market-value CDOs**, payments are made from collateral cash flows as well as sales of collateral. If the market value of the collateral falls below some level, payments to the equity tranche are suspended. This creates more flexibility for the portfolio manager.

Credit rating agencies analyze the quality of credit structures using **overcollateralization ratios** (OCs). This ratio measures how many times the collateral can cover the SPV liabilities. For a market-value CDO, define V as the market value of assets and D as the par value of liabilities. The OC ratio is then defined as:

$$\text{OC} = \frac{V}{D} \tag{23.6}$$

This must be high enough to ensure sufficient coverage of liabilities.

Alternatively, the par value of cumulative tranches, starting from the top, must be kept below the market value of assets times an **advance rate**. In our CDO example, the notional for the first tranche is \$800 million. This must be kept below the value of assets, say \$1,000 million, times an advance rate of 85%. Because $\$800 < \$850$, the first tranche in the structure passes the test. For the first two tranches, the advance rate is 95%, and the test gives $\$800 + \$100 < \$950$, so this passes the test as well. A structure that fails the overcolleralization test risks downgrading. Such failure can be cured by selling some of the assets and repaying some of the tranches or issuing more equity.

For a cash-flow CDO, the ratio uses the par value of total assets in the numerator. Another ratio, the **interest rate coverage ratio** (IC), is also used to assess the quality of a credit structure. This is computed as the total interest payment to be received by the collateral divided by the interest liability of each tranche and more senior tranches.

23.5.4 Static and Managed CDOs

Finally, CDOs differ in the management of the asset pool. In **static CDOs**, the asset pool is basically fixed. In contrast, with **managed CDOs**, a portfolio manager is allowed to trade actively the underlying assets.

This has all the usual benefits and disadvantages of active management. Benefits include the ability to unwind assets with decreasing credit quality, to buy undervalued securities, and to sell overvalued securities. With managed CDOs, investors face credit risk as well as poor management risk, however. In addition, they pay management fees.

23.5.5 Other Products

As the market for CDOs has expanded, new products have appeared. For instance, a CDO can invest in CDO tranches instead of individual credits. This is a **CDO-squared** structure. The main benefit of this structure is the greater degree of diversity. A typical single-layer CDO references 50 to 100 corporate credits. A CDO-squared has 5 to 10 one-layer CDOs, and is thus exposed to 250 to 1,000 names. There was even talk of a further structure, called a CDO-cubed.

The market now also trades credit default swaps on asset-backed security tranches, called **ABCDSs**. Most commonly, the assets are backed by home equity and commercial property loans. In the past, it was difficult to short such ABSs. Buying an ABCDS is equivalent to acquiring protection, or shorting the security. This opens up new possibilities to implement relative-value trades or to hedge this type of risk. The market has developed rapidly thanks to standardized ISDA documentation as well as the establishment of a benchmark index, the **ABX index**, which contains 20 home equity securities.

ABCDSs are complex instruments. A corporate CDS makes a payment if the underlying company suffers a credit event. In contrast, with an ABS, the issuing SPV cannot go bankrupt but defaults can occur for individual loans in the pool. Also, the notional amount is not fixed but amortizes over time as principal is paid back on the loans.

These ABCDSs are similar to **CDSs on CDOs**, which are credit default swaps on CDO tranches, usually the senior ones. These instruments provide an efficient way to short sell the market. Dealers who are arranging cash CDOs can buy the CDSs to hedge their exposure, for example.

Another recent innovation is the **constant proportional debt obligation** (CPDO). CPDOs offer protection on a portfolio of corporate credits such as the iTraxx European CDS index. The transaction is highly levered and dynamically adjusted, getting rid of the credits that deteriorate over time and changing the leverage as spreads vary. This creates a structure that is rated AAA yet pays LIBOR plus 200bp. These new instruments looked very attractive in a benign environment of stable or falling credit spreads. For risk managers, however, their risk profile is difficult to assess due to their dynamic nature. The credit crisis caused considerable losses to many CPDOs, some of which have already defaulted.

EXAMPLE 23.15: FRM EXAM 2008—QUESTION 3-29

In a synthetic CDO,

a. The SPV gains credit exposure by buying securities.
b. The SPV gains credit exposure by selling credit default swaps.
c. The SPV gains credit exposure by buying credit default swaps.
d. The SPV gains credit exposure by selling risk-free bonds.

EXAMPLE 23.16: FRM EXAM 2003—QUESTION 7

A standard synthetic CDO references a portfolio of 10 corporate names. Assume the following. The total reference notional is X, and the term is Y years. The reference notional per individual reference credit name is $X/10$. The default correlations between the individual credit names are all equal to one. The single-name CDS spread for each individual name is 100bp, for a term of Y years. The assumed recovery rate on default for all individual reference credits is zero in all cases. The synthetic CDO comprises two tranches, a 50% junior tranche priced at a spread J, and a 50% senior tranche priced at spread S. All else constant, if the default correlations between the individual reference credit names are reduced from 1.0 to 0.7, what is the effect on the relationship between the junior tranche spread J and the senior tranche spread S?

a. The relationship remains the same.
b. S increases relative to J.
c. J increases relative to S.
d. The effect cannot be determined given the data supplied.

EXAMPLE 23.17: FRM EXAM 2007—QUESTION 81

A bank is considering buying (i.e., selling protection on) an AAA-rated super-senior tranche [10%–11%] of a synthetic collateralized debt obligation (CDO) referencing an investment-grade portfolio. The pricing of the tranche assumes a fixed recovery of 40% for all names. All else being equal, which one of the following four changes will make the principal invested more risky?

a. An increase in subordination of 1% (i.e., investing in the [11%–12%] tranche)
b. An increase in the tranche thickness from 1% to 3% (i.e., investing in the [10%–13%] tranche)
c. Using a recovery rate assumption of 50%
d. An increase in default correlation between names in the portfolio

EXAMPLE 23.18: FRM EXAM 2007—QUESTION 10

Consider the following homogeneous reference portfolio in a synthetic CDO: number of reference entities, 100; CDS spread, $s = 150$bp; recovery rate $f = 50\%$. Assume that defaults are independent. On a single name the annual default probability is constant over five years and obeys the relation: $s = (1 - f)\mathrm{PD}$. What is the expected number of defaulting entities over the next five years, and which of the following tranches would be entirely wiped out (lose 100% of the principal invested) by the expected number of defaulting entities?

a. There would likely be 14 defaults and a [3%–14%] tranche would be wiped out.
b. There would likely be 3 defaults and a [0%–1%] tranche would be wiped out.
c. There would likely be 7 defaults and a [2%–3%] tranche would be wiped out.
d. There would likely be 14 defaults and a [6%–7%] tranche would be wiped out.

23.6 DISCUSSION

Credit products are by far the fastest growing segment of financial derivatives. Credit default swaps have become mainstream products, and now are actively traded for a large variety of names.

23.6.1 Risk Management Tools

The rapid growth of the credit derivatives market is the best testimony of their usefulness. These instruments are superior risk management tools, allowing the *transfer of risks*. Table 23.4 breaks down the market by participants, taken from a survey by the British Bankers' Association (BBA).

TABLE 23.4 Buyers and Sellers of Credit Protection

	Percentage		
Type of Institution	**Buyer**	**Seller**	**Net**
Banks	59	44	+15
Insurers	6	17	−11
Hedge funds	28	32	−4
Others	7	7	0
Total	100	100	0

Source: BBA Credit Derivatives Report 2006.

Banks are net buyers of credit protection, which is a hedge against their lending business. This helps explain why banks weathered the 2001 recession rather well, in spite of large corporate (WorldCom and Enron) and sovereign (Argentina) defaults. Most of these bank exposures had been sold. On the other side are insurance companies, which are net sellers of credit protection. This is akin to selling insurance.

23.6.2 Controversies

Structured products have spread risk all over the world, which has created contagion effects when subprime-backed assets started to go bad. In addition, insurance companies such as American International Group (AIG) have ended up selling too much protection. Because of its size, a failure by AIG probably would have caused systemic risk (the case of AIG is discussed in Chapter 26).

Standard credit derivatives such as credit default swaps, however, have many benefits. During 2007 and 2008, this market has remained fairly liquid, unlike the cash bond markets. This CDS market creates transaction prices that provide useful information about the cost of credit to outside observers. In other words, they provide *price discovery*. CDS contracts also allow *transactional efficiency*, because they have lower transaction costs than the cash markets.

On the downside, the growth of this market has created *operational risk* because of backlogs in the processing of trades. Regulators have pushed the industry to improve the operational infrastructures, including more automated trade processing.

In addition, *counterparty risk* has become an issue in the wake of Lehman's failure. Lehman was a major player in the CDS market. This explains the push for a *centralized clearinghouse*. As in the case of the CLS Bank discussed in Chapter 19, this would allow multilateral netting of contracts and generally decrease counterparty risk. This would also make trading more transparent and, for some contracts, more liquid.

Credit derivatives also introduce a new element of risk, which is *legal risk*. Parties may not agree on the terms of the trade in case of default. Even with full confirmation of the trade, parties sometimes squabble over the definition of a credit event. Such disagreement occurred during the Russian default as well as notable debt restructurings and demergers. The widespread use of ISDA confirmation agreements helps resolve some of this uncertainty.

Overall, credit default swaps are likely to continue to thrive because they provide many benefits to financial market participants. In contrast, the future of complex credit securitizations is more clouded. This market has evolved from *regulatory arbitrage*—that is, attempts to defeat capital requirements by laying off loan credit risk through securitizations. As discussed in Chapter 18, the recent credit crisis has revealed serious flaws in the securitization process, which created complex instruments. Evaluating these complex structures requires sophisticated portfolio credit risk models, which are covered in the next chapter. Indeed, the losses suffered on many of these structures were the trigger for the credit crisis that started in 2007.

23.6.3 Regulatory Changes

The credit crisis has led to major regulatory changes. In the United States, the **Dodd-Frank Wall Street Reform and Consumer Protection Act**, enacted in July 2010, will create fundamental changes in the derivatives markets.

The Dodd-Frank Act gives regulatory authority over **security-based swaps** (SBSs) to the **Securities and Exchange Commission** (SEC). These swaps are based on equities, bonds, or security-based indices, including, for example, single-name CDS and CDX contracts. Other swaps now fall under the jurisdiction of the **Commodity Futures Trading Commission** (CFTC). Currency derivatives are likely to be exempt.

The Act imposes new requirements on the most active OTC derivatives market participants, **swap dealers** (SDs) and **major swap participants** (MSPs). Narrow exemptions exist for nonfinancial end users.

The CFTC and SEC have the power to decide what OTC trades are subject to these new rules, in which case SDs and MSPs are required (1) to execute swap transactions through an exchange facility and (2) to clear these trades through a clearing organization.[9] This creates new reporting, capital, and margin requirements.

Clearing through **central counterparties** (CCPs) has been a key part of the regulatory effort to reduce the level of systemic risk in the OTC derivatives markets. This should reduce counterparty risk and in addition improve transparency, enhance levels of standardization, and encourage the greater use of collateral. As an additional incentive, regulators for commercial banks have proposed to reduce the capital requirements for trades that banks place through CCPs.

In fact, the industry had already made great progress in that area. For instance, SwapClear, a clearing service operated by London-based LCH.Clearnet, has been widely used to clear interest rate swaps. In addition, IntercontinentalExchange (ICE) and CME Group now provide client access to clearing for credit derivatives. Nonetheless, no clearing services currently exist for many products, including interest rate swaptions, equity swaps, and structured credit products.

CCPs allow automatic netting of identical contracts, provide daily valuation for the positions, and require the posting of margins. In addition, they make reporting easier, which will allow regulators to become more aware of systemically important financial institutions entering large positions.

Moving all OTC contracts to CCPs creates other difficulties, however. Counterparty risk is not totally eliminated but instead becomes concentrated in the CCPs. Thus it becomes crucial for them to manage their risks prudently. CCPs need to be able to exit a position of a defaulted dealer at a cost of no more than the margin posted by the dealer plus its contribution to the clearer's guarantee fund. This requires proper valuation models, sufficient margins, as well as liquid markets. Hopefully, these trade-offs will be recognized as regulators finalize the rules for a new financial order.

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[9] Exchanges are defined as either traditional **contract markets**, such as organized futures and options exchanges, or **swap execution facilities** (SEFs), such as nonexchange electronic trading venues.

23.7 IMPORTANT FORMULAS

Payoff on a credit default swap: Payment = Notional × Q × I(CE)

Payoff on credit spread forward contract:

Payment = $(S - F) \times \text{MD} \times$ Notional

Payment = $[P(y + F, \tau) - P(y + S, \tau)] \times$ Notional

Valuation of a CDS contract: $V = (\text{PV Payoff}) - s(\text{PV Spread}) = (\sum_{t=1}^{T} k_t(1 - f)\text{PV}_t) - s(\sum_{t=1}^{T} S_{t-1}\text{PV}_t)$

23.8 ANSWERS TO CHAPTER EXAMPLES

Example 23.1: FRM Exam 2004—Question 9

d. A long corporate bond position is equivalent to a long Treasury bond position plus a short CDS.

Example 23.2: FRM Exam 2009—Question 6-3

a. Because LIBOR is flat, the fixed-coupon yield is also 4.6%, creating a spread of 800 − 460 = 340bp on the bond. Going long the bond and short credit via buying the CDS yields an annual profit of 340 − 150 = 190bp.

Example 23.3: FRM Exam 2007—Question 120

b. The bank should buy the swap to protect against default. The quarterly payment will be \$10M × 0.50%/4 = \$12,500.

Example 23.4: FRM Exam 2004—Question 50

a. The payment is 200 × 113 − 300 × 27 − 100 × 43, which translates into \$1.02 million.

Example 23.5: FRM Exam 2004—Question 65

d. The protection buyer is exposed to the joint risk of default by the counterparty and underlying credit. If only one defaults, there is no credit risk.

Example 23.6: FRM Exam 2007—Question 85

d. For a loss to occur, both bank B and company R must default. The joint probability of default by B and R is 0.5% times 3.6%, which gives 0.018%.

Example 23.7: FRM Exam 2005—Question 111

c. If bank B buys company C, the two entities B and C will default at the same time. This increase in the default correlation makes the CDS contract less valuable. In Table 23.2, the fair CDS spread decreases when the correlation increases. Given that the existing CDS contract has a fixed spread, this event should decrease the value of the outstanding contract.

Example 23.8: FRM Exam 2005—Question 14

a. A TRS will provide protection against both interest rate and credit risk, as it is indexed to the bond portfolio value. A CDS or credit spread option provides protection only against credit risk. There is no currency risk in Yankee bonds, which are denominated in dollars anyway.

Example 23.9: FRM Exam 2008—Question 3-31

c. Note that this is a semiannual payment; hence all annual coupon rates must be divided by 2. Helman pays $300(6.5\%/2 + 2\%)$. In return, it gets $300(4.5\%/2)$. The net is $300(5.25\% - 2.25\%) = 300(3\%) = 9.0$.

Example 23.10: FRM Exam 2004—Question 63

c. The equity tranche, tranche 1, must have the highest yield, and is sometimes called "toxic waste" because it has the highest risk. Conversely, tranche 3 would have the highest credit rating and the lowest yield.

Example 23.11: FRM Exam 2002—Question 32

a. In the absence of transaction costs or fees, the yield on the underlying portfolio should be equal to the weighted average of the yields on the different tranches. With costs, however, the CBO yield will be slightly less. Otherwise, the super-senior tranche has the highest credit ratings and the lowest loss rate of all tranches, and absorbs the last loss on the structure.

Example 23.12: FRM Exam 2009—Question 6-6

a. The value of the senior tranche depends on the default correlation. If this goes down, the distribution of losses will be more diversified, or tighter, which makes it less likely that losses will wipe out the lower tranches. Hence, the value of senior tranche goes up. Selling default protection is equivalent to being long the senior tranche, which creates a gain under these conditions.

Example 23.13: FRM Exam 2007—Question 130

d. Because the current CDS spread is greater than the coupon, the CLN must be selling at a discount. The only solution is d. More precisely, we can use the spread duration from Equation (23.2), which is the sum of the present value factor over three years. Assuming a flat term structure, this is $\sum \mathrm{PV}_t = 0.952 + 0.907 + 0.864 = 2.72$ years. Multiplying by $(90 - 60) = 30$bp gives a fall of 0.81%, which gives \$99.19.

Example 23.14: FRM Exam 2009—Question 6-5

a. This is the sum of the value of the lower tranches, or \$190 million plus the overcollateralization, which is \$10 million.

Example 23.15: FRM Exam 2008—Question 3-29

b. The SPV can either buy credit-sensitive bonds or sell default swaps.

Example 23.16: FRM Exam 2003—Question 7

c. If the correlation is one, all names will default at the same time, and the junior and senior tranche will be equally affected. Hence their spread should be 100bp, which is the same as for the collateral. With lower correlations, the losses will be absorbed first by the junior tranche. Therefore, the spread on the junior tranche should be higher, which is offset by a lower spread for the senior tranches.

Example 23.17: FRM Exam 2007—Question 81

d. Increasing the subordination will make the senior tranche less risky because there is a thicker layer beneath to absorb losses. Increasing the thickness of the tranche will make it less likely to be wiped out, so is less risky. An increase in the default correlation will increase the risk. In the limit, if all assets default at the same time, all tranches will suffer a loss.

Example 23.18: FRM Exam 2007—Question 10

d. The annual marginal PD is $d = 1.5\%/(1 - 0.50) = 3.00\%$. Hence the cumulative PD for the five years is $d + S_1 d + S_2 d + S_3 d + S_4 d = 3\%(1 + 0.970 + 0.941 + 0.913 + 0.885) = 14.1\%$, where the survival rates are $S_1 = (1 - 3\%) = 0.970$, $S_2 = S_1(1 - 3\%) = 0.941$, and so on. The expected number of defaults is therefore $100 \times 14.1\%$, or 14. With a recovery rate of 50%, the expected loss is 7% of the notional. So, all the tranches up to the 7% point are wiped out.

CHAPTER 24

Managing Credit Risk

Previous chapters have explained how to estimate the various input to portfolio credit risk models, including default probabilities, credit exposures, and recovery rates for individual credits. We now turn to the measurement of credit risk for the overall portfolio.

In the past, credit risk was measured on a stand-alone basis, in terms of a "yes" or "no" decision by a credit officer. Some consideration was given to portfolio effects through very crude concentration limits at the overall level. Portfolio theory, however, teaches us that risk should be viewed in the context of the contribution to the total risk of a portfolio, not in isolation. Indeed, the new credit risk models measure risk on a portfolio basis.

While this focus on diversification also exists in market risk management, credit risk is markedly more complex. In particular, it is difficult to estimate probabilities and correlations of default events. These correlations, however, are essential drivers of diversification benefits, as we have seen in Chapter 19.

Section 24.1 introduces the distribution of credit losses. This has two major features. The first is the expected credit loss, which is essential information for pricing and reserving purposes, as explained in Section 24.2. The second component is the unexpected credit loss, or worst deviation from the expected loss at some confidence level. Section 24.3 shows how this credit value at risk (credit VAR), like market VAR, can be used to determine the amount of capital necessary to support a position. Section 24.4 then provides an overview of recently developed credit risk models, including CreditMetrics, CreditRisk+, the KMV model, and Credit Portfolio View. Finally, Section 24.5 gives some concluding comments. Given the complexity of these models, it is essential for risk managers to understand their weak spots and limitations.

24.1 MEASURING THE DISTRIBUTION OF CREDIT LOSSES

24.1.1 Steps

The previous chapters provided a detailed analysis of the various components of credit models, which include default probabilities, credit exposures, and recovery rates. We can now pool this information to measure the distribution of losses due to credit risk. For simplicity, we initially consider only losses in **default mode** (DM), that is, losses due to defaults instead of changes in market values.

For one instrument, the potential credit loss (CL) is

$$\text{CL} = b \times \text{CE} \times \text{LGD} \tag{24.1}$$

which involves the random variable b that takes on the value of 1 when the discrete state of default occurs, with probability of default (PD) p; the credit exposure (CE), also called exposure at default (EAD); and the loss given default (LGD). With this definition, the credit loss is positive.

For a portfolio of N counterparties, the credit loss (CL) is

$$\text{CL} = \sum_{i=1}^{N} b_i \times \text{CE}_i \times \text{LGD}_i \tag{24.2}$$

where CE_i is now the total credit exposure to counterparty i, across all contracts and taking into account netting agreements.

The distribution of credit loss is quite complex. Typically, information about credit risk is described by the **net replacement value** (NRV), which is

$$\text{NRV} = \sum_{i=1}^{N} \text{CE}_i \tag{24.3}$$

evaluated at the current time. This is the most that could be lost if all parties defaulted at the same time ($b_i = 1$) and if there was no recovery ($\text{LGD}_i = 1$). This is not very informative, however. The NRV, which is often disclosed in annual reports, is equivalent to using notionals to describe the risks of derivatives portfolios. It does not take into account the probability of default or correlations across defaults and exposures.

Chapter 19 gave an example of a loss distribution for a simple portfolio with three counterparties. This example was tractable, as we could enumerate all possible states. In general, we need to consider many more credit events. We also need to account for movements and comovements in risk factors, which drive exposures, uncertain recovery rates, and correlations among defaults. This can be done with the help of *Monte Carlo simulations*. Once this is performed for the entire portfolio, we obtain a distribution of credit losses on a target date. Figure 24.1 describes a typical distribution of credit profits and losses (P&L). A later section will illustrate the construction of this distribution as provided by commercial models.

We note that the distribution of credit P&L is *highly skewed to the left,* in contrast to that of market risk factors, which is in general roughly symmetrical. This credit distribution is similar to a short position in an option. This is one of the essential insights of the Merton model, which equates a risky bond to a risk-free bond plus a short position in an option.

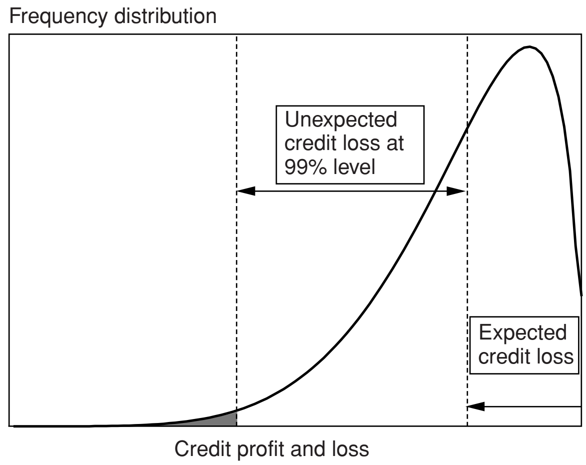

FIGURE 24.1 Distribution of Credit Losses

24.1.2 Major Features

This distribution can be described by:

- **Expected credit loss** (ECL). The **expected credit loss** represents the average credit loss. The *pricing* of the portfolio should be such that it covers the expected loss. In other words, the price should be advantageous enough to offset average credit losses. In the case of a bond, the price should be low enough, or the yield high enough, to compensate for expected losses. In the case of a derivative, the bank that takes on the credit risk should factor the expected loss into the pricing of its product. Loan loss reserves should be accumulated as a **credit provision** against expected losses. Focusing only on the default variables, the ECL depends solely on default probabilities.
- **Unexpected credit loss** (UCL). The **worst credit loss** represents the loss that will not be exceeded at some level of confidence, typically 99.9%. This is basically the quantile of the distribution. Taking the deviation from the expected loss gives the **unexpected credit loss**. The institution should have enough equity capital to cover the unexpected loss. Focusing only on the default variables, the UCL depends on both default probabilities and default correlations.

24.1.3 Effect of Correlations

The key to this approach is measuring risk at the top, portfolio level. This approach can also reveal the effect of correlations between and across risk types.

At the top of the list are the correlations across default events b_i. With low correlations and many obligors, the distribution will be narrow, as illustrated in Chapter 19. In this case, a bank could leverage up its equity several times. In its simplest version, the Basel Accord requires a minimum ratio of equity to assets of 8%, implying a maximum leverage ratio of 12.5. In contrast, high correlations will lead to simultaneous defaults, which extend the tail of the distribution and increase the UCL. In the limit, with perfect correlations, the worst loss at a fixed confidence level is the entire notional amount of the portfolio. In this case, the bank cannot have any leverage. Its assets must be covered by the same amount of equity.

Correlation can also occur across the default event and exposure (i.e., between b_i and CE). For example, **wrong-way trades** are positions where the exposure is positively correlated with the probability of default. Before the Asian crisis, for instance, many U.S. banks had lent to Asian companies in dollars, or entered equivalent swaps. Many of these Asian companies did not have dollar revenues but instead were speculating, reinvesting the funds in the local currency. When currencies devalued, the positions were in-the-money for the U.S. banks, but could not be collected because the counterparties had defaulted. Conversely, **right-way trades** occur when the transaction is a *hedge* for the counterparty—for instance, when a loss on its side of the trade offsets an operating gain.

KEY CONCEPT

Credit risk is lowered for right-way trades, where the counterparty is using the trade as a hedge. Conversely, wrong-way trades create a positive correlation between the credit exposure and the probability of default.

Another pernicious correlation is between the default event and the loss given default (i.e., between b_i and LGD). Figure 24.2 plots the recovery rate from Moody's over the period 1982 to 2009 against the speculative-grade default rate

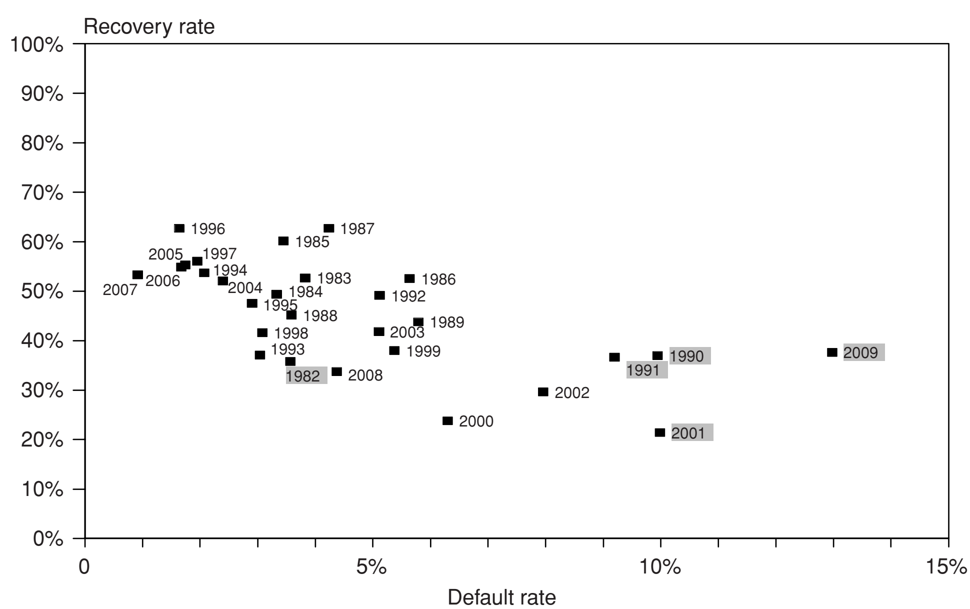

FIGURE 24.2 Recovery Rates and Default Rates

during the same years. During recession years, such as 1990, 1991, 2001, and 2009, the recovery rate on unsecured senior bonds was markedly lower than during other years. The default rate was also very high during these years. This effect will extend the tail of the credit loss distribution. If ignored, the credit VAR measure will underestimate risk.

EXAMPLE 24.1: CREDIT PROVISIONS

Credit provisions should be taken to cover all of the following *except*

a. Nonperforming loans
b. The expected loss of a loan portfolio
c. An amount equal to the VAR of the credit portfolio
d. Excess credit profits earned during below-average-loss years

EXAMPLE 24.2: FRM EXAM 2002—QUESTION 74

Following is a set of identical transactions. Assuming all counterparties have the same credit rating, which transaction should preferably be executed?

a. Buying gas from a trading firm
b. Buying gas from a gas producer
c. Buying gas from a distributor
d. Indifferent among a., b., and c.

24.2 MEASURING EXPECTED CREDIT LOSS

24.2.1 Expected Loss over a Target Horizon

For pricing purposes, we need to measure the expected credit loss, which is

$$E[\text{CL}] = \int f(b, \text{CE}, \text{LGD})(b \times \text{CE} \times \text{LGD})\, db\, d\text{CE}\, d\text{LGD} \tag{24.4}$$

If the random variables are independent, the joint density reduces to the product of densities. We have

$$E[\text{CL}] = \left[\int f(b)(b)\, db\right]\left[\int f(\text{CE})(\text{CE})\, d\text{CE}\right]\left[\int f(\text{LGD})(\text{LGD})\, d\text{LGD}\right] \tag{24.5}$$

which is the product of the expected values. In other words,

$$\text{ECL} = \text{Prob[default]} \times E[\text{CE}] \times E[\text{LGD}] \tag{24.6}$$

As an example, the actuarial expected credit loss on a BBB-rated \$100 million five-year bond with 47% recovery rate is $E[\text{CL}] = 2.28\% \times \$100{,}000{,}000 \times (1 - 47\%) = \$1.2$ million. Note that this expected loss is the same whether the bank has one \$100 million exposure or 100 exposures worth \$1 million each. The distributions, however, will be very different with more credits.

24.2.2 The Time Profile of Expected Loss

So far, we have focused on a fixed horizon, say a year. For pricing purposes, however, we need to consider the total credit loss over the life of the asset. This should involve the time profile of the exposure, the probability of default, and the discounting factor. Define PV_t as the present value of a dollar paid at time t.

The **present value of expected credit losses** (PVECL) is obtained as the sum of the discounted expected credit losses at each time step:

$$\text{PVECL} = \sum_t E[\text{CL}_t] \times \text{PV}_t = \sum_t [k_t \times \text{ECE}_t \times (1 - f)] \times \text{PV}_t \tag{24.7}$$

where the probability of default is $k_t = S_{t-1}\, d_t$, or the probability of defaulting at time t, conditional on not having defaulted before.

Alternatively, we could simplify by using the average default probability and average exposure over the life of the asset:

$$\text{PVECL}_A = \text{Ave}[k_t] \times \text{Ave}[\text{ECE}_t] \times (1 - f) \times \left[\sum_t \text{PV}_t\right] \tag{24.8}$$

This approach, however, may not be as good an approximation when default risk and exposure profile are correlated over time. For example, currency swaps with highly rated counterparties have an exposure and a default probability that increase with time. Due to this correlation, taking the product of the averages understates credit risk. In other cases, it could overstate credit risk.

An even simpler approach, when ECE is constant, considers the final maturity T only, using the cumulative default rate c_T and the discount factor PV_T:

$$\text{PVECL}_F = c_T \times \text{ECE} \times (1 - f) \times \text{PV}_T \tag{24.9}$$

24.2.3 Examples

Table 24.1 shows how to compute the PVECL. We consider a five-year interest rate swap with a counterparty initially rated BBB and a notional of \$100 million.

TABLE 24.1 Computation of Expected Credit Loss for a Swap

Year	*P*(default) (%)			Exposure	LGD	Discount	Total
t	c_t	d_t	k_t	ECE_t	$(1-f)$	PV_t	$PVECL_t$
1	0.22	0.220	0.220	\$1,660,000	0.55	0.9434	\$1,895
2	0.54	0.321	0.320	\$1,497,000	0.55	0.8900	\$2,345
3	0.88	0.342	0.340	\$1,069,000	0.55	0.8396	\$1,678
4	1.55	0.676	0.670	\$554,000	0.55	0.7921	\$1,617
5	2.28	0.741	0.730	\$0	0.55	0.7473	\$0
Total			2.280			4.2124	\$7,535
Average			0.456	\$956,000			
$PVECL_A$			0.456	\$956,000	0.55	4.2124	= \$10,100

The discount factor is 6% and the recovery rate 45%. We also assume that default can occur only at the end of each year. This analysis is similar to that for a credit default swap in Chapter 23. For simplicity, we use here real-world default probabilities.

In the first column, we have the cumulative default probability, c_t, for a BBB-rated credit from years 1 to 5, expressed as a percentage. The second column shows the marginal probability of defaulting during that year, d_t, and the third column shows the probability of defaulting in each year, conditional on not having defaulted before, $k_t = S_{t-1}d_t$. The fourth column reports the end-of-year expected credit exposure, ECE_t. The fifth column shows the constant LGD. The sixth column displays the present value factor, PV_t.

The final column gives the product $[k_t ECE_t(1-f)PV_t]$. The first entry, for example is $0.220\% \times \$1,660,000 \times 0.55 \times 0.9434 = \$1,895$. Summing across years gives \$7,535 on a swap with notional of \$100 million, or 0.007% of principal. This is very small, less than 1 basis point. So the expected credit loss on an interest rate swap is minuscule. Basically, the expected credit loss is very low due to the small exposure profile. For a regular bond or currency swap, the expected loss is much greater.

The last line shows a shortcut to the measurement of expected credit losses based on averages, from Equation (24.8). The average annual default probability is 0.456. Multiplying by the average exposure, \$956,000, the LGD, and the sum of the discount rates gives \$10,100. This is on the same order of magnitude as the exact calculation.

Table 24.2 details the computation for a bond assuming a constant exposure of \$100 million. The expected credit loss is \$1.197 million, about a hundred times larger than for the swap. This is because the exposure is also about a hundred times larger.

As in the previous table, the next line shows results based on averages. Here the expected credit loss is \$1.056 million, very close to the exact number, as there is no variation in credit exposures over time.

TABLE 24.2 Computation of Expected Credit Loss for a Bond

Year t	P(default) (%) c_t	d_t	k_t	Exposure ECE_t	LGD $(1-f)$	Discount PV_t	Total $PVECL_t$
1	0.22	0.220	0.220	\$100,000,000	0.55	0.9434	\$114,151
2	0.54	0.321	0.320	\$100,000,000	0.55	0.8900	\$156,639
3	0.88	0.342	0.340	\$100,000,000	0.55	0.8396	\$157,009
4	1.55	0.676	0.670	\$100,000,000	0.55	0.7921	\$291,887
5	2.28	0.741	0.730	\$100,000,000	0.55	0.7473	\$300,024
Total			2.280			4.2124	\$1,019,710
Average			0.456	\$100,000,000			
$PVECL_A$			0.456	×\$100,000,000	×0.55	×4.2124	= \$1,056,461
$PVECL_F$	2.280			×\$100,000,000	×0.55	×0.7473	= \$937,062

We could also take the usual shortcut and simply compute an expected credit loss given by the cumulative five-year default rate times \$100 million times the loss given default, which is \$1.254 million. Discounting to the present, we get \$0.937 million, close to the previous result.

EXAMPLE 24.3: FRM EXAM 2003—QUESTION 26

Which of the following loans has the lowest credit risk?

Loan	One-Year Probability of Default	Loss Given Default	Remaining Term (Months)
a.	1.99%	60%	3
b.	0.90%	70%	9
c.	1.00%	75%	6
d.	0.75%	50%	12

EXAMPLE 24.4: FRM EXAM 2007—QUESTION 38

Mr. Rosenqvist, asset manager, holds a portfolio of SEK 200 million, which consists of BBB-rated bonds. Assume that the one-year probability of default is 4%, the recovery rate is 60%, and defaults are uncorrelated over years. What is the two-year cumulative expected credit loss on Mr. Rosenqvist's portfolio?

a. SEK 6.40 million
b. SEK 6.27 million
c. SEK 9.60 million
d. SEK 9.48 million

24.3 MEASURING CREDIT VAR

24.3.1 Credit VAR over a Target Horizon

Credit VAR is defined as the unexpected credit loss at some confidence level. Using the measure of credit loss in Equation (24.1), we construct a distribution of the credit loss $f(\mathrm{CL})$ over a target horizon. At a given confidence c, the worst credit loss (WCL) is defined such that

$$1 - c = \int_{\mathrm{WCL}}^{\infty} f(x)dx \tag{24.10}$$

The credit VAR is then measured as the deviation from ECL

$$\text{Credit VAR} = \text{UCL} = \text{WCL} - \text{ECL} \tag{24.11}$$

where all losses are defined as positive numbers.

This credit VAR number should be viewed as the economic capital to be held as a buffer against *unexpected* losses. Its application is fundamentally different from the *expected* credit loss, which is additive across obligors and can be aggregated over time.

Instead, the credit VAR is measured over a target horizon, say one year, which is deemed sufficient for the bank to take corrective actions should credit problems start to develop. Corrective action can take the form of exposure reduction or adjustment of economic capital, both of which take considerably longer than the typical horizon for market risk.

24.3.2 Using Credit VAR to Manage the Portfolio

Once credit VAR is measured, it can be managed. The portfolio manager can examine the trades that contribute most to credit VAR. If these trades are not particularly profitable, they should be eliminated.

The **marginal contribution to risk** can also be used to analyze the incremental effect of a proposed trade on the total portfolio risk. As in the case of market risk, individual credits should be evaluated on the basis not only of their stand-alone risk, but also of their contribution to the portfolio risk. For the same expected return, a trade that lowers risk should be preferable over one that adds to the portfolio risk. Such trade-offs can be made only with a formal measurement of portfolio credit risk, however.

This marginal analysis can also help to establish the **remuneration of capital** required to support the position. Say the distribution has an ECL of \$1 billion and UCL of \$5 billion. The bank then needs to set aside \$5 billion just to cover random deviations from expected credit losses. This equity capital, however, will require remuneration. So, the pricing of loans should cover not only expected losses, but also the remuneration of this economic capital. This is what we call a *risk premium* and explains why observed credit spreads are larger than necessary simply to cover actuarial losses.

EXAMPLE 24.5: CREDIT VAR FOR ONE BOND

A risk analyst is trying to estimate the credit VAR for a risky bond. The credit VAR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one-month horizon. Assume that the bond is valued at $1,000,000 one month forward, and the one-year cumulative default probability is 2% for this bond. What is the best estimate of the credit VAR for the bond, assuming no recovery?

a. $20,000
b. $1,682
c. $998,318
d. $0

EXAMPLE 24.6: CREDIT VAR FOR TWO BONDS

A risk analyst is trying to estimate the credit VAR for a portfolio of two risky bonds. The credit VAR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one-month horizon. Assume that each bond is valued at $500,000 one month forward, and the one-year cumulative default probability is 2% for each of these bonds. What is the best estimate of the credit VAR for this portfolio, assuming no default correlation and no recovery?

a. $841
b. $1,682
c. $998,318
d. $498,318

EXAMPLE 24.7: FRM EXAM 2005—QUESTION 122

You are the credit risk manager for Bank Happy. Bank Happy holds Treasuries for USD 500 million: one large loan that has a positive probability of default for USD 400 million and another loan that has a positive probability of default for USD 100 million. The defaults are uncorrelated. The bank computes a credit VAR at 1% using CreditRisk+. Which of the following statements made about the VAR by the analyst who works for you is necessarily *wrong*?

a. The VAR or WCL can be equal to zero.
b. The expected loss on the portfolio exceeds the VAR.
c. The expected loss on the portfolio is necessarily smaller than the VAR.
d. None of the above statements is wrong.

24.4 PORTFOLIO CREDIT RISK MODELS

Portfolio credit risk models can be classified according to their approaches. This section also describes the four main portfolio credit models.

24.4.1 Approaches to Portfolio Credit Risk Models

Table 24.3 summarizes the essential features of portfolio credit risk models in the industry.

Model Type **Top-down models** group credit risks using single statistics. They aggregate many sources of risk viewed as *homogeneous* into an overall portfolio risk, without going into the details of individual transactions. This approach is appropriate for retail portfolios with large numbers of credits, but less so for corporate or sovereign loans. Even within retail portfolios, top-down models may hide specific risks, by industry or geographic location.

Bottom-up models account for features of each instrument. This approach is most similar to the structural decomposition of positions that characterizes market VAR systems. It is appropriate for corporate and capital market portfolios. Bottom-up models are also most useful for taking corrective action, because the risk structure can be reverse-engineered to modify the risk profile.

Risk Definitions **Default-mode models** consider only outright default as a credit event. Hence any movement in the market value of the bond or in the credit rating is irrelevant.

Mark-to-market (MTM) **models** consider changes in market values and ratings changes, including defaults. These fair market value models provide a better assessment of risk, which is consistent with the holding period defined in terms of the liquidation period.

Models of Default Probability **Conditional models** incorporate changing macroeconomic factors into the default probability through a functional relationship. Notably, we observe that the rate of default increases in a recession.

TABLE 24.3 Comparison of Credit Risk Models

	CreditMetrics	CreditRisk+	Moody's KMV	Credit Portfolio View
Originator	JPMorgan	Credit Suisse	KMV	McKinsey
Model type	Bottom-up	Bottom-up	Bottom-up	Top-down
Risk definition	Market value (MTM)	Default losses (DM)	Default losses (MTM/DM)	Market value (MTM)
Risk drivers	Asset values	Default rates	Asset values	Macro factors
Credit events	Rating change/ default	Default	Continuous default probability	Rating change/default
Probability	Unconditional	Unconditional	Conditional	Conditional
Volatility	Constant	Variable	Variable	Variable
Correlation	From equities (structural)	Default process (reduced-form)	From equities (structural)	From macro factors
Recovery rates	Random	Constant within band	Random	Random
Solution	Simulation/analytic	Analytic	Analytic	Simulation

Unconditional models have fixed default probabilities and tend to focus on borrower- or factor-specific information. Some changes in the environment, however, can be allowed by manually changing the input parameters.

Models of Default Correlations Because default correlations are not directly observed for the obligors in the portfolio, they must be inferred from a model.

Structural models explain correlations by the joint movements of assets—for example, stock prices. For each obligor, this price is the random variable that represents movements in default probabilities.

Reduced-form models explain correlations by assuming a particular functional relationship between the default probability and background factors. For example, the correlation between defaults across obligors can be modeled by the loadings on common risk factors—say, industrial and country.

24.4.2 CreditMetrics

CreditMetrics, published in April 1997 by JPMorgan, was an early portfolio credit risk model. The system is a bottom-up approach where credit risk is driven by movements in bond ratings taken from a transition matrix.

In this model, credit quality is measured by a *latent variable*, unobserved, which can be interpreted as the value of assets of the obligor. This is related to the value of the equity, which is the source of correlations across obligors because equity prices are observable. When the value of the assets falls below some floor, the obligor is assumed to be in a state of default. Thus, this class of models includes three types of random variables, (1) equity value, (2) asset value, and (3) default indicator.

The components of the system are described in Figure 24.3.

Measurement of Exposure by Instrument This starts from the user's portfolio, decomposing all instruments by their exposure and assessing the effect of market

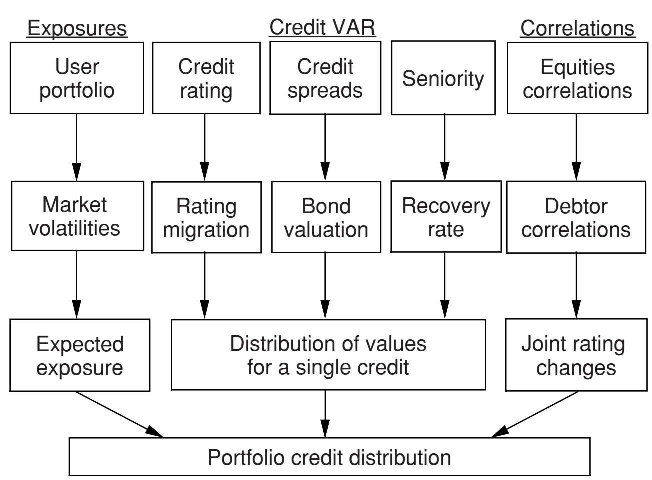

FIGURE 24.3 Structure of CreditMetrics

volatility on expected exposures on the target date. The range of covered instruments includes bonds and loans, swaps, receivables, commitments, and letters of credit.

Distribution of Individual Default Risk This step starts with assigning each instrument to a particular credit rating. Credit events are then defined by rating migrations, which include default, through a matrix of migration probabilities. Thus movements in default probabilities are discrete. After the credit event, the instrument is valued using credit spreads for each rating class. In the case of default, the distributions of recovery rates are used from historical data for various seniority classes.

This is illustrated in Figure 24.4. We start from a bond or credit instrument with an initial rating of BBB. Over the horizon, the rating can jump to eight values, including default. For each rating, the value of the instrument is recomputed—for example, to \$109.37 if the rating goes to AAA, or to the recovery value of \$51.13 in case of default. Given the state probabilities and associated values, we can compute an expected bond value, which is \$107.09, and a standard deviation of \$2.99.

Changes in the credit rating are driven by the latent factor, which is the asset value. Each asset value has a standard normal distribution with cutoff points selected to represent the probabilities of changes in credit ratings. Table 24.4 illustrates the computations for our BBB credit. From Figure 24.4, there is a 0.18% probability of going from BBB into the state of default. We choose z_1 such that the area to its left is $N(z_1) = 0.18\%$. This gives $z_1 = -2.91$. Next, we need to choose z_2 so that the probability of falling between z_1 and z_2 is 0.12%, or that the total left-tail probability is $N(z_2) = 0.18\% + 0.12\% = 0.30\%$. This gives $z_2 = -2.75$, and so on.

Correlations among Defaults Correlations among defaults are inferred from correlations between asset values. These in turn are taken from correlations across

	Probability (p_i)	Value (V_i)	Exp. $\Sigma(p_iV_i)$	Var. $\Sigma p_i(V_i\text{-}m)^2$
AAA	0.02%	\$109.37	0.02	0.00
AA	0.33%	\$109.19	0.36	0.01
A	5.95%	\$108.66	6.47	0.15
BBB	86.93%	\$107.55	93.49	0.19
BB	5.30%	\$102.02	5.41	1.36
B	1.17%	\$98.10	1.15	0.95
CCC	0.12%	\$83.64	0.10	0.66
Default	0.18%	\$51.13	0.09	5.64
	Sum = 100.00%		m = \$107.09	σ^2 = 8.95
				SD = \$2.99

FIGURE 24.4 Building the Distribution of Bond Values

TABLE 24.4 Cutoff Values for Simulations

Rating i	Probability p_i	Cumulative Probability $N(z_i)$	Cutoff z_i
AAA	0.02%	100.00%	
AA	0.33%	99.98%	3.54
A	5.95%	99.65%	2.70
BBB	86.93%	93.70%	1.53
BB	5.30%	6.77%	−1.49
B	1.17%	1.47%	−2.18
CCC	0.12%	0.30%	−2.75
Default	0.18%	0.18%	−2.91

equity indices. Each obligor is mapped to an industry and a geographical sector, using preassigned weights. Correlations are inferred from the comovements of the common risk factors, using a database with some 152 country-industry indices, 28 country indices, and 19 worldwide industry indices.

As an example, company 1 may be such that 90% of its volatility comes from the U.S. chemical industry. Using standardized returns, we can write

$$r_1 = 0.90\, r_{\text{US,Ch}} + k_1\epsilon_1$$

where the residual ϵ is uncorrelated with other variables. Because the total volatility is normalized to one, we must have $k_1 = \sqrt{1 - 0.9^2} = 0.44$.

Next, suppose that company 2 has a 74% weight on the German insurance index and 15% on the German banking index:

$$r_2 = 0.74\, r_{\text{GE,In}} + 0.15\, r_{\text{GE,Ba}} + k_2\epsilon_2$$

The correlation between asset values for the two companies is

$$\rho(r_1, r_2) = (0.90 \times 0.74)\rho(r_{\text{US,Ch}}, r_{\text{GE,In}}) + (0.90 \times 0.15)\rho(r_{\text{US,Ch}}, r_{\text{GE,Ba}})$$

$$\rho(r_1, r_2) = (0.90 \times 0.74)0.15 + (0.90 \times 0.15)0.08 = 0.11$$

CreditMetrics then uses simulations of the joint asset values, assuming a multivariate normal distribution with the prespecified correlations. Thus the approach relies on a normal copula. This gives a total value for the portfolio and a distribution of credit losses over an annual horizon.

These simulations can also be used to compute correlations among default events. Because defaults are much less common than rating changes, the correlation is typically much less than the correlation between asset values. CreditMetrics reports that asset correlations in the range of 40% to 60% will typically translate into default correlations of 2% to 4%.[1]

[1] This result, however, is driven by the joint normality assumption, which is not totally realistic. Other distributions can generate a greater likelihood of simultaneous defaults. This is obviously of major importance for credit portfolios.

Another drawback of this approach is that it does not integrate credit and market risk. Losses are generated only by changes in credit states, not by market movements. There is no uncertainty over market exposures. For swaps, for instance, the exposure on the target date is taken from the expected exposure. Bonds are revalued using today's forward rate and current credit spreads, applied to the credit rating on the horizon. So there is no interest rate risk.

24.4.3 CreditRisk+

CreditRisk+ was made public by Credit Suisse in October 1997. The approach is drastically different from CreditMetrics. It is based on a purely actuarial approach derived from the property insurance literature.

CreditRisk+ is a default mode (DM) model rather than a mark-to-market (MTM) model. Only two states of the world are considered—default and no-default.

The model starts with an assumption of a large number of identical loans n with independent default probability p. The total loss, $x = \sum_{i=1}^{n} b_i$, then follows a binomial distribution, which can be approximated by a Poisson distribution with intensity $\lambda = np$:

$$f(x) = e^{-\lambda} \frac{\lambda^x}{x!} \tag{24.12}$$

Default correlations are introduced by assuming that the intensity itself is random. A high draw in λ then increases the probability of default for each obligor. This default intensity can also be time-varying, in which case it is modeled as a function of factors that change over time. CreditRisk+ accounts for variability in default rates by dividing the portfolio into homogeneous sectors within which obligors share the same systematic risk factor.

The other component of the approach is the severity of losses. This is roughly modeled by sorting assets by severity bands, say loans around \$20,000 for the first band, \$40,000 for the second band, and so on. A distribution of losses is then obtained for each band. These distributions are combined across bands to generate an overall distribution of default losses.

The method provides a quick analytical solution to the distribution of credit losses with minimal data inputs. As with CreditMetrics, however, there is no uncertainty over market exposures.

24.4.4 Moody's KMV

Moody's KMV provides forecasts of estimated default frequencies (EDFs) for approximately 30,000 public firms globally.[2] Much of its technology is considered proprietary and is unpublished.

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[2] KMV was founded by S. Kealhofer, J. McQuown, and O. Vasicek (hence the abbreviation KMV) to provide credit risk services. KMV started as a private firm based in San Francisco in 1989 and was acquired by Moody's in April 2002.

The model is an application of the Merton approach, which views the firm's equity E as a call option on the firm's assets

$$E = c(A, K, r, \sigma_A, \tau) \tag{24.13}$$

In practice, KMV defines the floor K as the value of all short-term liabilities (one year and under) plus half the book value of all long-term debt. The value of assets is taken as the market value of equity plus the book value of all debt $A = nS + D$.

As seen in Chapter 21, this equation has to be iteratively estimated from observable variables, in particular the stock price S and its volatility σ_S to get the asset volatility. KMV computes a normalized **distance to default** (DD), which is essentially the distance between the current value of assets and the boundary point. Suppose, for instance, that $A = \$100$ million, $K = \$80$ million, and $\sigma_A = \$10$ million. We have

$$\text{DD} = z = \frac{A - K}{\sigma_A} = \frac{\$100 - \$80}{\$10} = 2 \tag{24.14}$$

The main drivers of DD are (1) the level of the stock price, (2) the amount of leverage, and (3) the volatility of asset value. Lower stock prices, higher leverage, and higher asset volatility will decrease the DD measure.

In the final step, KMV uses this information to report an estimated default frequency (EDF), or default probability. If we assume normally distributed returns, for example, the probability of a standard normal variate z falling below -2 is about PD = 2.3%. In practice, the EDFs are calibrated to actual default data, which gives objective (as opposed to risk-neutral) probabilities of default.

KMV generates default correlations between obligors directly from their equity prices, unlike CreditMetrics. First, returns on asset values are computed from changes in equity and debt values for the obligor. Second, these returns are regressed against a set of macroeconomic factors, country indices, and industry indices. Finally, this factor model is used to generate joint random variables representing obligor asset values, again using the industry standard normal copula.

The strength of this approach is that it relies on what is perhaps the best market data for a company—its stock price. Thus it works best for *public firms*. KMV also provides a model for *private companies*, which is based on accounting data for the firm and the industry as well as equity information, but only for public firms in the same industry. As expected, EDFs for private companies are considerably less accurate.

24.4.5 Credit Portfolio View

The last model we consider is **Credit Portfolio View** (CPV), published by the consulting firm McKinsey in 1997. The focus of this top-down model is on the effect of macroeconomic factors on portfolio credit risk.

This approach models loss distributions from the number and size of credits in subportfolios, typically consisting of customer segments. Instead of considering fixed transition probabilities, this model conditions the default probability on the state of the economy, allowing increases in defaults during recessions. The default probability p_t at time t is driven by a set of macroeconomic variables x^k for various countries and industries through a linear combination called y_t. The functional relationship to y_t, called the *logit model*, ensures that the probability is always between 0 and 1:

$$p_t = 1/[1 + \exp(y_t)], \qquad y_t = \alpha + \sum \beta^k x_t^k \tag{24.15}$$

Using a multifactor model, each debtor is assigned to a country, industry, and rating segment. Uncertainty in recovery rates is also factored in. The model uses numerical simulations to construct the distribution of default losses for the portfolio. While useful for modeling default probabilities conditioned on the state of the economy, this approach is mainly top-down and does not generate sufficient detail of credit risk for corporate portfolios.

24.4.6 Comparisons

The International Swaps and Derivatives Association (ISDA) conducted a comparative survey of credit risk models.[3] The empirical study consisted of three portfolios of one-year loans with a total notional of $66.3 billion each.

Portfolio A: High-credit-quality, diversified portfolio (500 names)

Portfolio B: High-credit-quality, concentrated portfolio (100 names)

Portfolio C: Low-credit-quality, diversified portfolio (500 names)

The models are listed in Table 24.5 and include CreditMetrics, CreditRisk+, and two internal models, all with a one-year horizon and 99% confidence level. Also reported are the charges from the Basel I standard rules, which are explained in Chapter 28. Suffice it to say that these rules make no allowance for variation in credit quality or diversification effects. Instead, the capital charge is based on 8% of the loan notional.

The top of the table examines the case of zero correlations. The Basel rules yield the same capital charge, irrespective of quality or diversification effects. The charge is also uniformly higher than most others, at $5,304 million, which is 8% of the notional.

Generally, the four credit portfolio models show remarkable consistency in capital charges. Portfolios A and B have the same credit quality, but B is more concentrated. Portfolio A has indeed lower credit VAR, approximately $800 million against $2,000 million for B. This reflects the benefit from greater diversification. Portfolios A and C have the same number of names, but C has lower credit quality. This increases credit VAR from around $800 million to $2,000 million.

The bottom panel assesses empirical correlations, which are typically positive. The Basel charges are unchanged, as expected, because they do not account for

[3] ISDA, *Credit Risk and Regulatory Capital* (New York: ISDA, 1998).

TABLE 24.5 Capital Charges from Various Credit Risk Models

	Assuming Zero Correlation		
	Portfolio A	**Portfolio B**	**Portfolio C**
CreditMetrics	777	2,093	1,989
CreditRisk+	789	2,020	2,074
Internal model 1	767	1,967	1,907
Internal model 2	724	1,906	1,756
Basel I rules	5,304	5,304	5,304
	Assessing Correlations		
	Portfolio A	**Portfolio B**	**Portfolio C**
CreditMetrics	2,264	2,941	11,436
CreditRisk+	1,638	2,574	10,000
Internal model 1	1,373	2,366	9,654
Basel I rules	5,304	5,304	5,304

correlations. Internal models show capital charges to be systematically higher than in the previous case. There is also more dispersion in results across models, however. It is interesting to see, in particular, that the economic capital charge for portfolio C, with low credit quality, is typically twice the Basel charge. Such results demonstrate that the Basel rules can lead to inappropriate credit risk charges. As a result, banks subject to these capital requirements may shift their risk profiles to lower-rated credits until their economic capital is in line with regulatory capital. This shift to lower credit quality was certainly not an objective of the original Basel rules. This lack of sensitivity is what led to the new Basel II Accord, which will be discussed in Chapter 28.

More recently, another study compared the capital charges from the new Basel II rules with three commercial models, taking great care to align parameters.[4] The base portfolio consists of $100 billion in loans to 3,000 obligors spread across different industries and countries, with an average credit rating of BBB.

Table 24.6 shows the results in default mode. The Basel II rules, under the advanced approach, require economic capital of about $3.3 billion, close to 4% of the notional. The three commercial models give remarkably close results. The report concludes that "if assumptions are aligned, there is not much difference between the valuation methods from PM and CM." Of course, this is also because

TABLE 24.6 Comparison of Credit Risk Models

	Expected Loss	Capital at 99.9%
KMV (PM)	563	3,791
CreditMetrics (CM)	562	3,533
CreditRisk+	564	3,662
Basel II	607	3,345

[4] IACPM and ISDA, *Convergence of Economic Capital Models* (New York: ISDA, 2006).

these models are based on the same joint density function, using a normal copula, and calibrated to the same historical data.

24.5 CONCLUSIONS

Portfolio credit risk models take market risk models one step further. Here again, the fundamental intuition is that diversification across obligors, regions, and industries should lower portfolio risk.

The problem with internal portfolio credit risk models, however, is their complexity. Unlike market risk, where the risk manager can observe a history of movements in risk factors, there is generally no history of default for a particular obligor. Hence, default probabilities have to be modeled indirectly. The problem is even more difficult for estimating default correlations, as well as the shape of the joint density of defaults. Prior to 2007, these models had not been tested over a full market cycle, which should include a recession. This is particularly important because downturns lead to higher default probabilities, higher default correlations, and lower recovery rates.[5]

Finally, it will be difficult to verify such models given that they generate measures of economic capital over a long horizon, typically one year, and at a very high confidence level, typically 99.9%. In contrast, market risk models produce a daily VAR at the 99% confidence level, which should produce on average two or three exceptions a year. Thus, backtests of credit risk VAR estimates are not feasible, unlike for market risk.

This explains why regulators had considerable doubts about the precision of these models and, as a result, did not allow commercial banks to use their internal portfolio models as the basis for their credit risk charge. Indeed, the losses suffered during the recession that started in 2007 have been much greater than the supposed worst-case scenarios. Even though considerable progress has been made in our understanding of credit risk, it is fair to conclude that much more work is needed to develop robust models.

EXAMPLE 24.8: FRM EXAM 2004—QUESTION 11

When determining the standard deviation of value due to credit quality changes for a single exposure, the CreditMetrics model uses three primary factors. Which of the following is *not* one of the factors used in this model?

a. Credit ratings
b. Seniority
c. Equity prices
d. Credit spreads

[5] The academic literature has long emphasized the sensitivity of credit risk models to correlations between defaults. See S. Das, D. Duffie, N. Kapadia, and L. Saita, "Common Failings: How Corporate Defaults are Correlated," *Journal of Finance,* 2007; P. Jorion and G. Zhang, "Credit Contagion from Counterparty Risk," *Journal of Finance,* 2009.

EXAMPLE 24.9: FRM EXAM 2002—QUESTION 129

A bank computes the distribution of its loan portfolio marked-to-market value one year from now using the CreditMetrics approach of computing values for rating transition outcomes using a rating agency transition matrix, current forward curves, and correlations among rating transition outcomes derived from stock returns of the obligors. In computing firmwide risk using this distribution of its loan portfolio, the bank is most likely to understate its risk because it ignores

a. The term structure of interest rates
b. Rating drift
c. Spread risk
d. The negative correlation between the Treasury rates and credit spreads

EXAMPLE 24.10: FRM EXAM 2003—QUESTION 92

KMV measures the normalized distance from default. How is this defined?

a. (Expected assets − Weighted debt)/(Volatility of assets)
b. Equity/(Volatility of equity)
c. Probability of stock price falling below a threshold
d. Leverage times stock price volatility

EXAMPLE 24.11: FRM EXAM 2004—QUESTION 20

A firm's assets are currently valued at $500 million and its current liabilities are $300 million. The standard deviation of asset values is $80 million. The firm has no other debt. What will be the approximate distance to default using the KMV calculation?

a. 2 standard deviations
b. 2.5 standard deviations
c. 6.25 standard deviations
d. Cannot be determined

EXAMPLE 24.12: FRM EXAM 2007—QUESTION 59

You are given the following information about a firm. The market value of assets at time 0 is 1,000; at time 1 is 1,200. Short-term debt is 500; long-term debt is 300. The annualized asset volatility is 10%. According to the KMV model, what are the default point and the distance to default at time 1?

a. 800 and 3.33
b. 650 and 7.50
c. 650 and 4.58
d. 500 and 5.83

EXAMPLE 24.13: FRM EXAM 2005—QUESTION 36

Which of the following credit risk models uses the option-theoretic approach for modeling correlation between the credit-risky assets?

a. CreditRisk+
b. CreditMetrics
c. KMV for public firms
d. Both CreditMetrics and KMV for public firms

EXAMPLE 24.14: FRM EXAM 2009—QUESTION 6-10

Which of the following statements correctly applies to the KMV model, CreditMetrics, and CreditRisk+ together?

a. In their original implementations these models do not take into account changes in interest rates or credit spreads.
b. All three models allow for changes in default probability only when ratings change, rather than continuously.
c. It is impossible to compute a VAR measure using these models.
d. Credit migrations from one ratings class to another are ignored by these models.

24.6 IMPORTANT FORMULAS

Credit loss: $\sum_{i=1}^{N} b_i \times \text{CE}_i \times \text{LGD}_i$

Expected credit loss: $\text{ECL} = \text{Prob[default]} \times E[\text{CE}] \times E[\text{LGD}]$

Present value of expected credit losses (PVECL):
$\text{PVECL} = \sum_t E[\text{CL}_t] \times \text{PV}_t = \sum_t [k_t \times \text{ECE}_t \times (1 - f)] \times \text{PV}_t$

Approximation to PVECL: $\text{PVECL}_F = c_T \times \text{ECE} \times (1 - f) \times \text{PV}_T$

Credit VAR: $\text{CVAR} = \text{WCL} - \text{ECL}$

KMV's normalized distance from default: $z = (A - K)/\sigma_A$

Credit portfolio view's default probability: $p_t = 1/[1 + \exp(y_t)]$, $y_t = \alpha + \sum \beta^k x_t^k$

24.7 ANSWERS TO CHAPTER EXAMPLES

Example 24.1: Credit Provisions

c. Credit provisions should be made for actual and expected losses. Capital, however, is supposed to provide a cushion against unexpected losses based on VAR.

Example 24.2: FRM Exam 2002—Question 74

b. This is an example of right-way trade. To have lower credit risk, it would be preferable to engage in a trade where there is a lower probability of a default by the counterparty when the contract is in-the-money. This will happen if the counterparty enters a transaction to hedge an operating exposure. For instance, a gas producer has a natural operating exposure to gas. If the producer sells gas at a fixed price, the swap will lose money if the market price of gas goes up. In this situation, however, there is little risk of default because the producer is sitting on an inventory of gas. A trading firm or distributor could go bankrupt if the transaction loses money.

Example 24.3: FRM Exam 2003—Question 26

a. The one-year PD needs to be adjusted to the maturity of the loan, using $(1 - d^m)^T$, where d^m is computed from $(1 - d^m)^{12} = (1 - d)$.

Loan	PD to Maturity	Loss Given Default	Expected Loss
a.	0.50%	60%	0.301%
b.	0.68%	70%	0.473%
c.	0.50%	75%	0.376%
d.	0.75%	50%	0.375%

Example 24.4: FRM Exam 2007—Question 38

b. The survival rate over two years is $S_2 = (1 - 4\%)^2 = 92.16\%$, which implies a cumulative two-year default rate of 7.84%. Put differently, the first-year PD is 4%, then $(1 - 4\%)4\% = 3.84\%$. Multiplying by 200 and 40% gives 6.27.

Example 24.5: Credit VAR for One Bond

c. First, we have to transform the annual default probability into a monthly probability. Using $(1 - 2\%) = (1 - d)^{12}$, we find $d = 0.00168$, which assumes a constant probability of default during the year. Next, we compute the expected credit loss, which is $d \times \$1{,}000{,}000 = \$1{,}682$. Finally, we calculate the WCL at the 99.9% confidence level, which is the lowest number CL_i such that $P(CL \leq CL_i) \geq 99.9\%$. We have $P(CL = 0) = 99.83\%$; $P(CL \leq 1{,}000{,}000) = 100.00\%$. Therefore, the WCL is \$1,000,000, and the CVAR is \$1,000,000 − \$1,682 = \$998,318.

Example 24.6: Credit VAR for Two Bonds

d. As in the previous question, the monthly default probability is 0.00168. The following table shows the distribution of credit losses.

Default	Probability (p_i)	Loss L_i	$p_i L_i$	$1 - \sum p_i$
2 bonds	$d^2 = 0.00000282$	\$1,000,000	\$2.8	100.00000%
1 bond	$2d(1 - d) = 0.00335862$	\$500,000	\$1,679.3	99.99972%
0 bonds	$(1 - d)^2 = 0.99663854$	\$0	\$0.0	99.66385%
Total	1.00000000		\$1,682.1	

This gives an expected loss of \$1,682, the same as before. Next, \$500,000 is the WCL at a minimum 99.9% confidence level because the total probability of observing a number equal to or lower than this is greater than 99.9%. The credit VAR is then \$500,000 − \$1,682 = \$498,318.

Example 24.7: FRM Exam 2005—Question 122

c. The credit VAR could be zero. For instance, assume that the PD is 0.003. The joint probability of no default is then $(1 - 0.003)(1 - 0.003) = 99.4\%$. Because this is greater than the 99% confidence level, the worst loss is zero. The expected loss, however, would be 0.3% assuming zero recovery, which is greater than VAR.

Example 24.8: FRM Exam 2004—Question 11

c. CreditMetrics uses credit ratings, the transition matrix, recovery rates, and LGD for various seniorities, but not equity prices for the obligor.

Example 24.9: FRM Exam 2002—Question 129

c. CreditMetrics ignores spread risk. It does account for rating drift and the term structure of interest rates, albeit not their volatility.

Example 24.10: FRM Exam 2003—Question 92

a. The distance-to-default measure is a standardized variable that measures how much the value of firm assets exceeds the liabilities.

Example 24.11: FRM Exam 2004—Question 20

b. Using Equation (24.14), the DD is $(500 - 300)/80 = 2.5$ standard deviations.

Example 24.12: FRM Exam 2007—Question 59

c. The default point is given by short-term liabilities plus half of long-term liabilities, which is $500 + 300/2 = 650$. The distance to default at point 1 is $(V - K)/\sigma_V = (1{,}200 - 650)/(1{,}200 \times 0.10) = 4.58$.

Example 24.13: FRM Exam 2005—Question 36

c. KMV estimates default probabilities using the Merton approach based on the company's stock price.

Example 24.14: FRM Exam 2009—Question 6-10

a. None of the models take into account changes in risk-free rates nor spreads, so answer a. is correct. Answer b. is incorrect, because the KMV model bases estimates of PD on the stock price, which moves continuously. Answer c. is incorrect, because the main purpose of all of these models is to estimate credit VAR measures. Answer d. is incorrect, for example, because CreditMetrics is based on credit ratings.

PART Seven

Operational and Integrated Risk Management

CHAPTER 25

Operational Risk

The financial industry has developed standard methods to measure and manage market and credit risks. The industry is turning next to operational risk, which has proved to be an important cause of financial losses. Indeed, most company-specific financial disasters can be attributed to a combination of market and credit risk along with some failure of controls, which is a form of operational risk.

As in the case of market and credit risk, the financial industry is being pushed in the direction of better control of operational risk by bank regulators. For the first time, the Basel Committee on Banking Supervision has established capital charges for operational risk, in exchange for lowering them on market and credit risk. This new charge would constitute approximately 12% of the total capital requirement.[1] As a result, this is forcing the banking industry to pay close attention to operational risk.

As with market and credit risk, the management of operational risk follows a sequence of logical steps: (1) identification, (2) assessment, (3) monitoring, and (4) control or mitigation.

Historically, operational risk has been managed by internal control mechanisms within business lines, supplemented by the audit function. The industry is now starting to use specific structures and control processes specifically tailored to operational risk.

To introduce operational risk, Section 25.1 summarizes lessons from well-known financial disasters. Given this information, Section 25.2 turns to definitions of operational risk. Various measurement approaches are discussed in Section 25.3. Section 25.4 shows how to use the distribution of operational losses to manage this risk better and offers some comments on conceptual problems. Finally, Section 25.5 explains the operational risk charge (ORC) established in 2004 by the Basel Committee.

25.1 IMPORTANCE OF OPERATIONAL RISK

The Basel Committee recently reported that "[a]n informal survey . . . highlights the growing realization of the significance of risks other than credit and market risks,

FRM Exam Part 2 topic.

[1] See Basel Committee on Banking Supervision, *Sound Practices for the Management and Supervision of Operational Risk* (Basel: BIS, 2003).

such as operational risk, which have been at the heart of some important banking problems in recent years." These problems are described in case histories next.

25.1.1 Case Histories

- *January 2008—SocGen (4.9 billion euros loss).* A trader, Jerome Kerviel, systematically deceives systems, taking unauthorized positions worth up to 49 billion euros in stock index futures. The bank has enough capital to absorb the loss but its reputation is damaged.
- *February 2002—Allied Irish Bank ($691 million loss).* A rogue trader, John Rusnack, hides three years of losing trades on the yen/dollar exchange rate at the U.S. subsidiary. The bank's reputation is damaged.
- *March 1997—NatWest ($127 million loss).* A swaption trader, Kyriacos Papouis, deliberately covers up losses by mispricing and overvaluing option contracts. The bank's reputation is damaged. NatWest is eventually taken over by the Royal Bank of Scotland.
- *September 1996—Morgan Grenfell Asset Management ($720 million loss).* A fund manager, Peter Young, exceeds his guidelines, leading to a large loss. Deutsche Bank, the German owner of MGAM, agrees to compensate the investors in the fund.
- *June 1996—Sumitomo ($2.6 billion loss).* A copper trader amasses unreported losses over three years. Yasuo Hamanaka, known as "Mr. Five Percent," after the proportion of the copper market he controlled, is sentenced to prison for forgery and fraud. The bank's reputation is severely damaged.
- *September 1995—Daiwa ($1.1 billion loss).* A bond trader, Toshihide Igushi, amasses unreported losses over 11 years at the U.S. subsidiary. The bank is declared insolvent.
- *February 1995—Barings ($1.3 billion loss).* Nick Leeson, a derivatives trader, amasses unreported losses over two years. Barings goes bankrupt.
- *October 1994—Bankers Trust ($150 million loss).* The bank becomes embroiled in a high-profile lawsuit with a customer that accuses it of improper selling practices. Bankers settles, but its reputation is badly damaged. It is later bought out by Deutsche Bank.

Many of these spectacular losses can be traced to a **rogue trader**, or a case of internal fraud. These failures involve a mix of market risk and operational risk (i.e., failure to supervise properly).

At times, the cost of these events has been quite high. They led to large direct monetary losses, sometimes even to bankruptcy. In addition to these direct costs, banks often suffer large indirect losses due to reputational damage. For instance, Perry and de Fontnouvelle (2005) find that, upon the announcement of an operational loss, equity market values fall one-for-one with losses when due to external events.[2] When losses are due to internal fraud, however, stock prices

[2] J. Perry and P. de Fontnouvelle, *Measuring Operational Risk: The Market Reaction to Operational Loss Announcements* (Boston: Federal Reserve Bank of Boston, 2005).

fall even more, because this is symptomatic of poor controls that could lead for further losses.

25.1.2 Implications

These failures have occurred across a variety of business lines. Some are more exposed than others to market risk or credit risk. All have some exposure to operational risk, however.

Commercial banks are exposed mainly to credit risk, less so to operational risk, and least to market risk. **Investment banks**, **proprietary trading**, and **treasury management** have greater exposure to market risk. In contrast, business lines such as retail brokerage and asset management are exposed primarily to operational risk. **Asset managers** assume no direct market risk since they act as agents for the investors. If they act in breach of guidelines, however, they may be liable to clients for their losses, which represents operational risk.

As a result, financial institutions have now put in place a formalized structure to assess and measure operational risk. In particular, they now attempt to measure the economic capital (EC) required to cover operational risk. The amounts are not small, which reflects the importance of this risk type. For instance, as of 2009, JPMorgan Chase estimated it needed \$8.5 billion to cover operational risk, or 11% of its total risk. Deutsche Bank's estimate was €3.5 billion, or 17% of its total risk. As will be seen in Chapter 28, this measurement effort has also been spurred by the Basel Committee, which now imposes a capital charge against operational risk. This is estimated at 12% of total capital, which is in line with the examples given here.

25.2 IDENTIFYING OPERATIONAL RISK

One could argue that operational risk has no clear-cut definition, unlike market risk and credit risk. There was a long debate as to the proper definition of operational risk, or even whether it makes sense to attempt to measure it.

After much industry consultation, the Basel Committee has settled on a definition that is becoming an industry standard. Operational risk is defined as

> *the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events.*

This includes the usual internal business events but also external events such as external fraud, security breaches, regulatory effects, or natural disasters. It includes legal risk, which arises when a transaction proves unenforceable in law. This definition, however, excludes strategic and reputational risk, which would be very difficult to measure anyway.

The British Bankers' Association provides further detail. Table 25.1 breaks down operational risk into categories of **people risk**, **processes risk**, **systems risk**, and **external risk**. Among these, a notable risk for complex products is **model risk**,

TABLE 25.1 Operational Risk Classification

Internal Risks		
People	**Processes**	**Systems**
Employee collusion/fraud	Accounting error	Data quality
Employee error	Capacity risk	Programming error
Employee misdeed	Contract risk	Security breach
Employer liability	Misselling/unsuitability	Strategic risk (platform/supplier)
Employment law	Product complexity	System capacity
Health and safety	Project risk	System compatibility
Industrial action	Reporting error	System delivery
Lack of knowledge/skills	Settlement/payment error	System failure
Loss of key personnel	Transaction error	System unsuitability
	Valuation error	

External Risks	
External	**Physical**
Legal	Fire
Money laundering	Natural disaster
Outsourcing	Physical security
Political	Terrorism
Regulatory	Theft
Supplier risk	
Tax	

Source: British Bankers' Association survey.

which is due to the use of wrong models for valuation and risk management. This is an internal risk that combines lack of knowledge (people) with product complexity/valuation errors (processes) and perhaps programming errors (systems).

The Basel Committee has further classified risk events according to seven event types.

1. **Internal fraud** (IF): Events intended to defraud, misappropriate property, or circumvent regulations or company policy, involving at least one internal party, categorized into unauthorized activity and internal theft and fraud.
2. **External fraud** (EF): Events intended to defraud, misappropriate property, or circumvent the law, by a third party, categorized into theft, fraud, and breach of system security.
3. **Employment practices and workplace safety** (EPWS): Acts inconsistent with employment, health, or safety laws or agreements, categorized into employee relations, safety of the environment, and diversity and discrimination.
4. **Clients, products, and business practices** (CPBP): Events due to failures to comply with a professional obligation to clients, or arising from the nature or design of a product, including disclosure and fiduciary, improper business and market practices, product flaws, and advisory activities.
5. **Damage to physical assets** (DPA): Events leading to loss or damage to physical assets from natural disasters or other events such as terrorism.

6. **Business disruption and system failures** (BDSF): Events causing disruption of business or system failures.
7. **Execution, delivery, and process management** (EDPM): Events due to failed transaction processing or process management that occur from relations with trade counterparties and vendors, classified into categories such as transaction execution and maintenance, customer intake and documentation, and account management.

The Basel II Accord also classifies the loss events according to eight business lines: (1) corporate finance, (2) trading and sales, (3) retail banking, (4) commercial banking, (5) payment and settlement, (6) agency services and custody, (7) asset management, and (8) retail brokerage.

These definitions have established industry standards for classifying operational risk events. These risk events are now collected, internally and externally, according to a **matrix classification** (i.e., by event type and business line). This facilitates the collection of operational losses in public databases. One example is the **Operational Riskdata eXchange Association** (ORX), which provides a platform for the anonymized exchange of operational risk loss data, which is collected in the ORX Global Loss Database.

EXAMPLE 25.1: FRM EXAM 2004—QUESTION 39

Which of the following is *not* a type of operational risk as defined by Basel II?

- **a.** Human error and internal fraud
- **b.** Destruction by fire or other external catastrophes
- **c.** Damaged reputation due to a failed merger
- **d.** Failure or breakdown in internal control processes

EXAMPLE 25.2: FRM EXAM 2003—QUESTION 65

Which of these outcomes is *not* associated with an operational risk process?

- **a.** The sale of call options is booked as a purchase.
- **b.** A monthly volatility is inputted in a model that requires a daily volatility.
- **c.** A loss is incurred on an option portfolio because *ex post* volatility exceeded expected volatility.
- **d.** A volatility estimate is based on a time series that includes a price that exceeds the other prices by a factor of 100.

EXAMPLE 25.3: FRM EXAM 2007—QUESTION 56

All the following are operational risk loss events, *except*:

a. An individual shows up at a branch presenting a check written by a customer for an amount substantially exceeding the customer's low checking account balance. When the bank calls the customer to ask him for the funds, the phone is disconnected and the bank cannot recover the funds.
b. A bank, acting as a trustee for a loan pool, receives less than the projected funds due to delayed repayment of certain loans.
c. During an adverse market movement, the computer network system becomes overwhelmed, and only intermittent pricing information is available to the bank's trading desk, leading to large losses as traders become unable to alter their hedges in response to falling prices.
d. A loan officer inaccurately enters client financial information into the bank's proprietary credit risk model.

EXAMPLE 25.4: FRM EXAM 2007—QUESTION 139

The risk of the occurrence of a significant difference between the mark-to-model value of a complex instrument and the price at which the same instrument is revealed to have traded in the market is referred to as:

a. Liquidity risk
b. Dynamic risk
c. Model risk
d. Mark-to-market risk

25.3 ASSESSING OPERATIONAL RISK

Once identified, operational risk should be measured, or rather assessed if it is less amenable to precise quantification than market or credit risk. Various approaches can be broadly classified into top-down models and bottom-up models.

25.3.1 Comparison of Approaches

Top-down models attempt to measure operational risk at the broadest level, that is, using firmwide or industry-wide data. Results are then used to determine the amount of capital that needs to be set aside as a buffer against this risk. This capital is allocated to business units.

Bottom-up models start at the individual business unit or process level. The results are then aggregated to determine the risk profile of the institution. The main benefit of bottom-up models is that they lead to a better understanding of the causes of operational losses, as in the case of value at risk (VAR)-based market risk systems.

Tools used to manage operational risk can be classified into six categories:

1. **Audit oversight.** This consists of reviews of business processes by an external audit department.
2. **Critical self-assessment.** Each business unit identifies the nature and degree of operational risk. These *subjective* evaluations include expected frequency and severity of losses, as well as a description of how risk is controlled. The tools used for this type of process include checklists, questionnaires, and facilitated workshops. The results are then aggregated, in a bottom-up approach.
3. **Key risk indicators.** This approach consist of simple measures that provide an indication of whether risks are changing over time. These *early warning signs* can include audit scores, staff turnover, trade volumes, and so on. The assumption is that operational risk events are more likely to occur when these indicators increase. These *objective* measures allow the risk manager to forecast losses through the application of regression techniques, for example.
4. **Earnings volatility.** This approach consists of taking a time series of earnings, after stripping the effect of market and credit risk, and computing its volatility. This risk measure is simple to use but has numerous problems. The measure also includes fluctuations due to business and macroeconomic risks, which fall outside of operational risk. Also, such a measure is backward-looking and does not account for improvement or degradation in the quality of controls.
5. **Causal networks.** Networks describe how losses can occur from a cascade of different causes. Causes and effects are linked through conditional probabilities. Simulations are then run on the network, generating a distribution of losses. Such bottom-up models improve the understanding of losses since they focus on drivers of risk. The process is explained in the appendix at the end of this chapter. Causal networks are best applied to processes involving complex **work flows** with many activities.
6. **Actuarial models.** These models combine the distribution of frequency of losses with their severity distribution to produce an *objective* distribution of losses due to operational risk. These can be either bottom-up or top-down models.

25.3.2 Actuarial Models: Loss Distribution Approach (LDA)

Actuarial models estimate the objective distribution of losses from historical data and are widely used in the insurance industry. Such models combine two distributions: loss frequencies and loss severities. The **loss frequency distribution** describes the number of loss events over a fixed interval of time. The **loss severity**

distribution describes the size of the loss once it occurs. This is called the **loss distribution approach** (LDA).

Loss severities can be tabulated from historical data—for instance, measures of the loss severity y_k, at time k. These measures can be adjusted for inflation and some measure of current business activity. Define P_k as the consumer price index at time k and V_k as a business activity measure such as the number of trades. We could assume that the severity is proportional to the volume of business V and to the price level. The *scaled* loss is measured as of time t as

$$x_t = y_k \times \frac{P_t}{P_k} \times \frac{V_t}{V_k} \tag{25.1}$$

Loss severity distributions have very long tails, representing the possibility of very large losses. Ideally, they should include internal and external data.

Internal data represent the actual control environment of the institution. However, the data may not be available in sufficient quantities. The data also suffer from a **survivorship bias** because the bank is still alive. As a result, the data will not have enough tail events, which are essential for modeling extreme losses.

To alleviate this problem, regulators require the use of **external data**. These have drawbacks as well. The scale and control systems of other banks may not provide a good match to the bank using the data. In addition, the reporting of operational losses to an external database can be incomplete because some banks may not like to reveal the extent of their weaknesses. Also, large losses are more difficult to hide. This creates a **data capture bias**. Another drawback is that databases only record losses beyond some minimum level, which can be as high as \$1 million. This creates a **truncation bias** in the frequency and severity distribution.

Finally, these data sources can be complemented by scenarios, which represent plausible and typically large losses that may not appear in internal or external data.

Next, define the loss frequency distribution by the variable n, which represents the number of occurrences of losses over the period. The density function is

$$\text{p.d.f. of loss frequency} = f(n), \quad n = 0, 1, 2, \ldots \tag{25.2}$$

This can be described by several analytical densities, such as the binomial, Poisson, negative binomial, or geometric, all of which require n to be integer and positive. If x (or X) is the loss severity when a loss occurs, its density is

$$\text{p.d.f. of loss severity} = g(x \mid n = 1), \quad x \geq 0 \tag{25.3}$$

This can be described by densities such as the lognormal, Weibull, gamma, or exponential distributions, all of which require x to be positive. The most common combination is Poisson and lognormaly.

Finally, the total loss over the period is given by the sum of individual losses over a random number of occurrences:

$$S_n = \sum_{i=1}^{n} X_i \tag{25.4}$$

Table 25.2 provides a simple example of two such distributions. Our task is now to combine these two distributions into one—that of total losses over the period.

Assuming that the frequency and severity of losses are independent, the two distributions can be combined into a distribution of aggregate loss through a process known as convolution. **Convolution** can be implemented, for instance, through tabulation. **Tabulation** consists of systematically recording all possible combinations with their associated probabilities and is illustrated in Table 25.3. Generally, convolution must be implemented by numerical methods, as there are too many combinations of variables for a systematic tabulation.

We start with the obvious case, no loss, which has probability 0.6. Next, we go through all possible realizations of one loss only. From Table 25.3, we see that a loss of \$1,000 can occur with total probability of $P(n = 1) \times P(x = \$1{,}000) = 0.3 \times 0.5 = 0.15$. Similarly, for one-time losses of \$10,000 and \$100,000, the probabilities are 0.09 and 0.06, respectively. We then go through all occurrences of two losses, which can result from many different combinations. For instance, a loss of \$1,000 can occur twice, for a total of \$2,000, with a probability of $0.1 \times 0.5 \times 0.5 = 0.025$. We can have a loss of \$1,000 and \$10,000, for a total of \$11,000, with probability $0.1 \times 0.5 \times 0.3 = 0.015$. We repeat these steps until we exhaust all combinations.

The resulting distribution is displayed in Figure 25.1, and in the lower panel of Table 25.3. As usual for operational risk, losses are recorded as positive values. It is interesting to note that the very simple distributions in Table 25.2, with only three realizations, create a complex loss distribution. We can compute the expected loss, which is simply the product of expected values for the two distributions, or $E[S] = E[N] \times E[X] = 0.5 \times \$23{,}500 = \$11{,}750$. Risk management, however, is about the unexpected. So, the risk manager should also report the lowest number such that the probability is greater than 95%. This is \$100,000, with a probability of 96.4%. Hence the unexpected loss is $\$100{,}000 - \$11{,}750 = \$88{,}250$. If **operational VAR** must include the expected loss, this is simply \$100,000. This is the default measure under Basel II, which measures VAR at the 99.9 percent level of confidence over one year.

TABLE 25.2 Sample Loss Frequency and Severity Distributions

Frequency Distribution		Severity Distribution	
Probability	Frequency	Probability	Severity
0.6	0	0.5	\$1,000
0.3	1	0.3	\$10,000
0.1	2	0.2	\$100,000
Expectation	0.5	Expectation	\$23,500

TABLE 25.3 Tabulation of Loss Distribution

Number of Losses	First Loss ($)	Second Loss ($)	Total Loss ($)	Probability
0	0	0	0	0.600
1	1,000	0	1,000	0.150
1	10,000	0	10,000	0.090
1	100,000	0	100,000	0.060
2	1,000	1,000	2,000	0.025
2	1,000	10,000	11,000	0.015
2	1,000	100,000	101,000	0.010
2	10,000	1,000	11,000	0.015
2	10,000	10,000	20,000	0.009
2	10,000	100,000	110,000	0.006
2	100,000	1,000	101,000	0.010
2	100,000	10,000	110,000	0.006
2	100,000	100,000	200,000	0.004

Sorted Losses	Cumulative Probability
0	60.0%
1,000	75.0%
2,000	77.5%
10,000	86.5%
11,000	89.5%
20,000	90.4%
100,000	96.4%
101,000	98.4%
110,000	99.6%
200,000	100.0%

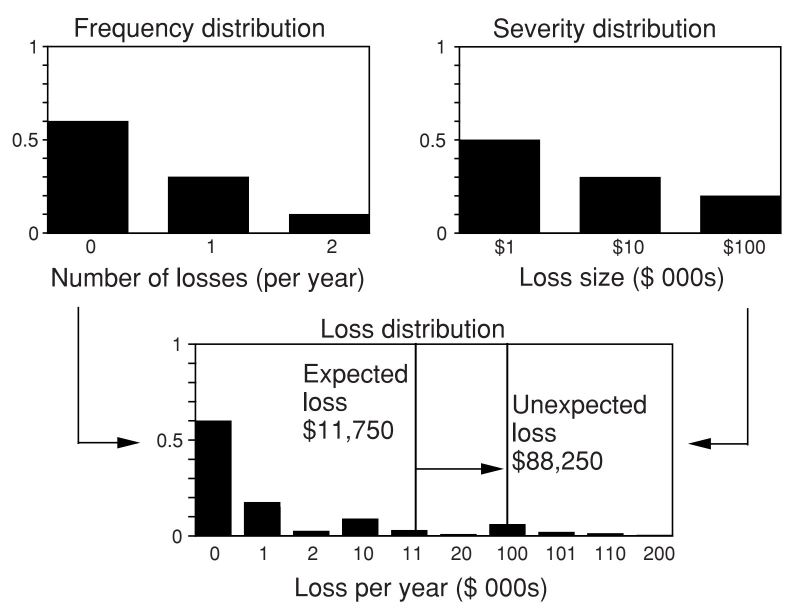

FIGURE 25.1 Construction of the Loss Distribution

25.3.3 LDA Implementation

This section now illustrates the implementation of the LDA for Deutsche Bank.[3] Table 25.4 presents the cells in the event type/business line matrix as defined by the bank. The bank groups the Basel categories for which it estimates that the loss distributions are similar. For instance, there is only one category for all infrastructure event types. This gives a total of 23 cells.

The next step is to model the frequency and loss distribution for each cell. The bank uses the frequency distribution from internal data for each cell, to which is fitted a Poisson distribution. Apparently, the final loss distribution is not too sensitive to the choice of this distribution. The modeling of the severity distribution is more difficult. It uses a combination of empirical data for the body and extreme value theory (EVT) for the tail, above €50 million. The distribution is fitted to a combination of internal data, external data, and scenarios.

The bank then needs to specify dependencies across events. Within cells, these can arise potentially between occurrences of losses, between severities, and between the frequency and severity. Across cells, dependencies can arise between the frequency distributions and between the severity distributions. Within cells, the bank ignores dependencies, which apparently is a good approximation. Instead, dependencies are modeled across cells between the frequency distributions. This simplifies the computations and seems consistent with the data. Dependencies between frequency distributions are modeled using a normal copula. In practice, correlations are positive but not very high. The highest correlations are found between the cells in EDPM (i.e., cells 16 to 22).

Losses from all these distributions are then generated via Monte Carlo simulations, which create the distribution of losses over the next year. The distribution can be summarized by a VAR measure, which was indeed €3.5 billion as of 2009 at the 99.98% level of confidence.

This risk measure can be split into a VAR contribution for each business line, which can be used for capital allocation purposes. Finally, changes in the

TABLE 25.4 Event Type/Business Line Matrix

Event Type		Business Line						
Basel	**Deutsche Bank**	**1**	**2**	**3**	**4**	**5**	**6**	**Group**
Internal fraud External fraud	Fraud	1	2	3	4	5	6	7
Damage to physical assets Business disruption	Infrastructure				8			
Clients, products, . . .	CPBP	9	10	11	12	13	14	15
Execution, delivery, . . .	EDPM	16	17	18	19	20	21	22
Employment practices, . . .	EPWS				23			

[3] F. Aue and M. Kalkbrener, *LDA at Work* (London: Deutsche Bank, 2007).

business and control environment are implemented by directly modifying the allocated risk measure. The bank has chosen this approach for simplicity and transparency.

EXAMPLE 25.5: FRM EXAM 2009—QUESTION 7-2

Gérard Kuper is modeling the number of operational risk loss events that could adversely impact Bank ABC in 2010. He expects the number of operational risk loss events for the year to be relatively small. Which type of distribution is he *least* likely to use?

a. Normal distribution
b. Binomial distribution
c. Negative binomial distribution
d. Poisson distribution

EXAMPLE 25.6: FRM EXAM 2007—QUESTION 138

The severity distribution of operational losses usually has the following shape:

a. Symmetrical with short tails
b. Long-tailed to the right
c. Uniform
d. Symmetrical with long tails

EXAMPLE 25.7: FRM EXAM 2008—QUESTION 4-10

Randy Bartell has collected operational loss data to calibrate frequency and severity distributions. Generally, he regards all data points as a sample from an underlying distribution and therefore gives each data point the same weight or probability in the statistical analysis. However, external loss data is inherently biased. Which of the following is *not* typically associated with external data?

a. Data capture bias
b. Scale bias
c. Truncation bias
d. Survivorship bias

EXAMPLE 25.8: FRM EXAM 2007—QUESTION 33

Suppose you are given the following information about the operational risk losses at your bank. What is the estimate of the VAR at the 95% confidence level, including expected loss (EL)?

Frequency Distribution	
Probability	Number
0.5	0
0.3	1
0.2	2

Severity Distribution	
Probability	Loss
0.6	USD 1,000
0.3	USD 10,000
0.1	USD 100,000

a. USD 100,000
b. USD 101,000
c. USD 200,000
d. USD 110,000

25.4 MANAGING OPERATIONAL RISK

25.4.1 Capital Allocation and Insurance

Like market VAR, the distribution of operational losses can be used to estimate expected losses as well as the amount of capital required to support this financial risk. Figure 25.2 highlights important attributes of a distribution of losses due to operational risk.

The **expected loss** (EL) represents the size of operational losses that should be expected to occur. Typically, this is dominated by high-frequency, low-severity

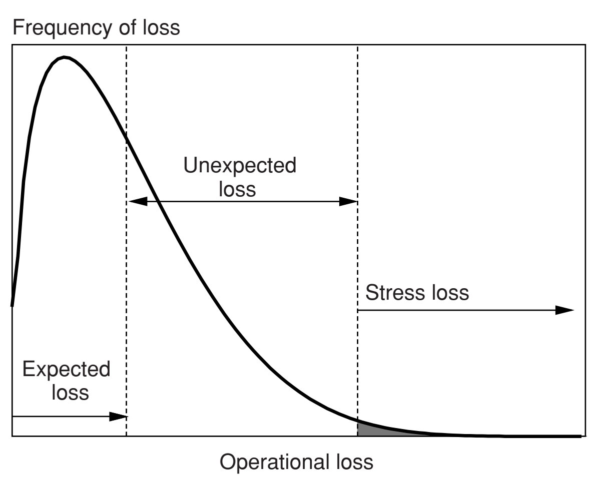

FIGURE 25.2 Distribution of Operational Losses

events. This type of loss is generally absorbed as an ongoing cost and managed through internal controls. Such losses are rarely disclosed.

The **unexpected loss** (UL) represents the deviation between the quantile loss at some confidence level and the expected loss. Typically, this represents lower-frequency, higher-severity events. This type of loss is generally offset against capital reserves or transferred to an outside insurance company, when available. The LDA approach can be easily modified to take into account the risk-mitigating effect of insurance.

The **stress loss** represents a loss in excess of the unexpected loss. By definition, such losses are very infrequent but extremely damaging to the institution. The Barings bankruptcy can be attributed, for instance, in large part to operational risk. This type of loss cannot be easily offset through capital allocation, as it would require too much capital. Ideally, it should be transferred to an insurance company. Due to their severity, such losses are disclosed publicly.

However, purchasing insurance is no panacea. The insurance payment would have to be made very quickly and in full. The bank could fail while waiting for payment or arguing over the size of compensation. In addition, the premium may be very high. This is because once the insurance is acquired, the purchaser has less incentive to control losses. This problem is called **moral hazard**. The insurer will be aware of this and will increase the premium accordingly. The premium may also be high because of the **adverse selection** problem. This describes a situation where banks vary in the quality of their controls. Banks with poor controls are more likely to purchase insurance than banks with good controls. Because the insurance company does not know what type of bank it is dealing with, it will increase the average premium. Insurance with a **deductible** amount (i.e., where the bank would have to share the first layer of losses) provides only a partial solution to these problems.

25.4.2 Mitigating Operational Risk

The approach so far has been to take operational risk as given and to try to measure it. This information is extremely useful because it highlights the extent of losses due to operational risk. Armed with this information, the institution can then decide whether it is worth spending resources on decreasing operational risk.

Say that a bank is wondering whether to install a **straight-through processing** system, which automatically captures trades in the front office and transmits them to the back office. Such a system eliminates manual intervention and the potential for human errors, thereby decreasing losses due to operational risk. The bank should purchase the system if its cost is less than its operational risk benefit.

More generally, reduction of operational risk can occur in terms of the frequency of losses and/or the size of losses when they occur. Operational risk is also contained by a firmwide risk management framework, which is covered in Chapter 27.

Consider, for instance, a transaction in a plain-vanilla, five-year interest rate swap. This simple instrument generates a large number of cash flows, each of which has the potential for errors. At initiation, the trade needs to be booked and confirmed with the counterparty. It must be valued so that a profit and loss (P&L) can be attributed to the trading unit. With biannual payments, the swap will generate 10 cash flows along with 10 rate resets and net payment computations. These payments need to be computed with absolute accuracy—that is, to the last cent. Errors can range from minor issues, such as paying a day late, to major problems, such as failure to hedge or fraudulent valuation by the trader.

The swap will also create some market risk, which may need to be hedged. The position needs to be transmitted to the market risk management system, which will monitor the total position and risk of the trader and of the institution as a whole. In addition, the current and potential credit exposure must be regularly measured and added to all other trades with the same counterparty. Errors in this risk measurement process can lead to excessive exposure to market and/or credit risk.

Operational risk can be minimized in a number of ways, through internal and external controls.[4] Internal control methods consist of:

- *Separation of functions*. Individuals responsible for committing transactions should not perform clearance and accounting functions.
- *Dual entries*. Entries (inputs) should be matched from two different sources—that is, the trade ticket and the confirmation by the back office.
- *Reconciliations*. Results (outputs) should be matched from different sources—for instance, the trader's profit estimate and the computation by the middle office.
- *Tickler systems*. Important dates for a transaction (e.g., settlement and exercise dates) should be entered into a calendar system that automatically generates a message before the due date.
- *Controls over amendments*. Any amendment to original deal tickets should be subject to the same strict controls as original trade tickets.

External control methods consist of:

- *Confirmations*. Trade tickets need to be confirmed with the counterparty, which provides an independent check on the transaction.
- *Verification of prices*. To value positions, prices should be obtained from external sources. This implies that an institution should have the capability of valuing a transaction in-house before entering it.
- *Authorization*. The counterparty should be provided with a list of personnel authorized to trade, as well as a list of allowed transactions.
- *Settlement*. The payment process itself can indicate if some of the terms of the transaction have been incorrectly recorded—for instance, if the first cash payments on a swap are not matched across counterparties.
- *Internal and external audits*. These examinations provide useful information on potential weakness areas in the organizational structure or business process.

[4] See W. Brewer, "Minimizing Operations Risk," in *Derivatives Handbook*, ed. R. Schwartz and C. Smith. (New York: John Wiley & Sons, 1997).

25.4.3 Model Risk

Model risk is a type of operational risk that particularly concerns risk managers. **Model risk** can be defined as the risk of losses due to inappropriate pricing or risk measurement models.

In some sense, all models are wrong. Models are only abstractions of reality. As Derman (1996) put it,[5]

> *[E]ven the finest model is only a model of the phenomena, not the real thing. A model is just a toy, though occasionally a very good one, in which case people call it a theory.*

Thus, models are just approximations. The key for risk managers is to understand conditions under which these approximations produce unacceptable results.

Figure 25.3 presents a taxonomy of sources of model risk. First, the input data can be wrong. Risk models rely on financial time series and other market data. The prices could have been observed with error or could be meaningless. Or option inputs such as the implied volatility could be biased.

Second, the parameters of the model can be incorrectly estimated. Risk models require a description of the statistical distribution of risk factors. These parameters are never estimated with perfect precision. As a result, outputs such as VAR measures must be contaminated by some error. They could be reported with confidence bands, for example, although this never happens. These bands should decrease with longer data series and with a lower confidence levels for VAR. In some cases, however, it may not be feasible to use longer series due to lack of data or structural changes.

Third, the choice of the model can be incorrect. For fixed-income options, for example, a simple Black-Scholes model may not be appropriate. Alternatively, the mapping process involves simplifications, which can prove deadly wrong in some situations. An example is that of the bank that mapped senior tranches of

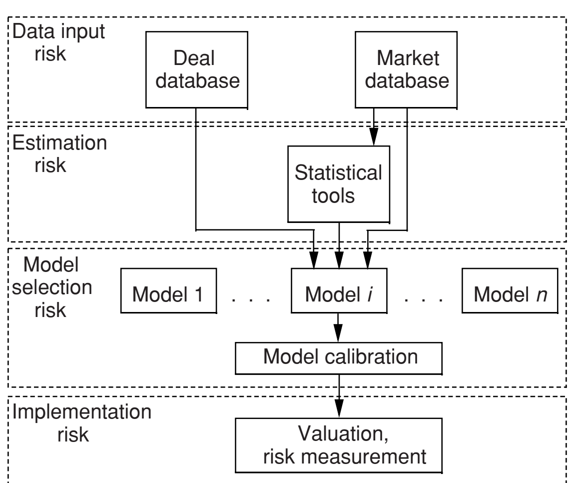

FIGURE 25.3 Model Risk

[5] E. Derman, *Model Risk* (New York: Goldman Sachs, 1996).

collateralized debt obligations (CDOs) backed by subprime loans on AAA-rated corporate yields. This was mainly for convenience in the absence of historical data on CDO yields. As a result, however, the bank thought that these securities had no risk and loaded up on them, only to realize billions of dollars in losses later. Another example is the practice of mapping corporate bonds and credit default swaps on the same credit to the same curve. This ignores **basis risk**, which caused severe losses during 2008.

Fourth, the model can be incorrectly implemented. This can be due to programming errors, to the selection of the wrong parameters, and so on.

Model risk is difficult to identify and even more difficult to measure. This is to be expected; otherwise, errors would have been corrected in the first place. Risk managers rely on ad hoc methods to protect themselves against model risk. First, they need to be aware of the comparative strengths and weaknesses of different models. In other words, intuition and experience are important. Second, they need to evaluate continuously whether the model's key assumptions are still valid. Third, models should be tested against simple problems for which one already knows the answer. For example, a model could be expanded to account for basis risk by mapping cash instruments and their derivatives on different curves. The risk manager would then check whether the model produces a risk measure that is compatible with a simple portfolio with two positions, for which the P&L is known.

Fourth, users should stress-test their models, changing the input data to check whether this substantially affects the output. Fifth, risk models should be backtested against actual data. This provides valuable guidance as to what part of the model can be improved.

For senior managers, the advice given in the previous section to minimize operational risk also applies to model risk. In addition, they should be aware of the fact that new products markets can cause model risk. These are situations where risk models may not have been fully developed because of time constraints or lack of data. Also, traders may purposefully game the risk systems to make their portfolios look more profitable and less risky. This is why independence of functions is a cornerstone of effective risk management.

25.4.4 Conceptual Issues

Even with all of these advances, the measurement of operational risk is still beset by conceptual problems.

First, unlike market and credit risk, operational risk is largely internal to financial institutions. Because institutions and business units are understandably reluctant to admit their mistakes, collecting data on operational losses creates challenges. As we have seen, internal data can be complemented by external data, but these have issues as well.

Second, market and credit risk can be conceptually separated into exposures and risk factors. Exposures can be easily measured and controlled. In contrast, the link between risk factors and the likelihood and size of operational losses is not so easy to establish. Here, the line of causation runs through internal controls. To be useful, the risk measure should account for the risk profile, or quality of control processes.

Third, very large operational losses that can threaten the stability of an institution are relatively rare (thankfully so). This leads to a very small number of observations in the tails. This thin tails problem makes it very difficult to come up with a robust value for operational risk at a high confidence level. In addition, backtesting is not as useful as a validation method. For market risk, VAR is typically measured over a daily horizon and compared to a daily P&L, which creates many observations to backtest the risk model. In contrast, the horizon is longer for operational risk, which creates fewer data points for backtesting.

As a result, there is still some skepticism as to whether operational risk can be subject to the same degree of quantification as market and credit risks.

EXAMPLE 25.9: FRM EXAM 2002—QUESTION 102

Capital is used to protect the bank from which of the following risks?

a. Risks with an extreme financial impact
b. High-frequency, low-loss events
c. Low-frequency risks with significant financial impact
d. High-frequency uncorrelated events

EXAMPLE 25.10: FRM EXAM 2001—QUESTION 49

Which of these terms is used in the insurance industry to refer to the effect of a reduction in the control of losses by an individual who is insured because of the protection provided by insurance?

a. Control trap
b. Moral hazard
c. Adverse selection
d. Control hazard

EXAMPLE 25.11: FRM EXAM 2003—QUESTION 48

Which of the following options does *not* describe a problem faced by banks when purchasing insurance as a hedge against operational risk?

a. The fact that the loss reimbursement period can take several years
b. The credit rating of insurers
c. The different perspective of operational risk between banks and insurers
d. Not having an operational VAR

EXAMPLE 25.12: FRM EXAM 2005—QUESTION 48

Insurance is an effective tool to transfer which of type of operational risk?

a. High frequency, low severity
b. Low frequency, high severity
c. Operational losses whose magnitude is affected by the actions of the company
d. Operational losses for which insurance companies only sell policies with low limits

EXAMPLE 25.13: FRM EXAM 2005—QUESTION 52

Which of the following statements are *valid* about hedging operational risk?

I. A primary disadvantage of insurance as an operational risk management tool is the limitation of policy coverage.
II. If an operational risk hedge works properly, a firm will avoid damage to its reputation from a high-severity operational risk event.
III. While all insurance contracts suffer from the problem of moral hazard, deductibles help reduce this problem.
IV. Catastrophe (cat) bonds allow a firm to hedge operational risks associated with natural disasters.

a. I, III, and IV only
b. I, II, and IV only
c. II and III only
d. III and IV only

EXAMPLE 25.14: FRM EXAM 2008—QUESTION 4-33

The following statements concern differences between market and operational risk VAR models. Which of the following statements is *false*?

a. Market risk models are primarily driven by historical data, whereas operational risk models are more flexible in this regard.
b. Market risk models typically define VAR as a specific quantile of the loss distribution, whereas operational risk models are more flexible in this regard.
c. Backtesting is generally a more useful form of validation for market risk models than for operational risk models.
d. The time horizon over which VAR is evaluated differs between market and operational risk models.

25.5 THE BASEL OPERATIONAL RISK CHARGE

One of the most significant additions to the Basel II Accord, finalized in 2004, is the operational risk charge (ORC). This establishes a minimum amount of capital that banks need to hold to cover their operational risk. The new rules allow three alternative methods to compute ORC.

25.5.1 Basic Indicator Approach

The simplest is called the **basic indicator approach** (BIA). This is based on an aggregate measure of business activity. The capital charge equals a fixed percentage, called the **alpha factor**, of the exposure indicator defined as gross income (GI):[6]

$$\text{ORC}^{BIA} = \alpha \times \text{GI} \tag{25.5}$$

where α has been set at 15%. The advantage of this method is that it is simple, is transparent, and uses readily available data. The problem is that it does not account for the quality of controls. As a result, this approach is expected to be mainly used by nonsophisticated banks.

25.5.2 Standardized Approach

The second method is the **standardized approach** (TSA).[7] Here, the bank's activities are divided into eight **business lines**. Within each business line, gross income is taken as an indicator of the scale of activity. The capital charge is then obtained by multiplying gross income by a fixed percentage, called the **beta factor**, and summing across business lines:

$$\text{ORC}^{TSA} = \sum_{i=1}^{8} \beta_i \times \text{GI}_i \tag{25.6}$$

The β factors are described in Table 25.5. This approach is still simple but better reflects varying risks across business lines.[8] The trading and sales line, for example,

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[6] This is taken as the average of positive gross income numbers over the past three years. Negative values are excluded.

[7] Banks are also allowed to use an alternative standardized approach, where gross income is replaced by loans and advances for retail and commercial banking, multiplied by a scaling factor $m = 0.035$.

[8] The formula is actually more complex and allows offsets for some negative GI numbers in a year with positive numbers in other business lines, up to a limit of zero. The exact formula is $ORC^{TSA} = \{\sum_{t=1}^{3} \text{Max}[\sum_{i=1}^{8}(\beta_i \times \text{GI}_i), 0]\}/3$.

TABLE 25.5 Basel Beta Factors for ORC

Business Line	Beta Factor
Corporate finance	18%
Trading and sales	18%
Retail banking	12%
Commercial banking	15%
Payment, settlement	18%
Agency services	15%
Asset management	12%
Retail brokerage	12%

carries a higher weight to reflect the possibility of high-severity losses due to trader fraud.

25.5.3 Advanced Measurement Approach

The third class of method is the **advanced measurement approach** (AMA). This allows banks to use their own internal models in the estimation of required capital using quantitative and qualitative criteria set by the Basel Accord. No particular method is prescribed, but AMA is allowed only if the bank demonstrates effective management and control of operational risk.

The qualitative criteria are similar to those for the use of internal market VAR systems.[9] Once these are satisfied, the risk charge is obtained from the unexpected loss (UL) at the 99.9% confidence level over a one-year horizon:

$$\text{ORC}^{AMA} = \text{UL}(\text{1-year},\ 99.9\%\ \text{confidence}) \tag{25.7}$$

Normally, the expected loss (EL) must be included in the capital charge, unless the bank can demonstrate that it adequately captures EL in its internal business practices.

Other quantitative criteria are as follows: (1) banks must track internal loss data measured over a minimum period of five years; (2) banks must use external data; (3) banks must use scenario analysis to evaluate their exposure to high-severity events; and (4) banks must take into account the business environment and internal control factors. Finally, insurance can be used to offset up to 20% of the operational risk charge. This approach offers the most refined measurement of operational risk and is expected to be used by more sophisticated institutions.

[9] Specifically, (1) the bank must have an independent operational risk function, (2) the system must be integrated in day-to-day management, (3) there must be regular reporting, (4) documentation must exist, (5) auditors must perform regular reviews, and (6) there must be external validation.

25.5.4 2008 Loss Data Collection Exercise

The Basel Committee undertook a loss data collection exercise in 2008, requesting information from 121 banks about their operational loss experience and modeling.[10] Out of this sample, 20 banks use the BIA, 51 the TSA, and 42 the AMA.

AMA banks have a higher frequency of large internal losses, even when scaling for size. This reflects the fact that these banks are more complex, or that they have a better collection process. Operational risk capital is lower for AMA banks, at 10.8% of gross income, versus 12.8% for non-AMA banks. Overall, these results indicate that the banking sector is making considerable progress in the collection and use of operational loss data.

EXAMPLE 25.15: FRM EXAM 2007—QUESTION 117

Which of the following approaches for calculating operational risk capital charges leads to a higher capital charge for a given accounting income as risk increases?

a. The basic indicator approach
b. The standardized approach
c. The advanced measurement approach
d. All of the above

EXAMPLE 25.16: FRM EXAM 2004—QUESTION 53

Which of the following statements about its methodology for calculating an operational risk capital charge in Basel II is *correct*?

a. The basic indicator approach is suitable for institutions with sophisticated operational risk profiles.
b. Under the standardized approach, the capital requirement is measured for each of the business lines.
c. Advanced measurement approaches will not allow an institution to adopt its own method of assessment of operational risk.
d. The AMA is less risk sensitive than the standardized approach.

[10] Basel Committee on Banking Supervision, *Results from the 2008 Loss Data Collection Exercise for Operational Risk* (Basel: BIS, 2009).

EXAMPLE 25.17: FRM EXAM 2007—QUESTION 6

Which of the following statements regarding Basel II nonadvanced approaches is *incorrect*?

a. The standardized approach makes it advantageous for a bank to book losses early if doing so reduces this year's gross income sufficiently to make it negative.
b. Corporate finance, trading and sales, and payment and settlement are the business lines with the highest regulatory capital requirements.
c. The standardized approach divides the bank into business lines and uses data from the last three years of a business line's gross income and a beta factor to obtain the regulatory capital for that business line.
d. The standardized approach uses data from the last three years of gross income to obtain a bank's operational risk capital charge.

25.6 IMPORTANT FORMULAS

Density function for loss frequency over the horizon: $f(n), \quad n = 0, 1, 2, \ldots$

Density function for loss severity: $g(x \mid n = 1), \quad x \geq 0$

Convolution: Combining loss frequency and severity into a distribution of total losses over the next year: $S_n = \sum_{i=1}^{n} X_i$

Operational VAR: Quantile of S_n, typically over a one-year horizon at 99.9% confidence level

Basel II operational risk charge:

Basic indicator approach: $ORC^{BIA} = \alpha \times GI$

Standardized approach: $ORC^{TSA} = \sum_{i=1}^{8} \beta_i \times GI_i$

AMA: $ORC^{AMA} = UL(\text{1-year, } 99.9\% \text{ confidence})$

25.7 ANSWERS TO CHAPTER EXAMPLES

Example 25.1: FRM Exam 2004—Question 39

c. Damaged reputation due to a failed merger is a business risk. Also, reputational risk is not a type of operational loss.

Example 25.2: FRM Exam 2003—Question 65

c. Choices a., b., and d. are operational losses. Answer c. is the result of a bet on volatility, which is market risk.

Example 25.3: FRM Exam 2007—Question 56

b. Statement a. represents external fraud, which is included in operational risk. Statement c. represents a systems failure. Statement d. is a failure in internal processes.

Example 25.4: FRM Exam 2007—Question 139

c. This is a situation where the model price is significantly different from the market price, which is model risk. Liquidity risk could also explain part of the difference, but this is less likely to be the case given the emphasis on the complexity of the instrument.

Example 25.5: FRM Exam 2009—Question 7-2

a. The last three distributions require the number n to be positive, which is not the case for the normal distribution.

Example 25.6: FRM Exam 2007—Question 138

b. Loss severity distributions are bounded by zero but should include very large losses. So, they are asymmetrical with long right tails.

Example 25.7: FRM Exam 2008—Question 4-10

d. Internal data certainly has a problem of survivorship bias because a bank where employees compute the operational risk distribution is still alive. This precludes a history of large, deadly losses.

Example 25.8: FRM Exam 2007—Question 33

a. Because VAR should include EL, there is no need to compute EL separately. The table shows that the smallest loss such that the cumulative probability is 95% or more is $100,000.

Loss ($)	Probability		Cumulative Probability
0	0.5	= 0.500	50.0%
1,000	0.3×0.6	= 0.180	68.0%
2,000	$0.2 \times 0.6 \times 0.6$	= 0.072	75.2%
10,000	0.3×0.3	= 0.090	84.2%
11,000	$0.2 \times 0.6 \times 0.3 \times 2$	= 0.072	91.4%
20,000	$0.2 \times 0.3 \times 0.3$	= 0.018	93.2%
100,000	0.3×0.1	= 0.030	96.2%
101,000	$0.2 \times 0.1 \times 0.6 \times 2$	= 0.024	98.6%
110,000	$0.2 \times 0.1 \times 0.3 \times 2$	= 0.012	99.8%
200,000	$0.2 \times 0.1 \times 0.1$	= 0.002	100.0%

Example 25.9: FRM Exam 2002—Question 102

c. Capital is supposed to absorb risks that have significant financial impact on the firm. Risks with extreme financial impact, such as systemic risk, cannot be absorbed by capital alone, so answer a. is wrong. Low-loss events are unimportant, so b. is wrong. Uncorrelated events tend to diversify, so d. is wrong.

Example 25.10: FRM Exam 2001—Question 49

b. Moral hazard arises when insured individuals have less incentive to control their losses because they are insured.

Example 25.11: FRM Exam 2003—Question 48

d. Answers a., b., and c. describe problems arising from the purchase of insurance against operational risk. This is irrespective of whether the bank has an operational VAR model.

Example 25.12: FRM Exam 2005—Question 48

b. The purpose of insurance is to reimburse large losses, or operational risk events with high severity. Answer c. is incorrect because this type of moral hazard should result in much higher premiums.

Example 25.13: FRM Exam 2005—Question 52

a. All the statements are valid, except for II. Even if a firm implements a hedge or purchases insurance, the news of a large operational loss will still damage its reputation.

Example 25.14: FRM Exam 2008—Question 4-33

b. Statement a. is true because operational risk models often rely heavily on scenario analysis. Backtesting is more difficult for operational risk models, so c. is true. VAR is usually evaluated over shorter horizons, so d. is true. Statement b. is false because both market and operational risk models use a quantile of the distribution.

Example 25.15: FRM Exam 2007—Question 117

c. The basic indicator approach uses a factor of $\alpha = 15\%$. The standardized approach uses a fixed factor ranging from 12% to 18%, so is not risk sensitive (except for changes across business lines). The AMA is the most risk-sensitive method.

Example 25.16: FRM Exam 2004—Question 53

b. The BIA is suitable for banks with basic risk profiles, so answer a. is incorrect. The AMA is an internal model, so answer c. is incorrect. The AMA is more risk sensitive than the standardized approach, so answer d. is incorrect.

Example 25.17: FRM Exam 2007—Question 6

a. Statement a. is incorrect because only positive income is considered. Statement b. is correct, given Table 25.5. Statements c. and d. are correct as well.

APPENDIX: CAUSAL NETWORKS

Causal networks explain losses in terms of a sequence of random variables. Each variable itself can be due to a combination of other variables. For instance, settlement losses can be viewed as caused by a combination of (1) exposure and (2) time delay. In turn, exposure depends on (1) the value of the transaction and (2) whether it is a buy or a sell. Next, the causal factor for time delay can be chosen as (1) the exchange, (2) the domicile, (3) the counterparty, (4) the product, and (5) daily volume.

These links are displayed through graphical models based on process work flows. One approach is the **Bayesian network**. Here each node represents a random variable and each arrow represents a causal link.

Causes and effects are related through conditional probabilities, an application of Bayes' theorem. For instance, suppose we want to predict the probability of a settlement failure, or *fail*. Set $y = 1$ if there is a failure and $y = 0$ otherwise. The causal factor is, say, the quality of the back-office team, which can be either good or bad. Set $x = 1$ if the team is bad. Assume there is a 20% probability that the team is bad. If the team is good, the conditional probability of a fail is $P(y = 1 \mid x = 0) = 0.1$. If the team is bad, this probability is higher, $P(y = 1 \mid x = 1) = 0.7$. We can now construct the unconditional probability of a fail, which is

$$P(y = 1) = P(y = 1 \mid x = 0)P(x = 0) + P(y = 1 \mid x = 1)P(x = 1) \qquad (25.8)$$

which is here $P(y = 1) = 0.1 \times (1 - 0.20) + 0.7 \times 0.20 = 0.22$. Armed with this information, we can now evaluate the benefit of changing the team from bad to good through training, for example, or through new hires. Or we could assess the probability that the team is bad given that a fail has occurred. Using Bayes' rule, this is

$$P(x = 1 \mid y = 1) = \frac{P(y = 1, x = 1)}{P(y = 1)} = \frac{P(y = 1 \mid x = 1)P(x = 1)}{P(y = 1)} \qquad (25.9)$$

which is here $P(x = 1 \mid y = 1) = \frac{0.7 \times 0.20}{0.22} = 0.64$. In other words, the probability that the team is bad has increased from 20% to 64% based on the observed fail. Such an observation is useful for process diagnostics.

Once all initial nodes have been assigned probabilities, the Bayesian network is complete. The bank can now perform Monte Carlo simulations over the network, starting from the initial variables and continuing to the operational loss, to derive a distribution of losses.

CHAPTER 26

Liquidity Risk

Liquidity risk is an important source of financial risk, as we have witnessed in the latest credit crisis. The crisis of confidence that started with subprime losses suddenly accelerated after the Lehman bankruptcy. Many debt holders refused to roll over their investments, creating massive funding problems for financial institutions. These problems were compounded by their difficulties in selling assets to meet funding needs.

Liquidity risk, unfortunately, is less amenable to formal risk measurement, unlike market risk, credit risk, and operational risk. This is why the Basel Committee on Banking Supervision (BCBS) did not institute formal capital charges against liquidity risk. Yet, it stated: "Liquidity is crucial to the ongoing viability of any banking organization. Banks' capital positions can have an effect on their ability to obtain liquidity, especially in a crisis."[1] Thus it is crucial for financial institutions to assess, monitor, and manage their liquidity risk.

Section 26.1 describes sources of liquidity risk, which involve both asset liquidity risk and funding risk. Section 26.2 analyzes asset liquidity risk. The ability to liquidate assets to generate cash depends on market conditions, including bid-ask spreads and market impact, as well as the liquidation time horizon. To some extent, value at risk (VAR) can be expanded into a liquidity-adjusted VAR. Section 26.3 then analyzes funding liquidity risk. This is illustrated using the example of Northern Rock, the failed British bank. Funding claims can be evaluated by examining the structure of liabilities as well as off-balance-sheet items. Finally, Section 26.4 discusses how banks can assess and control liquidity risk, primarily using gap analysis. It highlights the importance of contingency funding plans and illustrates disclosures about liquidity risk management.

26.1 SOURCES OF LIQUIDITY RISK

Lack of liquidity can cause the failure of an institution, even when it is technically solvent (i.e., when the value of its assets exceeds that of liabilities). Commercial

FRM Exam Part 2 topic.

[1] Paragraph 741 in BCBS, *International Convergence of Capital Measurement and Capital Standards* (Basel: BIS, 2006).

banks have an inherent liquidity imbalance between their assets (long-term loans) and their liabilities (retail deposits and capital market debt). As a result, a crisis of confidence might lead to depositors demanding their money right away. Even if the bank has sufficient assets to cover deposits, it might not be able to liquidate its assets fast enough, or at reasonable prices, to meet the redemption requests. Similarly, hedge funds need to manage carefully the liquidity risk inherent in some balance sheets.

Liquidity risk consists of both asset liquidity risk and funding liquidity risk. The **Committee of European Banking Supervisors** (CEBS) provides the following definitions.[2]

- **Asset liquidity risk**, also called **market/product liquidity risk**, is the risk that a position cannot easily be unwound or offset at short notice without significantly influencing the market price, because of inadequate market depth or market disruption.
- **Funding liquidity risk** is the current or prospective risk arising from an institution's inability to meet its liabilities and obligations as they come due without incurring unacceptable losses.

These two types of risk interact with each other if the portfolio contains illiquid assets that must be sold at distressed prices to meet funding requirements.

26.2 ASSET LIQUIDITY RISK

26.2.1 Assessing Asset Liquidity Risk

To evaluate asset liquidity risk, we start with a characterization of market conditions for the asset to be traded. The **bid-ask spread** measures the round-trip transaction cost of buying and selling an amount within **normal market size** (NMS). If $P(\text{ask})$ is the ask price, $P(\text{bid})$ the bid price, and $P(\text{mid}) = [P(\text{ask}) + P(\text{bid})]/2$ the mid price, the spread is defined in relative terms as

$$S = \frac{[P(\text{ask}) - P(\text{bid})]}{P(\text{mid})} \tag{26.1}$$

Assets with good liquidity will have tight bid-ask spreads. **Tightness** is a measure of the divergence between actual transaction prices and quoted midmarket prices. Liquid assets are also characterized by good **depth**, which is a measure of the volume of trades possible without affecting prices too much (e.g., at the bid-offer prices). This is in contrast to **thinness**.

[2] Committee of European Banking Supervisors, *Second Part of CEBS's Technical Advice to the European Commission on Liquidity Risk Management* (London: CEBS, 2008). Available at www.c-ebs.org.

For larger transactions, asset liquidity can be assessed by a price-quantity function, called **market impact**, which describes how the price is affected by the quantity transacted. Sometimes, this is called **endogenous liquidity**, meaning that the price drop depends on the size of the position. In contrast, positions within normal market sizes are characterized by **exogenous liquidity**.

When selling a large block of an asset in a liquid market, prices may drop temporarily but should recover quickly. **Resiliency** is a measure of the speed at which price fluctuations from trades are dissipated.

For liquid assets, such as the Treasury market, this function is rather flat, meaning that large volumes of transactions do not affect prices much. For instance, one can generally transact $10 million of a Treasury bond at a cost of one-half the bid-ask spread of 0.10%, which translates into a cost of $\$10,000,000 \times 0.10\%/2 = \$5,000$, which is very low.

In contrast, illiquid assets are those where spreads are wide and where transactions can quickly affect prices. For example, bank loans are traded over-the-counter (OTC) and can have spreads as wide as 10%. A sale of $10 million would then push prices down by 5%, which is a cost of $\$10,000,000 \times 10\%/2 = \$500,000$, which is much higher than in the previous example. A sale of twice this amount might incur an even larger price drop, such as 8%, to clear the market. Thus, the prices of illiquid assets are more affected by current demand and supply conditions. As a result, they are usually more volatile than liquid assets, provided trading occurs.

The latter example also shows that liquidity is a function of the **time horizon**. If the price-quantity function is steep, an immediate sale will force prices down by a large amount. A patient investor, in contrast, would fetch a better price by splitting the sale order over several days, thereby incurring a lower market impact.

Figure 26.1 compares the price-quantity functions for a liquid and an illiquid asset. For the liquid asset, the bid-ask spread is tight; the market has more depth,

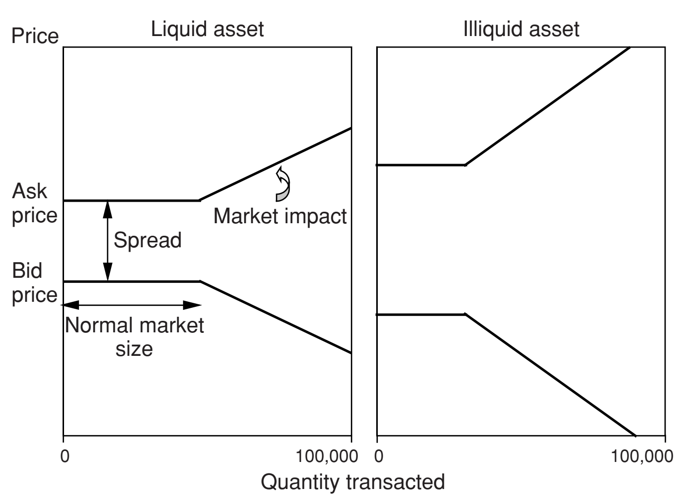

FIGURE 26.1 Comparison of Liquid and Illiquid Assets

implying larger normal market sizes; and the lines representing the market impact have lower slopes.

Generally, assets that have greater trading volumes are more liquid. Trading volume reflects differences of opinion across investors but also depends on the presence of active speculators. Hedge funds, in particular, actively trade in many markets, which increases market liquidity.

Assets that are simple to price are also more liquid. At one extreme are Treasury bonds with fixed coupons, which are simple instruments and therefore easy to evaluate. At the other extreme would be structured notes with complicated payoffs, which are harder for participants to evaluate and hedge. As a result, spreads on such notes will be much wider than on T-bonds.

Liquidity varies across asset classes and can be security-specific. Securities that have greater outstanding amounts or are issued more recently are generally more liquid. **On-the-run** securities are those that are issued most recently and hence are more active and liquid. Other securities are called **off-the-run**. Consider, for instance, the latest-issued 30-year U.S. Treasury bond. This is on-the-run until another 30-year bond is issued, at which time it becomes off-the-run. These two securities have the same credit risk (i.e., that of default by the U.S. government) and market risk (i.e., both have maturities close to 30 years). Because they are so similar, their yield spread must be a **liquidity premium**.

Asset liquidation costs also depend on **asset fungibility**. Contracts traded on centralized exchanges, such as futures or common stocks, can easily be resold to the highest bidder and therefore are fungible. In contrast, privately negotiated derivatives require the agreement of the original counterparty to unwind the trade. In this situation, the counterparty may demand a discount to cancel the position.

In summary, asset liquidity risk depends on several factors: (1) market conditions (bid-ask spreads and market impact), (2) liquidation time horizon, (3) asset and security type, and (4) asset fungibility.

Illiquidity can also be marketwide and time-varying. Large-scale changes in market liquidity seem to occur on a regular basis, including the bond market rout of 1994, the Russian/Long-Term Capital Management (LTCM) crisis of 1998, and the credit crisis that started in 2007. Such crises are characterized by a **flight to quality**, which occurs when there is a shift in demand away from low-grade securities toward high-grade securities, in particular government bonds. The low-grade market then becomes illiquid with depressed prices. This is reflected in an increase in the yield spread between corporate and government issues.

26.2.2 Liquidity-Adjusted VAR

Asset liquidity risk is less amenable to formal measurement than traditional market risk. Illiquidity can be loosely factored into VAR measures, by increasing the horizon or by selectively increasing volatilities. These adjustments, however, are mainly ad-hoc.

We can attempt to incorporate the effect of bid-ask spreads in risk measures. When the spread S is fixed, **liquidity-adjusted VAR** (LVAR) can be defined as:

$$\mathrm{LVAR} = \mathrm{VAR} + L_1 = W[\alpha\sigma + \frac{1}{2}(S)] \tag{26.2}$$

where W is the initial wealth, or portfolio value. If VAR is to be measured from zero (relative to the initial portfolio value), instead of away from the mean, we need to subtract μ from $\alpha\sigma$.

For instance, assume we have \$10 million invested in a 30-year Treasury bond, with daily volatility of $\sigma = 1\%$ and spread of $S = 0.10\%$. The one-day LVAR at the 95% confidence level is

$$\$10{,}000{,}000[(1.645 \times 0.01) + \frac{1}{2}(0.0010)] = \$164{,}500 + \$5{,}000 = \$169{,}500$$

Here, this correction term is small. In contrast, the correction term is \$500,000 for the bank loan in the previous example.

If bid-ask spreads vary substantially, Equation (26.2) can be adjusted to account for the worst increase in spread at some confidence level. The distribution of the spread can be described by its mean $\bar{S}$ and standard deviation σ_S. The worst-case LVAR is then:

$$\mathrm{LVAR} = \mathrm{VAR} + L_2 = W[\alpha\sigma + \frac{1}{2}(\bar{S} + \alpha'\sigma_S)] \tag{26.3}$$

By adding up the worst-case losses for market and liquidity risk, this effectively assumes a high correlation between these two risk drivers.

In practice, estimating the distribution of spreads is a challenge. Spreads tend to be stable for long periods and then explode in periods of crisis. Therefore, the distribution of spreads is highly nonnormal. To measure risk at the top level, the risk manager also needs estimates of correlations across spreads. In addition, this analysis assumes that the quantity transacted is within normal market sizes. Otherwise, an immediate forced sale of a large quantity would also incur market impact.

26.2.3 Illiquidity and Risk Measures

Asset illiquidity poses special problems for risk measurement. In illiquid markets, fewer trades imply that prices do not move much. As a result, prices at the end of a reporting period generally will not represent market-clearing transactions and will tend to be sluggish in the absence of regular trading. This creates downward biases in measures of volatility and correlations with other asset classes.

In addition, news slowly impacts prices, creating positive autocorrelation in returns, which invalidates the square root of time rule when extrapolating VAR to longer horizons. These effects are further discussed in Section 30.4.2, in the context of hedge funds.

EXAMPLE 26.1: FRM EXAM 2003—QUESTION 15

Which of the following statements regarding liquidity risk is *correct*?

a. Asset liquidity risk arises when a financial institution cannot meet payment obligations.
b. Flight to quality is usually reflected in a decrease in the yield spread between corporate and government issues.
c. Yield spread between on-the-run and off-the-run securities mainly captures the liquidity premium, and not the market and credit risk premium.
d. Funding liquidity risk can be managed by setting limits on certain asset markets or products and by means of diversification.

EXAMPLE 26.2: FRM EXAM 2002—QUESTION 36

The following statements compare a highly liquid asset against an (otherwise similar) illiquid asset. Which statement is most likely to be *false*?

a. It is possible to trade a larger quantity of the liquid asset without affecting the price.
b. The liquid asset has a smaller bid-ask spread.
c. The liquid asset has higher price volatility since it trades more often.
d. The liquid asset has higher trading volume.

EXAMPLE 26.3: FRM EXAM 2007—QUESTION 78

A mutual fund investing in common stocks has adopted a liquidity risk measure limiting each of its holdings to a maximum of 30% of its 30-day average value traded. If the fund size is USD 3 billion, what is the maximum weight that the fund can hold in a stock with a 30-day average value traded of USD 2.4 million?

a. 24.00%
b. 0.08%
c. 0.024%
d. 80.0%

EXAMPLE 26.4: FRM EXAM 2000—QUESTION 74

In a market crash, which the following are usually *true*?

I. Fixed-income portfolios hedged with short Treasury bonds and futures lose less than those hedged with interest rate swaps given equivalent durations.
II. Bid-offer spreads widen because of lower liquidity.
III. The spreads between off-the-run bonds and benchmark issues widen.

a. I, II, and III
b. II and III
c. I and III
d. None of the above

EXAMPLE 26.5: FRM EXAM 2007—QUESTION 116

You are holding 100 Wheelbarrow Company shares with a current price of \$50. The daily mean and volatility of the stock return is 1% and 2%, respectively. VAR should be measured relative to the initial wealth. The bid-ask spread of the stock varies over time. The daily mean and volatility of the spread are 0.5% and 1%, respectively. Both the return and spread are normally distributed. Calculate the daily liquidity-adjusted VAR (LVAR) at a 99% confidence level.

a. USD 254
b. USD 229
c. USD 325
d. USD 275

EXAMPLE 26.6: FRM EXAM 2009—QUESTION 7-7

You are a manager of a renowned hedge fund and are analyzing a 1,000-share position in an undervalued but illiquid stock BNA, which has a current stock price of USD 72 (expressed as the midpoint of the current bid-ask spread). Daily return for BNA has an estimated volatility of 1.24%. The average bid-ask spread is USD 0.16. Assuming returns of BNA are normally distributed, what is the estimated liquidity-adjusted daily 95% VAR, using the constant spread approach?

a. USD 1,389
b. USD 1,469
c. USD 1,549
d. USD 1,629

26.3 FUNDING LIQUIDITY RISK

26.3.1 Indicators of Liquidity Risk

Liquidity risk has been a major risk factor in the credit crisis that started in 2007. As commercial and investment banks started to accumulate losses, initially due to subprime asset-backed securities, banks were reluctant to lend in fear of counterparty default.

Conditions in money markets can be gauged, for example, by comparing the three-month Treasury bill rate, the three-month London Interbank Offered Rate (LIBOR), and the overnight **federal funds rate**.[3] To ensure comparability, all rates are in U.S. dollars. The T-bill rate has no credit risk—other than that of the U.S. government. In contrast, LIBOR and fed funds are for unsecured loans.

The difference between LIBOR and fed funds is a term spread. It can be viewed as the price of an option to call a loan. A bank that has lent overnight can choose not to renew the loan if bad news were to strike; conversely, a bank committed to a three-month loan has no such option. As usual, the value of an option increases in uncertain times, which explains why this term spread has sharply increased.

The credit spread between Eurodollar LIBOR and Treasuries is known as the **TED spread**. This reflects expected credit losses as well as a liquidity risk premium.

Figure 26.2 describes the behavior of these interest rates during 2007 and 2008. The sharp reduction in the fed funds rate shows that the Federal Reserve has aggressively eased monetary policy. Treasury yields have correspondingly gone down as well. LIBOR rates, however, have stayed stubbornly high, reflecting tight conditions in credit markets. In particular, the TED spread, which is usually around 25bp, has widened sharply, exceeding 500bp after the Lehman bankruptcy on September 15, 2008. Firms with a more shaky credit rating have had to face even higher rates—that is, when they have been able to obtain any funding at all.

26.3.2 Assessing Funding Liquidity Risk

The Basel Committee provides a comprehensive definition of this risk.[4]

> ***Funding liquidity risk*** *is the risk that the firm will not be able to meet efficiently both expected and unexpected current and future cash flow and collateral needs without affecting either daily operations or the financial condition of the firm.*

[3] Fed funds are unsecured loans of reserve balances at the Federal Reserve Bank that banking institutions make to one another, usually overnight. The rate at which these transactions occur is called the fed funds rate. The central bank sets a target level for the fed funds rate, which is its primary tool for monetary policy.

[4] Basel Committee on Banking Supervision, *Principles for Sound Liquidity Risk Management and Supervision* (Basel: BIS, 2008).

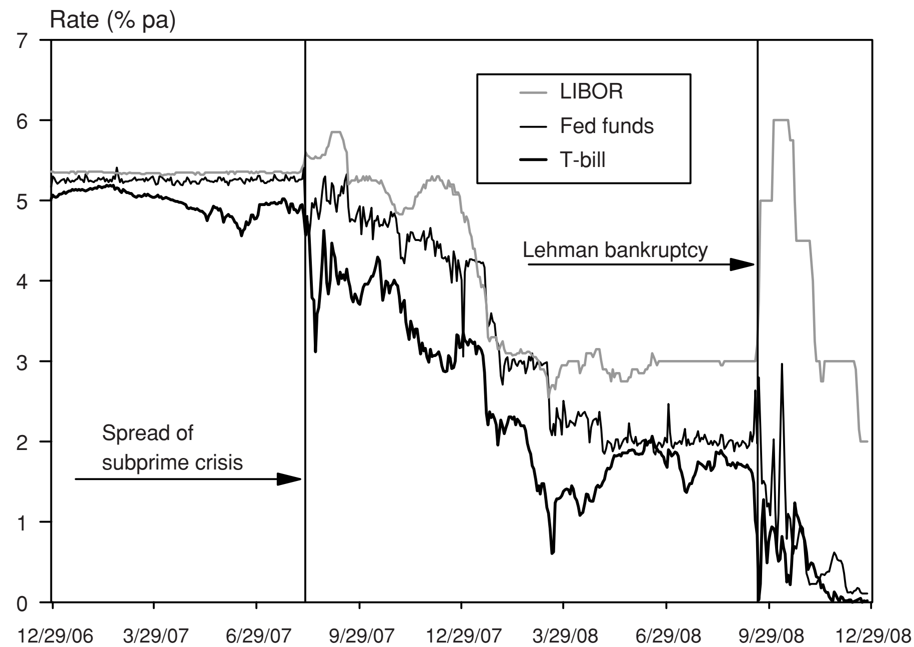

FIGURE 26.2 Comparison of Short-Term Dollar Interest Rates

The consequences of this risk can be illustrated by the Northern Rock example.

Example: Northern Rock's Liquidity Risk

Northern Rock (NR) is a bank that was counted among the top five mortgage lenders in Britain. As shown in the table, total assets added up to 113.5 billion British pounds (GBP) as of June 2007, about GBP 81 billion of which was funded through capital markets, and only GBP 30 billion through customer deposits. NR's business model was unusually reliant on funding from capital markets instead of retail deposits. Capital market funding, however, is more volatile than retail deposits. The bank had used this unusual structure to fuel its fast growth.

Northern Rock's Balance Sheet (GBP Billions)			
Assets		Debt	
Loans	96.7	Retail deposits	30.1
Cash	0.8	Debt securities	71.0
Securities	8.0	Other	10.1
Total	113.5	Total	111.2

Northern Rock stated that it was meeting Financial Services Authority (FSA) liquidity rules, which require enough liquidity for at least five business days. The bank had not expected such a long and widespread credit crisis, however.

During August 2007, NR started to run into difficulties rolling over its short-term debt and issuing securitized loans. Higher rates on new capital, when available, started to squeeze margins, leading to a free fall in the bank's share price.

Regulators believed that the bank was still solvent, however. On September 13, it was announced that the Bank of England had granted emergency financial support to Northern Rock. This news started a bank run. Because deposits are only partially insured in Britain, depositors panicked and withdrew billions of deposits in the following days. On Monday, September 17, the Chancellor of the Exchequer announced that the government would fully guarantee all deposits.

By the end of the year, the bank had been unable to roll over GBP 8 billion in short-term debt and had lost GBP 15 billion in customer accounts. The loan from the Bank of England had grown to GBP 27 billion. After two unsuccessful bids to sell Northern Rock, it was nationalized on February 22, 2008.

Northern Rock was victim of funding liquidity risk, as it had funded long-term loans by short-term debt that it could not roll over.

Table 26.1 provides a general framework for assessing liquidity risk. Funding liquidity risk arises from the liability side, for either on-balance-sheet or off-balance-sheet items. Liabilities can be classified into *stable* or *volatile*, where these terms refer to the predictability of cash flows.

For a public corporation, equity is stable.[5] Financial institutions can manage their equity liquidity profiles by means of dividend policies, share repurchases, and new issues.

Next, we turn to the debt, which can be divided into unsecured and secured funding. For instance, Northern Rock had issued GBP 45.7 billion in securitized notes and GBP 8.1 billion in covered bonds. Investors should be more willing to provide funds secured by assets. In contrast, unsecured funding is subject to default risk by the issuer.

Within the unsecured funding category, retail deposits are more stable than capital market instruments.[6] For example, under conditions of stress, investors in money market instruments may demand higher compensation for risk, or require to roll over their investments for shorter maturities, or even refuse to extend financing at all.

Among off-balance-sheet liabilities, loan commitments, letters of credit, and financial guarantees provided by a bank will create a contingent claim on liquidity if drawn. Derivatives may also create cash flow needs if counterparties demand more collateral as the position moves out-of-the-money, or if contracts contain credit trigger clauses that require additional collateral in the case of a downgrade. Indeed, **special-purpose vehicles** (SPVs) can also create contingent liquidity exposures. Some structures, such as bank-sponsored **conduits**, have explicit backing from the issuing bank and can draw upon liquidity lines if the SPV is not able to roll over its debt. Other structures, such as **structured investment vehicles** (SIVs), may not have explicit backing but the bank nevertheless may choose to provide liquidity support for business or reputational reasons.

[5] Hedge funds, however, need to worry about redemptions from equity investors.
[6] With the advent of Internet-based banking, however, deposits can move faster across banks.

TABLE 26.1 Managing Bank Liquidity Risk

Balance Sheet	
Assets	**Debt**
• Highly liquid (cash, etc.)	• Unsecured funding
• Other, unencumbered (pledgeable as collateral)	• Retail deposits
	• Capital markets
• Other, encumbered	• Secured funding
	Equity
	• New stock issues
Off Balance Sheet	
Assets	**Liabilities**
• Derivatives	• Derivatives
• Credit lines purchased	• Guarantees provided
	• Commitments
	• Special-purpose vehicles (SPVs)

Focusing now on the asset side of the balance sheet, **funding gaps** can be met with asset sales. Cash or liquid assets provide a cushion that can be used immediately. Unencumbered securities, defined as those that do not have claim against them, can be sold, perhaps at a discount that reflects asset liquidity risk. Alternatively, they can be sold through a repurchase agreement as collateral against cash either with a private counterparty or with the central bank if permitted.

In addition, cash can be forthcoming from derivatives positions that move in-the-money. Institutions can also establish bank credit lines, which can be drawn upon in case of liquidity needs.

Finally, this on- and off-balance-sheet information should be integrated with cash flows. In particular, the banking system has expanded securitization as a means to reduce assets on the balance sheet. During the recent credit crisis, however, banks were forced to postpone some securitization, leading to a buildup of a large loan inventory that had to be financed.

Example: AIG's Liquidity Risk

American International Group (AIG) is a global insurance conglomerate that once was among the largest public companies in the world. Due to its strong revenues and capital base, AIG long enjoyed a top credit rating of AAA. As a result, it had allowed its Financial Products division to take on increasingly large positions. It sold credit default swaps on the senior tranches of CDOs for which, because of its high credit rating, it did not have to post collateral.

On March 14, 2005, AIG's CEO was forced to step down amid allegations of questionable business practices. The next day, its credit rating was downgraded to AA+. The downgrade triggered provisions requiring posting $1.2 billion in additional collateral for its swaps. At the time, this was still manageable, given that AIG had equity of about $80 billion.

Its credit default swap (CDS) portfolio, however, continued to grow, reaching $500 billion. As the subprime crisis started to unfold, the CDO tranches lost value quickly. AIG announced that it had lost $13 billion in the first half of 2008.

On September 15, 2008, S&P lowered AIG's credit rating from AA− to A−. As a result, AIG was required to post an additional $20 billion in collateral, which it did not have. Because a collapse of AIG would have had systemic consequences, the U.S. government stepped in and provided an $85 billion loan. In October, this amount had to be increased by $38 billion. In November, the U.S. Treasury invested an additional $40 billion in newly issued AIG senior preferred stock under the **Troubled Asset Relief Program** (TARP). This was the largest public bailout of a private company. Apparently, leaders at the Financial Products division had failed to prepare for downgrades in AIG's credit rating.

EXAMPLE 26.7: FRM EXAM 2008—QUESTION 2-20

You are a risk manager for a hedge fund. You are told that the TED spread increased sharply. Which of the following statements best describes the change in your situation?

a. An increase in the TED spread indicates that the Federal Reserve will push interest rates up, so the duration of the portfolios should be reduced.
b. An increase in the TED spread indicates a bigger gap between the fed funds rate and Treasuries, so that the Fed will choose to increase liquidity in the markets, which will increase prices of securities as demand will increase.
c. An increase in the TED spread could indicate greater concerns about bank solvency, so that you should review your counterparty exposures and possibly hedge some exposure to banks.
d. An increase in the TED spread could indicate more willingness of banks to lend since they get paid more for lending, so we should use the opportunity to renegotiate lines of credit.

26.4 MANAGING LIQUIDITY RISK

26.4.1 Steps in Liquidity Risk Management

Liquidity risk management requires robust internal governance, implemented by adequate tools to identify, measure, monitor, and manage liquidity risk. The board of directors is ultimately responsible for the institution's liquidity strategy.

While there is no single measure of liquidity risk, a range of metrics can be used to assess liquidity risk. Liquidity risk management starts with **operational liquidity**, which lays out the *daily* payment queue, forecasting all cash inflows and outflows. This is no simple affair, however. In recent years, improvements in the design of payment and settlement systems (e.g., real-time gross settlement systems and the **CLS Bank** for netting foreign currency payments) have compressed the time between payments. While such systems reduce credit risk and operational risk, they do place more strain on liquidity management.

The next step is *tactical* management, which assesses access to unsecured funding sources and the liquidity characteristics of the asset inventory. This involves an evaluation of asset liquidity risk.

Finally, this information is integrated in a *strategic* perspective, which starts from current assets and liabilities as well as off-balance-sheet items. This information is used to build a **funding matrix**, which details the funding requirements for various maturities. Any **funding gap** should be covered by plans to raise additional funds, either through borrowing or by asset sales.

26.4.2 Funding Gaps

Table 26.2 gives a hypothetical example in a pure **run-off mode** (i.e., with no new business or rollover of funding). Here, the funding matrix starts from balance sheet items, loans, retail deposits, other short-term debt, and long-term debt.

These items have cash flows that are either fixed or stochastic and maturities that are either fixed or stochastic. For example, coupons and amortization payments of fixed-rate debt have both fixed cash flows and maturities. In a second

TABLE 26.2 Example of Funding Gap Analysis

		Time Profile						
	Balance	**O/N**	**7D**	**14D**	**1M**	**3M**	**1Y**	**Cumulative**
Funding matrix:								
Loans	100	5	5	3	15	5	5	38
Retail deposits	−50	−5	−5	−5	−8	−5	−5	−33
Short-term debt	−30	−10	−5	−5	−5	−5	0	−30
Long-term debt	−30	0	0	0	−5	0	0	−5
Total: Funding gap		−10	−5	−7	−3	−5	0	−30
Gap closure:								
Cash	5	5	0	0	0	0	0	5
Unencumbered securities	20	10	8	2	0	0	0	20
Total		15	8	2	0	0	0	25
Net funding gap:		5	3	−5	−3	−5	0	−5
Cumulative		5	8	3	0	−5	−5	

category are items such as floating-rate loans and bonds where cash flows are stochastic but maturities are deterministic. In a third category are items such as callable bonds or loans with flexible redemption schedules where maturities are stochastic but cash flows are deterministic. In the last category are items with stochastic cash flows and maturities, such as retail deposits, drawdowns on committed credit lines, and revolving loans. Stochastic cash flows or maturities require modeling based on market experience and product knowledge.

In Table 26.2, the initial balance of loans is 100. The table lays out the cash inflows from this amount for various maturity buckets. A portion of these loans normally will be paid back over the next year. Next are retail deposits, short-term and long-term debt, for which cash outflows are forecast over the horizon. Note that all of the short-term debt is expected to be repaid within one year. The total creates a time profile of the **funding gap**. In this case, the cumulative funding gap to one year is −30.

The next part of the table lays out **gap closure** items. For instance, cash can be used immediately to cover funding outflows. Unencumbered securities can be sold over time as a function of their asset liquidity risk. The sum of the funding gap and the gap closure items creates the expected net funding gap. The longer the period with a positive flow, the safer the bank. In this case, the **survival period**, which is the time until the cumulative net funding gap becomes negative, is one month.

26.4.3 Stress Tests

Risk management is about dealing with the unexpected, however. Hence, institutions should also evaluate stress scenarios where cash flows deviate from their expected paths and sources of funding are unexpectedly cut off.[7] Institutions should consider a broad range of scenarios, including institution-specific, country-specific, and marketwide scenarios. An example of a country-specific scenario would be sudden restrictions on currency convertibility.

In addition, in situations of extreme stress, funding liquidity risk is likely to interact negatively with asset liquidity risk, as it may become more difficult to sell assets under these conditions. Due to the self-fulfilling nature of reputation risk, an institution's perceived liquidity problems can undermine its ability to sell its assets at a reasonable cost.

26.4.4 Controlling Liquidity Risk

Liquidity risk can be controlled by various means, including reliance on more stable sources of funding and diversification across sources of funds, geographical

[7] In theory, the entire distribution of cash flows could be estimated instead of a stress test, which is just a particular realization. This leads to **liquidity at risk** (LAR) models. The problems, however, are that liquidity behavior is difficult to model and that recent historical data may not be relevant.

location, and debt maturities. Similarly, asset liquidity risk can be controlled by setting limits on certain markets or products and by means of diversification. Funding gaps should also be subject to limits over various horizons. Some regulators do require minimum levels of liquid assets, limits on maturity mismatches, or limits on the reliance on a particular funding source.

Another tool to control liquidity risk is to penalize business units or instruments that can generate claims on liquidity. According to the Senior Supervisor Group (SSG) 2008 report, institutions that performed better than others during the crisis had adopted a firmwide perspective that explicitly accounted for liquidity risk.[8] These firms had charged business lines appropriately for building contingent liquidity exposures to reflect the cost of obtaining liquidity in a more difficult market environment. Institutions that did well also had effective management of funding liquidity, capital, and their balance sheets.

26.4.5 Contingency Funding Plans

The goal of **contingency funding plans** (CFPs) is to establish a plan of action should one of the liquidity stress scenarios develop. In a crisis situation, management usually does not have much time to react, which is why a pre-established plan is useful. The CFP should define the trigger events, clear lines of responsibility for implementation, and a plan for alternative sources of funding. It should also take into account the reputational effects of announcement of the execution of the funding plan.

Generally, public disclosures about liquidity risk management procedures should help reassure investors that the institution has developed a process to deal with liquidity risk. A bank that is perceived as having liquidity is less likely to lose the confidence of funds providers. The following is an example of liquidity risk disclosures by Deutsche Bank.

Example: Deutsche Bank's Liquidity Management

Deutsche Bank (DB) is a leading German commercial bank. As of December 2007, it had 2,020 billion euros in assets and 329 billion euros in risk-weighted assets. With 37 billion euros in shareholder equity, its Tier 1 capital ratio was 8.6%.

DB's liquidity risk management approach starts at the *intraday level* forecasting daily cash flows and factoring in access to central banks. It then covers *tactical* liquidity risk management dealing with the access to unsecured funding sources and the liquidity characteristics of its assets. For example, the bank generates 25% of its unsecured funding from retail deposits and 20% from capital markets. In terms of asset liquidity, the bank assigns liquidity values to different assets; it also

[8] See Senior Supervisor Group, *Observations on Risk Management Practices during the Recent Market Turbulence* (Basel: BIS, 2008).

holds a portfolio of 25 billion euros in highly liquid securities to protect against short-term liquidity squeezes. Finally, the *strategic* perspective comprises the maturity profile of all assets and liabilities on its balance sheet as well as its issuance strategy.

The bank employs stress-testing and scenario analysis to evaluate the impact of sudden stress events on its liquidity position. The hypothetical events encompass external shocks, such as market risk events, emerging market crises, and systemic shocks, as well as internal shocks, such as operational risk events and ratings downgrades (e.g., from AA to AA− for a one-notch downgrade). Under each of these scenarios, the bank assumes that all maturing loans to customers will need to be rolled over and require funding, whereas rollover of liabilities will be partially impaired, resulting in a funding gap. The bank then models the steps it would take to counterbalance the resulting net shortfall in funding, which include selling assets and switching from unsecured to secured funding. For each scenario, the table shows the cumulative funding gap over an eight-week horizon, in billions of euros, and how much counterbalancing liquidity could be generated.

Scenario	Funding Gap	Gap Closure
Market risk	5.5	98.9
Emerging markets	27.7	117.1
Systemic shock	20.4	70.9
Operational risk	13.9	106.7
One-notch downgrade	28.1	129.3
Three-notch downgrade	108.6	129.3

EXAMPLE 26.8: FRM EXAM 2007—QUESTION 57

You have been asked to review a memo on how market liquidity is affected by shocks to the financial system. Which of the following observations made in the memo is *incorrect*?

a. In periods of acute market stress, market liquidity typically increases in the most liquid markets, creating a self-correcting loop that will ultimately remove downward pressure on asset prices.
b. Evaporation of market liquidity is an important factor in determining whether and at what speed financial disturbances become financial shocks with potentially systemic threats.
c. Market shocks may not be reflected in mark-to-market portfolio values immediately for portfolios with illiquid assets. As a result, it is possible for market shocks to have delayed effects on financial institutions.
d. The impact of a market shock on the liquidity of a specific asset depends on the characteristics of the investors who own the asset.

EXAMPLE 26.9: FRM EXAM 2009—QUESTION 7-12

Your CRO asks you to prepare a list of early warning indicators for liquidity problems for your bank. Which of the following are early warning indicators of a potential liquidity problem?

I. Rapid asset growth, especially when funded with potentially volatile liabilities
II. Growing concentrations in assets or liabilities
III. An increase of the weighted average maturity of liabilities
IV. Reduction in the frequency of positions approaching or breaching internal or regulatory limits
V. Narrowing debt or credit default swap spreads
VI. Counterparties that request additional collateral for credit exposures
VII. Increasing redemptions of CDs before maturity

a. I, II, VI, and VII
b. I, III, V, and VI
c. II, IV, V, and VII
d. I, V, VI, and VII

26.5 IMPORTANT FORMULAS

Relative bid-ask spread: $S = \frac{[P(\text{ask}) - P(\text{bid})]}{P(\text{mid})}$

Liquidity-adjusted VAR (LVAR), relative to the mean: $\text{LVAR} = \text{VAR} + L_1 = W[\alpha\sigma + \frac{1}{2}(S)]$

Worst-case liquidity-adjusted VAR (LVAR), relative to the mean: $\text{LVAR} = \text{VAR} + L_2 = W[\alpha\sigma + \frac{1}{2}(\bar{S} + \alpha'\sigma_S)]$

26.6 ANSWERS TO CHAPTER EXAMPLES

Example 26.1: FRM Exam 2003—Question 15

c. The yield spread between on-the-run and off-the-run securities reflects a liquidity premium because the bonds are otherwise nearly identical. In answers a. and d., asset and funding risk should be interchanged. Finally, for b., a flight to quality increases the yield spread.

Example 26.2: FRM Exam 2002—Question 36

c. Compare two stocks. The liquid stock typically has higher trading volumes and smaller bid-ask spreads, so b. and d. are true. It also has greater depth, meaning that large quantities can be traded without affecting prices too much, so a. is true. As a result, the remaining answer, c. must be wrong. There is no necessary relationship between trading activity and volatility.

Example 26.3 FRM Exam 2007—Question 78

c. The maximum weight w is given by $\$3,000 \times w = 30\% \times \$2.4$, or $w = 0.024\%$.

Example 26.4: FRM Exam 2000—Question 74

b. In a crash, bid-offer spreads widen, as do liquidity spreads. Statement I. is incorrect because Treasuries usually rally more than swaps, which leads to *greater* losses for a portfolio short Treasuries than swaps.

Example 26.5: FRM Exam 2007—Question 116

a. The regular VAR relative to the initial portfolio value is $\text{VAR} = W(\alpha\sigma - \mu) = \$5,000(2.33 \times 2\% - 1\%) = \$183$. (Note that this estimate of the mean is abnormally high.) To this must be added $L_2 = \frac{1}{2} W(\bar{S} + \alpha'\sigma_S) = \frac{1}{2}\$5,000(0.5\% + 2.33 \times 1\%) = \$70.75$, for a total of $254.

Example 26.6: FRM Exam 2009—Question 7-7

c. Conventional VAR is $\$72 \times 1,000 \times 1.24\% \times 1.645 = \$1,469$. The spread effect is $\$0.16 \times 1,000 = \$80$, for a total of $1,549. As usual, we see that the spread liquidity component is small.

Example 26.7: FRM Exam 2008—Question 2-20

c. Statement a. is not correct because a wider TED spread is consistent with the Fed lowering rates. Statement b. is not correct because the fed funds rate is for collateralized loans, whereas Eurodollar rates are for uncollateralized deposits. Statement d. is incorrect because a wider TED spread means that the cost of bank borrowing goes up, not down.

Example 26.8: FRM Exam 2007—Question 57

a. Statement b. is correct, as proved by the events of 2007. Statement c. correctly states that the prices of illiquid assets reflect a delayed reaction to events. Statement d. explains that asset liquidity depends on investor positions, which is correct; an asset that is mainly owned by leveraged investors can experience sharp swings in price if the investors are forced to sell.

Example 26.9: FRM Exam 2009—Question 7-12

a. Statement I. is correct; this is the Northern Rock story. Statement II. is also a problem because it means higher probability of either asset or funding risk. Statement III. is not a correct answer, because longer liabilities reduce the probability of a near-term funding problem. Statement IV. is not a correct answer, because this is market risk. Statement V. is not a correct answer, because a problem would arise from widening, not narrowing spreads. Statement VI. is correct because collateral demands create a claim on liquidity. Statement VII. is correct because this requires cash for repayment.

CHAPTER 27

Firmwide Risk Management

This chapter turns to best practices for firmwide management of financial risks. The financial industry has come to realize that risk management should be implemented on a firmwide basis, across business lines and types of risk. This is due to a number of factors, including (1) increased exposures to more global sources of risk as institutions expand their operations, (2) interactions between risk factors, and (3) linkages in products across types of market risks as well as types of financial risks. These linkages make it important to consider correlations among risks and products. Interactions between types of risk bear emphasis, as they are too often ignored.

There is another, more pragmatic reason for attempting to measure risk on a wider basis. The industry has made great strides in the measurement of market risk, credit risk, and even operational risk. Once measured, risk can be penalized, as with **risk-adjusted return on capital** (RAROC) measures. The danger with this approach, however, is that this creates an incentive to move risk to areas where it is not well measured or controlled.

All of these reasons explain the trend toward integrated, or firmwide, risk management. **Integrated risk management** provides a consistent and global picture of risk across the whole institution. This requires measuring risk across all business units and all risk factors, using consistent methodologies, systems, and data.

Section 27.1 presents the framework for integrated risk management and reviews different types of financial risks. Section 27.2 summarizes best practices reports that have shaped and continue to shape the risk management profession. Section 27.3 then turns to a description of organizational structures that are consistent with these best practices. Section 27.4 shows how traders can be controlled through compensation adjustment and limits. Finally, Section 27.5 explains how risk measures can be integrated in the measurement of performance of traders and business units through RAROC-type measures.

27.1 INTEGRATED RISK MANAGEMENT

Integrated risk management is founded on an **economic capital** (EC) framework, which can be defined as the methods and practices that allow institutions to

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assess risk consistently and to attribute capital to cover the economic effects of risk-taking activities.

27.1.1 Types of Risk

Risk aggregation begins with a classification of risk types along their economic nature.

- **Market risk** is the risk of loss due to movements in the level or volatility of market prices. This is covered in Chapters 12 to 18.
- **Credit risk** is the risk of loss due to the fact that counterparties may be unwilling or unable to fulfill their contractual obligations. This is covered in Chapters 19 to 24.
- **Operational risk** is generally defined as the risk of loss resulting from failed or inadequate internal processes, systems, and people, or from external events. This is covered in Chapter 25.
- **Business risk** is the risk of loss in the firm's earnings, which arises from declines in revenues that cannot be offset by decreases in costs.

Some banks also measure separately other risks, such as liquidity risk, which is covered in Chapter 26, or private equity risk.

Institutions should attempt to measure all of these risks in a consistent fashion and across the entire firm. Otherwise, risk will tend to flow to areas where it is least penalized, or with the weakest measures.

The previous chapter illustrated this point with liquidity risk during the recent credit crisis. Banks that did not properly assess this risk or penalize business units for the claims on liquidity they generated ended up with much worse liquidity problems than others did. Another example is the banks that had weak systems to measure the risk of senior tranches of subprime-backed debt. Because their systems showed very little risk, these institutions did not monitor large buildups in these securities. During 2007, for example, UBS ended up with losses of $19 billion on these securities, which were found in the collateralized debt obligation (CDO) warehousing book, in the trading book, in the liquid Treasury book, and in a hedge fund subsidiary.[1] Apparently, there was no monitoring of either net or gross concentrations of positions in this asset class at the firmwide level.

27.1.2 Risk Interactions

Risk categories do not always fit into neat, separate silos. Operational risk can create market and credit risk, and vice versa. For instance, collateral payments in swaps decrease credit risk by marking to market on a regular basis but create a greater need for cash flow management, which increases operational and liquidity risk. The reverse can also occur as an operational failure; for example, incorrect

[1] UBS, *Shareholder Report on UBS's Write-Downs* (Zurich: UBS, 2008).

confirmation of a trade can lead to inappropriate hedging or greater market risk. Incorrect data entry of swap terms can create incorrect market risk measurement as well as incorrect credit exposures.

Another important example is the interaction between market risk and credit risk. **Wrong-way trades** are those where market risk amplifies credit risk. Consider, for example, a swap between a bank and a speculator. If the bank loses money on the swap, credit risk is not an issue, because the bank has no credit exposure. If the bank makes a large profit on the swap, however, this must be at the expense of the speculator. If the loss to the other party is sufficiently large, the speculator could default precisely because of the swap. Therefore, such trades are inherently more dangerous than those where the counterparty is a hedger. In the case of a hedger, the loss on the swap should be offset by a gain on the hedged position. As a result, hedging trades are safer for the bank. Thus, complex interactions can arise across risk types.

27.1.3 Risk Aggregation

Once all risk categories have been properly identified and measured, the next step in the aggregation involves translating the distributions in the same risk currency and to the same horizon, usually one year. Typically, however, market risk is managed and measured over shorter periods, measured in days. The usual conversion uses scaling-up methods such as the square root of time rule. This assumes a constant risk profile over the year. The distributions can then be summarized by a value at risk (VAR) type of metric using the same confidence level.

The next step is the combination of distributions or risk measures across the risk factors. In theory, interactions between different types of risk should be taken into account.

In practice, most banks that now report VAR estimates for market, credit, and operational risk often simply add up the three risk measures to get an estimate of the bank's total risk. As an example, JPMorgan estimated that its economic capital was \$50 billion, \$15 billion, and \$9 billion for credit, market, and operational risk, respectively, which sum to \$74 billion. This **simple summation**, however, generally overstates the risk because it assumes that the worst loss will occur simultaneously across the three risk types.[2]

The next approach applies a **fixed diversification percentage**. This is a simple extension of the summation approach but applies a fixed diversification benefit. In the previous example, a fixed diversification percentage of 20% reduces economic capital from \$74 to \$59 billion. This method is simple and robust but is not sensitive to the actual interactions between components.

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[2] This is not necessarily the case, however. VAR may not be subadditive, as shown in Chapter 15. In practice, other types of risk, which are much harder to measure, can conspire to create more risk for the total entity than the sum of the individual risks. For very large institutions, liquidity is an example. As Long-Term Capital Management (LTCM) has shown, the potential loss from liquidating a \$100 billion position is greater than the sum of losses from liquidating 10 positions of \$10 billion separately. In addition, **reputational risk** can cause problems to one unit to spread and affect funding costs for the entire firm.

A third method of aggregation is to weigh the risk measures by an estimated **variance-covariance matrix**. This method is relatively simple and intuitive and accounts for variations in actual interactions; using correlations, however, implies linear relationships. Usually, banks assume a high correlation between market and credit risk, a lower correlation between business and other risks, and a very low correlation between operational and other risks. To be conservative, correlations are typically rounded up.

Few banks attempt more technically sophisticated approaches, such as using **copulas** to reflect dependencies between the risk factor distributions. This approach is more flexible and allows nonlinearities but is more difficult to implement.

This is related to the **full simulation approach**, which is potentially the most flexible and accurate method, but also the most computationally demanding.

27.1.4 Definitions of Capital

This aggregation provides a firmwide measure of risk. **Economic capital** (EC) is defined as the largest acceptable loss a firm is willing to suffer over a specified period and at a specified confidence level. In other words, EC is a VAR-type measure that represents the amount of capital a firm allocates to self-insurance. This should reflect the risk aversion of the board of directors and senior management.

Economic capital is distinct from **reserves**. Firms set aside reserves in preparation for *expected losses*. In contrast, capital is needed to provide a cushion against *unexpected losses*.

Normally, the amount of actual **equity capital** a firm carries on its balance sheet should exceed economic capital. It should also exceed the amount of **regulatory capital**, as required by the institution's regulator, which otherwise might close it down.

Economic capital can exceed regulatory capital or not. In the former case, regulations are neither binding nor relevant. In the second case, however, the firm is forced to carry too much capital, which amounts to a tax. With inconsistent capital regulations across the globe, this could create a serious competitive disadvantage if other countries have less onerous capital requirements.

27.1.5 Silo versus Integrated Approaches

Firmwide capital for financial conglomerates can be computed using a silo approach or building block approach. In the **silo approach**, capital requirements are computed separately for each risk type or business and then added together. This silo approach applies to regulatory capital. Consider, for example, a holding company that has a banking and insurance affiliate. The capital requirement at the top level is simply the sum of capital requirements for each subsidiary.

This has three disadvantages. First, the capital requirements do not take into account diversification effects, and could be too high or, in some cases, too low.

Second, different regulators could impose inconsistent capital requirements for the same economic risk, which is inefficient and could lead to regulatory arbitrage. As we will see in Chapter 28, the Basel II rules impose different capital charges for credit default swaps (CDSs) in the trading and banking books of commercial banks. Because the capital charge is lower in the trading book, this has led, not unexpectedly, to massive shifts in CDSs from the banking book to the trading book. Similarly, capital requirements could differ for the same risk in commercial banks or in insurance companies.

The third issue is that some affiliate could be subject to weaker or nonexistent capital requirements. A good example is that of American International Group (AIG), the U.S. insurance company that received a $160 billion bailout from the government. AIG exploited a huge gap in the regulatory system, with a **Financial Products** (FP) division that sold CDSs to insure $441 billion worth of securities that led to huge losses. This division was operating without effective regulatory oversight but was attached to a large and stable insurance company.[3] This regulatory oversight has been closed under the recently enacted **Dodd-Frank Wall Street Reform and Consumer Protection Act**, which has established a **Financial Stability Oversight Council**, which monitors institutions that could pose a threat to the financial system and may compel the Federal Reserve to assume oversight of institutions that pose a systemic risk.

In contrast, firmwide economic capital can take an **integrated approach**. This requires aggregation at three levels.

1. *Portfolio level*. These include, for example, commercial loans that are part of the credit book of a bank (this is the portfolio).
2. *Business unit level*. These include, for example, market, credit, and business risks within a banking subsidiary (this is the business unit).
3. *Holding company*. These include, for example, banking and insurance subsidiaries of a holding company.

The **building block** approach aggregates risk at these three consecutive levels. In general, the greatest diversification benefits are achieved at the lowest, portfolio level due to the large number of positions and low concentrations. Diversification benefits are typically 50% or more. Diversification effects at the next level are less because of fewer risk factors. Benefits are typically around 20% for banks but higher for insurers, around 40%. Finally, diversification benefits at the holding company level are the lowest, around 5% to 10%.

27.1.6 Illustration

Table 27.1 illustrates a firmwide economic capital analysis, as reported by Deutsche Bank. The bank estimates separately the worst loss at a 99.98% level of

[3] In fact, AIGFP was supervised by the **Office of Thrift Supervision** (OTS). The OTS has proved ineffective at regulating such a large and complex operation.

TABLE 27.1 Firmwide Economic Capital—Deutsche Bank (Millions of Euros)

By Source	2009	2008	2007	2006	2005	2004	2000
1. Credit risk	7,453	8,986	8,506	7,351	7,125	5,971	8,200
2. Market risk	12,515	8,794	3,481	2,994	3,042	5,476	3,700
Trading	4,613	5,547	1,763	1,605	1,595	1,581	
Nontrading	7,902	3,247	1,718	1,389	1,447	3,895	
3. Operational risk	3,493	4,147	3,974	3,323	2,270	2,243	2,800
Diversification	(3,166)	(3,134)	(2,651)	(2,158)	(563)	(870)	
Total	20,295	18,793	13,310	11,509	11,874	12,820	14,700
Actual core capital	34,406	31,094	28,320	23,539	21,898	18,727	23,504

confidence over one year for the three risk categories—market, credit, and operational risk. Market risk includes both trading risk as well as that of the banking book (e.g., the interest rate risk of loans and deposits). Nontrading market risk is sometimes referred to as **asset/liability management** (ALM) risk. For trading risk, the traditional one-day, 99% VAR measures have been extrapolated to the parameters for economic capital.

Except for an unusual 2009, credit risk accounts for the largest fraction of economic capital at Deutsche Bank, as is typical of a commercial bank or universal bank with substantial lending activities. Market risk is typically second in importance.[4] Operational risk is also high but in this case has decreased in recent years due to the addition of insurance coverage.

Recent surveys have confirmed that, for a broad sample of global banks, economic capital reflects primarily credit risk, then operational risk, then market risk.[5] In contrast, market risk is relatively more important for investment banks and life insurance companies.

The table also shows that the bank made no allowance for diversification effects across risk categories in 2000. This effectively assumes perfect correlations. Over time, the bank has refined its risk models, accounting for a diversification effect. As of 2009, total economic capital was estimated at 20 billion euros. The bank had actual core capital of 34 billion euros, which is well above its estimate of economic capital.[6]

For all their apparent elegance, however, these models have limitations. These are complex models that require many assumptions and simplifications, as well as fitting distributions and parameters. This creates model risk. In addition, because the confidence level is so high, the estimates of economic capital must be rather imprecisely measured. Reporting economic capital with five significant digits is rather ridiculous, and possibly misleading in terms of its implied precision. Yet

[4] In this case, the increase in 2009 was due to the acquisition of Deutsche PostBank, which contributed €4.3 billion to the increase in market nontrading risk.

[5] A. Kuritzkes, T. Schuermann, and S. Weiner, "Risk Measurement, Risk Management, and Capital Adequacy in Financial Institutions" (working paper, Wharton, 2003).

[6] Core capital is mainly book equity, as explained in Chapter 28.

financial reports never provide any information on the precision of risk numbers. Finally, these measures assume that all risks have been identified and properly measured. As discussed in Chapter 1, this is not likely to be the case. After all, risks due to "unknown unknowns" are by definition impossible to measure.

27.1.7 Integrated Risk Management and Value Added

Generally, **integrated risk management** is the process of managing all corporate risks in an integrated framework. This starts with an attempt to identify all major risks and to measure them on an aggregate basis.

Of course, the objective should not be to avoid all risks but rather to take risks where the firm has a comparative advantage. Other risks can be hedged, especially if it is inexpensive to do so in external capital markets. This will reduce the volatility of the firm value and the probability of financial distress. In addition, more stable earnings enable the firm to undertake worthy projects, which avoids the **underinvestment problem** that arises when a firm is cash constrained.

At the same time, individual projects and businesses should be evaluated in relation to their contribution to the total risk of the firm. Projects with greater contribution to the firm's risk should be penalized with a higher hurdle rate.

Therefore, integrated risk management helps guide the firm toward taking core risks, which are strategic, and business risks where the firm can add value because it knows more than outsiders. In contrast, noncore risks can be laid off in financial markets.

EXAMPLE 27.1: FRM EXAM 2008—QUESTION 4-24

Which of these statements about economic and regulatory capital are *valid*?

I. Regulatory capital seeks soundness and stability in the banking system by ensuring that there is enough capital in the banking system.
II. Economic capital is designed to keep a financial institution solvent at a specified confidence level.
III. For an individual bank, economic capital is always less than regulatory capital.
IV. The determination of economic capital, and its allocation to the various business units, is a strategic decision process that affects the risk/return performance of the business units and the bank as a whole.

a. II and IV only
b. I, II, III, and IV
c. I, II, and IV only
d. I and IV only

EXAMPLE 27.2: FRM EXAM 2009—QUESTION 7-9

Tower Bank approaches economic capital and risk aggregation by first estimating the stand-alone economic capital for individual risk factors. In a second step, the bank aggregates risks based on the relative amounts of economic capital allocated to these risks, taking into account the correlations between risk factors. Which of the following variables is *not* a primary driver of the diversification benefit that accrues from aggregation?

a. The number of risk positions
b. The size of the portfolio
c. The concentration of those risk positions, or their relative weights in a portfolio
d. The correlation between the positions

EXAMPLE 27.3: FRM EXAM 2002—QUESTION 103

Consider a bank that wants to have an amount of capital so that it can absorb unexpected losses corresponding to a firmwide VAR at the 1% level. It measures firmwide VAR by adding up the VARs for market risk, operational risk, and credit risk. There is a risk that the bank has too little capital because

a. It does not take into account the correlations among risks.
b. It ignores risks that are not market, operational, or credit risks.
c. It mistakenly uses VAR to measure operational risk because operational risks that matter are rare events.
d. It is meaningless to add VARs.

EXAMPLE 27.4: FRM EXAM 2006—QUESTION 109

Large banks typically allocate risk capital for credit, operational, and market/ALM risks. Which of the following statements ranks the typical amount of risk capital allocated to these different risks *correctly*?

a. Market/ALM risk requires more risk capital than credit risk.
b. Credit risk requires more risk capital than market/ALM risk, which requires more risk capital than operational risk.
c. Market/ALM risk requires more risk capital than operational risk but less than credit risk.
d. Credit risk requires more risk capital than operational risk, which requires more risk capital than market/ALM risk.

EXAMPLE 27.5: FRM EXAM 2005—QUESTION 33

Counterparty A is an American company with manufacturing operations in Indonesia and its main customers in the United States, while counterparty B is an American company that manufactures its goods domestically and exports solely to Indonesia. Which one of the following transactions with either counterparty will be a wrong-way exposure for a bank?

a. A five-year plain-vanilla IDR/USD cross-currency swap between the bank and counterparty A where the bank is USD interest rate receiver
b. A five-year plain-vanilla IDR/USD currency option sold by the bank to counterparty A for it to buy IDR at a certain rate
c. A five-year plain-vanilla IDR/USD cross-currency swap between the bank and counterparty B where the bank is USD interest rate receiver
d. A five-year plain-vanilla IDR/USD currency option bought by the bank from counterparty B for the bank to buy IDR at a certain rate

EXAMPLE 27.6: FRM EXAM 2008—QUESTION 4-29

Your bank calculates a one-day 95% VAR for market risk, a one-year 99% VAR for operational risk, and a one-year 99% VAR for credit risk. The measures are $100 million, $500 million, and $1 billion, respectively. Operational risk is defined to include all risks that are not market risks and credit risks, and these three categories are mutually uncorrelated. The market risk VAR assumes normally distributed returns, and the bank expects to be successful to keep its market risk VAR at that level for the whole year. Your boss wants your best estimate of a firmwide VAR at the 1% level. Among the following choices, your best estimate is:

a. $1.7 billion
b. $1.94 billion
c. $2.50 billion
d. It is impossible to aggregate risks with different distributions having only this information.

27.2 BEST PRACTICES REPORTS

Best practices in the industry have evolved from the lessons of financial disasters. Some well-publicized losses in the early 1990s led to the threat of regulatory action against derivatives.

Financial institutions then realized that it was in their best interests to promote a set of best practices to forestall regulatory action. This led to the Group of Thirty (G-30) report, which was issued in July 1993. The 1995 Barings failure was followed by an in-depth report from the Bank of England in July. Similarly, the 1998 near failure of Long-Term Capital Management (LTCM) was analyzed in a report produced by the Counterparty Risk Management Policy Group (CRMPG) in June 1999, followed by updates in 2005 and in 2008. These reports added to the collective wisdom about best practices.

27.2.1 The G-30 Report

The Group of Thirty (G-30) is a private, nonprofit association, consisting of senior representatives of the private and public sectors and academia. In the wake of the derivatives disasters of the early 1990s, the G-30 issued a report in 1993 that has become a milestone document for risk management.[7] The report provides a set of 24 sound management practices. The most important ones are summarized here.

- **Role of senior management.** *Dealers and end-users should use derivatives in a manner consistent with the overall risk management and capital policies approved by their boards of directors.... Policies governing derivatives use should be clearly defined, including the purposes for which these transactions are to be undertaken. Senior management should approve procedures and controls to implement these policies, and management at all levels should enforce them.*
- **Measuring market risk.** *Dealers should use a consistent measure to calculate daily the market risk of their derivatives positions and compare it with market risk limits. Market risk is best measured as "value at risk" using probability analysis based on a common confidence interval and time horizon.*
- **Stress simulations.** *Dealers should regularly perform simulations to determine how their portfolios would perform under stress conditions.*
- **Independent market risk management.** *Dealers should have a market risk management function, with clear independence and authority, to ensure that the following responsibilities are carried out: risk limits; stress tests; revenue reports; back-testing VAR; review of pricing models and reconciliation procedures.*
- **Independent credit risk management.** *Dealers and end-users should have a credit risk management function with clear independence and authority, ... responsible for: approving credit exposure measurement standards; setting credit limits and monitoring their use; reviewing credits and concentrations of credit risk; reviewing and monitoring risk reduction arrangements.*

These recommendations stress the need for risk management functions with "clear independence and authority."

[7] Group of Thirty, *Derivatives: Practices and Principles* (New York: Group of Thirty, 1993).

27.2.2 Bank of England Report on Barings

Violation of the fundamental principle of separation of functions was the primary cause of the Barings failure. Nick Leeson had control over both the front office and the back office. This organizational structure allowed him to falsify trading entries, hiding losses in a special account.

But new lessons were also described in the main report on Barings, produced by the Bank of England (BoE).[8] The report mentioned for the first time **reputational risk**. This is the risk of indirect losses to earnings arising from negative public opinion. These losses are distinct from the direct monetary loss ascribed to an event.

The BoE report listed several lessons from this disaster.

- **Duty to understand.** Management teams have a duty to understand fully the businesses they manage. Senior Barings management later claimed they did not fully understand the nature of their business.
- **Clear responsibility.** Responsibility for each business activity must be clearly established. Barings had a *matrix* structure, with responsibilities assigned by product and region, which made it harder to assign responsibility to one person.
- **Relevant internal controls.** Internal controls, including clear segregation of duties, is fundamental to any effective risk control system.
- **Quick resolution of weaknesses.** Any weakness identified by an internal or external audit must be addressed quickly. In the Barings case, an internal audit report in the summer of 1994 had identified the lack of segregation of duties as a significant weakness. Yet this was not addressed by Barings' top management.

27.2.3 The CRMPG Report on LTCM

The near failure of the hedge fund Long-Term Capital Management (LTCM) also led to useful lessons for the industry. The Counterparty Risk Management Policy Group (CRMPG) was established in the wake of the LTCM setback to strengthen practices related to the management of financial risks.

The CRMPG consists of senior-level practitioners from the financial industry, including many banks that provided funding to LTCM. The industry came under criticism for allowing LTCM to build up so much leverage. Apparently, loans to LTCM were fully collateralized as to their current, but not potential, exposure. In fact, it was fear of the disruption of markets and the potential for large losses that led the New York Federal Reserve Bank to orchestrate a bailout of LTCM.

In response, the CRMPG report provides a set of recommendations, which are summarized here.[9]

- **Information sharing.** Financial institutions should obtain more information from their counterparties, especially when significant credit exposures are

[8] Bank of England, *Report of the Board of Banking Supervision Inquiry into the Circumstances of the Collapse of Barings* (London: HMSO Publications, 1995).
[9] Counterparty Risk Management Policy Group, *Improving Counterparty Risk Management Practices* (New York: CRMPG, 1999).

involved. This includes the capital condition and market risk of the counterparty. This information should be kept confidential.

- **Leverage, market risk, and liquidity.** Financial risk managers should monitor the risks of large counterparties better, focusing on the interactions between leverage, liquidity, and market risk.
- **Liquidation-based estimates of exposure.** When exposures are large, information on exposures based on marked-to-market values should be supplemented by liquidation-based values. This should include current and potential exposures.
- **Stress-testing.** Institutions should stress-test their market and credit exposure, taking into account the concentration risk to groups of counterparties and the risk that liquidating positions could move the markets.
- **Collateralization.** Loans to highly leveraged institutions should require appropriate collateral, taking into account liquidation costs.
- **Management responsibilities.** Senior management should convey clearly its tolerance for risk, expressed in terms of potential losses. The function of risk managers is then to design a reporting system that enables senior management to monitor the risk profile.

Perhaps the most important lesson from LTCM for lenders is the relationship between market risk and credit risk. The G-30 report recommends the establishment of market and credit risk functions but does not discuss integration of these functions. When LTCM was about to fail, lenders realized that they had no protection for potential exposure and that many of their positions were similar to those of LTCM. Had LTCM defaulted (a credit event), lenders could have lost billions of dollars from market risk.

The second lesson from LTCM is the need for risk managers to make adjustments for large or illiquid positions. The third lesson from LTCM is that institutions should perform systematic stress tests, because VAR models based on recent history can fail to capture the extent of losses in a disrupted market. This seems obvious, as VAR purports to give only a first-order magnitude of the size of losses in a normal market environment.

27.2.4 The CRMPG II and III Reports

The CRMPG provided an update in a next report, called CRMPG II.[10] CRMPG II notes that many of the earlier recommendations had been put into practice. In particular, there is now a greater focus on liquidity-based adjustments to close-out values. However, the report notes that market developments have introduced new risks, including the dispersion of credit products with embedded leverage among industry participants, which lessens transparency.

[10] Counterparty Risk Management Policy Group, *Toward Greater Financial Stability: A Private Sector Perspective* (New York: CRMPG, 2005).

In response to the credit crisis that started in 2007, the CRMPG has provided a new set of recommendations, in a report called CRMPG III.[11]

- **Improved corporate governance.** Comparing how various institutions fared during the crisis, it has been apparent that the culture of corporate governance is important. Institutions that rely heavily on judgment, communication, and coordination across the entire firm did better than others. Also, the system of incentives has produced behavior that focused on short-tem profits at the expense of financial stability. Incentives need to be better aligned with the long-term success of institutions and their risk tolerance.
- **Risk monitoring.** Financial institutions should be able to monitor risk concentrations to asset classes on a net and gross basis and to provide coherent reports to top management.
- **Estimating risk appetite.** Financial institutions should regularly conduct comprehensive exercises aimed at estimating risk appetite, using stress tests and a combination of qualitative and quantitative factors.
- **Focusing on contagion.** Financial institutions should regularly assess the effect of **contagion**, or the channels and linkages through which local financial disturbances can take on systemic characteristics.

27.2.5 The Senior Supervisor Group Reports

In 2008, the **Senior Supervisor Group** (SSG) issued an influential report on risk management practices at major financial institutions.[12] A striking observation is the range of quality of risk management practices. Many institutions did very poorly during the crisis, whereas, others did relatively well. The SSG drew lessons from the comparison of winners and losers, summarized in Table 27.2.

In general, institutions that lost the most had a hierarchical business organization where decisions were made from the top with very little feedback. Top management made a strategic decision to expand into the structuring, warehousing, and trading of collateralized debt obligations (CDOs) in order to generate earnings.

This expansion, however, was accompanied with poor risk management practices, such as little industry analysis and the absence of notional or sector limits. In addition, many of the affected firms did not account for the risk of **pipeline positions** in asset-backed securities (ABSs) and loans. These arise when positions are warehoused, or kept temporarily, by the bank until securitized and sold. Finally, losers were heavily exposed to **conduits** and **structured investment vehicles** (SIVs) because they did not recognize the contingent liquidity risk they created.[13] As a result, they did not charge business lines for potential claims on the banks' balance

[11] Counterparty Risk Management Policy Group, *Containing Systemic Risk: The Road to Reform* (New York: CRMPG, 2008).
[12] Senior Supervisor Group, *Observations on Risk Management Practices during the Recent Market Turbulence* (Basel: BIS, 2008).
[13] SIVs and conduits, which are described in Chapter 26, constitute what is called the **shadow banking system**. Like banks, they invest in assets such as loans and ABSs and issue debt. As a result,

TABLE 27.2 Differences in Risk Management Practices

Practice	Winners	Losers
Organizational structure	• Cooperative	• Hierarchical
Business model	• Avoided CDOs, SIVs	• Exposed to CDOs, SIVs
Firmwide risk analysis	• Shared information across the firm	• No prompt discussion of risks across the firm
Valuations	• Developed in-house expertise	• Relied on credit ratings
Management of liquidity	• Charged business lines for liquidity risk	• Did not consider contingent exposures
Risk measurement	• Qualitative and quantitative analysis	• Strict model application
	• Varied assumptions	• Mapped to corporate AAA
	• Tested correlations	• No test of correlations

sheets, which encouraged expansion into structured credit. This was made worse due to the existence of organizational silos, which prevented the identification and aggregation of risks at the top level.

Among losers, top management often did not pay attention to warning signals given by risk managers. In contrast, at firms that avoided significant losses, "risk management had independence and authority but also considerable direct interaction with senior business managers (SSG (2008), P. 9)." Their managers used judgment as well as more quantitative techniques. They identified valuation anomalies that were early warning signals about model pricing errors and changing market conditions.

Many of the poorly performing institutions failed to develop their own valuations models for complex structured credits and instead relied mostly on credit ratings. These institutions blindly applied models without consideration of their weaknesses and did not stress-test correlations. Forward-looking scenarios were not seriously considered by top management because of their alleged implausibility.

In summary, the SSG identified four firmwide risk management practices that differentiated the winners from the losers.

1. Effective firmwide identification and analysis of risks
2. Informative and responsive risk measurement and management reporting
3. Consistent application of independent and rigorous valuation practices
4. Effective management of funding liquidity and capital

In 2009, the SSG provided an update on the progress made since the first report.[14] It reported that banks had not fully addressed the issues raised previously. This is in part due to the need for considerable investment and experience in

they are exposed to the same liquidity mismatch. They are not regulated, however. During the credit crisis, many banks that had sponsored or guaranteed these structures had to provide support, which further depleted their capital. In December 2007, for example, Citigroup announced that it would absorb $49 billion worth of SIV assets on its balance sheet.

[14] SSG, *Risk Management Lessons from the Global Banking Crisis of 2008* (Basel: BIS, 2009).

risk management. In belated response, "virtually all firms have strengthened the authority of the risk management function and increased the resources devoted to it." This is good news for the risk management profession.

In addition, many firms have reviewed their compensation practices for "revenue producers" (i.e., traders and investment bankers). Assuredly, the primary goal is to attract and retain talent. Even so, compensation packages now attempt to incorporate the cost of risk, liquidity, and capital. Many banks have also implemented deferred compensation plans with longer vesting, which would help avoid situations where traders focus on short-term revenues while creating long-term risks for the bank. Typical examples are the retained positions in senior tranches of CDOs, which embed short positions in out-of-the-money options. Such investments create a regular income, similar to the premium received from shorting an option, which traders get paid from. However, this comes at the cost of infrequent disasters, which are then paid by the bank, or by taxpayers. Finally, the latest SSG report also highlighted liquidity risks that arose in 2008 and are discussed in Chapter 26.

EXAMPLE 27.7: FRM EXAM 2004—QUESTION 47

The failure of Barings Bank is a typical example of a lack in control pertaining to which one of the following risks?

a. Liquidity risk
b. Credit risk
c. Operational risk
d. Foreign exchange risk

EXAMPLE 27.8: FRM EXAM 2008—QUESTION 4-34

According to both the CRMPG II Report and the Basel Committee, rigorous stress-testing should be an important component of risk measurement and management. To improve the value of stress-testing exercises, firms should consider all of the following *except*:

a. Asking risk managers to define and clearly express firm loss tolerance levels
b. Identifying a range of scenarios that could produce portfolio losses
c. Ranking the stress scenarios by level of potential adverse impact and assessing relative probabilities for scenarios
d. Ensuring that stress tests are plausible and consistent with the existing risk model framework

27.3 ORGANIZATIONAL STRUCTURE

To be effective, the organizational structure must be designed to reflect the policy of effective firmwide risk management. Figure 27.1 reflects a typical organizational structure of an old-style commercial bank.

Here risk is monitored mainly by the business lines. Within the credit function, the risk manager approves transactions, sets exposure limits, and monitors the exposure limits as well as the counterparty's financial health. Treasury and trading implement proprietary trading and hedging. Within this unit, the risk manager measures and monitors positions. Line management deals with business and product strategy. It also controls operations. Finally, the audit function, external or internal, provides an independent review of business processes.

There are numerous problems with such a structure. Perhaps the main one is that market risk management reports to trading, which violates the principle of independence of risk management. In addition, the decentralization of risk management among separate lines leads to a lack of coordination and failure to capture correlations between different types of risk. The credit risk manager, for instance, will prefer an instrument that transforms credit risk into operational risk, which is under another manager's watch. Situations where credit risk and market risk exacerbate each other (as in the case of LTCM) will also be missed. Finally, models and databases may be inconsistent across lines.

To maintain independence, risk managers should report not to traders but directly to top management. Ideally, the risk management function should be a firmwide function, covering market, credit, and operational risks. Such a structure will avoid situations where risks are pushed from one area, where they are well measured, toward other areas. Firmwide risk management should also be able to capture interactions between different types of risks.

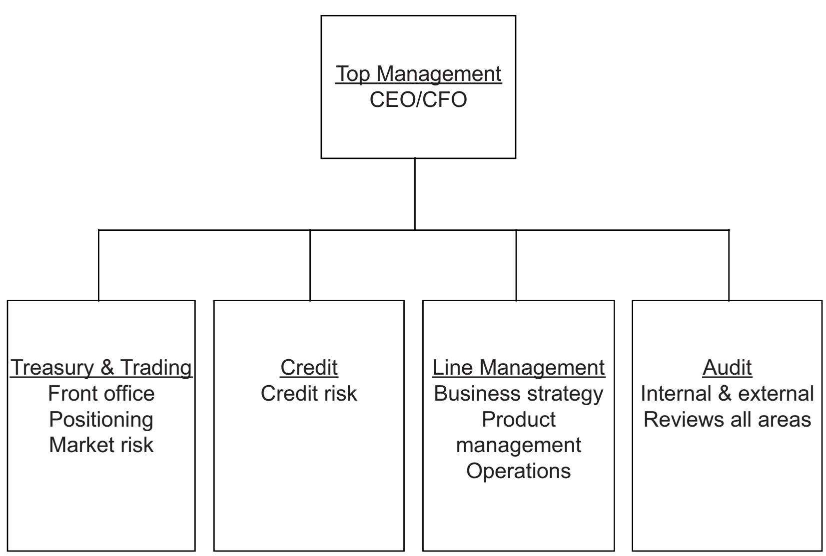

FIGURE 27.1 Old-Style Organizational Structure

The philosophy of separation of functions and independence of risk management must be embodied in the organizational structure of the institution. Figure 27.2 describes one such implementation. The most important aspect of this flowchart is that the risk management unit is independent of the trading unit.

The **front office** is concerned with trading, subject to position and VAR limits established by risk management. The **back office** deals with trade processing and reconciliation, as well as cash management. The **middle office** has expanded functions, which include risk measurement and control.

The **chief risk officer** (CRO) is responsible for:

- Establishing risk management policies, methodologies, and procedures consistent with firmwide policies
- Reviewing and approving models used for pricing and risk measurement
- Measuring risk on a global basis as well as monitoring exposures and movements in risk factors
- Enforcing risk limits with traders
- Communicating risk management results to senior management

Ideally, the risk management function should be centralized under a chief risk officer. To this officer report *market risk management*, which monitors risk in the trading book; *credit risk management*, which monitors risk in the banking and trading books; *operational risk management*, which monitors operational risks. Most advanced institutions have by now adopted the CRO model.[15]

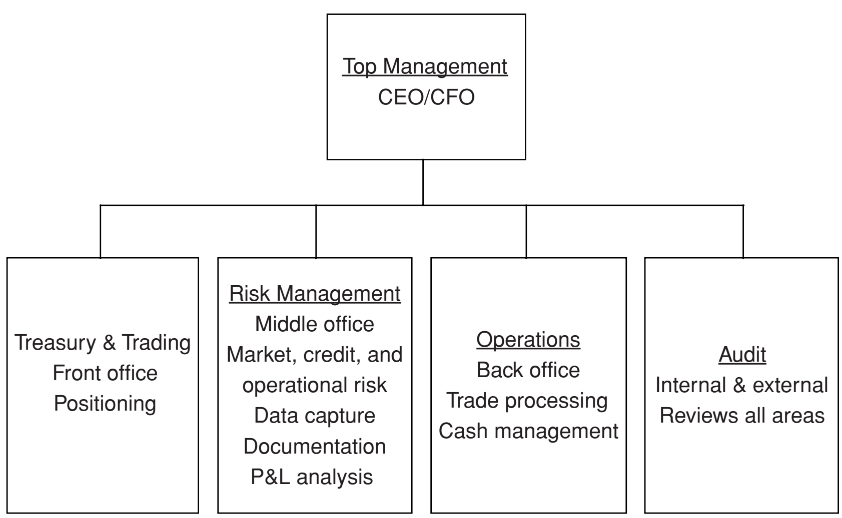

FIGURE 27.2 Modern Organizational Structure

[15] A recent report indicates that about 90% of the surveyed sample of financial institutions have a CRO. See Capital Markets Risk Advisors, *Risk Governance: A Benchmarking Survey* (New York: CMRA, 2010).

EXAMPLE 27.9: BEST PRACTICES

When would it be prudent for a trader to direct accounting entries?

a. Never
b. When senior management of the firm and the board of directors are aware and have approved the practice on an exception basis
c. When audit controls are such that the entries are reviewed on a regular basis to ensure detection of irregularities
d. Solely during such times as staffing turnover requires the trader to backfill until additional personnel can be hired and trained

EXAMPLE 27.10: FRM EXAM 2005—QUESTION 17

Which of the following is *not* a proper practice of risk management and control for a financial institution with assets in excess of $100 million?

a. A firm's sole mechanism to monitor the implementation of the control policies defined by the board is an external audit firm.
b. A subcommittee of the board is responsible for the approval of risk limits, risk management policies, and delegation of exceptional approval authorities.
c. Senior management is responsible for the day-to-day oversight of the firm's activities, implementing appropriate risk management and control policies, and monitoring the risks and exposures of the firm.
d. Senior management is responsible for establishing written documentation about control procedures at each level of the control hierarchy.

27.4 CONTROLLING TRADERS

27.4.1 Trader Compensation

The compensation structure for traders should also be given due thought. Usually, traders are paid a bonus that is directly related to their performance—for instance, 20% of profits—when positive. The design of such a compensation contract is asymmetrical, like that of an option. A trader who is successful can become a millionaire at a very young age. A trader who loses money is simply fired.

In many cases, the trader will find another employer since he or she now has experience.

Such a compensation scheme is designed to attract the very best talents into trading. The downside is that the trader, who is now long an option, has an incentive to increase the value of this option by increasing the risk of the positions. This, however, may not be in the best interests of the company.

Such a tendency for risk taking can be controlled by various means:

- By modifying the structure of the compensation contract to better align the interests of the trader and the company (e.g., by paying with company stock or tying compensation to longer-term performance)
- By subtracting a risk-based capital charge from trading profits, as in a RAROC-type system
- By appointing an independent risk manager

Last, recent regulatory changes are putting limits on compensation in the financial industry.

To be effective, the compensation structure for *risk managers* must be independent of how well traders perform. The compensation for risk managers needs to be attractive enough to draw talented individuals, however.

27.4.2 Trader Limits

To some extent, trading risk can be managed by appropriately altering the incentives of traders. Alternatively, this risk can be controlled by imposing limits. These can be separated into backward-looking and forward-looking limits. The former consist of stop-loss limits. The latter consist of exposure or VAR-type limits.

Stop-loss limits are restrictions on traders' positions that are imposed after a trader has accumulated losses. Because their design is backward-looking, they cannot prevent losses from occurring. What they do prevent, however, are attempts by traders who lose money to recover their losses by doubling their bets, that is, taking bigger bets in the hope that a future gain will be sufficient to wipe out a string of previous losses. These limits may also be useful if markets are trending, which would amplify the losses.

Exposure limits are systematically imposed on traders as a means to control losses before they occur. These are defined in terms of notional principal, duration, or other exposure measures. For example, the maximum position for a yen trader could be set at the equivalent of $10 million. These limits are typically set by considering the worst loss a unit could absorb, combined with an extreme move in the risk factor.

The problem with such limits is that they do not account for diversification or movements in market risks. Also, complex products for which the notional does not represent the worst loss lend themselves to a form of limit arbitrage, where the trader abides by the letter of the guideline but not its spirit. For instance, a trader may have a $10 million limit on notes with maturities up to five years. Typically, such notes will have duration of, say, four years. The spirit of the limit is to cap the interest rate exposure. The trader, however, may circumvent the spirit of the limit by investing in inverse floaters with a duration of 12 years.

VAR limits are becoming a more common addition to conventional limits. These account for diversification and time variation in risk. The same risk architecture can also be used to enforce stress-test limits, based on a series of scenarios. In practice, VAR limits are also susceptible to arbitrage, so they should be used together with exposure limits. Traders might move into positions with low estimated risks, taking advantage of weaknesses in risk models.

EXAMPLE 27.11: FRM EXAM 2002—QUESTION 132

The following is *not* a problem of having one employee perform trading functions and back-office functions.

a. The employee gets paid more because she performs two functions.
b. The employee can hide trading mistakes when processing the trades.
c. The employee can hide the size of her book.
d. The employee's firm may not know its true exposure.

EXAMPLE 27.12: FRM EXAM 2000—QUESTION 69

Which of the following strategies can contribute to minimizing operational risk?

I. Individuals responsible for committing to transactions should perform clearance and accounting functions.
II. To value current positions, price information should be obtained from external sources.
III. Compensation schemes for traders should be directly linked to calendar revenues.
IV. Trade tickets need to be confirmed with the counterparty.

a. I and II
b. II and IV
c. III and IV
d. I, II, and III

EXAMPLE 27.13: FRM EXAM 2007—QUESTION 36

To control risk taking by traders, your bank links trader compensation with their compliance with imposed VAR limits on their trading books. Why should your bank be careful in tying compensation to the VAR of each trader?

a. It encourages traders to select positions with high estimated risks, which leads to an underestimation of the VAR limits.
b. It encourages traders to select positions with high estimated risks, which leads to an overestimation of the VAR limits.
c. It encourages traders to select positions with low estimated risks, which leads to an underestimation of the VAR limits.
d. It encourages traders to select positions with low estimated risks, which leads to an overestimation of the VAR limits.

27.5 RISK-ADJUSTED PERFORMANCE AND RAROC

27.5.1 Risk Capital

The ability to measure risk has profound implications for performance measurement. In the past, performance was measured by yardsticks such as **return on assets** (**ROA**), which adjusts profits for the associated book value of assets, or **return on equity** (**ROE**), which adjusts profits for the associated book value of equity. Such measures are simple to compute but are fundamentally flawed because they ignore risks. As a result, these measures could lead to dangerous behavior, such as expanding operations in markets or lines of business where expected returns are high but where risks are also much higher.

Risk managers now have tools to control for this behavior. They can assess the total risk of an operation in terms of economic capital required to support all categories of risk, including market, credit, and operational risk. This capital, also called **risk capital**, is basically a value at risk (VAR) measure at a high confidence level.

Armed with this information, institutions can make better-informed decisions about business lines. Each activity should provide sufficient profit to compensate for the risks involved. Thus, product pricing should account not only for expected losses but also for the remuneration of risk capital.

This is the essence of **risk-adjusted return on capital** (**RAROC**) measures. RAROC was developed by Bankers Trust in the late 1970s. The bank was faced with the problem of evaluating traders involved in activities with different risk profiles.

27.5.2 Risk-Adjusted Performance Measures

RAROC is part of the family of **risk-adjusted performance measures** (**RAPMs**). Consider, for instance, two traders that each returned a profit of $10 million

TABLE 27.3 Computing RAPM

	Profit	Notional	Volatility	VAR	RAPM
FX trader	$10	$100	12%	$28	36%
Bond trader	$10	$200	4%	$19	54%

over the past year. The first is a foreign exchange (FX) trader, the second a bond trader. The question is, how do we compare their performance? This is important in providing appropriate compensation as well as deciding which line of activity to expand.

Assume the FX and bond traders have notional amounts and volatilities as described in Table 27.3. The notional amount is also the market value of the trader's book. The bond trader deals in larger amounts, $200 million, but in a market with lower volatility, at 4% per annum, against $100 million and 12% for the FX trader. The **risk capital** (RC) can be computed as a VAR measure, say at the 99% level over a year, as Bankers Trust did. Assuming normal distributions, this translates into a risk capital of

$$\text{RC} = \text{VAR} = \$100{,}000{,}000 \times 0.12 \times 2.33 = \$28 \text{ million}$$

for the FX trader and $19 million for the bond trader. More precisely, Bankers Trust computes risk capital from a weekly standard deviation σ_w as

$$\text{RC} = 2.33 \times \sigma_w \times \sqrt{52} \times (1 - \text{Tax Rate}) \times \text{Notional} \tag{27.1}$$

which includes a tax factor that determines the amount required on an after-tax basis.

The risk-adjusted performance is then measured as the dollar profit divided by the risk capital,

$$\text{RAPM} = \frac{\text{Profit}}{\text{RC}} \tag{27.2}$$

and is shown in the last column. Thus the bond trader is actually performing better than the FX trader, as the activity requires less risk capital. More generally, risk capital should account for credit risk, operational risk, and any interaction.

It should be noted that this approach views risk on a stand-alone basis, that is, using each product's volatility. In theory, for capital allocation purposes, risk should be viewed in the context of the bank's whole portfolio and measured in terms of its marginal contribution to the bank's overall risk. In practice, however, it is best to charge traders for risks under their control, which means the volatility of their portfolios.

27.5.3 RAROC

This RAROC methodology can be applied at the level of a transaction, of a desk, or of a business unit. In each case, the first step is to compute the **economic capital** (EC) required to support the operation. This includes market, credit, and operational risk.

RAROC is formally defined as

$$\text{RAROC} = \frac{\text{Net Profit}}{\text{EC}} = \frac{\text{Expected Profit} - \text{Costs} + k(\text{EC})}{\text{EC}} \tag{27.3}$$

where the net profit includes: (1) revenues, net of expected losses; (2) minus any direct operating cost; (3) minus financing costs; (4) plus the return on economic capital, which accrues at the rate k.

To illustrate the RAROC calculation, consider a loan portfolio with a principal of $1,000 million, paying an annual rate of 9%, for a revenue of $90 million. The economic capital is estimated at $75 million, which is invested in Treasury bills earning 6.5%, which gives $4.9 million. As a result, the difference of $925 million is raised by deposits with an interest rate of 6%, leading to a cost of $55.5 million. The bank has an operating cost of $15 million. The expected default rate on the loans is 1%. The RAROC is then

$$\text{RAROC} = \frac{90 - 55.5 - 15 - 10 + 4.9}{75} = 19.25\%$$

This number can be interpreted in terms of the annual expected rate of return on the equity that is required to support this loan portfolio. This can be compared to the firm's cost of equity capital. If higher (e.g., higher than 19%), the project adds value to shareholders and should go ahead.

Another version of this comparison uses the CAPM. Defining β_E as the firm equity's systematic risk, $\bar{R}_M$ as the market premium, and R_F as the risk-free rate, the project can be accepted when the adjusted RAROC (ARAROC) is

$$\text{ARAROC} = \frac{\text{RAROC} - \text{R}_\text{F}}{\beta_E} > \bar{R}_M - R_F \tag{27.4}$$

EXAMPLE 27.14: FRM EXAM 2006—QUESTION 3

A risk manager for ABC Bank has compiled the following data regarding a bond trader and an equity trader. Assume that the returns are normally distributed and that there are 52 trading weeks per year. ABC Bank computes its capital using a 99% VAR. Dollar amounts are in millions.

	After-Tax Profit	Net Book Market Value	Weekly Volatility	Tax Rate
Bond trader	USD 8	USD 120	1.10%	40%
Equity trader	USD 18	USD 180	1.94%	40%

Calculate the risk-adjusted performance measure (RAPM) for the bond trader.

a. 25.24%
b. 36.08%
c. 60.15%
d. 84.92%

EXAMPLE 27.15: FRM EXAM 2006—QUESTION 4

Continuing with the same ABC Bank data, which of the following statements is *correct* in relation to the equity trader?

I. The equity trader has an annual, after-tax VAR at a 99% confidence level of USD 33.2 million.
II. In comparing the RAROC for both traders, the equity trader is performing better than the bond trader.

a. I only
b. II only
c. Both
d. Neither

EXAMPLE 27.16: FRM EXAM 2007—QUESTION 124

The bank you work for has a RAROC model. The RAROC model, computed for each specific activity, measures the ratio of the expected yearly net income to the yearly VAR risk estimate. You are asked to estimate the RAROC of its $500 million loan business. The average interest rate is 10%. All loans have the same probability of default of 2% with a loss given default of 50%. Operating costs are $10 million. The funding cost of the business is $30 million. RAROC is estimated using a credit VAR for loan businesses, in this case 7.5%. The economic capital is invested and earns 6%. The RAROC is:

a. 19.33%
b. 46.00%
c. 32.67%
d. 13.33%

27.6 IMPORTANT FORMULAS

Economic risk capital (RC): RC = VAR

Risk-adjusted performance measure (RAPM): RAPM = Profit/RC

Risk-adjusted return on capital (RAROC): RAROC = [Expected Profit − Costs + k(EC)]/EC

27.7 ANSWERS TO CHAPTER EXAMPLES

Example 27.1: FRM Exam 2008—Question 4-24

c. All the statements are correct, except c., that economic capital must *always* be less than regulatory capital. This is too broad a statement. The two measures are not necessarily related, even though this is the goal of having more risk-sensitive capital requirements.

Example 27.2: FRM Exam 2009—Question 7-9

b. A portfolio is generally more diversified when it has many positions, which are not too large, and with low correlations. Hence answers a., c., and d. involve drivers of diversification. In contrast, risk measures are homogeneous with the size of the portfolio. Doubling all the positions will double the risk of the portfolio.

Example 27.3: FRM Exam 2002—Question 103

b. VAR can be added across different types of risk, but this will provide a conservative estimate of capital as diversification effects are ignored. So answer a. would be for *too much* capital. Answer c. is not correct because rare events can be factored into operational VAR. Most likely, the bank may have too little capital for other types of risk than those measured by these three categories.

Example 27.4: FRM Exam 2006—Question 109

d. For most global banks, the order of importance is, first, credit risk, then operational risk, then market/ALM risk. Also, answers b. and c. are the same.

Example 27.5: FRM Exam 2005—Question 33

c. This is an example of a wrong-way exposure, where a gain on the instrument for the bank is associated with a higher probability of default (PD) for its counterparty. If the IDR depreciates, counterparty A will make a profit because its costs will go down in dollars; conversely for counterparty B, because its dollar revenues will decrease. Under c., the company pays USD and receives IDR. This transaction will create a loss if the IDR depreciates. In this situation, counterparty B will lose money as well on its exports. Hence, this is a wrong-way trade.

Example 27.6: FRM Exam 2008—Question 4-29

c. First, we convert the daily VAR at the 95% level to the same parameters as the other. With the normality assumption, this is $VAR_{MKT} = \$100 \times (2.326/1.645)\sqrt{252} = \$2{,}245$. We then combine the three VARs by taking the square root of the sum of squares, which gives VAR $= \sqrt{\$2{,}245^2 + \$500^2 + \$1{,}000^2} =$ \$2,458.

Example 27.7: FRM Exam 2004—Question 47

c. The Barings failure falls in the category of operational risk because of a breakdown in procedures. The trader, Nick Leeson, had control of the back office.

Example 27.8: FRM Exam 2008—Question 4-34

a. Business managers or the board of directors should define the risk tolerance, not risk managers.

Example 27.9: Best Practices

a. As one risk manager has said, this is one of the few instances where *never* means *absolutely never*. Allowing traders to tabulate their own profits and losses is a recipe for disaster.

Example 27.10: FRM Exam 2005—Question 17

a. Control policies also need to be verified by an internal audit function.

Example 27.11: FRM Exam 2002—Question 132

a. Answers b., c., and d. all can lead to a situation where the trader loses money and hides the losses. Answer a. is not a problem per se.

Example 27.12: FRM Exam 2000—Question 69

b. Answer I. violates the principle of separation of functions. Answer III. may create problems of traders taking too much risk. Answer II. advises the use of external sources for valuing positions, as traders may affect internal price data.

Example 27.13: FRM Exam 2007—Question 36

c. Traders may engage in VAR arbitrage, trying to exploit weaknesses in VAR measures. With a VAR limit, they may seek positions that have low measured VAR, in which case the VAR limits will be less effective.

Example 27.14: FRM Exam 2006—Question 3

c. The 99% VAR is $2.33 \times 1.10\% \times \sqrt{52} \times (1 - 40\%) \times \$120 = \$13.3$ million. Hence RAPM $= 8/13.3 = 60.1\%$.

Example 27.15: FRM Exam 2006—Question 4

d. The equity trader's VAR is $2.33 \times 1.94\% \times \sqrt{52} \times (1 - 40\%) \times \$180 = \$35.2$ million, so statement I. is incorrect. The RAPM is 18/35.2, or 51.1%, which is worse than that of the bond trader, so statement II. is incorrect as well.

Example 27.16: FRM Exam 2007—Question 124

a. First, we compute the numerator. The net interest is, after expected losses, $\$500 \times [10\% - 2\%(1 - 50\%)] = \$45$. Next, we compute economic capital, or $\$500 \times 7.5\% = \$37.5$. To revenues, we then add the return on economic capital, or $\$37.5 \times 6\% = \$2.25$. From this, we deduct operating and funding costs, which gives $\$47.25 - 10 - 30 = \$7.25$. Finally, we divide by \$37.5 and get 19.33%.

CHAPTER 28

The Basel Accord

The **Basel Capital Accord**, concluded on July 15, 1988, by the **Basel Committee on Banking Supervision** (BCBS), is the cornerstone for the regulation of commercial banking.[1] It instituted for the first time minimum levels of capital to be held by internationally active commercial banks against financial risks.

This Accord, called **Basel I**, set capital charges against credit risk based on a set of relatively simple rules. To these were added capital adequacy requirements against market risk in 1996.

In June 2004, the Accord underwent a fundamental revision, which created more risk-sensitive capital requirements and added a charge against operational risks. The operational risk charge is explained in Chapter 25. As a result, the new Accord was called **Basel II**.

The credit crisis that started in 2007 revealed serious weaknesses in the regulatory framework. This led to a new round of revisions, which are informally called **Basel III**. The BCBS agreed in July 2009 to a major expansion in the capital requirements that was finalized in September 2010.

The Basel Committee formulates broad supervisory standards but its conclusions do not have legal force. Nevertheless, more than 100 countries have adopted the Basel Accord, with varying time lines and applications. For example, U.S. regulators applied Basel II to the largest U.S. banks only. In contrast, the European Union (EU) has incorporated the Basel II rules into EU law through a **Capital Adequacy Directive**, which applies to all banks within the Union.

Section 28.1 provides a broad overview of the Basel Accord. Section 28.2 turns to the definition of capital. Section 28.3 then details the original Basel capital requirements for credit risk. Section 28.4 illustrates the application of capital adequacy ratios for Citibank. Section 28.5 discusses major drawbacks of the original Basel Accord and describes the main components of the Basel II Accord, as well as work on Basel III. Section 28.6 is devoted to the market risk charge. Finally, Section 28.7 concludes with a general evaluation of these capital requirements.

FRM Exam Part 2 topic.

[1] The Basel Committee was established by the central bank governors of the G-10 countries in 1974. It now includes members from about 30 countries, who meet four times a year, usually in Basel, Switzerland, under the aegis of the Bank for International Settlements (BIS). The BCBS is sometimes referred to as the BIS, but these are two distinct entities.

28.1 STEPS IN THE BASEL ACCORD

28.1.1 The Basel I Accord

The original goal of the 1988 Basel Accord, which came into force in 1992, was to set minimum capital requirements for commercial banks as a buffer against financial losses. Thus its primary objective was to promote the safety and soundness of the global financial system. A secondary objective was to create a level playing field for internationally active banks by setting uniform minimum standards. The Accord applies to internationally active commercial banks whose failure could cause systemic risk and that compete across the globe. It is up to local supervisors whether to include other, smaller domestic banks.

Initially, the 1988 Basel Accord covered only credit risk. Basel I requires banks to hold a minimum level of capital of at least 8% of the total **risk-weighted assets** (RWA). RWA include on-balance-sheet and off-balance-sheet items, using risk weights that provided a rough classification of assets by credit risk. **Capital** includes the book value of equity on the balance sheet, with adjustments, as well as other entries such as subordinated debt. The purpose of this capital is to serve as a buffer against unexpected financial losses. Higher capital should decrease the probability of failure that could cause instability in financial markets, and should protect depositors that entrusted their money with the banks.

28.1.2 The 1996 Amendment

In 1996, the Basel Committee amended the Capital Accord to incorporate market risks. This was done because many banks were increasing the scale of their proprietary trading operations. This amendment came into force at the end of 1997 and added a capital charge for market risk. Banks are allowed to use either a standardized model or an **internal models approach** (IMA), based on their own risk management system.

The amendment separates the bank's assets into two categories, the trading book and the banking book. The **trading book** represents the bank portfolio with financial instruments that are intentionally held for short-term resale and typically marked to market. The **banking book** consists of other instruments, mainly loans, that are held to maturity and typically valued on a historical cost basis.

The 1996 amendment adds a capital charge for (1) the market risk of trading books and (2) the currency and commodity risk of the banking book. In exchange, the credit risk charge excludes debt and equity securities in the trading book and positions in commodities (apart from the specific risk charge). As before, it still includes all **over-the-counter** (OTC) derivatives, whether in the trading book or the banking book.

28.1.3 The Basel II Accord

Capital markets have undergone major changes since the initial Capital Accord of 1988. The design of the original credit risk changes had become increasingly outdated and, even worse, may have promoted unsound behavior by some banks.

In June 2004, the Basel Committee finalized a comprehensive revision to the Basel Accord. In the European Union, the new Capital Adequacy Directive implementing Basel II applies to all banks in the EU, starting in 2007, with the most advanced methods being available from 2008. U.S. regulators apply Basel II to a small number of large banks, with other banks subject to a revised version of Basel I because this is a simpler system. Basel II implementation started in 2008, with a three-year transition period during which U.S. regulators reserve the right to change the application of rules.

The new framework is based on **three pillars**, viewed as mutually reinforcing:

- *Pillar 1: Minimum capital requirement.* These are meant to cover credit, market, and operational risk. Relative to the 1988 Accord, banks now have a wider choice of models for computing their risk charges. The BCBS, however, still tried to keep constant the total level of capital in the global banking system, at 8% of risk-weighted assets.
- *Pillar 2: Supervisory review process.* Relative to the previous framework, supervisors are given an expanded role. Supervisors need to ensure that
  - Banks have a process in place for assessing their capital in relation to risks.
  - Banks indeed operate above the minimum regulatory capital ratios.
  - Corrective action is taken as soon as possible when problems develop.
- *Pillar 3: Market discipline.* The Basel II Accord emphasizes the importance of risk disclosures in financial statements. Such disclosures enable market participants to evaluate banks' risk profiles and the adequacy of their capital positions. The new framework sets out disclosure requirements and recommendations. Banks that fail to meet disclosure requirements will not qualify for using internal models. As internal models generally lead to lower capital charges, this provides a strong incentive for complying with disclosure requirements. In essence, the trade-off for greater reliance on a bank's own models is greater transparency.

Basel II provides for finer measurement of credit risk, which will generally lead to lower capital requirements. In order to maintain the overall level of bank capital, however, new capital charges are set against **operational risk**. Banks are required to carry enough capital to exceed the sum of the **credit risk charge** (CRC), the **market risk charge** (MRC), and the **operational risk charge** (ORC).

$$\text{Total Capital} > \text{CRC} + \text{MRC} + \text{ORC} \tag{28.1}$$

As before, the credit risk charge is 8% of credit risk-weighted assets. The MRC and ORC are computed using another approach.

For comparability across banks, capital strength is reported as a ratio:

$$\frac{\text{Total Capital}}{\text{Credit Risk} + \text{Market Risk} + \text{Operational Risk}} = \text{Bank's Capital Ratio} > 8\% \tag{28.2}$$

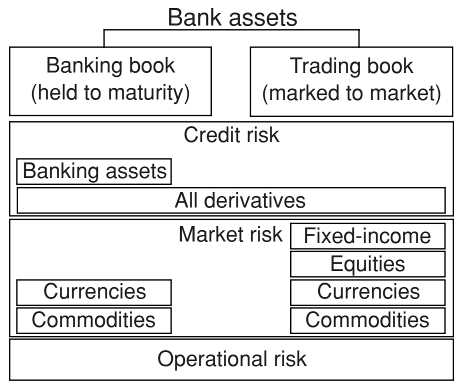

FIGURE 28.1 Summary of Basel II Risk Charges

Here, the last two components in the denominator are computed from the multiplication of the market risk charge (MRC) and operational risk charge (ORC) by $(1/8\%) = 12.5$. For instance, if a bank has \$875 in credit risk-weighted assets and MRC = \$10 and ORC = \$20, the denominator is computed as $\$875 + [(\$10 + \$20) \times 12.5] = \$1{,}250$. The bank then has to hold at least $8\% \times \$1{,}250 = \$100$ in capital to satisfy the minimum requirement. This is equivalent to saying that the total charge must be at least $8\% \times \$875 + \$10 + \$20 = \$70 + \$10 + \$20 = \$100$.

Figure 28.1 summarizes the coverage of credit, market, and operational risk charges for the banking and trading books. Banks will also have access to a menu of methods to compute their risk charges. These are described in Table 28.1.

We should note that Pillar 1 omits some important bank risks. In particular, **interest rate risk in the banking book** is not covered. Banks are indeed exposed to **repricing risk**, which arises from differences in the maturity and repricing of assets and liabilities. For instance, a bank funding a long-term fixed-rate loan by short-term deposits could suffer a repricing loss if interest rates increase. Measuring this risk, however, requires modeling the complex behavior of deposits, which do not have a fixed maturity date, as well as the prepayment optionality of loans. Due to these difficulties, there is substantial heterogeneity across banks in the methods used to monitor and manage this risk. As a result, the BCBS has decided that this

TABLE 28.1 Menu of Approaches to Measure Risk

Risk Category	Allowed Approach
Credit	Standardized approach (based on the 1988 Accord) Foundation internal ratings-based approach Advanced internal ratings-based approach
Market	Standardized approach Internal models approach
Operational	Basic indicator approach Standardized approach Advanced measurement approach

risk falls under Pillar 2. Institutions that are perceived to have more of this risk can be subject to higher capital charges.

Finally, there was no formal capital charge for **liquidity risk**, due to the difficulty of formal measurement of this risk. Yet, the Basel Committee recognizes that "liquidity is crucial to the ongoing viability of any banking organization."

28.1.4 Further Developments: Basel III

The credit crisis that started in 2007 has revealed weaknesses in the regulatory framework. Some banks that appeared sufficiently capitalized experienced major losses that in many cases required government support. In response, the Basel Committee implemented changes to the Basel II framework. The goal of this new set of rules, which has become known as **Basel III**, is to strengthen the resilience of the banking system by increasing the amount, quality, and coverage of capital.

In July 2009, the BCBS revised the market risk framework to increase the amount of capital required for trading risks, making it more consistent with the capital charge in the banking book for the same instrument.[2] In July 2010, additional major revisions were announced.[3]

- *Capital definition.* The definition of what constitutes acceptable capital was changed to exclude some components that turned out not to provide the desired protection during the credit crisis.
- *Leverage ratio.* The BCBS introduced a limit on the **leverage ratio**. This is because some banks had adequate capital using the Basel II rules but ran into difficulties because of their high leverage. Normally, the leverage ratio is defined as the ratio of assets to equity. For example, a bank with $50 billion in assets and $1 billion in equity would have a leverage ratio of 50 to 1 or, inverting, a leverage capital ratio of 2%. This measure is not risk-sensitive, because it ignores both the quality of the assets and hedging effects, but it is simple and robust. The Basel definition of leverage includes off-balance-sheet items. The current proposal is to require a minimum leverage capital ratio of 3%, to be included in Pillar 1 by January 1, 2018.
- *Liquidity requirements.* Throughout the financial crisis, many banks struggled to maintain adequate liquidity. As we have seen in Chapter 26, liquidity played a role in Northern Rock's failure. As a result, the BCBS introduced a global minimum **liquidity standard**. This includes a **30-day liquidity coverage ratio** (LCR), which would allow the bank to convert assets into cash to meet liquidity needs under a stress scenario. The ratio of the stock of high-quality liquid assets to the net cash outflows over 30 days must be greater than 100%. To that is added a **net stable funding ratio** (NSFR), which promotes better matching of assets and liabilities. This is defined as the ratio of the available amount of stable funding to the required amount, which must be greater than

[2] BCBS, *Revisions to the Basel II Market Risk Framework* (Basel: BIS, 2009).
[3] BCBS, *Broad Agreement on Basel Committee Capital and Liquidity Reform Package* (Basel: BIS, 2010), as well as BCBS, *Strengthening the Resilience of the Banking Sector* (Basel: BIS, 2009).

100%. Northern Rock would have failed this ratio because it was funding long-term requirements (i.e., long-term mortgage loans) by insufficient stable funding (e.g., long-term debt). After a period of observation, the new rules are expected to apply by January 1, 2015, for the LCR and by January 1, 2018, for the NSFR.

> **EXAMPLE 28.1: APPLICABLE MARKET RISKS**
>
> For regulatory capital calculation purposes, what market risks must be incorporated into a bank's VAR estimate?
>
> **a.** Risks in the trading account relating to interest rate risk and equity risk
> **b.** Risks in the trading account relating to interest rate risk and equity risk, and risks throughout the bank related to foreign exchange and commodity risks
> **c.** Risks throughout the bank related to interest rate risk, equity risk, foreign exchange risk, and commodity risk
> **d.** Interest rate risk, equity risk, foreign exchange risk, and commodity risk in the trading account only

28.2 DEFINITION OF CAPITAL

28.2.1 Basel I and II

The 1988 capital adequacy rules require any **internationally active bank** to carry capital of at least 8% of its total risk-weighted assets. This applies to commercial banks on a consolidated basis. So, for instance, holding companies that are parents of banking groups have to satisfy the capital adequacy requirements.

The starting point for the measurement of capital is a bank's financial statements. In the Basel Accord, "capital" has a broader interpretation than the book value of equity. The key purpose of capital is its ability to absorb losses, providing some protection to creditors and depositors. Hence, to be effective, capital must be permanent, must not impose mandatory fixed charges against earnings, and must allow for legal subordination to the rights of creditors and depositors.

The Basel Accord recognizes three forms of capital.

Tier 1 Capital, or Core Capital. **Tier 1 capital** includes equity capital and disclosed reserves, most notably after-tax retained earnings. Such capital is regarded as a buffer of the highest quality.

- **Equity capital** or shareholders' funds consist of issued and fully paid common stock and nonredeemable, noncumulative preference shares (also called preferred stock).
- **Disclosed reserves** correspond to share premiums, retained profits, and general reserves.

Goodwill is always subtracted from book equity. This is an accounting entry that, after an acquisition, goes into book equity to represent the excess of the purchase value over book value. It is omitted because it does not represent funds that can serve as a buffer against losses.

As a result of the credit crisis experience, there has been an increasing focus on narrower definitions of capital. Notably, **core tier 1 capital** excludes preferred equity. Next, **tangible common equity** (TCE), excludes preferred equity and intangible assets. **Intangible assets** are nonmonetary assets that cannot be touched, such as patents and trademarks.

$$\text{TCE} = \text{Equity} - \text{Intangible Assets} - \text{Goodwill} - \text{Preferred} \tag{28.3}$$

In the United States, for example, commercial banks are also subject to a maximum leverage requirement, defined as the ratio of assets, minus intangibles and goodwill, over TCE.

Tier 2 Capital, or Supplementary Capital. **Tier 2 capital** includes components of the balance sheet that provide some protection against losses but ultimately must be redeemed or contain a mandatory charge against future income. These include:

- **Undisclosed reserves**, or hidden reserves that are allowed by the accounting standards of some countries. These are reserves that passed through the earnings statement but remain unpublished. Due to this lack of transparency, as well as the fact that many countries refuse to recognize undisclosed reserves, undisclosed reserves are not part of core capital.
- **Asset revaluation reserves**, which arise, for instance, from long-term holdings of equity securities that are valued at historical acquisition costs. Such capital could be used to absorb losses on a going-concern basis, subject to some discount to reflect market volatility and future taxes in case of sales.
- **General provisions/loan loss reserves**, which are held against future unidentified losses. These are the result of **loan loss allowances**, which are deductions taken against interest income in anticipation of probable credit losses. These deductions reduce retained profits in tier 1 capital but may qualify as tier 2 capital to the extent that they do not reflect a known deterioration in particular assets (in which case they are "specific").[4]
- **Hybrid debt capital instruments**, which combine some characteristics of equity and of debt. When they are unsecured, subordinated, and fully paid up, they are allowed into supplementary capital. These include, for instance, **cumulative preference shares**.
- **Subordinated term debt**, with a minimum original maturity of five years, and subject to a discount of 20% during the last five years. Subordinated debt would be junior in right of payment to all other debt in the event of liquidation.

Tier 3 Capital, for Market Risk Only. **Tier 3 capital** consists of short-term subordinated debt with a maturity of at least two years. This is eligible to cover market risk only.

[4] As credit losses occur, they are charged against this reserve instead of profits, which helps to smooth out earnings.

There are additional restrictions on the relative amount of various categories. Of the 8% capital charge for credit risk, at least 50% must be covered by tier 1 capital. Next, the amount of tier 3 capital is limited to 250% of tier 1 capital allocated to support market risks (tier 2 capital can be substituted for tier 3 capital if needed). Other restrictions apply to various elements of the tiers.

KEY CONCEPT

The Basel capital adequacy rules require total capital (tiers 1 and 2) to be at least 8% of risk-weighted assets (RWA). In additional, tier 1 capital needs to be at least 4% of RWA. Local regulators can impose higher ratios and additional rules.

28.2.2 Basel III

The major goal of the so-called Basel III revisions to the Capital Accord was to increase the level and quality of bank capital. The primary focus was **common equity capital**, which is part of tier 1 capital but is viewed as having the best capacity to absorb losses.

In September 2010, the BCBS agreed to increase the required level of common equity capital to 4.5%. Tier 1 capital is increased from 4% to 6%. Both changes will start in 2013 and will be phased in by January 1, 2015. Total capital is still kept at a minimum of 8%.

The BCBS, however, also added a **capital conservation buffer** (CCB) of 2.5%. Banks will be allowed to draw on this buffer during periods of stress but will be then subject to constraints on earnings distribution (e.g., dividend and bonus payments). This buffer will start in January 2016 and become fully effective on January 1, 2019. As shown in Table 28.2, when fully effective, this will bring the minimum ratios for core tier 1, tier 1, and total capital to 7%, 8.5%, and 10.5%, respectively.

To this is added a **countercyclical buffer** consisting of common equity within a range of 0% to 2.5%, but according to "national circumstances." The goal of this buffer is to increase the amount of bank capital when there is excess credit growth that could result in a buildup of risk.

TABLE 28.2 Basel III Capital Requirements by Year (Jan. 1) Including Capital Conservation Buffer (CCB)

Capital	2010	2013	2014	2015	2016	2017	2018	2019
Total	8%	8.0%	8.0%	8.0%	8.625%	9.25%	9.875%	10.50%
Tier 1	4%	4.5%	5.5%	6.0%	6.625%	7.25%	7.875%	8.50%
Core Tier 1	2%	3.5%	4.0%	4.5%	5.125%	5.75%	6.375%	7.00%
CCB	0%	0.0%	0.0%	0.0%	0.625%	1.25%	1.875%	2.50%

Basel III introduces additional restrictions on what can count as part of tier 1 capital. For example, some portion of equity that represents minority interest in overseas subsidiaries has been disallowed. Other disallowances include some deferred tax assets and mortgage servicing rights. Last, Basel III abolishes tier 3 capital.

Note that these changes are introduced gradually. This reflects the worry that immediate changes would lead to a contraction of lending to the private sector, which would have negative effects on already weak economies. If banks cannot raise enough capital to comply with these new rules, they will be forced to cut down their exposures, which will constrain the expansion of credit.

As of 2010, most large global banks already have enough core tier 1 capital to satisfy the minimum 7% capital requirement, without taking into account the new disallowances. It is expected that retained earnings would be necessary to satisfy the countercyclical buffer.

KEY CONCEPT

Basel III increases the minimum capital ratio for core tier 1 capital from 2% to 4.5%, plus a 2.5% buffer, for a total of 7%, to be effective on January 1, 2019.

EXAMPLE 28.2: FRM EXAM 2002—QUESTION 71

What is the best definition of tier 1 regulatory capital?

- **a.** Equity capital, retained earnings, disclosed reserves
- **b.** Subordinated debt, undisclosed reserves
- **c.** Equity capital, subordinated debt with a maturity greater than five years
- **d.** Long-term debt, revaluation reserves

EXAMPLE 28.3: FRM EXAM 2007—QUESTION 53

Consider a bank balance sheet with (1) common stock of USD 600,000,000; (2) unrealized long-term marketable equity securities gain: USD 5,000,000; (3) allowance in anticipation of possible credit losses: USD 5,000,000; (4) goodwill: USD 30,000,000.

Based solely on this information, the tier 1 and tier 2 capital numbers are, respectively:

- **a.** USD 595,000,000, USD 45,000,000
- **b.** USD 570,000,000, USD 10,000,000
- **c.** USD 600,000,000, USD 15,000,000
- **d.** USD 630,000,000, USD 20,000,000

EXAMPLE 28.4: FRM EXAM 2004—QUESTION 29

Consider the following financial data for a bank, in millions of dollars: shareholders' funds: 627.4; retained earnings: 65.6; undisclosed reserves: 33.5; goodwill: 21.3; subordinated debt: 180.0; specific provisions: 11.7. The ratio of tier 2 to tier 1 capital is:

a. 30.81%
b. 31.78%
c. 33.53%
d. 34.03%

28.3 THE BASEL I CREDIT RISK CHARGE

We now turn to a review of the credit risk charges under Basel I.

28.3.1 On-Balance-Sheet Risk Charges

We first examine on-balance-sheet assets, which consist primarily of loans for most credit institutions. Ideally, the capital charges should recognize differences in asset credit quality.

Indeed, the 1988 Basel Accord applies to the notional of each asset a **risk capital weight** taken from four categories, as described in Table 28.3. Each dollar of risk-weighted notional exposure must be covered by 8% capital.

These categories provide an extremely rough classification of credit risk. For instance, claims on Organization for Economic Cooperation and Development (OECD) central governments, such as holdings of U.S. Treasuries, are assigned a

TABLE 28.3 Risk Capital Weights by Asset Class

Weights	Asset Type
0%	Cash held
	Claims on OECD central governments
	Claims on central governments in national currency
20%	Cash to be received
	Claims on OECD banks and regulated securities firms
	Claims on non-OECD banks below one year
	Claims on multilateral development banks
	Claims on foreign OECD public-sector entities
50%	Residential mortgage loans
100%	Claims on the private sector (corporate debt, equity, etc.)
	Claims on non-OECD banks above one year
	Real estate
	Plant and equipment

weight of zero since these assets have presumably no default risk. Cash held is also assigned a zero weight. At the other extreme, claims on corporations, including loans, bonds, and equities, receive a 100% weight, irrespective of the risk of default or maturity of the loan.

The **credit risk charge** (CRC) is then defined for balance sheet (BS) items as

$$\text{CRC(BS)} = 8\% \times (\text{RWA}) = 8\% \times \left(\sum_i \text{RW}_i \times \text{Notional}_i\right) \tag{28.4}$$

where RWA represents risk-weighted assets, and RW_i is the risk weight attached to asset i.

EXAMPLE 28.5: BASEL I CREDIT RISK CHARGE

A bank subject to the Basel I Accord makes a loan of $100 million to a firm with a risk weight of 50%. What is the basic on-balance-sheet credit risk charge?

a. $8 million
b. $4 million
c. $2 million
d. $1 million

28.3.2 Off-Balance-Sheet Risk Charges

By the late 1980s, focusing on balance sheet items only missed an important component of the credit risk of the banking system—the exposure to swaps. The first swaps were transacted in 1981. By 1990, the outstanding notional of open positions had grown to $3,500 billion, which seems enormous. Some allowance had to be made for the credit risk of swaps. Unlike loans, however, the notional amount does not represent the maximum loss.

To account for such **off-balance-sheet** items, the Basel Accord computes a **credit exposure** (CE) that is equivalent to the notional for a loan, through **credit conversion factors** (CCFs). The Accord identifies five broad categories.

1. Instruments that substitute for loans (e.g., guarantees, bankers' acceptances, and standby letters of credit serving as guarantees for loans and securities) carry the full 100% weight (or credit conversion factor). The rationale is that the exposure is not different from a loan. Take a **financial letter of credit** (LC), for instance, which provides irrevocable access to bank funds for a client. When the client approaches credit distress, it will almost assuredly draw down the letter of credit. Like a loan, the full notional is at risk. This category also

includes asset sales with recourse, where the credit risk remains with the bank, and forward asset purchases.

2. Transaction-related contingencies (e.g., performance bonds or **commercial letters of credit** related to particular transactions) carry a 50% factor. The rationale is that a performance letter of credit is typically secured by some income stream and has lower risk than a general financial LC.
3. Short-term, self-liquidating trade-related liabilities (e.g., documentary credits collateralized by the underlying shipments) carry a 20% factor.
4. Commitments with maturity greater than a year (such as credit lines), as well as note issuance facilities (NIFs), carry a 50% credit conversion factor. Shorter-term commitments or revocable commitments have a zero weight. Note that this applies to the unfunded portion of commitments only, as the funded portion is an outstanding loan and appears on the balance sheet. Under Basel II, shorter-term commitments now receive a CCF of 20%.
5. Other derivatives, such as swaps, forwards, and options on currency, interest rate, equity, and commodity products are given special treatment due to the complexity of their exposures.

For the first four categories, the position is replaced by a credit equivalent, or credit exposure, computed as

$$\text{Credit Exposure} = \text{Credit Conversion Factor} \times \text{Notional} \tag{28.5}$$

For the last category (derivatives), the credit exposure is computed as the sum of the current **net replacement value** (NRV) plus an **add-on** that is supposed to capture future or **potential exposure**:

$$\begin{aligned} \text{Credit Exposure} &= \text{NRV} + \text{Add-On} \\ \text{Add-On} &= \text{Notional} \times \text{Add-On Factor} \times (0.4 + 0.6 \times \text{NGR}) \end{aligned} \tag{28.6}$$

where NGR is the net-to-gross ratio (to be discussed later). Here, the add-on factor depends on the **tenor** (maturity) and type of contract, as listed in Table 28.4. It roughly accounts for the maximum credit exposure, which depends on the volatility of the risk factor and the maturity. As we have seen, volatility is highest for commodities, then equity, then currencies, then fixed-income instruments. This

TABLE 28.4 Add-On Factors for Potential Credit Exposure (Percent of Notional)

Residual Maturity (Tenor)	Contract				
	Interest Rate	Exchange Rate, Gold	Equity	Precious Metals	Other Commodities
< 1 year	0.0	1.0	6.0	7.0	10.0
1–5 years	0.5	5.0	8.0	7.0	12.0
> 5 years	1.5	7.5	10.0	8.0	15.0

explains why the add-on factor is greater for currency, equity, and commodity swaps than for interest rate instruments, and also increases with maturity.

More precisely, the numbers have been obtained from simulation experiments (such as those in Chapter 22) that measure the 80th percentile worst loss over the life of a matched pair of swaps. The matching of pairs reflects the hedging practice of swap dealers and effectively divides the exposure in two, since only one swap can be in-the-money. Take, for instance, a currency swap with five-year initial maturity. Assuming exchange rates are normally distributed and ignoring interest rate risk, the maximum credit exposure as a fraction of the notional should be

$$\text{WCE} = \frac{1}{2} \times 0.842 \times \sigma\sqrt{5} \tag{28.7}$$

where the $\frac{1}{2}$ factor reflects swap matching and the 0.842 factor corresponds to a one-sided 80% confidence level. Assuming a 10% annual volatility, this gives WCE = 9.4%. This is in line with the add-on of 7.5% in Table 28.4.

Further simulations by the Bank of England and the New York Fed have shown that these numbers also roughly correspond to a 95th percentile loss over a six-month horizon. In the case of a new five-year interest rate swap, for instance, the worst exposure over the life at the 80th percentile level is 1.49%; the worst exposure over six months at the 95th percentile level is 1.58%. This is in line with the add-on of 1.5% for this category.

Next, the NGR factor in Equation (28.6) represents the **net-to-gross ratio**, or ratio of current net market value to gross market value, which is always between 0 and 1. The purpose of this factor is to reduce the capital requirement for contracts that fall under a legally valid netting agreement. Without netting agreements in place (i.e., with NGR = 1), the multiplier $(0.4 + 0.6 \times \text{NGR})$ is equal to one. There is no reduction in the add-on.

In contrast, take a situation where a bank has two swaps with the same counterparty currently valued at +100 and at −60. The gross replacement value is the sum of positive values, which is 100. The net value is 40, creating an NGR ratio of 0.4. The multiplier $(0.4 + 0.6 \times \text{NGR})$ is equal to 0.64.

At the other extreme, if all contracts currently net out to zero, NGR = 0, and the multiplier $(0.4 + 0.6 \times \text{NGR})$ is equal to 0.4. The purpose of this minimum of 0.4 is to provide protection against *potential movements* in the NGR, which, even if currently zero, could change over time.

The computation of risk-weighted assets is then obtained by applying counterparty risk weights to the credit exposure in Equation (28.6). Since most counterparties for such transactions tend to have excellent credit, the risk weights from Table 28.4 are multiplied by 50%. The **credit risk charge** for off-balance-sheet (OBS) items is defined as

$$\text{CRC(OBS)} = 8\% \times \left(\sum_i \text{RW}_i \times 50\% \times \text{Credit Exposure}_i \right) \tag{28.8}$$

Example: Credit Charge for a Swap

Consider a \$100 million interest rate swap with a domestic corporation. Assume a residual maturity of four years and a current market value of \$1 million. What is the credit risk charge?

Answer

Since there is no netting, the factor $(0.4 + 0.6 \times \text{NGR}) = 1$. From Table 28.4, we find an add-on factor of 0.5. The credit exposure is then $\text{CE} = \$1{,}000{,}000 + \$100{,}000{,}000 \times 0.5\% \times 1 = \$1{,}500{,}000$. This number must be multiplied by the counterparty-specific risk weight and one-half of 8% to derive the minimum level of capital needed to support the swap. This gives \$60,000.

EXAMPLE 28.6: CREDIT EXPOSURE FOR OPTIONS

The Basel I Accord computes the credit exposure of derivatives using both replacement cost and an add-on to cover potential future exposure. Which of the following is the correct credit risk charge for a purchased seven-year OTC equity index option of \$50 million notional with a current mark to market of \$15 million with no netting and a counterparty weighting of 100%?

a. \$1.6 million
b. \$1.2 million
c. \$150,000
d. \$1 million

28.4 ILLUSTRATION: CITIBANK

As an illustration, let us examine the capital adequacy requirements for Citibank, which was once the biggest global commercial bank.

28.4.1 Risk-Weighted Assets

Table 28.5 summarizes on-balance-sheet and off-balance-sheet items as of December 2009. The bank has total assets of \$1,161 billion, consisting of cash equivalents, securities, loans, trading assets, and other assets. The notional for each asset is assigned to one of the four risk weight categories, ranging from 0% to 100%. For example, out of the \$257 billion in securities, \$129 billion have a zero risk weight because these represent, for instance, positions in OECD government bonds. Of the remainder, \$64 billion have a 20% weight, \$11 billion have a 50% weight, and \$63 billion have a 100% weight. Most of the loans carry a risk

TABLE 28.5 Citibank's Credit Risk-Weighted Assets: 2009

On-Balance-Sheet Assets ($ Billion)						
			Risk Weight Category			
Item	Notional	Not Covered	0%	20%	50%	100%
Cash and due	174.6	0.0	140.6	29.9	0.0	4.1
Securities	257.0	(10.6)	129.5	64.2	11.0	63.0
Loans and leases	464.7	(23.9)	10.5	77.0	110.1	291.1
Trading assets	156.0	156.0	0.0	0.0	0.0	0.0
All other assets	109.0	27.9	6.7	14.6	1.0	58.7
Total on-BS	1161.4	149.4	287.3	185.6	122.1	416.8

Off-Balance-Sheet Items ($ Billion)							
				Risk Weight Category			
Item	Notional	Conversion Factor	Credit Equivalent	0%	20%	50%	100%
Financial standby LC	85.2	1.00	85.2	12.8	24.2	2.2	45.9
Performance standby LC	13.0	0.50	6.5	0.9	0.8	0.0	4.8
Commercial LC	7.1	0.20	1.4	0.1	0.4	0.0	0.9
Securities lent	53.5	1.00	53.5	53.1	0.4	0.0	0.0
Other credit substitutes	0.2	—	2.6	0.0	0.0	0.0	2.6
Other off-balance-sheet	10.4	1.00	10.4	0.0	0.1	2.9	7.5
Unused commitmt. >1 year	111.5	0.50	55.7	1.2	13.9	0.8	39.9
Unused commitmt. <1 year	36.4	0.10	3.6	0.8	1.0	0.6	1.3
Derivative contracts	35,265.0		198.9	12.8	89.8	96.4	0.0
Total off-BS			417.8	81.7	130.6	102.9	102.9

weight of 100%. Trading assets are excluded from this computation because they carry a market risk charge only.

The second panel of the table displays off-balance-sheet information. The second column displays the notional, the third the conversion factor, and the fourth the credit equivalent, which is the product of the previous two. As described in the previous section, the conversion factors are 1.00 for financial LCs and securities lent, 0.50 for performance LCs and unused commitments greater than one year, and 0.20 for commercial LCs.[5]

Finally, note the huge size of the notional derivatives position. At $35,265 billion, it is several times the size of Citibank's total assets of $1,161 billion and dwarfs its equity of $117 billion. The notional amounts, however, give no indication of the risk. The credit equivalent amount, which consists of the net replacement value plus the add-on, is $199 billion, a much lower number.

[5] The category "credit substitutes" represent residual interests, such as the equity tranche from securitizations of assets, which are subject to a dollar-for-dollar capital requirement. This implies a credit conversion factor of (1/8%) = 12.50. U.S. regulators have imposed this high capital requirement to reflect the higher risk of such residual interests, whose value can be wiped out easily in case of losses on the underlying assets.

TABLE 28.6 Citibank's Risk-Weighted Assets: 2009

Risk-Weighted Assets ($ Billion)					
	Risk Weight Category				
Item	0%	20%	50%	100%	Total
On-BS and off-BS items	369.0	316.2	225.1	519.7	
Credit RW assets	0.0	63.2	112.6	519.7	695.5
Market RW assets					54.8
Others					−14.3
Total RW assets					736.0

28.4.2 Capital Adequacy

From this information, we can compute the total risk-weighted assets and capital adequacy ratios. These are shown in Tables 28.6 and 28.7. The first line of Table 28.6 adds up on-balance-sheet and off-balance-sheet items for each category. Multiplication by the risk weights gives the second line. The total RW assets for credit risk are $696 billion, which consists of $515 billion for on-BS items and $181 billion for off-BS items. To this, we add the RW assets for market risk, or $55 billion. Thus, most of Citibank's regulatory risk capital covers credit risk. Market risk represents less than 10% of the total.

The total RW assets add up to $736 billion. Applying the 8% ratio, we find a minimum regulatory capital of $59 billion. In fact, the available risk capital adds up to $111 billion, which represents a 15% ratio, well above the regulatory minimum. In the United States, the ratio for a **well-capitalized bank** is 10%. Apparently, the regulatory constraint is not binding.[6]

TABLE 28.7 Citibank's Capital Requirements: 2009

Capital	Amount ($ Billion)	Ratio (Percent)
Equity	116.6	
Goodwill	−11.3	
Others	−8.5	
Tier 1	96.8	13.2%
Sub.debt	4.0	
LL. allowance	9.4	
Others	0.4	
Tier 2	13.8	1.9%
Total	110.6	15.0%
Tier 1 leverage		8.3%

[6] In addition, Tier 1 leverage is 8.3%, which is above the minimum ratio of 3% set by the Federal Reserve Board.

As long as the regulatory capital ratio is not binding, the bank could decide on a target optimal capital ratio, based on a careful consideration of the trade-off between increasing expected returns and increasing risks. In other words, this requires a measure of **economic capital**. If the current capital ratio is viewed as too high relative to this target, the bank could shrink its capital base through dividend payments or share repurchases. Conversely, if the bank thinks it requires more capital, it could issue new shares. Like other major banks after the crisis, Citibank holds much more capital than the minimum regulatory standard of 8%.

28.4.3 Citigroup and TARP

Table 28.8 summarizes the financial statements for Citigroup from 2003 to 2009. This is the bank holding company that wholly owns Citibank, which is the commercial bank. From 2003 to 2006, the company has expanded aggressively, as indicated by the growth in assets, and has maintained a capital ratio in a narrow range. This was partly managed by dividend payments and share repurchases.

Walter Wriston, who was chief executive officer from 1967 to 1984, famously said that banks needed little capital as long as they were run well. In other words, income should be sufficient to cover losses. In fact, Citibank ran into serious trouble in the late 1980s due to third world loans that had turned sour and during the 1991 U.S. recession. This forced the bank to raise substantial capital in 1991, confirming the regulators' view that a bank needs to carry sufficient capital to absorb large, unexpected losses.

Starting in 2007, however, the company's situation deteriorated quickly, due to heavy exposure to troubled mortgages, including indirect exposure to off-balance-sheet **structured investment vehicles** (SIVs). In the fourth quarter of 2007, Citigroup reported a loss of $9.8 billion, which led to a sharp fall in capital ratios. In January 2008, Standard & Poor's lowered Citigroup's credit rating from AA to AA−. The situation continued to worsen in 2008.

After the Lehman bankruptcy in September 2008, Citigroup was forced to raise $25 billion from the U.S. Treasury **Troubled Asset Relief Program** (TARP)

TABLE 28.8 Citigroup's Summary Financials ($ Billion)

	2003	2004	2005	2006	2007	2008	2009
Assets	1,264	1,484	1,494	1,884	2,188	1,938	1,856
Risk-weighted assets	750	852	885	1,058	1,253	996	1,089
Equity	98	109	113	120	114	71	152
Tangible common equity					70	31	118
Capital ratio	12.0%	11.9%	12.0%	11.7%	10.7%	15.7%	15.3%
Tier 1 capital ratio	8.9%	8.7%	8.8%	8.6%	7.1%	11.9%	11.7%
Net income	17.8	17.0	24.6	21.5	3.6	−28.0	−1.5
Dividends	5.8	8.4	9.2	9.8	10.8	6.0	3.4
Share repurchase	2.4	1.8	12.8	7.0	0.7	0.0	0.0
Common stock issuance	0.7	0.9	1.4	1.8	1.1	6.9	17.5

in October 2008.[7] In exchange, the Treasury acquired perpetual preferred stock, which counts as tier 1 capital.

This government bailout was not enough, however. In December 2008, the government invested a further $20 billion in preferred stock. This was called TARP II. In addition, the government agreed to back about $306 billion in Citigroup's loans and securities, in exchange for a further $7.1 billion of preferred stock. Without this injection of $52 billion, Citigroup's equity would have shrunk to $19 billion, which means that the bank was essentially insolvent.

In July 2009, Citigroup implemented an exchange offer, where investors converted their preferred stock into $57 billion of common stock; the government converted $25 billion of TARP I money to common stock. As Vikram Pandit, Citi's CEO, said, "This securities exchange has one goal—to increase our tangible common equity," because "the markets also view tangible common equity as an important measure." By December 2009, Citigroup had raised approximately $20 billion in fresh equity and had repaid the $20 billion TARP II money. It also exited the loss-sharing agreement.

The substantial increase in capital ratios was driven by a combination of reduction in risk exposures and fund-raising. This came at the cost of diluting previous shareholders. Citi's share price, which had reached $55 at the start of 2007, now languishes around $4. Put differently, $250 billion in equity market value has been wiped out.

Citigroup's travails are widely attributed to bad management, poor oversight, and inadequate risk controls.[8] Top management purposefully expanded into structured credit in the pursuit of higher profits. Indeed, Chuck Prince, then Citi's CEO, infamously said in July 2007, referring to the firm's leveraged lending practices, "When the music stops, in terms of liquidity, things will be complicated. But as long as the music is playing, you've got to get up and dance. We're still dancing."

28.5 THE BASEL II ACCORD

The Basel Accord has been widely viewed as successful in raising banking capital ratios. As a result of the Accord, the aggregate tier 1 ratio increased from $840 billion to $1,500 billion from 1990 to 1998 for the 1,000 largest banks.

28.5.1 Issues with the 1988 Basel Accord

Over time, however, these regulations have shown their age. The system has led to **regulatory arbitrage**, which can be broadly defined as bank activities aimed at getting around these regulations. Lending patterns have been transformed,

[7] TARP is a U.S. government program enacted on October 3, 2008, which authorized the U.S. Treasury to spend $700 billion to stabilize the U.S. financial system.

[8] "Citigroup Saw No Red Flags Even as It Made Bolder Bets," *New York Times*, November 22, 2008.

generally in the direction of taking on more credit risk to drive the economic capital up to the level of regulatory capital.

To illustrate, consider a situation where a bank can make a loan of \$100 million to an investment-grade company rated AAA or to a speculative-grade company rated CCC. Irrespective of the credit quality, the bank is forced to hold regulatory capital of \$8 million, so it has to borrow \$92 million. Suppose the rate of return on the AAA loan is 6%, after expenses and expected losses. The cost of borrowing is close, at 5.7%. The net dollar return to shareholders is then $\$100{,}000{,}000 \times 6\% - \$92{,}000{,}000 \times 5.7\% = \$756{,}000$. Compared to a capital base of \$8 million, this represents a rate of return of 9.5% only, which may be insufficient for shareholders. The bank could support this loan with a much smaller capital base. For instance, a capital base of \$2 million would require borrowing \$98 million and would yield a return of $\$100{,}000{,}000 \times 6\% - \$98{,}000{,}000 \times 5.7\% = \$414{,}000$, assuming the cost of debt remains the same. This translates into a rate of return of 20.7%, which is much more acceptable. The bank, however, is unable to lower its capital due to the binding regulatory requirement.

Suppose now the rate of return on the CCC loan is 7%, after expenses and expected credit losses. The net dollar return to shareholders is now \$1.756 million, which represents a 22% rate of return. In this situation, the bank has an incentive to increase the risk of its loan in order to bring the economic capital more in line with its regulatory capital. This simple example has shown that regulation may perversely induce banks to shift lending to lower-rated borrowers.

In addition to inadequate differentiation of credit risk, the 1988 Accord did not recognize credit mitigation techniques, nor the effects of securitizations. Some of these drawbacks have been corrected with Basel II.

EXAMPLE 28.7: RETURN ON BANK EQUITY

A bank that funds itself at LIBOR − 5bp purchases an A+ rated corporate floating coupon loan paying LIBOR + 15bp. Based on the Basel I minimum capital requirements, what is the annualized return on regulatory capital for this loan?

a. 2.5%
b. 5.0%
c. 11%
d. None of the above

28.5.2 Definition of Capital

Basel II gives a choice between a standardized approach, which is a simple extension of the Basel I rules, and a more complex internal ratings-based (IRB) approach.

For the former, capital is still defined as before. However, **general provisions** or **loan loss reserves** can be included in tier 2 only subject to a limit of 1.25% of risk-weighted assets.

For the IRB approach, in contrast, Basel II distinguishes between expected loss (EL) and unexpected loss (UL). Capital is supposed to absorb unexpected losses, which means that it cannot support expected losses as well. Banks typically fund accounts called general provisions or loan loss reserves to absorb expected credit losses. Hence, Basel II withdraws general provisions from tier 2 capital.[9]

28.5.3 Approaches to the Credit Risk Charge

As before, the credit risk charge is computed as the sum of individual credit charges:

$$\text{CRC} = 8\% \times \left(\sum_i \text{RW}_i \times \text{CE}_i\right) \tag{28.9}$$

where RW is the risk weight and CE is the credit exposure. In general, the capital charges are calibrated to correspond to the amount of capital required to support a 99.9% confidence level over a one-year horizon.

Banks have now a choice of three approaches for the risk weights.

Standardized Approach. This is an extension of the 1988 Accord, but with finer classification of categories for credit risk, based on external credit ratings, provided by **external credit assessment institutions**. Table 28.9 describes the new weights, which now fall into five categories for banks and sovereigns, and four categories

TABLE 28.9 Risk Weights: Standardized Approach

	Credit Rating					
Claim	**AAA/AA−**	**A+/A−**	**BBB+/BBB−**	**BB+/B−**	**Below B−**	**Unrated**
Sovereign	0%	20%	50%	100%	150%	100%
Banks—option 1	20%	50%	100%	100%	150%	100%
Banks—option 2	20%	50%	50%	100%	150%	50%
Short-term	20%	20%	20%	50%	150%	20%
Claim	**AAA/AA−**	**A+/A−**	**BBB+/BB−**		**Below BB−**	**Unrated**
Corporates	20%	50%	100%		150%	100%

Note: Under option 1, the bank rating is based on the sovereign country in which it is incorporated. Under option 2, the bank rating is based on an external credit assessment. Short-term claims are defined as having an original maturity less than three months.

[9] If total expected losses are less than eligible provisions, however, the difference may be recognized in tier 2 capital, up to a maximum of 0.6% of risk-weighted assets. However, if total expected losses exceed eligible provisions, the bank must deduct the difference from capital (50% from tier 1 and 50% from tier 2).

TABLE 28.10 IRB Risk Weights

Probability of Default	Corporate	Residential Mortgage	Other Retail
0.03%	14.44%	4.15%	4.45%
0.10%	29.65%	10.69%	11.16%
0.25%	49.47%	21.30%	21.15%
0.50%	69.61%	35.08%	32.36%
0.75%	82.78%	46.46%	40.10%
1.00%	92.32%	56.40%	45.77%
2.00%	114.86%	87.94%	57.99%
3.00%	128.44%	111.99%	62.79%
4.00%	139.58%	131.63%	65.01%
5.00%	149.86%	148.22%	66.42%
10.00%	193.09%	204.41%	75.54%
20.00%	238.23%	253.12%	100.28%
50.00%	217.87%	226.62%	105.94%

Note: Illustrative weights for LGD = 45%, maturity of 2.5 years, and large corporate exposures (firms with turnover greater than 50 million euros).

for corporates. For sovereigns, OECD membership is no longer given preferential status. For banks, two options are available. The first assigns a risk weight one notch below that of the sovereign, whereas the other uses an external credit assessment. Basel II also removes the 50% risk weight cap on derivatives.

Foundation Internal Ratings-Based Approach (FIRB Approach). Under the **internal ratings-based approach** (IRB), banks are allowed to use their internal estimate of creditworthiness, subject to regulatory standards. Under the foundation IRB approach, banks estimate the **probability of default** (PD) and supervisors supply other inputs, which carry over from the standardized approach. Table 28.10 illustrates the links between PD and the risk weights for various asset classes. For instance, a corporate loan with a 1.00% probability of default would be assigned a risk weight of 92.32%, which is close to the standard risk weight of 100% from Basel I. Note that retail loans have much lower risk weights than the other categories, reflecting their greater diversification.[10]

Advanced Internal Ratings-Based Approach (AIRB Approach). Under the advanced IRB approach, banks can supply other inputs as well. These include **loss given default** (LGD) and **exposure at default** (EAD). The combined PDs and LGDs for all applicable exposures are then mapped into regulatory risk weights. The capital charge is obtained by multiplying the risk weight by EAD by 8%. The advanced IRB approach applies only to sovereign, bank, and corporate exposures and not to retail portfolios.

[10] Also, these weights cover unexpected losses; as the PD increases to very high levels, the weights start to decrease because most of the losses are expected and hence should be covered by general provisions.

Factor Model for Default Correlations. Basel II still computes the credit risk charge from the sum of individual credit charges. Indeed, Equation (28.9) is *additive*. This specification is simple and robust.

However, it is not clear why the addition of individual capital charges should lead to a capital charge that reflects a 99.9% VAR measure for the entire portfolio. After all, summing individual VARs certainly does not add up to the portfolio VAR.[11] It turns out, however, that when default correlations are generated by a single factor, this decomposition adds up to a good approximation of the portfolio risk. The capital charges have been chosen by the Basel Committee to represent bank portfolios of typical sizes with typical default correlations.

More precisely, the risk weight function for the IRB models is based on the so-called **asymptotic single risk factor** (ASRF) model, which assumes (1) a single common factor and (2) perfect diversification of the idiosyncratic risk. A well-diversified portfolio has a large number of positions without concentrations, in which case it is also called highly, or infinitely, **granular**. The ASRF model creates individual capital charges that are **portfolio invariant**, meaning that they do not depend on the rest of the portfolio.

The risk weight function in Table 28.9 is specified in terms of the PD, LGD, and maturity of the asset. It incorporates a correlation function $\rho(\text{PD})$ which decreases as the PD increases for corporate and retail credits. For corporate credits, the correlation starts around 0.24 for low PDs, then decreases to 0.12 for PDs above 10%. This reflects the observation that default probabilities for low credits are more idiosyncratic, in contrast with better credits, which tend to default due to the general state of the economy. Default correlations are lower for retail credits, varying from 0.16 to 0.03 to reflect their greater diversification.

Adoption of Approach. Banks with simple portfolios can follow the standardized approach. More advanced banks are expected to adopt an IRB approach. To be eligible for the IRB approach, a bank must demonstrate to its supervisor that it meets a set of minimum requirements. Most importantly, the internal rating system must be consistent and reliable. Also, banks cannot allocate borrowers across rating systems, or cherry-pick ratings to minimize capital requirements. In addition, the bank-developed rating system must be approved at the highest level and subject to independent oversight.

Once a bank adopts an IRB approach, it is expected to extend it eventually across all asset classes and across the entire banking group. Banks adopting the IRB approach are expected to continue to employ it; a voluntary return to the standardized approach is permitted only in special cases, as approved by the supervisor.

28.5.4 Extension: Credit Risk Mitigation

Basel II also recognizes **credit risk mitigation** (CRM) techniques, such as collateralization, third-party guarantees, credit derivatives, and netting. **Collateralized credit exposures** are those where the borrower has posted assets as collateral. Recognition is given only to cash, gold, listed equities, investment-grade debt,

[11] This analysis is similar to the **component VAR** decomposition presented in Chapter 29.

sovereign securities rated BB− or better, or mutual funds investing in the same assets.

Under the standardized approach, two treatments are possible. In the **simple approach**, the risk of the collateral is simply substituted for that of the counterparty, generally subject to a 20% floor. The second method is the **comprehensive approach**. This is more accurate and will lead to lower capital charges.

Even if the exposure is exactly matched by the collateral, there is some credit risk due to the volatility of values during a default. In the worst case, the value of the exposure could go up and that of the collateralized assets could go down. This volatility effect is measured by a **haircut** parameter (H) that is instrument-specific and approximates a 99% VAR measure over a 10-day period. For equities, for example, $H = 25\%$. For cash, this is zero.

The exposure after risk mitigation is then

$$E^* = E \times (1 + H_e) - C \times (1 - H_c - H_{fx}) \qquad (28.10)$$

if positive, where E is the value of the uncollateralized exposure, C is the current market value of the collateral held, H_e is the haircut appropriate to the exposure, H_c is the haircut appropriate to the collateral, and H_{fx} is the haircut appropriate for a currency mismatch between the two.

Under the foundation approach, only the comprehensive approach is allowed. The effective loss given default (LGD*) is derived from the usual LGD and the current value of the exposure E and the exposure after risk mitigation E^*

$$\text{LGD}^* = \text{LGD} \times (E^*/E) \qquad (28.11)$$

Other forms of CRM are **guarantees** and **credit derivatives**, which are a form of protection against obligor default provided by a third party, called the guarantor. Capital relief, however, is granted only if there is no uncertainty as to the quality of the guarantee. Protection must be direct, explicit, irrevocable, and unconditional. In such a situation, one can apply the principle of **substitution**. In other words, if Bank A buys credit protection against a default of Company B from Bank C, it may substitute C's credit risk for B's risk. It will do so if the credit rating of Bank C is better than that of B.

An allowance can be made, however, for the low probability of **double default**. In order for Bank A to incur a credit loss, both B and C must default. The likelihood of such a double default occurrence is generally very low. For instance, if defaults are independent, the probability of a credit loss is given by the *product* of the two default probabilities. In July 2005, the BCBS adopted new capital requirements that account for double default effects

$$\text{RW}_{\text{DD}} = \text{RW}_0(0.15 + 160 \times \text{PD}_g) \qquad (28.12)$$

where RW_0 is the original capital requirement and PD_g is the probability of default of the guarantor, Bank C in this case. For instance, assuming that Bank C is rated A, its default probability is about 0.1%, which gives $\text{RW}_{\text{DD}} = \text{RW}_0(0.31)$, which is lower than RW_0.

TABLE 28.11 Risk Weights for Securitizations: Standardized Approach

	AAA/AA−	A+/A−	BBB+/BBB−	BB+/BB−	B+ and Below or Unrated
Tranche	20%	50%	100%	350%	1,250% (deduction)

28.5.5 Extension: Securitization

Finally, Basel II also deals explicitly with **securitization**, which involves the economic or legal transfer of assets to a third party, typically called **special-purpose vehicle** (SPV). Examples are asset-backed securities such as collateralized loan obligations (CLOs), where the underlying asset is a pool of bank loans. Because of the high regulatory cost of keeping loans on their balance sheets, banks are now routinely transforming loans into tradable securities. The securitization process is explained in Chapter 18.

A bank can remove these assets from its balance sheet only after a **true sale**, which is defined using **clean break** criteria. These are satisfied if a number of conditions are all met: (1) significant credit risk must be transferred to third parties, (2) the seller does not maintain effective or indirect control over the assets,[12] (3) the securities are not an obligation of the seller, and (4) the holders of the SPV have the right to pledge or exchange those interests. Two other technical conditions are also involved.

If these conditions are all met, then the bank can remove the assets from its balance sheet and becomes subject to new risk weights for securitization tranches. These are described in Table 28.11 under the standardized approach. For example, the risk weight for a BBB-rated tranche is 100%. For the lowest-rated tranches, the bank must hold capital equal to the notional amount, which implies a risk weight of $(1/8\%) = 1{,}250\%$.

EXAMPLE 28.8: FRM EXAM 2004—QUESTION 67

Which of the following statements about the Basel II capital requirements is *false*?

- **a.** It increases the risk sensitivity of minimum capital requirements for internationally active banks.
- **b.** It addresses only credit risk and market risk.
- **c.** U.S. insurance companies are not required to comply with Basel II capital requirements.
- **d.** Banks are not allowed to use their internal models for credit risk in determining the capital requirements for credit risk.

[12] In particular, the transferred assets must be legally separated from the seller so that it does not have additional obligations in case the SPV goes bankrupt. Also, the seller cannot maintain effective control either by being able to repurchase the assets at a profit or by being obligated to retain the risk of the transferred assets.

EXAMPLE 28.9: FRM EXAM 2006—QUESTION 108

Which of the following statements is *not* correct about the foundation IRB and the advanced IRB approaches for credit risk capital charges in Basel II?

a. Under the advanced IRB approach, banks are allowed to use their own estimates of PD, LGD, EAD, and correlation coefficient but must use the risk weight functions provided by the supervisors.
b. Under the foundation IRB approach, banks provide their own estimates of PD and rely on supervisory estimates for other risk components.
c. Banks adopting the advanced IRB approach are expected to continue to employ this approach. A voluntary return to the standardized approach is permitted only in extraordinary circumstances.
d. Under both foundation IRB and advanced IRB approaches, the expected loss is not included in the credit risk capital charge.

EXAMPLE 28.10: FRM EXAM 2006—QUESTION 90

Under the comprehensive approach for the foundation internal ratings-based approach under Basel II, which of the following methods is used for calculating the effective loss given default (LGD^*) where: LGD^* is the effective loss given default (considering risk mitigation measures), LGD is that of the senior unsecured exposure before recognition of collateral, E is the current value of the exposure (i.e., cash lent or securities lent or posted), and E^* is the exposure value after risk mitigation?

a. $LGD^* = LGD \times (E^*/E)$
b. $LGD^* = LGD \times (E^*) * (E)$
c. $LGD^* = LGD \times (E^* + E)$
d. $LGD^* = LGD \times (E^* - E)$

EXAMPLE 28.11: FRM EXAM 2008—QUESTION 4-18

The Basel II risk weight function for the internal ratings-based (IRB) approach is based on the asymptotic single risk factor (ASRF) model, under which the the system-wide risks that affect all obligors are modeled with only one systematic risk factor. The major reason for using the ASRF is:

a. The model should not depend on the granularity of the portfolio.
b. The model should be portfolio invariant so that the capital required for any given loan depends only on the risk of that loan and does not depend on the portfolio it is added to.
c. The model should not be portfolio invariant and the capital required for any given loan should not depend on the risk of other loans.
d. The model corresponds to the one-year VAR at a 99.9% confidence level.

EXAMPLE 28.12: FRM EXAM 2008—QUESTION 4-3

Which of the following is *not* a drawback of the Basel II foundation internal ratings-based (IRB) approach?

a. PDs and LGDs are assumed to be uncorrelated.
b. Asset correlations decrease with increasing PDs.
c. The portfolio of the financial institution is assumed to be infinitely granular.
d. The approach uses a single risk factor portfolio model instead of a multiple risk factor model.

28.5.6 Evaluation

The BCBS has organized a large-scale analysis of the effect of the new capital requirements on the banking system. Table 28.12 reports the results for 228 banks in the G-10 countries. The table shows that the new capital charge will affect banks differentially. Smaller banks, with more retail exposures, will have lower capital requirements than before. Retail risks are indeed more diversified than other types.

Larger banks are more likely to adopt the AIRB approach because it leads to lower capital requirements than the standardized approach. The table lists the results for the standardized and most likely IRB approach. For instance, large

TABLE 28.12 Percentage Changes in Capital Requirements (Banks in G-10 Countries)

	Larger Banks		Smaller Banks	
	Method		Method	
Portfolio	Standardized	IRB	Standardized	IRB
Corporate	0.9%	−5.0%	−1.0%	−4.5%
Bank	1.5%	0.4%	0.2%	0.1%
Sovereign	0.2%	1.3%	−0.1%	0.6%
SME	−0.2%	−1.3%	−0.1%	−2.2%
Mortgage	−6.3%	−7.6%	−6.2%	−12.6%
Retail	−0.7%	−0.9%	−2.5%	−4.5%
Other	0.8%	2.6%	0.0%	1.5%
Overall credit risk	−3.8%	−10.5%	−9.7%	−21.6%
Operational risk	5.6%	6.1%	8.3%	7.5%
Overall change	1.8%	−4.4%	−1.4%	−14.1%

Source: QIS5 study conducted by the BCBS (2006).

banks will suffer a slightly higher capital charge (by 1.8%) under the standardized approach, which is primarily due to the addition of the operational risk charge. Under the AIRB approach, however, the credit risk charge drops by 10.5%, which leads to a net decrease in capital requirements of 4.4%. As indicated in Section 28.1, however, Basel III will require an increase in the amount and quality of bank capital.

28.6 THE MARKET RISK CHARGE

After the credit risk charges were instituted in 1988, the Basel Committee turned its attention to market risk in response to the increased proprietary trading activities of commercial banks. The Capital Accord was amended in 1996 to include a capital charge for market risk, which was implemented by January 1998.[13] This has become known as the **1996 Amendment**.

The capital charge can be computed using two methods. The first is based on a standardized method, similar to the credit risk system with add-ons determined by the Basel rules. Because diversification effects are not fully recognized, this method generates a high market risk charge. The second method is called the **internal models approach** (IMA) and is based on the banks' own risk management systems, which are more adaptable than the rigid set of standardized rules. This approach must be viewed as a breakthrough in financial regulation. For the first time, regulators relied on the banks' own VAR systems to determine the capital charge. Since banks may have an incentive to understate their market risk, however, the internal models approach also includes a strong system of verification, based on backtesting. Another motivation for the IMA was to provide incentives for banks to develop risk management systems. This is because the IMA approach leads to lower capital charges than the standardized approach.

The market risk framework was updated and fully incorporated in Basel II. During the recent credit crisis, however, a number of banks suffered very large losses not captured by the usual VAR numbers. As a result, the Basel Committee has made a number of changes to the market risk framework, to be implemented by December 31, 2011.[14]

28.6.1 The Standardized Method

This **building block approach** consists of attaching add-ons to all positions, which are summed up across the portfolio. The bank's market risk charge is first computed individually for portfolios exposed to interest rate risk (IR), equity risk

[13] BCBS, *Amendment to the Basel Capital Accord to Incorporate Market Risk* (Basel: BIS, 1996).
[14] See BCBS, *Revisions to the Basel II Market Risk Framework* (Basel: BIS, 2009, as updated in June 2010).

(EQ), foreign exchange (FX) risk, commodity risk (CO), and option risk (OP), using specific guidelines. The bank's total risk is then obtained from the summation of risks across the four categories. Because the construction of the risk charge follows a prespecified process, this approach is sometimes called the **standardized method**.

The bank's total risk is obtained from the summation of risks across different types of risks, j, on each day, t:

$$\mathrm{MRC}_t^{\mathrm{STD}} = \sum_{j=1}^{5} \mathrm{MRC}_t^{j} = \mathrm{MRC}_t^{\mathrm{IR}} + \mathrm{MRC}_t^{\mathrm{EQ}} + \mathrm{MRC}_t^{\mathrm{FX}} + \mathrm{MRC}_t^{\mathrm{CO}} + \mathrm{MRC}_t^{\mathrm{OP}} \tag{28.13}$$

The interest rate risk charge is the sum of a general market risk charge, which typically increases for longer-duration instruments, and a specific risk charge, which covers against issuer-specific risk. For instance, the weight for long-term investment-grade credits is 1.60%. For equity risk, the general market risk charge is 8% of the net positions; the specific risk charge is 8% of the gross positions, unless the portfolio is both liquid and well-diversified, in which case the weight is reduced to 4%. For currency risk, the market risk charge is 8% of the higher of either the net long currency positions or the net short currency positions. For commodity risk, several approaches are possible. In the simplified approach, the risk charge is 15% of the net position in each commodity. Finally, for option risk, several approaches are also possible. In the simplified approach, which applies when banks handle a limited range of purchased options, the capital charge is the lesser of the market risk charge for the underlying security and the option premium.

Thus, the standardized model is relatively straightforward to implement. It is also robust to model misspecification. The building block approach, however, has been criticized on several grounds. First, the risk classification is arbitrary. For instance, a capital charge of 8% is applied uniformly to equities and currencies without regard for their actual return volatilities. Different currencies have different volatilities relative to the dollar that also can change over time.

Second, the approach leads to high capital requirements because risk charges are systematically added up across different sources of risk, which ignores diversification. For instance, fixed-income charges are computed for each currency separately, then added up across markets. Implicitly, this approach is a worst-case scenario that assumes that the worst loss will occur at the same time across all sources of risk. In practice, these markets are not perfectly correlated, which means that the worst loss will be less than the sum of individual worst losses. Thus the standardized model fails to recognize the benefits of diversification, which gives no incentive for banks to diversify prudently. Recognition of these problems has led to another, more flexible approach based on internal models.

EXAMPLE 28.13: FRM EXAM 2007—QUESTION 63

You are an analyst at Bank Alpha. You were given the task to determine whether under Basel II your bank can use the standardized approach to report options exposure instead of the internal models approach. Which of the following criteria would your bank have to satisfy in order for it to use the simplified approach?

a. The bank writes options, but its options trading is insignificant in relation to its overall business activities.
b. The bank purchases and writes options and has significant option trading.
c. The bank solely purchases options, and its options trading is insignificant in relation to its overall business activities.
d. The bank purchases and writes options, but its option trading is insignificant.

28.6.2 The Internal Models Approach

In contrast to the simplistic standardized approach, the **internal models approach** (IMA) relies on internal risk management systems developed by banks themselves as the basis for the market risk charge. Regulators, however, have not given up their authority. A bank can use internal models only after it has been explicitly approved by the supervisory authority. The bank must satisfy qualitative requirements first. Second, the output is subject to a rigorous backtesting process. This is covered in Chapter 16.

Qualitative Requirements. Not all banks can use internal models, though. Regulators first must have some general reassurance that the bank's risk management system is sound. As a result, banks first have to satisfy various **qualitative standards**:

- *Independent risk control unit.* The bank must have a risk control unit that is independent of trading and reports to senior management. This structure minimizes potential conflicts of interest.
- *Backtesting.* The bank must conduct a regular backtesting program, which provides essential feedback on the accuracy of internal VAR models.
- *Involvement.* Senior management and the board need to be involved in the risk control process and devote sufficient resources to risk management.
- *Integration.* The bank's internal risk model must be integrated with day-to-day management. This avoids situations where a bank could compute its VAR simply for regulatory purposes and otherwise ignore it.

- *Use of limits*. The bank should use its risk measurement systems to set internal trading and exposure limits.
- *Stress testing*. The bank should conduct stress tests on a regular basis. Stress test results should be reviewed by senior management and be reflected in policies and limits set by management and the board of directors.
- *Compliance*. The bank should be compliant with a documented set of policies.
- *Independent review*. An independent review of the trading units and of the risk control unit should be performed regularly, at least once a year.

In addition to these requirements, the bank's risk model must contain a sufficient number of risk factors, where the definition of *sufficient* depends on the extent and complexity of trading activities. The 2009 revisions would require a bank to justify why any factor used in pricing is left out in the VAR computation.

For material exposures to interest rates, there should be at least six factors for yield curve risk plus separate factors to model spread risk. For equity risk, the model should consist of at least beta mapping on an index; a more detailed approach would have industry and even individual risk factor modeling. For active trading in commodities, the risk model should account for movements in spot rates plus convenience yields. Banks should also capture the nonlinear price characteristics of option positions, including vega risk. Correlations *within* broad risk categories are recognized explicitly. Regulators can also recognize correlations *across* risk categories provided the model is sound.

The IMA Market Risk Charge. Once these requirements are satisfied, the market risk charge is computed according to these rules:

- *Quantitative parameters*. The computation of daily VAR shall be based on a set of uniform quantitative inputs:
  - A horizon of 10 trading days, or two calendar weeks; banks can, however, scale their daily VAR by the square root of time.
  - A 99% confidence interval.
  - An observation period based on at least a year of historical data or, if a non-equal-weighting scheme is used, an average time lag of at least six months.[15]
  - At least quarterly updating, or whenever prices are subject to material changes (so that sudden increases in risk can be picked up); the 2009 revisions require a minimum monthly update.

[15] This is similar to a duration computation. For instance, with equal weights over the last 250 trading days, this average time lag is $\sum_{t=1}^{N} t(1/N) = N(N+1)/2\ (1/N) = (N+1)/2 = 125.5$ days, or six months. Note that this rules out models such as the GARCH process if the weight on more recent observations is too high. The 2009 revisions allow placing more weight on more recent data as long as the resulting VAR is higher than with the usual method.

- *Market risk charge*. The general market capital charge shall be set at the higher of the previous day's VAR, or the average VAR over the past 60 business days times a multiplicative factor k. The exact value of this **multiplicative factor** is to be determined by local regulators, subject to an absolute floor of 3.

  The purpose of this factor is twofold. Without this risk factor, a bank would be expected to have losses that exceed its capital in one 10-day period out of a hundred, or about once in four years. This does not seem prudent. Second, the factor serves as a buffer against model misspecifications, for instance assuming a normal distribution when the distribution really has fatter tails.
- *Plus factor*. A penalty component, called **plus factor**, shall be added to the multiplicative factor, k, if verification of the VAR forecasts reveals that the bank systematically underestimates its risks. The purpose of this factor is to penalize a bank that provides an overly optimistic projection of its market risk.

In summary, the market risk charge on any day t is

$$MRC_t^{IMA} = \text{Max}\left(k\frac{1}{60}\sum_{i=1}^{60} VAR_{t-i},\ VAR_{t-1}\right) + SRC_t \tag{28.14}$$

where VAR_{t-i} is the bank's VAR over a 10-day horizon at the 99% level of confidence and SRC is the specific risk charge (discussed later). Here, the factor k reflects both the multiplicative and the plus factors.

The first term consists of a multiplier k times the average VAR over the past 60 days. The second term uses yesterday's VAR, and will be binding only if the positions change dramatically. In practice, this is rarely the case.

The 2009 revisions add a **stress VAR** (SVAR) and an **incremental risk charge** (IRC):

$$\begin{aligned} MRC_t^{IMA} &= \text{Max}\left(k\tfrac{1}{60}\textstyle\sum_{i=1}^{60} VAR_{t-i},\ VAR_{t-1}\right) \\ &\quad + \text{Max}\left(k_S\tfrac{1}{60}\textstyle\sum_{i=1}^{60} SVAR_{t-i},\ SVAR_{t-1}\right) + SRC_t + IRC_t \end{aligned} \tag{28.15}$$

where SVAR is the loss on the current portfolio that corresponds to a 99% confidence level over a 10-day period, calibrated over a continuous 12-month period such as 2007/2008. The multiplier k_S, as with k, is subject to a floor of 3, in addition to a plus factor that depends on the results of backtesting the regular VAR, and not SVAR.

In Equation (28.14), SRC represents the **specific risk charge**, which is a buffer against idiosyncratic factors, including basis risk and event risk, related to individual bond and equity issuers. Event risk includes the risk of a downgrade and default. Banks that use internal models can incorporate specific risk in their VARs, as long as they (1) satisfy additional criteria and (2) can demonstrate that they can deal with event risk.[16]

Consider, for instance, a corporate bond issued by Ford Motor, which we assume has a credit rating of B. The general market risk component could capture the effect of movements in yields for an index of B-rated corporate bonds. In contrast, the SRC should capture the effect of basis risk between Ford bonds and the index, as well as a downgrade or default.

During the events of 2007 and 2008, a number of banks suffered very large losses not captured by the usual VAR numbers. Banks lost money on credit-related instruments, such as positions in CDOs of ABSs, which experienced extreme price moves. Many of these positions had been moved from the banking book to the trading book because of the lower capital charge in the trading book. In addition, positions such as bank loans became much less liquid. As a result, the Basel Committee has expanded the specific risk charge to two components.

The first is a more narrow specific risk charge, still based on a 10-day 99% VAR with a multiplier of 3 or more, which covers idiosyncratic risk. This reflects the risk that a security could move by more than the general market factors.

The second is an **incremental risk charge** (IRC) that covers (1) default risk and (2) credit migration risk for debt instruments. Correlations between defaults should be taken into account. In addition, portfolios with a higher concentration should have a higher capital charge. For equity instruments, this new charge should cover events such as merger breakups and takeovers. This charge does not apply to positions that depend on currencies, risk-free interest rates, and commodities. Securitized products cannot be included in the IRC model, even when used as a hedge, because they are subject to the capital charges of the banking book.

The IRC is based on an **incremental risk measure** (IRM), calibrated to a 99.9% confidence level over one year, computed on at least a weekly basis. These parameters are selected to avoid arbitrage with the banking book. The IRC is computed from the maximum of the 12-week average and the most recent value:

$$\mathrm{IRC}_t = \mathrm{Max}\left(\frac{1}{12}\sum_{i=1}^{12}\mathrm{IRM}_{t-i},\ \mathrm{IRM}_{t-1}\right) \tag{28.16}$$

Any position with a liquidity horizon shorter than a year is assumed replaced by a position with the same level of risk, or *rolled over*.

These changes lead to a substantial change in the market risk charge, which will typically increase by a factor of three. Half of this change is due to the addition of SVAR; the other half is from the new IRC. In addition, these changes will make

[16] The difficulty with event and default risk is that it is typically not reflected in historical data. When a bank cannot satisfy (2), a prudential surcharge is applied to the measure of specific risk.

the market risk charge less procyclical. When traditional VAR is based on a short window, benign market conditions lead to low estimated VAR measures, which may induce banks to increase exposures that may be hard to manage when market conditions deteriorate. SVAR eliminates this effect because the movements in the risk factors are more constant.

28.6.3 Combination of Approaches

The banks' market risk capital requirement will be either (1) the risk charge obtained by the standardized methodology, obtained from an arithmetic summation across the five risk categories, or (2) the risk charge obtained by the internal models approach, or (3) a mixture of (1) and (2) summed arithmetically.

EXAMPLE 28.14: FRM EXAM 2007—QUESTION 91

Under the Basel II Capital Accord, banks that have obtained prior regulatory approval can use the internal models approach to estimate their market risk capital requirement. What approach or methodology is used under the internal models approach to compute capital requirements?

a. Internal rating and vendor models
b. Stress-testing and backtesting
c. Expected tail loss, as VAR is not a coherent measure of risk
d. VAR methodology

EXAMPLE 28.15: FRM EXAM 2004—QUESTION 70

Under the 1996 market risk amendment to the Basel Accord, a bank can use its internal models to calculate its market risk charge subject to all the following provisions *except*:

a. A time horizon of 10 trading days
b. A 99% confidence level
c. One year of historical observations, which are updated semiannually
d. The market risk charge set at the higher of the previous day's VAR or the average VAR over the past 60 days scaled by a multiplicative factor

EXAMPLE 28.16: WEIGHING SCHEME

The 1996 amendment to the Capital Accord requires that internal models

a. Utilize at least six months of historical data
b. Utilize at least one year of equally weighted historical data
c. Utilize enough historical data so that the weighted average age of the data is at least six months
d. Utilize two years of historical data, unequally weighted

EXAMPLE 28.17: FRM EXAM 2001—QUESTION 42

Which of the following best describes the quantitative parameters of the 1996 internal models approach?

a. Ten-day trading horizon, 99% confidence interval, minimum one year of data, minimum quarterly updates
b. One-day trading horizon, 95% confidence interval, five years of data, updated weekly
c. One-day trading horizon, 99% confidence interval, minimum one year of data, updated monthly
d. Ten-day trading horizon, 97.5% confidence interval, minimum five years of data, updated daily

EXAMPLE 28.18: FRM EXAM 2009—QUESTION 7-4

As a risk manager for Bank ABC, John is asked to calculate the market risk capital charge of the bank's trading portfolio under the 1996 internal models approach. The VAR (95%, one-day) of the last trading day is USD 30,000; the average VAR (95%, one-day) for the last 60 trading days is USD 20,000. The multiplier is $k = 3$. Assuming the return of the bank's trading portfolio is normally distributed, what is the market risk capital charge of the trading portfolio?

a. USD 84,582
b. USD 189,737
c. USD 268,200
d. USD 134,594

EXAMPLE 28.19: FRM EXAM 2009—QUESTION 7-11

In the latest guidelines for computing capital for incremental risk in the trading book, the incremental risk charge (IRC) addresses a number of perceived shortcomings in the 99%/10-day VAR framework. Which of the following statements about the IRC are *correct*?

I. For all IRC-covered positions, the IRC model must measure losses due to default and migration over a one-year horizon at a 99% confidence level.
II. A bank can incorporate into its IRC model any securitization positions that hedge underlying credit instruments held in the trading account.
III. A bank must calculate the IRC measure at least weekly, or more frequently as directed by its supervisor.
IV. The incremental risk capital charge is the maximum of (1) the average of the IRC measures over 12 weeks and (2) the most recent IRC measure.

a. I and II
b. III and IV
c. I, II, and III
d. II, III, and IV

28.7 CONCLUSIONS

The Basel II Accord represents a major step forward for the measurement and management of banking risks. It creates more risk-sensitive capital charges for credit risk and, for the first time, attempts to account for operational risk.

Among winners will be banks that invest in risk management systems, banks with large retail portfolios, and banks with high-grade corporate credits. Indeed, all of these should have lower credit risk than the rest of the industry.

This new framework is certainly not perfect, however. The standardized approach has been criticized for the greater role given to credit rating agencies. The internal models approach is viewed as giving too much discretion to banks. These features, however, are certainly much better than the Basel I alternative and are the inevitable result of the move toward risk-sensitive capital charges.

Like any set of formal rules, the Basel II rules leave open some possibilities of regulatory arbitrage, due to discrepancies between economic and regulatory capital for some assets. The framework incorporates typical correlations in the construction of the credit risk charge. Institutions that have greater diversification than typical banks cannot enjoy lower capital charges. In theory, this could be corrected by using internal **portfolio credit risk models**, developed by the banks themselves. In practice, these models are not allowed for setting credit risk charges because they are not viewed as robust enough. Indeed, the losses suffered during the credit crisis that started in 2007 have highlighted major weaknesses in credit risk management.

More importantly, the system of capital charges has missed an important element of banking risk, which is liquidity risk. The need to recapitalize major banks by governments proves that capital levels were not adequate to protect against a major financial crisis. This justifies the addition of a liquidity risk charge under Basel III.

Finally, more risk-sensitive capital charges could create a **procyclical effect**. In a recession, defaults increase, leading to greater credit risk and greater capital charges. At the same time, the banking system suffers credit losses that erode its actual capital. Caught between lower actual capital and greater regulatory capital requirements, the banking system could respond by reducing its lending activities to the economy, which could aggravate the extent of the recession. This problem has long been identified as an inevitable by-product of setting more risk-sensitive capital charges. This explains why the Basel Committee has set up a **countercyclical buffer**. In effect, the capital charges would be higher during periods of expansion but would be lowered during a period of stress. In general, we should expect a continued flurry of activity from banking supervisors in the coming years.

EXAMPLE 28.20: FRM EXAM 2007—QUESTION 19

Your bank is implementing the AIRB approach for credit risk, the AMA for operational risk, and the internal models approach for market risk. The chief risk officer (CRO) wants to estimate the bank's total risk by adding up the regulatory capital for market risk, credit risk, and operational risk. The CRO asks you to identify the problems with using this approach to estimate the bank's total risk. Which of the following statements about this approach is *incorrect*?

a. It assumes market, credit, and operational risks have zero correlation.
b. It uses a 10-day horizon for market risk.
c. It ignores strategic risks.
d. It ignores the interest risk associated with the bank's loans.

28.8 IMPORTANT FORMULAS

Basel I credit risk charge: $CRC = 8\% \times RWA = 8\% \times \left(\sum_i RW_i \times CE_i\right)$

Derivatives credit exposure: Credit Exposure = NRV + Add-On

Add-On = Notional × Add-On Factor × $(0.4 + 0.6 \times NGR)$

Basel II total risk charge: $TRC = CRC + MRC + ORC$

Basel II credit risk charge:

Standardized approach: $RW = f(\text{credit rating})$

FIRB: $RW = f(PD)$

AIRB: $RW = f(PD, LGD)$

RW including double default effects: $RW_{DD} = RW_0(0.15 + 160 \times PD_g)$

Market risk charge, standardized method: $MRC_t^{STD} = MRC_t^{IR} + MRC_t^{EQ} + MRC_t^{FX} + MRC_t^{CO} + MRC_t^{OP}$

Market risk charge (1996), internal models approach: $MRC_t^{IMA} = \text{Max}\left(k\frac{1}{60}\sum_{i=1}^{60} VAR_{t-i}, VAR_{t-1}\right) + SRC_t$

New market risk charge (2009), internal models approach: $MRC_t^{IMA} = \text{Max}\left(k\frac{1}{60}\sum_{i=1}^{60} VAR_{t-i}, VAR_{t-1}\right) + \text{Max}\left(k_S\frac{1}{60}\sum_{i=1}^{60} SVAR_{t-i}, SVAR_{t-1}\right) + SRC_t + IRC_t$

28.9 ANSWERS TO CHAPTER EXAMPLES

Example 28.1: Applicable Market Risks

b. In addition to all the risks in the trading book (interest rate, equity, foreign exchange, commodity), the market capital charges also include foreign exchange and commodity risks in the banking book.

Example 28.2: FRM Exam 2002—Question 71

a. Tier 1 capital includes equity capital, disclosed reserves, and retained earnings. Tier 2 includes undisclosed reserves, hybrid debt, and subordinated debt.

Example 28.3: FRM Exam 2007—Question 53

b. Tier 1 capital consists of equity minus goodwill, or USD 570 million. Tier 2 capital includes asset revaluation reserves of \$5 million and loan loss reserves of \$5 million. For this question, it is sufficient to find the correct number for tier 1 capital.

Example 28.4: FRM Exam 2004—Question 29

b. Tier 1 capital consists of shareholders' funds plus retained earnings, minus goodwill, which is 671.7. Tier 2 capital consists of subordinated debt plus undisclosed reserves, or 213.5. The ratio is 31.78%. Specific provisions cannot be included in risk capital, because they are likely to be absorbed by specific bad loans.

Example 28.5: Basel I Credit Risk Charge

b. Under the Basel I rules, the charge is $\$100 \times 50\% \times 8\% = \$4$ million.

Example 28.6: Credit Exposure for Options

a. From Table 28.4, the add-on factor is 10%. This gives a credit exposure of $\$15 + \$50 \times 10\% = \$20$ million, and a credit risk charge of $\$20 \times 8\% = \$1.6$ million.

Example 28.7: Return on Bank Equity

a. An 8% capital charge applies to this bond. We buy \$100 worth of the bond, which is funded at the bank rate, for a net dollar return of $\$100[(L + 0.15\%) - (L - 0.05\%)] = \$0.20$. We need to keep \$8 in capital, which we assume is not invested. The rate of return is then $\$0.20/\$8 = 2.5\%$.

Example 28.8: FRM Exam 2004—Question 67

b. Statement b. is false because Basel II also covers operational risk. Banks can provide inputs but cannot use their internal models for credit risk, so statement d. is true.

Example 28.9: FRM Exam 2006—Question 108

a. Banks are never allowed to use their own correlations.

Example 28.10: FRM Exam 2006—Question 90

a. See Equation (28.11). Also, this answer is the only one that makes sense taking units into account because LGD is a unitless ratio.

Example 28.11: FRM Exam 2008—Question 4-18

b. Because the capital charges for individual credits are added together, it must be invariant to the rest of the portfolio. The model also assumes infinite granularity.

Example 28.12: FRM Exam 2008—Question 4-3

b. In practice, PDs and LGDs are positively correlated, so statement a. is a problem. Years with higher PDs are associated with higher LGDs. Portfolios may not be highly granular, so statement c. is a problem. The portfolio may be exposed to multiple common risk factors, so statement d. is a problem. In contrast, we do observe in practice that low credits tend to have more idiosyncratic risk, which means that high PDs have low correlations.

Example 28.13: FRM Exam 2007—Question 63

c. A bank can use the simplified approach only if it purchases options and its option trading is not significant. Otherwise, it is required to use the intermediate approach. Another way to look at the question is that answer c. contains the weakest conditions (i.e., those least likely to lead to a large loss).

Example 28.14: FRM Exam 2007—Question 91

d. The internal models approach is based on the banks' internal VAR methodology.

Example 28.15: FRM Exam 2004—Question 70

c. The IMA requires using one year of historical data updated at least quarterly, not semiannually.

Example 28.16: Weighing Scheme

c. Answer b. is correct if the bank uses fixed weights only. Otherwise, the average time lag of the observations cannot be less than six months.

Example 28.17: FRM Exam 2001—Question 42

a. The IMA is based on a 10-day horizon, 99% confidence level, one year of data, with at least quarterly updates.

Example 28.18: FRM Exam 2009—Question 7-4

c. The average VAR times 3 is USD 60,000. Because this is higher than yesterday's VAR, this is the binding number. Multiplying by $\sqrt{10} \times 2.323/1.645 = 4.47$ gives USD 268,200.

Example 28.19: FRM Exam 2009—Question 7-11

b. Statement I. is incorrect because the confidence level is 99.9%. Statement II. is incorrect because securitizations are subject to the banking book capital requirements. The other two statements are correct.

Example 28.20: FRM Exam 2007—Question 19

a. Adding up the capital charges assumes perfect correlations (or at least high correlations, implying extreme shocks happen at the same time), not zero correlations. The market risk charge uses a 10-day horizon, so statement b. is correct. The Basel capital charges do ignore strategic risk and interest rate risk in the banking book, so statements c. and d. are correct.

PART

Eight

Investment Risk Management

CHAPTER 29

Portfolio Risk Management

We now turn to risk management in the context of the portfolio management process. Investors assume risk because they expect to be compensated for it in the form of higher returns. The real issue is how to balance risk against expected return. This trade-off is the subject of **portfolio management**. This requires formal risk measurement, however.

In recent years, institutional investors have placed a much sharper focus on the total risk of their portfolio. This has led to the widespread use of **risk budgeting**. The process starts with a broad portfolio allocation into asset classes that reflects the best trade-off between risk and return. Once a total risk budget is decided upon, this can be allocated to various asset classes and managers. Thus, risk budgeting reflects a top-down view of the total portfolio risk.

At the end of the investment process, it is important to assess whether realized returns were in line with the risks assumed. The purpose of **performance evaluation** methods is to decompose the investment performance into various components. The goal is to identify whether the active manager really adds value. Part of the returns generally represents general market factors, also called "beta bets"; the remainder represents true value added, or "alpha bets."

The purpose of this chapter is to present risk and performance evaluation tools used in the investment management industry. Section 29.1 gives a brief introduction to institutional investors. Performance evaluation techniques are developed in Section 29.2. Finally, Section 29.3 discusses risk budgeting. Hedge funds, because of their importance, will be covered in the next chapter.

29.1 INSTITUTIONAL INVESTORS

Institutional investors are entities that have large amounts of funds to invest for an organization, or on behalf of others. This is in contrast with *private* investors.[1] As shown in Table 29.1, institutional investors can be classified into investment companies, pension funds, insurance funds, and others. The latter category includes endowment funds, bank-managed funds, and private partnerships, also known as

FRM Exam Part 2 topic.

[1] The Securities and Exchange Commission (SEC) has formal definitions of, for example, qualified institutional buyers under Rule 144a.

TABLE 29.1 Classification of Institutional Investors

Investment companies	Open-end funds Closed-end funds
Pension funds	Defined benefit Defined contribution
Insurance funds	Life Nonlife
Others	Foundations and endowment funds Nonpension funds managed by banks Private partnerships

hedge funds. **Hedge funds** are private partnership funds that can take long and short positions in various markets and are accessible only to large investors.

Even though institutional investors and bank proprietary desks are generally exposed to similar risk factors, their philosophies are quite different. Bank trading desks employ high leverage and are aggressive investors. They typically have short horizons and engage in active trading in generally liquid markets. Financial institutions, such as commercial banks, investment banks, and broker-dealers, are sometimes called the **sell side** because they are primarily geared toward selling financial services.

Institutional investors are part of the **buy side** because they are buying financial services from the sell side, in other words Wall Street for the United States. In contrast to the sell side, institutional investors have little or no leverage and are more conservative. Most have longer time horizons and can invest in less liquid markets. Many hedge funds, however, have greater leverage and trade actively.

29.2 PERFORMANCE EVALUATION

Performance evaluation is the process by which investment management decisions are measured and assessed. It can be broken down into three steps.

1. **Performance measurement** starts with the computation of total return, which is then compared to the **benchmark** in terms of total and relative risk.
2. **Performance attribution** decomposes the performance of a portfolio and its benchmark to sources of differential returns. This typically includes asset allocation, currency selection, industry selection, and security selection. Risk is also attributed to these factors.
3. **Performance appraisal** is the evaluation of risk-adjusted performance and investment skill. This involves making a judgment as to whether the outperformance is due to luck or to skill.

Performance measurement should properly adjust for the risks taken. This can be done with a number of metrics, typically based on the standard deviation and regression coefficients. At an even more basic level, however, the first question is how to define the risks that matter to the investor or the manager. In particular, should risk be measured in absolute terms or relative to some benchmark?

29.2.1 Return Measurement

The first step of performance evaluation is the proper measurement of periodic returns. This is not always obvious due to the fact that a portfolio's value is affected by cash inflows and outflows, which are outside the control of the investment manager. By now, the industry standard is the **time-weighted rate of return** (TWRR). This method involves valuing the portfolio on a regular basis, and certainly before cash flows. Returns are then linked by compounding over the period. Assume, for instance, that R_t is the daily return. The monthly return is then over T days

$$(1 + R) = [(1 + R_1)(1 + R_2)\ldots(1 + R_T)] \tag{29.1}$$

The TWRR method provides a measure of performance that is not sensitive to the timing or amount of cash flows.[2]

29.2.2 Risk Measurement

Next, risk can be measured from returns or from positions. As in Chapter 1, we can take two views of risk. Let us use the standard deviation as the risk measure, as an example.

- **Absolute risk** is measured in terms of shortfall relative to the initial value of the investment, or perhaps an investment in cash. Let us use the standard deviation as the risk measure and define P as the initial portfolio value and R_P as the rate of return. Absolute risk in dollar terms is

$$\sigma(\Delta P) = \sigma(\Delta P/P) \times P = \sigma(R_P) \times P \tag{29.2}$$

- **Relative risk** is measured relative to a benchmark index and represents active management risk. Defining B as the benchmark, the deviation is $e = R_P - R_B$, which is also known as the **tracking error**. In dollar terms, this is $e \times P$. The risk is

$$\sigma(e)P = [\sigma(R_P - R_B)] \times P = [\sigma(\Delta P/P - \Delta B/B)] \times P = \omega \times P \tag{29.3}$$

  where ω is called **tracking error volatility** (TEV), or sometimes **active risk**. Defining σ_P and σ_B as the volatility of the portfolio and the benchmark and ρ as their correlation, the variance of the difference is

$$\omega^2 = \sigma_P^2 - 2\rho\sigma_P\sigma_B + \sigma_B^2 \tag{29.4}$$

Consider the example of a fund with volatility of $\sigma_P = 22\%$. This fund is compared to a benchmark with volatility of $\sigma_B = 20\%$, to which the correlation is $\rho = 0.9864$. What is the tracking error volatility? Using Equation

[2] This method differs from the **money-weighted rate of return** (MWRR), which is the internal rate of return on a portfolio taking into account all cash flows. This is similar to a yield to maturity on a bond and is easy to compute. The MWRR method, however, does not revalue the portfolio at intermediate steps and hence provides performance numbers that depend on the timing and size of cash flows. As such, it is considered inferior to the TWRR.

(29.4), we have $\omega^2 = 22\%^2 - 2 \times 0.9864 \times 22\% \times 20\% + 20\%^2 = 0.0016$, giving $\omega = 4.0\%$. Thus the tracking error is much smaller than the absolute risk. This is due to the fact that the portfolio is highly correlated with the benchmark. If the correlation were zero, ω would be 30%.

One of the advantages of focusing on relative risk is that tests of outperformance are more powerful because the volatility measure is lower. Assume, for example, that the portfolio returns 4% in excess of the benchmark, and 9% in excess of cash, all in percent per annum. After $T = 4$ years, we can compute a t-statistic for the hypothesis that the active manager adds no value, or that any outperformance is due to luck. The statistic is

$$\frac{(\bar{R}_P - \bar{R}_B)}{(\omega/\sqrt{T})} = \frac{(4\%)}{(4\%/\sqrt{4})} = \frac{4\%}{2\%} = 2.0$$

Because this is greater than 1.96, we can reject the null hypothesis. However the same computation using absolute returns gives $t = 0.82$, in which case we cannot reject the null hypothesis. Thus, we can conclude that the manager has skill when using relative returns but not when using absolute returns.

Also note that the t-statistic is a simple transformation of the information ratio[3]

$$t = \frac{(\bar{R}_P - \bar{R}_B)}{(\omega/\sqrt{T})} = \frac{(\bar{R}_P - \bar{R}_B)}{\omega}\sqrt{T} = \text{IR}\sqrt{T} \tag{29.5}$$

Using absolute or relative risk depends on how the trading or investment operation is judged. For bank trading portfolios or hedge funds, market risk is measured in absolute terms. These are sometimes called **total return funds**. In contrast portfolio managers that are given the task of beating a benchmark or peer group measure risk in relative terms.

EXAMPLE 29.1: FRM EXAM 2008—QUESTION 5-9

Over the past year, the HIR Fund had a return of 7.8%, while its benchmark, the S&P 500 index, had a return of 7.2%. Over this period, the fund's volatility was 11.3%, while the S&P index's volatility was 10.7% and the fund's TEV was 1.25%. Assume a risk-free rate of 3%. What is the information ratio for the HIR Fund and for how many years must this performance persist to be statistically significant at a 95% confidence level?

a. 0.480 and approximately 16.7 years
b. 0.425 and approximately 21.3 years
c. 3.840 and approximately 0.2 years
d. 1.200 and approximately 1.9 years

[3] This was defined in Chapter 1 and will be reviewed in the next section.

29.2.3 Surplus Risk

As is sometimes said, "risk is in the eye of the beholder." For investors with fixed future liabilities, the risk is not being able to perform on these liabilities. For pension funds with **defined benefits**, these liabilities consist of promised payments to current and future pensioners, and are called **defined benefit obligations**. In this case, the investment risk falls on the entity promising the benefits. In contrast, employees covered by a **defined contribution** plan are subject to investment risk.

For life insurance companies, these liabilities represent the likely pattern of future claim payments. These liabilities can be represented by their net present value. In general, the present value of long-term fixed payments behaves very much like a *short position in a fixed-rate bond*. If the payments are indexed to inflation, the analogous instrument is an inflation-protected bond.

The difference between the current values of assets and liabilities is called the **surplus**, S, defined as the difference between the value of assets A and liabilities L. The change is then $\Delta S = \Delta A - \Delta L$. Normalizing by the initial value of assets, we have

$$R_S = \frac{\Delta S}{A} = \frac{\Delta A}{A} - \frac{\Delta L}{L}\frac{L}{A} = R_{\text{assets}} - R_{\text{liabilities}}\frac{L}{A} \tag{29.6}$$

The duration of liabilities is long, typically 12 years. Using the duration approximation, the return on liabilities can be measured from changes in yields y, as $R_{\text{liabilities}} = -D^{*}\Delta y$. The worst combination of movements in market values is when assets fall due to a fall in equities, in a year when yields decrease. **Immunization** occurs when the asset portfolio, or part of it, provides a perfect hedge against changes in the value of the liabilities. Thus, investments in long-term bonds help to hedge movements in liabilities.

In this case, risk should be measured as the potential shortfall in surplus over the horizon. This is sometimes called **surplus at risk**. This value at risk (VAR)-type measure is an application of relative risk, where the benchmark is the present value of liabilities.

EXAMPLE 29.2: PENSION FUND LIABILITIES

The AT&T pension plan reports a projected benefit obligation of \$17.4 billion. If the discount rate decreases by 0.5%, this liability will increase by \$0.8 billion. Based on this information, the liabilities behave like a

a. Short position in the stock market
b. Short position in cash
c. Short position in a bond with maturity of about nine years
d. Short position in a bond with duration of about nine years

EXAMPLE 29.3: PENSION FUND RISK

The AT&T pension fund reports total assets worth $19.6 billion and liabilities of $17.4 billion. Assume the surplus has a normal distribution and volatility of 10% per annum. The 95% surplus at risk over the next year is

a. $360 million
b. $513 million
c. $2,860 million
d. $3,220 million

EXAMPLE 29.4: FRM EXAM 2006—QUESTION 25

The DataSoft Corporation has an employee pension scheme with fixed liabilities and a long time horizon reflecting its young workforce. The fund's assets are $9 billion and the present value of its liabilities is $8.8 billion. Which of the following statements are *incorrect*?

I. The present value of long-term fixed payments behaves very much like a long position in a fixed-rate bond.
II. Surplus at risk is a measure of relative risk.
III. The DataSoft Corporation will be able to immunize its liabilities by investing $8 billion in long-term fixed-rate bonds.

a. I and II
b. II and III
c. I and III
d. I, II and III

29.2.4 Returns-Based and Position-Based Risk Measures

Traditionally, risk has been measured from returns-based information, i.e. (from the time series of historical returns on the portfolio $R_{P,t}$). On the one hand, a returns-based risk system is easy and cheap to implement. On the other hand, returns-based measures suffer from severe drawbacks. They are ineffective for new instruments, markets, and managers because there is no history. They do not capture **style drift**, which is the divergence of a portfolio manager from its stated investment style. They may not reveal hidden risks such as out-of-the-money short positions in options. Such positions capture a stable premium but may not reveal the occurrence of a big loss.

Most of these drawbacks are addressed by position-based risk measures. They can be applied to new instruments, markets, and managers. These use the most current position information, which should reveal style drift or hidden risks.

Position-based risk systems, however, can be challenging to implement and have drawbacks that risk managers must understand. First, they require more resources and are expensive to implement. A large bank could have several million positions, in which case aggregation at the top level is a major technology challenge. Second, position-based risk measures assume that the portfolio is frozen over the time horizon considered, thus ignoring any active trading that takes place in practice. Finally, position-based systems are susceptible to errors and approximations in data and models. They require modeling all positions from the ground up, repricing instruments as a function of movements in the risk factors. The modeling of some instruments can be complex, leading to model risk.

Even so, position-based risk measures are vastly more informative than returns-based risk measures. This explains why modern risk management systems are built from position-level information.

29.2.5 Risk-Adjusted Performance Measurement

This dichotomy, absolute versus relative returns, carries through performance measurement, which evaluates the risk-adjusted performance of the fund. The **Sharpe ratio** (SR) measures the ratio of the average rate of return, $\mu(R_P)$, in excess of the risk-free rate R_F, to the absolute risk

$$\mathrm{SR} = \frac{[\mu(R_P) - R_F]}{\sigma(R_P)} \tag{29.7}$$

The Sharpe ratio focuses on total risk measured in absolute terms. Because total risk includes both systematic and idiosyncratic risk, this measure is appropriate for portfolios that are not very diversified (i.e., that have large idiosyncratic risk).

A related measure is the **Sortino ratio** (SOR). This replaces the standard deviation in the denominator by the semistandard deviation, $\sigma_L(R_P)$, which considers only data points that represent a loss. The ratio is

$$\mathrm{SOR} = \frac{[\mu(R_P) - R_F]}{\sigma_L(R_P)} \tag{29.8}$$

where $\sigma_L(R_P) = \sqrt{\frac{1}{(N_L)} \sum_{i=1}^{N} [\mathrm{Min}(R_{P,i}, 0)]^2}$, and N_L is the number of observed losses. The Sortino ratio is more relevant than the Sharpe ratio when the return distribution is skewed to the left. It is much less widely used, however.

In contrast, the **information ratio** (IR) measures the ratio of the average rate of return in excess of the benchmark to the TEV

$$\mathrm{IR} = \frac{[\mu(R_P) - \mu(R_B)]}{\omega} \tag{29.9}$$

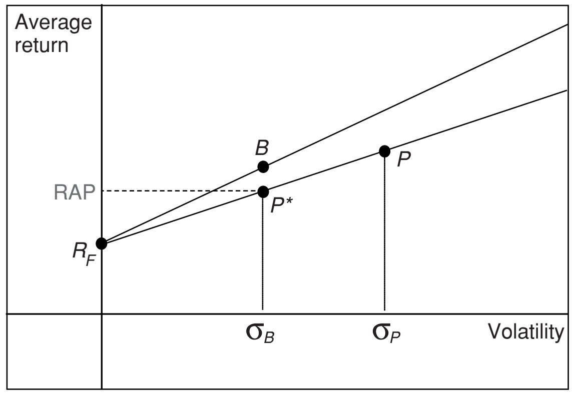

FIGURE 29.1 Risk-Adjusted Performance

Dealing with ratios, however, is rather abstract. It is more intuitive to express performance in terms of a rate of return, adjusted for risk. Suppose we use a reference benchmark, R_B, for which we measure first its average return and risk. We can leverage up or down the portfolio P so as to bring its volatility in line with B. The **risk-adjusted performance** (RAP) is then[4]

$$\text{RAP}_P = R_F + \frac{\sigma_B}{\sigma_P}[\mu(R_P) - R_F] \tag{29.10}$$

This is illustrated in Figure 29.1. The average return on portfolio P is greater than that of B. Its volatility, however, is much higher. The straight line going from R_F to R_P represents portfolios that mix the risk-free asset with P. For example, an investment of 50% in each will give an average return that is the mean of R_F and $\mu(R_P)$, and a volatility that is half that of P. The slope of this line represents the Sharpe ratio, given by Equation (29.7).

Portfolio P^* has the same level of risk as B; its performance is given by Equation (29.10). We can then compare directly RAP_P and $\mu(R_B)$. In this case, portfolio P underperforms B on a risk-adjusted basis. We obtain the same ranking between P and B, however, using the Sharpe ratio.

EXAMPLE 29.5: FRM EXAM 2009—QUESTION 8-2

Your firm hired Vikram Mehra as an active manager for its pension fund. His benchmark is the Russell 2000 growth index. Which of the following statistics are most suitable to evaluating Vikram's performance and risk?

a. VAR and Sharpe ratio
b. Tracking error and information ratio
c. Tracking error and Sharpe ratio
d. VAR and information ratio

[4] This performance measure is sometimes called *M*-square.

29.2.6 Performance Attribution: Returns-Based

So far, we have implemented a simple adjustment for risk that takes into account a volatility measure. To evaluate the performance of investment managers, however, it is crucial to decompose the total return into a component due to market risk premiums and to other factors. Exposure to the stock market is widely believed to reward investors with a long-term premium, called the **equity premium**. Assume that this premium is EP = 4 % annually. This is the expected return in excess of the risk-free rate. For simplicity, it is usually assumed that the same rate applies to lending and borrowing.

Now take the example of an investment fund of \$1 million. A long position of \$1.5 million, or 150% in passive equities financed by 50% cash borrowing, should have an *excess return* composed of the total return on the 150% equity position, minus the cost of borrowing 50%, minus the risk-free rate. This gives

$$[150\% \times (\mathrm{EP} + R_F) - 50\% R_F] - R_F = 1.5 \times \mathrm{EP} = 6\%$$

This could be also achieved by taking a notional position of \$1.5 million in stock index futures and parking the investment in cash, including the margin. So, an investment manager who returns 6% in excess of the risk-free rate in this way is not really delivering any value added because this extra amount is simply due to exposure to the market. Therefore, it is crucial to account for factors that are known to generate risk premiums.

Define $R_{M,t}$ as the rate of return in period t on the stock market, say the S&P 500 for U.S. equities; define $R_{F,t}$ as the risk-free rate; and $R_{P,t}$ is the return on the portfolio. The general specification for this adjustment consists of estimating the regression

$$R_{P,t} - R_{F,t} = \alpha_P + \beta_P[R_{M,t} - R_{F,t}] + \epsilon_{P,t}, \quad t = 1, \ldots, T \tag{29.11}$$

where β_P is the exposure of portfolio P to the market factor, or **systematic risk**, and α_P is the abnormal performance after taking into account the exposure to the market. The intercept is also known as **Jensen's alpha**. This term is widely used in the investment management industry to describe the performance adjusted for market factors.

Denoting $\bar{R} = (1/T)\sum_{t=1}^{T}(R_t - R_{F,t})$ as the average over the sample period, the estimated alpha is

$$\widehat{\alpha} = \bar{R} - \widehat{\beta}\bar{R}_M \tag{29.12}$$

If there is no exposure to the market ($\beta = 0$), Equation (29.12) shows that alpha is the sample average of the investment returns. More generally, Equation (29.12) properly accounts for the exposure to the systematic risk factor. In the case of our investment fund, we have $\bar{R} = 6\%$, and $\beta = 1.5$. So, the alpha is

$$\widehat{\alpha} = 6\% - 1.5 \times 4\% = 0$$

which correctly indicates that there is no value added.

KEY CONCEPT

Performance evaluation must take into account the component of returns that can be attributed to exposures on general market factors (or risk premiums). An investment manager adds value only if the residual return, called alpha, is positive.

This specification can be generalized to multiple factors. Assume we believe that in addition to the market premium, a premium is earned for *value* (or for low price-to-book companies) and *size* (or for small firms). We need to take this information into account in evaluating the manager; otherwise he or she may load up on factors that are priced but not recorded in the performance attribution system. With K factors, Equation (29.11) can be generalized to

$$R_i = \alpha_i + \beta_{i1} y_1 + \cdots + \beta_{iK} y_K + \epsilon_i \tag{29.13}$$

This decomposition is also useful to detect **timing ability**, which consists of adding value by changing exposures on risk factors. A manager could, for example, move into stocks with higher betas in anticipation of the market going up. Timing ability can be detected by adding another term to Equation (29.11):

$$R_{P,t} - R_{F,t} = \alpha_P + \beta_P[R_{M,t} - R_{F,t}] + \delta_P[R_{M,t} - R_{F,t}]D_t + \epsilon_{P,t} \tag{29.14}$$

where D_t is a dummy variable that is equal to $D_t = +1$ for up markets and $D_t = 0$ for down markets. A positive coefficient δ_P indicates that the manager has added value from market timing, implying that beta is positively correlated with the market.

Another popular tool is **style analysis**.[5] The objective is to explain the fund returns by predefined asset classes, $F_1, F_2, \ldots, F_K$. These could be, for instance, (1) large U.S. stocks, (2) small U.S. stocks, and (3) international stocks. We then run a regression on these three factors:

$$R_i = \beta_{i1} F_1 + \beta_{i2} F_2 + \beta_{i3} F_3 + \epsilon_i \tag{29.15}$$

This should be set up in a way that constrains the estimated coefficients to be positive $\beta \geq 0$ and to add up to one, $\beta_{i1} + \beta_{i2} + \beta_{i3} = 1$. Here, the exposures can be interpreted as weights on the asset classes. This identifies the combination of long positions in passive indices that would have most closely replicated the actual performance of the fund. For example, suppose the regression yields weights of 25%, 25%, and 50%, with an R-squared of 92%. In other words, this passive allocation to three indices was associated with 92% of the return variance, with the remainder due to other factors. This is useful to understand the drivers of the

[5] W. Sharpe, "Asset Allocation: Management Style and Performance Measurement," *Journal of Portfolio Management* 18 (1992): 7–19.

performance of the portfolio. In addition, these weights could be used to construct a benchmark to which future portfolio performance can be compared.

Finally, consider the effect of leverage on the portfolio. Long leverage is the ratio of assets to equity. For instance, a fund may have \$100 million in investors' money, borrow \$100 million, and invest \$200 million in stocks. The short position (i.e., the loan) has no market risk. In this case, the leverage is 2 to 1. A profit of 10% on the portfolio assets, or \$20 million, now represents a 20% return on the equity. Thus, leverage multiplies asset returns to give equity returns. In Equation (29.11), leverage multiplies α_P, β_P, and ϵ_P. From the regression decomposition, $V(R_P) = \beta_P^2 V(R_M) + V(\epsilon_P^2)$. Thus leverage increases proportionately the total volatility σ_P as well as the volatility of the residual $\sigma_{\epsilon,P}$.

EXAMPLE 29.6: FRM EXAM 2008—QUESTION 5-13

Portfolio Q has a beta of 0.7, an expected return of 12.8%, and an equity risk premium of 5.25%. The risk-free rate is 4.85%. Calculate Jensen's alpha measure for portfolio Q.

a. 7.67%
b. 2.70%
c. 5.73%
d. 4.27%

EXAMPLE 29.7: PERFORMANCE EVALUATION

Assume that a hedge fund provides a large positive alpha. The fund can take leveraged long and short positions in stocks. The market went up over the period. Based on this information,

a. If the fund has net positive beta, all of the alpha must come from the market.
b. If the fund has net negative beta, part of the alpha comes from the market.
c. If the fund has net positive beta, part of the alpha comes from the market.
d. If the fund has net negative beta, all of the alpha must come from the market.

29.2.7 Performance Attribution: Position-Based

So far, the analysis has used actual portfolio returns. The performance evaluation process, however, could be deepened by using positions.

Suppose, for example, that we want to understand the performance drivers for a portfolio of global stocks. The fund invests in three asset classes, (1) large U.S. stocks, (2) small U.S. stocks, and (3) international stocks. At the beginning of the month, the weights w_i are 30%, 30%, and 40%, respectively. We then decompose the return into a component due to a set of passive factor returns and to a specific return

$$R_P = w_{P1}F_1 + w_{P2}F_2 + w_{P3}F_3 + \epsilon_P \tag{29.16}$$

The same analysis can be performed in terms of the active return, which is the portfolio return minus the benchmark return:

$$R_P - R_B = [(w_{P1} - w_{B1})F_1 + (w_{P2} - w_{B2})F_2 + (w_{P3} - w_{B3})F_3] + \epsilon_P \tag{29.17}$$

Here, the term between brackets represents the active return due to the asset allocation decision. For example, a finding that the first term is regularly positive but that the specific term is not would suggest that the manager has skill in asset allocation but not in other decisions for this fund. This type of analysis can be extended to evaluate the value added from country selection, currency selection, and security selection.

29.2.8 Performance Evaluation and Survivorship

Another key issue when evaluating the performance of a group of investment managers is **survivorship**. This occurs when funds are dropped from the investment universe for reasons related to poor performance and survivors only are considered. Commercial databases often give information on funds that are alive only, because clients are no longer interested in dead funds.

The problem is that the average performance of the group of funds under examination becomes subject to **survivorship bias**. In other words, the apparent performance of the existing funds is too high, or biased upward relative to the true performance of the underlying population, due to the omission of some poorly performing funds.

The extent of this bias depends on the attrition rate of the funds and can be very severe. Mutual fund studies, for example, report an **attrition rate** of 3.6% per year. This represents the fraction of funds existing at the beginning of the year that becomes dead during the year. In this sample, the survivorship bias is estimated at approximately 0.70% per annum.[6] This represents the difference between the performance of the survived sample and that of the true population. This is a significant number because it is on the order of management fees, which are around 1% of assets per annum. Samples with higher attrition rates have larger biases. For example, **Commodity Trading Advisors** (CTAs), a category of hedge funds, are reported to have an attrition rate of 16% per year, leading to survivorship biases on the order of 5.2% per annum, which is very high.[7]

Other sources of bias can be introduced, due to the inclusion criteria and the voluntary reporting of returns. A fund with excellent performance is more likely

[6] Mark Carhart, Jennifer Carpenter, Anthony Lynch, and David Musto, "Mutual Fund Survivorship," *Review of Financial Studies* 15 (2002): 1355–1381.
[7] CTAs are investment managers that trade futures and options. In the United States, they are regulated by the Commodity Futures Trading Commission (CFTC).

to be chosen for inclusion by the database vendor, or the investment manager of such a fund may be more inclined to submit the fund returns to the database. Consequently, there is a bias toward adding funds with better returns. Or a fund may decide to stop reporting returns if its performance drops. This is called **selection bias**. This bias differs from the previous one because it also exists when dead funds are included in the sample.

Finally, another subtle bias arises when firms incubate different types of funds before making them available to outsiders. Say 10 different funds are started by the same company over a two-year period. Some will do well and others will not, partly due to chance. The best-performing fund is then opened to the public, with its performance instantly backfilled for the previous two years. The other funds are ignored or disbanded. As a result, the performance of the public fund is not representative of the entire sample. This is called **instant-history bias**. The difference between this bias and selection bias is that the fund was not open to investors during the reported period.

KEY CONCEPT

Performance evaluation can be overly optimistic if based on a sample of funds affected by survivorship, selection, or instant-history bias. The extent of survivorship bias increases with the attrition rate.

EXAMPLE 29.8: FRM EXAM 2005—QUESTION 103

A database of hedge fund returns is constructed as follows. The first year of the database is 1994. All funds existing as of the end of 1994 that a willing to report their verified returns for that year are included in that year. The database is extended by asking the funds for verified returns before 1994. Subsequently, funds are added as they are willing to report verified returns to the database. If a fund stops reporting returns, its returns are deleted from the database, but the database has an agreement with funds that they will keep reporting verified returns even if they stop being open to new investors. Which of the following four statements are correct?

I. The database suffers from backfilling bias.
II. The database suffers from survivorship bias.
III. The database suffers from an errors-in-variables bias.
IV. The equally weighted annual return average of fund returns will underestimate the performance one would expect from a hedge fund.

a. All the above statements are correct.
b. Statements I and II are correct.
c. Statements I, II, and III are correct.
d. Statements II and IV are correct.

29.3 RISK BUDGETING

The revolution in risk management reflects the recognition that risk should be measured at the highest level—that is, firmwide or portfolio-wide. This ability to measure total risk has led to a top-down allocation of risk, called **risk budgeting**. Risk budgeting is the process of parceling out the total risk of the fund, or risk budget, to various assets classes and managers.

This concept is being implemented by institutional investors as a follow-up to their **asset allocation process**. Asset allocation consists of finding the optimal allocation into major asset classes (i.e., the allocation that provides the best risk/return trade-off for the investor). This choice defines the total risk profile of the portfolio.

29.3.1 Illustration

Consider, for instance, an investor having to decide how much to invest in U.S. stocks, in U.S. bonds, and in non-U.S. bonds. Risk is measured in absolute terms, assuming returns have a joint normal distribution. More generally, this could be extended to other distributions or to a historical simulation method. The allocation will depend on the expected return and volatility of each asset class, as well as their correlations. Table 29.2 illustrates these data, which are based on historical dollar returns measured over the period 1978 to 2003.

Say the investor decides that the portfolio with the best risk/return trade-off has an expected return of 12.0% with total risk of 10.3%. Table 29.2 shows a portfolio allocation of 60.0%, 7.7%, and 32.3% to U.S. stocks, U.S. bonds, and non-U.S. bonds, respectively.

The volatility can be measured in terms of a 95% annual VAR. This defines a total risk budget of $\text{VAR} = \alpha\sigma W = 1.645 \times 10.3\% \times \$100 = \$16.9$ million. This VAR budget can then be parceled out to various asset classes and active managers within asset classes.

Risk budgeting is the process by which these efficient portfolio allocations are transformed into VAR assignments. At the asset class level, the individual VARs are \$15.3, \$0.9, and \$5.9 million, respectively. For instance, the VAR budget for U.S. stocks is $60.0\% \times (1.645 \times 15.50\% \times \$100) = \$15.3$ million. Note that the sum of individual VARs is \$22.1 million, which is greater than the portfolio VAR of \$16.9 million due to diversification effects.

TABLE 29.2 Risk Budgeting

Asset		Expected Return	Volatility	Correlations 1	2	3	Percentage Allocation	VAR
U.S. stocks	1	13.80%	15.50%	1.00			60.0	\$15.3
U.S. bonds	2	8.40%	7.40%	0.20	1.00		7.7	\$0.9
Non-U.S. bonds	3	9.60%	11.10%	0.04	0.40	1.00	32.3	\$5.9
Portfolio		12.00%	10.30%				100.0	\$16.9

The process can be repeated at the next level. The fund has a risk budget of \$15.3 million devoted to U.S. equities, with an allocation of \$60 million. This allocation could be split equally between two active equity managers. Assume that the two managers are equally good, with a correlation of returns of 0.5. The optimal risk budget for each is then \$8.83 million. We can verify that the total risk budget is

$$\sqrt{8.83^2 + 8.83^2 + 2 \times 0.5 \times 8.83 \times 8.83} = \sqrt{233.91} = \$15.3$$

Note that, as in the previous step, the sum of the risk budgets, which is \$8.83 + \$8.83 = \$17.66 million, is greater than the total risk budget of \$15.3 million. This is because the latter takes into account diversification effects. If the two managers were perfectly correlated with each other, the risk budget would have to be \$15.3/2 = \$7.65 million for each. This higher risk budget is beneficial for the investor because it creates more opportunities to take advantage of the managers' positive alphas.

The risk budgeting process highlights the importance of correlations across managers. To control their risk better, institutional investors often choose equity managers that follow different market segments or strategies. For example, the first manager could invest in small growth stocks, the second in medium-size value stocks. Or the first manager could follow momentum-based strategies, the second value-based strategies. The first type tends to buy more of a stock after its price has gone up, and the second after the price has become more attractive (i.e., low). Different styles lead to low correlations across managers. For a given total risk budget, low correlations mean that each manager can be assigned a higher risk budget, leading to a greater value added for the fund.

These low correlations explain why investors must watch for **style drift**, which refers to a situation where an investment manager changes investment style. This is a problem for the investor because it can change the total portfolio risk. If all the managers, for instance, drift into the small growth stocks category, the total risk of the fund will increase. Style drift is controlled by the choice of benchmarks with different characteristics, such as small growth and medium value indices, and by controls on the tracking error volatility for each manager.

In conclusion, this risk budgeting approach is spreading rapidly to the field of investment management. This approach provides a consistent measure of risk across all subportfolios. It forces managers and investors to confront squarely the amount of risk they are willing to assume. It gives them tools to monitor their risk in real time.

29.3.2 Marginal Risk and Contribution to Risk

A well-designed risk system should also provide tools to understand how to manage risk. A risk report should display measures of **marginal risk**. This represents the change in risk due to a small increase in one of the allocations. Using the volatility of returns as the risk measure, this is

$$\text{MRISK} = \frac{\partial \sigma_P}{\partial w_i} = \frac{\text{cov}(R_i, R_p)}{\sigma_P} = \beta_{i,P}\sigma_P \quad (29.18)$$

Thus, beta represents the marginal contribution to the risk of the total portfolio P. A large value for β indicates that a small addition to this position will have a relatively large effect on the portfolio risk. Conversely, positions with large betas should be cut first because they will lead to the greatest reduction in risk. **Marginal VAR** is a similar measure, except that MRISK is multiplied by the α that corresponds to the confidence level.

This can be expanded to measure contributions to the portfolio risk. The **risk contribution**, or **risk allocation**, is obtained by multiplying the marginal risk for position i by its weight i

$$\text{CRISK} = w_i\beta_{i,P}\sigma_P \quad (29.19)$$

Because the beta of a portfolio with itself is one, the sum of $w_i\beta_{i,P}$ is guaranteed to be one. Hence, the sum of the risk contributions adds up exactly to the total portfolio risk, σ_P. When risk is expressed in terms of VAR, this measure is called **component VAR**.

Table 29.3 gives an example, expanding on the previous table. The marginal risk column shows that U.S. stocks are the asset class with the greatest marginal contribution to the risk of the portfolio. As an example, increasing the allocation from 60% to 61% increases the portfolio risk from 10.30% to 10.44%, which is an increase of 0.14%. This is precisely the marginal risk number of 0.14 multiplied by the 1% weight increase.

The last column shows the risk contribution, or allocation. Out of a total portfolio risk of 10.30%, 8.63% is attributed to U.S. stocks. This high number reflects the high volatility of this asset class, its high weight in the portfolio, as well as correlations. Reporting systems should therefore display not only the conventional weights, or market allocations, but also risk allocations.

TABLE 29.3 Risk Analysis

Asset	Volatility	Market Allocation	Marginal Risk	Risk Allocation
U.S. stocks	15.50%	60.0%	0.1438	8.63%
U.S. bonds	7.40%	7.7%	0.0278	0.21%
Non-U.S. bonds	11.10%	32.3%	0.0451	1.46%
Portfolio	10.30%	100.0%		10.30%

Such analysis provides useful insights into the structure of the portfolio. Given a scarce risk budget, high risk allocations can be justified only by expected returns that are high relative to other assets. In fact, an exact relationship holds for portfolios that are mean-variance efficient (i.e., maximize the Sharpe ratio). If this is the case with portfolio P, then the ratio of excess returns on all assets to their marginal risk, which is also proportional to the Treynor ratio, must be the same. However if P is not efficient, then we should be able to improve its performance by tilting toward assets that provide a greater ratio of expected return to their contribution to risk. Thus, this top-down analysis of portfolio risk can help investors improve the performance of their portfolios, given a set of risk measures and asset class forecasts.

EXAMPLE 29.9: RISK BUDGETING

The AT&T pension fund has 68%, or about $13 billion, invested in equities. Assume a normal distribution and volatility of 15% per annum. The fund measures absolute risk with a 95%, one-year VAR, which gives $3.2 billion. The pension plan wants to allocate this risk to two equity managers, each with the same VAR budget. Given that the correlation between managers is 0.5, the VAR budget for each should be

a. $3.2 billion
b. $2.4 billion
c. $1.9 billion
d. $1.6 billion

EXAMPLE 29.10: FRM EXAM 2005—QUESTION 140

Suppose a portfolio consists of four assets. The risk contribution of each asset is as follows: UK large cap, 3.9%; UK small cap, 4.2%; UK bonds, 0.9%; non-UK bonds, 1.1%. Which of the following would *not* be a possible explanation for the relatively high risk contribution values for UK equities?

a. High expected returns on UK equities
b. High weights on UK equities
c. High volatilities of UK equities
d. High correlation of UK equities with all other assets in the portfolio

EXAMPLE 29.11: FRM EXAM 2009—QUESTION 8-9

A risk manager assumes that the joint distribution of returns is multivariate normal and calculates the following risk measures for a two-asset portfolio:

Asset	Position	Individual VAR	Marginal VAR	VAR Contribution
1	USD 100	USD 23.3	0.176	USD 17.6
2	USD 100	USD 46.6	0.440	USD 44.0
Total	USD 200	USD 61.6		USD 62.6

If asset 2 is dropped from the portfolio, what is the reduction in portfolio VAR?

a. USD 15.0
b. USD 38.3
c. USD 44.0
d. USD 46.6

EXAMPLE 29.12: FRM EXAM 2009—QUESTION 8-10

Continue with the previous question. Let $\beta_{ip} = \rho_{ip}\sigma_i/\sigma_p$, where ρ_{ip} denotes the correlation between the return of asset i and the return of the portfolio, σ_i is the volatility of the return of asset i, and σ_p is the volatility of the return of the portfolio. What are β_1 and β_2?

a. $\beta_1 = 0.571$, $\beta_2 = 1.429$
b. $\beta_1 = 0.756$, $\beta_2 = 1.513$
c. $\beta_1 = 0.286$, $\beta_2 = 0.714$
d. Cannot determine from information provided

EXAMPLE 29.13: FRM EXAM 2009—QUESTION 8-12

The pension management analysts at Big Inc. use a two-step process to manage the assets and risk in the pension portfolio. First, they use a VAR-based risk budgeting process to determine the asset allocation across four broad asset classes. Then, within each asset class, they set a maximum tracking error allowance from a benchmark index and determine an active risk budget to distribute among individual managers. Assume the returns are all normally distributed. From the first step in the process, the following information is available.

	Expected Return (%)	Volatility (%)	Asset Allocation (%)	Individual VAR (USD)	Marginal VAR
Small cap	0.20%	2.66%	35.0%	6,491	0.055
Large cap	0.15%	2.33%	40.0%	6,497	0.044
Commodities	0.10%	1.91%	16.7%	2,216	0.020
Emerging markets	0.15%	2.70%	8.3%	1,570	0.047
			Total VAR:	13,322	

Which of the following statements is/are *correct*?

I. Using VAR as the risk budgeting measure, the emerging markets class has the smallest risk budget.
II. If an additional dollar were added to the portfolio, the marginal impact on portfolio VAR would be greatest if it were invested in small caps.
III. As the maximum tracking error allowance is lowered, the individual managers have more freedom to achieve greater excess returns.
IV. Setting well-defined risk limits and closely monitoring risk levels guarantee that risk limits will not be exceeded.

a. I and II only
b. I, II, III, and IV
c. II and III
d. I only

29.4 IMPORTANT FORMULAS

Absolute risk: $\sigma(\Delta P) = \sigma(\Delta P/P) \times P = \sigma(R_P) \times P$

Relative risk: $\sigma(e)P = [\sigma(R_P - R_B)] \times P = \omega \times P$

Tracking error volatility (TEV): $\omega = \sigma(\Delta P/P - \Delta B/B)$

Sharpe ratio (SR): $\text{SR} = [\mu(R_P) - R_F]/\sigma(R_P)$

Risk-adjusted performance (RAP): $\text{RAP}_P = R_F + \frac{\sigma_B}{\sigma_P}[\mu(R_P) - R_F]$

Information ratio (IR): $\text{IR} = [\mu(R_P) - \mu(R_B)]/\omega$

Alpha, from the intercept in: $R_{P,t} - R_{F,t} = \alpha + \beta_P[R_{M,t} - R_{F,t}] + \epsilon_{P,t}$

Treynor ratio (TR): $\text{TR} = [\mu(R_P) - R_F]/\beta_P$

Market timing skill, positive δ in: $R_{P,t} - R_{F,t} = \alpha_P + \beta_P[R_{M,t} - R_{F,t}] + \delta_P[R_{M,t} - R_{F,t}]^2 + \epsilon_{P,t}$

Marginal risk: the change in total portfolio risk due to a small change in position i: $\text{MRISK} = \frac{\partial \sigma_P}{\partial w_i} = \frac{\text{cov}(R_i, R_p)}{\sigma_P} = \beta_{i,P}\sigma_P$

Risk contribution: a component of total portfolio risk due to one position: $\text{CRISK} = w_i \beta_{i,P}\sigma_P$

29.5 ANSWERS TO CHAPTER EXAMPLES

Example 29.1: FRM Exam 2008—Question 5-9

a. The information ratio is $(7.8 - 7.2)/1.25 = 0.48$. Statistical significance is achieved when the t-statistic is above the usual value of 1.96. By Equation (29.5), the minimum number of years T for statistical significance is $(\text{IR}1.96)^2 = 16.7$. Note, however, that there is no need to perform the second computation because there is only one correct answer for the IR question.

Example 29.2: Pension Fund Liabilities

d. We can compute the modified duration of the liabilities as $D^* = -(\Delta P/P)/\Delta y = -(0.8/17.4)/0.0005 = 9.2$ years. So, the liabilities behave like a short position in a bond with a duration around nine years. Answers a. and b. are incorrect because the liabilities have fixed future payoffs, which do not resemble cash flow patterns on equities or cash. Answer c. is incorrect because the duration of a bond with a nine-year maturity is less than nine years. For example, the duration of a 6% coupon par bond with nine-year maturity is seven years only.

Example 29.3: Pension Fund Risk

a. The fund's surplus is the excess of assets over liabilities, which \$19.6 − \$17.4 = \$2.2 billion. The surplus at risk at the 95% level over one year is, assuming a normal distribution, 1.645 × 10% × \$2,200 = \$360 million. Answer b. is incorrect because it uses a 99% confidence level. Answers c. and d. are incorrect because they apply the risk to the liabilities and assets instead of the surplus.

Example 29.4: FRM Exam 2006—Question 25

c. Statement I. is incorrect because this liability is similar to a *short* (not long) position in a bond. Statement II. is correct because surplus at risk is a relative risk measure, assets minus liabilities. Statement III. is incorrect because the company needs to invest \$8.8 billion, not \$8 billion.

Example 29.5: FRM Exam 2009—Question 8-2

b. Because the active manager is compared to a benchmark, your firm should use relative performance measures (i.e., tracking error volatility and the information ratio).

Example 29.6: FRM Exam 2008—Question 5-13

d. The alpha is $(12.8\% - 4.85\%) - 0.7(5.25\%) = 4.27\%$.

Example 29.7: Performance Evaluation

c. Because the market went up, a portfolio with positive beta will have part of its positive performance due to the market effect. A portfolio with negative beta will have in part a negative performance due to the market. Answer a. is incorrect because the fund manager could still have generated some of its alpha through judicious stock picking. Answers b. and d. are incorrect because a negative beta combined with a market going up should lead to a negative, not positive, return.

Example 29.8: FRM Exam 2005—Question 103

b. The database includes histories before 1994 and therefore suffers from backfill bias. Next, funds that stop reporting are deleted from the database, so it has survivorship bias. Errors-in-variables bias arises in other contexts, such as regression. Finally, the average of fund returns will be too high (not too low) because of these two biases. Hence, I. and II. are correct.

Example 29.9: Risk Budgeting

c. Call x the risk budget allocation to each manager. This should be such that $x^2 + x^2 + 2\rho\ xx = \$3.2^2$. Solving for $x\sqrt{1+1+2\rho} = x\sqrt{3} = \$3.2$, we find $x =$ \$1.85 billion. Answer a. is incorrect because it refers to the total VAR. Answer b. is incorrect because it assumes a correlation of zero. Answer d. is incorrect because it simply divides the \$3.2 billion VAR by 2, which ignores diversification effects.

Example 29.10: FRM Exam 2005—Question 140

a. The risk contribution is proportional to the weight times the beta. The latter involves the correlation between the asset and the portfolio, as well as the volatility of the asset. Higher weight, correlation, and volatility would create higher risk contribution. In contrast, high expected returns would explain a high weight, but not a high risk contribution.

Example 29.11: FRM Exam 2009—Question 8-9

b. This is 61.6 minus the portfolio VAR of asset 1 alone, which is USD 23.3, for a difference of 38.3.

Example 29.12: FRM Exam 2009—Question 8-10

a. From Equation (29.18), beta is proportional to the marginal risk. Alternatively, the VAR contribution is proportional to beta times the weight times the portfolio VAR. Hence, $\beta_1 = 17.6/(0.5 \times 61.6) = 0.57$ and $\beta_2 = 44.0/(0.5 \times 61.6) = 1.43$.

Example 29.13: FRM Exam 2009—Question 8-12

a. Risk budget is represented by the individual VAR, which is the smallest for emerging markets, so statement I. is correct. The marginal VAR is highest for small caps, so adding one dollar to that asset class would have the largest impact on the portfolio. Statement III. is incorrect, as lowering TEV would give less, not more freedom to manages. Finally, setting risk limits does not ensure they will not be exceeded. Bad luck and exceptions can happen, even if the risk model is correct.

CHAPTER 30

Hedge Fund Risk Management

The first hedge fund was started by A. W. Jones in 1949. Unlike the typical equity mutual fund, the fund took long *and* short positions in equities. Over the subsequent decades, the hedge fund industry has undergone exponential growth. As of December 2009, hedge funds accounted for more than $1,600 billion in equity capital, called **assets under management** (AUM).

Hedge funds are organized as private partnerships and as a result differ in a number of essential ways from mutual funds. They provide more flexible investment opportunities and are less regulated. They have very few limitations on their investment strategies. In particular, they can take long and short positions in various markets and can use leverage. Due to this leverage, the assets they control are greater than their AUMs. Hedge funds have become an important force in financial markets, accounting for the bulk of trading in some markets.

Unlike mutual funds, which are open to any investor, hedge funds are accessible only to accredited investors, perhaps due to their perceived risks. To control their risk, most hedge funds have adopted risk controls using position-based, value at risk (VAR)-type techniques. Because some types of hedge fund strategies are very similar to those of proprietary trading desks of commercial banks, it was only natural for hedge funds to adopt similar risk management tools.

The purpose of this chapter is to provide an overview of risk management for the hedge fund industry. Section 30.1 gives an introduction to the hedge fund industry. Section 30.2 presents the mechanics of shorting and various measures of leverage. Section 30.3 then analyzes commonly used strategies for hedge funds and shows how to identify and measure their risk. The risk factors that are largely specific to hedge funds are presented in Section 30.4. Last, Section 30.5 shows how to deal with hedge fund risk.

30.1 THE HEDGE FUND INDUSTRY

The growth of the hedge fund industry is described in Figure 30.1. By now, there are close to 9,000 hedge fund managers[1] controlling close to $1,600 billion in

FRM Exam Part 2 topic.

[1] This number includes funds of funds, which add up to approximately 2,200.

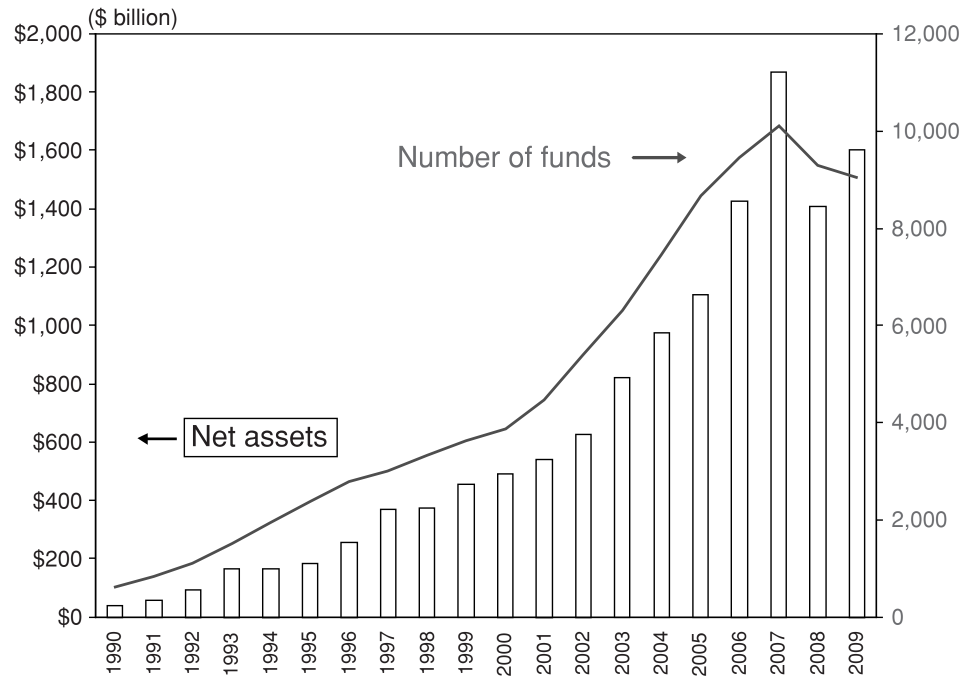

FIGURE 30.1 Growth of Hedge Fund Industry
Source: Hedge Fund Research. Data as of December of each year.

equity capital, also called net assets, up from $40 billion in 1990. This represents an annualized rate of growth of more than 20%. In comparison, U.S. mutual funds currently manage $11,121 billion, up from $1,065 billion in 1990. This represents an annualized rate of growth of 13%. Thus, hedge funds have grown much faster than mutual funds over the same period.

The growth of this industry is due to a number of factors. On the investor side, the performance of hedge funds has been attractive, especially compared to the poor record of stock markets during the 2000–2002 period. Hedge funds also claim to have low betas, which makes them useful as diversifiers.

On the manager side, hedge funds provide greater remuneration than traditional investment funds. Typical investment management fees for mutual funds range from a fixed 0.5% to 2% of AUM. In contrast, hedge funds commonly charge a fixed management fee of 1% to 2% of assets plus an incentive fee of 20% of profits.

Hedge funds also typically have fewer restrictions on their investment strategies and are less regulated, giving more leeway to portfolio managers. More flexible investment opportunities include the ability to short securities, to leverage the portfolio, to invest in derivatives, and generally to invest across a broader pool of assets. The lighter regulatory environment creates an ability to set performance fees, lockup periods, or other forms of managerial discretion.

The unprecedented turbulence of 2008, however, has hit the hedge fund industry hard. Many funds posted poor performance and suffered widespread investor redemptions, leading to many hedge fund closures. Even so, hedge funds in general suffered only half the loss of equities.

Example: Computing Fees

The most common fee structure consists of (1) a management fee that is a fixed proportion of assets under management, typically 2%, and (2) an incentive fee, which is typically 20% of the profits over a year when positive. Sometimes, the incentive fee is paid after the performance exceeds a **hurdle rate**, such as LIBOR.

As an example, assume that the net asset value (NAV) goes from 100 to 120 over the year before fees (gross) and that LIBOR is at 5%. Fees are at the usual 2 and 20. The management fee is deducted first from the ending asset value, which drops from 120 to 118. The total fee is then $100 \times 2\% + [(118 - 100) - 5\% \times 100] \times 20\%$. In this case, the gross return is 20%, the total fees to the manager are 4.6%, and the net return is 15.4%. Without a hurdle, the net return would be 14.4%.

30.2 LEVERAGE, LONG POSITIONS, AND SHORT POSITIONS

Hedge funds can achieve leverage and implement short sales through their **prime broker** (PB). PBs provide various back-office services to hedge funds, including trade reconciliation (clearing and settlement), custody services, risk management, as well as record keeping. In addition, they provide credit lines for financing leverage and short-selling capabilities.

To understand the mechanics of hedge funds, we need to describe how stock borrowing and margins work. In typical corporate balance sheet analysis, **balance sheet leverage** is defined as the ratio of balance sheet assets over equity. This simplistic measure, however, assumes that all the risk is coming from the assets, or that the future value of liabilities is known. Such a definition is not adequate for hedge funds, or most financial institutions, for that matter. In these cases, both assets and liabilities, long and short positions, are risky.

In what follows, we illustrate the use of long and short positions in stocks. This analysis, however, can be extended to any asset that can be shorted, subject to its own specific margin requirements.

30.2.1 Long Position

Let us start with the simplest case, which is a long position in a risky asset. Consider an investor with \$100 (say millions) invested in one stock. This can be achieved with \$100 of investor equity. Or the investor can borrow. Suppose the broker requires a 50% margin deposit, which is the minimum requirement under **Regulation T** in the United States. The investor needs to invest only \$50 and the remainder is provided by the broker, who gives a \$50 loan. The balance sheet of the position is as follows, with the risky entry in bold. Defining leverage as the ratio of assets over equity, the leverage of this position is 2 to 1.

Assets	Liabilities
\$100 **Long stock**	\$50 Broker loan
	\$50 Equity

The risk is that of a *decrease* in the value of the stock. For instance, a loss of \$1, which is 1% of the value of the stock, translates into a \$1 loss in the value of equity, which is a 2% loss in relative terms. Thus, movements in the asset value are magnified by the leverage factor L. If there is no leverage ($L = 1$), the worst loss occurs when the stock price goes to zero.

The rate of return on the equity is the summation of L times the rate of return on the long stock position R_S minus $(L - 1)$ times the cost of the loan, R_F:

$$R_E = LR_S - (L - 1)R_F = R_F + L(R_S - R_F) \tag{30.1}$$

Hence the volatility of equity will be L times that of the stock position. Similarly, the beta and idiosyncratic volatility are multiplied by the leverage:

$$\beta_E = L\beta_S \tag{30.2}$$

Leverage amplifies returns but also creates more risk.

Note that leverage can also be obtained by using derivatives, instead of cash instruments. This includes single stock futures, contracts for differences, or equity swaps. If a stock futures position can be entered with a margin of only 10%, the economically equivalent liabilities would consist of a loan of \$90 plus equity of \$10. The dollar exposure is still the same, at \$100, but the embedded leverage is now much higher than before, at 10 to 1.

30.2.2 Short Position

Consider next a situation where the investor wants to short the stock instead. Under a stock loan agreement, the owner of a stock lends the stock to our investor in exchange for cash and a future demand to get the stock back. In the meantime, the investor must pass along any cash flow on the stock, such as dividends, to the original owner.[2] When the operation is reversed, the stock lender returns the cash plus the short-term interest rate minus a **stock loan fee**. This is typically 20 basis points (bp) for most stocks but can reach 400 bp for stocks that are hard to borrow (said to be "on special"). In the meantime, the stock lender will have invested the cash, thus earning a net fee of 20bp. From the viewpoint of the stock lender, this is an easy way to increase the return on the stock by a modest amount.

The stock borrower will now sell the stock in anticipation of a fall in the price. The sale, however, will go through a broker, who will not allow the seller to have full access to the sales proceeds. In the United States, under Regulation T, the broker keeps 50% of the sales proceeds. This margin, which can be posted as any security owned clear by the investor, imposes a limit on the investor's leverage.

So, the investor receives \$100 worth of stocks, sells it, and keeps at least \$50 as margin with the broker. The hope is for a fall in the stock price, so that the stock can be repurchased later at a lower price.

All of the cash flows are arranged at the same time. The investor needs to send \$100 to the stock lender, half of which will come from the remaining proceeds and

[2] Traditional stock loans are made on a day-to-day basis. The lender can demand the return of the stock at any time, with a three-day period for delivery.

the other half from the equity invested, or the investor's own funds. The balance sheet for the short position is as follows, with the risky entry in bold. Here, leverage can be defined as the ratio of the absolute value of the short position to the equity, which is 2 to 1. As in the previous long-only case, we have a position of $50 in equity leveraged into a position of $100 in stocks. Regulation T imposes a maximum leverage ratio of 2, which is the inverse of the 50% of the short-sales proceeds kept by the broker.

Assets	Liabilities
$100 Cash lent to stock owner	$100 **Short stock**
$50 Margin at the broker	$50 Equity

Here, the risk is that of an *increase* in the value of the stock. If the stock price goes up by $1, or 1%, the equity loses $1, which is 2% in relative terms. This ratio equals the leverage of 2. The beta of the equity is now negatively related to that of the stock:

$$\beta_E = -L\beta_S \tag{30.3}$$

Short positions are intrinsically more risky than long positions, however. This is because the distribution of prices is asymmetrical. The price has a lower bound of zero but has unlimited upper values, albeit with decreasing probabilities. With a long position, the most that could be lost is $100 million. With a short position, the price could go from $100 to $200 million or even higher, in which case the dollar loss would exceed $100 million.

30.2.3 Long and Short Positions

Consider now a typical hedge fund, which has both long and short positions. Say the initial capital is $100. This is the equity, or **net asset value** (NAV). The fund could buy $100 worth of stocks and short $100 worth of stocks as before. Part of the long stock position can be used to satisfy the broker's minimum margin requirement of $50 for shorting the stock. The balance sheet for the long and short positions is as follows, with the risky entries in bold.

Assets	Liabilities
$100 **Long stock**	$100 **Short stock**
$100 Cash lent to stock owner	$100 Equity

Let us now turn to traditional risk measures. Define V_L, V_S, and V_E as the (absolute) dollar values of the long stock positions, short stock positions, and equity, respectively. V_A is the value of total assets. If β_L and β_S are the betas of the long and short stock positions, the total dollar beta is

$$(\beta_L\ V_L - \beta_S V_S) = \beta_E V_E \tag{30.4}$$

which defines the net beta of equity, or β_E. This net measure of systematic risk, however, ignores idiosyncratic risk.

Traditional **leverage** is commonly used as a risk measure

$$\text{Leverage} = \frac{V_A}{V_E} = \frac{\text{Long Stock Positions plus Cash}}{\text{Equity}} \tag{30.5}$$

In our example, this is (\$100 + \$100)/(\$100) = 2. Usually, cash is ignored, and **long leverage** is 1. This, however, ignores the hedging effect of short stock positions, so it is inadequate.

Using gross amounts, **gross leverage** is

$$\begin{aligned}\text{Gross Leverage} &= \frac{V_L + V_S}{V_E} \\ &= \frac{\text{Long Positions plus Absolute Value of Short Positions}}{\text{Equity}}\end{aligned} \tag{30.6}$$

In our example, this is (\$100 + \$100)/(\$100) = 2.

Gross leverage is often used as a rough measure of hedge fund risk. This measure, however, fails to capture the systematic risk of the equity position adequately. If the long and short positions have the same value and market beta, the net beta is zero, so there is no directional market risk. In the limit (even though there would be no reason to do so), if the long and short positions are invested in the same stock, there is no risk. Yet, gross leverage is high.

Another definition often used is **net leverage**, which is

$$\begin{aligned}\text{Net Leverage} &= \frac{V_L - V_S}{V_E} \\ &= \frac{\text{Long Positions minus Absolute Value of Short Positions}}{\text{Equity}}\end{aligned} \tag{30.7}$$

In our example, this is (\$100 − \$100)/(\$100) = 0.

Net leverage is also inadequate as a risk measure. Although it roughly accounts for systematic risk, it fails to take into account potential divergences in the value of the long and short positions. It is appropriate only under restrictive assumptions. For example, if the betas of the long and short positions are the same, then the equity beta is

$$\beta_E = \frac{\beta_L(V_L - V_S)}{V_E} = \beta_L \times \text{Net Leverage} \tag{30.8}$$

so this net leverage term measures the multiplier applied to the beta of the long position. This totally ignores idiosyncratic risk, however, which is precisely the type of risk that the hedge fund manager should take views on. In conclusion, these leverage measures should be viewed as only rough indicators of risk. They are robust and easy to compute, however.

This is why the industry has moved to more comprehensive position-based risk measures. VAR, for example, accounts for the size of positions and volatilities, as well as correlations between assets and liabilities. As such, it is a superior measure of the risk of loss.

EXAMPLE 30.1: FRM EXAM 2006—QUESTION 41

A hedge fund is long \$315 million in certain stocks and short \$225 million in other stocks. The hedge fund's equity is \$185 million. The fund's overall beta is 0.75. Calculate the gross and net leverage.

a. 2.91 and 0.48
b. 2.18 and 0.36
c. 2.91 and 0.36
d. 2.18 and 0.48

EXAMPLE 30.2: HEDGING AND RETURNS

Continuing with the previous question, assume the stock market went up by 20% last year. Ignore the risk-free rate and idiosyncratic risk, and assume the average beta of both long and short positions is 1. Over the same period, the return on the fund should be about

a. 20%
b. 15%
c. 10%
d. 5%

EXAMPLE 30.3: FRM EXAM 2004—QUESTION 2

A relative value hedge fund manager holds a long position in asset A and a short position in asset B of roughly equal principal amounts. Asset A currently has a correlation with asset B of 0.97. The risk manager decides to overwrite this correlation assumption in the variance-covariance-based VAR model to a level of 0.30. What effect will this change have on the resulting VAR measure?

a. It increases VAR.
b. It decreases VAR.
c. It has no effect on VAR, but changes profit or loss of strategy.
d. There is not enough information to answer.

30.3 HEDGE FUNDS: MARKET RISKS

30.3.1 Types of Market Risks

Hedge funds are a much more heterogeneous group of investment managers than others. They follow a great variety of strategies, which can be classified into different styles. More generally, they can be categorized into taking directional or nondirectional risks.

- **Directional risks** involve exposures to the direction of movements in major financial market variables. These directional exposures are measured by first-order or linear approximations such as
  - **Beta** for exposure to general stock market movements
  - **Duration** for exposure to the level of interest rates
  - **Spread duration** for exposure to movements in credit spreads
  - **Delta** for exposure of options to the price of the underlying asset
- **Nondirectional risks** involve other remaining exposures, such as nonlinear exposures, exposures to hedged positions, and exposures to volatilities. These nondirectional exposures are measured by exposures to differences in price movements, or quadratic exposures such as
  - **Basis risk**, which involves differences in prices of related assets
  - **Convexity risk**, which involves quadratic effects for interest rates
  - **Gamma risk**, which involves quadratic effects with options
  - **Volatility risk**, which involves movements in volatility

Directional trades can take long or short positions on the major risk factors, such as equities, currencies, fixed-income instruments, and commodities. As a result, directional positions have greater volatilities than nondirectional ones. For funds that take directional risks, total portfolio risk is controlled through diversification across sources of risks, across trading strategies, and with risk limits.

Many categories of hedge funds are hedged against directional risks. As a result, they are exposed to nondirectional risks. Such strategies need to take long *and* short positions in directional trades. The example we gave in the previous section was long \$100 in a stock offset by a short position worth \$100 in another stock. Such a strategy has little directional risk to the stock market, but is exposed to changes in the relative value of the two stocks. Limiting risk also limits rewards, however. As a result, nondirectional strategies are often highly leveraged in order to multiply gains from taking nondirectional bets.

30.3.2 Hedge Fund Styles

Hedge funds can be classified into various styles, reflecting the types of trading and markets they are exposed to. Table 30.1 lists various hedge fund styles. To some extent, this classification is arbitrary. Definitions of categories vary within the industry. Different hedge fund index providers, for example, use different classifications, even though the underlying pool of hedge funds is similar.

TABLE 30.1 Hedge Fund Styles

Style	AUM ($b)	Number of Funds	Risk (% pa)	Description
Directional Strategies				
Long/short equity	424	975	11%	Combination of long and short equity positions with net long bias
Emerging markets	115	205	12%	Equity and bond positions in emerging countries, with net long bias
Global macro	122	148	11%	Long and/or short positions across all asset classes
Nondirectional Strategies				
Relative value:				
Equity market neutral	71	183	6%	Combination of long and short equity positions with net beta close to zero
Fixed-income arbitrage	59	138	6%	Offsetting long and short positions in fixed-income securities
Convertible arbitrage	39	79	5%	Long positions in convertible bonds hedged for stock risk and interest risk
Event driven: Merger arbitrage, distressed securities, credit hedging	318	289	6%	Positions driven by corporate events such as mergers, reorganizations, and bankruptcy proceedings
Fund Structure				
Managed futures	57	235	17%	Positions in futures and option contracts (includes CTAs)
Multistrategy	181	262	9%	Combinations of hedge fund strategies in the same fund
Funds of funds		872	6%	Diversified portfolios of hedge funds

Source: TASS database, sample of live funds reporting in U.S. dollars as of December 2007. Risk is cross-sectional average of annualized volatility over the past four years.

Classifications can also lose meaning if hedge fund managers change strategies over time.

The table also reports the number of existing funds in each group, as well as their typical risk, measured as the annual standard deviation averaged across all funds.[3] Styles are generally listed in order of decreasing risk.

Table 30.2 presents the performance of typical hedge fund indices, measured over the period 1994 to 2009. Credit Suisse First Boston (CSFB) builds each sector index as a value-weighted average of eligible funds. The table shows the compound growth, volatility, beta to the S&P 500 stock index, return in excess of cash, and the alpha from a regression on equities.

The table shows that the overall hedge fund index returned 9.3% over this period, which is above the 7.6% return of the S&P 500 index, with much lower volatility. The index, however, has slightly positive beta of 0.27, so part of its performance is due to the equity premium. The last column shows an alpha of 4.2%, which is significant.

[3] Note that the risk measures are for live funds only. Hence, the data are subject to survivorship bias. The risk of existing funds is less than that of dead funds.

TABLE 30.2 Hedge Fund Performance: CSFB Indices, 1994 to 2009

	Growth	Volatility	Beta	Excess Return	Alpha
Overall index	9.3%	7.8%	0.27	5.2%	4.2%
Sectors:					
Long/short equity	10.3%	10.0%	0.41	6.2%	4.8%
Short biased	−2.5%	16.9%	−0.80	−6.6%	−3.8%
Emerging markets	8.0%	15.6%	0.53	3.9%	2.1%
Global macro	12.4%	10.3%	0.16	8.3%	7.7%
Equity market neutral	5.5%	10.8%	0.18	1.3%	0.7%
Fixed-income arbitrage	4.8%	6.1%	0.14	0.7%	0.2%
Convertible arbitrage	7.7%	7.2%	0.16	3.6%	3.0%
ED—merger arbitrage	11.2%	6.7%	0.26	7.1%	6.2%
ED—distressed securities	9.8%	6.4%	0.22	5.7%	4.9%
ED—credit hedging	7.4%	4.2%	0.13	3.2%	2.8%
Managed futures	6.3%	11.8%	−0.11	2.2%	2.5%
Multistrategy	8.4%	5.5%	0.11	4.2%	3.9%
Benchmarks:					
Cash	4.1%	0.0%	0.00	0.0%	0.0%
S&P 500 index	7.6%	15.5%	1.00	3.5%	0.0%
Treasury index	5.9%	4.8%	−0.03	1.8%	
High-yield index	7.0%	9.5%	0.37	2.9%	

Note: Excess returns are measured relative to one-month London Interbank Bid Rate (LIBID). Alpha is measured from a market model regression on the S&P 500 index. ED—event driven.

Also note that the volatility in this table is not directly comparable to that in Table 30.1, because this is the volatility of a portfolio, instead of the average fund volatility. In contrast, the beta of a portfolio is a weighted average of the fund betas, so the portfolio beta gives a good indication of the typical beta of individual funds.[4]

Long/Short Equity The first category consists of directional strategies. These include **long/short equity funds**, which, as Table 30.1 shows, is the most prevalent strategy. These funds are not market neutral. Most have a long bias (e.g., 100% of NAV in long positions, and 50% in short positions). Table 30.2 indeed shows a beta of 0.41.

A related category consists of **short biased funds**, which are net short. Table 30.2 shows a negative beta, close to −1. Another category is **emerging markets funds**, which consists of equity and bond positions in emerging countries, such as Brazil, Russia, India, and China.

These funds are exposed to the general market risk factor, in addition to sector and idiosyncratic risks. Because of leverage, volatility is high, at 11% on average across all such funds. This is on the order of the volatility of an unleveraged position in the S&P 500.

[4] In addition, the numbers are not directly comparable across the two tables because they are measured over different periods.

Global Macro Next are **global macro funds**, which take directional, leveraged bets on global asset classes, equities, fixed-income securities, currencies, and commodities. Because they span so many markets, these funds do not have a homogeneous risk profile. An example is George Soros's fund that shorted the British pound against the German mark just before the pound's devaluation, leading to a reported gain of $1 billion for the hedge fund. This group is close to **global tactical asset allocation** (GTAA), which is a traditional investment manager category. GTAA managers take positions across national stock markets, fixed-income markets, and currencies to take advantage of short-term views, often through derivatives.

These funds are exposed to a number of general market risk factors, in addition to sector and idiosyncratic risks. The average volatility is 11%. This is less than the previous category because these funds also invest in other markets, which are less volatile than equities.

We now turn to nondirectional strategies. The next three categories are sometimes called **relative value funds**, because they rely on comparisons of securities with similar characteristics, buying the cheap ones while selling the expensive ones in the hope of future convergence.

Equity Market Neutral The first group is **equity market neutral funds**, which attempt to maintain zero beta through balanced long and short positions in equity markets. These funds may or may not be neutral across other risk factors, including industries, styles, and countries.

So, these funds are exposed to these other risk factors (industries, styles, countries) in addition to idiosyncratic, stock-specific risks. Balance sheet leverage is typically three times on each side; that is, both longs and shorts add up to 300% of equity. The average volatility is 6%, which is much less than that of equity indices, due to the hedging effect of the short positions.

Fixed-Income Arbitrage The next group is **fixed-income arbitrage funds**. This is a generic term for a number of strategies that involve fixed-income securities and derivatives. The hedge fund manager assesses the relative value of various fixed-income instruments.

For instance, on-the-run bonds are the most recently issued bonds within a maturity range and hence the most liquid; otherwise, the bonds are called off-the-run. If the on-the-run bond is very expensive relative to the off-the-run bond, the fund would buy the undervalued security and sell the expensive one. This position has a net duration close to zero but is exposed to the spread between the two securities. Other examples include taking positions in swap spreads, or in asset-backed securities when their option-adjusted spread is high. This group includes mortgage arbitrage.

These funds avoid directional exposures to interest rates but are exposed to other nondirectional risks, such as spread risk. Due to the small expected profit of each trade, fixed-income arbitrage funds are highly leveraged, with leverage ratios ranging from 10 to 25.

Example: LTCM's Bet

Long-Term Capital Management (LTCM) started as a fixed-income arbitrage fund, taking positions in relative value trades, such as duration-matched positions in long swaps, short Treasuries. It started the year 1998 with $4.7 billion in equity capital.

On August 21, 1998, the 10-year Treasury yield dropped from 5.38% to 5.32%. The swap rate, in contrast, increased from 6.01% to 6.05%. This divergence was highly unusual. Assuming a notional position of $50 billion and modified duration of eight years, this leads to a value change of $-8 \times (5.32 - 5.38)/100 \times \$50{,}000 = +\$240$ million on a long Treasury position and $-8 \times (6.05 - 6.01)/100 \times \$50{,}000 = -\$160$ million on a long swap position. As the spread position is long the swap and short Treasuries, this leads to a total loss of $400 million, close to 10% of capital.

LTCM also took positions in option markets, selling options when they were considered expensive and dynamically hedging to maintain a net delta of zero. Implied volatilities went up sharply on August 21, leading to further losses on the option positions. On that day, LTCM's reported loss was $550 million.

The average volatility of this group is 6%. The distribution of payoffs is typically asymmetric, however. Swap spreads, for example, cannot narrow below zero but can increase to very large values, and have done so. This asymmetry in the distribution of spreads is reflected in that of profits. Such hedge funds have negatively skewed distribution. When they lose money, they can lose large amounts.

Convertible Arbitrage The last group in the relative value category is **convertible arbitrage funds**. The hedge fund manager assesses the relative value of convertible bonds using proprietary option pricing models. If the convertible bond is cheap, the hedge fund buys the bond while hedging the major risks.

Because a convertible bond involves a long call option position, it has positive delta with respect to the underlying stock. Therefore, the manager should short the stock to bring the net delta of the position close to zero. Typically, interest rate risk is hedged by shorting Treasury bonds, or T-bond futures. Sometimes, credit spread risk is hedged by buying credit default swaps.

These funds avoid directional exposures to interest rates but are exposed to other nondirectional risks, such as spread risk. Being typically long convertible bonds, the long option position creates positive gamma and vega (long implied volatility). The bond position creates positive convexity, unless the bond is callable. This strategy is also exposed to corporate event risk, such as default (if not hedged) and takeover. Leverage is moderate. Typically, the long convertible bond position is no more than three times equity. The average volatility of this group is 5%, which is fairly low, in part because of illiquidity.

Event Driven The next group includes **event-driven funds**, which attempt to capitalize on the occurrence of specific corporate events. This group includes **merger arbitrage funds** and **distressed securities funds**.

Let us focus first on merger arbitrage funds, also known as **risk arbitrage funds**. **Mergers and acquisitions** are transactions that combine two firms into one new firm.[5] The parties can be classified as the **acquiring firm**, or bidder, which initiates the offer, and the **target firm**, or acquired firm, which receives the offer. The bidder offers to buy the target at a **takeover premium**, which is the difference between the offer price and the target's stock price before the bid. This premium is typically high, averaging 50% of the initial share price.

Upon the announcement of the merger, the price of the target firm reacts strongly, increasing by, say, 40%. This still falls short of the takeover price, due to the uncertainty as to whether the transaction will occur. The completion rate is 83% on average, so there is always a possibility the transaction could fail. When this happens, the target firm typically suffers a large price drop. As a result, it is important to diversify by spreading the portfolio over many deals.

Offers can take the form of cash or stock of the bidding company. For a cash deal, the risk arbitrage position simply consists of buying the target's stock, and hoping the price will eventually move to the takeover price. For a stock deal, the bidder offers to exchange each target share for Δ shares of the bidder. The risk arbitrage position then consists of a long position in the target offset by a short position of Δ in the bidder's stock. These positions generate an average annualized excess return of 10%.[6]

The volatility of this group is relatively low. Because the stochastic process for the target's stock price changes after the announcement, traditional position-based risk measures are not appropriate measures of risk.[7]

Example: Exxon–Mobil Merger

On December 1, 1998, Exxon confirmed that it had agreed to buy Mobil, another major oil company, in a transaction valued at $85 billion, which was the biggest acquisition ever. The deal created the world's largest traded oil company, with a market capitalization of $250 billion. Under the terms of the agreement, each shareholder of Mobil would receive $\Delta = 1.32015$ shares of Exxon in exchange.

[5] These are sometimes called takeovers. Takeovers can take the form of mergers or tender offers. Mergers are negotiated directly with the target managers, approved by the board of directors, and then approved by shareholder vote. Tender offers are offers to buy shares made directly to target shareholders.

[6] This is a risk-adjusted excess return. These profits, however, seem to be related to limits to arbitrage, as they are lower for firms that are large and have low idiosyncratic risk. See M. Baker and S. Savasoglu, "Limited Arbitrage in Mergers and Acquisitions," *Journal of Financial Economics* 64 (2002): 91–115.

[7] See P. Jorion, "Risk Management for Event-Driven Funds," *Financial Analysts Journal* 64 (2008): 61–73.

Before the announcement, the initial prices of Mobil and Exxon were \$78.4 and \$72.7, respectively, which implies a modest premium of (1.32016 × \$72.7)/\$78.4 − 1 = 22%. Over the three days around the announcement, Mobil's stock price went up by +6.9% to \$84.2 and Exxon's price went down by −1.5% to \$71.6. This stock price reaction is typical of acquisition announcements.

The exchange was consummated on November 30, 1999, after regulatory and shareholder approval. On that day, the respective stock prices for Mobil and Exxon were \$104.4 and \$79.3. Multiplying the latter by 1.32015, we get \$104.7, which is close to the final stock price for Mobil. So, the two prices converged to the same converted value. The profit from the risk arbitrage trade was (\$104.4 − \$84.2) − 1.32016(\$79.3 − \$71.6) = \$10.0 per share.

Event-driven funds also include **distressed securities funds**, which take positions in securities, debt or equity, of firms in financial difficulty. In such situations, the hedge fund manager needs to assess the effect of restructuring or the bankruptcy process on the market price of the securities. This requires an evaluation of the financial situation of the firm, as well as a good understanding of legal issues involved. If, for instance, the debt of a bankrupt company trades at 40 cents on the dollar, the hedge fund would benefit if the total payment after reorganization is 50 cents. Such funds are also actively involved in the bankruptcy processes and the reorganization plans.

These funds are exposed to event risks, that is, that the takeover or reorganization fails. They may also be exposed to equity market risk and interest rate risk if these exposures are not hedged. Because distressed securities do not trade actively, there is also liquidity risk.

Leverage for event-driven funds is low to moderate, no more than two times. The average volatility for event-driven funds is 6%, which is fairly low. This, however, hides the fact that the distribution of payoffs is asymmetric. Typically, the upside is more limited than the downside, should the takeover or reorganization fail. So, these funds are short volatility, or exposed to rare events. Because of the unusual nature of the event, measures of risk based on historical returns can be inaccurate for forecasting risk.

Managed Futures Funds The next category of hedge funds differs from others on the basis of the fund structure. **Managed futures funds** consist of managers who use commodity and financial futures and options traded on organized exchanges. Trading strategies often involve **technical trading**, where positions depend on patterns in price histories. Leverage is high, leading to high volatility.

These funds have directional exposures to all the markets that have listed futures contracts. Their risk factors overlap with global macro funds. GTAA strategies, for instance, often involve stock index and currency futures. The average volatility of this group is 17%, which is fairly high.

Multistrategy Funds Next, **multistrategy funds** are hedge funds that cover combinations of previously described fund strategies. One advantage of such funds is that they can reallocate capital quickly from one strategy to another. They also provide automatic diversification across strategies. The average volatility of this group is 9%.

However, multistrategy funds tend to be more concentrated in one type of strategy, and do not provide as much diversification as funds of funds, described next. Amaranth, for example, initially started as a multistrategy fund focused on convertible bond arbitrage, then morphed into a natural gas trading operation, and eventually blew up. Because all strategies are run within the same fund, a large loss in one strategy may affect the capital of other strategies. In other words, strategies are not firewalled, unlike in the fund of funds structure.

Funds of Funds Finally, **funds of funds**, also called **multimanager funds**, are portfolios of hedge funds. These add value by careful selection of styles and investment managers. They also perform essential functions, such as the due diligence process when evaluating new managers and their continuous monitoring. Funds of funds can take views on strategies, increasing allocation to strategies that are expected to perform better.

Funds of funds charge additional management fees on top of those levied by the underlying funds, typically around 1%. However, because of their size, funds of funds can negotiate lower fees from the hedge fund managers.

Relative to multistrategy funds, funds of funds have higher fees. This cost difference, however, is offset by the fact that a fund of funds has access to the best managers, who generally want to run their own fund, thus creating better performance.

Also, funds of funds have lower risk of losses due to blowups than multistrategy funds, where the entire investment can be lost, as in the case of Amaranth. There are two reasons for this. First, funds of funds are generally better diversified across strategies than are multistrategy funds. Second, the hedge funds in a fund of funds pool are legally separate from each other (i.e., are firewalled). As a result, a blowup in one hedge fund will not contaminate the rest of the portfolio, unlike what can happen in the case of a multistrategy fund.

Funds of funds provide convenient access to a diversified portfolio of hedge funds. Because hedge funds have minimum investment amounts, such diversification is difficult to achieve for small mandates allocated to hedge funds. For example, a $100 million allocation to hedge funds can be realistically invested in at most 10 hedge funds. A typical fund of funds, in contrast, will invest in 50 funds. Funds of funds provide economies of scale in the due diligence and risk monitoring processes. Funds of funds can also negotiate greater capacity and better liquidity than other investors.

Table 30.1 shows that the average volatility of this group is 6%. This low number reflects effective diversification across managers and styles.

This list makes it clear that hedge funds are a very heterogeneous group. They are exposed to a wide variety of risk factors, follow different trading rules, and have varying levels of leverage and risk. The common element, however, is the need to manage risk.

EXAMPLE 30.4: FRM EXAM 2009—QUESTION 8-7

A fund of hedge funds combines a mix of strategy sectors, managers, and styles, and therefore fund of funds risk managers need to understand the common attributes of hedge fund strategies. Which of the following statements is *incorrect*?

a. Equity market neutral funds aim to generate returns that have low correlation to the overall equity market and to insulate their portfolios from broad market risk factors.
b. Convertible arbitrage funds typically purchase securities that are convertible into the issuer's stock and simultaneously short the underlying stock. These funds earn returns in part from gamma trading on the stock's volatility.
c. Merger arbitrage funds buy the stock of an acquisition target company and simultaneously short the bidding company's stock. These funds have large exposure to deal risk.
d. Equity short-selling funds sell stocks not currently owned by the seller in order to take a directional bet that the stock price will decline. These funds tend to be uncorrelated with traditional long-only equity portfolios.

EXAMPLE 30.5: RISKS IN FIXED-INCOME ARBITRAGE

Identify the risks in a fixed-income arbitrage strategy that takes long positions in interest rate swaps hedged with short positions in Treasuries.

a. The strategy could lose from decreases in the swap-Treasury spread.
b. The strategy could lose from increases in the Treasury rate, all else fixed.
c. The payoff in the strategy has negative skewness.
d. The payoff in the strategy has positive skewness.

EXAMPLE 30.6: RISKS IN CONVERTIBLE ARBITRAGE

Identify the risk in a convertible arbitrage strategy that takes long positions in convertible bonds hedged with short positions in Treasuries and the underlying stock.

a. Short implied volatility
b. Long duration
c. Long stock delta
d. Positive gamma

EXAMPLE 30.7: RISKS IN MERGER ARBITRAGE—I

A major acquisition has just been announced, targeting company B. The bid from company A is an exchange offer with a ratio of 2. Just after the announcement, the prices of A and B are $50 and $90, respectively. A hedge fund takes a long position in company B hedged with A's stock. After the acquisition goes through, the prices move to $60 and $120. For each share of B, the gain is

a. $30
b. $20
c. $10
d. $0 since the acquisition is successful

EXAMPLE 30.8: RISKS IN MERGER ARBITRAGE—II

Suppose the payoff from a merger arbitrage operation is $5 million if successful, −$20 million if not. The probability of success is 83%. The expected payoff on on the operation is

a. $5 million
b. $0.75 million
c. $0 since markets are efficient
d. Symmetrically distributed

EXAMPLE 30.9: FRM EXAM 2005—QUESTION 47

The Big Bucks hedge fund has the following description of its activities. It uses simultaneous long and short positions in equity with a net beta close to zero. Which of the following statements about Big Bucks is/are *correct*?

I. It uses a directional strategy.
II. It is a relative value hedge fund.
III. This fund is exposed to idiosyncratic risks.

a. I and II
b. II and III
c. I and III
d. II only

30.4 HEDGE FUNDS: SPECIFIC RISKS

30.4.1 Agency Risk

Hedge fund managers act as agents for investors. This can cause misalignment of incentives, however. Incentive fees make a payment that is a fraction of the profits, if positive. As a result, the hedge fund manager is long an option. Because the value of an option increases with the volatility, the fund manager may have an incentive to increase the risk of the fund.

Another potential problem is that of **style drift**, which occurs when managers change their investment patterns or stray into new markets.

This type of behavior can be minimized in a number of ways. Most importantly, hedge fund managers should invest a large fraction of their personal wealth in the fund they manage. This lessens the incentive to take on too much risk.

Some risk monitoring occurs at the level of the prime broker (PB). Because the PB is primarily concerned about the risk of loss from lending to the hedge fund, however, its interests do not align with those of fund investors. For example, a lender may use margin calls to force liquidation of the fund assets at distressed prices. As long as there is excess collateral, the lender would be protected, but at the expense of investors.

The incentive to take risk is also lessened with **high-water marks**, also known as loss carryforward provisions.[8] The manager receives performance fees only to the extent that the current net asset value (NAV) exceeds the highest NAV previously achieved. Suppose, for instance, that the NAV changes from $100 to $130 to $120 to $140 in four consecutive years. The $130 NAV year, the performance fee would apply to $30. The next year, there is no performance fee because the fund lost money. The final year, the performance fee applies only to the portion of $140 in excess of $130, which is the highest previous NAV. This mechanism, however, may not provide complete protection if the watermark is too high. In this case, the fund manager may choose to close the fund and to start a new one (if investors can be found).

30.4.2 Liquidity and Leverage Risk

Hedge funds take leveraged positions to increase returns, especially with nondirectional trades such as fixed-income arbitrage, where the expected return on individual trades is generally low.

Perversely, this creates other types of risks, including **liquidity risk**. This strategy indeed failed for Long-Term Capital Management (LTCM), a highly leveraged hedge fund that purported to avoid directional risks. LTCM had a leverage ratio of 25 to 1. It had grown to $125 billion in assets, four times the asset size of the next largest hedge fund. Once the fund started to accumulate losses, it became

[8] Sometimes, a **claw-back** provision is included, which requires the fund manager to pay back performance fees when the value of the fund drops.

difficult to cut positions given its size. LTCM also had to meet margin calls from brokers. The fund ended up losing $4.4 billion, or 92% of its equity.

Table 30.3 links sources of liquidity risk to a hedge fund balance sheet. Liquidity risk arises on the asset side and is a function of the size of the positions as well as of the price impact of a trade. On the liabilities side, funding risk arises when the hedge fund cannot roll over funding from its broker, or when losses in marked-to-market positions or increases in haircuts lead to cash outflows. This is often a major source of risk for hedge funds, as the failure to meet margin calls can lead the lender to seize the collateral, forcing liquidation of the fund. Finally, funding risk also arises when the fund faces investor redemptions.

The price impact function is instrument-specific. For example, major currencies, large stocks, Treasury bills, and Treasury bonds are very liquid, meaning that large amounts can be transacted without too much effect on the price. Other markets are by nature less liquid. For instance, minor currencies, small stocks, and most corporate debt instruments are generally illiquid.

LTCM dealt with mostly liquid instruments but was exposed to liquidity risk due to the sheer size of its positions. This is why hedge funds often say they have a **maximum capacity**. Beyond that optimal size, trading becomes difficult due to market impact.

Another type of risk that is exacerbated by leverage is **model risk**. This occurs when the investment strategy relies on valuation or risk models that are flawed. Due to leverage, small errors in the model can create big errors in the risk measure. Indeed, LTCM's risk measurement system was deficient, leading to a fatal underestimation of the amount of capital required to support its positions.

Some categories of hedge funds have intrinsic liquidity risk because the instruments are thinly traded, implying a large price impact for most trades. This is the case with convertible bonds and especially so with distressed securities. Because these funds invest in thinly traded securities, liquidity risk arises even for small funds.

Typically, funds with greater liquidity risk impose a longer **lockup period** and **redemption notice period**. The former refers to the minimum time period during which investor money is to be held in the fund. The latter refers to the period required to notify the fund of an intended redemption. Lockup periods average three months, and can extend to five years. Advance-notice periods average 30 days. Funds also often have **gates**, which limit the amount of withdrawals to a fraction of the net assets. In the extreme, funds might be able to impose an outright **suspension** of redemptions.

TABLE 30.3 Sources of Liquidity Risk

Assets	Liabilities
Size of position	Funding
Price impact	Mark-to-market, haircuts
	Equity
	Investor redemptions

Instrument liquidity risk creates a major problem for the measurement of risk. Typically, funds report their **net asset value** at the end of each month. If transaction prices are not observed at the end of the month, the valuation may be using a price from a trade that occurred in the middle of the month. This price is called a **stale price** because it is old and does not reflect a market-clearing trade on the day of reporting. Unfortunately, this will distort the reported NAVs as well as the risk measures.

The first effect is that the reported monthly volatility will be less than the true volatility. This is because prices are based on trades during the month, which is similar to an averaging process. Averages are less volatile than end-of-period values.

The second effect is that monthly changes will display positive autocorrelation. A movement in one direction will be only partially captured using prices measured during the month. The following month, part of the same movement will show up in the return. This positive autocorrelation substantially increases the risk over longer horizons. Consider, for instance, the extrapolation of a one-month volatility to two months. The usual adjustment factor is $\sqrt{T} = \sqrt{2} = 1.41$. With autocorrelation of $\rho = 0.5$, this adjustment factor is instead $\sqrt{(1 + 1 + 2\rho)} = \sqrt{2(1 + 0.5)} = 1.73$. The true risk is understated by $(1.73 - 1.41)/1.73 = 18\%$. This effect increases with the length of the horizon. As a result, the annualized volatility presented in Table 30.1, which extrapolates monthly volatility using the square root of time, may understate the true annual risk. Long-term measures of risk must specifically account for the observed autocorrelation.

A third, related effect is that measures of systematic risk will be systematically biased downward. If the market goes up during a month, only a fraction of this increase will be reflected in the NAV, leading to beta measures that are too low. Corrections to the beta involve measuring the portfolio's beta with the contemporaneous market return plus the beta with respect to the one-month lagged return plus the beta with respect to the one-month future return. With thin trading, the sum of these three betas should be higher than the contemporaneous beta, and also closer to the true systematic risk.[9]

Leverage can create other problems, which can be classified as **crowded trade risk**. This arises when many leveraged investors are on the same side of a trade.[10] A loss in their portfolios may require them to post additional margin, which may be satisfied by several funds selling similar assets at the same time, which can create disruptions in markets. Apparently, this explains why many **quant funds**, which are generally equity market neutral (EMN) strategies driven by quantitative models, suffered heavy losses in August 2007. The story is that

[9] This correction is called the *Dimson beta*. See E. Dimson, "Risk Measurement When Shares Are Subject to Infrequent Trading," *Journal of Financial Economics* 7 (1979): 197–226.

[10] Of course, other investors must be on the other side of the trade. This classification supposes that the other side is not so leveraged; otherwise, trades could be crossed without much effect on prices. In practice, positions are confidential, and it is impossible to know who is on which side of a trade, except anecdotally.

a large multistrategy fund lost money on credit trades and then liquidated its equity positions, because these are more liquid than others. This caused large losses in EMN portfolios, on both the long and the short sides, which was highly unusual.

This is not just a problem with leverage, however. Any mechanistic trading rule that involves cutting positions after a loss is incurred may have similar effects if there is an imbalance between demand and supply. This includes, for instance, stop losses, which are equivalent to synthetically replicating long positions in options.

30.4.3 Leverage and Counterparty Risk

Leverage also creates another type of risk, which is **counterparty risk**. Hedge funds that use leverage give collateral to prime brokers. **Hypothecation** is the pledge of client-owned securities in a margin account to secure a loan. The broker then has the right to **rehypothecate** the securities to another party. If the broker goes bankrupt, however, the rehypothecated assets become part of the claims against the broker.

In the case of the Lehman failure, $22 billion out of the $40 billion held by Lehman's European prime brokerage had been rehypothecated. As a result, hedge funds trying to reclaim these assets found themselves in the line of general creditors with claims against Lehman.

30.4.4 Fraud Risk

A last issue, especially with complex or illiquid assets, is **improper valuation of assets**. This problem arises when assets do not have market-clearing prices at the end of the reporting period and when fund managers calculate the NAV themselves. As a result, some unscrupulous hedge fund managers have succumbed to the temptation to misreport the value of the fund's assets in order to hide their trading losses.[11] Others have even stolen investors' assets.

Indeed, a recent study has shown that valuation problems played a role in 35% of hedge fund failures, and that 57% of those valuation problems were caused by fraud or misrepresentation.[12] The growth of the hedge fund industry, along with well-publicized occurrences of fraud, explains the trend toward requiring funds to **register** with the Securities and Exchange Commission (SEC) in the United States.[13]

[11] A 2003 report by the SEC, however, notes that there is no evidence that hedge fund advisers engage *disproportionately* in fraudulent activities.

[12] See C. Kundro and S. Feffer, "Valuation Issues and Operational Risk in Hedge Funds" (Capco white paper, 2003).

[13] The SEC issued a new rule in December 2004 that required hedge funds to register as investment advisers. This rule applied to U.S.-based hedge funds, and to non-U.S. funds that have at least 14 U.S. investors. Funds with less than $25 million under management did not have to comply. In June 2006, however, this registration requirement was annulled by the U.S. Court of Appeals. Even so, Congress created a new law in 2010 that requires most hedge funds to register with the SEC.

Registration gives the SEC the authority to conduct examinations of hedge fund activities. The goal is to help to identify compliance problems at an early stage and to provide a deterrent to fraud. Registration also requires the hedge fund to designate a **chief compliance officer**. In practice, however, the majority of U.S.-based hedge funds have voluntarily registered as investment advisers. Registration is often required by investors as a precondition for investing.

The possibility of fraud can be lessened when a fund has an independent **administrator**. Administrators perform day-to-day administrative duties associated with running a fund, in particular financial and tax reporting. They calculate net asset values, maintain the statutory books and records, and provide shareholder services. In addition, an outside **auditor** provides important additional information. Auditors issue a written opinion on the fair presentation of the fund's financial statements, typically on an annual basis. To protect against theft, investors should insist on an external **custodian**, which is a financial institution such as a bank or trust company that holds the fund's assets. Usually a fund's prime broker will perform the role of custodian.

Example: Ponzi Scheme

The term **Ponzi scheme** is attributed to Charles Ponzi, who in 1919 established an inventive pyramid scheme using new investor funds to repay earlier investors. The investment was based on a relative-value trade, in which postal coupons were bought overseas for the equivalent of one U.S. cent and resold for six American one-cent stamps. After transaction costs were factored in, however, the trade was unprofitable. Nevertheless, thousands of people invested with him, lured by a promise of 50% return in 90 days. Ultimately, he lost $140 million of investor funds, in today's dollars, and was jailed for fraud.

The most famous case of a Ponzi scheme since then was perpetrated by **Bernard Madoff**, who was arrested in December 2008 after admitting to defrauding investors of perhaps $50 billion through his brokerage firm, Bernard Madoff Investment Securities (BMIS). BMIS was established in 1960 and by the end of 2007 was managing about $17 billion in hedge fund investments. Returns to initial investors were paid using new investments. The scheme collapsed when investors sought about $7 billion in redemptions during 2008, which could not be met. Ponzi schemes work only as long as new money flows in.

Investors had been attracted by the high and steady returns from Madoff's funds, which turned out to have been fabricated. Many leading funds of funds, however, had performed due diligence on Madoff and had spotted enough warning signs to stay clear of him. BMIS acted as custodian of investor assets, had no external administrator, and relied on an unknown three-person audit firm. In addition, some people who had analyzed the options-trading strategy had concluded that it was infeasible because it would have implied trading volumes far in excess of exchange-traded volumes. This illustrates the value of a thorough due diligence process.

30.4.5 Regulatory Risks

Finally, hedge funds are also subject to **regulatory risks**. This is the risk of loss due to regulatory changes. In September 2008, for example, many countries put in place outright bans on short-selling equities.

This ban created havoc with strategies that rely on short sales to hedge. For example, the convertible bond arbitrage sector suffered extreme losses during these months. Statistical arbitrage funds, which take long and short positions in stocks, had to withdraw from many markets.

There is broad consensus, however, that such bans are ineffective at stopping prices from falling.[14] Such bans also have far-reaching consequences. In the case of convertible bonds, for example, the market freeze prevented many institutions, including banks, that traditionally issue convertible bond debt from raising new funds. In addition, such bans cause investors to withdraw from markets, sapping liquidity, which actually increases volatility. Bans on short-selling removed hedge funds' main mechanism for risk management, which led to an acceleration of withdrawal of capital when the market needed it most.

EXAMPLE 30.10: LIQUIDITY RISKS

Asset liquidity risk is most pronounced for

a. A $10 million position in distressed securities
b. A $10 million position in Treasury bonds
c. A $100 million position in distressed securities
d. A $100 million position in Treasury bonds

EXAMPLE 30.11: FRM EXAM 2006—QUESTION 112

For a portfolio of illiquid assets, hedge fund managers often have considerable discretion in portfolio valuation at the end of each month and may have incentives to smooth returns by marking values below actual, in high-return months and above actual, in low-return months. Which of the following is *not* a consequence of return smoothing over time?

a. Higher Sharpe ratio
b. Lower volatility
c. Higher serial correlation
d. Higher market beta

[14] I. Marsh and N. Niemer, "The Impact of Short Sales Restrictions" (working paper, Cass Business School, London, 2008).

EXAMPLE 30.12: FRM EXAM 2007—QUESTION 62

You are asked to estimate the exposure of a hedge fund to the S&P 500. Though the fund claims to mark to market weekly, it does not do so and merely marks to market once a month. The fund also does not tell investors that it simply holds an exchange-traded fund (ETF) indexed to the S&P 500. Because of the claims of the hedge fund, you decide to estimate the market exposure by regressing weekly returns of the fund on the weekly return of the S&P 500. Which of the following *correctly* describes a property of your regression estimates?

a. The intercept of your regression will be positive, showing that the fund has a positive alpha when estimated using an ordinary least squares (OLS) regression.
b. The beta will be misestimated because hedge fund exposures are nonlinear.
c. The beta of your regression will be one because the fund holds the S&P 500.
d. The beta of your regression will be zero because the fund returns are not synchronous with the S&P 500 returns.

30.5 DEALING WITH HEDGE FUND RISKS

Because of these risks, hedge funds need to be monitored closely. This starts with **due diligence**, which is the process of systematically investigating the fund before investing. On the operational side, this involves an analysis of the fund documents; of the key personnel (including background checks); of the fund service providers (administrator, prime broker, legal counsel, auditor); of the regulatory registration; and of the operations and valuation procedures. On the investment side, this involves an analysis of the investment strategy, of the risk factors, and of the risk control systems. Once a hedge fund manager is hired, some components of this due diligence process need to be verified periodically.

Without information about the positions, however, this process is rather incomplete. It is very difficult to detect style drift, for example, from historical returns. Because returns are typically provided at monthly intervals, structural breaks can generally be identified only after a few years.

30.5.1 Hedge Fund Transparency Issues

Hedge funds are generally reluctant to reveal information about their positions. This lack of transparency has serious disadvantages for investors.

Disclosure allows *risk monitoring* of the hedge fund, which is especially useful with active trading. This can help an investor avoid situations where the hedge fund manager unexpectedly increases leverage or changes style. Closer monitoring of the fund can also decrease the probability of fraud or misvaluation of assets.

Disclosure is also important for *risk aggregation*. The investor should know how the hedge fund interacts with other assets in the portfolio. Whether the hedge fund has a positive or negative correlation with the rest of the portfolio affects the total portfolio risk.

Example: Why Risk Aggregation?

Aggregation of positions is important to identify potential concentrations to individual names (companies). The story is told of a large pension fund that had allocated assets to outside managers investing in corporate bonds, growth equities, and value equities. In 2000, Enron was rated investment-grade and viewed as a growth stock, reflecting its high stock price of $90. The fund had positions in Enron corporate bonds, and in Enron stock through the growth manager.

As negative news unfolded the following year, the stock price fell to $15 by October 2001. Many saw this decline as a great opportunity to buy the stock. The pension fund's value managers started to buy the stock. As the same time, its other managers did not have the discipline to sell. By December 2001, the stock price had fallen to $0.03. The pension fund ended up with large holdings of Enron stock and bonds, and a huge loss due to its failure to identify this concentration risk.

Greater disclosure is often resisted on the grounds that it would disclose *proprietary information*, leading to the possibility of a third-party trading against the hedge fund. This threat, however, comes from the broker-dealer community, generally not from investors. If this is an issue, confidentiality agreements should prevent leakages of sensitive information. Hedge funds generally prefer to release such information to investors with no trading operations of their own, whether directly or through affiliates, who would not be able to profit from this information.

Another argument sometimes advanced is the *lack of investor sophistication*. In other words, disclosing positions would give too much information to investors who might not able to use it.

30.5.2 Solutions for Transparency

These arguments can be addressed with a number of solutions. The first consists of external risk measurement services. These firms have access to the individual positions of hedge funds, with the proper confidentiality agreements, and provide aggregate risk measures to investors. They release only aggregate information such as gross and net leverage, asset, industry, and geographic allocations, as well as factor exposures. This solves the risk aggregation problem.

Another solution is to go through a fund of funds that has position-level information. The fund of funds can use this information to monitor the managers and to provide aggregate statistics to the investors. Thus this approach solves both problems, risk monitoring and risk aggregation.

EXAMPLE 30.13: TRANSPARENCY

Investors should insist on learning about the positions of hedge funds because

a. They want to trade ahead of the hedge fund.
b. They do not understand the trading strategies behind the positions.
c. They want to aggregate the risk of hedge funds with the rest of their portfolio.
d. They receive the information from the prime broker anyway.

EXAMPLE 30.14: FRM EXAM 2009—QUESTION 8-8

Risk management of hedge funds has challenges not generally faced in traditional investment management companies. Which of the following statements are *correct* about hedge fund risk management?

I. Because hedge funds can hold long and short positions, and can use derivatives and leverage, their exposure to market risks can experience large and rapid changes that make it difficult to assess these exposures using only monthly returns.
II. Many hedge funds use over-the-counter derivatives, which are valued by models or quoted prices and often hold illiquid assets; as a result, the returns of these strategies generally exhibit much lower serial correlation than mutual fund returns.
III. For hedge fund strategies that use leverage to amplify returns and rely on their ability to move out of trades quickly when they turn against them, liquidity risk must be closely monitored and managed.
IV. Hedge fund returns are often similar to the return of a basket of exotic derivatives with nonlinear payoffs, and therefore assessing risk based on past performance can be misleading.

a. I, II, III, and IV
b. I, III, and IV
c. I and III
d. II and IV

30.6 IMPORTANT FORMULAS

Net beta: $(\beta_L\ V_L - \beta_S V_S) = \beta_E V_E$

Long leverage: $\frac{V_A}{V_E} = \dfrac{\text{Long Stock Positions plus Cash}}{\text{Equity}}$

Gross leverage: $\frac{V_L+V_S}{V_E} = \dfrac{\text{Long Positions plus Absolute Value of Short Positions}}{\text{Equity}}$

Net leverage: $\frac{V_L-V_S}{V_E} = \dfrac{\text{Long Positions minus Absolute Value of Short Positions}}{\text{Equity}}$

30.7 ANSWERS TO CHAPTER EXAMPLES

Example 30.1: FRM Exam 2006—Question 41

a. The gross leverage is $(315 + 225)/185 = 2.9$. The net leverage is $(315 - 225)/185 = 0.5$. Note that beta is not needed for this calculation.

Example 30.2: Leverage and Returns

c. The net return on the stock portfolio is $(\beta_L\$315 - \beta_S\$225) \times 20\%$. With betas of 1, this is \$18 million. Given that the equity is \$185 million, the rate of return is about 10%. The rate of return is less than that on the market because most of the exposure to the market is hedged.

Example 30.3: FRM Exam 2004—Question 2

a. Because the position is both long and short, high correlation implies low risk. Conversely, lowering correlation increases risk.

Example 30.4: FRM Exam 2009—Question 8-7

d. Statements a., b., and c. are correct. Funds that short-sell, however, have negative correlation with long-only portfolios. They cannot be uncorrelated.

Example 30.5: Risks in Fixed-Income Arbitrage

c. The strategy has no exposure to the level of rates but is exposed to a widening of the swap-Treasury spread. Assume, for instance, that the swap and Treasury rates are initially 5.5% and 5%. If these rates change to 5.3% and 4.5%, for example, values for both the swap and the Treasury bond would increase. Because the drop in the Treasury rate is larger, however, the price of the Treasury bond would fall more than the swap, leading to a net loss on the position. The strategy should *gain* from decreases in the swap-Treasury spread, so a. is wrong. The strategy should *gain* from increases in the Treasury rate, all else equal, so b. is wrong. Finally, the distribution of the payoff depends on the distribution of the swap-Treasury spread. Because this cannot go below zero, there is a limit on the upside. The position has negative skewness, so c. is correct.

Example 30.6: Risks in Convertible Arbitrage

d. This position is hedged against interest rate risk, so b. is wrong. It is also hedged against directional movements in the stock, so c. is wrong. The position is long an option (the option to convert the bond into the stock) so is long implied volatility, so a. is wrong. Long options positions have positive gamma.

Example 30.7: Risks in Merger Arbitrage—I

c. The position is long one share of company B offset by a short position in two shares of company A. The payoff is $(\$120 - \$90) - 2(\$60 - \$50) = \$30 - \$20 = \$10$.

Example 30.8: Risks in Merger Arbitrage—II

b. The expected payoff is the sum of probabilities times the payoff in each state of the world, or $83\% \times \$5 + 17\% \times (-\$20) = \$4.15 - \$3.40 = \$0.75$. Note that the distribution is highly asymmetric, with a small probability of a large loss.

Example 30.9: FRM Exam 2005—Question 47

b. This fund has zero beta, so is a relative value fund. It is, however, exposed to idiosyncratic, stock-specific risk.

Example 30.10: Liquidity Risks

c. Asset liquidity risk is a function of the size of the position and the intrinsic liquidity of the instrument. Distressed securities trade much less than Treasury bonds, so have more liquidity risk. A $100 million position is more illiquid than a $10 million position in the same instrument.

Example 30.11: FRM Exam 2006—Question 112

d. Illiquidity creates an *understatement* of the total risk measure; as a result, the Sharpe ratio will be artificially higher. Illiquidity creates trends in returns (higher serial correlation), as market shocks during a month will be partially recorded in two consecutive months. Illiquidity, however, biases the market beta downward.

Example 30.12: FRM Exam 2007—Question 62

d. The weekly returns are not synchronized with those of the S&P. As a result, the estimate of beta from weekly data will be zero.

Example 30.13: Transparency

c. Risk aggregation is an important reason for investors to learn about the positions of their investments in hedge funds. Answer a. is incorrect because front-running the hedge fund would be a reason *not* to disclose position information. Answer b. is incorrect because misunderstanding the trading strategies would be a reason *not* to require position information. Answer d. is incorrect because they do not receive position information from the prime broker.

Example 30.14: FRM Exam 2009—Question 8-8

b. Statements I., III., and IV. are correct. Statement II. is false because illiquid assets create higher serial correlation.

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