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Mathematics and Its Applications

Ordinary Differential Equations with Applications to Mechanics

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CONTENTS

PREFACE

The present book has its source in the authors' wish to create a bridge between mathematics and the technical disciplines that need a good knowledge of a strong mathematical tool. The authors tried to reflect a common experience of the University of Bucharest, Faculty of Mathematics and of the Technical University of Civil Engineering of Bucharest.

The necessity of such an interdisciplinary work drove the authors to publish a first book with this aim ("Ecuații diferențiale cu aplicații în mecanica construcțiilor" – Ordinary differential equations with applications to the mechanics of constructions, Editura Tehnică, Bucharest, Romania).

The present book is a new edition of the volume published in 1999. Unfortunately, the first author (M.V. Soare) passed away shortly before the publication of the Romanian edition, so that the present work is only due to the other two authors. It contains many improvements concerning the theoretical (mathematical) information, as well as new topics, using enlarged and updated references.

We considered only ordinary differential equations and their solutions in an analytical frame, leaving aside their numerical approach.

Compared to the Romanian edition, this volume presents the applications in a new way. The *problem* is firstly stated in its mechanical frame. Then the *mathematical model* is set up, emphasizing on the one hand the physical magnitude playing the part of the unknown function and on the other hand the laws of mechanics that lead to an ordinary differential equation or system. The *solution* is then obtained by specifying the mathematical methods described in the corresponding theoretical presentation. Finally – last, but not least – a *mechanical interpretation* of the solution is provided, this giving rise to a complete knowledge of the studied phenomenon; after all, this is the main goal of any scientific approach. In most of cases, the solution is interpreted by using a parametrical study, which better emphasizes the core of the phenomenon. Sometimes, we pointed out the influence of a certain parameter or presented auxiliary diagrams and tables, whence, by interpolation, one can immediately get effective numerical values of the solution.

The number of the applications was increased; in order to keep the volume within a reasonable number of pages and also, not to exaggerate the interference between mathematics and engineering, we did not exhaustively introduce and present the mathematical model. It must be pointed out that many of these problems currently appear in engineering.

The book is organized in seven chapters. Each of them begins with a theoretical presentation, which insists on the practical computation – the "know-how" of the mathematical method – and ends with a rich range of applications. Unlike the standard presentations, we introduced separately the linear case, which is exposed in the first three chapters. The reason of this is that in the linear case one can use not only general methods, fitted for any differential equation, but also specific methods. The non-linear case forms the object of the next two chapters. The sixth chapter treats problems in a variational frame. Finally, the last chapter is devoted to an initiation in the modern domain of stability.

It should be mentioned that the book contains some personal results of the authors, published in scientific reviews of wide circulation.

The prerequisites of this book are courses of elementary analysis and algebra, acquired by a student in a technical university. It is addressed to a large audience, to all those interested in using mathematical models and methods in various fields, like: mechanics, civil and mechanical engineering, people involved in teaching or design as well as students.

P.P.TEODORESCU and ILEANA TOMA

INTRODUCTION

1. Generalities

The study of physical phenomena becomes consistent and applicable by establishing mathematical relationships between the involved physical quantities. Sometimes, these relationships are algebraic. But in most cases, algebra is not enough to characterize the phenomenon. The involved quantities may depend on other quantities, considered as independent variables, and the relationships are no more algebraic, containing both the unknown function and its derivatives. In the case of functions depending only on one variable, these are called *ordinary differential equations* (ODEs). If the unknown function depends on several variables, the equations will also contain its partial derivatives; such equations are called *partial differential equations* (PDEs).
In this book, only ODEs will be considered. Solving them is not only formally necessary, but also leads to physical interpretations in the frame of the considered phenomenon.
To emphasize the above considerations, let us take an example.

The parabolic mirror

Problem. Find the profile of an axially symmetric reflector (mirror), such that all the luminous radiations coming from a point-source O be reflected as a parallel beam, of given direction.

Solution. We choose O as origin for the system of co-ordinate axes and as Ox-axis – the direction of the parallel beam and we search the equation of the generating curve in the form

$$y = \varphi(x). \tag{1.1}$$

We admit that this unknown curve is contained in the xOy plane (Fig.1)

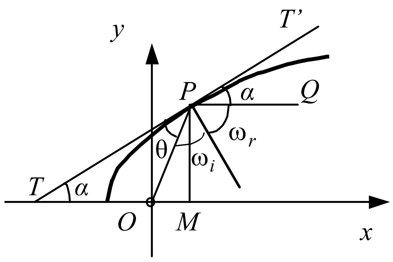

Figure 1. The parabolic mirror

Let $P(x, y)$ be a point on this curve. Draw the tangent TT' at P and consider a luminous beam OP, issued from O and reflected in PQ. By hypothesis, $PQ \parallel Ox$, so that the angles OTP and QPT' are both equal to α. As the incidence angle ω_i must equal the reflection angle ω_r, we see that the angles θ and α are equal. Looking at the triangle *POT*, we see that the angle *MOP* is 2α. From the right triangle *MOP* we deduce

$$\tan 2\alpha = \frac{y}{x} . \tag{1.2}$$

On the other hand,

$$\tan \alpha = y' = \frac{\mathrm{d}y}{\mathrm{d}x} . \tag{1.3}$$

From (1.2) and (1.3), using the tangent of a double arc, it follows

$$\frac{y}{x} = \frac{2y'}{1 - y'^2} . \tag{1.4}$$

Equation (1.4) is an *ODE*, representing precisely the mathematical model for the curve $y = \varphi(x)$.
To get the form of φ, one must find the solutions of this equation.
Let us leave aside – for the moment – the physical phenomenon and provide these solutions in a mathematical frame. This is by no means an easy task; we shall use the idea of differentiating with respect to *y*. Step by step, we thus get

$$\begin{gathered} 2x = y\left(\frac{1}{y'} - y'\right), \\ 2\frac{\mathrm{d}x}{\mathrm{d}y} = y\frac{\mathrm{d}y'}{\mathrm{d}y}\left(-\frac{1}{y'^2} - 1\right) + \frac{1}{y'} - y' , \end{gathered} \tag{1.5}$$

or else

$$\frac{2}{y'} = \frac{1}{y'} - y' - y\frac{\mathrm{d}y'}{\mathrm{d}y}\left(\frac{1}{y'^2} + 1\right); \tag{1.6}$$

after canceling and simplifying by $1 + y'^2$, $y' \neq 0$, $y \neq 0$, it is obtained

$$\frac{\mathrm{d}y'}{y'} = -\frac{\mathrm{d}y}{y} . \tag{1.7}$$

But equation (1.7) is equivalent to

$$\ln|y'| = -\ln|y| + \ln C' , \tag{1.8}$$

where $C' > 0$ is an arbitrary constant. It then follows

$$y' = \frac{C}{y}, \qquad C = \pm C', \tag{1.9}$$

and finally

$$\frac{y^2}{2} = Cx + K, \tag{1.10}$$

K being a new arbitrary constant.
Note that in this case there were obtained two arbitrary constants only because of the differentiation with respect to y; one of them may be determined from the other. Indeed, at the point of intersection of the curve with the Oy-axis one has $x = 0$, therefore, by the first equation (1.5) $\mathrm{d}y / \mathrm{d}x = 1$ and so $\alpha = 45^0$; it follows that $K = C^2 / 2$. Thus, the final form of the solution is

$$y^2 = 2Cx + C^2, \tag{1.11}$$

that is a family of parabolae of axis Ox, of common focal point O and focal distance $C / 2$.
Conclusion. The internal surface of the silvered mirror is a paraboloid of revolution.
This simple example emphasizes the necessity of an organized study of the ODEs in an appropriate mathematical frame.

2. Ordinary Differential Equations

An *ODE* is defined by an equality of the form

$$F\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right) = 0, \tag{2.1}$$

where the unknown function y also appears through its derivatives $y^{(i)}, i = \overline{1, n}$. The variable x is also called *independent variable*.
It is considered that it belongs to a real interval I, on which the function y is defined; this last one is supposed of class $\mathrm{C}^n(I)$, meaning that y is continuous on I, together with its derivatives up to n–th order inclusive. The function F is supposedly defined on the Cartesian product $I \times \Omega$, where $\Omega \subseteq \Re^{n+1}$ is such that $I \times \Omega$ be compact in the space of co-ordinates $\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right)$. In most of the standard applications, F is continuous in its arguments. The maximum order of differentiation of the unknown function is called the *order* of the differential equation. For instance, the equation (2.1) is of order n.
Under the previous conditions, the equation (2.1) may be developed with respect to $y^{(n)}$ to give

$$y^{(n)} = f\left(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\right); \tag{2.2}$$

we call this form *normal*.

A *particular solution* of (2.1) is a function of class $C^n(I)$ that satisfies (2.1) for any $x \in I$.

The *general solution* (or *general integral*) of (2.1) is a function

$$y = y(x, C_1, C_2, \ldots, C_n) \tag{2.3}$$

of class $C^n(I)$, depending on *n* arbitrary constants $C_1, C_2, \ldots, C_n$, corresponding to the order of equation, and satisfying (2.1) on *I*, for any set of admissible constants.

Thus, in the previous example, the function (1.11) is the general solution of equation (1.4).

The particular solutions of a differential equation are obtained from the general one by giving particular values to the constants $C_1, C_2, \ldots, C_n$. The solutions that cannot be obtained in this way are called *singular*.

If we represent the general solution (2.3) in a system of rectangular axes xOy, we shall obtain a family of plane curves – the parabolae (1.11) in the case of the previous example. This justifies the denomination of *integral curve* for any particular solution of (1.4).

A very important class of ODEs is the class of linear differential equations. In order to make things clear, denote by

$$-b(x) = F(x, 0, 0, \ldots, 0).$$

Then

$$F\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right) = G\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right) - b(x).$$

We call the *n*-th order differential equation (1.12) *linear* if its left member satisfies

$$\begin{aligned} &G\left(x, \alpha y + \beta z, \alpha y' + \beta z', \ldots, \alpha y^{(n)} + \beta z^{(n)}\right) = \\ &\quad = \alpha G\left(x, y, y', \ldots, y^{(n)}\right) + \beta G\left(x, z, z', \ldots, z^{(n)}\right), \end{aligned} \tag{2.4}$$

for any $\alpha, \beta \in \Re$ and any $y, z \in C^n(I)$.

If *G* has an analytic expression and is linear, then necessarily

$$G\left(x, y(x), y'(x), \ldots, y^{(n)}(x)\right) = \sum_{i=0}^{n} a_i(x) y^{(i)}, \tag{2.5}$$

where $a_i(x)$ are real functions, defined on *I*. So, *G* is a homogeneous first degree polynomial with respect to the unknown function *y* and its derivatives. Consequently, a linear *n*-th order ODE has the general form

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \ldots + a_{n-1}(x) y' + a_n(x) y = b(x), \quad b : I \to \Re, \tag{2.6}$$

If, in particular, $b(x) = 0,\ x \in I$, then (2.6) is called *homogeneous*.

Remark. While the linear ODE of order *n* could be directly defined by (2.6), we preferred to express the linearity in the form (2.5) from various reasons. First of all, the definition (2.5) of the linearity is extremely useful in applications and mostly effective to establish the general representation of the solutions. Secondly, from (2.4) it is immediately seen that the linearity of an ODE means linearity with respect to the unknown function and not at all with respect to the independent variable *x*, the last confusion being a common error.

Obviously, the differential equation (1.4) is non-linear, as it does not satisfy condition (2.6). Let us note, in particular, that we used an artifice to solve this equation; it could not be easily solved by using a standard method.

Actually, the non-linear case has not the advantage of a general method, leading to satisfactory representations of the solution. Unlike this, in the linear case there were found general techniques effectively leading to the solutions, in many cases expressed in closed form.

This is why, in the present book, we decided to treat separately the case of linear ODEs, starting with the first order ones.

3. Supplementary Conditions Associated to ODEs

We saw that the general integral of an ODE does not represent a well defined integral curve. For instance, (1.11) represents a family of parabolae. This means that the solution of a differential equation is not *unique*. As the classical physical phenomena are deterministic, this means that some supplementary conditions must be added to the equation such that the whole problem should allow a unique solution. Such conditions are naturally imposed in the process of modelling itself. More precisely, the mathematical model must be *well posed* in the sense of Hadamard. This means that its solution

a) must exists, in a certain class of function C_1;

b) must be unique, in a certain class of functions C_2;

c) must be continuous with respect to the given data.

Again, according to the involved phenomenon, we may distinguish two standard types of such conditions:

- the Cauchy (or initial) conditions;
- the boundary conditions.

3.1 THE CAUCHY (INITIAL) PROBLEM

Consider, for the moment, the ODE of first order

$$F(x, y(x), y'(x)) = 0, \quad x \in I, \tag{3.1}$$

whose general solution is

$$y = \varphi(x, C), \quad x \in I. \tag{3.2}$$

In other words, every particular solution of (3.1) may be found among the curves of the family (3.2), in which we take *C* as a parameter. A possible choice would be to get the curve passing through a certain point $(x_0, y_0) \in I \times \Omega$, therefore, for which

$$y(x_0) = y_0\,; \tag{3.3}$$

this yields

$$\varphi(x_0, C) = y_0\,. \tag{3.4}$$

From (3.4), we get *C*.
The condition (3.3) is a Cauchy (initial) condition associated to (3.1).
The equation (3.1) and the Cauchy condition (3.3)

$$\begin{cases} F(x, y(x), y'(x)) = 0, & x \in I, \\ y(x_0) = y_0, \end{cases} \tag{3.5}$$

form together a Cauchy (initial) problem.
If the problem is well posed (i.e., the solution exists and is unique), then the functional equation (3.4) allows only one solution $C = C(x_{0,}\, y_0)$ and the unique integral curve, satisfying (3.5), is

$$y = y(x, C(x_0, y_0)). \tag{3.6}$$

Suppose now that we deal with a second order ODE. A common physical phenomenon leading to such equations is e.g. the motion of a particle. Let us give an example.

Problem. Study the free fall of a body of mass *m*.

Mathematical model. We must firstly set up a mathematical model for this phenomenon. To do this, we must observe two steps:
1) establish the physical quantity/quantities representing the unknown function, whose knowledge should give us an exact and complete idea of the phenomenon evolution
2) find the physical law/laws governing the considered phenomenon.
In the case of a free fall, the body, modelled as a particle (material point), moves on a vertical to the earth. To know the motion is therefore to know at every moment *t* the distance *y* from the impact point. Thus, the unknown function will be the displacement $y = y(t)$ along the vertical; this is a real function, of one independent variable: the time *t*. As for the law of mechanics governing the free fall, we can obviously use Newton's law

$$m\mathbf{a} = \mathbf{F}\,,$$

where **a** is the acceleration and **F** is the resultant of the forces acting upon the body. According to the problem, we only deal with the force of gravity **G**, expressed as

$$\mathbf{G} = -m\mathbf{g}\,,$$

g being the gravity acceleration.
The sign minus is meant to indicate that **G** acts downwards, unlike *y*, which is upwards directed.

Note that all the involved vectors have only one component-dimensional, as the motion evolves along one direction: the vertical. Consequently, the forces acting upon the body are expressed in the form $-mg$, with $g = 9.81$ m/s^2.

The velocity of the body – also one-dimensional – will be expressed as the derivative of the displacement y with respect to the independent variable t

$$\frac{\mathrm{d}y}{\mathrm{d}t} \equiv \dot{y} .$$

The acceleration will be the first derivative of the velocity with respect to t, and therefore the second derivative of the displacement

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} \equiv \ddot{y} .$$

In the above expressions, we used the dot for the derivative with respect to the time, as it is a standard notation in mechanics.
Introducing this in Newton's law, we get

$$m\ddot{y} = -mg .$$

After simplifying with m, we finally obtain

$$\ddot{y} = -g , \tag{3.7}$$

which represents the mathematical model for the free fall.

Solution. This is a second order ODE.As g is a constant, we can immediately integrate once both sides, to get

$$\dot{y} = -gt + C_1 , \tag{3.8}$$

where C_1 is an arbitrary constant. It is possible to integrate once more and we obtain

$$y = -g\frac{t^2}{2} + C_1 t + C_2 , \tag{3.9}$$

where C_2 is a new arbitrary constant.

According to the previously defined notions, (3.9) is precisely the general solution of (3.7) and it is seen that it depends on two arbitrary constants. So, clearly, we need two supplementary conditions in order to specify these constants. In this case, it is natural to define more accurately

- the position of the body at the beginning of the motion (initial position)
- its velocity at the same moment (initial velocity).

If the motion starts at the moment $t = 0$, then these conditions read

$$\begin{cases} y(0) = y_0 & \text{– the initial position,} \\ \dot{y}(0) = v_0 & \text{– the initial velocity,} \end{cases} \tag{3.10}$$

with y_0, v_0 previously given. The conditions (3.10) are called *Cauchy* or *initial conditions*.
If we now make $t = 0$ in (3.8), we get

$$\dot{y}(0) = C_1 \quad \Rightarrow \quad C_1 = v_0 . \tag{3.11}$$

Taking $t = 0$ in (3.9) yields

$$y(0) = C_2 \quad \Rightarrow \quad C_2 = y_0 , \tag{3.12}$$

therefore both constants are perfectly determined from the supplementary conditions (3.10).
The problem formed by equation (3.7) and the initial conditions (3.10)

$$\begin{cases} \ddot{y} = -g, \\ y(0) = y_0, \\ \dot{y}(0) = v_0, \end{cases} \tag{3.13}$$

is a *Cauchy* or *initial problem*.
In general, the motion problems may be modelled by using Newton's law. If we consider the case of a single particle, the unknown function will be its displacement, say, $x = x(t)$. As in the previous example, the particle velocity will be $\dot{x} = \mathrm{d}x / \mathrm{d}t$ and the particle acceleration, $\ddot{x} = \mathrm{d}^2 x / \mathrm{d}t^2$. As for the resultant of the forces acting upon the particle, we usually find expressions depending on x and $\dot{x}$. Thus, the unidimensional equation of motion of a particle is usually expressed in the form

$$m\ddot{x} = F(t, x, \dot{x}). \tag{3.14}$$

The initial conditions read now

$$\begin{cases} x(0) = x_0, \\ \dot{x}(0) = \dot{x}_0. \end{cases} \tag{3.15}$$

The equation (3.14) together with the conditions (3.15) form a *Cauchy* or *initial problem*.
We observe that a first order ODE requires one Cauchy condition, while a second order – two such conditions.
In the general case, to the equation (2.1) we associate *n* Cauchy conditions

$$y(x_0) = y_{10}, \quad y^{(i)}(x_0) = y_{i+1,0}, \quad i = \overline{1, n-1}, \quad x_0 \in I , \tag{3.16}$$

where $y_{i0}, i = \overline{1, n}$ are previously defined constants, usually known as *Cauchy* or *initial data*. It is important to note that, in this case, all the involved conditions are given at the same point x_0. Obviously, the point $(x_0, y_{10}, \ldots, y_{n0})$ must belong to the domain of definition of *F*.
If the general solution (or integral) of equation (2.1)

$$y = y(x, C_1, C_2, \ldots, C_n) \tag{3.17}$$

is known, then the Cauchy problem (2.1), (3.16) is reduced to get the constants $C_i, i=\overline{1,n}$ from the algebraic (or functional) system of *n* relationships and *n* unknowns

$$\begin{cases} y(x_0, C_1, C_2, \ldots, C_n) = y_{10}, \\ y'(x_0, C_1, C_2, \ldots, C_n) = y_{20}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ y^{(n-1)}(x_0, C_1, C_2, \ldots, C_n) = y_{n0.} \end{cases} \tag{3.18}$$

Under certain conditions, ensuring the existence and uniqueness of the solution of the Cauchy problem (2.1), (3.16), this system allows a unique solution.

3.2 THE TWO-POINT PROBLEM

Like the Cauchy problem, this kind of problem has it source in the modelling of the physical phenomena.

A classic example is the simply supported bar. Let us study the deflection of the bar axis $y(x)$ with respect to its rest position Ox, when the bar is acted upon by some known forces or loadings.

The unknown function $y(x)$ satisfies, under certain physical hypotheses, the Bernoulli – Euler equation

$$y'' = f(x)\left(1+y'^2\right)^{\frac{3}{2}}, \tag{3.19}$$

where *f* depends on the bending moment and on the bar rigidity. This is a second order non-linear ODE, in normal form. From the physical point of view, a bar is simply supported if the bar deflection is null at the bar ends *a* and *b*, say, laying on the Ox - axis. Translated in mathematical terms, this reads

$$y(a) = 0, \quad y(b) = 0\,. \tag{3.20}$$

These are no more Cauchy conditions, because the unknown function must be known at two different points: *a* and *b*.

The simply supported bar problem (3.19), (3.20) is therefore a *two-point* (or bilocal) *problem*.

The two-point conditions may be generalized for the *n*-th order equation (2.1)

$$\sum_{j=0}^{n-1}\left[\alpha_{ij}\, y^{(j)}(a) + \beta_{ij}\, y^{(j)}(b)\right] = \gamma_i, \quad i=\overline{0,n}, \tag{3.21}$$

where $y^{(0)} = y$, $\alpha_{ij}, \beta_{ij}, \gamma_i, i, j = \overline{0, n-1}$ are given constants and $a, b \in I$.

Let us note that, while for the Cauchy problem (2.1), (3.16) we have the benefit of appropriate theorems, ensuring the existence and uniqueness of the solution under sufficiently large hypotheses, for the two-point problem such general theorems are no more available, even for linear equations. This is why we shall not try to find convenient hypotheses ensuring the existence and uniqueness of the solution of a two-point problem; they will be specified on particular cases, when applied.

Another important generalization of the two-point problem is the *polylocal* (or *n-point*) *problem*, which consists in getting those solutions of (2.1) that also take given values at n different points $a_i \in I, i = \overline{1,n}$

$$y(a_i) = y_i, \quad i = \overline{1,n}, \quad a_j < a_k, \quad j,k = \overline{1,n}. \tag{3.22}$$

To find convenient theorems of existence and uniqueness of the solution of the polylocal problem (2.1), (21) is not an easy task. Yet, the polylocal conditions may be considered, in a certain sense, a generalization of the Cauchy conditions (3.16). Indeed, if the points $a_i, i = \overline{2,n}$, are getting closer to a_{i-1}, then the ratios $\dfrac{y(a_i) - y(a_{i-1})}{a_i - a_{i-1}}$ tend to the derivative of y at a_{i-1}. All the involved constants being previously known, it follows that the limit $y'(a_i), i = \overline{1,n-1}$ is also known. Further, the points $a_i, i = \overline{2,n-1}$ are again moving to the left; this yields $y''(a_i), i = \overline{1,n-2}$. After $n-1$ such steps, we know all the values $y^{(i)}(a_1), i = \overline{0,n-1}$.

This interpretation is intuitive and might be somewhat formal, but it serves as a foundation for some general considerations in the study of polylocal problems. As we do not consider here applications involving polylocal problems, we shall not treat such problems in detail.

Chapter 1

LINEAR ODEs OF FIRST AND SECOND ORDER

1. Linear First Order ODEs

As it was already specified in the introduction, the general form of such equations is

$$y' + p(x)y = f(x), \tag{1.1.1}$$

where *f* and *g* are functions defined and supposed continuous on the real interval *I*. The function $f(x)$ is usually called the *free term*.
We shall study this equation starting from the most simple up to the most general case, which is (1.1.1).

1.1 EQUATIONS OF THE FORM $y' = f(x)$

This is the simplest form of (1.1.1). The solutions of this equation may be obviously regarded as primitives of *f*. Consequently, its general solution (integral) is

$$y(x) = \int f(x)dx + C\,, \tag{1.1.2}$$

where $\int f(x)dx$ is one of the primitives of f and C is an arbitrary constant. The representation (1.1.2) is obviously obtained by integrating both members of $y' = f(x)$.
If we wish to get the solution passing through the point (x_0, y_0), where $x_0 \in I$, then it is convenient to choose $\int_{x_0}^{x} f(\xi)\mathrm{d}\xi$ among the primitives of *f*. Indeed, with this choice, the solution passes through (x_0, y_0) if

$$C + \int_{x_0}^{x_0} f(\xi)\mathrm{d}\xi = y_0, \tag{1.1.3}$$

therefore if $C = y_0$. This yields

$$y(x) = \int_{x_0}^{x} f(\xi)\mathrm{d}\xi + y_0. \tag{1.1.4}$$

1.2 THE LINEAR HOMOGENEOUS EQUATION

This equation is also a particular case of (1.1.1), where the free term is identically null, that is

$$y' + p(x)y = 0 . \tag{1.1.5}$$

Dividing by y both terms of this equation, we immediately get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln|y|\right) = -p(x) . \tag{1.1.6}$$

This means that $\ln|y|$ satisfies an equation of the previously considered type. Thus, the general solution of (1.1.6) is, by using directly (1.1.2),

$$\ln|y| = \widetilde{C} - \int p(x)\mathrm{d}x , \tag{1.1.7}$$

where $\widetilde{C}$ is an arbitrary constant and $\int p(x)\mathrm{d}x$ – one of the primitives of p. From (1.1.7) we see that y is the general solution of (1.1.5) and is expressed by

$$y(x) = C\mathrm{e}^{-\int p(x)\mathrm{d}x} , \tag{1.1.8}$$

with C arbitrary constant.
As previously, to get a particular solution, passing through the point (x_0, y_0), we shall choose $-\int_{x_0}^{x} p(\xi)\mathrm{d}\xi$ among the primitives of p. Then (1.1.8) immediately yields $C = y_0$. Consequently, the solution passing through (x_0, y_0) is given by

$$y(x) = y_0\mathrm{e}^{-\int_{x_0}^{x} p(\xi)\mathrm{d}\xi} . \tag{1.1.9}$$

1.3 THE GENERAL CASE

Let us get back to the equation (1.1.1), in which the functions f and p, defined on $I \subseteq \Re$, are not identically null. Suppose that we know a particular solution of (1.1.1), $Y(x)$ say, and let us perform the change of function

$$y(x) = z(x) + Y(x) . \tag{1.1.10}$$

Introducing this in (1.1.1) immediately involves

$$z' + p(x)z + Y' + p(x)Y = f(x); \tag{1.1.11}$$

thus, z satisfies the homogeneous equation

$$z' + p(x)z = 0\,, \tag{1.1.12}$$

which was studied at Sec.1.2 and whose general solution is

$$z(x) = C\mathrm{e}^{-\int p(x)\mathrm{d}x}. \tag{1.1.13}$$

Getting back to (1.1.10), we see that the general solution of (1.1.1) may be expressed in the form

$$y(x) = C\mathrm{e}^{-\int p(x)\mathrm{d}x} + Y(x), \tag{1.1.14}$$

where $Y(x)$ is a particular solution of the non-homogeneous equation (1.1.1). This form is very important, as it is characteristic for linear ODEs in general; we shall discuss it further.

1.4 THE METHOD OF VARIATION OF PARAMETERS (LAGRANGE'S METHOD)

Except for $Y(x)$, formula (1.1.14) refers only to the coefficients of (1.1.1). Lagrange remarked that $Y(x)$ can be obtained in terms of these coefficients if we search it under the form

$$Y(x) = C(x)\mathrm{e}^{-\int p(x)\mathrm{d}x}, \tag{1.1.15}$$

that is, shaping it according to the general solution of the associated to (1.1.1) homogeneous equation. Introducing this in (1.1.1) yields

$$C'(x)\mathrm{e}^{-\int p(x)\mathrm{d}x} - p(x)C(x)\mathrm{e}^{-\int p(x)\mathrm{d}x} + p(x)C(x)\mathrm{e}^{-\int p(x)\mathrm{d}x} = f(x), \tag{1.1.16}$$

from which we deduce that $C(x)$ must satisfy

$$C'(x)\mathrm{e}^{-\int p(x)\mathrm{d}x} = f(x), \tag{1.1.17}$$

which leads to

$$C'(x) = f(x)\mathrm{e}^{\int p(x)\mathrm{d}x}. \tag{1.1.18}$$

This is an equation considered at Sec.1.1. It follows that the general integral of (1.1.18) is written in the form

$$C(x) = K + \int f(x) e^{\int p(x)\mathrm{d}x} \mathrm{d}x . \tag{1.1.19}$$

In this expression, K is an arbitrary constant and the integral in the right member is a primitive of the function $f(x) e^{\int p(x)\mathrm{d}x}$. Actually, we don't need the general solution of (1.1.18) for our purpose; all we need is a particular solution, which can be found giving to K an arbitrarily chosen value, e.g. $K = 0$. With this, we get

$$Y(x) = e^{-\int p(x)\mathrm{d}x} \int f(x) e^{\int p(x)\mathrm{d}x} \mathrm{d}x . \tag{1.1.20}$$

We replace now this particular solution in (1.1.14). The final form of the general solution of the linear non-homogeneous equation (1.1.1) is thus

$$y(x) = e^{-\int p(x)\mathrm{d}x} \left(K + \int f(x) e^{\int p(x)\mathrm{d}x} \mathrm{d}x \right). \tag{1.1.21}$$

It is seen that this expression contains only primitives involving the coefficients of the equation.
To find the integral curve passing through a given point (x_0, y_0) we conveniently choose the primitives. The solution of this Cauchy problem will be

$$y(x) = e^{-\int_{x_0}^{x} p(\xi)\mathrm{d}\xi} \left(y_0 + \int_{x_0}^{x} f(\eta) e^{\int_{x_0}^{x} p(\xi)\mathrm{d}\xi} \mathrm{d}\eta \right). \tag{1.1.22}$$

From the above considerations, we point out the following two aspects, particularly important in the study of linear ODEs:

(i) The general integral $y(x)$ of the non-homogeneous equation (1.1.1) may be put under the form (1.1.10), i.e., a sum between a particular solution of (1.1.1) and the general solution of the associated to (1.1.1) homogeneous equation.

(ii) We succeeded to find a particular solution of the non-homogeneous equation shaping it in the form of the general solution of the associated homogeneous equation, in which the constant C was replaced by a function $C(x)$. This method is called *the method of variation of parameters* or *Lagrange's method*.

The representation (1.1.10), as well as Lagrange's method, are extremely important and useful tools for the study of linear ODEs and systems; they will be also used for higher order linear ODEs.

Let us think of the property of linearity in an algebraic frame. Denote by

$$\mathrm{L}y \equiv y' + p(x)y \tag{1.1.23}$$

the left member of (1.1.1). Actually, we can think of L as being a succession of functional operations executed on $\mathrm{C}^1(I)$ - class functions y.

Example. For $\mathrm{L}y \equiv y' - xy$, let us take $y_1 = x^2$. According to the operations indicated by the definition of L, we have $\mathrm{L}y_1 \equiv 2x - x \cdot x^2 = 2x - x^3$, therefore the result is a function. If, for instance we take $y_2 = \mathrm{e}^{2x}$, then $\mathrm{L}y_2 \equiv 2\mathrm{e}^{2x} - x \cdot \mathrm{e}^{2x} = (2-x)\mathrm{e}^{2x}$. Taking $y_3 = \mathrm{e}^{x^2/2}$, we get $\mathrm{L}y_3 \equiv x\mathrm{e}^{x^2/2} - x \cdot \mathrm{e}^{x^2/2} = 0$, therefore the null function.

We can say that L is an *operator*, as it realizes a function-to-function correspondence. Moreover, we say that it is defined on $\mathrm{C}^1(I)$, with the range in $\mathrm{C}^0(I)$.

In general, an operator $\mathrm{L}: \mathrm{Y} \to \mathrm{Z}$, where Y and Z are spaces of functions, is called *linear* if

$$\mathrm{L}(\alpha y_1 + \beta y_2) = \alpha \mathrm{L}y_1 + \beta \mathrm{L}y_2, \quad \forall \alpha, \beta \in \Re, \forall y_1, y_2 \in \mathrm{Y}. \tag{1.1.24}$$

With this definition, we can easily prove that the differential operator introduced in (1.1.23) is linear. Indeed, we have

$$\begin{aligned} \mathrm{L}(\alpha y_1 + \beta y_2) &= (\alpha y_1 + \beta y_2)' + p(x)(\alpha y_1 + \beta y_2) \\ &= \alpha[y_1' + p(x)y_1] + \beta[y_2' + p(x)y_2] \\ &= \alpha L y_1 + \beta L y_2. \end{aligned} \tag{1.1.25}$$

Let us get back to the general case. The *kernel* of an operator $\mathrm{L}: \mathrm{Y} \to \mathrm{Z}$ is a subset of Y, containing functions cancelled by L

$$\ker \mathrm{L} = \{ y \in \mathrm{Y} \mid \mathrm{L}y = 0 \}. \tag{1.1.26}$$

As Y is a linear vector space, $\ker \mathrm{L}$ will be a linear subspace of Y. Indeed, if $y_1, y_2 \in \ker \mathrm{L}$, then $\mathrm{L}(\alpha y_1 + \beta y_2) = \alpha \mathrm{L}y_1 + \beta \mathrm{L}y_2 = 0 \ \forall \alpha, \beta \in \Re$, therefore $\alpha y_1 + \beta y_2 \in \ker \mathrm{L}$. It is seen that finding solutions for the homogeneous ODE (1.1.5) means in fact to get $\ker \mathrm{L}$. From the form (1.1.8) of the general solution we deduce that the dimension of $\ker \mathrm{L}$ is 1, for first order ODEs. This is not a casualty; we shall see that the kernels of linear *n*-th order ODEs have the dimension *n*.

1.5 DIFFERENTIAL POLYNOMIALS

Let us denote by D the operator indicating the derivative of first order of a function

$$\mathrm{D} \equiv \frac{\mathrm{d}}{\mathrm{d}x} \tag{1.1.27}$$

and by E the identity

$$\mathrm{E}y = y \tag{1.1.28}$$

Then L may be also expressed as

$$\mathrm{L}y = \mathrm{P}_1(x, \mathrm{D})y, \quad \mathrm{P}_1(x, \mathrm{D}) \equiv \mathrm{D} + p(x)\mathrm{E}\,. \tag{1.1.29}$$

The operator defined in (1.1.29) is a formal polynomial of first order in D and it is called a *differential polynomial*.
Let now $\mathbf{y} = \lfloor y_j \rfloor_{j=\overline{1,n}}$, $\mathbf{f} = \lfloor f_j \rfloor_{j=\overline{1,n}}$ be vector functions and assume that we must solve the vector equation

$$\mathrm{L}\mathbf{y} \equiv \dot{\mathbf{y}} + p(x)\mathbf{y} = \mathbf{f}, \quad p \in \mathrm{C}^0(I), \mathbf{f} \in \left(\mathrm{C}^0(I)\right)^n. \tag{1.1.30}$$

Writing (1.1.30) componentwisely, this means, in fact, that one has to solve n uncoupled ODEs

$$\mathrm{L}y_j \equiv \dot{y}_j + p(x)y_j = f_j, \quad j = \overline{1,n}\,. \tag{1.1.31}$$

These first order ODEs are linear and non-homogeneous, therefore their general solution can be written, following formula (1.1.21)

$$y_j(x) = \mathrm{e}^{-\int p(x)\mathrm{d}x}\left(K_j + \int f_j(x)\mathrm{e}^{\int p(x)\mathrm{d}x}\mathrm{d}x\right), \tag{1.1.32}$$

or, in vector form

$$\mathbf{y}(x) = \mathrm{e}^{-\int p(x)\mathrm{d}x}\left(\mathbf{K} + \int \mathbf{f}(x)\mathrm{e}^{\int p(x)\mathrm{d}x}\mathrm{d}x\right), \quad \mathbf{K} = \left[K_j\right]_{j=\overline{1,n}}. \tag{1.1.33}$$

2. Linear Second Order ODEs

The general form of such equations is, according to the introduction (see e.g.(15))

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x)\,, \tag{1.2.1}$$

where a_0, a_1, a_2, b are real functions defined on a real interval $I \subseteq \mathfrak{R}$. We may consider these functions continuous on I.
If $a_0(x) \neq 0, \forall x \in I$, we can divide both members of (1.2.1) by it, thus getting an equation whose leading coefficient is 1

$$y'' + p(x)y' + q(x)y = f(x), \tag{1.2.2}$$

where we used the notations $p(x) = \dfrac{a_1(x)}{a_0(x)}$, $q(x) = \dfrac{a_2(x)}{a_0(x)}$, $f(x) = \dfrac{b(x)}{a_0(x)}$. Obviously, if the coefficients of (1.2.1) are of class $C^0(I)$, so are *p*, *q* and *f*.

We see that, if $a_0(x) = 0, \forall x \in I$, the equation is no more of second order, and, at the points at which $a_0(x) = 0$, it has singularities. For the moment, we shall not deal with such situations, such that we consider that the given equation may be brought to the form (1.2.2).

Let us denote by

$$\mathrm{L}y \equiv y'' + p(x)y' + q(x)y \ . \tag{1.2.3}$$

The operator L is defined on $C^2(I)$, with range in $C^0(I)$, and we can easily prove that it is linear.

The kernel of this operator is a subset of $C^2(I)$, containing functions cancelled by L

$$\ker \mathrm{L} = \{ y \in C^2(I) \,|\, \mathrm{L}y = 0 \}. \tag{1.2.4}$$

In other terms, $\ker \mathrm{L}$ is the set of all the solutions of the homogeneous ODE $y'' + p(x)y' + q(x)y = 0$.

As in the case of first order ODEs, by using the notations we can express L in terms of the second degree differential polynomial

$$\mathrm{L}y = \mathrm{P}_2(x, \mathrm{D})y, \quad \mathrm{P}_2(x, \mathrm{D}) \equiv \mathrm{D}^2 + p(x)\mathrm{D} + q(x)\mathrm{E} \tag{1.2.5}$$

In (1.2.5), the formal power D^2 means to apply twice the operator D, in other words, to differentiate twice

$$\mathrm{D}^2 y = \left(\frac{\mathrm{d}}{\mathrm{d}x} \right)^2 y = \frac{\mathrm{d}^2}{\mathrm{d}x^2} y = y''. \tag{1.2.6}$$

2.1 HOMOGENEOUS EQUATIONS

Let us take the associated to (1.2.1) homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \ . \tag{1.2.7}$$

If we know a particular solution of this equation, say $Y(x)$, we can completely solve (1.2.7). Indeed, let us perform the change of function

$$y(x) = z(x)Y(x), \tag{1.2.8}$$

$z(x)$ being the new unknown function. Replacing this in (1.2.7), we get

$$a_0(x)Yz'' + [2a_0(x)Y' + a_1 Y]z' + [a_0(x)Y'' + a_1(x)Y' + a_2(x)Y]z = 0 . \tag{1.2.9}$$

As Y is a solution of (1.2.7), it follows that $u = z'$ must satisfy

$$a_0(x)Yu' + [2a_0(x)Y' + a_1 Y]u = 0 ; \tag{1.2.10}$$

this is a linear first order ODE.
We conclude that if we know a particular solution, we can reduce the order of the given equation by one unit.
Suppose now that $Y_1(x)$ is a known particular solution of the homogeneous equation, associated to (1.2.2)

$$y'' + p(x)y' + q(x)y = 0 \tag{1.2.11}$$

and suppose moreover that Y_1 does not vanish on I. Using the change of function $y = Y_1 z$, we find that $u = z'$ must satisfy

$$u' + \left(2\frac{Y_1'(x)}{Y_1(x)} + p(x) \right) u = 0 , \tag{1.2.12}$$

i.e., a linear first order homogeneous ordinary differential equation. According to Sec.1.2, it allows the general integral

$$u(x) = C_1 \frac{e^{-\int p(x)\mathrm{d}x}}{Y_1^2(x)} , \tag{1.2.13}$$

where $\int p(x)\mathrm{d}x$ is a primitive of $p(x)$ and C_1 is an arbitrary constant. Getting back to y, we deduce

$$y(x) = C_1 Y_1(x) \int \frac{e^{-\int p(x)\mathrm{d}x}}{Y_1^2(x)} \mathrm{d}x . \tag{1.2.14}$$

The path we followed so far, as well as the linearity of the homogeneous equation, involve that any solution of (1.2.11) is a linear combination between the function $Y_1(x)$ and the function

$$Y_2(x) = Y_1(x) \int \frac{e^{-\int p(x)\mathrm{d}x}}{Y_1^2(x)} \mathrm{d}x . \tag{1.2.15}$$

The two particular solutions $Y_1(x), Y_2(x)$ are linearly independent, i.e.

$$k_1 Y_1(x) + k_2 Y_2(x) = 0, \forall x \in I \quad \Rightarrow \quad k_1 = 0, k_2 = 0 . \tag{1.2.16}$$

We can check this either directly, or, better, by introducing the *Wronskian*

$$W[Y_1, Y_2] \underset{def}{\equiv} \det\begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} = Y_1 Y_2' - Y_2 Y_1' . \tag{1.2.17}$$

We can prove that if Y_1, Y_2 are linearly dependent, then $W(Y_1, Y_2) \equiv 0$ on I, and if Y_1, Y_2 are linearly independent, then $W(Y_1, Y_2)(x) \neq 0, \forall x \in I$.

A *fundamental system* of solutions of (1.2.7), or, accordingly, of (1.2.11), is a pair of linearly independent particular solutions of (1.2.7) or (1.2.11), nonvanishing identically on I.

Following this definition, we can say that the above mentioned functions $Y_1(x), Y_2(x)$ form a fundamental system of solutions for the equation (1.2.11) and the general integral of this equation will be expressed in the form

$$y(x) = C_1 Y_1(x) + C_2 Y_2(x), \tag{1.2.18}$$

i.e., in the form of a linear combination of the functions of the fundamental system with arbitrary constant coefficients.

Otherwise speaking, in this case the dimension of $\ker \mathrm{L}$ is 2 and any fundamental system of solutions represent a basis for it.

We can choose the functions of the fundamental system such that, at a given point $x_0 \in I$, the following Cauchy conditions be satisfied

$$\begin{aligned} &Y_1(x_0) = 1, \quad Y_2(x_0) = 0, \\ &Y_1(x_0) = 0, \quad Y_2(x_0) = 1. \end{aligned} \tag{1.2.19}$$

The corresponding system of solution will be called in this case *normal*; it is a fundamental system for the equation (1.2.11). Indeed, suppose that

$$k_1 Y_1(x) + k_2 Y_2(x) = 0, \forall x \in I.$$

As $Y_1(x), Y_2(x)$ are differentiable on I, we can also write

$$k_1 Y_1'(x) + k_2 Y_2'(x) = 0, \forall x \in I.$$

These two relationships may be written at any point of I, therefore also at $x_0 \in I$,

$$\begin{aligned} &k_1 Y_1(x_0) + k_2 Y_2(x_0) = 0, \\ &k_1 Y_1'(x_0) + k_2 Y_2'(x_0) = 0, \end{aligned} \tag{1.2.20}$$

and, taking (1.2.19) into account, it results that $k_1 = 0, k_2 = 0$.

This might be more effectively proved by using the Wronskian. Indeed,

$$W[Y_1, Y_2](x_0) = \begin{vmatrix} Y_1(x_0) & Y_2(x_0) \\ Y_1'(x_0) & Y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 , \tag{1.2.21}$$

which means that $Y_1(x), Y_2(x)$ are linearly independent on I.

The Wronskian also has a special property, very useful in practice.
Let us differentiate it, by using the well-known rules of differentiating determinants. We have

$$\begin{aligned}\frac{\mathrm{d}W[Y_1,Y_2]}{\mathrm{d}x}&=\begin{vmatrix}Y_1 & Y_2\\ Y_1'' & Y_2''\end{vmatrix}=-\begin{vmatrix}Y_1 & Y_2\\ pY_1'+qY_1 & pY_2'+qY_2\end{vmatrix}=-p\begin{vmatrix}Y_1 & Y_2\\ Y_1' & Y_2'\end{vmatrix}\\&=-pW[Y_1,Y_2].\end{aligned}\tag{1.2.22}$$

This means that the Wronskian satisfies the first order linear homogeneous ODE

$$\frac{\mathrm{d}W}{\mathrm{d}x}+pW=0.\tag{1.2.23}$$

According to Sec.1.2, the general solution of this equation is

$$W[Y_1,Y_2]=C\mathrm{e}^{-\int p(x)\mathrm{d}x},\tag{1.2.24}$$

where C is an arbitrary constant.
If we know the value of W at a point $x_0\in I$, then (1.2.24) may be also written in the form

$$W(x)=W(x_0)\mathrm{e}^{-\int_{x_0}^{x}p(t)\mathrm{d}t}.\tag{1.2.25}$$

From this, it follows that, if the Wronskian vanishes at some point $x_0\in I$, then it vanishes identically.
Formula (1.2.24), or, equivalently, (1.2.25), is known as *Liouville's formula*.

2.2 NON-HOMOGENEOUS EQUATIONS. LAGRANGE'S METHOD

To solve the non-homogeneous equation (1.2.2), we shall use again the previous ideas, exposed for first order ODEs.
Suppose that we know a particular solution of (1.2.2), say $Y(x)$. Let us perform the change of function $y=z+Y$, where z is the new unknown function. Introducing this in (1.2.2), we get for z

$$z''+p(x)z'+q(x)z=0,\tag{1.2.26}$$

which is precisely the associated to (1.2.2) homogeneous equation. Therefore, the general solution of (1.2.2) is the sum between one of its particular solutions and the general solution of the associated homgeneous equation, exactly as in the case of first order equations.
If we also know a fundamental system $Y_1(x), Y_2(x)$ for (1.2.26), we can write the general solution of this equation in the form of the linear combination

$$z(x) = C_1 Y_1(x) + C_2 Y_2(x). \tag{1.2.27}$$

Thus, the general solution of the non-homogeneous equation (1.2.2) is

$$y(x) = C_1 Y_1(x) + C_2 Y_2(x) + Y(x). \tag{1.2.28}$$

To write this, we must therefore know three functions: Y, Y_1, Y_2.
But from the above considerations it follows that we need to know *only one particular solution of the homogeneous equation* (1.2.26), say $Y_1(x)$. Indeed, in this case we immediately get another particular solution, $Y_2(x)$, linearly independent on $Y_1(x)$, as it was previously shown. The functions $Y_1(x), Y_2(x)$ form a fundamental system of solutions for (1.2.26).
According to Lagrange's idea, we can search now for a particular solution of (1.2.2) under the form

$$Y(x) = C_1(x) Y_1(x) + C_2(x) Y_2(x). \tag{1.2.29}$$

Differentiating this once, it is obtained

$$Y'(x) = C_1' Y_1(x) + C_2' Y_2(x) + C_1 Y_1'(x) + C_2 Y_2'(x). \tag{1.2.30}$$

We can choose C_1, C_2 such that

$$C_1' Y_1(x) + C_2' Y_2(x) = 0, \tag{1.2.31}$$

therefore

$$Y'(x) = C_1 Y_1'(x) + C_2 Y_2'(x). \tag{1.2.32}$$

We differentiate this once more

$$Y''(x) = C_1' Y_1'(x) + C_2' Y_2'(x) + C_1 Y_1''(x) + C_2 Y_2''(x). \tag{1.2.33}$$

To retrieve the non-homogeneous equation (1.1.24) we shall multiply (1.2.32) by $p(x)$, (1.2.29) by $q(x)$ and then add them to (1.2.33). We obtain

$$\mathrm{L}Y(x) = C_1' Y_1'(x) + C_2' Y_2'(x) + C_1 \mathrm{L}Y_1(x) + C_2 \mathrm{L}Y_2(x) \tag{1.2.34}$$

or

$$C_1' Y_1'(x) + C_2' Y_2'(x) = \mathrm{L}Y(x). \tag{1.2.35}$$

But $\mathrm{L}Y(x) = f(x)$ and so we get for C_1', C_2' a linear algebraic system

$$\begin{cases} C_1' Y_1(x) + C_2' Y_2(x) = 0, \\ C_1' Y_1'(x) + C_2' Y_2'(x) = f(x). \end{cases} \tag{1.2.36}$$

The associated determinant is precisely the Wronskian of Y_1, Y_2, therefore it does not vanish all over *I*. Solving (1.2.36), we find for C_1', C_2'

$$\begin{cases} C_1' = -\dfrac{Y_2(x)f(x)}{W(x)}, \\ C_2' = \dfrac{Y_1(x)f(x)}{W(x)}, \end{cases} \tag{1.2.37}$$

that, integrated once, yield

$$\begin{cases} C_1 = -\displaystyle\int \dfrac{Y_2(x)f(x)}{W(x)}\,\mathrm{d}x, \\ C_2 = \displaystyle\int \dfrac{Y_1(x)f(x)}{W(x)}\,\mathrm{d}x. \end{cases} \tag{1.2.38}$$

We did not add arbitrary constants to the primitives C_1, C_2, because we only need a particular solution of (1.1.24). This particular solution is precisely

$$Y(x) = Y_2(x)\int \frac{Y_1(x)f(x)}{W(x)}\,\mathrm{d}x - Y_1(x)\int \frac{Y_2(x)f(x)}{W(x)}\,\mathrm{d}x\,. \tag{1.2.39}$$

The conclusion is that if we know $Y_1 \in \ker \mathrm{L}$, then the non-homogeneous ODE (1.1.24) is completely solved. Its general solution may be put in the form

$$y(x) = C_1 Y_1(x) + C_2 Y_2(x) + Y_2(x)\int \frac{Y_1(x)f(x)}{W(x)}\,\mathrm{d}x - Y_1(x)\int \frac{Y_2(x)f(x)}{W(x)}\,\mathrm{d}x, \tag{1.2.40}$$

where $Y_2(x)$ is expressed e.g. by (1.2.15).

Lagrange's idea of finding a particular solution *Y* for a non-homogeneous linear ODE may be applied in various ways. Thus, we can search for *Y* in the form

$$Y(x) = C_1(x)Y_1(x) + C_2(x)Y_2(x) + z(x), \tag{1.2.41}$$

or

$$Y(x) = [C_1(x)Y_1(x) + C_2(x)Y_2(x)]z(x), \tag{1.2.42}$$

if these forms present more advantages in computation.

Naturally, the algebraic system in C_1', C_2' will differ from (1.2.36). Thus, for instance, applying (1.2.41), we find, if $q(x) \neq 0,\ x \in I$

$$\begin{cases} C_1' Y_1(x) + C_2' Y_2(x) = -z', \\ C_1' Y_1'(x) + C_2' Y_2'(x) = 0, \\ bz = f. \end{cases} \tag{1.2.43}$$

The general integral (1.2.40) may be written in a compact form by introducing the function

$$k(x,t) = \pm \frac{Y_1(x)Y_2(t) - Y_2(x)Y_1(t)}{W(t)}, \tag{1.2.44}$$

where we take the sign + for $x < t$ and the sign – for $x > t$. We observe that this function has the following properties

(i) k is of class $C^1(I)$, for $x \neq t$;

(ii) $k(x,t)$ satisfies the homogeneous ODE (1.2.11), for $x \neq t, (x,t) \in I \times I$;

(iii) for any $s \in I$, the first derivative of k with respect to x has a jump equal to one unit

$$\frac{\partial k}{\partial x}(s+0,s) - \frac{\partial k}{\partial x}(s-0,s) = 1. \tag{1.2.45}$$

By definition, a function with the above properties is called a *fundamental solution* of the equation (1.2.11).

The fundamental solution is not unique. Yet, one can prove that the set of all the fundamental solutions of (1.2.11) is given by the function of the form

$$k(x,t) + C_1(t)Y_1(x) + C_2(t)Y_2(x), \tag{1.2.46}$$

with $k(x,t)$ defined by (1.2.44) and C_1, C_2 continuous with respect to t.

By means of the fundamental solution, we can express the general solution of the non-homogeneous equation (1.2.2) in the form

$$y(x) = k_1 Y_1(x) + k_2 Y_2(x) + \int_\alpha^\beta k(x,t) f(t) \mathrm{d}t, \tag{1.2.47}$$

α and β being the extremities of I.

The idea of fundamental solution may be also applied to first order ODEs and may be extended to PDEs, but this last exceeds the topics of the present book. The natural mathematical frame for the fundamental solutions is the *theory of distributions*.

Let now $\mathbf{y} = \lfloor y_j \rfloor_{j=\overline{1,n}}$, $\mathbf{f} = \lfloor f_j \rfloor_{j=\overline{1,n}}$ be vector functions and assume that we must solve the vector equation

$$\mathrm{L}\mathbf{y} \equiv \ddot{\mathbf{y}} + p(x)\dot{\mathbf{y}} + q(x)\mathbf{y} = \mathbf{f}, \quad p,q \in C^0(I), \mathbf{f} \in \left(C^0(I)\right)^n. \tag{1.2.48}$$

Writing (1.1.30) componentwisely, this means, in fact, that one has to solve n uncoupled ODEs

$$\mathrm{L}y_j \equiv \ddot{y}_j + p(x)\dot{y}_j + q(x)y_j = f_j, \quad j = \overline{1,n}. \tag{1.2.49}$$

These second order ODEs are linear and non-homogeneous, therefore, knowing the fundamental system of solutions Y_1, Y_2, their general solution can be written, following formula (1.2.40),

$$
\begin{aligned}
y_j(x) &= k_{j1}Y_1(x) + k_{j2}Y_2(x) \\
&+ Y_2(x)\int \frac{Y_1(x) f_j(x)}{W(x)}\,\mathrm{d}x - Y_1(x)\int \frac{Y_2(x) f_j(x)}{W(x)}\,\mathrm{d}x, \quad j = \overline{1,n}.
\end{aligned}
\tag{1.2.50}
$$

Eventually, the solution of the vector equation (1.2.48), written in vector form, is

$$
\begin{gathered}
\mathbf{y}(x) = \mathbf{k}_1 Y_1(x) + \mathbf{k}_2 Y_2(x) + Y_2(x)\int \frac{Y_1(x)\mathbf{f}(x)}{W(x)}\,\mathrm{d}x - Y_1(x)\int \frac{Y_2(x)\mathbf{f}(x)}{W(x)}\,\mathrm{d}x, \\
\mathbf{k}_1 = \left[k_{j1}\right]_{j=\overline{1,n}}, \quad \mathbf{k}_2 = \left[k_{j2}\right]_{j=\overline{1,n}}.
\end{gathered}
\tag{1.2.51}
$$

2.3 ODEs WITH CONSTANT COEFFICIENTS

In this case, we can always find easily a fundamental system of solutions for the given ODE.

Indeed, consider the second order homogeneous ODE with constant coefficients

$$
\mathrm{L}y \equiv a_0 y'' + a_1 y' + a_2 y = 0, \quad a_0, a_1, a_2 \in \Re,\ a_0 \neq 0 \tag{1.2.52}
$$

and, naturally, $I \equiv \Re$.

Euler's idea was to search for solutions in exponential form, i.e.

$$
y(x) = \mathrm{e}^{\alpha x} \tag{1.2.53}
$$

with α constant. Introducing this in (1.2.52) yields for α an algebraic equation

$$
a_0\alpha^2 + a_1\alpha + a_2 = 0\,, \tag{1.2.54}
$$

also called the *characteristic equation*. The second degree polynomial in the left member is the *characteristic polynomial*.

This equation allows two roots, α_1, α_2, say, that might be

i) real and distinct,

ii) complex-conjugate,

iii) double.

Let us analyse one by one the above mentioned cases.

i) There are two distinct solutions of the exponential form (1.2.53)

$$
Y_1(x) = \mathrm{e}^{\alpha_1 x}, \ Y_2(x) = \mathrm{e}^{\alpha_2 x}\,. \tag{1.2.55}
$$

We notice that Y_1, Y_2 are linearly independent. Indeed, their Wronskian

$$W[Y_1, Y_2] = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = \begin{vmatrix} \mathrm{e}^{\alpha_1 x} & \mathrm{e}^{\alpha_2 x} \\ \alpha_1 \mathrm{e}^{\alpha_1 x} & \alpha_2 \mathrm{e}^{\alpha_2 x} \end{vmatrix} = (\alpha_2 - \alpha_1)\mathrm{e}^{(\alpha_1 + \alpha_2)x}, \tag{1.2.56}$$

does not vanish, as the roots are distinct.

So, the functions $\mathrm{e}^{\alpha_1}, \mathrm{e}^{\alpha_2}$ form a fundamental system and the general solution may be written as

$$y(x) = C_1 \mathrm{e}^{\alpha_1 x} + C_2 \mathrm{e}^{\alpha_2 x}. \tag{1.2.57}$$

Let us get now a normal system, at some point $x_0 \in \Re$.

We must find the solutions Z_1, Z_2 of (1.2.52) satisfying the Cauchy conditions

$$\begin{aligned} &Z_1(x_0) = 1, \quad Z_2(x_0) = 0, \\ &Z_1'(x_0) = 0, \quad Z_2'(x_0) = 1. \end{aligned} \tag{1.2.58}$$

Inspired by (1.2.57), we express Z_1, Z_2 in the form

$$\begin{aligned} Z_1(x) &= c_1 \mathrm{e}^{\alpha_1 (x - x_0)} + c_2 \mathrm{e}^{\alpha_2 (x - x_0)}, \\ Z_2(x) &= d_1 \mathrm{e}^{\alpha_1 (x - x_0)} + d_2 \mathrm{e}^{\alpha_2 (x - x_0)}. \end{aligned} \tag{1.2.59}$$

Imposing now the conditions (1.2.58), we obtain for the coefficients c_1, c_2 the following linear algebraic system

$$\begin{aligned} &c_1 + c_2 = 1, \\ &\alpha_1 c_1 + \alpha_2 c_2 = 0, \end{aligned} \tag{1.2.60}$$

and for d_1, d_2 the system

$$\begin{aligned} &d_1 + d_2 = 0, \\ &\alpha_1 d_1 + \alpha_2 d_2 = 1. \end{aligned} \tag{1.2.61}$$

Both systems allow the unique solution

$$\begin{aligned} &c_1 = \frac{\alpha_2}{\alpha_2 - \alpha_1}, \quad c_2 = \frac{\alpha_1}{\alpha_2 - \alpha_1}, \\ &d_1 = -\frac{1}{\alpha_2 - \alpha_1}, \quad d_2 = \frac{1}{\alpha_2 - \alpha_1}. \end{aligned} \tag{1.2.62}$$

Introducing this in (1.2.59), we obtain

$$Z_1(x)=\frac{\alpha_2 e^{\alpha_1(x-x_0)}-\alpha_1 e^{\alpha_2(x-x_0)}}{\alpha_2-\alpha_1},$$
$$Z_2(x)=\frac{-e^{\alpha_1(x-x_0)}+e^{\alpha_2(x-x_0)}}{\alpha_2-\alpha_1}. \tag{1.2.63}$$

The general solution of (1.2.52) in terms of this normal system is

$$y(x)=C_1\frac{\alpha_2 e^{\alpha_1(x-x_0)}-\alpha_1 e^{\alpha_2(x-x_0)}}{\alpha_2-\alpha_1}+C_2\frac{-e^{\alpha_1(x-x_0)}+e^{\alpha_2(x-x_0)}}{\alpha_2-\alpha_1}. \tag{1.2.64}$$

If $\alpha_1=-\alpha_2=-1$, then $Z_1(x)=\cosh(x-x_0)$, $Z_2(x)=\sinh(x-x_0)$ and (1.2.64) becomes

$$y(x)=C_1\cosh(x-x_0)+C_2\sinh(x-x_0). \tag{1.2.65}$$

ii) In this case, the two roots are complex-conjugate. Putting $\alpha_1=\rho+\mathrm{i}\theta, \alpha_2=\rho-\mathrm{i}\theta$, with $\theta\neq 0$, we see that the functions $e^{(\rho+\mathrm{i}\theta)x}, e^{(\rho-\mathrm{i}\theta)x}$ form a fundamental system for (1.2.52). But this equation is linear, and, as $\ker L$ is a vector space, their linear combinations, which, according to Euler's formulae, are real functions, also belong to $\ker L$

$$Y_1=\frac{e^{(\rho+\mathrm{i}\theta)x}+e^{(\rho-\mathrm{i}\theta)x}}{2}=e^{\rho x}\frac{e^{\mathrm{i}\theta x}+e^{-\mathrm{i}\theta x}}{2}=e^{\rho x}\cos\theta x,$$
$$Y_2=\frac{e^{(\rho+\mathrm{i}\theta)x}-e^{(\rho-\mathrm{i}\theta)x}}{2\mathrm{i}}=e^{\rho x}\frac{e^{\mathrm{i}\theta x}-e^{-\mathrm{i}\theta x}}{2\mathrm{i}}=e^{\rho x}\sin\theta x. \tag{1.2.66}$$

Besides, they form a fundamental system, as their Wronskian

$$W[Y_1,Y_2]=\begin{vmatrix} e^{\rho x}\cos\theta x & e^{\rho x}\sin\theta x \\ e^{\rho x}(\rho\cos\theta x-\theta\sin\theta x) & e^{\rho x}(\rho\sin\theta x+\theta\cos\theta x)\end{vmatrix}=\theta e^{2\rho x} \tag{1.2.67}$$

is obviously non-zero.
The general solution of (1.77) is therefore

$$y(x)=e^{\rho x}(C_1\cos\theta x+C_2\sin\theta x), \tag{1.2.68}$$

with C_1, C_2 arbitrary constants.

iii) Let us denote by α the double root of the characteristic equation (1.2.54). Obviously, (1.77) allows

$$Y_1(x)=e^{\alpha x} \tag{1.2.69}$$

as a particular solution. A second particular solution cannot coincide with Y_1. So, in order to get a new particular solution, linearly independent from Y_1, we suppose, for the moment, that the characteristic equation allows two distinct solutions, α, α', very close to each other. To these roots, according to i), we put into correspondence the solutions $e^{\alpha x}, e^{\alpha' x}$, which, so far, are linearly independent. But this does not hold if $\alpha' \to \alpha$. We then replace $e^{\alpha' x}$ by the linear combination $\dfrac{e^{\alpha' x} - e^{\alpha x}}{\alpha' - \alpha}$ and, passing to the limit as $\alpha' \to \alpha$, we obtain a second particular solution, distinct from Y_1,

$$Y_2(x) = \lim_{\alpha' \to \alpha} \frac{e^{\alpha' x} - e^{\alpha x}}{\alpha' - \alpha} = \lim_{\alpha' \to \alpha} \frac{x e^{\alpha' x}}{1} = x e^{\alpha x} . \tag{1.2.70}$$

To get Y_2 we used L'Hospital rule.

The functions Y_1, Y_2 form a fundamental system. Indeed,

$$W[Y_1, Y_2] = \begin{vmatrix} e^{\alpha x} & x e^{\alpha x} \\ \alpha e^{\alpha x} & (\alpha x + 1) e^{\alpha x} \end{vmatrix} = e^{2\alpha x} \neq 0 . \tag{1.2.71}$$

The general solution of (1.2.52) is then

$$y(x) = e^{\alpha x}(C_1 + C_2 x), \tag{1.2.72}$$

with C_1, C_2 arbitrary constants.

The general integral of the non-homogeneous ODE

$$\mathrm{L}y \equiv a_0 y'' + a_1 y' + a_2 y = f(x), \quad a_0, a_1, a_2 \in \Re, a_0 \neq 0, f \in \mathrm{C}^0(I), \tag{1.2.73}$$

obviously allows the representation (1.2.47), where Y_1, Y_2 are determined as it was shown in the cases i), ii) or iii). But this kind of formula often leads to cumbrous computation, because of the integral in the right side. We have another option, if *f* is an elementary function – polynomial, exponential, trigonometric function etc. In this case, the particular solution of (1.2.73) is searched under a form similar to *f*. In applications, we shall make use of this idea.

2.4 ORDER REDUCTION

Let us get back to the linear ODE(1.2.7), whose coefficients are supposed continuous on the real interval I and $a_0(x) \neq 0, \forall x \in I$. We already proved that, once we know a particular solution, one can completely solve this equation. But there are cases in which we don't even need this. Let us mention some of them.

a) If $a_2(x) = a_1' - a_0''$, then (1.2.7) may be integrated once, to give

$$a_0 y' + (a_1 - a_0')y = C , \tag{1.2.74}$$

where C is a real arbitrary constant. The linear first order ODE (1.2.74) was already completely solved at Sec.1.3.

b) By the change of function $u(x) = y(x)\mathrm{e}^{\frac{1}{2}\int \frac{a_1(x)}{a_0(x)}\mathrm{d}x}$, the equation (1.2.7) is brought to Liouville's normal form

$$u'' + K(x)u = 0 . \tag{1.2.75}$$

The function $K(x)$ is defined by the formula

$$K(x) \equiv \frac{a_2}{a_0} - \frac{1}{4}\left(\frac{a_1}{a_0}\right)^2 - \frac{1}{2}\left(\frac{a_1}{a_0}\right)' \tag{1.2.76}$$

and is called the *invariant* of (1.2.7). We see that K has a sense only if $a_1 / a_0 \in \mathrm{C}^1(I)$.

c) Let us consider, together with (1.2.7), a similar ODE, fulfilling the same conditions

$$b_0(x)u'' + b_1(x)u' + b_2(x)u = 0 . \tag{1.2.77}$$

It can be proved that their solutions are connected by the relationship

$$y(x) = u(x)p(x), \quad p \in C^2(I), \tag{1.2.78}$$

with p nonvanishing on I, if and only if the two corresponding ODEs have the same invariant $K(x)$. In this case, let Y_1, Y_2 and, accordingly, U_1, U_2, be two fundamental systems for these ODEs. It can be proved that the ratio

$$s(x) = \frac{Y_1(x)}{Y_2(x)} = \frac{U_1(x)}{U_2(x)}, \tag{1.2.79}$$

with $s'(x) \neq 0$, satisfies the non-linear ODE

$$\frac{s'''}{s'} - \frac{3}{2}\left(\frac{s''}{s'}\right)^2 = 2K(x). \tag{1.2.80}$$

The differential expression in the left member is called the *Schwarz derivative* of s and plays an important part in the study of stability of the solutions of ODEs.

d) By means of the transformation

$$u = \frac{y'}{y}, \tag{1.2.81}$$

the equation (1.2.7) becomes

$$a_0(x)u' + a_1(x)u + a_0(x)u^2 + a_2(x) = 0 , \tag{1.2.82}$$

which is a first order non-linear ODE, of Riccati type (see Chap. 4, Sec. 1.4).

2.5 THE CAUCHY PROBLEM. ANALYTICAL METHODS TO OBTAIN THE SOLUTION

We shall briefly expose several of the mostly used and most general analytical methods to get the solutions of Cauchy problems for linear second order ODEs.
Let us associate to the equation (1.2.2) the following initial (or Cauchy) conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \qquad x_0 \in I. \tag{1.2.83}$$

Suppose that the coefficients and the free term allow derivatives of any order on *I*, therefore $p, q, f \in \mathrm{C}^{\infty}(I)$.

a) *The method of the Taylor series expansion*
We write Taylor's formula for $y(x)$

$$\begin{aligned} y(x) = y(x_0) + \frac{x - x_0}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \ldots \\ + \frac{(x - x_0)^k}{k!} y^{(k)}(x_0) + R_k(x, x_0), \end{aligned} \tag{1.2.84}$$

where

$$R_k(x, x_0) = \frac{(x - x_0)^{k+1}}{(k+1)!} y^{(k)}(x_0 + \theta(x - x_0)), \quad |\theta| < 1,$$

is the Lagrange remainder. The first $(k+1)$ terms of Taylor's formula form a k-th degree polynomial, usually called *Taylor's polynomial*. The value of $y(x)$ around x_0 may thus be approximated by Taylor's polynomial. To obtain the coefficients $y^{(j)}(x_0)$, $j = \overline{2, k}$, we differentiate step by step, also using the ODE (1.2.2). This yields

$$\begin{aligned} y''(x_0) &= -p(x_0)y'(x_0) - q(x_0)y(x_0) - f(x_0) = -p(x_0)y'_0 - q(x_0)y_0 - f(x_0), \\ y'''(x_0) &= \left[p^2(x_0) - p'(x_0) - q(x_0)\right]y'(x_0) + \left[p(x_0)q(x_0) - q'(x_0)\right]y(x_0) \\ &\quad - f'(x_0) + p(x_0)f(x_0) = \left[p^2(x_0) - p'(x_0) - q(x_0)\right]y'_0 \\ &\quad + \left[p(x_0)q(x_0) - q'(x_0)\right]y_0 - f'(x_0) + p(x_0)f(x_0), \end{aligned} \tag{1.2.85}$$

....

This way, we expressed the values of higher derivatives of y at x_0, required by Taylor's polynomial, in terms of the Cauchy data y_0, y'_0.
Remark. The Taylor's formula is currently used especially to solve Cauchy problems. Yet, the obtained approximation has a local character and for this reason it serves to set up one-step methods in the frame of the numerical analysis.

b) *The method of the successive approximations (Picard)*
Let y_1 be a linear function, satisfying the Cauchy conditions (1.2.52).

Picard's method for the linear second order ODE consists of determining the sequence of functions $y_k(x)$ by the recurrence relationship

$$y_k''(x) = -p(x)y_{k-1}'(x) - q(x)y_{k-1} + f(x), \quad k \geq 2, \tag{1.2.86}$$

starting from y_1. It can be proved that the (convergent) series

$$y(x) = y_1(x) + y_2(x) + y_3(x) + \ldots \tag{1.2.87}$$

is the desired solution.

c) *The method of continued fraction expansion*

This method can be applied to the homogeneous ODEs (1.2.11) for which $q(x) \neq 0$. The equation may be written in the form

$$y = p_0(x)y' + q_0(x)y. \tag{1.2.88}$$

If p_0, q_0 allow derivatives of any order on *I*, then, differentiating (1.2.88), we get

$$y' = p_1(x)y'' + q_1(x)y''', \quad p_1 = \frac{p_0 + q_0'}{1 - q_0'}, \; q_1 = \frac{q_0}{1 - q_0'}, \tag{1.2.89}$$

and, in general,

$$y^{(k)} = p_k(x)y^{(k+1)} + q_k(x)y^{(k+2)}, \quad p_k = \frac{p_{k-1} + q_{k-1}'}{1 - q_{k-1}'}, \; q_k = \frac{q_{k-1}}{1 - q_{k-1}'}, \tag{1.2.90}$$

if the denominators do not vanish.

The relationships (1.2.88) and (1.2.89) involve

$$\frac{y}{y'} = p_0(x) + q_0(x)\frac{y''}{y'}. \tag{1.2.91}$$

Dividing (1.2.89) by y'', we deduce

$$\frac{y}{y'} = p_0(x) + \frac{q_0(x)}{p_1 + q_1 \dfrac{y'''}{y''}}. \tag{1.2.92}$$

Eventually, we obtain the continued fraction

$$\frac{y}{y'} = p_0 + \frac{q_0|}{|p_1} + \frac{q_1|}{|p_2} + \frac{q_2|}{|p_3} + \ldots, \tag{1.2.93}$$

where $\dfrac{q|}{|p}$ stands for

$$\frac{q|}{|p} + Q = \frac{q}{p+Q}. \tag{1.2.94}$$

If the expression in the right side of (1.2.93) converges, then it may be either determined or conveniently approximated. In both cases, (1.2.93) becomes a first order linear ODE, which can be straightforwardly integrated with the method described at Sec.1.2.
An interesting application of this method will be presented in Sec.2.7, where we shall study the hypergeometric series.

2.6 TWO-POINT PROBLEMS (PICARD)

Another kind of problem, very interesting for applications to mechanics is the two-point (bilocal) problem. The (semi-homogeneous) linear two-point problem consists of finding a solution of (1.2.2) that satisfies the homogeneous conditions

$$y(\alpha) = 0, \quad y(\beta) = 0, \quad \alpha, \beta \in I, \quad \alpha < \beta. \tag{1.2.95}$$

This problem may be solved in many ways, among which we chose two, that are connected with the previously exposed facts.

a) The general solution of the ODE (1.2.2) allows the representation (1.2.47), based on the fundamental solution $k(x,t)$. Therefore, to get the solution of the above two-point problem, it is enough to find $C_1(t), C_2(t)$ such that the fundamental solution match (1.2.95).

The Green function for the two-point problem (1.2.2), (1.2.95) is that fundamental solution of (1.2.2) that satisfies (1.2.95).
Remark. The Green function is defined provided the homogeneous two-point problem (for $f = 0$) allows only the null solution.
Let us suppose now that, instead of (1.2.95), the solution y of (1.2.2) must satisfy some non-zero conditions

$$y(\alpha) = A, \quad y(\beta) = B, \quad \alpha, \beta \in I, \quad \alpha < \beta. \tag{1.2.96}$$

In this case, we make the change of function $y(x) = z(x) + h(x)$, with h chosen such that $h(\alpha) = A, h(\beta) = B$. The new unknown function $z(x)$ will obviously satisfy a semi-homogeneous two-point problem.

Examples of Green functions

1. The fundamental solution for the ODE $y'' = 0$ is $\frac{1}{2}|x-t|$. Consequently, the Green function for the associated semi-homogeneous two-point problem is

$$K(x,t) = C_1(t) + xC_2(t) + \frac{1}{2}|x-t|. \tag{1.2.97}$$

2. Take the ODE $y'' - y = 0$. The Green function for the associated semi-homogeneous two-point problem is

$$K(x,t) = C_1(t)\mathrm{e}^{x} + C_2(t)\mathrm{e}^{-x} + \frac{1}{2}\sinh|x-t|\,, \tag{1.2.98}$$

where $C_1(t), C_2(t)$ match (1.2.95).

b) The general solution of the ODE (1.2.2) can be written in the form

$$y(x) = C_1 Y_1(x) + C_2 Y_2(x) + Y(x), \tag{1.2.99}$$

where Y_1, Y_2 form a fundamental system for the associated to (1.2.2) homogeneous equation and Y is a particular solution of (1.2.2); suppose that these three functions could be obtained by the previously described methods.

Imposing now to y the two-point conditions (1.2.95), we get for C_1, C_2 the following algebraic system

$$\begin{cases} C_1 Y_1(\alpha) + C_2 Y_2(\alpha) = A - Y(\alpha), \\ C_1 Y_1(\beta) + C_2 Y_2(\beta) = B - Y(\beta). \end{cases} \tag{1.2.100}$$

There are two possibilities:

i) The determinant $d \equiv \begin{vmatrix} Y_1(\alpha) & Y_2(\alpha) \\ Y_1(\beta) & Y_2(\beta) \end{vmatrix} \neq 0$.

In this case, the two-point problem allows a unique solution of the form (1.2.99), with C_1, C_2 uniquely determined from (1.2.100). We observe that, in this case, the homogeneous two-point problem allows only the null solution.

ii) The determinant $d = 0$. According to Rouché's theorem,

a. either the bordered matrix has the rank 1, in which case the two-point problem allows infinitely many solutions;
b. or the bordered matrix has the rank 2, and the two-point problem has no solution.

In conclusion, the following alternative works

Alternative. Either $d \neq 0$ and the non-homogeneous problem (1.2.2), (1.2.96) allows a unique solution, and the homogeneous one – only the null solution, or $d = 0$, case in which the homogeneous problem allows also non-null solutions. In this last situation, the non-homogenoues problem has no solutions, in general, except for some special cases.

It may be also proved that, if $p, q, f \in \mathrm{C}^0(I)$ and if there is a strictly negative constant Q such that $q(x) \leq Q < 0, \forall x \in I$, then the two-point problem (1.2.2), (1.2.96) allows a unique solution.

2.7 STURM-LIOUVILLE PROBLEMS

Another class of boundary value problems that might be associated to ODEs are the eigenvalue and eigenfunction problems; these are by no means simple artificial mathematical generalizations, but, on the contrary, they come from the study of physical models. We shall illustrate the way these problems appear by a notorious example.
Let us study the longitudinal oscillations of a non-homogeneous thread, fixed at its ends α and β. It is known that these oscillations are described by the linear second order PDE

$$\frac{\partial}{\partial x}\left(E\frac{\partial u}{\partial x}\right)=\rho\frac{\partial^2 u}{\partial t^2}, \tag{1.2.101}$$

where $E(x)$ is the modulus of elasticity and $\rho(x)$ is the volume density. The unknown function $u(x,t)$ represents the displacement of the point of abscissa x of the thread at the moment t, with respect to its rest position.
Suppose we know the initial displacement and velocity

$$u(x,0)=f(x),\quad \frac{\partial u}{\partial t}(x,0)=g(x),\quad x\in I\equiv[\alpha,\beta]. \tag{1.2.102}$$

As the thread is fixed at its ends, we shall also have

$$u(\alpha,t)=0,\quad u(\beta,t)=0,\quad t\geq 0\ . \tag{1.2.103}$$

By reason of continuity, the functions f and g must satisfy the compatibility conditions

$$f(\alpha)=0,\quad g(\alpha)=0,\quad f(\beta)=0,\quad g(\beta)=0\ . \tag{1.2.104}$$

Let us search for solutions of (1.2.101) in the form

$$u(x,t)=X(x)T(t)\ . \tag{1.2.105}$$

Replacing this in (1.2.101) we obtain

$$\frac{\dfrac{\mathrm{d}}{\mathrm{d}x}[E(x)X']}{\rho(x)X}=\frac{\ddot{T}}{T}, \tag{1.2.106}$$

where the primes stand for the derivatives with respect to x and the points – for the derivatives with respect to t.
The right member of this equation does not depend on x and the left one does not depend on t. Consequently, the above ratio must have a constant value, say λ. Otherwise, the relationship (1.2.106) would represent a functional dependence between the temporal and the spatial variables t and x. From (1.2.106) we thus get two linear second order ODEs: one in x

$$[E(x)X']'=\lambda\rho(x)X\ , \tag{1.2.107}$$

and one in t

$$\ddot{T} = \lambda T \ . \tag{1.2.108}$$

Obviously, the ODE (1.2.107) is defined for $E \in C^1(I), \rho \in C^0(I)$ at least. The boundary conditions (1.2.103) yield for X the two-point conditions

$$X(\alpha) = 0, \quad X(\beta) = 0 \ . \tag{1.2.109}$$

If $\rho(x) = 1, x \in I$, then the ODE (1.2.107) becomes more simple. Even if this is not true, but $\rho(x)$ does not cancel on I and is of class $C^0(I)$, we may perform the change of function

$$y = X\sqrt{\rho(x)} \ , \tag{1.2.110}$$

by which (1.2.107) becomes

$$\mathrm{L}y \equiv [p(x)y']' + q(x)y = \lambda y \ . \tag{1.2.111}$$

In (1.2.111) we introduced the linear operator L and used the notations

$$p(x) = \frac{E(x)}{\rho(x)}, \ q(x) = \frac{1}{\sqrt{\rho(x)}}\left[E(x)\left(\frac{1}{\sqrt{\rho(x)}}\right)'\right]' \ .$$

The two-point conditions (1.2.109) obviously will not change for y

$$y(\alpha) = 0, \quad y(\beta) = 0 \ . \tag{1.2.112}$$

So, our problem was transformed into a homogeneous two-point problem

$$\begin{cases} \mathrm{L}y = \lambda y, \\ y(\alpha) = 0, \quad y(\beta) = 0. \end{cases} \tag{1.2.113}$$

It is easily seen that the null function satisfies (1.2.113), for any constant λ. But, naturally, this is not a convenient issue. Thus, we are led, by the above considerations, to the following problem:

Problem. Find λ such that the solutions of the homogeneous two-point problem (1.2.113) allow at least another solution, different from the null one.
These particular values of λ are called *eigenvalues*; they are included in the *spectrum* of L. The corresponding non-zero solutions are called *eigenfunctions*.
In order to get a representation of $u(x,t)$, we must prove that the eigenfunctions for an *infinite* and *complete system* in $\mathrm{L}_2(I)$ – the space of measurable and square-integrable on I functions. This property ensures the representation of any function of $\mathrm{L}_2(I)$ as a series of eigenfunctions.

In the case of the thread, the boundary conditions are precisely the two-point conditions. But there are other physical models leading to the more general problem

$$\begin{cases} \mathrm{L}y = \lambda r(x) y, \\ a_{11} y(\alpha) + a_{12} y'(\alpha) = 0, \qquad |a_{j1}| + |a_{j2}| \neq 0, \quad j = 1,2 \\ a_{21} y(\beta) + a_{22} y'(\beta) = 0, \end{cases} \tag{1.2.114}$$

The boundary conditions of (1.2.114) are usually called the *Sturm conditions*.

The problem of finding the eigenvalues and eigenfunctions of (1.2.114), as well as of proving the closure and completeness of the eigenfunction system is called the *Sturm-Liouville problem*.

It can be proved that two eigenfunctions y_1, y_2, corresponding to two distinct eigenvalues λ_1, λ_2, are orthogonal with weight r on I

$$\int_\alpha^\beta r(x) y_1(x) y_2(x) \mathrm{d}x = 0 \,. \tag{1.2.115}$$

If the sign of r does not change on I, then all the eigenvalues of (1.2.114) are real.

By using Sturm's oscillation theorem, one can prove that, if $p \in \mathrm{C}^1(I)$, $q, r \in \mathrm{C}^0(I)$ and if p, r do not vanish on I, then

i) the set of eigenvalues $\{\lambda_j\}_{j \in \mathcal{N}}$ form a monotonically decreasing sequence;

ii) the eigenvalues are simple (their order of multiplicity is 1);

iii) any eigenfunction has, in I, only n zeros.

These general facts are helpful in proving e.g. the completeness of the eigenfunction system.

Getting back to the Sturm-Liouville problem (1.2.113), let us try to solve it in the particular case of a homogeneous thread, i.e., $\rho = \mathrm{const}$. If $E = \mathrm{const}$ too and $\alpha = 0, \beta = l$, l being the thread length, the problem (1.2.113) becomes

$$\begin{cases} y'' - \mu y = 0, \\ y(0) = 0, \quad y(l) = 0, \end{cases} \quad \mu = \lambda \frac{E}{\rho} \,. \tag{1.2.116}$$

We see that for $\mu \geq 0$ the problem allows only the null solution. So, the only possibility is that $\mu = -\nu^2, \nu > 0$. The involved ODE becomes $y'' + \nu^2 y = 0$. It is with constant coefficients and its associated characteristic equation allows only the purely imaginary roots $\pm \mathrm{i}\nu$. So, its general solution is the linear combination

$$y(x) = C_1 \cos \nu x + C_2 \sin \nu x \,.$$

Introducing the boundary conditions, we obtain for C_1, C_2 the linear algebraic system

$$\begin{cases} C_1 \cdot 1 + C_2 \cdot 0 = 0, \\ C_1 \cos \nu l + C_2 \sin \nu l = 0. \end{cases}$$

This leads to $C_1 = 0, \quad C_2 \sin \nu l = 0$. The only option in order to get non-zero solutions is that $\sin \nu l = 0$, which yields

$$\nu_k = \frac{k\pi}{l}, \quad k \in \mathcal{N}. \tag{1.2.117}$$

The eigenvalues of the Sturm-Liouville problem (1.2.116) are

$$\lambda_k = -\frac{k^2\pi^2}{l^2}, \quad k \in \mathcal{N}, \tag{1.2.118}$$

and the corresponding eigenfunction

$$y_k(x) = \sin\frac{k\pi x}{l}, \quad k \in \mathcal{N}. \tag{1.2.119}$$

Thus, $\{y_k(x)\}_{k\in\mathcal{N}}$ forms an infinite system of eigenfunctions for the problem (1.2.116). The spectrum of L is, in this case, composed of the eigenvalues $\lambda_k = -\frac{k^2\pi^2}{l^2}, k \in \mathcal{N}$. It is seen that $\{y_k(x)\}_{k\in\mathcal{N}}$ is orthogonal on I, with weight 1, as $r = 1$, i.e.

$$\int_0^l \sin\frac{k\pi x}{l}\sin\frac{j\pi x}{l}\mathrm{d}x = \begin{cases} 0, & k \neq j, \\ \frac{1}{2}, & k = j, \end{cases} \quad k, j \in \mathcal{N}. \tag{1.2.120}$$

The final solution of the thread problem exceeds the frame of this book. However, this example emphasizes the natural way in which a Sturm-Liouville system may occur and serve to solve a physical problem.

2.8 LINEAR ODEs OF SPECIAL FORM

In what follows, we shall consider two ODEs, leading to the introduction of several special functions.

1. *Gauss' equation. The hypergeometric function (series)*

There are various physical models that lead to a second order ODE of the form

$$\left(t^2 + at + b\right)\ddot{y} + (ct + d)\dot{y} + ey = 0, \quad a,b,c,d,e \in \Re, \tag{1.2.121}$$

where the dot means differentiation with respect to t.

Let us assume that the polynomial $t^2 + at + b$ allows the distinct roots, t_1, t_2. Then, by using the change of variable $x = (t - t_1)/(t_2 - t_1)$ we can reduce this ODE to a standard form, *Gauss' equation*,

$$x(1-x)y'' + [\gamma - (\alpha + \beta + 1)x]y' - \alpha\beta y = 0, \quad \gamma \neq -n, n \in \mathrm{N}. \tag{1.2.122}$$

In (1.2.122), the primes mean differentiation with respect to *x*.
The invariant of this equation, computed by formula (1.2.76), is

$$K(x)=\frac{1-(\gamma-1)^2}{4x^2}+\frac{1-(\alpha+\beta-\gamma)^2}{4(x-1)^2}+\frac{(\alpha+\beta-\gamma)^2+(\gamma-1)^2-(\alpha-\beta)^2-1}{4x(x-1)}\,. \qquad (1.2.123)$$

Searching for a solution in the form of a power series, we obtain the *hypergeometric series* or *function*

$$\begin{aligned} y_1(x)\equiv F(\alpha,\beta,\gamma,x)&=1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma\cdot(\gamma+1)}x^2+\ldots \\ &+\frac{\alpha(\alpha+1)\ldots(\alpha+n)\beta(\beta+1)\ldots(\beta+n)}{(n+1)!\cdot\gamma\cdot(\gamma+1)\ldots(\gamma+n)}x^{n+1}+\ldots\,. \end{aligned} \qquad (1.2.124)$$

A second solution of Gauss' equation, independent of $y_1(x)$, is a series as well

$$y_2(x)\equiv x^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;x)\,. \qquad (1.2.125)$$

Both series are convergent for $|x|<1$. Obviously, the series are breaking off if α or β are zero or negative integers. Some of the polynomials obtained this way have various applications. Thus,

$$\begin{aligned} F\left(-n,n+\frac{1}{2},\frac{1}{2};x^2\right)&=(-1)^n\,\frac{2^n\,n!}{1\cdot 3\cdot 5\ldots(2n-1)}P_{2n}(x), \\ xF\left(-n,n+\frac{3}{2},\frac{3}{2};x^2\right)&=(-1)^n\,\frac{2^n\,n!}{1\cdot 3\cdot 5\ldots(2n+1)}P_{2n+1}(x), \end{aligned} \qquad (1.2.126)$$

where $P_j(x)$ are Legendre's polynomials, satisfying the equation

$$(x^2-1)y''+2xy'-n(n+1)y=0\,. \qquad (1.2.127)$$

Jacobi's polynomials, more general than Legendre's, are obtained by considering

$$Q_n(x)\equiv F(n,-n+\alpha,\beta;x)=\frac{x^{1-\beta}(1-x)^{\beta-\alpha}}{\beta(\beta+1)\ldots(\beta+n-1)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[x^{\beta+n-1}(1-x)^{\alpha+n-\beta}\right]. \qquad (1.2.128)$$

The function systems $\{P_n\}_{n\in\mathcal{N}}$ and $\{Q_n\}_{n\in\mathcal{N}}$ are orthogonal and complete.
Other particular cases of hypergeometric series, leading to elementary functions, are, e.g.,

$$F(-n,\beta,\beta;-x)=(1+x)^n\,,\qquad F\left(\frac{1}{2},1,1;\sin x\right)=\sec x,$$
$$F(1,\beta,\beta;x)=\frac{1}{1-x}\,,\qquad xF\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right)=\arcsin x,. \tag{1.2.129}$$
$$F(1,1,2;-x)=\frac{1}{x}\ln x,\qquad xF\left(\frac{1}{2},1,\frac{3}{2};-x^2\right)=\arctan x\,.$$

One can also get convergent numerical series. For instance, a numerical series, converging to π, is

$$2F\left(1,1,\frac{3}{2};\frac{1}{2}\right)=\pi\,. \tag{1.2.130}$$

Other elementary functions could be obtained from the hypergeometric series passing to the limit

$$\lim_{\beta\to\infty}F(1,\beta,1;x)=\mathrm{e}^x\,,$$
$$\lim_{\alpha,\beta\to\infty}xF\left(\alpha,\beta,\frac{3}{2};-\frac{x^2}{4\alpha\beta}\right)=\sin x. \tag{1.2.131}$$

From the identity

$$F(\alpha,\beta,\gamma;1)=F(-\alpha,-\beta,\gamma-\alpha-\beta;1),\quad \gamma>0. \tag{1.2.132}$$

we get the recurrence formula

$$F(\alpha,\beta,\gamma;x)=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1;x),\quad \gamma\neq -n. \tag{1.2.133}$$

The Wronskian of a fundamental system of solutions Y_1, Y_2 of Gauss' equation is

$$W[Y_1,Y_2]=C|x|^{-\gamma}|1-x|^{\gamma-\alpha-\beta-1}\,, \tag{1.2.134}$$

according to Liouville's formula (1.2.24).

Finally, let us mention the continued fraction expansion for the hypergeometric function, obtained by the method exposed at Sec.2.4

$$\frac{F(\alpha,\beta+1,\gamma+1;x)}{F(\alpha,\beta,\gamma;x)}=\frac{1|}{|1}+\frac{a_1x|}{|1}+\frac{a_2x|}{|1}+\ldots\,, \tag{1.2.135}$$

where

$$a_{2n} = \frac{(\beta+n)(\alpha-\gamma-n)}{(\gamma+2n-1)(\gamma+2n)}, \quad a_{2n+1} = \frac{(\alpha+n)(\beta-\gamma-n)}{(\gamma+2n)(\gamma+2n+1)}. \tag{1.2.136}$$

This expansion converges on the whole complex plane, with a slit from +1 to $+\infty$, except for the zeros of $F(\alpha, \beta, \gamma; x)$.

2. *Euler's gamma function*

To get another remarkable representation of the hypergeometric function, we need to introduce the gamma function, which is also important in itself.
The gamma function is defined, for real arguments, by means of the integral

$$\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x} dx, \tag{1.2.137}$$

and makes sense for $\operatorname{Re}\nu > 0$.
From the definition, we get, integrating by parts,

$$\Gamma(\nu+1) = \int_0^\infty x^\nu e^{-x} dx = -x^\nu e^{-x}\Big|_0^\infty + \nu\int_0^\infty x^{\nu-1} e^{-x} dx, \tag{1.2.138}$$

The term $x^\nu e^{-x}$ is null for $x = 0$ and

$$\lim_{x\to\infty} \frac{x^\nu}{e^x} = 0,$$

therefore

$$\Gamma(\nu+1) = \nu\Gamma(\nu). \tag{1.2.139}$$

This recurrence relationship extends the factorials of positive numbers. Indeed, by integration, we have

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1. \tag{1.2.140}$$

Applying now the recurrence formula, it results

$$\begin{aligned}
&\Gamma(2) = 1\cdot\Gamma(1) = 1 = 1!,\\
&\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1! = 2!,\\
&\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2! = 3!,\\
&\dots\dots\dots\dots\\
&\Gamma(n+1) = n\cdot\Gamma(n) = n\cdot(n-1)! = n!,
\end{aligned} \tag{1.2.141}$$

where n is a positive integer.

The values of $\Gamma(\nu)$ for all positive ν may be deduced from the above recurrence formula, once $\Gamma(\nu)$ known between two consecutive integers, e.g., between 1 and 2. For instance, as $\Gamma(0.5)=\sqrt{\pi}$, we have, step by step

$$\Gamma(4.5)=3.5\cdot\Gamma(3.5)=3.5\cdot 2.5\cdot\Gamma(2.5)=3.5\cdot 2.5\cdot 1.5\cdot 0.5\cdot\Gamma(0.5)\cong 11.63\,,$$

and, in general, for a positive integer *r*,

$$\Gamma(\nu+r+1)=(\nu+r)\Gamma(\nu+r)=\ldots=(\nu+r)(\nu+r-1)\ldots(\nu+1)\ \nu\Gamma(\nu)\,. \qquad (1.2.142)$$

Taking this formula into account, the well-known combinatorial formula

$$C_n^k=\frac{n!}{k!(n-k)!}$$

may be expressed in terms of gamma functions as

$$C_n^k=\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}\,; \qquad (1.2.143)$$

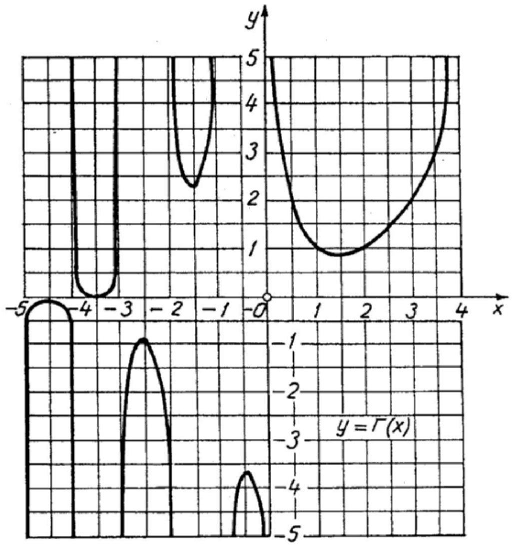

Figure 1. 1. The graph of the gamma function

This formula is useful for the calculus of C_n^k for great *n* and *k*.

It should be mentioned that the classic definition of the gamma function may be extended to complex arguments

$$\Gamma(\nu) = -\frac{1}{2\mathrm{i}\sin\pi\nu}\int_c (-t)^{\nu-1}\mathrm{e}^{-t}\,\mathrm{d}t\,, \tag{1.2.144}$$

where c is a given contour.
It can be proved that $\Gamma(\nu)$ is a rational analytical function with respect to its argument, having simple poles at $\nu = -n, n \in \mathcal{N}$, at which the corresponding residues are $(-1)^n / n$.
The graph of Γ as a function of ν is represented in Fig.1.1.
The gamma function also satisfies other recurrence relationships; we mention here two of them

$$\Gamma(1-\nu)\Gamma(\nu) = \frac{\pi}{\sin\pi\nu}\,,\quad \Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\nu\right) = \frac{\pi}{\cos\pi\nu}\,, \tag{1.2.145}$$

useful for applications.
Getting back to the hypergeometric series, we see that, by using gamma functions, one obtains the following representation formula

$$F(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \tag{1.2.146}$$

3. *Bessel's equation*

The ODE

$$x^2 y'' + xy' + \left(x^2 - \nu^2\right)y = 0\,, \tag{1.2.147}$$

where ν is a real/complex parameter and x may be real or complex, is called *Bessel's equation*. Its solutions are called *Bessel functions* and also *cylindrical functions*, as they usually appear when solving boundary values problems on domains with cylindrical symmetry; such models appear e.g. in the frame of the potential theory.
Searching for solutions of (1.2.147) in the form of a power series, we find the Bessel functions of order ν and first kind

$$\begin{aligned} J_\nu(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^{2n+\nu}, \\ J_{-\nu}(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(-\nu+n+1)}\left(\frac{x}{2}\right)^{2n-\nu}, \end{aligned} \tag{1.2.148}$$

for $\nu \notin \mathcal{N}$. The expansions (1.2.148) converge on $\Re$ (even on the complex space $\mathcal{C}$), but not at the infinity. The Wronskian of the system $J_\nu, J_{-\nu}$ is

$$W[J_\nu, J_{-\nu}] = -\frac{2}{\pi x}\sin\pi\nu\,. \tag{1.2.149}$$

As $W[J_\nu, J_{-\nu}] \neq 0$ for $\nu \notin \mathcal{N}$, it follows that the general solution of Bessel's equation may be written in the form of the linear combination

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x), \quad \nu \notin \mathcal{N} . \tag{1.2.150}$$

If $\nu = n \in \mathcal{N}$, then the Wronskian vanishes. In fact, it can be shown that

$$J_{-n}(x) = (-1)^n J_n(x), \quad n \in \mathcal{N} . \tag{1.2.151}$$

So J_n, J_{-n} form no more a fundamental system. To avoid this, one introduces the *second kind Bessel functions*, also called *Neumann* or *Weber functions*

$$Y_\nu(x) = \frac{J_\nu(x)\cos \nu\pi - J_{-\nu}(x)}{\sin \pi\nu} . \tag{1.2.152}$$

It is seen that

$$W[J_\nu, Y_\nu] = \frac{2}{\pi x}, \quad \nu \in \Re / \mathcal{C}, x \neq 0 . \tag{1.2.153}$$

It can be proved that

$$Y_n(x) \equiv \lim_{\nu \to n} \frac{J_\nu(x)\cos \nu\pi - J_{-\nu}(x)}{\sin \pi\nu} = \frac{1}{\pi} \lim_{\nu \to n} \left[\frac{\partial}{\partial \nu} J_\nu(x) - (-1)^n \frac{\partial}{\partial \nu} J_{-\nu}(x) \right] \tag{1.2.154}$$

satisfies Bessel's equation of integer order n. It follows that J_ν, Y_ν are linearly independent for any ν, so they form in any case a fundamental system. Thus, the general solution of Bessel's equation may be expressed as

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x), \tag{1.2.155}$$

for any ν.

The Bessel functions may be obtained as the coefficients of the development of their generating function

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{+\infty} J_n(x) t^n . \tag{1.2.156}$$

The expansions

$$\begin{aligned} J_0(x) &= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots, \\ J_1(x) &= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 - \frac{1}{2!3!}\left(\frac{x}{2}\right)^5 - \ldots \end{aligned} \tag{1.2.157}$$

represent the Bessel functions of orders 0 and 1.

The recurrence relationships, e.g.

$$J_{\nu+1}(x)+J_{\nu-1}(x)=\frac{2\nu}{x}J_\nu(x),\qquad J_{\nu+1}(x)-J_{\nu-1}(x)=-2J'_\nu(x) \tag{1.2.158}$$

simplify the calculus of Bessel's functions. By using them, one can get e.g. the expressions of $J_n(x)$, $n\in\mathcal{N}$ starting from $J_0(x), J_1(x)$.

The only orders for which $J_\nu(x)$ is converted into an elementary function are, according to Liouville's theorem, $\nu = n+\frac{1}{2}$, with *n* integer ; for instance,

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x,\quad J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x\,. \tag{1.2.159}$$

Starting from (1.2.159), one can get step by step $J_{n+\frac{1}{2}}(x)$, by using the recurrence relationships (1.2.158).

All the other $J_\nu(x)$ are *new* functions.

The invariant of Bessel's equation, computed by formula (1.2.76), is

$$K(x)=1+\frac{1-4\nu^2}{4x^2}\,. \tag{1.2.160}$$

A property very useful in applications is the orthogonality.

Let us firstly note that, for $\operatorname{Re}\nu>-1$, $J_\nu(x)$ allows infinitely many real and simple zeros, $\pm\mu_1,\pm\mu_2,\pm\mu_3,\dots,\pm\mu_n,\dots$, symmetric with respect to the origin. For $\operatorname{Re}\nu>-1$, the functions $\left\{J_\nu\left(\frac{\mu_k}{a}x\right)\right\}_{k\in\mathcal{N}}$ form an orthogonal system, with weight *x*, on the real interval $[0,a]$

$$\int_0^a xJ_\nu\left(\frac{\mu_k}{a}x\right)J_\nu\left(\frac{\mu_n}{a}x\right)\mathrm{d}x=\begin{cases}0, & k\neq n,\\ \dfrac{a^2}{2}J'^2_{\nu+1}(\mu_n), & k=n,\end{cases} \tag{1.2.161}$$

if $\mu_k^2\neq\mu_n^2$

3. Applications

Application 1.1

Problem. Consider a symmetric membrane state of efforts in a thin shell of rotation, acted upon by the external loads *Y* and *Z*, along the tangent to the meridian line and the normal to the median surface, accordingly. Find the general expressions of the meridian and the annular efforts N_φ, N_θ, respectively (Fig.1.2).

Mathematical model. The equations of equilibrium of a shell element read

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(N_\varphi r_0\right) - N_\theta r_1 \cos\varphi + Y r_0 r_1 = 0, \tag{a}$$

$$\frac{N_\varphi}{r_1} + \frac{N_\theta}{r_2} + Z = 0\,. \tag{b}$$

The independent variable for this problem is the meridian angle φ, measured clock-wise from the top, while θ is the angle along the parallel circle. Other intervening quantities are: the radius r_0 of the parallel circle, the curvature radius $r_1 = (1/\cos\varphi)(\mathrm{d}r_0/\mathrm{d}\varphi)$ of the meridian curve – the first principal radius of curvature of the median surface – and the second principal curvature radius of the median surface, $r_2 = r_0/\sin\varphi$.

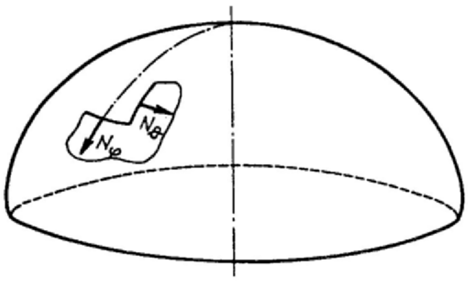

Figure 1. 2. Membrane efforts in a thin shell of rotation

Solution. The equation (b) is algebraic, therefore we find for N_θ

$$N_\theta = -\frac{r_2}{r_1} N_\varphi - Z r_2\,, \tag{c}$$

which introduced in (a) yields

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(N_\varphi r_0\right) + N_\varphi r_2 \cos\varphi + Y r_0 r_1 + Z r_1 r_2 = 0\,.$$

Taking into account the relationship between the radii r_2 and r_0, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(N_\varphi r_0 \sin\varphi\right) = -\left(Y\sin\varphi + Z\cos\varphi\right) r_0 r_1\,;$$

denoting by $y(\varphi) = N_\varphi r_0 \sin\varphi$, we get for y a linear and non-homogeneous first order ODE, studied at the Sec.1.1

$$\frac{\mathrm{d}y}{\mathrm{d}\varphi} = -(Y \sin\varphi + Z \cos\varphi) r_0 r_1 \,. \tag{d}$$

By straightforward integration, it follows

$$N_\varphi = -\frac{1}{r_0 \sin\varphi} \int (Y \sin\varphi + Z \cos\varphi) r_0 r_1 \mathrm{d}\varphi + C \,,$$

C being an arbitrary constant.
The annular effort is directly obtained from (c)

$$N_\theta = -Zr_2 + \frac{1}{r_1 \sin^2\varphi} \int (Y \sin\varphi + Z \cos\varphi) r_0 r_1 \mathrm{d}\varphi - C\frac{r_2}{r_1} \,.$$

The constant C may be determined from a condition imposed at the superior edge ($\varphi = \varphi_s$), or at the vertex ($\varphi = 0$).

Application 1.2

Problem. Find the general expression of the normal stress, as a function of time, for a Maxwell body.

Mathematical model. To explain the relaxation, one sets up the Maxwell model by a series combination of a Hooke (elastic) and a Newton (viscous) model (Fig.1.3, a). The stress results as a sum of the states of strain of the two bodies; thus, the total strain $\varepsilon_0 = \text{const}$ is composed of

- the elastic strain of the arc, expressed as

$$\varepsilon_{\text{elastic}} = \sigma / E \,, \tag{a}$$

where E is the longitudinal modulus of elasticity, and

- the viscous strain, $\varepsilon_{\text{viscous}}$.

Consequently (Fig.1.3, a)

$$\varepsilon_0 = \frac{\sigma}{E} + \varepsilon_{\text{viscous}} \,.$$

Differentiating this with respect to the time t ($\dot{\varepsilon}_0 = 0$), we get

$$\frac{\dot{\sigma}}{E} + \dot{\varepsilon}_{\text{viscous}} = 0 \,. \tag{b}$$

By Newton's law,

$$\dot{\varepsilon}_{\text{viscous}} = \frac{\sigma}{\eta} \,,$$

where η is the coefficient of dynamic viscosity, which is constant. Thus, (b) becomes

$$\dot{\sigma} + \frac{E}{\eta}\sigma = 0 \,. \tag{c}$$

Solution. The equation (c) is a first order linear and homogeneous ODE, of the type studied at Sec.1.2. Separating the variables, we get

$$\frac{\mathrm{d}\sigma}{\sigma} = -\frac{E}{\eta}\mathrm{d}t \,,$$

involving

$$\ln|\sigma| = \ln C - \frac{E}{\eta}t \,,$$

where C is an arbitrary constant.

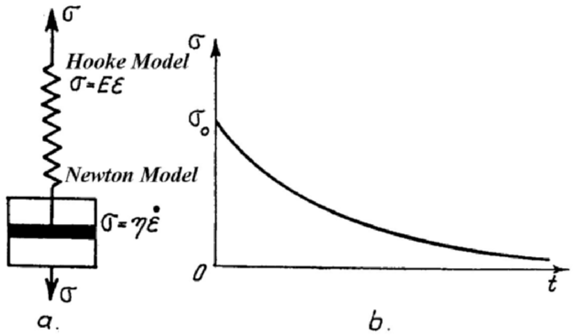

Figure 1. 3. Maxwell model (a). Diagram σ vs. t (b)

The general solution of the homogeneous ODE is therefore

$$\sigma = Ce^{-\frac{E}{\eta}t} \,.$$

To this equation, we can add the initial condition

$$\sigma(0) = \sigma_0 \,.$$

yielding $C = \sigma_0$. The solution of the above Cauchy problem is thus

$$\sigma = \sigma_0 \mathrm{e}^{-\frac{R}{\eta}t} \,. \tag{d}$$

The variation of σ as a function of t is given in Fig.1.3, b. The diagram represents a decreasing exponential, having as asymptote the time axis.

Application 1.3

Problem. A thread is wrapped round a rough circular fixed pulley, of radius R (Fig.1.4). If the thread end P_1 is acted upon by a tension $\mathbf{T}_1$, then what tension $\mathbf{T}_2$ must be applied to the other end P_2 , such that the thread slide on the pulley?

Mathematical model. As the pulley is rough, the reaction $\mathbf{R}(s)\mathrm{d}s$ upon an element of thread will have, along with a normal component $\mathbf{N}(s)\mathrm{d}s$, a tangential one, $\mathbf{\Phi}(s)\mathrm{d}s$, called force of sliding friction.
The equilibrium of an element of thread (Fig.1.5) leads to the vector equation

$$\mathrm{d}\mathbf{T} + \mathbf{R}(s)\mathrm{d}s = \mathbf{0}\,; \tag{a}$$

we can also write

$$\frac{\mathrm{d}}{\mathrm{d}s}(T\boldsymbol{\tau}) - N\boldsymbol{\nu} - fN\boldsymbol{\tau} = \mathbf{0}\,, \tag{b}$$

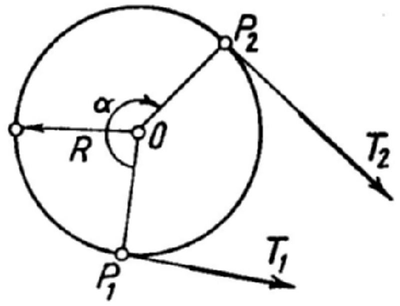

Figure 1. 4. Equilibrium of a thread on a pulley

In (b), N is the normal reaction along the unit vector $\boldsymbol{\nu}$ and fN is the tangential reaction at the limit – along the unit vector $\boldsymbol{\tau}, f$ being the coefficient of sliding friction.

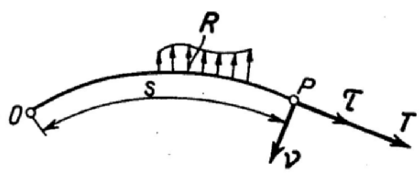

Figure 1. 5. Efforts acting on a thread

Finally, using Frenet's formula $\mathrm{d}\boldsymbol{\tau} / \mathrm{d}s = \boldsymbol{\nu} / R$, we can write the system that models the phenomenon

$$\frac{\mathrm{d}T}{\mathrm{d}s} - fN = 0,$$
$$\frac{T}{R} - N = 0. \tag{c}$$

Solution. Eliminating the normal reaction *N*, we find the first order linear homogeneous ODE ($\mathrm{d}s = R\mathrm{d}\theta$)

$$\frac{\mathrm{d}T(\theta)}{\mathrm{d}\theta} - fT = 0 . \tag{d}$$

According to the hypotheses, this equation must be integrated under the initial condition

$$T(0) = T_1 . \tag{e}$$

Integrating this, as shown at the Sec.1.2, we get the general solution of (d)

$$T = C\mathrm{e}^{f\theta} , \tag{f}$$

where *C* is an arbitrary constant.
The initial condition leads to the solution of the Cauchy problem (d), (e)

$$T = T_1\mathrm{e}^{f\theta} . \tag{g}$$

For $\theta = \alpha$, one can write $T_2 = T_1\mathrm{e}^{f\alpha}$, where the tensions at the thread ends were emphasized.

The equilibrium may also occur for $T_2 < T_1$; in this case, the force of sliding friction changes the sense and we have $T_1 = T_2\mathrm{e}^{f\alpha}$. Thus*, Euler's condition of equilibrium* is obtained

$$\mathrm{e}^{-f\alpha} < \frac{T_2}{T_1} < \mathrm{e}^{f\alpha} . \tag{h}$$

If the ratio T_2 / T_1 is outside this interval, the thread begins to slide.

Application 1.4

Problem. Find the general expression of the strain $\varepsilon = \varepsilon(t)$ in the case of a Voigt-Kelvin model and determine it in the particular case $\varepsilon(0) = 0$.

Mathematical model. To explain the creep phenomenon, one sets up the Voigt-Kelvin model, by combining in parallel a Hooke and a Newton body (Fig.1.6, a). The strain state is then a sum between the states of stress of the two bodies

$$\sigma_0 = \sigma_1 + \sigma_2 ,$$

where σ_0 represents the resultant stress, supposedly known; $\sigma_1 = E\varepsilon$ corresponds to Hooke's, while $\sigma_2 = \eta\dot{\varepsilon}$ corresponds to Newton's model. In the last two relationships, E is the modulus of longitudinal elasticity, η is the coefficient of dynamic viscosity and $\dot{\varepsilon} = \mathrm{d}\varepsilon / \mathrm{d}t$ is the velocity of deformation.

It immediately follows $\sigma_0 = E\varepsilon + \eta\dot{\varepsilon}$, that may be also written in the form

$$\dot{\varepsilon} + \frac{E}{\eta}\varepsilon = \frac{\sigma_0}{\eta} . \tag{a}$$

Consequently, in a Voigt-Kelvin model the strain $\varepsilon = \varepsilon(t)$ must satisfy the equation (a).

Solution. The first order ODE (a) is linear and non-homogeneous; this type was treated at the Sec.1.3. The associated homogeneous equation $\dot{\varepsilon} + \frac{E}{\eta}\varepsilon = 0$ allows the general solution

$$\varepsilon_{\text{homog}} = C\mathrm{e}^{-\frac{E}{\eta}t} . \tag{b}$$

As the free term is constant, we can directly search for a particular solution of (a) in the form of a constant, $\varepsilon_{\text{part}} = K$. Finally, $\varepsilon_{\text{part}} = \sigma_0 / E$ and the general solution of (a) is

$$\varepsilon(t) = C\mathrm{e}^{-\frac{E}{\eta}t} + \frac{\sigma_0}{E} . \tag{c}$$

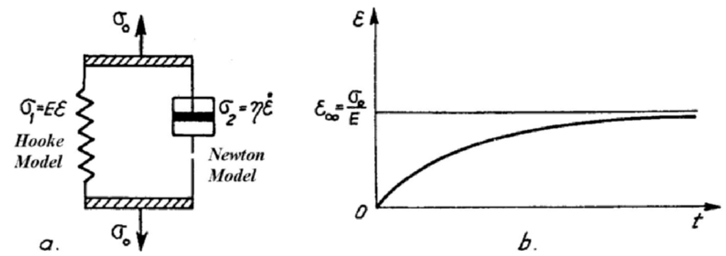

Figure 1. 6. Voigt-Kelvin model (a). Diagram ε vs. t (b)

This formula gives the strain in the case of a Voigt-Kelvin model. To get the solution of (a) corresponding to null Cauchy conditions, we put $t = 0$ in (c); it results

$$\varepsilon = \frac{\sigma_0}{E}\left(1 - \mathrm{e}^{-\frac{E}{\eta}t}\right) . \tag{d}$$

The variation of ε as a function of t is presented in Fig.1.6, b. It is seen that the graph of the function allows an asymptote $\varepsilon_\infty = \sigma_0 / E$, parallel to the t – axis; this means that the deformation is damped in time. The tangent at the origin is $\dot{\varepsilon} = \sigma_0 / \eta$.
The time-dependent function

$$\varphi(t) = 1 - \mathrm{e}^{-\frac{E}{\eta}t}$$

is called the *creep function*.

Application 1.5

Problem. Determine the general meridian displacements w of a thin shell of rotation. Particular case: the spherical dome of radius a, acted upon by its own weight g.

Mathematical model. The meridian displacements of a shell of rotation are described, in the membrane theory, by the ODE (see e.g. Flügge)

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} - w\cot\varphi = f(\varphi), \tag{a}$$

where φ is the angular variable (the meridian angle) and $f(\varphi)$ is a function depending on the external loading.

Solution. The equation (a) is a linear first order non-homogeneous ODE, of the kind treated at Sec.1.3. The associated homogeneous equation

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} - w\cot\varphi = 0\,, \tag{b}$$

allows, according to Sec.1.2, the general solution

$$w_{\text{homog}} = C\sin\varphi\,.$$

To get a particular solution of (a), we use the variation of parameters, searching for it in the form

$$w_{\text{part}} = C(\varphi)\sin\varphi\,.$$

Replacing this in (a) yields

$$w_{\text{part}} = \sin\varphi\int\frac{f(\varphi)}{\sin\varphi}\mathrm{d}\varphi\,.$$

Thus, the general solution of (a) is

$$w(\varphi) = \left(C + \int\frac{f(\varphi)}{\sin\varphi}\mathrm{d}\varphi\right)\sin\varphi, \quad C \in \Re\,. \tag{c}$$

In the case of the spherical dome, one has

$$f(\varphi) = \frac{ga^2(1+\nu)}{E\delta}\left(\cos\varphi - \frac{2}{1+\cos\varphi}\right), \tag{d}$$

where E represents the modulus of longitudinal elasticity, v is Poisson's ratio and δ is the thickness of the shell, assumed constant.
In the particular case of the loading (d), we directly replace the expression of f in (c). After integration, we get the closed formula

$$w(\varphi) = \frac{ga^2(1+\nu)}{E\delta}\left[\ln(1+\cos\varphi) - \frac{1}{1+\cos\varphi}\right]\sin\varphi + C\sin\varphi, \quad C \in \Re\,. \tag{e}$$

We get the constant C requiring null displacements along the inferior circle of support, defined by the angle $\varphi = \varphi_i$

$$w(\varphi_i) = 0\,. \tag{f}$$

This is a Cauchy condition, associated to the ODE (a). We deduce

$$C = -\frac{ga^2(1+\nu)}{E\delta}\left[\ln(1+\cos\varphi_i) - \frac{1}{1+\cos\varphi_i}\right]. \tag{g}$$

Finally, the solution of the Cauchy problem (a), (f) is

$$w(\varphi) = \frac{ga^2(1+\nu)}{E\delta}\left[\ln\frac{1+\cos\varphi}{1+\cos\varphi_i} - \frac{1}{1+\cos\varphi} + \frac{1}{1+\cos\varphi_i}\right]\sin\varphi\,.$$

Application 1.6

Problem. Let P be a particle of mass m,acted upon by an elastic force of attraction $\mathbf{F} = -k\mathbf{r}$, where $\mathbf{r}$ is the position vector and $k > 0$ is a coefficient of elasticity. Study the motion of P.

Mathematical model. Newton's equation of motion

$$m\ddot{\mathbf{r}} = \mathbf{F} = -k\mathbf{r}\,, \tag{a}$$

may be written in the form of a second order vector ODE

$$\ddot{\mathbf{r}} + \omega^2\mathbf{r} = \mathbf{0}\,, \tag{b}$$

where $\omega^2 = k/m$.
To (b) we can add the initial (Cauchy) conditions

$$\mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0\,. \tag{c}$$

Solution. The mathematical model (b) is a second order homogeneous vector ODE, of a type studied at Sec.2.2. We must find a fundamental system of solutions for the scalar equation $\ddot{y} + \omega^2 y = 0$, searching them in the exponential form $e^{\alpha t}$. We get the characteristic equation $\alpha^2 + \omega^2 = 0$, allowing only the purely imaginary roots $\pm i\omega$. Using Euler's formulae, we obtain the solutions $\cos \omega t, \sin \omega t$, that form a fundamental system. The vector solution of (b) is

$$\mathbf{r}(t) = \mathbf{A} \cos \omega t + \mathbf{B} \sin \omega t , \tag{d}$$

with **A** and **B** arbitrary constant vectors. Imposing the initial conditions (c), we find (see Fig.1.7)

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 \cos \omega t + \frac{\mathbf{v}_0}{\omega} \sin \omega t, \\ \mathbf{v}(t) &= \mathbf{v}_0 \cos \omega t - \omega \mathbf{r}_0 \sin \omega t. \end{aligned} \tag{e}$$

Mechanical interpretation. We observe that $\mathbf{r} \times \mathbf{v} = \mathbf{r}_0 \times \mathbf{v}_0$, corresponding to the first integral of areas. The vector $\mathbf{r}$ is a linear combination of $\mathbf{r}_0, \mathbf{v}_0$; consequently, the trajectory is a *plane curve*, except for the case $\mathbf{r}_0 \times \mathbf{v}_0 = \mathbf{0}$, which means that $\mathbf{r}_0, \mathbf{v}_0$ are collinear. The trajectory does not pass through the origin, because $\mathbf{r}(t) \neq \mathbf{0}$ for any t.

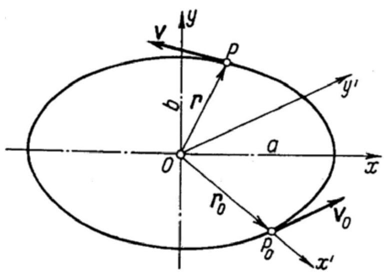

Figure 1. 7. The elliptic oscillator

We see that $|\mathbf{r}| \leq |\mathbf{r}_0| + |\mathbf{v}_0 / \omega|$ for any t, so that all the points of the trajectory lay at *finite distance*. The trajectory is a *closed curve*, surrounding the centre O, which is a stable position of equilibrium; the orbit can be included in an arbitrarily small circle and the particle velocity can be also arbitrarily small. The motion is periodic, as the particle returns at the same point $\mathbf{r}(t+T) = \mathbf{r}(t)$ with the same velocity $\mathbf{v}(t+T) = \mathbf{v}(t)$, after the same period of time

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} .$$

The pole O is a *centre of symmetry* of the motion, because $\mathbf{r}(t + T/2) = -\mathbf{r}(t)$, $\mathbf{v}(t + T/2) = -\mathbf{v}(t)$. The velocity vector is finite too, is continuous and is different from zero no matter *t*. Using the oblique co-ordinate system *Ox'y'*, determined by the conjugate diameters corresponding to the vectors $\mathbf{r}_0$ and $\mathbf{v}_0$, we get the parametric equations of the trajectory

$$x' = r_0 \cos \omega t, \qquad y' = \frac{v_0}{\omega} \sin \omega t, \tag{f}$$

which is an *ellipse* of equation

$$\frac{x'^2}{r_0^2} + \frac{y'^2}{(v_0 / \omega)^2} = 1, \tag{g}$$

known as the elliptic oscillator.
We notice that

$$\mathbf{r} \cdot \mathbf{v} = \mathbf{r}_0 \cdot \mathbf{v}_0 \cos \omega t + \frac{1}{2\omega}\left(v_0^2 - r_0^2 \omega^2\right) \sin 2\omega t \, ; \tag{h}$$

hence, to obtain a circular oscillator ($\mathbf{r} \cdot \mathbf{v} = 0, \forall t$) it is necessary and sufficient that the initial conditions of the elliptic case verify the relationships $\mathbf{r}_0 \cdot \mathbf{v}_0 = 0$ and $v_0 = r_0 \omega$. In the case of the circular oscillator, the motion is uniform (v = const), because one has

$$v^2 = v_0^2 + \left(\omega^2 r_0^2 - v_0^2\right) \sin^2 \omega t - \omega \mathbf{r}_0 \cdot \mathbf{v}_0 \sin 2\omega t \, , \tag{i}$$

as a consequence of the above conditions.
The number which shows how many times the particle travels through the whole trajectory in a unit time is called the *frequency* of the motion and is given by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \, ; \tag{j}$$

we notice that the pulsation $\omega = 2\pi f$ represents the number of periods in 2π units of time , and the denomination of *circular frequency*, also used, is thus justified.

Application 1.7

Problem. Study the motion of a particle *P* of mass *m*, acted upon by an elastic repulsive force, $\mathbf{F} = k\mathbf{r}$, where **r** is the position vector and $k > 0$ is a coefficient of elasticity.

Mathematical model. Newton's equation of motion may be written in the form of a second order vector ODE

$$\ddot{\mathbf{r}} - \omega^2 \mathbf{r} = \mathbf{0} \, , \tag{a}$$

where $\omega^2 = k / m$.
To (a) we can add the initial (Cauchy) conditions

$$\mathbf{r}(0) = \mathbf{r}_0, \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0 . \tag{b}$$

Solution. The mathematical model (a) is a second order homogeneous vector ODE, of a type studied at Sec.2.2. We must find a fundamental system of solutions for the scalar equation $\ddot{y} - \omega^2 y = 0$, searching them in the exponential form $e^{\alpha t}$. We get the characteristic equation $\alpha^2 - \omega^2 = 0$, allowing the real and distinct roots $\pm\omega$. Using the *hyperbolic functions*, we obtain the solutions $\cosh \omega t, \sinh \omega t$, that form a fundamental system. The vector solution of (a) is

$$\mathbf{r}(t) = \mathbf{A} \cosh \omega t + \mathbf{B} \sinh \omega t , \tag{c}$$

with **A** and **B** arbitrary constant vectors.
The initial conditions (b) lead to (Fig.1.8)

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}_0 \cosh \omega t + \frac{\mathbf{v}_0}{\omega} \sinh \omega t, \\ \mathbf{v}(t) &= \mathbf{v}_0 \cosh \omega t + \omega \mathbf{r}_0 \sinh \omega t. \end{aligned} \tag{d}$$

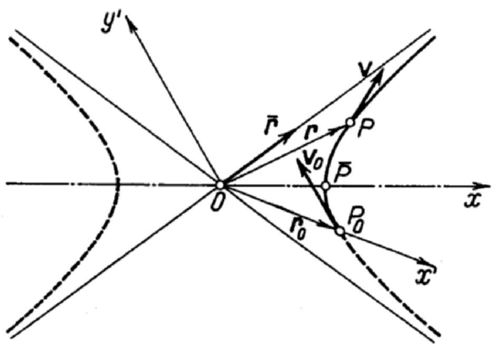

Figure 1. 8. The motion on a hyperbola under the action of repulsive forces

Mechanical interpretation. With respect to an oblique co-ordinate system *Ox'y'*, determined by the conjugate diameters corresponding to the vectors $\mathbf{r}_0$ and $\mathbf{v}_0$, we get the parametric equations of the trajectory

$$x' = r_0 \cosh \omega t, \qquad y' = \frac{v_0}{\omega} \sinh \omega t, \tag{e}$$

which is an *arc of hyperbola*, of equation

$$\frac{x'^2}{r_0^2} - \frac{y'^2}{(v_0 / \omega)^2} = 1 . \tag{f}$$

It is seen that the centre O is a *labile position of equilibrium*, as the orbit cannot be contained inside an arbitrarily small circle and the velocity of the particle may increase indefinitely.
The particle travels only once through the trajectory and does not return to the initial position. Putting (d) in the form

$$\mathbf{r}(t) = \left(\mathbf{r}_0 + \frac{\mathbf{v}_0}{\omega}\tanh\omega t\right)\cosh\omega t,$$
$$\mathbf{v}(t) = \omega\left(\mathbf{r}_0 \tanh\omega t + \frac{\mathbf{v}_0}{\omega}\right)\cosh\omega t \tag{g}$$

and noticing that $\lim_{t\to\infty}\tanh\omega t = 1$, it follows that the trajectory tends asymptotically to

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{v}_0}{\omega}\,. \tag{h}$$

Let us note that the velocity also tends to a vector with the same direction.

Application 1.8

Problem. Study the oscillatory motion of a heavy particle P on a cycloid $\mathcal{C}$ of horizontal basis, laying in a vertical plane, of concavity directed upwards (*the cycloidal pendulum*).

Mathematical model. Let us take the tangent to the cycloid at its lowest point as Ox- axis and the Oy-axis be the symmetry axis of the cycloid, being ascendent (Fig.1.9). The parametric equations of the cycloid $\mathcal{C}$ are then

$$x = a(\theta + \sin\theta),\quad y = a(1 - \cos\theta),\quad \theta \in [-\pi, \pi], \tag{a}$$

where $y = 2a$ is the right line along which the cycloid generating circle – of centre O' and radius a – is rolling without sliding. Starting from $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, we find $\mathrm{d}s = 2a\cos(\theta/2)\mathrm{d}\theta = \sqrt{2a/y}\,\mathrm{d}y$. Integrating this with respect to y, it is obtained $s = 2\sqrt{2ay} = 4a\sin(\theta/2)$, so that $\mathrm{d}y/\mathrm{d}s = s/4a$.
If m is the particle mass and g – the gravity acceleration, the motion equations read

$$m\dot{v} = m\ddot{s} = F_{\boldsymbol{\tau}} = -mg\frac{\mathrm{d}y}{\mathrm{d}s},\quad \frac{mv^2}{\rho} = F_{\boldsymbol{\nu}} + R\,, \tag{b}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are the unit vectors of the tangent and, accordingly, of the principal normal, and ρ is the curvature radius of the cycloid. It results

$$\ddot{s} + \omega^2 s = 0,\quad \omega^2 = \frac{g}{4a}\,; \tag{c}$$

This is a linear and homogeneous second order ODE with constant coefficients. Searching for solutions of the exponential type $e^{\alpha t}$, we firstly get the characteristic equation $\alpha^2 + \omega^2 = 0$, allowing only the purely imaginary roots $\pm i\omega$. We then find the fundamental system of solutions $\cos\omega t, \sin\omega t$ by using Euler's formulae. The general solution of the ODE (b) is thus

$$s(t) = c_1 \cos\omega t + c_2 \sin\omega t . \qquad \text{(d)}$$

Assume now that the particle is launched with null initial velocity from the point P_0 of curvilinear co-ordinate s_0 at the moment $t_0 = 0$; this corresponds to the Cauchy conditions

$$s(0) = s_0, \qquad \dot{s}(0) = 0 . \qquad \text{(e)}$$

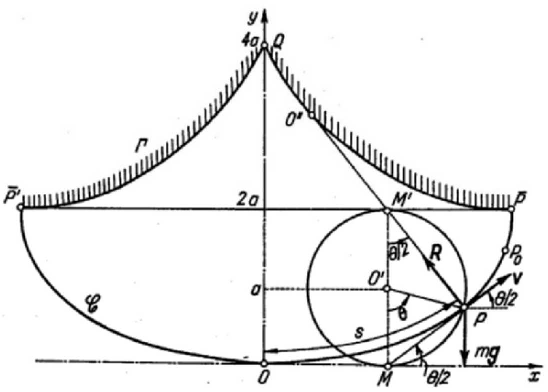

Figure 1. 9. The cycloidal pendulum

The solution of the Cauchy problem (c), (e) is then

$$s(t) = s_0 \cos\omega t . \qquad \text{(f)}$$

Mechanical interpretation. The *period* of the motion is

$$T = \frac{2\pi}{\omega} = 4\pi\sqrt{\frac{a}{g}} = 2\pi\sqrt{\frac{4a}{g}} ; \qquad \text{(g)}$$

this period does not depend on the amplitude *s*, so that the oscillations are isochronic (immaterial of their magnitude). On the other hand, the particle in free falling from the point P_0 attains the point O – the lowest point of the cycloid – in a time of $T/4$, no matter s_0, therefore it is independent on the initial position; this is the property of *tautochronism* of the cycloid. We say that the motion is *tautochronous*, i.e., immaterial of the magnitude of oscillations, the cycloid being thus a *tautochronous curve*. This property was emphasized by Huygens, who realized a cycloidal pendulum by means of the evolute Γ of a cycloid, Γ being itself a cycloid. The thread linking the particle *P* (unilateral constraint) is fixed at the point *Q*, the cuspidal point of a set up cycloid (Fig.1.9); but the occurring resistance considerably modifies the motion.

Integrating the first equation (b), we get $v^2 = 2g(y_0 - y)$, where the ordinate y_0 corresponds to the initial position P_0. As

$$\mathrm{d}x = a(1+\cos\theta)\mathrm{d}\theta = 2a\cos^2(\theta/2)\mathrm{d}\theta = \cos(\theta/2)\mathrm{d}s\,,$$

we have $F_\mathrm{v} = -mg\mathrm{d}x/\mathrm{d}s = -mg\cos(\theta/2)$; but $\rho = \overline{PO''} = 2\overline{PM'} = 4a\cos(\theta/2)$ and thus the second equation (b) provides the constraint force in the form

$$R = mg\left(\cos\frac{\theta}{2} + \frac{\cos\theta - \cos\theta_0}{2\cos\dfrac{\theta}{2}}\right). \tag{h}$$

If, in particular, $\theta_0 = \pm\pi$, meaning that the particle is left to travel along the cycloid with null initial velocity, starting from one of the cuspidal points $\overline{P}$ or $\overline{P'}$, it results

$$R = 2mg\cos\left(\frac{\theta}{2}\right) = -2F_\mathrm{v}\,; \tag{i}$$

we can state in this case, following Euler, that *the modulus of the constraint force is twice as much as the modulus of the normal component of the particle weight.*

Application 1.9

Problem. Study the motion of a heavy particle *P* acted upon by an elastic force of attraction of fixed support (the *Ox* – axis), of the form $F(x) = -kx$, $k > 0$ being an elastic constant.

Mathematical model. Using the results of Appl.1.6, we may write the equation of motion in the form

$$\ddot{x} + \omega^2 x = 0\,, \tag{a}$$

with the initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = v(0) = v_0\,. \tag{b}$$

Solution. The second order linear and homogeneous ODE (a) is with constant coefficients and was already solved at Appl.1.6. We thus find

$$\begin{aligned} x(t) &= x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t = a \cos(\omega t - \varphi), \\ \dot{x}(t) &= v(t) = v_0 \cos \omega t - \omega x_0 \sin \omega t = -a\omega \sin(\omega t - \varphi). \end{aligned} \tag{c}$$

where

$$a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \varphi = \arctan \frac{v_0}{\omega x_0}. \tag{d}$$

Mechanical interpretation. In (c), a is the *amplitude* of the oscillation, i.e., the maximum elongation, *the elongation* $|x|$ being the distance from the centre O to an arbitrary position of the particle; φ is the phase difference, computed with respect to the phase ωt, such as the whole argument $\omega t - \varphi$ represents the phase at the moment t. The trajectory is the segment of a line $\overline{A}A$, travelled through back and forth during the period $T = 2\pi / \omega$, starting with the initial position P_0 (Fig.1.10, a).

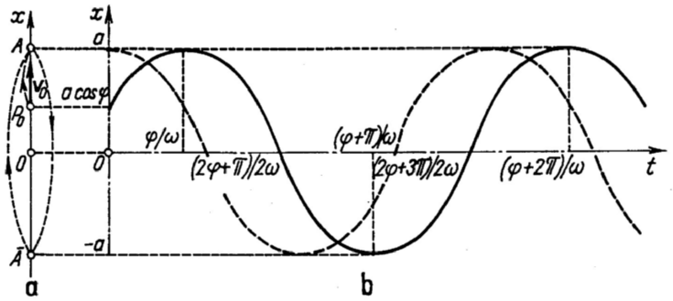

Figure 1. 10. The linear oscillator (a). Diagram of the motion (b)

Therefore, the motion is oscillatory, around the oscillation centre O, which is a stable position of equilibrium. Because the period T, as well as the frequency $f = 1/T$, are independent on the amplitude, it results that the free linear oscillations of a particle with one degree of freedom are isochronic; on the other hand, the interval of time $T/4$ in which the segment AO is travelled through does not depend on the initial position A (more precisely, does not depend on a), the velocity vanishing at this point, so that the motion is tautochronous too. The diagram of the motion is given in Fig.1.10, b, where the phase difference effect is also emphasized.

The mechanical system formed by a particle which describes a segment of a line under the action of an elastic force is called a *linear oscillator*; this one can be also considered as a limit case of an elliptic oscillator, i.e., the case in which one of the semiaxes of the ellipse tends to zero.

We also notice a connection with the *circular oscillator*, that is, a particle of velocity $\mathbf{v}$, of constant modulus $|\mathbf{v}| = a\omega$, uniformly travelling along a circle; ω is the angular velocity. The above linear oscillator may be obtained by projecting the motion of this circular oscillator on one of its diameters $\overline{A}A$. Let the diameter $\overline{A}A$ be positioned by

the angle φ with respect to the Ox – axis and let the radius OP' be positioned by the angle $\theta = \omega t$; then we get the equation (b) (Fig.1.11).
Let us consider now two harmonic vibrations having the same direction

$$x_j(t) = a_0 \cos(\omega_j t - \varphi_j), \quad j = 1,2 . \tag{e}$$

Each $x_j(t)$ satisfies the equation

$$\ddot{x}_j + \omega_j^2 x_j = 0, \quad j = 1,2 , \tag{f}$$

in which $a_j, \omega_j, \varphi_j, \quad j = 1,2$, are the corresponding amplitudes, pulsations and phase differences. We shall study the motion obtained by their superposition.

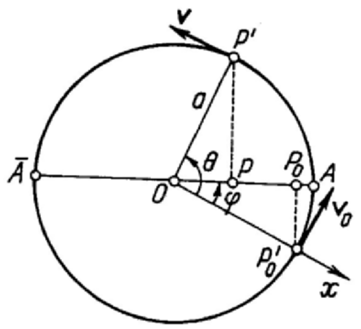

Figure 1. 11. The linear oscillator as projection of a circular one

We firstly consider two harmonic vibrations of the same pulsation $\omega_1 = \omega_2 = \omega$; the composition of these vibrations, usually called *interference* in the case of acoustic or light waves, also results in a harmonic vibration

$$x = x_1 + x_2 = a\cos(\omega t - \varphi) . \tag{g}$$

where, by identification,

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varphi_2 - \varphi_1)}, \quad \varphi = \arctan\frac{a_1 \sin\varphi_1 + a_2 \sin\varphi_2}{a_1 \cos\varphi_1 + a_2 \cos\varphi_2} . \tag{h}$$

The term $2a_1 a_2 \cos(\varphi_2 - \varphi_1)$ is called *the interference term*, producing the effect of *interference fringes*. If $\varphi_2 - \varphi_1 = 2n\pi$, then $a = a_1 + a_2$ and the interference is *constructive*, while if $\varphi_2 - \varphi_1 = (2n+1)\pi, n \in \mathbb{Z}$, then $a = |a_1 - a_2|$ and the interference is *destructive*. Finally, if $a_1 = a_2$, the destructive interference leads to *extinction* (zones in which the sound disappears, in case of acoustic waves, or zones of darkness, in case of light waves). If $\varphi_2 - \varphi_1 = \frac{(2n-1)\pi}{2}$, then $a = \sqrt{a_1^2 + a_2^2}, \quad \varphi = \arctan(a_1 / a_2)$.
In the case of composition of a certain number of harmonic vibrations, one can make an analogous computation.

If the two harmonic vibrations have not the same pulsation, then their composition, still called interference, by extension, leads to an expression of the same form, modulated in amplitude

$$a(t) = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \left[\cos(\omega_1 - \omega_2)t - \cos(\varphi_1 - \varphi_2)\right]}\,, \tag{i}$$

as well as in phase

$$\varphi = \arctan \frac{-a_1 \sin(\overline{\omega}t - \varphi_1) + a_2 \sin(\overline{\omega}t + \varphi_2)}{a_1 \cos(\overline{\omega}t - \varphi_1) + a_2 \cos(\overline{\omega}t + \varphi_2)}, \qquad \overline{\omega} = \frac{\omega_1 - \omega_2}{2}\,, \tag{j}$$

where $\omega = (\omega_1 + \omega_2)/2$. The motion thus obtained is no more harmonic, as its form depends on the amplitude, on the ratio of frequencies and on the phase differences; it is periodic only if the periods of the two motions have a common multiple, i.e. if $2\pi n_1 / \omega_1 = 2\pi n_2 / \omega_2$, $n_1, n_2 \in \mathbb{N}$, or, equivalently, $\omega_1 / \omega_2 = q \in \mathbb{Q}$ (Fig.1.12, a, b).

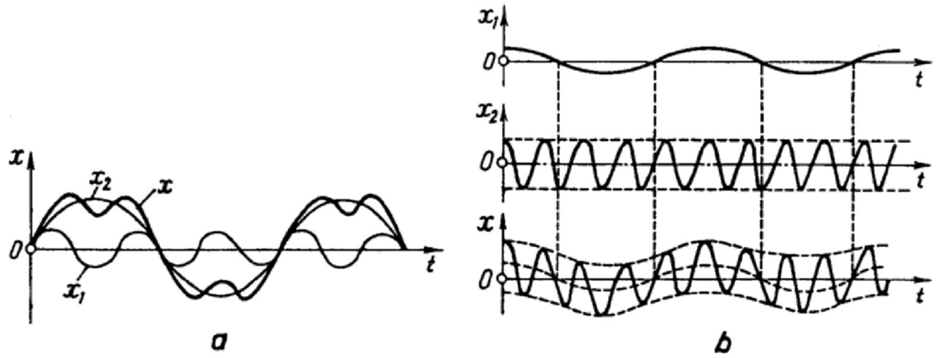

Figure 1. 12. The resultant motion of two collinear periodic motions: non-periodic case (a); periodic case (b)

The amplitude $a(t)$ varies between $a_{\min} = |a_1 - a_2|$ and $a_{\max} = a_1 + a_2$. Its maximal values are attained at intervals of time given by the periods $T_b = 2\pi / |\omega_1 - \omega_2|$ and are called *beats* (Fig.1.13), in the case of acoustic waves; the corresponding frequency will be

$$f_b = \frac{1}{2\pi}|\omega_1 - \omega_2| = |f_1 - f_2|\,, \tag{k}$$

hence it equals the modulus of the difference of the frequencies of the component motions. One may thus tune two musical instruments: the period of the beats tends to infinity if the frequencies of the two instruments tend to become equal.

One can take notice of this phenomenon the more so as the two amplitudes are close in magnitude. If $a_1 = a_2$, the formulae (i) and (j) lead to

$$x = 2a \cos\left(\frac{\omega_1 - \omega_2}{2} t - \frac{\varphi_1 - \varphi_2}{2}\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t - \frac{\varphi_1 + \varphi_2}{2}\right), \quad (l)$$

i.e., to a product of two harmonic functions.

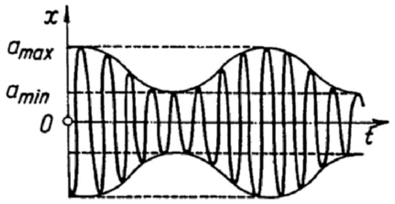

Figure 1. 13. Beats

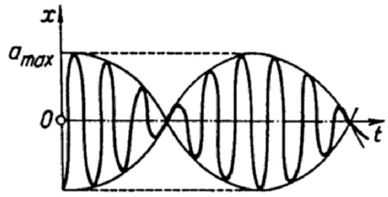

Figure 1. 14. Simple beats

In this case, $a_{\max} = 2a$, while $a_{\min} = 0$, for which one gets the *nodes* of the beats (Fig.1.14). It is seen that in this case one has *simple beats*.

Application 1.10

Problem. Study the motion of a simple pendulum in a resistent medium.

Mathematical model. Consider the simple pendulum of Appl.4.33. We introduce the resistance **R** of the medium, tangent to the trajectory and of a direction opposite to that of the velocity; the equation of motion along the tangent reads, with the notations used in the above mentioned application,

$$ml\ddot{\theta} = -mg \sin\theta - Rt . \quad \text{(a)}$$

Consider the case of small oscillations ($\sin\theta \cong \theta$); if we assume that the resistance is proportional to the velocity (*viscous damping*), of the form $R = 2\lambda ml\dot{\theta}, \lambda = \text{const}, \lambda > 0$, then the equation (a) becomes

$$\ddot{\theta} + 2\lambda\dot{\theta} + \omega^2\theta = 0 . \quad \text{(b)}$$

If, in the case of oscillations of finite amplitude, we consider a resistance proportional to the square of the velocity (*aerodynamic damping*), i.e., $R = mlk^2\dot{\theta}^2, k^2 = \text{const}$, then the equation (a) becomes

$$\ddot{\theta} + k^2\dot{\theta} + \omega^2 \sin^2\theta = 0 \quad \text{(c)}$$

for an ascendant motion; in the case of a descendant motion, k^2 will be replaced by $-k^2$.

Solution. The second order linear and homogeneous ODE (b) is with constant coefficients. Assuming that $\omega^2 > \lambda^2$ and denoting by $\mu^2 = \omega^2 - \lambda^2$, we get the general solution in the form

$$\theta = \mathrm{e}^{-\lambda t}\left(A\cos\mu t + B\sin\mu t\right), \tag{d}$$

where the constants A and B may be determined from the initial conditions $\theta(t_0) = \theta_0$, $\dot\theta(t_0) = \dot\theta_0$. Taking $t_0 = 0$ for the sake of simplicity, we may thus write

$$\theta = \mathrm{e}^{-\lambda t}\left[\theta_0\cos\mu t + \frac{1}{\mu}\left(\lambda\theta_0 + \dot\theta_0\right)\sin\mu t\right], \tag{e}$$

$$\dot\theta = \mathrm{e}^{-\lambda t}\left[\dot\theta_0\cos\mu t - \frac{1}{\mu}\left(\omega^2\theta_0 + \lambda\dot\theta_0\right)\sin\mu t\right]. \tag{f}$$

If, in particular, we take $\dot\theta_0 = 0$, then the particle moves without initial velocity from the point P_0 and attains the point P_1, where the velocity $\dot\theta = -(1/\mu)\omega^2\theta_0\mathrm{e}^{-\lambda t}\sin\mu t$ vanishes at the moment $t_1 = \pi/\mu$ (Fig.1.15); the motion continues following the same law, the particle returning till the point P_2 after a time $t_2 = 2\pi/\mu$ a.s.o.

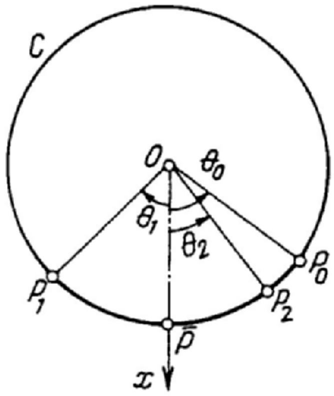

Figure 1. 15. Simple pendulum in a resistent medium

The oscillations are isochronic and the period

$$T = 2\pi/\mu = 2\pi/\sqrt{\omega^2 - \lambda^2}$$

(greater than that of the motion in vacuum) does not depend on the amplitudes

$$\theta_0 > |\theta_1| > \theta_2 > |\theta_3| > \ldots\ ;$$

we also notice that

$$|\theta_1|/\theta_0 = \theta_2/|\theta_1| = |\theta_3|/\theta_2 = \ldots = \mathrm{e}^{-\pi\lambda/\mu},$$

so that the absolute values of the amplitudes form a geometric series of ratio $\mathrm{e}^{-\pi\lambda/\mu}$. Hence, the motion is damped after an infinite time and the particle attains its lowest position (the stable position of equilibrium).

In what concerns the equation (c), we notice that $\ddot{\theta} = \dot{\theta}\mathrm{d}\dot{\theta}/\mathrm{d}\theta = \mathrm{d}(\dot{\theta}^2)/2\mathrm{d}\theta$, so that we get

$$\frac{1}{2}\frac{\mathrm{d}(\dot{\theta}^2)}{\mathrm{d}\theta} \pm k^2\dot{\theta}^2 + \omega^2 \sin^2\theta = 0, \tag{g}$$

a linear and non-homogeneous first order ODE, whose general solution is

$$\dot{\theta}^2 = Ce^{\mp 2k^2\theta} + \frac{2\omega^2}{4k^4+1}\left(\cos\theta \mp 2k^2\sin\theta\right), \tag{h}$$

where C is a constant to be determined. Actually, the relationship (h) represents an ODE with separable variables; the quadrature is easily performed for small amplitudes.

Application 1.11

Problem. Consider a particle acted upon by an elastic force of attraction $\mathbf{F} = -k\mathbf{r}$, $k > 0$ and by a damping force $\mathbf{\Phi} = -\Phi \operatorname{vers} \mathbf{v}$, tangent to the trajectory and whose direction is opposite to the direction of motion. Study the motion in case of a *viscous damping force* $\Phi = -k'v$, $k' = \text{const}$, $k' > 0$ being a damping coefficient.

Mathematical model. The equation motion in Appl.1.6 is completed in the form

$$\ddot{\mathbf{r}} + 2\lambda\dot{\mathbf{r}} + \omega^2\mathbf{r} = \mathbf{0}, \tag{a}$$

introducing the constant $\lambda = k'/2m > 0$. The damping coefficient corresponding to the relation $\omega = \lambda$ is the *coefficient of critical damping* k'_c, that does not depend on k'; we notice that, in this case,

$$k'_c = 2m\omega = 2\sqrt{km}. \tag{b}$$

We also introduce *the non-dimensional ratio of damping*

$$\chi = \frac{k'}{k'_c} = \frac{\lambda}{\omega}. \tag{c}$$

Solution. The vector ODE (a) with constant coefficients will be solved as shown at Sec.2.2. With the initial conditions $\mathbf{r}(0) = \mathbf{r}_0$, $\mathbf{v}(0) = \mathbf{v}_0$, the solution reads

$$\mathbf{r}(t) = \mathrm{e}^{-\lambda t}\left[\mathbf{r}_0 \cos\omega' t + \frac{1}{\omega'}(\mathbf{v}_0 + \lambda\mathbf{r}_0)\sin\omega' t\right], \tag{d}$$

$$\mathbf{v}(t) = \mathrm{e}^{-\lambda t}\left[\mathbf{v}_0 \cos\omega' t - \frac{1}{\omega'}(\omega^2\mathbf{r}_0 + \lambda\mathbf{v}_0)\sin\omega' t\right], \tag{e}$$

where we introduced *the pseudopulsation*

$$\omega' = \sqrt{\omega^2 - \lambda^2} = \omega\sqrt{1-\chi^2}, \tag{f}$$

assuming that $\chi < 1$, hence $\omega > \lambda$ (*subcritical damping*). The damping factor $\mathrm{e}^{-\lambda t}$ transforms the trajectory, which, in its absence, would be an ellipse, in a spiral (the vector radius $\mathbf{r}(t)$ diminishes continuously in magnitude), the particle tending in an infinite time to the origin O, with a velocity tending to zero (Fig.1.16, a).

This mechanical system is called a *damped pseudoelliptic oscillator*, the respective motion of the particle being a *pseudoperiodic damped motion*. After intervals of time equal to the pseudoperiod

$$T = \frac{2\pi}{\omega'} = \frac{2\pi}{\omega\sqrt{1-\chi^2}} = \frac{2\pi}{\omega}\left(1 + \frac{1}{2}\chi^2 + \frac{3}{8}\chi^4 - \dots\right), \tag{g}$$

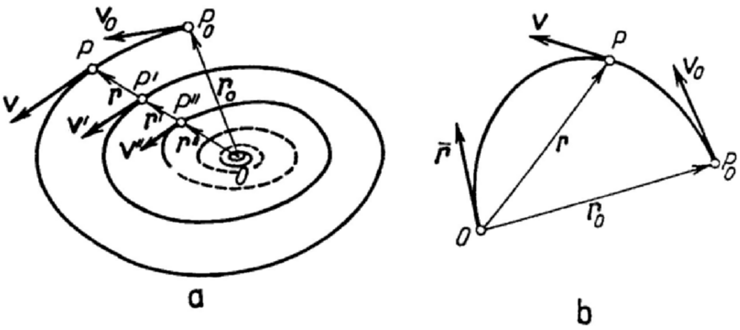

Figure 1. 16. Pseudoelliptic damped oscillator (a). Critical and supercritical damping (b)

the particle attains the points $P', P'', \dots$, all of them situated on the common support of the position vectors $\mathbf{r}'(t), \mathbf{r}''(t), \dots$, with the velocities $\mathbf{v}'(t), \mathbf{v}''(t), \dots$, of the same direction. We notice that $r'/r = r''/r' = \dots = \mathrm{e}^{-\lambda T}$, $v'/v = v''/v' = \dots = \mathrm{e}^{-\lambda T}$, thus obtaining a geometric progression of decreasing ratio $\mathrm{e}^{-\lambda T}$ of the vector radius and of the velocities; the number

$$\delta = -\lambda T = -\frac{2\pi\lambda'}{\omega'} = -\frac{2\pi\chi}{\sqrt{1-\chi^2}} = -2\pi\left(\chi + \frac{1}{2}\chi^3 + \frac{3}{8}\chi^5 - \ldots\right), \tag{h}$$

is called *the logarithmic decrement* (for $\chi << 1$ we may take $-\lambda T \cong -2\pi\chi$), being equal to $\ln(r'/r) = \ln(v'/v) = \ldots$.

If $\chi = 1$, hence if $\omega = \lambda$ (*critical damping*), then we may write

$$\mathbf{r}(t) = \mathrm{e}^{-\lambda t}\left[\mathbf{r}_0 + (\mathbf{v}_0 + \lambda\mathbf{r}_0)t\right], \quad \mathbf{v}(t) = \mathrm{e}^{-\lambda t}\left[\mathbf{v}_0 - \lambda(\mathbf{v}_0 + \lambda\mathbf{r}_0)t\right]. \tag{i}$$

The corresponding motion is damped; the trajectory starts from the point P_0 and tends, in an infinite time, with a velocity tending to zero, to the centre *O*, which is an asymptotic point (Fig.1.16, b). Noting that we may write

$$\mathbf{r}(t) = t\mathrm{e}^{-\lambda t}\left[\frac{\mathbf{r}_0}{t} + (\mathbf{v}_0 + \lambda\mathbf{r}_0)\right], \quad \mathbf{v}(t) = t\mathrm{e}^{-\lambda t}\left[\frac{\mathbf{v}_0}{t} - \lambda(\mathbf{v}_0 + \lambda\mathbf{r}_0)\right] \tag{j}$$

and that we have $\lim_{t\to\infty} t\mathrm{e}^{-\lambda t} = 0$, it results that the tangent at *O* to the trajectory is specified by the vector

$$\mathbf{r} = \mathbf{r}_0 + \frac{\mathbf{v}_0}{\lambda}. \tag{k}$$

If $\chi > 1$, hence if $\omega < \lambda$ (*supercritical damping*), then we use the notation

$$\omega'' = \sqrt{\lambda^2 - \omega^2} = \omega\sqrt{\chi^2 - 1}, \tag{l}$$

and we obtain

$$\mathbf{r}(t) = \mathrm{e}^{-\lambda t}\left[\mathbf{r}_0 \cosh\omega'' t + \frac{1}{\omega''}(\mathbf{v}_0 + \lambda\mathbf{r}_0)\sinh\omega'' t\right], \tag{m}$$

$$\mathbf{v}(t) = \mathrm{e}^{-\lambda t}\left[\mathbf{v}_0 \cosh\omega'' t - \frac{1}{\omega''}\left(\omega^2\mathbf{r}_0 + \lambda\mathbf{v}_0\right)\sinh\omega'' t\right]. \tag{n}$$

Observing that we may write

$$\mathbf{r}(t) = \mathrm{e}^{-\lambda t}\cosh\omega'' t\left[\mathbf{r}_0 + \frac{1}{\omega''}(\mathbf{v}_0 + \lambda\mathbf{r}_0)\tanh\omega'' t\right], \tag{o}$$

$$\mathbf{v}(t) = \mathrm{e}^{-\lambda t}\cosh\omega'' t\left[\mathbf{v}_0 - \frac{1}{\omega''}\left(\omega^2\mathbf{r}_0 + \lambda\mathbf{v}_0\right)\tanh\omega'' t\right] \tag{p}$$

and that

$$2\lim_{t\to\infty} e^{-\lambda t}\cosh\omega'' t = \lim_{t\to\infty} e^{-(\lambda-\omega'')t}\left(1+e^{-2\omega'' t}\right) = 0, \quad \lim_{t\to\infty}\tanh\omega'' t = 1,$$

it results that the trajectory of the particle has the same form as in the previous case (Fig.1.16, b); the tangent at the asymptotic point O will be specified by the vector

$$\overline{\mathbf{r}}(t) = \mathbf{r}_0 + \frac{1}{\omega''}(\mathbf{v}_0 + \lambda\mathbf{r}_0). \tag{q}$$

The corresponding motion is a strongly damped motion. More precisely, we may say that the last two cases correspond to *aperiodic motions*.

Application 1.12

Problem. Study the motion of a damped linear oscillator.

Mathematical model. Using the notations in Appl.1.11, we get the equation of motion (along the Ox-axis)

$$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0; \tag{a}$$

with the initial conditions $x(0) = x_0$, $v(0) = v_0$, we obtain the solution

$$x(t) = e^{-\lambda t}\left[x_0\cos\omega' t + \frac{1}{\omega'}(v_0 + \lambda x_0)\sin\omega' t\right] = a e^{-\lambda t}\cos(\omega' t - \varphi), \tag{b}$$

where we used the notations

$$a = \sqrt{x_0^2 + \frac{1}{\omega'^2}(v_0 + \lambda x_0)^2}, \quad \varphi = \arctan\frac{v_0 + \lambda x_0}{x_0\omega'}, \tag{c}$$

corresponding to a *subcritical damping* ($\chi < 1$). The motion is a pseudoperiodic damped motion of pseudoperiod $T = 2\pi/\omega'$, the trajectory – which starts from the point P_0 being contained in the segment of a line $\overline{A}A$ and tending to the asymptotic point O after an infinity of oscillations around this pole (Fig.1.17, a).

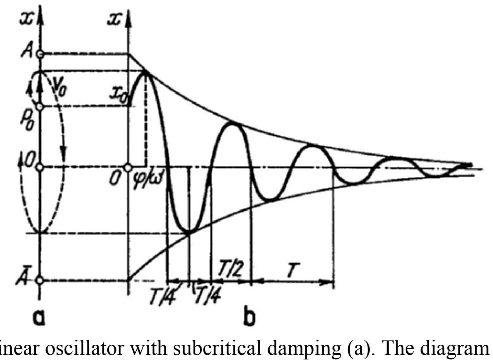

Figure 1. 17. Linear oscillator with subcritical damping (a). The diagram of the motion (b)

This motion constitutes a modulated vibration in amplitude, being strongly damped; the diagram of motion has the form of a cosinusoid contained between the curves $x = \pm a e^{-\lambda t}$ and the tangents to it at the points $t = \varphi / \omega'$, $t = \varphi / \omega' + T$,..., and $t = \varphi / \omega' + T / 2$, $t = \varphi / \omega' + 3T / 2$,..., respectively, where $T = 2\pi / \omega'$ (Fig.1.17, b).

In the case of a *critical damping* ($\chi = 1$), we obtain an aperiodic damped motion given by

$$x(t) = e^{-\lambda t}\left[x_0 + (v_0 + \lambda x_0)t\right]. \tag{d}$$

If $v_0 > 0$, then the particle starts from the point P_0, attains A at the moment $t' = v_0 / \lambda(v_0 + \lambda x_0)$ and then changes of direction, tending asymptotically to the centre O (Fig.1.18, a, b); the diagram of motion has a maximum for $t = t'$, tending then asymptotically to zero.

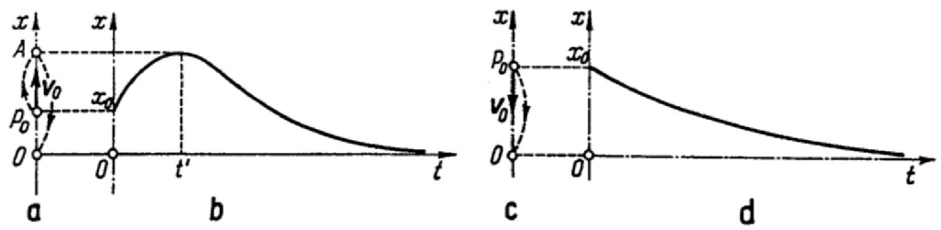

Figure 1. 18. Linear oscillator with critical damping (a) and the diagram of the motion (b). Linear oscillator with critical and sub critical damping (c) and the diagram of the motion (d)

If $-\lambda x_0 \le v_0 \le 0$, then the particle starts from the point P_0 and tends asymptotically to O (Fig.1.18, c, d); the corresponding diagram has no zeros and no extrema, yet if $-\lambda x_0 / 2 \le v_0 \le 0$ a point of inflection appears.

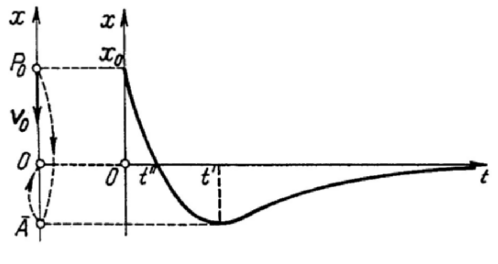

Figure 1. 19. Linear oscillator with critical and subcritical damping

If $v_0 < -\lambda x_0$, then the particle starts from P_0, passes through the centre O at the moment $t'' = -x_0 / (v_0 + \lambda x_0)$, attains $\overline{A}$ at the moment t' and then turns back asymptotically to the centre O (Fig.1.19); the diagram of motion pierces the Ot-axis at the point $t = t''$, has a minimum for $t = t'$, and tends asymptotically to zero with

negative values. If the point P_0 is on the other side of the pole *O*, hence if $x_0 < 0$, then – by symmetry – one obtains analogous results.
A *supercritical damping* ($\chi > 1$) leads to an aperiodic damped motion of the form

$$x(t) = \mathrm{e}^{-\lambda t}\left[x_0 \cosh \omega'' t + \frac{1}{\omega''}(v_0 + \lambda x_0)\sinh \omega'' t \right]. \tag{e}$$

In what concerns the trajectory and the diagram of motion, one obtains the same results as before, as $v_0 > 0$, $-(\lambda + \omega'')x_0 \le v_0 \le 0$ or as $v_0 < -(\lambda + \omega'')x_0$ (Fig.1.18, 1.19); we notice that

$$t' = \frac{1}{\omega''}\,\mathrm{arg\,tanh}\,\frac{\omega'' v_0}{\omega^2 x_0 + \lambda v_0}, \quad t'' = \frac{1}{\omega''}\,\mathrm{arg\,tanh}\,\frac{-\omega'' x_0}{v_0 + \lambda x_0}. \tag{f}$$

Application 1.13

Problem. Determine the oscillation period of a liquid in a curved pipe.

Mathematical model. By means of Bernoulli's conservation theorem of mechanical energy one can write

$$z_1 = z_2 + \frac{1}{g}\int_0^l \frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}s = z_2 + \frac{l}{g}\frac{\mathrm{d}v}{\mathrm{d}t},$$

where the data of the problem are given in Fig.1.20; it is supposed that the velocity v depends only on the time.

Solution. Noting that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2},$$

one obtains

$$\frac{l}{g}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + z_2 - z_1 = 0.$$

Using $z_1 = x\sin\alpha$, $z_2 = x\sin\beta$ (the angles α and β are given), the differential equation of the problem reads

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{g}{l}(\sin\alpha + \sin\beta)x = 0.$$

Noting $\omega^2 = (g/l)(\sin\alpha + \sin\beta)$, this equation becomes a linear second order ODE with constant coefficients

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0 \,, \tag{a}$$

whose general solution is

$$x = A\cos(\omega t - \varphi) \,.$$

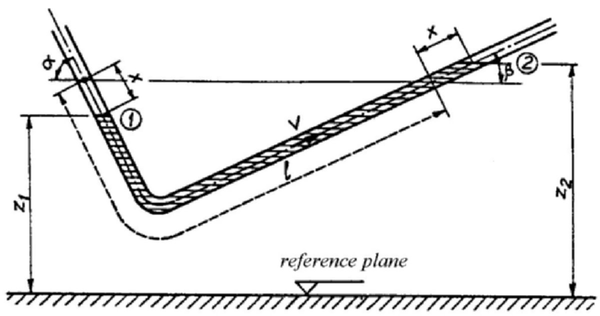

Figure 1. 20. Oscillations of a liquid in a curved pipe

The period of the proper oscillations of the liquid is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g(\sin\alpha + \sin\beta)}} \,. \tag{b}$$

Application 1.14

Problem. Study the motion of a heavy particle P (the motion of a particle in gravitational field of the Earth) of mass m, in vacuum.

Mathematical model. Newton's equation of motion is of the form

$$m\ddot{\mathbf{r}} = m\mathbf{g} \,, \tag{a}$$

where $\mathbf{g}$ is the gravitational acceleration.

Solution. By direct integration, we get

$$\mathbf{r} = \frac{1}{2}\mathbf{g}(t - t_0)^2 + \mathbf{v}_0(t - t_0) + \mathbf{r}_0, \quad \mathbf{v} = \mathbf{g}(t - t_0) + \mathbf{v}_0 \,, \tag{b}$$

where we took into account the initial conditions $\mathbf{r}(t_0) = \mathbf{r}_0$, $\mathbf{v}(t_0) = \mathbf{v}_0$; noting that $\mathbf{r} - \mathbf{r}_0$ is a linear combination of the constant vectors $\mathbf{g}$ and $\mathbf{v}_0$, it results that the trajectory is a plane curve. Without any loss of generality, we may assume that $\mathbf{r}_0 = \mathbf{0}$, $t_0 = 0$, so that

$$\mathbf{r} = \frac{1}{2}\mathbf{g}t^2 + \mathbf{v}_0 t, \quad \mathbf{v} = \mathbf{g}t + \mathbf{v}_0\,; \tag{c}$$

and also obtain the remarkable relation

$$\mathbf{r} = -\frac{1}{2}\mathbf{g}t^2 + \mathbf{v}t = \frac{1}{2}\left(\mathbf{v} + \mathbf{v}_0\right)t\,. \tag{d}$$

Mechanical intepretation. We suppose that $\mathbf{v}_0 \neq \mathbf{0}$ and has not the same direction as **g**; the velocity **v** cannot vanish in this case, and the relation (d) allows a simple graphic construction of the velocity of the particle P if its position is known or allows to set up graphically its position if one knows the velocity **v**. Projecting the equations (c) on the co-ordinate axes Ox and Oy (Oy is the ascendent vertical, while α is the angle made by the initial velocity with the Ox- axis), we get the parametric equations of the trajectory (Fig.1.21)

$$x = v_0 t\cos\alpha,\ y = -\frac{1}{2}gt^2 + v_0 t\sin\alpha \tag{e}$$

and the components of the velocity

$$v_x = v_0 \cos\alpha,\ v_y = -gt + v_0 \sin\alpha\,. \tag{f}$$

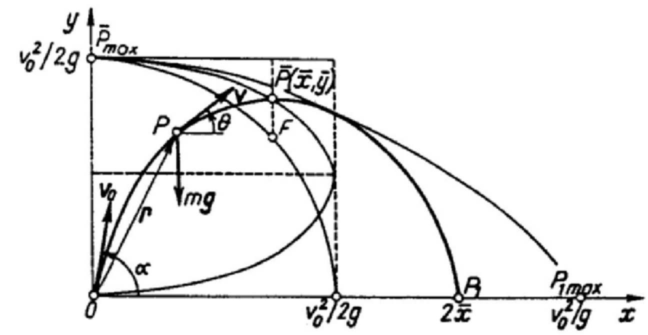

Figure 1. 21. Motion of a heavy particle in vacuum – Cauchy's problem

Eliminating the time t between the equations (e), we obtain

$$y = \frac{g}{2v_0^2\cos^2\alpha}x^2 + x\tan\alpha\,, \tag{g}$$

hence, the trajectory of the particle is a parabola. Further, taking into account (e), we may write the magnitude of the velocity in the form

$$v = \sqrt{v_0^2 - 2gy}\,. \tag{h}$$

For $\alpha > 0$, we are in *the basic problem of external ballistics* neglecting the friction with the air. The particle (an eventual projectile) obtains the highest point of the trajectory $\overline{P}$ if $v_y = 0$, hence at the moment $\bar{t} = v_0 \sin\alpha/g = v_y^0/g$; the co-ordinates of the point are

$$\bar{x} = \frac{v_0^2}{2g}\sin 2\alpha = \frac{v_x^0 v_y^0}{g}, \quad \bar{y} = \frac{v_0^2}{2g}\sin^2\alpha = \frac{\left(v_y^0\right)^2}{2g}, \tag{i}$$

and *Torricelli's formula* is given by

$$v_y^0 = \sqrt{2gh}, \quad h = \bar{y}\,. \tag{j}$$

We obtain thus the component v_y^0 of the velocity by which one must launch a projectile to attain the height *h*, immaterial of the angle α; the formula (j) takes place for the motion along the vertical ($\alpha = \pi/2$) too. The formula (h) may be written also in the form $v^2 = 2g\left(v_0^2/2g - y\right)$; one can thus state that the magnitude of the velocity at a given moment is equal to that of a particle falling, without initial velocity, from a height $v_0^2/2g$.

If $\alpha < 0$, then the particle starts from a point situated on the descending branch of the parabola.

The point P_1 of abscissa $2\bar{x} = \left(v_0^2/g\right)\sin 2\alpha$ is the most distant point attained by the projectile on a horizontal plane, at the moment $2\bar{t}$, the magnitude of the velocity being the same as that of the initial moment; the range of throw is maximal for $\alpha = \pi/4$, namely $2\bar{x}_{\max} = v_0^2/g$. If we wish to attain a point P_1 of abscissa $2\bar{x}$, the initial conditions must verify the relation $v_0^2 \sin 2\alpha = 2g\bar{x}$ (*the two-point problem*). To the same magnitude v_0 of the initial velocity correspond two angles: $\alpha < \pi/4$ and $\pi/2 - \alpha$ (symmetric with respect to the angle $\pi/4$, because $\pi/4 - \alpha = (\pi/2 - \alpha) - \pi/4$) under which one may attain the same point P_1 (Fig.1.22); in particular, if $v_0 = \sqrt{2g\bar{x}}$, then we have $\alpha = \pi/4$. To the two shooting angles there correspond the shooting heights $h = \left(v_0^2/2g\right)\sin^2\alpha$ and $h = \left(v_0^2/2g\right)\cos^2\alpha$.

If we wish that the projectile do pass through the point $P(\xi, \eta)$, then we find the condition

$$\frac{g\xi^2}{2v_0^2}\tan^2\alpha - \xi\tan\alpha + \eta + \frac{g\xi^2}{2v_0^2} = 0\,; \tag{k}$$

as in the above considered particular case, one may reach the point P, shooting a projectile under two angles specified by

$$\tan\alpha = \frac{v_0^2}{g\xi}\left[1 \pm \sqrt{1 - \frac{2g}{v_0^2}\left(\eta + \frac{g\xi^2}{2v_0^2}\right)}\right]. \tag{l}$$

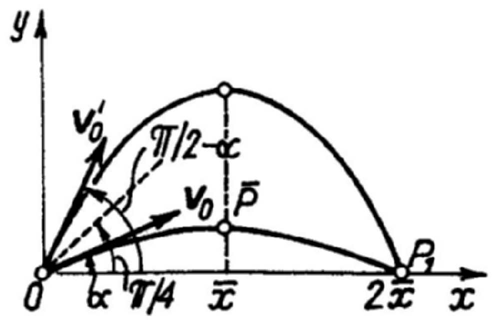

Figure 1. 22. Motion of a heavy particle in vacuum – two-point problem

To reach a point *P* by a projectile, that one must be in the interior of *the safety parabola* (Fig.1.21)

$$y = -\frac{g}{2v_0^2}x^2 + \frac{v_0^2}{2g}, \tag{m}$$

which passes through the points $\overline{P}_{\max}\left(0, v_0^2/2g\right)$ and $P_{1\max}\left(v_0^2/2g, 0\right)$; no point in the exterior of this parabola may be reached by an initial velocity of magnitude v_0. This parabola is the envelope of the family of trajectories (g) for $v_0 = \text{const}$ and α variable.

The parameter of the parabola (g) is $p = \left(v_0^2/2g\right)\cos^2\alpha$, so that the locus of the focus $F\left(\left(v_0^2/2g\right)\sin 2\alpha, -\left(v_0^2/2g\right)\cos 2\alpha\right)$ is the quarter of a circle (Fig.1.21)

$$x^2 + y^2 = \frac{v_0^4}{4g^2}, \tag{n}$$

the centre of which is the origin and which passes through the point $\overline{P}_{\max}$; all these parabolas have as directrix a parallel to the *Ox*- axis of equation $y = v_0^2/2g$, which passes through the vertex of the safety parabola. The locus of the vertices of the trajectories (g) is the ellipse (Fig.1.21)

$$x^2 + 4y^2 = \frac{2v_0^2}{g} y \,, \tag{o}$$

the minor axis of which is $\overline{OP}_{max}$, the major axis being parallel to the *Ox*- axis (the half of it is equal to $v_0^2/2g$).

Application 1.15

Problem. Consider a cantilever bar of a variable cross section, the height h being constant, while the width has a linear variation (b_0 at the free end and b_1 at the built-in cross section), which is acted upon by a concentrated force *P*. It is required:

i) to determine the rotation of the free end;

ii) to determine the deformed axis of the bar and its maximal deflection.

Particular case: $b_1 = 2b_0$ (Fig.1.23).

Mathematical model. Let $I_0 = b_0 h^3/12$ and $I_1 = b_1 h^3/12$ be the moments of inertia of the cross section with respect to the neutral axis for the free end and for the built-in end, respectively. The moment of inertia of an arbitrary cross section of abscissa *x* is given by

$$I(x) = I_0\left[\left(\frac{I_1}{I_0} - 1\right)\frac{x}{l} + 1\right] = I_0\left[\left(\frac{b_1}{b_0} - 1\right)\frac{x}{l} + 1\right] = \frac{I_0}{\beta l}(x + \beta l) \,, \tag{a}$$

with the notation

$$\frac{1}{\beta} = \frac{b_1}{b_0} - 1 \,. \tag{b}$$

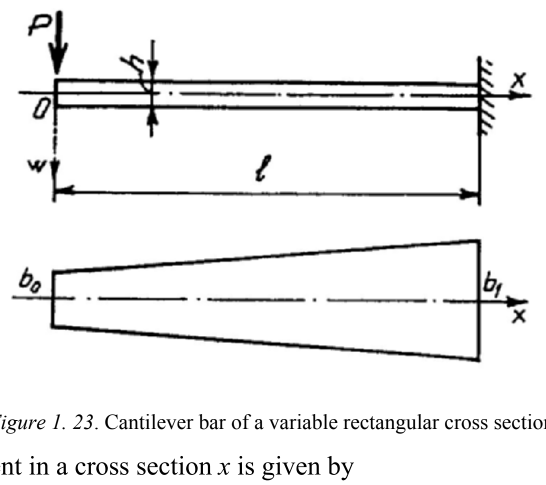

Figure 1. 23. Cantilever bar of a variable rectangular cross section

The bending moment in a cross section *x* is given by

$$M(x) = -Px \,, \tag{c}$$

so that the approximate differential equation of the bar axis is of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M(x)}{EI(x)} = \frac{Pl\beta}{EI_0}\frac{x}{x+\beta l},$$

where w is the deflection.

Solution. The rotation of the cross section is given by $\varphi = \dfrac{\mathrm{d}w}{\mathrm{d}x}$. Thus, we get

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{Pl\beta}{EI_0}\left(1 - \frac{\beta l}{x+\beta l}\right), \tag{d}$$

from which, by straightforward integration,

$$\varphi = \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{Pl\beta}{EI_0}\int\left(1 - \frac{\beta l}{x+\beta l}\right)\mathrm{d}x = \frac{Pl\beta}{EI_0}\left[x - \beta l \ln(x+\beta l) + C_1\right],$$

C_1 being an arbitrary constant. The condition $\varphi(l) = 0$ leads to

$$C_1 = -\frac{Pl\beta}{EI_0}\left[l - \beta l \ln(1+\beta)l\right],$$

and the rotation is given by

$$\varphi = \frac{Pl\beta}{EI_0}\left[x - l - \beta l \ln\frac{x+\beta l}{(1+\beta)l}\right] = \frac{Pl^2\beta}{EI_0}\left(\frac{x}{l} - 1 - \beta \ln\frac{\frac{x}{l}+\beta}{1+\beta}\right). \tag{e}$$

At the free end we have

$$\varphi_0 = \varphi_{\max} = \frac{Pl^2\beta}{EI_0}\left(-1 - \beta\ln\frac{\beta}{1+\beta}\right). \tag{f}$$

A new integration leads to the deflections

$$w = \frac{Pl\beta}{EI_0}\int\left[x - l - \beta l \ln\frac{x+\beta l}{(1+\beta)l}\right]\mathrm{d}x,$$

which gives

$$w = \frac{Pl\beta}{EI_0}\left\{\frac{x^2}{2} - lx - \beta l\left[x\ln\frac{x+\beta l}{(1+\beta)l} - x + \beta l \ln\frac{x+\beta l}{(1+\beta)l}\right] + C_2\right\}.$$

The condition $w(l)=0$ determines the new integration constant

$$C_2=\left(\frac{1}{2}-\beta\right)l^2,$$

so that the expression of the deflections reads

$$w=\frac{Pl^3}{EI_0}\beta\left[\frac{1}{2}\left(1-\frac{x}{l}\right)^2-\beta\left(1-\frac{x}{l}\right)-\beta\left(\frac{x}{l}+\beta\right)\ln\frac{\frac{x}{l}+\beta}{1+\beta}\right]. \tag{g}$$

The maximal deflection is obtained at the free end $(x=0)$ and reads

$$w_0=w_{\max}=\frac{Pl^3}{EI_0}\beta\left(\frac{1}{2}-\beta-\beta^2\ln\frac{\beta}{1+\beta}\right). \tag{h}$$

In the particular case $(b_1/b_0=2)$ it results $\beta=1$ and the maximal rotation (f) becomes

$$\varphi_{\max}=\frac{Pl^2}{EI_0}\left(-1-\ln\frac{1}{2}\right)=\frac{Pl^2}{EI_0}(-1+0.69314718)=-0.306852819\frac{Pl^2}{EI_0},$$

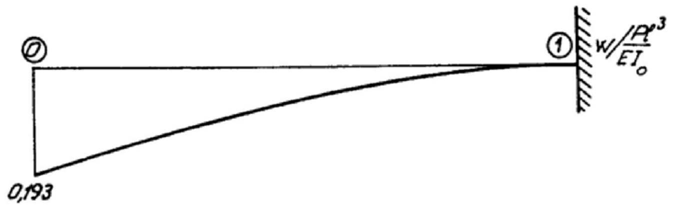

Figure 1. 24. Diagram of the deflections *w*

while the deflection (g) is given by

$$w=\frac{Pl^3}{EI_0}\left[-\frac{1}{2}\left(1-\frac{x^2}{l^2}\right)-\left(1+\frac{x}{l}\right)\ln\frac{\frac{x}{l}+1}{2}\right];$$

their diagram is plotted in Fig.1.24.

Application 1.16

Problem. A cantilever bar of span *l* has a variable circular cross section of radius r. We have to determine:

i) the profile of the bar so as to be of equal resistance for a concentrated force *P* acting upon the free end;

ii) the maximal deflection at the same cross section (Fig.1.25).

Mathematical model. Let $r = r(x)$ be the radius of the cross section at the abscissa x (Fig.1.25, a). The moment of inertia and the modulus of resistance are given by

$$I(x) = \frac{\pi r^4}{4},\ W(x) = \frac{\pi r^3}{4},$$

respectively. For the bending moment $M(x) = -Px$ (Fig.1.25, b) the normal stress (in absolute value) is given by Navier's formula

$$\sigma_{\max} = \frac{M(x)}{W(x)} = \frac{Px}{\pi r^3} = \frac{4P}{\pi}\frac{x}{r^3} = \sigma_a, \tag{a}$$

and is equated to the admissible stress σ_a.
The approximate differential equation of the deflection w is of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M(x)}{EI(x)} = \frac{Px}{EI_0}\left(\frac{l}{x}\right)^{4/3} = \frac{Pl^{4/3}}{EI_0}x^{-1/3}, \tag{b}$$

where E is the modulus of longitudinal elasticity.

Solution. From (a) we get

$$r^3 = \frac{4Px}{\pi\sigma_a},\ r = \sqrt[3]{\frac{4Pl}{\pi\sigma_a}}\sqrt[3]{\frac{x}{l}};$$

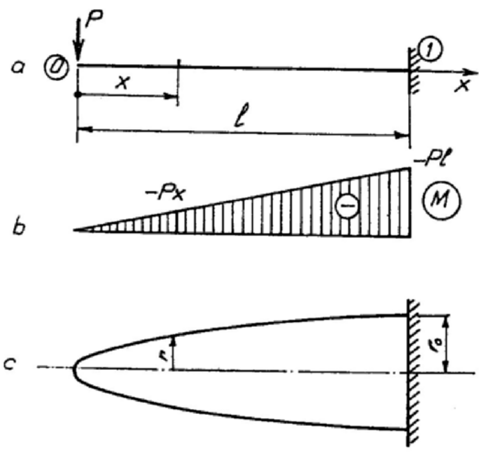

Figure 1. 25. Cantilever bar with a variable circular cross section (a). Diagram of bending moments (b). Variation of the radius r (c)

denoting by

$$r_0 = \sqrt[3]{\frac{4Pl}{\pi\sigma_a}}$$

the radius of the built-in cross section, it results $r = r_0 \sqrt[3]{x/l}$, so that one obtains a cubic parabola (Fig.1.25, c). Numerical values for the ratio r/r_0 as a function of the non-dimensional abscissa x/l are given in Table 1.1.

Table 1. 1. The values of r/r_0

x/l	r/r_0	x/l	r/r_0	x/l	r/r_0
0	0	0.333…	0.6934	0.70	0.8879
0.10	0.4642	0.40	0.7368	0.75	0.9086
0.20	0.5848	0.50	0.7937	0.80	0.9283
0.25	0.6300	0.60	0.8434	0.90	0.9655
0.30	0.6694	0.666…	0.8736	1	1

The moment of inertia becomes

$$I(x) = \frac{\pi r_0^4}{4}\left(\frac{x}{l}\right)^{4/3} = I_0\left(\frac{x}{l}\right)^{4/3}, I_0 = \frac{\pi r_0^4}{4} .$$

Integrating the equation, one obtains successively

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{Pl^{4/3}}{EI_0}\left(\frac{3}{2}x^{2/3} + C_1\right), w = \frac{Pl^{4/3}}{EI_0}\left(\frac{9}{10}x^{5/3} + C_1 x + C_2\right),$$

where C_1 and C_2 are integration constants determined by the conditions

$$\frac{\mathrm{d}w}{\mathrm{d}x} = 0, w = 0 \text{ for } x = l .$$

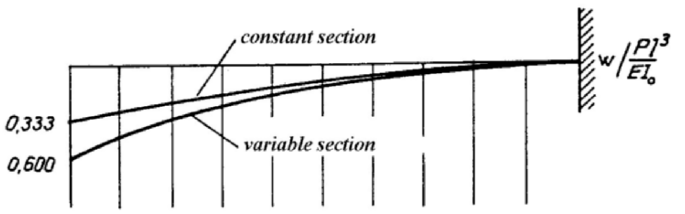

Figure 1. 26. Diagram of the deflections *w*

It is easily seen that $C_1 = -(3/2)l^{2/3}$, $C_2 = (3/5)l^{5/3}$. The deflection becomes (Fig.1.26)

$$w = \frac{Pl^{4/3}}{EI_0}\left(\frac{9}{10}x^{5/3} - \frac{15}{10}l^{2/3}x + \frac{6}{10}l^{5/3}\right) = \frac{3}{10}\frac{Pl^3}{EI_0}\left[3\left(\frac{x}{l}\right)^{5/3} - 5\frac{x}{l} + 2\right].$$

The maximal deflection is obtained at the free end $(x = 0)$

$$w_{\max} = C_2 = \frac{3}{5}\frac{Pl^3}{EI_0} = 1.8\frac{Pl^3}{EI_0},$$

and is with 80% greater than the maximal deflection of the cantilever bar of constant circular cross section of moment of inertia I_0.

Application 1.17

Problem. Study the motion of a particle of mass *m* subjected to the action of a force of Newtonian attraction

$$F = -f\frac{mM}{r^2}, \tag{a}$$

where *M* is the mass of the attracting particle, *r* the distance between the two particles and $f = 6.6732\cdot 10^{-8} \cong 1/3871^2$ cm/$g\cdot s^2$ is a coefficient of universal attraction.

Mathematical model. The force *F* is a central force (the particle of mass *M* is considered fixed), so that we may consider *Binet's theory* (see Appl.4.25); one obtains the equation

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{fM}{C^2}, \tag{b}$$

in polar co-ordinates r, θ, where *C* is the constant of areas.

Solution. The associated equation is a non-homogeneaous linear second order ODE with constant coefficients, with respect to $1/r$. Integrating, we get

$$\frac{1}{r} = C_1\cos(\theta - C_2) + \frac{fM}{C^2}, \tag{c}$$

where C_1, C_2 are two scalar integration constants.

Using the notations $C_1 = e/p$, $C_2 = \theta_1$, $p = C^2/fM$, we find the equation of a conic, in polar co-ordinates, with respect to the focus *F* and to an axis inclined by θ_1 with respect to the apsidal line in the form

$$r = \frac{p}{1 + e\cos(\theta - \theta_1)}. \tag{d}$$

The conditions at the initial moment $t = t_0$ (as in Appl.4.25) lead to the parameter of the conic

$$p = \frac{C^2}{fM} = \frac{r_0^2 v_0^2 \sin^2 \alpha_0}{fM}, \tag{e}$$

where r_0, v_0 correspond to the initial conditions, while α_0 is the angle formed by those vectors.
Analogously, the eccentricity *e* and the angle θ_1 are given by

$$1 + e\cos(\theta - \theta_1) = \frac{p}{r_0},\ e\sin(\theta - \theta_1) = \frac{p}{r_0}\cot\alpha_0, \tag{f}$$

whence

$$\begin{aligned} e^2 &= \left(\frac{p}{r_0} - 1\right)^2 + \frac{p^2}{r_0^2}\cot^2\alpha_0 = 1 + \frac{p}{r_0}\left(\frac{p}{r_0 \sin^2\alpha_0} - 2\right) \\ &= 1 + \frac{r_0 v_0^2 \sin^2\alpha_0}{fM}\left(\frac{r_0 v_0^2}{fM} - 2\right), \end{aligned} \tag{g}$$

$$\tan(\theta - \theta_1) = \frac{p\cot\alpha_0}{p - r_0} = \frac{r_0 v_0^2 \sin\alpha_0 \cos\alpha_0}{r_0 v_0^2 \sin^2\alpha_0 - fM}. \tag{h}$$

Hence, the trajectory is an *ellipse* if $r_0 v_0^2 < 2fM$, a *parabola* if $r_0 v_0^2 = 2fM$ or a *hyperbola* if $r_0 v_0^2 > 2fM$.

Mechanical interpretation. The genus of the conic depends only on the initial distance to the centre of attraction (radius r_0), on the intensity of this centre (the mass *M*), and on the intensity of the initial velocity (the velocity v_0), but does not depend on the direction of this velocity (angle α_0). As (f) yields $p = r_0$ and $\alpha_0 = \pi/2$, the condition $e = 0$ leads to

$$\frac{r_0 v_0^2}{fM}\left(\frac{r_0 v_0^2}{fM} - 2\right) = -1;$$

hence, the orbit is circular if $r_0 v_0^2 = fM$ (one can see that $\alpha_0 = \pi/2$ is now a consequence).
Using the results of Appl.4.25 , we notice that

$$U(r) = \frac{fmM}{r}, \overline{U}(r) = \frac{fmM}{r} - \frac{mC^2}{2r^2}, \varphi(r) = \frac{2fM}{r} - \frac{C^2}{r^2} + \frac{2h}{m}. \tag{i}$$

We choose the Ox- axis so as to be an apsidal line; we are thus led to the equation of the trajectory in polar co-ordinates (we take $\theta_0 = 0$)

$$\theta = C\int_{r_{\min}}^{r} \frac{\mathrm{d}\rho}{\sqrt{\frac{2fM}{\rho} - \frac{C^2}{\rho^2} + \frac{2h}{m}}} = \int_{1/r}^{1/r_{\min}} \frac{\mathrm{d}(1/\rho)}{\frac{f^2M^2}{C^4} + \frac{2h}{mC^2} - \left(\frac{1}{\rho} - \frac{fM}{C^2}\right)^2}$$

$$= \arccos \frac{\frac{1}{r} - \frac{fM}{C^2}}{\frac{1}{\frac{1}{mC}\sqrt{\frac{f^2m^2M^2}{C^2} + 2mh}}},$$

where we noticed that $r = r_{\min}$ corresponds to $\theta = 0$. We find again the equation (d) of a conic, with

$$p = \frac{C^2}{fM}, e = \sqrt{1 + \frac{2C^2h}{f^2mM^2}}. \tag{j}$$

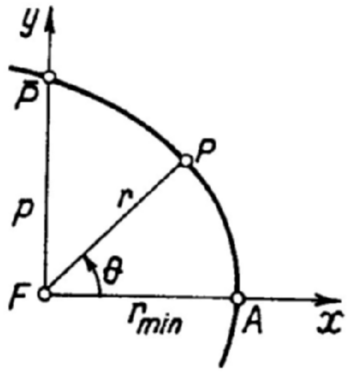

Figure 1. 27. The orbit in case of a force of Newtonian attraction

It results (we observe that $x = r\cos\theta$) (Fig.1.27)

$$x^2 + y^2 - (ex - p)^2 = 0 \tag{k}$$

in Cartesian co-ordinates; the conic pierces the co-ordinate axes at the points $(r_{\min}, 0)$ and $(0, p)$, obtaining thus a geometric interpretation for the parameter of the conic. From the expression of the eccentricity one sees that the trajectory is an *ellipse*, a *parabola* or a *hyperbola* as $h < 0$, $h = 0$, $h > 0$, respectively; in particular, if $h = -f^2mM/2C^2$, then $e = 0$ and the ellipse is a *circle*.

The law of motion of the particle on the trajectory is of the form

$$t = t_0 + \int_{r_{\min}}^{r} \frac{d\rho}{\sqrt{\frac{2fM}{\rho} - \frac{C^2}{\rho^2} + \frac{2h}{m}}}. \tag{l}$$

In the case of an elliptic motion we write the equation (k) in the form (we have $0 \le e < 1$)

$$\left(1-e^2\right)x^2 + y^2 + 2pex = p^2.$$

We notice that one may write this equation also in the form

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{m}$$

where the semiaxes

$$a = \frac{p}{1-e^2} = -\frac{fmM}{2h},\ b = \frac{p}{\sqrt{1-e^2}} = C\sqrt{-\frac{m}{2k}} \tag{n}$$

and the focal distance (Fig.1.28)

$$c = ae = \frac{pe}{1-e^2} = -\frac{fmMe}{2h} = \sqrt{a^2 - b^2} \tag{o}$$

are emphasized; we mention that, for a given potential – fmM is given – the semi-major axis of the ellipse depends only on the constant mechanical energy h. We may express

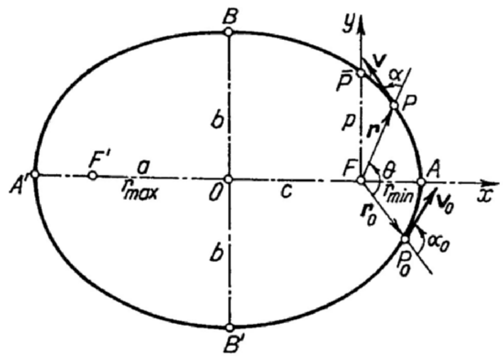

Figure 1. 28. Elliptic orbit

the semiaxes of the ellipse in the form

$$a = -\frac{1}{2\left(\frac{1}{r_0} - \frac{v_0{}^2}{2fM}\right)}, b = \frac{r_0 v_0 \sin\alpha_0}{\sqrt{\frac{2fM}{r_0} - v_0^2}}, \tag{p}$$

with respect to the initial conditions; we notice thus that a does not depend on the direction of the initial velocity.

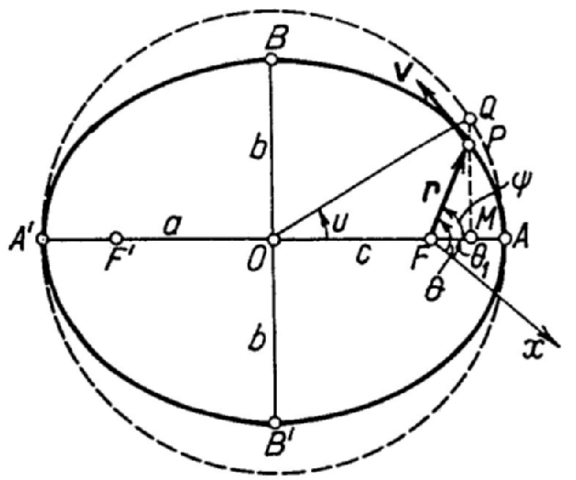

Figure 1. 29. Kepler's ellipse

Taking into account the above used notations, we see that the relations

$$fmM = -2\alpha h, C^2 = -\frac{2h}{m}\frac{p^2}{1-e^2} = -\frac{2h}{m}a^2\left(1-e^2\right), r_{\min} = a(1-e) \tag{q}$$

hold true; we thus obtain the law of motion of the particle along the ellipse (l) in the form

$$t = t_0 + \sqrt{-\frac{m}{2h}}\int_{(1-e)a}^{r}\frac{\rho\mathrm{d}\rho}{\sqrt{a^2e^2-(a-\rho)^2}}; \tag{r}$$

by the change of variable $\rho = a(1 - e\cos u)$, we may write

$$t = t_0 + a\sqrt{-\frac{m}{2h}}\int_0^u\left(1-e\cos\overline{u}\right)\mathrm{d}\overline{u}, \tag{s}$$

so that *Kepler's equation* reads

$$u - e\sin u = n(t - t_0), \tag{t}$$

with the notation

$$n = \sqrt{\frac{fM}{a^3}}\,, \tag{u}$$

usual in celestial mechanics.
We assume now that, in general, the Fx- axis does not coincide with the aspidal line ($\theta_1 \neq 0$, Fig.1.29), $\psi = \theta - \theta_1$ being the *true anomaly*. The equation of the conic takes the form $r(1 + e\cos\psi) = p = b^2/a = (a^2 - c^2)/a = a - ce$, where we used the above notations; it results $c + r\cos\psi = (a - r)/e = a\cos u$, if we take into account the previous change of variable. The ordinate of the point P pierces the director circle of the ellipse at Q; denoting by u the angle QOF, we notice that $\overline{OQ}\cos u = a\cos u$ is given by $\overline{QF} + \overline{FM} = c + r\cos\psi$, hence just by the expression obtained above. The angle u has thus a simple geometric interpretation, being called *eccentric anomaly*.
In the case in which the centre of attraction of mass M, considered fixed, is the Sun, the particle in motion (relative to the fixed centre) being a planet, we have to do with the solar system. Analogously, one may consider the motion of a satellite of a planet with respect to the planet itself, e.g., the motion of the Moon around the Earth. One may state *Kepler's laws*, obtained as a synthesis of astronomic observations, i.e.:

Law I. The motion of a planet around the Sun takes place along an elliptic orbit, the Sun being in one of the foci.
As a consequence of the first integral of areas (see Appl.4.25) one may state

Law II. (the law of areas). In the motion of a planet around the Sun, the vector radius drawn from the Sun to the planet sweeps over equal areas in equal times.
We notice that to a variation 2π of the true anomaly ψ corresponds the same variation of the eccentric anomaly u. Kepler's equation (t) leads to the period T in which the planet describes the whole ellipse, hence a motion of revolution is effected (the vector radius describes the whole area of the ellipse), in the form

$$T = \frac{2\pi}{n} = 2\pi a\sqrt{\frac{a}{fM}}\,; \tag{v}$$

it results that n represents the circular frequency (called *mean motion*). We may write

$$\frac{T^2}{a^3} = \frac{4\pi^2}{fM}\,, \tag{w}$$

too, stating thus (the ratio $4\pi^2/fM$) depends only on the mass of the Sun).

Law III. In the motion of planets around the Sun, the ratio of the square of the time of revolution and the cube of the semi-major axis is the same for all the planets.
By astronomical observations, these results represent a particularly important check of the Newtonian model of mechanics.

Application 1.18

Problem. Study the motion of a linear non-damped oscillator subjected to the action of a perturbing force of the form $f(t) = \alpha\cos(pt - \varphi)$.

Mathematical model. Using the notations of Appl.1.9, we may write the equation of motion (along the *Ox*-axis) in the form

$$\ddot{x} + \omega^2 x = f(t) = \alpha\cos(pt - \varphi)\,. \tag{a}$$

Solution. This is a non-homogeneous linear second order ODE, with constant coefficients. Its general solution is written as a sum between the general solution of the associated homogeneous equation and a particular solution of the non-homogeneous equation. The solution corresponding to the Cauchy data $x(0) = x_0, v(0) = v_0$ is then

$$\begin{aligned} x(t) &= x_0 \cos\omega\,\mathrm{t} + \frac{v_0}{\omega}\sin\omega\,\mathrm{t} \\ &- \frac{\alpha}{\omega^2 - p^2}\left[\cos\varphi\cos\omega\,\mathrm{t} + \frac{\mathrm{p}}{\omega}\sin\varphi\sin\omega\,\mathrm{t} - \cos(pt - \varphi)\right], \end{aligned} \tag{b}$$

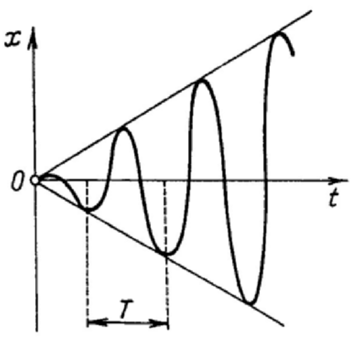

Figure 1. 30. Phenomenon of resonance

We may write

$$x(t) = a\cos(\omega t - \psi) + \frac{\alpha}{\omega^2 - p^2}\cos(pt - \varphi)\,, \tag{c}$$

where

$$a = \sqrt{\left(x_0 - \frac{\alpha\cos\varphi}{\omega^2 - p^2}\right)^2 + \frac{1}{\omega^2}\left(v_0 - \frac{\alpha p\sin\varphi}{\omega^2 - p^2}\right)^2}\,, \tag{d}$$

$$\psi = \arctan \frac{v_0 - \dfrac{\alpha p \sin\varphi}{\omega^2 - p^2}}{\omega\left(x_0 - \dfrac{\alpha\cos\varphi}{\omega^2 - p^2}\right)} . \tag{e}$$

It is thus seen that the motion of the particle is obtained as an interference of two harmonic vibrations: the eigen vibration of pulsation ω and the forced vibration of pulsation p.

If, in particular, we assume null initial conditions $(x_0 = v_0 = 0)$ and if the difference phase of the perturbing force vanishes $(\varphi = 0)$, then it results

$$x(t) = \frac{\alpha}{\omega^2 - p^2}\cos(pt - \omega t). \tag{f}$$

If the pulsation p differs greatly from the pulsation ω ($p << \omega$ or $p >> \omega$), then the diagram of the motion is that of Fig.1.12, b (the case $p << \omega$, hence an eigen vibration of great pulsation, "carried on" by a forced vibration of small pulsation); we notice that the maximal elongation of the resultant motion is practically equal to the double of the amplitude of one of the motions $\left(x_{\max} \cong 2\alpha/\left(\omega^2 - p^2\right)\right)$. If the two magnitudes of the pulsations are close, then one obtains the phenomenon of "beats" (Fig.1.13, 1.14).

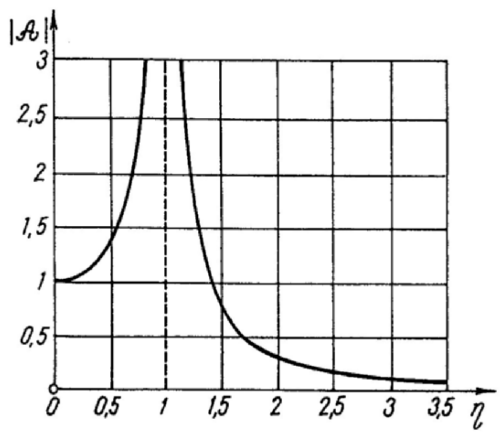

Figure 1. 31. The diagram of the amplitude $\mathcal{A}$ vs. η in the case of the phenomenon of resonance

In the case $p = \omega$, it results an indeterminedness in (b), as well as in (f). For $p \to \omega$ we obtain at the limit, according to the theorem of l' Hospital),

$$x(t) = \frac{\alpha}{2\omega} t \sin \omega t , \tag{g}$$

for the law of motion (f). In the case of the law of motion (b) one obtains an analogous result (one adds supplementary harmonic vibrations). The diagram of the motion (g) is a sinusoid of modulated amplitude after the straight lines $x = \pm \alpha t/2\omega$ and of pseudoperiod $T = 2\pi/\omega$ (Fig.1.30). The amplitude increases very much, in arithmetic progression, and the phenomenon is called resonance, being particularly dangerous for civil and industrial constructions or for mechanical ones.
The amplitude of the forced vibration (g) is proportional to the amplification factor

$$\mathcal{A} = \frac{1}{1-\eta^2} , \tag{h}$$

where we have introduced the relative pulsation $\eta = p/\omega$, which is a non-dimensional ratio. The graphic of the absolute value $\mathcal{A}$ is given in Fig.1.31.

Application 1.19

Problem. Study the motion of the previous case for a damped linear oscillator.

Mathematical model. Assuming a viscous damping (as in Appl.1.12), we are led to the equation of motion

$$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = \alpha \cos pt , \tag{a}$$

with the notations introduced in the mentioned application; to simplify, we admit that $\varphi = 0$. To fix the ideas, we assume to be in the case of a *subcritical damping* $(\chi < 1)$.

Solution. The solution of the linear second ordre ODE with constant coefficients (a) is of the form

$$x(t) = ae^{-\omega t} \cos(\omega' t - \psi) + C_1 \cos pt + C_2 \sin pt , \tag{b}$$

where

$$C_1 = \frac{(\omega^2 - p^2)\alpha}{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}, C_2 = \frac{2\lambda p\alpha}{(\omega^2 - p^2)^2 + 4\lambda^2 p^2} , \tag{c}$$

the last two terms corresponding to the forced motion.

Mechanical interpretation. Taking into account the exponential term, the proper motion is rapidly damped, so that we may consider the forced motion in the form

$$x(t) = A\cos(pt - \varphi), \tag{d}$$

with

$$A = \frac{\alpha}{\sqrt{(\omega^2 - p^2)^2 + 4\lambda^2 p^2}}, \varphi = \arctan\frac{2\lambda p}{\omega^2 - p^2} . \tag{e}$$

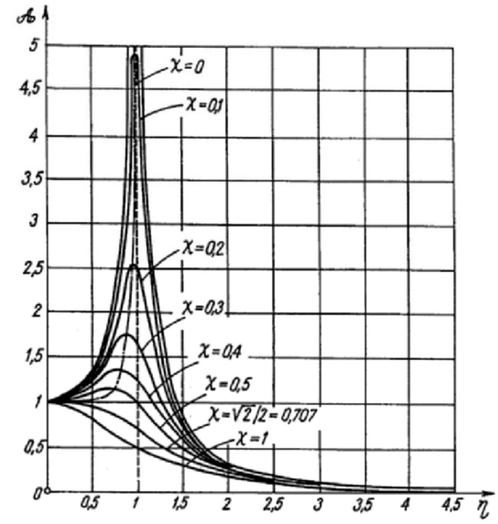

Figure 1. 32. Diagram of the amplification factor of the amplitude ($\mathscr{A}$ vs. η)

By means of the notations introduced in Appl.1.12 and of the relative pulsation $\eta = p/\omega$, we may also write

$$\mathscr{A} = \frac{1}{\sqrt{\left(1-\eta^2\right)^2 + 4\chi^2\eta^2}}, \varphi = \arctan\frac{2\chi\eta}{1-\eta^2}, \tag{f}$$

the amplitude A being proportional to the *amplification factor* $\mathscr{A} = \mathscr{A}(\eta)$, the diagram of which is given in Fig.1.32 as a function of various values of the damping factor χ. We notice that $\mathscr{A}(1) = 1/2\chi$.

We define an amplitude resonance for the values

$$\eta = \eta_{res} = \sqrt{1-2\chi^2} < 1, \chi \leq 1/\sqrt{2}, \tag{g}$$

for which the amplification factor has a maximum

$$\mathscr{A}_{\max} = \frac{1}{2\chi\sqrt{1-2\chi^2}} > \frac{1}{2\chi}. \tag{h}$$

One observes that the resonance amplitude is smaller as damping is greater, the graphic of the function becoming oblate for a great damping; the effect of the damping is particularly important in the vicinity of the resonance zone $(\eta \cong 1)$. If the damping is very small $(\chi << 1)$, then the amplitude resonance appears for $\eta \cong 1$, the amplitude factor being $\mathscr{A}_{\max} \cong 1/2\chi$. Eliminating χ between (g) and (h), we get

$$\mathscr{A}_{\max} = 1\Big/\sqrt{2 - \eta_{res}^4}\ , \tag{i}$$

that is the locus of the points of maximum of the graphics for various values of χ (represented by a broken line); these points are at the left of the line $\eta = 1$.

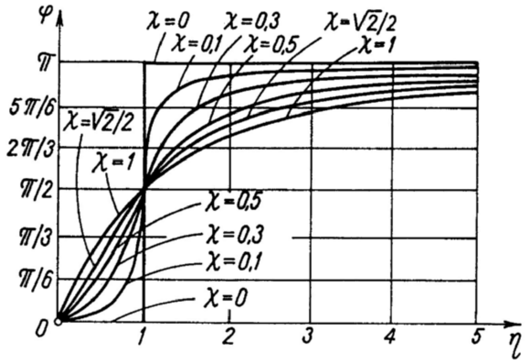

Figure 1. 33. Diagram of the phase function (φ vs. η)

The diagram of the phase function $\varphi = \varphi(\eta)$ is given Fig.1.33 for various values of the coefficient χ. We notice that for the non-damped system the phase is $\varphi = 0$ below the resonance $(\eta < 1)$, the vibration being in phase with the perturbing force, and $\varphi = \pi$, over the resonance $(\eta > 1)$, the vibration being in phase opposition with respect to the perturbing force; at the damped system there always exists a phase difference between the perturbing force and the vibration. For $\eta < 1$, χ (hence, the damping) increases as the phase shift between the motion and the perturbing force increases, the motion remaining after that force. For $\eta > 1$, χ increases as the phase shift decreases, the motion remaining after the perturbing force too. For a very great η, the phase shift increases immaterial the perturbing force. But the opposition is rigorously obtained only in the absence of the damping $(\eta = 0)$. For $\eta = 1$ one obtains $\varphi = \pi/2$, immaterial of the damping coefficient χ; one may thus define a phase resonance for which the vibration is in quadrature with the perturbing force.

Application 1.20

Problem. Determine the bending deflections w of a circular ring of radius a acted upon by two diametral concentrated forces *P* (Fig.1.34, a).

Mathematical model. The deflections *w* satisfy the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} + w = -\frac{Pa^3}{2EI}\left(\frac{2}{\pi} - \cos\varphi\right), \tag{a}$$

in polar co-ordinates, where EI represents the bending rigidity (E is the modulus of longitudinal elasticity and I is the moment of inertia of the cross section with respect to the neutral axis).

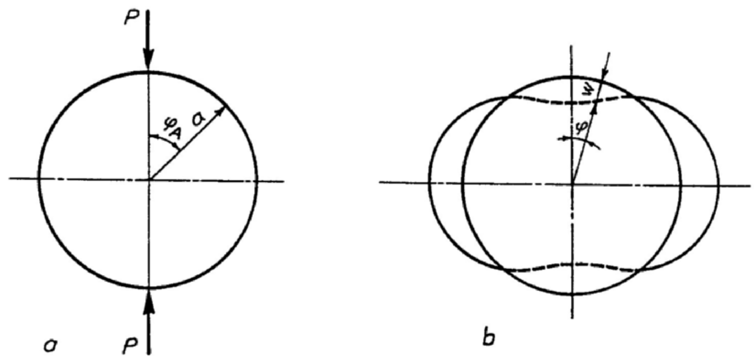

Figure 1. 34. Loading of a ring with diametral concentrated forces P (a). The deformation of the ring (b)

Solution. This is a second order linear, non-homogeneous ODE. The general solution of the associated homogenous equation is

$$w_h = C_1 \sin\varphi + C_2 \cos\varphi \,,$$

and a particular solution of the non-homogeneous ODE may be searched in the form

$$w_p = A + \varphi\left(B\sin\varphi + C\cos\varphi\right);$$

introducing this in (a), we get the coefficients

$$A = -\frac{Pa^3}{\pi EI}, B = \frac{Pa^3}{4EI}, C = 0 \,.$$

The general solution of the above ODE is thus

$$w = w_p + w_h = -\frac{Pa^3}{\pi EI} + \frac{Pa^3}{4EI}\varphi\sin\varphi + C_1\sin\varphi + C_2\cos\varphi \,.$$

By differentiation, we get

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} = \frac{Pa^3}{4EI}\sin\varphi + \frac{Pa^3}{4EI}\varphi\cos\varphi + C_1\cos\varphi - C_2\sin\varphi \,.$$

The integration constants are specified by the symmetry condition for $\varphi = 0$ and $\varphi = \pi/2$. It results $C_1 = 0$ and $C_2 = Pa^3/4EI$. Finally, the deflections read

$$w = \frac{Pa^3}{4EI}\left(\cos\varphi + \varphi\sin\varphi - \frac{4}{\pi}\right),$$

the deformation of the ring axis being drawn in Fig.1.34, b.

Application 1.21

Problem. The tram of a cable railroad moves downwards (Fig.1.35) with a velocity v_0. The driving wheel is braked by a band brake, so that after a time t_0 it remains blocked (the delayed acceleration may be considered constant). Determine the frequency and the amplitude a of the free longitudinal vibrations of the tram hanged down, due to the brake.

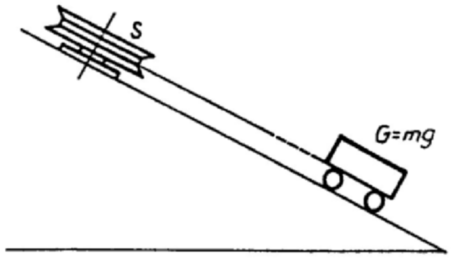

Figure 1. 35. Circulation of a tram

Numerical data: $v_0 = 3.6\,\mathrm{km/h}$, $t_0 = 3\,\mathrm{s}$, the modulus of longitudinal elasticity $E = 1.3 \cdot 10^6\,\mathrm{daN/cm^2}$, the area of the active cross section of the cable $A = 3\,\mathrm{cm}^2$, the length of the cable in rest $l = 1.3\,\mathrm{km}$, the weight of the tram $G = 29.4\,\mathrm{kN}$.

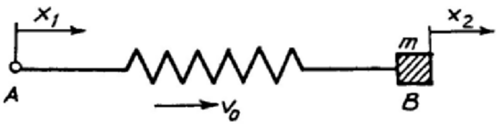

Figure 1. 36. The equivalent mechanical system

Mathematical model. The cable subjected to traction may be modelled by an elastic string; due to the small linear strain

$$\Delta l = v_0 t = 3\,\mathrm{m}, \frac{\Delta l}{l} = \frac{3}{1300} = 0.0023\,,$$

the elastic constant of the string may be considered invariable. To study the problem enounced above, we may consider the equivalent mechanical system (Fig.1.36), corresponding to the following

Equivalent problem. Two points *A* and *B*, moving with a constant velocity v_0, are connected by a string. In *B* there is a particle of mass $m = G/g$. Starting from the moment $t = 0$, the velocity $v_A = (\mathrm{d}x/\mathrm{d}t)_{x=A}$ of the point *A* is reduced from v_0 to zero in t_0 seconds, by a constant delayed acceleration, and then the point *A* remains in rest. Study the motion of the particle *B*.

The velocity v_A has a linear variation in time (Fig.1.37).

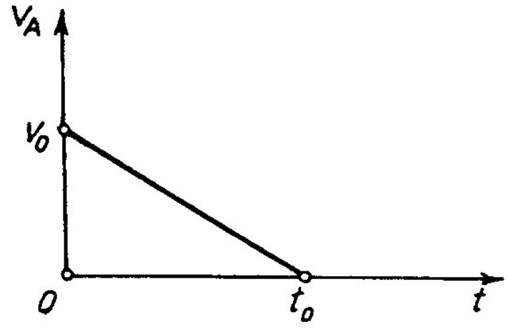

Figure 1. 37. The linear variation of the velocity v_A

If the link between the points A and B would be rigid, then the displacements, for $t \in [0, t_0]$, would be equal to

$$x_A = x_B = v_0 t - \frac{v_0 t^2}{2t_0} . \tag{a}$$

Due to the elastic connection, the differential equation of motion of the mass m in the interval of time $[0, t_0]$ is

$$m \frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} + k(x_B - x_A) = 0 , \tag{b}$$

where k is an elastic constant, x_A is given by (a), while x_B is the unknown of the problem.

Solution. Introducing the notation $\beta = \sqrt{kg/G}$ for the pulsation of the free vibrations, the equation (b) becomes

$$\frac{\mathrm{d}^2 x_B}{\mathrm{d}t^2} + \beta^2 x_B = \beta^2 \left(v_0 t - \frac{v_0 t^2}{2t_0} \right) ; \tag{c}$$

hence, the perturbing term is no more periodic. The general solution of the associated homogeneous equation is

$$x_{B,h} = C_1 \sin \beta t + C_2 \cos \beta t$$

and a particular solution of the non-homogeneous equation (c) is of the form

$$x_{B,p} = c_1 + c_2 t + c_3 t^2 ,$$

where c_1, c_2, c_3 are constants to be determined by identification; one obtains thus

$$c_1 = \frac{v_0}{\beta^2 t_0}, c_2 = v_0, c_3 = -\frac{v_0}{2t_0},$$

so that

$$x_{B,p} = \frac{v_0}{\beta^2 t_0} + v_0 t - \frac{v_0}{2t_0} t^2 .$$

The general solution of (c) is the sum

$$x_B = x_{B,h} + x_{B,p} = C_1 \sin \beta t + C_2 \cos \beta t + \frac{v_0}{\beta^2 t_0} + v_0 t - \frac{v_0}{2t_0} t^2 . \tag{d}$$

The initial conditions for $t = 0$ are $x_B = 0$, $\mathrm{d}x_B / \mathrm{d}t = v_0$ and lead to

$$x_B = v_0 t - \frac{v_0}{2t_0} t^2 + \frac{v_0}{\beta^2 t_0}(1 - \cos \beta t),$$

or, taking (a) into account, to

$$x_B = x_A + \frac{v_0}{\beta^2 t_0}(1 - \cos \beta t). \tag{e}$$

If $\xi = x_B - x_A$ is the deviation from the rest position, one obtains

$$\xi = \frac{v_0}{\beta^2 t_0}(1 - \cos \beta t), \frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{v_0}{\beta t_0} \sin \beta t ,$$

and for $t = t_0$, the initial values read

$$\xi_0 = \xi(t_0) = \frac{v_0}{\beta^2 t_0}(1 - \cos \beta t_0), \frac{\mathrm{d}\xi_0}{\mathrm{d}t} = \frac{v_0}{\beta t_0} \sin \beta t_0 . \tag{f}$$

For $t > t_0$, the differential equation of motion becomes

$$m \frac{\mathrm{d}^2 \xi}{\mathrm{d}t^2} + \beta^2 \xi = 0 , \tag{g}$$

so that the motion of the point B is a free harmonic vibration, the amplitude of which must be determined by using the initial values ξ_0 and $\mathrm{d}\xi_0 / \mathrm{d}t$.
The equation (g) leads to

$$\xi = A \sin \beta (t - t_0) + B \cos \beta (t - t_0),$$
$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \beta A \cos \beta (t - t_0) - \beta B \sin \beta (t - t_0).$$

One obtains the harmonic motion

$$\xi = \frac{v_0}{\beta^2 t_0}\left[(1-\cos\beta t_0)\cos\beta(t-t_0)+\sin\beta t_0 \sin\beta(t-t_0)\right]$$

$$= \frac{v_0}{\beta^2 t_0}\left[\cos\beta(t-t_0)-\cos\beta t\right] = \frac{2v_0}{\beta^2 t_0}\sin\frac{\beta t_0}{2}\sin\beta\left(t-\frac{t_0}{2}\right) = a\sin\beta\left(t-\frac{t_0}{2}\right),$$

where the amplitude and the proper period of vibration are accordingly given by

$$a = \frac{2v_0}{\beta^2 t_0}\sin\frac{\beta t_0}{2}\,,\; T = \frac{2\pi}{\beta}\,.$$

Introducing numerical values, one has

$$k = \frac{EA}{l} = \frac{1.3\cdot 10^6\cdot 3}{1.3\cdot 10^5} = 30\,\text{daN/cm}\,,\; m = \frac{G}{g} = \frac{2940}{981} = 2.99694\text{daNcm}^{-1}s^2\,,$$

so that the pulsation is

$$\beta = \sqrt{\frac{kg}{G}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{30}{3}} = 3.162\text{s}^{-1}\,,$$

the amplitude is given by

$$a = \frac{2v_0}{\beta^2 t_0}\sin\frac{\beta t_0}{2} = \frac{2\cdot 100}{10\cdot 3}\sin\frac{3.162\cdot 3}{2} = 6.66\text{cm}\,,$$

while the proper period is

$$T = \frac{2\pi}{\beta} = \frac{2\pi}{3.162} = 1.987\text{s}\,.$$

Application 1.22

Problem. Determine the deflections w of a hanged up structure.

Mathematical model. The deflections w satisfy the linear second order non-homogeneous ODE with constant coefficients

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \beta^2 \mathrm{w} = -\frac{1}{EI}\left[M_p + M_1\left(1-\frac{x}{l}\right) + M_2\frac{x}{l} - 4fH_p\frac{x}{l}\left(1-\frac{x}{l}\right)\right], \tag{a}$$

where x is the abscissa, M_p, M_1, M_2, H_p are dimensional constants, EI is the bending rigidity, while f and l are the bending deflection and the span of the cable, respectively.

Solution. The general solution of the associated homogeneous ODE is of the form

$$w_h = A\mathrm{e}^{\beta x} + B\mathrm{e}^{-\beta x}\,, \tag{b}$$

and a particular solution of the non-homogeneous equation is a trinomial of second degree

$$w_p = C_1 \frac{x^2}{l^2} + C_2 \frac{x}{l} + C_3\,; \tag{c}$$

by identifying the coefficients, it results

$$C_1 = \frac{4fH_p}{\beta^2 EI}, C_1 = \frac{M_1 + M_2 + 4fH_p}{\beta^2 EI}, C_3 = \frac{M_p + M_1}{\beta^2 EI} + \frac{8fH_p}{\beta^4 l^2 EI}\,.$$

Hence, the general solution of (a) is of the form

$$w = Ae^{\beta x} + Be^{-\beta x} + \frac{1}{\beta^2 EI}\left[M_p + M_1\left(1 - \frac{x}{l}\right) + M_2 \frac{x}{l} - 4fH_p \frac{x}{l}\left(1 - \frac{x}{l}\right) - \frac{8fH_p}{\beta^2 l^2}\right]. \tag{d}$$

The solution (b) may be also written in the form

$$w_h = A' \cosh \beta x + B' \sinh \beta x\,,$$

where $A' = A + B, B' = A - B$ are new integration constants.

Application 1.23

Problem. Determine the amplitude and the period of the water oscillations in the cylindrical equilibrium tank, of (horizontal) cross section *F*, of hydroenergetical conduit, having the length *L* and the cross section *A* (Fig.1.38). The frictions are neglected and the suddenly vanishing of the rate of flow of the turbine Q_t is assumed, the initial conditions being $v = v_0$, $z = 0$, $Q_t = Q_0$.

Mathematical model. Bernoulli's conservation theorem of mechanical energy, written between the storage basin and the equilibrium tank, leads to

$$z + \frac{L}{g}\frac{\mathrm{d}v}{\mathrm{d}t} = 0\,, \tag{a}$$

where *g* is the gravitational acceleration, and the equation of continuity reads

$$Av = F\frac{\mathrm{d}z}{\mathrm{d}t}\,. \tag{b}$$

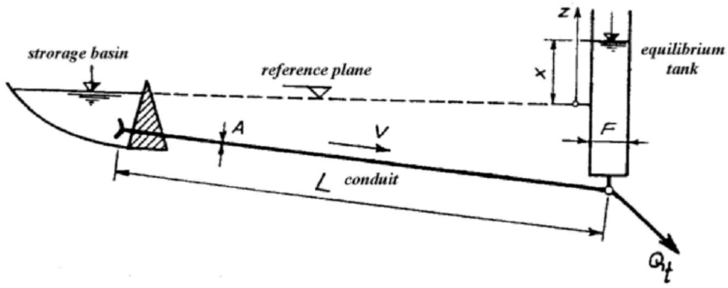

Figure 1. 38. Schema of the hydroenergetical conduit

Eliminating the velocity *v* between (a), (b), one obtains finally

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \omega^2 z = 0\,, \tag{c}$$

with the notation

$$\omega = \sqrt{\frac{gA}{FL}}\,. \tag{d}$$

Solution. The solution of the linear and homogeneous ODE with constant coefficients (c) may be put in the form

$$z = z_0 \sin(\omega t + \varphi)\,, \tag{e}$$

where z_0 is the oscillation amplitude; the period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{FL}{gA}}\,,$$

so that

$$z = z_0 \sin\left(\frac{2\pi}{T}t + \varphi\right). \tag{f}$$

The initial condition $z(0) = 0$ leads to $\varphi = 0$. The condition $v(0) = v_0$ at the initial moment, the rate of flow of the turbine Q_t vanishing, all the rate of flow in the conduit enters in the tank, so that

$$Av_0 = F\left(\frac{\mathrm{dz}}{\mathrm{d}t}\right)_{t=0}.$$

On the other hand

$$\frac{\mathrm{dz}}{\mathrm{d}t}=\frac{2\pi}{T}z_0\cos\left(\frac{2\pi}{T}t\right),\left(\frac{\mathrm{dz}}{\mathrm{d}t}\right)_{t=0}=\frac{2\pi}{T}z_0 .$$

Equating the two expressions of $\left(\frac{\mathrm{dz}}{\mathrm{d}t}\right)_{t=0}$, it follows

$$z_0=v_0\sqrt{\frac{LA}{gF}} .$$

Application 1.24

Problem. A bar of steel subjected to traction is formed by joining two bands by two longitudinal welding seams (Fig.1.39). Determine the effort S in one of the bands and the effort T in the welding seams.

Mathematical model. The searched efforts are given by the differential equations

$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2}-\frac{c}{E}\frac{A_1+A_2}{A_1A_2}S+P\frac{c}{EA_2}=0 , \tag{a}$$

$$T=-\frac{1}{2}\frac{\mathrm{d}S}{\mathrm{d}x} , \tag{b}$$

where A_1 and A_2 represent the areas of the cross sections of the joining bands, P is the effort of traction in the bar, and E and G are the moduli of longitudinal and transverse elasticity, respectively, of the material; the coefficient of deformation due to shifting is

$$\frac{1}{c}=\frac{1}{2G}\left(1+\frac{b_1}{6t_1}+\frac{b_2}{6t_2}\right),$$

where b_1,b_2 and t_1,t_2 are the width and the thickness of the two bands, respectively.

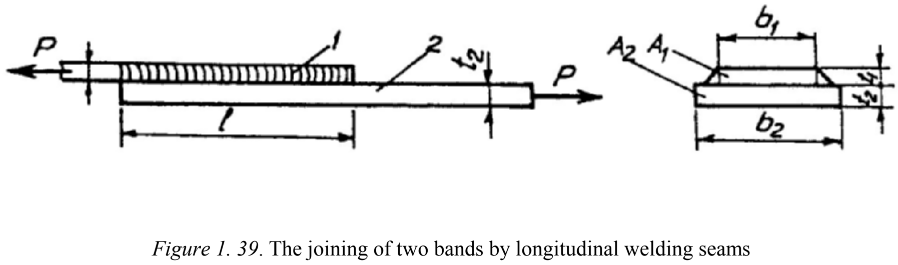

Figure 1. 39. The joining of two bands by longitudinal welding seams

Using the notation

$$\omega^2=\frac{c}{E}\frac{A_1+A_2}{A_1A_2} , \tag{c}$$

the differential equation (a) becomes

$$\frac{d^2S}{dx^2} - \omega^2 S = -P\frac{c}{EA_2}. \tag{d}$$

Solution. The model represents a linear second order non-homogeneous ODE with constant coefficients. The associated two-point conditions are $S(0) = P$ and $S(l) = 0$. Noting that the free term is constant, the general solution of (d) is

$$S = C_1 \cosh \omega x + C_2 \sinh \omega x + \frac{A_1}{A_1 + A_2} P .$$

The two-point conditions lead to

$$C_1 + \frac{A_1}{A_1 + A_2} P = P,$$
$$C_1 \cosh \omega l + C_2 \sinh \omega l + \frac{A_1}{A_1 + A_2} P = 0,$$

so that

$$C_1 = \frac{A_2}{A_1 + A_2} P,\, C_2 = -\frac{P}{A_1 + A_2}\frac{A_1 + A_2 \cosh \omega l}{\sinh \omega l}.$$

The final form of S is thus

$$\begin{aligned} S &= \frac{A_2}{A_1 + A_2} P \cosh \omega x - \frac{P}{A_1 + A_2}\frac{A_1 + A_2 \cosh \omega l}{\sinh \omega l}\sinh \omega x + \frac{A_2}{A_1 + A_2} P \\ &= \frac{P}{A_1 + A_2}\left[A_1\left(1 - \frac{\sinh \omega x}{\sinh \omega l}\right) + A_2 \frac{\sinh \omega(l - x)}{\sinh \omega l}\right]. \end{aligned} \tag{e}$$

Differentiating (e) and taking (b) into account, it results

$$\begin{aligned} T &= -\frac{\omega}{2}\frac{P}{A_1 + A_2}\left(A_2 \sinh \omega x - \frac{A_1 + A_2 \cosh \omega l}{\sinh \omega l}\cosh \omega x \right) \\ &= \frac{\omega}{2}\frac{P}{A_1 + A_2}\frac{A_1 \cosh \omega x - A_2 \cosh \omega(l - x)}{\sinh \omega l}. \end{aligned}$$

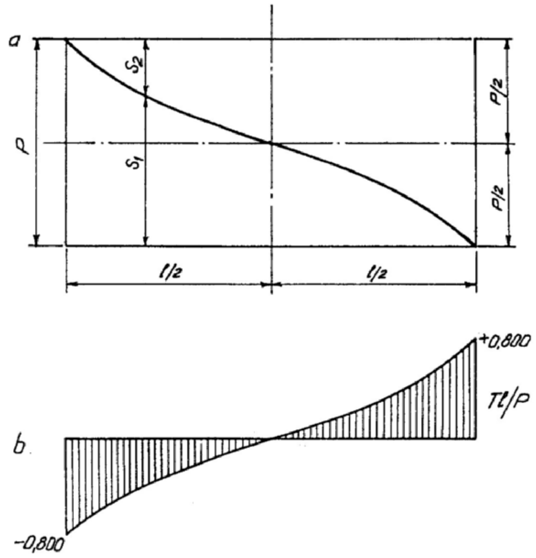

Figure 1. 40. Variation of S_1 and S_2 (a). Variation of T (b)

For the numerical data: $b_1 = 100\text{mm}$, $b_2 = 120\text{mm}$, $t_1 = 12\text{mm}$, $t_2 = 10\text{mm}$, $A_1 = A_2 = 12\text{cm}^2$, $G = E/2.6$, $E = 2.1 \cdot 10^6 \text{ daN/cm}^2$, $l = 20\text{cm}$, one obtains

$$\frac{1}{c} = \frac{1}{2G}\left(1 + \frac{10}{6 \cdot 1.2} + \frac{12}{6 \cdot 1.0}\right) = \frac{79}{36G},\ c = \frac{36}{79}G,$$
$$\omega^2 = \frac{36}{79}\frac{G}{E}\frac{12+12}{12 \cdot 12} = \frac{36}{79}\frac{1}{2.6}\frac{24}{144} = \frac{6}{79 \cdot 2.6} = 0.029211295,$$
$$\omega = 0.170913121\text{cm}^{-1},\ \omega l = 3.148262426,\ \sinh \omega l = 15.2417876.$$

The solution (e) is given by

$$S = \frac{P}{12+12}\left[12\left(1 - \frac{\sinh \omega x}{\sinh \omega l}\right) + 12\frac{\sinh \omega(l-x)}{\sinh \omega l}\right]$$
$$= \frac{12}{24 \cdot 15.2417876}\left[15.2417876 - \sinh \omega x + \sinh \omega(l-x)\right]P$$
$$= 0.032804551\left[15.2417876 - \sinh \omega x + \sinh \omega(l-x)\right]P.$$

The variations of S_1 and $S_2 = P - S_1$ are given in Fig.1.40, a, and the variation of Tl/P in Fig.1.40, b.

Application 1.25

Problem. Consider a bar of axis not perfectly rectilinear (we say that the bar has an *initial curvature*). Study the influence of this curvature, supposing that the bar is doubly hinged and is acted upon by compression forces *P*. One assumes that the initial curvilinear form of the axis is given by

$$\overline{w} = w_0 \sin \frac{\pi x}{l}, \tag{a}$$

where w_0 is known (Fig.1.41).

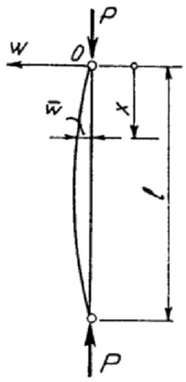

Figure 1. 41. The influence of the initial curvature in the stability of a bar

Mathematical model. The bending moment in the deformed state is given by

$$M = P\left(w + \overline{w}\right) = Pw + Pw_0 \sin \frac{\pi x}{l},$$

where *w* is the bending deflection. The differential equation of the deformed bar axis becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \beta^2 w = -\frac{P}{EI} w_0 \sin \frac{\pi x}{l}, \tag{b}$$

where *EI* is the bending rigidity.

Solution. This is a linear non-homogeneous ODE with constant coefficients. A particular solution is searched in the form

$$w = C \sin \frac{\pi x}{l}; \tag{c}$$

it satisfies the two-point conditions $w(0) = w(l) = 0$. Introducing this in (b), one gets

$$C = -\frac{Pw_0}{EI}\frac{1}{\beta^2 - \frac{\pi^2}{l^2}} = \frac{Pw_0}{P_E - P}, \tag{d}$$

where

$$\beta^2 = \frac{P}{EI},\ P_E = \frac{\pi^2 EI}{l^2}.$$

In this case, we get

$$w = \frac{P}{P_E - P} w_0 \sin\frac{\pi x}{l} = \frac{P/P_E}{1 - P/P_E} w_0 \sin\frac{\pi x}{l}, \tag{e}$$

emphasizing an amplification of the initial geometric line (a).
The bending moment is given by

$$\begin{aligned} M &= -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{\pi^2 EI}{l^2}\frac{P}{P_E - P} w_0 \sin\frac{\pi x}{l} = \frac{P_E}{P_E - P} P w_0 \sin\frac{\pi x}{l} \\ &= \frac{1}{1 - P/P_E} P w_0 \sin\frac{\pi x}{l}. \end{aligned} \tag{f}$$

We notice that for $P \to P_E$ (Euler's load), the deflection w and the bending moment M tend to infinity, independently on the initial curvature w_0 (instability by divergence).

Application 1.26

Problem. Consider a doubly hinged bar, of length l, acted upon by the compression forces P and transversally by a sinusoidal load $p(x) = p_0 \sin(\pi x/l)$ (Fig.1.42). Determine the deflection w and the bending moment M.

Mathematical model. The bending moment is given by

$$M = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2},$$

where EI is the bending rigidity, and the deflection w satisfies the differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{P}{EI} w = -\frac{p_0 l^2}{\pi^2 EI}\sin\frac{\pi x}{l}. \tag{a}$$

Solution. Denoting by

$$\beta^2 = \frac{P}{EI}, \tag{b}$$

the solution of the associated homogeneous equation is given by

$$w_h = A\sin\beta x + B\cos\beta x\ . \qquad \text{(c)}$$

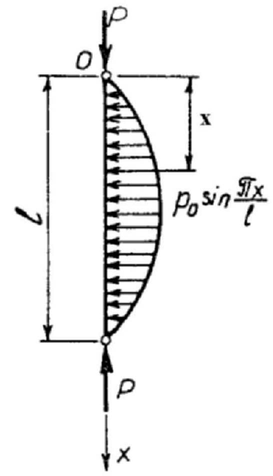

Figure 1. 42. Doubly hinged bar acted upon by axial forces *P* and by a sinusoidal transverse load

A particular solution of the non-homogeneous ODE (a) is searched of the same form as the free term

$$w_p = W\frac{\sin\pi x}{l}\ . \qquad \text{(d)}$$

As (d) must satisfy the ODE (a), it follows

$$W = -\frac{p_0 l^2}{\pi^2}\frac{1}{P - \dfrac{\pi^2 EI}{l^2}} = \frac{p_0 l^2}{\pi^2}\frac{1}{P_E - P},$$

where

$$P_E = \frac{\pi^2 EI}{l^2}$$

is Euler's load.

Because the particular solution satisfies the two-point conditions $w(0) = w(l) = 0$, the constants *A* and *B* of (c) vanish. We thus get

$$w = \frac{p_0 l^2}{\pi^2}\frac{1}{P_E - P}\frac{\sin\pi x}{l},$$

$$M=-EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=\frac{P_E}{P_E-P}\frac{p_0l^2}{\pi^2}\frac{\sin\pi x}{l}=\frac{1}{1-\dfrac{P}{P_E}}\frac{p_0l^2}{\pi^2}\frac{\sin\pi x}{l}.$$

For $P\to P_E$, the quantities w and M tend to infinity, immaterial of the intensity of the load $p(x)$(instability by divergence).

Application 1.27

Problem. Consider a doubly hinged bar, of length l, acted upon by compression forces P and transversally by a uniform load p (Fig.1.43). Determine the deflection w and the bending moment M.

Mathematical model. The deflection w satisfies the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\frac{P}{EI}w=-\frac{px(l-x)}{2EI}, \tag{a}$$

and the bending moment reads $M=-EI\,\mathrm{d}^2w/\mathrm{d}x^2$ (*EI* is the bending rigidity).

Solution. The model represents a second order linear non-homogeneous ODE with constant coefficients. Searching a particular solution of the form

$$w_p=c_0+c_1x+c_2x^2,$$

one obtains, by identification, the coefficients

$$c_0=-\frac{pl}{2P},c_1=\frac{p}{2P},c_2=-\frac{pEI}{P^2}.$$

Denoting by $\beta^2=P/EI$, the general solution of the ODE (a) reads

$$w=-\frac{pEI}{P^2}-\frac{px(l-x)}{2P}+A\sin\beta x+B\cos\beta x. \tag{b}$$

By using the two-point conditions $w(0)=w(l)=0$, we get

$$A=\frac{pEI}{P^2}\frac{1-\cos\beta l}{\sin\beta l},B=\frac{pEI}{P^2}.$$

We obtain the deflection w and the bending moment M

$$w=\frac{pEI}{P^2}\frac{\sin\beta(l-x)-(\sin\beta l-\sin\beta x)}{\sin\beta l}-\frac{px(l-x)}{2P},$$

$$M = \frac{pEI}{P}\left[\frac{\sin\beta x}{\sin\beta l} + \frac{\sin\beta(l-x)}{\sin\beta l} - 1\right] = \frac{p}{\beta^2}\left[-1 + \frac{\cos\beta\left(\frac{l}{2} - x\right)}{\cos\frac{\beta l}{2}}\right],$$

respectively.

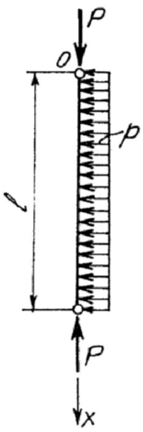

Figure 1. 43. Doubly hinged bar acted upon by axial forces *P* and by a uniform distributed transverse load

The maximal values (at the middle of the span $x = l/2$) are

$$w_{\max} = \frac{pEI}{P^2}\frac{1-\cos\frac{\beta l}{2}}{\cos\frac{\beta l}{2}} - \frac{pl^2}{8P}, \quad M = \frac{P}{\beta^2}\left(-1 + \frac{1}{\cos\frac{\beta l}{2}}\right).$$

For $P \to P_E = \pi^2 EI/l^2$ (Euler's force), the quantities w and M tend to infinity, immaterial of the load p (instability by divergence).

Application 1.28

Problem. Study the influence of the eccentricity of application of the normal force P to a bar free at the upper end and perfectly built-in at the lower end.

Mathematical model. We denote by e the initial eccentricity (Fig.1.44); the bending deflections satisfy the second order linear non-homogeneous ODE with constant coefficients

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{P}{EI}w = -\frac{pe}{EI}. \qquad \text{(a)}$$

Solution. The general solution of the non-homogeneous equation is the sum between the general solution of the associated homogeneous equation and a particular solution of the non-homogeneous one, therefore it is of the form

$$w = -e + A\sin\beta x + B\cos\beta x\,;$$

we also have

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \beta\left(A\cos\beta x - B\sin\beta x\right).$$

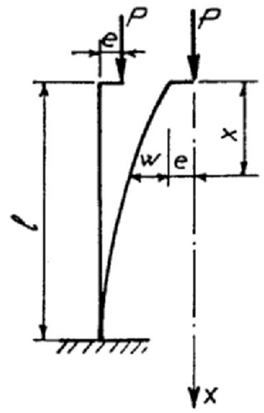

Figure 1. 44. Eccentricity of application of the normal force *P*

The boundary conditions $w(0) = 0$, $\left(\mathrm{d}w/\mathrm{d}x\right)_{x=l} = 0$ lead to $A = e\tan\beta l, B = e$.
Hence, the deflections become

$$w = e\left(-1 + \cos\beta x + \tan\beta l \sin\beta x\right) = e\left[-1 + \frac{\cos\beta(l - x)}{\cos\beta l}\right],$$

and the bending moments are given by

$$M = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{\beta^2 EIe}{\cos\beta l}\cos\beta(l - x) = Pe\frac{\cos\beta(l - x)}{\cos\beta l}.$$

For $\beta l \to \pi/2$ we have $\cos\beta l = 0$, so that the deflection and the bending moment tend to infinity (instability by divergence). In this case, the normal force P tends to the value of the critical buckling force (see Appl.1.31).

Application 1.29

Problem. Let be a doubly hinged bar, of length *l*, acted upon by compression forces *P* and transversally by a concentrated force *F* at the middle of the span (Fig.1.45). Determine the bending deflections *w*.

Mathematical model. For an arbitrary cross-section of abscissa x one may write the bending moment

$$M = Pw + \frac{1}{2}Fx \, .$$

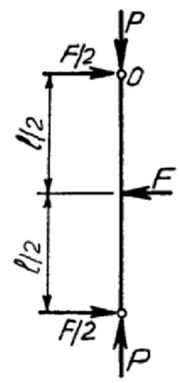

Figure 1. 45. Doubly hinged bar acted upon by axial forces P and by a transverse force F

The model is thus a linear second order ODE of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \beta^2 w = -\frac{F}{2EI}x, \; x \in \left[0, \frac{l}{2}\right], \quad \beta^2 = \frac{P}{EI} \, ,$$

to which one associates the two-point conditions $w(0) = 0$, $(\mathrm{d}w/\mathrm{d}x)_{x=l/2} = 0$; the last one is a symmetry condition.

Solution. The general solution of the above ODE is

$$w = -\frac{F}{2P}x + A\sin\beta x + B\cos\beta x \, ,$$

and therefore

$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{F}{2P} + \beta A\cos\beta x - \beta B\sin\beta x \, .$$

Taking into account the boundary conditions, we get

$$A = \frac{F}{2P\beta\cos\frac{\beta l}{2}}, \; B = 0 \, ,$$

so that

$$w = \frac{F}{2P\beta}\left(-\beta x + \frac{\sin \beta x}{\cos \frac{\beta l}{2}} \right).$$

The bending moment becomes

$$M = \frac{F}{2\beta} \frac{\sin \beta x}{\cos \frac{\beta l}{2}}.$$

For $P \to P_E = \pi^2 EI/l^2$ (Euler's force) we have $\cos(\beta l/2) = 0$, so that w and M tend to infinity, independently of the transverse force F (instability by divergence).

Application 1.30

Problem. A cable BOC passes over a mast OA, of height l (Fig.1.46, a). A tension in the cable introduces a compression force in the mast. Determine the value of the critical force for which the mast loses its stable form.

Mathematical model. Let α be the inclination angle of the cable with respect to a horizontal line, in the initial position (Fig.1.46, a). We suppose that, due to the buckling phenomenon, the upper edge O has a lateral displacement f. Then, the inclination angle of the left part of the cable is reduced with $\Delta\alpha$, while the inclination angle of the right part of it increases with $\Delta\alpha$ (Fig.1.46, b).
If N is the effort of tension in the cable, the initial position of equilibrium leads to

$$N = \frac{P}{2\sin \alpha}. \tag{a}$$

Due to the deformation of the mechanical system, a horizontal force arises

$$H = N\cos(\alpha - \Delta\alpha) - N\cos(\alpha + \Delta\alpha) = 2N \sin\alpha \sin\Delta\alpha;$$

as $\Delta\alpha$ is very small with respect to α ($\sin\Delta\alpha \cong \Delta\alpha$), we may write

$$H = 2N \sin\alpha\Delta\alpha = P\Delta\alpha. \tag{b}$$

If D is the projection of O on BO' (O' is the point reached by O by buckling), then from the triangle ODO' it results (Fig.1.46, c)

$$\overline{OD} = \overline{BO}\Delta\alpha = f\sin\alpha;$$

as $\overline{BO} = l/\sin\alpha$, one obtains

$$\Delta\alpha = \frac{f\sin\alpha}{\dfrac{l}{\sin\alpha}} = \frac{f}{l}\sin^2\alpha\ , \tag{c}$$

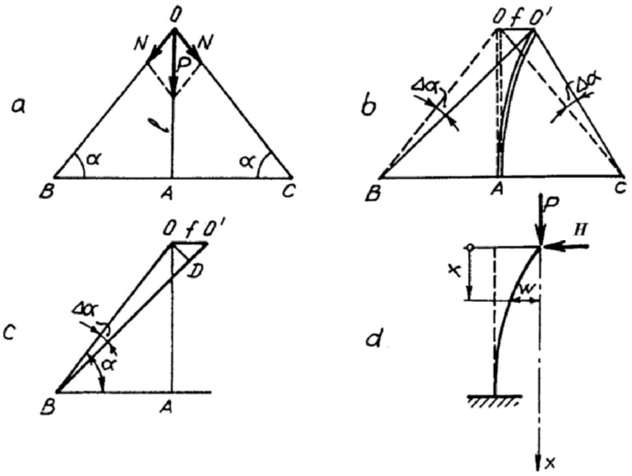

Figure 1. 46. Geometric schema of the mast and of the cable (a). Lateral displacement *f* (b). Displacement of the upper edge (c). Static schema of the mast (d)

so that

$$H = P\frac{f}{l}\sin^2\alpha\ . \tag{d}$$

One must thus determine the critical force for a cantilever bar OA acted upon at the free end by the forces *P* and *H* (Fig.1.46, d).
We choose the origin of the x-axis at the upper edge of the bar, so that for an arbitrary x we obtain the bending moment

$$M(x) = Pw - Hx = P\left(w - x\frac{f}{l}\sin^2\alpha\right). \tag{e}$$

The differential equation of the deformed bar axis is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{P}{EI}\left(w - x\frac{f}{l}\sin^2\alpha\right),$$

where *EI* is the bending rigidity, or

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \beta^2 w = \beta^2 x\frac{f}{l}\sin^2\alpha\ , \tag{f}$$

with the usual notation

$$\beta^2 = \frac{P}{EI} . \tag{g}$$

Solution. The general solution of the linear second order ODE is of the form

$$w = A\sin\beta x + B\cos\beta x + x\frac{f}{l}\sin^2\alpha , \tag{h}$$

and the rotation of the cross section is given by

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \beta A\cos\beta x - \beta B\sin\beta x + \frac{f}{l}\sin^2\alpha . \tag{i}$$

The boundary conditions $w(0) = 0$, $w(l) = 0$, $(\mathrm{d}w/\mathrm{d}x)_{x=l} = 0$ lead to

$$\begin{aligned} &B = 0, \\ &A\sin\beta l + B\cos\beta l + f\sin^2\alpha = f, \\ &\beta A\cos\beta l - \beta B\sin\beta l + \frac{f}{l}\sin^2\alpha = 0. \end{aligned} \tag{j}$$

The linear system of algebraic equation in A, B, f has non- zero solutions (which correspond to the stable position of equilibrium) if

$$\det\begin{bmatrix} \sin\beta l & -\cos^2\alpha \\ \beta\cos\beta l & \frac{1}{l}\sin^2\alpha \end{bmatrix} = 0 ;$$

thus, the following characteristic equation is obtained

$$\frac{\tan\beta l}{\beta l} = -\cot^2\alpha ,$$

whose solution can be obtained only numerically. The Table 1.2 may be used to this goal. For instance, for $\alpha = \pi/4$ one obtains $\beta l = 2.02876$, so that the critical force is given by

$$P_{cr} = 2.02876^2\frac{EI}{l^2} = \frac{\pi^2 EI}{(1.5485l)^2} .$$

Table 1. 2. The values of the function $f(u) = \tan u / u$

u	$\frac{\tan u}{u}$	u	$\frac{\tan u}{u}$	u	$\frac{\tan u}{u}$	u	$\frac{\tan u}{u}$
0	1.0000	1.800	-2.3813	3.600	0.1371	5.300	-0.2833
0.050	1.0008	1.850	-1.8854	3.650	0.1527	5.350	-0.2523
0.100	1.0033	1.900	-1.5406	3.700	0.1688	5.400	-0.2255
0.150	1.0076	1.950	-1.2869	3.750	0.1857	5.450	-0.2019
0.200	1.0136	2.000	-1.0925	3.800	0.2036	5.500	-0.1810
0.250	1.0214	2.050	-0.9388	3.850	0.2225	5.550	-0.1623
0.300	1.0311	2.100	-0.8142	3.900	0.2429	5.600	-0.1433
0.350	1.0429	2.150	-0.7112	3.950	0.2651	5.650	-0.1299
0.400	1.0570	2.200	-0.6245	4.000	0.2895	5.700	-0.1157
0.450	1.0735	2.250	-0.5505	4.050	0.3166	5.750	-0.1026
0.500	1.0926	2.300	-0.4866	4.100	0.3472	5.800	-0.0905
0.550	1.1147	2.350	-0.4308	4.150	0.3823	5.850	-0.0791
0.600	1.1402	2.400	-0.3817	4.200	0.4233	5.900	-0.0683
0.650	1.1695	2.450	-0.3380	4.250	0.4721	5.950	-0.0582
0.700	1.2033	2.500	-0.2988	4.300	0.5316	6.000	-0.0485
0.750	1.2421	2.550	-0.2635	4.350	0.6063	6.050	-0.0393
0.800	1.2870	2.600	-0.2314	4.400	0.7037	6.100	-0.0304
0.850	1.3392	2.650	-0.2021	4.450	0.8367	6.150	-0.0218
0.900	1.4002	2.700	-0.1751	4.4934	1.0000	6.200	-0.0134
0.950	1.4720	2.750	-0.1502	4.500	1.0305	6.250	-0.0053
1.000	1.5574	2.800	-0.1270	4.550	1.3415	2π	0
1.050	1.6605	2.850	-0.1053	4.600	1.9261	6.300	0.0027
1.100	1.7861	2.900	-0.0850	4.6042	2.0000	6.350	0.0105
1.150	1.9430	2.950	-0.0658	4.650	3.4425	6.400	0.0183
1.200	2.1435	3.000	-0.0475	4.700	17.1729	6.450	0.0261
1.250	2.4077	3.050	-0.0301	$3\pi/2$	$\pm\infty$	6.500	0.0339
1.300	2.7708	3.100	-0.0134	4.750	-5.5948	6.550	0.0417
1.350	3.3002	π	0	4.800	-2.3718	6.600	0.0497
1.400	4.1413	3.150	0.0027	4.850	-1.4889	6.650	0.0578
1.450	5.6814	3.200	0.0183	4.900	-1.0750	6.700	0.0661
1.500	9.4009	3.250	0.0335	4.950	-0.8342	6.750	0.0747
1.550	31.0184	3.300	0.0484	5.000	-0.6761	6.800	0.0836
$\pi/2$	$\pm\infty$	3.350	0.0631	5.050	-0.5641	6.850	0.0929
1.600	-21.3953	3.400	0.0777	5.100	-0.4803	6.900	0.1028
1.650	-7.6359	3.450	0.0923	5.150	-0.4150	6.950	0.1132
1.700	-4.5274	3.500	0.1070	5.200	-0.3626	7.000	0.1245
1.750	-3.1545	3.550	0.1219	5.250	-0.3195	7.050	0.1369

Application 1.31

Problem. A slender doubly hinged bar is subjected to compression by two axial forces P. If a critical force P_{cr} is attained, then the bar does no more remain in the rectilinear form of equilibrium. Determine the first two values of this force. To solve the problem, one considers the moment in which the bar leaves its rectilinear form of equilibrium and takes a new curvilinear form, very close to the initial position.

Mathematical model. For a cross section of arbitrary abscissa *x*, the bending moment is given by $M = Pw$, where *w* is the deflection; the differential equation of the deformed axis becomes

$$\frac{d^2 w}{dx^2} + \frac{P}{EI} w = 0 \,, \tag{a}$$

where *EI* is the minimal bending rigidity of the cross section.
Choosing the origin of the *x*-axis at the upper edge, the two-point conditions are

$$w(0) = w(l) = 0 \,, \tag{b}$$

whith *l* the bar length (Fig.1.47).

Solution. This is a Sturm-Liouville problem, as the linear ODE (a) and the boundary conditions (b) are homogeneous; a non-zero solution is only possible for certain eigenvalues of the parameter *P*.

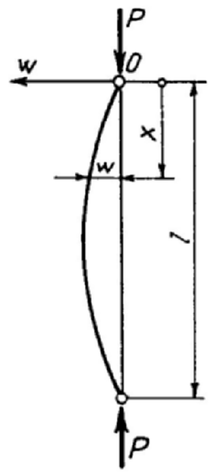

Figure 1. 47. Buckling of a doubly hinged bar

For the sake of simplicity, with the notation (g) from Appl.1.30, the equation (a) becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \beta^2 w = 0 . \tag{c}$$

Searching a solution of the form $w = \mathrm{e}^{\lambda x}$, we find the characteristic equation $\lambda^2 + \beta^2 = 0$, with the roots $\lambda_{1,2} = \pm \mathrm{i}\beta$. The general solution is

$$w = A \sin \beta x + B \cos \beta x , \tag{d}$$

A and B being integration constants.

The boundary conditions $w(0) = w(l) = 0$ yield $B = 0$ and $A \sin \beta l = 0$. But $A \neq 0$, or else the bar remains rectilinear. Also, $\beta \neq 0$, or else the bar should not be loaded. Thus, it follows that $\sin \beta l = 0$, with the roots $\beta l = n\pi,\ n = 1,2,3\ldots$.

Turning back to the notation (g) for β, one obtains the eigenvalues

$$P_{cr} = \beta^2 EI = n^2 \frac{\pi^2 EI}{l^2},\ n = 1,2,3\ldots$$

and the equations of the deformed axis

$$w = A \sin \frac{n\pi x}{l},\ n = 1,2,3\ldots \tag{e}$$

We notice that the amplitude A of the deformed axis remains non-determinate; as a matter of fact, the model we used was an approximate (linearized) form of the ODE satisfied by the deformed bar axis. The solution (e) represents a sinusoid of semi-wave l/n.

Practically, the minimal value of the critical force (for $n = 1$) is of particular interest. This one is called the Eulerian critical force

$$P_{cr} = P_E = \frac{\pi^2 EI}{l^2} , \tag{f}$$

for which the deformed axis of semi-wave l is given by

$$w = w_{\max} \frac{\pi x}{l}, \tag{g}$$

where $w_{\max}$ corresponds to the middle of the span.

For greater values of n, e.g., $n = 2$, the next critical force is obtained

$$P_{cr,2} = 2^2 \cdot \frac{\pi^2 EI}{l^2} = 4P_{cr} , \tag{h}$$

corresponding to another form of equilibrium; this situation also matches to a supplementary simple support at the middle of the span.

Application 1.32

Problem. Study the buckling problem for a straight bar built-in at one end and free at the other end (a cantilever bar).

Mathematical model. The problem being similar with the previous one, we use the same ODE (a), or, likewise, (c), for the deflection of the bar axis; we also use the same notations. The difference between the two problems is mathematically expressed by the differences between the two-point conditions, which, in this case, are (Fig.1.48): $w(0) = 0$, $(\mathrm{d}w/\mathrm{d}x)_{x=l} = 0$.

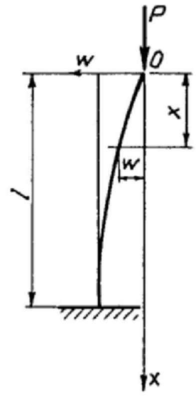

Figure 1. 48. Buckling of a cantilever bar

Solution. As in the previous application, the model is a Sturm-Liouville problem. The general solution of the ODE (c) and its derivative read accordingly

$$w = A\sin\beta x + B\cos\beta x\,,$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = \beta A\cos\beta x - \beta B\sin\beta x\,.$$

The boundary conditions involve $B = 0$ and $\cos\beta l = 0$, with the eigenvalues

$$\beta_n = \frac{n\pi}{2l}\,.$$

The minimal value of the critical force $(n = 1)$ is

$$P_{cr} = \frac{\pi^2 EI}{4l^2}$$

and the equation of the corresponding deformed axis is given by

$$w = A\sin\frac{\pi x}{2l}\,.$$

This represents a sinusoid whose period is twice as much as that of the previous case; the amplitude A is non-determinate.

Application 1.33

Problem. Study the buckling problem for a straight bar of length l, built-in at one end and hinged at the other end.

Mathematical model. Due to the built-in mounting, a reaction H – playing the rôle of a non-determinate parameter – also appears in the hinge, normal to the bar axis (Fig.1.49).

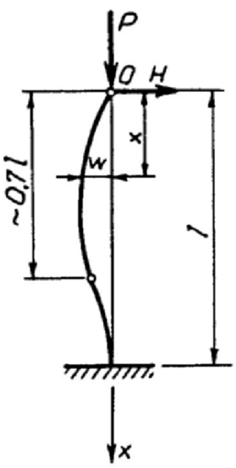

Figure 1. 49. Buckling of a bar built-in at one end and hinged at the other end

The bending moment in a cross section of abscissa x of the deformed axis is given by $M = Pw + Hx$, so that the differential equation of the problem is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{P}{EI} w = -\frac{H}{EI} x \,, \tag{a}$$

where P is the compression force and EI is the bending rigidity.
With the notation (g), Appl.1.30, the equation (a) becomes

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \beta^2 w = -\frac{H}{EI} x \,. \tag{b}$$

The boundary conditions are

$$w(0) = 0 \,, \; w(l) = 0 \,, \; (\mathrm{d}w/\mathrm{d}x)_{x=l} = 0 \,. \tag{c}$$

Solution. The above model is a Sturm-Liouville problem. The general solution of the linear second order ODE (b) and its derivative are, accordingly,

$$w = -\frac{H}{P}x + A\sin\beta x + B\cos\beta x\,,$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{H}{P} + \beta\left(A\cos\beta x - B\sin\beta x\right). \tag{d}$$

The boundary conditions yield $B = 0$ and the algebraic linear system, satisfied by $\left(-H/P\right)$ and A,

$$-\frac{H}{P}l + A\sin\beta l = 0,\ -\frac{H}{P} + A\cos\beta l = 0\,. \tag{e}$$

This system is also homogeneous, therefore it has non-vanishing solutions only if

$$\det\begin{bmatrix} l & \sin\beta l \\ l & \beta\cos\beta l \end{bmatrix} = 0\,.$$

Computing this determinant, we obtain the transcendental characteristic equation

$$\tan\beta l = \beta l\,. \tag{f}$$

The minimal root of this equation (corresponding to the Table 1.2)

$$\beta l = 4.4934095 = \frac{\pi}{0.699155653} \cong \frac{\pi}{0.7}$$

leads to the minimal critical force

$$P_{cr} \cong \frac{\pi^2 EI}{\left(0.7l\right)^2}\,. \tag{g}$$

Application 1.34

Problem. Determine the critical buckling force for a doubly built-in bar.

Mathematical model. The differential equation of the problem is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{P}{EI}w = -\frac{H}{EI}x - \frac{M_0}{EI}\,, \tag{a}$$

where P is the compression force, H and M_0 (the reaction normal to the bar axis and the moment at the built-in cross section, respectively) are non-determined parameters, EI is the bending rigidity and w is the unknown deflection (Fig.1.50). The two-point conditions are, in this case,

$$w(0) = w(l) = 0\,,\ \left(\mathrm{d}w/\mathrm{d}x\right)_{x=0} = \left(\mathrm{d}w/\mathrm{d}x\right)_{x=l} = 0\,, \tag{b}$$

where l is the bar length.

Solution. To solve this Sturm-Liouville problem, we firstly get the general solution of the linear ODE and the corresponding derivative

$$w = -\frac{H}{P}x - \frac{M_0}{P} + A\sin\beta x + B\cos\beta x,$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{H}{P} + \beta\left(A\cos\beta x - B\sin\beta x\right). \tag{c}$$

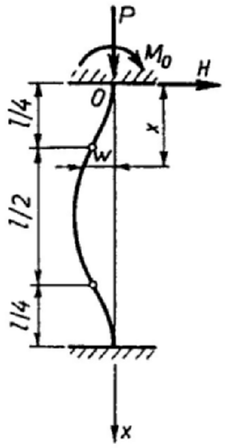

Figure 1. 50. Buckling of a doubly built-in bar

The two-point conditions lead to the linear homogeneous algebraic system, written in matrix form

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ \beta & 0 & -1 & 0 \\ \sin\beta l & \cos\beta l & -l & -1 \\ \beta\cos\beta l & -\beta\sin\beta l & -1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ H/P \\ M_0/P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{d}$$

To have non-zero solutions, we must equate to zero the determinant of the associated matrix, thus obtaining the characteristic equation

$$2\left(1-\cos\beta l\right) - \beta l\sin\beta l = 2\sin\beta l\left(\tan\frac{\beta l}{2} - \frac{\beta l}{2}\right) = 0\,, \tag{e}$$

of roots $\beta l = 2n\pi,\ n = 1,2,3\dots$. The root $\beta_1 l = 2\pi$ leads to the minimal buckling force

$$P_{cr} = \frac{4\pi^2 EI}{l^2} = \frac{\pi^2 EI}{(0.5l)^2}\,. \tag{f}$$

The minimal root corresponding to the second factor is greater than $\beta_1 l / 2$.

Application 1.35

Problem. Study the lateral buckling of a slender beam subjected to bending.

Mathematical model. It is possible for a slender beam subjected to bending to lose its plane form of equilibrium if the bending moment attains a *critical value*. (Fig.1.51). The beam loses its stability in the compressed zone; the beam axis becomes curvilinear in its plane of minimal rigidity while various cross sections of the beam rotate around the axis.

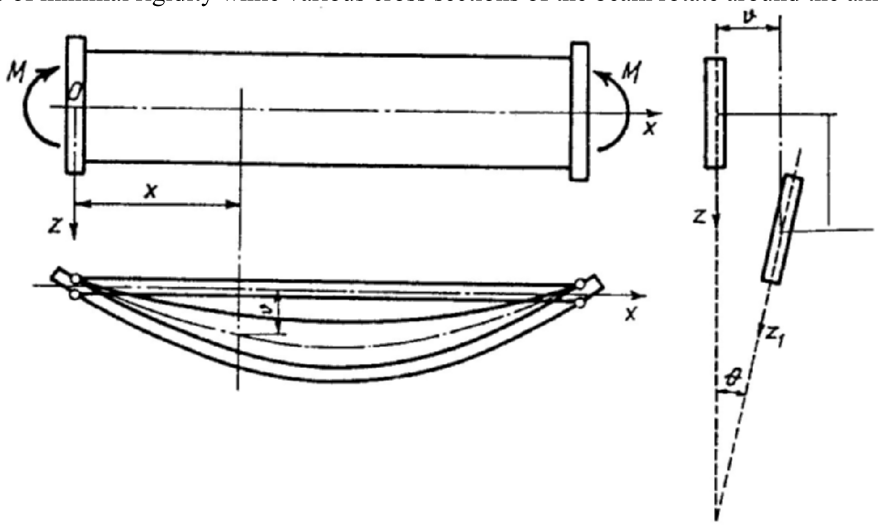

Figure 1. 51. Lateral buckling of a beam of simple cross section

This phenomenon of losing the stability of the equilibrium form of a beam subjected to bending is called *lateral buckling* (or *buckling due to bending*).
The study of the lateral buckling leads to the differential equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} + \frac{M^2}{EI_z GI_t}\theta = 0\,, \tag{a}$$

where EI_z and GI_t are the rigidities by bending in the z-plane or torsion (of the cross section), respectively, M is the bending moment in the y-plane, while θ is the unknown rotation of torsion of the cross section (simple, without booms) .
Introducing the notation

$$\beta = \frac{M}{\sqrt{EI_z GI_t}} , \tag{b}$$

the equation (a) becomes

$$\frac{d^2\theta}{dx^2} + \beta^2\theta = 0 , \tag{c}$$

analogous to that of the axial buckling (see e.g. Appl.1.31). To this ODE one associates the two-point conditions

$$\theta(0) = \theta(l) = 0 , \tag{d}$$

therefore a Sturm-Liouville problem.

Solution. The general solution of the ODE (c) is

$$\theta = A \sin \beta x + B \cos \beta x .$$

Making use of the two-point conditions, one obtains the minimal eigenvalue $\beta = \pi/l$, so that

$$M_{cr} = \frac{\pi}{l}\sqrt{EI_z GI_t} .$$

Application 1.36

Problem. Consider a steel bar built-in at one end and elastically supported at the other end. Determine the critical buckling force P_{cr} .

Mathematical model. The bending moment in a cross section of abscissa x is given by (Fig.1.52)

$$M = P(f - w) - cf(l - x), \tag{a}$$

where P is the axial force, f the deflection of the elastically supportel end (the elastic coefficient is c), and l is the bar length. Using the notation (g), Appl.1.30, it results the differential equation of the deflection

$$\frac{d^2 w}{dx^2} + \beta^2 w = \beta^2\left[f - \frac{cf}{P}(l - x)\right] , \tag{b}$$

to which we must add the conditions

$$w(0) = (dw/dx)_{x=0} = 0 , \; w(l) = f , \tag{c}$$

therefore, again an eigenvalue problem.

Solution. The general solution of the differential equation is

$$w = A\sin\beta x + B\cos\beta x + f\left(1 - \frac{cl}{P}\right) + \frac{cf}{P}x\,. \qquad \text{(d)}$$

The initial conditions lead to

$$A = -f\left(1 - \frac{cl}{\beta^2 EI}\right),\ B = \frac{cl}{\beta^3 EI}\,, \qquad \text{(e)}$$

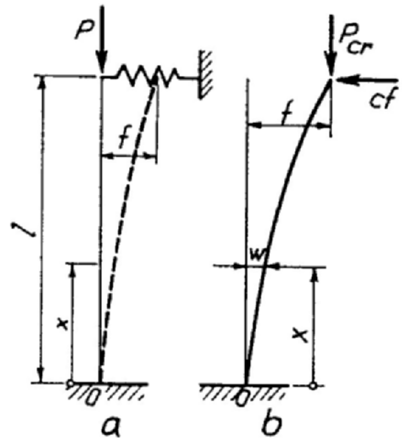

Figure 1. 52. Buckling of a bar built-in at one end and elastically supported at the other end

so that the deflection is given by

$$w(x) = f\left[\left(1 - \frac{cl}{\beta^2 EI}\right)(1 - \cos\beta x) + \frac{c}{\beta^2 EI}\left(x - \frac{1}{\beta}\sin\beta x\right)\right]. \qquad \text{(f)}$$

The condition $w(l) = f$ leads to the characteristic equation

$$\left(1 - \frac{cl}{\beta^2 EI}\right)(1 - \cos\beta l) + \frac{c}{\beta^2 EI}\left(l - \frac{1}{\beta}\sin\beta l\right) = 1\,, \qquad \text{(g)}$$

which can also be written as

$$\beta l - k(\beta l)^3 = \tan\beta l\,, \qquad \text{(h)}$$

where

$$k = \frac{EI}{cl^3}\,. \qquad \text{(i)}$$

For a bar of circular cross section of diameter d and the numerical data $E = 2.1\cdot 10^6\ \mathrm{daN/cm^2}$, $l = 2\mathrm{m}$, $c = 5\,\mathrm{daN/cm}$, $d = 4\mathrm{cm}$, we get

$$k = \frac{2.1\cdot 10^6 \cdot \dfrac{\pi\cdot 4^4}{64}}{500\cdot 200^3} = 0.659734457\ .$$

Table 1. 3. The values of *k* fas a function of *βl*

βl	k	βl	k
π/2	+∞	2.05	0.461347
1.60	8.748177	2.10	0.411386
1.65	3.172054	2.15	0.370179
1.70	1.912600	2.20	0.335633
1.75	1.356572	2.25	0.306272
1.80	1.043598	2.30	0.281024
1.85	0.843079	2.35	0.259092
1.90	0.703761	2.40	0.239874
1.95	0.601423	2.45	0.222901

The minimal root of the equation (h) is, in this case,

$$\beta l = 1.9197825\,.$$

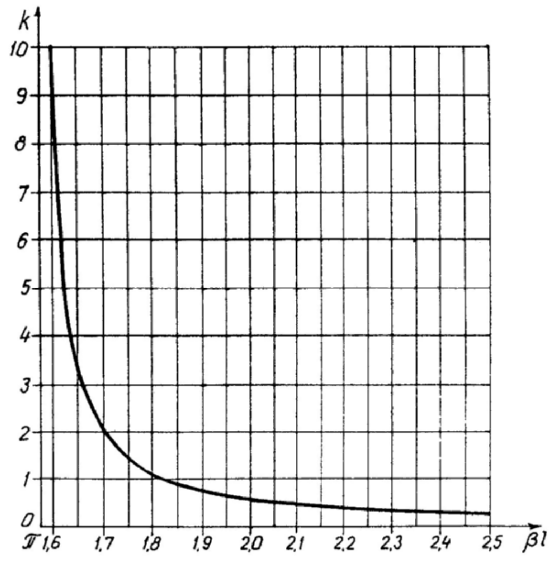

Figure 1. 53. The diagram of the function $k = f(\beta l)$

The critical force becomes

$$P_{cr} = 1.9197825^2 \frac{EI}{l^2} = \frac{\pi^2 EI}{(1.6364l)^2},$$

where the buckling length $l_f = 1.6364l$ was emphasized.
Various values of

$$k = f(\beta l) = \frac{\beta l - \tan\beta l}{(\beta l)^3} = \frac{1}{(\beta l)^2}\left(1 - \frac{\tan\beta l}{\beta l}\right) \tag{j}$$

are given in Table 1.3 and are plotted into a diagram (Fig.1.53). Both the table and the figure are useful to obtain the root βl for a given *k*.

Application 1.37

Problem. Search a solution by power series for the buckling of a doubly hinged bar.

Mathematical model. The deflection w satisfies the linear second order ODE

$$\frac{d^2 w}{dx^2} + \frac{P}{EI} w = 0, \tag{a}$$

where *EI* is the bending rigidity and *w* is the unknown deflection. To this ODE, we must add the boundary conditions

$$w(0) = 0, \quad w(l) = 0. \tag{b}$$

Solution. As the deformed axis of the bar has an antisymmetric form with respect to the origin *O*, we use an odd series expansion

$$w = \alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5 + \ldots + \alpha_{2n-1} x^{2n-1} + \alpha_{2n+1} x^{2n+1} + \ldots. \tag{c}$$

The second derivative of (c) is

$$\frac{d^2 w}{dx^2} = 2 \cdot 3\alpha_3 x + 4 \cdot 5\alpha_5 x^3 + 6 \cdot 7\alpha_7 x^5 + \ldots + 2n(2n+1)\alpha_{2n+1} x^{2n-1} + \ldots. \tag{d}$$

As it is seen, the boundary condition $w(0) = 0$ is fulfilled.
Introducing (c) and (d) in (a), it follows

$$\begin{aligned}&\left[2 \cdot 3\alpha_3 x + 4 \cdot 5\alpha_5 x^3 + 6 \cdot 7\alpha_7 x^5 + \ldots + 2n(2n+1)\alpha_{2n+1} x^{2n-1} + \ldots\right] + \\ &+ \frac{P}{EI}\left(\alpha_1 x + \alpha_3 x^3 + \alpha_5 x^5 + \ldots + \alpha_{2n-1} x^{2n-1} + \alpha_{2n+1} x^{2n+1} + \ldots\right) = 0.\end{aligned} \tag{e}$$

The value of the force *P* must be P_{cr}, so that $P/EI = \beta^2$ must be positive; the polynomials in (e) must differ by a constant factor, and the ratio of two homologous coefficients (of the same power) must be negative

$$\frac{2\cdot 3\alpha_3}{\alpha_1} = -\beta^2, \frac{4\cdot 5\alpha_5}{\alpha_3} = -\beta^2, \frac{6\cdot 7\alpha_7}{\alpha_5} = -\beta^2, \frac{2n(2n+1)\alpha_{2n+1}}{\alpha_{2n-1}} = -\beta^2, \tag{f}$$

so that

$$\alpha_3 = -\frac{\beta^2}{2\cdot 3}\alpha_1, \alpha_5 = -\frac{\beta^2}{4\cdot 5}\alpha_3 = \frac{\beta^4}{5!}\alpha_1, \ldots, \alpha_{2n+1} = (-1)^n \frac{\beta^{2n}}{(2n+1)!}\alpha_1, \ldots . \tag{g}$$

Finally, we get

$$w = \frac{\alpha_1}{\beta}\left[\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \ldots + (-1)^n \frac{(\beta x)^{2n+1}}{(2n+1)!} + \ldots\right], \tag{h}$$

which is precisely the series expansion of the sinus

$$w = a \sin \beta x, \;\; a = \frac{\alpha_1}{\beta}. \tag{i}$$

The boundary condition $w(l) = 0$ is satisfied if $\beta = n\pi/l$; hence,

$$w = a \sin \frac{n\pi x}{l}; \tag{j}$$

thus, we found again the classical solution.
We also obtain $P_{cr} = \beta^2 EI = n^2 \pi^2 EI/l^2$.
Using the same development (c), one may study the buckling of a bar free at one end and built-in at the other end, a.s.o.

Application 1.38

Problem. Determine the buckling critical force of a cantilever bar, of moment of inertia varying as $I_x = I_0 (x/a)^4$ (e.g., for a circular cross section, Fig.1.54).

Mathematical model. The deflection w of the bar axis due to the compression force P is governed by the differential equation

$$EI_x \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + Pw = 0, \tag{a}$$

where EI_x is the bending rigidity.

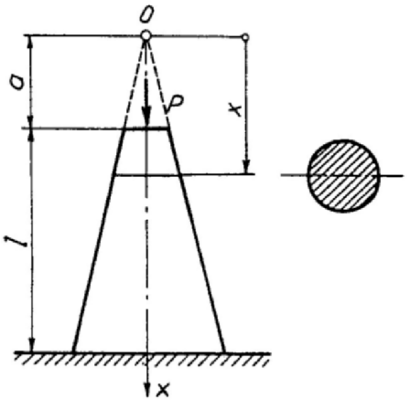

Figure 1. 54. Buckling of a cantilever bar with a variable moment of inertia

Taking into account the expression of I_x, the equation (a) becomes

$$x^4 \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \beta^2 w = 0 , \tag{b}$$

where

$$\beta = a^2 \sqrt{\frac{P}{EI_0}} . \tag{c}$$

To (b) we must associate the boundary conditions

$$w(a) = 0 , \ (\mathrm{d}w/\mathrm{d}x)_{x=a+l} = 0 .$$

Solution. The ODE (b) is linear, but it has no more constant coefficients. Yet, we can obtain its general solution by means of Bessel's functions of the first species and order $\gamma = 1/2$; in this case, applying Liouville's theorem, we conclude that it can be expressed by elementary functions (see Sec.2.7)

$$w = x\left(A\cos\frac{\beta}{x} + B\sin\frac{\beta}{x} \right), \tag{d}$$

where A and B are integration constants.
The derivative reads

$$\frac{\mathrm{d}w}{\mathrm{d}x} = A\cos\frac{\beta}{x} + B\sin\frac{\beta}{x} + \frac{\beta}{x}\left(A\sin\frac{\beta}{x} - B\cos\frac{\beta}{x} \right). \tag{e}$$

Applying now the boundary conditions (c), we get for A and B a homogeneous linear algebraic system

$$\begin{bmatrix} \cos\dfrac{\beta}{a} & \sin\dfrac{\beta}{a} \\ \cos\dfrac{\beta}{a+l}+\dfrac{\beta}{a+l}\sin\dfrac{\beta}{a+l} & \sin\dfrac{\beta}{a+l}-\dfrac{\beta}{a+l}\cos\dfrac{\beta}{a+l} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix}=0\,. \tag{f}$$

To obtain non-zero solutions, the determinant of (f) must vanish and we get the characteristic equation

$$\sin\frac{\beta l}{a(a+l)}+\frac{\beta l}{a+l}\cos\frac{\beta l}{a(a+l)}=0\,, \tag{g}$$

or

$$\tan\frac{\beta l}{a(a+l)}=-\frac{\beta l}{a(a+l)}\frac{a}{l}\,. \tag{h}$$

Table 1. 4. The values of u and μ for various ratios a/l

a/l	u	μ
0.2	2.65366	0,19731
0.5	2.28893	0.45751
1	2.02876	0.77426
2	1.83660	1.14037
3	1.75186	1.34014
5	1.68868	1.55032
10	1.63199	1.69126
∞	$\pi/2$	2

With the notation

$$u=\frac{\beta l}{a(a+l)}=\frac{al}{a+l}\sqrt{\frac{P}{EI_0}}\,, \tag{i}$$

the equation (h) becomes

$$\frac{\tan u}{u}=-\frac{a}{l}\,, \tag{j}$$

which is solved using the Table 1.2.
From (i) one obtains the critical force

$$P_{cr}=\gamma^2 EI_0\left(\frac{1}{a}+\frac{1}{1}\right)^2=\frac{\pi^2 EI_0}{(\mu l)^2}\,, \tag{k}$$

where

$$\mu = \frac{\pi}{u\left(1 + \frac{l}{a}\right)}. \qquad (1)$$

Table 1.4 contains the numerical values of u and μ for various ratios $a/l \in [0.2, \infty)$.
The variation of the buckling length $l_f = \mu l$ as function of the ratio a/l is plotted in a diagram (Fig.1.55). For $a/l \to \infty$, one obtains $\mu = 2$, that is, the value corresponding to a cantilever of constant cross section.

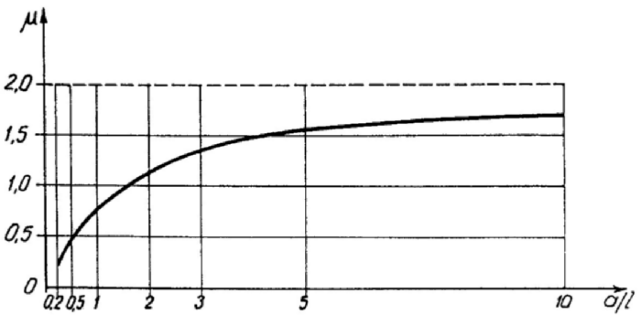

Figure 1. 55. The diagram of the function $\mu = f(a/l)$

Application 1.39

Problem. Determine the buckling critical load of a bar of length l, free at the upper end and built-in at the bottom; the axial load p is supposed to be uniformly distributed along the bar axis (Fig.1.56).

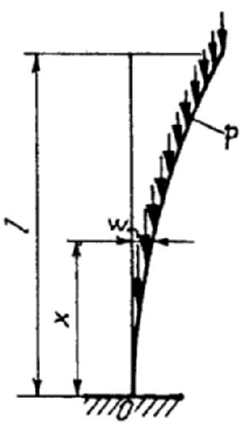

Figure 1. 56. Buckling of a cantilever bar acted upon by an axial uniformly distributed load

Mathematical model. The deflection w satisfies the differential equation

$$\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + \frac{P}{EI}(l - x)\frac{\mathrm{d}w}{\mathrm{d}x} = 0\,, \tag{a}$$

where EI is the bending rigidity. The boundary conditions are

$$w(0) = 0\,,\ (\mathrm{d}w/\mathrm{d}x)_{x=0} = 0\,, \tag{b}$$

$$\left(\mathrm{d}^2 w/\mathrm{d}x^2\right)_{x=l} = 0\,. \tag{c}$$

Solution. We notice that the order of the equation (a) may be easily reduced by a unit. But first of all, we make a change of variable

$$z = \frac{2}{3}\sqrt{\frac{P}{EI}}(l - x)^{3/2}\,,\ x = l - \sqrt[3]{\frac{9EI}{4p}}z^{2/3}\,. \tag{d}$$

Step by step differentiation yields

$$\begin{aligned}
\frac{\mathrm{d}w}{\mathrm{d}x} &= -\frac{\mathrm{d}w}{\mathrm{d}z}\sqrt[3]{\frac{3p}{2EI}z}\,,\\
\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} &= \left(\frac{3p}{2EI}\right)^{2/3}\left(\frac{1}{3}z^{-1/3}\frac{\mathrm{d}w}{\mathrm{d}z} + z^{2/3}\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}\right),\\
\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} &= \frac{3p}{2EI}\left(\frac{1}{9}z^{-1}\frac{\mathrm{d}w}{\mathrm{d}z} - \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} - z\frac{\mathrm{d}^3 w}{\mathrm{d}z^3}\right).
\end{aligned} \tag{e}$$

Introducing this in (a) and using the notation

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u\,, \tag{f}$$

we get

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + \frac{1}{z}\frac{\mathrm{d}u}{\mathrm{d}z} + \left(1 - \frac{1}{9z^2}\right)u = 0, \tag{g}$$

i.e. a differential equation of Bessel type of first species and order $\gamma = 1/3$ (see Sec.2.7). The general solution of this ODE is

$$u = C_1 J_{1/3}(z) + C_2 J_{-1/3}(z), \tag{h}$$

where

$$J_{1/3}(z) = z^{1/3}\left(1 - \frac{3}{16}z^2 + \frac{9}{896}z^4 - \ldots\right),\ J_{-1/3}(z) = z^{-1/3}\left(1 - \frac{3}{8}z^2 + \frac{9}{320}z^4 - \ldots\right). \tag{i}$$

In the new variables, the boundary condition (c) becomes $(1/3)z^{-1/3}u + z^{2/3}\,\mathrm{d}u/\mathrm{d}z = 0$ for $z = 0$, and we obtain $C_2 = 0$. The second boundary condition (b) becomes $u = 0$ for $x = 0$, so that $z = (2/3)\sqrt{pl^3/EI}$.

The transcendental equation which leads to $(pl)_{cr}$ becomes

$$\left(\frac{2}{3}\sqrt{\frac{pl^3}{EI}}\right)^{-4/3}\left[1 - \frac{3}{8}\left(\frac{4}{9}\frac{pl^3}{EI}\right) + \frac{9}{320}\left(\frac{4}{9}\frac{pl^3}{EI}\right)^2 - \ldots\right] = 0. \tag{j}$$

The smallest root of this equation is $(2/3)\sqrt{pl^3/EI} = 1.866$, so that

$$(pl)_{cr} = \left(\frac{3}{2}\right)^2 \cdot 1.866^2 \cdot \frac{EI}{l^2} = \frac{7.834EI}{l^2} = \frac{\pi^2 EI}{(1.122l)^2}. \tag{k}$$

The deflection w is obtained from (f), by integration, taking $C_2 = 0$, while C_1 remains non-determinate. The value $l_f = 1.122l$ represents the buckling length of the bar.

Application 1.40

Problem. Consider a circular cylindrical vessel of wall thickness varying linearly with the height. Determine its axially symmetric deformation due to an interior loading with liquid (Fig.1.57).

Mathematical model. Let us take the origin of the Ox -axis (Fig.1.57) at the theoretical applicate corresponding to a vanishing wall thickness. Then the differential equation of the deflection is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) + \frac{12(1-\nu^2)}{\alpha^3 a^2}xw = -\frac{12(1-\nu^2)(x - x_0)}{E\alpha^3}, \tag{a}$$

where the variation law of the thickness of the wall is given by

$$h = \alpha x\,. \tag{b}$$

The free edges ($x = x_0$ and $x = x_0 + h$) are thus specified; the constants E and ν are the modulus of longitudinal elasticity and Poisson's ratio, respectively.

The problem requires the general solution of (a).

Solution. A particular solution of the linear fourth order ODE (a) is

$$w_p = -\frac{\gamma a^2}{E}\frac{x - x_0}{x}\,, \tag{c}$$

where γ is the unit weight of the liquid; it represents the radial dilatation of the cylinder.

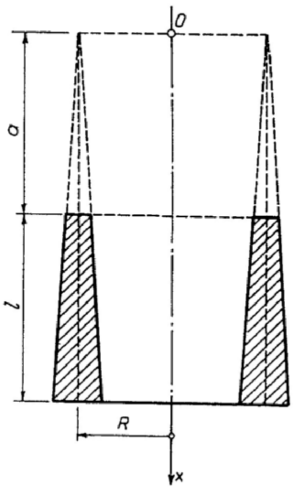

Figure 1. 57. Cylindrical tank the wall thickness of which has a linear variation

Further, it is necessary to search the general solution of the homogeneous equation

$$\frac{1}{x}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) + \rho^4 w = 0\,, \tag{d}$$

where we used the notation

$$\rho^4 = \frac{12\left(1 - \nu^2\right)}{\alpha^2 a^2}\,. \tag{e}$$

We mention that the first term in (d) may be written in the form

$$\frac{1}{x}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\left(x^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^2\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}w}{\mathrm{d}x}\right)\right]\right\}.$$

Introducing the differential operator

$$\mathrm{L}(w) = \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = x\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 2\frac{\mathrm{d}w}{\mathrm{d}x}\,, \tag{f}$$

the equation (d) becomes

$$\mathrm{L}[\mathrm{L}(w)] + \rho^4 w = 0 . \tag{g}$$

We search solutions of the form $\mathrm{L}(w) = \lambda w$, $\lambda = \mathrm{const}$; introducing this in (g), we get, step by step,

$$\mathrm{L}(\lambda w) + \rho^4 w = \lambda \mathrm{L}(w) + \rho^4 w = \lambda^2 w + \rho^4 w = \left(\lambda^2 + \rho^4\right) w = 0 . \tag{h}$$

If $\lambda^2 + \rho^4 = 0$, that is $\lambda = \pm \mathrm{i}\rho^2$, then the differential equation (h) may be split into the following two differential equations

$$\mathrm{L}(w) + \mathrm{i}\rho^2 w = 0 , \tag{i}$$

$$\mathrm{L}(w) - \mathrm{i}\rho^2 w = 0 . \tag{j}$$

Let

$$w_1 = \varphi_1 + \mathrm{i}\varphi_2 ,\ w_2 = \varphi_3 + \mathrm{i}\varphi_4 , \tag{k}$$

be two independent linear solutions of the equation (i); then

$$w_3 = \varphi_1 - \mathrm{i}\varphi_2 ,\ w_4 = \varphi_3 - \mathrm{i}\varphi_4 , \tag{l}$$

are two linearly independent solutions of the equation (j).
By a convenient choice of the integration constants, the general solution of the differential equation (d) may be put in the form

$$w = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 + C_4\varphi_4 , \tag{m}$$

where C_1, C_2, C_3, C_4 are four integration constants.
Thus, the problem is reduced to searching the four functions φ_i, $i = \overline{1,4}$; hence, one must search the solution of one of the equations (i) or (j).
Choosing e.g. the equation (i) and replacing $\mathrm{L}(w)$ by (f), it results

$$x\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 2\frac{\mathrm{d}w}{\mathrm{d}x} + \mathrm{i}\rho^2 w = 0 . \tag{n}$$

By the change of variable

$$\eta = 2\rho\sqrt{\mathrm{i}x},\ \xi = w\sqrt{x} , \tag{o}$$

the equation (n) becomes

$$\eta^2\frac{\mathrm{d}^2\xi}{\mathrm{d}\eta^2} + \eta\frac{\mathrm{d}\xi}{\mathrm{d}\eta} + \left(\eta^2 - 1\right)\xi = 0 . \tag{p}$$

One may search a solution of the equation (p) in the form of a power series

$$\xi_1 = a_0 + a_1\eta + a_2\eta^2 + \ldots .$$

Introducing this in (p) and taking $a_0 = 0$, we obtain

$$\xi_1 = \frac{\eta}{2}\left[1 - \frac{\eta^2}{2\cdot 4} + \frac{\eta^4}{2\cdot 4^2\cdot 6} - \frac{\eta^6}{2\cdot(4\cdot 6)^2\cdot 8} + \ldots\right] = J_1(\eta), \tag{q}$$

where $J_1(\eta)$ is Bessel's function of first species and order 1. This expression may be also written in the form

$$\xi_1 = J_1(\eta) = -\frac{\mathrm{d}}{\mathrm{d}\eta}\left[1 - \frac{\eta^2}{2^2} + \frac{\eta^4}{(2\cdot 4)^2} - \frac{\eta^6}{(2\cdot 4\cdot 6)^2} + \ldots\right] = -\frac{\mathrm{d}J_0}{\mathrm{d}\eta}, \tag{r}$$

where J_0 is Bessel's function of first species and order zero

$$J_0(\eta) = 1 - \frac{\eta^2}{2^2} + \frac{\eta^4}{(2\cdot 4)^2} - \frac{\eta^6}{(2\cdot 4\cdot 6)^2} + \ldots$$

Replacing η in the first expression (o) and separating the real and imaginary parts, one may write

$$J_0(\eta) = \psi_1\left(2\rho\sqrt{x}\right) + \mathrm{i}\psi_2\left(2\rho\sqrt{x}\right),$$

where

$$\psi_1\left(2\rho\sqrt{x}\right) = 1 - \frac{\left(2\rho\sqrt{x}\right)^4}{(2\cdot 4)^2} + \frac{\left(2\rho\sqrt{x}\right)^8}{(2\cdot 4\cdot 6\cdot 8)^2} - \ldots,$$

$$\psi_2\left(2\rho\sqrt{x}\right) = -\frac{\left(2\rho\sqrt{x}\right)^2}{2^2} + \frac{\left(2\rho\sqrt{x}\right)^6}{(2\cdot 4\cdot 6)^2} - \frac{\left(2\rho\sqrt{x}\right)^{10}}{(2\cdot 4\cdot 6\cdot 8\cdot 10)^2} + \ldots;$$

in this case the solution (q) reads

$$\xi_1 = -\psi_1'\left(2\rho\sqrt{x}\right) - \mathrm{i}\psi_2'\left(2\rho\sqrt{x}\right). \tag{s}$$

A second solution of the equation (p) may be obtain in the form

$$\xi_2 = -\psi_3'\left(2\rho\sqrt{x}\right) + \mathrm{i}\psi_4'\left(2\rho\sqrt{x}\right), \tag{t}$$

where

$$\psi_3\left(2\rho\sqrt{x}\right) = \frac{1}{2}\psi_1\left(2\rho\sqrt{x}\right) - \frac{2}{\pi}\left[R_1 + \ln\left(\rho\beta\sqrt{x}\right)\psi_2\left(2\rho\sqrt{x}\right)\right],$$

$$\psi_4\left(2\rho\sqrt{x}\right) = \frac{1}{2}\psi_2\left(2\rho\sqrt{x}\right) + \frac{2}{\pi}\left[R_2 + \ln\left(\rho\beta\sqrt{x}\right)\psi_1\left(2\rho\sqrt{x}\right)\right],$$

with

$$R_1 = \left(\rho\sqrt{x}\right)^2 - \frac{S(3)}{(3!)^2}\left(\rho\sqrt{x}\right)^6 + \frac{S(5)}{(5!)^2}\left(\rho\sqrt{x}\right)^{10} - \ldots,$$
$$R_2 = \frac{S(2)}{2^2}\left(\rho\sqrt{x}\right)^4 - \frac{S(4)}{(4!)^2}\left(\rho\sqrt{x}\right)^8 + \frac{S(6)}{(6!)^2}\left(\rho\sqrt{x}\right)^{12} - \ldots$$

and

$$S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$
$$\ln\beta = 0.57722\ldots \text{ (Euler's constant).}$$

The general solution of the equation (m) becomes

$$w = \frac{\xi}{\sqrt{x}} = \frac{1}{\sqrt{x}}\left[C_1\psi_1'\left(2\rho\sqrt{x}\right) + C_2\psi_2'\left(2\rho\sqrt{x}\right) + C_3\psi_3'\left(2\rho\sqrt{x}\right) + C_4\psi_4'\left(2\rho\sqrt{x}\right)\right].$$

Numerical values of the functions $\psi_i, i = \overline{1,4}$, and of their derivatives of first order may be found in F. Schleicher. These functions are connected also to Klein's functions.

Chapter 2

LINEAR ODEs OF HIGHER ORDER ($n > 2$)

1. The General Study of Linear ODEs of Order $n > 2$

1.1 GENERALITIES

The linear ODE of order *n* is of the form (see also the Introduction)

$$\mathrm{L}y \equiv a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + ... + a_{n-1}(x)y' + a_n(x)y = F(x), \tag{2.1.1}$$

where the functions $a_j(x), F(x)$ are defined and supposedly continuous on a real interval *I*.
Obviously, in a classical frame we search for solutions of (2.1.1) in the class $\mathrm{C}^n(I)$.
If $a_0(x) \neq 0$ for $x \in I$, we can divide both members of (2.1.1) by $a_0(x)$. We obtain

$$\mathrm{L}_1 y \equiv y^{(n)} + p_1(x)y^{(n-1)} + ... + p_{n-1}(x)y' + p_n(x)y = f(x), \tag{2.1.2}$$

where

$$p_j(x) = \frac{a_j(x)}{a_0(x)}, f(x) = \frac{F(x)}{a_0(x)}. \tag{2.1.3}$$

The ODEs (2.1.1), (2.1.2) are *non-homogeneous*. If the right member is null, then they are called *homogeneous*. The homogeneous ODE associated to (2.1.1) is

$$\mathrm{L}y \equiv a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + ... + a_{n-1}(x)y' + a_n(x)y = 0. \tag{2.1.4}$$

A linear ODE is still linear for any change of variable and for any linear change of function.

1.2 LINEAR HOMOGENEOUS ODEs

The operator L, defined by the left member of (2.1.1), is linear, i.e.,

$$\mathrm{L}(\alpha y + \beta z) = \alpha \mathrm{L}y + \beta \mathrm{L}z, \tag{2.1.5}$$

for any real/complex α, β and any $y, z \in \mathrm{C}^n(I)$.
The operator L, and, consequently, also L_1, may be put in the form of a *differential polynomial*, as shown in Sec.1.5,

$$\mathrm{L} \equiv \mathrm{P}(x, \mathrm{D}) \equiv a_0(x)\mathrm{D}^n + a_1(x)\mathrm{D}^{n-1} + \ldots + a_{n-1}(x)\mathrm{D} + a_n(x)\mathrm{E}, \quad \mathrm{D} = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{2.1.6}$$

E being the identity on $\mathrm{C}^n(I)$. By using the well-known Leibniz formula

$$\mathrm{D}(uv) = u\mathrm{D}v + v\mathrm{D}u, \tag{2.1.7}$$

we can prove the following formula, useful for applications,

$$\begin{aligned}\mathrm{P}(x,\mathrm{D})(uv) = u\mathrm{P}(x,\mathrm{D})v + \frac{1}{1!}\mathrm{D}u\,\mathrm{P}^{(1)}(x,\mathrm{D})v + \frac{1}{2!}\mathrm{D}^2u\,\mathrm{P}^{(2)}(x,\mathrm{D})v \\ + \ldots + \frac{1}{j!}\mathrm{D}^j u\,\mathrm{P}^{(j)}(x,\mathrm{D})v + \ldots + \frac{1}{n!}\mathrm{D}^n u\,\mathrm{P}^{(n)}(x,\mathrm{D})v,\end{aligned} \tag{2.1.8}$$

in which $\mathrm{P}^{(j)}(x,\mathrm{D})$ are the formal derivatives with respect to D of the differential polynomial P

$$\begin{aligned}\mathrm{P}^{(j)}(x,\mathrm{D}) = a_0(x)\cdot n(n-1)\ldots(n-j+1)\mathrm{D}^{n-j} \\ + a_1(x)\cdot(n-1)\ldots(n-j)\mathrm{D}^{n-j-1} + \ldots + a_j(x)\cdot j!\,\mathrm{E}.\end{aligned} \tag{2.1.9}$$

Indeed, we have

$$\begin{array}{ll}\mathrm{E}(uv) = uv, & \times a_n(x) \\ \mathrm{D}(uv) = u\,\mathrm{D}v + v\,\mathrm{D}u, & \times a_{n-1}(x) \\ \mathrm{D}^2(uv) = u\,\mathrm{D}^2 v + C_2^1 \mathrm{D}u\,\mathrm{D}v + C_2^2 v\,\mathrm{D}^2 u, & \times a_{n-2}(x) \\ \mathrm{D}^3(uv) = u\,\mathrm{D}^3 v + C_3^1 \mathrm{D}u\,\mathrm{D}^2 v + C_3^2 \mathrm{D}^2 u\,\mathrm{D}v + C_3^3 v\,\mathrm{D}^3 u, & \times a_{n-3}(x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \mathrm{D}^j(uv) = u\,\mathrm{D}^j v + C_j^1 \mathrm{D}u\,\mathrm{D}^{j-1} v + \ldots + C_j^{j-1}\mathrm{D}^{j-1}u\,\mathrm{D}v + C_j^j v\,\mathrm{D}^j u, & \times a_{n-j}(x) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \mathrm{D}^n(uv) = u\,\mathrm{D}^n v + C_n^1 \mathrm{D}u\,\mathrm{D}^{n-1} v + \ldots + C_n^{n-1}\mathrm{D}^{n-1}u\,\mathrm{D}v + C_n^n v\,\mathrm{D}^j. & \times a_0(x)\end{array} \tag{2.1.10}$$

We then perform the multiplication with the coefficients, indicated on the right hand of (2.1.10) and we sum up both members of these relationships. Also observing the common factors u, Du, a.s.o., we finally get (2.1.8).
As previously, in the case of lower order ODEs, y is a solution of the homogeneous equation (2.1.4) if and only if y is an element of the kernel of L

$$\ker \mathrm{L} = \{y \in \mathrm{C}^n(I) \,|\, \mathrm{L}y = 0\}. \tag{2.1.11}$$

As L is linear, it immediately follows that if y_1, y_2 are solutions of the homogeneous ODE (2.1.4), then any of their linear combination is also a solution of the same equation.

It results that $\ker L$ is a vector subspace of $C^n(I)$. Obviously, one can immediately prove that if $y_1, y_2, ..., y_n$ are solutions of (2.1.4), then any linear combination

$$y = c_1 y_1 + c_2 y_2 ... + c_n y_n \tag{2.1.12}$$

is also a solution, i.e., it belongs to $\ker L$.
One can prove that

$$\dim \ker L = n. \tag{2.1.13}$$

A basis in $\ker L$ is called a *fundamental system* of solutions of (2.1.4). In other words, a fundamental system of solutions for (2.1.4) is a system of *n* linearly independent solutions of (2.1.4).
In general, a system of *n* functions $\{\varphi_k\}_{k=\overline{1,n}}$ defined on some real set A is called *linearly independent* if any linear combination $\sum_{j=1}^{n} c_j \varphi_j(x)$ vanishing identically on *I* – i.e., $\sum_{j=1}^{n} c_j \varphi_j(x) = 0$, $\forall x \in A$ – involves $c_j = 0, j = \overline{1,n}$.
It can be proved that the necessary and sufficient condition that a system of *n* solutions $\{\varphi_k\}_{k=\overline{1,n}}$ of (2.1.1) be fundamental is that its *Wronskian*, defined by the determinant

$$W[y_1, y_2, ..., y_n] \underset{def}{\equiv} \begin{vmatrix} y_1 & y_2 & ... & y_n \\ y_1' & y_2' & ... & y_n' \\ ... & ... & ... & ... \\ y_1^{(n-1)} & y_2^{(n-1)} & ... & y_n^{(n-1)} \end{vmatrix}, \tag{2.1.14}$$

be non-zero on *I*.
We previously mentioned Liouville's formula for linear second order ODEs. This result may be generalized to get Liouville's formula for linear *n*-th order ODEs, which is

$$W(x) = C e^{-\int \frac{a_1(x)}{a_0(x)} dx}, \tag{2.1.15}$$

or, for an arbitrary $x_0 \in I$

$$W(x) = W(x_0) e^{\int_{x_0}^{x} \frac{a_1(t)}{a_0(t)} dt}. \tag{2.1.16}$$

From the last formula, we see that if the Wronskian cancels at a point of *I*, then it vanishes identically on *I*. Hence, given a system of *n* solutions of the homogeneous ODE (2.1.4), if their Wronskian cancels in a point of *I*, the system is not fundamental. If the Wronskian is not nul on the whole *I*, then the system is fundamental.
Also by using the Wronskian, it can be proved that a linear *n*-th order ODE with continuous coefficients always allows a fundamental system of solutions.

From the above considerations, it follows that if $\{y_j\}_{j=\overline{1,n}}$ form a fundamental system for a linear homogeneous ODE, then every solution y of this ODE may be written as a linear combination of the functions of the system, i.e.

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x). \tag{2.1.17}$$

Thus, if we know a fundamental system of solutions for a linear homogeneous ODE, then this equation is completely solved.

Consider now the problem of finding a solution of (2.1.4) that also satisfies the Cauchy conditions

$$\begin{aligned} &y(x_0) = y_0, \\ &y'(x_0) = y'_0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &y^{(n-1)}(x_0) = y_0^{(n-1)}, \qquad x_0 \in I. \end{aligned} \tag{2.1.18}$$

The solution of the Cauchy problem (2.1.4), (2.1.18) may be written in the form (2.1.17). Differentiating this expression $n-1$ times and taking then into account the initial conditions (2.1.18), we get a linear algebraic system for the constants c_j, $j = \overline{1,n}$,

$$\begin{aligned} &c_1 y_{10}^0 + c_2 y_{20}^0 + \ldots + c_n y_{n0}^0 = y_0, \\ &c_1 y_{10}^1 + c_2 y_{20}^1 + \ldots + c_n y_{n0}^1 = y'_0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &c_1 y_{10}^{n-1} + c_2 y_{n0}^{n-1} + \ldots + c_n y_{n0}^{n-1} = y_0^{(n-1)}, \end{aligned} \tag{2.1.19}$$

in which we used the notations $y_{j0}^k = y_j^{(k)}(x_0)$, $k = \overline{0, n-1}$, $j = \overline{1,n}$, for the values of the functions belongong to the fundamental system, obtained for $k = 0$, and of their derivatives, all of them taken at $x_0 \in I$. We see that the determinant of the system (2.1.19) is precisely the Wronskian $W(x_0)$of the considered fundamental system, taken at x_0. As the system is fundamental, its Wronskian never vanishes on I, therefore $W(x_0) \neq 0$. It follows that the Cauchy problem (2.1.4), (2.1.18) is unique. We get this unique solution by replacing the solution c_j, $j = \overline{1,n}$, of the algebraic system (2.1.19) in the expression (2.1.17).

The calculus of the coefficients may be considerably simplified if the functions of the fundamental system were determined such that they satisfy the initial conditions

$$y_j(x_0) = 0,\ y'_j(x_0) = 0, \ldots, y_j^{(j)}(x_0) = 1,\ y_{j+1}^{(j)}(x_0) = 0, \ldots, y_n^{(j)}(x_0) = 0, \quad j = \overline{1,n}. \tag{2.1.20}$$

Indeed, in this case the system (2.1.19) straightforwardly yields $c_j = y_0^{(j-1)}$, $j = \overline{1.n}$, where $y_0^{(0)} = y_0$. The solution of the Cauchy problem (2.1.4), (2.1.18) is then

$$y(x) = y_0 y_1(x) + y_0' y_2(x) + \ldots + y_0^{(n-1)} y_n(x). \tag{2.1.21}$$

A fundamental system of solutions satisfying the Cauchy conditions (2.1.20) is called a *normal system*. As we see, a normal system allows to write directly the solution of a Cauchy problem replacing the initial data in the formula (2.1.17).

At Sec.2.1, Chap.1, we considered a normal system in the particular case of the second order ODEs. At Sec.2.3, same chapter, we determined the functions Z_1, Z_2, representing the normal system for the second order linear ODE with constant coefficients, if the associated characteristic equation allows real and distinct roots (formulae (1.2.63), (1.2.65)).

Given a linear ODE, we can get for it infinitely many fundamental systems. Conversely, a fundamental system $\{y_j\}_{j=\overline{1,n}}$ corresponds to a unique linear *n*-th order ODE, except for a multiplicative factor. This ODE is found by using the functions $y_j, j = \overline{1,n}$ of the fundamental system. Indeed, if y is an arbitrary solution of the ODE, then $y_1, y_2, \ldots, y_n, y$ are linearly dependent, i.e. their Wronskian is identically null on *I*. We thus have

$$W[y_1, y_2, \ldots, y_n, y] \equiv \begin{vmatrix} y & y_1 & y_2 & \cdots & y_n \\ y' & y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y^{(n-2)} & y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y^{(n-1)} & y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = 0. \tag{2.1.22}$$

This is the ODE we are looking for. It is linear, as we can see developing the above determinant following the first column and it is of order *n*, as the coefficient of $y^{(n)}$ is precisely the Wronskian $W[y_1, y_2, \ldots, y_n]$ of the given fundamental system, which does not vanish on *I*.

Example. Let us find the homogeneous ODE allowing $y_1 = \cosh x, y_2 = \sinh x$ as a fundamental system.

The searched ODE is of second order and the Wronskian of the given fundamental system is

$$W[y_1, y_2] = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1 \neq 0.$$

As any solution y of the searched equation is linearly dependent on y_1, y_2, we shall have

$$\begin{vmatrix} y & \cosh x & \sinh x \\ y' & \sinh x & \cosh x \\ y'' & \cosh x & \sinh x \end{vmatrix} = 0 .$$

Developing this determinant following its first column, we immediately find

$$y'' - y = 0 . \tag{2.1.23}$$

1.3 THE GENERAL SOLUTION OF THE NON-HOMOGENEOUS ODE

The general solution of (2.1.1) is written as a sum between the general solution of the associated homogeneous equation and a particular solution $y_p(x)$ of the non-homogeneous ODE, i.e., in terms of a fundamental system $\{y_j\}_{j=\overline{1,n}}$, also taking (2.1.17) into account,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x) + y_p(x) . \tag{2.1.24}$$

To get the particular solution $y_p(x)$, we can use again the fundamental system. Following Lagrange, we can use *the method of variation of parameters*, also called *Lagrange's method*. The particular solution is searched in the form

$$y_p(x) = K_1(x) y_1(x) + K_2(x) y_2(x) + \ldots + K_n(x) y_n(x) . \tag{2.1.25}$$

After step-by-step differentiations and replacements in (2.1.1), we get for $K'_j(x)$ the following linear algebraic system

$$\begin{aligned} &K'_1 y_1 + K'_2 y_2 + \ldots + K'_n y_n = 0, \\ &K'_1 y'_1 + K'_2 y'_2 + \ldots + K'_n y'_n = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &K'_1 y_1^{(n-1)} + K'_2 y_2^{(n-1)} + \ldots + K'_n y_n^{(n-1)} = \frac{F(x)}{a_0(x)}. \end{aligned} \tag{2.1.26}$$

The determinant of this system is precisely the Wronskian of the fundamental system, hence it does not vanish. Solving algebraically the system (2.1.26), we get $K'_j(x),\ j = \overline{1,n}$, and by integration, we finally obtain the expressions of $K_j(x),\ j = \overline{1,n}$.

1.4 ORDER REDUCTION

Let $y_p(x)$ be a particular solution of the homogeneous ODE (2.1.4), i.e.

$$\mathrm{L} y_p(x) = 0 ; \tag{2.1.27}$$

let us perform the change of function

$$y(x) = y_p(x)z(x). \qquad (2.1.28)$$

To apply it, we must compute $\mathrm{L}(y_p z) = \mathrm{P}(x,\mathrm{D})(y_p z)$, where $\mathrm{P}(x,\mathrm{D})$ is the associated to L differential polynomial; to perform this computation, we can use formula (2.1.8), with the same notations, explained in (2.1.9). Taking $u = z,\, v = y_p$, we get

$$\begin{aligned}\mathrm{L}(zy_p) = \mathrm{P}(x,\mathrm{D})(zy_p) = z\,\mathrm{P}(x,\mathrm{D})y_p + \frac{1}{1!}\mathrm{D}z\,\mathrm{P}^{(1)}(x,\mathrm{D})y_p + \frac{1}{2!}\mathrm{D}^2 z\,\mathrm{P}^{(2)}(x,\mathrm{D})y_p \\ + \ldots + \frac{1}{j!}\mathrm{D}^j u\,\mathrm{P}^{(j)}(x,\mathrm{D})y_p + \ldots + \frac{1}{n!}\mathrm{D}^n z\,\mathrm{P}^{(n)}(x,\mathrm{D})y_p,\end{aligned} \qquad (2.1.29)$$

or, as $\mathrm{L}y = 0$ and $\mathrm{P}(x,\mathrm{D})y_p = 0$,

$$a_0(x)z^{(n)} + \frac{\mathrm{P}^{(n-1)}(x,\mathrm{D})y_p}{(n-1)!}z^{(n-1)} + \ldots + \frac{\mathrm{P}^{(2)}(x,\mathrm{D})y_p}{2!}z'' + \mathrm{P}^{(1)}(x,\mathrm{D})y_p z' = 0. \qquad (2.1.30)$$

In this ODE we perform again the change of function $z' = u$, thus obtaining another ODE, of order $n-1$ with respect to the new unknown function *u*.
By using the same pattern, one can prove that if we previously know *r* particular solutions of the homogeneous ODE (2.1.4), which are linearly independent, the order of the ODE may be reduced by *r* units.
At Sec.2.1 and 2.2, Chap.1, we treated the case of second order ODEs, for which one knows a particular solution, say $Y_1(x)$, of the associated homogeneous equation. In this case, it was obtained the representation (1.2.40), in which $Y_2(x)$ is given by (1.2.15).

2. Linear ODEs with Constant Coefficients

The general form of such equations is (2.1.1), with $a_j,\, j = \overline{1,n}$ real constants and $a_0 \neq 0$. More precisely, we have

$$\mathrm{L}y \equiv a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = f(x), \quad f : I \subseteq \Re \to \Re. \qquad (2.2.1)$$

This equation may be written in terms of differential polynomials

$$\mathrm{L}y \equiv \mathrm{P}(\mathrm{D})y \equiv \left(a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \ldots + a_{n-1}\mathrm{D} + a_n\mathrm{E}\right)y = f(x), \quad f : I \subseteq \Re \to \Re. \qquad (2.2.2)$$

From the above considerations, it follows that the solution of this equation depends on the effective determination of a fundamental system of solutions, i.e., of *n* linearly independent solutions of the associated homogeneous ODE

$$\mathrm{L}y \equiv a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}y' + a_n y = 0, \qquad (2.2.3)$$

or, equivalently, of

$$\mathrm{P}(\mathrm{D})y \equiv \left(a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \ldots + a_{n-1}\mathrm{D} + a_n\mathrm{E}\right)y = 0\,. \tag{2.2.4}$$

2.1 THE GENERAL SOLUTION OF THE HOMOGENEOUS EQUATION

Following Euler's idea, one searches for solutions of the exponential form

$$y(x) = \mathrm{e}^{\lambda x}\,, \tag{2.2.5}$$

where λ is a parameter, so far undetermined. Replacing this in (2.2.3), or, better, in (2.2.4), we immediately see that $\mathrm{P}(\mathrm{D})\mathrm{e}^{\lambda x} = \mathrm{e}^{\lambda x}\mathrm{P}(\lambda)$, hence we find for λ the algebraic equation $\mathrm{e}^{\lambda x}\mathrm{P}(\lambda) = 0$; as this must hold for any real x and as $\mathrm{e}^{\lambda x} \neq 0$, we eventually obtain the algebraic equation

$$\mathrm{P}(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n = 0\,, \tag{2.2.6}$$

known as *the characteristic equation*. The polynomial $\mathrm{P}(\lambda)$ is called *the characteristic polynomial*. It is easily seen that it may be formally written replacing the j-th derivative of y in the given ODE by λ^j. The solutions of the ODE (2.2.3), or, equivalently, (2.2.4), depend on the roots of the characteristic polynomial. We must therefore examine the cases *a) – d)*. The set $\mathcal{C}$ of the complex numbers form an algebraically closed field, therefore the characteristic polynomial allows n roots, all of them contained in $\mathcal{C}$. Let us denote them by λ_j, $j = \overline{1,n}$.

a) λ_j are real and distinct. In this case, we obtain the system of n particular solutions of (2.2.3)

$$y_1(x) = \mathrm{e}^{\lambda_1 x}\,,\ y_2(x) = \mathrm{e}^{\lambda_2 x}\,,\ y_n(x) = \mathrm{e}^{\lambda_n x}\,, \tag{2.2.7}$$

which is fundamental, as their Wronskian

$$W[y_1, y_2, \ldots, y_n] = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \ldots & \lambda_n^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix} \mathrm{e}^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)x} \tag{2.2.8}$$

is non-zero. Indeed, the determinant in (2.2.8) is of Vandermonde type and does not vanish, as $\lambda_j \neq \lambda_k$ for $j \neq k,\ j,k = \overline{1,n}$.

The general solution of the homogeneous ODE is thus

$$y(x) = c_1\mathrm{e}^{\lambda_1 x} + c_2\mathrm{e}^{\lambda_2 x} + \ldots + c_n\mathrm{e}^{\lambda_n x}\,, \tag{2.2.9}$$

where c_j, $j=\overline{1,n}$, are arbitrary constants.

b) The characteristic equation allows complex roots. Let, for instance, $\lambda_1=\alpha+\mathrm{i}\beta$, with α,β real and $\beta\neq 0$. As the characteristic equation has real coefficients, once λ_1 is a root, its complex-conjugate $\lambda_2=\alpha-\mathrm{i}\beta$ will also be a root. For the sake of simplicity, suppose now that the remaining roots λ_j, $j=\overline{3,n}$ are real and distinct. Then, according to the previous considerations, the system

$$y_1(x)=\mathrm{e}^{(\alpha+\mathrm{i}\beta)x},\ y_2(x)=\mathrm{e}^{(\alpha-\mathrm{i}\beta)x},\ y_3(x)=\mathrm{e}^{\lambda_3 x},\ldots,y_n(x)=\mathrm{e}^{\lambda_n x} \tag{2.2.10}$$

is fundamental. To avoid complex calculus, we consider, instead of the first two functions of this system, two linear combinations of them, which are also solutions of the ODE (2.2.3)

$$\begin{aligned} Y_1(x)&=\frac{y_1+y_2}{2}=\frac{\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}+\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}}{2}=\mathrm{e}^{\alpha x}\cos\beta x,\\ Y_2(x)&=\frac{y_1-y_2}{2\mathrm{i}}=\frac{\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}-\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}}{2\mathrm{i}}=\mathrm{e}^{\alpha x}\sin\beta x.\end{aligned} \tag{2.2.11}$$

In (2.2.11), we used Euler's formulae

$$\cos\beta x=\frac{\mathrm{e}^{\mathrm{i}\beta x}+\mathrm{e}^{-\mathrm{i}\beta x}}{2},\quad \sin\beta x=\frac{\mathrm{e}^{\mathrm{i}\beta x}-\mathrm{e}^{-\mathrm{i}\beta x}}{2\mathrm{i}}. \tag{2.2.12}$$

Finally, the general solution reads, in this case,

$$y(x)=\mathrm{e}^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)+c_3\,\mathrm{e}^{\lambda_3 x}+\ldots+\mathrm{c}_n\mathrm{e}^{\lambda_n x}, \tag{2.2.13}$$

with c_j, $j=\overline{1,n}$, arbitrary constants.

c) The characteristic equation allows multiple roots. Suppose e.g. that λ_1 has the order of multiplicity *m*. We cannot take $\mathrm{e}^{\lambda_1 x}$ *m* times in the fundamental system, because it should not be linearly independent. We can take it just once.
To complete the fundamental system, we use the following remark. Let $n=2$ and suppose for now that $\lambda_1\neq\lambda_2$. We can choose for the corresponding second order ODE the fundamental system

$$y_1(x)=\mathrm{e}^{\lambda_1 x},\ y_2(x)=\frac{\mathrm{e}^{\lambda_2 x}-\mathrm{e}^{\lambda_1 x}}{\lambda_2-\lambda_1}. \tag{2.2.14}$$

If the associated characteristic equation allows the double root λ_1, we can consider for it

$$y_1(x) = \mathrm{e}^{\lambda_1 x},\ y_2(x) = \lim_{\lambda_2 \to \lambda_1} \frac{\mathrm{e}^{\lambda_2 x} - \mathrm{e}^{\lambda_1 x}}{\lambda_2 - \lambda_1} = x\mathrm{e}^{\lambda_1 x}; \tag{2.2.15}$$

to compute the above limit, we used l'Hospital rule.

Getting back to arbitrary n, we see that $y_j(x) = x^j \mathrm{e}^{\lambda_1 x}$ satisfy the linear ODE with constant coefficients for $j = \overline{1, m-1}$. To prove this, we use again formulae (2.1.8), (2.1.9), taking $u = x^j$, $v = \mathrm{e}^{\lambda_1 x}$

$$\begin{aligned} \mathrm{P(D)}\left(x^j \mathrm{e}^{\lambda_1 x}\right) &= x^j \mathrm{P(D)}\mathrm{e}^{\lambda_1 x} + \frac{1}{1!}\mathrm{D}x^j\, \mathrm{P}^{(1)}(\mathrm{D})\mathrm{e}^{\lambda_1 x} + \frac{1}{2!}\mathrm{D}^2 x^j\, \mathrm{P}^{(2)}(\mathrm{D})\mathrm{e}^{\lambda_1 x} \\ &+ \ldots + \frac{1}{j!}\mathrm{D}^j x^j\, \mathrm{P}^{(j)}(\mathrm{D})\mathrm{e}^{\lambda_1 x} + \ldots + \frac{1}{n!}\mathrm{D}^n x^j\, \mathrm{P}^{(n)}(\mathrm{D})\mathrm{e}^{\lambda_1 x}, \end{aligned} \tag{2.2.16}$$

in which $\mathrm{P}^{(k)}(\mathrm{D})\mathrm{e}^{\lambda_1 x} = \mathrm{P}^{(k)}(\lambda_1)\mathrm{e}^{\lambda_1 x}$ and $\mathrm{D}^k x^j = j(j-1)\ldots(j-k+1)x^{j-k}$, $k \le j$. We eventually get

$$\mathrm{P(D)}\left(x^j \mathrm{e}^{\lambda_1 x}\right) = x^j \mathrm{P}(\lambda_1) + \frac{1}{1!} j x^{j\ 1}\, \mathrm{P}^{(1)}(\lambda_1)\mathrm{e}^{\lambda_1 x} + \ldots + \frac{1}{j!} j!\, \mathrm{P}^{(j)}(\lambda_1)\mathrm{e}^{\lambda_1 x}; \tag{2.2.17}$$

The other terms in the sum (2.2.16) vanish, because $\mathrm{D}^k x^j = 0, k > j$. As the order of multiplicity of λ_1 is m, we obviously have $\mathrm{P}(\lambda_1) = 0,\ \mathrm{P}^{(1)}(\lambda_1) = 0, \ldots, \mathrm{P}^{(m-1)}(\lambda_1) = 0$. From (2.2.17) it then follows that $\mathrm{P(D)}\left(x^j \mathrm{e}^{\lambda_1 x}\right) = 0,\ j = \overline{0, m\ \ 1}$. One can easily see that $x^j \mathrm{e}^{\lambda_1 x}$, $j = \overline{0, m-1}$, are linearly independent. Again for the sake of simplicity, suppose that the other roots of the characteristic equation λ_j, $j = \overline{m+1, n}$ are real and distinct. Then $\mathrm{e}^{\lambda_1 x}, x\mathrm{e}^{\lambda_1 x}, \ldots, x^{m-1}\mathrm{e}^{\lambda_1 x}, \mathrm{e}^{\lambda_{m+1} x}, \mathrm{e}^{\lambda_{m+2} x}, \ldots, \mathrm{e}^{\lambda_n x}$ form a fundamental system for the given ODE and its general solution is

$$y(x) = \left(c_1 + c_2\, x + \ldots + c_m x^{m\ \ 1}\right)\mathrm{e}^{\lambda_1 x} + c_{m+1}\, \mathrm{e}^{\lambda_{m+1} x} + c_{m+2}\, \mathrm{e}^{\lambda_{m+2} x} + \ldots + c_n \mathrm{e}^{\lambda_n x}. \tag{2.2.18}$$

d) The characteristic equation allows multiple complex roots. Let $\lambda_1 = \lambda_2 = \ldots = \lambda_m = \alpha + \mathrm{i}\beta$ be a multiple root of order m. Then $\lambda_{m+1} = \lambda_{m+2} = \ldots = \lambda_{2m} = \alpha - \mathrm{i}\beta$ is also a root with the same order of multiplicity. Exactly as before, we deduce $2m$ linearly independent solutions of the given ODE

$$\begin{aligned} &\mathrm{e}^{\alpha x}\cos\beta x,\ \mathrm{e}^{\alpha x}\sin\beta x, \\ &x\mathrm{e}^{\alpha x}\cos\beta x,\ x\mathrm{e}^{\alpha x}\sin\beta x, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &x^{m-1}\mathrm{e}^{\alpha x}\cos\beta x,\ x^{m-1}\mathrm{e}^{\alpha x}\sin\beta x, \end{aligned} \tag{2.2.19}$$

that, together with the other $n-2m$ linearly independent solutions – determined by taking into account the nature of the roots of $\mathrm{P}(\lambda)$ – form a fundamental system for (2.2.3). The general solution of this ODE is then a linear combination of the functions of this fundamental system, with arbitrary constants as coefficients.

2.2 THE NON-HOMOGENEOUS ODE

At the previous section, we showed how to find a fundamental system for a homogeneous ODE with constant coefficients, by using the roots of the characteristic equation. According to Sec.1.3, we can get particular solutions of the non-homogeneous ODE by using Lagrange's method. Yet this method lead to cumbrous computation, the more so as the order of the ODE is greater. If the free term is an elementary function, or a linear combination of such functions, then there exists a direct method of obtaining particular solutions, which is more efficient than the method of variation of parameters.
Let us note firstly that if $f(x)=f_1(x)+f_2(x)+\ldots+f_p(x)$ and if we determine $y_1, y_2, \ldots, y_p$ such that

$$\mathrm{L}y_1 = f_1,\, \mathrm{L}y_2 = f_2, \ldots, \mathrm{L}y_p = f_p, \tag{2.2.20}$$

then their sum $Y = y_1 + y_2 + \ldots + y_p$ is a particular solution of the non-homogeneous ODE ((2.2.1), i.e.,

$$\mathrm{L}Y = \mathrm{L}(y_1 + y_2 + \ldots + y_p) = f_1 + f_2 + \ldots + f_p = f(x). \tag{2.2.21}$$

Now let us get particular solutions for non-homogeneous ODEs with free terms composed of elementary functions, currently met in applications.

a) $f(x)=b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m$. We search for a particular solution shaping $f(x)$

$$Y(x) = x^r\left(q_0x^m + q_1x^{m-1} + \ldots + q_{m-1}x + q_m\right), \tag{2.2.22}$$

where r is the order of multiplicity of 0 as a root of the associated characteristic polynomial. Naturally, if 0 does not satisfy the characteristic equation, then $r = 0$. Introducing the above expression in (2.2.1) and identifying the coefficients of the same powers of x, we find $q_j, j=\overline{0,m}$. The algebraic system obtained for $q_j, j=\overline{0,m}$, is linear and allows a unique solution.

b) $f(x)=\mathrm{e}^{\alpha x}$. We search for solutions of the form

$$Y(x) = Ax^r\mathrm{e}^{\alpha x}, \tag{2.2.23}$$

where r is the order of multiplicity of α as root of the characteristic equation. Again, if α does not satisfy the characteristic equation, then $r = 0$. To introduce (2.2.23) in the ODE (2.2.1), we use formula (2.2.17), for $j = r, \lambda_1 = \alpha$. We get

$$\mathrm{P}(\mathrm{D})\left(Ax^r e^{\alpha x}\right) = A\mathrm{P}(\mathrm{D})\left(x^r e^{\alpha x}\right)$$
$$= A\left[x^r \mathrm{P}(\alpha) + \frac{1}{1!} r x^{r-1} \mathrm{P}^{(1)}(\alpha) + \ldots + \frac{1}{r!} r! \mathrm{P}^{(r)}(\alpha)\right] e^{\alpha x}. \tag{2.2.24}$$

By virtue of multiplicity, $\mathrm{P}^{(j)}(\alpha) = 0,\ j = \overline{0, r-1}$, with $\mathrm{P}^{(0)}(\alpha) \equiv \mathrm{P}(\alpha)$, but $\mathrm{P}^{(r)}(\alpha) \neq 0$. We finally get for Y

$$Y(x) = \frac{1}{\mathrm{P}^{(r)}(\alpha)} x^r e^{\alpha x}. \tag{2.2.25}$$

c) $f(x) = e^{\alpha x}\left(b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m\right)$. If α is not a root of the characteristic equation, then we search for Y in the form

$$Y(x) = e^{\alpha x}\left(q_0 x^m + q_1 x^{m-1} + \ldots + q_{m-1} x + q_m\right). \tag{2.2.26}$$

The coefficients q_j are found by identification.

If α is a multiple root of order r of the characteristic equation, then we search for Y in the form

$$Y(x) = x^r e^{\alpha x}\left(q_0 x^m + q_1 x^{m-1} + \ldots + q_{m-1} x + q_m\right). \tag{2.2.27}$$

The introduction of this expression in the given ODE leads to tiresome computation. This is why it is recommended to perform firstly the change of function

$$y(x) = z(x) e^{\alpha x}, \tag{2.2.28}$$

where $z(x)$ is a new unknown function. Applying formula (2.1.8) for $u = z(x),\ v = e^{\alpha x}$, we get for z the following ODE

$$\left(A_n z^{(n)} + A_{n-1} z^{(n-1)} + \ldots + A_r z^{(r)}\right) e^{\alpha x} = \left(b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m\right) e^{\alpha x}, \tag{2.2. 29}$$

where

$$A_j = \frac{1}{j!} \mathrm{P}^{(j)}(\alpha), \quad j = \overline{r, n}. \tag{2.2.30}$$

Simplifying with $e^{\alpha x}$, this case is reduced to a).

d) Suppose now that

$$f(x) = \left(b_0 x^m + b_1 x^{m-1} + \ldots + b_{m-1} x + b_m\right)\cos \beta x$$
$$+ \left(d_0 x^k + d_1 x^{k-1} + \ldots + d_{k-1} x + d_k\right)\sin \beta x.$$

Let us denote by $s = \max\{m, k\}$. If β is not a root of the characteristic equation, then we search for Y in the form

$$\begin{aligned} Y(x) = &\left(q_0 x^s + q_1 x^{s-1} + \ldots + q_{s-1} x + q_s\right)\cos \beta x \\ &+ \left(p_0 x^s + p_1 x^{s-1} + \ldots + p_{s-1} x + p_s\right)\sin \beta x. \end{aligned} \tag{2.2.31}$$

Replacing this in the given ODE, we get by identification the coefficients $p_j, q_j, j = \overline{0, s}$. If β is a multiple root of order r of the characteristic equation, then we search for Y in the form

$$\begin{aligned} Y(x) = x^r &\left[\left(q_0 x^s + q_1 x^{s-1} + \ldots + q_{s-1} x + q_s\right)\cos \beta x \right. \\ &\left. + \left(p_0 x^s + p_1 x^{s-1} + \ldots + p_{s-1} x + p_s\right)\sin \beta x\right], \end{aligned} \tag{2.2.32}$$

the coefficients $p_j, q_j, j = \overline{0, s}$, being obtained, as previously, by identification.

2.3 EULER TYPE ODEs

These ODEs are also linear, but with variable coefficients. Yet, by a change of variable, they can be reduced to ODEs with constant coefficients. Euler's ODEs are of the form

$$\mathrm{L}y \equiv a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_{n-1} x y' + a_n y = 0\,. \tag{2.2.33}$$

Applying the change of variable $x = \mathrm{e}^t$, we immediately get

$$y' = \mathrm{e}^{-t}\mathrm{D}y,\ y'' = \mathrm{e}^{-2t}\mathrm{D}(\mathrm{D}-\mathrm{E})y,\ y''' = \mathrm{e}^{-3t}\mathrm{D}(\mathrm{D}-\mathrm{E})(\mathrm{D}-2\mathrm{E})y, \ldots\,. \tag{2.2.34}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}t$ and E is the identity operator. Introducing this in (2.2.33), we get an ODE with constant coefficients. In this new equation, searching for solutions of exponential type $y = \mathrm{e}^{rt}$, we get the characteristic equation

$$a_0 r(r-1)\ldots(r-n+1) + a_1 r(r-1)\ldots(r-n+2) + \ldots + a_{n-1} r + a_n = 0\,, \tag{2.2.35}$$

whose roots lead to a fundamental system of solutions for (2.2.33). We see that we can get the same characteristic equation by searching directly for y in the form $y = \mathrm{e}^{r \ln x} = x^r$.

3. Fundamental Solution. Green Function

3.1 THE FUNDAMENTAL SOLUTION

By definition, a *fundamental solution* of the ODE (2.1.1) is a function $E(x, t)$ with the following properties:

i) $E \in \mathrm{C}^n(D \setminus J)$, where $D = I \times I$, $I \equiv [a,b]$ and J is the diagonal of the square D, i.e., $J = \{(x,x), x \in I\}$,

ii) as a function of x, E satisfies the ODE in $D \setminus J$,

iii) $E \in \mathrm{C}^{n-2}(D)$ and $\dfrac{\partial^{n-1} E}{\partial x^{n-1}}(t_+, t) - \dfrac{\partial^{n-1} E}{\partial x^{n-1}}(t_-, t) = \dfrac{1}{a_0(x)}$.

An ODE of type (2.1.1) always allows fundamental solutions. For instance, if $\{y_j\}_{j=\overline{1,n}}$ is a fundamental system for (2.1.1), then

$$E(x,t) = \frac{\operatorname{sgn}(x-t)}{2a_0(t)W(t)} \begin{vmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \dots & y_n^{(n-2)}(t) \\ y_1^{(n)}(t) & y_2^{(n)}(t) & \dots & y_n^{(n)}(t) \end{vmatrix} \tag{2.3.1}$$

is a fundamental solution for (2.1.1). This solution has the important property

$$E(t,t) = \frac{\partial E}{\partial x}(t,t) = \dots = \frac{\partial^{n-2} E}{\partial x^{n-2}}(t,t) = 0\,. \tag{2.3.2}$$

The set of the fundamental solutions of the ODE (2.1.1) is given by

$$E(x,t) + \sum_{j=1}^{n} c_j(t) y_j(x), \tag{2.3.3}$$

where $c_j(t)$ are continuous functions. By using the fundamental solution, one can immediately put the solution of (2.1.1) in the form

$$y(x) = \int_a^b E(x,t) F(t)\,\mathrm{d}t\,. \tag{2.3.4}$$

In the case of constant coefficients, we can easily find a fundamental system of solutions as shown at Sec.2.2. Then, the corresponding fundamental solution will be obtained by replacing the expressions of y_j in formula (2.3.1).

3.2 THE GREEN FUNCTION

Let us consider the generalized two-point problem

$$\mathrm{L}y = 0\,, \tag{2.3.5}$$

$$
\begin{aligned}
&\mathrm{U}_1 y \equiv \sum_{k=0}^{n-1}\left[A_{1k}\, y^{(k)}(a)+B_{1k}\, y^{(k)}(b)\right]=0,\\
&\mathrm{U}_2 y \equiv \sum_{k=0}^{n-1}\left[A_{2k}\, y^{(k)}(a)+B_{2k}\, y^{(k)}(b)\right]=0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\mathrm{U}_n y \equiv \sum_{k=0}^{n-1}\left[A_{nk}\, y^{(k)}(a)+B_{nk}\, y^{(k)}(b)\right]=0.
\end{aligned}
\tag{2.3.6}
$$

The operator L is given by (2.1.1); the coefficients A_{jk}, B_{jk} must be such that the rank of the matrix

$$
\begin{bmatrix}
A_{10} & A_{11} & A_{12} & \dots & A_{1,n-1} & B_{10} & B_{11} & B_{12} & \dots & B_{1,n-1}\\
A_{20} & A_{21} & A_{22} & \dots & A_{2,n-1} & B_{20} & B_{21} & B_{22} & \dots & B_{2,n-1}\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
A_{n0} & A_{n1} & A_{n2} & \dots & A_{n,n-1} & B_{n0} & B_{n1} & B_{n2} & \dots & B_{n,n-1}
\end{bmatrix}
\tag{2.3.7}
$$

be *n*.

As previously, we shall consider only coefficients $a_j(x)$ continuous on $I=[a,b]$.

By definition, we call *Green function* or *influence function* for the problem (2.3.5), (2.3.6) a fundamental solution $G(x,t)$ for the ODE (2.3.6) also satisfying the boundary conditions (2.3.7).

The two-point problem (2.3.5), (2.3.6) may allow other solutions besides the trivial one. We say that the two-point problem has the *index k* if every one of its solutions may be written as a linear combination of *k* solutions of a fundamental system of the ODE (2.3.5).

If the boundary problem (2.3.5), (2.3.6) allows only the trivial solution, then the associated Green function is unique.

The Green function may be effectively set up if one knows a fundamental system of solutions for the given ODE, which is always possible in the case of constant coefficients. If $y_1, y_2, \dots, y_n$ form a fundamental system of (2.3.5), then the associated Green function is given by

$$
G(x,t)=\frac{H(x,t)}{\Delta}
\tag{2.3.8}
$$

where

$$
H(x,t)=\begin{vmatrix}
E(x,t) & y_1(x) & y_2(x) & \dots & y_n(x)\\
\mathrm{U}_1 E & \mathrm{U}_1 y_1 & \mathrm{U}_1 y_2 & \dots & \mathrm{U}_1 y_n\\
\dots & \dots & \dots & \dots & \dots\\
\mathrm{U}_n E & \mathrm{U}_n y_1 & \mathrm{U}_n y_2 & \dots & \mathrm{U}_n y_n
\end{vmatrix},
\tag{2.3.9}
$$

$E(x,t)$ is given by formula (2.3.1) and Δ is the determinant

$$\Delta = \begin{vmatrix} \mathrm{U}_1 y_1 & \mathrm{U}_1 y_2 & \dots & \mathrm{U}_1 y_n \\ \mathrm{U}_2 y_1 & \mathrm{U}_2 y_2 & \dots & \mathrm{U}_2 y_n \\ \dots & \dots & \dots & \dots \\ \mathrm{U}_n y_1 & \mathrm{U}_n y_2 & \dots & \mathrm{U}_n y_n \end{vmatrix} . \tag{2.3.10}$$

The representation (2.3.8) is generally not valid at the ends a and b of the interval I. At these points, one takes $G(x,a) = \lim_{t \to a} G(x,t)$, $G(x,b) = \lim_{t \to b} G(x,t)$.

3.3 THE NON-HOMOGENEOUS PROBLEM

Consider firstly the semi-homogeneous problem

$$\begin{aligned} &\mathrm{L}y = f(x), \\ &\mathrm{U}_j y = 0, \quad j = \overline{1,n} . \end{aligned} \tag{2.3.11}$$

Its solution is represented in the form

$$y(x) = \int_a^b G(x,t) f(t) \mathrm{d}t . \tag{2.3.12}$$

Finally, the solution of the non-homogeneous problem

$$\begin{aligned} &\mathrm{L}y = f(x), \\ &\mathrm{U}_j y = K_j, \quad j = \overline{1,n}, K_j \in \Re \end{aligned} \tag{2.3.13}$$

reads

$$y(x) = \int_a^b G(x,t) f(t) \mathrm{d}t + \sum_{j=1}^n K_j \varphi_j(x), \tag{2.3.14}$$

where $\varphi_j(x)$ are the unique solutions of the "elementary" Cauchy problems

$$\begin{aligned} &\mathrm{L}\varphi_j = 0, \\ &\mathrm{U}_1 \varphi_j = 0, \\ &\mathrm{U}_2 \varphi_j = 0, \\ &\dots\dots\dots\dots\dots\dots \qquad j = \overline{1,n} . \\ &\mathrm{U}_j \varphi_j = 1, \\ &\dots\dots\dots\dots\dots\dots \\ &\mathrm{U}_n \varphi_j = 0, \end{aligned} \tag{2.3.15}$$

To get $\varphi_j(x)$, we may write them as linear combinations of the functions $y_j(x),\ j=\overline{1,n}$ of a fundamental system

$$\varphi_j = c_{1j}y_1 + c_{2j}y_2 + \ldots + c_{nj}y_n, \quad j=\overline{1,n} \ . \tag{2.3.16}$$

The constants c_{kj} are, for every *j*, the solutions of the linear algebraic system

$$\begin{aligned}
&c_{1j}\mathrm{U}_1 y_1 + c_{2j}\mathrm{U}_1 y_2 + \ldots + c_{nj}\mathrm{U}_1 y_n = 0,\\
&c_{1j}\mathrm{U}_2 y_1 + c_{2j}\mathrm{U}_2 y_2 + \ldots + c_{nj}\mathrm{U}_2 y_n = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&c_{1j}\mathrm{U}_j y_1 + c_{2j}\mathrm{U}_j y_2 + \ldots + c_{nj}\mathrm{U}_j y_n = 1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&c_{1j}\mathrm{U}_n y_1 + c_{2j}\mathrm{U}_n y_2 + \ldots + c_{nj}\mathrm{U}_n y_n = 0.
\end{aligned} \tag{2.3.17}$$

For $j=\overline{1,n}$ we get *n* such systems, whose associated determinant is defined by formula (2.3.10). The solutions of these systems are then replaced in (2.3.16) and, eventually, the functions $\varphi_j(x)$ are introduced in (2.3.14).

3.4 THE HOMOGENEOUS TWO-POINT PROBLEM. EIGENVALUES

Let us consider now the homogeneous problem

$$\begin{aligned}
&\mathrm{L}y + \lambda g(x)y = 0,\\
&\mathrm{U}_j y = 0, \quad j=\overline{1,n} \ ,
\end{aligned} \tag{2.3.18}$$

where λ is a parameter and $g \in \mathrm{C}^0(I)$. Obviously, this problem always allows the trivial solution.

The values of λ for which (2.3.18) allows non-zero solutions are called *eigenvalues*; they are included in the *spectrum* of the problem. The non-trivial solutions corresponding to every λ are called *eigenfunctions* (see also Sec.2.7, Chap.1).

We say that an eigenvalue λ has the order *k* of multiplicity if for that λ the two-point problem (2.3.18) has the index *k*.

Let $\{y_j(x,\lambda)\}_{j=\overline{1,n}}$ be a fundamental system for the ODE $\mathrm{L}y + \lambda g(x)y = 0$, for every fixed λ. Also suppose that this system is normal (for $x_0 = a$), i.e.

$$y_j^{(m-1)}(a) = \delta_j^m, \quad m, j = \overline{1,n} \ , \tag{2.3.19}$$

where δ_j^m is the *Kronecker delta*, defined as follows

$$\delta_j^m = \begin{cases} 1, & m = j, \\ 0, & m \neq j. \end{cases} \tag{2.3.20}$$

The determinant

$$\Delta(\lambda) = \begin{vmatrix} U_1 y_1(x,\lambda) & U_1 y_2(x,\lambda) & \dots & U_1 y_n(x,\lambda) \\ U_2 y_1(x,\lambda) & U_2 y_2(x,\lambda) & \dots & U_2 y_n(x,\lambda) \\ \dots & \dots & \dots & \dots \\ U_n y_1(x,\lambda) & U_n y_2(x,\lambda) & \dots & U_n y_n(x,\lambda) \end{vmatrix} \tag{2.3.21}$$

is called the characteristic determinant. According to the previous considerations, the zeros of this determinant will be the eigenvalues of the problem. The order of multiplicity of an eigenvalue is less or equal to its order of multiplicity, considered as a root of the characteristic determinant.
In applications, we shall treat each problem by using specific methods, that, in general, could not be considered as particular cases of the above exposed theory.

4. Applications

Application 2.1

Problem. Study the wire drawing.

Mathematical model. Modelling the drawing phenomenon, one obtains an ODE of the form

$$u''' + u'' \cot\theta + \left(6 - \frac{1}{\sin^2\theta}\right)u' = 0 . \tag{a}$$

Solution. Putting $u' = y$, the equation (a) becomes

$$y'' + y' \cot\theta + \left(6 - \frac{1}{\sin^2\theta}\right)y = 0 . \tag{b}$$

Observing that

$$\frac{1}{\sin^2\theta} = 1 + \cot^2\theta ,$$

the equation (b) may be written further

$$y'' + y' \cot\theta + \left(5 - \cot^2\theta\right)y = 0 , \tag{c}$$

or in the form

$$y'' + 4y + y' \cot\theta + \left(1 - \cot^2\theta\right) y = 0 \,; \tag{d}$$

it also reads

$$y'' + 4y + y \cot\theta \left(\frac{y'}{y} + \frac{1 - \cot^2\theta}{\cot\theta} \right) = 0 \,.$$

On the other hand,

$$\frac{1 - \cot^2\theta}{\cot\theta} = -2\cot 2\theta = \frac{-2\cot 2\theta}{\sin 2\theta} = -\frac{\left(\sin 2\theta\right)'}{\sin 2\theta} \,,$$

and the equation (b) may take the form

$$y'' + 4y + y \cot\theta \left(\frac{y'}{y} - \frac{\left(\sin 2\theta\right)'}{\sin 2\theta} \right) = 0 \,. \tag{e}$$

Because the differential equation $y'' + 4y = 0$ has the general integral

$$y_1 = A \sin 2\theta + B \cos 2\theta \,,$$

it results that the equation (b) has the particular solution $y_1 = \sin 2\theta$.
The other particular solution of the equation (b) is thus reduced to quadratures. Let y_2 be this particular solution.
In the particular case of homogeneous linear equations of second order, Liouville's formula is of the form

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = C \mathrm{e}^{-\int \frac{a_1(\theta)}{a_0(\theta)} \mathrm{d}\theta} \,;$$

hence,

$$\frac{y_1 y_2' - y_1' y_2}{y_1^2} = \left(\frac{y_2}{y_1} \right)' = \frac{C}{y_1^2} \mathrm{e}^{-\int \frac{a_1(\theta)}{a_0(\theta)} \mathrm{d}\theta} \,. \tag{f}$$

In our case $y_1 = \sin 2\theta$, $a_1(\theta) = \cot\theta$, $a_0(\theta) = 1$, so that the relation (f) becomes

$$\left(\frac{y_2}{\sin 2\theta} \right)' = \frac{C}{4 \sin^2\theta \cos^2\theta} \mathrm{e}^{-\int \frac{\cos\theta}{\sin\theta} \mathrm{d}\theta} = \frac{C}{\sin^3\theta \cos^2\theta} \,,$$

therefore

$$y_2 = \frac{C}{2} \sin\theta \cos\theta \int \frac{\mathrm{d}\theta}{\sin^3\theta \cos^2\theta} \,.$$

Neglecting the multiplicative constant, the second particular solution is given by

$$y_2 = \sin\theta\cos\theta \int \frac{d\theta}{\sin^3\theta\cos^2\theta},$$

hence, by integration, one obtains

$$y_2 = \frac{3}{2}\sin\theta + \frac{3}{2}\sin\theta\cos\theta\ln\tan\frac{\theta}{2} - \frac{1}{2\sin\theta}. \tag{g}$$

Let us also notice that

$$\int y_1 d\theta = -\frac{1}{2}\cos 2\theta,$$

$$\int y_2 d\theta = -\frac{3}{8}\left[2\cos\theta + \left(\frac{1}{3} + \cos 2\theta\right)\ln\tan\frac{\theta}{2}\right].$$

By a slight modification of the arbitrary constants, the general integral of the differential equation (a) reads

$$u(\theta) = A + B\cos 2\theta + C\left[2\cos\theta + \left(\frac{1}{3} + \cos 2\theta\right)\ln\tan\frac{\theta}{2}\right].$$

Application 2.2

Problem. Study the deformation and the state of stress of a circular gallery, surrounded by an elastic medium. Determine the efforts in a cross section and the bending deflection w. Particular case: a lateral uniform pressure of the medium.

Mathematical model. The equations of equilibrium are of the form

$$\frac{1}{a}\frac{dN}{d\varphi} - \frac{T}{a} + Y = 0, \tag{a}$$

$$\frac{1}{a}\frac{dT}{d\varphi} + \frac{N}{a} + Z - kw = 0, \tag{b}$$

$$T = \frac{1}{a}\frac{dM}{d\varphi}; \tag{c}$$

and the equation of deformation is given by

$$\frac{1}{a^2}\left(\frac{d^2w}{d\varphi^2} + w\right) = -\frac{M}{E_0 I}, \tag{d}$$

where N, T, M are the axial force, the shearing force, and the bending moment, respectively, in a cross section specified by the angle φ (the angular variable, measured clockwise from the vertex), k is the foundation modulus (representing the pressure which leads to a deflection $w = 1$), a is the median radius of the pipe (supposed of constant thickness), $E_0 I$ is the bending rigidity of a span of a pipe of unit width, and Y, Z are the tangential and normal components of the external bending, respectively. From (b), one obtains

$$N = -\frac{\mathrm{d}T}{\mathrm{d}\varphi} - Za + kaw, \tag{e}$$

i.e.

$$\frac{\mathrm{d}N}{\mathrm{d}\varphi} = -\frac{\mathrm{d}^2 T}{\mathrm{d}\varphi^2} - \frac{\mathrm{d}Z}{\mathrm{d}\varphi} a + ka \frac{\mathrm{d}w}{\mathrm{d}\varphi}; \tag{f}$$

introducing this in (a), it results

$$\frac{\mathrm{d}^2 T}{\mathrm{d}\varphi^2} + T - ka \frac{\mathrm{d}w}{\mathrm{d}\varphi} + \left(\frac{\mathrm{d}Z}{\mathrm{d}\varphi} - Y \right) a = 0. \tag{g}$$

Further, we eliminate T between (c) and (g)

$$\frac{\mathrm{d}^3 M}{\mathrm{d}\varphi^3} + \frac{\mathrm{d}M}{\mathrm{d}\varphi} - ka^2 \frac{\mathrm{d}w}{\mathrm{d}\varphi} + \left(\frac{\mathrm{d}Z}{\mathrm{d}\varphi} - Y \right) a = 0. \tag{h}$$

Finally, M given by (d) is introduced in (h)

$$\frac{\mathrm{d}^5 w}{\mathrm{d}\varphi^5} + 2 \frac{\mathrm{d}^3 w}{\mathrm{d}\varphi^3} + \left(1 + \frac{ka^4}{E_0 I} \right) \frac{\mathrm{d}w}{\mathrm{d}\varphi} = \left(-Y + \frac{\mathrm{d}Z}{\mathrm{d}\varphi} \right) \frac{a^4}{E_0 I}, \tag{i}$$

obtaining thus the searched differential equation.
The efforts on the cross section may be thus expressed by means of w in the form

$$M = -\frac{E_0 I}{a^2} \left(\frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} + w \right), \tag{j}$$

$$T = -\frac{E_0 I}{a^2} \left(\frac{\mathrm{d}^3 w}{\mathrm{d}\varphi^3} + \frac{\mathrm{d}w}{\mathrm{d}\varphi} \right), \tag{k}$$

$$N = \frac{E_0 I}{a^3} \left(\frac{\mathrm{d}^4 w}{\mathrm{d}\varphi^4} + \frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} \right) + kaw - Za. \tag{l}$$

Solution. To obtain a solution in the form of a trigonometric series for the equation (i), we suppose that the components of the external loading are of the form

$$Y = \sum_n Y_n \sin n\varphi, \; Z = \sum_n Z_n \cos n\varphi,$$

where Y_n and Z_n are dimensional factors specifying that loading. We denote

$$\psi = -Y + \frac{\mathrm{d}Z}{\mathrm{d}\varphi} = \sum_n \left(-Y_n + nZ_n\right)\sin n\varphi = \sum_n p_n \sin n\varphi . \tag{m}$$

Thus, the equation (i) reads

$$\frac{\mathrm{d}^5 w}{\mathrm{d}\varphi^5} + 2\frac{\mathrm{d}^3 w}{\mathrm{d}\varphi^3} + \left(1 + \frac{ka^4}{E_0 I}\right)\frac{\mathrm{d}w}{\mathrm{d}\varphi} = \frac{a^4}{E_0 I}\sum_n p_n \sin n\varphi , \tag{n}$$

where

$$p_n = -Y_n + nZ_n . \tag{o}$$

Taking into account the trigonometric form of the right member in (m), we search for a similar solution

$$w = \sum_n w_n \cos n\varphi . \tag{p}$$

Introducing it in the equation (i), it results

$$\sum_n \left\{\left[n^5 - 2n^3 + \left(1 + \frac{ka^4}{E_0 I}\right)n\right]w_n + \frac{p_n a^4}{E_0 I}\right\}\sin n\varphi = 0 .$$

Hence, the coefficients of the series must vanish, so that

$$w_n = -\frac{a^4}{E_0 I}\frac{p_n}{n\left[\left(n^2 - 1\right)^2 + \dfrac{ka^4}{E_0 I}\right]} . \tag{q}$$

In the particular case of a lateral pressure of the medium one has $Y = p\sin\varphi\cos\varphi$, $Z = p\sin^2\varphi$, leading to

$$\psi = \frac{3}{2}p\sin 2\varphi .$$

We thus obtain

$$w = -\frac{3}{4}\frac{pa^4}{E_0 I}\frac{p_n}{9 + \dfrac{ka^4}{E_0 I}}\cos 2\varphi ,$$

$$M = -\frac{9}{4}pa^2\frac{p_n}{9 + \dfrac{ka^4}{E_0 I}}\cos 2\varphi ,$$

$$T = -\frac{9}{2}pa\frac{p_n}{9+\dfrac{ka^4}{E_0 I}}\sin 2\varphi\,,$$

$$N = \frac{pa}{4}\left(\frac{3}{9+\dfrac{ka^4}{E_0 I}} - 1 + \cos 2\varphi\right).$$

Application 2.3

Problem. Determine the angle φ of relative rotation in the starting of an engine under the action of a variable driving moment.

Mathematical model. The ODE governing the above enounced problem is of the form

$$\frac{d^3\varphi}{dt^3} + \frac{M_0}{J_1\omega_0}\frac{d^2\varphi}{dt^2} + \frac{k(J_1+J_2)}{J_1 J_2}\frac{d\varphi}{dt} + \frac{M_0 k}{J_1 J_2 \omega_0}\varphi = 0\,, \qquad \text{(a)}$$

where J_1 and J_2 are the moments of inertia of the mass of the rotor, of the driving motor and of the coupling and of the reduced mass of the mechanism of transmission of motion of the work organ and of the connected loads, respectively, k is the rigidity coefficient of the elastic element of connection between the disks J_1 and J_2 (Fig.2.1), $\varphi = \varphi_1 - \varphi_2$ is the relative rotation angle, M_0 is the starting moment of the motor, and ω_0 is the angular velocity in loose running.

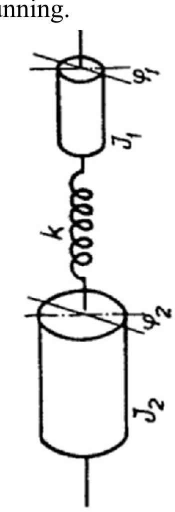

Figure 2. 1. Geometric schema of the elastic element and of the disks J_1 and J_2

One assumes the initial conditions of Cauchy type

$$\varphi(0) = 0, \frac{\mathrm{d}\varphi(0)}{\mathrm{d}t} = 0, \frac{\mathrm{d}^2\varphi(0)}{\mathrm{d}t^2} = 0 . \tag{b}$$

Solution. The linear differential equation (a) is homogeneous and with constant coefficients; we search for solutions of the form $\varphi = \mathrm{e}^{\lambda t}$, being thus led to the characteristic equation

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0 , \tag{c}$$

where

$$a_1 = \frac{M_0}{J_1\omega_0}, a_2 = \frac{k(J_1 + J_2)}{J_1 J_2}, a_3 = \frac{M_0 k}{J_1 J_2 \omega_0} .$$

By the substitution $\lambda = y - a_1/3$, the equation (c) is reduced to the canonical form

$$y^3 + 3py + 2q = 0 , \tag{d}$$

where

$$p = \frac{1}{3}\left(a_2 - \frac{a_1^2}{3}\right) = \frac{k(J_1 + J_2)}{3J_1 J_2} - \frac{M_0^2 k^2}{9J_1^2\omega_0^2} , \tag{e}$$

$$q = \frac{a_1}{6}\left(\frac{2a_1^2}{9} - a_2\right) + \frac{a_3}{2} = \frac{M_0}{2J_1\omega_0}\left[\frac{M_0^2}{9J_1^2\omega_0^2} + \frac{(2J_1 - J_2)k}{2J_1 J_2}\right]. \tag{f}$$

In the case of the considered mechanical system, $p^3 + q^3 > 0$, so that the equation (d) has one real and two complex conjugate roots, that is

$$y_1 = u + v,\ y_2 = \varepsilon_1 u + \varepsilon_2 v, y_3 = \varepsilon_2 u + \varepsilon_1 v ,$$

where

$$u = \sqrt[3]{- q + \sqrt{q^2 + p^3}} ,\ v = -\sqrt[3]{q + \sqrt{q^2 + p^3}} ,$$

ε_1 and ε_2 being the roots of the equation $\varepsilon^2 + \varepsilon + 1 = 0$, that is

$$\varepsilon_1 = -\frac{1}{2}\left(1 + \mathrm{i}\sqrt{3}\right),\ \varepsilon_2 = -\frac{1}{2}\left(1 - \mathrm{i}\sqrt{3}\right).$$

The roots of the equation (d) become

$$\lambda_1 = -\frac{a_1}{3} + u + v,\quad \lambda_2 = -\frac{a_1}{3} - \frac{u+v}{2} + \mathrm{i}\frac{\sqrt{3}}{2}(u - v),\quad \lambda_3 = -\frac{a_1}{3} - \frac{u+v}{2} - \mathrm{i}\frac{\sqrt{3}}{2}(u - v).$$

Introducing the notations

$$\alpha = -\frac{a_1}{3} - \frac{u+v}{2} = -\frac{M_0}{3J_1\omega_0} - \frac{1}{2}\left(\sqrt[3]{-q+\sqrt{q^2+p^3}} - \sqrt[3]{q+\sqrt{q^2+p^3}}\right),$$

$$\beta = \frac{\sqrt{3}}{2}(u-v) = \frac{\sqrt{3}}{2}\left(\sqrt[3]{-q+\sqrt{q^2+p^3}} + \sqrt[3]{q+\sqrt{q^2+p^3}}\right),$$

the roots of the characteristic equation read

$$\lambda_1 = \lambda = -2\alpha - \frac{M_0}{J_1\omega_0},$$
$$\lambda_2 = \alpha + \mathrm{i}\beta, \quad \lambda_3 = \alpha - \mathrm{i}\beta .$$

The general solution of the differential equation (a) becomes

$$\varphi = C_1 \mathrm{e}^{\lambda t} + \mathrm{e}^{\alpha t}\left(C_2 \sin\beta t + C_3 \cos\beta t\right). \tag{h}$$

To determine the integration constants, one must compute the first two derivatives of φ with respect to time, i.e.

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = C_1\lambda \mathrm{e}^{\lambda t} + \mathrm{e}^{\alpha t}\left[(C_2\alpha - C_3\beta)\sin\beta t + (C_2\beta + C_3\alpha)\cos\beta t\right], \tag{i}$$

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = C_1\lambda^2 \mathrm{e}^{\lambda t} + \mathrm{e}^{\alpha t}\left\{\left[\left(\alpha^2-\beta^2\right)C_2 - 2\alpha\beta C_3\right]\sin\beta t + \left[2\alpha\beta C_2 + \left(\alpha^2-\beta^2\right)C_3\right]\cos\beta t\right\}. \tag{j}$$

The initial conditions (b) lead to the linear system of algebraic equations

$$\begin{bmatrix} 1 & 0 & 1 \\ \lambda & \beta & \alpha \\ \lambda^2 & 2\alpha\beta & \alpha^2-\beta^2 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \frac{M_0}{J_1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

therefore

$$C_1 = \frac{M_0}{J_1\left[(\alpha-\lambda)^2+\beta^2\right]} = \frac{\beta}{\alpha-\lambda}C_2 = -C_3,$$

$$C_2 = \frac{\alpha-\lambda}{\beta}\frac{M_0}{J_1\left[(\alpha-\lambda)^2+\beta^2\right]} = \frac{\alpha-\lambda}{\beta}C_1.$$

The solution (h) becomes

$$\varphi = \frac{M_0}{J_1\left[(\alpha-\lambda)^2+\beta^2\right]}\left[\mathrm{e}^{\lambda t} + \mathrm{e}^{\alpha t}\left(\frac{\alpha-\lambda}{\beta}\sin\beta t - \cos\beta t\right)\right].$$

Application 2.4

Problem. Determine the buckling critical force of a doubly hinged bar in an elastic medium, the coefficient of soil reaction of which is k .

Mathematical model. The deflection w satisfies the differential equation

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + P\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + kw = 0\,, \tag{a}$$

and is obtained combining the bending equation of a beam on elastic medium with buckling; we denote by P the compression forces and by EI the bending rigidity. Taking into account that $M(x) = EI\,\mathrm{d}^2 w/\mathrm{d}x^2$, the following boundary conditions must be added to this ODE

$$w(0) = 0,\quad w(l) = 0,\quad \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(0) = 0,\quad \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(l) = 0\,.$$

Solution. The above model reprezents an eigenvalue problem. The linear homogeneous differential equation

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \frac{P}{EI}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{k}{EI}w = 0 \tag{b}$$

is of fourth order with constant coefficients. Searching a solution of the form $w = \mathrm{e}^{rx}$, we get the characteristic equation

$$r^4 + \frac{P}{EI}r^2 + \frac{k}{EI} = 0\,,$$

of roots

$$r_1, r_2, r_3, r_4 = \pm\sqrt{-\frac{P}{2EI} \pm \sqrt{\left(\frac{P}{2EI}\right)^2 - \frac{k}{EI}}}\,, \tag{c}$$

the solution depending on the sign of the expression $P^2 - 4kEI$.

If $P^2 < 4kEI$, then the roots of the characteristic equation are complex conjugate, i.e. $r_1, r_2, r_3, r_4 = \pm a \pm \mathrm{i}b$, where

$$a = \sqrt{\sqrt{\frac{k}{4EI}} - \frac{P}{4EI}}\,,\quad b = \sqrt{\sqrt{\frac{k}{4EI}} + \frac{P}{4EI}}\,, \tag{d}$$

and the general solution is of the form

$$w = C_1 \cosh ax \cos bx + C_2 \cosh ax \sin bx + C_3 \sinh ax \cos bx + C_4 \sinh ax \sin bx\,. \tag{e}$$

If $P^2 > 4kEI$, then the roots are imaginary and may be written in the form $r_1, r_2 = \pm ik_1$, $r_3, r_4 = \pm \mathrm{i} k_2$, where

$$k_1 = \sqrt{\frac{P}{2EI} - \sqrt{\left(\frac{P}{2EI}\right)^2 - \frac{k}{EI}}}\,,\ k_2 = \sqrt{\frac{P}{2EI} + \sqrt{\left(\frac{P}{2EI}\right)^2 - \frac{k}{EI}}}\,. \tag{f}$$

The general solution takes the form

$$w = C_1 \sin k_1 x + C_2 \cos k_1 x + C_3 \sin k_2 x + C_4 \cos k_2 x\,. \tag{g}$$

For both solutions (e) and (g), the constants C_1, C_2, C_3, C_4 are determined by two-point conditions, which lead to a homogeneous system of four linear algebraic equations; to get non-zero solutions, its associated determinant must vanish. It is thus obtained a characteristic equation, of mimimal root corresponding to the critical force.
We consider the solution (g) (the solution (e) does not lead to real values for P_{cr}), then we put the conditions $w = 0$ and $M = EI\,\mathrm{d}^2 w/\mathrm{d}x^2 = 0$ for $x = 0$ and $x = l$, which lead to the system

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & k_1^2 & 0 & k_2^2 \\ \sin k_1 l & \cos k_1 l & \sin k_2 l & \cos k_2 l \\ k_1^2 \sin k_1 l & k_1^2 \cos k_1 l & k_2^2 \sin k_2 l & k_2^2 \cos k_2 l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0\,.$$

The characteristic equation becomes

$$\left(k_2^2 - k_1^2\right)^2 \sin k_1 l \sin k_2 l = 0$$

and it is satisfied for

$$k_1, k_2 = \frac{n\pi}{l}\left(n = 1, 2, 3, \ldots\right). \tag{h}$$

The solution (g) is reduced to

$$w = C\,\frac{n\pi x}{l}\,;$$

the relationships (f) and (h) lead to

$$\pm\sqrt{\left(\frac{P}{2EI}\right)^2 - \frac{k}{EI}} = \frac{n^2\pi^2}{l^2} - \frac{P}{2EI}\,,$$

whence

$$P_{cr} = \frac{\pi^2 EI}{l^2}\left(n^2 + \frac{kl^4}{n^2\pi^4 EI}\right). \tag{i}$$

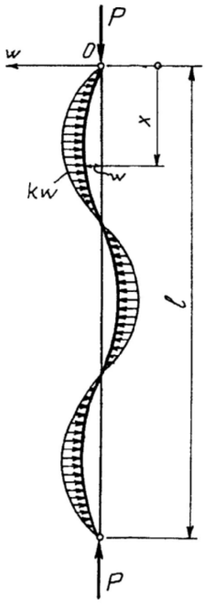

Figure 2. 2. Buckling of a beam in an elastic medium

The critical force is given by the minimal value in (i). Denoting

$$\gamma = \frac{kl^4}{n^2\pi^4 EI}, \tag{j}$$

we may write

$$P_{cr} = \frac{\pi^2 EI}{l^2}\left(n^2 + \frac{\gamma}{n^2}\right). \tag{k}$$

Hence, one must determine the integer number n of semiwaves which minimizes (k) (Fig.2.2). In the absence of the elastic medium $(k = 0)$ we get $\gamma = 0$ and the minimum takes place for $n = 1$, obtaining again Euler's critical force

$$P_{cr} = P_E = \frac{\pi^2 EI}{l^2}.$$

For increasing γ, the minimum takes place for $n = 1, 2, 3, \ldots$, hence if the deformed axis has one, two, three or more semiwaves.

The value P_{cr} must be smaller for *n* than for $n-1$ or $n+1$, hence we must have

$$(n-1)^2 + \frac{\gamma}{(n-1)^2} > n^2 + \frac{\gamma}{n^2} < (n+1)^2 + \frac{\gamma}{(n+1)^2}. \tag{l}$$

The second inequality leads to

$$\frac{\gamma}{n^2} - \frac{\gamma}{(n+1)^2} < (n+1)^2 - n^2$$

or to

$$\gamma < n^2 (n+1)^2 .$$

Likewise, from the first inequality it results

$$\gamma > (n-1)^2 n^2 ,$$

so that

$$(n-1)^2 n^2 < \gamma < n^2 (n+1)^2 , \tag{m}$$

therefore: $0 \le \gamma \le 4$ for $n = 1$, $4 \le \gamma \le 36$ for $n = 2$, $36 \le \gamma \le 144$ for $n = 3$, $144 \le \gamma \le 400$ for $n = 4$, a.s.o.; thus, one, two, three or four semiwaves are obtained. In general, if $\gamma = n^2 (n+1)^2$, the deformed curve may have n or $n+1$ semiwaves.
If γ is great, that is if the coefficient of soil reaction k is great or if the bar length is great, then the number of semiwaves is also great. In these cases, the inequalities (l) are reduced to the approximate relation $\gamma = n^4$; hence, one sees that

$$P_{cr} = \frac{\pi^2 EI}{l^2} \left(\sqrt{\gamma} + \frac{\gamma}{\sqrt{\gamma}} \right) = 2\sqrt{\gamma}\, \frac{\pi^2 EI}{l^2} = 2\sqrt{\frac{kl^4}{\pi^2 EI}}\, \frac{\pi^2 EI}{l^2} ,$$

or, finally,

$$P_{cr} = 2\sqrt{kEI} , \tag{n}$$

the critical force being thus independent of the bar length. One obtains the same result by differentiating the relation (k)

$$\frac{\mathrm{d}P_{cr}}{\mathrm{d}n} = \frac{\pi^2 EI}{l^2} \left(2n - \frac{2\gamma}{n^3} \right) = 0 ,$$

as if *n* would take continuous values. One obtains the same value for γ.
Practically, one determines first the non-dimensional quantity γ, then the consecutive integers between which is situated the value $n = \sqrt[4]{\gamma}$. The minimal value of P_{cr} is then given by (k) for *n* thus obtained.

Application 2.5

Problem. Study the deformation of circular pipe in an elastic medium, assuming that it is acted upon by a uniformly distributed load along the vertex generator (Fig.2.3, a). Discussion.

Mathematical model. One starts from the results and notations in Appl.2.2, i.e. from the differential equation

$$\frac{\mathrm{d}^5 w}{\mathrm{d}\varphi^5} + 2\frac{\mathrm{d}^3 w}{\mathrm{d}\varphi^3} + \left(1 + \frac{ka^4}{E_0 I}\right)\frac{\mathrm{d}w}{\mathrm{d}\varphi} = \left(-Y + \frac{\mathrm{d}Z}{\mathrm{d}\varphi}\right)\frac{a^4}{E_0 I}\,. \tag{a}$$

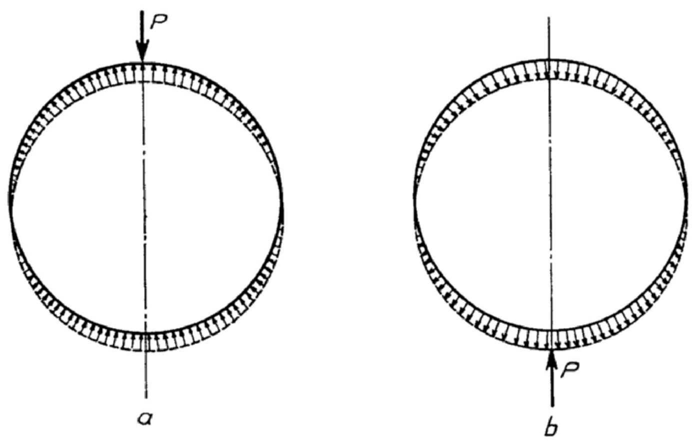

Figure 2. 3. Circular pipe in an elastic medium: geometric and static schema. Loading with an uniformly distributed force along the vertex genetrix (a); the case of a loading acting in antigravitational direction (b)

Assuming that the length of the pipe is great (theoretically infinite), the study is made on a span of unit length (the case of a plane state of deformation).

Solution. The reaction of the elastic medium is specified by the foundation modulus k; because there are not other distributed external loads, we may take $Y = Z = 0$.
With the notation

$$\eta = \sqrt{1 + \frac{ka^4}{E_0 I}}\,, \tag{b}$$

the ODE (a) becomes

$$\frac{\mathrm{d}^5 w}{\mathrm{d}\varphi^5} + 2\frac{\mathrm{d}^3 w}{\mathrm{d}\varphi^3} + \eta^2\frac{\mathrm{d}w}{\mathrm{d}\varphi} = 0\,, \tag{c}$$

hence a linear homogeneous differential equation of 5th order with constant coefficients. Searching for a solution of the form $w = \mathrm{e}^{m\varphi}$, one obtains the characteristic equation

$$m^5 + 2m^3 + \eta^2 m = 0 ,$$

with the roots $m_1 = 0$, $m_2, m_3, m_4, m_5 = \pm\alpha \pm \mathrm{i}\beta$, where

$$\alpha = \sqrt{\frac{\eta - 1}{2}} , \ \beta = \sqrt{\frac{\eta + 1}{2}} . \tag{d}$$

The general solution of the equation (b) is thus of the form

$$w = C_0 + \left(C_1 \cosh \alpha\varphi + C_2 \sinh \alpha\varphi\right)\cos \beta\varphi + \left(C_3 \cosh \alpha\varphi + C_4 \sinh \alpha\varphi\right)\sin \beta\varphi , \tag{e}$$

where C_0, C_1, C_2, C_3, C_4 are five integration constants which must be determined from the boundary conditions.
The sectional efforts are given by (see Appl. 2.2)

$$\begin{aligned} M &= -\frac{E_0 I}{a^2}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} + w\right), \\ T &= -\frac{E_0 I}{a^3}\left(\frac{\mathrm{d}^3 w}{\mathrm{d}\varphi^3} + \frac{\mathrm{d}w}{\mathrm{d}\varphi}\right), \\ N &= \frac{E_0 I}{a^3}\left(\frac{\mathrm{d}^4 w}{\mathrm{d}\varphi^4} + \frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2}\right) + kaw. \end{aligned} \tag{f}$$

Taking into account the displacement *w*, we get

$$\begin{aligned} M = -\frac{E_0 I}{a^2}\{C_0 - 2\alpha\beta[(C_1 \sinh \alpha\varphi + C_2 \cosh \alpha\varphi)\sin \beta\varphi \\ -(C_3 \sinh \alpha\varphi + C_4 \cosh \alpha\varphi)\cos \beta\varphi]\}, \end{aligned} \tag{g}$$

$$\begin{aligned} T = \frac{E_0 I}{a^3}\cdot 2\alpha\beta[(\alpha C_1 + \beta C_4)\cosh \alpha\varphi \sin \beta\varphi + (\beta C_1 - \alpha C_4)\sinh \alpha\varphi \cos \beta\varphi \\ + (\alpha C_2 + \beta C_3)\sinh \alpha\varphi \sin \beta\varphi + (\beta C_2 - \alpha C_3)\cosh \alpha\varphi \cos \beta\varphi], \end{aligned} \tag{h}$$

$$\begin{aligned} N = kaC_0 + \frac{E_0 I}{a^3}[(C_1 \sinh \alpha\varphi + C_2 \cosh \alpha\varphi)\sin \beta\varphi \\ -(C_3 \sinh \alpha\varphi + C_4 \cosh \alpha\varphi)\cos \beta\varphi]. \end{aligned} \tag{i}$$

Let us notice that the functions $\cosh \alpha\varphi \cos \beta\varphi$ and $\sinh \alpha\varphi \sin \beta\varphi$ are even, while the functions $\sinh \alpha\varphi \cos \beta\varphi$ and $\cosh \alpha\varphi \sin \beta\varphi$ are odd.

In our case, the origin of the variable φ is at the vertex. Considering the symmetry and the antisymmetry with respect to the vertical axis, respectively ($\varphi = 0$ and $\varphi = \pi$), there result the boundary conditions

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} = 0 , \tag{j}$$

$$T = \frac{P}{2} , \tag{k}$$

for $\varphi = 0$ and

$$\frac{\mathrm{d}w}{\mathrm{d}\varphi} = 0 , \tag{l}$$

$$T = 0 , \tag{m}$$

$$\int_0^\pi \frac{Ma}{E_0 I} \mathrm{d}\varphi = -\frac{1}{a}\int_0^\pi \left(\frac{\mathrm{d}^2 w}{\mathrm{d}\varphi^2} + w \right) \mathrm{d}\varphi = -\frac{1}{a}\left|\frac{\mathrm{d}w}{\mathrm{d}\varphi}\right|_0^\pi - \frac{1}{a}\int_0^\pi w \mathrm{d}\varphi = 0 \tag{n}$$

for $\varphi = \pi$.

The last condition shows that there is no relative rotation between $\varphi = 0$ and $\varphi = \pi$; taking into account the conditions (j) and (l), the condition (n) reduces to

$$\int_0^\pi w \mathrm{d}\varphi = 0 . \tag{o}$$

To simplify the calculus, we consider also the pipe loaded by the force P applied in the antigravitational direction at the bottom (Fig.2.3, b). In this case, the boundary conditions are: condition (j) and

$$T = 0 , \tag{k'}$$

for $\varphi = 0$ and (k) and

$$T = -\frac{P}{2} , \tag{m'}$$

for $\varphi = \pi$, as well as the condition (o).

Conditions (j) and (k') yield $\alpha C_2 + \beta C_3 = 0$ and $\beta C_2 - \alpha C_3 = 0$, accordingly, consequently $C_2 = C_3 = 0$, so that the solution (e) contains only even terms.

The other three conditions lead to

$$C_1 \left(\alpha \sinh \alpha\pi \cos \beta\pi - \beta \cosh \alpha\pi \sin \beta\pi \right)$$
$$+ C_4 \left(\alpha \cosh \alpha\pi \sin \beta\pi + \beta \sinh \alpha\pi \cos \beta\pi \right) = 0,$$

$$C_1\left(\alpha\cosh\alpha\pi\sin\beta\pi+\beta\sinh\alpha\pi\cos\beta\pi\right)$$
$$-C_4\left(\alpha\sinh\alpha\pi\cos\beta\pi-\beta\cosh\alpha\pi\sin\beta\pi\right)=\frac{Pa^3}{4\alpha\beta E_0 I},$$

$$\int_0^\pi\left(C_0\,\frac{1}{2\alpha\beta}-C_1\sinh\alpha\varphi\sin\beta\varphi+C_4\cosh\alpha\varphi\cos\beta\varphi\right)\mathrm{d}\varphi=0\,.$$

The first two relationships involve

$$C_1=\frac{Pa^3}{4\alpha\beta E_0 I}\,\frac{\alpha\cosh\alpha\pi\sin\beta\pi+\beta\sinh\alpha\pi\cos\beta\pi}{\eta\left(\sinh^2\alpha\pi+\sin^2\beta\pi\right)},$$

$$C_4=\frac{Pa^3}{4\alpha\beta E_0 I}\,\frac{\alpha\sinh\alpha\pi\cos\beta\pi-\beta\cosh\alpha\pi\sin\beta\pi}{\eta\left(\sinh^2\alpha\pi+\sin^2\beta\pi\right)};$$

introducing these expressions in the third condition, we get

$$C_0=-\frac{Pa^3}{2\pi\eta^2 E_0 I}\,.$$

By means of the notations

$$A=\frac{\alpha\cosh\alpha\pi\sin\beta\pi+\beta\sinh\alpha\pi\cos\beta\pi}{4\alpha\beta\eta\left(\sinh^2\alpha\pi+\sin^2\beta\pi\right)},$$

$$B=\frac{\alpha\sinh\alpha\pi\cos\beta\pi-\beta\cosh\alpha\pi\sin\beta\pi}{4\alpha\beta\eta\left(\sinh^2\alpha\pi+\sin^2\beta\pi\right)},$$

where α, β, η are given by (b) and (d), the final expressions of w, M, T and N become

$$w=-\frac{Pa^3}{E_0 I}\left(\frac{1}{2\pi\eta^2}-A\cosh\alpha\varphi\cos\beta\pi+B\sinh\alpha\varphi\sin\beta\varphi\right),$$

$$M=Pa\left(\frac{1}{2\pi\eta^2}+\frac{A}{2}\sinh\alpha\varphi\sin\beta\pi+\frac{B}{2}\cosh\alpha\varphi\cos\beta\varphi\right),$$

$$T=\frac{P}{2}\left[\left(\alpha A-\beta B\right)\cosh\alpha\varphi\sin\beta\varphi+\left(\beta A+\alpha B\right)\sinh\alpha\varphi\cos\beta\varphi\right],$$

$$N=-\frac{P}{2}\left(\frac{\eta^2-1}{\pi\eta^2}-A\sinh\alpha\varphi\sin\beta\varphi-B\cosh\alpha\varphi\cos\beta\varphi\right).$$

In the previous expressions, the sectional efforts appear as product of a dimensional factor $Pa^3/E_0 I$ (for the deflections), Pa (for the bending moment), and P (for the shearing force and the axial force) by a factor which is a function of the angular velocity φ. As it is seen, only one parameter η intervenes, which depends on the geometry of the pipe (the radius a and the moment of inertia I), on the elastic medium (the foundation modulus k), and on the material of the pipe (the modulus of elasticity E_0, corresponding to a state of plane deformation).

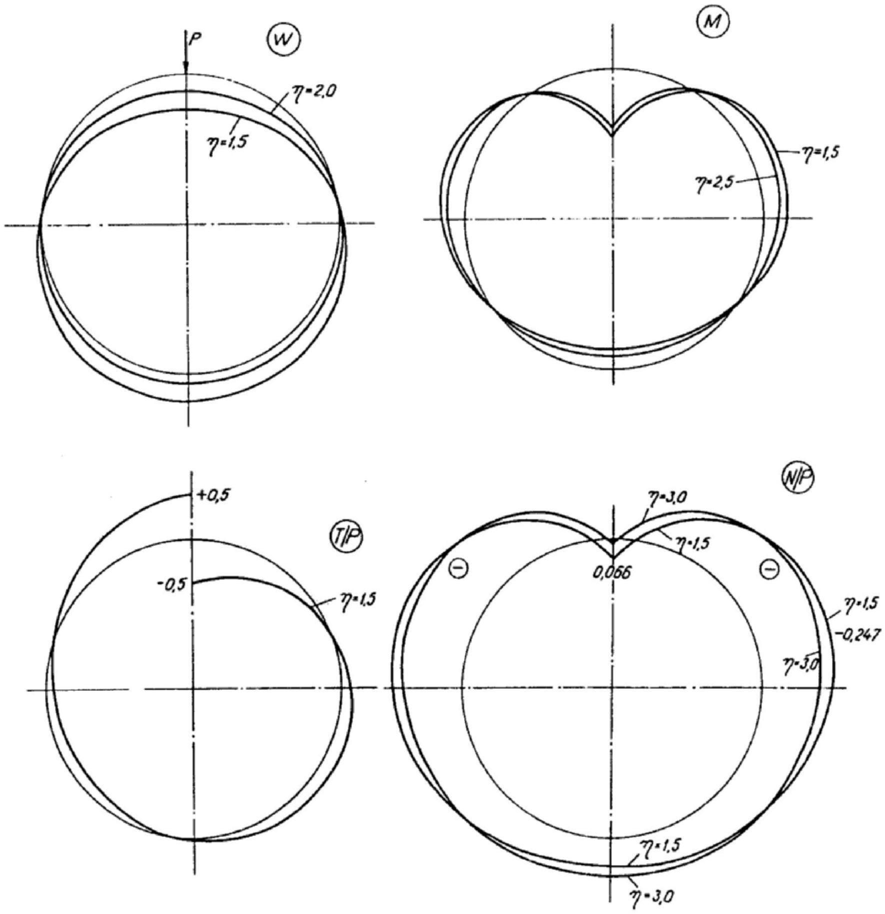

Figure 2. 4. The *w*-diagram (a); the *M*-diagram (b); the *T*-diagram (c); the *N*-diagram (d)

Tables 2.1, 2.2, 2.3, 2.4 contain the values of w, M, T, N as functions of φ for various values of the parameter η. These values are plotted into the diagrams 2.4, a, b, c, d, corresponding to the values $\eta = 1.5, 2.0, 2.5, 3.0$ and 5.0.

Table 2.1. The values of $w / \frac{Pa^3}{E_0 I}$

φ^0 \ η	1.5	2.0	2.5	3.0	5.0	10.0
0	+1.173	+0.560	+0.361	+0.262	+0.116	+0.041
10	+1.143	+0.540	+0.345	+0.249	+0.107	+0.035
20	+1.061	+0.489	+0.305	+0.215	+0.086	+0.024
30	+0.938	+0.414	+0.249	+0.169	+0.059	+0.012
40	+0.785	+0.327	+0.185	+0.119	+0.033	+0.003
50	+0.611	+0.233	+0.120	+0.069	+0.010	- 0.004
60	+0.427	+0.139	+0.058	+0.024	- 0.008	- 0.007
70	+0.239	+0.051	+0.002	- 0.015	- 0.021	- 0.009
80	+0.056	- 0.030	- 0.046	- 0.047	- 0.030	- 0.009
90	- 0.119	- 0.101	- 0.085	- 0.070	- 0.034	- 0.009
100	- 0.281	- 0.161	- 0.114	- 0.087	- 0.036	- 0.008
110	- 0.427	- 0.211	- 0.136	- 0.097	- 0.036	- 0.007
120	- 0.555	- 0.249	- 0.151	- 0.102	- 0.034	- 0.007
130	- 0.664	- 0.279	- 0.160	- 0.104	- 0.032	- 0.006
140	- 0.753	- 0.301	- 0.165	- 0.104	- 0.030	- 0.006
150	- 0.822	- 0.316	- 0.167	- 0.102	- 0.028	- 0.006
160	- 0.871	- 0.326	- 0.168	- 0.100	- 0.027	- 0.006
170	- 0.900	- 0.331	- 0.168	- 0.099	- 0.026	- 0.006
180	- 0.910	- 0.333	- 0.168	- 0.098	- 0.025	- 0.006

Table 2.2. The values of M / Pa

φ^0 \ η	1.5	2.0	2.5	3.0	5.0	10.0
0	+0.225	+0.210	+0.196	+0.183	+0.148	+0.108
10	+0.143	+0.129	+0.116	+0.104	+0.072	+0.037
20	+0.072	+0.061	+0.050	+0.041	+0.017	- 0.005
30	+0.013	+0.007	+0.001	- 0.005	- 0.017	- 0.023
40	- 0.032	- 0.033	- 0.035	- 0.035	- 0.035	- 0.027
50	- 0.063	- 0.060	- 0.056	- 0.053	- 0.042	- 0.022
60	- 0.082	- 0.075	- 0.067	- 0.060	- 0.040	- 0.015
70	- 0.090	- 0.079	- 0.068	- 0.059	- 0.033	- 0.008
80	- 0.088	- 0.075	- 0.063	- 0.052	- 0.024	- 0.003
90	- 0.078	- 0.065	- 0.052	- 0.042	- 0.016	+0.001
100	- 0.062	- 0.050	- 0.039	- 0.029	- 0.008	+0.002
110	- 0.042	- 0.032	- 0.023	- 0.016	- 0.001	+0.003
120	- 0.019	- 0.013	- 0.008	- 0.004	+0.004	+0.003
130	+0.003	+0.005	+0.006	+0.007	+0.007	+0.003
140	+0.024	+0.021	+0.019	+0.016	+0.009	+0.002
150	+0.042	+0.035	+0.029	+0.024	+0.011	+0.002
160	+0.056	+0.046	+0.037	+0.029	+0.011	+0.002
170	+0.064	+0.052	+0.042	+0.032	+0.012	+0.002
180	+0.067	+0.055	+0.043	+0.033	+0.012	+0.001

Table 2.3. The values of T / P

φ^0 \ η	1.5	2.0	2.5	3.0	5.0	10.0
0	- 0.500	- 0.500	- 0.500	- 0.500	- 0.500	- 0.500
10	- 0.441	- 0.429	- 0.418	- 0.407	- 0.373	- 0.317
20	- 0.371	- 0.350	- 0.330	- 0.311	- 0.251	- 0.163
30	- 0.296	- 0.269	- 0.242	- 0.219	- 0.147	- 0.055
40	- 0.219	- 0.190	- 0.162	- 0.137	- 0.066	+0.008
50	- 0.145	- 0.117	- 0.091	- 0.068	- 0.009	+0.036
60	- 0.076	- 0.053	- 0.032	- 0.014	+0.027	+0.042
70	- 0.015	+0.001	+0.014	+0.025	+0.076	+0.036
80	+0.037	+0.043	+0.048	+0.052	+0.052	+0.026
90	+0.077	+0.075	+0.071	+0.067	+0.049	+0.015
100	+0.106	+0.095	+0.084	+0.073	+0.042	+0.007
110	+0.124	+0.105	+0.088	+0.073	+0.033	+0.002
120	+0.130	+0.107	+0.086	+0.068	+0.024	- 0.001
130	+0.126	+0.101	+0.078	+0.059	+0.016	- 0.002
140	+0.113	+0.088	+0.066	+0.048	+0.009	- 0.003
150	+0.091	+0.071	+0.052	+0.037	+0.005	- 0.002
160	+0.064	+0.049	+0.036	+0.025	+0.002	- 0.001
170	+0.033	+0.025	+0.018	+0.012	+0.001	- 0.001
180	0	0	0	0	0	0

Table 2.4. The values of N / P

φ^0 \ η	1.5	2.0	2.5	3.0	5.0	10.0
0	+0.066	+0.051	+0.037	+0.024	- 0.011	- 0.051
10	- 0.017	- 0.030	- 0.043	- 0.055	- 0.088	- 0.123
20	- 0.088	- 0.098	- 0.109	- 0.118	- 0.142	- 0.164
30	- 0.146	- 0.152	- 0.159	- 0.164	- 0.176	- 0.182
40	- 0.191	- 0.192	- 0.194	- 0.195	- 0.195	- 0.186
50	- 0.222	- 0.219	- 0.216	- 0.212	- 0.201	- 0.182
60	- 0.242	- 0.234	- 0.226	- 0.219	- 0.199	- 0.175
70	- 0.249	- 0.238	- 0.228	- 0.218	- 0.192	- 0.168
80	- 0.247	- 0.234	- 0.222	- 0.211	- 0.184	- 0.162
90	- 0.237	- 0.224	- 0.211	- 0.201	- 0.175	- 0.159
100	- 0.221	- 0.209	- 0.198	- 0.188	- 0.167	- 0.157
110	- 0.201	- 0.191	- 0.183	- 0.175	- 0.160	- 0.156
120	- 0.178	- 0.172	- 0.167	- 0.163	- 0.155	- 0.156
130	- 0.156	- 0.154	- 0.153	- 0.152	- 0.152	- 0.156
140	- 0.135	- 0.138	- 0.140	- 0.143	- 0.150	- 0.157
150	- 0.117	- 0.124	- 0.130	- 0.135	- 0.148	- 0.157
160	- 0.103	- 0.113	- 0.122	- 0.130	- 0.148	- 0.157
170	- 0.095	- 0.107	- 0.118	- 0.127	- 0.148	- 0.158
180	- 0.092	- 0.105	- 0.116	- 0.126	- 0.148	- 0.158

Application 2.6

Problem. Study the buckling of a straight bar in a general case of support at both ends.

Mathematical model. In a general case, the buckling of a straigth bar of length l, acted upon by compression forces *P*, leads to a linear ODE of fourth order

$$\frac{d^4 w}{dx^4} + \beta^2 \frac{d^2 w}{dx^2} = 0, \qquad \text{(a)}$$

where the parameter β is given by $\beta^2 = P/EI$, *EI* being the bending rigidity. Particular case: a doubly built-in bar (Fig.1.49).

Solution. Searching a solution of the form $e^{\beta x}$, we get the characteristic equation

$$\lambda^4 + \beta^2 \lambda^2 = 0,$$

of roots $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = \pm i\beta$. The general solution of (a) and its derivative are, accordingly,

$$w = A \sin \beta x + B \cos \beta x + Cx + D,$$

$$\frac{dw}{dx} = \beta(A \cos \beta x - B \sin \beta x) + C.$$

Choosing the origin of x-coordinates at the upper end of the bar, the boundary conditions in the particular case mentioned above are $w = 0$, $dw/dx = 0$ for $x = 0$ and $x = l$. The four conditions lead to

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & 1 & 0 \\ \sin \beta l & \cos \beta l & l & 1 \\ \beta \cos \beta l & -\beta \sin \beta l & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0.$$

This system has non-zero solutions only if

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & 1 & 0 \\ \sin \beta l & \cos \beta l & l & 1 \\ \beta \cos \beta l & -\beta \sin \beta l & 1 & 0 \end{vmatrix} = -\beta\left[2(1 - \cos \beta l) - \beta l \sin \beta l\right] = 0.$$

As $\beta \neq 0$, one obtains again the solution given in Appl.1.34.

Application 2.7

Problem. Determine the deflections of a beam in an elastic medium, assuming Winkler's hypothesis ($p = kw$, the pressure p is proportional to the displacement w, $k = const$ being the coefficient of soil reaction).

Mathematical model. The ODE which governs the deformation of the bar is of the form

$$\frac{\mathrm{d}^4 w}{\mathrm{d}\varphi^4} + 4\beta^2 w = 0\,, \tag{a}$$

where the parameter β depends on the elasticity of the medium.

Solution. Searching for solutions of the exponential form $e^{\beta x}$, one obtains the characteristic equation

$$\lambda^4 + 4\beta^2 = 0\,,$$

of roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4 = (\pm 1 \pm \mathrm{i})\beta$.
The general solution may be expressed in one of the following forms

$$w = A_1 \cosh\beta x \cos\beta x + A_2 \sinh\beta x \cos\beta x + A_3 \cosh\beta x \sin\beta x + A_4 \sinh\beta x \sin\beta x\,, \tag{b}$$

$$w = e^{-\beta x}\left(A\cos\beta x + B\sin\beta x\right) + e^{\beta x}\left(C\cos\beta x + D\sin\beta x\right), \tag{c}$$

where A_1, A_2, A_3, A_4 and A, B, C, D, respectively, represent integration constants.
Starting from formula (b), we can introduce new integration constants, with a physical significance (initial parameters), i.e. : w_0, φ_0, M_0, T_0, representing the deflection, the rotation, the bending moment, and the shearing force, respectively, at the left end of the bar (chosen as origin of x-co-ordinates).
Introducing the functions

$$\begin{aligned} f_1(\beta x) &= \cosh\beta x \cos\beta x, \\ f_2(\beta x) &= \sinh\beta x \cos\beta x + \cosh\beta x \sin\beta x, \\ f_3(\beta x) &= \sinh\beta x \sin\beta x, \\ f_4(\beta x) &= \sinh\beta x \cos\beta x - \cosh\beta x \sin\beta x, \end{aligned} \tag{d}$$

we may express the deflection, the rotation, the bending moment, and the shearing force in the form

$$\begin{aligned} w &= w_0 f_1(\beta x) + \frac{\varphi_0}{2\beta} f_2(\beta x) - \frac{2M_0\beta^2}{k} f_3(\beta x) + \frac{T_0\beta}{k} f_4(\beta x), \\ \varphi &= \frac{\mathrm{d}w}{\mathrm{d}x} = \beta w_0 f_4(\beta x) + \varphi_0 f_1(\beta x) - \frac{2M_0\beta^3}{k} f_2(\beta x) + \frac{2T_0\beta^2}{k} f_3(\beta x), \end{aligned} \tag{e}$$

$$M = -EI\frac{d^2w}{dx^2} = \frac{kw_0}{2\beta^2}f_3(\beta x) - \frac{k\varphi_0}{4\beta^3}f_4(\beta x) + M_0 f_1(\beta x) + \frac{T_0}{2\beta}f_2(\beta x),$$

$$T = -EI\frac{d^3w}{dx^3} = \frac{kw_0}{2\beta}f_2(\beta x) + \frac{k\varphi_0}{2\beta^2}f_3(\beta x) + \beta M_0 f_4(\beta x) + T_0 f_1(\beta x).$$

The above defined functions $f_i(\beta x)$, $i = 1, 2, 3, 4$, are often met in the mechanics of deformable solids. Their diagrams are given in Fig.2.5.

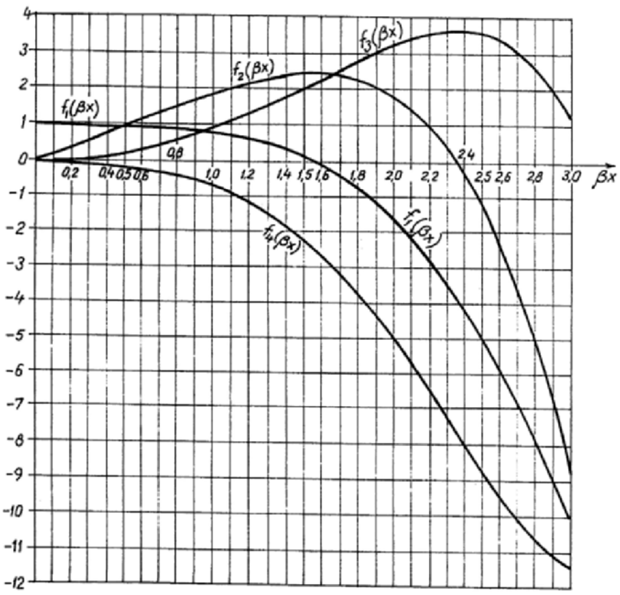

Figure 2. 5. Graphics of functions $f_i(\beta x)$, $i = 1, 2, 3, 4$

Application 2.8

Problem. Determine the critical moment M_{cr} in the lateral buckling of a doubly hinged beam (Fig.2.6).

Mathematical model. The lateral buckling of a beam subjected to pure bending in a vertical plane is governed by the differential equation

$$C_1\frac{d^4\theta}{dx^4} - C\frac{d^2\theta}{dx^2} - \frac{M_0^2}{EI_z}\theta = 0, \tag{a}$$

where θ is he rotation of the transverse section in its plane, EI_z is the bending rigidity with respect to the minimal neutral axis (vertical), C and C_1 are the torsion and hindered torsion rigidities, respectively, and M_0 are the bending moments applied at the end sections of the beam.

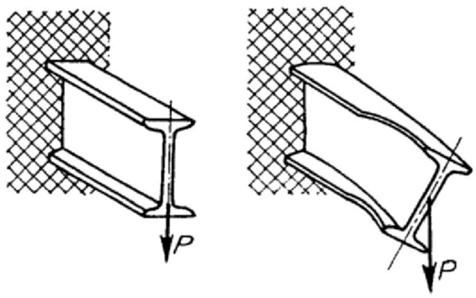

Figure 2. 6. Lateral buckling of a doubly hinged beam of rotation

The boundary conditions are

$$\theta(0)=0,\ \theta(l)=0\,,\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}(0)=0,\ \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}(l)=0\,. \tag{b}$$

Solution. Using the notations

$$\alpha=\frac{C}{2C_1}\,,\ \ \beta^2=\frac{M_0^2}{EI_2C_1}\,,\ \ \alpha,\beta>0\,, \tag{c}$$

the equation (a) becomes

$$\frac{\mathrm{d}^4\theta}{\mathrm{d}x^4}-2\alpha\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-\beta^2\theta=0\,, \tag{d}$$

that is a linear, homogeneous differential equation with constant coefficients. The roots of the characteristic equation

$$\lambda^4-2\alpha\lambda^2-\beta^2=0$$

are $\lambda_1,\lambda_2=\pm\mathrm{i}m$, $\lambda_3,\lambda_4=\pm n$, with

$$m=\sqrt{-\alpha+\sqrt{\alpha^2+\beta^2}}\ ,\ n=\sqrt{\alpha+\sqrt{\alpha^2+\beta^2}}\ , \tag{e}$$

yielding the general solution

$$\theta=A_1\sin mx+A_2\cos mx+A_3\sinh nx+A_4\cosh nx\,,$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}=-m^2\left(A_1\sin mx+A_2\cos mx\right)+n^2\left(A_3\sinh nx+A_4\cosh nx\right).$$

Introducing the boundary conditions (b), we get the homogeneous algebraic system

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -m^2 & 0 & n^2 \\ \sin ml & \cos ml & \sinh nl & \cosh nl \\ -m^2 \sin ml & -m^2 \cos ml & n^2 \sinh nl & n^2 \cosh nl \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0 .$$

Equating to zero the determinant of the coefficients, we get

$$2n^2 \left(m^2 + n^2 \right) \sin ml \sinh nl = 0 .$$

The only factor which can vanish is

$$\sin ml = 0 . \tag{f}$$

The equation of the rotations of the cross sections is then given by

$$\theta = A_1 \sin mx ,$$

where the constant A_1 remains non-determinate.

From (f) one obtains the minimal value $m = \pi/l$, and – returning to (c) – one has

$$-\alpha + \sqrt{\alpha^2 + \beta^2} = \frac{\pi^2}{l^2} .$$

Introducing the relations (c), the critical moment becomes

$$M_{cr} = \frac{\pi}{l} \sqrt{EI_z C_1 + \frac{\pi^2 C_1}{l^2 C}} .$$

Application 2.9

Problem. Determine the critical rotative speed of a simply supported driving shaft.

Mathematical model. The deflections w satisfy the homogeneous differential equation

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - \frac{\gamma A \omega^2}{gEI} w = 0 , \tag{a}$$

where ω is the angular velocity, A is the area of the cross section of the shaft, γ is the unit weight of the material and g is the gravitational acceleration. The boundary conditions are $w = 0$, $\mathrm{d}w / \mathrm{d}x$ for $x = 0$, $x = l$.

Solution. Introducing the notation

$$\beta^4 = \frac{\gamma A \omega^2}{gEI} , \tag{b}$$

the equation (a) becomes

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} - \beta^4 w = 0\,, \tag{c}$$

with the general solution

$$w = C_1 \cosh \beta x + C_2 \sinh \beta x + C_3 \cos \beta x + C_4 \sin \beta x\,, \tag{d}$$

where C_1, C_2, C_3, C_4 are integration constants. The second derivative is given by

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \beta^2 \left(C_1 \cosh \beta x + C_2 \sinh \beta x - C_3 \cos \beta x - C_4 \sin \beta x \right).$$

Taking into account the boundary conditions, we get the homogeneous algebraic system

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \beta^2 & 0 & -\beta^2 & 0 \\ \cosh \beta l & \sinh \beta l & \cos \beta l & \sin \beta l \\ \beta^2 \cosh \beta l & \beta^2 \sinh \beta l & -\beta^2 \cos \beta l & -\beta^2 \sin \beta l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0\,. \tag{e}$$

It has non-zero solutions only if the determinant of the coefficients vanishes. The characteristic equation thus obtained leads to $\sin \beta l = 0$, that involves $\beta l = n\pi$, $n = 1, 2, \ldots$. From the relation (b), we obtain thus the critical rotation speed

$$\omega_{cr} = \beta^2 \sqrt[4]{\frac{gEI}{\gamma A}} = n^2 \frac{\pi^2}{l^2} \sqrt[4]{\frac{gEI}{\gamma A}}\,. \tag{f}$$

The solution (d) is currently met in the mechanics of deformable solids.

Application 2.10

Problem. A very long beam (theoretically infinite) stays on an elastic medium and is acted upon by a concentrated transverse force P. Determine the deflection w, the rotation φ, the bending moment M and the sharing force T in an arbitrary cross section.

Mathematical model. The origin of the x-co-ordinates may be chosen in any point, because the beam is of infinite length; but it is convenient to choose the point of application of the force P, to obtain diagrams with properties of symmetry or antisymmetry with respect to this point (Fig.2.7, a).

The deflection is given by the general solution (see Appl.2.7)

$$w = \mathrm{e}^{-\beta x} \left(A \cos \beta x + B \sin \beta x \right) + \mathrm{e}^{\beta x} \left(C \cos \beta x + D \sin \beta x \right), \tag{a}$$

where β is a dimensional constant given by $\beta^4 = k / 4EI$, where k is the response of the elastic medium and EI is the bending rigidity of the beam.

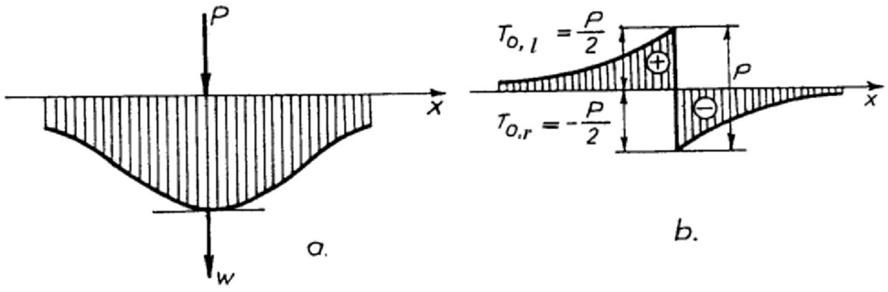

Figure 2. 7. Beam of infinite length on elastic medium. Diagram of deflections (a). Diagram of shearing forces in the vicinity of the origin (b)

Solution. At the origin we have $\mathrm{d}w/\mathrm{d}x = 0$, while the shearing force has a jump; at the right, we have $T_\mathrm{r} = -P/2$ (Fig.2.7, b). At infinity, the influence of the concentrated force vanishes, so that w, M and T tend to zero. As the factor $\mathrm{e}^{\beta x}$ increases indefinitely for $x \to \infty$, we take $C = D = 0$. We therefore get

$$w = \mathrm{e}^{-\beta x}\left(A\cos\beta x + B\sin\beta x\right),$$

$$\varphi = \frac{\mathrm{d}w}{\mathrm{d}x} = \beta\mathrm{e}^{-\beta x}\left[\left(B - A\right)\cos\beta x - \left(A + B\right)\sin\beta x\right],$$

$$M = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 2\beta^2 EI\mathrm{e}^{-\beta x}\left(B\cos\beta x - A\sin\beta x\right), \tag{b}$$

$$T = -EI\frac{\mathrm{d}^3 w}{\mathrm{dx}^3} = -2\beta^3 EI\mathrm{e}^{-\beta x}\left[\left(A + B\right)\cos\beta x - \left(B - A\right)\sin\beta x\right].$$

Introducing the above mentioned boundary conditions, it results $A = B = P\beta/2k$, so that

$$w = \frac{P\beta}{2k}\mathrm{e}^{-\beta x}\left(\cos\beta x + \sin\beta x\right),$$

$$\varphi = -\frac{P\beta^2}{k}\mathrm{e}^{-\beta x}\sin\beta x\,,$$

$$M = \frac{P}{4\beta}\mathrm{e}^{-\beta x}\left(\cos\beta x - \sin\beta x\right), \tag{c}$$

$$T = -\frac{P}{2}\mathrm{e}^{-\beta x}\cos\beta x \;\text{ for }\; x \geq 0\,.$$

We notice that formula (c) contains four functions of argument βx, i.e.:

$$\begin{aligned}
\psi_1(\beta x) &= e^{-\beta x}\cos\beta x,\\
\psi_2(\beta x) &= e^{-\beta x}\sin\beta x,\\
\psi_3(\beta x) &= e^{-\beta x}(\cos\beta x + \sin\beta x) = \psi_1(\beta x) + \psi_2(\beta x),\\
\psi_4(\beta x) &= e^{-\beta x}(\cos\beta x - \sin\beta x) = \psi_1(\beta x) - \psi_2(\beta x).
\end{aligned} \tag{d}$$

The functions $\psi_i(\beta x), i = \overline{1,4}$, are usual in the mechanics of deformable solids; they are plotted into diagrams (Fig.2.8).

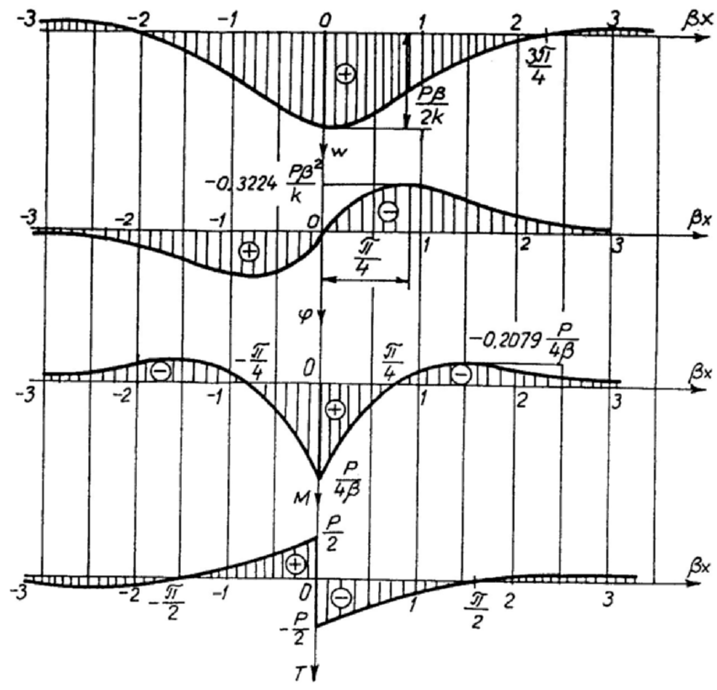

Figure 2.8. Graphics of functions w, φ, M, T

Application 2.11

Problem. Determine the deflection, the bending moment M and the shearing force T for a beam on an elastic medium of elastic response k, acted upon by moments M_0 at its free ends.

Mathematical model. We use the functions $f_i(\beta x), i = \overline{1,4}$, introduced in Appl.2.7 (see Fig.2.9), as well as the solution (e). The boundary conditions are $M = M_0, T = T_0 = 0$

for $x = 0$ and $x = l$. In the above mentioned formulae we therefore take $T_0 = 0$, M_0 being a known value.

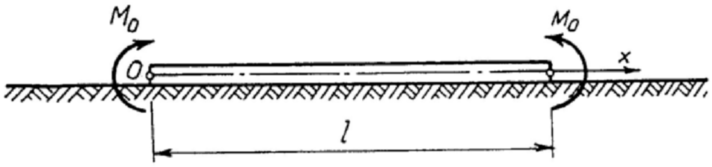

Figure 2. 9. Beam on elastic medium acted upon at its ends by moments M_0

Solution. From the previous considerations, we obtain

$$w = w_0 f_1(\beta x) + \frac{\varphi_0}{2\beta} f_2(\beta x) - \frac{2M_0\beta^2}{k} f_3(\beta x), \tag{a}$$

$$M = \frac{kw_0}{2\beta^2} f_3(\beta x) - \frac{k\varphi_0}{4\beta^3} f_4(\beta x) + M_0 f_1(\beta x), \tag{b}$$

$$T = \frac{kw_0}{2\beta} f_2(\beta x) + \frac{k\varphi_0}{2\beta^2} f_3(\beta x) + \beta M_0 f_4(\beta x). \tag{c}$$

Introducing the boundary conditions in (b) and (c), we are led to

$$\frac{kw_0}{2\beta^2} f_3(\beta x) - \frac{k\varphi_0}{4\beta^3} f_4(\beta x) = -M_0 f_1(\beta x),$$
$$\frac{kw_0}{2\beta^2} f_2(\beta x) - \frac{k\varphi_0}{2\beta^2} f_3(\beta x) = -\beta M_0 f_4(\beta x);$$

the solution is given by

$$w_0 = \frac{2M_0\beta^2}{k} \frac{f_1(\beta l) f_3(\beta l) + f_4^2(\beta l)}{f_2(\beta l) f_4(\beta l) + f_3^2(\beta l)},$$

$$\varphi_0 = \frac{4M_0\beta^3}{k} \frac{f_1(\beta l) f_2(\beta l) - f_3(\beta l) f_4(\beta l)}{f_2(\beta l) f_4(\beta l) + f_3^2(\beta l)},$$

where $\beta^4 = k/4EI$, EI being the bending rigidity.

The values of the bending moment M are plotted into diagrams for various values of the argument βl (Fig.2.10). As it can be seen, βl has a strong influence on M; the greater βl, the more the variation of M has the character of a local perturbation (at the bar ends).

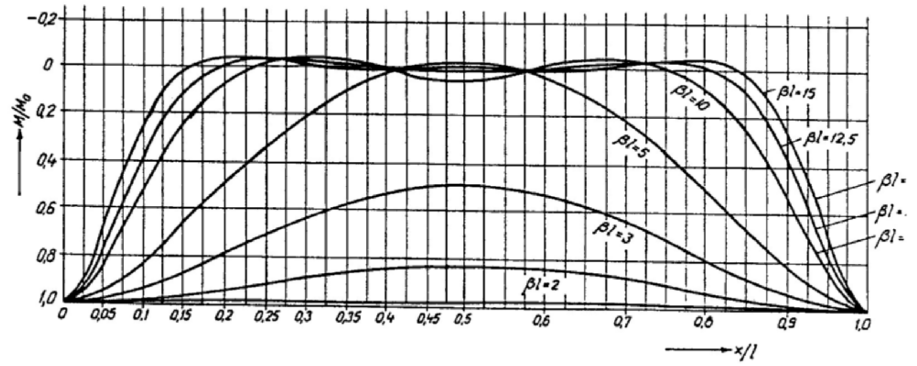

Figure 2. 10. Variation of the bending moments M for various values of the parameter βl

We notice that the boundary conditions for $x = l$ may be replaced by $\varphi = 0$ and $T = 0$ for $x = l/2$, and the initial parameters become

$$w_0 = -\frac{2M_0\beta^2}{k}\frac{f_1\left(\frac{\beta l}{2}\right)f_4\left(\frac{\beta l}{2}\right)+f_2\left(\frac{\beta l}{2}\right)f_3\left(\frac{\beta l}{2}\right)}{f_1\left(\frac{\beta l}{2}\right)f_2\left(\frac{\beta l}{2}\right)-f_3\left(\frac{\beta l}{2}\right)f_4\left(\frac{\beta l}{2}\right)},$$

$$\varphi_0 = \frac{2M_0\beta^3}{k}\frac{f_2^2\left(\frac{\beta l}{2}\right)+f_4^2\left(\frac{\beta l}{2}\right)}{f_1\left(\frac{\beta l}{2}\right)f_2\left(\frac{\beta l}{2}\right)-f_3\left(\frac{\beta l}{2}\right)f_4\left(\frac{\beta l}{2}\right)}.$$

One may also take into account the geometric and the loading symmetry of the beam, choosing the origin of the x-co-ordinates at the middle of the span. In this case, the initial parameters φ_0 and T_0 vanish. We put the boundary conditions $M = \overline{M}$ and $T = 0$ for $x = l/2$ (we denote by $\overline{M}$ the moments at the beam ends); by means of

$$w = w_0 f_1(\beta x) - \frac{2M_0\beta^2}{k} f_3(\beta x),$$

$$M = \frac{kw_0}{2\beta^2} f_3(\beta x) + M_0 f_1(\beta x),$$

$$T = \frac{kw_0}{2\beta} f_2(\beta x) + \beta M_0 f_4(\beta x),$$

we obtain the initial parameters

$$w_0 = -\frac{2\overline{M}\beta^2}{k}\frac{f_4\left(\frac{\beta l}{2}\right)}{f_1\left(\frac{\beta l}{2}\right)f_2\left(\frac{\beta l}{2}\right)-f_3\left(\frac{\beta l}{2}\right)f_4\left(\frac{\beta l}{2}\right)},$$

$$M_0 = \overline{M}\frac{f_2\left(\frac{\beta l}{2}\right)}{f_1\left(\frac{\beta l}{2}\right)f_2\left(\frac{\beta l}{2}\right)-f_3\left(\frac{\beta l}{2}\right)f_4\left(\frac{\beta l}{2}\right)}.$$

Application 2.12

Problem. Determine the general expression of the deflections of a beam of span *l*, acted upon by two uniformly distributed loads of intensities p_1 and p_2 (Fig.2.11).

Mathematical model. The deflections w are given by the differential equation

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = \frac{p(x)}{EI}, \tag{a}$$

where $p(x)$ is the transverse load and *EI* is the bending rigidity.

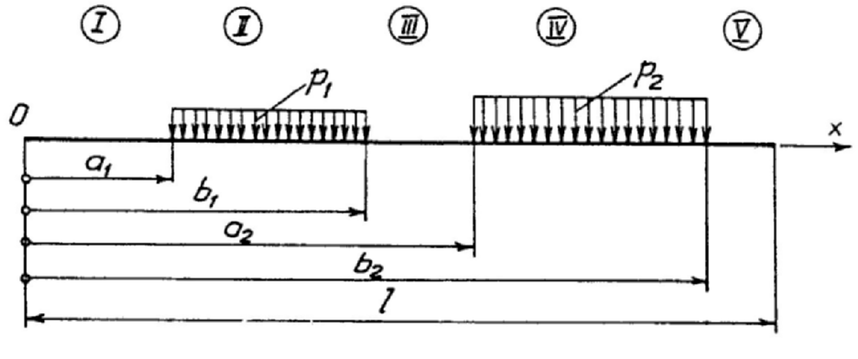

Figure 2. 11. Beam acted upon by two uniformly distributed loads

Solution. The general solution of the homogeneous differential equation is given by

$$EIw = C_1\frac{x^3}{3!} + C_2\frac{x^2}{2!} + C_3 x + C_4 . \tag{b}$$

To obtain a particular solution of the non-homogeneous ODE, we use Cauchy's integral relationship, i.e.

$$\int_0^x \mathrm{d}x \int_0^x \mathrm{d}x \int_0^x \mathrm{d}x \ldots \int_0^x p(x)\mathrm{d}x = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} p(t)\mathrm{d}t\ ; \tag{c}$$

in our case, $n = 4$ and the particular solution becomes

$$EIw_p = \frac{1}{6}\int_0^x (x-t)^3 p(t)\mathrm{d}t\ . \tag{d}$$

We notice that the load is expressed in the form

$$p(x) = \begin{cases} 0, & x \in [0, a_1), \\ p_1, & x \in (a_1, b_1), \\ 0, & x \in (b_1, a_2), \\ p_2, & x \in (a_2, b_2), \\ 0, & x \in (b_2, l]\ . \end{cases} \tag{e}$$

In this case

$$EIw_p = \frac{1}{6}\int_0^x (x-t)^3 \cdot 0 \cdot \mathrm{d}t = 0 \quad \text{for } x \in [0, a_1),$$

$$EIw_p = \frac{1}{6}\int_0^x (x-t)^3 \mathrm{p(t)d}t = \frac{1}{6}\int_0^{a_1} (x-t)^3 \cdot 0 \cdot \mathrm{d}t + \frac{1}{6}\int_{a_1}^x (x-t)^3 p_1 \mathrm{d}t$$
$$= \frac{p_1 (x-a_1)^4}{24} \quad \text{for } x \in [a_1, b_1),$$

$$\begin{aligned} EIw_p &= \frac{1}{6}\int_0^x (x-t)^3 p(t)\mathrm{d}t = \frac{1}{6}\int_0^{a_1} (x-t)^3 \cdot 0 \cdot \mathrm{d}t + \frac{1}{6}\int_{a_1}^{b_1} (x-t)^3 p_1 \mathrm{d}t \\ &+ \frac{1}{6}\int_{b_1}^x (x-t)^3 \cdot 0 \cdot \mathrm{d}t = \frac{p_1(x-a_1)^4}{24} - \frac{p_1(x-b_1)^4}{24} \qquad \text{for } x \in [b_1, a_2), \end{aligned} \tag{f}$$

$$EIw_p = \frac{1}{6}\int_0^x (x-t)^3 p(t)\mathrm{d}t = \frac{1}{6}\int_0^{a_1} (x-t)^3 \cdot 0 \cdot \mathrm{d}t + \frac{1}{6}\int_{a_1}^{b_1} (x-t)^3 p_1 \mathrm{d}t$$
$$+ \frac{1}{6}\int_{b_1}^{a_2} (x-t)^3 \cdot 0 \cdot \mathrm{d}t + \frac{1}{6}\int_{a_2}^x (x-t)^3 p_2 \mathrm{d}t$$
$$= \frac{p_1(x-a_1)^4}{24} - \frac{p_1(x-b_1)^4}{24} + \frac{p_2(x-a_2)^4}{24} \quad \text{for } x \in [a_2, b_2],$$

$$EIw_p = \frac{1}{6}\int_0^x (x-t)^3 p(t)\mathrm{d}t = \frac{1}{6}\int_0^{a_1}(x-t)^3\cdot 0\cdot \mathrm{d}t + \frac{1}{6}\int_{a_1}^{b_1}(x-t)^3 p_1 \mathrm{d}t$$

$$+\frac{1}{6}\int_{b_1}^{a_2}(x-t)^3\cdot 0\cdot\mathrm{d}t + \frac{1}{6}\int_{a_2}^{b_2}(x-t)^3 p_2\mathrm{d}t + \frac{1}{6}\int_{b_2}^{x}(x-t)^3\cdot 0\cdot\mathrm{d}t$$

$$= \frac{p_1(x-a_1)^4}{24} - \frac{p_1(x-b_1)^4}{24} + \frac{p_2(x-a_2)^4}{24} - \frac{p_2(x-b_2)^4}{24} \quad \text{for } x\in[b_2, l].$$

We observe that at the common end of two intervals the deflections are continuous, while in the expression of w_p appears a supplementary term.

Introducing the Macaulay brackets $\langle\ldots\rangle$, the general expression of w_p reads

$$EIw_p = 0 + \left\langle \frac{p_1(x-a_1)^4}{24}\right\rangle_{x\geq a_1} - \left\langle \frac{p_1(x-b_1)^4}{24}\right\rangle_{x\geq b_1} + \left\langle \frac{p_2(x-a_2)^4}{24}\right\rangle_{x\geq a_2} - \left\langle \frac{p_2(x-b_2)^4}{24}\right\rangle_{x\geq b_2}, \tag{g}$$

with the convention that the respective term must be considered only for the positive argument.

Application 2.13

Problem. Determine the traiectory of an electrized particle in an electromagnetic field of intensity **E** and induction **B**.

Mathematical model. The components of the two forces are represented in Fig.2.12 with respect to an orthogonal reference trihedron *Oxyz*. The resultant force is $\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$, where q is the electric load, **v** is the velocity of the particle, and the second term is Lorenz's force. We have

$$\mathbf{v}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = v_y B\mathbf{i} - v_x B\mathbf{j};$$

to study the motion, we introduce Newton's equation

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}},$$

where m is the mass of the particle.

Solution. Projecting on the three axes of co-ordinates, we obtain the equations of motion

$$m\ddot{x} = qv_y B, \tag{a}$$

$$m\ddot{y} = qE_y - qv_x B, \tag{b}$$

$$m\ddot{z} = qE_z. \qquad \text{(c)}$$

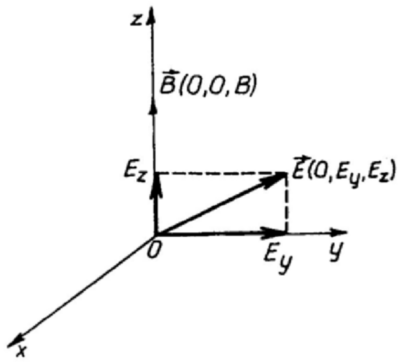

Figure 2. 8. Electrized particle in an electromagnetic field

The equation (c) may be considered separately. By integration, we get

$$\dot{z} = \int \frac{qE_z}{m}\,\mathrm{d}t = \frac{qE_z}{m}t + C_1,$$

the constant C_1 being determined by the initial condition $\dot{z}(0) = v_z^0$. One obtains $C_1 = v_z^0$ and

$$\dot{z} = \frac{qE_z}{m}t + v_z^0.$$

A new integration gives

$$z = \frac{qE_z}{2m}t^2 + v_z^0 t + C_2.$$

The condition $z(0) = 0$ leads to $C_2 = 0$, so that

$$z = \frac{qE_z}{2m}t^2 + v_z^0 t \qquad \text{(d)}$$

represents a uniformly accelerated motion along the *z*-axis, of acceleration $a_z = qE_z/m$. For the other two axes, we may write the equations (a) and (b) in the form

$$m\ddot{x} = qB\dot{y},\ m\ddot{y} = qB - qB\dot{x}. \qquad \text{(e)}$$

Eliminating the function y, we obtain

$$\dddot{x} + \frac{q^2 B^2}{m^2} \dot{x} = \frac{q^2 B E_y}{m^2}, \tag{f}$$

hence a linear non-homogeneous third order ODE with constant coefficients.
The ratio q/m represents the elastic load on the unit mass.
Denoting by

$$\frac{q^2 B^2}{m^2} = \omega^2, \frac{q^2 B E_y}{m^2} = R^2, \tag{g}$$

the equation (f) becomes

$$\dddot{x} + \omega^2 \dot{x} = R^2.$$

As the right member of this ODE is a constant, we find easily the particular solution $x_p = E_y t / B$. Searching for an exponential solution $x = \mathrm{e}^{\lambda t}$ of the associated homogeneous ODE, one obtains the characteristic equation

$$\lambda^3 + \omega^2 \lambda = \lambda\left(\lambda^2 + \omega^2\right) = 0,$$

having three roots, $\lambda_1 = 0$, $\lambda_2, \lambda_3 = \pm \mathrm{i}\omega$. Thus, the general solution of the associated homogeneous ODE is

$$x_h = D_1 + D_2 \cos \omega t + D_3 \sin \omega t,$$

hence, the general solution of the non-homogeneous ODE is finally given by

$$x = D_1 + D_2 \cos \omega t + D_3 \sin \omega t + \frac{E_y}{B} t. \tag{h}$$

The first equation (e) gives

$$y = \frac{m}{qB} \dot{x} + \text{const} = \frac{m}{qB}\left(\frac{E_y}{B} - D_2 \omega \sin \omega t + D_3 \omega \cos \omega t \right) + \text{const}$$

or, finally,

$$y = \frac{m\omega}{qB}\left(- D_2 \sin \omega t + D_3 \cos \omega t\right) + \text{const}. \tag{i}$$

The four integration constants are determined by the initial conditions $x(0) = 0$, $y(0) = 0$, $\dot{x}(0) = v_x^0$, $\dot{y}(0) = v_y^0$. They are

$$D_1 = \frac{qBv_y^0}{m\omega^2}, \; D_2 = -\frac{qBv_y^0}{m\omega^2}, \; D_3 = \frac{v_x^0}{\omega} - \frac{E_y}{B\omega}, \; D_4 = -\frac{mv_x^0}{qB} + \frac{mE_y}{qB^2},$$

so that

$$x = \frac{qBv_y^0}{m\omega^2}(1-\cos\omega t) + \frac{E_y}{B\omega}(\omega t - \sin\omega t) + \frac{v_x^0}{\omega}\sin\omega t\,,$$

$$y = \frac{mE_y}{qB^2}(1-\cos\omega t) + \frac{v_y^0}{\omega}\sin\omega t\,.$$

Taking into account the notation (g), the displacements x and y read

$$x = \frac{m}{qB}\left[v_x^0\sin\omega t + v_y^0(1-\cos\omega t) + \frac{E_y}{B}(\omega t - \sin\omega t)\right],$$

$$y = \frac{m}{qB}\left[v_y^0\sin\omega t + \frac{E_y}{B}(1-\cos\omega t)\right] = \omega\dot{x}\,.$$

We obtained the parametric equations of the projection of the trajectory on the Oxy-plane; this projection is a *trochoid*.

Application 2.14

Problem. A circular cylindrical tank, of a very great length is subjected to an internal constant pressure p. Assuming that at the bottom end the envelope is articulated to the corresponding circular plate, determine the deflection w and the bending moment M_x.

Mathematical model. The ODE of the deflection is

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = -\frac{p}{K}\,,$$

where β is a constant damping coefficient, while K is the bending rigidity of the cylindrical shell.

We consider the cylindrical tank of semi-infinite length. Choosing the origin of the x-co-ordinates at the bottom (Fig.2.13), the boundary conditions are $w = 0$, $d^2 w / dx^2 = 0$ for $x = 0$.

Solution. Obviously, a particular solution of the above non-homogeneous ODE with constant coefficients is

$$w_p = -\frac{p}{4K\beta^4}\,;$$

searching for solutions of the associated homogeneous equation of the exponential form $e^{\lambda x}$, we get the characteristic equation

$$\lambda^4 + 4\beta^4 = 0\,,$$

of roots $\lambda_1, \lambda_2, \lambda_3, \lambda_4 = (\pm 1 \pm \mathrm{i})\beta$.

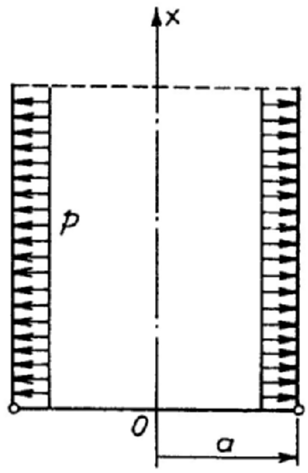

Figure 2. 9. Circular cylindrical bunker

Hence the general solution of the non-homogeneous equation is

$$w = -\frac{p}{4K\beta^4}\left[1 + \mathrm{e}^{-\beta x}\left(A\cos\beta x + B\sin\beta x\right) + \mathrm{e}^{\beta x}\left(C\cos\beta x + D\sin\beta x\right)\right],$$

where A, B, C, D are integration constants.
For $x \to \infty$, the conditions are satisfied only if $C = D = 0$. One obtains thus

$$w = -\frac{p}{4K\beta^4}\left[1 + \mathrm{e}^{-\beta x}\left(A\cos\beta x + B\sin\beta x\right)\right],$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{p}{4K\beta^3}\mathrm{e}^{-\beta x}\left[(A - B)\cos\beta x + (A + B)\sin\beta x\right],$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{p}{2K\beta^2}\mathrm{e}^{-\beta x}\left(B\cos\beta x - A\sin\beta x\right).$$

The conditions at the bottom $x = 0$ lead to

$$w(0) = -\frac{p}{4K\beta^4}(1 + A) = 0,$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(0) = \frac{p}{2K\beta^2}B = 0,$$

yielding $A = -1$, $B = 0$. The general solution becomes

$$w = -\frac{p}{4K\beta^4}\left(1 - \mathrm{e}^{-\beta x}\cos\beta x\right),$$

$$M_x = \frac{p}{2\beta^2} e^{-\beta x} \sin \beta x \,.$$

Application 2.15

Problem. A circular cylindrical tank of vertical axis is subjected to an internal pressure p. Assuming that at both the bottom ($x = 0$) and the upper end ($x = l$) the envelope is articulated to rigid plates, determine the deflections w and the bending moments $M_x = -K\mathrm{d}^2 w / \mathrm{d}x^2$, K being the bending rigidity of the plate.

Mathematical model. The differential equation of the deflection is given by (see also the previous application)

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + 4\beta^4 w = -\frac{p}{K}, \tag{a}$$

where β is a constant damping coefficient, and the boundary conditions are $w = 0$, $\mathrm{d}^2 w / \mathrm{d}x^2 = 0$ for $x = 0$ and $x = l$.

Solution. We may choose the particular solution $w_p = -p / 4K\beta^4$ for the non-homogeneous equation. In this case, it is convenient to express the general solution of the associated homogeneous equation in terms of hyperbolic functions, which are linear combinations of the exp-functions. We finally get the general solution of the non-homogeneous equation (a) in the form

$$\begin{aligned} w = -\frac{p}{4K\beta^4}\big(&1 + C_1 \cosh \beta x \cos \beta x + C_2 \cosh \beta x \sin \beta x \\ &+ C_3 \sinh \beta x cos \beta x + C_4 \sinh \beta x \sin \beta x\big). \end{aligned} \tag{b}$$

Its second derivative is

$$\begin{aligned} \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{p}{2K\beta^2}\big(&C_4 \cosh \beta x \cos \beta x - C_3 \cosh \beta x \sin \beta x \\ &+ C_2 \sinh \beta x \cos \beta x - C_1 \sinh \beta x \sin \beta x\big). \end{aligned} \tag{c}$$

From $w(0) = 0$, we obtain $C_1 = -1$.

Analogously, the condition $\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2}(0) = 0$ leads to $C_4 = 0$.

The same conditions for $x = l$ yield the relationships

$$\begin{aligned} C_2 \cosh \beta l \sin \beta l + C_3 \sinh \beta l \cos \beta l &= -1 + \cosh \beta l \cos \beta l, \\ C_2 \sinh \beta l \cos \beta l - C_3 \cosh \beta l \sin \beta l &= -\sinh \beta l \sin \beta l, \end{aligned}$$

therefore

$$C_2 = -\frac{\sin\beta l(\cosh\beta l - \cos\beta l)}{\sinh^2\beta l + \sin^2\beta l},$$
$$C_3 = -\frac{\sinh\beta l(\cosh\beta l - \cos\beta l)}{\sinh^2\beta l + \sin^2\beta l}. \tag{d}$$

Eventually, one obtains

$$w = -\frac{p}{4K\beta^4}\left(1 - \cosh\beta x\cos\beta x + C_2\cosh\beta x\sin\beta x + C_3\sinh\beta x\cos\beta x\right),$$

$$M_x = \frac{p}{2\beta^2}\left(-C_3\cosh\beta x\sin\beta x + C_2\sinh\beta x\cos\beta x + \sinh\beta x\sin\beta x\right),$$

where C_2 and C_3 are given by (d).

Choosing the origin of the co-ordinates x at the middle of the height (for the sake of symmetry), we take $C_2 = C_3 = 0$ in (b) and we have

$$w = -\frac{p}{4K\beta^4}\left(1 + C_1\cosh\beta x\cos\beta x + C_4\sinh\beta x\sin\beta x\right),$$
$$\frac{d^2w}{dx^2} = -\frac{p}{4K\beta^4}\left(C_4\cosh\beta x\cos\beta x - C_1\sinh\beta x\sin\beta x\right). \tag{e}$$

Applying now the conditions $w(l/2) = 0, \dfrac{d^2w}{dx^2}(l/2) = 0$, it results

$$C_1\cosh\frac{\beta l}{2}\cos\frac{\beta l}{2} + C_4\sinh\frac{\beta l}{2}\sin\frac{\beta l}{2} = -1,$$
$$-C_1\sinh\frac{\beta l}{2}\sin\frac{\beta l}{2} + C_4\cosh\frac{\beta l}{2}\cos\frac{\beta l}{2} = 0,$$

which means

$$C_1 = -\frac{\cosh^2\dfrac{\beta l}{2}\cos^2\dfrac{\beta l}{2}}{\cosh^2\dfrac{\beta l}{2}\cos^2\dfrac{\beta l}{2} + \sinh^2\dfrac{\beta l}{2}\sin^2\dfrac{\beta l}{2}},$$
$$C_4 = -\frac{\sinh\dfrac{\beta l}{2}\cosh\dfrac{\beta l}{2}\sin\dfrac{\beta l}{2}\cos\dfrac{\beta l}{2}}{\cosh^2\dfrac{\beta l}{2}\cos^2\dfrac{\beta l}{2} + \sinh^2\dfrac{\beta l}{2}\sin^2\dfrac{\beta l}{2}}.$$

Application 2.16

Problem. Determine the radial displacements u, the radial stress σ_r and the annular stress σ_φ for a circular (or annular) disk acted upon by an axially symmetric load. Application for the annular disk in Fig.2.14.

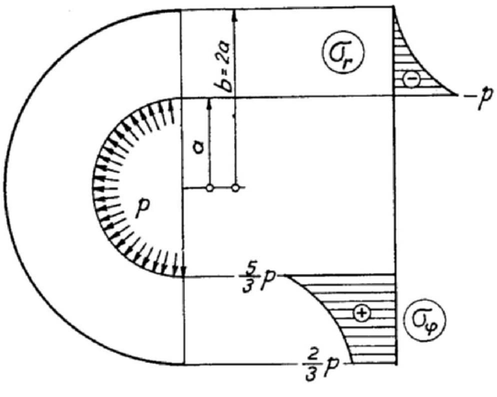

Figure 2. 10. Annular disk

Mathematical model. The displacements u verify the linear second order ODE

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0\,, \tag{a}$$

where r is the vector radius. The radial stress and the annular stress are given by

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) = 0\,,\ \sigma_\varphi = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) = 0\,, \tag{b}$$

respectively.

Solution. The homogeneous linear ODE is of Euler type; we search for solutions of the form $u = r^\lambda$. The characteristic equation is

$$\lambda^2 - 1 = 0$$

and has the roots $\lambda_1, \lambda_2 = \pm 1$. The general solution for the radial displacement is therefore

$$u = Ar + \frac{B}{r}\,.$$

The stresses are given accordingly by

$$\sigma_r = \frac{E}{1-\nu^2}\left(A(1+\nu) - B(1-\nu)\frac{1}{r^2} \right), \tag{c}$$

$$\sigma_\varphi = \frac{E}{1-\nu^2}\left(A(1+\nu) + B(1-\nu)\frac{1}{r^2} \right). \tag{d}$$

In the particular case of an annular disk (Fig.2.14), one determines the integration constants using the boundary conditions $\sigma_r(a) = -p$ and $\sigma_r(b) = 0$. It results

$$A = \frac{(1-\nu)p}{E}\frac{a^2}{b^2 - a^2}, \; B = \frac{(1+\nu)p}{E}\frac{a^2 b^2}{b^2 - a^2},$$

so that the stresses read

$$\sigma_r = \frac{pa^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right), \; \sigma_\varphi = \frac{pa^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right),$$

Their distribution is given in Fig.2.14.

Application 2.17

Problem. Determine the buckling critical force of a cantilever bar the moment of inertia of which has a variation given by $I_x = I_0 (x/a)^2$ (Fig.2.15).

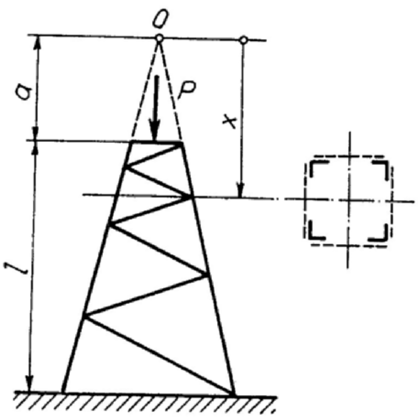

Figure 2. 11. Pillar of variable cross section subjected to compression

Mathematical model. The differential equation of the deformed axis is

$$EI_x \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + Pw = 0; \tag{a}$$

taking into account the expression of I_x, one obtains

$$x^2 \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \frac{Pa^2}{EI_0} w = 0 . \tag{b}$$

To this ODE we add the boundary conditions $w(a) = 0$ and $\frac{\mathrm{d}w}{\mathrm{d}x}(a+l) = 0$.

Solution. The linear homogeneous ODE is of Euler type, thus we search for solutions of the form $w = x^{\lambda}$; the corresponding characteristic equation

$$\lambda(\lambda - 1) + \frac{Pa^2}{EI_0} = 0$$

has the roots $\lambda_1, \lambda_2 = 1/2 \pm \mathrm{i}\beta$, with the notation

$$\beta = \sqrt{\frac{Pa^2}{EI_0} - \frac{1}{4}} . \tag{c}$$

Thus, the solution may be expressed in the form

$$w = \sqrt{\frac{x}{a}} \left[A \sin\left(\beta \ln \frac{x}{a} \right) + B \cos\left(\beta \ln \frac{x}{a} \right) \right],$$

$$\frac{\mathrm{d}w}{\mathrm{d}x} = \frac{1}{\sqrt{ax}} \left[\left(\frac{A}{2} - B\beta \right) \sin\left(\beta \ln \frac{x}{a} \right) + \left(\frac{B}{2} + A\beta \right) \cos\left(\beta \ln \frac{x}{a} \right) \right].$$

The boundary conditions lead to $B = 0$ and to

$$\frac{A}{2\sqrt{a(a+l)}} \left[\sin\left(\beta \ln \frac{a+l}{a} \right) + 2\beta \cos\left(\beta \ln \frac{a+l}{a} \right) \right] = 0 .$$

As $A \neq 0$, one obtains the equation

$$\tan\left(\beta \ln \frac{a+l}{a} \right) + 2\beta = 0 . \tag{d}$$

For a given ratio l/a, one obtains the minimal value β from (d); the relationship (c) determines the critical force

$$P_{cr} = \frac{EI_0}{a^2} \left(\beta^2 + \frac{1}{4} \right) = \frac{\pi^2 EI_0}{(\mu l)^2} . \tag{e}$$

To solve numerically the equation (d), we write it in the form

$$\tan\left(\beta \ln \frac{a+l}{a}\right) + \beta \ln \frac{a+l}{a} \cdot \frac{2}{\ln \frac{a+l}{a}} = 0 .$$

Denoting by

$$u = \beta \ln \frac{a+l}{a} , \ \gamma = -\frac{2}{\ln \frac{a+l}{a}} ,$$

the above equation is put in the form

$$\frac{\tan u}{u} = \gamma ;$$

to solve it, one may use the Table 1.2. For various values of the ratio a/l, the values of u, β and μ are given in Table 2.5.

Table 2.5. The values of u, β and μ for $a / l \in [0.2, 10]$

a/l	$u_{\min}$ (rad)	β	μ
0.2	1.993206	1.112429	0.51517
0.5	1.858220	1.691425	0.89059
1	1.764719	2.545951	1.21083
2	1.690173	4.168480	1.49658
3	1.657368	5.761110	1.62980
5	1.626775	8.922560	1.75772
10	1.600561	16.793180	1.86993

The variation of the buckling length $l_b = \mu l$ as a function of the ratio a/l is given in Fig.2.16.

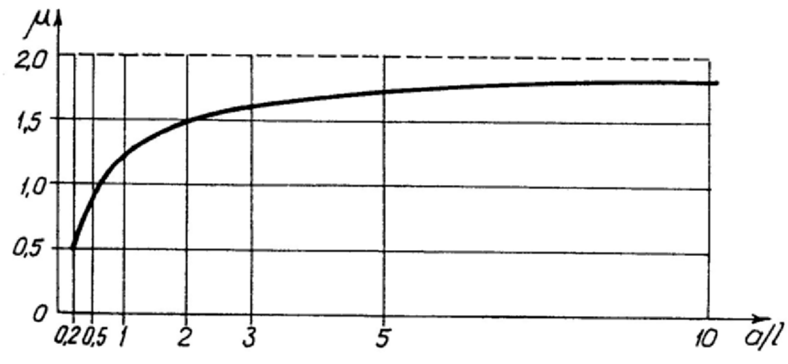

Figure 2. 12. Variation of the buckling length as a function of the ratio a/l

For $a/l \to \infty$ one obtains $\mu = 2$, that is the value corresponding to a cantilever bar of constant cross section.
The variable moment of inertia considered above corresponds approximately to a cross section formed by four corner irons, the moment of inertia of which with respect to the own axis is negligible with respect to the product $A d^2/4$ (A is the area of the cross section of a corner iron and d is the distance between the centers of gravity of two adjacent corner irons).

Application 2.18

Problem. Study the symmetric state of stress with respect to the pole in plane elasticity.

Mathematical model. The plane state of stress in an axially symmetric case is governed by the differential equation

$$\Delta\Delta F = 0 , \tag{a}$$

where $F = F(r)$ is a potential function.

Solution. In polar co-ordinates, Laplace's operator is

$$\Delta = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} ,$$

in the axially symmetric case. Successive differentiations

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}\right) = \frac{\mathrm{d}^3 F}{\mathrm{d}r^3} + \frac{1}{r}\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} - \frac{1}{r^2}\frac{\mathrm{d}F}{\mathrm{d}r} ,$$

$$\frac{\mathrm{d}^2 F}{\mathrm{d}r^2}\left(\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}\right) = \frac{\mathrm{d}^4 F}{\mathrm{d}r^4} + \frac{1}{r}\frac{\mathrm{d}^3 F}{\mathrm{d}r^3} - \frac{2}{r^2}\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{2}{r^3}\frac{\mathrm{d}F}{\mathrm{d}r}$$

lead to the biharmonic equation

$$\frac{\mathrm{d}^4 F}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3 F}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}F}{\mathrm{d}r} = 0 ,$$

which finally yields an ODE of Euler type

$$r^4 \frac{\mathrm{d}^4 F}{\mathrm{d}r^4} + 2r^3 \frac{\mathrm{d}^3 F}{\mathrm{d}r^3} - r^2 \frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + r\frac{\mathrm{d}F}{\mathrm{d}r} = 0 . \tag{b}$$

Searching for solutions of the form r^λ, one obtains the characteristic equation

$$\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) + 2\lambda(\lambda - 1)(\lambda - 2) - \lambda(\lambda - 1) + \lambda = \lambda^2(\lambda - 2)^2 = 0 ,$$

with the double roots $\lambda_1, \lambda_2 = 0$, $\lambda_3, \lambda_4 = 2$.
The general solution is then

$$F(r) = A \ln r + Br^2 \ln r + Cr^2 + D\,,$$

where the integration constants A, B, C, D must be determined from the boundary conditions.
Due to the logarithmic terms, $r = 0$ is a pole. In the case of a circular disk one must take $A = B = 0$, while in the case of an annular disk one has $A, B \neq 0$.

Application 2.19

Problem. Determine the deflections w of a circular (or annular) plate of constant thickness in case of an axially symmetric state of stress and strain with respect to its centre.
Particular case: a simply supported circular plate acted upon by a uniformly distributed load $p = const$ (Fig.2.17).

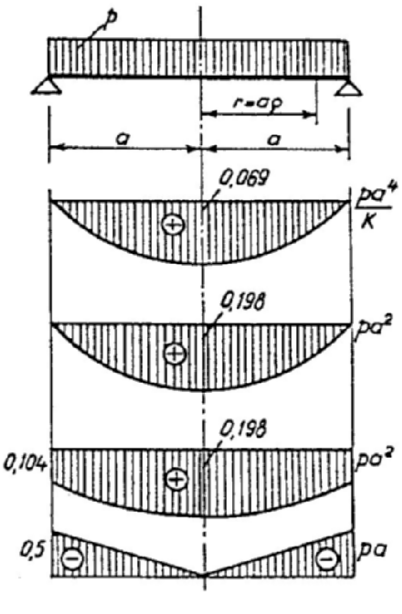

Figure 2. 13. Circular plate acted upon by a uniformly distributed load. Diagrams of the deflections w, the radial moments M_r, the annular moments M_φ and the shearing force T_r

Mathematical model. The deflections satisfy the ODE

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right) = \frac{p(r)}{K}\,, \tag{a}$$

where $p(r)$ is the external load and $K = \text{const}$ is the bending rigidity of the plate.
The bending moments are given by

$$M_r = -K\left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{\nu}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right),\ M_\varphi = -K\left(\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} + \nu\frac{\mathrm{d}^2 w}{\mathrm{d}r^2}\right), \tag{b}$$

where ν is the coefficient of transverse contraction of the material.

Solution. As we have seen in Appl.2.18, the general solution of the homogeneous equation (a) is of the form

$$w_c = A\ln r + Br^2\ln r + Cr^2 + D.$$

To obtain a particular solution of the non-homogeneous equation, we notice that

$$\Delta w = \frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}w}{\mathrm{d}r}$$

or

$$r\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{\mathrm{d}w}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}w}{\mathrm{d}r}.$$

The equation (a) may be thus written

$$r\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2 w_p}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w_p}{\mathrm{d}r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}^2 w_p}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w_p}{\mathrm{d}r}\right) = \frac{p(r)r}{K};$$

after integration, we get

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}^2 w_p}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}w_p}{\mathrm{d}r}\right) = \frac{1}{K}\int p(r)\mathrm{d}r.$$

Multiplying both members by r, it follows

$$r\frac{\mathrm{d}^2 w_p}{\mathrm{d}r^2} + \frac{\mathrm{d}w_p}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}w_p}{\mathrm{d}r}\right) = \frac{1}{K}r\int\frac{\mathrm{d}r}{r}\int p(r)r\,\mathrm{d}r.$$

A new integration leads to

$$r\frac{\mathrm{d}w_p}{\mathrm{d}r} = \frac{1}{K}\int r\mathrm{d}r\int\frac{\mathrm{d}r}{r}\int p(r)r\,\mathrm{d}r.$$

Thus, the general solution of the equation (a) reads

$$w(r) = w_p + A\ln r + Br^2\ln r + Cr^2 + D,$$

whence

$$w_p = \frac{1}{K}\int \frac{\mathrm{d}r}{r}\int r\mathrm{d}r\int \frac{\mathrm{d}r}{r}\int p(r)r\,\mathrm{d}r \,. \tag{c}$$

The bending moments are of the form

$$M_r = -K\left[w_p'' + \frac{\nu}{r}w_p' - (1-\nu)\frac{A}{r^2} + 2(1+\nu)B\ln r + (3+\nu)B + 2(1+\nu)C \right],$$

$$M_\varphi = -K\left[\frac{1}{r}w_p' + \nu w_p'' + (1-\nu)\frac{A}{r^2} + 2(1+\nu)B\ln r + (1+3\nu)B + 2(1+\nu)C \right]. \tag{d}$$

In the particular case of a simply supported plate one must take $A = B = 0$, to have a finite displacement for $r = 0$; on the boundary one has $w(a) = 0$, $M_r(a) = 0$.
The general solution becomes

$$w = w_p + Cr^2 + D\,,$$

$$M_r = -K\left[w_p'' + \frac{\nu}{r}w_p' + 2(1+\nu)C \right],$$

$$M_\varphi = -K\left[\frac{1}{r}w_p' + \nu w_p'' + 2(1+\nu)C \right]. \tag{e}$$

If $p(r) = p = \mathrm{const}$, then the particular solution becomes

$$w_p = \frac{p}{K}\int \frac{\mathrm{d}r}{r}\int r\,\mathrm{d}r\int \frac{\mathrm{d}r}{r}\int r\,\mathrm{d}r = \frac{pr^4}{64K}\,,$$

$$w_p' = \frac{pr^3}{16K}\,,\quad w_p'' = \frac{3pr^2}{16K}\,.$$

The formulae (e) take the form

$$w = \frac{pr^4}{64K} + Cr^2 + D\,,$$

$$M_r = -K\left[(3+\nu)\frac{pr^2}{16K} + 2(1+\nu)C \right],$$

$$M_\varphi = -K\left[(1+3\nu)\frac{pr^2}{16K} + 2(1+\nu)C \right]. \tag{f}$$

Introducing the boundary conditions

$$(3+\nu)\frac{pa^2}{16K}+2(1+\nu)C=0\,,\quad \frac{pa^4}{64K}+Ca^2+D=0\,,$$

one obtains the integration constants

$$C=-\frac{3+\nu}{1+\nu}\frac{pa^3}{32K}\,,\ D=\frac{5+\nu}{1+\nu}\frac{pa^4}{64K}\,.$$

The formulae (f) thus become

$$w=\frac{pr^4}{64K}-\frac{3+\nu}{1+\nu}\frac{pa^2r^2}{32K}+\frac{5+\nu}{1+\nu}\frac{pa^4}{64K}=\frac{pa^4}{64K}\left(1-\frac{r^2}{a^2}\right)\left(\frac{5+\nu}{1+\nu}-\frac{r^2}{a^2}\right),$$

$$M_r=(3+\nu)\frac{pa^2}{16K}\left(1-\frac{r^2}{a^2}\right),$$

$$M_\varphi=\frac{pa^2}{16}\left[3+\nu-(1+3\nu)\frac{r^2}{a^2}\right],\ r\in[0,a]\,.$$

A global equilibrium (for a plate δ of radius r)

$$T_r\cdot 2\pi r=-p\pi a^2$$

leads to

$$T_r=-\frac{pr}{2}\,.$$

The variations of w, M_r, M_φ, and T_r are given in Fig.2.17.

Application 2.20

Problem. Study the deflection w of a beam on an elastic medium, of variable response depending on the law

$$k=\frac{\alpha l^4}{(x+\beta l)^4}\,, \tag{a}$$

where x is the abscissa (measured from the left end of the beam), l is the bar length, α and β are parameters characterizing the variability of k.

Mathematical model. In the absence of the distributed loads, the differential equation of the problem is of the form

$$\frac{\mathrm{d}^4w}{\mathrm{d}x^4}+\frac{k}{EI}w=0\,, \tag{b}$$

where $EI = \text{const}$ is the bending rigidity of the beam. We assume that

$$\gamma = \alpha l^4 / EI \in [0,1]. \tag{c}$$

Solution. Replacing (a) in (b), one obtains the linear ODE

$$(x + \beta l)^4 \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \gamma w = 0. \tag{d}$$

The equation (d) is of Euler type; assuming a solution of the form $w = (x + \beta l)^r$, we get the characteristic equation

$$r(r-1)(r-2)(r-3) + \gamma = 0.$$

If we write this equation in the form

$$(r^2 - 3r)(r^2 - 3r + 2) + 1 = (r^2 - 3r)^2 + 2(r^2 - 3r) + 1 = (r^2 - 3r + 1)^2 = 1 - \gamma,$$

then, taking into account the interval mentioned in (c), we get the real and distinct roots

$$r_1, r_2 = \frac{3}{2} \pm \sqrt{\frac{5}{4} + \sqrt{1-\gamma}}, \; r_3, r_4 = \frac{3}{2} \pm \sqrt{\frac{5}{4} - \sqrt{1-\gamma}}.$$

The general solution of the ODE (d) reads

$$w = C_1 (x + \beta l)^{r_1} + C_2 (x + \beta l)^{r_2} + C_3 (x + \beta l)^{r_3} + C_4 (x + \beta l)^{r_4}, \tag{e}$$

where the integration constants C_i, $i = 1,2,3,4$, must be determined by boundary conditions.

Application 2.21

Problem. Consider a circular cylindrical tank of radius R and height l, the thickness of which has a parabolic variation between the values δ_s (at the upper end) and δ_0 (at the bottom of the tank). Determine the general expression of the deflection w.

Mathematical model. The deformation of tank walls is governed by the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(K_x \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \right) + \frac{E\delta_x}{a^2} w = -Z, \tag{a}$$

where Z is the normal component of the internal load, due to a liquid of unit weight γ, δ_x is the wall thickness and $K_x = E\delta_x^3 / 12(1-\nu^2)$ is the bending rigidity at the abscissa x and E, ν are the elastic constants of the material.

Solution. One chooses the origin of the *x*-co-ordinates at the section where – theoretically – the wall thickness vanishes (Fig.2.18). The thickness of the wall is given by the parabolic law

$$\delta_x = \delta_0 \frac{x^2}{(a+l)^2}, \tag{b}$$

where a is the distance from the origin to the upper end of the tank. Taking into account that $\delta_x = \delta_s$ for $x = a$, it results

$$a = \frac{1}{\sqrt{\frac{\delta_0}{\delta_x}} - 1}. \tag{c}$$

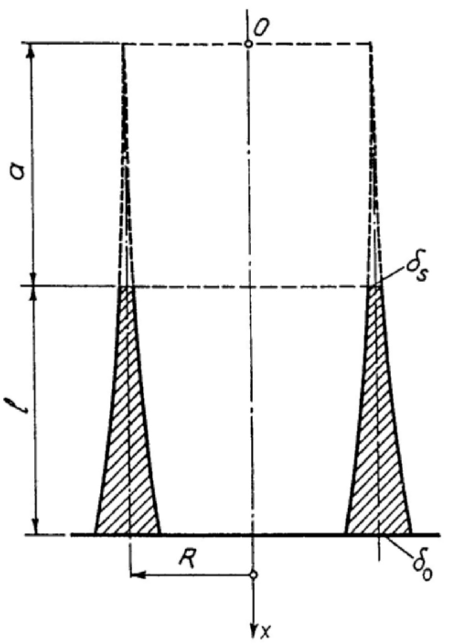

Figure 2. 14. Circular cylindrical tank the thickness of which has a parabolic variation

The loading *Z* is of the form

$$Z = \gamma(a - x), \tag{d}$$

as we have $Z(a) = 0$ and for $x = a + l$ it results $Z = -\gamma l$, corresponding to the hydrostatic pressure ($Z > 0$ towards the interior of the tank).
The bending rigidity becomes

$$K_x = \frac{E\delta_0^3}{12\left(1-\nu^2\right)} \frac{x^6}{(a+l)^6} = \frac{K_0}{(a+l)^6} x^6 , \; K_0 = \frac{E\delta_0^3}{12\left(1-\nu^2\right)} , \tag{e}$$

so that the differential equation (a) reads

$$\frac{K_0}{(a+l)^6} \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(x^6 \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \right) + \frac{E\delta_0}{R^2} \frac{1}{(a+l)^2} x^2 w = \gamma(a-x)$$

and further

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(x^6 \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \right) + \frac{E\delta_0}{K_0 R^2} (a+l)^4 x^2 w = \frac{\gamma(a+l)^6}{K_0} (a-x).$$

Differentiating the first term in the previous equation, we have

$$x^4 \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + 12x^3 \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} + 30x^2 \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 4\beta^4 w = \frac{\gamma(a+l)^6}{K_0} \frac{a-x}{x^2} , \tag{f}$$

with the notation

$$4\beta^4 = \frac{E\delta_0}{K_0 R^2} (a+l)^4 = \frac{12\left(1-\nu^2\right)}{R^2 \delta_0^2} (a+l)^4 . \tag{g}$$

The equation (f) is a linear, non-homogeneous ODE of Euler type.
For this equations, we firstly search a particular solution of the form

$$w_p = \frac{A(a-x)}{x^2} , \tag{h}$$

where A is a constant to be specified. Introducing this in (f), we obtain

$$A = \frac{\gamma(a+l)^6}{K_0} \frac{1}{4\left(3+\beta^4\right)} , \tag{i}$$

so that the particular solution is

$$w_p = \frac{\gamma(a+l)^6}{4K_0\left(3+\beta^4\right)} \frac{a-x}{x^2} . \tag{j}$$

At the upper end ($x = a$) we have $w_{p,s} = 0$, while at the bottom ($x = a + l$) we have

$$w_{p,0} = -\frac{\gamma l (a+l)^4}{4K_0\left(3+\beta^4\right)} .$$

The general solution w_0 of the homogeneous equation

$$x^4 \frac{d^4 w_0}{dx^4} + 12x^3 \frac{d^3 w_0}{dx^3} + 30x^2 \frac{d^2 w_0}{dx^2} + 4\beta^4 w_0 = 0 \qquad (k)$$

is searched in the form $w_0 = x^r$, where the parameter r must be specified. Differentiating

$$\frac{d^2 w_0}{dx^2} = r(r-1)x^{r-2}, \ \frac{d^3 w_0}{dx^3} = r(r-1)(r-2)x^{r-3}, \ \frac{d^4 w_0}{dx^4} = r(r-1)(r-2)(r-3)x^{r-4}$$

and introducing in (k), we get the associated characteristic equation

$$r(r-1)(r-2)(r-3) + 12r(r-1)(r-2) + 30r(r-1) + 4\beta^4 = 0,$$

which may be also written in the form

$$\left(r^2 + 3r\right)^2 - 4\left(r^2 + 3r\right) + 4\beta^4 = 0.$$

To solve this equation, we write further

$$\left(r^2 + 3r\right)^2 - 4\left(r^2 + 3r\right) + 4 = 4\left(1 - \beta^4\right) = -4\left(\beta^4 - 1\right).$$

So, the roots are $r_1, r_2 = p_1 \pm \mathrm{i}q$, $r_3, r_4 = p_2 \pm \mathrm{i}q$, where

$$p_1 = -\frac{3}{2} + \frac{1}{2\sqrt{2}}\sqrt{\sqrt{225 + 64\beta^4} + 17} > 0,$$

$$p_2 = -\frac{3}{2} - \frac{1}{2\sqrt{2}}\sqrt{\sqrt{225 + 64\beta^4} + 17} < 0,$$

$$q = \frac{1}{2\sqrt{2}}\sqrt{\sqrt{225 + 64\beta^4} - 17} > 0.$$

Taking into account the complex form of the roots $r_i, i = \overline{1,4}$, this solution may be written as

$$\begin{aligned} w_0 &= C_1' x^{p_1 + \mathrm{i}q} + C_2' x^{p_1 - \mathrm{i}q} + C_3' x^{p_2 + \mathrm{i}q} + C_4' x^{p_2 - \mathrm{i}q} \\ &= x^{p_1}\left(C_1' x^{\mathrm{i}q} + C_2' x^{-\mathrm{i}q}\right) + x^{p_2}\left(C_3' x^{\mathrm{i}q} + C_4' x^{-\mathrm{i}q}\right), \end{aligned}$$

where $C_i', i = \overline{1,4}$, are integration constants, or, equivalently,

$$w_0 = x^{p_1}\left[C_1 \cos(q \ln x) + C_2 \sin(q \ln x)\right] + x^{p_2}\left[C_3 \cos(q \ln x) + C_4 \sin(q \ln x)\right]. \qquad (l)$$

The general solution of the ODE (a) is thus $w = w_0 + w_p$, where w_0 is given by (l) and w_p, by (j). The constants $C_i, i = \overline{1,4}$, will be determined from convenient boundary conditions, put at both ends of the tank.

The above solution is suggested by a study of E. Steuermann.

Application 2.22

Problem. Consider a cantilever column of length l and minimal flexural rigidity *EI*, subjected to axial compression forces *P* and immersed in an elastic medium of response constant k, corresponding to a Winkler model. Determine the deformed axis and the critical load (see).

Mathematical model. The action of the elastic medium is equivalent to the transverse load

$$g(x) = -kw, \qquad k \geq 0, \tag{a}$$

where for $k = 0$ the elastic medium does not exist and for $k \to \infty$ this a rigid one and the bifurcation of the equilibrium does no more take place; $w(x)$ is the transverse displacement of the axis of the column in the cross section of abscissa x (Fig.2.19). This model corresponds e.g. to piles driven in the earth, which can suffer displacements at the upper end.

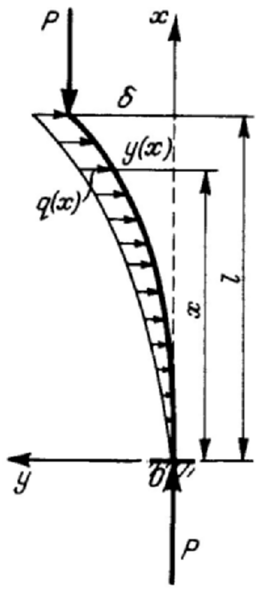

Figure 2. 15. Cantilever column in an elastic medium

The bending moment in a current cross section is given by

$$M(x) = -P(\delta - y) + M_q(x),$$

where M_q is the bending moment due to the load *q*; noting that

$$EIy'' = -M, \quad M''_q = -q = ky$$

we can write

$$EIy^{IV} + Py'' + ky = 0 .$$

By using the notations

$$\frac{P}{EI} = 2a^2, \quad \frac{k}{EI} = b^4, \tag{b}$$

we obtain the ODE of the problem

$$y^{IV} + 2a^2 y'' + b^4 y = 0 . \tag{c}$$

The boundary conditions are two-point conditions of the form

$$y(0) = 0, \quad y'(0) = 0 \tag{d}$$

for the built-in cross section and of the form

$$M = 0, \quad T + Py' = 0 ,$$

where $T' = -M'$ is the shearing force, i.e. of the form

$$y''(l) = 0, \quad y'''(l) + 2a^2 y'(l) = 0 \tag{d$'$}$$

at the free end of the column.
The deflection curve of the column is only specified up to a multiplicative factor, because the phenomenon has been linearized.

Solution. We introduce the conventional load

$$P_0 = 2\sqrt{EIk} \tag{e}$$

and the critical Euler type load

$$P_{\text{cr}} = \frac{\pi^2 EI}{l_{\text{cr}}^2}, \tag{e$'$}$$

where l_{cr} is the critical length in case of buckling.
If $P_{cr} < P_0$, i.e. if $a^2 < b^2$, then we can use the solution

$$\begin{aligned} y(x) = C_1 \cosh \beta_2 x \cos \beta_1 x + C_2 \cosh \beta_2 x \sin \beta_1 x \\ + C_3 \sinh \beta_2 x \cos \beta_1 x + C_4 \sinh \beta_2 x \sin \beta_1 x, \end{aligned} \tag{f}$$

where

$$\beta_{1,2} = \frac{1}{2}\sqrt{2\sqrt{\frac{k}{EI}} \pm \frac{P_{\mathrm{cr}}}{EI}} = \frac{1}{\sqrt{2}}\sqrt{b^2 \pm a^2} = \frac{1}{\sqrt{2}}\, b\sqrt{1 \pm \frac{P_{\mathrm{cr}}}{P_0}}\ ; \qquad (\mathrm{f}')$$

the boundary conditions (d′) become

$$y''(l) = 0, \quad y'''(l) + 2\left(\beta_1^2 - \beta_2^2\right) y'(l) = 0\ . \qquad (\mathrm{d}'')$$

The boundary conditions (d) lead to $C_1 = 0$, $\beta_1 C_2 + \beta_2 C_3 = 0$, so that

$$\begin{aligned} y'(x) &= (\beta_2 C_2 - \beta_1 C_3)\sinh\beta_2 x \sin\beta_1 x + C_4(\beta_2 \cosh\beta_2 x \sin\beta_1 x + \beta_1 \sinh\beta_2 x \cos\beta_1 x) \\ &= \frac{1}{\beta_2}\left(\beta_1^2 + \beta_2^2\right) C_2 \sinh\beta_2 x \sin\beta_1 x + C_4(\beta_2 \cosh\beta_2 x \sin\beta_1 x + \beta_1 \sinh\beta_2 x \cos\beta_1 x); \end{aligned}$$

we obtain

$$\begin{aligned} y''(x) &= \frac{1}{\beta_2}\left(\beta_1^2 + \beta_2^2\right) C_2 (\beta_2 \cosh\beta_2 x \sin\beta_1 x + \beta_1 \sinh\beta_2 x \cos\beta_1 x) \\ &\quad - C_4\left[\left(\beta_1^2 - \beta_2^2\right)\sinh\beta_2 x \cos\beta_1 x - 2\beta_1\beta_2 \cosh\beta_2 x \cos\beta_1 x\right], \\ y'''(x) &= -\frac{1}{\beta_2}\left(\beta_1^2 + \beta_2^2\right) C_2 \left[\left(\beta_1^2 - \beta_2^2\right)\sinh\beta_2 x \sin\beta_1 x - 2\beta_1\beta_2 \cosh\beta_2 x \cos\beta_1 x\right] \\ &\quad - C_4\left[\left(3\beta_1^2 - \beta_2^2\right)\beta_2 \cosh\beta_2 x \sin\beta_1 x + \left(\beta_1^2 - 3\beta_2^2\right)\beta_1 \sinh\beta_2 x \cos\beta_1 x\right]. \end{aligned}$$

The conditions (d″) lead to a linear algebraic system

$$\begin{aligned} &\left(\beta_1^2 + \beta_2^2\right) C_2 (\beta_2 \cosh\beta_2 l \sin\beta_1 l + \beta_1 \sinh\beta_2 l \cos\beta_1 l) \\ &- \beta_2 C_4\left[\left(\beta_1^2 - \beta_2^2\right)\sinh\beta_2 l \sin\beta_1 l - 2\beta_1\beta_2 \cosh\beta_2 l \cos\beta_1 l\right] = 0, \\ &C_2\left[\left(\beta_1^2 - \beta_2^2\right)\sinh\beta_2 l \sin\beta_1 l + 2\beta_1\beta_2 \cosh\beta_2 l \cos\beta_1 l\right] \\ &- \beta_2\, C_4(\beta_2 \cosh\beta_2 l \sin\beta_1 l - \beta_1 \sinh\beta_2 l \cos\beta_1 l) = 0\,, \end{aligned}$$

where we admit that $\beta_1^2 + \beta_2^2 = b^2 \neq 0$ (if not, we have $k = 0$, that is absence of the elastic medium). To obtain a bifurcation of the equilibrium, i.e. a deformation of the axis of the column ($C_2, C_4 \neq 0$), the equation

$$\begin{aligned} &\left(\beta_1^2 + \beta_2^2\right)\left(\beta_2^2 \cosh^2\beta_2 l \sin^2\beta_1 l - \beta_1^2 \sinh^2\beta_2 l \cos^2\beta_1 l\right) \\ &= \left(\beta_1^2 - \beta_2^2\right)\sinh^2\beta_2 l \sin^2\beta_1 l - 4\beta_1^2\beta_2^2 \cosh^2\beta_2 l \cos^2\beta_1 l \end{aligned} \qquad (\mathrm{g})$$

must be verified; this is the characteristic equation, which leads to the critical load.
The deflection curve is given by

$$\begin{aligned}y(x)=&\frac{\delta}{\beta_1^2\sinh^2\beta_2 l+\beta_2^2\sin^2\beta_1 l}\{\beta_1[\beta_1\sinh\beta_2 l\cos\beta_1(l-x)\\&-\sinh\beta_2 x-\beta_2[\beta_1\cos\beta_1 l\sinh\beta_2(l-x)\\&-\beta_2\sin\beta_1 l\cosh\beta_2(l-x)]\sin\beta_1 x\},\end{aligned}\tag{h}$$

where we have introduced the deflection $\delta = y(l)$ at the free end of the column; the bending moment is

$$\begin{aligned}M(x)=&\frac{P\delta}{2(\beta_1^2-\beta_2^2)(\beta_1^2\sinh^2\beta_2 l+\beta_2^2\sin^2\beta_1 l)}\\&\times\{(\beta_1^2+\beta_2^2)(\beta_2\cosh\beta_2 l\sin\beta_1 l-\beta_1\sinh\beta_2 l\cos\beta_1 l)\\&\times(\beta_2\cosh\beta_2 x\sin\beta_1 x+\beta_1\sinh\beta_2 x\cos\beta_1 x)\\&-[(\beta_1^2-\beta_2^2)\sinh\beta_2 l\sin\beta_1 l+2\beta_1\beta_2\cosh\beta_2 l\cos\beta_1 l][(\beta_1^2-\beta_2^2)\\&\times\sinh\beta_2 x\sin\beta_1 x-2\beta_1\beta_2\cosh\beta_2 x\cos\beta_1 x]\}.\end{aligned}\tag{i}$$

The condition $M(l) = 0$ is verified if we take into account the equation (g); the moment in the built-in cross section is given by

$$M(x)=\frac{\beta_1\beta_2 P\delta}{(\beta_1^2-\beta_2^2)(\beta_1^2\sinh^2\beta_2 l+\beta_2^2\sin^2\beta_1 l)}.\tag{i$'$}$$

Dividing by $\cosh^2\beta_2 l\cosh^2\beta_1 l$, we notice that the characteristic equation (g) can be written in the form

$$(\beta_1^2+\beta_2^2)(\beta_2^2\tan^2\beta_1 l-\beta_1^2\tanh^2\beta_2 l)=(\beta_1^2-\beta_2^2)^2\tanh^2\beta_2 l\tan^2\beta_1 l-4\beta_1^2\beta_2^2,\tag{g$'$}$$

which is more convenient for computation; taking into account the relations $\cosh^2\beta_2 l = 1 + \sinh^2\beta_2 l$, $\cos^2\beta_1 l = 1-\sin^2\beta_1 l$, we can write

$$4\beta_1^2\beta_2^2=(\beta_1^2-3\beta_2^2)\beta_1^2\sinh^2\beta_1 l+(3\beta_1^2-\beta_2^2)\beta_2^2\sin^2\beta_1 l\tag{g$''$}$$

or

$$(\beta_1^2-\beta_2^2)^2+(3\beta_1^2-\beta_2^2)\beta_2^2\cos^2\beta_1 l=(\beta_1^2-3\beta_2^2)\beta_1^2\cosh^2\beta_2 l.\tag{g$'''$}$$

With the notation (f$'$), we have

$$4\beta_1^2\beta_2^2=b^4-a^4,\ (\beta_1^2-\beta_2^2)^2=a^4,\ \beta_1^2-3\beta_2^2=2a^2-b^2,\ 3\beta_1^2-\beta_2^2=2a^2+b^2;$$

using also the notations (b), the critical load will be determined by the equation

$$2\left[1-\left(\frac{P_{\rm cr}}{P_0}\right)^2\right]=\left(2\frac{P_{\rm cr}}{P_0}-1\right)\left(1+\frac{P_{\rm cr}}{P_0}\right)\sinh^2\beta_2 l+\left(2\frac{P_{\rm cr}}{P_0}+1\right)\left(1-\frac{P_{\rm cr}}{P_0}\right)\sin^2\beta_1 l\,, \qquad \text{(j)}$$

which can be written in the form

$$\left(1+\frac{1}{1+\dfrac{P_0}{P_{\rm cr}}}\right)\sin^2\beta_1 l-\left(1+\frac{1}{1-\dfrac{P_0}{P_{\rm cr}}}\right)\sinh^2\beta_2 l=2 \qquad \text{(j}'\text{)}$$

or by the equation

$$2\left(\frac{P_{\rm cr}}{P_0}\right)^2+\left(2\frac{P_{\rm cr}}{P_0}+1\right)\left(1-\frac{P_{\rm cr}}{P_0}\right)\cos^2\beta_1 l=\left(2\frac{P_{\rm cr}}{P_0}-1\right)\left(1+\frac{P_{\rm cr}}{P_0}\right)\cosh^2\beta_2 l\,. \qquad \text{(j}''\text{)}$$

Taking into account the condition $a^2 < b^2$ of validity of the solution (f'), we notice that

$$\frac{1}{2}<\frac{P_{\rm cr}}{P_0}<1\,. \qquad \text{(k)}$$

The equations (j), (j'') allow to determine the ratio $P_{\rm cr}/P_0$ (the reduced critical load) as a function of the non-dimensional magnitude $bl = l\sqrt[4]{k/EI}$, i.e. as a function of the data of the problem (rigidity of the elastic medium and rigidity and length of the column); starting from

$$\frac{P_{\rm cr}}{P_0}=\frac{\pi^2 EI}{l_{\rm cr}^2}\frac{1}{2\sqrt{EIk}}\,,$$

the critical length is given by

$$\frac{l_{\rm cr}}{l}=\frac{\pi}{\sqrt{2}}\frac{1}{bl}\frac{1}{\sqrt{\dfrac{P_{\rm cr}}{P_0}}}\,. \qquad \text{(l)}$$

We can write

$$\frac{P_{\rm cr}}{P_0}=\frac{P_{\rm cr}}{P_{\rm cr}^0}\frac{P_{\rm cr}^0}{P_0}\,,$$

where $P_{\rm cr}^0$ is the critical load in the absence of the elastic medium; observing that $P_{\rm cr} > P_{\rm cr}^0$ (in the absence of the elastic medium, $P_{\rm cr}^0$ is smaller) and that $P_{\rm cr} < P_{\rm cr}^0$ (the condition of validity of the solution), it results

$$\frac{P_{\text{cr}}^0}{P_0} = \frac{\pi^2 EI}{4l^2} \frac{1}{2\sqrt{EIk}} = \frac{\pi^2}{8} \frac{1}{(bl)^2} < 1,$$

hence $bl > \pi/2\sqrt{2} \cong 1.1107207$.

In the limit case $P_{\text{cr}} = P_0$, i.e. in the case $a^2 = b^2$, the general solution of the equation (c) is of the form

$$y(x) = (C_1 x + C_2)\cos bx + (C_3 x + C_4)\sin bx\,; \tag{m}$$

the two-point conditions (d), (d′) lead to the relations $C_2 = 0$, $C_1 + aC_4 = 0$, as well as to

$$C_1(\sin bl + bl\cos bl) - C_3(2\cos bl - bl\sin bl) = 0,$$
$$C_1(2\cos bl + bl\sin bl) + C_3(\sin bl - bl\cos bl) = 0.$$

The corresponding characteristic equation will be

$$\cos^2 bl = \frac{1}{3}\left(b^2 l^2 - 1\right), \tag{n}$$

leading to $bl \cong 1.1896$, whence, taking into account the notations (b), we obtain the critical load

$$P_{\text{cr}} \cong \frac{2.830EI}{l^2} \cong \frac{\pi^2 EI}{(1.867l)^2}, \tag{o}$$

as well as

$$k_{\text{cr}} \cong \frac{2.00EI}{l^4}. \tag{o′}$$

In the absence of the elastic medium, the critical load is given by

$$\frac{\pi^2 EI}{4l^2} \cong \frac{2.467EI}{l^2};$$

this load is somewhat smaller than the critical load (o).

Taking into account the continuity of the solution with respect to the coefficients of the differential equation, the case $P_{\text{cr}} < P_0$ can take place only for $bl > 1.1896$.

The general solution

$$y(x) = C_1 \cos\alpha_1 x + C_2 \sin\alpha_1 x + C_3 \cos\alpha_2 x + C_4 \cos\alpha_2 x\,, \tag{p}$$

where

$$\alpha_{1,2} = \sqrt{a^2 \pm \sqrt{a^4 - b^4}} = b\sqrt{\frac{P_{cr}}{P_0} \pm \sqrt{\left(\frac{P_{cr}}{P_0}\right)^2 - 1}}\ , \qquad (\mathrm{p}')$$

corresponds to the case $P_{cr} > P_0$, i.e. $a^2 > b^2$; the boundary conditions (d') become

$$y''(l) = 0, \quad y'''(l) + \left(\alpha_1^2 + \alpha_2^2\right) y'(l) = 0\ . \qquad (\mathrm{d}''')$$

As in the former cases, the conditions (d), (d''') lead to a system of linear equations for the constants C_1, C_2, C_3, and C_4, hence to the characteristic equation

$$\alpha_1 \alpha_2 \left(\alpha_1^2 + \alpha_2^2\right) \sin \alpha_1 l \sin \alpha_2 l + \left(\alpha_1^4 + \alpha_2^4\right) \cos \alpha_1 l \cos \alpha_2 l = 2\alpha_1^2 \alpha_2^2\ . \qquad (\mathrm{q})$$

Noting that

$$\alpha_1 \alpha_2 = b^2,\ \alpha_1^2 + \alpha_2^2 = 2a^2,\ \alpha_1^4 + \alpha_2^4 = 2\left(2a^4 - b^4\right),$$

we can write

$$\frac{P_{cr}}{P_0} \sin \alpha_1 l \sin \alpha_2 l + \left[2\left(\frac{P_{cr}}{P_0}\right)^2 - 1\right] \cos \alpha_1 l \cos \alpha_2 l = 1 \qquad (\mathrm{q}')$$

or

$$\frac{P_{cr}}{P_0} \cos(\alpha_1 - \alpha_2) l + \left(2\frac{P_{cr}}{P_0} + 1\right)\left(\frac{P_{cr}}{P_0} - 1\right) \cos \alpha_1 l \cos \alpha_2 l = 1\ . \qquad (\mathrm{q}'')$$

The equations (q), (q'') allow to determine the reduced critical force P_{cr}/P_0 as a function of the non-dimensional magnitude bl, i.e. as a function of the data of the problem; the reduced critical length l_{cr}/l is given by the formula (l).

Table 2.6. The values of P_{cr} / P_0 and l_{cr} / l

bl	P_{cr} / P_0	l_{cr} / l	bl	P_{cr} / P_0	l_{cr} / l	bl	P_{cr} / P_0	l_{cr} / l
0	∞	2.000	1.5	0.748	1.712	4.5	0.516	0.687
0.2	30.846	2.000	1.8	0.654	1.526	5.0	0.506	0.625
0.5	4.958	1.995	2.0	0.636	1.393	6.0	0.502	0.523
0.8	1.986	1.970	2.5	0.629	1.120	7.0	0.502	0.448
1.0	1.325	1.930	3.0	0.661	0.947	8.0	0.501	0.392
1.1896	1.000	1.867	3.5	0.573	0.838	8.5	0.500	0.370
1.2	0.987	1.863	4.0	0.538	0.757	∞	0.500	0

For $bl \to 1.1896$ we have $P_{cr}/P_0 \to 1$, hence the solution corresponding to the general integral (p) tends to the solution corresponding to the general integral (m); for $bl \to 0$, we obtain $P_{\text{cr}}/P_0 \to \infty$. The reduced critical load approaches the unity, remaining greater than the unity for an infinity of values of the non-dimensional magnitude bl (e.g., for $bl \cong 142$). In conclusion, till $bl \cong 1.1896$ we have one P_{cr} and from this value further two critical loads; but we take into account only the smallest one.

The values of the reduced critical load P_{cr}/P_0 (P_{cr} is the smallest critical load) and of the reduced critical load are listed in Table 2.6 and plotted into diagrams (Figs.2.20 and 2.21).

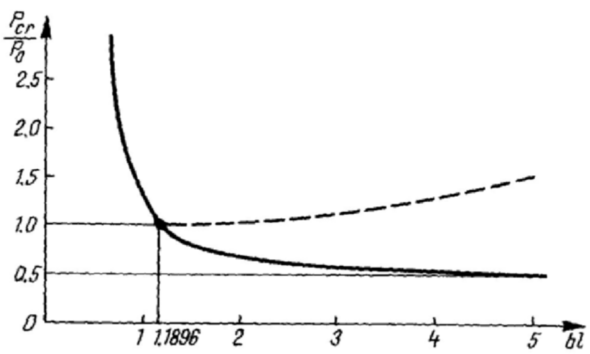

Figure 2. 16. Diagram of the reduced critical load P_{cr}/P_0 vs. bl

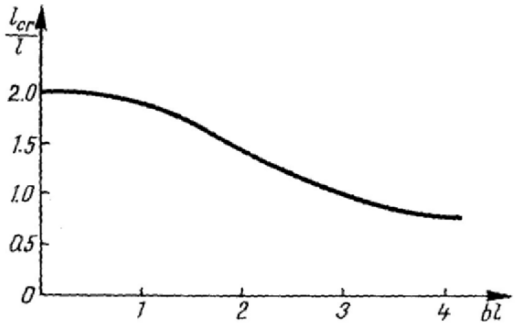

Figure 2. 17. Diagram of the reduced critical length l_{cr}/l_0 vs. .bl

For

$$bl = l\sqrt[4]{\frac{k}{EI}} > 8.5$$

we can use the asymptotic formulae

$$P_{\mathrm{cr}} \cong 0.500 P_0 = \sqrt{EIk} = (bl)^2 \frac{EI}{l^2} = \left(\frac{bl}{\pi}\right)^2 P_E \tag{r}$$

and

$$\frac{l_{\mathrm{cr}}}{l} \cong \frac{\pi}{bl}\,, \tag{r'}$$

where

$$P_E = \frac{\pi^2 EI}{l^2} \tag{s}$$

is the Euler critical load corresponding to a simply supported column in the absence of the elastic medium.

We observe that in case of a column for which the bilocal conditions allow a greater deformation the elastic medium has a smaller influence on the critical load. Indeed, in case of the cantilever column, this load grows (in comparison to the critical load corresponding to the absence of the elastic medium) less than in case of the simply supported one. A calculus shows that the critical load is much greater in case of a double built-in column in an elastic medium and much smaller in case of a free column immersed in such a medium.

Chapter 3

LINEAR ODSs OF FIRST ORDER

1. The General Study of Linear First Order ODSs

1.1 GENERALITIES

The canonical form of a linear non-homogeneous first order ODS with n unknown functions is

$$\begin{aligned} y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \ldots + a_{1n}(x)y_n + f_1(x), \\ y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \ldots + a_{2n}(x)y_n + f_2(x), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n' &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \ldots + a_{nn}(x)y_n + f_n(x), \end{aligned} \tag{3.1.1}$$

where the primes mean differentiation with respect to x, $f_i, a_{ij}, i, j = \overline{1,n}$, are functions considered of class $C^0(I), I \equiv [a,b] \in \Re$ and y_j, $j = \overline{1,n}$, are the unknown functions. In a classic frame, we shall search for solutions of class $C^1(I)$.
If f_i vanish identically on I, then we get the associated homogeneous ODS

$$\begin{aligned} y_1' &= a_{11}(x)y_1 + a_{12}(x)y_2 + \ldots + a_{1n}(x)y_n , \\ y_2' &= a_{21}(x)y_1 + a_{22}(x)y_2 + \ldots + a_{2n}(x)y_n , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n' &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \ldots + a_{nn}(x)y_n . \end{aligned} \tag{3.1.2}$$

To simplify the writing and, also, to emphasize certain useful properties, we shall introduce the following vector functions

$$\mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \quad \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ \vdots \\ y_n'(x) \end{bmatrix}, \quad \mathbf{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix} \tag{3.1.3}$$

and the associated to (3.1.1) – or, also to (3.1.2) – matrix

$$\mathbf{A}(x)=\begin{bmatrix} a_{11}(x) & a_{12}(x) & \dots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \dots & a_{2n}(x) \\ \dots & \dots & \dots & \dots \\ a_{n1}(x) & a_{n2}(x) & \dots & a_{nn}(x) \end{bmatrix}. \tag{3.1.4}$$

Then the above non-homogeneous system is written in matrix form

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}=\mathbf{A}(x)\mathbf{y}+\mathbf{f}\,, \tag{3.1.5}$$

while the associated homogeneous ODS takes the form

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}=\mathbf{A}(x)\mathbf{y}\,. \tag{3.1.6}$$

1.2 THE GENERAL SOLUTION OF THE HOMOGENEOUS ODS

Consider n linearly independent vector-solutions of the homogeneus system (3.1.6)

$$\mathbf{Y}_1(x)=\begin{bmatrix} y_{11}(x) \\ y_{21}(x) \\ \vdots \\ y_{n1}(x) \end{bmatrix},\quad \mathbf{Y}_2(x)=\begin{bmatrix} y_{12}(x) \\ y_{22}(x) \\ \vdots \\ y_{n2}(x) \end{bmatrix},\dots,\mathbf{Y}_n(x)=\begin{bmatrix} y_{1n}(x) \\ y_{2n}(x) \\ \vdots \\ y_{nn}(x) \end{bmatrix}. \tag{3.1.7}$$

Such a system is called a *fundamental system of solutions*. Exactly as in the case of the linear ODEs, we consider the Wronskian of this system

$$W[\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_n](x)\equiv\begin{vmatrix} y_{11}(x) & y_{12}(x) & \dots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \dots & y_{2n}(x) \\ \dots & \dots & \dots & \dots \\ y_{n1}(x) & y_{n2}(x) & \dots & y_{nn}(x) \end{vmatrix}=W(x), \tag{3.1.8}$$

which is non-zero if and only if the system $\{\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_n\}$ is linearly independent. We also can prove Liouville's theorem and formula

$$W(x)=W(x_0)\mathrm{e}^{\int_{x_0}^{x}\mathrm{tr}\,\mathbf{A}(t)\mathrm{d}t}\,,\quad \mathrm{tr}\,\mathbf{A}=a_{11}+a_{22}+\dots+a_{nn}\,. \tag{3.1.9}$$

More specific, tr **A** is *the trace* of the matrix **A**, that is the sum of the entries of the main diagonal. In the above formula (3.1.9), x_0 is an arbitrary point in *I*. Exactly as in the case of higher order ODEs, one can prove that any linear ODS with continuous coefficients always allows a fundamental system of solutions.

Any solution **y** of the homogeneous ODS may thus be written as a linear combination with constant coefficients of the functions belonging to a fundamental system, i.e.

$$\mathbf{y}(x) = c_1\mathbf{Y}_1(x) + c_2\mathbf{Y}_2(x) + ... + c_n\mathbf{Y}_n(x). \tag{3.1.10}$$

The Cauchy – or initial – problem associated either to the non-homogeneous ODS (3.1.1) or to the homogeneous ODS (3.1.2) consists of finding a solution of the given ODS that also satisfies the initial (Cauchy) conditions

$$y_1(x_0) = y_{10},\, y_2(x_0) = y_{20}, ..., y_n(x_0) = y_{n0}, \quad x_0 \in I, \tag{3.1.11}$$

or, using vector-functions

$$\mathbf{y}(x_0) = \mathbf{y}_0, \quad \mathbf{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{n0} \end{bmatrix}. \tag{3.1.12}$$

If the vector-functions of the fundamental system satisfy the initial conditions

$$\mathbf{Y}_1(x_0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{Y}_2(x_0) = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, ..., \mathbf{Y}_n(x_0) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{3.1.13}$$

then the system is called *normal*. Knowing a normal system, the solution of the Cauchy problem (3.1.6), (3.1.13) is written using directly the Cauchy data

$$\mathbf{y}(x) = y_{10}\mathbf{Y}_1(x) + y_{20}\mathbf{Y}_2(x) + ... + y_{n0}\mathbf{Y}_n(x). \tag{3.1.14}$$

1.3 THE GENERAL SOLUTION OF THE NON-HOMOGENEOUS ODS

The general solution of the linear non-homogeneous ODS (3.1.1) – or, equivalently, (3.1.6) – is the sum between one of its particular solutions

$$\mathbf{y}_p(x) = \begin{bmatrix} y_{1p}(x) \\ y_{2p}(x) \\ \vdots \\ y_{np}(x) \end{bmatrix}$$

and the general solution of the associated homogeneous ODS, i.e.,

$$\mathbf{y}(x) = c_1\mathbf{Y}_1(x) + c_2\mathbf{Y}_2(x) + ... + c_n\mathbf{Y}_n(x) + \mathbf{y}_p(x). \tag{3.1.15}$$

Knowing a fundamental system of solutions for the given ODS enables us to get a particular solution for the non-homogeneous ODS by using, as previously, the method of variation of parameters (Lagrange's method). We search for a solution shaping (3.1.10), but with variable – instead of constant – coefficients

$$\mathbf{y}_p(x) = c_1(x)\mathbf{Y}_1(x) + c_2(x)\mathbf{Y}_2(x) + \ldots + c_n(x)\mathbf{Y}_n(x). \tag{3.1.16}$$

The functions $c_j(x)$ are supposedly of class $\mathrm{C}^1(I)$. Introducing the above expression in the non-homogeneous ODS (3.1.5), we get

$$\begin{aligned}\frac{\mathrm{d}\mathbf{y}_p}{\mathrm{d}x} &= c_1'(x)\mathbf{Y}_1(x) + c_2'(x)\mathbf{Y}_2(x) + \ldots + c_n'(x)\mathbf{Y}_n(x) \\ &+ c_1(x)\frac{\mathrm{d}\mathbf{Y}_1}{\mathrm{d}x} + c_2(x)\frac{\mathrm{d}\mathbf{Y}_2}{\mathrm{d}x} + \ldots + c_n(x)\frac{\mathrm{d}\mathbf{Y}_n}{\mathrm{d}x} \\ &= c_1'(x)\mathbf{Y}_1(x) + c_2'(x)\mathbf{Y}_2(x) + \ldots + c_n'(x)\mathbf{Y}_n(x) \\ &+ c_1(x)\mathbf{A}(x)\mathbf{Y}_1 + c_2(x)\mathbf{A}(x)\mathbf{Y}_2 + \ldots + c_n(x)\mathbf{A}(x)\mathbf{Y}_n,\end{aligned} \tag{3.1.17}$$

or, taking (3.1.16) into account,

$$\frac{\mathrm{d}\mathbf{y}_p}{\mathrm{d}x} = c_1'(x)\mathbf{Y}_1(x) + c_2'(x)\mathbf{Y}_2(x) + \ldots + c_n'(x)\mathbf{Y}_n(x) + \mathbf{A}(x)\mathbf{y}_p(x). \tag{3.1.18}$$

But $\mathbf{y}_p(x)$ must satisfy the ODS (3.1.5), whence we get for c_j', $j = \overline{1,n}$,

$$c_1'(x)\mathbf{Y}_1(x) + c_2'(x)\mathbf{Y}_2(x) + \ldots + c_n'(x)\mathbf{Y}_n(x) = \mathbf{f}(x), \tag{3.1.19}$$

that may be written componentwise

$$\begin{aligned}c_1'y_{11} + c_2'y_{12} + \ldots + c_n'y_{1n} &= f_1(x), \\ c_1'y_{21} + c_2'y_{22} + \ldots + c_n'y_{2n} &= f_2(x), \\ &\ldots\ldots\ldots\ldots \\ c_1'y_{n1} + c_2'y_{n2} + \ldots + c_n'y_{nn} &= f_n(x).\end{aligned} \tag{3.1.20}$$

This represents, in fact, a linear algebraic system, whose associated determinant is precisely the Wronskian of the fundamental system. As the Wronskian does not vanish on *I*, it follows that the algebraic system (3.1.20) allowe a unique solution. We therefore get

$$c_j'(x) = p_j(x), \quad j = \overline{1,n}, \tag{3.1.21}$$

whence $\mathbf{y}_p(x)$ results by direct integration.

1.4 ORDER REDUCTION OF HOMOGENEOUS ODSs

To simplify the presentation, we shall take the case $n = 3$, i.e., we consider the ODS

$$y_1' = a_{11}(x)y_1 + a_{12}(x)y_2 + a_{13}(x)y_3 ,$$
$$y_2' = a_{21}(x)y_1 + a_{22}(x)y_2 + a_{23}(x)y_3 , \qquad (3.1.22)$$
$$y_3' = a_{31}(x)y_1 + a_{32}(x)y_2 + a_{33}(x)y_3 ,$$

for which we know a particular solution, say

$$\mathbf{y}_p(x) = \begin{bmatrix} y_{1p}(x) \\ y_{2p}(x) \\ y_{3p}(x) \end{bmatrix} .$$

We perform the change of functions

$$y_1 = y_{1p}u_1 ,$$
$$y_2 = y_{2p}u_1 + u_2 , \qquad (3.1.23)$$
$$y_3 = y_{3p}u_1 + u_3 ,$$

where u_1, u_2, u_3 are the new unknown functions.

We differentiate the above expressions and we replace them in (3.1.23), getting the degenerate ODS

$$u_1' = \frac{a_{12}(x)}{y_{1p}}u_2 + \frac{a_{13}(x)}{y_{1p}}u_3 ,$$
$$u_2' = \left[a_{22}(x) - a_{12}(x)\frac{y_{2p}}{y_{1p}} \right]u_2 + \left[a_{23}(x) - a_{13}(x)\frac{y_{2p}}{y_{1p}} \right]u_3 , \qquad (3.1.24)$$
$$u_3' = \left[a_{32}(x) - a_{12}(x)\frac{y_{3p}}{y_{1p}} \right]u_2 + \left[a_{33}(x) - a_{13}(x)\frac{y_{3p}}{y_{1p}} \right]u_3 .$$

The last two equations of this system may be solved separately. As a result, we obtain the functions u_2, u_3, which, introduced in the first equation (3.1.24), determine u_1. As the system formed by the last two equations has only two unknown functions, it follows that the order of the ODS (3.1.22) was reduced by one unit.

1.5 BOUNDARY VALUE PROBLEMS FOR ODSs

Consider again the ODS

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{A}(x)\mathbf{y} + \mathbf{f} , \qquad (3.1.25)$$

with $a_{ij}, f_j, j = \overline{1,n}$, defined and continuous on the real interval $I = [a,b]$. Also consider the real matrix of rank n

$$\mathbf{M} \equiv \begin{bmatrix} A_{10} & A_{11} & A_{12} & \dots & A_{1,n-1} & B_{10} & B_{11} & B_{12} & \dots & B_{1,n-1} \\ A_{20} & A_{21} & A_{22} & \dots & A_{2,n-1} & B_{20} & B_{21} & B_{22} & \dots & B_{2,n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{n0} & A_{n1} & A_{n2} & \dots & A_{n,n-1} & B_{n0} & B_{n1} & B_{n2} & \dots & B_{n,n-1} \end{bmatrix} \tag{3.1.26}$$

with n rows and $2n$ columns. By using this matrix, we form the two-point conditions

$$\mathrm{U}_i \mathbf{y} \equiv \sum_{j=1}^{n} \left[A_{ij} y_j(a) + B_{ij} y_j(b) \right] = K_i, \quad i = \overline{1,n}, \tag{3.1.27}$$

where K_i are given real constants.

The boundary value (*two-point*) *problem* for the ODS (3.1.25) consists of finding a solution of this system that also satisfies the two-point conditions (3.1.27). The *semi-homogeneous problem* consists of finding a solution of (3.1.25) that satisfies (3.1.27) for $K_i = 0, i = \overline{1,n}$. The *homogeneous problem* is defined as

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{A}(x)\mathbf{y}, \\ &\mathrm{U}_i \mathbf{y} = 0, \quad i = \overline{1,n}. \end{aligned} \tag{3.1.28}$$

If the homogeneous problem allows only the trivial solution, then we say that it has the *index* 0.

If the homogeneous problem allows k linearly independent non-trivial solutions, then this problem is called of *index k*. Any other solution may be written as a linear combination of these k solutions.

If $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_n$ are n vector functions, solutions of the homogeneous ODS, then the homogeneous boundary value problem allows non-trivial solutions if and only if the determinant

$$\Delta \equiv \begin{vmatrix} \mathrm{U}_1(\mathbf{Y}_1) & \mathrm{U}_1(\mathbf{Y}_2) & \dots & \mathrm{U}_1(\mathbf{Y}_n) \\ \mathrm{U}_2(\mathbf{Y}_1) & \mathrm{U}_2(\mathbf{Y}_2) & \dots & \mathrm{U}_2(\mathbf{Y}_n) \\ \dots & \dots & \dots & \dots \\ \mathrm{U}_n(\mathbf{Y}_1) & \mathrm{U}_n(\mathbf{Y}_2) & \dots & \mathrm{U}_n(\mathbf{Y}_n) \end{vmatrix} \tag{3.1.29}$$

Is identically null. If the associated matrix has the rank r, then the index of the corresponding boundary value problem is $n - r$.

One can introduce the notions of fundamental solution and Green function for the ODS (3.1.25) by considering its correponding adjoint system.

2. ODSs with Constant Coefficients

2.1 THE GENERAL SOLUTION OF THE HOMOGENEOUS ODS

Consider the homogeneous ODS with constant coefficients

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \left[a_{ij}\right]_{i,j=\overline{1,n}}, \quad a_{ij} \in \Re, i, j = \overline{1,n}. \tag{3.2.1}$$

As in the case of linear ODEs with constant coefficients, we shall search for solutions of exponential form

$$\mathbf{y}(x) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \mathrm{e}^{\lambda x} \equiv \mathbf{C}\mathrm{e}^{\lambda x}, \tag{3.2.2}$$

where λ is a parameter and

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \tag{3.2.3}$$

is a constant vector. Introducing this in (3.2.1), we get

$$\mathbf{C}\lambda\mathrm{e}^{\lambda x} = \mathbf{A}\mathbf{C}\mathrm{e}^{\lambda x} \tag{3.2.4}$$

or, denoting by **E** the $n \times n$ unit matrix

$$\mathbf{E} \equiv \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}, \tag{3.2.5}$$

we deduce

$$(\mathbf{A} - \lambda\mathbf{E})\mathbf{C} = \mathbf{0}, \tag{3.2.6}$$

i.e., an algebraic linear homogeneous system, that must be fulfilled by the components of **C**. Componentwise, this system reads

$$\begin{aligned}&(a_{11}-\lambda)c_1+a_{12}c_2+\ldots+a_{1n}c_n=0,\\&a_{21}c_1+(a_{22}-\lambda)c_2+\ldots+a_{2n}c_n=0,\\&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\&a_{n1}c_1+a_{n2}c_2+\ldots+(a_{nn}-\lambda)c_n=0.\end{aligned} \tag{3.2.7}$$

It allows non-zero solutions if and only if its associated determinant vanishes

$$\Delta(\lambda)\equiv\det[\mathbf{A}-\lambda\mathbf{E}]=0. \tag{3.2.8}$$

The n-th degree polynomial $\Delta(\lambda)$ is called the *characteristic polynomial* and the equation (3.2.8) – the *characteristic equation*. The solutions of the homogeneous ODS depend on the nature of the roots of the characteristic equation. It is seen that these roots are precisely the eigenvalues of the matrix **A** and that **C** are the corresponding eigenvectors.

Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the roots of the characteristic equation. As for the ODEs, we must consider several distinct cases.

a) Real and distinct roots. Denote by

$$\mathbf{C}_j=\begin{bmatrix}c_{1j}\\c_{2j}\\\vdots\\c_{nj}\end{bmatrix},\quad j=\overline{1,n}, \tag{3.2.9}$$

the corresponding eigenvectors. As the eigenvalues are distinct, the Wronskian of the vector-functions $\mathbf{C}_j\mathrm{e}^{\lambda_j x}$, $j=\overline{1,n}$, does not vanish. Consequently, these vectors are linearly independent and the general solution of the homogeneous ODS (3.2.1) is

$$\mathbf{y}(x)=k_1\mathbf{C}_1\mathrm{e}^{\lambda_1 x}+k_2\mathbf{C}_2\mathrm{e}^{\lambda_2 x}+\ldots+k_n\mathbf{C}_n\mathrm{e}^{\lambda_n x} \tag{3.2.10}$$

or, componentwise,

$$\begin{bmatrix}y_1(x)\\y_2(x)\\\vdots\\y_n(x)\end{bmatrix}=k_1\begin{bmatrix}c_{11}\\c_{21}\\\vdots\\c_{n1}\end{bmatrix}\mathrm{e}^{\lambda_1 x}+k_2\begin{bmatrix}c_{12}\\c_{22}\\\vdots\\c_{n2}\end{bmatrix}\mathrm{e}^{\lambda_2 x}+\ldots+k_n\begin{bmatrix}c_{1n}\\c_{2n}\\\vdots\\c_{nn}\end{bmatrix}\mathrm{e}^{\lambda_n x}, \tag{3.2.11}$$

where k_j, $j=\overline{1,n}$, are arbitrary constants.

b) Complex-conjugate roots. Suppose that all the roots are distinct, but some of them are complex. Let $\lambda_1=\alpha+\mathrm{i}\beta$, with $\beta\neq 0$. The characteristic equation has real coefficients, it will allow also the solution $\lambda_2=\alpha-\mathrm{i}\beta$. If $\mathbf{C}=\mathbf{C}_1+\mathrm{i}\mathbf{C}_2\mathbf{C}$ is the (complex) eigenvector corresponding to λ_1, then $\overline{\mathbf{C}} = \mathbf{C}_1 - \mathrm{i}\mathbf{C}_2$ will be the eigenvector corresponding to λ_2, $\mathbf{C}_1, \mathbf{C}_2$ being constant real vectors. As the ODS is linear, once with the solutions $\mathbf{C}\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}, \overline{\mathbf{C}}\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}$ it also allows as solutions their linear combinations

$$\begin{aligned} \mathbf{y}_1(x) &= \frac{\mathbf{C}\mathrm{e}^{(\alpha+\mathrm{i}\beta)x} + \overline{\mathbf{C}}\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}}{2} = \mathrm{e}^{\alpha x}\mathbf{C}_1 \frac{\mathrm{e}^{\mathrm{i}\beta x} + \mathrm{e}^{-\mathrm{i}\beta x}}{2} = \mathbf{C}_1 \mathrm{e}^{\alpha x} \cos\beta x, \\ \mathbf{y}_2(x) &= \frac{\mathbf{C}\mathrm{e}^{(\alpha+\mathrm{i}\beta)x} - \overline{\mathbf{C}}\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}}{2\mathrm{i}} = \mathrm{e}^{\alpha x}\mathbf{C}_2 \frac{\mathrm{e}^{\mathrm{i}\beta x} - \mathrm{e}^{-\mathrm{i}\beta x}}{2\mathrm{i}} = \mathbf{C}_2 \mathrm{e}^{\alpha x} \sin\beta x; \end{aligned} \tag{3.2.12}$$

to get the above expressions, we used Euler's formulae (2.2.5). These real solutions are linearly independent too. Therefore, they may replace $\mathbf{C}\mathrm{e}^{(\alpha+\mathrm{i}\beta)x}, \overline{\mathbf{C}}\mathrm{e}^{(\alpha-\mathrm{i}\beta)x}$ in the corresponding fundamental system. So, if the remaining roots of the characteristic equation are real and distinct, the general solution of (3.2.1) reads

$$\mathbf{y}(x) = k_1\mathbf{C}_1\mathrm{e}^{\alpha x}\cos\beta x + k_2\mathbf{C}_2\mathrm{e}^{\alpha x}\sin\beta x + k_3\mathbf{C}_3\mathrm{e}^{\lambda_3 x} + \ldots + k_n\mathbf{C}_n\mathrm{e}^{\lambda_n x} \tag{3.2.13}$$

c) Multiple roots. To simplify the presentations, let us suppose that λ_1 has the order of multiplicity *m* and that the other roots of the characteristic equation are all of them real and distinct. As in the case of ODEs, we search for solutions of the form

$$\mathbf{C}_1\mathrm{e}^{\lambda_1 x}, \mathbf{C}_2 x\mathrm{e}^{\lambda_1 x}, \ldots\ldots, \mathbf{C}_m x^{m-1}\mathrm{e}^{\lambda_1 x}, \tag{3.2.14}$$

where $\mathbf{C}_j$, $j = \overline{1,m}$, are constant vectors, whose components are determined by identification, after replacing (3.2.14) in the ODS (3.2.1). Thus, the general solution of (3.2.1) reads

$$\begin{aligned} \mathbf{y}(x) = &\left(k_1\mathbf{C}_1 + k_2\,\mathbf{C}_2 x + \ldots + k_m\mathbf{C}_m x^{m-1}\right)\mathrm{e}^{\lambda_1 x} \\ &+ k_{m+1}\mathbf{C}_{m+1}\mathrm{e}^{\lambda_{m+1}x} + \ldots + k_n\mathbf{C}_n\mathrm{e}^{\lambda_n x}. \end{aligned} \tag{3.2.15}$$

2.2 SOLUTIONS IN MATRIX FORM FOR LINEAR ODSs WITH CONSTANT COEFFICIENTS

Let us start from the Cauchy problem associated to the linear ODS with constant coefficients (3.2.1)

$$\mathbf{y}(x_0) = \mathbf{y}_0, \quad \mathbf{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{n0} \end{bmatrix}, \quad x_0 \in \Re\,. \tag{3.2.16}$$

We can expand the solution in a Taylor series around x_0

$$\mathbf{y}(x)=\mathbf{y}(x_0)+\frac{x-x_0}{1!}\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x_0)+\frac{(x-x_0)^2}{2!}\frac{\mathrm{d}^2\mathbf{y}}{\mathrm{d}x^2}(x_0)$$
$$+\ldots+\frac{(x-x_0)^k}{k!}\frac{\mathrm{d}^k\mathbf{y}}{\mathrm{d}x^k}(x_0)+\ldots \tag{3.2.17}$$

Replacing this expansion in (3.2.1) and taking into account the initial conditions (3.2.16), we obtain, step by step,

$$\begin{aligned}
&\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x_0)=\mathbf{A}\mathbf{y}(x_0)=\mathbf{A}\mathbf{y}_0,\\
&\frac{\mathrm{d}^2\mathbf{y}}{\mathrm{d}x^2}(x_0)=\mathbf{A}\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x}(x_0)=\mathbf{A}^2\mathbf{y}_0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{\mathrm{d}^k\mathbf{y}}{\mathrm{d}x^k}(x_0)=\mathbf{A}^k\mathbf{y}_0,\\
&\ldots\ldots\ldots\ldots
\end{aligned} \tag{3.2.18}$$

We thus get for **y** the expansion

$$\mathbf{y}(x)=\mathbf{y}_0+\frac{x-x_0}{1!}\mathbf{A}\mathbf{y}_0+\frac{(x-x_0)^2}{2!}\mathbf{A}^2\mathbf{y}_0+\ldots+\frac{(x-x_0)^k}{k!}\mathbf{A}^k\mathbf{y}_0+\ldots \tag{3.2.19}$$

or

$$\mathbf{y}(x)=\left[\mathbf{E}+\frac{x-x_0}{1!}\mathbf{A}+\frac{(x-x_0)^2}{2!}\mathbf{A}^2+\ldots+\frac{(x-x_0)^k}{k!}\mathbf{A}^k+\ldots\right]\mathbf{y}_0. \tag{3.2.20}$$

By analogy with the scalar functions, we define

$$\mathrm{e}^{\mathbf{A}(x-x_0)}=\mathbf{E}+\frac{x-x_0}{1!}\mathbf{A}+\frac{(x-x_0)^2}{2!}\mathbf{A}^2+\ldots+\frac{(x-x_0)^k}{k!}\mathbf{A}^k+\ldots \tag{3.2.21}$$

Thus, the solution of the Cauchy problem (3.2.1), (3.2.16) finally reads

$$\mathbf{y}(x)=\mathrm{e}^{\mathbf{A}(x-x_0)}\mathbf{y}_0. \tag{3.2.22}$$

The problem of solving the system (3.2.1) is ultimately reduced to the calculus of the exponential matrix (3.2.21).

Let $\lambda_1,\lambda_2,\ldots,\lambda_r$ be the eigenvalues of the matrix **A**, i.e., the roots of the characteristic equation

$$\det[\mathbf{A}-\lambda\mathbf{E}]=0, \tag{3.2.23}$$

each of them having the order of multiplicity $m_1, m_2, ..., m_r$, accordingly. Hence $\sum_{k=1}^{r} m_k = n$. Let us denote by $\mathbf{J}_k$ the $m_k \times m_k$ matrix

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_k & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_k & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \lambda_k & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & \lambda_k \end{bmatrix}, \qquad k = \overline{1, r}, \tag{3.2.24}$$

and by $\mathbf{J}$ the block matrix

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{J}_r \end{bmatrix}. \tag{3.2.25}$$

The matrix $\mathbf{J}$ is called *the normal Jordan form* for $\mathbf{A}$; $\mathbf{J}_k$ are called *Jordan cells*. If $m_k = 1$, then the corresponding Jordan cell is reduced to the 1×1 matrix $\mathbf{J}_k = [\lambda_k]$. From the matrix theory, it is known that one can find a non-degenerate matrix $\mathbf{D}$ such that

$$\mathbf{A} = \mathbf{D}\mathbf{J}\mathbf{D}^{-1}. \tag{3.2.26}$$

The ODS (3.2.1) may then be written in the form

$$\mathbf{D}^{-1}\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{D}^{-1}\mathbf{A}\mathbf{y}. \tag{3.2.27}$$

Let us apply now the change of vector-function $\mathbf{y}(x) = \mathbf{D}\mathbf{z}(x)$. The ODS (3.2.27) becomes

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}\mathbf{z} \tag{3.2.28}$$

or, taking (3.2.26) into account,

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}x} = \mathbf{J}\mathbf{z}. \tag{3.2.29}$$

For the new unknown vector-function $\mathbf{z}$, one has the initial conditions

$$\mathbf{z}(x_0) = \mathbf{D}^{-1}\mathbf{y}_0, \tag{3.2.30}$$

deduced from (3.2.16).

Applying now formula (3.2.22) to the ODS (3.2.29), also considering (3.2.30), we deduce for $\mathbf{z}$ the following representation

$$\mathbf{z}(x) = \mathrm{e}^{\mathbf{J}(x-x_0)}\mathbf{D}^{-1}\mathbf{y}_0 . \tag{3.2.31}$$

But the exponential matrix of this formula is easily computed, due to the particular form of $\mathbf{J}$. Indeed, we find immediately

$$\mathrm{e}^{\mathbf{J}(x-x_0)} = \begin{bmatrix} \mathrm{e}^{\mathbf{J}_1(x-x_0)} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathrm{e}^{\mathbf{J}_2(x-x_0)} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathrm{e}^{\mathbf{J}_r(x-x_0)} \end{bmatrix}, \tag{3.2.32}$$

where

$$\mathrm{e}^{\mathbf{J}_k(x-x_0)} = \begin{bmatrix} 1 & (x-x_0) & \dfrac{(x-x_0)^2}{2!} & \dots & \dfrac{(x-x_0)^{m_k-1}}{(m_k-1)!} \\ 0 & 1 & (x-x_0) & \dots & \dfrac{(x-x_0)^{m_k-2}}{(m_k-2)!} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \mathrm{e}^{\lambda_k(x-x_0)}, \quad k = \overline{1,r} . \tag{3.2.33}$$

Eventually, the solution of the initial problem (3.2.1), (3.2.16) reads

$$\mathbf{y}(x) = \mathbf{D}\mathrm{e}^{\mathbf{J}(x-x_0)}\mathbf{D}^{-1}\mathbf{y}_0 . \tag{3.2.34}$$

Remarks. 1)Another practical possibility to solve a linear ODS with constant coefficients is to eliminate the unknown functions, all but one, by successive differentiations, thus reducing it to a linear ODE with constant coefficients, which can be solved by the methods exposed at Chap.2.

2) To solve a non-homogeneous ODS with constant coefficients, one can use, as in the case of linear ODEs, either the general method of variation of parameters (Lagrange), as exposed at Sec.1.3, or to search for solutions in the form of the free term, if this term is formed by elementary functions.

3. Applications

Application 3.1

Problem. Consider two masses m_1 and m_2, sliding frictionless along a vertical axe, being connected with springs of elastic constants k_1 and k_2 (Fig.3.1). Study the motion of the two springs.

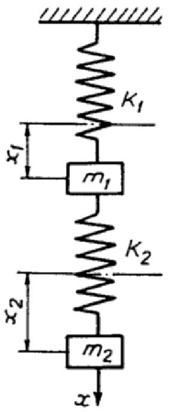

Figure 3. 1. Oscillation of two masses m_1 and m_2 connected with springs of elastic constants k_1 and k_2

Mathematical model. We specify the positions of the two masses at the moment t by the displacements x_1 and x_2, measured from the static positions of equilibrium, when the springs are not acted upon. Taking into account Newton's equation of motion, we may write

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 \left(x_2 - x_1 \right), \tag{a}$$

$$m_2 \ddot{x}_2 = -k_2 \left(x_2 - x_1 \right). \tag{b}$$

Introducing the notations

$$\frac{k_1 + k_2}{m_1} = a \,, \; \frac{k_2}{m_1} = b \,, \; \frac{k_2}{m_2} = c \,, \tag{c}$$

these equations read

$$\begin{aligned} &\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + a x_1 - b x_2 = 0 \,, \\ &\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} - c x_1 + c x_2 = 0 \,, \end{aligned} \tag{d}$$

that is, they form a linear and homogeneous ODS, of unknown functions x_1 and x_2 and of independent variable *t*.

Solution. The solution of the problem can be obtained by two methods: 1) the method of elimination and 2) the standard method.

1) In the first method we eliminate one of the unknown functions, e.g., x_2. To this goal, we write the system (a), (b) in the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\right)x_1 - bx_2 = 0\,,$$
$$-cx_1 + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\right)x_2 = 0\,. \tag{e}$$

The differential operators $\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c$ are prime between them, so that – eliminating x_2 – one obtains

$$\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\right) - bc\right]x_1 = 0 \tag{f}$$

or

$$\frac{\mathrm{d}^4 x_1}{\mathrm{d}t^4} + (a + c)\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + c(a - b)x_1 = 0\,. \tag{g}$$

We get thus a linear differential equation of fourth order, homogeneous and with constant coefficients. Searching solutions of the form $x_1 = \mathrm{e}^{\gamma t}$, we obtain the characteristic equation

$$\gamma^4 + (a + c)\gamma^2 + c(a - b) = 0\,, \tag{h}$$

of roots

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4 = \pm\sqrt{-\frac{a + c}{2} \pm \sqrt{\left(\frac{a - c}{2}\right)^2 + bc}}\,. \tag{i}$$

The quantity under the second radical must be positive

$$\left(\frac{a - c}{2}\right)^2 + bc > 0\,.$$

Further, the notations (c) lead to $a - b > 0$, hence the value of the second radical is always less than $(a + c)/2$. In this case, we may write $\gamma = \mathrm{i}p$, where

$$p_1, p_2, p_3, p_4 = \pm\sqrt{\frac{a+c}{2} \pm \sqrt{\left(\frac{a-c}{2}\right)^2 + bc}}\ . \tag{j}$$

Taking into account Euler's formula ($e^{ip} = \cos p + i \sin p$), the general solution may be written in a real form

$$x_1 = C_1 \cos p_1 t + C_2 \sin p_1 t + C_3 \cos p_2 t + C_4 \sin p_2 t\ . \tag{k}$$

The second function x_2 may be determined by the first relation (d)

$$x_2 = \frac{m_1}{m_2}\ddot{x}_1 + \frac{k_1 + k_2}{k_2} x_1 . \tag{l}$$

Noting that $p_3 = -p_1$ and $p_4 = -p_2$, the relation (k) may take the form

$$x_1 = A_1 \sin(p_1 t + \alpha') + A_2 \sin(p_2 t + \alpha''), \tag{m}$$

and (l) takes the corresponding form

$$x_2 = \lambda' A_1 \sin(p_1 t + \alpha') + \lambda'' A_2 \sin(p_2 t + \alpha''), \tag{n}$$

where

$$\lambda' = \frac{a - p_1^2}{b} = \frac{c}{c - p_1^2}\ ,\ \lambda'' = \frac{a - p_2^2}{b} = \frac{c}{c - p_2^2}\ . \tag{o}$$

2) To apply the standard method exposed in Sec.2.2, we firslty write the system (a), (b) in the form of a first order ODS, introducing two new unknown auxiliary functions u and v,

$$\begin{aligned} \dot{x}_1 &= u, \\ \dot{u} &= -ax_1 + bx_2, \\ \dot{x}_2 &= v, \\ \dot{v} &= cx_1 - cx_2. \end{aligned} \tag{p}$$

According to the results in Sec.2.2, we determine the eigenvalues of the matrix $\mathbf{P}$ of the system, which satisfy

$$\det[\mathbf{P} - \lambda\mathbf{E}] = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -a & -\lambda & b & 0 \\ 0 & 0 & -\lambda & 1 \\ c & 0 & -c & -\lambda \end{vmatrix} = 0\ , \tag{q}$$

leading to the biquadratic equation

$$\lambda^4 + (a + c)\lambda^2 + c(a - b) = 0\,, \tag{r}$$

the same as (b). One obtains thus the imaginary roots given by (j).
The eigenvector corresponding to the eigenvalue $\mathrm{i}p_1$ is

$$\begin{bmatrix} 1 \\ \mathrm{i}p_1 \\ \dfrac{a-p_1^2}{b} \\ \mathrm{i}p_1\dfrac{a-p_1^2}{b} \end{bmatrix}.$$

By means of all the four eigenvectors, corresponding to the eigenvalues $\pm\mathrm{i}p_1, \pm\mathrm{i}p_2$, we obtain the general solution of the system (p) in the form

$$\begin{aligned} \begin{bmatrix} x_1 \\ u \\ x_2 \\ v \end{bmatrix} &= \alpha \mathrm{e}^{\mathrm{i}p_1 t}\begin{bmatrix} 1 \\ \mathrm{i}p_1 \\ \dfrac{a-p_1^2}{b} \\ \mathrm{i}p_1\dfrac{a-p_1^2}{b} \end{bmatrix} + \overline{\alpha}\mathrm{e}^{-\mathrm{i}p_1 t}\begin{bmatrix} 1 \\ -\mathrm{i}p_1 \\ \dfrac{a-p_1^2}{b} \\ -\mathrm{i}p_1\dfrac{a-p_1^2}{b} \end{bmatrix} \\ &+ \beta \mathrm{e}^{\mathrm{i}p_2 t}\begin{bmatrix} 1 \\ \mathrm{i}p_2 \\ \dfrac{a-p_2^2}{b} \\ \mathrm{i}p_2\dfrac{a-p_2^2}{b} \end{bmatrix} + \overline{\beta}\mathrm{e}^{-\mathrm{i}p_2 t}\begin{bmatrix} 1 \\ -\mathrm{i}p_2 \\ \dfrac{a-p_2^2}{b} \\ -\mathrm{i}p_2\dfrac{a-p_2^2}{b} \end{bmatrix}, \end{aligned} \tag{s}$$

where $\alpha = A + \mathrm{i}B$, $\beta = C + \mathrm{i}D$, and A, B, C, D are arbitrary real constants. It follows

$$\begin{aligned} x_1 &= A\cos p_1 t - B\sin p_1 t + C\cos p_2 t - D_4 \sin p_2 t\,, \\ x_2 &= \frac{a - p_1^2}{b}\left(A\cos p_1 t - B\sin p_1 t\right) + \frac{a - p_2^2}{b}\left(C\cos p_2 t - D_4\sin p_2 t\right). \end{aligned} \tag{t}$$

If we take $A_1 = -B$, $\alpha' = \arctan(-A/B)$, $A_2 = -D$, $\alpha'' = \arctan(-C/D)$, then we obtain the form (m), (n) of the solution.
Finally, we notice that we may assume from the very beginning a trigonometric form of the solution, taking into account that we have to do with a problem of oscillations. For the sake of simplicity of the calculation we search for x_1 and x_2 solutions of the form

$$\begin{aligned} x_1 &= A\sin(pt+\alpha), \\ x_2 &= B\sin(pt+\alpha), \end{aligned} \tag{u}$$

where A, B, p, α are indeterminate constants. Introducing the solution (u) in the differential system (a), (b), one obtains the homogeneous algebraic equations

$$\begin{aligned} A(a-p^2) - Bb &= 0, \\ -Ac + B(c-p^2) &= 0. \end{aligned} \tag{v}$$

The trivial solution $A = B = 0$ defines the condition of equilibrium. A non- trivial solution is obtained by equating to zero the determinant

$$\begin{vmatrix} a-p^2 & -b \\ -c & c-p^2 \end{vmatrix} = 0, \tag{w}$$

yielding the biquadratic equation

$$p^4 - (a+c)p^2 + c(a-b) = 0,$$

which coincides with the equation (h) in γ, of roots (j).

Due to the homogeneity of the algebraic system, we may determine only the ratio B/A; the calculus corresponding to the two values p_1^2 and p_2^2 results in $B_1/A_1 = \lambda'$ and $B_2/A_2 = \lambda''$, with the values (o) previously given.

Application 3.2

Problem. Consider a vertical string strongly tensioned by a force S. On the string are fixed three masses m at equal distances (Fig.3.2, a). Determine the various types of vibration, assuming that the tension does not change very much four small transverse displacements.

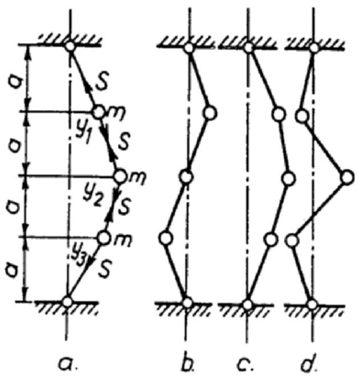

Figure 3. 2. String acted upon by the tension *S* and having three equal masses *m* fixed (a); three types of vibration (b, c, d)

Mathematical model. We denote by a the distances between the three masses and by y_1, y_2, y_3 the transverse displacements of those masses. The equations of motion of the three masses are

$$\begin{aligned} m\ddot{y}_1 &= -\frac{S}{a}\left(2y_1 - y_2\right), \\ m\ddot{y}_2 &= -\frac{S}{a}\left(-y_1 + 2y_2 - y_3\right), \\ m\ddot{y}_3 &= -\frac{S}{a}\left(-y_2 + 2y_3\right). \end{aligned} \tag{a}$$

Solution. The above linear and homogeneous ODS may be written in the form

$$\begin{aligned} \dot{y}_1 &= bu_1, \\ \dot{u}_1 &= -b^2\left(2y_1 - y_2\right), \\ \dot{y}_2 &= bu_2, \\ \dot{u}_2 &= -b^2\left(-y_1 + 2y_2 - y_3\right), \\ \dot{y}_3 &= bu_3, \\ \dot{u}_3 &= -b^2\left(-y_1 + 2y_3\right), \end{aligned} \tag{b}$$

where u_1, u_2, u_3 are new auxiliary unknown functions, while b is given by

$$b = \sqrt{\frac{S}{am}}\ . \tag{c}$$

Introducing the variable

$$\tau = bt\ , \tag{d}$$

one simplifies the system to the form

$$\begin{aligned} y_1' &= u_1, \\ u_1' &= -2y_1 + y_2, \\ y_2' &= u_2, \\ u_2' &= y_1 - 2y_2 + y_3, \\ y_3' &= u_3, \\ u_3' &= y_2 - 2y_3. \end{aligned} \tag{e}$$

where primes mean differentiation with respect to τ. The matrix associated to (e) is

$$\mathbf{P}=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ -2 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & -2 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 & -2 & 0\end{bmatrix}. \tag{f}$$

The eigenvalues of this matrix are given by

$$\det\left[\mathbf{P}-\lambda\mathbf{E}\right]=\begin{vmatrix}-\lambda & 1 & 0 & 0 & 0 & 0\\ -2 & -\lambda & 1 & 0 & 0 & 0\\ 0 & 0 & -\lambda & 1 & 0 & 0\\ 1 & 0 & -2 & -\lambda & 1 & 0\\ 0 & 0 & 0 & 0 & -\lambda & 1\\ 0 & 0 & 1 & 0 & -2 & -\lambda\end{vmatrix}=0. \tag{g}$$

The biquadratic equation (g) has purely imaginary roots, i.e.

$$p_1=\mathrm{i}\sqrt{2},\ p_2=\mathrm{i}\sqrt{2-\sqrt{2}},\ p_3=\mathrm{i}\sqrt{2+\sqrt{2}}\ , \tag{h}$$

the other three being their conjugates. After computing the corresponding eigenvectors, the general solution of the system (e) is given by

$$\begin{bmatrix}y_1\\ u_1\\ y_2\\ u_2\\ y_3\\ u_3\end{bmatrix}=\alpha\mathrm{e}^{\mathrm{i}\sqrt{2}\tau}\begin{bmatrix}1\\ \mathrm{i}\sqrt{2}\\ 0\\ 0\\ -1\\ -\mathrm{i}\sqrt{2}\end{bmatrix}+\overline{\alpha}\mathrm{e}^{-\mathrm{i}\sqrt{2}\tau}\begin{bmatrix}1\\ -\mathrm{i}\sqrt{2}\\ 0\\ 0\\ -1\\ -\mathrm{i}\sqrt{2}\end{bmatrix}+\beta\mathrm{e}^{\mathrm{i}\sqrt{2-\sqrt{2}}\tau}\begin{bmatrix}1\\ \mathrm{i}\sqrt{2-\sqrt{2}}\\ 2\\ \mathrm{i}\sqrt{4-2\sqrt{2}}\\ 1\\ \mathrm{i}\sqrt{2-\sqrt{2}}\end{bmatrix}$$
$$+\overline{\beta}\mathrm{e}^{-\mathrm{i}\sqrt{2-\sqrt{2}}\tau}\begin{bmatrix}1\\ -\mathrm{i}\sqrt{2-\sqrt{2}}i\\ 2\\ -\mathrm{i}\sqrt{4-2\sqrt{2}}\\ 1\\ -\mathrm{i}\sqrt{2-\sqrt{2}}\end{bmatrix}+\gamma\mathrm{e}^{\mathrm{i}\sqrt{2+\sqrt{2}}\tau}\begin{bmatrix}1\\ \mathrm{i}\sqrt{2+\sqrt{2}}\\ -\sqrt{2}\\ -\mathrm{i}\sqrt{4+2\sqrt{2}}\\ 1\\ \mathrm{i}\sqrt{2+\sqrt{2}}\end{bmatrix}+\overline{\gamma}\mathrm{e}^{-\mathrm{i}\sqrt{2+\sqrt{2}}\tau}\begin{bmatrix}1\\ -\mathrm{i}\sqrt{2+\sqrt{2}}\\ -\sqrt{2}\\ \mathrm{i}\sqrt{4+2\sqrt{2}}\\ 1\\ -\mathrm{i}\sqrt{2+\sqrt{2}}\end{bmatrix}, \tag{i}$$

where $2\alpha=A_1+\mathrm{i}A_2$, $2\beta=B_1+\mathrm{i}B_2$, $2\gamma=C_1+\mathrm{i}C_2$, A_j, B_j, C_j, $j=1,2$, being arbitrary real constants. Finally, we return to the variable t and choose from the representation (i) only the components of odd index, corresponding to the unknown functions y_1, y_2, y_3, of interest for our problem; we thus get

$$
\begin{aligned}
y_1(t) &= A_1 \cos\sqrt{2}bt - A_2 \sin\sqrt{2}bt + B_1 \cos\left(\sqrt{2-\sqrt{2}}bt\right) - B_2 \sin\left(\sqrt{2-\sqrt{2}}bt\right) + \\
&\quad + C_1 \cos\left(\sqrt{2+\sqrt{2}}bt\right) - C_2 \sin\left(\sqrt{2+\sqrt{2}}bt\right), \\
y_2(t) &= \sqrt{2}\left[B_1 \cos\left(\sqrt{2-\sqrt{2}}bt\right) - B_2 \sin\left(\sqrt{2-\sqrt{2}}bt\right)\right] \\
&\quad - \sqrt{2}\left[C_1 \cos\left(\sqrt{2+\sqrt{2}}bt\right) - C_2 \sin\left(\sqrt{2+\sqrt{2}}bt\right)\right] \\
y_3(t) &= -\left(A_1 \cos\sqrt{2}bt - A_2 \sin\sqrt{2}bt\right) + B_1 \cos\left(\sqrt{2-\sqrt{2}}bt\right) - B_2 \sin\left(\sqrt{2-\sqrt{2}}bt\right) + \\
&\quad + C_1 \cos\left(\sqrt{2+\sqrt{2}}bt\right) - C_2 \sin\left(\sqrt{2+\sqrt{2}}bt\right)
\end{aligned}
\tag{j}
$$

or

$$
\begin{aligned}
y_1 &= \lambda_1 \cos\left(\sqrt{2}bt - \delta_1\right) + \lambda_2 \cos\left(\sqrt{2-\sqrt{2}}bt - \delta_2\right) + \lambda_3 \cos\left(\sqrt{2+\sqrt{2}}bt - \delta_3\right), \\
y_2 &= \sqrt{2}\left[\lambda_2 \cos\left(\sqrt{2-\sqrt{2}}bt - \delta_2\right) - \lambda_3 \cos\left(\sqrt{2+\sqrt{2}}bt - \delta_3\right)\right], \\
y_3 &= -\lambda_1 \cos\left(\sqrt{2}bt - \delta_1\right) + \lambda_2 \cos\left(\sqrt{2-\sqrt{2}}bt - \delta_2\right) + \lambda_3 \cos\left(\sqrt{2+\sqrt{2}}bt - \delta_3\right),
\end{aligned}
\tag{k}
$$

with the notations

$$
\begin{aligned}
&\lambda_1 = \frac{A_1}{\cos\delta_1}, \quad \lambda_2 = \frac{B_1}{\cos\delta_2}, \quad \lambda_3 = \frac{C_1}{\cos\delta_3}, \\
&\tan\delta_1 = -\frac{A_2}{A_1}, \quad \tan\delta_2 = -\frac{B_2}{B_1}, \quad \tan\delta_3 = -\frac{C_2}{C_1}.
\end{aligned}
\tag{l}
$$

The standard method used above often leads to cumbrous computation, despite its generality. In the above considered particular case one may simplify the computation; thus, the system (e) can be directly written in the form

$$
\begin{aligned}
\frac{\mathrm{d}^2 y_1}{\mathrm{d}\tau^2} &= -2y_1 + y_2, \\
\frac{\mathrm{d}^2 y_2}{\mathrm{d}\tau^2} &= y_1 - 2y_2 + y_3, \\
\frac{\mathrm{d}^2 y_3}{\mathrm{d}\tau^2} &= y_2 - 2y_3;
\end{aligned}
\tag{m}
$$

then, subtracting the last equation from the first one, we find out that the function $\varphi = y_1 - y_3$ satisfies the second order ODE with constant coefficients

$$\frac{d^2\varphi}{d\tau^2} + 2\varphi = 0 . \tag{n}$$

The characteristic equation associated to (n) is

$$\lambda^2 + 2 = 0 , \tag{o}$$

and thus

$$\varphi = y_1 - y_3 = \alpha_1 \cos \sqrt{2}\tau + \beta_1 \sin \sqrt{2}\tau = A_1 \cos\left(\sqrt{2} - \delta_1\right), \tag{p}$$

with notations of the form (l).
Further, we add the first and the last equation (m) and get

$$\begin{aligned} &\frac{d^2}{d\tau^2}\left(y_1 + y_3\right) = -2\left(y_1 + y_3\right) + 2y_2 , \\ &\frac{d^2 y_2}{d\tau^2} + 2y_2 = y_1 + y_3 . \end{aligned} \tag{q}$$

Eliminating y_1 between the above two equations, one obtains

$$\frac{d^4 y_2}{d\tau^4} + 4\frac{d^2 y_2}{d\tau^2} + 2y_2 = 0 . \tag{r}$$

The corresponding characteristic equation is

$$\lambda^4 + 4\lambda^2 + 2 = 0 , \tag{s}$$

with the roots $\pm i\sqrt{2-\sqrt{2}}$, $\pm i\sqrt{2+\sqrt{2}}$. Hence, the general solution of the equation (r) is

$$y_2(\tau) = A_2 \cos\left(\sqrt{2 - \sqrt{2}}\tau - \delta_2\right) + A_3 \cos\left(\sqrt{2 + \sqrt{2}}\tau - \delta_3\right). \tag{t}$$

From the second equation (q) we get

$$y_1 + y_3 = \sqrt{2}\left[A_2 \cos\left(\sqrt{2 - \sqrt{2}}\tau - \delta_2\right) - A_3 \cos\left(\sqrt{2 + \sqrt{2}}\tau - \delta_3\right)\right], \tag{u}$$

and, together with (p), the unknowns y_1 and y_3 read

$$y_1 = \frac{A_1}{2}\cos\left(\sqrt{2}\tau - \delta_1\right)$$
$$+\frac{\sqrt{2}}{2}\left[A_2\cos\left(\sqrt{2-\sqrt{2}}\tau - \delta_2\right) - A_3\cos\left(\sqrt{2+\sqrt{2}}\tau - \delta_3\right)\right],$$
$$y_3 = -\frac{A_1}{2}\cos\left(\sqrt{2}\tau - \delta_1\right)$$
$$+\frac{\sqrt{2}}{2}\left[A_2\cos\left(\sqrt{2-\sqrt{2}}\tau - \delta_2\right) - A_3\cos\left(\sqrt{2+\sqrt{2}}\tau - \delta_3\right)\right]. \tag{v}$$

The formulae (t) and (v) represent the general solution of the system in τ, where A_j, δ_j, $j = 1,2,3$, are arbitrary constants. It coincides with the formulae (k) if we return to the variable t and denote by

$$\lambda_1 = \frac{A_1}{2},\ \lambda_2 = \frac{\sqrt{2}}{2}A_2,\ \lambda_3 = -\frac{\sqrt{2}}{2}A_3,$$

without any loss of generality.
The three types of oscillation are indicated in Fig.3.2, b, c, d.

Application 3.3

Problem. Study the translation and the rotation vibrations of a foundation block on an elastic ground.

Mathematical model. The differential equations governing this phenomenon are

$$m\ddot{x} + k_x x - k_x h\varphi = 0, \tag{a}$$

$$J\ddot{\varphi} + \left(k_\varphi - Gh + k_x h^2\right)\varphi - k_x hx = 0, \tag{b}$$

in a plane Ozx, where J is the moment of inertia of the assembly foundation-engine with respect to the *Oy*-axis (normal to the plane *Ozx*), passing through the centre of gravity. $G = mg$ is the weight of the block on the elastic ground, h is the applicate of the centre of gravity with respect to the ground, x is the translation displacement in the direction of the *Ox*-axis, φ is the rotation about the *Oy*-axis, k_x is the horizontal force due to a unit displacement and k_φ is the moment in the plane *Ozx* due to a unit rotation (Fig.3.3).

Solution. The equations (a) and (b) may be written in the form

$$\left(m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + k_x\right)x - k_x h\varphi = 0, \tag{c}$$

$$-k_x h x + \left(J \frac{d^2}{dt^2} + k_x h^2 + h\varphi - Gh \right) \varphi = 0 \,. \tag{d}$$

Eliminating the displacement x between these equations one obtains a linear, homogeneous differential equation of fourth order with constant coefficients for the rotation φ.

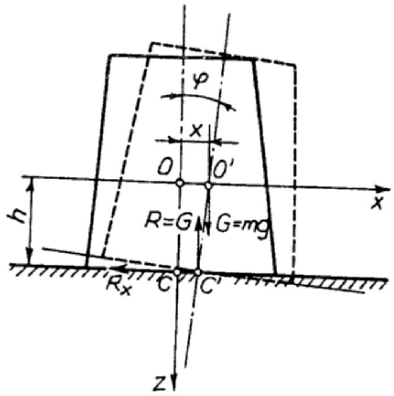

Figure 3. 3. Foundation block on an elastic ground

Searching a solution of the form e^{rt}, we find for r the characteristic equation

$$r^4 + \left(\frac{k_x h^2 + h\varphi - Gh}{J} + \frac{k_x}{m} \right) r^2 + \frac{k_x \left(k_\varphi - Gh \right)}{mJ} = 0 \,. \tag{f}$$

The notations

$$p_x^2 = \frac{k_x}{m} \,, \; p_\varphi^2 = \frac{k_\varphi - Gh}{J + mh^2} \,, \; \gamma = \frac{J}{J + mh^2} \,, \; \gamma \in [0, 1] \,, \tag{g}$$

where p_x is the limit pulsation of the translation vibrations in the absence of rotations, while p_φ is the limit pulsation of the rotaion vibration in the absence of sliding, are introduced.

The biquadratic equation (f) becomes

$$\gamma r^4 + \left(p_x^2 + p_\varphi^2 \right) r^2 + p_x^2 p_\varphi^2 = 0 \,;$$

its roots are given by

$$r^2 = \frac{1}{2\gamma} \left[-\left(p_x^2 + p_\varphi^2 \right) \pm \sqrt{\left(p_x^2 + p_\varphi^2 \right)^2 - 4\gamma p_x^2 p_\varphi^2} \right]$$

and are all imaginary. Hence, the solution of the equations (a) and (b) may be obtained directly in the form

$$\begin{aligned}\varphi &= B\sin(pt+\alpha),\\ x &= C\sin(pt+\alpha),\end{aligned} \qquad \text{(h)}$$

where B, C, α are constants to be determined from the initial conditions. Introducing (h) in (a) and (b), one obtains the linear algebraic system

$$\begin{bmatrix} k_x - mp^2 & -k_x h \\ -k_x h & k_\varphi - Gh + -k_x h^2 - Jp^2 \end{bmatrix}\begin{bmatrix} C \\ B \end{bmatrix} = 0 . \qquad \text{(i)}$$

The system is homogeneous, so that the determinant of the coefficients must be equated to zero, to get non-zero solutions; this leads to the equation of the pulsation p

$$\gamma p^4 + \left(p_x^2 + p_\varphi^2\right)p^2 + p_x^2 p_\varphi^2 = 0 ,$$

which differs from the corresponding equation for r only by a sign (the change of r in p_i). The roots of this equation are

$$p_1^2, p_2^2 = \frac{1}{2\gamma}\left[p_x^2 + p_\varphi^2 \pm \sqrt{\left(p_x^2 + p_\varphi^2\right)^2 - 4\gamma\, p_x^2 p_\varphi^2}\,\right]. \qquad \text{(j)}$$

Hence, in motions with two degrees of freedom the system engine-foundation may oscillate with one of the principal pulsations p_1 or p_2, given by (j).
The ratio of the amplitudes B and C of the two vibrations is of the form

$$\frac{C}{B} = \frac{k_x h}{k_x - mp^2} = \frac{\frac{k_x}{m}h}{\frac{k_x}{m} - p^2} = \frac{p_x^2 h}{p_x^2 - p^2} .$$

The system (a), (b) may be also solved directly, using the standard method for the linear first order ODS, without reducing to only one differential equation (of fourth order in this case). By means of the notations (g) and introducing the auxiliary functions y and ψ , the system (a), (b) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -p_x^2 x + hp_x^2, \\ \dot{\varphi} &= \psi, \\ \dot{\psi} &= \frac{hm}{J}p_x^2 x - \left(\frac{p_\varphi^2}{\gamma} + p_x^2\frac{mh^2}{J}\right)\varphi. \end{aligned} \qquad \text{(k)}$$

The associated characteristic determinant is

$$\det[\mathbf{P}-\lambda\mathbf{E}]=\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -p_x^2 & -\lambda & hp_x^2 & 1 \\ 0 & 0 & -\lambda & 1 \\ \dfrac{hm}{J}p_x^2 & 0 & -\left(\dfrac{p_\varphi^2}{\gamma}+p_x^2\dfrac{mh^2}{J}\right) & -\lambda \end{vmatrix}=\lambda^4+\frac{p_x^2+p_\varphi^2}{\gamma}\lambda^2+\frac{p_x^2 p_\varphi^2}{\gamma}.$$

Equating it with zero, we find the eigenvalues of the matrix **P** of the system (k) which are purely imaginary and coincide with the roots of the equation in r. Taking into account the form of the system, we may search for the solution, as in the previous method, in the form

$$\begin{bmatrix} x \\ y \\ \varphi \\ \psi \end{bmatrix}=\begin{bmatrix} A\sin(rt+\alpha) \\ B\cos(rt+\alpha) \\ C\sin(rt+\alpha) \\ D\cos(rt+\alpha) \end{bmatrix}. \tag{l}$$

From now on, the solution of the problem follows the same way as before.

Application 3.4

Problem. The foundation of an engine of weight Q lays on an elastic medium (Fig.3.4). The area of the foundation basis is S and the coefficient of elasticity of the medium is k_s. To avoid the resonance which may appear during the working, the engine is placed on a rigid bed, connected to the foundation by springs of elastic constant k_1. The weight of the engine and of the bed is P. Determine the pulsation of the system foundation-engine. Numerical data: $Q = 9.8\cdot 10^6 N$, $S = 17m^2$, $k_s = 58.8\cdot 10^6\ N/m^3$, $k_1 = 49\cdot 10^6\ N/m$, $P = 48.02\cdot 10^3 N$.

Mathematical model. The differential equations of the motion are

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = 0, \tag{a}$$

$$m_2\ddot{x}_2 + (k_1 + k)x_2 - k_1 x_1 = 0, \tag{b}$$

where the displacements x_1 and x_2 are measured from the static position of equilibrium of the system and $k = k_s S$.

Solution. The second order ODS given by (a) and (b) may be expressed as a system of first order, introducing auxiliary unknown functions. One may search directly the unknown functions in the form

$$x_1 = C_1 e^{\beta t},\ x_2 = C_2 e^{\beta t};$$

it follows that

$$m_1\beta^2 C_1 + k_1(C_1 - C_2) = 0,$$
$$m_2\beta^2 C_2 + (k_1 + k)C_2 - k_1 C_1 = 0.$$

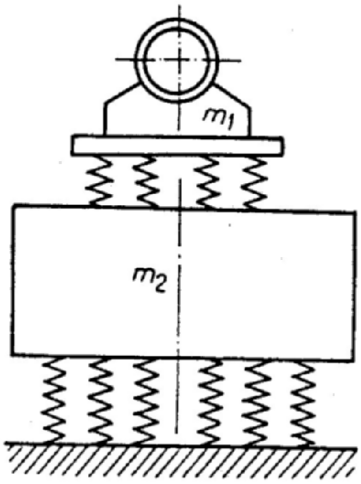

Figure 3. 4. The foundation of an engine on a rigid bed

This is a linear and homogeneous algebraic system in C_1, C_2. To get non-zero solutions, the associated determinant must vanish, i.e.

$$\Delta = \begin{vmatrix} m_1\beta^2 + k_1 & -k_1 \\ -k_1 & m_2\beta^2 + k_1 + k \end{vmatrix} = 0,$$

or

$$\beta^4 + \left(\frac{k_1}{m_1} + \frac{k_1 + k}{m_2}\right)\beta^2 + \frac{k_1 k}{m_1 m_2} = 0. \qquad \text{(c)}$$

Taking into account that $m_1 = P/g$ and $m_2 = Q/g$, the equations (c) becomes

$$\beta^4 + g\left(\frac{k_1}{P} + \frac{k_s S + k_1}{Q}\right)\beta^2 + \frac{k_1 k_s S g^2}{PQ} = 0. \qquad \text{(d)}$$

The roots of this equation are

$$\beta_1^2, \beta_2^2 = -\frac{g}{2}\left[\frac{k_1}{2} + \frac{k_s S + k_1}{Q} \pm \sqrt{\left(\frac{k_1}{P} + \frac{k_s S + k_1}{Q}\right)^2 - \frac{4k_1 k_s S}{PQ}}\right].$$

We denote by $\beta_i^2 = -p_i^2$, $i = 1, 2$. Introducing numerical data, we have

$$p_1^2, p_2^2 = \frac{9.81}{2}\left[\frac{49\cdot 10^6}{48.02\cdot 10^3} + \frac{58.8\cdot 10^6\cdot 17 + 49\cdot 10^6}{9.8\cdot 10^6} \pm \right.$$

$$\left. \pm\sqrt{\left(\frac{49\cdot 10^6}{48.02\cdot 10^3} + \frac{58.8\cdot 10^6\cdot 17 + 49\cdot 10^6}{9.8\cdot 10^6}\right)^2 - \frac{4.49\cdot 10^6\cdot 458.810^6\cdot 17}{48.02\cdot 10^3\cdot 9.8\cdot 10^6}}\right] =$$

$$= 4.905\left(1020.408163 + 107 \pm \sqrt{1127.408163^2 - 416326.5308}\right)$$

$$= 4.905(1127.408163 + 107 \pm 924.512107),$$

therefore

$$p_1^2 = 995.2051568\ s^{-2},\ p_2^2 = 10064.66892 s^{-2},$$

and

$$p_1 = 31.547 s^{-1},\ p_2 = 100.323 s^{-1}.$$

Because the roots of the characteristic equation (d) are purely imaginary in this case, the solution of the system (a), (b) is of the form

$$x_1 = A_1 \sin(p_1 t + \alpha_1) + A_2 \sin(p_2 t + \alpha_2),$$

$$x_2 = A_1\left(1 - \frac{m_1 p_1^2}{k_1}\right)\sin(p_1 t + \alpha_1) + A_2\left(1 - \frac{m_1 p_2^2}{k_l}\right)\sin(p_2 t + \alpha_2).$$

Application 3.5

Problem. An engine of mass *M*, staying on an elastic spring of constant *K* is subjected to a vertical pulsatory force $F = F_0 \sin \omega t$. Because, for a certain velocity of running of the engine, the frequence of the pulsatory force may become equal to the frequence of the eigenvibrations of the system (M, K) it appears the risk of resonance (Fig.3.5, a); it is useful to fit out the equipment by a dynamic damper, formed by a mass m linked to the engine M by a spring of elastic constant k (Fig.3.5, b). The system thus obtained has two degrees of freedom.

Mathematical model. The ODS modelling the phenomenon is of the form

$$m\ddot{y} = -k(y - x), \tag{a}$$

$$M\ddot{x} = k(y - x) - Kx + F_0 \sin \omega t, \tag{b}$$

and the boundary conditions are

$$x(0) = 0,\ y(0) = 0,\ \dot{x}(0) = 0,\ \dot{y}(0) = 0. \tag{c}$$

Solution. By using the notations

$$\frac{K}{M} = \alpha^2, \ \frac{k}{m} = \beta^2, \ \frac{k}{M} = \gamma^2, \ \frac{F_0}{M} = f_0, \tag{d}$$

the differential equations (a), (b) become

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \beta^2\right)y - \beta^2 x = 0, \tag{e}$$

$$-\gamma^2 y + \left(J\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \alpha^2 + \gamma^2\right)x = f_0 \sin\omega t. \tag{f}$$

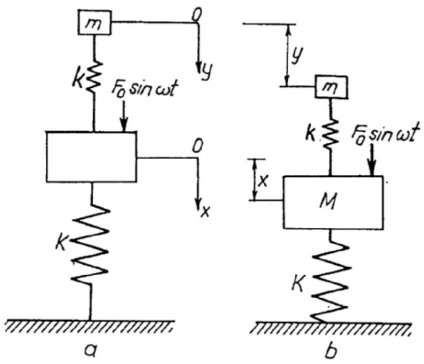

Figure 3. 5. Resonance of the mechanical system (a). Dynamic damper (b)

Eliminating the function y between these equations, we find

$$\left[\frac{\mathrm{d}^4}{\mathrm{d}t^4} + \left(\alpha^2 + \beta^2 + \gamma^2\right) + \alpha^2\beta^2\right]x = f_0\left(\beta^2 - \omega^2\right)\sin\omega t. \tag{g}$$

Similarly, the function x may be eliminated and it results

$$\left[\frac{\mathrm{d}^4}{\mathrm{d}t^4} + \left(\alpha^2 + \beta^2 + \gamma^2\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \alpha^2\beta^2\right]y = f_0\beta^2 \sin\omega t. \tag{h}$$

As it should be expected, the differential operator applied to the functions x and y is the same, because the system (e), (f) is linear with constant coefficients.

Noting that (g), (h) contain only derivatives of even order, we may search particular solutions of the form

$$x_p = A\sin\omega t, \ y_p = B\sin\omega t. \tag{i}$$

Introducing (i) in (g) and (h), we get

$$x_p = \frac{1}{N} f_0 \left(\beta^2 - \omega^2\right) \sin \omega t \,, \ y_p = \frac{1}{N} f_0 \beta^2 \sin \omega t \,, \tag{j}$$

with the notation

$$N = \omega^4 - \left(\alpha^2 + \beta^2 + \gamma^2\right)\omega^2 + \alpha^2 \beta^2 \,. \tag{k}$$

The eigenvibrations (represented by the solution of the homogeneous equations) may be neglected, remaining only the forced vibrations (represented by the particular solution (i)). From (i), one observes that the masses m and M have a simple harmonic motion after the eigenvibrations tend to zero.

The reaction between the pulsatory force F and the system (M, K) works when the frequency ω or F and the eigenfrequency $\alpha = \sqrt{K/M}$ of the system (M, K) are equal. Taking $\alpha = \omega$, the expressions (i) become

$$x_p = -\frac{1}{\omega^2 \gamma^2} f_0 \left(\beta^2 - \omega^2\right) \sin \omega t \,, \ y_p = -\frac{1}{\omega^2 \gamma^2} f_0 \beta^2 \sin \omega t \,; \tag{l}$$

it is thus proved that the amplitude of x_p, which – normally – tends to infinity, is reduced – due to the damper – to the finite value $f_0 \left(\beta^2 - \omega^2\right) / \omega^2 \gamma^2$.

If the values of k and m of the damper are such that $\alpha = \beta = \omega$, then the relations (l) are reduced to

$$x_p = 0 \,, \ y_p = -\frac{1}{\gamma^2} f_0 \sin \omega t \,;$$

this proves that the *damper* called *syntonized*, completely cancels the vibrations of *M*.

Chapter 4

NON-LINEAR ODEs OF FIRST AND SECOND ORDER

1. First Order Non-Linear ODEs

1.1 FORMS OF FIRST ORDER ODEs AND OF THEIR SOLUTIONS

1.1.1 Forms of ODEs

A first order ODE may appear in various forms, according to the modelled physical phenomenon and it also may be put in forms better suited to the method of solving it.

a) The general form

$$F(x, y, y') = 0, \quad y' = \frac{\mathrm{d}y}{\mathrm{d}x}, \tag{4.1.1}$$

also called *the implicit form*.

If $\partial F / \partial y' \neq 0$, then, according to the implicit function theorem, we can express y' as a function of *x* and *y*, thus getting

b) The canonic/normal/explicit form

$$y' = f(x, y), \quad y' = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{4.1.2}$$

Writing this as $\mathrm{d}y = f(x, y)\mathrm{d}x$, we observe that a first order ODE may also be expressed in

c) The differential form

$$P(x, y)dx + Q(x, y)dy = 0. \tag{4.1.3}$$

Dividing by the product *PQ* and re-noting the functions, this can also be written in

d) The symmetric form

$$\frac{\mathrm{d}x}{X(x, y)} = \frac{\mathrm{d}y}{Y(x, y)}. \tag{4.1.4}$$

1.1.2 Forms of the solutions

We firstly define the types of solutions of first order ODEs.

1. A *solution* of a first order ODE is a function of class $C^1(I)$, $I \subseteq \Re$, identically satisfying the ODE for any $x \in I$.

2. *The general solution* is a function $y = \varphi(x, C)$, depending on the arbitrary constant *C*, that satisfies the given ODE for any admissible *C*.
3. *The particular solutions* are obtained from the general one by giving numerical values to *C*.
4. *The singular solutions* are those solutions of the ODE that cannot be obtained from the general one by particularizing the constant *C*.

The constant *C* is determined imposing a supplementary condition. For instance, it is required that $y(x_0) = y_0$, where x_0, y_0 are previously given. This is a *Cauchy condition* (see also the Introduction).
The forms in which there can be obtained the solutions of first order ODEs are

a) the explicit form

$$y = \varphi(x), \tag{4.1.5}$$

b) the implicit form

$$\Phi(x, y) = 0 \tag{4.1.6}$$

and

c) the parametric form

$$\begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}, \quad t \in [a, b] \subset \Re . \tag{4.1.7}$$

Example. Consider the ODE $y' = -\dfrac{x}{y}$, defined for $y > 0$. Then

a) the function $y = \sqrt{1 - x^2}$, $x \in (-1, 1)$ is an *explicit* solution of the ODE;

b) the function $x^2 + y^2 = 1$ is an *implicit* solution. Indeed, differentiating both members, we get $2x\mathrm{d}x + 2y\mathrm{d}y = 0$, or $\mathrm{d}y / \mathrm{d}x = -x / y$;

c) the functions

$$\begin{cases} x = \cos t, \\ y = \sin t, \end{cases} \quad t \in (-\pi, \pi),$$

determine a *parametric* solution. Indeed,

$$\begin{cases} \mathrm{d}x = -\sin t \,\mathrm{d}t, \\ \mathrm{d}y = \cos t \,\mathrm{d}t, \end{cases}$$

whence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos t \,\mathrm{d}t}{-\sin t \,\mathrm{d}t} = -\frac{\cos t}{\sin t},$$

which coincides with $-x / y = -\cos t / \sin t$, therefore the parametric solution identically satisfies the given ODE.

1.2 GEOMETRIC INTERPRETATION. THE THEOREM OF EXISTENCE AND UNIQUENESS

Let $f(x, y)$ be a function depending on the real variables x and y; suppose that the point $P(x, y)$ belongs to an open set $\Omega \subset \Re^2$. The function $f(x, y)$ defines an ODE with one unknown function, of first order with respect to the independent variable x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y). \tag{4.1.8}$$

To solve this ODE means to find all its solutions and to study their behaviour.
We call *solution* or *integral curve* or, simply, *integral* of (4.1.1), a function $y = \varphi(x)$, defined on a real open interval $I \equiv [a, b] \in \Re$, of class $\mathrm{C}^1(I)$, that satisfies

$$\varphi'(x) = f(x, \varphi(x)), \quad \forall x \in I, \tag{4.1.9}$$

if, moreover, the points $(x, \varphi(x))$ belong to Ω for any $x \in I$.
To solve the associated *Cauchy* (or *initial*) *problem* means to find those solutions of (4.1.1) that satisfy

$$y(x_0) = y_0, \tag{4.1.10}$$

where (x_0, y_0) is a given point, belonging to Ω.

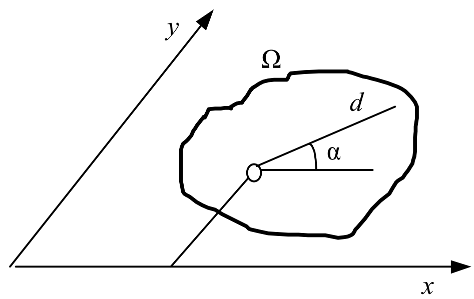

Figure 4. 1. The contact element

In what follows, we shall see that, under certain convenient hypotheses concerning the regularity of *f*, the Cauchy problem (4.1.1), (4.1.10) allows at least one solution; the uniqueness is ensured only if *f* satisfies some supplementary conditions.
Let us define, for every point $P(x, y)$ belonging to Ω, the angle α, by the formula

$$\tan\alpha = f(x, y). \tag{4.1.11}$$

The point $P(x, y)$ forms, together with the angle α, the so called *contact element* or *linear element*. The set of all contact elements is called *field of directions*; this field of directions defines the differential ODE (4.1.1).
Hence, a solution – or, equivalently, an integral curve – of the ODE (4.1.1) is a curve possessing a tangent of slope α at each of its points $P(x, y)$, with the property that $P(x, y)$ and α are, all of them, contact elements of (4.1.1).
In Fig.4.1 we give an intuitive representation of the contact element.
We shall give two classical examples that are significant for the importance of these notions.

Example 1. Consider the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2. \tag{4.1.12}$$

To draw the integral curves, we firstly shall draw the curves for which the slope is the same; these curves are called *isoclines*. For example, if $y' = 0$, it follows that $x = 0, y = 0$. For $y' = 1/2$, we find $x^2 + y^2 = 1/2$, i.e. a circle centered at the origin, of radius $1/\sqrt{2}$; the unit circle corresponds to $y' = 1$, a.s.o. (see Fig.4.2, a). We then choose in the plane a point of co-ordinates (x_0, y_0) and we draw a curve passing through this point and has, at any of its points, a tangent parallel to the field direction; according to the previous considerations, this will be an integral curve of the ODE (4.1.12). Choosing another point, we find another integral curve. In the Fig.4.2, a there are drawn those integral curves passing through the points $(0,0)$, $(0,-1/2)$, $(\sqrt{2},0)$. One finally obtain a family of integral curves depending on a parameter.

Example 2. The first order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}. \tag{4.1.13}$$

defines a field of directions in the whole plane, except for the origin. In the current point $M(x, y)$ the field direction is perpendicular on the vector radius *OM*. Due to this property, the integral curves will be circles centered at the origin, of arbitrary radii, and they will be represented analitically by the expression

$$y = \pm\sqrt{C^2 - x^2},$$

where C is an arbitrary real constant.

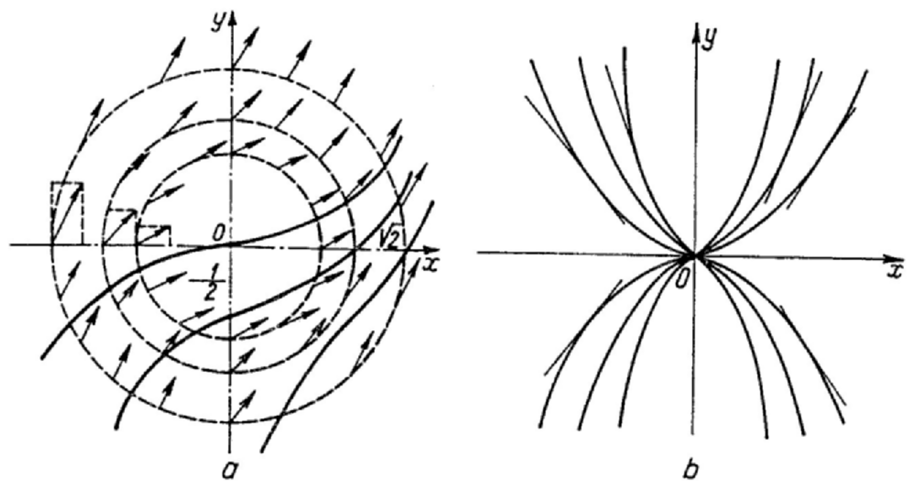

Figure 4. 2. Field of directions and integral curves for the ODE (4.1.12) (a); Field of directions and integral curves for the ODE (4.1.14) (b)

Example 3. In Fig.4.2, b it is represented the field of directions corresponding to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\frac{y}{x}, \tag{4.1.14}$$

formed by the tangents to the parabolae $y = Cx^2$.

A study of uniqueness and existence of the solution of the Cauchy problem associated to the ODE (4.1.1) may be tackled in many ways, following the functional frame in use.

To enounce the classic theorem of existence and uniqueness some preliminary notions must be introduced: the maximal solution and the Lipschitz property.

If $y = \varphi(x), x \in I$, is a solution of (4.1.1), then any of its restrictions to a subinterval of I is also a solution. This remark permits the introduction of an order relationship on the set of the solutions of (4.1.1); more precisely, if $\varphi_1, x \in I_1$, and $\varphi_2, x \in I_2$, are two solutions, then we say that φ_1 is "smaller" than φ_2 and we write $\varphi_1 \prec \varphi_2$ if $I_1 \subset I_2$ and $\varphi_1(x) = \varphi_2(x)$ for any $x \in I_1$. In fact, $\varphi_1 \prec \varphi_2$ means that φ_2 is the prolongation of φ_1. Any maximal element of the set of solutions is called a *maximal solution*. According to this definition, such a solution cannot be anymore prolonged in Ω. One can also prove that any solution is "smaller" than a certain maximal solution.

We say that the function $f(x, y)$ is *Lipschitzian* with respect to *y* if one can find a constant $K > 0$ such that

$$|f(x, y_1) - f(x, y_2)| < K|y_1 - y_2|, \quad (x, y_1) \in \Omega, \ (x, y_2) \in \Omega. \tag{4.1.15}$$

The function $f(x, y)$ is called *locally Lipschitzian* if any point of Ω has a neighbourhood on which *f* is Lipschitzian.

There are large classes of functions with Lipschitz's propriety; e.g., the analytic functions and, in general, the functions of bounded derivatives with respect to y are also Lipschitzian.

A function f may be Lipshitzian in y without being continuous with respect to (x, y). Indeed, let $f(x, y) = g(x) + y$; this function is obviously Lipschitz with respect to y, independently of the continuity of g.

Let us also note that a locally Lipschitz function is not necessarily Lipschitz on its whole domain of definition; as an example, let us take $f(x, y) = y^2, (x, y) \in \Omega \equiv \Re^2$. With these preparations, one can state

Theorem 4.1. *Let* $f(x, y)$ *be defined and continuous on the open set* $\Omega \subset \Re^2$. *Then there is a unique maximal solution of* (4.1.1) *passing through any arbitrary point of* Ω.

Yet, there are simple ODEs that do not fit the conditions of this theorem and for which the uniqueness of the solution is not ensured. Indeed, let us consider the ODE

$$y' = 3y^{2/3}. \tag{4.1.16}$$

The right member is defined and continuous on $\Re^2$. Yet, there are at least two solutions, $y_1 = 0, y_2 = x^3$, passing through $(0,0)$. Actually, there are infinitely many solutions passing through any point of the plane. The most general form of the solutions passing through the origin is represented by the function

$$y(x) = \begin{cases} (x-a)^3, & x \le a, \\ 0, & a < x \le b,. \\ (x-b)^3, & x > b, \end{cases} \tag{4.1.17}$$

where $a \le 0, b \ge 0$.

Intuitively, the Lipschitz propriety plays an important part in what concerns the uniqueness of the solution. What does this mean from the geometric point of view?

Let $P_1(x, y_1)$ and $P_2(x, y_2)$ be two points in Ω and let let Q be the piercing point of the right lines corresponding to the elements of contact of P_1 and P_2 (Fig.4.3).

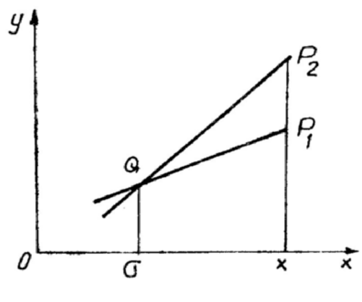

Figure 4. 3. The Lipschitz propriety

From the figure, it is seen that

$$\frac{f(x, y_1) - f(x, y_2)}{y_1 - y_2} = \frac{1}{x - \sigma} \equiv \rho . \tag{4.1.18}$$

If f is Lipschitz, then it satisfies the inequality (4.1.12) and thus the relationship (4.1.15) involves $|\rho| \le K$, for all the points of Ω. Consequently, any condition imposing to ρ values greater than $1/x$ automatically yields the uniqueness of the solution.

This remark may represent a starting point in considering some hypotheses – other than the Lipschitz propriety – yielding the existence and uniqueness of solutions.

In what concerns the Cauchy problem, the (local) existence and the uniqueness of the solutions are ensured by

Theorem 4.2 (Cauchy-Picard-Lipschitz). *If*

i) $f \in C^0(D)$, where $D = \{(x, y) \in \Re^2, |x - x_0| \le a, |y - y_0| \le b\}$,

ii) *f is Lipschitz in y, i.e.* $\exists K > 0 : |f(x, Y) - f(x, Z)| < K|Y - Z|, (x, Y), (x, Z) \in D$,

then the Cauchy problem (4.1.1), (4.1.10) *allows a unique solution* $y = y(x)$, *of class* $C^1(I), I = [x_0 - h, x_0 + h]$, *where* $h = \min\{a, b / M\}, M = \sup_{(x,y)\in D} |f(x, y)|$.

Remark. If f is only continuous in D, then one can only ensure the existence of the solution (the Cauchy-Peano theorem), but uniqueness may fail, as in the case of equation (4.1.16).

The proof of Theorem 4.2 is constructive, being based on *the method of successive approximations*, also called *the Picard-Lindelöff method*; by using it, one can get analytic approximates of the solution of the Cauchy problem (4.1.1), (4.1.10).

1.3 ANALYTIC METHODS FOR SOLVING FIRST ORDER NON-LINEAR ODEs

a) *The method of successive approximations*

Suppose that f satisfies the hypotheses of theorem 4.2. Then one sets up on the rectangle D the recurrent sequence of functions, defined as follows

$$\begin{aligned} & y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,\mathrm{d}t, \\ & y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,\mathrm{d}t, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,\mathrm{d}t, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{4.1.19}$$

It is proved that the sequence $\{y_n\}_{n\in\mathfrak{N}}$ is uniformly and absolutely convergent to the solution of the Cauchy problem (4.1.1), (4.1.10) on the interval *I*, centred at x_0, of length *h*, defined in theorem 4.2. More precisely, it is shown that the following inequality holds true

$$|y_n(x)-y(x)|\le\frac{M}{K}\sum_{j=n+1}^{\infty}\frac{K^j}{j!}|x-x_0|^j,\qquad |x-x_0|\le h. \tag{4.1.20}$$

The above inequality allows a good enough evaluation of the distance between the approximate and the solution itself.

b) *The method of power series expansion*

If *f* is infinitely many differentiable with respect to both its arguments, then $y(x)$ will alow a Taylor series expansion around x_0

$$\begin{aligned}y(x)&=y(x_0)+\frac{(x-x_0)}{1!}y'(x_0)+\frac{(x-x_0)^2}{2!}y''(x_0)\\&+\ldots+\frac{(x-x_0)^n}{n!}y^{(n)}(x_0)+R_n(x,x_0),\end{aligned} \tag{4.1.21}$$

where $R_n(x,x_0)$ is the remainder. The Lagrange's form for the remainder reads

$$R_n(x,x_0)=\frac{(x-x_0)^{n+1}}{(n+1)!}y^{(n+1)}(\xi),\quad \xi\in(x_0,x), \tag{4.1.22}$$

so that, if $\sup\limits_{(x,y(x))\in D}\left|f^{(n)}(x,y)\right|\le M$, then

$$|R_n(x,x_0)|\le M\frac{|x-x_0|^{n+1}}{(n+1)!}y^{(n+1)}(\xi),\quad \xi\in(x_0,x). \tag{4.1.23}$$

Therefore, in a close neighbourhood of x_0 the remainder is small enough to be neglected; thus, the solution of (4.1.1), (4.1.10) can be approximated by *Taylor's polynomial*

$$y(x)=y(x_0)+\frac{(x-x_0)}{1!}y'(x_0)+\ldots+\frac{(x-x_0)^n}{n!}y^{(n)}(x_0), \tag{4.1.24}$$

whose coefficients $y^{(k)}(x_0)$ are computed step by step, by using the chain rule

$$
\begin{aligned}
&y(x_0) = y_0\,,\\
&y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0),\\
&y''(x_0) = \frac{\partial f}{\partial x}(x_0, y_0) + \frac{\partial f}{\partial y}(x_0, y_0)y'(x_0) = \frac{\partial f}{\partial x}(x_0, y_0) + f(x_0, y_0)\frac{\partial f}{\partial y}(x_0, y_0),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{4.1.25}
$$

In the particular case in which f can be developd as a double power series in x and y around (x_0, y_0), i.e.,

$$
f(x, y) = \sum_{j,k=0}^{\infty} a_{jk} (x - x_0)^j (y - y_0)^k\,, \tag{4.1.26}
$$

also expanding y in a power series around x_0

$$
y(x) = y_0 + \sum_{n=1}^{\infty} c_n (x - x_0)^n\,, \tag{4.1.27}
$$

convergent for $|x - x_0| \leq h$, therefore on the interval I from Theorem 4.2, we get, introducing both developments in the ODE (4.1.1),

$$
\sum_{j,k=0}^{\infty} a_{jk} (x - x_0)^j \left[\sum_{n=1}^{\infty} c_n (x - x_0)^n \right]^k = \sum_{m=1}^{\infty} c_m (x - x_0)^{m-1}\,. \tag{4.1.28}
$$

From (4.1.28) we obtain by identification the coefficients c_m

$$
\begin{aligned}
&c_1 = a_{00}\,,\\
&2c_2 = a_{10} + a_{01}c_1\,,\\
&3c_3 = a_{20} + a_{11}c_1 + a_{02}c_1^2 + a_{01}c_2\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{4.1.29}
$$

1.4 FIRST ORDER ODEs INTEGRABLE BY QUADRATURES

There are several types of ODEs of first order that may be solved by special methods, leading to general solutions expressed in terms of first integrals of known functions. We shall give here some of the most usual such types.

1.4.1 ODEs with separate variables

The ODEs with separate variables are of the form

$$
X(x)\mathrm{d}x + Y(y)\mathrm{d}y = 0\,, \tag{4.1.30}
$$

where the functions X and Y are supposedly continuous with respect to the variables x and y respectively. In this case, the ODE can be integrated directly, obtaining the general solution in the form

$$\int X(x)\mathrm{d}x + \int Y(y)\mathrm{d}y = C\,, \tag{4.1.31}$$

or else

$$\int_{x_0}^{x} X(t)\mathrm{d}t + \int_{y_0}^{y} Y(t)\mathrm{d}t = C\,. \tag{4.1.32}$$

The Cauchy problem for (4.1.30) consists of finding the integral curve that passes through the point (x_0, y_0); this solution reads

$$\int_{x_0}^{x} X(t)\mathrm{d}t + \int_{y_0}^{y} Y(t)\mathrm{d}t = 0\,. \tag{4.1.33}$$

1.4.2 ODEs with separable variables

These ODEs are of the form

$$P(x)q(y)\mathrm{d}x + p(x)Q(y)\mathrm{d}y = 0. \tag{4.1.34}$$

If $p(x), q(y)$ do not vanish, then we divide the ODE by the product $p(x)q(y)$, thus getting

$$\frac{P(x)}{p(x)}\mathrm{d}x + \frac{Q(y)}{q(y)}\mathrm{d}y = 0\,, \tag{4.1.35}$$

which is an ODE with separate variables. The general solution of (4.1.34) is then

$$\int \frac{P(x)}{p(x)}\mathrm{d}x + \int \frac{Q(y)}{q(y)}\mathrm{d}y = C\,. \tag{4.1.36}$$

1.4.3 Homogeneous first order ODEs

A function $f : \Re^2 \to \Re$ is called *homogeneous of degree m* if

$$f(tx, ty) = t^m f(x, y), \forall t, x, y \in \Re\,.$$

The ODE

$$P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = 0. \tag{4.1.37}$$

in which $P, Q \in \mathrm{C}^0(D), D \subseteq \Re^2$ are homogeneous functions of the same degree *m*, is called *homogeneous*.

By using the change of function

$$y = zx\,. \tag{4.1.38}$$

the ODE (4.1.37) becomes, after simplification by x^m (we take $t = 1/x$)

$$[P(1,z)+zQ(1,z)]\mathrm{d}x + xQ(1,z)\mathrm{d}z = 0, \tag{4.1.39}$$

i.e., an ODE with separable variables. Its general solution is therefore

$$x = C\mathrm{e}^{\psi(z)}, \quad \psi(z) = -\int \frac{Q(1,z)}{P(1,z)+zQ(1,z)}\mathrm{d}z = 0 \tag{4.1.40}$$

and getting back to the variables *x*, *y*, the general solution of (4.1.37) is

$$x = C\mathrm{e}^{\psi\left(\frac{y}{x}\right)}, \tag{4.1.41}$$

where ψ is not defined for $x = 0$.

1.4.4 ODEs of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\right). \tag{4.1.42}$$

a) If $\Delta \equiv a\beta - b\alpha \neq 0$, then the linear algebraic system

$$\begin{aligned} ax+by+c &= 0,\\ \alpha x+\beta y+\gamma &= 0, \end{aligned} \tag{4.1.43}$$

allows the unique solution (x_0, y_0), as its determinant Δ is not null. By using the change of variables $u = x - x_0, v = y - y_0$, we reduce (4.1.41) to the 0-degree homogeneous ODE

$$\frac{\mathrm{d}v}{\mathrm{d}u} = f\left(\frac{au+bv}{\alpha u+\beta v}\right). \tag{4.1.44}$$

b) If $\Delta \equiv a\beta - b\alpha = 0$, then $a/\alpha = b/\beta = \lambda$, and therefore $ax+by = \lambda(\alpha x+\beta y)$. Denoting by $t = ax+by$, we get $\mathrm{d}t/dx = a + b\mathrm{d}y/\mathrm{d}x$, whence

$$\frac{\mathrm{d}t}{\mathrm{d}x} = a + bf\left(\frac{\lambda t+\lambda c}{t+\gamma\lambda}\right), \tag{4.1.45}$$

i.e., an ODE with separable variables.

1.4.5 Total differential ODEs

By definition, an ODE

$$P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = 0, \; P,Q \in \mathrm{C}^0(D), D \subseteq \Re^2, \tag{4.1.46}$$

is called a *total differential ODE* if there exists a differentiable function $F = F(x, y)$ such that $\mathrm{d}F \equiv P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$. Consequently, the general solution of a total differentiable ODE is

$$F(x, y) = C\,, \tag{4.1.47}$$

where C is an arbitrary constant. So, solving such an ODE is equivalent to finding a function of two real variables given its first order differential.

It is well known that, if $P, Q \in \mathrm{C}^0(D)$, then $\mathrm{d}F \equiv P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$ if and only if

$$\frac{\partial P}{\partial y}(x, y) = \frac{\partial Q}{\partial x}(x, y), \quad (x, y) \in D\,. \tag{4.1.48}$$

Thus, to solve a total differential ODE one must observe the following two steps:

1) One computes the partial derivatives $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$; if they coincide, then the ODE is with total differentials, i.e., there exists F such that $\mathrm{d}F \equiv P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y$.

2) As the first differential of a function F is

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y\,, \tag{4.1.49}$$

one must have

$$\frac{\partial F}{\partial x}(x, y) = P(x, y), \quad \frac{\partial F}{\partial y}(x, y) = Q\,, \quad (x, y) \in I\,. \tag{4.1.50}$$

Integrating the first relationship with respect to x, we find

$$F(x, y) = \int_{x_0}^{x} P(t, y)\mathrm{d}t + \varphi(y)\,, \tag{4.1.51}$$

where φ is an arbitrary function depending only on y. Differentiating both members of (4.1.51) with respect to y, we get

$$\frac{\partial F}{\partial y} = \int_{\mathrm{x}_0}^{x} \frac{\partial P}{\partial y}(t, x)\mathrm{d}t + \varphi'(y)\,, \tag{4.1.52}$$

where x_0 is fixed up, but arbitrarily chosen, such that (x_0, y) belong to D. Taking now (4.1.50) into account, it results

$$\frac{\partial F}{\partial y} = \int_{\mathrm{x}_0}^{x} \frac{\partial Q}{\partial t}(t, x)\mathrm{d}t + \varphi'(y) = Q(x, y) - Q(x_0, y) + \varphi'(y)\,. \tag{4.1.53}$$

Comparing this with the expression of $\partial F / \partial y$ from (4.1.52), it follows

$$Q(x,y) - Q(x_0,y) + \varphi'(y) = Q(x,y), \tag{4.1.54}$$

whence

$$\varphi'(y) = Q(x_0,y); \tag{4.1.55}$$

thus, φ is given by

$$\varphi(y) = \int_{y_0}^{y} Q(x_0,t)\mathrm{d}t, \tag{4.1.56}$$

y_0 being chosen in the same conditions as x_0. Eventually, we find F in the form

$$F(x,y) = \int_{x_0}^{x} P(t,y)\mathrm{d}t + \int_{y_0}^{y} Q(x_0,t)\mathrm{d}t. \tag{4.1.57}$$

The general solution of the ODE with total differentials (4.1.46) is

$$\int_{x_0}^{x} P(t,y)\mathrm{d}t + \int_{y_0}^{y} Q(x_0,t)\mathrm{d}t = C, \tag{4.1.58}$$

where C is an arbitrary constant.
If we firstly integrate the second relation (4.1.50) with respect to y, we obtain the general solution of (4.1.46) in an equivalent form

$$\int_{x_0}^{x} P(x,y_0)\mathrm{d}t + \int_{y_0}^{y} Q(x,t)\mathrm{d}t = C. \tag{4.1.59}$$

1.4.6 Integrant factor

In most of cases, an ODE is not a total differential one. In this case, we can still use this idea by looking for a function $\mu = \mu(x,y)$ such that

$$\mu(x,y)[P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y] = 0, \quad P,Q \in \mathrm{C}^1(D), D \subseteq \Re^2 \tag{4.1.60}$$

be a total differential ODE.
The function $\mu = \mu(x,y)$ is called an *integrant factor*. One can prove several important fact, ensuring the existence and the form of the integrant factors of a given ODE.

a) *One can always find an integrant factor for a given first order ODE*. Indeed, the general solution of the ODE (4.1.46) may be written in the implicit form

$$F(x,y) = C. \tag{4.1.61}$$

Then

$$\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y = 0\,, \tag{4.1.62}$$

which means

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}}\,. \tag{4.1.63}$$

On the other hand, from the ODE (4.1.46), we also get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{P(x,y)}{Q(x,y)}\,. \tag{4.1.64}$$

This yields

$$\frac{\dfrac{\partial F}{\partial x}}{P(x,y)} = \frac{\dfrac{\partial F}{\partial y}}{Q(x,y)} = \mu(x,y) \tag{4.1.65}$$

and so

$$\frac{\partial F}{\partial x} = \mu(x,y)P(x,y), \quad \frac{\partial F}{\partial y} = \mu(x,y)Q(x,y)\,. \tag{4.1.66}$$

This means that μ is an integrant factor for (4.1.46).

b) *A first order ODE allows infinitely many integrant factors*. Indeed, if μ is an integrant factor for (4.1.46) and $F(x,y) = C$, for some C, is one of its integral curves, then any $\lambda(x,y) = \varphi(F(x,y))\mu(x,y)$ is also an integrant factor, as

$$\lambda(x,y)[P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y] = \varphi(F)[\mu(x,y)P(x,y)\mathrm{d}x + \mu(x,y)Q(x,y)\mathrm{d}y] = 0. \tag{4.1.67}$$

Thus, $\lambda(x,y)P(x,y)\mathrm{d}x + \lambda(x,y)Q(x,y)\mathrm{d}y$ is the differential of the function

$$\Phi(F) = \int \varphi(F)\mathrm{d}F\,, \tag{4.1.68}$$

i.e., $\lambda(x,y)$ is an integrant factor for (4.1.46).

c) *Any integrant factor of* (4.1.46) *is of the form* $\varphi(F(x,y))\mu(x,y)$. Let λ be another integrant factor, different from μ. Then we have

$$\begin{aligned}\mu(x,y)P(x,y)\mathrm{d}x + \mu(x,y)Q(x,y)\mathrm{d}y &= \mathrm{d}F(x,y),\\ \lambda(x,y)P(x,y)\mathrm{d}x + \lambda(x,y)Q(x,y)\mathrm{d}y &= \mathrm{d}G(x,y).\end{aligned} \tag{4.1.69}$$

Therefore, according to the properties of the first order differentials, we have

$$\mu P = \frac{\partial F}{\partial x}, \quad \mu Q = \frac{\partial F}{\partial y},$$
$$\lambda P = \frac{\partial G}{\partial x}, \quad \lambda Q = \frac{\partial G}{\partial y}, \tag{4.1.70}$$

involving

$$\frac{D(F,G)}{D(x,y)} \equiv \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{vmatrix} = 0, \quad (x,y) \in D . \tag{4.1.71}$$

According to the properties of the Jacobian, it exists then $\Phi = \Phi(F)$ such that $G(x,y) = \Phi(F(x,y))$. So,

$$\begin{aligned} &\lambda(x,y)[P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y] = \lambda(x,y)P(x,y)\mathrm{d}x + \lambda(x,y)Q(x,y)\mathrm{d}y = \mathrm{d}G(x,y) \\ &= \Phi'(F)\mathrm{d}F(x,y) = \Phi'(F)[\mu(x,y)P(x,y)\mathrm{d}x + \mu(x,y)Q(x,y)\mathrm{d}y] \\ &= \Phi'(F)\mu(x,y)[P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y]. \end{aligned} \tag{4.1.72}$$

This yields precisely that

$$\lambda(x,y) = \Phi'(F)\mu(x,y). \tag{4.1.73}$$

Consequence. If one knows two qualitatively different integrant factors, say λ and μ, of a first order ODE, then its general solution is written without quadrature

$$\frac{\lambda(x,y)}{\mu(x,y)} = C . \tag{4.1.74}$$

d) *Getting an integrant factor*. If (4.1.60) is a total differential ODE, then

$$\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q), \tag{4.1.75}$$

or

$$Q\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial y} = \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right). \tag{4.1.76}$$

Let us find for (4.1.76) solutions of the form $\mu = \mu(\omega)$, where ω is a known functionm depending on x and y. As

$$\frac{\partial \mu}{\partial x} = \frac{\mathrm{d}\mu}{\mathrm{d}\omega}\frac{\partial \omega}{\partial x}, \qquad \frac{\partial \mu}{\partial y} = \frac{\mathrm{d}\mu}{\mathrm{d}\omega}\frac{\partial \omega}{\partial y},$$

we deduce that

$$\left(Q\frac{\partial\omega}{\partial x}-P\frac{\partial\omega}{\partial y}\right)\frac{\mathrm{d}\mu}{\mathrm{d}\omega}=\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right). \tag{4.1.77}$$

Suppose now that the expression

$$E(x,y)\equiv\frac{\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}}{Q\dfrac{\partial\omega}{\partial x}-P\dfrac{\partial\omega}{\partial y}} \tag{4.1.78}$$

depends explicitely only on ω, i.e. $E(x,y)=\psi(\omega)$. Then μ satisfies the linear ODE

$$\frac{\mathrm{d}\mu}{\mathrm{d}\omega}=\psi(\omega)\mu\ , \tag{4.1.79}$$

allowing the solution

$$\mu=\mathrm{e}^{\int\psi(\omega)\mathrm{d}\omega}. \tag{4.1.80}$$

This the integrant factor we are looking for. Note that we only need a particular solution of (4.1.79) and not its general solution.

Particular cases. A) If $\omega=x$, then

$$\mu=\mathrm{e}^{\int\psi(x)\mathrm{d}x},\quad \psi(x)=\frac{\dfrac{\partial P}{\partial y}-\dfrac{\partial Q}{\partial x}}{Q}; \tag{4.1.81}$$

B) If $\omega=y$, then

$$\mu=\mathrm{e}^{\int\psi(y)\mathrm{d}y},\quad \psi(y)=\frac{\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y}}{P}\ . \tag{4.1.82}$$

1.4.7 Clairaut's equation

This ODE is of the form

$$y=xy'+\varphi(y'). \tag{4.1.83}$$

We see that the ODE is linear in both *x* and *y*, but it is not explicit with respect to y'. Using the change $y'=p$, (4.1.83) reads

$$y=xp+\varphi(p). \tag{4.1.84}$$

As $\mathrm{d}y = p\mathrm{d}x$, differentiating (4.1.84) we get

$$p\mathrm{d}x = p\mathrm{d}x + x\mathrm{d}p + \varphi'(p)\mathrm{d}p\,, \tag{4.1.85}$$

whence

$$[x + \varphi'(p)]\mathrm{d}p = 0\,, \tag{4.1.86}$$

If $\mathrm{d}p = 0$, then $p = C$; introducing this in (4.1.84), we get the *general solution* of Clairaut's equation

$$y = Cx + \varphi(C)\,. \tag{4.1.87}$$

The second possibility, $x = -\varphi'(p)$, yields the *singular solution*, expressed in parametric form

$$\begin{aligned} x &= -\varphi'(p), \\ y &= -p\varphi'(p) + \varphi(p). \end{aligned} \tag{4.1.88}$$

Indeed, it is easily seen that this solution cannot be obtained from the general one by giving particular values to *C*.
From the geometric point of view, the general solution of Clairaut's equation always represents a pencil of straight lines; the envelope of this pencil can be obtained eliminating *C* from the algebraic system formed of the general solution and its partial derivative with respect to *C*, i.e.

$$\begin{aligned} &F(x, y, C) \equiv y - Cx - \varphi(C) = 0\,, \\ &\frac{\partial F}{\partial C} \equiv -x - \varphi'(C) = 0\,, \end{aligned} \tag{4.1.89}$$

which is precisely, apart from the notation, the singular solution. We deduce that the singular solution of Clairaut's equation always represents the envelope of the pencil of straight lines giving its general solution.

1.4.8 Lagrange's equation

This is, in fact, a generalization of Clairaut's equation

$$A(y')y + B(y')x + C(y') = 0\,. \tag{4.1.90}$$

Supposing that $A(y') \neq 0$, we divide by it and thus (4.1.90) reads

$$y = \varphi(y')x + \psi(y')\,. \tag{4.1.91}$$

In order to avoid Clairaut's equation, previously treated, we also suppose $\varphi(y') \neq y'$. The method of solving (4.1.91) is the same: we use the change $y' = p$, thus getting

$$y = \varphi(p)x + \psi(p)\,, \tag{4.1.92}$$

then we differentiate this, and, taking into account that $\mathrm{d}y = p\mathrm{d}x$, we deduce

$$[\varphi(p)-p]\mathrm{d}x+[x\varphi'(p)+\psi'(p)]\mathrm{d}p=0\,. \tag{4.1.93}$$

If $\varphi(p)$ is a constant, then (4.1.93) is an ODE with separable variables. If $\varphi(p)$ is not constant, then, as $\varphi(p)\neq p$, (4.1.93) may be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}p}+\frac{\varphi'(p)}{\varphi(p)-p}x=\frac{\psi'(p)}{p-\varphi(p)}, \tag{4.1.94}$$

which is a first order linear non-homogeneous ODE, that can be easily solved by using the method described in Chap.1, Sec.1. We get x as a function of p

$$x=\alpha(p)C+\beta(p), \tag{4.1.95}$$

with C an arbitrary constant. Getting back to (4.1.92), we deduce

$$y=\varphi(p)[\alpha(p)C+\beta(p)]+\psi(p), \tag{4.1.96}$$

so that the general solution of Lagrange's equation, written in parametric form, is

$$\begin{aligned} x&=\alpha(p)C+\beta(p),\\ y&=\gamma(p)C+\delta(p). \end{aligned} \tag{4.1.97}$$

Let us consider now the case $\varphi(p)=p$. Generally speaking, this represents a transcendental equation. Denoting by p_i its solutions, we find the equations of some straight lines

$$y=p_i x+\psi(p_i), \tag{4.1.98}$$

also representing solutions of Lagrange's equation, possibly singular.

1.4.9 Bernoulli's equation

This ODE is of the form

$$y'+P(x)y+Q(x)y^{\alpha}=0, \tag{4.1.99}$$

with $P,Q\in C^0(I), I\subseteq\Re$. If $\alpha=0$, then (4.1.99) is a linear non-homogeneous first order ODE; if $\alpha=1$, then (4.1.99) becomes also a linear first order ODE, but in this case it is homogeneous. As both these cases were treated in Chap.1, we shall consider $\alpha\notin\{0,1\}$.

By using the change of function

$$u=y^{1-\alpha}, \tag{4.1.100}$$

the Bernoulli ODE becomes

$$\frac{u'}{1-\alpha} + P(x)u + Q(x) = 0\,, \tag{4.1.101}$$

i.e., again a linear ODE, that can be solved as shown in Chap.1. After obtaining u, we return to y by using (4.1.100).

1.4.10 Riccati's equation

This widely studied ODE is of the form

$$y' = P(x)y + Q(x)y^2 + R(x), \tag{4.1.102}$$

where $P, Q, R \in \mathrm{C}^0(I), I \subseteq \Re$. If R or Q vanish identically on I, then (4.1.102) is reduced either to a Bernoulli equation for $\alpha = 2$ or to a linear first order ODE, both of them previously studied.

Riccati's equation is of great interest as it models important classes of physical phenomena. We shall emphasize several important properties of this equation and of its solutions, along with methods of solving it.

a) *If we know one of its particular solutions, say Y, then Riccati's equation may be solved by quadratures*.

Indeed, by the change of function

$$y(x) = z(x) + Y(x), \tag{4.1.103}$$

we find out that the new unknown function $z(x)$ must satisfy Bernoulli's equation

$$z' = [P(x) + 2Y(x)Q(x)]z(x) + Q(x)z^2\,, \tag{4.1.104}$$

and therefore the function $u = 1/z = 1/(y - Y)$ satisfies the linear non-homogeneous ODE

$$u' + [P(x) + 2Y(x)Q(x)]u = Q(x)\,. \tag{4.1.105}$$

b) *The solution of a Riccati equation is a homographic function of an arbitrary constant C.*

The solution of (4.1.105) may be written in the form

$$u = C\mathrm{e}^{\int [P(x)+2Y(x)Q(x)]\mathrm{d}x} + U(x), \tag{4.1.106}$$

where U is a particular solution of the non-homogeneous ODE. Note that (4.1.106) may be also written in the form

$$u = C\varphi(x) + \psi(x)\,, \tag{4.1.107}$$

putting C into evidence. Getting back to y, we find the general solution of Riccati's equation in the form

$$y(x) = Y(x) + \frac{1}{C\varphi(x) + \psi(x)} = \frac{C\varphi(x)Y(x) + \psi(x)Y(x)}{C\varphi(x) + \psi(x)} \tag{4.1.108}$$

or, with obvious notations,

$$y(x) = \frac{C\alpha(x) + \beta(x)}{C\gamma(x) + \delta(x)}, \tag{4.1.109}$$

meaning that y is a homographic function of *C*.
We can prove, conversely, that

a) *Any homographic function* (4.1.109) *represents the general solution of a certain Riccati equation.*
Indeed, from (4.1.109) it follows that

$$C = \frac{\beta(x) - \delta(x)y(x)}{\gamma(x)y(x) - \alpha(x)}. \tag{4.1.110}$$

Differentiating this with respect to *x*, we find that *y* satisfies a Riccati equation.

d) *If we know two particular solutions, say* Y_1, Y_2*, then Riccati's equation can be solved by using only one quadrature*.
By using the same changes of function as before, we find out that the function

$$u(x) = \frac{1}{Y_1(x) - Y_2(x)} \tag{4.1.111}$$

is a particular solution of the linear non-homogeneous ODE

$$u' + [P(x) + 2Y_1(x)Q(x)]u = Q(x). \tag{4.1.112}$$

To find the general solution of (4.1.112) we need only the general solution of its associated homogeneous ODE, which is

$$u_h = C\mathrm{e}^{\int [P(x) + 2Y_1(x)Q(x)]\mathrm{d}x}, \tag{4.1.113}$$

yielding only one quadrature.

e) *If we know three particular solutions, say* Y_1, Y_2, Y_3*, then Riccati's equation can be solved without quadratures*.
Indeed, in this case, the functions

$$u_1(x) = \frac{1}{Y_2(x) - Y_1(x)},\ u_2(x) = \frac{1}{Y_3(x) - Y_1(x)} \tag{4.1.114}$$

are both particular solutions of the linear non-homogeneous ODE (4.1.112). Their difference will satisfy the associated homogeneous ODE. Therefore, the general solution of (4.1.112) is obtained without quadratures

$$u(x) = u_1(x) + C[u_2(x) - u_1(x)]$$
$$= \frac{1}{Y_2(x) - Y_1(x)} + C\left[\frac{1}{Y_3(x) - Y_1(x)} - \frac{1}{Y_2(x) - Y_1(x)}\right]. \tag{4.1.115}$$

Thus, turning back to y, we obtain

$$\frac{1}{y(x) - Y_1(x)} = \frac{1}{Y_2(x) - Y_1(x)} + C\left[\frac{1}{Y_3(x) - Y_1(x)} - \frac{1}{Y_2(x) - Y_1(x)}\right] \tag{4.1.116}$$

or

$$\frac{y(x) - Y_2(x)}{y(x) - Y_1(x)} : \frac{Y_3(x) - Y_2(x)}{Y_2(x) - Y_1(x)} = C\ . \tag{4.1.117}$$

The general solution of Riccati's equation can therefore be written in the form of a constant anharmonic ratio. This immediately yields the following property.

f) *The anharmonic ratio of any four particular solution of Riccati's quation is always constant.*

As it was previously shown, there is a tight connection between the Riccati's equation and the linear second order ODE; this connection is useful if this linear ODE is easier solved. Let us mention some particular cases of interest.

1) If $P(x) + Q(x) + R(x) = 0$ on I, then the general solution of Riccati's equation is

$$y(x) = \frac{C + \int [Q(x) + R(x)]\varphi(x)\mathrm{d}x - \varphi(x)}{C + \int [Q(x) + R(x)]\varphi(x)\mathrm{d}x + \varphi(x)}, \quad \varphi(\mathrm{x}) = \mathrm{e}^{\int [Q(x) - R(x)]\mathrm{d}x}. \tag{4.1.118}$$

2) In the more general case $a^2 P(x) + abQ(x) + b^2 R(x) = 0$, $x \in I$, where the constants a and b are not simultaneously null; if $b \neq 0$, we can use the change of function $y(x) = a/b + u(x)$, obtaining for the new unknown function a Bernoulli-type equation

$$u' = Q(x)u^2 + \left[\frac{2a}{b} Q(x) + P(x)\right]u\ . \tag{4.1.119}$$

3) If P and R are polynomials satisfying $\Delta = P^2 - 2P' - 4R = \text{const}$, then $Y_1(x) = -\frac{1}{2}\left[P(x) + \sqrt{\Delta}\right]$ and $Y_2(x) = -\frac{1}{2}\left[P(x) - \sqrt{\Delta}\right]$ are both of them solutions of Riccati's equation

$$y' = P(x)y + y^2 + R(x)\ . \tag{4.1.120}$$

2. Non-Linear Second Order ODEs

The general form of a second order ODE is

$$F(x, y, y', y'') = 0. \tag{4.2.1}$$

If $\partial F / \partial y'' \neq 0$ on the domain of definition of F and if F is sufficiently regular, then, by the implicit function theorem we can explicit y'', thus getting the *normal/canonic form*

$$y'' = f(x, y, y'), \quad f : I \times D_y \times D_{y'} \to \Re . \tag{4.2.2}$$

2.1 CAUCHY PROBLEMS

In the examples given in the Introduction, we saw that the study of a motion, for whoch Newton's second law represents a fundamental principle in the classical mechanics leads to an ODE of form (4.2.2). To determine completely the trajectory of the moving body one must know its position and its initial velocity. The mathematical correspondent of the velocity is the derivative of the displacement with respect to time. Consequently, to an ODE of type (4.2.2) one can naturally associate the following supplementary conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad (x_0, y_0, y'_0) \in I \times D_y \times D_{y'}, \tag{4.2.3}$$

called initial or Cauchy conditions.
As in a study of motion the initial position and velocity perfectly determine the trajectory of the body, we should expect that the initial problem (4.2.2), (4.2.3) allow unique solution, under certain hypotheses on *f*.
One can easily prove an existence and uniqueness theorem for the solution of this problem, similar to theorem 4.2. But, as we previously saw, a second order ODE can be reduced to a first order ODS with two unknown functions, we shall rediscover this theorem in Chap.5, as a particular case of the corresponding theorem for ODSs.

2.2 TWO-POINT PROBLEMS

We already saw that, if we associate to an ODE the Cauchy (or initial) conditions, this means that the values of the unknown function and of its derivative at the same point x_0 are supposedly known. Such conditions do not match to all mathematical models; for instance, they do not fit to the simply supported bar, as in this case the physical problem requires the values of the displacement at two distinct point: the bar ends. The simplest conditions of this type are

$$y(a) = A, \quad y(b) = B, \qquad a, b \in I, \quad A, B \in \Re , \tag{4.2.4}$$

which, associated to the ODE (4.2.2), form *the two-point* (*bilocal*) *problem*.

The first difficulty in tackling this problem is to get appropriate hypotheses ensuring the existence and uniqueness of the solution, as in this case we have no more the benefit of such powerful a tool as the Cauchy-Picard theorem.

We shall suppose $f \in \mathrm{C}^0([a,b])$, for any $y, y' \in \Re$. Obviously, there are infinitely many integral curves passing through the point (a, A). But it is possible, even in simple cases as that of the ODE $y'' = 2y'^3$, that none of these integral curves reach the point (b, B). In other words, it is possible that the solution of the two-point problem not even exist.

Hereafter, we give some of the most common conditions, each of them ensuring the existence and uniqueness of the solution of the two-point problem (4.2.1), (4.2.4):

1. $f(x, y, y')$ bounded.
2. $|f| < C|y|$ for sufficiently great values of $|y|$; here, $C < \sqrt{3\pi^3}\,/(b-a)^2$.
3. f is Lipschitzian with respect to y, y' on any finite interval and $f(x, y, y')/(|y| + |y'|)$ tends to 0, uniformly on $[a,b]$, if $(|y| + |y'|) \to \infty$.
4. f is Lipschitzian with respect to y, y' on any finite interval and has the form

$$f(x, y, y') \equiv \varphi(x, y) + \psi(x, y, y'), \tag{4.2.5}$$

where $\psi(x, y, y')/(|y| + |y'|)$ tends to 0 uniformly on $[a,b]$, if $(|y| + |y'|) \to \infty$.

5. f allows continuous partial derivatives with respect to y, y' and

$$\left|\frac{\partial f}{\partial y}\right| < \alpha, \quad \left|\frac{\partial f}{\partial y'}\right| < \beta, \quad \alpha + \beta < 1, \tag{4.2.6}$$

or $\partial f / \partial y \geq 0$.

6. A particular case of interest is that of the two-point problem

$$\begin{aligned} &y'' = f(x, y), \\ &y(0) = 0, \quad y(a) = 0. \end{aligned} \tag{4.2.7}$$

One can prove the existence and uniqueness of its solution provided $f \in C^0([0,a] \times \Re)$, and there exist two numbers $c_0 \geq 0, c_1 > 0$ such that

$$\int_0^y f(x,t)\mathrm{d}t \geq -c_1 y^2 - c_0, \quad a \in \left(0, \frac{\pi}{\sqrt{2c_1}}\right). \tag{4.2.8}$$

2.3 ORDER REDUCTION OF SECOND ORDER ODEs

There are particular cases in which the second order ODEs may be easier solved by reducing their order. In what follows, we shall present some of these cases, frequently met in applications.

a) If the ODE is of the form

$$F(x, y', y'') = 0 , \tag{4.2.9}$$

i.e., F does not explicitly depend on y, then, by the change of function $y' = p$, we get

$$F(x, p, p') = 0 , \tag{4.2.10}$$

which is a first order ODE, that can be solved by using the above presented methods. Let $p = p(x, C_1)$ be its general solution. Then the general solution of (4.2.9) is

$$y(x) = \int p(x, C_1)\mathrm{d}x + C_2 , \tag{4.2.11}$$

C_1 and C_2 being arbitrary constants.

b) If the ODE does not depend explicitly on x, i.e., if

$$F(y, y', y'') = 0 , \tag{4.2.12}$$

then, using again the change $y' = p$, we obtain

$$y'' = \frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = p\frac{\mathrm{d}p}{\mathrm{d}y} ; \tag{4.2.13}$$

this means that (4.2.12) becomes a first order ODE, having p as unknown function and y as independent variable

$$F\left(y, p, \frac{\mathrm{d}p}{\mathrm{d}y} \right) = 0 . \tag{4.2.14}$$

The general solution of this ODE reads $p = p(y, C_1)$, whence we get another first order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p(y, C_1), \tag{4.2.15}$$

which can be solved by separation of variables, thus getting

$$x = \int \frac{1}{p(y, C_1)}\mathrm{d}y + C_2 . \tag{4.2.16}$$

This is precisely the general solution of (4.2.12).

c) If the function $F(x, y, y', y'')$ is homogeneous of degree m with respect to y, y', y'', that is, if

$$F(x, ty, ty', ty'') = t^m F(x, y, y', y''), \tag{4.2.17}$$

then we can use the change

$$u = \frac{y'}{y} . \tag{4.2.18}$$

This yields

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{y}\right) = \frac{y''y - y'^2}{y^2} , \tag{4.2.19}$$

whence

$$\frac{y''}{y} = u' + u^2 . \tag{4.2.20}$$

Eventually, the second order ODE (4.2.1) is replaced by the first order ODE

$$F\left(x, 1, u, u' + u^2\right) = 0 . \tag{4.2.21}$$

Let $u = u(x, C_1)$ be its general solution. Introducing it in (4.2.18), we get a new first order ODE, linear and homogeneous,

$$y' - u(x, C_1)y = 0 , \tag{4.2.22}$$

whose general solution reads

$$y(x) = C_2 \mathrm{e}^{\int u(x, C_1)\mathrm{d}x} ; \tag{4.2.23}$$

this is also the general solution of (4.2.1) in this particular case.

2.4 THE BERNOULLI-EULER EQUATION

This ODE is of greatest importance in the mechanics of constructions, as it represents the mathematical model of an elastic bar deformation by bending.
We shall consider later on the physical hypotheses under which this model is set up. The Bernoulli-Euler equation reads

$$y'' = f(x)\left(1 + y'^2\right)^{3/2} , \tag{4.2.24}$$

where y corresponds to the deflection of the bar axis, and the independent variable x is considered along the ideal non-deflected bar. The function $f(x) = M / EI$, where M is the bending moment and the rigidity EI is expressed by the product between the modulus of elasticity E and the moment of inertia I of the cross section with respect to the neutral bar axis.
The ODE (4.2.24) is of the form a) from Sec.2.3. Therefore, by using the change $y' = z$, it becomes

$$z' = f(x)\left(1 + z^2\right)^{3/2} . \tag{4.2.25}$$

Introducing the function $h(x)$ as a primitive of $f(x)$, i.e.

$$\frac{\mathrm{d}h}{\mathrm{d}x} = f(x), \quad h(x) = \int_0^x f(t)\mathrm{d}t, \tag{4.2.26}$$

we can simplify the form of (4.2.25), applying the change of variable $z = z(h)$, that leads to

$$\frac{dz}{\mathrm{d}h} = \left(1 + z^2\right)^{3/2}. \tag{4.2.27}$$

This ODE is invariant on the class of Bernoulli-Euler type equations. Its form does not depend on the physical bar characteristics; one can say that it represents the intrinsic mathematical structure of an elastic bar model.

Using the change of function $z = \sinh u$, we can integrate (4.2.27), obtaining its general solution in the form

$$\frac{z}{\sqrt{1 + z^2}} = h + C, \tag{4.2.28}$$

where C is an arbitrary constant; it results

$$z = \frac{h + C}{\sqrt{1 - (h + C)^2}}, \qquad h + C < 1. \tag{4.2.29}$$

In the particular case $f(x) = 1/R = \text{const}$, we deduce $h(x) = x/R$, therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dfrac{x}{R} + C}{\sqrt{1 - \left(\dfrac{x}{R} + C\right)^2}}, \tag{4.2.30}$$

whence we get the general solution of the Bernoulli-Euler equation in the form of a pencil of circles

$$(x + CR)^2 + (y - b)^2 = R^2, \qquad b, C = \text{const}. \tag{4.2.31}$$

This was to be expected, taking into account the physical interpretation of the function $f(x)$.

Expanding now (4.2.29) in a power series with respect to $(h + C)$, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (h + C) + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \prod_{j=1}^{k} (2j - 1)(h + C)^{2k+1}, \tag{4.2.32}$$

whence

$$y(x) = \int [h(x) + C]\mathrm{d}x + \sum_{k=1}^{\infty} \frac{1}{2^k k!} \prod_{j=1}^{k} (2j-1) \int [h(x) + C]^{2k+1} \mathrm{d}x\,, \qquad (4.2.33)$$

valid for $h + C < 1$.

2.5 ELLIPTIC INTEGRALS

From (4.2.29) it follows that the general solution of the Bernoulli-Euler ODE can also be written in integral form

$$y(x) = \int \frac{h(x) + C}{\sqrt{1 - [h(x) + C]^2}} \mathrm{d}x + b\,. \qquad (4.2.34)$$

If $h(x)$ is a polynomial $P(x)$, then the Bernoulli-Euler equation would be reduced to the study of an integral of the type

$$\int \Phi\left(x, \sqrt{P(x)}\right) \mathrm{d}x. \qquad (4.2.35)$$

where $\Phi(\alpha, \beta)$ is rational with respect to its arguments.
If $P(x)$ has the degree 3 ar 4, then this integral can be reduced to integrals of rational functions and to three other integrals, called *elliptic integrals of first, second and third species* accordingly, in normal Legendre form

$$\int \frac{1}{\sqrt{\left(1 - x^2\right)\left(1 - k^2 x^2\right)}} \mathrm{d}x, \quad \int \frac{\sqrt{2 - k^2 x^2}}{\sqrt{1 - x^2}} \mathrm{d}x, \quad \int \frac{1}{\left(1 + nx^2\right)\sqrt{\left(1 - x^2\right)\left(1 - k^2 x^2\right)}} \mathrm{d}x\,. \qquad (4.2.36)$$

The number k is called the *modulus* of the integrals and $k' = \sqrt{1 - k^2}$ is the complementary modulus ($k < 1$); the number n is the parameter of the integral of third species.
By the substitution $x = \sin\varphi$ one obtains the elliptic integrals in *normal trigonometric form*; thus

$$F(\varphi, k) = \int_0^{\sin\varphi} \frac{1}{\sqrt{\left(1 - x^2\right)\left(1 - k^2 x^2\right)}} \mathrm{d}x = \int_0^{\varphi} \frac{1}{\sqrt{1 - k^2 \sin^2 \psi}} \mathrm{d}\psi\,. \qquad (4.2.37)$$

is the elliptic integral of first species,

$$E(\varphi, k) = \int_0^{\sin\varphi} \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}} \mathrm{d}x = \int_0^{\varphi} \sqrt{1 - k^2 \sin^2 \psi}\, \mathrm{d}\psi\,, \qquad (4.2.38)$$

is the elliptic integral of second species and

$$\varepsilon(\varphi,n,k)=\int_0^{\sin\varphi}\frac{1}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}\mathrm{d}x=\int_0^{\varphi}\frac{1}{(1+n\sin^2\psi)\sqrt{1-k^2\sin^2\psi}}\mathrm{d}\psi \qquad (4.2.39)$$

is the elliptic integral of third species.
We must also mention the following combination of the elliptic integrals, useful in applications

$$D(\varphi,k)=\frac{F(\varphi,k)-E(\varphi,k)}{k^2}=\int_0^{\sin\varphi}\frac{x^2}{\sqrt{(1-x^2)(1-k^2x^2)}}\mathrm{d}x=\int_0^{\varphi}\frac{\sin^2\psi}{\sqrt{1-k^2\sin^2\psi}}\mathrm{d}\psi\,. \qquad (4.2.40)$$

For $\varphi=\pi/2$, we get *the complete elliptic integrals*

$$F\left(\frac{\pi}{2},k\right)\equiv K(k)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k^2\sin^2\varphi}}\mathrm{d}\varphi,$$
$$E\left(\frac{\pi}{2},k\right)\equiv E(k)=\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\varphi}\,\mathrm{d}\varphi, \qquad . \qquad (4.2.41)$$
$$D\left(\frac{\pi}{2},k\right)\equiv\frac{K(k)-E(k)}{k^2}=\int_0^{\frac{\pi}{2}}\frac{\sin^2\varphi}{\sqrt{1-k^2\sin^2\varphi}}\mathrm{d}\varphi\,.$$

The current notations when we use the complementary modulus are

$$K'(k)=F\left(\frac{\pi}{2},k'\right)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-k'^2\sin^2\varphi}}\mathrm{d}\varphi,$$
$$E'(k)=E\left(\frac{\pi}{2},k'\right)=\int_0^{\frac{\pi}{2}}\sqrt{1-k'^2\sin^2\varphi}\,\mathrm{d}\varphi. \qquad (4.2.42)$$

In practice, the modulus k is usually omitted; for instance, we can write E instead of $E(k)$, E' instead of $E(k')$, a.s.o.
In most of cases, the elliptic integrals, whether they are complete or not, cannot be computed in terms of elementary functions. This is why series expansions were used, leading to accurate approximations. This approximations were then used to set up tables of values for the elliptic integrals.
We give several of the most useful series developments for the calculus of complete elliptic integrals:

$$K(k)=\frac{\pi}{2}\left\{1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\ldots+\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2n}+\ldots\right\},$$

$$E(k)=\frac{\pi}{2}\left\{1-\frac{1}{2^2}k^2-\frac{1^2\cdot 3}{2^2\cdot 4^2}k-\ldots-\left[\frac{(2n-1)!!}{2^n n!}\right]^2\frac{k^{2n}}{2n-1}+\ldots\right\}, \tag{4.2.43}$$

$$D(k)=\pi\left\{\frac{1}{1}\left(\frac{1}{2}\right)^2+\frac{2}{3}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^2+\ldots+\frac{n}{2n-1}\left[\frac{(2n-1)!!}{2^n n!}\right]^2 k^{2(n-1)}+\ldots\right\}.$$

For the elliptic integrals $F(\varphi,k), E(\varphi,k)$ there were also found trigonometric series expansion. It should be mentioned that the complete elliptic integrals can be decomposed in Legendre's polynomials.

The calculus of the elliptic integrals is considerably simplified by certain functional relationships between them. The more currently in use are

$$\begin{aligned}
&F(-\varphi,k)=-F(\varphi,k),\\
&E(-\varphi,k)=-E(\varphi,k),\\
&F(n\pi\pm\varphi,k)=2nK(k)+F(\varphi,k),\\
&E(n\pi\pm\varphi,k)=2nE(k)+E(\varphi,k),\\
&\frac{\partial E}{\partial k}=\frac{E-F}{k},\\
&\frac{\partial F}{\partial k}=\frac{1}{k'^2}\left(\frac{E-k'^2F}{k}-\frac{\sin\varphi\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right).
\end{aligned} \tag{4.2.44}$$

For the complete elliptic integral, we emphasize the following representative relationships

$$\begin{aligned}
&E(k)K(k)+E'(k)K(k)+K(k)E'(k)=\frac{\pi}{2},\\
&\frac{\mathrm{d}K(k)}{\mathrm{d}k}=\frac{E(k)}{kk'^2}-\frac{K(k)}{k},\\
&\frac{\mathrm{d}E(k)}{\mathrm{d}k}=\frac{E(k)-K(k)}{k},
\end{aligned} \tag{4.2.45}$$

and also

$$\begin{aligned}
&K\left(\frac{1-k'}{1+k'}\right)=\frac{1+k'}{2}K(k),\\
&E\left(\frac{1-k'}{1+k'}\right)=\frac{1}{1+k'}\left[E(k)+k'K(k)\right].
\end{aligned} \tag{4.2.46}$$

Finally, let us mention that the functions K and K' satisfy the ODE of independent variable k

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(kk'^2\frac{\mathrm{d}u}{\mathrm{d}k}\right)-ku=0, \tag{4.2.47}$$

and the functions E and $(E'-K')$ are particular solutions of the ODE

$$k'^2\frac{\mathrm{d}}{\mathrm{d}k}\left(k\frac{\mathrm{d}u}{\mathrm{d}k}\right)+ku=0\,. \tag{4.2.48}$$

3. Applications

Application 4.1

Problem. Study the motion of a heavy particle P (the motion of a particle in the gravitational field of the Earth), of mass m, in a resistant medium. Such a particle is, for instance, a projectile in motion, which has a spherical form and is not subjected to rotations; from the point of view of the mathematical modelling, the projectile is reduced to its centre of gravity.

Mathematical model. We assume that, besides the given force (in our case the gravitational force $m\mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration) intervenes also a force $\mathbf{R}$, called *resistance*,

$$\mathbf{R}=-mg\varphi(v)\operatorname{vers}\mathbf{v}\,,\ \ \varphi(0)=0\,,\ \ \lim_{v\to\infty}\varphi(v)=\infty\,, \tag{a}$$

where $\varphi(v)$ is a strictly increasing function (the resistance of the air increases together with the velocity v); there exists – obviously – a value v^* and only one for which $\varphi(v^*)=1$.

Solution. Newton's equation of motion is

$$m\ddot{\mathbf{r}}=\mathbf{g}-mg\overline{\varphi}(v)\dot{\mathbf{r}}\,, \tag{b}$$

where $\overline{\varphi}(v)=\varphi(v)/v$; we assume that, in general, the initial velocity $\mathbf{v}_0$ is not directed along the vertical of the launching position ($\mathbf{v}_0$ is not collinear with $\mathbf{g}$); the trajectory is a plane curve (contained in a vertical plane). Using Frenet's trihedron, we may write

$$\dot{v}=-g[\sin\theta+\varphi(v)],\quad \frac{v^2}{\rho}=g\cos\theta\,, \tag{c}$$

where θ is the angle made by velocity $\mathbf{v}$ with the x-axis, while ρ is the curvature radius of the trajectory. We notice that $\cos\theta\geq 0$, hence $-\pi/2\leq\theta\leq\pi/2$; the concavity of the trajectory is directed towards the negative ordinates (Fig.4.4), so that to $\mathrm{d}s > 0$ corresponds $\mathrm{d}\theta < 0$ (the angle θ is decreasing). It follows $\rho = -\mathrm{d}s/\mathrm{d}\theta = -v\,\mathrm{d}t/\mathrm{d}\theta$, so that the second equation (c) takes the form

$$v\dot{\theta} = -g\cos\theta\,. \tag{d}$$

We have thus obtained a system of two differential equations (c), (d) for the unknown functions $v = v(t)$ and $\theta = \theta(t)$, with the initial conditions $v(t_0) = v_0$, $\theta(t_0) = \theta_0$. Eliminating the time t, we may write the equation

$$\frac{\mathrm{d}v}{\mathrm{d}\theta} = v\left[\tan\theta + \frac{\varphi(v)}{\cos\theta}\right], \tag{e}$$

which defines the function $v = v(\theta)$ with the initial condition $v(\theta_0) = v_0$. This equation of the hodograph of velocities, which can be written in the form

$$\frac{\mathrm{d}(v\cos\theta)}{\mathrm{d}\,\theta} = v\varphi(v) \tag{f}$$

too, is *the basic equation of the external ballistics*. The equation (d) allows then to determine (usually, one takes $t_0 = 0$)

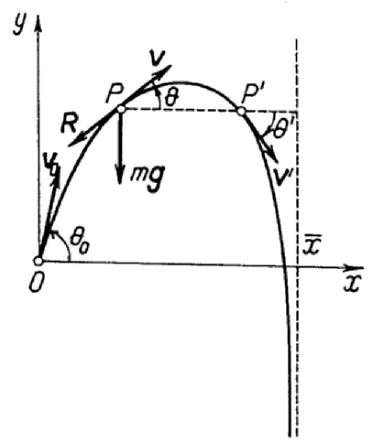

Figure 4. 4. Motion of a heavy particle in a resistant medium

$$t = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta}\frac{v(\vartheta)}{\cos\vartheta}\mathrm{d}\vartheta\,, \tag{g}$$

whence – afterwards – we may obtain $\theta = \theta(t)$. Noting that $\mathrm{d}x = v\cos\theta\,\mathrm{d}t$, $\mathrm{d}y = v\sin\theta\,\mathrm{d}t$, there result the parametric equations of the trajectory in the form

$$x = x_0 - \frac{1}{g}\int_{\theta_0}^{\theta} v^2(\vartheta)\mathrm{d}\vartheta,\ y = y_0 - \frac{1}{g}\int_{\theta_0}^{\theta} v^2(\vartheta)\tan\vartheta\mathrm{d}\vartheta, \tag{h}$$

where we take $x_0 = y_0 = 0$ if the particle (the projectile) is launched from the origin 0. In the case of an object launched from an airplane at the height h we take $x_0 = 0$, $y_0 = h$; the initial velocity is the velocity of the airplane at the moment of launching the object.

From the second equation (c) one observes that (θ is only decreasing and greater than $-\pi/2$ for t finite, hence $\cos\theta > 0$) the velocity v is finite and non-zero. An extreme value of v is given by $\mathrm{d}v/\mathrm{d}t = 0$; we obtain thus $\varphi(v) = -\sin\theta$. Because the velocity v is finite, from (d) it follows that θ has an extreme value for $\mathrm{d}\theta/\mathrm{d}t = 0$, hence for $\cos\theta = 0$; but the angle θ is decreasing, so that we have $\lim_{t\to\infty}\theta = -\pi/2$. We notice that for $v > v^*$, $\varphi(v^*) = 1$, we have $\dot{v} < 0$, the function $\varphi(v)$ being monotone decreasing. Hence, the velocity v has a lower limit ($v > 0$) and a upper limit ($v \le v^*$). The trajectory has a vertical asymptote $x = \bar{x}$, with

$$\bar{x} = \lim_{\theta\to-\pi/2+0} x = \frac{1}{g}\int_{-\pi/2}^{\theta_0} v^2(\vartheta)\mathrm{d}\vartheta, \tag{i}$$

and the corresponding velocity is given by $\lim_{\theta\to-\pi/2+0} v(\theta) = v^*$. Because of the resistance of the air, we notice that the range of throw of the projectile is smaller. Besides, for two points P and P', which have the same ordinate y, it results $|\theta| < |\theta'|$; hence, the two branches (increasing and decreasing) of the trajectory are not symmetric.

Multiplying the first equation (c) by v and noting that $\mathrm{d}y = v\sin\theta\mathrm{d}t$, we may write $\mathrm{d}(v^2/2) = -g\mathrm{d}y - g\varphi(v)v\mathrm{d}t$, so that, integrating between the points $P(t)$ and $P'(t')$, we obtain

$$\frac{1}{2}\left(v'^2 - v^2\right) = -g\int_t^{t'}\varphi(v(\tau))v(\tau)\mathrm{d}\tau < 0,$$

whence $v > v' > 0$.

Modelling the projectile as a rigid solid, one can take into account also its rotation, being led to a deviation from the vertical plane of the trajectory.

In particular, d'Alembert has considered the law of resistance $\varphi(v) = \lambda v^n$, $n > 0$, λ being a positive constant with dimension. The equation (f) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(v\cos\theta) = \frac{\lambda(v\cos\theta)^{n+1}}{\cos^{n+1}\theta}; \tag{j}$$

integrating, we get

$$v\cos\theta = \frac{v_0\cos\theta_0}{\left\{1 - n\lambda\left[\varepsilon_n(\theta)\varepsilon_n(\theta_0)\right]\left(v_0\cos\theta_0\right)^n\right\}^{1/n}}\,, \tag{k}$$

where we have introduced the integral

$$\varepsilon_n(\theta) = \int_0^\theta \frac{\mathrm{d}\,\vartheta}{\cos^{n+1}\vartheta}\,. \tag{l}$$

For small velocities one can use Stokes' law ($n = 1$); thus, we obtain $\varepsilon_1(\theta) = \tan\theta$, so that

$$v(\theta) = \frac{v_0\cos\theta_0}{\cos\theta_0 - \lambda v_0\sin(\theta - \theta_0)}\,. \tag{m}$$

For velocities till $250\,\mathrm{m/s}$, one may take $n = 2$, obtaining Euler's law; we notice that

$$\varepsilon_2(\theta) = \frac{1}{2}\left[\frac{\tan\theta}{\cos\theta} + \ln\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]. \tag{n}$$

Let us consider now the case $n \in \mathfrak{N}$; for n odd ($n = 2p - 1$), we have

$$\varepsilon_n(\theta) = \frac{\sin\theta}{2p-1}\left[\sec^{2p-1}\theta + \sum_{k=1}^{p-1}\frac{2^k(p-1)(p-2)\ldots(p-k)}{(2p-3)(2p-5)\ldots(2p-2k-1)}\sec^{2p-2k-1}\theta\right], \tag{o}$$

while for n even ($n = 2p$) we may write

$$\begin{aligned}\varepsilon_n(\theta) = {}& \frac{\sin\theta}{2p}\left[\sec^{2p}\theta + \sum_{k=1}^{p-1}\frac{(2p-1)(2p-2)\ldots(2p-2k+1)}{2^k(p-1)(p-2)\ldots(p-k)}\sec^{2p-2k}\theta\right] \\ & + \frac{(2p-1)!!}{2^{\mathrm{p}}\,p!}\ln\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right).\end{aligned} \tag{p}$$

The velocity $v(\theta)$ is easily obtained from the formula (k), getting the time t and the parametric equations of the trajectory from the formulae (g) and (h).
We observe that, by the substitution $v[\sin\theta + \varphi(v)] = 1/y$, the equation (e) reads

$$\frac{\mathrm{d}y}{\mathrm{d}v} = v\left[\varphi^2(v) - 1\right]y^3 - \left[2\varphi(v) + v\frac{\mathrm{d}\varphi(v)}{\mathrm{d}v}\right]y^2\,. \tag{q}$$

Drach has determined all the forms of the function $\varphi(v)$ for which the solution of the equation may be obtained by quadratures.

Application 4.2

Problem. Study the motion of a heavy solid body of weight P_0 which is moving on a plane inclined by the angle α with respect to the horizontal and is tied by a chain wrapped up frictionless on a pulley in A (Cayley's problem, 1857) (Fig.4.5).

Mathematical model. Applying the theorem of momentum, one obtains the differential equation

$$\frac{P}{g}\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{p}{g}\left(v-v_0\right)=X\,, \tag{a}$$

where P/g is the total mass of the mechanical system at the moment t, g being the gravitational acceleration, p/g is the accumulation of mass, X is the external force, v is the velocity at the moment t, while v_0 is the initial velocity of the additional mass, one obtains the model of a mechanical system of variable mass.
Let be q the weight of the chain on the unit length; in this case, for a displacement x of the weight P_0 the total mass is

$$P = P_0 + qx\,. \tag{b}$$

We notice that

$$p=\frac{\mathrm{d}P}{\mathrm{d}t}=qv\,. \tag{c}$$

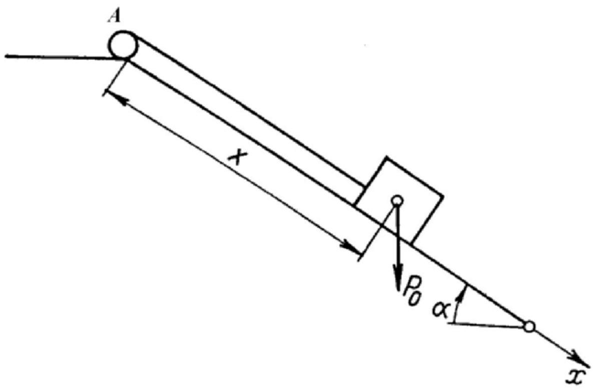

Figure 4. 5. Mechanical system of variable mass

The portion of the chain wrapped up on the pulley being in rest, we may consider that the initial velocity of the additional mass is zero ($v_0 = 0$). The external force X is the component along the inclined plane of the force P, so that $X = \left(P_0 + qx\right)\sin\alpha$. Thus, the equation (a) becomes

$$\frac{1}{g}\left(P\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}P}{\mathrm{d}t} \right) = \left(P_0 + qx \right)\sin\alpha \,.$$

Solution. The equation governing the problem becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Pv \right) = Pg\sin\alpha = \left(P_0 + qx \right)g\sin\alpha \,. \tag{d}$$

Multiplying at the left by Pv and at the right by $\left(P_0 + qx \right)\mathrm{d}x/\mathrm{d}t$ and integrating, we get

$$\frac{1}{2}\left(Pv \right)^2 = \frac{g}{3q}\left(P_0 + qx \right)^3 \sin\alpha + C \,. \tag{e}$$

If we assume that for $t = 0$ the mechanical system is at rest at the upper part of the inclined plane, then the condition $x(0) = 0$ leads to $C = -\left(g/3q \right)P_0{}^3 \sin\alpha$ and the velocity is given by

$$v^2 = \frac{2g}{3q}\frac{\left(P_0 + qx \right)^3 - P_0^3}{\left(P_0 + qx \right)^2}\sin\alpha = \frac{2gx}{3}\frac{3P_0\left(P_0 + qx \right) + q^2x^2}{\left(P_0 + qx \right)^2}\sin\alpha \,. \tag{f}$$

In the particular case $P_0 = 0$ (the chain is free to fall), one obtains

$$v^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 = \frac{2gx}{3}\sin\alpha \,, \tag{g}$$

whence

$$\frac{\mathrm{d}x}{\sqrt{x}} = \sqrt{\frac{2g}{3}\sin\alpha}\,\mathrm{d}t \,;$$

then

$$\sqrt{6x} = \sqrt{g\sin\alpha}\,t + C_1 \,,$$

so that ($x(0) = 0$)

$$x(t) = \frac{g}{6}t^2 \sin\alpha \,,\; v(t) = \frac{g}{3}t\sin\alpha \,,\; a(t) = \frac{g}{3}\sin\alpha \,, \tag{h}$$

the motion of the chain being uniformly accelerated.

Application 4.3

Problem. Study the motion in air along the vertical of a body of mass m, launched with an initial velocity v_0, if the resistance of the air is given by $R = -kv^2$, v being the velocity and k a constant coefficient. Determine the maximal height attained by the body.

Mathematical model. Modelling the body as a particle, Newton's equation of motion $m\ddot{x} = -mg - kv^2$ becomes

$$\ddot{x} + \omega^2 v^2 = -g\,, \tag{a}$$

where $\omega^2 = k/m$.

Solution. Noting that

$$\ddot{x} = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}\,,$$

the equation of motion becomes

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -\left(g + \omega^2 v^2\right) \tag{b}$$

and is a differential equation with separable variables. Separating the variables and integrating, we have

$$\frac{v\mathrm{d}v}{g + \omega^2 v^2} = -\mathrm{d}x\,,$$

$$x = -\frac{1}{2\omega^2}\ln\left(g + \omega^2 v^2\right) + \ln C\,;$$

because $x = 0$ and $v = v_0$ for $t = 0$, we obtain

$$x = \frac{1}{2\omega^2}\ln\frac{g + \omega^2 v_0^2}{g + \omega^2 v^2}\,;$$

the maximal value $x_{\max}$ is obtained for $v = 0$ and is given by

$$x_{\max} = \frac{1}{2\omega^2}\ln\frac{g + \omega^2 v_0^2}{g} = \frac{1}{2\omega^2}\ln\left(1 + \frac{\omega^2 v_0^2}{g}\right).$$

Application 4.4

Problem. Study the motion of a heavy particle on a surface of rotation.

Mathematical model. Let be a heavy particle P of mass m, constrained to move on a surface of rotation the symmetry axis of which is vertical (Fig.4.6). The own weight of the particle $m\mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration, and the constraint force $\mathbf{R}$ (the support of which pierces the Oz-axis) act in the meridian plane, their moments with respect to the symmetry axis vanishing; hence, we may write the first integral of areas for the projection P' of particle P on the plane Oxy (for the particle P too) in the form (we use cylindrical co-ordinates r, θ, z)

$$r^2\dot{\theta} = r_0^2\dot{\theta}_0 = C, \tag{a}$$

where $r(t_0) = r_0$, $\dot{\theta}(t_0) = \dot{\theta}_0$. Because the constraint is scleronomic and the given force is conservative, we may use the first integral of energy

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 = v_0^2 + 2g(z_0 - z) \tag{b}$$

too, where $z(t_0) = z_0$, $v(t_0) = v_0$.

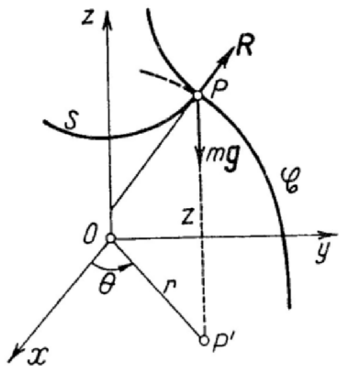

Figure 4. 6. Motion of a heavy particle on a surface of rotation

Solution. If the surface of rotation is specified by the equation $r = f(z)$ (the equation of the meridian curve C), we can eliminate the functions $r = r(t)$ and $\theta = \theta(t)$ from (a) and (b), obtaining the equation with separate variables

$$\dot{z}^2\left(1 + f'^2\right) = v_0^2 + 2g(z_0 - z) - \frac{C^2}{f^2},\ f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \tag{c}$$

which determines the applicate $z = z(t)$ by a quadrature; returning to the equation of the rotation surface and to the first integral of areas, we obtain the other co-ordinates of the point P.

In case of a *circular cylinder* of radius l, the equation (c) becomes ($f = l$)

$$\dot{z}_0^2 = v_0^2 + 2g(z_0 - z) - \frac{C^2}{l^2}, \tag{d}$$

in case of a *circular cone* of equation $r = kz$, we may write

$$\left(1 + k^2\right)\dot{z}^2 = v_0^2 + 2g(z_0 - z) - \frac{C^2}{k^2z^2} \tag{e}$$

and in case of a *sphere* of radius l, we obtain ($r^2 + z^2 = l^2$)

$$l^2 \dot{z}^2 = \left[v_0^2 + 2g(z_0 - z)\right]\left(l^2 - z^2\right) - C^2 . \tag{f}$$

If we represent the rotation surface by the equation $z = \varphi(r)$, then we may eliminate the functions $z = z(t)$ and $\theta = \theta(t)$; it results

$$\dot{r}^2\left(1 + \varphi'^2\right) = v_0^2 + 2g(z_0 - z) - \frac{C^2}{r^2}, \quad \varphi' = \frac{\mathrm{d}\varphi}{\mathrm{d}r}, \tag{g}$$

which specifies the radius $r = r(t)$ by a quadrature too.

Eliminating the time, we get the equation of the trajectory of the point P' in the form

$$\theta = \theta_0 + C\int_{r_0}^{r} \frac{\mathrm{d}\rho}{\rho}\sqrt{\frac{1+[\varphi'(\rho)]^2}{\{v_0^2 + 2g[z_0 - \varphi(\rho)]\}\rho^2 - C^2}}, \tag{h}$$

where $\theta(t_0) = \theta_0$; assuming that the surface is algebraic, we may put in evidence the cases in which the function $\theta = \theta(t)$ is expressed by means of elliptic functions.

In the case of a conservative force the potential of which depends only on r, the problem may be solved also only by quadratures.

Application 4.5

Problem. Study the motion of a heavy particle of weight $m\mathbf{g}$ (m is the mass, $\mathbf{g}$ is the gravitational acceleration), which moves frictionless on a sphere of radius l (*spherical pendulum*).

Mathematical model. The constraint may be bilateral or unilateral in the considered problem; we consider the case of a bilateral constraint. We choose the equatorial plane of the sphere as *Oxy*-plane, the *Oz*-axis being directed towards the descendent vertical; it is convenient to use cylindrical co-ordinates (Fig.4.7). If the constant C in the first integral of areas (a) (see Appl.4.4) vanishes, then $\dot{\theta} = 0$ and $\theta = \text{const}$; the trajectory of the particle is contained in a meridian plane of the sphere, hence it is a great circle of it. The spherical pendulum is, in this case, a simple pendulum (see Appl.4.33). If the constant C is non-zero, then we have to do with a non-degenerate spherical pendulum. The equation (f) of Appl.4.4 becomes

$$l^2 \dot{z}^2 = P(z), \; P(z) = \left[v_0^2 + 2g(z - z_0)\right]\left(l^2 - z^2\right) - C^2. \tag{a}$$

Solution. From (a), we get

$$t = t_0 \pm l\int_{z_0}^{z} \frac{\mathrm{d}\,\xi}{\sqrt{P(\xi)}}; \tag{b}$$

the first integral (a) of Appl.4.4 allows to determine the angle θ in the form

$$\theta = \theta_0 \pm Cl \int_{t_0}^{t} \frac{\mathrm{d}\xi}{\left(l^2 - \xi^2\right)\sqrt{P(\xi)}} . \tag{c}$$

Assuming that $\dot{z}_0 \neq 0$, we take the sign of $\dot{z}(t_0) = \dot{z}_0$ in the two above formulae. If $\dot{z}_0 = 0$, then we search if z is increasing or decreasing, starting from the initial value z_0.

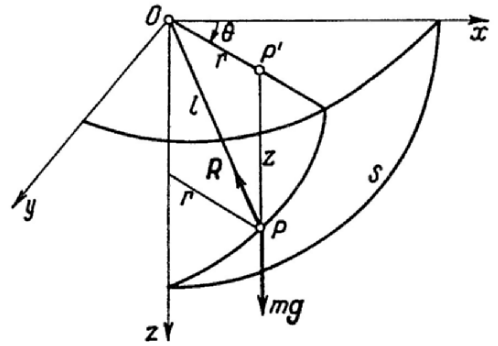

Figure 4. 7. Spherical pendulum

Let us suppose firstly that $\dot{z}_0 \neq 0$; in this case, $P(z_0) > 0$ (from (a) and (b), Appl.4.4 it results $P(z_0) = r_0^2(\dot{r}_0^2 + \dot{z}_0^2)$).

However, during the motion we must have $P(z) \geq 0$ so that the integrals (b) and (c) be real. Noting that $|z_0| < l$ (if $|z_0| = l$, then we have a simple pendulum) and $P(-\infty) = \infty$, $P(\pm l) = -C^2$, it results that the polynomial $P(z)$ is of the form

$$P(z) = -2g(z - z_1)(z - z_2)(z - z_3),\ -\infty < z_3 < -l < z_2 < z_0 < z_1 < l . \tag{d}$$

Hence, the particle P oscillates on the spherical zone between the parallel circles specified by $z = z_1$ and $z = z_2$ (to have $P(z) \geq 0$).

Application 4.6

Problem. Study the motion in Appl.4.3 , assuming a resistance of the form $R = -kv^{\alpha}$. The case $\alpha = 2$ may be considered for a simplification of the computation; it is a satisfactory approximation in case of motions at small velocities.

Mathematical model. If $\alpha > 2$ the equation of motion becomes

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -\left(\frac{g}{v} + \omega^2 v^{\alpha-1}\right),\ v(x_0) = v_0 \neq 0 . \tag{a}$$

Solution. In this form, the equation may be easily solved by Taylor series. Corresponding to the relations (4.16), we write

$$v(x_0) = v_0,$$
$$v'(x_0) = -\left(\frac{g}{v_0} + \omega^2 v_0^{\alpha-1}\right),$$
$$v''(x_0) = \left[\frac{g}{v_0^2} - (\alpha-1)\omega^2 v_0^{\alpha-2}\right]\left(\frac{g}{v_0} + \frac{\omega^2}{v_0}\right),$$

so that, taking only the first three terms of the series

$$v(x) \cong v_0 - \left(\frac{g}{v_0} + \omega^2 v_0^{-1}\right)\left\{x - x_0 - \frac{1}{2}\left[\frac{g}{v_0^2} - (\alpha-1)\omega^2 v_0^{\alpha-2}\right](x-x_0)^2\right\}. \qquad \text{(b)}$$

Application 4.7

Problem. To eliminate the unfortunate effect of the centrifugal force which appears in case of a curvilinear motion of a vehicle, between the straight- way and the arc of circle an arc of curve having a progressive curvature is inserted. Determine this curve, called *clothoid* (or *spiral curve of Cornu*) (Fig.4.8).

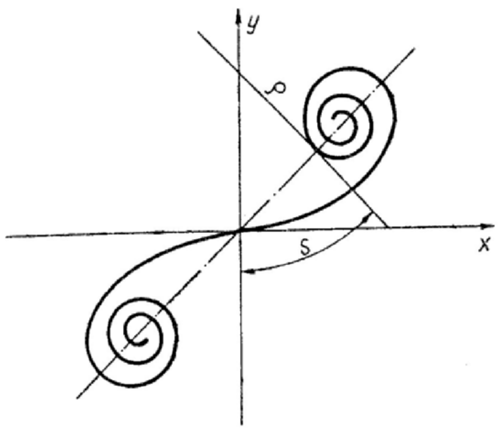

Figure 4. 8. Clothoid

Mathematical model. The intrinsic equation of the clothoid is of the form

$$\rho s = k^2, \qquad \text{(a)}$$

where ρ is the curvature radius, s is the length of the arc measured from the point of zero curvature ($1/\rho = 0$) and $k = \text{const}$ is the modulus of the clothoid. The curve will be determined by its parametric equations in the form of a power series in x and y.

Solution. If we denote by α the angle made by the tangent at a point of the clothoid with the $\mathrm{O}x$ - axis, then the curvature is expressed by means of the relation $\rho = \mathrm{d}s/\mathrm{d}\alpha$, so that the equation (a) becomes

$$s\frac{\mathrm{d}s}{\mathrm{d}\alpha} = k^2 . \tag{b}$$

A direct integration leads to $s^2 = 2k^2\alpha + C$; noting that $\alpha = 0$ for $s = 0$, it results $C = 0$, so that

$$s^2 = 2k^2\alpha \tag{c}$$

or

$$s = k\sqrt{2}\sqrt{\alpha} . \tag{d}$$

One can write the ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\alpha , \; \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\alpha , \tag{e}$$

allowing the determination of x and y when s and α are known. From (d) one obtains, by differentiation,

$$\mathrm{d}s = \frac{k}{\sqrt{2}}\frac{\mathrm{d}\alpha}{\sqrt{\alpha}} . \tag{f}$$

Further, the substitution

$$\alpha = t^2 , \mathrm{d}\alpha = 2t\mathrm{d}t , \tag{g}$$

leads to

$$\mathrm{d}s = \frac{k}{\sqrt{2}}\frac{2t\mathrm{d}t}{\sqrt{t}} = k\sqrt{2}\mathrm{d}t . \tag{h}$$

One obtains thus

$$\mathrm{d}x = \mathrm{d}s\cos\alpha = k\sqrt{2}\cos t^2\mathrm{d}t,$$
$$\mathrm{d}y = \mathrm{d}s\sin\alpha = k\sqrt{2}\sin t^2\mathrm{d}t,$$

whence, by integration,

$$x = k\sqrt{2}\int_0^t \cos t^2 \mathrm{d}t\,,\quad y = k\sqrt{2}\int_0^t \sin t^2 \mathrm{d}t\,. \tag{i}$$

Developing in a power series

$$\cos t^2 = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \ldots + (-1)^n \frac{t^{4n}}{(2n)!} + \ldots,$$

$$\sin t^2 = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \ldots + (-1)^n \frac{t^{4n+2}}{(2n+1)!} + \ldots,$$

and integrating term by term, we get

$$x = k\sqrt{2}\left[t - \frac{t^5}{5\cdot 2!} + \frac{t^9}{9\cdot 4!} - \ldots + (-1)^n \frac{t^{4n+1}}{(4n+1)(2n)!} + \ldots\right],$$

$$y = k\sqrt{2}\left[\frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \ldots + (-1)^n \frac{t^{4n+3}}{(4n+3)(2n+1)!} + \ldots\right]. \tag{j}$$

Application 4.8

Problem. Determine the curve for which the length of the segment of tangent from the contact point to the curve till the intersection with the O x - axis is constant.

Mathematical model. Let $P(x_0, y_0)$ be a point on the curve; the tangent to it is given by

$$y - y_0 = y_0'(x - x_0)\,, \tag{a}$$

and pierces the *Ox*-axis in A, of abscissa $x_A = x_0 - y_0/y_0'$. The condition imposed ($\overline{PA} = a = \text{const}$) leads to

$$a^2 = (x_A - x_0)^2 + y_0^2\,,$$

whence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{y}{\sqrt{a^2 - y^2}}$$

or

$$\mathrm{d}x = \pm\frac{\sqrt{a^2 - y^2}}{y}\mathrm{d}y\,.$$

Solution. We obtain thus a differential equation with separate variables. By integration, we get

$$x = \sqrt{a^2 - y^2} + a \ln\left(\frac{a}{y} - \sqrt{\frac{a^2}{y^2} - 1}\right) + C\,. \tag{b}$$

The curve thus obtained is called *tractrix*. The graphic of the function (b) is given in Fig.4.9 for $C = 0$.

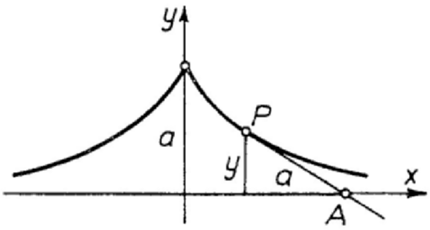

Figure 4. 9. Tractrix

Application 4.9

Problem. Determine the families of principal normal stresses in the case of an elastic half- plane acted upon by a concentrated force P normal to the separaton line.

Mathematical model. The searched families of lines are defined by the differential equation of first order

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \frac{\sigma_x - \sigma_y}{\tau_{xy}}\frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0\,, \tag{a}$$

where σ_x, σ_y and τ_{xy} are the normal stresses and the tangential stress (supposed known), respectively, at the point (x, y), given by

$$\sigma_x = -\frac{2P}{\pi b}\frac{x^3}{\left(x^2 + y^2\right)^2}\,,$$

$$\sigma_y = -\frac{2P}{\pi b}\frac{xy^2}{\left(x^2 + y^2\right)^2}\,, \tag{b}$$

$$\tau_{xy} = -\frac{2P}{\pi b}\frac{x^2 y}{\left(x^2 + y^2\right)^2}\,,$$

where P/b is known.

Solution. The differential equation is of second degree with respect to $\mathrm{d}y/\mathrm{d}x$ and may be decomposed in two differential equations of first order. The product of the roots is equal to -1, so that the two families of trajectories are orthogonal. Solving the equation (a) with respect to $\mathrm{d}y/\mathrm{d}x$, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sigma_x-\sigma_y}{2\tau_{xy}}\pm\sqrt{\left(\frac{\sigma_x-\sigma_y}{2\tau_{xy}}\right)^2+1}\,. \tag{c}$$

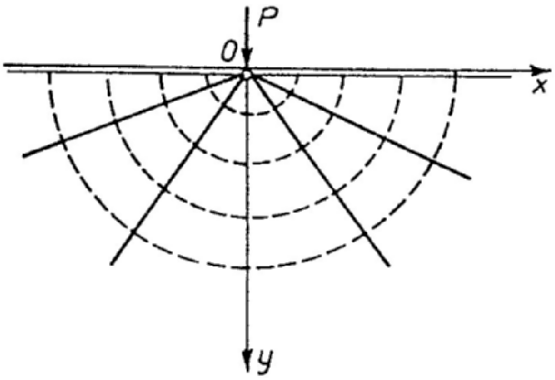

Figure 4. 10. Trajectories of the principal normal stress in case of an elastic half- plane acted upon by a concentrated force normal to the separation line

The relations (b) lead to

$$\frac{\sigma_x-\sigma_y}{2\tau_{xy}}=\frac{-x^3+xy^2}{-2x^2y}=\frac{x^2-y^2}{2xy},$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x^2-y^2}{2xy}\pm\sqrt{\left(\frac{x^2-y^2}{2xy}\right)^2+1}=-\frac{x^2-y^2}{2xy}\pm\frac{x^2+y^2}{2xy},$$

and may be decomposed in the equations

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}, \tag{d}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{x}\,. \tag{e}$$

The equation (d) is a differential equation with separate variables

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}$$

and has the general solution $\ln x = \ln y - \ln m$, $m = \text{const}$. One obtains thus $y = mx$, which represents a family of radial semi-lines (passing through the point O of application of the force).

The equation (e) may be also written in the form of a differential equation with constant coefficients

$$x\mathrm{d}x + y\mathrm{d}y = 0 ;$$

to the general solution $x^2 + y^2 = R^2$ corresponds a family of semicircles with the centre O (the integration constant is R^2). The two nets are represented in Fig.4.10.

If we wish to determine the trajectories passing through the point (x_0, y_0) (the Cauchy problem), it results

$$m = \frac{y_0}{x_0},\ R^2 = x_0^2 + y_0^2 .$$

Application 4.10

Problem. The vessel of a storage basin is asimilated to a parallelepiped the transverse (horizontal) section area of which is A. The discharge of the water at the downhill is made with the aid of an overflow, the flow rate of which is given by the formula $Q_d = Ch^{3/2}$, where C is a constant and h is the charge of the overfall, defined in the Fig.4.11. Study the variation in time of the water level if the flow rate of the entrance stream Q_e is given by

$$Q_e = \begin{cases} Q_0 & \text{for } t \in [0, T], \\ 0 & \text{for } t > T, \end{cases}$$

where Q_0 and T are constants.

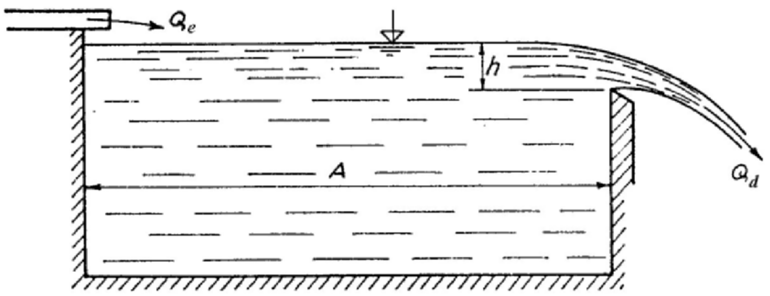

Figure 4. 11. The vessel of a storage basin

Mathematical model. To obtain the differential equation governing the motion, we notice that, in a time interval dt, the sum of the stored volume and the evacuated volume is equal to the entrance volume

$$A\mathrm{d}h + Ch^{3/2}\mathrm{d}t = Q_e \mathrm{d}t\ . \tag{a}$$

Solution. For the first interval, we write the equation (a) in the form

$$\frac{A\mathrm{d}h}{Q_e - Ch^{3/2}} = \mathrm{d}t\ . \tag{b}$$

Introducing the notation $Q_e / C = \beta^3$, the change of function

$$h = y^2,\ \mathrm{d}h = 2y\mathrm{d}y \tag{c}$$

leads to the differential equation with separate variables

$$\frac{2A}{C}\frac{y\mathrm{d}y}{\beta^3 - y^3} = \mathrm{d}t\ . \tag{d}$$

Decomposing the previous fraction in simple fractions

$$\frac{y}{\beta^3 - y^3} = \frac{1}{3\beta}\left(\frac{1}{\beta - y} + \frac{\beta - y}{\beta^2 + \beta y + y^2}\right)$$
$$= \frac{1}{3\beta}\left(\frac{1}{\beta - y} + \frac{1}{2}\frac{2y + \beta}{\beta^2 + \beta y + y^2} - \frac{3\beta}{2}\frac{1}{\beta^2 + \beta y + y^2}\right),$$

the differential equation becomes

$$\frac{2A}{3C\beta}\left(\frac{1}{\beta - y} + \frac{1}{2}\frac{2y + \beta}{\beta^2 + \beta y + y^2} - \frac{3\beta}{2}\frac{1}{\beta^2 + \beta y + y^2}\right)\mathrm{d}y = \mathrm{d}t\ .$$

Integrating, we get

$$\frac{2A}{3C\beta}\left[-\ln(\beta - y) + \frac{1}{2}\ln\left(y^2 + \beta y + \beta^2\right) - \sqrt{3}\arctan\frac{2y + \beta}{\sqrt{3}\beta}\right] = t + t_0\ ,$$

where t_0 is an integration constant.
The previous solution reads

$$\frac{2A}{3C\beta}\left(\ln\frac{\sqrt{y^2 + \beta y + \beta^2}}{\beta - y} - \sqrt{3}\arctan\frac{2y + \beta}{\sqrt{3}\beta}\right) = t + t_0$$

too; returning to the initial function h, we obtain

$$\frac{2A}{3C\beta}\left(\ln\frac{\sqrt{h+\beta\sqrt{h}+\beta^2}}{\beta-\sqrt{h}}-\sqrt{3}\arctan\frac{2\sqrt{h}+\beta}{\sqrt{3}\beta}\right)=t+t_0\,. \tag{e}$$

For $t = 0$ we have $h = 0$, so that

$$-\frac{2A}{3C\beta}\sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)=t_0\,.$$

Finally, we obtain

$$\frac{2A}{3C\beta}\left(\ln\frac{\sqrt{h+\beta\sqrt{h}+\beta^2}}{\beta-\sqrt{h}}-\sqrt{3}\arctan\frac{\sqrt{3h}}{\sqrt{h}+2\beta}\right)=t,\ t\in[0,T]. \tag{f}$$

At the time $t = T$, (f) becomes a transcendental equation

$$\frac{2A}{3C\beta}\left(\ln\frac{\sqrt{h_T+\beta\sqrt{h_T}+\beta^2}}{\beta-\sqrt{h_T}}-\sqrt{3}\arctan\frac{\sqrt{3h_T}}{\sqrt{h_T}+2\beta}\right)=T\,, \tag{g}$$

which determines the level h_T of the water.
For $t > T$ we have $Q_e = 0$ and the equation (a) reads

$$A\mathrm{d}h+Ch^{3/2}\mathrm{d}t=0 \tag{h}$$

or

$$\frac{A}{C}h^{-3/2}\mathrm{d}h+\mathrm{d}t=0\,.$$

Integrating, one obtains

$$-\frac{2A}{C}h^{-1/2}+t=t_1\,, \tag{i}$$

where t_1 is an integration constant, which is determined by the condition $h(T) = h_T$. In this case

$$-\frac{2A}{C}h_T^{-1/2}+T=t_1\,.$$

Hence, we get the formal solution

$$t=\frac{2A}{C}\left(h^{-1/2}-h_T^{-1/2}\right)+T,\ t\geq T\,,$$

whence

$$h = \frac{1}{\left[h_T^{-1/2} + \frac{C}{2A}(t-T)\right]^2}, \; t \geq T.$$

Application 4.11

Problem. A vessel the transverse (horizontal) section area of which is A has at the bottom an outflow orifice which may evacuate a flow rate $Q_\mathrm{d} = Ch^{1/2}$, where C is a constant and h is the depth of the water in the vessel. Study the variation in time of the level h of the water in the vessel if the flow rate of the inflow is Q_e (initially the vessel is empty, that is we have $h = 0$ for $t = 0$). One considers two cases:

$$\text{a) } Q_\mathrm{e} = \begin{cases} Q_0 & \text{for } t \in [0, T], \\ 0 & \text{for } t > T, \end{cases}$$

$$\text{b) } Q_\mathrm{e} = \begin{cases} Q_0 \dfrac{4t}{T} & \text{for } t \in \left[0, \dfrac{T}{4}\right], \\ Q_0\left(2 - \dfrac{4t}{T}\right) & \text{for } t \in \left[\dfrac{T}{4}, \dfrac{T}{2}\right], \end{cases}$$

where Q_0 and T are constants.

The computation schema is given in the Fig.4.12, a and the two variation laws of Q_i are given in Fig.4.12, b.

Mathematical model. To obtain the differential equation governing the motion, we notice that, in a time interval dt, the sum of the stored volume and the evacuated volume is equal to the inflow volume

$$A\frac{\mathrm{d}h}{\mathrm{d}t} + Ch^{1/2} = Q_\mathrm{e}. \tag{a}$$

This is a non-linear, non-homogeneous differential equation.

Solution. By the change of function

$$h = y^2 \quad \Rightarrow \quad \mathrm{d}h = 2y\mathrm{d}y \tag{b}$$

the equation (a) becomes

$$2Ay\frac{\mathrm{d}y}{\mathrm{d}t} + Cy = Q_\mathrm{e}, \tag{c}$$

and we may consider the two cases for Q_e.

a) For $t \in [0, T]$ the equation (c) is with separate variables

$$\mathrm{d}t = \frac{2Ay\mathrm{d}y}{Q_0 - Cy}.$$

Introducing the notation $Q_0/C = \beta$, the general solution of the previous equation becomes

$$y + \beta \ln(\beta - y) = -\frac{C}{2A}(t + \tau_0),$$

where τ_0 is an integration constant; returning to the variable h, the solution becomes

$$h^{1/2} + \frac{Q_0}{C} \ln\left(\frac{Q_0}{C} - h^{1/2}\right) = -\frac{C}{2A}(t + \tau_0). \tag{d}$$

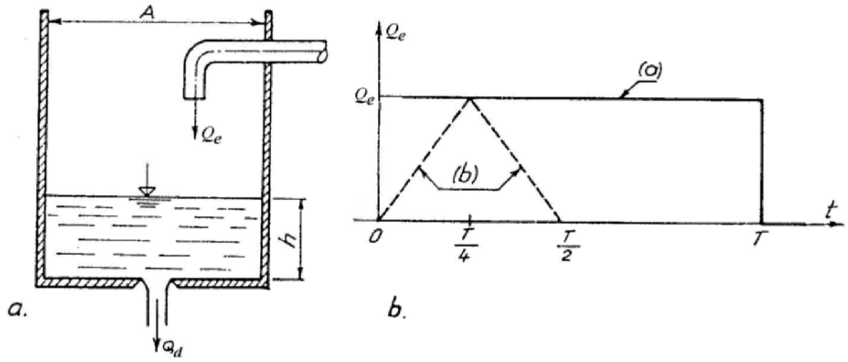

Figure 4. 12. Vessel with orifice. Computation schema (a). Variation laws of Q_e (b)

Introducing the initial condition ($h = 0$ for $t = 0$), it results

$$\tau_0 = -\frac{2A}{C}\frac{Q_0}{C} \ln \frac{Q_0}{C},$$

so that (d) becomes

$$h^{1/2} + \frac{Q_0}{C} \ln\left(1 - \frac{Ch^{1/2}}{Q_0}\right) = -\frac{C}{2A} t, \; t \in [0, T]. \tag{e}$$

In particular, at the moment $t = T$, we have

$$h_T^{1/2} + \frac{Q_0}{C} \ln\left(1 - \frac{Ch_T^{1/2}}{Q_0}\right) = -\frac{C}{2A} T, \tag{f}$$

obtaining the height h_T.

For the interval $t > T$, the differential equation (a) takes the form

$$A\frac{\mathrm{d}h}{\mathrm{d}t} + Ch^{1/2} = 0$$

or

$$A\frac{\mathrm{d}h}{h^{1/2}} + C\mathrm{d}t = 0\,,$$

with the general solution

$$2Ah^{1/2} + Ct = \tau_1\,, \tag{g}$$

where τ_1 is an integration constant, which must be determined from the condition of continuity; for $t = T$ we must have $h = h_T$, so that

$$2Ah_T^{1/2} + CT = \tau_1\,. \tag{h}$$

Introducing in (g), one obtains

$$2Ah^{1/2} + Ct = 2Ah_T^{1/2} + CT\,,$$

so that the level h is determined by

$$h = \left[h_T^{-1/2} - \frac{C}{2A}\left(t - T\right)\right]^2,\ t > T\,. \tag{i}$$

The time t is thus given by

$$t = \begin{cases} -\dfrac{2A}{C}\left[h^{1/2} + \dfrac{Q_0}{C}\ln\left(1 - \dfrac{Ch^{1/2}}{Q_0}\right)\right] & \text{for } t \in [0, T], \\ T + \dfrac{2A}{C}\left(h_T^{1/2} - h^{1/2}\right) & \text{for } t > T. \end{cases}$$

b) The differential equation (a) becomes

$$A\frac{\mathrm{d}h}{\mathrm{d}t} + Ch^{1/2} = Q_0\,\frac{4t}{T}$$

for the first interval; by means of the change of function $h = t^2 u$, the equation reads

$$A\left(2tu + t^2 u'\right) + Ct\sqrt{u} = Q_0\,\frac{4t}{T}\,.$$

Simplifying by t, we get the equation with separate variables

$$\frac{A\mathrm{d}u}{\frac{4Q_0}{T}-C\sqrt{u}-2Au}=\frac{\mathrm{d}t}{t},$$

hence

$$\ln t=\int\frac{A\mathrm{d}u}{\frac{4Q_0}{T}-C\sqrt{u}-2Au}+\ln K\equiv F(u)+\ln K\ ,$$

K being an arbitrary positive constant.
The primitive F in the right member may be written

$$F(u)=-\int\frac{2Av\mathrm{d}v}{2Av^2+Cv-\frac{4Q_0}{T}}=-\int\frac{2Av\mathrm{d}v}{2A(v-v_1)(v-v_2)}$$
$$=\int\frac{v_1}{v_2-v_1}\frac{1}{v-v_1}\mathrm{d}v-\int\frac{v_2}{v_2-v_1}\frac{1}{v-v_2}\mathrm{d}v$$
$$=\frac{v_1}{v_2-v_1}\ln|v-v_1|-\frac{v_2}{v_2-v_1}\ln|v-v_2|,$$

where $v=\sqrt{u}$ and v_1, v_2 are the roots of the algebraic equation

$$2Av^2+Cv-\frac{4Q_0}{T}=0,$$

which are always real. Hence,

$$v_{1,2}=\frac{-C\pm\sqrt{C^2+\frac{16Q_0}{T}2A}}{4A},\quad v_1>0,\ v_2<0.$$

The solution is thus of the form

$$\frac{v_1}{v_2-v_1}\ln\left|\frac{\sqrt{h}}{t}-v_1\right|-\frac{v_2}{v_2-v_1}\ln\left|\frac{\sqrt{h}}{t}-v_2\right|=\ln t-\ln K$$

or

$$\left(\sqrt{h}-v_1t\right)^{v_1}\left(\sqrt{h}-v_2t\right)^{-v_2}=K_1,$$

where K_1 is a new arbitrary constant.
If $h(0)=0$, it results $h=v_1^2t^2$ on the first interval.
For the second interval, we use the same method.
In the equation

$$A\frac{\mathrm{d}h}{\mathrm{d}t}+Ch^{1/2}=Q_0\left(2-\frac{4t}{T}\right)$$

we make the change of function $h=(2-4t/T)^2u$. Thus

$$A\left[-\frac{8}{T}\left(2-\frac{4t}{T}\right)u+\left(2-\frac{4t}{T}\right)^2u'\right]+C\left(2-\frac{4t}{T}\right)\sqrt{u}=Q_0\left(2-\frac{4t}{T}\right).$$

Simplifying by $2-(4t/T)$, we obtain again a differential equation with separable variables

$$-\frac{8A}{T}u+\left(2-\frac{4t}{T}\right)\frac{\mathrm{d}u}{\mathrm{d}t}+C\sqrt{u}=Q_0,$$

that is

$$\frac{\mathrm{d}t}{2-\frac{4t}{T}}=\frac{\mathrm{d}u}{Q_0-C\sqrt{u}+\frac{8A}{T}u}.$$

Application 4.12

Problem. Study the variation of the velocity of the water in a simple pipe filled in from a tank by the sudden oppening of the slide valve (Fig.4.13).

Mathematical model. The energetically relation of Bernoulli between the bunker and the slide valve leads to

$$H_0=(a+\xi)\frac{v^2}{2g}+\frac{L}{g}\frac{\mathrm{d}v}{\mathrm{d}t} \tag{a}$$

for the case of the non-permanent motion (transitory regime), and to

$$H_0=(a+\xi)\frac{v_0^2}{2g} \tag{b}$$

where $v_0=\text{const}$ is the velocity in a permanent regime, for the case of the permanent motion (stabilized regime).

Solution. Subtracting the relation (b) from (a), it results the differential equation

$$\frac{a+\xi}{2g}\left(v^2-v_0^2\right)+\frac{L}{g}\frac{\mathrm{d}v}{\mathrm{d}t}=0;$$

simplifying by g and introducing the notation

$$B = \frac{a + \xi}{2L} = \frac{g}{L} \frac{H_0}{v_0^2} , \tag{c}$$

we may write

$$B\mathrm{d}t = \frac{\mathrm{d}v}{v^2 - v_0^2} = \frac{1}{2v_0} \left(\frac{1}{v_0 + v} + \frac{1}{v_0 - v} \right) \mathrm{d}v . \tag{d}$$

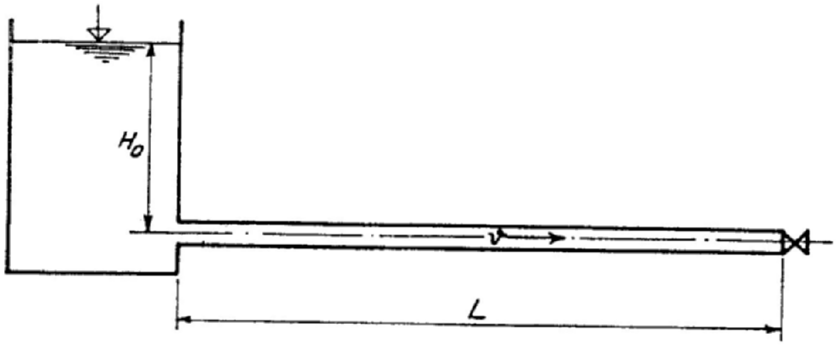

Figure 4. 13. Geometric schema of the tank and of the pipe

The general solution of the differential equation with separate variables (d) is

$$t = \frac{1}{2Bv_0} \ln \frac{v_0 + v}{v_0 - v} + C , \tag{e}$$

where C is an integration constant. We put $v(0) = 0$; it results $C = 0$, so that we have

$$t = \frac{1}{2Bv_0} \ln \frac{v_0 + v}{v_0 - v} = \frac{v_0 L}{2gH_0} n \frac{v_0 + v}{v_0 - v} , \tag{f}$$

as well as

$$v = v_0 \tanh\left(\frac{gH_0}{v_0 L} t \right) . \tag{f}$$

Application 4.13

Problem. Study the form of the free surface of water which flows through a pervious layer on a tight bed of inclination *i*. The velocity *v* of aparent flow through an arbitrary section (the flow rate with respect to the whole section) is proportional to the inclination of the free surface of water in that section (Darcy's law). Particular case: $i = 0$.

Mathematical model. The computation schema is given in Fig.4.14, where *q* is the unit flow rate (corresponding to a section of unit breadth), z is the applicate of the tight bed with respect to a horizontal plane of reference, *z* is the applicate of the free surface of water, measured from the inclined tight bed, and h_0 is the constant depth in the uniform motion.

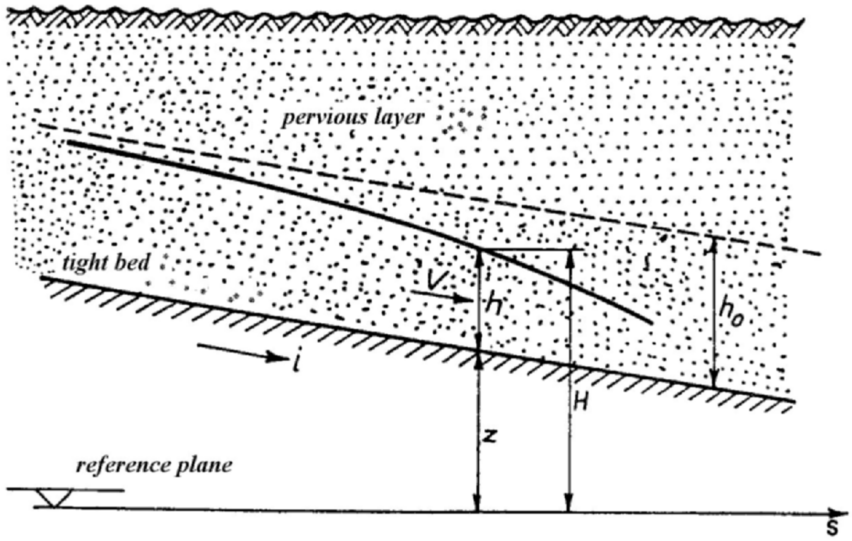

Figure 4. 14. Flow through a pervious layer

Hence, $i = -\mathrm{d}z/\mathrm{d}s$ is the inclination of the tight bed and $j = -\mathrm{d}H/\mathrm{d}s$ is the inclinationt of the free surface, where

$$H = z + h\,. \tag{a}$$

Darcy's law reads

$$v = kj\,, \tag{b}$$

where k is the proportionality constant.

Solution. The velocity may be written in the following forms:

$$v = \frac{q}{h\cdot 1} = kj = -k\frac{\mathrm{d}H}{\mathrm{d}s} = -k\frac{\mathrm{d}(z+h)}{\mathrm{d}s} = -k\frac{\mathrm{d}z}{\mathrm{d}s} - k\frac{\mathrm{d}h}{\mathrm{d}s} = ki - k\frac{\mathrm{d}h}{\mathrm{d}s}\,.$$

From the second and the last member, we get

$$\frac{\mathrm{d}h}{\mathrm{d}s} = i - \frac{q}{kh}\,. \tag{c}$$

In the case of a uniform motion we have $v = v_0 = q/h_0$ and $j = i$, hence $q/h_0 = ki$; it results $q = kih_0$. Replacing in (c), one obtains

$$\frac{\mathrm{d}h}{\mathrm{d}s} = i\left(1 - \frac{h_0}{h}\right) \tag{d}$$

or, separating the variables,

$$\left(1 - \frac{h_0}{h_0 - h}\right)\mathrm{d}h = i\mathrm{d}s \; .$$

Integrating, it results

$$h + h_0 \ln\left(h_0 - h\right) = is + C \; , \tag{e}$$

where C is an integration constant. To determine it, we put the condition that the applicate of the free surface is $h = h_1$ in a section $s = s_1$; the relation (e) becomes

$$h_1 + h_0 \ln\left(h_0 - h_1\right) = is_1 + C \; . \tag{f}$$

Subtracting (f) from (e), we have, finally,

$$h - h_1 + h_0 \ln \frac{h_0 - h}{h_0 - h_1} = i\left(s - s_1\right), \tag{g}$$

whence

$$s = s_1 + \frac{h - h_1}{i} + \frac{h_0}{i} \ln \frac{h_0 - h}{h_0 - h_1} \, . \tag{h}$$

To obtain h , one must solve numerically the transcendental equation (g).
In the particular case $i = 0$, the equation (c) has a simpler form

$$\frac{\mathrm{d}h}{\mathrm{d}s} = -\frac{q}{kh} \, ,$$

and, separating the variables, we get

$$h\mathrm{d}h = -\frac{q}{k}\mathrm{d}s \; .$$

Noting that for $s = s_1$ we have $h = h_1$, it results, eliminating the constant C ,

$$s = s_1 - \frac{k}{2q}\left(h^2 - h_1^2\right) \tag{i}$$

or

$$h = \sqrt{h_1^2 - \frac{2q}{k}\left(s - s_1\right)} \, . \tag{j}$$

In this case, the free surface is a parabolical cylinder.

Application 4.14

Problem. Establish the equation of the meridian curve of the free surface of water which flows through a pervious layer with horizontal bed towards a circular fountain (Fig.4.15). One assumes that the perfect fountain attains the bottom tight layer.

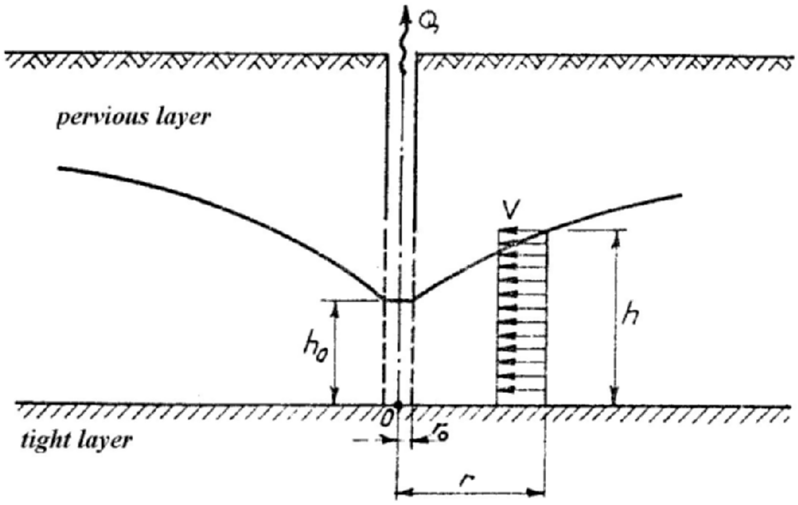

Figure 4. 15. Free surface of the water in flow through a pervious layer

Mathematical model. The problem is axi-symmetrical, so that the free surface of water is a surface of rotation defined by its meridian curve.

We denote by Q the flow rate extracted from the fountain, by r_0 the radius of the fountain, by r the radius of the cylinder of height h through which the water flows, by $v = k\,\mathrm{d}h/\mathrm{d}r$ the velocity (given by d'Arcy's law), where k is a proportionality constant, and by h_0 the free depth of water in the fountain.

To put the problem in equation, we notice that the flow rate extracted from the fountain is equal to the flow rate which flows through the pervious layer towards the fountain. We may write

$$Q = 2\pi rhv = 2\pi rhk\frac{\mathrm{d}h}{\mathrm{d}r},$$

obtaining thus a differential equation with separate variables

$$\frac{Q}{2\pi k}\frac{\mathrm{d}r}{r} = h\mathrm{d}h\,. \tag{a}$$

Solution. Integrating, one obtains

$$\frac{Q}{2\pi k}\ln r = \frac{h^2}{2} + C\,, \tag{b}$$

where C is an integration constant, determined by the condition $h = h_0$ for $r = r_0$; hence

$$\frac{Q}{2\pi k} \ln r_0 = \frac{h_0^2}{2} + C\,. \tag{c}$$

Subtracting (c) from (b), we get

$$\frac{Q}{2\pi k} \ln \frac{r}{r_0} = \frac{1}{2}\left(h^2 - h_0^2\right).$$

The flow rate through a cylinder of radius r and height h is thus given by

$$Q = \frac{\pi k\left(h^2 - h_0^2\right)}{\ln \dfrac{r}{r_0}}\,, \tag{d}$$

whence

$$h = \sqrt{h_0^2 + \frac{Q}{\pi k} \ln \frac{r}{r_0}} \tag{e}$$

and

$$r = r_0 \mathrm{e}^{\pi k\left(h^2 - h_0^2\right)/Q}\,,\ r \in \left[r_0, \infty\right), \tag{f}$$

respectively.

The formula (f) may be written more conveniently if a point of the curve, e.g. $h = h_1$ for $r = r_1$, is known. From (d), one obtains

$$\frac{Q}{\pi k} = \frac{h_1^2 - h_0^2}{\ln \dfrac{r_1}{r_0}}\,;$$

introducing this in (f), we eventually have

$$r = r_0 \exp\left(\frac{h^2 - h_0^2}{h_1^2 - h_0^2} \ln \frac{r_1}{r_0}\right). \tag{g}$$

Application 4.15

Problem. Study the curve of the free surface of water in a prismatic channel of rectangular cross section, the longitudinal gradient being i.

Mathematical model. The computation schema is given in Fig.4.16 and the differential equation of the problem is

$$\frac{\mathrm{d}h}{\mathrm{d}s} = i\,\frac{h^3 - h_0^3}{h^3 - h_{cr}^3}\,, \tag{a}$$

where h is the depth of water at the distance s and h_0 and h_{cr} are the normal and the critical depth, respectively. The two heights h_0 and h_{cr} may be in any ratio($h_0 < h_{\mathrm{cr}}$ or $h_0 > h_{\mathrm{cr}}$); in Fig.4.16 it has been considered the case $h_0 > h_{cr}$.

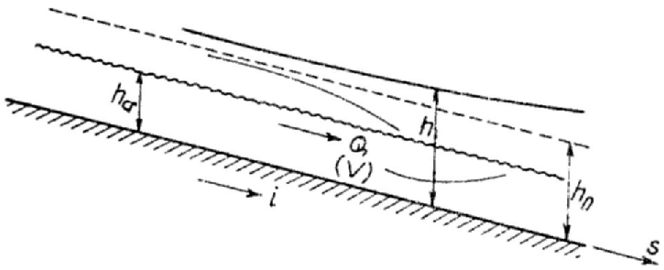

Figure 4. 16. The curve of the surface of water in an inclined channel

Solution. The equation (a) may be written in the form

$$\frac{h^3 - h_{cr}^3}{h^3 - h_0^3}\,\mathrm{d}h = i\mathrm{d}s\,, \tag{b}$$

hence an ODE with separate variables.
The ratio in the left member may be written successively

$$\frac{h^3 - h_{cr}^3}{h^3 - h_0^3} = \frac{h^3 - h_0^3 + h_0^3 - h_{cr}^3}{h^3 - h_0^3} = 1 + \frac{h_0^3 - h_{cr}^3}{h^3 - h_0^3}$$

$$= 1 + \frac{h_0^3 - h_{cr}^3}{3h_0^2}\left(\frac{1}{h - h_0} - \frac{h + 2h_0}{h^2 + h_0 h + h_0^2}\right)$$

$$= 1 + \frac{h_0^3 - h_{cr}^3}{3h_0^2}\left(\frac{1}{h - h_0} - \frac{1}{2}\,\frac{2h + h_0}{h^2 + h_0 h + h_0^2} - \frac{3}{2}\,h_0\,\frac{1}{h^2 + h_0 h + h_0^2}\right)$$

and the equation (b) becomes

$$\left[1 + \frac{h_0^3 - h_{cr}^3}{3h_0^2}\left(\frac{1}{h - h_0} - \frac{1}{2}\,\frac{2h + h_0}{h^2 + h_0 h + h_0^2} - \frac{3}{2}\,h_0\,\frac{1}{h^2 + h_0 h + h_0^2}\right)\right]\mathrm{d}h = i\mathrm{d}s\,.$$

Integrating, it results

$$h+\frac{h_0^3-h_{cr}^3}{3h_0^2}\ln(h-h_0)-\frac{h_0^3-h_{cr}^3}{6h_0^2}\ln\left(h^2+h_0h+h_0^2\right)$$
$$-\sqrt{3}\arctan\frac{h_0+2h}{\sqrt{3}h_0}=i(s+C),$$

where C is an integration constant; the solution may be written

$$h+\frac{h_0^3-h_{cr}^3}{3h_0^2}\ln\frac{h-h_0}{\sqrt{h^2+h_0h+h_0^2}}-\sqrt{3}\arctan\frac{2h+h_0}{\sqrt{3}h_0}=i(s+C) \tag{c}$$

too, where the constant is determined supposing that, downhill, we have $h=h_1$ for $s=s_1$, that is

$$h_1+\frac{h_0^3-h_{cr}^3}{3h_0^2}\ln\frac{h_1-h_0}{\sqrt{h_1^2+h_0h_1+h_0^2}}-\sqrt{3}\arctan\frac{2h_1+h_0}{\sqrt{3}h_0}=i(s_1+C), \tag{d}$$

Subtracting (d) from (c), we finally get

$$h-h_1+\frac{h_0^3-h_{cr}^3}{3h_0^2}\ln\left(\frac{h-h_0}{h_1-h_0}\sqrt{\frac{h_1^2+h_0h_1+h_0^2}{h^2+h_0h+h_0^2}}\right),$$
$$-\sqrt{3}\arctan\frac{\sqrt{3}(h-h_1)}{h+2h_0+3h_1}=i(s-s_1). \tag{e}$$

The formula (e) allows to determine the free surface upstream the section $s=s_1$.

Application 4.16

Problem. Study the flow rate of water from a vessel the form of which is a rotation surface of vertical axis. Consider the particular case of a semi-sphere vessel of radius a, with an orifice of area A at the bottom (we assume that the radius of the orifice may be neglected with respect to the dimensions of the vessel). Determine the interval of time in which the full vessel becomes empty. Numerical data: $a=100\,\text{cm}$, $A=1\,\text{cm}^2$.

Mathematical model. In hydro-dynamics, the velocity of flow of water through an orifice at the depth h from the free surface of the liquid is given by Galilei's formula, in the form

$$v=k_1\sqrt{2gh}=k\sqrt{h}, \tag{a}$$

where k_1 is a viscosity coefficient (for water, $k_1\cong 0.6$).

We suppose that the equation of the meridian curve of the vessel is $r^2=r^2(h)$ (Fig.4.17) and we must determine the height h of water at a given moment t.

The velocity v of the flow is also a function of time (by the agency of h, as it can be seen in (a)).

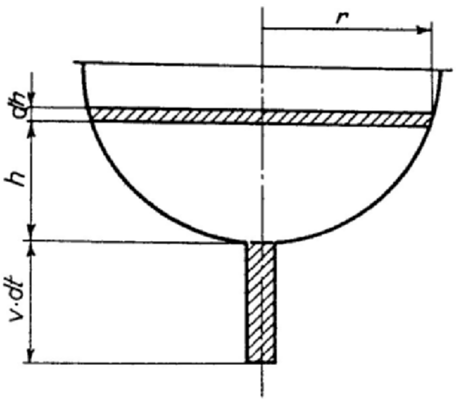

Figure 4. 17. The flow of water from a vessel the form of which is a rotation surface

We calculate now the volume of water which flows in the interval of time dt. First of all, through the orifice flows the liquid contained in a cylinder of basis area A and height vdt , hence

$$\mathrm{d}V = Av\mathrm{d}t = Akh^{1/2}\mathrm{d}t \ . \tag{b}$$

On the other hand, the height in the vessel lowers with dh; the corresponding volume is

$$\mathrm{d}V = -\pi r^2 \mathrm{d}h \ . \tag{c}$$

Equating the expressions (b) and (c) of dV, it results the ODE of the problem

$$-\pi r^2 \mathrm{d}h = Akh^{1/2}\mathrm{d}t \ .$$

Solution. Separating the variables, we get

$$\mathrm{d}\,t = -\frac{\pi}{Ak}\frac{r^2}{h^{1/2}}\,\mathrm{d}\,h \ ,$$

whence, by integration

$$t = -\frac{\pi}{Ak}\int\frac{r^2}{h^{1/2}}\mathrm{d}h + C \ . \tag{d}$$

The integration constant C is determined by the initial condition $h = h_{\max}$ for $t = 0$. Then it results

$$t = \frac{\pi}{Ak}\int_{h_{\max}}^{h}\frac{r^2}{h^{1/2}}\mathrm{d}h \ . \tag{e}$$

In the particular case of a semi-sphere, the equation of the meridian curve is (with $h_{\max} = a$)

$$r^2 = h(2a - h).$$

Introducing in (e), we have successively

$$t = \frac{\pi}{Ak}\int_a^h \frac{h(2a-h)}{h^{1/2}}\,\mathrm{d}h = \frac{\pi}{Ak}\int_a^h \left(2ah^{1/2} - h^{3/2}\right)\mathrm{d}h = \frac{\pi}{Ak}\left[\frac{4}{3}ah^{3/2} - \frac{2}{5}h^{5/2}\right]_a^h$$
$$= \frac{\pi}{Ak}\left(\frac{14}{15}a^{5/2} - \frac{4}{3}ah^{3/2} + \frac{2}{5}h^{5/2}\right).$$

The vessel is completely empty for $h = 0$; it corresponds the interval of time

$$t_0 = \frac{\pi}{Ak}\frac{14}{15}a^{5/2}.$$

With the numerical data of the problem and taking $g = 981\,\mathrm{cm/s^2}$, we get

$$t_0 = \frac{14}{15}\cdot\frac{\pi\cdot 100^{5/2}}{1\cdot 0.6\sqrt{2\cdot 981}} = 11033'' = 183'53'' = 3\mathrm{h}\ 03'\ 53''.$$

Application 4.17

Problem. To cross a river, a swimmer starts from a point $P(x_0, y_0)$ situated on a bank and wishes to reach the point $Q(0,0)$ on the other bank. The velocity of the water flow is **a** and the velocity of displacement of the swimmer is **b**. Which is the trajectory described by the swimmer if the relative velocity is directed all the time towards the point *Q* ?

Mathematical model. Let be M the position of the swimmer at the moment *t* (Fig.4.18). The components of the absolute velocity along the two axes *Ox* and *Oy* $(O \equiv Q)$ are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a - b\frac{x}{\sqrt{x^2 + y^2}},$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -b\frac{y}{\sqrt{x^2 + y^2}}; \qquad \text{(a)}$$

eliminating d*t*, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{x}{y} - \frac{a}{b}\sqrt{1 + \frac{x^2}{y^2}}, \qquad \text{(b)}$$

which is the differential equation of the searched trajectory.

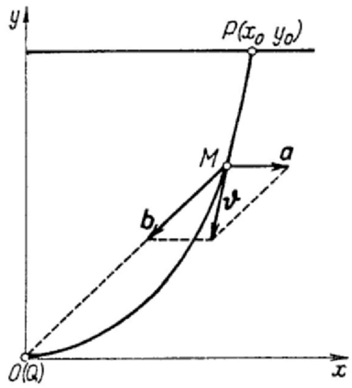

Figure 4. 18. Swimmer's problem

Solution. The equation (b) is a homogeneous one, so that we make the substitution

$$x = uy \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}y} = u + y\frac{\mathrm{d}u}{\mathrm{d}y},$$

and it becomes

$$y\frac{\mathrm{d}u}{\mathrm{d}y} = -\frac{a}{b}\sqrt{1+u^2}\ . \tag{c}$$

Introducing the ratio $m = a/b$ of the velocities, we get

$$-m\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}u}{\sqrt{1+u^2}}\ .$$

By integration, one obtains

$$-m\ln y + m\ln c = \ln\left(u + \sqrt{1+u^2}\right),$$

where c is an integration constant, or

$$\left(\frac{c}{y}\right)^m = \frac{x}{y} + \sqrt{1 + \frac{x^2}{y^2}}\ .$$

Thus, we get

$$x = \frac{y}{2}\left[\left(\frac{c}{y}\right)^m - \left(\frac{y}{c}\right)^m\right], \tag{d}$$

and the problem has a solution only for $m \in (0,1)$. The constant may be determined by imposing the condition that the trajectory passes through the points *P* and *Q*.

Application 4.18

Problem. Determine the families of the trajectories of the extreme tangential stresses in case of an elastic half-plane acted upon by a concentrated force P normal to the separation line.

Mathematical model. The searched trajectories are defined by the first order ODE

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{4\tau_{xy}}{\sigma_x - \sigma_y}\frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0\,, \tag{a}$$

where σ_x, σ_y and τ_{xy} are the normal and tangential stresses (supposedly known), respectively, at a point (x, y) (Fig.4.19). The state of stress is given by

$$\begin{aligned} \sigma_x &= -\frac{2P}{\pi b}\frac{x^3}{\left(x^2 + y^2\right)^2}\,, \\ \sigma_y &= -\frac{2P}{\pi b}\frac{xy^2}{\left(x^2 + y^2\right)^2}\,, \\ \tau_{xy} &= -\frac{2P}{\pi b}\frac{x^2 y}{\left(x^2 + y^2\right)^2}\,, \end{aligned} \tag{b}$$

where b is the constant thickness of the plate and $P/b = \mathrm{const}$.

Solution. The differential equation is of second degree and may be decomposed in two differential equations of first order. The product of the roots is -1, so that the two families of curves are orthogonal. Solving the algebraic equation of second degree (a), one obtains

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \pm \sqrt{\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)^2 + 1}\,, \tag{c}$$

By means of relations (b), we obtain

$$\frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2xy}{x^2 - y^2}\,,$$

so that the differential equation of the trajectories becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy}{x^2 - y^2} \pm \sqrt{\left(\frac{2xy}{x^2 - y^2}\right)^2 + 1} = \frac{2xy}{x^2 - y^2} \pm \frac{x^2 + y^2}{x^2 - y^2}$$

and may be decomposed in two equations

$$\frac{dy}{dx} = \frac{x+y}{x-y}, \tag{d}$$

$$\frac{dy}{dx} = -\frac{x-y}{x+y}, \tag{e}$$

The equation (d) may be written as a homogeneous equation

$$\frac{dy}{dx} = \frac{1+\dfrac{y}{x}}{1-\dfrac{y}{x}}. \tag{f}$$

By the substitution $u = y/x$, the equation (f) reads

$$\frac{dx}{x} = \frac{du}{\dfrac{1+u}{1-u} - u} = \frac{1-u}{1+u^2}du = \frac{du}{1+u^2} - \frac{1}{2}\frac{2u du}{1+u^2};$$

integrating, we get

$$\ln x = \arctan u - \frac{1}{2}\ln\left(1 + u^2\right) + \ln C,$$

where C is an integration constant.
The solution is obtained in a simpler form in polar co-ordinates; we have successively (with $x = r\cos\varphi$, $y = r\sin\varphi$, $y/x = \tan\varphi$)

$$\ln x + \ln\sqrt{1 + \frac{y^2}{x^2}} = \arctan\frac{y}{x} + \ln C_1,$$

$$\ln\sqrt{x^2 + y^2} = \arctan\frac{y}{x} + \ln C_1,$$

$$\ln r = \varphi + \ln C_1$$

and, finally,

$$r = C_1 e^{\varphi}. \tag{g}$$

The curve (g) represents the equation of a family of logarithmic spirals which pierce the radial half-lines in the Appl.4.9 under angles of $\pi/4$.
The equation (e) may be written in the homogeneous form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{y}{x} - 1}{\frac{y}{x} + 1}. \tag{h}$$

The same substitution $u = y/x$ leads to

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}u}{\frac{u-1}{u+1} - u} = -\frac{1+u}{1+u^2}\mathrm{d}u = -\frac{\mathrm{d}u}{1+u^2} - \frac{1}{2}\frac{2u\mathrm{d}u}{1+u^2}.$$

By integration, it results

$$\ln x = -\arctan u - \frac{1}{2}\ln\left(1 + u^2\right) + \ln C_2$$

or, finally,

$$r = C_2 \mathrm{e}^{-\varphi}. \tag{i}$$

which represents a family logarithmic spirals, orthogonal to the first one.
Let us determine the constants C_1 and C_2. Consider the point $A(x_0, y_0)$ through which pass the trajectories of the principal normal stresses and the trajectories of extreme tangential stresses.
The equation of the trajectory σ_1 reads

$$y = x \tan \varphi_0, \ \tan \varphi_0 = \frac{y_0}{x_0}. \tag{j}$$

and the trajectory σ_2 may be written

$$x^2 + y^2 = r_0^2, \ r_0 = \sqrt{x_0^2 + y_0^2}. \tag{k}$$

Let us consider further the solution (g). The condition that this logarithmic spiral passes through the point A leads to

$$C_1 = r_0 \mathrm{e}^{-\varphi_0};$$

hence,

$$r = r_0 \mathrm{e}^{\varphi - \varphi_0}. \tag{l}$$

For the second trajectory we may write

$$r = r_0 \mathrm{e}^{\varphi_0 - \varphi}. \tag{l}$$

To represent graphically the trajectories, we take $x_0 = y_0 = 1$. It follows that $\varphi_0 = \pi/4$ and $r_0 = \sqrt{2}$.

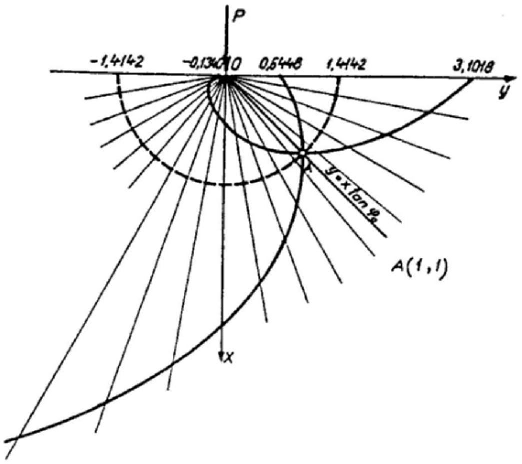

Figure 4. 19. The trajectories of the extreme tangential stresses in case of an elastic half- plane acted upon by a concentrated force normal to the separation line

The curves (l) and (m) have been represented in Fig.4.19, together with the trajectories (j) and (k). We notice that the trajectories of the extreme tangential stress pierce the straight line (j) and the semicircle (m) under angles of $\pi/4$. From the two trajectories we retain the arcs corresponding to $x > 0$.

Application 4.19

Problem. Find the isogonal trajectories of the family of straight lines passing through a fixed point; the angle of intersection is α.

Mathematical model. As it is known, in general, if a family of curves is given by the differential equation

$$F(x, y, \mathrm{d}y/\mathrm{d}x) = 0, \tag{a}$$

then the family of isogonal trajectories is defined by the differential equation

$$F\left(x, y, \frac{\dfrac{\mathrm{d}y}{\mathrm{d}x} - k}{k\dfrac{\mathrm{d}y}{\mathrm{d}x} + 1}\right) = 0, \tag{b}$$

where $k = \tan\alpha$.

Solution. In our case, choosing the origin of the co-ordinate axes at the fixed point, the equation of the family of straight lines is given by $y = mx$, whence $\mathrm{d}y/\mathrm{d}x = m$, so that the differential equation of the family of straight lines is

$$\frac{y}{x} = \frac{\mathrm{d}y}{\mathrm{d}x} . \tag{c}$$

In this case, the equation (b) leads to

$$\frac{y}{x} = \frac{\dfrac{\mathrm{d}y}{\mathrm{d}x} - k}{k\dfrac{\mathrm{d}y}{\mathrm{d}x} + 1} ,$$

which represents the ODE of the isogonal trajectories.
It may be reduced to the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx + y}{x - ky} , \tag{d}$$

hence to a homogeneous equation with separate variables

$$\frac{\mathrm{d}x}{x} = \frac{1}{k}\frac{1 - ku}{1 + u^2}\,\mathrm{d}u ,$$

the general solution of which is

$$\ln x = \frac{1}{k}\arctan u - \frac{1}{2}\ln\left(1 + u^2\right) + \ln C ;$$

returning to the initial variables, we get

$$\ln x + \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = \frac{1}{k}\arctan\frac{y}{x} + \ln C .$$

After some transformations, the previous expression reads

$$\sqrt{x^2 + y^2} = C\mathrm{e}^{(1/k)\arctan(y/x)} \tag{e}$$

In polar co-ordinates $r = \sqrt{x^2 + y^2}$, $y = \arctan(y/x)$, one obtains

$$r = C\mathrm{e}^{\varphi/k} , \tag{f}$$

which is the equation of a family of logarithmic spirals.
The previous application is thus generalized.

Application 4.20

Problem. Determine the trajectories of the principal normal stresses in a gravity dam, the upstream face of which is vertical, while the downhill face is inclined of angle α. The unit weights of concrete and of water are γ_1 and γ, respectively (Fig.4.20).

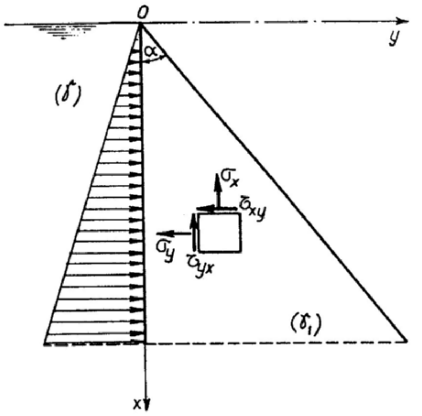

Figure 4. 20. Cross section in a gravity dam

Mathematical model. As a study of plane elasticity, one may express the state of stress in a gravity dam in the form

$$\begin{aligned} \sigma_x &= \left(\frac{\gamma}{\tan^2\alpha} - \gamma_1\right)x + \left(\frac{\gamma_1}{\tan\alpha} - \frac{2\gamma}{\tan^3\alpha}\right)y, \\ \sigma_y &= -\gamma x, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(a)} \\ \tau_{xy} &= \tau_{yx} = -\frac{\gamma y}{\tan^2\alpha}. \end{aligned}$$

where σ_x, σ_y and τ_{xy} are the normal and the tangential stresses, respectively. The differential equations of the searched trajectories are given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sigma_y - \sigma_x}{2\tau_{xy}} \pm \sqrt{\left(\frac{\sigma_y - \sigma_x}{2\tau_{xy}}\right)^2 + 1}\,. \qquad \text{(b)}$$

Solution. We compute the ratio

$$\frac{\sigma_y - \sigma_x}{2\tau_{xy}} = \frac{\gamma_1}{2\gamma}\tan\alpha - \frac{1}{\tan\alpha} + \left(\frac{1+\tan^2\alpha}{2} - \frac{\gamma_1}{2\gamma}\tan^2\alpha\right)\frac{x}{y} = a + b\frac{x}{y}\,, \qquad \text{(c)}$$

where we used the notations

$$a = \frac{\gamma_1}{2\gamma}\tan\alpha - \frac{1}{\tan\alpha}\,,$$
$$b = \frac{1+\tan^2\alpha}{2} - \frac{\gamma_1}{2\gamma}\tan^2\alpha\,. \tag{d}$$

The ODE of the trajectories become

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\left(a + b\frac{x}{y}\right) \pm \sqrt{\left(a + b\frac{x}{y}\right)^2 + 1}\,. \tag{e}$$

From the equation (e) we may separate the two trajectories. We take first of all the sign $+$ before the radical, hence

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\left(a + b\frac{x}{y}\right) + \sqrt{\left(a + b\frac{x}{y}\right)^2 + 1}\,. \tag{f}$$

For $y \to 0$ it results $\mathrm{d}x/\mathrm{d}y = 0$, hence $\mathrm{d}y/\mathrm{d}x \to \infty$, characterizing thus the family of the trajectories of the compression stresses σ_2 which start normal to the upstream face. To see this, one multiplies the equation by its conjugate and, after a reduction of terms, one makes $y = 0$.
For $y = x\tan\alpha$ (the inclined downhill face) it results

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\tan\alpha}\,,\ \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\alpha\,,$$

and the trajectories of σ_2 become asymptotically tangent to the downhill face.
Returning to the differential equation (f), the substitution

$$\frac{x}{y} = \frac{t-a}{b}\,,\ t = a + b\frac{x}{y}\,, \tag{g}$$

leads to the equation with separate variables

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}t}{a - (b+1)t + b\sqrt{t^2+1}}\,. \tag{h}$$

A new change of variables

$$t = \frac{u^2-1}{2u} \Rightarrow \mathrm{d}t = \frac{u^2+1}{2u}\mathrm{d}u\,,\quad u = t + \sqrt{t^2+1}\,,\ \sqrt{t^2+1} = \frac{u^2+1}{2u}\,, \tag{i}$$

transforms the equation (h) in another equation with separate variables

$$\frac{\mathrm{d}y}{y} = \frac{\left(u^2+1\right)\mathrm{d}u}{u\left(u^2-2au-2b-1\right)}. \tag{j}$$

We notice that

$$a^2+2b+1 = \left(\frac{\gamma_1}{2\gamma}\tan\alpha - \frac{1}{\tan\alpha} - \tan\alpha\right)^2 = \left(2a-\tan\alpha\right)^2,$$

so that

$$u^2-2au-2b-1 = \left(u-\tan\alpha\right)\left(u-2a+\tan\alpha\right).$$

Decomposing in simple fractions

$$\begin{aligned}\frac{u^2+1}{u\left(u^2-2au-2b-1\right)} &= \frac{u^2+1}{\left(u-\tan\alpha\right)\left(u-2a+\tan\alpha\right)} \\ &= -\frac{A}{u} + \frac{B}{u-\tan\alpha} + \frac{C}{u-2a+\tan\alpha},\end{aligned} \tag{k}$$

we obtain

$$\begin{aligned}A &= \frac{1}{\tan\alpha\left(\tan\alpha-2a\right)}, \\ B &= \frac{1+\tan^2\alpha}{2\tan\alpha\left(\tan\alpha-a\right)}, \\ C &= 1+A-B = 1+\frac{1}{\tan\alpha\left(\tan\alpha-2a\right)} - \frac{1+\tan^2\alpha}{2\tan\alpha\left(\tan\alpha-a\right)}.\end{aligned} \tag{l}$$

Integrating now the equation (j), we get

$$\ln\frac{y}{C_1} = A\ln u - B\ln\left(u-\tan\alpha\right) - C\ln\left(u-2a+\tan\alpha\right),$$

where the constants A, B, C are given by (l); further, we may write

$$y = C_1\frac{u^A}{\left(u-\tan\alpha\right)^B\left(u-2a+\tan\alpha\right)^C}, \tag{m}$$

where C_1 is a new integration constant, which can be determined by the condition that the trajectory passes through a given point.

The variable u may be expressed by means of the variables x and y in the form

$$u = a + b\frac{x}{y} + \sqrt{\left(a + b\frac{x}{y}\right)^2 + 1} \ . \tag{n}$$

In particular, for $y = x \tan \alpha$ we have

$$a + b\frac{x}{y} = \frac{\tan^2 \alpha - 1}{2 \tan \alpha} \ , \ u = \tan \alpha \ .$$

For this value of u, the denominator in the right member of (m) vanishes, so that $y = x \tan \alpha$ is an asymptote of the trajectory.

Taking into account (n), the solution (m) may be written in its final form

$$y = C_1 \frac{\left(u + \sqrt{u^2 + 1}\right)^A}{\left(u + \sqrt{u^2 + 1} - \tan \alpha\right)^B \left(-2a + u + \sqrt{u^2 + 1} + \tan \alpha\right)^C} \ , \quad u(x, y) = a + b\frac{x}{y} \ . \tag{o}$$

For the trajectories of the tension stress σ_2, the differential equation is

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\left(a + b\frac{x}{y}\right) - \sqrt{\left(a + b\frac{x}{y}\right)^2 + 1} \ . \tag{p}$$

For $y \to 0$ it results $\mathrm{d}x/\mathrm{d}y \to \infty$, hence $\mathrm{d}y/\mathrm{d}x \to 0$, and the trajectories are asymptotic to the upstream face; for $y = x \tan \alpha$ it results $\mathrm{d}x/\mathrm{d}y = -\tan \alpha$, hence $\mathrm{d}y/\mathrm{d}x = -1/\tan \alpha$, so that the trajectories are normal to the downhill face.

By the same substitutions (g) and (i) one obtains the differential equation with separate variables

$$\frac{\mathrm{d}y}{y} = \frac{\left(u^2 - 1\right)\mathrm{d}u}{u(u \tan \alpha + 1)[u(\tan \alpha - 2a) - 1]} = \frac{\mathrm{d}u}{u} - \frac{D\mathrm{d}u}{u \tan \alpha + 1} - \frac{E\mathrm{d}u}{u(\tan \alpha - 2a) - 1} \ , \tag{q}$$

where

$$D = \frac{1 + \tan^2 \alpha}{2(\tan \alpha - a)} \ , \ E = D - 2a = \frac{1 + \tan^2 \alpha}{2(\tan \alpha - a)} - 2a \ . \tag{r}$$

As in the first case, the final solution is

$$y = C_2 \frac{u + \sqrt{u^2 + 1}}{\left[\left(u + \sqrt{u^2 + 1}\right)\tan \alpha + 1\right]^D \left[\left(u + \sqrt{u^2 + 1}\right)(\tan \alpha - 2a) - 1\right]^E} \ , \tag{s}$$

where C_2 is a second integration constant and u has the same significance as in the formula (o).

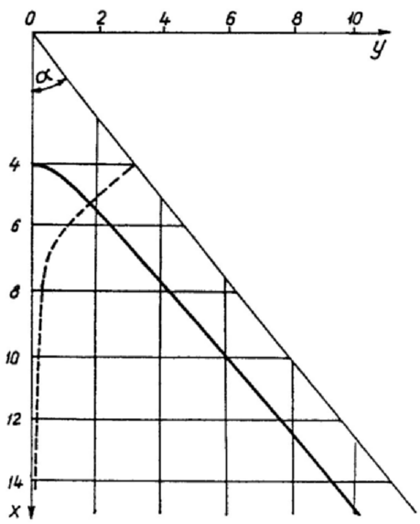

Figure 4. 21. Trajectories of principal normal stresses passing through a point in a gravity dam

The two trajectories passing through a certain point are given in Fig.4.21; this analytical solution is quite difficult to use in practice.

Application 4.21

Problem. Study the surface of coincidence in case of a surface of the form

$$\frac{\cos\varphi}{\frac{\mathrm{d}r_0}{\mathrm{d}\varphi}}+\frac{\sin\varphi}{r_0}=-\frac{Z(r_0)}{N_0}, \tag{a}$$

where r_0 is the radius of the parallel circle, φ is the meridian angle between the axis of rotation and the support of the curvature radii, $Z(r_0)$ is the normal component of the load (uniform distributed along the parallel circles) and N_0 is the constant value of the meridian and annular efforts (supposed to be known).

Solution. The equation (a) may be written in the form

$$r_0\cos\varphi\mathrm{d}\varphi+\left(\sin\varphi+\frac{Z(r_0)r_0}{N_0}\right)\mathrm{d}r_0=0\,. \tag{b}$$

It is an ODE with total differentials, because

$$\frac{\partial}{\partial r_0}(r_0 \cos\varphi) = \frac{\partial}{\partial\varphi}\left(\sin\varphi + \frac{Z(r_0)r_0}{N_0}\right) = \cos\varphi\,.$$

Hence, there exists a function $F = F(r_0, \varphi)$ so that

$$\mathrm{d}F = \frac{\partial F}{\partial\varphi}\mathrm{d}\varphi + \frac{\partial F}{\partial r_0}\mathrm{d}r_0\,, \tag{c}$$

with

$$\begin{aligned}\frac{\partial F}{\partial\varphi} &= r_0 \cos\varphi,\\ \frac{\partial F}{\partial r_0} &= \sin\varphi + \frac{Z(r_0)r_0}{N_0}\,.\end{aligned} \tag{d}$$

The function F is thus of the form

$$F(r_0, \varphi) = r_0 \sin\varphi + f(r_0);$$

introducing in (d), we are led to $f'(r_0) = Z(r_0)\,r_0/N_0$.
Hence, the general solution of the equation (b) is given by

$$r_0 \sin\varphi + \int \frac{Z(r_0)r_0}{N_0}\mathrm{d}r_0 + C\,. \tag{e}$$

Application 4.22

Problem. Search the solution of the equation (d) in Appl.4.19 by means of an integrating factor.

Solution. The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx + y}{x - ky}$$

may be written in the form

$$x\mathrm{d}x + y\mathrm{d}y = \frac{1}{k}(x\mathrm{d}y - y\mathrm{d}x) \tag{a}$$

too. An integrant factor is $1/(x^2 + y^2)$, so that the equation (a) becomes

$$\frac{x\mathrm{d}x + y\mathrm{d}y}{x^2 + y^2} = \frac{1}{k}\frac{x\mathrm{d}y - y\mathrm{d}x}{x^2 + y^2}\,;$$

each member of this equation is a total differential, the general solution being of the form

$$\frac{1}{2}\ln\left(x^2 + y^2\right) = \frac{1}{k}\arctan\frac{y}{x} + \ln C \, .$$

We find thus again the solution in orthogonal Cartesian co-ordinates or in polar co-ordinates, respectively.

Application 4.23

Problem. Determine a curve so that the portion of a tangent to it between two rectangular straight lines be of constant length *a*.

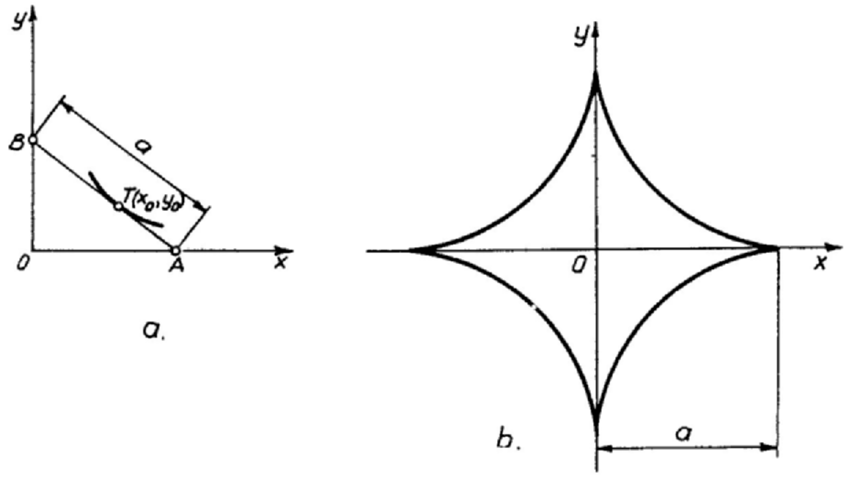

Figure 4. 22. The segment of line of length *a* with the ends leaning on two rectangular axes (a). Astroid (b)

Mathematical model. We choose the two orthogonal straight lines as Ox- and Oy-axes and be (x_0, y_0) a point on the searched curve. The equation of the tangent reads

$$y - y_0 = y_0'(x - x_0).$$

The segments $\overline{OA}$ and $\overline{OB}$ determined by the tangent on the two axes (Fig.4.22, a) are

$$\overline{OA} = x_0 - \frac{y_0}{y_0'} = -\frac{y_0 - x_0 y_0'}{y_0'}, \; \overline{OB} = y_0 - x_0 y_0' \, .$$

Applying Pythagoras' theorem, we obtain

$$\left(\frac{y_0 - x_0 y_0'}{y_0'}\right)^2 + \left(y_0 - x_0 y_0'\right)^2 = a^2 \, . \tag{a}$$

Passing to the co-ordinates x, y and denoting $p = y'$, the equation (a) becomes $(y - px)^2\left(1 + p^2\right) = a^2 p^2$, whence

$$y = px \pm \frac{ap}{\sqrt{1 + p^2}}, \tag{b}$$

obtaining an equation of Clairaut type.

Solution. The general solution of the equation (b) is

$$y = Cx \pm \frac{ac}{\sqrt{1 + C^2}}, \tag{c}$$

representing a family of straight lines.
The solution is obtained by eliminating the constant C between the equation (c) and its derivative with respect to C

$$x \pm \frac{a}{\left(1 + C^2\right)^{3/2}} = 0 .$$

Denoting $C = \tan\varphi$, it results

$$y = \pm a \sin^3 \varphi , \; x = \pm a \cos^3 \varphi ;$$

eliminating the parameter φ between the two above relations, we have

$$x^{2/3} + y^{2/3} = a^{2/3},$$

obtaining thus an *astroid* (Fig.4.22, b).
Technically, one may find such a situation in case of the door of a rectangular shower bath, from the open to the closed position.

Application 4.24

Problem. Study the differential equation of thin shells of rotation in a theory of membrane. Particular cases: spherical and parabolical dome.

Mathematical model. The function efforts in the membrane shell $U = U(\varphi)$ is of the form

$$\frac{\mathrm{d}U}{\mathrm{d}\varphi} + \left(\frac{1}{r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\varphi} - \cot\varphi\right)U - \frac{n}{\cos\varphi}\frac{1}{r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\varphi}U^2 + \frac{n}{\sin\varphi} = 0, \tag{a}$$

where φ is the meridian angle (independent variable), $r_0 = r_0(\varphi)$ is the radius of the parallel circle of the rotation surface, and $n \geq 2$ is an integer number.

Solution. The equation (a) is of Riccati type and its solution may be obtained by quadratures only if a particular integral is known; this is possible only in particular cases, specifying the form of the meridian curve.
In case of a spherical dome for which a is the radius of the sphere; it results $r_0 = a \sin\varphi$, whence $\left(1/r_0\right)\mathrm{d}r_0/\mathrm{d}\varphi = \cot\varphi$. The equation (a) takes the simpler form

$$\frac{\mathrm{d}U}{\mathrm{d}\varphi}+\frac{n}{\sin\varphi}\left(1-U^2\right)=0\,. \tag{b}$$

We may write this equation as a differential equation with separate variables

$$\frac{\mathrm{d}U}{1-U^2}+\frac{n\mathrm{d}\varphi}{\sin\varphi}=0\,, \tag{c}$$

the solution of which is

$$U=\frac{C+\tan^{2n}\dfrac{\varphi}{2}}{C-\tan^{2n}\dfrac{\varphi}{2}}\,, \tag{d}$$

where C is an integration constant.
In case of a parabolical dome for which a is the curvature radius at the vertex of the paraboloid, we have $r_0=a\tan\varphi$ and $\mathrm{d}r_0/\mathrm{d}\varphi=a/\cos^2\varphi$, so that the equation (a) becomes

$$\frac{\mathrm{d}U}{\mathrm{d}\varphi}+U\frac{\sin\varphi}{\cos\varphi}-\frac{n}{\sin\varphi\cos^2\varphi}U^2+\frac{n}{\sin\varphi}=0\,. \tag{e}$$

It may be written also in the form

$$\cos\varphi\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{U}{\cos\varphi}\right)+\frac{n}{\sin\varphi}\left(1-\frac{U^2}{\cos^2\varphi}\right)=0\,; \tag{f}$$

we notice thus that it has two particular solutions $U_1, U_2=\pm\cos\varphi$.
From now on, one may follow two ways to get the solution.

i) We introduce the notation $v=U/\cos\varphi$ and the equation (f) becomes

$$\frac{\mathrm{d}v}{\mathrm{d}\varphi}+\frac{n}{\cos\varphi\sin\varphi}\left(1-v^2\right)=0\,, \tag{g}$$

hence an equation with separable variables of the same type as (b). We may write

$$\frac{\mathrm{d}\,v}{1-v^2}=-\frac{2n\,\mathrm{d}\,\varphi}{\sin 2\varphi}\,,$$

whence

$$v=\frac{c+\tan^{2n}\varphi}{c-\tan^{2n}\varphi}\,;$$

finally, we have

$$U = \cos\varphi \frac{c + \tan^{2n}\varphi}{c - \tan^{2n}\varphi},$$

where c is an arbitrary constant.

ii) Another way is to use the fact that the equation (e) is of Riccati type; if we know a particular solution we are led to a complete solution. Indeed, by a change of function

$$U = v + \cos\varphi, \tag{i}$$

we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}\varphi} + v\left(\frac{\sin\varphi}{\cos\varphi} - \frac{2n}{\sin\varphi\cos\varphi}\right) - \frac{n}{\sin\varphi\cos^2\varphi}v^2 = 0, \tag{j}$$

hence an equation of Bernoulli type with $\alpha = 2$. Denoting $z = 1/v$, it results for the new unknown function z the non-homogeneous linear equation

$$-\frac{\mathrm{d}z}{\mathrm{d}\varphi} + z\left(\frac{\sin\varphi}{\cos\varphi} - \frac{2n}{\sin\varphi\cos\varphi}\right) - \frac{n}{\sin\varphi\cos^2\varphi} = 0, \tag{k}$$

which may be solved by the method presented in Sec.1.6, c. The solution is the sum of the general solution of the associated non-homogeneous equation

$$z_0 = \frac{c}{\cos\varphi}(\tan\varphi)^{-2n}$$

and a particular solution of the non-homogeneous equation, which may be obtained by the method of variation of parameters. Finally, we have

$$z = \frac{c}{\cos\varphi}(\tan\varphi)^{-2n} - \frac{1}{2\cos\varphi}.$$

Returning to v and then to U, we get

$$U = \frac{\cos\varphi}{c(\tan\varphi)^{-2n} - \frac{1}{2}} + \cos\varphi = \cos\varphi\left[\frac{c(\tan\varphi)^{-2n} + \frac{1}{2}}{c(\tan\varphi)^{-2n} - \frac{1}{2}}\right] = \cos\varphi\frac{K + (\tan\varphi)^{2n}}{K - (\tan\varphi)^{2n}}, \tag{l}$$

where we denoted $K = 1/2c$. The forms (h) and (l) of the solution are identical.

A possibility to integrate the equation (a) appears if its coefficients satisfy the condition in Sec.1.6. d, case 1, that is

$$\frac{1}{r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\varphi} - \cot\varphi - \frac{n}{\cos\varphi}\frac{1}{r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\varphi} + \frac{n}{\sin\varphi} = 0, \tag{m}$$

whence

$$\frac{\frac{\mathrm{d}r_0}{\mathrm{d}\varphi}}{r_0} = \frac{\cos\varphi}{\sin\varphi}. \tag{n}$$

As we have seen, this condition is satisfied for a spherical dome $(r_0 = a\sin\varphi)$.
In the more general case indicated in Sec.1.6. d, case 2, the Riccati equation (a) may be integrated if there exist two non- simultaneous non-zero constants a, b, so that

$$a^2\frac{n}{\cos\varphi}\frac{1}{r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\varphi} + ab\left(-\frac{1}{r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\varphi} + \cot\varphi\right) - b^2\frac{n}{\sin\varphi} = 0\,, \tag{o}$$

whence

$$\frac{1}{r_0}\frac{\mathrm{d}r_0}{\mathrm{d}\varphi} = \cot\varphi\frac{b^2 n - ab\cos\varphi}{a^2 n - ab\cos\varphi}. \tag{p}$$

Application 4.25

Problem. Study the motion of a particle P of mass m, acted upon by a *central force* $\mathbf{F}$, which passes through the fixed point O.

Mathematical model. Newton's equation of motion is of the form

$$m\ddot{\mathbf{r}} = F\frac{\mathbf{r}}{r}, \tag{a}$$

where $\mathbf{r}$ is the position vector of the point P (Fig.4.23). A cross product by $\mathbf{r}$ in both members leads to $\ddot{\mathbf{r}}\times\mathbf{r} = \mathbf{0}$, so that $\mathrm{d}(\dot{\mathbf{r}}\times\mathbf{r})/\mathrm{d}t = \mathbf{0}$, whence $\dot{\mathbf{r}}\times\mathbf{r} = \mathbf{C}$, $\mathbf{C} = \overrightarrow{\mathrm{const}}$; we effect now a scalar product by $\mathbf{r}$ in both members and obtain $\mathbf{C}\cdot\mathbf{r} = 0$ (the triple scalar product in the left member vanishes). We may thus state that the trajectory is a plane curve $\mathscr{C}$; taking the corresponding plane as plane Oxy, we may write the equations of motion in polar co-ordinates r, θ in the form

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = F, \quad m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0\,. \tag{b}$$

Solution. The second equation (b) leads to the *first integral of areas*

$$2\Omega = r^2\dot{\theta} = rv_\theta = C,\, C = r_0^2\dot{\theta}_0 = r_0 v_\theta^0 = r_0 v_0 \sin\alpha_0 = \mathrm{const}\,, \tag{c}$$

where Ω is the *areal velocity* of the particle P and the constant C is specified by the initial conditions (Fig.4.23)

$$r_0 = r(t_0),\; v_0 = v(t_0),\; \theta_0 = \theta(t_0),\; \dot{\theta}_0 = \dot{\theta}(t_0),$$

where α_0 is the angle between the vectors $\mathbf{r}_0$ and $\mathbf{v}_0$.

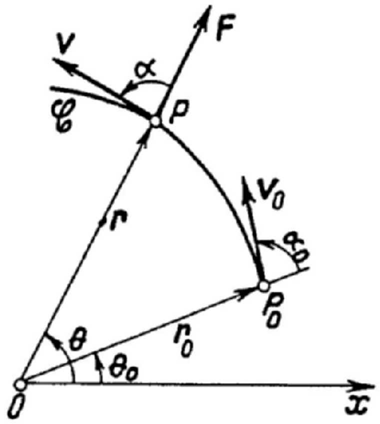

Figure 4. 23. The motion of a particle subjected to the action of a central force

Taking into account (c), the first equation (b) may be written in the form

$$m\ddot{r} = \overline{F}(r, \theta, \dot{r}, \dot{\theta}; t) = F(r, \theta, \dot{r}, \dot{\theta}; t) + \frac{mC^2}{r^3} = F(r, \theta, \dot{r}, \dot{\theta}; t) + \frac{mv_\theta^2}{r}, \tag{d}$$

where we have introduced the *apparent force* $\overline{F}$ (we notice that the supplementary force mv_0^2/r is of the nature of a centrifugal force); the system of differential equations (c), (d) determines the functions $r = r(t)$, $\theta = \theta(t)$, the three integration constants which appear being specified by the initial conditions. If $F = F(r, \dot{r}; t)$, then the motion along the vector radius is given by Newton's one-dimensional equation, where the apparent force $\overline{F}$ is used, the angle θ being then obtained from the integral areas.
Successive differentiations lead to

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = \frac{C}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -C\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right),$$

$$\ddot{r} = -C\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)\dot{\theta} = -\frac{C^2}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right);$$

replacing in the equation (d), we obtain *Binet's equation* (we assume that $\dot{F} = \partial F/\partial t = 0$)

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{Fr^2}{mC^2}, \quad F = F(r, \theta, \dot{r}, \dot{\theta}); \tag{e}$$

eliminating, analogously, $\dot{r}$ and $\dot{\theta}$ from the expression of the force F, one obtains a differential equation of the second order, which determines the trajectory of the motion in the form

$$\frac{1}{r} = f(\theta; C_1, C_2). \tag{f}$$

The initial conditions

$$f(\theta_0; C_1, C_2) = \frac{1}{r_0},\ f'(\theta_0; C_1, C_2) = -\frac{\dot{r}_0}{C} = -\frac{\cot\alpha_0}{r_0}, \tag{g}$$

where $\dot{r}_0 = v_0 \cos\alpha_0$, $f' \equiv \partial f/\partial\theta$ allow to determine the integration constants C_1 and C_2. The integral of areas specifies the motion on the trajectory in the form

$$t = t_0 + \frac{1}{C}\int_{\theta_0}^{\theta} \frac{\mathrm{d}\vartheta}{f^2(\vartheta; C_1, C_2)}. \tag{h}$$

If we notice that $\mathbf{F}\cdot\mathrm{d}\mathbf{r} = F\frac{\mathbf{r}}{r}\cdot\mathrm{d}\left(r\frac{\mathbf{r}}{r}\right) = F\mathrm{d}r$, the theorem of kinetic energy leads to

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} = \int_{r_0}^{r} F(\rho, \theta, \dot{\rho}, \dot{\theta}; t)\mathrm{d}\rho. \tag{i}$$

If $F = F(r, \dot{\theta})$, hence if $F = F(r)$, then we may write a first integral of Binet's equation in the form

$$\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right)\right]^2 + \frac{1}{r^2} = \frac{1}{C^2}\left[v_0^2 + 2m\int_{r_0}^{r} F(\rho)\mathrm{d}\rho\right], \tag{j}$$

noting that $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + C^2/r^2$; one may obtain this result multiplying both members of Binet's equation by $\mathrm{d}(1/r)/\mathrm{d}\theta$ and integrating. The given force is, in this case, conservative and we can introduce the simple potential $U = U(r)$, so that $F(r) = U'(r) = \mathrm{d}U/\mathrm{d}r$. The first integral (f) becomes

$$\begin{gathered}\frac{m\dot{r}^2}{2} = \frac{mC^2}{2}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right)\right]^2 = \frac{mC^2}{2r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \overline{U}(r) + h, \\ \overline{U}(r) = U(r) - \frac{mC^2}{2r^2},\ h = \frac{mv_0^2}{2} - U(r_0),\end{gathered} \tag{k}$$

where we have introduced the *apparent potential* $\overline{U}(r)$ and the *energy constant h*; we obtain thus

$$\theta = \theta_0 \pm C\int_{r_0}^{r}\frac{\mathrm{d}(1/\rho)}{\sqrt{\varphi(\rho)}} = \theta_0 \mp C\int_{r_0}^{r}\frac{\mathrm{d}\rho}{\rho^2\sqrt{\varphi(\rho)}}, \quad \varphi(r) = \frac{2}{m}\left[\overline{U}(r) + h\right], \tag{l}$$

the trajectory being determined in polar co-ordinates. The two first integral used allow, at the same time, to put in evidence the motion of the particle along the trajectory, establishing the parametric equations of that one in the form

$$t = t_0 \pm C\int_{\theta_0}^{\theta} r^2(\vartheta)\mathrm{d}\vartheta, \quad t = t_0 \pm \int_{r_0}^{r}\frac{\mathrm{d}\rho}{\sqrt{\varphi(\rho)}}\,. \tag{m}$$

If the potential is of the form $U(r) = k/r^s$, $k = \mathrm{const}$, $s \in \mathbb{Q}$, then the above integrals may be expressed by elementary functions only if $s = -2$ (harmonic oscillator), $s = -1$, $s = 1$ (Keplerian motion), and $s = 2$; if $s = -6, -4, 3, 4, 6$, then these integrals may be expressed by means of elliptic functions.

The sign of the radical is determined by the sign of the initial velocity $\dot{r}_0 = \dot{r}(t_0)$ if $\varphi(r) > 0$. If $\varphi(r) = 0$, then $v_r^0 = \dot{r}_0 = 0$, so that the velocity is normal to the vector radius at the initial moment; the motion along the vector radius takes place as if the radius would be fixed, the force acting upon the particle being $\overline{F}$. If the apparent force is positive (repulsive force), then r is increasing and one takes the sign $+$; otherwise one takes the sign –. Let us suppose, in particular, that $\overline{F} = 0$ at the initial moment; in this case the particle remains immovable for an observer of the vector radius, because the particle moves on this radius as it would be fixed, the particle being launched without initial velocity from a point at which the apparent force vanishes. Hence, the trajectory is a circle of radius r_0, the motion being uniform (because the areal velocity is constant).

To have a circular trajectory we must have $\alpha_0 = \pm\pi/2$ (the velocity must be normal to the vector radius at the initial moment so that $C = \pm r_0 v_0$) and $F(r_0) + mC^2/r_0^3 = 0$. If $r = r_0$ (circular motion) and $\dot{\theta} = \dot{\theta}_0$ (uniform motion) during the motion, then the equation (e) is identically verified; because the initial conditions are fulfilled, the theorem of uniqueness ensures us about the searched solution. The velocity at the initial moment must have the modulus

$$v_0 = \sqrt{\frac{-F(r_0)r_0}{m}}\;; \tag{n}$$

hence, at the initial moment, the force F must be of attraction ($F(r_0) < 0$.

The relation (e) may be written also in the form

$$F = -\frac{mC^2}{r^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]; \tag{o}$$

we obtain thus *Binet's formula*, which allows to solve the inverse problem: determinate the central force which, applied to a given particle, leads to a plane trajectory, after the areas law with respect to a fixed pole. Taking into account the equation (f) of the trajectory, we may write

$$F = -\frac{mC^2}{r^2}\left[f''(\theta) + f(\theta)\right] \tag{p}$$

too, where $f'' \equiv \partial^2 f/\partial\theta^2$. If a given form of the expression F is not previously imposed, then that one has a certain indetermination, taking into account the equation of the trajectory (the equation which links r to θ); eliminating θ, one obtains $F = F(r)$, a form used the most times.

For example, in the case of trajectories for which corresponds the equation

$$r^k = a\cos k\theta + b\,,\ a, b, k = \text{const}\,, \tag{q}$$

we obtain

$$F(r) = -\frac{C^2}{r^{k+3}}\left[\frac{(k+1)(a^2 - b^2)}{r^k} + (k+2)b\right], \tag{r}$$

choosing the origin as a fixed pole; in particular, these trajectories may be conics having the pole as focus $(k = -1)$ or as centre $(k = 1)$, Pascal's limaçons $(k = 2, b = 0)$, lemniscates etc.

The trajectory of a particle in a field of central forces is usually called *orbit* (even if it is not a closed curve). The relations (k) – (m) determine the orbit and the motion on the orbit only if $\dot{r}$, θ, and t are real quantities, that is if $\varphi(r) \geq 0$; the apparent potential must verify the condition $\overline{U}(r) + h \geq 0$, which determines the domain of variation of r, corresponding to the motion of the particle; the solutions of the equation

$$\overline{U}(r) + h = 0 \tag{s}$$

specify the frontier of the domain. From (k) it is seen that the radial velocity vanishes on the frontier $(\dot{r} = 0)$, the angular velocity being non-zero ($\dot{\theta} \neq 0$; if we have $\dot{\theta} = 0$ at a point different of the origin, then, from the first integral of areas, it results $C = 0$, hence the trajectory is rectilinear); the velocity is normal to the vector radius at the respective points. On the frontier, $r(t)$ changes of sign, the respective point corresponding to a relative extremum for $r(t)$. The relation (c) shows that $\dot{\theta}(t)$ has a constant sign, so that $\theta(t)$ is a monotone function; the integrals (l) and (m) must be calculated on intervals of monotony, the sign being chosen correspondingly. Let be $r_{\min}$ and $r_{\max}$ the extreme values which may be taken by r; the corresponding points of the orbit are called *apsides*. In this case $0 \leq r_{\min} \leq r \leq r_{\max}$.

The radius $r_{\max}$ is finite, hence the orbit is bounded and the trajectory is contained in the annulus determined by the circles $r = r_{\min}$ and $r = r_{\max}$ (we suppose at the beginning that $r_{\min} > 0$); the radii $r_{\min}$ and $r_{\max}$ are called *apsidal distances*. The points for which $r = r_{\min}$ are called *pericentres*, while those for which $r = r_{\max}$ are called *apocentres*. Taking into account that at an apsidal point the velocity is normal to the vector radius, which is the radius of a circle, it results that *the trajectory is tangent to the concentric circles at the corresponding apsides* (Fig.4.24). Choosing as origin of angles θ the radius of an apsidal point $\theta = 0$, called *apsidal line*, we may use the relation (m) for two points of same vector radius r of the trajectory, of one and the other part of that line, r_0 being $r_{\min}$ or $r_{\max}$; it results that *the trajectory of the particle is symmetric with respect to an apsidal line. The angle χ at the centre between two consecutive apsidal lines is constant*; it is called *apsidal angle* and is given by

$$\chi = C \int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{r^2 \sqrt{\varphi(r)}} . \tag{t}$$

It results that the angle at the centre between two consecutive pericentres (apocentres) is equal to 2χ.

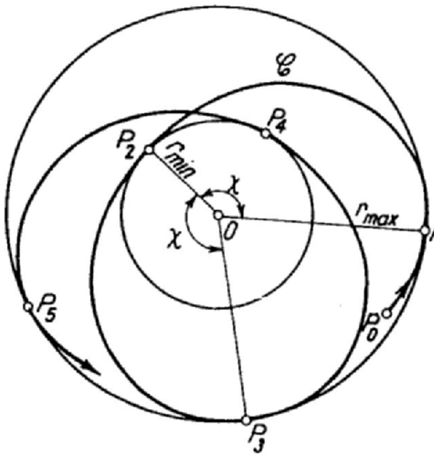

Figure 4. 24. Orbit of a particle subjected to the action of a central force

From the above mentioned properties it results that, *if the arc of trajectory between two consecutive apsides is known, then one may set up geometrically the whole trajectory* (Fig.4.24). From (c) it results that $\dot{\theta}$ has a constant sign, so that the particle rotates always in the same direction around the point O. To have a closed bound trajectory, it is necessary that, after a finite number of such rotations, the particle returns at a previous position; hence, the condition $2\chi = 2\pi q$, $q \in \mathbb{Q}$, must be satisfied. In the contrary case, the orbit is open and covers the annulus $r \in [r_{\min}, r_{\max}]$. We observe that the apparent potential $\overline{U}(r)$ has a maximum at a point in the interior of the annulus, corresponding to $\overline{F}(r) = \mathrm{d}\overline{U}/\mathrm{d}r = 0$. It is possible that the equation $\varphi(r) = 0$ may have more than two roots. In this case, we obtain two possible annular domains; the motion takes place in that domain which contains the given initial position $r_0 = r(t_0)$. If $r_{\min} = 0$, then the particle passes through the pole O or stops at this point. Assuming that $C \neq 0$ (otherwise, the trajectory is rectilinear), the term $-mC^2/2r^2$ leads to $\lim_{r\to 0} \overline{U}(r) = -\infty$, "the fall" towards O being thus hindered. The condition of "falling" towards O is obtained from the condition $\overline{U} \geq -h$, written in the form $r^2 U(r) - mC^2/2 \geq -hr^2$. To have $r_{\min} = 0$ we must have $\lim_{r\to 0+0} \left[r^2 U(r) \right] \geq mC^2/2$, hence $U(r)$ must tend to zero at least as A/r^2, $A > mC^2/2$ or as A/r^n, $A > 0$, $n > 2$.

If $r_{\min} = r_{\max} = r_0$, the trajectory is a circle of radius r_0, corresponding to $\overline{F}(r) = 0$ and the energy constant $h = -U_{\max}$.

One may prove the following

Theorem 4.3 (J. Bertrand). *The only closed orbits corresponding to central forces are those for which* $s = -2$, $k < 0$ *for any initial conditions or* $s = 1$, $k > 0$ *for certain initial conditions, assuming a potential of the form* $U(r) = k/r^s$, $k = \text{const}$, $s \in \mathbb{Q}$.

Jacobi considered the case in which the central force is of the form $F = \gamma(\theta)/r^2$, hence it is inverse proportional to the square of the distance to the point O. Binet's equation (c) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\gamma(\theta)}{mC^2}; \tag{u}$$

integrating, we obtain

$$\frac{1}{r} = C_1 \cos\theta + C_2 \sin\theta + \overline{\gamma}(\theta), \tag{v}$$

where $\overline{\gamma}(\theta)$ is a particular integral, which may be always obtained by quadratures. The integration constants are easily obtained by initial conditions of Cauchy type.

Analogously, we may consider central forces of the form k/r^3, $k = \text{const}$, leading to the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \left(1 + \frac{k}{mC^2}\right)\frac{1}{r} = 0, \tag{w}$$

whence the general integral

$$\frac{1}{r} = C_1 \cos\beta\theta + C_2 \sin\beta\theta, \beta = \sqrt{1 + \frac{k}{mC^2}}. \tag{x}$$

Application 4.26

Problem. Study the motion of rotation of a simple pendulum around a vertical axis.

Mathematical model. Let us assume that the vertical circle on which mores a heavy particle, in particular the mathematical pendulum considered in Appl.4.33, rotates with a constant angular velocity ω around its vertical diameter. The co-ordinates of the particle are thus: $x = l \sin\theta \cos \varpi t,\ y = l \cos\theta \sin \varpi t,\ z = -l \cos\theta$, where the applicate z has been taken along the ascendent local vertical (Fig.4.25).

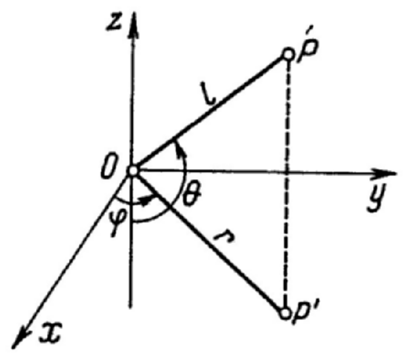

Figure 4. 25. Simple pendulum in a motion of rotation

The constraint is rheonomic, so that we use Lagrange's equation (see Appl.2, formula (m)), where

$$T = \frac{1}{2} ml^2 \left(\dot{\theta}^2 + \omega^2 \sin^2 \omega\right),\ Q = m\mathbf{g} \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = -mgl \sin\theta\,; \tag{a}$$

we obtain thus

$$\ddot{\theta} - \omega^2 \sin\theta \cos\theta + \frac{g}{l} \sin\theta = 0\,. \tag{b}$$

Solution. Introducing the non-dimensional variable $\varphi = \omega t$, we may write the equation (b) in the form ($\mathrm{d}\varphi = \mathrm{d}\omega \mathrm{d}t,\ \theta' = \mathrm{d}\theta/\mathrm{d}\varphi$)

$$\theta'' = \left(\cos\theta - \lambda\right)\sin\theta,\ \lambda = \frac{g}{l\omega^2}\,; \tag{c}$$

multiplying by $2\dot{\theta}$ and integrating, it results the first integral

$$\dot{\theta}^2 - \left(\sin^2\theta + 2\lambda\cos\theta\right) = \text{const}\,, \tag{d}$$

and the equation $\theta = \theta(\varphi)$ of the trajectory is obtained by a quadrature.

The above considerations are valid for a pendular motion as well as for a circular motion.

Application 4.27

Problem. Determine the motion of a particle constrained to stay on a straight line which rotates around one of its points, the tangent of the rotation angle being proportional to the time t (Fig.4.26).

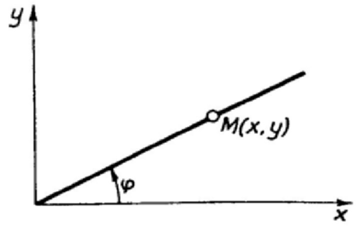

Figure 4. 26. Motion of a particle on a straight line which is rotating

Mathematical model. Let us consider $\tan\varphi = kt$, where k is a constant of proportionality. At the time t, the equation of the straight line is

$$y = ktx\,, \tag{a}$$

The components of the acceleration along the two axes are $a_x = \mathrm{d}^2x/\mathrm{d}t^2$, $a_y = \mathrm{d}^2y/\mathrm{d}t^2$. By a virtual displacement of the particle, of components δx and δy, the condition of compatibility, deduced from (a), leads to

$$\delta y = kt\delta x\,; \tag{b}$$

the virtual work is (m is the mass of the particle)

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\delta x + m\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\delta y = 0\,. \tag{c}$$

Simplifying by m and taking into account (b), the relation (c) becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kt\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0\,. \tag{d}$$

Solution. From (a), one obtains (differentiating twice)

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 2k\frac{\mathrm{d}x}{\mathrm{d}t} + kt\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,. \tag{e}$$

Eliminating $\mathrm{d}^2y/\mathrm{d}t^2$ between (d) and (e), we get

$$\left(1+k^2t^2\right)\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2k^2t\frac{\mathrm{d}x}{\mathrm{d}t}=0\,. \tag{f}$$

Noting that in this differential equation we have not a term in x, we make the substitution

$$\frac{\mathrm{d}x}{\mathrm{d}t}=u \quad \Rightarrow \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{\mathrm{d}u}{\mathrm{d}t}$$

and the equation (f) becomes successively

$$\left(1+k^2t^2\right)\frac{\mathrm{d}u}{\mathrm{d}t}+2k^2tu=0\,,$$

$$\frac{\mathrm{d}u}{u}=-\frac{2k^2t}{1+k^2t^2}\,\mathrm{d}t\,.$$

The variables are separated and, by integration,

$$\ln u = \ln\left(1+k^2t^2\right)+\ln C_1k\,;$$

hence, we deduce

$$u=\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{C_1k}{1+k^2t^2}\,,$$

$$\mathrm{d}x=C_1k\frac{\mathrm{d}t}{1+k^2t^2}\,,$$

whence

$$x = C_1 \arctan kt + C_2\,. \tag{g}$$

From (a), it results

$$y = kt\left(C_1 \arctan kt + C_2\right), \tag{h}$$

so that (g) and (h) are the parametric equations of the trajectory.
Eliminating the parameter t between the two relations, it results

$$\arctan\frac{y}{x} = \frac{x-C_2}{C_1}$$

or

$$y = x\tan\frac{x-C_2}{C_1}\,. \tag{i}$$

We notice that (f) is a linear, homogeneous ODE; one may apply the results in Sec.1.2 after the change of function $u=\mathrm{d}x/\mathrm{d}t$, obtaining

$$u(t) = C \exp\left(-\int \frac{2k^2 t}{1 + k^2 t^2} \, \mathrm{d}t \right) = C \exp\left[-\ln\left(1 + k^2 t^2\right)\right] = \frac{C}{1 + k^2 t^2} .$$

Integrating once more with respect to t, one obtains the formula (g).

Application 4.28

Problem. An electron is situated in an electrostatic field of a very long wire (theoretically infinite) with a positive charge and starts from rest at the moment $t = 0$. The electron has a negative charge and is attracted towards the wire. Evaluate the time T necessary for the electron to reach the wire.

Mathematical model. Corresponding to Coulomb's law, two particles of electric charges q_1 and q_2, of opposite sign, respectively, situated at a distance r, are attracted by a force $F = q_1 q_2 / kr^2$, where k is the dielectric constant of the medium.

If e is the charge of the electron, λ is the charge per unit length of the wire, y is the distance from the electron to the wire, and dz is the elementary length of wire (Fig.4.27), then the attraction exerted by the charge $\lambda \mathrm{d}z$ upon e is

$$\mathrm{d}F = \frac{e\lambda \mathrm{d}z}{kr^2} , \tag{a}$$

where r is the distance between the electron and the element dz. Denoting by θ the angle between the Oy-axis and the straight line connecting the electron to the element dz, we have

$$r = \frac{y}{\cos\theta} , \ \mathrm{d}z = \frac{r\mathrm{d}\theta}{\cos\theta} ,$$

and the relation (a) becomes

$$\mathrm{d}F = \frac{e\lambda r\mathrm{d}\theta}{kr^2 \cos\theta} = \frac{e\lambda \mathrm{d}\theta}{ky} .$$

The element dz symmetric with respect to the origin, hence situated at the distance $-z$, acts upon e with a force of the same magnitude; the components parallel to $\mathrm{O}z$ of these forces are equal in modulus and of opposite directions, their sum vanishing. The non-zero resultant, parallel to the Oy-axis, is

$$\mathrm{d}F \cos\theta = \frac{2e\lambda}{ky} \cos\theta \mathrm{d}\theta .$$

Summing all these elementary forces, we obtain the force by which the wire acts upon the electron, i.e.

$$F = \frac{2e\lambda}{ky}\int_0^{\pi/2}\cos\theta\,\mathrm{d}\theta = \frac{2e\lambda}{k}\frac{1}{y}\,. \tag{b}$$

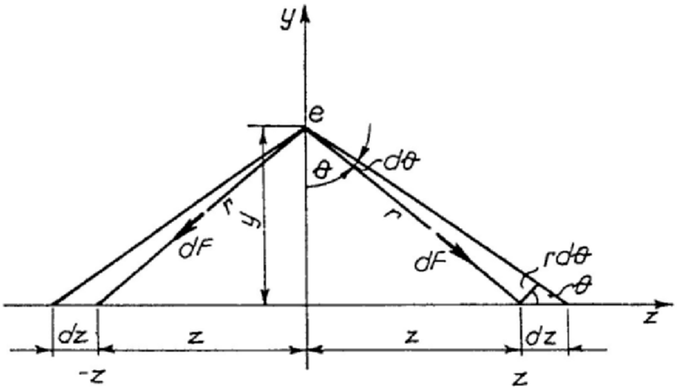

Figure 4. 27. Motion of an electron in the electrostatic field of a wire

Under the action of this force, the motion of the electron of mass m is governed by Newton's law

$$m\ddot{y} = -\frac{2e\lambda}{k}\frac{1}{y}\,, \tag{c}$$

where the sign minus in the second member takes into account the fact that, for a positive y, the force acts in the negative direction of O y, and inversely.

Denoting

$$K = \frac{2e\lambda}{km}\,, \tag{d}$$

the equation of motion is a non-linear equation of second order

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{K}{y}\,. \tag{e}$$

The initial conditions are

$$y(0) = h,\ y(0) = 0\,, \tag{f}$$

where h represents the initial distance of the electron to the wire.

Solution. Noting that the equation (e) does not contain the independent variable, we make the substitution

$$p = \frac{\mathrm{d}y}{\mathrm{d}t}\,; \tag{g}$$

hence,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}y}{\mathrm{d}t} = p\frac{\mathrm{d}p}{\mathrm{d}y}\,. \tag{h}$$

Substituting in (e), we obtain

$$p\frac{\mathrm{d}p}{\mathrm{d}y} = -\frac{K}{y}\,. \tag{i}$$

Separating the variables, it results $p\mathrm{d}p = -K\,\mathrm{d}y/y$; integrating this, we get $p^2/2 = -K\ln y + C$. From the homogeneous initial conditions $y(0) = 0$ and $\dot{y}(0) = p(0) = 0$, it results $0 = -K\ln h + C$ or $C = K\ln h$, so that

$$\frac{p^2}{2} = K(\ln h - \ln y) = K\ln\frac{h}{y}\,.$$

One obtains thus

$$p = \frac{\mathrm{d}y}{\mathrm{d}t} = \pm\sqrt{2K\ln\frac{h}{y}}\,.$$

In the previous relation one takes the sign minus, because the velocity is directed towards a negative y for a positive h, so that

$$\mathrm{d}t = -\frac{\mathrm{d}y}{\sqrt{2K\ln\dfrac{h}{y}}}\,.$$

Integrating in the left member between 0 and T and in the right member between the corresponding limit h and 0, we get finally,

$$T = -\frac{1}{\sqrt{2K}}\int_h^0 \frac{\mathrm{d}y}{\sqrt{\ln\dfrac{h}{y}}} = \frac{1}{\sqrt{2K}}\int_0^h \frac{\mathrm{d}y}{\sqrt{\ln\dfrac{h}{y}}}\,. \tag{j}$$

The integrals in (j) may be calculated by the change of variable

$$x = \ln\frac{h}{y}\,, \tag{k}$$

whence

$$\frac{h}{y} = \mathrm{e}^x \ , \ y = h\mathrm{e}^{-x} , \ \mathrm{d}y = -h\mathrm{e}^{-x}\mathrm{d}x \ ,$$

where $x \to \infty$ for $y = 0$ and $x = 0$ for $y = h$.

The expression (j) of T becomes

$$T = -\frac{1}{\sqrt{2K}} \int_{\infty}^{0} h\mathrm{e}^{-x} x^{-1/2} \mathrm{d}x = \frac{h}{\sqrt{2K}} \int_{0}^{\infty} x^{-1/2} \mathrm{e}^{-x} \mathrm{d}x \, . \tag{l}$$

The integral (l) may not be calculated by means of elementary functions in a finite form. Developing the integrand into a power series

$$x^{-1/2} \mathrm{e}^{-x} = x^{-1/2} \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots \right) = x^{-1/2} - x^{1/2} + \frac{x^{3/2}}{2!} - \frac{x^{5/2}}{3!} + \ldots$$

and integrating, one obtains the power series of primitive

$$\begin{aligned} \int x^{-1/2} \mathrm{e}^{-x} \mathrm{d}x &= 2x^{-1/2} - \frac{2}{3} x^{3/2} + \frac{2}{5 \cdot 2!} x^{5/2} - \frac{2}{7 \cdot 3!} x^{7/2} + \ldots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{\frac{2n+1}{2}}}{\frac{2n+1}{2} n!} . \end{aligned} \tag{m}$$

The series (m) is convergent for any x and may be used to calculate

$$\int_0^{\infty} x^{-1/2} e^{-x} \, \mathrm{d}\, x = \lim_{n \to \infty} \sum_{n=0}^{\infty} (-1)^n \frac{x^{\frac{2n+1}{2}}}{\frac{2n+1}{2} n!} , \tag{n}$$

hence a Gamma function Γ (see the Chap.1, Subsec.2.8).

Application 4.29

Problem. The form of a directrix curve of a surface of translation is given by the differential equation

$$\frac{y''}{1 + y'^2} = -\frac{1}{a} . \tag{a}$$

Determine the general solution of the equation (a) and the integration constants assuming the bilocal homogeneous conditions $y(0) = y(l) = 0$. Discussion.

Solution. The equation (a) is of the form $F(y', y'') = 0$ and, by a change of variable $y' = p$, we obtain the differential equation with separate variables

$$\frac{p'}{1+p^2} = -\frac{1}{a}$$

or

$$\frac{\mathrm{d}p}{1+p^2} = -\frac{\mathrm{d}x}{a}\,.$$

The general integral is

$$\arctan p = -\frac{x}{a} - C_1\,,\ C_1 = \text{const}\,,$$

whence

$$p = \frac{\mathrm{d}y}{\mathrm{d}x} = -\tan\left(\frac{x}{a} + C_1\right) = -\frac{\sin\left(\frac{x}{a} + C_1\right)}{\cos\left(\frac{x}{a} + C_1\right)}\,.$$

A last integration leads to

$$y = a \ln \cos\left(\frac{x}{a} + C_1\right) + C_2\,, \tag{b}$$

where C_2 is a second integration constant.
From the boundary condition $y(0) = 0$, we get

$$C_2 = -a \ln \cos C_1\,,$$

hence

$$y = -a \ln \frac{\cos C_1}{\cos\left(\frac{x}{a} + C_1\right)}\,. \tag{c}$$

The boundary condition $y(l) = 0$ leads to

$$\cos C_1 = \cos\left(\frac{l}{a} + C_1\right) = \cos\frac{l}{a} \cos C_1 - \sin\frac{l}{a} \sin C_1\,,$$

whence

$$\tan C_1 = \frac{1 - \cos\dfrac{l}{a}}{\sin\dfrac{l}{a}} = \tan\frac{l}{2a}$$

or

$$C_1 = -\frac{l}{2a}\,.$$

Finally, the equation of the directrix curve reads

$$\frac{y}{a} = -\ln\frac{\cos\dfrac{l}{2a}}{\cos\left(\dfrac{l}{2a} - \dfrac{x}{a}\right)}\,. \tag{d}$$

To have a real solution, the condition $l/2a \in [0, \pi/2]$ must be fulfilled. We consider the particular cases $a/l = 3/\pi, 2/\pi, 3/2\pi$.
For $a/l = 3/\pi$, the equation (d) becomes

$$\frac{y}{a} = -\ln\frac{\cos\dfrac{\pi}{6}}{\cos\left(\dfrac{\pi}{6} - \dfrac{\pi x}{3l}\right)} = -\ln\frac{\sqrt{3}}{\sqrt{3}\cos\dfrac{\pi x}{3l} + \sin\dfrac{\pi x}{3l}} = \ln\left(\cos\frac{\pi x}{3l} + \frac{1}{\sqrt{3}}\sin\frac{\pi x}{3l}\right).$$

We notice that all the curves defined by the equation (d) are symmetric with respect to the middle of the span. To set up the curve, we divide the span in 10 equal intervals. The ordinates thus obtains are listed in Table 4.1 and are plotted into diagrams in Fig.4.28.
In the limit case $a/l = \pi$ the co-ordinates y tend to infinity, while for $a/l \to \infty$ we have $y \to 0$, that is the graphic of the curve is reduced to the segment of a line *l*.

Table 4.1. The values of *y/l* for various *a/l*

x/l	*y/l*		
	$a/l = 3/\pi$	$a/l = 2/\pi$	$a/l = 3/2\pi$
0	0	0	0
0.1	0.05101	0.08571	0.13912
0.2	0.08944	0.14717	0.22976
0.3	0.11626	0.18869	0.28779
0.4	0.13211	0.21275	0.32040
0.5	0.13736	0.22064	0.33095

Another way to solve the equation takes into account the fact that the equation (a) does not contain explicitly the function y. As it was shown in Sec.2.3, b, we may make a change of function $y' = p$, considering then p as function of y. We obtain

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}p}{\mathrm{d}y} y' = p \frac{\mathrm{d}p}{\mathrm{d}y} ,$$

and the equation (a) becomes

$$\frac{p \dfrac{\mathrm{d}p}{\mathrm{d}y}}{1 + p^2} = -\frac{1}{a} , \tag{e}$$

hence a non-linear equation of first order with separate variables, the general solution of which is of the form

$$y' = p = \pm\sqrt{\mathrm{e}^{-\frac{2y}{a}+C_1} - 1} , \tag{f}$$

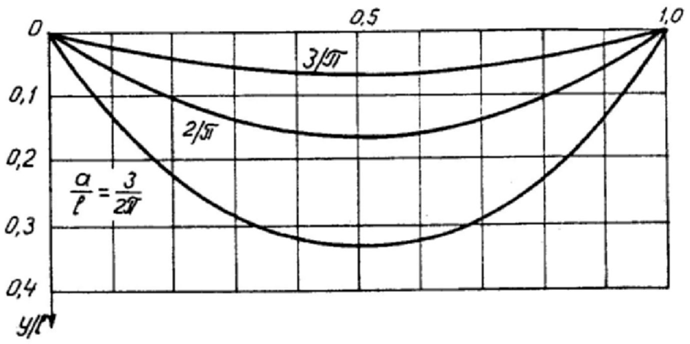

Figure 4. 28. The directrix curve for various *a*/*l*

where C_1 is an arbitrary constant. The equation (f) is also with separate variables; integrating it, we get

$$\int \frac{\mathrm{d}y}{\sqrt{\mathrm{e}^{-\frac{2y}{a}+C_1} - 1}} = x + k,\ k \in \Re .$$

Calculating the primitive in the left member, we obtain, successively,

$$\int \frac{\mathrm{d}y}{\sqrt{\mathrm{e}^{-\frac{2y}{a}+C_1} - 1}} = \int \frac{a\mathrm{d}u}{u\sqrt{u^2 - 1}} = \int \frac{-a\mathrm{d}v}{\sqrt{\cosh v}} = -2a \arctan \mathrm{e}^v ,$$

by the change of variable $u = \mathrm{e}^{-x/a + C_1/2}$, $u = \cosh v$. Hence, we get

$$\mathrm{e}^v = \tan\left(-\frac{x}{2a} + k \right)$$

$$\mathrm{e}^{-\frac{2y}{a}+C_1} = \frac{1}{\sin\left(-\frac{x}{a}+k\right)} = \frac{1}{\cos\left(\frac{x}{a}+C_2\right)};$$

finally,

$$y = a\ln\cos\left(\frac{x}{a}+C_2\right)+C_1, \tag{h}$$

i.e. the same formula as that previously obtained.
But the first method is more convenient, because it is quite direct. In this case, the second method led to an intricate computation of the primitive (g).

Application 4.30

Problem. Determine the form of equilibrium of an elastic thread suspended between two points; the area of the cross section is A and the modulus of longitudinal elasticity is E. The thread is acted upon by its own weight $m\mathbf{g}$.

Mathematical model. Let be S the tension in the thread and $S\,\mathrm{d}x/\mathrm{d}s$, $S\,\mathrm{d}y/\mathrm{d}s$ its components along the axes Ox and Oy, respectively (Fig.4.29).
In the deformed form, the equations of projection on the two axes are

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(S\frac{\mathrm{d}x}{\mathrm{d}s}\right) = 0, \tag{a}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(S\frac{\mathrm{d}y}{\mathrm{d}s}\right)\left(1+\frac{S}{EA}\right) = g, \tag{b}$$

where g is the own weight on unit length (we take the mass equal to unity). From (a) it results

$$S\frac{\mathrm{d}x}{\mathrm{d}s} = S_0 = \text{const}, \; S = S_0\frac{\mathrm{d}s}{\mathrm{d}x}, \tag{c}$$

and introducing in (b) we get

$$S_0\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)\left(1+\frac{S_0}{EA}\frac{\mathrm{d}s}{\mathrm{d}x}\right) = g. \tag{d}$$

Taking into account the relations $\mathrm{d}s = \sqrt{1+y'^2}\,\mathrm{d}x$, $\mathrm{d}y/\mathrm{d}x = y'$, we obtain the differential equation

$$S_0\frac{\mathrm{d}y'}{\mathrm{d}x}\left(\frac{S_0}{EA}+\frac{1}{\sqrt{1+y'^2}}\right) = g. \tag{e}$$

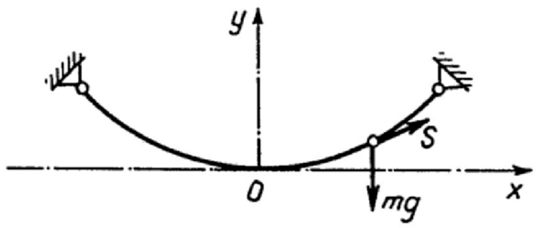

Figure 4. 29. Deformation of an elastic thread suspended between two points

Solution. We denote $y' = p$ and consider p as independent variable; we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}p} = \frac{S_0}{g}\left(\frac{S_0}{EA} + \frac{1}{\sqrt{1+p^2}}\right). \tag{f}$$

Integrating, it results

$$x = \frac{S_0}{g}\left[\frac{S_0}{EA}p + \ln\left(p + \sqrt{p^2+1}\right) + C_1\right]. \tag{g}$$

Because for $x = 0$ we have $y' = p = 0$, we obtain $C_1 = 0$ and

$$x = \frac{S_0}{g}\left[\frac{S_0}{EA}p + \ln\left(p + \sqrt{p^2+1}\right)\right]. \tag{h}$$

Multiplying (f) by $p = \mathrm{d}y/\mathrm{d}x$, we get

$$\frac{\mathrm{d}y}{\mathrm{d}p} = \frac{S_0}{g}\left(\frac{S_0}{EA}p + \frac{p}{\sqrt{1+p^2}}\right). \tag{i}$$

Integrating, it results

$$y = \frac{S_0}{g}\left(\frac{S_0}{2EA}p^2 + \sqrt{1+p^2} - 1\right) + C_2.$$

Because we have $y' = p = 0$ for $y \equiv 0$, we obtain $C_2 = 0$, so that

$$y = \frac{S_0}{g}\left(\frac{S_0}{2EA}p^2 + \sqrt{1+p^2} - 1\right). \tag{j}$$

The relations (h) and (j) constitute the parametric representation of the deformed thread. If $EA \to \infty$ (inextensible thread) one finds the catenary curve.

The tension S_0 may be obtained from a geometric condition connected to the total length of the thread.

Application 4.31

Problem. Determine the deflections of a cantilever bar of length *l* acted upon at the free end by a couple M_0 (Fig.4.30).

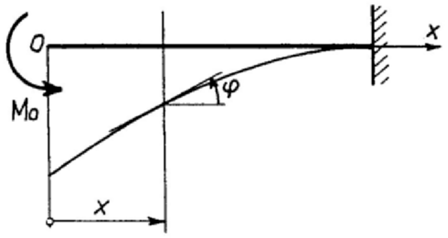

Figure 4. 30. Cantilever bar acted upon by a concentrated moment at the free end

Mathematical model. The bending moment along the bar axis is $M = M_0$, so that the equation of the curvature is given by

$$\frac{1}{\rho} = \frac{M_0}{EI} = \text{const}\,, \tag{a}$$

where ρ is the curvature radius and *EI* is the bending rigidity. The curvature is given by

$$\frac{\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]^{3/2}} = -\frac{1}{\rho}\,, \tag{b}$$

where *w* is the deflection. The equations (a) and (b) leads to the differential equation of the problem.

Solution. By the substitution $u = \mathrm{d}w/\mathrm{d}x$, $\mathrm{d}u/\mathrm{d}x = \mathrm{d}^2 w/\mathrm{d}x^2$, the relation (b) becomes

$$\frac{\mathrm{d}u}{\left(1+u^2\right)^{3/2}} = -\frac{\mathrm{d}x}{\rho}\,.$$

Integrating once, we obtain

$$\frac{u}{\sqrt{1+u^2}} = C_1 - \frac{x}{\rho}\,. \tag{c}$$

The condition of built-in section $u(l) = 0$ determines the integration constant $C_1 = 1/\rho$, so that the relation (c) becomes

$$u = \frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{l-x}{\sqrt{\rho^2 - (l-x)^2}}.$$

The sign minus appears because to positive deflections correspond negative slopes (see Fig.4.30).
The deflection w is obtained by integration in the form

$$w = -\int \frac{l-x}{\sqrt{\rho^2 - (l-x)^2}}\,\mathrm{d}x + C_2\,, \tag{d}$$

where C_2 is a second constant of integration.
To calculate the integral in (d), one makes the change of variable

$$l - x = \rho \sin\varphi \quad \Rightarrow \quad \mathrm{d}x = -\rho\cos\varphi\,\mathrm{d}\,\varphi,\ \cos\varphi = \frac{1}{\rho}\sqrt{\rho^2 - (l-x)^2}\,,$$

so that

$$w = \int \frac{\sin\varphi}{\sqrt{\rho^2 - \rho^2\sin^2\varphi}}\,\rho\cos\varphi\mathrm{d}\varphi + C_2 = \rho\int \sin\varphi\mathrm{d}\varphi + C_2 = -\rho\cos\varphi + C_2\,;$$

returning to the variable x, we get

$$w = C_2 - \sqrt{\rho^2 - (l-x)^2}\,.$$

The constant C_2 is determined by the condition that, in the built-in cross section, the deflection be zero; hence, $w = 0$ for $x = l$ and one obtains $C_2 = \rho$ so that

$$w = \rho - \sqrt{\rho^2 - (l-x)^2}\,. \tag{e}$$

From (a) it results, obviously, that the deformed axis in an arc of circle of radius ρ. The relation (e) leads to the equation of this circle in Cartesian co-ordinates

$$(l-x)^2 + (w-\rho)^2 = \rho^2.$$

The expression (e) is not convenient for the computation, because it is a difference of two great quantities of near values. We may write

$$w = \rho\left[1 - \sqrt{1 - \left(\frac{l-x}{\rho}\right)^2}\right]; \tag{f}$$

developing the radical into power series after Newton's binomial we have

$$\left[1-\sqrt{1-\left(\frac{l-x}{\rho}\right)^2}\right]^{1/2}=1-\frac{1}{2}\left(\frac{l-x}{\rho}\right)^2+\frac{1\cdot 3}{2\cdot 4}\left(\frac{l-x}{\rho}\right)^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{l-x}{\rho}\right)^6+\ldots .$$

Thus, the expression (f) becomes

$$w\cong\frac{(l-x)^2}{2\rho}-\frac{3(l-x)^4}{8\rho^3}+\frac{5(l-x)^6}{16\rho^5}-\ldots .$$

Taking only the first term in the power series, it results

$$w\cong\frac{(l-x)^2}{2\rho}=\frac{M_0(l-x)^2}{2EI},$$

hence an arc of parabola; this solution coincides with that obtained if we start from the approximate differential equation of the deformed axis

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=-\frac{M_0}{EI},$$

in the case of infinitesimal strains and of rotations negligible with respect to unity.

Application 4.32

Problem. Determine the deflections of a cantilever bar of length l, acted upon by a uniformly distributed normal load p (Fig.4.31).

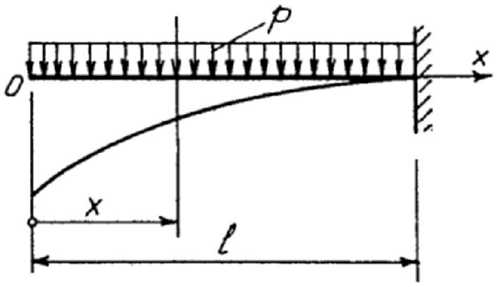

Figure 4. 31. Cantilever bar acted upon by a normal uniformly distributed load *p*

Mathematical model. We search a solution by means of a power series. The bending moment in a section of abscissa x is $M=-px^2/2$; the differential equation of the deformed axis is given by

$$\frac{\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]^{3/2}} = -\frac{M}{EI} = \frac{px^2}{2EI}\,. \tag{a}$$

Solution. The function w does not effectively appear in (a); by the substitution $u = \mathrm{d}w/\mathrm{d}x$, $\mathrm{d}u/\mathrm{d}x = \mathrm{d}^2 w/\mathrm{d}x^2$, the equation (a) becomes

$$\frac{\mathrm{d}u}{\left(1+u^2\right)^{3/2}} = \frac{px^2}{2EI}\,\mathrm{d}x\,.$$

We integrate once

$$\frac{u}{\sqrt{1+u^2}} = C_1 + \frac{px^3}{6EI}\,. \tag{b}$$

The condition $u(l) = 0$ in the built-in cross section determines the integration constant

$$C_1 = \frac{pl^3}{6EI}\,,$$

so that (b) becomes

$$\frac{u}{\sqrt{1+u^2}} = -\frac{p}{6EI}\left(l^3 - x^3\right),$$

whence

$$u = \frac{\mathrm{d}w}{\mathrm{d}x} = -\frac{p}{6EI}\frac{l^3 - x^3}{\sqrt{1-\left[\dfrac{p}{6EI}\left(l^3 - x^3\right)\right]^2}}\,.$$

Integrating the previous relation, it results

$$w = C_2 - \frac{p}{6EI}\int \frac{l^3 - x^3}{\sqrt{1-\left[\dfrac{p}{6EI}\left(l^3 - x^3\right)\right]^2}}\,\mathrm{d}x\,.$$

We denote

$$\xi = \frac{p}{6EI}\left(l^3 - x^3\right),$$

so that the integrand becomes

$$\frac{\xi}{\sqrt{1-\xi^2}} = \left(1-\xi^2\right)^{-1/2}.$$

Because $\xi << 1$, one may develop in a power series

$$\left(1-\xi^2\right)^{-1/2} = \xi\left(1+\frac{1}{2}\xi^2+\frac{1\cdot 3}{2\cdot 4}\xi^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\xi^6+\ldots\right)$$
$$= \xi+\frac{1}{2}\xi^3+\frac{3}{8}\xi^5+\frac{5}{16}\xi^7+\ldots$$

Taking into account the substitution (e) and returning to the variable *x*, we have

$$w = C_2 - \int\left\{\frac{p}{6EI}\left(l^3-x^3\right)+\frac{1}{2}\left[\frac{p}{6EI}\left(l^3-x^3\right)\right]^3\right.$$
$$\left.+\frac{3}{8}\left[\frac{p}{6EI}\left(l^3-x^3\right)\right]^5+\frac{5}{16}\left[\frac{p}{6EI}\left(l^3-x^3\right)\right]^7+\ldots\right\}\mathrm{d}x$$

or, developing the parentheses and integrating, we get

$$w = C_2 - \frac{p}{6EI}\left(l^3x-\frac{x^4}{4}\right)+\frac{1}{2}\left(\frac{p}{6EI}\right)^3\left(l^9x-\frac{3}{4}l^6x^4+\frac{3}{7}l^3x^7-\frac{1}{10}x^{10}\right)+$$
$$+\frac{3}{8}\left(\frac{p}{6EI}\right)^5\left(l^{15}x-\frac{5}{4}l^{12}x^4+\frac{10}{7}l^9x^7-l^6x^{10}+\frac{5}{13}l^3x^{13}-\frac{1}{16}x^{16}\right)+\ldots$$

The condition $w(l) = 0$ in the built-in cross section leads to

$$C_2 = \frac{pl^4}{8EI}-\frac{81l}{280}\left(\frac{pl^3}{6EI}\right)^3-\frac{2187l}{11648}\left(\frac{pl^3}{6EI}\right)^5+\ldots$$
$$= \frac{pl^4}{8EI}\left[1-\frac{27}{70}\left(\frac{pl^3}{6EI}\right)^2-\frac{729}{2912}\left(\frac{pl^3}{6EI}\right)^4+\ldots\right]$$

The constant C_2 represents the maximal deflection (at the free end $x = 0$) of the cantilever bar. The first term in the development into series corresponds to the approximate solution, which is given by the simplified differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = -\frac{M}{EI}.$$

Application 4.33

Problem. Study the motion of a heavy particle P of mass, frictionless, on a circle C of radius l, situated in a vertical plane (*mathematical pendulum*).

Mathematical model. We choose the Ox-axis in the direction of the gravitational acceleration **g** (Fig.4.32); the theorem of kinetic energy, applied between the points P_0 and P, allows to write

$$v^2 = v_0^2 - 2g(x_0 - x) = v_0^2 - 2gl(\cos\theta_0 - \cos\theta) = -2g(a - x), \qquad \text{(a)}$$

where $a = x_0 - v_0^2/2g$, $v_0 = l\dot{\theta}_0$ being the initial velocity at the point P_0 at the initial moment t_0.

The equation $x = a$ is the equation of the straight line till which a particle may rise if it is launched after the local vertical, with the initial velocity v_0; the values of the constant a determine the character of the motion in case of a bilateral constraint. Indeed, if the straight line $x = a$ pierces the circle C $(-l < a < l)$, then the motion is *oscillatory*, if this straight line is tangent to the circle $(a = -l)$, then we have $v_0 = 0$, corresponding a stable position of equilibrium, hence an *asymptotic* motion, while if the straight line does not pierce the circle $(a < -l)$, then the motion is *circular*. We cannot have $a > l$.

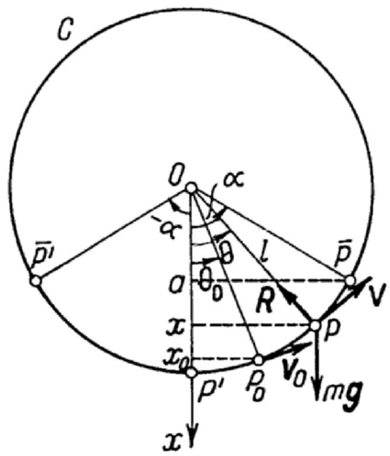

Figure 4. 32. Mathematical pendulum

From the relation (a) it results that the velocity $v = l\dot{\theta}$ may vanish for an angle given by $\cos\theta = \cos\theta_0 - v_0^2/2gl$ or by $\sin^2(\theta/2) = \sin^2(\theta_0/2) + v_0^2/4gl$. This condition can never be satisfied if $v_0^2 > 4gl$ (or $\dot{\theta}_0^2 > 4\omega^2, \omega^2 = g/l$), the motion being circular. If $v_0^2 < 4gl$, then the condition may be fulfilled for certain values of the angle θ_0, hence for some initial positions, e.g. for $\theta_0 = 0$, the motion being, in this case, oscillatory. If $v_0^2 = 4gl$ we must have $\theta_0 = 0$, the motion being asymptotic.

Solution. We assume that the motion is oscillatory, we denote $a = l\cos\alpha$, where $0 < \alpha < \pi$ is the angle corresponding to the limit position $\overline{P}$ (for which $v = 0$) of the particle *P*, specifying the amplitude of the motion. The relation (a) takes the form

$$\dot{\theta}^2 = 2\omega^2(\cos\theta - \cos\alpha); \tag{b}$$

differentiating with respect to time, we may write (we notice that $\theta_0 \neq 0$)

$$\ddot{\theta}^2 + \omega^2 \sin\theta = 0 \tag{c}$$

too. This equation (called *the equation of mathematical pendulum*) is often encountered in problems of mechanics in one of the two forms mentioned above; in fact, the relation (b) corresponds to a first integral of the equation of motion (c).

The particle *P* starts from the initial position P_0 with the velocity v_0 and mounts on the circle with a velocity of diminished intensity; at the extreme position $\overline{P}$ the velocity vanishes. Returning on the arc of circle, the velocity increases; the particle passes over the initial position P_0 and reaches the lowest point P', where it has the maximal velocity; then, the velocity decreases till the particle attains the point $\overline{P}'$ for which $\theta = -\alpha$. The particle returns then at P', at P_0, at $\overline{P}$, a.s.o. Hence the motion is oscillatory. From the relation (b) we observe also that the velocity $v(t)$ depends only on the position of the particle, being a periodic function of this position (of angle θ); integrating this equation with separate variables, we may write (during the motion $\cos\theta > \cos\alpha$)

$$t = t^0 + \frac{1}{\omega\sqrt{2}} \int_{\theta^0}^{\theta} \frac{\mathrm{d}\vartheta}{\sqrt{\cos\vartheta - \cos\alpha}}, \tag{d}$$

where θ^0 corresponds to the position at the arbitrary moment t^0 (which may be different from the initial moment t_0). As one may see, the interval of time $t - t^0$ depends only on the corresponding positions of the two moments; it results that the oscillatory motion is periodical, of period *T*. We notice further that, if we change the direction of the motion on the arc of circle, then the sign of the velocity is changed, its modulus remaining the same by passing through the same point; hence, the arc PP' is traveled through in an interval of time $T/2$. Because the relation (b) is even with respect to θ, it results that for symmetric points with respect to the *Ox*-axis we have the same velocity (by up, or down travel); hence the arc $P'P$ is traveled through in a quarter of period. In this case, the period *T* is given by

$$T = \frac{2\sqrt{2}}{\omega} \int_0^{\alpha} \frac{\mathrm{d}\vartheta}{\sqrt{\cos\vartheta - \cos\alpha}}. \tag{e}$$

We notice that $\cos\theta - \cos\alpha = 2\left[\sin^2(\alpha/2) - \sin^2(\theta/2)\right]$ and denoting by $\sin(\theta/2) = k\sin\varphi$, $k = \sin(\alpha/2)$, we may write

$$t = t^0 + \frac{1}{\omega}\int_{\varphi^0}^{\varphi}\frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}}, \tag{f}$$

where φ^0 is given by $\sin(\theta^0/2) = k\sin\varphi^0$; denoting $\sin\varphi^0 = z$, we can write

$$t = t^0 + \frac{1}{\omega}\int_{z^0}^{z}\frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta)(1-k^2\zeta^2)}}, \tag{g}$$

too, where z^0 is specified by $\sin\varphi^0 = z^0$. Introducing, after Legendre, the elliptic integral of first species

$$F(\varphi, k) = \int_0^{\varphi}\frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}} = \int_0^{\sin\varphi}\frac{\mathrm{d}z}{\sqrt{(1-z)(1-k^2z^2)}}, \tag{h}$$

where φ is *the amplitude* and k is *the modulus of the integral*, we obtain

$$t = t^0 + \frac{1}{\omega}\left[F(\varphi, k) - F(\varphi^0, k)\right]. \tag{i}$$

By the notation $u = \omega t$, we can write

$$u - u^0 = F(\varphi, k) - F(\varphi^0, k), \tag{j}$$

where $u^0 = \omega t^0$. Taking $t^0 = 0$, without any loss of generality, and if we assume that $\theta^0 = 0$, then it results $\varphi^0 = z^0 = u^0 = F(\varphi^0, k) = 0$, so that

$$u = F(\varphi, k), \tag{k}$$

As it was noticed by Abel, we may express the angle φ as a function of the variable u in the form

$$\sin\varphi = \mathrm{sn}\, u, \tag{l}$$

where sn is the symbol of the *elliptic sinus* (*the amplitude sinus*), one of the *elliptic functions of Jacobi*; analogously, we may use the *elliptic cosinus* (*the amplitude cosinus*), denoted by the symbol cn $(\cos\varphi = \mathrm{cn}\, u)$.

Starting from the formula (a), the period of motion is given by

$$T = \frac{4}{\omega}K(k) = \frac{4}{\omega}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{4}{\omega}\int_0^{l}\frac{\mathrm{d}z}{\sqrt{(1-z)(1-k^2z^2)}}, \tag{m}$$

with $\omega = \sqrt{g/l}$, where $K(k) = F(\pi/2, k)$ is the *elliptic integral of first species*. Noting that $k^2 < 1$, it results the development (we use *Newton's binomial series*)

$$\left(1 - k^2 \sin^2 \varphi\right)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} k^{2n} \sin^{2n} \varphi \,. \tag{n}$$

This series is absolutely and uniform convergent in the interval $[0, \pi/2]$, so that we may integrate, taking into account Wallis' formula

$$\int_0^{\pi/2} \sin^{2n} \varphi \mathrm{d}\varphi = \frac{(2n)\,!}{2^{2n}(n!)^2} \frac{\pi}{2}, \tag{o}$$

and obtain the period

$$T = 2\pi \sqrt{\frac{l}{g}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{[(2n)!]^2}{2^{4n}(n!)^4} k^{2n} \sin^{2n} \frac{\alpha}{2} \right\}. \tag{p}$$

Because we may develop $\sin(\alpha/2)$ too into an absolutely convergent series with respect to α, we obtain also for the period *T* such a development, which takes the form

$$T = 2\pi \sqrt{\frac{l}{g}} \left\{ 1 + \frac{\alpha^2}{16} + \frac{11}{12} \frac{\alpha^4}{16^2} + \ldots \right\}. \tag{q}$$

We notice that the ratio between the second and the first term of series is equal to $\alpha^2/16$; as well, the ratio between the third and the second term is given by $(11/12)\alpha^2/16 < \alpha^2/16$, a.s.o. This series is rapidly convergent; practically, we may take

$$T = 2\pi \sqrt{\frac{l}{g}} \left\{ 1 + \frac{\alpha^2}{16} \right\}. \tag{r}$$

If $\alpha = 0.4$ (corresponding to an angle of $22^0 55' 06''$), then the correction brought by the second term of the development is not greater than 1%. The astronomical clocks have penduli with amplitudes not greater than $1^0 30'$, corresponding a correction of approximate 0.05%. In general, the period T depends on the angle α, but is independent of the mass *m* of the particle.

In case of small oscillations around a stable position of equilibrium, the equation (c) becomes the form (we approximate $\sin\theta$ by θ)

$$\ddot{\theta} + \omega^2 \theta = 0, \tag{s}$$

whence

$$\theta(t) = \alpha \cos(\omega t + \varphi), \tag{t}$$

the angle φ being specified by the initial conditions, the period being given by Galilei's formula

$$T = 2\pi\sqrt{\frac{l}{g}}\,; \tag{u}$$

we observe that this result approximates the development into series (q). The period T thus obtained depends on the length l of the pendulum and on the gravitational acceleration **g** at the respective place on the Earth. Because this period does not depend on the amplitude α, we say that the respective motion is *isochronic* (the small motions around a stable position of equilibrium take place in the same interval of time). A particle P left to fall from $\overline{P}$ without initial velocity reaches the lowest position P' in an interval of time $T/4$, which does not depend on the initial position (angle α); hence, the respective motion is called *tautochronous*.

Application 4.34

Problem. Study the motion of a system with one degree of freedom which begins to move from the initial position with the velocity v_0 at the moment $t = 0$ and oscillates under the action of a non- linear spring.

Mathematical model. The motion is governed by *Duffing's equation*

$$\ddot{x} = -\left(\frac{k_0}{m}x + \frac{r}{m}x^3\right), \tag{a}$$

with the initial conditions $x(0) = 0,\ \dot{x}(0) = v_0 = 0$.

Solution. Multiplying both members of the equation (a) by $2\,\mathrm{d}\,x = 2\dot{x}\,\mathrm{d}\,t$, we have successively

$$2\ddot{x}\mathrm{d}x = -2\left(\frac{k_0}{m}x + \frac{r}{m}x^3\right)\dot{x}\mathrm{d}t\,,$$

$$\mathrm{d}(\dot{x})^2 = -\left(\frac{2k_0}{m}x + \frac{2r}{m}x^3\right)\mathrm{d}x\,.$$

Integrating between v_0 and $\dot{x}$ in the left member and between 0 and x in the right member, we get

$$\dot{x}^2 - v_0^2 = -\left(\frac{k_0}{m}x^2 + \frac{r}{2m}x^4\right),$$

whence one obtains

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \sqrt{v_0^2 - \left(\frac{k_0}{m}x^2 + \frac{r}{2m}x^4\right)} = v_0\sqrt{1 - \left(\frac{k_0}{mv_0^2}x^2 + \frac{r}{2mv_0^2}x^4\right)}\;;$$

it was taken the sign $+$ because $v_0 > 0$.
Separating the variables and integrating with respect to time between 0 and t, we may write

$$t = \frac{1}{v_0}\int_0^x \frac{\mathrm{d}z}{\sqrt{1 - \left(\frac{k_0}{mv_0^2}z^2 + \frac{r}{2mv_0^2}z^4\right)}}\,, \tag{b}$$

where z is an integration variable.
The above integral may be reduced to elliptic integrals. We denote

$$1 - \left(\frac{k_0}{mv_0^2}z^2 + \frac{r}{2mv_0^2}z^4\right) = \left(1 - a^2z^2\right)\left(1 + b^2z^2\right) = 1 + \left(b^2 - a^2\right)z^2 - a^2b^2z^4\,,$$

where a^2 and b^2 are constants given by the relations

$$b^2 - a^2 = -\frac{k_0}{mv_0^2}\,,\; a^2b^2 = \frac{r}{2mv_0^2}\,,$$

with the solutions

$$a^2 = \frac{k_0}{mv_0^2} + \sqrt{\left(\frac{k_0}{2mv_0^2}\right)^2 + \frac{r}{2mv_0^2}}\,,$$

$$b^2 = -\frac{k_0}{mv_0^2} + \sqrt{\left(\frac{k_0}{2mv_0^2}\right)^2 + \frac{r}{2mv_0^2}}\,.$$

Using the new notations, the integral in relation (b) becomes

$$t = \frac{1}{v_0}\int_0^x \frac{\mathrm{d}z}{\sqrt{\left(1 - a^2z^2\right)\left(1 + b^2z^2\right)}}\,.$$

Denoting

$$u = az,\; \frac{b^2}{a^2} = c^2,\; \mathrm{d}z = \frac{\mathrm{d}u}{a}\,, \tag{c}$$

we may write

$$t = \frac{1}{av_0} \int_0^{ax} \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1+c^2u^2)}} . \tag{d}$$

Denoting further

$$\frac{c^2}{1+c^2} = \frac{b^2}{a^2+b^2} = k^2 ,\ u = \cos\psi , \tag{e}$$

we obtain

$$\mathrm{d}u = -\sin\psi \mathrm{d}\psi ,\ 1-u^2 = \sin^2\psi ,$$

$$\begin{aligned} 1+c^2u^2 &= 1+c^2\cos^2\psi = 1+c^2-c^2\sin^2\psi \\ &= (1+c^2)\left(1-\frac{c^2}{1+c^2}\sin^2\psi\right) = \frac{a^2+b^2}{a^2}(1-k^2\sin^2\psi) \end{aligned}$$

and the integration limits become

$$\begin{aligned} &\psi = \frac{\pi}{2} && \text{for } u = 0, \\ &\psi = \arccos u = \arccos ax = \varphi && \text{for } u = ax. \end{aligned} \tag{f}$$

So, from (b), (c) and (d) we get

$$\begin{aligned} t &= \frac{1}{av_0}\int_{\pi/2}^{\varphi} \frac{-\sin\psi \mathrm{d}\psi}{\sin\psi\sqrt{\frac{a^2+b^2}{a^2}(1-k^2\sin^2\psi)}} = \frac{1}{v_0\sqrt{a^2+b^2}}\int_{\pi/2}^{\varphi}\frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}} \\ &= \frac{1}{v_0\sqrt{a^2+b^2}}\left[\int_0^{\pi/2}\frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}} - \int_0^{\varphi}\frac{\mathrm{d}\psi}{\sqrt{1-k^2\sin^2\psi}}\right] \\ &= \frac{1}{v_0\sqrt{a^2+b^2}}[K(k)-F(k,\varphi)]. \end{aligned} \tag{g}$$

To obtain x as a function of t, we express, first of all,

$$F(k,\varphi) = K(k) - v_0\sqrt{a^2+b^2}\,t .$$

Using the inverse function $\operatorname{am} u$, we get

$$\varphi = \operatorname{am}\left(K(k) - v_0\sqrt{a^2+b^2}\,t\right)$$

and, finally,

$$x(t) = \frac{1}{a}\cos\varphi = \frac{1}{a}\,\mathrm{cn}\left(K(k) - v_0\sqrt{a^2 + b^2}\,t\right).$$

The displacement $x(t)$ is thus a periodic function of amplitude $1/a$, the maximal value of which is obtained for

$$t = \frac{K(k)}{v_0\sqrt{a^2 + b^2}}.$$

The period T of he motion is four times greater and is given by

$$T = \frac{4K(k)}{v_0\sqrt{a^2 + b^2}}. \tag{h}$$

Application 4.35

Problem. Study the non- linear problem of buckling of a doubly hinged straight bar of length *l*, subjected to compression by forces *P*, also taking into account the shortening of the bar.

Mathematical model. The second order ODE which governs the deformation of the bar is

$$\varphi'' + \lambda_1 \sin\varphi - \lambda_2 \sin\varphi\cos\varphi = 0, \tag{a}$$

where φ is the slope of the deformed axis and

$$\lambda_1 = \frac{P}{EI},\ \lambda_2 = \frac{P^2}{EA \cdot EI},$$

EA and *EI* being the axial and the bending rigidities, respectively. The abscissa along the initial bar axis will be denoted by *x*.

Solution. We do not consider the solution $\varphi = 0$, which corresponds to the non-deformed state of the bar.
Multiplying the equation (a) by the integrating factor $2\varphi'$, we get

$$2\varphi'\varphi'' + 2\lambda_1 \sin\varphi\varphi' - 2\lambda_2 \sin\varphi\cos\varphi\varphi' = 0,$$

whence, by integration, we may write

$$(\varphi')^2 = 2\lambda_1\cos\varphi - \frac{\lambda_2}{2}\cos 2\varphi + C_1, \tag{b}$$

where C_1 is a first integration constant.
The maximal value of φ is obtained for $x = 0$; let be φ_m this value. For the doubly articulated bar, the bending moment at both ends must vanish, hence $\varphi' = 0$. One obtains thus from (b)

$$C_1 = -\lambda_1 \cos\varphi_m + \frac{\lambda_2}{4}\cos 2\varphi_m ,$$

so that, introducing in (b), one obtains

$$\varphi' = \pm\sqrt{2\lambda_1(\cos\varphi - \cos\varphi_m) - \frac{\lambda_2}{2}(\cos 2\varphi - \cos 2\varphi_m)} . \tag{c}$$

The signs $\pm$ indicate that the buckling may take place on both sides of the bar.
Noting that $\varphi' = \mathrm{d}\varphi/\mathrm{d}x$, from (c) one obtains

$$x = \pm\int \frac{\mathrm{d}\varphi}{\sqrt{2\lambda_1(\cos\varphi - \cos\varphi_m) - \frac{\lambda_2}{2}(\cos 2\varphi - \cos 2\varphi_m)}} + C_2 ,$$

where C_2 is a second integration constant.
The solution of the differential constant is thus reduced to a quadrature. It cannot be performed in a finite form by means of elementary functions, but may reduced to elliptic integrals listed in tables.
Further, we make the substitutions

$$\cos\varphi = 1 - 2\sin^2\frac{\varphi}{2} ,\ \cos 2\varphi = 1 - 8\sin^2\frac{\varphi}{2} + 8\sin^4\frac{\varphi}{2} .$$

The relation (b) becomes

$$x = \pm\int \frac{\mathrm{d}\varphi}{\sqrt{4(\lambda_1 - \lambda_2)\left(\sin^2\frac{\varphi_m}{2} - \sin^2\frac{\varphi}{2}\right)\left[1 + \frac{\lambda_2}{\lambda_1 - \lambda_2}\left(\sin^2\frac{\varphi_m}{2} - \sin^2\frac{\varphi}{2}\right)\right]}} + C_2 . \tag{e}$$

Using the notation

$$p = \sin^2\frac{\varphi_m}{2} ,\ q = \frac{\lambda_2}{\lambda_1 - \lambda_2}\sin^2\frac{\varphi_m}{2}$$

and introducing a new variable defined by

$$\sin\frac{\varphi}{2} = \sqrt{pz} \quad\Rightarrow\quad \mathrm{d}\varphi = \frac{p}{\sqrt{pz}}\frac{1}{\sqrt{1 - pz}}\mathrm{d}z ,$$

the expression (e) becomes

$$x = \pm\frac{1}{2\sqrt{\lambda_1 - \lambda_2}}\int \frac{\mathrm{d}z}{\sqrt{z(1-z)(1-pz)(1+q+qz)}} + C_2 .$$

A new substitution

$$z = \frac{u}{a+bu} \quad \Rightarrow \quad \mathrm{d}z = \frac{\mathrm{d}z}{(a+bu)^2}\mathrm{d}u \,,$$

with the notations

$$a = \frac{1+2q}{1+q} \,, \; b = -\frac{q}{1+q} \,,$$

leads to

$$x = \pm \frac{1}{2\sqrt{(\lambda_1-\lambda_2)(1+2q)}} \int \frac{\mathrm{d}u}{\sqrt{u(1-u)\left(1-\dfrac{p-b}{a}u\right)}} + C_2 \ .$$

Finally, taking $u = \sin^2\theta$, which yields $\mathrm{d}u = 2\sin\theta\cos\theta\mathrm{d}\theta$ and denoting

$$K = \frac{1}{2\sqrt{(\lambda_1-\lambda_2)(1+2q)}} \,, \; k^2 = \frac{p-b}{a} \,,$$

one obtains

$$x = \pm K \int \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} + C_2 \,. \qquad \text{(f)}$$

Expressing now the last notations K and k^2 in terms of the first ones, we get

$$K = \frac{1}{\sqrt{(\lambda_1-\lambda_2)\left(1+2\dfrac{\lambda_2}{\lambda_1-\lambda_2}\sin^2\dfrac{\varphi_m}{2}\right)}} \,,$$

$$k^2 = \sin^2\frac{\varphi_m}{2} \, \frac{\lambda_1 - \lambda_2 \sin^2\dfrac{\varphi_m}{2}}{\lambda_1 - \lambda_2 + 2\lambda_2 \sin^2\dfrac{\varphi_m}{2}} \,,$$

$$\sin\theta = \sin\frac{\varphi}{2}\sqrt{\frac{1}{\sin^2\dfrac{\varphi_m}{2}} \, \frac{1+2\dfrac{\lambda_2}{\lambda_1-\lambda_2}\sin^2\dfrac{\varphi_m}{2}}{1+\dfrac{\lambda_2}{\lambda_1-\lambda_2}\left(\sin^2\dfrac{\varphi_m}{2}+\sin^2\dfrac{\varphi}{2}\right)}} \ .$$

We determine now the constant C_2. Introducing the limits of the primitive (f) and taking into account the boundary conditions $\varphi(l) = \varphi_m$ and $\theta(l) = \pi/2$, we obtain

$$C_2 = K\int_0^{\theta} \frac{\mathrm{d}\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} ,$$

and the final result is

$$x = K\left(\int_0^{\pi/2} \frac{\mathrm{d}\,\theta}{\sqrt{1-k^2\sin^2\theta}} \pm \int_0^{\theta} \frac{\mathrm{d}\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} \right). \tag{g}$$

These two integrals represent the Legendre's normal form of the elliptic integral of first species.
For given values of the rigidity, of the loading and of the angle φ_m, we may determine, with the aid of tables, firstly x as a function of θ and secondly φ as a function of x.
We notice that this is a boundary value problem and not a problem of eigenvalues as that in Appl.1.31.

Application 4.36

Problem. Study the previous buckling problem, assuming that the axial rigidity of the bar is neglected.

Mathematical model. The differential equation of second order which governs the deformation of the bar is given by the equation (a), Appl.4.35

$$\varphi'' + \lambda_1 \sin\varphi = 0 , \tag{a}$$

where we made $EA \to \infty$, hence $\lambda_2 = 0$.

Solution. We denote $\lambda_1 = p^2$. The above equation may be thus written in the form

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} = -p^2 \sin\varphi , \tag{b}$$

where s represents a linear variable, measured along the deformed axis. Multiplying both members by $(\mathrm{d}\varphi/\mathrm{d}s)\mathrm{d}s$, it results

$$\int \frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2}\frac{\mathrm{d}\varphi}{\mathrm{d}s}\mathrm{d}s = -p^2 \int \sin\varphi \mathrm{d}\varphi$$

or still

$$\frac{1}{2}\int \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 \mathrm{d}s = -p^2 \int \sin\varphi \mathrm{d}\varphi .$$

Integrating, we obtain

$$\frac{1}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = p^2 \cos\varphi + C_1,$$

where C_1 is an integration constant, which is determined by the boundary condition at one end of the bar. Thus, for $x = 0$, we must have $\mathrm{d}\varphi/\mathrm{d}s = 0$, because the bending moment vanishes, while the slope is $\varphi = \varphi_m$. We obtain thus $C_1 = -p^2 \cos\varphi_m$ so that

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = 2p^2(\cos\varphi - \cos\varphi_m)$$

or

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \pm\sqrt{2}\,p\sqrt{\cos\varphi - \cos\varphi_m}\ .$$

Solving with respect to *s*, we may write

$$\mathrm{d}s = \pm\frac{\mathrm{d}\varphi}{\sqrt{2}\,p\sqrt{\cos\varphi - \cos\varphi_m}}\ .$$

The total length of the bar remains unchanged, so that

$$l = \int_0^l \mathrm{d}s = \int_0^{\varphi_m} \frac{\mathrm{d}\varphi}{\sqrt{2}\,p\sqrt{\cos\varphi - \cos\varphi_m}} = \frac{1}{2p}\int_0^{\varphi_m} \frac{\mathrm{d}\varphi}{\sqrt{\sin^2\frac{\varphi_m}{2} - \sin^2\frac{\varphi}{2}}}\ .$$

The integral may be written in a simpler form, denoting $k = \sin(\varphi_m/2)$ and introducing a new variable θ, so that

$$\sin\frac{\varphi}{2} = k\sin\theta = \sin\frac{\varphi_m}{2}\sin\theta\ .$$

Thus, if φ variates between 0 and φ_m, then θ variates between 0 and $\pi/2$. Differentiating, we obtain

$$\mathrm{d}\varphi = \frac{2k\cos\theta\mathrm{d}\theta}{\cos\frac{\varphi}{2}} = \frac{2k\cos\theta\mathrm{d}\theta}{\sqrt{1 - k^2\sin^2\theta}}\ .$$

Introducing in the expression of the length l, and noting that

$$\sqrt{\sin^2\frac{\varphi_m}{2} - \sin^2\frac{\varphi}{2}} = k\sin\theta\,,$$

we get

$$l = \frac{1}{p}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{p}K(k),$$

where $K(k)$ is the complete elliptic integral of the first species.

Application 4.37

Problem. Compute the deformed axis of a cantilever bar acted upon at the free end by a normal concentrated force *P*. The length of the bar is *l* and the bending rigidity is *EI*.

Mathematical model. The curvature of the deformed axis of the bar is given by

$$\frac{1}{\rho} = \frac{\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]^{3/2}} = -\frac{M}{EI},$$

where *w* is the deflection and the bending moment is $M = P(l - x)$ (the origin of the *Ox*-axis is chosen at the built-in cross section, Fig.4.33). We obtain thus the non-linear second order ODE

$$\frac{\dfrac{\mathrm{d}^2 w}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]^{3/2}} = \frac{P}{EI}(l-x). \tag{a}$$

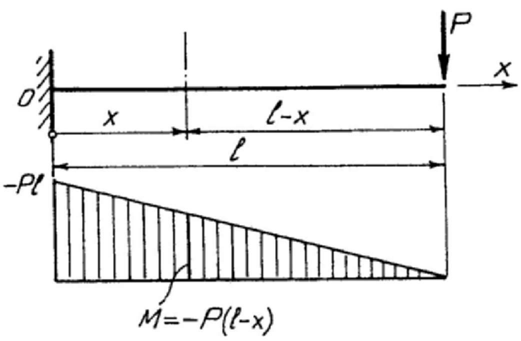

Figure 4. 33. Cantilever bar acted upon by a normal concentrated force *P* at the free end

Solution. By the substitution $p = \mathrm{d}w/\mathrm{d}x$, the equation (a) becomes an ODE with separable variables

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{P}{EI}(l - x)\left(1 + p^2\right)^{3/2}$$

or

$$\frac{\mathrm{d}p}{\left(1 + p^2\right)^{3/2}} = \frac{P}{EI}(l - x)\mathrm{d}x \,.$$

Integrating once, we get

$$\frac{p}{\sqrt{1 + p^2}} = \frac{P}{EI}\left(lx - \frac{x^2}{2}\right) + C_1 \,.$$

We put the condition $p = \mathrm{d}w/\mathrm{d}x = 0$ for $x = 0$ and obtain $C_1 = 0$; hence

$$\frac{p}{\sqrt{1 + p^2}} = \frac{P}{EI}\left(lx - \frac{x^2}{2}\right).$$

From (b) we obtain

$$p = \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{\dfrac{P}{EI}\left(lx - \dfrac{x^2}{2}\right)}{\sqrt{1 - \left(\dfrac{P}{EI}\right)^2\left(lx - \dfrac{x^2}{2}\right)^2}}$$

and, by a new integration, we may write

$$w = \int_0^x \frac{\dfrac{P}{EI}\left(lx - \dfrac{x^2}{2}\right)}{\sqrt{1 - \left[\dfrac{P}{EI}\left(lx - \dfrac{x^2}{2}\right)\right]^2}}\,\mathrm{d}x + C_2 \,.$$

By means of the condition $w(0) = 0$, we obtain $C_2 = 0$, so that

$$w = \int_0^x \frac{\dfrac{P}{EI}\left(l\xi - \dfrac{\xi^2}{2}\right)}{\sqrt{1 - \left[\dfrac{P}{EI}\left(l\xi - \dfrac{\xi^2}{2}\right)\right]^2}}\,\mathrm{d}\xi \,.$$

The substitution

$$\zeta = \frac{P}{EI}\left(lx - \frac{x^2}{2} \right)$$

leads to

$$x = l \pm \sqrt{l^2 - \frac{2EI}{P}\zeta}, \quad \mathrm{d}x = \pm \frac{\beta \,\mathrm{d}\,\zeta}{2\sqrt{\alpha - \beta\zeta}} \ ,$$

introducing the notations $\alpha = l^2$, $\beta = 2\,EI/P$, $k^2 = 2EI/Pl^2 = \beta/\alpha$.
Finally, we have

$$w_{\max} = -\frac{EI}{Pl} \int_0^{Pl^2/2EI} \frac{\zeta \mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} .$$

Application 4.38

Problem. Establish the equation of the deformed axis of a simply supported beam acted upon by a concentrated moment M_1 at the fixed end. The span is L and the bending rigidity is *EI* (Fig.4.34, a,b)

Mathematical model. Taking into account the bending curvature of the beam and the linear variation of the moment diagram (Fig.4.34, c), the solution may be obtained with the aid of elliptic integrals. In the previous application the problem was directly treated, noting that the boundary value problem was a problem of initial values. In the present case we have to do with a bilocal problem.
The equilibrium of an element of beam (Fig.4.34, d) leads to

$$\frac{\mathrm{d}M}{\mathrm{d}s} - \frac{M_1}{l}\frac{\mathrm{d}x}{\mathrm{d}s} = 0 . \tag{a}$$

From the geometry of an element we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta , \tag{b}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \sin\theta . \tag{c}$$

The relation between the bending moment and the slope θ of the tangent is (Fig.4.34, b)

$$M = EI\frac{\mathrm{d}\theta}{\mathrm{d}s} . \tag{d}$$

Differentiating M with respect to s and introducing in (a) we get

$$EI\frac{d^2\theta}{ds^2}-\frac{M_1}{l}\cos\theta=0$$

or

$$\frac{d^2\theta}{ds^2}=\frac{M_1}{EIl}\cos\theta\,. \tag{e}$$

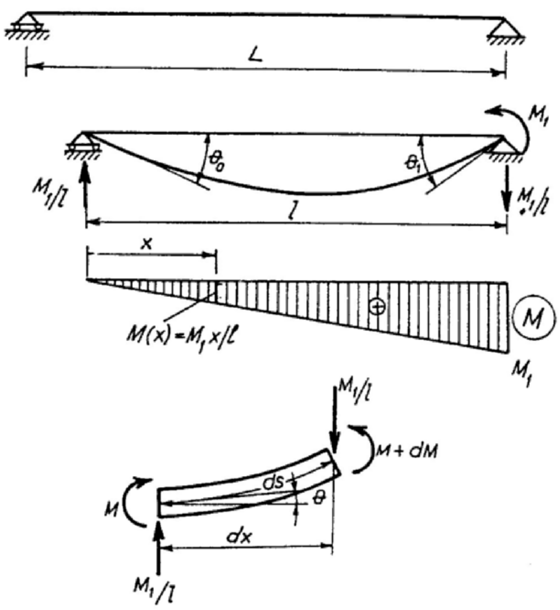

Figure 4. 34. Simply supported beam (a). Deformation of the beam axis acted upon by a concentrated moment M_1 (b). *M*-diagram (c). Equilibrium of an element d*s* (d)

Solution. Multiplying both members by $(d\theta/ds)ds=d\theta$, we get

$$\frac{d^2\theta}{ds^2}\frac{d\theta}{ds}ds=\frac{M_1}{EIl}\cos\theta d\theta$$

or further

$$\frac{1}{2}\frac{d}{ds}\left(\frac{d\theta}{ds}\right)^2ds=\frac{M_1}{EIl}\cos\theta d\theta\,.$$

Integrating both members, it results

$$\frac{1}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}s}\right)^2 = \frac{M_1}{EIl}\sin\theta + C_1 . \tag{f}$$

The integration constant may be determined if we put the condition that the bending moment does vanish at the left end, hence where $\mathrm{d}\theta/\mathrm{d}s = 0$ and $\theta = -\theta_0$ (still unknown).
The relation (f) may be written in the form

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = 2\sqrt{\frac{M_1}{EIl}}\sqrt{\sin^2\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} . \tag{g}$$

where the rotation θ is positive if it takes place in the anticlockwise direction. We make a change of variable

$$\sin\varphi = \frac{\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)} , \tag{h}$$

whence

$$\mathrm{d}\theta = -\frac{2\sin\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\cos\varphi}{\sqrt{1-\sin^2\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\sin^2\varphi}}\,\mathrm{d}\varphi . \tag{i}$$

Thus, the relation (g) becomes

$$\mathrm{d}s = \sqrt{\frac{EIl}{M_1}}\frac{-\mathrm{d}\varphi}{\sqrt{1-\sin^2\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\sin^2\varphi}} . \tag{j}$$

If we suppose that the axial rigidity of the beam is infinite (the length of the axis does not change by bending), we my write

$$\begin{aligned} L &= \sqrt{\frac{EIl}{M_1}}\int_{\varphi_1}^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-\sin^2\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\sin^2\varphi}} , \\ &= \sqrt{\frac{EIl}{M_1}}\left[K\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right) - F\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)\right], \end{aligned} \tag{k}$$

where F and K are the elliptic and the complete elliptic integrals of the first species, respectively, φ_1 is the value of the variable φ given by

$$\sin\varphi_1 = \frac{\sin\left(\frac{\pi}{4}+\frac{\theta_1}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)},\ \varphi_1 = \arcsin\left[\frac{\sin\left(\frac{\pi}{4}+\frac{\theta_1}{2}\right)}{\sin\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)}\right], \tag{l}$$

and θ_1 is the slope of the deformed axis at the right (fixed) end. The quantities θ_0, θ_1 and l are the unknowns of the problem.

From the relations (h), (i) and (j), it results

$$dx = -2\sqrt{\frac{EIl}{M_1}}\sin\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\sin\varphi d\varphi, \tag{m}$$

$$dy = -\sqrt{\frac{EIl}{M_1}}\left[2\sqrt{1-\sin^2\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\sin^2\varphi} - \frac{1}{\sqrt{1-\sin^2\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)\sin^2\varphi}}\right]d\varphi. \tag{n}$$

By integration, one obtains

$$l = \int_{\theta_0}^{-\theta_1} dy = \sqrt{\frac{EIl}{M_1}}\sqrt{2(\sin\theta_1 + \sin\theta_0)}, \tag{o}$$

$$0 = \int_{\theta_0}^{-\theta_1} dy = 2\left[E\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)+E\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)\right]-\left[K\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)-F\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)\right]. \tag{p}$$

The equations which determine the three unknowns are (p), (n) and (o); it results

$$\frac{M_1 L}{EI} = \sqrt{2(\sin\theta_1 + \sin\theta_0)}\left[K\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)-F\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)\right], \tag{q}$$

obtaining the slope θ_1 as function of $M_1 L/EI$.

The distance between the supports si given by the the relation

$$\frac{l}{L} = \frac{\sqrt{2(\sin\theta_1 + \sin\theta_0)}}{K\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)-F\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)},$$

while the parametric equations of a point of the deformed axis are

$$\frac{x}{L}=\frac{1}{L}\int_{\theta_1}^{\theta}\mathrm{d}x=\frac{\sqrt{2(\sin\theta+\sin\theta_0)}-\sqrt{2(\sin\theta_1+\sin\theta_0)}}{K\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)-F\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)},$$

$$\frac{y}{L}=\frac{1}{L}\int_{\theta_1}^{\theta}\mathrm{d}y=\frac{2E\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)-K\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)-2E\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)+F\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)}{K\left(\frac{\pi}{4}+\frac{\theta_0}{2}\right)-F\left(\frac{\pi}{4}+\frac{\theta_0}{2},\varphi_1\right)}.$$

Application 4.39

Problem. A flexible band of length l and unit breadth, in a vertical position, is subjected to a hydrostatic pressure of a liquid of unit weight γ. Determine the deformation of the band. Discussion.

Mathematical model. A flexible bar cannot take over bending moments and shearing forces; thus, the equations of equilibrium are reduced to

$$\frac{\mathrm{d}N}{\mathrm{d}s}=0\,, \tag{a}$$

$$\frac{N}{\rho}=p_n\,, \tag{b}$$

where N represents the axial effort, s is the arc of curve,

$$\rho=\frac{\left(1+y'^2\right)^{3/2}}{y''} \tag{c}$$

is the curvature radius of he deformed axis, and p_n is the normal pressure exerted by the liquid.

From (a) we deduce $N(s)=\text{const}$, while from (b), we have

$$N=p_n\rho=\text{const}\,. \tag{d}$$

The relation (d) determines the curvature radius ρ, hence the form of the funicular curve, for a given p_n.

The hydrostatic pressure is given by $p_n=\gamma x$, $x\in[0,l]$, where we suppose that the water plane is at the upper end of the band.

From (c) and p_n we get

$$\frac{y''}{\left(1+y'^2\right)^{3/2}}=\frac{\gamma x}{N}\,. \tag{e}$$

Integrating once, we may write

$$\frac{y'}{\sqrt{1 + y'^2}} = \frac{\gamma}{2N}\left(x^2 + C_1\right), \tag{f}$$

where C_1 is an integration constant. The first derivative becomes

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\gamma}{2N}\left(x^2 + C_1\right)}{\sqrt{1 - \frac{\gamma^2}{4N^2}\left(x^2 + C_1\right)^2}}\,. \tag{g}$$

Taking into account the mechanical significance of the problem, that is that along the span l there is a section for which the tangent to the deformed axis is parallel to the non-deformed axis, it results that $C_1 < 0$; we take $C_1 = -c^2/l^2$, where $c \in [0, 1]$.
By the notation, the relation (g) becomes

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\gamma}{2N}\left(x^2 - c^2 l^2\right)}{\sqrt{1 - \frac{\gamma^2}{4N^2}\left(x^2 - c^2 l^2\right)^2}}\,. \tag{h}$$

Integrating once the previous relation, we get

$$y = \int_0^x \frac{\frac{\gamma}{2N}\left(x^2 - c^2 l^2\right)}{\sqrt{1 - \frac{\gamma^2}{4N^2}\left(x^2 - c^2 l^2\right)^2}}\mathrm{d}x\,. \tag{i}$$

Fixing the inferior limit of the integral, the condition of support at the origin, that is $y = 0$ for $x = 0$, is satisfied. The constant c is then specified by $y(l) = 0$.
The integral in (i) is an *elliptic integral*. To obtain a canonical form of it, one makes the substitutions

$$\frac{\gamma}{2N}\left(x^2 - c^2 l^2\right) = 1 - 2k^2 \sin^2 \varphi\,, \tag{j}$$

$$c^2 = -\frac{2N}{\gamma l^2}\left(1 - 2k^2\right). \tag{k}$$

It results

$$x = 2\sqrt{\frac{N}{\gamma}}k \cos\varphi,\ \mathrm{d}x = -2\sqrt{\frac{N}{\gamma}}k \sin\varphi \mathrm{d}\varphi\,, \tag{l}$$

and the limits become $x = 0$ for $\varphi = \pi/2$ and $x \to \varphi$. Thus, we obtain

$$y=\sqrt{\frac{N}{\gamma}}\int_0^{\pi/2}\frac{1-2k^2\sin^2\varphi}{1-k^2\sin^2\varphi}\mathrm{d}\varphi=2\sqrt{\frac{N}{\gamma}}\int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\mathrm{d}\varphi$$
$$-2\sqrt{\frac{N}{\gamma}}\int_0^{\varphi}\sqrt{1-k^2\sin^2\varphi}\mathrm{d}\varphi-\sqrt{\frac{N}{\gamma}}\int_0^{\pi/2}\frac{\mathrm{d}\,\varphi}{1-k^2\sin^2\varphi}+\sqrt{\frac{N}{\gamma}}\int_0^{\varphi}\frac{\mathrm{d}\xi}{1-k^2\sin^2\xi}. \tag{m}$$

Introducing the elliptic integral of second species, of amplitude φ and of modulus k,

$$E(\varphi,k)=\int_0^{\varphi}\sqrt{1-k^2\sin^2\xi}\mathrm{d}\xi,$$

the complete elliptic integral of second species,

$$E\left(\frac{\pi}{2},k\right)=\int_0^{\pi/2}\sqrt{1-k^2\sin^2\varphi}\mathrm{d}\varphi,$$

the elliptic integral of first species, of amplitude φ, and of modulus k,

$$F(\varphi,k)=\int_0^{\varphi}\frac{\mathrm{d}\xi}{\sqrt{1-k^2\sin^2\xi}}$$

and the complete elliptic integral of first species

$$F\left(\frac{\pi}{2},k\right)=\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

the equation of the funicular curve becomes

$$y=\sqrt{\frac{N}{\gamma}}\left[2E\left(\frac{\pi}{2},k\right)-2E(\varphi,k)-F\left(\frac{\pi}{2},k\right)+F(\varphi,k)\right]. \tag{n}$$

The elliptic functions are listed in tables as functions of φ and θ (where $k=\sin\theta$).
To put the condition at the end $x=l$, we notice that from (l) we get

$$\cos\varphi_1=\frac{1}{2k}\sqrt{\frac{\gamma l^2}{N}},\quad \varphi_1=\arccos\left(\frac{1}{2k}\sqrt{\frac{\gamma l^2}{N}}\right). \tag{o}$$

If we take $\varphi=\varphi_1$ in (n), that one is transformed in the transcendental equation

$$2E\left(\frac{\pi}{2},k\right)-2E(\varphi_1,k)-F\left(\frac{\pi}{2},k\right)+F(\varphi_1,k)=0, \tag{p}$$

whose solution finally yields the constant c.

As the direct solution of the transcendental equation (p) is difficultly obtained, one may search a numerical solution, starting from the equation

$$\int_0^l \frac{\gamma\left(x^2 - c^2 l^2\right)}{2N\sqrt{1 - \frac{\gamma^2}{4N^2}\left(x^2 - c^2 l^2\right)^2}} \mathrm{d}x = 0 . \tag{q}$$

To this goal, the length l was divided in 20 equal intervals. To obtain non-dimensional expressions, we denote

$$N = \alpha \gamma l^2 \tag{r}$$

and make the change of variable $x = \xi l \Rightarrow \mathrm{d}x = l\mathrm{d}\xi$, $\xi \in [0, 1]$; thus, the equation (q) becomes

$$\int_0^l \frac{\xi^2 - c^2}{2\alpha\sqrt{1 - \frac{1}{4\alpha^2}\left(\xi^2 - c^2\right)}} \mathrm{d}\xi = 0 , \tag{s}$$

where the roots c are determined for various values of the parameter α. If α takes very great values (great efforts in the band), then the value of c may be directly obtained considering under the radical that $1/4\alpha^2 \to 0$, hence neglecting the paranthesis with respect to unity. It is left to compute the integral

$$\int_0^l \left(\xi^2 - c^2\right) \mathrm{d}\xi = \left.\left(\frac{\xi^3}{3} - c^2 \xi\right)\right|_0^1 = \frac{1}{3} - c^2 ,$$

whence, equating to zero the last member, it results the convenient root $c = 1/\sqrt{3} \cong 0.57735$ (value which determines the position of the maximal bending moment in a simply supported beam), acted upon by a triangular distributed load.
The problem of the inferior limit is more difficult. From the condition of existence of a real solution we may write the inequality

$$1 - \frac{1}{4\alpha^2}\left(\xi^2 - c^2\right) > 0 ,$$

equivalent to the inequalities

$$-1 < -\frac{c^2}{2\alpha} \le \frac{1}{2\alpha}\left(\xi^2 - c^2\right) \le \frac{1}{2\alpha}\left(1 - c^2\right) < 1 .$$

One obtains the conditions

$$c^2 > 1 - 2\alpha, \; c^2 < 2\alpha .$$

To the limit, if the inequalities become equalities, then the equations $c^2 = 1 - 2\alpha$ and $c^2 = 2\alpha$ represent two parabolas; their graphics, c vs α, are represented in Fig.4.35. The two parabolas have a piercing point, i.e. $\alpha = 1/4$, $c = 1/\sqrt{2}$.

We try to represent the graphic of the function $f(c, \alpha) = 0$, corresponding to the equation (s). As it was established before, the graphic admits an asymptote parallel to the axis $O\alpha$, that is $c = 1/\sqrt{3}$.

The numerical calculation provides values of c for $\alpha > 0.35$. If $\alpha < 0.35$, then the equation (s) must be solved directly.

We search the limit point of the curve, situated on the parabola $c^2 = 1 - 2\alpha$. Associating the relation (k), in which we replace N by the expression (r), we have

$$c^2 = 1 - 2\alpha = -2\alpha\left(1 - 2k^2\right),$$

whence we obtain $4k^2\alpha = 1$ or $2k\sqrt{\alpha} = 1$.

Further, the relation (o) becomes $\cos\varphi_1 = 1/2k\sqrt{\alpha} = 1$ or $\alpha_1 = 0$.

Because $E(0, \theta) = F(0, \theta) = 0$, the transcendental equation (p) is reduced to

$$2E\left(\frac{\pi}{2}, k\right) - F\left(\frac{\pi}{2}, k\right) = 0$$

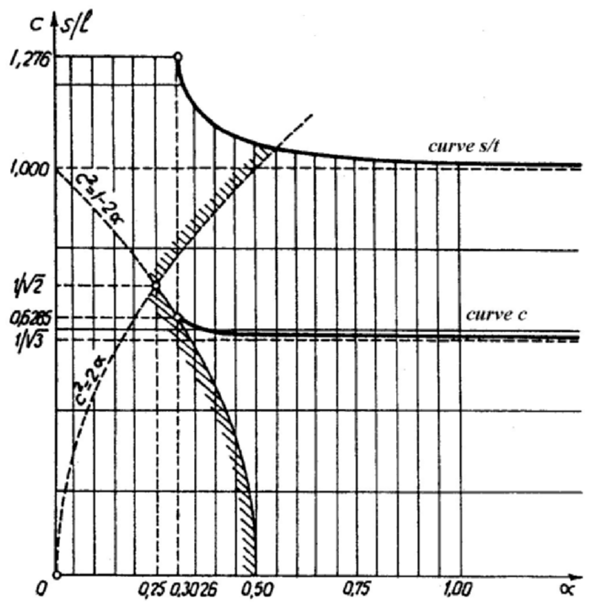

Figure 4. 35. Diagrams of the two parabolas (c vs. α)

or

$$2E\left(\frac{\pi}{2}, \theta\right) - F\left(\frac{\pi}{2}, \theta\right) = 0\,,$$

by the substitution $k \to \theta$.
Using the tables and the linear interpolation, it results $\theta = 65{,}315^0$ and successively

$$k = \sin\theta = 0.90891,\ \alpha = \frac{1}{4k^2} = 0.30262\,,$$

$$c^2 = 1 - 2\alpha = 0.39476, \quad c = 0.62830\,.$$

The solution may be obtained in a direct way, using the developments into power series of the functions $E(\pi/2\,, k)$ and $F(\pi/2\,, k)$; finally, we get the equation

$$l = \sum_{m=1}^{\infty} \frac{2m+1}{2m-1}\left(\frac{1\cdot 3\cdot 5\ldots(2m-1)}{2\cdot 4\cdot 6\ldots(2m)}\right)^2 k^{2m}\,,$$

obtaining the same value of the root k.

Application 4.40

Problem. To compute the contour of a section with thin walls of constant thickness and of maximal rigidity of the cross section area, one must solve the ODE

$$2y + 2yy'^2 - y^2 y'' - \lambda y'' = 0\,, \tag{a}$$

where y is the applicate of the median line of the cross section, x is the abscissa in the cross section, while λ is a given constant. Determine the general solution of (a).

Solution. The equation (a) may be written

$$2y\left(1 + y'^2\right) - y''\left(y^2 + \lambda\right) = 0$$

or

$$\frac{y''}{1 + y'^2} = \frac{2y}{y^2 + \lambda}\,.$$

Supposing that $y \neq 0$, we multiply by $2y'$ and obtain

$$\frac{2y'y''}{1 + y'^2} = \frac{4yy'}{y^2 + \lambda}\,;$$

integrating once we have

$$\ln\left(1 + y'^2\right) = 2\ln\left(y^2 + \lambda\right) - \ln C_1^2$$

or

$$1 + y'^2 = \frac{\left(y^2 + \lambda\right)^2}{C_1^2}\,.$$

Further, we may write

$$y' = \frac{\sqrt{\left(y^2 + \lambda\right)^2 - C_1^2}}{C_1}\,,\ \mathrm{d}x = \frac{C_1}{\sqrt{\left(y^2 + \lambda\right)^2 - C_1^2}}\,\mathrm{d}y\,,$$

$$x + C_2 = I = C_1 \int \frac{\mathrm{d}y}{\sqrt{\left(y^2 + \lambda + C_1\right)\left(y^2 + \lambda - C_1\right)}}\,.$$

By the change of variable $y = \sqrt{-\left(C_1 + \lambda\right)}z$, we obtain

$$I = \int \frac{C_1 \mathrm{d}z}{\sqrt{z^2 - 1}\sqrt{-\left(C_1 + \lambda\right)z^2 + \lambda - C_1}}\,;$$

further

$$I = \int \frac{C_1 \mathrm{d}z}{\sqrt{z^2 - 1}\sqrt{C_1 - \lambda}\sqrt{\dfrac{C_1 + \lambda}{C_1 - \lambda}z^2 + 1}}\,.$$

If we denote $k^2 = -\left(C_1 + \lambda\right)/\left(C_1 - \lambda\right)$, it results

$$\frac{\sqrt{C_1 - \lambda}}{C_1}\left(x + C_2\right) = \int \frac{\mathrm{d}z}{\sqrt{\left(1 - z^2\right)\left(1 - k^2 z^2\right)}}\,.$$

The integral in the second member is the inverse Legendre's elliptic integral of first species; we obtain $z = \mathrm{sn}\left(\sqrt{C_1 - \lambda}/C_1\right)\left(x + C_2\right)$ and, having in view the value of z,

$$y = \sqrt{-C_1 - \lambda}\ \mathrm{sn}\,\frac{\sqrt{C_1 - \lambda}}{C_1}\left(x + C_2\right),$$

where sn is the Legendre's sinus-amplitude function.

Chapter 5

NON-LINEAR ODSs OF FIRST ORDER

1. Generalities

1.1 THE GENERAL FORM OF A FIRST ORDER ODS

The general form of a first order ODS with *n* unknown functions is

$$\begin{aligned} &F_1(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n') = 0, \\ &F_2(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n') = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &F_n(x, y_1, y_2, ..., y_n, y_1', y_2', ..., y_n') = 0, \end{aligned} \tag{5.1.1}$$

where F_j are defined on the same $(2n+1)$-dimensional domain and are considered sufficiently regular.

If the hypotheses of the theorem of implicit systems are fulfilled, then one can get y_j' explicitely from (5.1.1), thus obtaining the *canonic*/*normal* form of a first order ODS

$$\begin{aligned} &y_1' = f_1(x, y_1, y_2, ..., y_n), \\ &y_2' = f_2(x, y_1, y_2, ..., y_n), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &y_n' = f_n(x, y_1, y_2, ..., y_n). \end{aligned} \tag{5.1.2}$$

In what follows, we shall consider only first order ODSs of canonic form.

These ODSs can also be written in compact form. Indeed, by using the notations

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}, \quad \mathbf{f}(x, \mathbf{y}) = \begin{bmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \\ \vdots \\ f_n(x, \mathbf{y}) \end{bmatrix}, \tag{5.1.3}$$

the ODS (5.1.2) may be written in the vector form

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}x} = \mathbf{f}(x, \mathbf{y}). \tag{5.1.4}$$

Let us also note that any *n*-th order ODE

$$y^{(n)} = f\left(x, y, y', y''..., y^{(n-1)}\right), \tag{5.1.5}$$

can be written in the form of a first order ODS with n unknown functions. Indeed, with the notations

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'',..., y_n = y^{(n-1)}, \tag{5.1.6}$$

the ODE (5.1.5) becomes the following first order ODS with n unknown functions

$$\begin{aligned} &y_1' = y_2, \\ &y_2' = y_3, \\ &\dots\dots\dots\dots \\ &y_{n-1}' = y_n, \\ &y_n' = f\left(x, y_1, y_2,..., y_n\right). \end{aligned} \tag{5.1.7}$$

One can prove that, conversely, a normal (canonical) first order ODS with n unknown functions can be reduced to a n-th order ODE, under certain regularity conditions.

1.2 THE EXISTENCE AND UNIQUENESS THEOREM FOR THE SOLUTION OF THE CAUCHY PROBLEM

Exactly as in the case of linear ODSs, we can consider the problem of determining that solution of (5.1.2) that satisfies *the initial* or *Cauchy conditions*

$$\begin{aligned} &y_1(x_0) = y_{10}, \\ &y_2(x_0) = y_{20}, \\ &\dots\dots\dots\dots \\ &y_n(x_0) = y_{n0} \end{aligned} \tag{5.1.8}$$

or, in vector form,

$$\mathbf{y}(x_0) = \mathbf{y}_0, \qquad \mathbf{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{n0} \end{bmatrix}. \tag{5.1.9}$$

The point $(x_0, y_{10}, y_{20},..., y_{n0}) \equiv (x_0, \mathbf{y}_0)$ belongs to the $(n+1)$-dimensional domain on which (5.1.2), or, equivalently, (5.1.4), makes sense. We can generalize to ODSs the Cauchy-Picard theorem 4.2 from Chap.4.

Theorem 5.1. *Suppose that* **f** *satisfies the following conditions:*

i) $\mathbf{f} \in \left(\mathrm{C}^0(D)\right)^n$, where $D = \left\{(x, \mathbf{y}) \in \Re^n, |x - x_0| \le a, \left|y_j - y_{j0}\right| \le b, j = \overline{1, n}\right\}$,

ii) $f_j, j = \overline{1, n}$ are *Lipschitz with respect to* **y**, *i.e*

$$\exists K_j > 0 : \left| f_j(x, \mathbf{Y}) - f(x, \mathbf{Z}) \right| < K_j \sum_{m=1}^{n} \left| Y_j - Z_j \right|, (x, \mathbf{Y}), (x, \mathbf{Z}) \in D, \quad j = \overline{1, n}\,.$$

Then the Cauchy problem (5.1.4), (5.1.9) *allows a unique solution* $\mathbf{y} = \mathbf{y}(x)$, *of class* $\left(\mathrm{C}^1(I)\right)^n$, $I = [x_0 - h, x_0 + h]$, *where* $h = \min\{a, b / M\}$, $M = \max\limits_{j=1,n} \left\{ \sup\limits_{(x,\mathbf{y}) \in D} \left| f_j(x, \mathbf{y}) \right| \right\}$.

The proof of this theorem is also based on successive approximations; this method offers a practical and efficient possibility of getting solutions of ODSs.

Let us note that, if the general solution of a first order ODE depends on an arbitrary constant, the general solution of a first order ODS with *n* unknown functions and *n* equations depends on *n* arbitrary constants. In both cases, the constants can be fixed up by adding Cauchy conditions to the ODE or to the ODS, accordingly.

The general solution of an ODS of type (5.1.2) or (5.1.4) can thus be written in the explicit form

$$\begin{aligned} y_1 &= \varphi_1(x, C_1, C_2, \ldots, C_n), \\ y_2 &= \varphi_2(x, C_1, C_2, \ldots, C_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n &= \varphi_n(x, C_1, C_2, \ldots, C_n). \end{aligned} \tag{5.1.10}$$

If we think of (5.1.10) as a functional system with respect to $C_1, C_2, \ldots, C_n$, then, supposing that this system fulfills the hypotheses of the implicit function theorem, we can explicit $C_1, C_2, \ldots, C_n$ from (5.1.10), thus obtaining the general solution of the ODS (5.1.2) in the implicit form

$$\begin{aligned} \psi_1(x, y_1, y_2, \ldots, y_n) &= C_1, \\ \psi_2(x, y_1, y_2, \ldots, y_n) &= C_2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_n(x, y_1, y_2, \ldots, y_n) &= C_n\,. \end{aligned} \tag{5.1.11}$$

1.3 THE PARTICLE DYNAMICS

The classical study of mechanical motions is generally based on Newton's second law, according to which the acceleration of a moving particle is determined by the resultant of the forces acting upon it, i.e.

$$m\mathbf{a} = \mathbf{F} \tag{5.1.12}$$

where *m* is the mass of the particle.

A moving body can be thought as a particle of co-ordinates (x, y, z) with respect to a fixed up system of co-ordinates; obviously, as the position of the particle changes every moment, *x*, *y* and *z* will be functions depending on time.

The co-ordinates of the velocity vector of the particle will be given by

$$\mathbf{V} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k}, \tag{5.1.13}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the versors of the co-ordinate axes and the point signifies differentiation with respect to the time t. Also, the acceleration vector is represented in the form

$$\mathbf{a} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k}. \tag{5.1.14}$$

Suppose that the resultant of the forces acting upon the particle, determining its motion, is known, that is

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}, \tag{5.1.15}$$

where X, Y and Z are given functions that might depend on the time t, on the particle position and also on its velocity, so that the motion of the particle is finally described by the second order ODS

$$\begin{aligned} m\ddot{x} &= X(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \\ m\ddot{y} &= Y(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \\ m\ddot{z} &= Z(t, x, y, z, \dot{x}, \dot{y}, \dot{z}). \end{aligned} \tag{5.1.16}$$

This system can be reduced to a first order ODS, by introducing the functions

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = w. \tag{5.1.17}$$

The new first order ODS will have six equations and six unknown functions, x, y, z, u, v, w

$$\begin{aligned} \dot{x} &= u, \\ \dot{y} &= v, \\ \dot{z} &= w, \\ m\dot{u} &= X(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \\ m\dot{v} &= Y(t, x, y, z, \dot{x}, \dot{y}, \dot{z}), \\ m\dot{w} &= Z(t, x, y, z, \dot{x}, \dot{y}, \dot{z}). \end{aligned} \tag{5.1.18}$$

The general integral of this system will be written in the form

$$\begin{aligned} x &= f_1(x, C_1, C_2, C_3, C_4, C_5, C_6), \\ y &= f_2(x, C_1, C_2, C_3, C_4, C_5, C_6), \\ z &= f_3(x, C_1, C_2, C_3, C_4, C_5, C_6), \\ u &= \varphi_1(x, C_1, C_2, C_3, C_4, C_5, C_6), \\ v &= \varphi_2(x, C_1, C_2, C_3, C_4, C_5, C_6), \\ w &= \varphi_3(x, C_1, C_2, C_3, C_4, C_5, C_6), \end{aligned} \tag{5.1.19}$$

obvioulsy depending on six arbitrary constants $C_1, C_2, C_3, C_4, C_5, C_6$.

The first three relations (5.1.19) refer to the particle position and define its *trajectory* and the last three, concerning its velocity, represent *the law of motion*.
The system (5.1.18) allows infinitely many solutions. But if the initial position and the initial velocity of the particle are fixed up, this particle will follow a unique trajectory. This physical fact is mathematically justified by applying the Theorem 5.1 of local existence and uniqueness of the solution of the Cauchy problem associated to the ODS (5.1.18); indeed, knowing the initial position and velocity of the particle means in fact that there are satisfied the Cauchy (initial) conditions

$$\begin{aligned} x(t_0) &= x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0 , \\ \dot{x}(t_0) &= \dot{x}_0, \quad \dot{y}(t_0) = \dot{y}_0, \quad \dot{z}(t_0) = \dot{z}_0 , \end{aligned} \tag{5.1.20}$$

where t_0 marks the beginning of the motion. If, moreover, X, Y, Z are continuous with respect to their arguments and Lipschitz-ian in x, y, z, u, v, w, then by Theorem 5.1 the solution of the Cauchy problem (5.1.18), (5.1.20) allows a unique solution.

2. First Integrals of an ODS

2.1 GENERALITIES

The left members ψ_j of the relations (5.1.11) obviously become identically constant if we replace y_j by their corrresponding expressions (5.1.10).
We call *first integral* of the ODS (5.1.2) a C^1-class function, depending on the independent variable and on the unknown functions, which becomes identically constant if we replace the unknown functions by an arbitrary solution of the system.
With this definition, we see that any of the relations (5.1.11) is a first integral of the ODS (5.1.2). Also from the definition we deduce that a given ODS allows infinitely many first integrals. Indeed, the relation

$$\Phi(\psi_1(x, y_1, y_2, \ldots, y_n), \psi_2(x, y_1, y_2, \ldots, y_n), \ldots, \psi_n(x, y_1, y_2, \ldots, y_n)) = C , \tag{5.2.1}$$

where Φ is an arbitrary C^1-class function in its arguments and C is an arbitrary constant, is obviously a first integral of (5.1.2).
Suppose now that in one of the first integrals of the ODS (5.1.2)

$$\psi(x, y_1, y_2, \ldots, y_n) = C \tag{5.2.2}$$

we replaced $y_1, y_2, \ldots, y_n$ by an arbitrary solution of the system. If ψ allows a total differential, then $\mathrm{d}\psi = 0$, whence

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y_1}\frac{\mathrm{d}y_1}{\mathrm{d}x} + \frac{\partial \psi}{\partial y_2}\frac{\mathrm{d}y_2}{\mathrm{d}x} + \ldots + \frac{\partial \psi}{\partial y_n}\frac{\mathrm{d}y_n}{\mathrm{d}x} = 0 \; ; \tag{5.2.3}$$

as y_j also satisfy the ODS (5.1.2), we find out that

$$\begin{aligned}&\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y_1}f_1(x,y_1,y_2,...,y_n)+\frac{\partial\psi}{\partial y_2}f_2(x,y_1,y_2,...,y_n)\\&+...+\frac{\partial\psi}{\partial y_n}f_n(x,y_1,y_2,...,y_n)=0.\end{aligned} \tag{5.2.4}$$

Thus, the left members of every first integral of (5.1.2) satisfies (5.2.4).
We observe that (5.2.4) may represent a linear first order PDE, having ψ as unknown function.
Suppose now that, conversely, ψ satisfies (5.2.4). Then along any integral hypersurface of the system (5.1.2) the relation (5.2.3) holds true, therefore (5.2.4) is also true. We thus get the following results:

A. *The PDE* (5.2.4) *is the necessary and sufficient condition for* (5.2.2) *to be a first integral of the ODS* (5.1.2).

Suppose now that we could find n functionally independent first integrals of (5.1.2), $\psi_1,\psi_2,...,\psi_n$. This means that their Jacobian with respect to $y_1,y_2,...,y_n$ does no vanish identically, i.e.

$$\frac{D(\psi_1,\psi_2,...,\psi_n)}{D(y_1,y_2,...,y_n)}=\begin{bmatrix}\frac{\partial\psi_1}{\partial y_1}&\frac{\partial\psi_1}{\partial y_2}&\cdots&\frac{\partial\psi_1}{\partial y_n}\\\frac{\partial\psi_2}{\partial y_1}&\frac{\partial\psi_2}{\partial y_2}&\cdots&\frac{\partial\psi_2}{\partial y_n}\\\cdots&\cdots&\cdots&\cdots\\\frac{\partial\psi_n}{\partial y_1}&\frac{\partial\psi_n}{\partial y_2}&\cdots&\frac{\partial\psi_n}{\partial y_n}\end{bmatrix}\neq 0\,. \tag{5.2.5}$$

Let us show that, once determined a system of n functionally independent first integrals, one can determine any solution of the ODS (5.1.2). Indeed, suppose known the system of first integrals and let $y_1,y_2,...,y_n$ be an arbitrary solution of (5.1.2). Let x_0 be an arbitrary point of the domain of definition of the solution and denote by $y_{j0}=y_j(x_0),\ j=\overline{1,n}$. Also suppose that x_0 was chosen such that the Jacobian (5.2.5) be not null at the point $(y_{10},y_{20},...,y_{n0})$. According to the implicit function theorem, we can invert (5.1.11) around this point, thus obtaining

$$\begin{aligned}&y_1=y_1(x,C_1,C_2,...,C_n),\\&y_2=y_2(x,C_1,C_2,...,C_n),\\&.....................................\\&y_n=y_n(x,C_1,C_2,...,C_n),\end{aligned} \tag{5.2.6}$$

where y_j are continuous and univocal functions of $x,C_1,C_2,...,C_n$. These relations will become identities if we replace every C_j by $\psi_j(x,y_1,y_2,...,y_n)$. Now, replacing each C_j by $C_{j0} = \psi_j(x_0, y_{10}, y_{20},..., y_{n0})$ accordingly, then, obviously, the vector of components

$$\begin{aligned} y_1 &= y_1(x, C_{10}, C_{20},..., C_{n0}), \\ y_2 &= y_2(x, C_{10}, C_{20},..., C_{n0}), \\ &\cdots\cdots\cdots\cdots \\ y_n &= y_n(x, C_{10}, C_{20},..., C_{n0}), \end{aligned} \tag{5.2.7}$$

satisfies the Cauchy conditions $y_j(x_0) = y_{j0}$, $j = \overline{1,n}$, as well as the system (5.1.2). But the arbitrary solution we started from satisfies the same conditions; therefore, by theorem 5.1, this solution (locally) coincides with the solution (5.2.7), determined by first integrals. Consequently,

B. *Knowing n functionally independent first integrals is equivalent to the integration of the ODS* (5.1.2).

Suppose that we succeeded to find only one first integral of (5.1.2)

$$\psi(x, y_1, y_2,..., y_n) = C. \tag{5.2.8}$$

From this relation we can express one of the unknown functions – say y_n, if $\partial\psi / \partial y_n \neq 0$ – as a function of $x, y_1, y_2,..., y_{n-1}$ and *C*, therefore

$$y_n = \varphi(x, y_1, y_2,..., y_{n-1}, C). \tag{5.2.9}$$

Introducing y_n in (5.1.2), we obtain a new first order ODS, with $(n-1)$ unknown functions $y_1, y_2,..., y_{n-1}$. Thus, the number of the unknown functions of the ODS was diminished by one unit. Integrating this new system, its solution will depend on $(n-1)$ arbitrary constants, which, together with C, will complete the set of *n* arbitrary constants corresponding to (5.1.2). Similarly, we can prove that

C. *If we know k functionally independent first integrals of* (5.1.2) *, then the number of the unknown functions can be reduced by k units.*

2.2 THE THEOREM OF CONSERVATION OF THE KINETIC ENERGY

Let us write the ODEs (5.1.16) in the form

$$\begin{aligned} F &\equiv m\ddot{x} - X(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0, \\ G &\equiv m\ddot{y} - Y(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0, \\ H &\equiv m\ddot{z} - Z(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0. \end{aligned} \tag{5.2.10}$$

We suppose for now that the functions *X*, *Y*, *Z* do not depend on $t, \dot{x}, \dot{y}, \dot{z}$, therefore they only depend on the particle position (x, y, z). Also suppose that the vector of components (X, Y, Z) may be written as the gradient of a scalar function *U*, i.e.

$$X = \frac{\partial U}{\partial x}, \quad Y = \frac{\partial U}{\partial y}, \quad Z = \frac{\partial U}{\partial z}. \tag{5.2.11}$$

The function U is called *force function* or *potential*. The derivative of U with respect to the time t is expressed in the form

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z} = X\dot{x} + Y\dot{y} + Z\dot{z}. \tag{5.2.12}$$

Multiplying the above ODEs (5.2.10) by $\dot{x}, \dot{y}, \dot{z}$ respectively and adding them member by member, we get

$$F\dot{x} + G\dot{y} + H\dot{z} = \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - X\dot{x} + Y\dot{y} + Z\dot{z} = 0. \tag{5.2.13}$$

or, taking (5.2.12) into account,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(x, y, z)\right] = 0, \tag{5.2.14}$$

whence, by integration

$$\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U(x, y, z) = C. \tag{5.2.15}$$

This is, in fact, a first integral of the ODS (5.2.10); actually, in terms of the above definition for first integrals, (5.2.15) is a first integral of the equivalent first order ODS (5.1.18).

The mechanical interpretation of this first integral is extreemly important. It practically proves *the theorem of energy conservation*. Indeed, denoting by v the modulus of the particle velocity, it results

$$\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{mv^2}{2}, \tag{5.2.16}$$

so that the first term of the sum in (5.2.15) has the significance of *kinetic energy*.

2.3 THE SYMMETRIC FORM OF AN ODS. INTEGRAL COMBINATIONS

The system (5.1.2) may be written in the differential form

$$\frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y_1}{f_1(x, y_1, y_2, \ldots, y_n)} = \frac{\mathrm{d}y_2}{f_2(x, y_1, y_2, \ldots, y_n)} = \ldots = \frac{\mathrm{d}y_n}{f_n(x, y_1, y_2, \ldots, y_n)} \tag{5.2.17}$$

This system is equivalent to (5.1.2) if we multiply the denominators with the same non-zero factor. We can thus suppose from the beginning that, instead of 1, the differential dx is divided by an arbitrary function. To take advantage of a symmetric writing, we shall

put $x_1, x_2, ..., x_n$ instead of $x, y_1, y_2, ..., y_n$, re-noting the number of variables with *n*, instead of $(n+1)$.
In conclusion, the symmetric form of a first order ODS is

$$\frac{\mathrm{d}x_1}{X_1(x, y_1, y_2, ..., y_n)} = \frac{\mathrm{d}x_2}{X_2(x, y_1, y_2, ..., y_n)} = ... = \frac{\mathrm{d}x_n}{X_n(x, y_1, y_2, ..., y_n)}. \quad (5.2.18)$$

Let us note that the values cancelling all the functions X_j cannot be chosen as initial data for an associated Cauchy problem. These values are called *singular* and they corrrespond to the *critica*l (*singular*) *points* of the system. In this case, the Cauchy-Picard theorem 5.1. does not work.
The symmetric form of a first order ODS may be useful to emphasize first integrals. Indeed, if we can determine *n* functions $\lambda_j(x_1, x_2, ..., x_n)$, $j = \overline{1, n}$ such that

$$\lambda_1 X_1 + \lambda_2 X_2 + ... + \lambda_n X_n = 0 \quad (5.2.19)$$

and if the expression

$$\lambda_1 \mathrm{d}x_1 + \lambda_2 \mathrm{d}x_2 + ... + \lambda_n \mathrm{d}x_n = \mathrm{d}\Phi \quad (5.2.20)$$

is a total differential, then the sum $\sum_{j=1}^{n} \lambda_j X_j$ is called *integral combination* and

$$\Phi(x_1, x_2, ..., x_n) = C \quad (5.2.21)$$

is a first integral of the system (5.2.18).

2.4 JACOBI'S MULTIPLIER. THE METHOD OF THE LAST MULTIPLIER

The necessary and sufficient condition that a function $f(x_1, x_2, ..., x_n)$ be a first integral of the ODS (5.2.18) is

$$\mathrm{d}f = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \mathrm{d}x_j = 0. \quad (5.2.22)$$

Taking the system into account, this can be written in the form

$$X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + .. + X_n \frac{\partial f}{\partial x_n} = 0. \quad (5.2.23)$$

This is a linear and homogeneous first order PDE. Without insisting on details, we shall only note that the characteristic system associated to this PDE is precisely (5.2.18). We already showed that if one knows $(n-1)$ functionally independent first integrals $\varphi_1, \varphi_2, ..., \varphi_{n-1}$ of (5.2.18), then any other first integral *f* is functionally dependent on them, i.e.

$$\Delta \equiv \frac{D(f, \varphi_1, \varphi_2, ..., \varphi_{n-1})}{D(x_1, x_2, ..., x_n)} \equiv \begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi_{n-1}}{\partial x_1} & \frac{\partial \varphi_{n-1}}{\partial x_2} & \cdots & \frac{\partial \varphi_{n-1}}{\partial x_n} \end{vmatrix} = 0 . \tag{5.2.24}$$

Expanding this determinant with respect to its first row, we get

$$\Delta = \sum_{j=1}^{n} \Delta_j \frac{\partial f}{\partial x_j} = 0 , \tag{5.2.25}$$

where Δ_j is the algebraic complement corresponding to $\partial f / \partial x_j$. As both (5.2.23) and (5.2.25) must be fulfilled, it follows

$$\Delta_j = MX_j , \quad j = \overline{1, n} . \tag{5.2.26}$$

The function M is called *Jacobi's multiplier*. We cam also write

$$M\left(X_1 \frac{\partial f}{\partial x_1} + X_2 \frac{\partial f}{\partial x_2} + .. + X_n \frac{\partial f}{\partial x_n} \right) = \Delta . \tag{5.2.27}$$

Starting, with Jacobi, from the determinant

$$U \equiv \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_2} & \cdots & \frac{\partial \varphi_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \varphi_{n-1}}{\partial x_1} & \frac{\partial \varphi_{n-1}}{\partial x_2} & \cdots & \frac{\partial \varphi_{n-1}}{\partial x_n} \end{vmatrix} , \tag{5.2.28}$$

we observe that $U = \sum_{j=1}^{n} a_j \Delta_j$, again developing U with respect to its first row. Differentiating now Δ_j with respect to x_j, we deduce

$$\sum_{j=1}^{n} \frac{\partial \Delta_j}{\partial x_j} = 0 , \tag{5.2.29}$$

which, together with (5.2.25), involves

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(MX_j \right) = 0 . \tag{5.2.30}$$

This is the equation of Jacobi's multiplier.
Computing the expression

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(MfX_j\right)=f\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(MX_j\right)+M\sum_{j=1}^{n}X_j\,\frac{\partial f}{\partial x_j}\,, \tag{5.2.31}$$

whence, taking (5.2.23) and (5.2.30) into account, we obtain

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(MfX_j\right)=0\,. \tag{5.2.32}$$

This means that the product between a Jacobi multiplier of the ODS (5.2.18) and any of its first integrals is also a Jacobi multiplier. If

$$\sum_{j=1}^{n}\frac{\partial X_j}{\partial x_j}=0\,, \tag{5.2.33}$$

in other words, if the divergence of the vector of components $X_1, X_2, ..., X_n$ is null, then, obviously, any constant is a Jacobi multiplier for (5.2.18). Consequently, for such systems any non-constant Jacobi multiplier is a first integral.
Suppose that we know $(n-2)$ independent first integrals $\varphi_1, \varphi_2, ..., \varphi_{n-2}$ of (5.2.18). Should we know another first integral φ_{n-1}, functionally independent on the previous ones, we could express $(n-1)$ variables as functions of the n-th one and of $(n-1)$ arbitrary constant, thus obtaining the general integral of the ODS (5.2.18).
According to the previous remarks, the system would be then reduced to

$$\frac{\mathrm{d}x_1}{X_1}=\frac{\mathrm{d}x_2}{X_2}\,, \tag{5.2.34}$$

which is in fact a first order ODE, that can be written in differential form

$$X_2\mathrm{d}x_1 - X_1\mathrm{d}x_2 = 0\,. \tag{5.2.35}$$

Multiplying this equation by the integrant factor μ, we get a total differential equation

$$\mathrm{d}\Phi\left(x_1, x_2\right)=0\,. \tag{5.2.36}$$

The integrant factor must satisfy the condition (see also Chap.4, Sec.1.4)

$$\frac{\partial\left(\mu X_1\right)}{\partial x_1}+\frac{\partial\left(\mu X_2\right)}{\partial x_2}=0\,. \tag{5.2.37}$$

This integrant factor is also called *the last multiplier* for the system (5.2.18).
Consequently, if we know a multiplier for the ODS (5.2.18), then it is enough to know $(n-1)$ first integrals in order to integrate it. If the ODS satisfies the condition (5.2.33), then $M=1$ is a Jacobi multiplier, so that to integrate the system we need only $(n-2)$ first integrals.

3. Analytical Methods of Solving the Cauchy Problem for non-Linear ODSs

3.1 THE METHOD OF SUCCESSIVE APPROXIMATIONS (PICARD-LINDELÕFF)

Let us consider again the Cauchy problem (5.1.2), (5.1.8) or, equivalently, (5.1.4), (5.1.9) and suppose satisfied the hypothesis of the existence Theorem 5.1. The Picard-Lindelöff method, also called the successive approximation method, which serves to prove the local existence and uniqueness of the solutions, is a strong tool for setting up this solution. It is proved that the sequence of the approximations

$$y_j^m(x) = y_{j0} + \int_{x_0}^{x} f_j\left(t, y_1^{m-1}(t), y_2^{m-1}(t), \ldots, y_n^{m-1}(t)\right)\mathrm{d}t, \quad y_j^0(x) = y_{j0}, \quad j = \overline{1,n}, \tag{5.3.1}$$

uniformly converges on $[x_0 - h, x_0 + h]$ to the solution of the considered Cauchy problem. The length $2h$ of this interval is determined by Theorem 5.1.
The construction of the n recurrent sequences tending to the unique solution of the Cauchy problem is a straightforward generalization of that exposed at Sec.1.2 for the case $n = 1$. More precisely, the functions defined by recurrence in (5.3.1) satisfy the inequality

$$\left|y_j^m(x) - y_j(x)\right| \leq \frac{M}{nK} \sum_{p=m+1}^{\infty} \frac{\left(nK\left|x - x_0\right|\right)^p}{p!}, \tag{5.3.2}$$

for $\left|x - x_0\right| < h$. The constant K is the Lipschitz constant.

3.2 THE METHOD OF THE TAYLOR SERIES EXPANSION

If the functions $f_j(x, y_1, y_2, \ldots, y_n)$ are of class C^∞ in a neighbourhood of $(x_0, y_{10}, y_{20}, \ldots, y_{n0})$, then the solution of (5.1.2), (5.1.8) may be searched in the form of a Taylor series, i.e.

$$\begin{aligned}
y_1(x) &= y_{10} + \frac{x - x_0}{1!} y_1'(x_0) + \frac{(x - x_0)^2}{2!} y_1''(x_0) + \ldots + \frac{(x - x_0)^m}{m!} y_1^{(m)}(x_0) + R_m^1, \\
y_2(x) &= y_{20} + \frac{x - x_0}{1!} y_2'(x_0) + \frac{(x - x_0)^2}{2!} y_2''(x_0) + \ldots + \frac{(x - x_0)^m}{m!} y_2^{(m)}(x_0) + R_m^2, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_n(x) &= y_{n0} + \frac{x - x_0}{1!} y_n'(x_0) + \frac{(x - x_0)^2}{2!} y_n''(x_0) + \ldots + \frac{(x - x_0)^m}{m!} y_n^{(m)}(x_0) + R_m^n,
\end{aligned} \tag{5.3.3}$$

where $R_m^j = R_m^j(x, x_0)$ are the corresponding remainders of the above developments. The Lagrange estimations of these remainders are

$$R_m^j(x,x_0)=\frac{(x-x_0)^{m+1}}{(m+1)!}y_j^{(m+1)}(\xi),\quad \xi\in(x_0,x), \tag{5.3.4}$$

therefore

$$\left|R_m^j(x,x_0)\right|\le M\frac{|x-x_0|^{m+1}}{(m+1)!}, \tag{5.3.5}$$

where

$$M=\max_{j=\overline{1,n}}\left(\sup_{(x,\mathbf{y})\in\mathfrak{D}}\left|f_j(x,\mathbf{y})\right|\right),\quad \mathfrak{D}=\left\{(x,\mathbf{y})\big||x-x_0|<a,\left|y_j-y_{j0}\right|<b,\ j=\overline{1,n}\right\}. \tag{5.3.6}$$

Precisely as in the one-dimensional case (see Sec.1.2), the solution of the Cauchy problem (5.1.2), (5.1.8) may be approximated by Taylor's polynomials in the right sides of (5.3.3), neglecting the remainder, therefore

$$y_j(x)=y_{j0}+\frac{x-x_0}{1!}y_j'(x_0)+\frac{(x-x_0)^2}{2!}y_j''(x_0)+\ldots+\frac{(x-x_0)^m}{m!}y_j^{(m)}(x_0), \tag{5.3.7}$$

for $j=\overline{1,n}$. The corresponding coefficients are computed successively, as follows

$$\begin{aligned}
y_j'(x_0)&=f_j(x_0,\mathbf{y}(x_0))=f_j(x_0,\mathbf{y}_0),\\
y_j''(x_0)&=\frac{\partial f_j}{\partial x}(x_0,\mathbf{y}_0)+\frac{\partial f_j}{\partial y_1}(x_0,\mathbf{y}_0)y_1'(x_0)\\
&\quad+\frac{\partial f_j}{\partial y_2}(x_0,\mathbf{y}_0)y_2'(x_0)+\ldots+\frac{\partial f_j}{\partial y_n}(x_0,\mathbf{y}_0)y_n'(x_0)\\
&=\frac{\partial f_j}{\partial x}(x_0,\mathbf{y}_0)+\sum_{k=1}^n f_k(x_0,\mathbf{y}_0)\frac{\partial f_j}{\partial y_k}(x_0,\mathbf{y}_0),
\end{aligned} \tag{5.3.8}$$

....

We can also consider the case in which f_j allow a series expansion in the form

$$\begin{aligned}
&f_j(x,\mathbf{y}(x))=\sum_{|\nu|\ge 1}\frac{\rho^{\nu_0}}{\nu_0!\nu_1!\ldots\nu_n!}f_{j\nu}(x)\mathbf{y}^\nu,\\
&\nu=(\nu_0,\nu_1,\ldots,\nu_n),\quad |\nu|=\nu_0+\nu_1+\ldots+\nu_n,\quad \mathbf{y}^\nu=y_1^{\nu_1}y_2^{\nu_2}\ldots y_n^{\nu_n},
\end{aligned} \tag{5.3.9}$$

where ν is a $(n+1)$-dimensional multi-index and the coefficients $f_{j\nu}(x)$ are continuous on $[0,a]$. We suppose that they also satisfy in this interval the inequality

$$\left|f_{j\nu}(x)\right| = \frac{|\nu|!}{\nu_0!\nu_1!...\nu_n!}\frac{Ar}{r_0^{\nu_0}r_1^{\nu_1}...r_n^{\nu_n}}, \quad j=\overline{1,n}, \quad |\nu| \in \mathfrak{N}, \tag{5.3.10}$$

A and r_k, $k=\overline{0,n}$ being determined constants.

Then the solution of the initial problem (5.1.2), (5.1.8) can be developed in a series expansion

$$y_j(x) = \sum_{|\nu|\geq 1} \varphi_{j\nu}(x)\rho^{\nu_0} y_{10}^{\nu_1} y_{20}^{\nu_2}...y_{n0}^{\nu_n}, \quad j=\overline{1,n}, \tag{5.3.11}$$

which is absolutely convergent in the domain determined by the inequality

$$\frac{|\rho|}{r_0} + \frac{|y_{10}|}{r_1} + ... + \frac{|y_{n0}|}{r_n} < \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \mathrm{e}^{-k(\alpha An+1)} . \tag{5.3.12}$$

In the particular case $f_{j\nu}(x) = a_{j\nu}x^{\nu_0}$, in the above series of y_j we also take $\varphi_{j\nu}(x) = c_{j\nu}x^{\nu_0}$. The coefficients $c_{j\nu}$ are obtained by identification.

Yet this procedure is very difficultly applied in practice. In the next section, dedicated to the linear equivalence method, we shall present an efficient way of deducing the coefficients $c_{j\nu}$ by using the normal LEM representation.

3.3 THE LINEAR EQUIVALENCE METHOD (LEM)

The linear equivalence method, or, briefly, LEM, was introduced to find convenient, both quantitative and qualitative representations of the solutions of non-linear ODSs via the methods in use for the linear ones. The method, initially introduced for first order polynomial differential systems , was extended to first order ODSs, with right side analytic with respect to the unknown functions. The case of polynomial operators involves some simplified formulae for the LEM representations and even more simplifications are emphasized in the case of constant coefficients.

Consider therefore the system

$$\dot{\mathbf{y}} = \mathbf{f}(t,\mathbf{y}), \qquad \mathbf{f}(t,\mathbf{y}) \equiv \left[f_j(t,\mathbf{y})\right]_{j=\overline{1,n}}, \quad \mathbf{y} \in \left(\mathrm{C}^1(\mathrm{I})\right)^n, \quad \mathrm{I} = [a,b] \subseteq \Re, \tag{5.3.13}$$

where $f_j(t,\mathbf{y})$ are analytic functions, uniformly with respect to $t \in \mathrm{I}$, i.e.

$$f_j(t,\mathbf{y}) = \sum_{|\mu|=1}^{\infty} f_{j\mu}(t)\mathbf{y}^{\mu}, \quad j=\overline{1,n}, \ \mu \in (\mathfrak{N} \cup \{0\})^n, \tag{5.3.14}$$

$\mu = (\mu_1, \mu_2, ..., \mu_n)$ is a multi-index, $|\mu| = \mu_1 + \mu_2 + ... + \mu_n$ and

$$\mathbf{y}^{\mu} \equiv y_1^{\mu_1} y_2^{\mu_2} ... y_n^{\mu_n} . \tag{5.3.15}$$

The coefficients $f_{j\mu}:\mathrm{I}\to\Re$ are supposedly at least of class $\mathrm{C}^0(\mathrm{I})$. The applications presented here deal only with ODS with null free term; this is why the sums in (5.3.14) are starting from 1 on.
This system may be also written putting into evidence the differential operator $\mathcal{F}(\mathbf{y})$,

$$\mathcal{F}(\mathbf{y})\equiv\dot{\mathbf{y}}-\mathbf{f}(t,\mathbf{y})=\mathbf{0}\,. \tag{5.3.16}$$

LEM considers an exponential mapping depending on n parameters, $\boldsymbol{\xi}\in\Re^n$, $\boldsymbol{\xi}=(\xi_1,\xi_2,\dots,\xi_n)$, namely

$$v(x,\boldsymbol{\xi})\equiv\mathrm{e}^{\langle\boldsymbol{\xi},\mathbf{y}\rangle},\quad \langle\boldsymbol{\xi},\mathbf{y}\rangle=\sum_{j=1}^{n}\xi_j y_j\,, \tag{5.3.17}$$

that associates to the above non-linear ODS two linear equivalents :

1. a linear PDE, always of first order with respect to x

$$\mathcal{L}v(x,\boldsymbol{\xi})\equiv\frac{\partial v}{\partial t}-\langle\boldsymbol{\xi},\mathbf{f}(x,\mathrm{D})\rangle v=0\,, \tag{5.3.18}$$

2. and a linear, while infinite, first order ODS

$$\frac{\mathrm{d}v_\gamma}{\mathrm{d}t}=\sum_{j=1}^{n}\gamma_j\sum_{|\mu|=1}^{\infty}f_{j\mu}(t)v_{\gamma+\mu-\mathbf{e}_j}\,,\quad \mathbf{e}_j=\left(\delta_i^j\right)_{i=\overline{1,n}},\quad |\gamma|\in\mathfrak{N}\,. \tag{5.3.19}$$

The operator $\mathcal{L}$ introduces the linear PDE (5.3.18), always of first order with respect to t; in (5.3.18), the formal operators

$$\langle\boldsymbol{\xi},\mathbf{f}(t,\mathrm{D})\rangle\equiv\sum_{j=1}^{n}\xi_j f_j\left(t,\mathrm{D}_\xi\right),\quad f_j\left(t,\mathrm{D}_\xi\right)\equiv\sum_{|\mu|=1}^{\infty}f_{j\mu}(t)\frac{\partial^{|\mu|}v}{\partial\xi_1^{\mu_1}\partial\xi_2^{\mu_2}\dots\partial\xi_n^{\mu_n}} \tag{5.3.20}$$

make sense on Exp-type spaces.
The LEM equivalent (5.3.18) was obtained by differentiating (5.3.17) with respect to t and replacing the derivatives $\dot{y}_j$ from the non-linear system (5.3.16).
The usual notation $f_j\left(t,D_\xi\right)$ stands for the differential polynomial associated to $f_j(t,\mathbf{y})$. The second LEM equivalent, the system (5.3.19), is obtained from the first one, by searching the unknown function v in the class of analytic with respect to $\boldsymbol{\xi}$ functions

$$v(t,\boldsymbol{\xi})=1+\sum_{|\gamma|=1}^{\infty}v_\gamma(t)\frac{\boldsymbol{\xi}^\gamma}{\gamma!}\,. \tag{5.3.21}$$

The LEM system (5.3.19) may be also written in matrix form

$$\mathfrak{S}\mathbf{V}\equiv\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}x}-\mathbf{A}(t)\mathbf{V}=\mathbf{0},\quad \mathbf{V}=\left(\mathbf{V}_j\right)_{j\in\mathfrak{N}},\quad \mathbf{V}_j=\left(v_\gamma\right)_{|\gamma|=j}\,. \tag{5.3.22}$$

The LEM matrix **A** has a special form, being always row-finite and, in the case of polynomial operators, also column-finite

$$\mathbf{A}(t) \equiv \begin{bmatrix} \mathbf{A}_{11}(t) & \mathbf{A}_{12}(t) & \mathbf{A}_{13}(t) & \dots & \mathbf{A}_{1,m-1}(t) & \mathbf{A}_{1m}(t) & \dots \\ \mathbf{0} & \mathbf{A}_{22}(t) & \mathbf{A}_{23}(t) & \dots & \mathbf{A}_{2,m-1}(t) & \mathbf{A}_{2m}(t) & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33}(t) & \dots & \mathbf{A}_{3,m-1}(t) & \mathbf{A}_{3m}(t) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{A}_{mm}(t) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{5.3.23}$$

The cells $\mathbf{A}_{ss}(t)$ on the main diagonal are square, of $s+1$ rows and columns, and are generated by the coefficients of the linear part of the operator – namely, those $f_{j\mu}(t)$ for which $|\mu| = 1$. The other cells $\mathbf{A}_{k,k+s}(t)$ contain only those $f_{j\mu}(t)$ with $|\mu| = s+1$. More precisely, the diagonal cells contain the coefficients of the linear part, on the next upper diagonal we find the coefficients of the second degree in **y** etc. In the case of polynomial operators of degree *m*, the associated LEM matrix is band-diagonal, the band being made up of *m* lines.

It should be mentioned that this particular form of the LEM matrix permits the calculus by block partitioning, which represents a considerable simplification.

Consider now for (5.3.13) or, equivalently, (5.3.16), the initial conditions

$$\mathbf{y}(t_0) = \mathbf{y}_0, \quad t_0 \in \mathrm{I}. \tag{5.3.24}$$

By LEM, they are transferred to

$$v(t_0, \xi) = \mathrm{e}^{\langle \xi, \mathbf{y}_0 \rangle}, \quad \xi \in \Re^n, \tag{5.3.25}$$

a condition that must be associated to the PDE , and

$$v_\gamma(t_0) = \mathbf{y}_0^\gamma, \quad |\gamma| \in \mathcal{N}, \tag{5.3.26}$$

indicating an initial condition for the system (5.3.19) or, equivalently, (5.3.22). For the matrix form, the initial conditions (5.3.26) become

$$\mathbf{V}(t_0) = \left(\mathbf{y}_0^\gamma\right)_{|\gamma| \in \mathcal{N}}. \tag{5.3.27}$$

Let us note that, in order to get back to the solutions of the polynomial Cauchy problem (5.3.16), (5.3.24), the PDE should be conveniently defined on some space of analytic with respect to ξ functions, uniformly for $t \in \mathrm{I}$. To this aim, we introduce

$$\mathcal{A}_n^k(\mathrm{I}) \equiv \left\{ v : \mathrm{I} \times \Re^n \to \Re; v(x, \xi) = \sum_{|\gamma| \geq 0} v_\gamma(x) \frac{\xi^\gamma}{\gamma!}, \left\| v_\gamma \right\|_k \leq K M^{|\gamma|}, |\gamma| \in \mathcal{N} \right\}. \tag{5.3.28}$$

where $\|\cdot\|$ is the "sup" norm in $\mathrm{C}^0(\mathrm{I})$ and $\|f\|_m = \max\left\{\left\|f^{(j)}\right\|, j = \overline{0,m}\right\}$ is the norm in $\mathrm{C}^m(\mathrm{I})$.

Another space may be similarly introduced, $\mathcal{B}_n^k(\mathrm{I})$ – the isomorphic with $\mathcal{C}_n^k(\mathrm{I})$ space of infinite vectors **V**, of components satisfying the same inequalities as in (5.3.28). The isomorphism is emphasized by the application $\tau : \mathcal{C}_n^k(\mathrm{I}) \to \mathcal{B}_n^k(\mathrm{I})$ that associates to v the infinite vector of the coefficients in the development, i.e. $\tau(v(t,\xi)) = \mathbf{V}(t)$.

The relationships among the above-introduced operators are suggestively explained in the following diagram

$$\begin{array}{ll} \mathcal{P} : \left(\mathrm{C}^1(\mathrm{I})\right)^n & \to \left(\mathrm{C}^0(\mathrm{I})\right)^n \\ \mathrm{e}^{\langle \xi, \bullet \rangle} \ \downarrow & \\ \mathcal{L} : \mathcal{C}_n^1(\mathrm{I}) & \to \mathcal{C}_n^0(\mathrm{I}) \\ \tau \quad \downarrow & \\ \mathcal{S} : \mathcal{B}_n^1(\mathrm{I}) & \to \mathcal{B}_n^0(\mathrm{I}) \end{array}$$

3.3.1. Solutions of non-linear ODSs by LEM

We note that the above diagram is not closed; yet, it may be used to turn back to the solutions of the polynomial system. In this respect, it was proved

Theorem 5.2. *Suppose that* $f_{j\mu} \in C^{\infty}(\mathrm{I})$. *Then the solution of the initial problem*, (5.3.27) *formally allows the representation*

$$\mathbf{V}(t) = \mathbf{\Pi}(t - t_0)\mathbf{V}(t_0), \tag{5.3.29}$$

where the infinite matrix $\mathbf{\Pi}$ *is given by*

$$\mathbf{\Pi}(t - t_0) \equiv \sum_{k \geq 0} \mathbf{A}^{(k)}(t_0) \frac{(t - t_0)^k}{k!} . \tag{5.3.30}$$

The matrices $\mathbf{A}^{(k)}$ *are determined by the recurrence*

$$\mathbf{A}^{(k)}(t) = \frac{\mathrm{d}\mathbf{A}^{(k-1)}}{\mathrm{d}t}(t) + \mathbf{A}^{(k-1)}(t)\mathbf{A}(t), \quad \mathbf{A}^{(0)}(t) = \mathbf{E} , \tag{5.3.31}$$

where **E** *is the infinite unit matrix. The components* v_γ *of* **V** *are consistent on intervals* $\mathrm{I}_{|\gamma|}$, $|\gamma| \in \mathfrak{N}$, *centred at* t_0, *whose length depends on* $f_{j\mu}$, *on* γ *and on* $|y_0|$.

In particular, the first n components of **V** *coincide with the Taylor series expansion of the solution of the Cauchy problem* (5.3.16), (5.3.24) *around* t_0.

The first n rows of $\mathbf{\Pi}$ represent in fact the inverse of the non-linear operator $\mathcal{F}$ in matrix form. Thus, the representation separates the contribution of the operator from that of the initial data.

Let us mention that, as the series form cannot be completely computed, if we wish to stop at some level k, all the involved computation up to this level is finite.

In the case of constant coefficients, the following representations were found:

Theorem 5.3. *If the coefficients* $f_{j\mu}$, $j=\overline{1,n}$*, are constant, then the solution of the non-linear initial problem* (5.3.16), (5.3.24)
i) coincides with the first n components of the infinite vector

$$\mathbf{V}(t)=\mathrm{e}^{\mathbf{A}(t-t_0)}\mathbf{V}_0\,, \tag{5.3.32}$$

where the exponential matrix

$$\mathrm{e}^{\mathbf{A}(t-t_0)}=\mathbf{E}+\frac{(t-t_0)}{1!}\mathbf{A}+\frac{(t-t_0)^2}{2!}\mathbf{A}^2+\ldots+\frac{(t-t_0)^k}{k!}\mathbf{A}^k+\ldots\,, \tag{5.3.33}$$

can be computed by block partitioning, each step involving finite sums;
ii) coincides with the series

$$y_j(t)=y_{j0}+\sum_{l=1}^{\infty}\sum_{|\gamma|=l}u_{j\gamma}(t)y_0^{\gamma},\quad j=\overline{1,n}\,, \tag{5.3.34}$$

where $u_{j\gamma}(t)$ *are solutions of the finite linear ODS*

$$\frac{\mathrm{d}\mathbf{U}_k}{\mathrm{d}t}=\mathbf{A}_{1k}^{\mathrm{T}}\mathbf{U}_1+\mathbf{A}_{2k}^{\mathrm{T}}\mathbf{U}_2+\ldots+\mathbf{A}_{kk}^{\mathrm{T}}\mathbf{U}_k\,,\quad k=\overline{1,l},\quad \mathbf{U}_s=\left(u_\gamma(t)\right)_{|\gamma|=s}\,, \tag{5.3.35}$$

that satisfy the Cauchy conditions

$$\mathbf{U}_1(t_0)=\mathbf{e}_j\,,\quad \mathbf{U}_s(x_0)=\mathbf{0},\, s=\overline{2,l}\,. \tag{5.3.36}$$

The representation is very much alike a solution of a linear ODS with constant coefficients. There is more: the computation is even easier due to the fact that the eigenvalues of the diagonal cells are always known . The representation (5.3.34) is called the *normal LEM representation* and was used in many applications requiring the qualitative behavior of the solution.

3.3.2. New LEM representations in the case of polynomial coefficients

Suppose now that the coefficients $f_{j\mu}$ of the non-linear operator are also polynomials, of maximum degree q, written in the form

$$f_{j\mu}(t)=\sum_{k=0}^{q}f_{j\mu}^{k}(t-t_0)^k,\quad j=\overline{1,n},\ |\mu|\in\mathfrak{N}\,. \tag{5.3.37}$$

Then, the linear equivalent system becomes

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}=\left[\mathbf{A}_0+(t-t_0)\mathbf{A}_1+(t-t_0)^2\mathbf{A}_2+\ldots+(t-t_0)^q\mathbf{A}_q\right]\mathbf{V},$$
$$\mathbf{V}=\left(\mathbf{V}_j\right)_{j\in\mathfrak{N}},\quad \mathbf{V}_j=\left(v_\gamma\right)_{|\gamma|=j}. \tag{5.3.38}$$

Let us mention that in (5.3.38) the matrices $\mathbf{A}_k$ are all of them constant and, obviously, of LEM construction. Each of the LEM matrices $\mathbf{A}_k$ is set up by using only the coefficients $f_{j\mu}^{k}$. One can formally write (5.3.38) in integral form

$$\mathbf{V}(t)=\mathbf{V}_0+\int_{t_0}^{t}\left[\mathbf{A}_0+(u-t_0)\mathbf{A}_1+(u-t_0)^2\mathbf{A}_2+\ldots+(u-t_0)^q\mathbf{A}_q\right]\mathbf{V}(u)\mathrm{d}u\,, \tag{5.3.39}$$

and apply to this linear integral equation the successive approximations

$$\mathbf{V}^{(0)} = \mathbf{V}_0,$$

$$\mathbf{V}^{(l)}(t) = \mathbf{V}_0 + \int_{t_0}^{t} \left[\sum_{j=0}^{q} (u - t_0)^q \mathbf{A}_q \right] \mathbf{V}^{(l-1)}(u) \mathrm{d}u. \tag{5.3.40}$$

With these preparations, using the same techniques as in Theorem 5.1, one can obtain LEM representations in the case of polynomial coefficients.

The representation (5.3.29) is more suitable for numerical applications, while the normal LEM representation suits better to study the qualitative behavior of the solution.

LEM was used to many applications: to set up a transport matrix for REBs (relativistic electron beams), to get asymptotic estimations for the solution of Troesch's plasma problem, to a qualitative study of the oscillatory solution of Belousov-Zhabotinskij's chemical reaction, to the Lotka-Volterra prey-predator model and even in the theory of graphs.

The most pertinent results were obtained in the frame of mechanics, studying the Bernoulli-Euler bar and the non-linear rigid pendulum; some of them will be presented in this book.

It must be mentioned that during the last several years, the interest in applying LEM to various mechanical and technical problems was continuously increasing. Thus, the application of LEM was extended to modern high-tech modelling for shape memory alloys for non-linear mesoscopic materials and to domains like damping in machine tools .

LEM was applied to the non-linear coupled pendulum, comparing the LEM representations and the cnoidal ones, comparison also sustained by the numerical results obtained via wavelets. In it is suggested the application of LEM to equations like Korteweg de Vries. In an excellent book, recently appeared, there are opened the perspectives of applying LEM to non-linear models in nanotechnologies. This may give rise to a fruitful feedback between the development of the method itself, on the one hand, and the specific results obtained by applying it, on the other hand.

4. Applications

Application 5.1

Problem. Study the motion of a discrete mechanical system formed by two particles P_1 and P_2 of masses m_1 and m_2, respectively, subjected to the reciprocal action of forces of Newtonian attraction (*the problem of the two particles*).

Mathematical model. Consider the particles P_1 and P_2 of position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, acted upon by forces of Newtonian attraction (Fig.5.1)

$$\mathbf{F}_{12} = f \frac{m_1 m_2}{r^3} \mathbf{r} = -\mathbf{F}_{21}, \tag{a}$$

where $\mathbf{r} = \overrightarrow{P_1 P_2}$. Newton's equations of motion are

$$m_1\ddot{\mathbf{r}}_1 = f\frac{m_1 m_2}{r^3}\mathbf{r}\,,\ m_2\ddot{\mathbf{r}}_2 = -f\frac{m_1 m_2}{r^3}\mathbf{r}\,, \tag{b}$$

$$\ddot{\mathbf{r}}_1 = f\frac{m_2}{r^3}\mathbf{r}\,,\quad \ddot{\mathbf{r}}_2 = -f\frac{m_1}{r^3}\mathbf{r}\,. \tag{c}$$

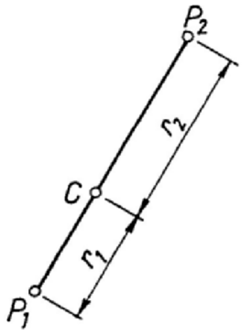

Figure 5. 1. The problem of the two particles

Noting that $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1$ and subtracting the equations (c) we obtain

$$\ddot{\mathbf{r}} = -f\frac{m_1 + m_2}{r^3}\mathbf{r}\,, \tag{d}$$

hence the equation of motion of the particle P_2 with respect to the particle P_1; analogously, we may determine the equation of motion of the particle P_1 with respect to the particle P_2.

Solution. The equation (d) has been considered in Appl. 1.17 for Kepler's problem: motion of a planet of mass $m_2 = m$ subjected to the action of a central force of Newtonian attraction, the Sun, of mass $m_1 = M$, considered fixed. The equation (d) becomes

$$\ddot{\mathbf{r}} = -f\frac{M\left(1+\dfrac{m}{M}\right)}{r^3}\mathbf{r}\,. \tag{e}$$

The first two Kepler's laws are verified. The planet describes an ellipse, the Sun being at one of the foci; but also the Sun describes an ellipse, the planet being at one of its foci. Concerning the third law, we are led to

$$\frac{T^2}{a^3} = \frac{4\pi^2}{fM}\frac{1}{1+\dfrac{m}{M}} = \frac{4\pi^2}{fM}\left(1+\frac{m}{M}\right)^{-1} = \frac{4\pi^2}{fM}\left(1-\frac{m}{M}+\ldots\right);$$

this law is verified with good approximation too, because $m / M << 1$.
This problem plays an important rôle both for the macrocosm (in celestial mechanics) an for the microcosm (in atomic mechanics).

Application 5.2

Problem. A man goes along the straight line $O'y'$ with a uniform velocity v_1. At the moment $t = 0$ he is at O' and calls his dog; that one runs towards the master with the uniform velocity $v_2 = kv_1$, $k > 1$, so that it is directed at any moment towards the man. Determine the trajectory of the dog and the interval of time in which it reaches the master (*problem of the meeting*). Discussion.

Mathematical model. We model mathematically the man and the dog by two particles P_1 and P_2, respectively; the velocity $\mathbf{v}_2$ of the particle P_2 is collinear with the vector $\overrightarrow{PP_1}$, tangent to its trajectory. We choose a fixed frame of reference $O'x'y'$ linked to the initial position O' of the particle P_1 and a movable frame of reference P_1xy, linked to a momentary position of the particle P_1, in uniform translation with respect to the fixed frame; in fact, its motion is specified by the equations (we admit that the two particles start from the points $P_1^0 \equiv O'$ and P_2^0 at the initial moment $t = 0$) (Fig.5.2)

$$x' = x\,,\ y' = v_1 t + y\,. \tag{a}$$

We notice that $\mathbf{v}_2 = v_2 \operatorname{vers} \overrightarrow{P_2P_1}$, obtaining the differential equations which determine the trajectory of the particle P_2 in the movable frame in the form ($\mathbf{v}_2$ has the components $\dot{x}$ and $\dot{y}$)

$$\dot{x} = -\frac{v_2 x}{\sqrt{x^2 + y^2}}\,,\ \dot{y} = -v_1 - \frac{v_2 y}{\sqrt{x^2 + y^2}}\,. \tag{b}$$

Solution. Dividing the two equations (b), member by member (we eliminate the time t), we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{v_1}{v_2}\sqrt{1 + \left(\frac{y}{x}\right)^2}\,. \tag{c}$$

Noting that $\mathrm{d}y / \mathrm{d}x = y / x + (x / \mathrm{d}x)\mathrm{d}(y / x)$ and integrating, we may write the equation of the trajectory with respect to the movable frame in the form

$$y = \frac{x}{2}\left[\left(\frac{x}{a}\right)^{v_1 / v_2} - \left(\frac{x}{a}\right)^{-v_1 / v_2}\right]. \tag{d}$$

Eliminating $\sqrt{x^2+y^2}$ between the two differential equations, we may write $\dot{x}=(v_1+\dot{y})(x/y)$; taking into account the previous observation and (d), it results

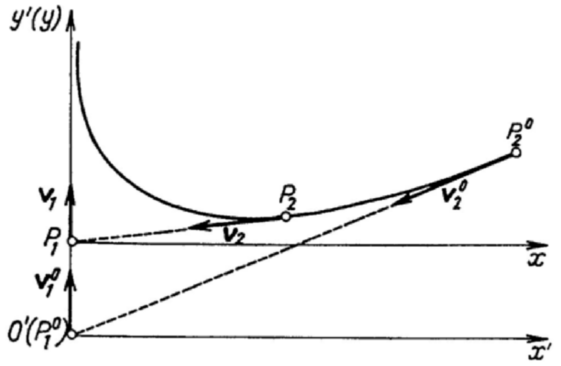

Figure 5. 2. The problem of the meeting

$$\mathrm{d}t=\frac{1}{v_1}\left(\frac{y}{x}\mathrm{d}x-\mathrm{d}y\right)=-\frac{x}{v_1}\mathrm{d}\left(\frac{y}{x}\right)=-\frac{a}{2\sqrt{2}}\left[\left(\frac{x}{a}\right)^{v_1/v_2}+\left(\frac{x}{a}\right)^{-v_1/v_2}\right]\mathrm{d}\left(\frac{x}{a}\right),$$

whence , assuming that $v_1\neq v_2$,

$$t=t_1-\frac{x}{2}\left[\frac{1}{v_1+v_2}\left(\frac{x}{a}\right)^{v_1/v_2}-\frac{1}{v_1-v_2}\left(\frac{x}{a}\right)^{-v_1/v_2}\right]. \tag{e}$$

Calculating $x=x(t)$ and then $y=y(t)$, we obtain the parametric equations of motion of the particle P_2 on the trajectory. The equation of the trajectory with respect to the fixed frame is of the form

$$y'=v_1t_1+\frac{v_2x'}{2}\left[\frac{1}{v_1+v_2}\left(\frac{x'}{a}\right)^{v_1/v_2}+\frac{1}{v_1-v_2}\left(\frac{x'}{a}\right)^{-v_1/v_2}\right]. \tag{f}$$

The constants a an t_1 are specified by (d) and (e), if we put the initial conditions $x=x_0=x_0'$, $y=y_0=y_0'$ for $t=0$. For $x\to 0$ we have $y\to 0$, $t\to t_1$ if $v_1<v_2$, and $y\to-\infty$, $t\to\infty$ if $v_1>v_2$. As a consequence, if $v_1<v_2$, then the particle P_2 meets the particle P_1 at the moment $t=t_1$ at the point o co-ordinates $x'=0$, $y'=v_1t_1$ with respect to the fixed frame. If $v_1=v_2$, then the two particles do not meet. The distance between them is given by $\overline{P_1P_2}^2=x^2+y^2$, so that

$$\overline{P_2P_1} = \frac{x}{2}\left[\left(\frac{x}{a}\right)^{v_1/v_2} + \left(\frac{x}{a}\right)^{-v_1/v_2}\right] = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1+v_1/v_2} + \left(\frac{x}{a}\right)^{1-v_1/v_2}\right]; \tag{g}$$

the minimum of this distance is obtained for

$$x = a\left(\frac{v_1 - v_2}{v_1 + v_2}\right)^{v_2/2v_1}, \tag{h}$$

so that

$$\overline{P_2P_1}_{\min} = \frac{av_2}{\sqrt{v_1^2 - v_2^2}}\left(\frac{v_1 - v_2}{v_1 + v_2}\right)^{v_2/2v_1} \tag{i}$$

at the moment

$$t = t_1 + \frac{2av_1v_2}{\left(v_1^2 - v_2^2\right)\sqrt{v_1^2 - v_2^2}}\left(\frac{v_1 - v_2}{v_1 + v_2}\right)^{v_2/2v_1}. \tag{j}$$

In particular, for $y_0 = 0$, we obtain $x = a$ and

$$t_1 = -\frac{av_2}{v_1^2 - v_2^2}, \tag{k}$$

being led to

$$y' = \frac{ak}{2}\left[\frac{1}{k+1}\left(\frac{x'}{a}\right)^{(k+1)/k} - \frac{1}{k-1}\left(\frac{x'}{a}\right)^{(k-1)/k} + \frac{2}{k^2 - 1}\right]. \tag{l}$$

For instance, for $k = 2$ the trajectory is an arc of semicubic parabola

$$y' = \frac{a}{3}\left[\left(\frac{x'}{a}\right)^{3/2} - 3\left(\frac{x'}{a}\right)^{1/2} + 2\right] = \frac{a}{3}\left(\sqrt{\frac{x'}{a}} - 1\right)^2\left(\sqrt{\frac{x'}{a}} + 2\right). \tag{m}$$

If $v_1 = v_2 = v$, $k = 1$, then the equation of the trajectory with respect to the movable frame is given by

$$y = \frac{1}{2a}\left(x^2 - a^2\right), \tag{n}$$

the motion being specified by

$$t = t_1 - \frac{a}{2v}\left[\frac{1}{2}\left(\frac{x}{a}\right)^2 + \ln\frac{x}{a}\right]; \tag{o}$$

as well

$$\overline{P_2 P_1} = \frac{1}{2a}\left(x^2 + a^2\right). \tag{p}$$

The two particles do not meet, the minimal distance between them being obtained for $x = 0$ at the moment $t \to \infty$; it is equal to $a/2$.

Application 5.3

Problem. Determine the first integrals in the motion of a discrete mechanical system S , expressed by Lagrange's equations in the space o configurations

Mathematical model. In case of discrete mechanical system S of n particles, the equations of motion of *the representative point* P in *the space of configurations* Λ_s are of the form (see Appl.6.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial T}{\partial q_k} = Q_k\,,\ k = \overline{1, s}\,, \tag{a}$$

where T is *the kinetic energy*, Q_k are *the generalized forces* and s is the number of *degrees of freedom* of the system S .

Solution. Multiplying by $\dot{q}_k$ and summing from 1 to s, we obtain

$$\sum_{k=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\dot{q}_k - \frac{\partial T}{\partial q_k}\dot{q}_k\right] = \sum_{k=1}^{s} Q_k \dot{q}_k\ .$$

But

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\dot{q}_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\dot{q}_k\right) - \frac{\partial T}{\partial \dot{q}_k}\ddot{q}_k\ .$$

Taking into account the relation (c) in Appl.6.3 and applying *Euler's theorem* concerning homogeneous functions, we may write

$$\sum_{k=1}^{s}\frac{\partial T}{\partial \dot{q}_k}\dot{q}_k = \sum_{k=1}^{s}\left(\frac{\partial T_2}{\partial \dot{q}_k}\dot{q}_k + \frac{\partial T_1}{\partial \dot{q}_k}\dot{q}_k\right) = 2T_2 + T_1\,,$$

where we noticed that T_2 and T_1 are homogeneous forms of the second and first degree with respect to the generalized velocities, respectively, while T_0 is a constant with respect to these velocities. As well,

$$\frac{\mathrm{d}T}{\mathrm{d}t}=\sum_{k=1}^{s}\left(\frac{\partial T}{\partial q_k}\dot{q}_k+\frac{\partial T}{\partial \dot{q}_k}\ddot{q}_k+\dot{T}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(T_2+T_1+T_0\right).$$

Finally, we may write

$$\mathrm{d}\left(T_2-T_0\right)=\sum_{k=1}^{s}Q_k\,\mathrm{d}q_k-\dot{T}\mathrm{d}t\,. \tag{b}$$

If there exists a function W so that

$$\mathrm{d}W=\sum_{k=1}^{s}Q_k\,\mathrm{d}q_k-\dot{T}\mathrm{d}t\,,$$

then

$$T_2-T_0-W=h=\text{const} \tag{c}$$

is a first integral of Lagrange's equations (*the first integral of Painlevé*). We observe that $W=W\left(q_1,q_2,\ldots,q_s;t\right)$ and cannot depend on the generalized velocities too. Indeed, in this case, in the total derivative $\mathrm{d}W/\mathrm{d}t$ would appear also the generalized acceleration $\ddot{q}_k$; but the given forces cannot depend on accelerations (second principle of Newton), hence neither the generalized forces cannot depend on the generalized accelerations, so that neither in the expression $\mathrm{d}W/\mathrm{d}t$ cannot appear such accelerations. We may write

$$\frac{\mathrm{d}W}{\mathrm{d}t}=\sum_{k=1}^{s}\frac{\partial W}{\partial q_k}\dot{q}_k+\dot{W}=\sum_{k=1}^{s}Q_k\dot{q}_k-\left(\dot{T}_2+\dot{T}_1+\dot{T}_0\right);$$

because only the functions W and T_0 do not depend on the generalized velocities, it results that $\dot{W}=-\dot{T}_0$. Differentiating partially the relation (c) with respect to the time, we obtain also $\dot{T}_2=0$. We may thus state that Painlevé's first integral does not depend explicitly on time, neither in the case of rheonomic constraints and of forces which depend explicitly on time.

In case of quasi-conservative generalized forces $Q_k=\partial U/\partial q_k\,,U=U\left(q_1,q_2,\ldots,q_s;t\right)$, we may introduce the kinetic potential $\mathsf{L}=T+U$, and the equations (a) take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathsf{L}}{\partial \dot{q}_k}\right)-\frac{\partial \mathsf{L}}{\partial q_k}=0\,,\ k=\overline{1,s}\,. \tag{d}$$

We follow now an analogous procedure. We multiply the equation by $\dot{q}_k$ and sum for k from 1 to s; we obtain

$$\sum_{k=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathsf{L}}{\partial \dot{q}_k}\right)\dot{q}_k-\frac{\partial \mathsf{L}}{\partial q_k}\dot{q}_k\right]=0\,.$$

By an analogous calculation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{s}\frac{\partial \mathsf{L}}{\partial \dot{q}_k}\dot{q}_k - \mathsf{L}\right) + \dot{\mathsf{L}} = 0 .$$

Hence, if $\dot{\mathsf{L}} = 0$, we may write the first integral

$$\sum_{k=1}^{s}\frac{\partial \mathsf{L}}{\partial \dot{q}_k}\dot{q}_k - \mathsf{L} = h = \text{const} , \tag{e}$$

called the first integral of Jacobi. Noting that $\mathsf{L} = T_2 + T_1 + T_0 + U$, we may also write

$$\mathsf{E} = T_2 - T_0 - U = h = \text{const} , \tag{f}$$

where E is *the generalized mechanical energy*. In case of scleronomic constraints (which do no depend explicitly on time), we have $T_1 = T_0 = 0$, while $T = T_2$, so that $\mathsf{E} = T - U = E$; the generalized mechanical energy coincides with the mechanical energy E. We get again *the conservation theorem of mechanical energy*.
If $\partial T / \partial qk = 0$, then the respective co-ordinate q_k is called a *hidden co-ordinate*; in this case, the corresponding equation (a) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = Q_k .$$

If we have $Q_k = 0$ too, then we obtain

$$\frac{\partial T}{\partial \dot{q}_k} = C_k , \tag{g}$$

which is also a first integral.
If $\partial \mathsf{L} / \partial q_k = 0$, then the respective co-ordinate is called an *ignorable co-ordinate*, and the equations (d) lead to the first integral

$$\frac{\partial \mathsf{L}}{\partial \dot{q}_k} = c_k ; \tag{h}$$

in fact, $\partial \mathsf{L} / \partial \dot{q}_k$ is just *the generalized momentum* p_k (see Appl. 6.3), so that $p_k = c_k$.

Application 5.4

Problem. Determine the first integrals in the motion of a discrete mechanical system S, expressed with the aid of Hamilton's equations in the phase space.

Mathematical model. In case of a discrete mechanical system S of n particles, the equations of motion of the representative point P in *the phase space* Γ_{2s} may be written in the form (see Appl.6.3)

$$\dot{q}_k = \frac{\partial H}{\partial p_k}\,;\ \dot{p}_k = -\frac{\partial H}{\partial q_k}\,,\ k = \overline{1,s}\,, \tag{a}$$

where $H = H(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$ is *Hamilton's function*, while s is the number of degrees of freedom of the system S .

Solution. The total derivative of the Hamiltonian H takes the form

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_{k=1}^{s}\left(\frac{\partial H}{\partial q_k}\dot{q}_k + \frac{\partial H}{\partial p_k}\dot{p}_k\right) + \dot{H}\,.$$

If the canonical equations are verified, then it results

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \dot{H}\,; \tag{b}$$

we may thus state that along the trajectory of the representative point P (when Hamilton's equations, which govern the motion of this point, take place) the total derivative of the function H does not depend explicitly on time (e.g., in case of scleronomic constraints), therefore $\mathrm{d}H / \mathrm{d}t = 0$ and H is a first integral of the system of canonical equations. Taking into account the definition of Hamilton's function (see Appl.6.3), we may write

$$H = \sum_{k=1}^{s} p_k \dot{q}_k - \mathsf{L} = \sum_{k=1}^{s}\frac{\partial \mathsf{L}}{\partial \dot{q}_k}\dot{q}_k - (T + U) = \sum_{k=1}^{s}\frac{\partial T}{\partial \dot{q}_k}\dot{q}_k - (T + U)$$
$$= 2T_2 + T_1 - (T_2 + T_1 + T_0 + U)$$

so that

$$H = T_2 - T_0 - U = \mathsf{E}\,, \tag{c}$$

where E is *the generalized mechanical energy* (see Appl. 5.3). We find thus again *the first integral of Jacobi* in canonical co-ordinates. Indeed, the link between the functions H and L puts in evidence the equivalence of the conditions $\dot{\mathsf{L}} = 0$ and $\dot{H} = 0$. If the constraints are scleronomic, we get $H = E = \text{const}$, hence *the conservation theorem of the mechanical energy* (as in case of Lagrange's equations).

Let be $\varphi(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$ and $\psi(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$ two functions of class C^1 ; the expression

$$(\varphi, \psi) = \begin{vmatrix} \dfrac{\partial \varphi}{\partial q_k} & \dfrac{\partial \varphi}{\partial p_k} \\ \dfrac{\partial \psi}{\partial q_k} & \dfrac{\partial \psi}{\partial p_k} \end{vmatrix} = \sum_{k=1}^{s}\left(\frac{\partial \varphi}{\partial q_k}\frac{\partial \psi}{\partial p_k} - \frac{\partial \varphi}{\partial p_k}\frac{\partial \psi}{\partial q_k}\right) \tag{d}$$

is called *the Poisson bracket* corresponding to the functions φ and ψ .

The obvious properties

$$(\varphi, C) = 0\,,\ C = \text{const}\,,\ (\varphi, \psi) = -(\psi, \varphi)\,,\ (-\varphi, \psi) = -(\varphi, \psi) \tag{e}$$

take place; taking into account the definition relation (d) it may be shown that

$$\frac{\partial}{\partial t}(\varphi, \psi) = (\dot{\varphi}, \psi) + (\varphi, \dot{\psi})\,. \tag{f}$$

Let us consider now a first integral of the canonical system (a), hence a function of class C^1, which is identically reduced to a constant if one replaces the generalized co-ordinates q_k and the generalized momenta p_k by the solutions of this system. Hence, $f = \text{const}$ along the trajectory of the representative point P; it results that $\mathrm{d}f / \mathrm{d}t = 0$ or

$$\sum_{k=1}^{s} \left(\frac{\partial f}{\partial q_k} \dot{q}_k + \frac{\partial f}{\partial p_k} \dot{p}_k \right) + \dot{f} = 0\,.$$

Because the equations (a) take place, we may also write

$$\sum_{k=1}^{s} \left(\frac{\partial f}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial H}{\partial q_k} \right) + \dot{f} = 0$$

or

$$(f, H) + \dot{f} = 0\,. \tag{g}$$

Hence, if f is a first integral, then the relation (g) takes place. Reciprocally, supposing that the relation (g) holds, let us write *the sequence of Lagrange-Charpit differential equations* attached to this partial derivative equation of first order

$$\frac{\mathrm{d}q_1}{\dfrac{\partial H}{\partial p_1}} = \frac{\mathrm{d}q_2}{\dfrac{\partial H}{\partial p_2}} = \ldots = \frac{\mathrm{d}q_s}{\dfrac{\partial H}{\partial p_s}} = \frac{\mathrm{d}p_1}{-\dfrac{\partial H}{\partial q_1}} = \frac{\mathrm{d}p_2}{-\dfrac{\partial H}{\partial q_2}} = \ldots = \frac{\mathrm{d}p_s}{-\dfrac{\partial H}{\partial q_1}} = \frac{\mathrm{d}t}{1}\,; \tag{h}$$

but this sequence is just the system of canonical equations (a). We may thus state the relation (g) represents the necessary and sufficient condition so that the function f be a first integral of Hamilton's equations.

By partial differentiation of the relation (g) with respect to time, taking into account the property (f), we obtain

$$\frac{\partial}{\partial t}(f, H) + \ddot{f} = (\dot{f}, H) + (f, \dot{H}) + \ddot{f} = 0\,.$$

If $\dot{H} = 0$, then we have

$$\left(\dot{f}, H\right) + \frac{\partial}{\partial t}\dot{f} = 0 ; \tag{i}$$

hence, if H and f are first integrals of the canonical system, then $\dot{f}$ is also a first integral of this system (*Poisson's theorem*). Analogously, $\ddot{f}$, $\dddot{f}$, … are first integrals too.

If $\varphi(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$, $\psi(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$ and $\chi(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$ are functions of class C^2, then *the Poisson-Jacobi identity*

$$((\varphi, \psi), \chi) + ((\psi, \chi), \varphi) + ((\chi, \varphi), \psi) = 0 , \tag{j}$$

expressed by means of Poisson's brackets, holds true; indeed, using the definition relation (d), one obtains mixed derivatives of second order of the functions φ, ψ and χ, the coefficients of which vanish. Let us suppose now that φ and ψ are first integrals of the system (a); then the relations

$$(\varphi, H) + \dot{\varphi} = 0 , \ (\psi, H) + \dot{\psi} = 0 \tag{k}$$

take place. If $\chi = H$ the Poisson-Jacobi identity (j) becomes

$$((\varphi, \psi), H) + ((\psi, H), \varphi) + ((H, \varphi), \psi) = 0 .$$

Taking into account (k) and the properties (e), it results

$$((\varphi, \psi), H) + \frac{\partial}{\partial t}(\varphi, \psi) = 0 . \tag{l}$$

Hence, if φ and ψ are first integrals of the canonical system (relations (k)), then their Poisson bracket (φ, ψ) is a first integral of this system (relation (l)) (*Jacobi-Poisson theorem*).

Assuming that φ, ψ and H are first integrals of the system (a) and using Poisson's theorem and the Poisson-Jacobi theorem, one may obtain various first integrals for this system i.e.: $\partial(\varphi, \psi)/\partial t$, $((\varphi, \psi), H)$, (φ, H), (ψ, H), $(\dot{\varphi}, \psi)$, $(\varphi, \dot{\psi})$ etc. We notice that one may obtain at the most $2s$ distinct first integrals (linear independent), any other first integral being a linear combination of the other ones. Often, we find again a first integral previously obtained or a combination of such first integrals or we obtain a constant (which may be zero).

If $\partial H / \partial q_k = 0$, then the corresponding co-ordinate is called cyclic co-ordinate; in this case, the second sequence of equations (a) leads to $p_k = c_k = \text{const}$. Let us suppose that the co-ordinates q_1, q_2, …, q_h, $h \leq s$, are cyclic co-ordinates. In this case

$$p_k = c_k , \ k = \overline{1, h} , \tag{m}$$

and we have h first integrals, while

$$H = H(q_{h+1}, q_{h+2}, \ldots, q_s, c_1, c_2, \ldots, c_h, p_{h+1}, p_{h+2}, \ldots, p_s; t).$$

The system of canonical equations (a) is reduced to

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \ \dot{p}_j = -\frac{\partial H}{\partial q_j}, \ j = \overline{k+1, s}, \tag{n}$$

hence to a system of $2(s-h)$ equations with $2(s-h)$ unknown functions $q_j = q_j(t)$, $p_j = p_j(t)$, $j = \overline{h+1, s}$. The functions once determined are introduced in the Hamiltonian H, which becomes thus a function depending only on the time t. There remain the equations

$$\mathrm{d}q_k = \frac{\partial H}{\partial q_k} \mathrm{d}t, \ k = \overline{1, h}, \tag{o}$$

which specify the cyclic co-ordinates $q_k = q_k(t)$, $k = \overline{1, h}$.

If $h = s$, hence if all the co-ordinates are cyclic $q_k = q_k(t)$, $k = \overline{1, s}$, then we have

$$p_k = c_k, \ k = \overline{1, s}, \tag{p}$$

hence s first integrals, the Hamiltonian being thus of the form $H = H(c_1, c_2, \ldots, c_s; t)$, hence a function of time. The cyclic co-ordinates are thus given by

$$q_k = \int \frac{\partial H}{\partial q_k} \mathrm{d}t + \gamma_k, \ \gamma_k = \mathrm{const}, \ k = \overline{1, s}. \tag{q}$$

Particularly, if H is a first integral $(\dot{H} = 0)$, denoting $\partial H / \partial c_k = \omega_k = \mathrm{const}$, we have

$$q_k = \omega_k t + \gamma_k, \ k = \overline{1, s}. \tag{r}$$

If $s = 1$ one obtains the equation of motion on a circle, q_1 being an angle and ω_1 the angular velocity; the denomination of cyclic co-ordinate is just justified. Hence, the integration of the canonical system (a) is equivalent to the finding of a transformation of co-ordinates so that all generalized co-ordinates be cyclic.

Application 5.5

Problem. Determine the first integrals in the motion of a discrete mechanical system S, expressed by Hamilton's equations in the phase space, by the Hamilton-Jacobi method.

Mathematical model. In the case of a discrete mechanical system S of n particles, the equations of motion of the representative point P in *the phase space* Γ_{2s} are written in the form (see Appl.6.3)

$$\dot{q}_k = \frac{\partial H}{\partial p_k}\,;\ \dot{p}_k = -\frac{\partial H}{\partial q_k}\,,\ k = \overline{1,s}\,, \tag{a}$$

where $H = H(q_1, q_2, \dots, q_s, p_1, p_2, \dots, p_s; t)$ is *Hamilton's function*, while s is the number of degrees of freedom of the system S .

Solution. Let us build up the partial differential equation

$$\dot{S} + H\left(q_1, q_2, \dots, q_s, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \dots, \frac{\partial S}{\partial q_s}; t \right) = 0\,, \tag{b}$$

where we replace the generalized momenta p_k by the partial derivatives of first order $\partial S / \partial q_k$, $k = \overline{1,s}$, in the expression of the Hamiltonian. We assume that $S = S(q_1, q_2, \dots, q_s; t; a_1, a_2, \dots, a_s)$ is a *complete integral* of the equation (b) (an integral which contains s *essential constants of integration* a_1, a_2, …, a_s and which may be obtained, for instance, as a combination of s particular integrals), which verifies the condition

$$\det\left[\frac{\partial^2 S}{\partial q_j \partial a_k} \right] \neq 0\,. \tag{c}$$

The partial differential equation (b) is called *the Hamilton-Jacobi equation* or the equation in S .

Let us set up the sequences of relations

$$\frac{\partial S}{\partial a_k} = b_k\,,\ \frac{\partial S}{\partial q_k} = p_k\,,\ k = \overline{1,s}\,, \tag{d}$$

where b_1, b_2, …, b_s are arbitrary constants. The total derivatives of these sequences with respect to time (condition of first integral) lead to

$$\frac{\partial^2 S}{\partial t \partial a_k} + \sum_{j=1}^{s} \frac{\partial^2 S}{\partial q_j \partial a_k} \dot{q}_j = 0\,,\ \frac{\partial^2 S}{\partial t \partial q_k} + \sum_{j=1}^{s} \frac{\partial^2 S}{\partial q_j \partial q_k} \dot{q}_j = \dot{p}_k\,,\ k = \overline{1,s}\,. \tag{e}$$

As well, noting that by introducing the complete integral S in the equation (b) we obtain an identically zero expression (which does not depends on a_k and q_k) the equation (b) leads to

$$\frac{\partial^2 S}{\partial a_k \partial t} + \sum_{j=1}^{s} \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial a_k \partial q_j} = 0\,,\ \frac{\partial^2 S}{\partial q_k \partial t} + \sum_{j=1}^{s} \frac{\partial H}{\partial p_j} \frac{\partial^2 S}{\partial q_k \partial q_j} + \frac{\partial H}{\partial p_k} = 0\,,\ k = \overline{1,s}\,. \tag{f}$$

Subtracting the relations (e) and (f) member by member an noting that S is a function of class C^2 (the mixed derivatives of the second order do not depend on the order of differentiation, corresponding to the *theorem of Schwartz*), we obtain the conditions of first integral, equivalent to (e), in the form

$$\sum_{j=1}^{s} \frac{\partial^2 S}{\partial q_j \partial a_k}\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right) = 0\,,\ \sum_{j=1}^{s} \frac{\partial^2 S}{\partial q_j \partial q_k}\left(\dot{q}_j - \frac{\partial H}{\partial p_j}\right) = \dot{p}_k + \frac{\partial H}{\partial q_k}\,,\ k = \overline{1,s}\,. \tag{g}$$

If the canonical equations (a) take place, the conditions (g) are identically verified. Let us assume now that the relations (g) take place. The first of these relations may be considered as a system of homogeneous algebraic linear equations in the parentheses $\dot{q}_j - \partial H / \partial p_j$; noting that the determinant (c) of the coefficients is non-zero, it results that we can have only vanishing solutions, corresponding the first subsystem (a) of equations of Hamilton. If we replace in the second relation (g), we find the second subsystem (a). We may thus state:

The sequences of relations (d) represent $2s$ first integrals of the canonical system (a) (*Hamilton-Jacobi theorem*). The first sequence of relations (d) specifies the trajectory of the representative point in the configuration space Λ_s (the condition (c) allows to apply the theorem of implicit functions for the determination of the generalized co-ordinates), while the second sequence of relations (d) determines the generalized momenta, hence also the trajectory of the representative point in the phase space Γ_{2s}.

The Hamilton-Jacobi method may be simplified in some particular cases. Thus, if $\dot{H} = 0$ (e.g. in case of scleronomic constraints) the equation (b) leads to $\ddot{S} = 0$, where, by integration,

$$S = -ht + \overline{S}(q_1, q_2, \ldots, q_s; a_1, a_2, \ldots, a_s), \tag{h}$$

where we take, for instance, $a_s = h$. The Hamilton-Jacobi equation takes the reduced form

$$H\left(q_1, q_2, \ldots, q_s; \frac{\partial \overline{S}}{\partial q_1}, \frac{\partial \overline{S}}{\partial q_2}, \ldots, \frac{\partial \overline{S}}{\partial q_s}\right) = h\,, \tag{i}$$

hence the sequences of first integrals are written in the form

$$\frac{\partial \overline{S}}{\partial a_j} = b_j\,,\ j = \overline{1, s-1}\,,\ \frac{\partial \overline{S}}{\partial a_s} = b_s + t\,,\ \frac{\partial \overline{S}}{\partial q_k} = p_k\,,\ k = \overline{1,s}\,. \tag{j}$$

If one of the generalized co-ordinates is cyclic (for instance, q_1), then we have $\dot{p}_1 = 0$, hence $p_1 = a_1 = \text{const}$. It results $\partial^2 S / \partial q_1 \partial t = 0$; integrating, we obtain

$$S = a_1 q_1 + S_0(q_2, q_3, \ldots, q_s; t; a_2, a_3, \ldots, a_s), \tag{k}$$

where S_0 verifies the equation

$$\dot{S}_0 + H\left(q_2, q_3, \ldots, q_s; a_1, \frac{\partial S_0}{\partial q_2}, \frac{\partial S_0}{\partial q_3}, \ldots, \frac{\partial S_0}{\partial q_s}; t \right) = 0\,, \tag{l}$$

hence an equation which contains only $s-1$ generalized co-ordinates.

Application 5.6

Problem. Study the motion of a rigid solid with a fixed point O subjected to the action of the own weight in *the Euler-Poinsot case* (*first case of integrability*).

Mathematical model. In the Euler-Poinsot case (see Appl. 5.7 too) the system of differential equations of motion is written in the form (Euler's kinetic equations)

$$\begin{aligned} I_x \dot{\omega}_x + \left(I_z - I_y \right) \omega_y \omega_z &= 0\,, \\ I_y \dot{\omega}_y + \left(I_x - I_z \right) \omega_z \omega_x &= 0\,, \\ I_z \dot{\omega}_z + \left(I_y - I_x \right) \omega_x \omega_y &= 0\,, \end{aligned} \tag{a}$$

where I_x, I_y, I_z are the moments of inertia with respect to the axes of the movable frame $Oxyz$, rigidly linked to the rigid, while ω_x, ω_y, ω_z are the components of the rotation vector of the movable frame (of the rigid solid) with respect to the fixed frame $Ox'y'z'$. The principal axes of inertia are taken as axes Ox, Oy and Oz, respectively.
Noting that multiplying the first equation (a) by $I_x \omega_x$, the second by $I_y \omega_y$ and the third one by $I_z \omega_z$ and summing, we obtain a first integral of the form

$$I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2 = K_O'^2 = \text{const}\,, \tag{b}$$

where K_O' is the moment of momentum of the rigid solid with respect to the pole of the fixed frame, in that frame. Analogously, multiplying the first equation by ω_x, the second one by ω_y and the third one by ω_z and summing, it results a first integral given by

$$I_x \omega_x^2 + I_y \omega_y^2 + I_z \omega_z^2 = 2T' = \text{const}\,, \tag{c}$$

where T' is the kinetic energy of the rigid solid with respect to the fixed frame. The constants K_O' and T' which intervene in these first integrals are, obviously, positive; we denote them $K_O' = I\Omega$, $2T = I\Omega^2$, where I is a quantity of the nature of moment of inertia, while Ω is a quantity of the nature of an angular velocity ($I = K_O'^2 / 2T'$, $\Omega = 2T' / K_O'$).
In this case, the motion is governed by the dynamical system

$$I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2 = I^2\Omega^2 \,,$$
$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = I\Omega^2 \,, \tag{d}$$

$$I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = 0 \,, \tag{e}$$

the equation (e) being one of the three equation (a). We associate to these equations the initial conditions $\omega_x(t_0) = \omega_x^0$, $\omega_y(t_0) = \omega_y^0$, $\omega_z(t_0) = \omega_z^0$. The ratio of the two relations (d) are written in the form

$$\frac{I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2}{I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2} = I \,. \tag{f}$$

Assuming that the principal moments of inertia are ordered in the form $I_x > I_y > I_z$, we may write (the ellipsoid of inertia is not of rotation)

$$I_x > \frac{I_x^2\omega_x^2 + I_y^2\omega_y^2}{I_x\omega_x^2 + I_y\omega_x^2} > I > \frac{I_y^2\omega_y^2 + I_z^2\omega_z^2}{I_y\omega_y^2 + I_z\omega_z^2} > I_z \,. \tag{g}$$

From the equations (d) we get

$$\omega_x^2 = \frac{I_y(I_y - I_z)}{I_x(I_x - I_z)}\left(\beta_y^2 - \omega_y^2\right),\ \beta_y^2 = \frac{I(I - I_z)}{I_y(I_y - I_z)}\Omega^2 \,,$$
$$\omega_z^2 = \frac{I_y(I_x - I_y)}{I_z(I_x - I_z)}\left(\overline{\beta}_y^2 - \omega_y^2\right),\ \overline{\beta}_y^2 = \frac{I(I_x - I)}{I_y(I_x - I_y)}\Omega^2 \tag{h}$$

and the differential equation (e) becomes

$$\dot{\omega}_y^2 = \frac{(I_x - I_y)(I_y - I_z)}{I_x I_z}\left(\beta_y^2 - \omega_y^2\right)\left(\overline{\beta}_y^2 - \omega_y^2\right), \tag{i}$$

hence a differential equation of the first order with separate variables for the unknown function $\omega_y = \omega_y(t)$. We obtain thus

$$t - t_0 = \frac{1}{p}\int_{\omega_y^0/\beta_y}^{\omega_y/\beta_y} \frac{\mathrm{d}z}{\sqrt{(1 - z^2)(1 - k^2z^2)}} \,, \tag{j}$$

where

$$p^2 = \frac{(I_y - I_z)(I_x - I)}{I_x I_y I_z}\Omega^2 ,\ k^2 = \left(\frac{\beta_y}{\overline{\beta}_y}\right)^2 = \frac{(I - I_z)(I_x - I_y)}{(I_x - I)(I_y - I_z)}. \quad \text{(k)}$$

Denoting $\omega_y = \beta_y \sin \kappa$ and introducing *the elliptic integral of the first species* $F(\kappa, k)$, given by

$$F(\kappa, k) = \int_0^\kappa \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \int_0^\kappa \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} , \quad \text{(l)}$$

where κ is *the amplitude* and k is *the modulus of the integral*, we may write the relation (j) in the form

$$t = t_0 + \frac{1}{p}\left[F(\kappa, k) - F(\kappa^0, k)\right], \quad \text{(m)}$$

where $\sin \kappa^0 = \omega_y^0 / \beta_y$. One obtains thus $\omega_y = \omega_y(t)$ and then $\omega_x = \omega_x(t)$, $\omega_z = \omega_z(t)$, using the relations (h).

Denoting $u = p(t - t_0)$, we may also write

$$u = F(\kappa, k) - (\kappa^0, k). \quad \text{(n)}$$

Without any loss of generality, we assume that $\omega_y^0 = 0$; it results $\kappa^0 = F(\kappa^0, k) = 0$, so that

$$u = F(\kappa, k), \quad \text{(o)}$$

where $u = \arg \kappa$, $\kappa = \operatorname{am} u$. Introducing *Jacobi's elliptic functions*: *the amplitude sinus* $(\operatorname{sn} u = \sin \kappa)$, *the amplitude cosine* $(\operatorname{cn} u = \cos \kappa)$ and *the amplitude delta* $\left(\operatorname{dn} u = \sqrt{1 - k^2 \sin^2 \kappa}\right)$, we may express the components of the vector angular velocity of rotation in the form

$$\omega_x(t) = -\overline{\beta}_x \operatorname{cn} p(t - t_0),\ \omega_y(t) = \beta_y \operatorname{sn} p(t - t_0),\ \omega_z(t) = \beta_z \operatorname{dn} p(t - t_0), \quad \text{(p)}$$

observing that $\omega_x^2 = -\overline{\beta}_x$, $\omega_y^0 = 0$ and $\omega_z^0 = \beta_z$, where

$$\overline{\beta}_x^2 = \frac{I(I - I_z)}{I_x(I_x - I_z)}\Omega^2 < \beta_y^2,\ \beta_z^2 = \frac{I(I_x - I)}{I_z(I_x - I_z)}\Omega^2 < \overline{\beta}_y^2 . \quad \text{(q)}$$

Application 5.7

Problem. Study the motion of a rigid solid with a fixed point O acted upon by its own weight $M\mathbf{g}$, where M is the mass and $\mathbf{g}$ the gravitational acceleration.

Mathematical model. We consider a fixed frame of reference $Ox'y'z'$ and a movable frame of reference $Oxyz$, the last one rigidly linked to the rigid solid S and having as axes the principal axes of inertia of S. The own weight $M\mathbf{g} = -Mg\mathbf{k}'$, where $\mathbf{k}'$ is the unit vector of the Oz'-axes, acts a the centre of gravity C, of position vector $\boldsymbol{\rho}$ with respect to the movable frame. The equation of motion, corresponding to the principle of moment o momentum, is written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}_O\boldsymbol{\omega}) = \mathbf{I}_O\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbf{I}_O\boldsymbol{\omega}) = -Mg\boldsymbol{\rho}\times\mathbf{k}', \tag{a}$$

where we put into evidence the derivative with respect to time in the fixed and in the movable frames of reference of the contracted tensor product $\mathbf{I}_O\boldsymbol{\omega}$; $\mathbf{I}_O$ is the moment of inertia tensor and $\boldsymbol{\omega}$ is the rotation vector of the rigid solid (of the movable frame). Projecting on the axes of the movable frame $Oxyz$, we find Euler's kinetic equations

$$\begin{aligned} I_x\dot{\omega}_x + (I_z - I_y)\omega_y\omega_z &= Mg(\rho_z\alpha_y - \rho_y\alpha_z), \\ I_y\dot{\omega}_y + (I_x - I_z)\omega_z\omega_x &= Mg(\rho_x\alpha_z - \rho_z\alpha_x), \\ I_z\dot{\omega}_z + (I_y - I_x)\omega_x\omega_y &= Mg(\rho_y\alpha_x - \rho_x\alpha_y), \end{aligned} \tag{b}$$

where α_x, α_y, α_z are the components (direction cosines) of the unit vector $\mathbf{k}'$ with respect to the same movable frame. We may establish the vector equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{k}' = \mathbf{k}' + \boldsymbol{\omega}\times\mathbf{k}' = \mathbf{0} \tag{c}$$

too, which shows that the unit vector $\mathbf{k}'$ has a fixed direction; projecting on the axes of the same frame, we may write

$$\begin{aligned} \dot{\alpha}_x + \omega_y\alpha_z - \omega_z\alpha_y &= 0, \\ \dot{\alpha}_y + \omega_z\alpha_x - \omega_x\alpha_z &= 0, \\ \dot{\alpha}_z + \omega_x\alpha_y - \omega_y\alpha_x &= 0. \end{aligned} \tag{d}$$

We have thus obtained a system of six differential equations of first order, formed by the subsystems (b) and (d) for the unknown functions $\omega_x = \omega_x(t)$, $\omega_y = \omega_y(t)$, $\omega_z = \omega_z(t)$, $\alpha_x = \alpha_x(t)$, $\alpha_y = \alpha_y(t)$ and $\alpha_z = \alpha_z(t)$.

Solution. Introducing the notations $x_1 = \omega_x$, $x_2 = \omega_y$, $x_3 = \omega_z$, $x_4 = \alpha_x$, $x_5 = \alpha_y$, $x_6 = \alpha_z$, as well as

$$X_1 = \frac{1}{I_x}\left[(I_y - I_z)\omega_y\omega_z + Mg(\rho_z\alpha_y - \rho_y\alpha_z)\right],$$

$$X_2 = \frac{1}{I_y}\left[\left(I_z - I_x\right)\omega_z\omega_x + Mg\left(\rho_x\alpha_z - \rho_z\alpha_y\right)\right],$$

$$X_3 = \frac{1}{I_z}\left[\left(I_x - I_y\right)\omega_x\omega_y + Mg\left(\rho_y\alpha_x - \rho_x\alpha_y\right)\right],$$

$$X_4 = \omega_z\alpha_y - \omega_y\alpha_z,$$

$$X_5 = \omega_x\alpha_z - \omega_z\alpha_x,$$

$$X_6 = \omega_y\alpha_x - \omega_x\alpha_y,$$

we may write the system (b), (d) in the form (5.35), that is

$$\frac{\mathrm{d}x_1}{X_1} = \frac{\mathrm{d}x_2}{X_2} = \ldots = \frac{\mathrm{d}x_6}{X_6} = \mathrm{d}t . \tag{e}$$

Let us suppose now that for the system

$$\frac{\mathrm{d}x_1}{X_1} = \frac{\mathrm{d}x_2}{X_2} = \ldots = \frac{\mathrm{d}x_6}{X_6}, \tag{f}$$

which does not contain the time explicitly, we succeed in obtaining the independent first integrals $f_k\left(x_1, x_2, \ldots, x_6\right) = C_k$, $k = 1, 2, \ldots, 5$, $C_k = \mathrm{const}$, which form a basic system of first integrals (the rank of the matrix $\left[\partial f_k / \partial x_j\right]$, $k = 1, 2, \ldots, 5$, $j = 1, 2, \ldots, 6$); we may thus express five of the variables as functions of the sixth one (e.g. $x_k = x_k\left(x_6, C_1, C_2, \ldots, C_5\right)$, $k = 1, 2, \ldots, 5$, so that the system (e) is reduced to the differential equations with separate variables $\mathrm{d}x_6 = X_6\left(x_6, C_1, C_2, \ldots, C_5\right)\mathrm{d}t$. By a quadrature, we obtain $f\left(x_6\right) = t + \tau$, $\tau = \mathrm{const}$, noting that $\mathrm{d}f / \mathrm{d}x_6 = \mathrm{d}t / \mathrm{d}x_6 = 1 / X_6 \neq 0$. The theorem of implicit functions leads to $x_6 = x_6\left(t + \tau\right)$, obtaining also $x_k = x_k\left(t + \tau, C_1, C_2, \ldots, C_5\right)$, $k = 1, 2, \ldots, 5$, too. Hence, to integrate the system of differential equations (b) and (d) it is sufficient to determine five independent first integrals, which do not depend on time. We notice that a condition of the form (5.50) is verified, that is

$$\sum_{i=1}^{6} \frac{\partial X_i}{\partial x_i} = 0 ; \tag{g}$$

using *the method of the last multiplier*, it results that it is sufficient to know four independent first integrals f_1, f_2, f_3, f_4 of the considered differential system to may determine a fifth first integral, independent of the other ones; we obtain then an integrating factor, which allows to determine all the unknown functions of the problem.

A scalar product of the vector equation (a) by $\mathbf{k}'$ leads to $[\mathrm{d}(\mathbf{I}_O\boldsymbol{\omega})/\mathrm{d}t]\cdot\mathbf{k}'=0$ or $\mathrm{d}[(\mathbf{I}_O\boldsymbol{\omega})\cdot\mathbf{k}']/\mathrm{d}t=0$, $\mathbf{k}'$ being a constant unit vector, so that

$$(\mathbf{I}_O\boldsymbol{\omega})\cdot\mathbf{k}'=K'_{Oz'}, \tag{h}$$

where $K'_{Oz'}$ represents the constant projection of the moment of momentum on the fixed Oz'; we obtain thus a *scalar first integral of the moment of momentum* (the conservation of the moment of momentum along the local vertical) in the form

$$I_x\omega_x\alpha_x+I_y\omega_y\alpha_y+I_z\omega_z\alpha_z=K'_{Oz'}. \tag{i}$$

A scalar product of the equation (a) by $\boldsymbol{\omega}$, leads to

$$(\mathbf{I}_O\dot{\boldsymbol{\omega}})\boldsymbol{\omega}=Mg(\boldsymbol{\omega},\mathbf{k}',\boldsymbol{\rho})=Mg(\boldsymbol{\omega}\times\mathbf{k}')\cdot\boldsymbol{\rho}=-Mg\dot{\mathbf{k}}'\cdot\boldsymbol{\rho}=-Mg\frac{\partial}{\partial t}(\mathbf{k}'\cdot\boldsymbol{\rho}),$$

the derivative being taken with respect to the movable frame; integrating we get

$$(\mathbf{I}_O\boldsymbol{\omega})\cdot\boldsymbol{\omega}=-2Mg\mathbf{k}'\cdot\boldsymbol{\rho}+2h, \tag{j}$$

where h represents the constant of mechanical energy. It results thus *the first integral of mechanical energy* in the form

$$I_x\omega_x^2+I_y\omega_y^2+I_z\omega_z^2=-2Mg(\rho_x\alpha_x+\rho_y\alpha_y+\rho_z\alpha_z)+2h. \tag{k}$$

A third first integral is

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1, \tag{l}$$

which is justified because $\mathbf{k}'$ is a unit vector.
Taking into account the above results, we may state that the problem of integration of the system of equations (b), (d) reduces to the problem of finding a fourth first integral of this system. Ed. Husson proved in his doctor thesis (1906) that, in the problem of the rigid solid with a fixed point O, governed by the mechanical equations (b) and by the geometric equations (d), in case of arbitrary initial conditions, excepting the first integrals (I), (k), (l), there exists a fourth first integral, algebraic function of ω_x, ω_y, ω_z, α_x, α_y, α_z, non-depending explicitly on t, if an only if the fixed point is just the centre of mass ($O\equiv C$, hence $\boldsymbol{\rho}=\mathbf{0}$, Euler-Poinsot case) or if the ellipsoid of inertia is of rotation ($I_x=I_y$ and $\rho_x=\rho_y=0$, Lagrange-Poisson case; $I_x=I_y=2I_z$, $\rho_z=0$, Sonya Krukovsky (Sophia Kovalévsky) case). If we renounce to the generality concerning the initial conditions, we may find also other cases of integrability (by quadratures).

Application 5.8

Problem. Study the motion of a rigid solid with a fixed point O subjected to the action of its own weight *in the Lagrange-Poisson case* (*second case of integrability*).

Mathematical model. In the Lagrange-Poisson case (see Appl. 5.7 too), the ellipsoid of inertia with respect to the axes of the movable frame of reference $Oxyz$, rigidly linked to the rigid solid, verifies the relations $I_x = I_y = J > I_z$ (hence, the *oblate* case); the principal axes of inertia are taken as Ox, Oy and Oz axes. The co-ordinates of the centre of mass C with respect to the movable frame verify the relations $\rho_x = \rho_y = 0$, $\rho_z = \rho_0 > 0$, this centre being on the principal axis Oz, which is thus a central principal axis of inertia. The differential equations of motion (*Euler's kinetic equations*) are written in the form

$$\begin{aligned} &J\dot{\omega}_x - (J - I_z)\omega_y\omega_z = Mg\rho_z\alpha_y, \\ &J\dot{\omega}_x - (J - I_z)\omega_z\omega_x = Mg\rho_z\alpha_x, \\ &\dot{\omega}_z = 0, \end{aligned} \tag{a}$$

where ω_x, ω_y, ω_z are the components of the rotation vector of the movable frame (of the rigid solid) with respect to a fixed frame $Ox'y'z'$, $M\mathbf{g} = -Mg\mathbf{k}'$, $\mathbf{k}'$ being the unit vector of the Oz'-axis, is the own weight which acts at the centre C (M is the mass, and $\mathbf{g}$ is the gravitational acceleration); as well α_x, α_y, α_z are the components (direction cosines) of the unit vector $\mathbf{k}'$ with respect to the same movable frame.

Solution. We obtain

$$\omega_z(t) = \omega_z^0, \tag{b}$$

the constant ω_z^0 is called *spin*, that one being the fourth integral in Husson's theorem (see Appl. 5.7). The first integrals (i) an (k) (see the same application) become

$$\begin{aligned} &J(\omega_x\alpha_x + \omega_y\alpha_y) + I_z\omega_z^0\alpha_z = K'_{Oz'}, \\ &J(\omega_x^2 + \omega_y^2) + I(\omega_z^0)^2 = -2Mg\rho_z\alpha_z + 2h, \end{aligned} \tag{c}$$

where we took into account the first integral (b).
It is useful to introduce *Euler's cinematic equations*

$$\begin{aligned} &\omega_x = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi, \\ &\omega_y = -\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi, \\ &\omega_z = \dot{\psi} + \dot{\varphi}\cos\theta, \end{aligned} \tag{d}$$

where *Euler's angles* appear: *the angle of precession* ψ, *the angle of nutation* θ and *the angle of proper rotation* φ, which specify the position of the movable frame with respect to the fixed frame. We mention, as well, the relations

$$\begin{aligned} \alpha_x &= \sin\theta\sin\varphi, \\ \alpha_y &= \sin\theta\cos\varphi, \\ \alpha_z &= \cos\theta, \end{aligned} \tag{e}$$

which link the components of the vector $\boldsymbol{\omega}$ to the components of the unit vector $\mathbf{k}'$; thus we may determine Euler's angles when we know the direction cosines α_x, α_y, α_z.

Using the relations (e), we may write the first integrals (c) in the form

$$\begin{aligned} &J\left(\omega_x \sin\varphi + \omega_y \cos\varphi\right) + I_z\omega_z^0 \cos\theta = K'_{Oz'}, \\ &J\left(\omega_x^2 + \omega_y^2\right) + I\left(\omega_z^0\right)^2 = -2Mg\rho_z \cos\theta + 2h. \end{aligned} \tag{f}$$

Using the equations (d), the first integrals (b) and (f) lead to the system of equations

$$\begin{aligned} &\dot\psi \sin^2\theta = \alpha - a\omega_z^0 \cos\theta, \\ &\dot\psi \sin^2\theta + \dot\theta^2 = \beta - b\cos\theta, \\ &\dot\psi \cos\theta + \dot\varphi = \omega_z^0, \end{aligned} \tag{g}$$

where we have introduced the notations $\alpha = K'_{Oz'}/J$, $\beta = \left[2 - I_z\left(\omega_z^0\right)^2\right]/J$, $a = I_z/J > 0$, $b = 2mg\rho_z/J > 0$; we observe that α and β are constants which depend on the initial conditions, while the constants a and b are functions depending only on the geometry and on the mechanical properties of the rigid solid.

The system of differential equations (g) will determine Euler's angles $\psi = \psi(t)$, $\theta = \theta(t)$ and $\varphi = \varphi(t)$. Eliminating $\dot\psi$ between the first two equations, we obtain

$$\left(\alpha - a\omega_z^0\cos\theta\right)^2 = (\beta - b\cos\theta)\sin^2\theta - \dot\theta\sin^2\theta. \tag{h}$$

Denoting $u = \cos\theta$, it results the differential equation

$$\dot u^2 = P(u),\ P(u) = (\beta - bu)\left(1 - u^2\right) - \left(\alpha - a\omega_z^0 u\right)^2, \tag{i}$$

whence

$$t = t_0 + \int_{u_0}^{u} \frac{\mathrm{d}\xi}{\sqrt{P(\xi)}}, \tag{j}$$

with $u_0 = \cos\theta_0$, $\theta_0 = \theta(t_0)$; assuming that $\dot{u}(t_0) \neq 0$ ($\dot{u}(t)$ has a continuous variation, beginning with $\dot{u}(t_0)$), the radical is taken with the sign of $\dot{u}(t_0)$.

If $\theta_0 \neq 0$ and $\theta_0 \neq \pi$, then we have $u_0 \in (-1,1)$, hence $K'_{Oz'} \neq \pm I_z \omega_z^0$. Because the equation (i) admits a solution only if $P(u_0) \geq 0$, it results that the polynomial $P(u)$ is of the form

$$P(u) = b(u - u_1)(u - u_2)(u - u_3), \tag{k}$$

where u_1, u_2, u_3 are the real zeros of the polynomial of third degree $P(u)$, so that $-1 < u_1 \leq u_0 \leq u_2 \leq u_3 < \infty$. One may thus show that $u(t)$ varies between u_1 and u_2, the duration of a complete period being

$$T = 2\int_{u_1}^{u_2} \frac{\mathrm{d}u}{\sqrt{P(u)}}. \tag{l}$$

Hence, $u(t+T) = u(t)$ and $\dot{u}(t+T) = \dot{u}(t)$; it results $\theta(t+T) = \theta(t)$ too.

We may introduce a new variable κ by the relation

$$\begin{aligned} u &= u_1 \cos^2\kappa + u_2 \sin^2\kappa = u_1 + (u_1 - u_2)\sin^2\kappa \\ &= u_2 - (u_2 - u_1)\cos^2\kappa = u_3 - (u_3 - u_1)\left(1 - k^2 \sin^2\kappa\right), \end{aligned} \tag{m}$$

where

$$k = \frac{u_2 - u_1}{u_3 - u_1} < 1;$$

introducing this in (i) and (j), we obtain

$$t - t_0 = \frac{1}{p}\int_{\kappa_0}^{\kappa} \frac{\mathrm{d}\chi}{\sqrt{1 - k^2 \sin^2\chi}}, \quad p = \frac{1}{2}\sqrt{b(u_3 - u_1)}, \tag{n}$$

where $\kappa = \kappa_0$ corresponds to $u = u_0$. Using the notation $w = \sin\kappa$, we may also write

$$t - t_0 = \frac{1}{p}\int_{w_0}^{w} \frac{\mathrm{d}\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \qquad w_0 = \sin\kappa_0. \tag{o}$$

Introducing now *Jacobi's elliptic functions* (see Appl.5.6), it also results

$$\begin{aligned} u(t) &= u_1 \,\mathrm{cn}^2 p(t - t_0) + u_2 \,\mathrm{sn}^2 p(t - t_0) = u_1 + (u_2 - u_1)\mathrm{sn}^2 p(t - t_0) \\ &= u_2 - (u_2 - u_1)\mathrm{cn}^2 p(t - t_0) = u_3 - (u_3 - u_1)\mathrm{dn}^2 p(t - t_0), \end{aligned} \tag{p}$$

the nutation angle being thus completely determined.

The other angles of Euler are given by the equations (g) in the form

$$\dot{\psi} = \frac{\alpha - a\omega_z^0 u}{1 - u^2}\,,\ \dot{\varphi} = \omega_z^0 - \frac{\left(\alpha - a\omega_z^0 u\right)u}{1 - u^2}\,; \tag{q}$$

hence, it results $\dot{\psi}(t + T) = \dot{\psi}(t)$ and $\dot{\varphi}(t + T) = \dot{\varphi}(t)$, so that

$$\psi(t + T) = \psi(t) + \Psi_0\,,\ \varphi(t + T) = \varphi(t) + \Phi_0\,, \tag{r}$$

where Ψ_0 and Φ_0 are arbitrary constants.

Application 5.9

Problem. Study the circular thin plates acted upon by axially symmetric loads, in the hypothesis of great deformations.

Mathematical model. We take into consideration the equations of equilibrium

$$N_r - N_t + r\frac{\mathrm{d}N_t}{\mathrm{d}r} = 0\,, \tag{a}$$

$$T_r = -N_t\frac{\mathrm{d}w}{\mathrm{d}r} - \frac{1}{r}\int_0^r qr\mathrm{d}r\,, \tag{b}$$

the equations of deformation

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{r}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2, \tag{c}$$

$$\varepsilon_t = \frac{u}{r} \tag{d}$$

and the relations of elasticity

$$N_r = \frac{Et}{1 - \nu^2}\left(\varepsilon_r + \nu\varepsilon_t\right) = \frac{Et}{1 - \nu^2}\left[\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}r}\right)^2 + \nu\frac{u}{r}\right], \tag{e}$$

$$N_t = \frac{Et}{1 - \nu^2}\left(\varepsilon_t + \nu\varepsilon_r\right) = \frac{Et}{1 - \nu^2}\left[\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{\nu}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2\right], \tag{f}$$

$$T = -\frac{Et^3}{12\left(1 - \nu^2\right)}\left(\frac{\mathrm{d}^3 w}{\mathrm{d}r^3} + \frac{1}{r}\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right), \tag{g}$$

where N_r, N_t represent the radial and the annular efforts in the plate, respectively, $T_r = T$ is the shearing force, u and w are the displacement and the deflection of a point of the plate in the radial and transverse direction, respectively, ε_r, ε_t are the linear strains in the radial and annular directions, respectively, E, ν are the elastic constants of the material, $t = \text{const}$ and a are the thickness and the radius of the plate, respectively, and q is the transverse load (supposed to be constant). Application for the circular built-in plate.

Solution. Introducing the shearing force (g) in (b), we obtain

$$\frac{Et^3}{12(1-\nu^2)}\left(\frac{\mathrm{d}^3 w}{\mathrm{d}r^3}+\frac{1}{r}\frac{\mathrm{d}^2 w}{\mathrm{d}r^2}-\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right)=N_r\frac{\mathrm{d}w}{\mathrm{d}r}+\frac{qr}{2}, \tag{h}$$

the last term representing the integral in (b) for $q = \text{const}$.
Eliminating u between (c) and (d), we get the equation of compatibility

$$\varepsilon_r = \varepsilon_t + r\frac{\mathrm{d}\varepsilon_t}{\mathrm{d}r}+\frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2,$$

or, replacing

$$\varepsilon_r = \frac{1}{Et}(N_r - \nu N_t),\ \varepsilon_t = \frac{1}{Et}(N_t - \nu N_r)$$

and using the equation (a),

$$r\frac{\mathrm{d}}{\mathrm{d}r}(N_r + N_t)+\frac{Et}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 = 0. \tag{i}$$

The equations (a), (b), and (i) contain the unknown functions N_r, N_t and w and will be considered as the general equations of the problem.
We introduce the non-dimensional unknowns

$$p = \frac{q}{E},\ \xi = \frac{r}{t},\ S_r = \frac{N_r}{Et},\ S_t = \frac{N_t}{Et}. \tag{j}$$

With these notations, the equations (a), (b), (i) become

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi S_r) - S_t = 0, \tag{k}$$

$$\frac{1}{12(1-\nu^2)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}w}{\mathrm{d}r}\right)\right]=\frac{p\xi}{2}+S_r\frac{\mathrm{d}w}{\mathrm{d}r}, \tag{l}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(S_r + S_t) + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 = 0 . \tag{m}$$

From the equation (d), one obtains

$$u = r\varepsilon_t = \frac{r}{Et}(N_t - \nu N_r) = r(S_r - \nu S_t) .$$

The boundary conditions are $u = 0$, $\mathrm{d}w/\mathrm{d}r = 0$, $w = 0$ for $r = a$. Thus, on the contour $(r = a)$ we have

$$(S_t - \nu S_r)\big|_{r=a} = 0 . \tag{n}$$

We may assume that S_r is a symmetric function, while $\mathrm{d}w/\mathrm{d}r$ is an antisymmetric function with respect to ξ, so that one may introduce the power series

$$S_r = B_0 + B_2\xi^2 + B_4\xi^4 + \ldots , \tag{o}$$

$$\frac{\mathrm{d}w}{\mathrm{d}r} = \sqrt{8}\left(C_1\xi + C_3\xi^3 + C_5\xi^5 + \ldots\right), \tag{p}$$

where B_0, B_2, B_4, … and C_1, C_3, C_5, … are constants which must be determined. Introducing the series (o) in (k), it results

$$S_t = B_0 + 3B_2\xi^2 + 5B_4\xi^4 + \ldots \tag{q}$$

Differentiating the relation (p) with respect to ξ, we get

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right) = \sqrt{8}\left(C_1 + 3C_2\xi^2 + 5C_5\xi^4 + \ldots\right). \tag{r}$$

It is seen that all the quantities of interest may be obtained if we know the constants B_0, B_2, B_4, …, and C_1, C_3, C_5, … Introducing the series (o), (p), (q) in the equations (l) and (m) and noting that all these equations must be satisfied for any ξ, we find following relations between the constants B and C

$$\begin{gathered} B_{2k} = -\frac{4}{2k(2k+2)}\sum_{m=1}^{k} C_{2m-1}C_{2k-2m+1} , \ k = 1, 2, 3, \ldots , \\ C_{2k+3} = \frac{12\left(1-\nu^2\right)}{k^2 - 1}\sum_{m=0}^{k} B_{2m}C_{2k-2m+1} , \ k = 1, 2, 3, \ldots , \\ C_3 = \frac{3}{2}\left(1-\nu^2\right)\left(\frac{p}{2\sqrt{8}} + B_0C_1\right) . \end{gathered} \tag{s}$$

We notice that if we choose certain values for B_0 and C_1, all the other constants may be determined with the aid of the relations (s). We observe also that choosing S_r is equivalent to choosing B_0 and C_1 and the curvature at the centre of the plate.
The problem is extremely difficult from the point of view of the numerical computation. Practically, there are chosen values for ν and $p = q / E$ and then, for the values which are chosen for B_0 and C_1, are determined the radii of the plates, so as to satisfy the boundary conditions $\mathrm{d}w / \mathrm{d}r = 0$ for $r = a$.
The boundary values for S_r and S_t have been thus calculated, as well as the radial displacement u for $r = a$. The condition (n) is not generally satisfied, but all the data which are necessary for plates, if both boundary conditions are satisfied, may be obtained.

Application 5.10

Problem. Study the critical and postcritical behaviour of a cantilever bar acted upon by an axial force P.

Mathematical model. The deformed axis of the bar, denoted by y and supposed to be a function depending on the arc s, satisfies the system of non-linear ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin\theta, \quad \frac{\mathrm{d}\theta}{\mathrm{d}s} = -\alpha^2 (f - y), \tag{a}$$

where α^2, f, θ have the signification mentioned in Chap. 4, Sec. 2.4. The functions y and θ must verify also the Cauchy conditions

$$y(0) = 0\,, \; \theta(0) = 0\,. \tag{b}$$

Solution. We apply the LEM, presented in the Section 3.3. In this case, the LEM exponential transformation depends on two parameters σ and ξ

$$v(s, \sigma, \xi) = \mathrm{e}^{\sigma\tilde{y} + \xi\theta}, \; \tilde{y} = y - f\,. \tag{c}$$

We preferred the function $\tilde{y}$ as unknown function, because the LEM is easier applied to homogeneous non-linear systems; indeed, we notice that $\tilde{y}$ and θ satisfy the differential system

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}s} = \sin\theta, \quad \frac{\mathrm{d}\theta}{\mathrm{d}s} = -\alpha^2\tilde{y} \tag{d}$$

and the initial conditions

$$\tilde{y}(0) = -f\ ,\ \theta'(0) = 0\ . \tag{e}$$

The first linear equivalent equation corresponding to the transformation (c) is

$$\frac{\partial v}{\partial s} - \sigma \sin D_\xi v + \alpha^2 \xi \frac{\partial v}{\partial \xi} = 0\ , \tag{f}$$

where $\sin D_\xi$ is the operator

$$\sin D_\xi = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} \frac{\partial^{2k-1}}{\partial \xi^{2k-1}}\ .$$

Consider now for v the development

$$v(s, \sigma, \xi) = 1 + \sum_{i+j=1}^{\infty} v_{ij}(s) \frac{\sigma^i}{i!} \frac{\xi^j}{j!}\ .$$

Then we obtain from (f) the infinite linear ODS of first order for the coefficients $v_{ij}(s)$

$$\frac{\mathrm{d}v_{ij}}{\mathrm{d}s} - i \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!} v_{i-1,j+2k-1} + \alpha^2 j v_{i+1,j-1} = 0\ ,\ i, j \in \mathfrak{N}\ . \tag{g}$$

In vector form, we get

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}s} = \mathbf{AV}\ ,\ \mathbf{V} = [\mathbf{V}_{2m-l}]_{m \in \mathbf{N}}\ ,\ \mathbf{V}_{2m-l} = \lfloor v_{ij} \rfloor_{0 \le i+j \le 2m-1}\ . \tag{h}$$

The linear equivalence matrix $\mathbf{A}$ is of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{13} & \mathbf{A}_{15} & \cdots & \mathbf{A}_{1,2j-1} & \mathbf{A}_{1,2j+1} & \cdots \\ \mathbf{0} & \mathbf{A}_{33} & \mathbf{A}_{35} & \cdots & \mathbf{A}_{3,2j-1} & \mathbf{A}_{1,2j+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{2j-1,2j-1} & \mathbf{A}_{2j-1,2j+1} & \cdots \\ \cdots & \cdots & v & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{i}$$

the cells $\mathbf{A}_{kl}$ being given by

$$\mathbf{A}_{11} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix},$$

$$\mathbf{A}_{2j-1,2j-1} = \begin{bmatrix} 0 & 2j-1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\alpha^2 & 0 & 2j-2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -2\alpha^2 & 0 & 2j-3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -(2j-2)\alpha^2 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -(2j-1)\alpha^2 & 0 \end{bmatrix} \tag{j}$$

$$\mathbf{A}_{2j-1,2j-2k-1} = \frac{(-1)^k}{(2k+1)!}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 2j-1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 2j-2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2j-3 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

We specify that the matrix $\mathbf{A}_{2j-1,2j+2k-1}$ has $2j$ lines and $2j+2k$ columns.

As we have shown in the Sec.3.3, the solution of the non-linear problem (d), (e) assumes the following normal LEM representation

$$y(s) = -f + \sum_{j=1}^{\infty} u_{2j-1,0}(s) f^{2j-1}, \tag{k}$$

where $u_{2j-1,0}(s)$ are the first components of the finite vectors $\mathbf{U}_{2j-1}$, satisfying the finite systems of ODEs, written by blocks

$$\begin{aligned} &\frac{d\mathbf{U}_1}{ds} = \mathbf{A}_{11}^* \mathbf{U}_1, \\ &\frac{d\mathbf{U}_1}{ds} = \mathbf{A}_{33}^* \mathbf{U}_3 + \mathbf{A}_{13}^* \mathbf{U}_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\frac{d\mathbf{U}_{2j-1}}{ds} = \mathbf{A}_{2j-1,2j-1}^* \mathbf{U}_{2j-1} + \mathbf{A}_{2j-3,2j-1}^* \mathbf{U}_{2j-3} + \ldots + \mathbf{A}_{1,2j-1}^* \mathbf{U}_1, \end{aligned} \tag{l}$$

as well as the initial conditions

$$\mathbf{U}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{U}_{2m+1}(0) = [\mathbf{0}], \ m \in \mathbf{N} \setminus \{1\}. \tag{m}$$

We will solve these systems on blocks. Firstly, we look for the vector $\mathbf{U}_1$, using the methods presented in Chap.3. We have

$$\frac{d\mathbf{U}_1}{ds} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix} \mathbf{U}_1, \ \mathbf{U}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{n}$$

The characteristic determinant is $\det[\mathbf{A}_{11} - \lambda \mathbf{E}]$, hence $\lambda^2 + \alpha^2 = 1$, where $\lambda_{1,2} = \pm i\alpha$. The matrix of the corresponding eigenvectors is

$$\begin{bmatrix} 1 & 1 \\ i\alpha & -i\alpha \end{bmatrix};$$

its inverse reads

$$\frac{1}{2}\begin{bmatrix} 1 & -\frac{\mathrm{i}}{\alpha} \\ 1 & \frac{\mathrm{i}}{\alpha} \end{bmatrix}.$$

Hence, the solution of the linear Cauchy problem deduced for $\mathbf{U}_1$ from (l) and (m) is written as

$$\mathbf{U}_1(s)=\frac{1}{2}\begin{bmatrix} 1 & -\frac{\mathrm{i}}{\alpha} \\ 1 & \frac{\mathrm{i}}{\alpha} \end{bmatrix}\begin{bmatrix} \mathrm{e}^{\mathrm{i}\alpha s} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\alpha s} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \mathrm{i}\alpha & -\mathrm{i}\alpha \end{bmatrix}=\begin{bmatrix} \cos\alpha s & \alpha\sin\alpha s \\ \frac{\sin\alpha s}{\alpha} & \cos\alpha s \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} \cos\alpha s \\ \frac{\sin\alpha s}{\alpha} \end{bmatrix}. \tag{o}$$

We look now for the second block of equations, corresponding to $\mathbf{U}_3$. Similarly, we get

$$u_{30}=\frac{\alpha^2}{16}\left[\frac{3}{4}\left(-\cos 3\alpha s+\cos\alpha s\right)-3\alpha s\sin\alpha s\right].$$

We can stop at $\mathbf{U}_3$, for instance, including thus the 3rd order effects; this yields

$$\begin{aligned} y(s) &\cong -f(\cos\alpha s-1) \\ &-f^3\alpha^2\cdot\frac{1}{6}\left[\frac{3}{4}\left(-\cos 3\alpha s+\cos\alpha s\right)-3\alpha s\sin\alpha s\right]-f^5\alpha^4\psi(\alpha s). \end{aligned} \tag{p}$$

The criticity condition is determined by the relation $y(l)=f$. From (p) we get

$$\cos\alpha l+(\alpha f)^2\cdot\frac{1}{16}\left[\frac{3}{4}\left(-\cos 3\alpha l+\cos\alpha l\right)-3\alpha s\sin\alpha l\right]\cong 0, \tag{q}$$

which leads to the critical values

$$\alpha l=(2k-1)\frac{\pi}{2},\ k\in\mathfrak{N}, \tag{r}$$

corresponding to the critical charges

$$P_{\mathrm{cr}}=\frac{\pi^2 EI}{4l^2}(2k-1). \tag{s}$$

Turninig back to (q), we obtain an approximate formula for the postcritical behaviour of the cantilever bar, i.e.

$$\frac{f}{l}\cong\frac{4}{\alpha l}\sqrt{\frac{2\cot\alpha l}{\sin 2\alpha l-2\alpha l}},\ \frac{\pi}{2}<\alpha l<\pi. \tag{t}$$

This formula leads to numerical results closer to the solution expressed by elliptic integrals than other approximate postcritical formulae (e.g. to Grashof's or Steiner's formula).

Application 5.11

Problem. Study the criticity problem of the built-in bar, with small geometric imperfections

Mathematical model. Let us suppose that the bar is not perfectly built-in, so that its axis forms a small angle θ_0 with the ideal direction. Let be $\beta_0 = \tan\theta_0$. In this case, the deformation y of the mean fiber of the bar satisfies the Bernoulli-Euler equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \alpha^2 (f - y)\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{3/2} = 0\,, \tag{a}$$

as well as Cauchy's conditions

$$y(0) = 0\,,\ \frac{\mathrm{d}y}{\mathrm{d}x}(0) = \beta_0\,. \tag{b}$$

By the translation $\widetilde{y}(x) = y - f$, where f is the deflection, we get for $\widetilde{y}$ the non-linear problem with initial values

$$\begin{gathered}\widetilde{y}'' + \alpha^2 \widetilde{y}\left(1 + \widetilde{y}'^2\right)^{3/2} = 0\,,\\ \widetilde{y}(0) = -f\,,\ \widetilde{y}'(0) = \beta_0\,.\end{gathered} \tag{c}$$

Solution. Applying LEM to this problem, same way as previously, we obtain for y

$$\begin{aligned} y(x) \cong{}& \frac{\beta_0}{\gamma_0}\sin\gamma_0 x - \frac{9\beta_0^2}{16\left(1+\beta_0^2\right)\gamma_0}\left[\frac{1}{12}\left(\sin 3\gamma_0 + 9\sin\gamma_0 x\right) - \gamma_0 x\cos\gamma_0 x\right] \\ &+ f\left[1 - \cos\gamma_0 x - \frac{9\beta_0^2}{16\left(1+\beta_0^2\right)}\left(\gamma_0 x - \sin\gamma_0 x\cos\gamma_0 x\right)\sin\gamma_0 x\right], \end{aligned} \tag{d}$$

with $\gamma_0 = \alpha\left(1+\beta_0^2\right)^{3/4}$. The condition $y(l) = f$ involves

$$\frac{f}{f_0} \cong \frac{\tan\gamma_0 l}{\gamma_0 l}\,\frac{1 + \dfrac{9\beta_0^2}{16\left(1+\beta_0^2\right)}\left[1 - \dfrac{1}{3}\sin^2\gamma_0 l - \gamma_0 l\cot\gamma_0 l\right]}{1 + \dfrac{9\beta_0^2}{16\left(1+\beta_0^2\right)}\left(\gamma_0 l - \sin\gamma_0 l\cos\gamma_0 l\right)\tan\gamma_0 l}\,, \tag{e}$$

where $f_0 = \beta_0 l$ is the deformation due to the imperfection of the built-in cross section. We note that

$$\alpha l = \frac{\pi}{2}\sqrt{\frac{P}{P_E}}\,, \tag{f}$$

where Euler's force is given by the formula

$$P_E = \frac{\pi^2 EI}{4l^2} . \tag{g}$$

The formula (e) represents the non-dimensional fraction f / f_0 as a function of the ratio P / P_E. Neglecting β_0^2 with respect to unity, we obtain the linear classical result

$$\frac{f}{f_0} = \frac{\tan \alpha l}{\alpha l} , \tag{h}$$

which leads to the critical value

$$P_{\text{cr}} = P_E , \tag{i}$$

corresponding to $\alpha l = \pi / 2$.

Writing the formula (e) in the form

$$\begin{aligned} &\frac{f}{f_0}\left\{1 + \frac{9\beta_0}{16\left(1+\beta_0^2\right)}\left[\beta_0\left(1 - \frac{1}{3}\sin^2\gamma_0 l\right) - \frac{f}{f_0}\gamma_0 l\left(\gamma_0 l - \sin\gamma_0 l\cos\gamma_0 l\right)\right]\right\} \\ &\qquad = \frac{\gamma_0 l}{\tan\gamma_0 l}\left[1 + \frac{9\beta_0^2}{16\left(1+\beta_0^2\right)}\frac{f_0}{f}\right], \end{aligned} \tag{j}$$

we observe that the arrow f cannot be greater than the length l of the bar and grows indefinitely for $\gamma_0 l \to \pi / 2$.

One obtains immediately the critical load for the cantilever bar with small geometrical imperfections

$$P_{\text{cr}} \cong \frac{P_E}{\left(1+\beta_0^2\right)^{3/2}} \tag{k}$$

or

$$P_{\text{cr}} \cong \frac{P_E}{1 + \frac{3}{2}\beta_0^2} . \tag{l}$$

Unlike the previous application, in which the instability is obtained by *bifurcation*, in this case it is obtained by *divergence*. By LEM, we got a complete picture of the criticity, i.e. a picture which definitely cannot be obtained by a linear study. We also notice that the formulae (k) and (l) lead to Euler's critical load for $\beta_0 = 0$.

Chapter 6

VARIATIONAL CALCULUS

1. Necessary Condition of Extremum for Functionals of Integral Type

1.1 GENERALITIES

In various cases, the mathematical models associated to mechanical phenomena are presented in integral form. This form naturally appears e.g. when we are searching for a minimum energy.

If the energy depends only on one physical magnitude, corresponding to a function $y(x)$, as well as on its derivative $y'(x)$, then one can enounce the following

Minimum problem. Find the function $y \in \mathrm{C}^2([x_1, x_2])$ for which the integral

$$\mathrm{I}[y] \equiv \int_{x_1}^{x_2} F(x, y(x), y'(x))\,\mathrm{d}x \tag{6.1.1}$$

has a mimimal value.

If the mechanical problem involves other restrictions on *y*, then the minimum of $\mathrm{I}[y]$ must be searched for in *the set of the admissible functions*, i.e., of the functions satisfying these restrictions.

We admit that the integrand of $\mathrm{I}[y]$ – the function F – is of class C^2 with respect to its arguments x, y, y'; the ends x_1, x_2 of the interval of integration are supposedly fixed up. Obviously, the integral I has a well-determined value for each $y \in \mathrm{C}^1([x_1, x_2])$. It thus associates to any such function a real number.

We say the I is *a real functional*. We can also say that I is defined on $\mathrm{C}^1([x_1, x_2])$.

In what follows, we shall denote by $\mathcal{F} \subseteq \mathrm{C}^1([x_1, x_2])$ the domain of definition of I and by $\mathcal{U} \subset \mathcal{F}$ the set of the admissible functions that satisfy the supplementary conditions imposed by the considered mechanical problem.

Denote by $\|f\| = \sup_{x \in [x_1, x_2]} |f(x)|$ the norm in $\mathrm{C}^0([x_1, x_2])$ and by $\|f\|_1 = \max\{\|f\|, \|f'\|\}$ the norm in $\mathrm{C}^1([x_1, x_2])$.

Let now $y \in \mathcal{F}$. We call $V_0 = \{Y \in \mathcal{F}, \|Y - y\| \le \varepsilon\}$ a *neighbourhood of order* 0 *of* *y*. The set $V_1 = \{Y \in \mathcal{F}, \|Y - y\|_1 \le \varepsilon\}$ *is a neighbourhood of order* 1 *of* *y*. Obviously, a neighbourhood of order 0 is richer than one of order 1.

We say that $\mathrm{I} : \mathcal{F} \to \Re$ allows *an absolute maximum* at $y \in \mathcal{U}$ if

$$\mathrm{I}[Y] \leq \mathrm{I}[y] \text{ for any } Y \in \mathcal{U}\ ; \tag{6.1.2}$$

Similarly, we say that $\mathrm{I}: \mathcal{F} \to \Re$ allows *an absolute minimum* at $y \in \mathcal{U}$ if

$$\mathrm{I}[Y] \geq \mathrm{I}[y] \text{ for any } Y \in \mathcal{U}\ . \tag{6.1.3}$$

The maxima and minima are also called *extrema*. Relaxing the above conditions, we obtain the definitions of the relatively strong/weak extrema.
We say that $\mathrm{I}: \mathcal{F} \to \Re$ allows *a relatively strong maximum* at $y \in \mathcal{U}$ if there exists a neighbourhood V_0 of order 0 of y such that

$$\mathrm{I}[Y] \leq \mathrm{I}[y] \text{ for any } Y \in V_0 \cap \mathcal{U} \tag{6.1.4}$$

and allows *a relatively weak maximum* at $y \in \mathcal{U}$ if there exists a neighbourhood V_1 of order 1 of y such that

$$\mathrm{I}[Y] \leq \mathrm{I}[y] \text{ for any } Y \in V_1 \cap \mathcal{U}\ . \tag{6.1.5}$$

The relatively strong/weak minima are defined exactly sameway; the only difference is that one changes the sense of the inequalities (6.1.4), (6.1.5).
From the above definitions, we see that an absolute extremum is also both relatively strong and weak; a relatively strong extremum is also relatively weak.
To get necessary conditions of extremum for relatively weak extrema one must prove the following essential result

Lemma 6.1 (fundamental). *Let* $f \in \mathrm{C}^0([x_1, x_2])$. *If*

$$\int_{x_1}^{x_2} f(x)\eta(x)\mathrm{d}x = 0 \tag{6.1.6}$$

for any $\eta \in \mathrm{C}^2([x_1, x_2]),\ \eta(x_1) = 0,\ \eta(x_2) = 0$, *then* $f(x) = 0,\ \forall x \in [x_1, x_2]$.

The proof is by reductio ad absurdum. We firstly note that, due to the continuity of f, if $f(x) = 0,\ \forall x \in (x_1, x_2)$, then $f(x_1) = 0,\ f(x_2) = 0$. So, we only need to prove that $f(x) = 0$ in the open interval (x_1, x_2). Let, for instance $a \in (x_1, x_2)$ such that $f(a) > 0$. Then, again by the continuity of f, one can find $\varepsilon > 0$ such that $f(x) > 0,\ \forall x \in (a - \varepsilon, a + \varepsilon)$. Let us consider the function η defined as follows

$$\eta(x) = \begin{cases} \left[(x-a)^2 - \varepsilon^2\right]^3, & x \in (a - \varepsilon, a + \varepsilon), \\ 0, & x \notin (a - \varepsilon, a + \varepsilon). \end{cases} \tag{6.1.7}$$

Obviously, η satisfies the hypotheses of the fundamental lemma. For this choice, we have

$$\int_{x_1}^{x_2} f(x)\eta(x)\mathrm{d}x = \int_{a-\varepsilon}^{a+\varepsilon} f(x)\eta(x)\mathrm{d}x < 0; \tag{6.1.8}$$

this contradicts the hypothesis, the lemma being thus proved.

1.2 FUNCTIONALS OF THE FORM $\mathrm{I}[y] \equiv \int_{x_1}^{x_2} F(x, y(x), y'(x))\mathrm{d}x$

Let us consider the functional (6.1.1), defined in the introduction of this chapter. We wish to determine the relatively weak extrema of I. Suppose that F is of class C^2 with respect to its arguments and that y realizes an extremum on a set of admissible functions $\mathcal{U}$, defined by

$$\mathcal{U} = \left\{ Y \in C^2([x_1, x_2]), Y(x_1) = y_1, Y(x_2) = y_2 \right\}, \tag{6.1.9}$$

where y_1, y_2 are given real numbers. It is natural to search for this extremum among the functions in a neighbourhood of order 1 of y. In particular, the functions of the type

$$Y(x) = y(x) + \varepsilon\eta(x), \tag{6.1.10}$$

where η is a function of class $C^2([x_1, x_2])$, vanishing at x_1, x_2, belong to such a neighbourhood, as

$$\|Y - y\| < \varepsilon\|\eta\|. \tag{6.1.11}$$

Moreover, due to the continuity of the derivatives of the three functions y, Y, η, we also have the inequality

$$\|Y - y\|_1 < \varepsilon\|\eta\|_1. \tag{6.1.12}$$

Let us replace y by Y in (6.1.1). For a fixed up η, we get an integral depending on the parameter ε

$$J(\varepsilon) \equiv \int_{x_1}^{x_2} F(x, y(x) + \varepsilon\eta(x), y'(x) + \varepsilon\eta'(x))\mathrm{d}x, \tag{6.1.13}$$

that must be maximum or minimum at $\varepsilon = 0$, as a function of ε. Therefore, the becessary condition of extremum is

$$\left.\frac{\mathrm{d}J(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0. \tag{6.1.14}$$

As the conditions of differentiation of the integral (6.1.13) with respect to the parameter ε are fulfilled, we can write

$$\left.\frac{\mathrm{d}J(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0}=\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\eta(x)+\frac{\partial F}{\partial y'}\eta'(x)\right]\mathrm{d}x\,. \tag{6.1.15}$$

Integration by parts yields

$$\int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\eta'(x)\mathrm{d}x=\left[\eta(x)\frac{\partial F}{\partial y'}\right]_{x=x_1}^{x=x_2}-\int_{x_1}^{x_2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\eta(x)\mathrm{d}x\,. \tag{6.1.16}$$

As $\eta(x_1)=0,\ \eta(x_2)=0$, the first term in the right member of (6.1.16) vanishes and the condition (6.1.14) eventually becomes

$$\int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\mathrm{d}x=0\,. \tag{6.1.17}$$

This equality is satisfied for any η of class $\mathrm{C}^2([x_1,x_2])$, vanishing at x_1,x_2. The function in square brackets is also continuous, by our initial assumption on *F*. Therefore we can apply the fundamental lemma and it results that *y* must satisfy

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)=0,\quad x\in[x_1,x_2]\,. \tag{6.1.18}$$

This is a second order ODE, called *Euler's equation*. So, we proved the following theorem

Theorem 6.1 (Euler*). Suppose that F is of class C^2 with respect to its arguments and that y realizes an extremum on a set of admissible functions* $\mathcal{U}$, *defined by* (6.1.9). *Then y must satisfy Euler's equation* (6.1.18).

The reciprocal of this theorem is not always true. The solutions of Euler's equation are called *extremals*, even if they do not realize an extremum for (6.1.1).

1.3 FUNCTIONALS OF THE FORM $\mathrm{I}[y]\equiv\int_{x_1}^{x_2}F\left(x,y,y',y'',\ldots,y^{(n)}\right)\mathrm{d}x$

Let us consider now the case of an integrand depending on higher order derivatives of *y*. Let $\mathrm{I}[y]$ be of the form

$$\mathrm{I}[y]\equiv\int_{x_1}^{x_2}F\left(x,y(x),y'(x),y''(x),\ldots,y^{(n)}(x)\right)\mathrm{d}x\,, \tag{6.1.19}$$

where *F* is of class C^{n+1} in its arguments. We wish to get relatively weak extrema for $\mathrm{I}[y]$ on the set $\mathcal{U}$ of the function of class $\mathrm{C}^{2n}([x_1,x_2])$, satisfying the conditions

$$\begin{aligned} &y(x_1)=y_{11},\, y'(x_1)=y_{12},\, y''(x_1)=y_{13},\dots, y^{(n-1)}(x_1)=y_{1n},\\ &y(x_2)=y_{21},\, y'(x_2)=y_{22},\, y''(x_2)=y_{23},\dots, y^{(n-1)}(x_2)=y_{2n}, \end{aligned} , \tag{6.1.20}$$

where y_{jk}, $j=1,2,\ k=\overline{1,n}$, are given constants.

Let $y \in \mathcal{U}$ realizing a relatively weak extremum of (6.1.19). As in the previous case, we shall consider variations of the function *y* of the form

$$Y(x)=y(x)+\varepsilon\eta(x), \tag{6.1.21}$$

where $\eta \in \mathrm{C}^{2n}([x_1,x_2])$ is an arbitrary function, vanishing together with its first $(n-1)$ derivatives at x_1, x_2. Replacing *y* by *Y* in (6.1.19), we get, for a fixed up η, the integral depending on the parameter ε

$$J(\varepsilon) \equiv \int_{x_1}^{x_2} F\left(x, y+\varepsilon\eta, y'+\varepsilon\eta', \dots, y^{(n-1)}+\varepsilon\eta^{(n-1)}\right)\mathrm{d}x\,; \tag{6.1.22}$$

This integral allows, as a function of ε, an extremum at $\varepsilon = 0$, therefore

$$\left.\frac{\mathrm{d}J(\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y}\eta(x)+\frac{\partial F}{\partial y'}\eta'(x)+\dots+\frac{\partial F}{\partial y^{(n)}}\eta^{(n)}(x)\right]\mathrm{d}x = 0\,. \tag{6.1.23}$$

Integrating by parts, taking into account the fundamental lemma and the conditions satisfied by η, we deduce for the functional (6.1.19) the following ODE, of order $2n$

$$\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial y''}\right)-\dots+(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)=0, \quad x\in[x_1,x_2]. \tag{6.1.24}$$

This is *Euler-Poisson's equation*. Thus, we have proved

Theorem 6.2 (Euler-Poisson). *Suppose that F is of class* C^{n+1} *with respect to its arguments and that* $y \in \mathrm{C}^{2n}([x_1,x_2])$ *y realizes an extremum on the set* $\mathcal{U}$ *of admissible functions, defined by* (6.1.20) . *Then y must satisfy Euler-Poisson's equation* (6.1.24).

Let us note that Euler's equation is a particular case (for $n=1$) of Euler-Poisson's equation.

1.4 FUNCTIONALS OF INTEGRAL TYPE, DEPENDING ON *n* FUNCTIONS

Let us consider necessary conditions of extrema for functionals of the type

$$\mathrm{I}[y_1,y_2,\dots,y_n] \equiv \int_{x_1}^{x_2} F(x, y_1, y_2,\dots,y_n, y_1', y_2',\dots,y_n')\mathrm{d}x\,, \tag{6.1.25}$$

where the integrand F depends on n unknown functions $y_1, y_2, ..., y_n$ and on their derivatives of first order.
Considering the vector functions

$$\mathbf{y}(x) = [y_1(x), y_2(x), ..., y_n(x)], \quad \mathbf{y}'(x) = [y_1'(x), y_2'(x), ..., y_n'(x)], \quad x \in [x_1, x_2], \tag{6.1.26}$$

we can simplify the form of (6.1.25)

$$\mathrm{I}[\mathbf{y}] \equiv \int_{x_1}^{x_2} F(x, \mathbf{y}(x), \mathbf{y}'(x))\,\mathrm{d}x\,. \tag{6.1.27}$$

We shall search for the relatively weak extrema of I on the class $\mathcal{U}$ of admissible functions, defined by

$$\mathcal{U} = \left\{ \mathbf{Y} \in \left(\mathrm{C}^2([x_1, x_2])\right)^n, \mathbf{Y}(x_1) = \mathbf{y}_1, \mathbf{Y}(x_2) = \mathbf{y}_2 \right\}, \tag{6.1.28}$$

where the constant vectors

$$\mathbf{y}_1 = [y_{11}, y_{12}, ..., y_{1n}], \quad \mathbf{y}_2 = [y_{21}, y_{22}, ..., y_{2n}] \tag{6.1.29}$$

are considered known.
Suppose that $\mathbf{y} \in \mathcal{U}$ realizes a relatively weak extremum for I. As previously, consider variations of $\mathbf{y}$ of the form

$$\mathbf{Y}(x) \equiv [Y_1(x), Y_2(x), ..., Y_n(x)], \quad Y_j(x) = y_j(x) + \varepsilon_j \eta_j(x), \; j = \overline{1, n}, \tag{6.1.30}$$

where ε_j are small parameters and η_j are arbitrary $\mathrm{C}^2([x_1, x_2])$-functions, vanishing at x_1, x_2. Replacing $\mathbf{y}$ by $\mathbf{Y}$ in (6.1.27), we get, for fiexd up η_j, a function J depending on the n parameters $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$, written as $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, ..., \varepsilon_n]$

$$J(\boldsymbol{\varepsilon}) \equiv \int_{x_1}^{x_2} F(x, y_1 + \varepsilon_1\eta_1, y_2 + \varepsilon_2\eta_2, ..., y_n + \varepsilon_n\eta_n, ..., y_1' + \varepsilon_1\eta_1', ..., y_n' + \varepsilon_n\eta_n')\,\mathrm{d}x\,. \tag{6.1.31}$$

This function allows an extremum for $\boldsymbol{\varepsilon} = \mathbf{0}$, therefore

$$\left.\frac{\partial J(\boldsymbol{\varepsilon})}{\partial \varepsilon_k}\right|_{\boldsymbol{\varepsilon}=0} = 0, \quad k = \overline{1, n}\,; \tag{6.1.32}$$

this immediately yields

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y_k} \eta_k(x) + \frac{\partial F}{\partial y_k'} \eta_k'(x) \right] \mathrm{d}x = 0, \quad k = \overline{1, n}\,. \tag{6.1.33}$$

Integrating by parts the terms containing $\eta'_k(x)$, taking into account that $\eta_k(x_1)=0,\ \eta_k(x_2)=0,\ k=\overline{1,n}$ and, eventually, applying the fundamental lemma, we deduce for **y** the second order ODS

$$
\begin{aligned}
&\frac{\partial F}{\partial y_1}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'_1}\right)=0,\\
&\frac{\partial F}{\partial y_2}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'_2}\right)=0,\\
&\dots\dots\dots\dots\dots\dots\\
&\frac{\partial F}{\partial y_n}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'_n}\right)=0,
\end{aligned}
\tag{6.1.34}
$$

that must be satisfied on $x\in[x_1,x_2]$. This is called *the Euler-Lagrange system*.
Thus, we proved the

Theorem 6.3 (Euler-Lagrange). *Suppose that F is of class* C^2 *with respect to its arguments and that* $\mathbf{y}\in\left(\mathrm{C}^2([x_1,x_2])\right)^n$ *y realizes an extremum on the set* $\mathcal{U}$ *of admissible functions*, *defined by* (6.1.28). *Then* **y** *must satisfy Euler-Lagrange'system* (6.1.34).

As in the previous cases, all the solutions of the Euler-Lagrange system will be called *extremals*.
Let us note that the necessary conditions of extrema emphasized in this chapter may be also expressed in a significant form by introducing the notions of variation of a functional, of Gâteaux and Fréchet derivatives.

2. Conditional Extrema

In certain cases, one must search for extrema of functionals on classes of admissible functions that must satisfy supplementary conditions, expressed in terms of functions or integrals. We shall tackle here variational problems of isoperimetric and Lagrange type.

2.1 ISOPERIMETRIC PROBLEMS

The isoperimetric problem consists of finding the extrema of a functional

$$\mathrm{I}[\mathbf{y}]\equiv\int_{x_1}^{x_2}F(x,\mathbf{y}(x),\mathbf{y}'(x))\,\mathrm{d}x\,, \tag{6.2.1}$$

where

$$\mathbf{y}(x)=[y_1(x),y_2(x),\dots,y_n(x)],\quad \mathbf{y}'(x)=[y'_1(x),y'_2(x),\dots,y'_n(x)],\quad x\in[x_1,x_2]\,, \tag{6.2.2}$$

on a set $\mathcal{U}$ of admissible functions, satisfying the standard conditions

$$\mathbf{y}_1 = [y_{11}, y_{12}, ..., y_{1n}], \quad \mathbf{y}_2 = [y_{21}, y_{22}, ..., y_{2n}] \tag{6.2.3}$$

and also the supplementary conditions

$$\mathrm{I}_j[\mathbf{y}] \equiv \int_{x_1}^{x_2} G_j(x, \mathbf{y}(x), \mathbf{y}'(x))\,\mathrm{d}x = a_j, \quad j = \overline{1, p}, \tag{6.2.4}$$

a_j being given constants.

The term isoperimetric comes from Greek, meaning the same perimeter.
This type of problem is also called Dido's problem; this denomination has roots of history and legend. Dido – or Didona – the legendary founder of Carthago (Carthagena) was a Phoenician queen, obliged to leave hastily her country because of a plot put in application by her brother. Once on the African coast, Dido and her faithful servants required hospitality and a place to settle up from the natives. The local king's diplomatic answer was positive, but, in fact, he offered them as much land as could be held by a bull's skin. The fugitives were highly disappointed, but Dido did not immediately reject the offer; she promised a firm answer for the next morning. During the night, she cut the bull skin in thin stripes and, joining them one by one, she succeded to cover a great piece of land, with the skin stripes as a perimeter. So, the natives gave up and Dido settled up on that land, building Carthagena after a while.
While her idea was fruitful, obviously, Dido was not initiated in modern variational calculus. Yet, her problem can be easily put in mathematical terms. In the *xOy* plane, denote by Γ the smooth closed curve that limits the plane domain *D*. The area $\mathcal{A}$ of *D* is then given by Green's formula

$$\mathcal{A} = \frac{1}{2}\oint_{\Gamma} x dy - y dx\,. \tag{6.2.5}$$

The curve Γ has a fixed length *l*, as, according to the problem, the perimeter is the same, therefore

$$\oint_{\Gamma} \mathrm{d}s = l\,, \tag{6.2.6}$$

where d*s* is the element of arclength on Γ. So, Dido's problem consists of getting a maximum value of (6.2.5) if (6.2.6) is fulfilled. Considering a parametrization of Γ, we immediately obtain a variational problem of isoperimetric type, whose solution should be a circle. In general, in the plane, $\mathcal{A} \leq l^2 / 4\pi$.
Suppose that the functional I depends only on one argument, i.e.

$$\mathrm{I}[y] \equiv \int_{x_1}^{x_2} F(x, y(x), y'(x))\,\mathrm{d}x \tag{6.2.7}$$

and only one supplementary condition must be fulfilled

$$\mathrm{I}[y] \equiv \int_{x_1}^{x_2} G(x, y(x), y'(x))\mathrm{d}x = a . \tag{6.2.8}$$

Then one can prove by using, as a rule, the same techniques as in the previous section, that if y realizes an extremum for I and satisfies the condition (6.2.8) and also the conditions $y(x_1) = y_1$, $y(x_2) = y_2$, then one can find a constant λ such that y be a free extremum for the functional

$$\mathrm{K}[y] \equiv \int_{x_1}^{x_2} [F(x, y(x), y'(x)) + \lambda G(x, y(x), y'(x))]\mathrm{d}x . \tag{6.2.9}$$

Thus, the problem of a conditional extremum was reduced to that of a free one, similar to those treated at Sec.1. According to Theorem 6.1, if y realizes an extremum for K, then y satisfies

$$\frac{\partial F}{\partial y} + \lambda \frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'} + \lambda \frac{\partial G}{\partial y'}\right) = 0 . \tag{6.2.10}$$

a) Consider now the general isoperimetric problem, stated at the beginning of this section. Suppose that the admissible functions $y_1, y_2, ..., y_n$ realize an extremum for the functional (6.2.1), but not for any of the functionals (6.2.4). In this case, one can prove, again by using the calculus of variations, that one can find p constants, $\lambda_1, \lambda_2, ..., \lambda_p$, such that $y_1, y_2, ..., y_n$ realize a free extremum for the functional

$$\mathrm{K}[\mathbf{y}] \equiv \int_{x_1}^{x_2} \left[F(x, \mathbf{y}(x), \mathbf{y}'(x)) + \sum_{j=1}^{p} \lambda_j G_j(x, \mathbf{y}(x), \mathbf{y}'(x)) \right] \mathrm{d}x . \tag{6.2.11}$$

Applying to the integrand the Euler-Lagrange system (6.1.34), we find the ODS

$$\frac{\partial F}{\partial y_j} + \sum_{k=1}^{p} \lambda_k \frac{\partial G_k}{\partial y_j} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'_j}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=1}^{p} \lambda_k \frac{\partial G_k}{\partial y'_j}\right) = 0, \quad j = \overline{1, n}, \tag{6.2.12}$$

whose solutions are the extremals of the considered isoperimetric problem.

2.2 LAGRANGE'S PROBLEM

a) We shall firstly state this problem for the functional

$$\mathrm{I}[y_1, y_2] \equiv \int_{x_1}^{x_2} F(x, y_1(x), y_2(x), y'_1(x), y'_2(x))\mathrm{d}x . \tag{6.2.13}$$

Let us find an arc C, of equations $y_1 = y_1(x), y_2 = y_2(x), x \in [x_1, x_2]$, laying on the surface S of equation

$$G(x, y_1, y_2) = 0 \tag{6.2.14}$$

and for which the functional (6.2.13) realizes an extremum. The co-ordinates of the arc ends will be $(x_1, y_1(x_1), y_2(x_1)), (x_2, y_1(x_2), y_2(x_2))$. Let us denote

$$\begin{aligned} y_1(x_1) &= y_{11}, \quad y_1(x_2) = y_{12}, \\ y_2(x_1) &= y_{21}, \quad y_2(x_2) = y_{22}. \end{aligned} \tag{6.2.15}$$

As C lays on S, so will its extremities, therefore

$$G(x_1, y_{11}, y_{21}) = 0, \quad G(x_2, y_{12}, y_{22}) = 0. \tag{6.2.16}$$

If $\partial G / \partial y_2 \neq 0$ along the extremal, then we can explicit y_2 from (6.2.14)

$$y_2 = \varphi(x, y_1). \tag{6.2.17}$$

Introducing this in (6.2.13), we finally get a new functional, depending only on the argument y_1

$$\tilde{\mathrm{I}}[y_1] \equiv \int_{x_1}^{x_2} \Phi(x, y_1(x), y_1'(x))\mathrm{d}x. \tag{6.2.18}$$

with Φ given by

$$\Phi(x, y_1(x), y_1'(x)) = F\left(x, y_1, \varphi(x, y_1), y_1', \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y_1} y_1'\right). \tag{6.2.19}$$

The corresponding Euler equation is immediately brought to the form

$$\frac{\partial F}{\partial y_1} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_1'}\right) + \frac{\partial\varphi}{\partial y_1}\left[\frac{\partial F}{\partial y_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_2'}\right)\right] = 0. \tag{6.2.20}$$

Replacing y_2 by $\varphi(x, y_1)$ in (6.2.13), we deduce

$$\frac{\partial G}{\partial y_1} + \frac{\partial F}{\partial y_2}\frac{\partial\varphi}{\partial y_1} = 0 \tag{6.2.21}$$

and, eliminating $\partial\varphi / \partial y_1$ between (6.2.20) and (6.2.21), we get

$$\frac{\dfrac{\partial F}{\partial y_1} - \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\partial F}{\partial y_1'}\right)}{\dfrac{\partial G}{\partial y_1}} = \frac{\dfrac{\partial F}{\partial y_2} - \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\partial F}{\partial y_2'}\right)}{\dfrac{\partial G}{\partial y_2}}. \tag{6.2.22}$$

Denoting by $-\lambda(x)$ the common value of the ratios (6.2.22) along the extremal, we have

$$\begin{aligned}&\frac{\partial F}{\partial y_1}+\lambda(x)\frac{\partial G}{\partial y_1}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_1'}\right)=0,\\&\frac{\partial F}{\partial y_2}+\lambda(x)\frac{\partial G}{\partial y_2}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y_2'}\right)=0.\end{aligned} \qquad (6.2.23)$$

As $\partial G/\partial y_j'=0,\ j=1,2$, the above relations may be also written in the form

$$\frac{\partial H}{\partial y_j}-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial H}{\partial y_j'}\right)=0,\quad j=1,2,\quad H=F+\lambda(x)G. \qquad (6.2.24)$$

These are the necessary conditions of extremum for Lagrange's problem in the case of a functional depending on two arguments. Let us note that the ODS (6.2.24) is in fact the Euler-Lagrange system, written for the functional

$$\mathrm{K}[y_1,y_2]\equiv\int_{x_1}^{x_2}[F(x,y_1(x),y_2(x),y_1'(x),y_2'(x))+\lambda(x)G(x,y_1(x),y_2(x))]\mathrm{d}x. \qquad (6.2.25)$$

b) Lagrange's problem for functionals

$$\mathrm{I}[\mathbf{y}]\equiv\int_{x_1}^{x_2}F(x,\mathbf{y}(x),\mathbf{y}'(x))\mathrm{d}x, \qquad (6.2.26)$$

depending on several arguments – or, equivalently, on a vector function $\mathbf{y}$ – consists of finding a vector function $\mathbf{y}=(y_1,y_2,...,y_n)$, at least of class $\left(\mathrm{C}^2([x_1,x_2])\right)^n$, satisfying (6.2.15) as well as the supplementary conditions

$$G_j(x,\mathbf{y})=a_j,\quad j=\overline{1,p}. \qquad (6.2.27)$$

As previously, this problem may be reduced to a problem of free extremum for a certain functional. More precisely, if $\mathbf{y}$ realizes an extremum for the Lagrange problem b), then one can find p functions $\lambda_1(x),\lambda_2(x),...,\lambda_p(x)$ such that y is an extremal for the functional

$$\mathrm{K}[\mathbf{y}]\equiv\int_{x_1}^{x_2}\left[F(x,\mathbf{y}(x),\mathbf{y}'(x))+\sum_{j=1}^{p}\lambda_j(x)G_j(x,\mathbf{y}(x))\right]\mathrm{d}x. \qquad (6.2.28)$$

The equations of the extremals are, in this case

$$\frac{\partial H}{\partial y_j}-\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial H}{\partial y_j'}\right]=0,\quad j=\overline{1,n},\quad H=F+\sum_{j=1}^{p}\lambda_j(x)G_j. \qquad (6.2.29)$$

3. Applications

Application 6.1

Problem. Study the motion of a discrete mechanical system of n particles subjected to holonomic constraints and situated in a field of quasi-conservative forces, using Hamilton's variational principle.

Mathematical model. Let be a system of n particles $P_j(x, y_j, z_j)$ of masses m_j, $j = \overline{1, n}$, subjected to m *holonomic* (geometric) *constraints*

$$f_k = f_k(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n; t) = 0\,,\ k = \overline{1, m}\,, \tag{a}$$

and acted upon by *quasi-conservative forces*

$$\mathbf{F}_j = \mathbf{F}_j(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n; t),\ j = \overline{1, n}\,,$$

which derives from *the simple quasi-potential*

$$U = U(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n; t);$$

hence, the components of the forces $\mathbf{F}_j$ which acts upon the particle P_j are

$$X_j = \frac{\partial U}{\partial x_j}\,,\ Y_j = \frac{\partial U}{\partial y_j}\,,\ Z_j = \frac{\partial U}{\partial z_j}\,. \tag{b}$$

In general, the constraints (a) are *rheonomic* (time appears explicitly); if $\dot{f}_k = \partial f_k / \partial t = 0$, hence if it does not depend explicitly on time, then the constraints are *scleronomic*. As well, if $\dot{U} = \partial U / \partial t = 0$, hence if U does not depend explicitly on time, then the quasi-potential is a *simple potential* and the given forces are *conservative*.

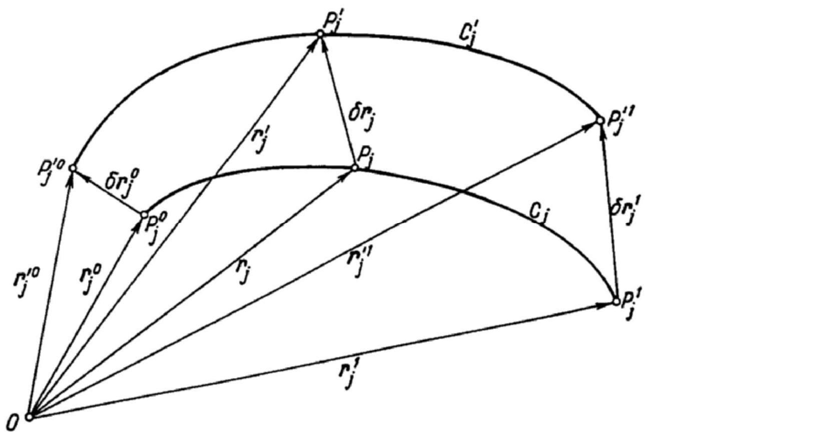

Figure 6. 1. True curve. Various paths

We introduce the kinetic energy

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j v_j^2 = \frac{1}{2}\sum_{j=1}^{n} m_j \left(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2\right) \tag{c}$$

too, where $\mathbf{v}_j$ is the velocity of the particle P_j.
The sum

$$\mathsf{L} = T + U \tag{d}$$

represents *the kinetic potential of Lagrange* (the Lagrangian) in the absence of holonomic constraints. The integral

$$\mathsf{A} = \\ = \int_{t_0}^{t_1} \mathsf{L}\left(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n, \dot{x}_1, \dot{y}_1, \dot{z}_1, \dot{x}_2, \dot{y}_2, \dot{z}_2, \ldots, \dot{x}_n, \dot{y}_n, \dot{z}_n; t\right) \mathrm{d}t \tag{e}$$

is called *Lagrangian action* and represents a functional which plays an important role in mechanics. We state that:
The motion of a discrete mechanical system of free particles takes place only if the Lagrangian action has a minimum (*Hamilton's principle*).

Solution. *The Euler-Lagrange equations* corresponding to an extremum of this functional are written in the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathsf{L}}{\partial \dot{x}_j}\right) - \frac{\partial \mathsf{L}}{\partial x_j} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathsf{L}}{\partial \dot{y}_j}\right) - \frac{\partial \mathsf{L}}{\partial y_j} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathsf{L}}{\partial \dot{z}_j}\right) - \frac{\partial \mathsf{L}}{\partial z_j} = 0, \end{aligned} \tag{f}$$

for $j = \overline{1, n}$. Taking into account (b), (c) and (d), we find Newton's equations of motion (second principle of mechanics) in the form of a theorem

$$m_j \ddot{x}_j = X_j,\ m_j \ddot{y}_j = Y_j,\ m_j \ddot{z}_j = Z_j,\ j = \overline{1, n}. \tag{g}$$

If we take into account the holonomic constraints (a), then we may introduce a Lagrangian of the form

$$\overline{\mathsf{L}} = T + U + \sum_{i=1}^{m} \lambda_i f_i, \tag{h}$$

where λ_i are *Lagrange's multipliers*. The corresponding Euler-Lagrange equations lead to

$$m_j\ddot{x}_j = X_j + \sum_{i=1}^{m}\lambda_i\frac{\partial f_i}{\partial \dot{x}_j},$$
$$m_j\ddot{y}_j = Y_j + \sum_{i=1}^{m}\lambda_i\frac{\partial f_i}{\partial \dot{y}_j}, \tag{i}$$
$$m_j\ddot{z}_j = Z_j + \sum_{i=1}^{m}\lambda_i\frac{\partial f_i}{\partial \dot{z}_j},$$

for $j=\overline{1,n}$.

Let us consider now the particle P_j of position vector $\mathbf{r}_j$, the trajectory of which, due to the given forces which act upon it, is the arc of curve C_j, contained between the points P_j^0 and P_j^1, corresponding to the initial moment t_0 and to the final moment t_1, respectively. By a virtual displacement $\delta\mathbf{r}_j$, we obtain the point P_j'; from the set of virtual displacements $\delta\mathbf{r}_j$ we choose those which are uniquely obtained, travelling through from P_j^0 to P_j^1, the locus of the points P_j' being a *varied path* C_j' (Fig.6.1).

An infinity of varied paths are thus obtained and we may write

$$\mathbf{r}_j' = \mathbf{r}_j + \delta\mathbf{r}_j. \tag{j}$$

Starting from Newton's equations, in the case of holonomic constraints, we obtain the principle of virtual work (*the d'Alembert-Lagrange principle*) in the form of a theorem, i.e.

$$\sum_{j=1}^{n}\mathbf{\Phi}_j\cdot\delta\mathbf{r}_j = 0, \tag{k}$$

where we have introduced *the lost forces of d'Alembert*

$$\mathbf{\Phi}_j = \mathbf{F}_j - m_j\ddot{\mathbf{r}}_j,\ j=\overline{1,n}; \tag{l}$$

thus, the dynamical problem has been reduced to a statical one, by eliminating from computation the constraint forces. We may also write

$$\sum_{j=1}^{n}\int_{t_0}^{t_1}\mathbf{\Phi}_j\cdot\delta\mathbf{r}_j\mathrm{d}t = 0. \tag{m}$$

The fundamental lemma of the variational calculus allows to show that the relations (k) and (m) are equivalent.

We calculate

$$\sum_{j=1}^{n}\int_{t_0}^{t_1} m_j \ddot{\mathbf{r}}_j \cdot \delta\mathbf{r}_j \mathrm{d}t = \sum_{j=1}^{m}\int_{t_0}^{t_1} m_j \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\mathbf{r}}_j \cdot \delta\mathbf{r}_j\right)\mathrm{d}t - \sum_{j=1}^{m}\int_{t_0}^{t_1} m_j \dot{\mathbf{r}}_j \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta\mathbf{r}_j\right)\mathrm{d}t$$

$$= \sum_{j=1}^{n} m_j \dot{\mathbf{r}}_j \cdot \delta\mathbf{r}_j \Bigg|_{t_0}^{t_1} - \sum_{j=1}^{n}\int_{t_0}^{t_1} m_j \mathbf{v}_j \cdot \delta\mathbf{v}_j \mathrm{d}t = \sum_{j=1}^{n} m_j \mathbf{v}_j \cdot \delta\mathbf{r}_j \Bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \delta T \mathrm{d}t \,,$$

taking into account the operational relation $(\mathrm{d}/\mathrm{d}t)\delta = \delta(\mathrm{d}/\mathrm{d}t)$; we may also write

$$\sum_{j=1}^{n} \mathbf{F}_j \cdot \delta\mathbf{r}_j = \delta W \,, \tag{n}$$

where δW is the virtual work of the given forces. The relation leads thus to

$$\int_{t_0}^{t_1} \left(\delta T + \delta W\right)\mathrm{d}t = \sum_{j=1}^{n} m_j \mathbf{v}_j \cdot \delta\mathbf{r}_j \Bigg|_{t_0}^{t_1} . \tag{o}$$

This relation represents a general integral theorem; starting from this theorem (considered as to be a principle) one may obtain various integral and variational principles. The relation (o) corresponds to a *synchronous case*, in which the chronology (hence the time variable) is the same for all the varied paths.

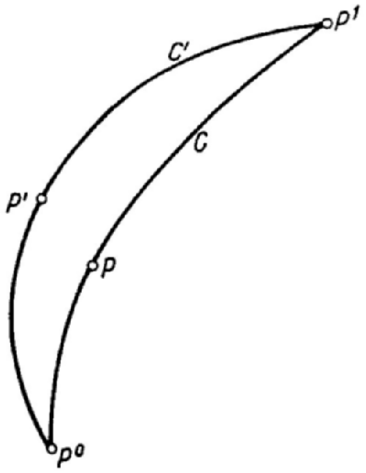

Figure 6. 2. Various paths with fixed ends

In the particular case of *the varied paths with fixed ends* (Fig.6.2) we have $\delta\mathbf{r}_j^0 = \delta\mathbf{r}_j^1 = \mathbf{0}$, so that the general integral principle becomes

$$\int_{t_0}^{t_1} \left(\delta T + \delta W\right)\mathrm{d}t = 0 \,. \tag{p}$$

In the case of given conservative forces (or, more general, quasi-conservative) we have $U = U(\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n; t)$ so that $\delta W = \delta U$. Introducing the Lagrangian (d) and taking into account the permutability of the operator δ with the operator integral, we obtain

$$\delta \mathsf{A} = \delta \int_{t_0}^{t_1} \mathsf{L} \, \mathrm{d}t = 0 ; \qquad \text{(q)}$$

we can thus state *Hamilton's principle* in the form:

Among all possible motions of a discrete mechanical system subjected to holonomic and ideal constraints and acted upon by quasi-conservative forces on synchronous varied paths with fixed ends, only and only the motion for which the variation of the Lagrangian action vanishes (the extremal curves) takes place.

We have obtained Hamilton's principle in the form of a theorem; starting from this result, considered as to be a principle, we find again Newton's equations as a theorem. We mention that Newton's equations have a general character, while Hamilton's principle may be applied only in the case of the existence of a Lagrangian of the mechanical system.

This principle was enunciated in 1834 by W. R. Hamilton for scleronomic constraints; it was extended by M. V. Ostrogradski in 1848 to the case of rheonomic constraints. One observes that, unlike differential principles (in which, to establish the motion at a given moment, one considers only the motion in the vicinity of this one), in case of variational principles the motion of the mechanical system at a given moment is specified by its motion in the whole (finite) interval of time.

Application 6.2

Problem. Establish Lagrange's equations of motion, corresponding to a discrete mechanical system *S* of n particles, subjected to holonomic constraints and situated in a field of quasi-conservative forces, in the configuration space Λ_s, using Hamilton's variational principle.

Mathematical model. Let be a system S of n particles $P_j(x_j, y_j, z_j)$, $j = \overline{1,n}$, subjected to m holonomic (geometric) constraints

$$f_i = f_i(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_n, y_n, z_n; t) = 0 , \ i = \overline{1,m} . \qquad \text{(a)}$$

In the space E_3, the system of particles, considered as to be free, has $3n$ degrees of freedom, being necessary $3n$ parameters to fix its position. But we may introduce a representative space E_{3n} with $3n$ dimensions, in which the position of a representative point P is specified by $3n$ co-ordinates X_k, $k = \overline{1,3n}$, which may be chosen, e.g., in the form $X_1 = x_1$, $X_2 = y_1$, $X_3 = z_1$, $X_4 = x_2$, …, $X_{3n} = z_n$. Hence, the position of the mechanical system S in the space E_3 is specified by the position of the representative point P in the space E_{3n}. The presence of m holonomic constraints (a), expressed in a finite form, diminishes the number of degrees of freedom of the system S

to $s = 3n - m$; hence, there are necessary s parameters to specify the position of this system. Let q_1, q_2, …, q_s be such a set of parameters, obtained by eliminating the holonomic constraints. We introduce now a space Λ_s with s dimensions, called *Lagrange's space*, in which the position of a representative point P is specified by the generalized co-ordinates q_1, q_2, …, q_s. If we know the position of the representative point P in the space Λ_s, then we know the position (or the configuration) of the mechanical system S in the space E_3; hence the space Λ_s is called also *the space of configurations*. A great advantage is the fact that the representative point P is a free point (non-subjected to any constraints, which have been eliminated) in the space Λ_s; as well, we notice that $s \leq 3n$.

The kinetic potential L of Lagrange (*the Lagrangian*) introduced in Appl.6.1, is a function of the position of the particles of the system and of their velocities. We notice that one passes from the space E_3 to the space Λ_s by relations of the form

$$\mathbf{r}_j = \mathbf{r}_j\left(q_1, q_2, \ldots, q_s; t\right),\ j = \overline{1, n}\,; \tag{b}$$

for velocities, we may write

$$\mathbf{v}_j = \frac{\mathrm{d}\mathbf{r}_j}{\mathrm{d}t} = \sum_{k=1}^{s} \frac{\partial \mathbf{r}_j}{\partial q_k} \frac{\mathrm{d}q_k}{\mathrm{d}t} + \frac{\partial \mathbf{r}_j}{\partial t} = \sum_{k=1}^{s} \frac{\partial \mathbf{r}_j}{\partial q_k} \dot{q}_k + \dot{\mathbf{r}}_j,\ j = \overline{1, n}, \tag{c}$$

where, by analogy, $\dot{q}_k$ are *the generalized velocities*. In this case, the Lagrangian L is expressed in the space Λ_s in he form

$$\mathsf{L} = \mathsf{L}\left(q_1, q_2, \ldots q_s, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_s\right),\ k = \overline{1, s}, \tag{d}$$

where $q_k = q_k(t)$. To obtain the extremum of the Lagrangian action A, given by the formula (c) in Appl.6.1, we may write the corresponding Euler-Lagrange equations in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathsf{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathsf{L}}{\partial q_k} = 0,\ k = \overline{1, s}. \tag{e}$$

These equations are Lagrange's equations of second species (shortly, Lagrange's equations) which specify the motion of the representative point P in the space Λ_s. It is a system of s differential equations of second order in the unknown functions $q_k = q_k(t)$, $k = \overline{1, s}$. By integration, one introduces $2s$ arbitrary constants which are determined by conditions of Cauchy type (at the initial moment t_0)

$$q_k\left(t_0\right) = q_k^0,\ \dot{q}_k\left(t_0\right) = \dot{q}_k^0,\ k = \overline{1, s}. \tag{f}$$

Solution. As it was asked, the first variation of the Lagrangian action may be directly calculated in the form

$$\delta\int_{t_0}^{t_1}\mathsf{L}(q_1,q_2,\ldots q_s,\dot q_1,\dot q_2,\ldots\dot q_s;t)\mathrm{d}t=\int_{t_0}^{t_1}\delta\mathsf{L}\mathrm{d}t=\int_{t_0}^{t_1}\sum_{k=1}^{s}\left(\frac{\partial\mathsf{L}}{\partial q_k}\delta q_k+\frac{\partial\mathsf{L}}{\partial\dot q_k}\delta\dot q_k\right)\mathrm{d}t\,;$$

but

$$\sum_{k=1}^{s}\frac{\partial\mathsf{L}}{\partial\dot q_k}\delta\dot q_k=\sum_{k=1}^{s}\frac{\partial\mathsf{L}}{\partial\dot q_k}\frac{\mathrm{d}}{\mathrm{d}t}(\delta q_k)=\sum_{k=1}^{s}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathsf{L}}{\partial\dot q_k}\delta q_k\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathsf{L}}{\partial\dot q_k}\right)\delta q_k\right]$$

and

$$\int_{t_0}^{t_1}\sum_{k=1}^{s}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathsf{L}}{\partial\dot q_k}\delta q_k\right)\mathrm{d}t=\sum_{k=1}^{s}\frac{\partial\mathsf{L}}{\partial\dot q_k}\delta q_k\Bigg|_{t_0}^{t_1}=0\,,$$

because the varied paths are wit fixed ends. Consequently, we remain with

$$\int_{t_0}^{t_1}\sum_{k=1}^{s}\left[\frac{\partial\mathsf{L}}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathsf{L}}{\partial\dot q_k}\right)\right]\delta q_k\mathrm{d}t=0\,.$$

The generalized virtual displacements δq_k are independent (the holonomic constraints have been eliminated); we can thus take, in turn, one of them different of zero, the other ones being taken equal to zero, and we obtain just the equations (e).
Introducing the operator

$$[\ ]_k=\frac{\partial}{\partial q_k}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot q_k} \tag{g}$$

which generalises the operator of partial differentiation, we may write Lagrange's equations also in the form

$$[\mathsf{L}]_k=0\,,\ k=\overline{1,s}\,. \tag{h}$$

Starting from the relations (b), we notice that

$$\delta\mathbf{r}_j=\sum_{k=1}^{s}\frac{\partial\mathbf{r}_j}{\partial q_k}\delta q_k\ ,\ j=\overline{1,n}\,; \tag{i}$$

in this case, the virtual work, expressed by the relation (n) in the Appl.6.1, becomes

$$\delta W=\sum_{k=1}^{s}Q_k\delta q_k\ , \tag{k}$$

where we have introduced *the generalized forces*

$$Q_k = \sum_{j=1}^{n} \mathbf{F}_j \cdot \frac{\partial \mathbf{r}_j}{\partial q_k}, \; k = \overline{1,s} . \tag{l}$$

If we apply the above methodology of computation to the general integral principle (o) in the mentioned application, then we find

$$\int_{t_0}^{t_1} \sum_{k=1}^{s} \left[\frac{\partial T}{\partial q_k} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k} \right) + Q_k \right] \delta q_k \,\mathrm{d}t = 0 ;$$

on the basis of considerations analogous to those above, we obtain Lagrange's equations in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k , \; k = \overline{1,s} . \tag{m}$$

These equations have a more general character than the equations (e), because they correspond to arbitrary given forces. In the case of quasi-conservative generalized forces

$$Q_k = \frac{\partial U}{\partial q_k}, \; U = U(q_1, q_2, \dots, q_s; t), \; k = \overline{1,s} \tag{n}$$

we find again the equations (e).

Application 6.3

Problem. Establish Hamilton's equations of motion, corresponding to a discrete mechanical system of n particles subjected to holonomic constraints and situated in a field of conservative forces, in the phase space Γ_{2s}, using Hamilton's variational principle.

Mathematical model. Let be a system of n particles P_j, $j = \overline{1,n}$, subjected to m holonomic (geometric) constraints, which may be, generally, rheonomic. If there exists a kinetic potential $\mathsf{L} = T + U$, where T is the kinetic energy and U is the potential of quasi-conservative forces (which depend explicitly on the time), then we may write Lagrange's equations of motion in the configuration space in the form (see Appl.6.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathsf{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathsf{L}}{\partial q_k} = 0 , \; k = \overline{1,s} , \tag{a}$$

where $q_k = q_k(t)$ are the generalized co-ordinates, $\dot{q}_k = \dot{q}_k(t)$ are the generalized velocities, and $s = 3n - m$. We introduce the notation

$$p_k = \frac{\partial \mathsf{L}}{\partial \dot{q}_k}, \; k = \overline{1,s} , \tag{b}$$

where p_k are the generalized momenta; this denomination is given because, in case of only one particle $P(x, y, z)$, the Lagrangian is given by

$$\mathsf{L} = m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)/2 + U(x, y, z; t),$$

so that $p_1 = m\dot{x}$, $p_2 = m\dot{y}$, $p_3 = m\dot{z}$. Noting that

$$T = \frac{1}{2}\sum_{j=1}^{m} m_j v_j^2$$

and taking into account the expression (c) in Appl.6.2 of the velocity $\mathbf{v}_j$, we may express *the kinetic energy* in the form

$$T = T_2 + T_1 + T_0, \tag{c}$$

where T_2 is a quadratic form, positive definite in the generalized velocities, T_1 is a linear form, while T_0 is a constant with respect to those velocities. Thus, the relation (b) may be seen as a system of s linear algebraic equations, the unknowns being the generalized velocities $\dot{q}_k$; because T_2 is a positive definite quadratic form, the determinant of the coefficients of this system is just the discriminant of the quadratic form, which is non-zero. Hence, we may solve the system of equations (b) with respect to $\dot{q}_k$, obtaining $\dot{q}_k = \dot{q}_k(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$. In general, we get the Lagrangian $\mathsf{L} = \mathsf{L}(q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s; t)$; taking into account the solutions of the system of equations (b), it finally results that $\mathsf{L} = \mathsf{L}(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$.

Solution. Hamilton has introduced the space Γ_{2s} with $2s$ dimensions, called *the phase space* (or *Gibb's space*), in which the position of a representative point P_j is specified by *the canonical co-ordinates* q_1, q_2, …, q_s, p_1, p_2, …, p_s, in the given order. From the above considerations, it is seen that, by knowing the position of a representative point in the phase space Γ_{2s}, one knows the position and the velocity of a representative point in the configuration space Λ_s, hence the position of the mechanical system S in the space E_3.

We introduce *Hamilton's function* H in the form

$$H = \sum_{k=1}^{s} p_k \dot{q}_k - \mathsf{L}; \tag{d}$$

taking into account the transformation (b) (denoted also *the Legendre's transformation*), it results $H = H(q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s; t)$. In this case, Hamilton's principle (q) in Appl.6.1 is written in the form

$$\delta\int_{t_0}^{t_1}\left[\sum_{k=1}^{s} p_k\dot{q}_k - H(q_1,q_2,\ldots,q_s,p_1,p_2,\ldots,p_s;t)\right]\mathrm{d}t = 0\,, \tag{e}$$

called *the canonical form of Hamilton's principle*.
Writing the Euler-Lagrange equations corresponding to this functional, we find the equations of motion of the representative point P in the form

$$\dot{q}_k = \frac{\partial H}{\partial p_k}\,,\ \dot{p}_k = -\frac{\partial H}{\partial q_k}\,,\ k = \overline{1,s}\,; \tag{f}$$

these equations form the canonical system of equations of analytical mechanics (Hamilton's equations), hence a system of $2s$ differential equations of first order with $2s$ unknown functions $q_k = q_k(t)$, $p_k = p_k(t)$, $k = \overline{1,s}$. One introduces $2s$ integration constants determined by conditions of Cauchy type (at the initial moment t_0)

$$q_k(t_0) = q_k^0\,,\ p_k(t_0) = p_k^0\,,\ k = \overline{1,s}\,. \tag{g}$$

Passing from Lagrangian mechanics (space Λ_s) to Hamiltonian mechanics (space Γ_{2s}) the number of equations becomes double; in exchange, these ones are no more of second order, but of first one. As well, the initial conditions are homogeneous (only for the position of the representative point).
If a certain position of the representative point is given (for instance, the initial position), the canonical equations allow to determine the position of this point at any moment; thus the deterministic character of Hamiltonian mechanics is put into evidence (in fact – in general – of Newtonian mechanics).

Application 6.4

Problem. Study the problem of two particles, using Lagrange's equations in the configuration space.

Mathematical model. Consider the particles P_1 and P_2 of masses m_1 and m_2, their positions being specified by the spherical co-ordinates r_1, θ, φ, and r_2, $\pi-\theta$, $\pi+\varphi$, respectively, with respect to he centre of mass O, situated on the segment of straight line P_1P_2, so that

$$m_1 r_1 = m_2 r_2\,,\ m_2 > m_1\,. \tag{a}$$

Because, in this problem, are acting only internal forces of Newtonian attraction $\mathbf{F}_{12} = -\mathbf{F}_{21} = \left(f m_1 m_2 / r^3\right)\mathbf{r}$ (see Appl.5.1), then the centre of mass has a rectilinear and uniform motion with respect to an inertial (fixed) frame of reference. We study the motion of the two particles with respect to this point.
The kinetic energy is expressed in the form

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$
$$= \frac{1}{2} m_1 \left[\dot{r}_1^2 + r_1^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right)\right] + \frac{1}{2} m_2 \left[\dot{r}_2^2 + r_2^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right)\right], \tag{b}$$

where the velocities are expressed by spherical co-ordinates. The forces of Newtonian attraction are conservative and derive from the potential

$$U = f\,\frac{m_1 m_2}{r}\,, \tag{c}$$

where

$$r = r_1 + r_2\,. \tag{d}$$

We notice thus that the positions of the particles P_1 and P_2 are specified by the parameters r_1, r_2, θ and φ. Taking into account the relations (a) and (b), we may write

$$m_1 r_1^2 + m_2 r_2^2 = m r^2\,,\ m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 = m \dot{r}^2\,,$$

if we introduce the notation

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}\,. \tag{e}$$

Thus, the Lagrangian corresponding to this problem may be written with the aid of three generalized co-ordinates r, θ and φ, respectively, in the form

$$\mathsf{L} = \frac{1}{2} m \left[\dot{r}^2 + r^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right)\right] + f\,\frac{m_1 m_2}{r}\,. \tag{f}$$

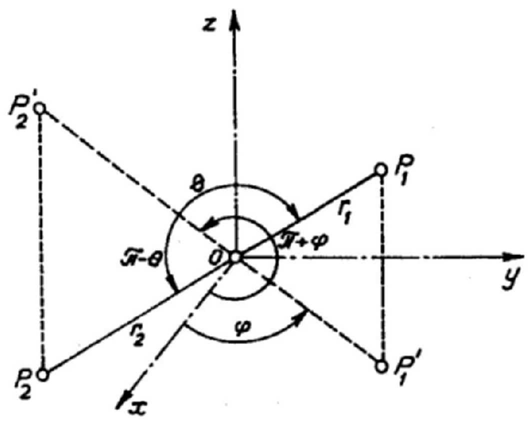

Figure 6. 3. Problem of two particles

Because

$$\frac{\partial \mathsf{L}}{\partial \dot{r}} = m\dot{r}\,,\ \frac{\partial \mathsf{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}\,,\ \frac{\partial \mathsf{L}}{\partial \dot{\varphi}} = mr^2 \sin^2\theta\dot{\varphi} \tag{g}$$

and

$$\frac{\partial \mathsf{L}}{\partial r} = mr\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right) - f\frac{m_1 m_2}{r^2}\,,\ \frac{\partial \mathsf{L}}{\partial \theta} = mr^2\dot{\varphi}^2 \sin\theta\cos\theta\,,\ \frac{\partial \mathsf{L}}{\partial \varphi} = 0\,,$$

Lagrange's equations (see Appl.6.2) read

$$\begin{aligned} &m\left[\ddot{r} - r\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right)\right] + f\frac{m_1 m_2}{r^2} = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) - r^2\dot{\varphi}^2 \sin\theta\cos\theta = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2 \sin^2\theta\dot{\varphi}\right) = 0\,; \end{aligned} \tag{h}$$

hence, we obtain a system of three differential equations with the unknown functions $r = r(t)$, $\theta = \theta(t)$ and $\varphi = \varphi(t)$.

Solution. From the very beginning, we notice that

$$r^2 \sin^2\theta\dot{\varphi} = a_1 = \text{const} \tag{i}$$

represents a first integral of the system of equations (h). Taking into account (i), the second equation (h) is written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = \frac{a_1 \cos\theta}{r^2 \sin^3\theta}\,;$$

multiplying by $2r^2\dot{\theta}$ and integrating, we obtain a new first integral

$$r^4\dot{\theta}^2 + \frac{a_1^2}{\sin^2\theta} = a_2 = \text{const}\,. \tag{j}$$

Eliminating the terms $r^2\dot{\theta}^2$ and $r\sin^2\theta\dot{\varphi}$ from the first equation (h) by means of the two first integrals obtained above, we get

$$\ddot{r} - \frac{a_2}{r^3} + f\frac{m_1 m_2}{mr^2} = 0\,;$$

multiplying then by $2\dot{r}$ and integrating, it results the third first integral

$$\dot{r}^2 + \frac{a_2}{r^2} - 2f\frac{m_1 m_2}{mr} = a_3 = \text{const}\,. \tag{k}$$

This first integral contains only one space variable, so that

$$\mathrm{d}t = \pm \frac{\mathrm{d}r}{f(r)}, \ f(r) = 2f\frac{m_1 m_2}{mr} - \frac{a_2}{r^2} + a_3 ; \tag{l}$$

hence, by a quadrature, one obtains t as a function of r and then $r = r(t)$, introducing a fourth integration constant a_4. Analogously, the first integral (j) leads to

$$\frac{\mathrm{d}r}{r^2\sqrt{f(r)}} = \pm \frac{\mathrm{d}\theta}{\sqrt{g(\theta)}}, \ g(\theta) = a_2 - \frac{a_1^2}{\sin^2\theta} ; \tag{m}$$

we obtain $\theta = \theta(r)$ and then $\varphi = \varphi(t)$, by two quadratures, introducing a new integration constant a_5. Finally, the first integral (i) allows to write

$$\mathrm{d}\varphi = \pm \frac{a_1 \mathrm{d}\theta}{\sin^2\theta\sqrt{g(\theta)}}, \tag{n}$$

where we used the previous results; hence, we obtain $\varphi = \varphi(\theta)$ and then $\varphi = \varphi(t)$, introducing the integration constant a_6. The integration constants a_k, $k = 1, 2, \ldots, 6$ are then determined with the aid of the initial conditions.

Application 6.5

Problem. Study the problem of two particles, using Hamilton's equations, in the phase space.

Mathematical model. Let us consider the particles P_1 and P_2 of masses m_1 and m_2, respectively, the positions of which are specified by the generalized co-ordinates r, θ, φ (see Appl.6.4, with the results and the corresponding notations). We notice that the generalized momenta (given by formulae of the form $p_k = \partial \mathrm{L} / \partial \dot{q}_k$) are expressed by (see formulae (g) in the mentioned application)

$$p_r = m\dot{r}, \ p_\theta = mr^2\dot{\theta}, \ p_\varphi = mr^2 \sin^2\theta\dot{\varphi} ; \tag{a}$$

hence, it results

$$\dot{r} = \frac{p_r}{m}, \ \dot{\theta} = \frac{p_\theta}{mr^2}, \ \dot{\varphi} = \frac{p_\varphi}{mr^2\sin^2\theta}. \tag{b}$$

Having to do with holonomic and scleronomic constraints, the Hamiltonian is of the form $H = T - U$ (see Appl.5.4), so that

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta} - 2f\frac{m_1 m_2 m}{r} \right). \tag{c}$$

The first subsystem of canonical equations (see Appl.6.3) is given by (b), while the second subsystem reads

$$\dot{p}_r = \frac{1}{mr^2}\left(\frac{p_\theta^2}{r} + \frac{p_\varphi^2}{r\sin^2\theta} - fm_1m_2m\right),\ \dot{p}_\theta = \frac{p_\varphi^2\cos\theta}{mr^2\sin^3\theta},\ \dot{p}_\varphi = 0. \tag{d}$$

Solution. We will use *the Hamilton-Jacobi method*. We notice that φ is a cyclic coordinate, so that

$$p_\varphi = a_1 = \text{const} \tag{e}$$

represents a first integral of the system of canonical equations. Because the constraints are scleronomic, the function S is of the form

$$S = -ht + a_1\varphi + S_0(r,\theta,a_2), \tag{f}$$

h, a_1 and a_2 being the three integration constants. The Hamilton-Jacobi equation becomes

$$H\left(r,\theta,\frac{\partial S_0}{\partial\theta},a_1\right) = h \tag{g}$$

or, taking into account the expression (c) of the Hamiltonian

$$\frac{1}{2m}\left[\left(\frac{\partial S_0}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S_0}{\partial\theta}\right)^2 + \frac{a_1^2}{r^2\sin^2\theta} - 2f\frac{m_1m_2m}{r}\right] = h. \tag{h}$$

Choosing S_0 as a sum of two functions of a single variable

$$S_0(r,\theta) = R(r) + \Theta(\theta), \tag{i}$$

the equation (h) takes the form (we denote $R' = \mathrm{d}R/\mathrm{d}r$, $\Theta' = \mathrm{d}\Theta/\mathrm{d}\theta$)

$$R'^2 + \frac{1}{r^2}\Theta'^2 + \frac{a_1^2}{r^2\sin^2\theta} - 2f\frac{m_1m_2m}{r} = 2mh; \tag{j}$$

we may also write

$$r^2R'^2 - 2fm_1m_2mr - 2mhr^2 = -\Theta'^2 - \frac{a_1^2}{\sin^2\theta} = -a_2,\ a_2 > 0,\ a_2 = \text{const}.$$

Hence, one obtains the equations

$$\mathrm{d}R = \pm\sqrt{g_1(r)}\mathrm{d}r,\ g_1(r) = 2mh + 2f\frac{m_1m_2m}{r} + \frac{a_2^2}{r^2}, \tag{k}$$

$$\mathrm{d}\Theta = \pm\sqrt{g_2(\theta)}\mathrm{d}\theta\,,\; g_2(\theta) = a_2 - \frac{a_1^2}{\sin^2\theta} \geq 0\,. \tag{l}$$

By two quadratures, we get the functions $R(r)$ and $\Theta(\theta)$, hence the function $S_0 = S_0(r,\theta)$ and, finally, the function S.

Applying the Hamilton-Jacobi theorem (see Appl.5.5), one may write two sequences of three first integrals, which allow the solution of the problem.

Application 6.6

Problem. Let be a doubly hinged straight bar. Determine the variation of the radius of the circular cross section $r = r(x)$ so that, for a given volume V of material, to obtain the maximal resistance to buckling.

Mathematical model. The moment of inertia of the cross section is $I_x = (\pi/4)r^4$, so that the differential equation of the deformed axis in the first state of buckling is

$$w'' + \frac{P}{EI_x}w = w'' + \frac{4P}{\pi E}\frac{w}{r^4} = 0\,. \tag{a}$$

By the notation

$$\lambda^2 = \frac{4P}{\pi E}\,, \tag{b}$$

the equation (a) becomes

$$r^4 = -\lambda^2\frac{w}{w''}\,. \tag{c}$$

The volume of the bar of length l is given by

$$V = \pi\int_0^l r^2\mathrm{d}x\,, \tag{d}$$

so that the equation (c) reads

$$\frac{V}{\pi\lambda} = \int_0^l \sqrt{-\frac{w}{w''}}\mathrm{d}x\,.$$

This expression attains a minimum when λ (and at the same time P too) will attain a maximum. We are thus led to the variational problem

$$I = \int_0^l F(w, w'')\mathrm{d}x = \text{extremum}\,,\; F = \sqrt{-\frac{w}{w''}}\,, \tag{e}$$

with the bilocal conditions $w(0) = w(l) = 0$.

Solution. The Euler-Poisson equation reads

$$\frac{\partial F}{\partial w} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial F}{\partial w''}\right) = 0 .$$

We have further

$$\frac{\partial F}{\partial w} = \frac{1}{\sqrt{-w''}}\frac{1}{2}w^{-1/2} = \frac{1}{2\sqrt{-ww''}} ,$$

$$\frac{\partial F}{\partial w''} = -\frac{1}{2}\sqrt{-w}(w'')^{-3/2} = -\frac{1}{2}\sqrt{-\frac{w}{w''^3}} ,$$

so that the Euler-Poisson equation becomes

$$\frac{1}{2\sqrt{-ww''}} - \frac{1}{2}\left(\sqrt{-\frac{w}{w''^3}}\right)'' = 0$$

or

$$\frac{1}{\sqrt{-ww''}} - \left(\sqrt{-\frac{w}{w''^3}}\right)'' = 0 .$$

If we denote

$$v^2 = -\frac{w}{w''^3} ,$$

the previous equation may be written

$$\frac{1}{\sqrt{-ww''}} - v'' = 0 .$$

Multiplying by w and amplifying the first term by w'', it results

$$\frac{ww''}{\sqrt{-ww''^3}} - v''w = 0$$

or

$$vw'' - v''w = 0 .$$

Integrating, we have

$$vw' - v'w = C_1 .$$

From the conditions $w(0)=0$, hence $v(0)=0$, we get $C_1=0$.
The relation $vw'-v'w=0$ is equivalent to

$$\frac{vw'-v'w}{w^2}=\left(\frac{v}{w}\right)'=0\,,$$

whence $v/w=C_2$ or $v=C_2w$.
Because in the hypothesis of infinitesimal deformations the amplitude of the deformed axis is non-determinate, one may take $C_2=1$, so that $v=w$ or $v^2=w^2$; it results $ww''^3=-1$.
But the independent variable is missing in this last equation, so that one may take $w'=p$ and $w''=p\mathrm{d}p/\mathrm{d}w$. Thus, the differential equation becomes

$$\left(p\frac{\mathrm{d}p}{\mathrm{d}w}\right)^3=-\frac{1}{w}$$

or

$$p\mathrm{d}p=-w^{-1/3}\mathrm{d}w\,.$$

Integrating the equation with separate variables, we get

$$p^2=3\left(a^2-w^{2/3}\right),$$

where a is a new integration constant. Hence, one may deduce

$$\frac{\mathrm{d}w}{\mathrm{d}x}=\sqrt{3}\sqrt{a^2-w^{2/3}}\,.$$

A first substitution $w^{1/3}=u$, hence $w=u^3$ and $\mathrm{d}w=3u^2\mathrm{d}u$, leads to the equation

$$\mathrm{d}x=\frac{3u^2\mathrm{d}u}{\sqrt{3}\sqrt{a^2-u^2}}=\sqrt{3}\frac{u^2\mathrm{d}u}{\sqrt{a^2-u^2}}\,.$$

A new substitution $u=a\sin\varphi$, hence $\mathrm{d}u=a\cos\varphi\mathrm{d}\varphi$ allows us to write

$$\mathrm{d}x=\sqrt{3}\frac{a^2\sin^2\varphi\cdot a\cos\varphi\mathrm{d}\varphi}{\sqrt{a^2-a^2\sin^2\varphi}}=a^2\sqrt{3}\sin^2\varphi\,\mathrm{d}\varphi=\frac{\sqrt{3}a^2}{2}(1-\cos 2\varphi)\mathrm{d}\varphi\,,$$

whence, by integration,

$$x = \frac{\sqrt{3}}{2} a^2 \int (1 - \cos 2\varphi) \mathrm{d}\varphi + C_4 = \frac{\sqrt{3}}{2} a^2 \left(\varphi - \frac{1}{2} \sin 2\varphi \right) + C_4$$

$$= \frac{\sqrt{3}}{2} a^2 \left(\arcsin \frac{u}{a} - \frac{u}{a} \sqrt{1 - \frac{u^2}{a^2}} \right) + C_4 ,$$

where C_4 is the fourth integration constant.
Returning to the variable w, we get

$$x = \frac{\sqrt{3}}{2} a^2 \left(\arcsin \frac{w^{1/3}}{a} - \frac{w^{1/3}}{a} \sqrt{1 - \frac{w^{2/3}}{a^2}} \right) + C_4 .$$

Because $w = 0$ for $x = 0$, it results $C_4 = 0$ and

$$x = \frac{\sqrt{3}}{2} a^2 \left(\arcsin \frac{w^{1/3}}{a} - \frac{w^{1/3}}{a} \sqrt{1 - \frac{w^{2/3}}{a^2}} \right) .$$

The boundary condition at the second end $w(l) = 0$ leads to

$$l = \frac{\sqrt{3}}{2} a^2 (\pi - 0) \Rightarrow \frac{l}{\pi} = \frac{\sqrt{3}}{2} a^2 ;$$

hence

$$x = \frac{l}{\pi} \left(\arcsin \frac{w^{1/3}}{a} - \frac{w^{1/3}}{a} \sqrt{1 - \frac{w^{2/3}}{a^2}} \right) .$$

From $r^4 = -\lambda^2 \dfrac{w}{w''}$ and $w = -\dfrac{1}{w''^3}$ it results

$$r^4 = \frac{\lambda^2}{w''^4}$$

and then

$$r^3 = \frac{\lambda^{3/2}}{w''^3} = \lambda^{3/2} w , \ w^{1/3} = \frac{r}{\sqrt{\lambda}} .$$

Therefore

$$x = \frac{l}{\pi} \left(\arcsin \frac{r}{a\sqrt{\lambda}} - \frac{r}{a\sqrt{\lambda}} \sqrt{1 - \frac{r^2}{a^2 \lambda}} \right) .$$

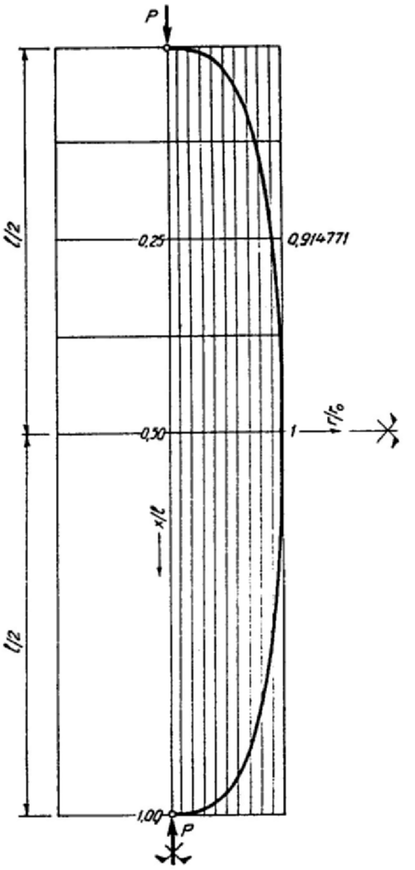

Figure 6. 4. Meridian curve of a bar of given volume V

But we have

$$a\sqrt{\lambda} = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{l}}{\sqrt{\pi}} \frac{\sqrt{2}}{\sqrt[4]{\pi}} \sqrt[4]{\frac{P}{E}} = \frac{2}{\left(3\pi^3\right)^{1/4}} \sqrt{l} \sqrt[4]{\frac{P}{E}} = r_0 ,$$

where r_0 is the maximal radius of the cross section; hence

$$r_0^4 = \frac{16}{3\pi^3} \frac{Pl^2}{E} .$$

The notation introduced above allows us to write, finally,

$$\frac{x}{l} = \frac{1}{\pi}\left(\arcsin\frac{r}{r_0} - \frac{r}{r_0}\sqrt{1-\frac{r^2}{r_0^2}}\right).$$

The values of x/l vs. $r/r_0 \in [0,1]$ are listed in Table 6.1 (in fact, the inverse problem is solved).
By solving the transcendental equation we get a solution of the form $r = r_0 f(x)$. The meridian curve of the cross section is drawn in Fig.6.4.
Further, the volume of the bar is given by

$$V = \pi\int_0^l r^2(x)\mathrm{d}x = \pi r_0^2 \int_0^l f^2(x)\mathrm{d}x\,,$$

whence

$$r_0^2 = \frac{V}{\pi\int_0^l f^2(x)\mathrm{d}x}.$$

The critical buckling force is obtained from r_0^4 in the form

$$P_{cr} = \frac{3\pi^3}{16} r_0^4 \frac{E}{l^2} = \frac{\pi r_0^4}{4}\frac{3\pi^2}{4}\frac{E}{l^2} = \frac{3}{4}\frac{\pi^2 EI_0}{l^2} = \frac{3}{4}P_{0,cr}\,,$$

hence the critical force for a bar of given volume represents three quarters of the critical force which corresponds to a bar of constant cross section and moment of inertia I_0.

Table 6.1.

r/r_0	x/l
0	0
0.1	0.000213
0.2	0.001718
0.3	0.005892
0.4	0.014296
0.5	0.028883
0.6	0.052044
0.7	0.087694
0.8	0.142378
0.85	0.180870
0.90	0.231560
0.914771	0.250
0.95	0.304495
1	0.500000

Application 6.7

Problem. Let be a cantilever bar in the form of a solid body of revolution, acted upon at the free end by a concentrated force P. Determine the variation of the radius of the cross section along the span, for a given volume V of material, so that the deflection be maximal or minimal.

Mathematical model. Let $r = r(x)$ be the variable radius and $I_x = \pi r^4 / 4$ the moment of inertia. The bending moment at a cross section of abscissa x is $M_x = -Px$, so that the approximate differential equation of the deformed axis is of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{4P}{\pi E} \frac{x}{r^4} . \tag{a}$$

The boundary conditions are of Cauchy type, hence: $w(l) = w'(l) = 0$.

Solution. Integrating once the equation (a), one obtains

$$w'(x) = \frac{4P}{\pi E} \int_x^l \frac{x'}{r^4(x')} \mathrm{d}x' + w'(l);$$

because $w'(l) = 0$, we obtain

$$w'(x) = \frac{4P}{\pi E} \int_x^l \frac{x'}{r^4(x')} \mathrm{d}x'$$

which, integrating once more, leads to

$$w(x) = \frac{4P}{\pi E} \int_x^l \left(\int_x^l \frac{x''}{r^4(x'')} \mathrm{d}x'' \right) \mathrm{d}x' + w(l)$$

and, because $w(l) = 0$, we have

$$w(x) = \frac{4P}{\pi E} \int_x^l \left(\int_x^l \frac{x''}{r^4(x'')} \mathrm{d}x'' \right) \mathrm{d}x' .$$

Integrating by parts, we obtain, finally,

$$w(x) = \frac{4Px}{\pi E} \int_x^l \frac{x'}{r^4(x')} \mathrm{d}x' + \frac{4P}{\pi E} \int_x^l \frac{x'^2}{r^4(x')} \mathrm{d}x' .$$

Obviously, w attains its maximum at $x = 0$. Therefore

$$w_{\max} = w(0) = \frac{4P}{\pi E} \int_0^l \frac{x^2}{r^4(x)} \mathrm{d}x . \tag{b}$$

The volume of the cantilever bar is

$$V = \pi \int_0^l r^2 \mathrm{d}x \,. \tag{c}$$

It results thus the variational problem

$$F = \frac{\pi E}{4P} w_{\max} = \int_0^l \frac{x^2}{r^4} \mathrm{d}x = \min \,, \tag{d}$$

with the condition

$$G = \frac{V}{\pi} = \int_0^l r^2 \mathrm{d}x = \mathrm{const} \,. \tag{e}$$

The variational problem (d), (e) is a problem of isoperimetric type. According to Sec.2.1, the solution is among the extrema of the functional $F + \lambda G$. Euler's equation for the integrand of this functional

$$\varphi(x, r, \lambda) \equiv \frac{x^2}{r^4} + \lambda r^2 \tag{f}$$

is

$$\frac{\partial \varphi}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \varphi}{\partial r'}\right) = 0 \,.$$

But φ does not depend on r', so that the above equation is reduced to

$$\frac{\partial \varphi}{\partial r} = -\frac{4x^2}{r^5} + 2\lambda r = 0 \,. \tag{g}$$

Therefore $r^6 = 2x^2 / \lambda$ or

$$r^2 = \sqrt[3]{\frac{2x^2}{\lambda}} \,. \tag{h}$$

Introducing in the expression (c) of the volume, we get

$$V = \pi \int_0^l \sqrt[3]{\frac{2}{\lambda}} x^2 \mathrm{d}x = \frac{3\pi}{5} \sqrt[3]{\frac{2}{\lambda}} l^{5/3} \,, \tag{i}$$

whence

$$\sqrt[3]{\frac{2}{\lambda}} = \frac{5}{3\pi} \frac{V}{l^{5/3}} \,.$$

Thus, the relation (h) leads to

$$r = r(x) = \sqrt{\frac{5}{3\pi}}\sqrt{\frac{V}{l}}\sqrt[3]{\frac{x}{l}},$$

the variation of the radius being after a cubic parabola.
If we denote by r_0 the radius at the built-in cross section, then we have

$$r_0 = \sqrt{\frac{5}{3\pi}}\sqrt{\frac{V}{l}}$$

and we find again the same expression listed in Table 6.1.

Application 6.8

Problem. A filling of gelatine dynamite is placed on a circular surface of radius a at a depth h in the ground. Determine the meridian curve of the funnel of earth ejected due to the detonation (Fig.6.5, a).

Mathematical model. We assume that the component of the outbreak force along the normal to the meridian curve is proportional to the element of area and that the total explosion force is minimal.

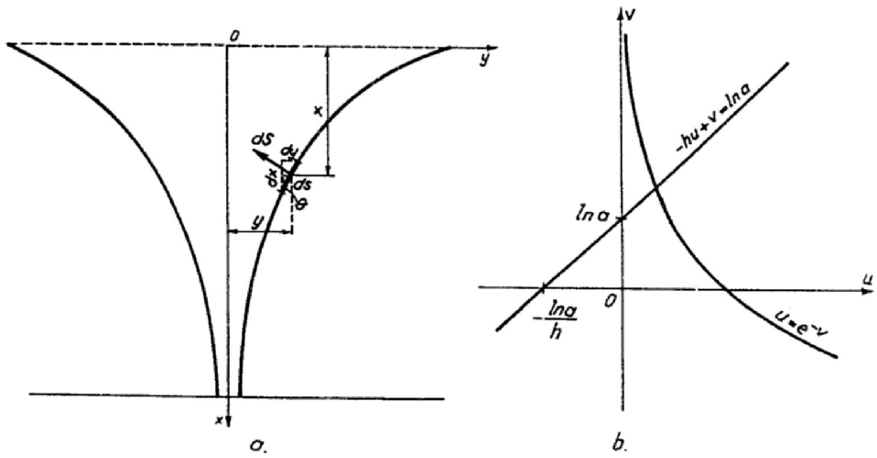

Figure 6. 5. Evaluation of the force dS(a). Graphic solution of the transcendental system (b)

Let θ be the inclination of the tangent to the meridian curve with respect to the vertical line; we may write $\sin\theta = \mathrm{d}y/\mathrm{d}x$. We have thus $\mathrm{d}S\sin\theta = \lambda\cdot 2\pi y\mathrm{d}x$, where λ is a factor of proportionality; it results

$$\mathrm{d}S = 2\pi\lambda y \frac{(\mathrm{d}s)^2}{\mathrm{d}y} = 2\pi\lambda y\left(\frac{\mathrm{d}x}{\mathrm{d}y} + \frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x .$$

From the elementary rectangular triangle, we get

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2 = \mathrm{d}x\mathrm{d}y\left(\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}x}{\mathrm{d}y}\right) = \mathrm{d}x\mathrm{d}y\left(y' + \frac{1}{y'}\right).$$

One obtains a variational condition

$$S = 2\pi\lambda \int_{-h}^{0} y\left(y' + \frac{1}{y'}\right)\mathrm{d}x = \min$$

for a functional of (6.1) type.

Solution. As it was shown in § 1, the extrema are, in this case, solutions of Euler's equation (6.1.8) for $F(x, y, y') = y\left(y' + \frac{1}{y'}\right)$. We easily obtain

$$\frac{yy'' - y'^2}{y'^2} = 0 ,$$

whence $y / y' = C_1 / 2$, where C_1 is an integration constant. Integrating once more, it results the equation of the meridian curve

$$y = \mathrm{e}^{2(x+C_2)/C_1} , \tag{a}$$

where C_2 is a second integration constant.
To determine the constants, we may write a first condition $y = a$ for $x = -h$, i.e.

$$\mathrm{e}^{2(-h+C_2)/C_1} = a . \tag{b}$$

A second natural boundary condition reads

$$\left[\frac{\partial F}{\partial y'}\right]_{x=0} = \left[y\left(1 - \frac{1}{y'^2}\right)\right]_{x=0} = 0 . \tag{c}$$

This condition is obtained from the relation (6.1.6). Indeed, if $\partial J(\varepsilon)/\partial\varepsilon = 0$ for $\varepsilon = 0$, then we have

$$\left[\frac{\partial F}{\partial y'}\eta(x)\right]_{x=x_1}^{x=x_2} = 0;$$

a non-zero η at the ends x_1, x_2 yields the condition (c).

Further, the condition (c) implies

$$y'(0) = \pm 1 = \frac{2}{C_1} e^{2C_2 / C_1} . \tag{d}$$

The constants C_1 and C_2 are obtained from (b) and (d). We deduce thus the transcendental system

$$\frac{2(-h + C_2)}{C_1} = \ln a , \ \frac{2}{C_1} e^{2C_2 / C_1} = \pm 1 . \tag{e}$$

If we denote $2 / C_1 = u$ and $2C_2 / C_1 = v$, the system (e) reads

$$-hu + v = \ln a , \ u e^{v} = \pm 1 . \tag{f}$$

In the system of the axes Ouv, the two equations represent (Fig.6.5, b) a straight line

$$\frac{u}{-\frac{\ln a}{h}} + \frac{v}{\ln a} = 1$$

and an exponential $u = \pm e^{-v}$, respectively.

For given values of h and a, the solutions u and v may be determinate numerically, and then

$$C_1 = \frac{u}{2} , \ C_2 = \frac{v}{u} .$$

Chapter 7

STABILITY

1. Lyapunov Stability

1.1 GENERALITIES

Let us consider first order ODSs of the form

$$\dot{x}_i = f_i(t, x_1, x_2, ..., x_n), \quad i = \overline{1, n}, \quad \dot{x}_i \equiv \frac{\mathrm{d}x_i}{\mathrm{d}t}, \tag{7.1.1}$$

where the point stands for the derivative with respect to the time t; this is a usual notation in mechanics. The functions f_i are of class $\mathrm{C}^1([t_0, \infty))$.

The system $x_1, x_2, ..., x_n$ may be interpreted as representing the co-ordinates of a particle in motion, the independent variable being the time t. If we denote by $\mathbf{x}$ the vector function $\mathbf{x} = (x_1, x_2, ..., x_n)$, then the system may be written in the equivalent compact form

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{f}(t, \mathbf{x}) \equiv (f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), ..., f_n(t, \mathbf{x})). \tag{7.1.2}$$

The system (7.1.1) or, equivalently, (7.1.2), is called *autonomous* (*dynamical*) if $\mathbf{f}$ does not explicitly depend on t and non-autonomous in the opposite case. With this interpretation, the particular solutions of the above ODSs will represent displacements of the particle.

Consider now that the co-ordinates of the particle are given at t_0, i.e.

$$x_1(t_0) = x_{10}, x_2(t_0) = x_{20}, ..., x_n(t_0) = x_{n0} \tag{7.1.3}$$

or, in vector terms

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}_0 = (x_{10}, x_{20}, ..., x_{n0}). \tag{7.1.4}$$

The Cauchy-Picard theorem applied to the Cauchy problem (7.1.1), (7.1.3) or, similarly, to (7.1.2), (7.1.4) ensures the local existence and uniqueness of the solution

$$\mathbf{x}(t) = \mathbf{x}(t, t_0, \mathbf{x}_0). \tag{7.1.5}$$

A problem of great importance is the long term behaviour of the solution. If the considered ODS represents a dynamical system, then the analysis of the asymptotic behaviour of the solution leads to the knowledge of the successive states of the motions, up to its annihilation, according to the principles of thermodynamics. If the initial data are slightly perturbed, e.g. $\tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0$, then we should expect that the perturbed solution $\tilde{\mathbf{x}}(t, t_0, \tilde{\mathbf{x}}_0)$ be close to $\mathbf{x}(t, t_0, \mathbf{x}_0)$. In this case, obviously, the behaviour of the solution would be predictible. Such a solution will be called *stable*.

In order to make things clearer, let us denote by $\left|\mathbf{x}(t)\right| = \sqrt{x_1^2(t) + x_2^2(t) + \ldots + x_n^2(t)}$. If for any $\varepsilon > 0$ one can find $\delta(\varepsilon)$ such that, as soon as $\left|\mathbf{x}_0 - \tilde{\mathbf{x}}_0\right| < \delta(\varepsilon)$, we have

$$\left|\mathbf{x}(t, t_0, \mathbf{x}_0) - \tilde{\mathbf{x}}(t, t_0, \tilde{\mathbf{x}}_0)\right| < \varepsilon \tag{7.1.6}$$

for any $t > t_0$, then we say that $\mathbf{x}(t, t_0, \mathbf{x}_0)$ is *stable in the sense of Lyapunov*. The solutions that are not sstable are called *unstable* (Fig.7.1 a, c)

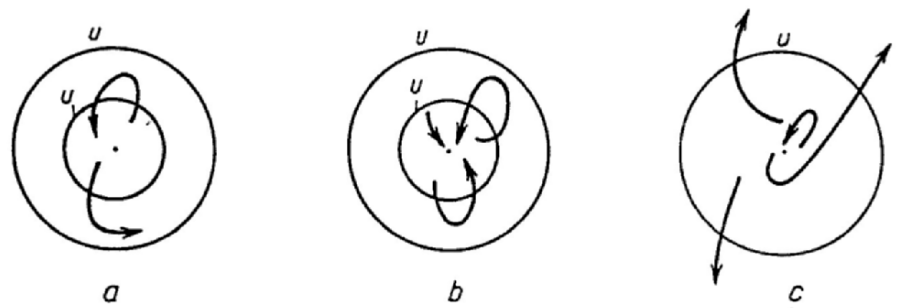

Figure 7. 1. Lyapunov stability for an equilibrium state; stable (a); asymptotically stable (b); unstable (c)

A solution $\mathbf{x}(t)$ is called asymptotically stable if it is stable and, moreover, $\lim_{t\to\infty}\left|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\right| = 0$ for any solution $\tilde{\mathbf{x}}(t)$ which is such that $\left|\mathbf{x}(t_0) - \tilde{\mathbf{x}}(t_0)\right| < \varepsilon$ (Fig.7.1 b)

1.2 LYAPUNOV'S THEOREM OF STABILITY

Besides the above conditions of regularity imposed on **f**, let us suppose that the components f_i allow constant partial derivatives along the trivial solution

$$\frac{\partial f_i}{\partial x_j}(t, \mathbf{0}) = a_{ij}, \quad i, j = \overline{1, n}. \tag{7.1.7}$$

The functions f_i may then be represented in the form

$$f_i(t, \mathbf{x}) = \sum_{j=1}^{n} a_{ij} x_j + \varphi_i(t, \mathbf{x}), \quad i = \overline{1, n}; \tag{7.1.8}$$

if, moreover, $f_i(t, \mathbf{0}) = 0, i = \overline{1, n}$, then φ_i tend to zero once $x_j \to 0, j = \overline{1, n}$. Hence we can neglect the non-linear terms φ_i, keeping in (7.1.8) only the linear part of f_i. We thus get the following linear and homogeneous ODS with constant coefficients

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = \overline{1, n}; \tag{7.1.9}$$

let us denote by $\mathbf{A} = \left[a_{ij}\right]_{i,j=\overline{1,n}}$ the matrix of its coefficients.

The ODS (7.1.9), associated to is called the system of the first (or linear) approximation of (7.1.1). Initially, this system was considered satisfactory for a qualitative study of the solutions of (7.1.1). This idea was infirmed by Lyapunov, who proved

Theorem 7.1. *If the roots of the characteristic equation*

$$P_n(\lambda) = \det[\mathbf{A} - \lambda\mathbf{E}] \tag{7.1.10}$$

of the linear approximation of (7.1.1) *have all of them strictly negative real part and if the functions* φ_i *satisfy*

$$\left|\varphi_i(t, \mathbf{x})\right| < M\left|\mathbf{x}\right|^{1+\alpha}, \tag{7.1.11}$$

where M is a constant and $\alpha > 0$, *then the trivial solution of* (7.1.1) *is stable. If at least one of the roots of the characteristic equation* (7.1.10) *has a positive real part, then the trivial solution of* (7.1.1) *is unstable*.

The above theorem studies the stability around **0**; it can be directly applied to ODSs allowing **0** as a solution. If we wish to study the stability around another critical point of (7.1.1) – say, $\overline{\mathbf{x}} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)$ – then, by using the change of functions

$$\mathbf{X} = \mathbf{x} - \overline{\mathbf{x}}, \quad \mathbf{X} = (X_1, X_2, ..., X_n), \tag{7.1.12}$$

the solution $\overline{\mathbf{x}}$ will be translated to the origin. The problem of the stability of $\overline{\mathbf{x}}$ is thus reduced to the study of the stability of the trivial solution for the transformed system

$$\dot{\mathbf{X}} = \mathbf{g}(t, \mathbf{X}), \quad \mathbf{g}(t, \mathbf{X}) = (g_1(t, \mathbf{X}), g_2(t, \mathbf{X}), ..., g_n(t, \mathbf{X})), \tag{7.1.13}$$

where

$$g_j(t, \mathbf{X}) = f_j(t, \mathbf{X} + \overline{\mathbf{x}}) - \dot{\overline{x}}_j, \quad j = \overline{1,n}. \tag{7.1.14}$$

If the system is autonomous, the solutions of the functional system

$$\begin{aligned} &f_1(\mathbf{x}) = 0, \\ &f_2(\mathbf{x}) = 0, \\ &.................. \\ &f_n(\mathbf{x}) = 0 \end{aligned} \tag{7.1.15}$$

or

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{7.1.16}$$

are called *critical points* or *equilibrium points* or else *stationary solutions* of the ODS (7.1.1) or of its equivalent (7.1.2).

2. The Stability of the Solutions of Dynamical Systems

2.1 AUTONOMOUS DYNAMICAL SYSTEMS

In the case of autonomous systems, if **f** satisfies the hypothesis of Theorem 7.1, then the trivial solution is not only stable, but also asymptotically stable. If at least one of the eigenvalues of the matrix **A** – or, otherwise speaking, a root of the characteristic polynomial $P_n(\lambda)$ – has a positive real part, then the trivial solution is unstable.

Using the Hurwitz matrix associated to the polynomial $P_n(\lambda)$, one can straightforwardly check if the real part of the eigenvalues is or is not strictly negative.

If the eigenvalues of **A** have zero real part, then the stability of the null solution cannot be checked in the frame of the first approximation of the given ODS. It may be tackled in another frame – for instance, in the frame of the central manifold theory.

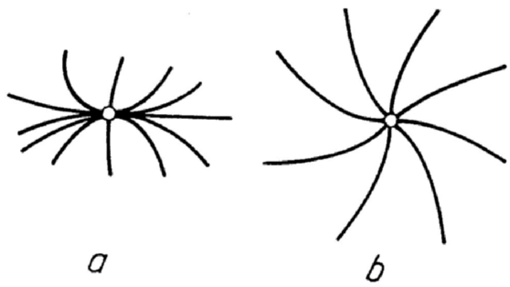

Figure 7. 2. Nodes

Consider the autonomous system (case $n = 2$)

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y). \tag{7.2.1}$$

At the points at which $f(x, y) \neq 0$, this ODS can be reduced to a single ODE

$$y' = \frac{f(x, y)}{g(x, y)}, \quad y' = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{7.2.2}$$

The stationary solutions of (7.2.1) will be singular points of a special type for the ODE (7.2.2).

Let us admit that the system (7.2.1) is defined for $(x, y) \in \Omega \subseteq \Re^2$. Also suppose that Ω is simply connected, i.e., together with any closed curve $\Gamma \subset \Omega$, it also contains the domain limited by Γ.

If Ω contains a unique criticity point $P_0(x_0, y_0)$ of (7.2.1), then the trajectories belonging to Ω behave in a few qualitatively distinct ways; these types of behaviour also represent criteria of classification of the critical point P_0, as follows:

a) *node* – if the trajectories passing through P_0 have a well-defined tangent (Fig.7.2);

b) *focus* – if the trajectories tend asymptotically to P_0, spiraling towards it (Fig.7.3);

c) *centre* – if it is surrounded only by closed trajectories (Fig.7.4);
d) *saddle point* (Fig.7.5).

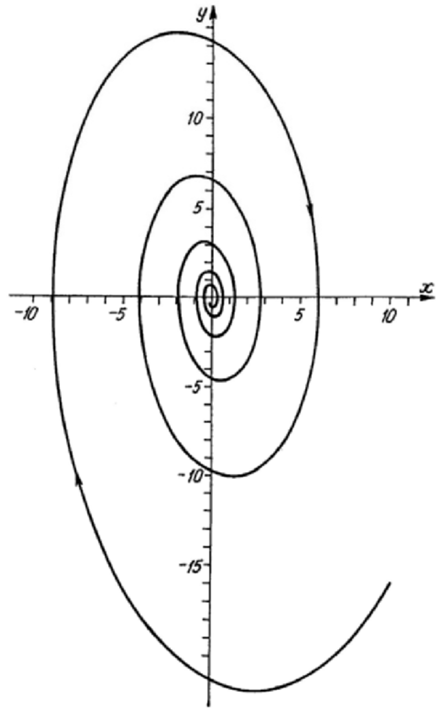

Figure 7. 3. A focus

More precisely, let

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy \tag{7.2.3}$$

be the linear approximation of (7.2.1) after a translation of type (7.1.12) of the equilibrium point to the origin. Denote by λ_1, λ_2 the eigenvalues of the associated characteristic equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0 \,, \tag{7.2.4}$$

i.e., the eigenvalues of the matrix

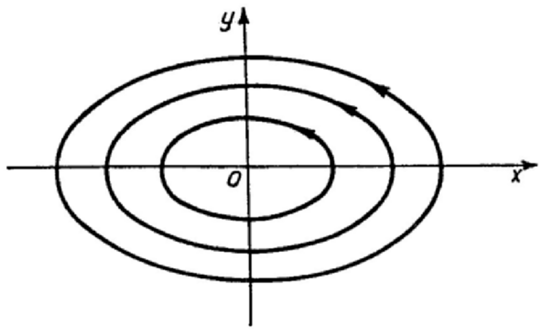

Figure 7. 4. A centre at the origin

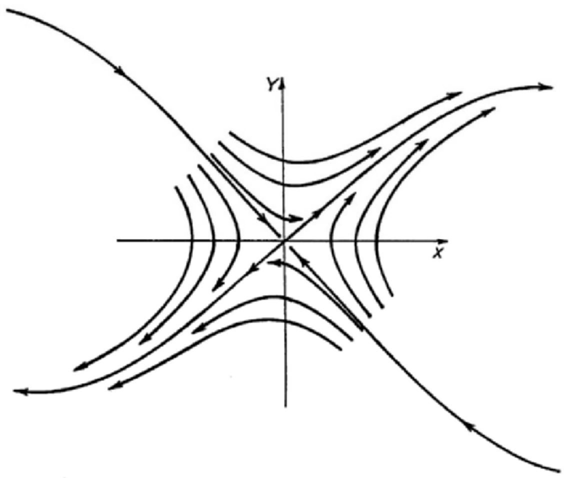

Figure 7. 5. A saddle point

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{7.2.5}$$

H. Poincaré explicitly found the type of the equilibrium point, according to the nature of λ_1, λ_2.

2.2 LONG TERM BEHAVIOUR OF THE SOLUTIONS

Usually, the solutions of the dynamical systems show firstly a transient state, after which the motion tends to a stable state for a long period of time. The neighbouring motions, with initial data close to each other, converge to these stable "attraction basins".
The simplest case is that of the equilibrium point at which any motion stops. A typical example is that of the non-linear rigid pendulum, which, after several damped

oscillations, gets back to the vertical position, no matter the initial data. In the phase space – of co-ordinates position and velocity – the *portraits* of these motions appear as non-intersecting spirals, converging to a unique point: the equilibrium point. This is a *point attractor*.

Another type of attractor is the *periodic attractor*. A classical example of such an attractor is a thin and flexible steel rod, in resonance with an electromagnet subjected to alternating current. After a short transient state, the rod motion will be stabilized to a forced oscillation. A change in the initial conditions will generate a distinct periodic motion, also stabilized after a short transient.

Two *limit cycles* are thus emphasized, each of them attracting certain motions; it is noticed that the attraction basins are separated by a curve called *separatrix*.

We thus conclude that a linear analysis is unsatisfactory for a qualitative study of the solutions of a non-linear dynamical system.

A third type of attractor, recently discovered, is *the strange* (or *haotic*) *attractor*, which collects the motion of a perfectly determined dynamical system in a bounded domain of the phase space; apparently, the motion is in a perpetual haos. While some values of the solution may repeat at irregular periods of time, one cannot say that the motion is periodic. Even if the phase portrait seems to be haotic, this attractor shows some particularities and properties that may lead to deeply know the structure of the solutions.

Among the first discoverers of such attractors one may quote Lorenz and Hénon. Hénon's attractor (Fig.7.6) was put into evidence on the occasion of an astronomic study, and Lorenz's attractor was emphasized in a study of some meteorologic phenomena (Fig.7.7).

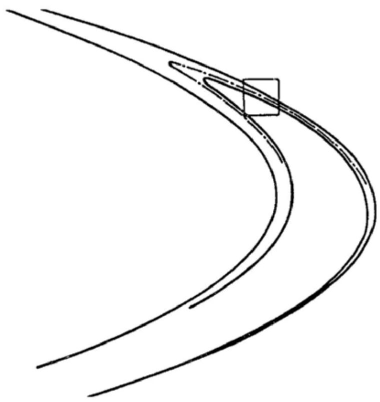

Figure 7. 6. Hénon's attractor

A dynamical system often depends on some parameters with a physical significance. It was noticed that not only some variations of the initial data, but also the variations of this parameters lead to qualitative modifications of the solutions. In this sense, there are serious perspectives of explaining the phenomena of turbulence by using the analysis of the structures of the solutions of non-linear ODS generating strange attractors.

The above remarks point out several steps in a study of the long term behaviour of the solutions of a non-linear dynamical system.

First of all, one must identify all the possible attractors of the given dynamical system. The non-linear systems may allow various attractors of different types, which might coexist (e.g., periodic and haotic attractors).

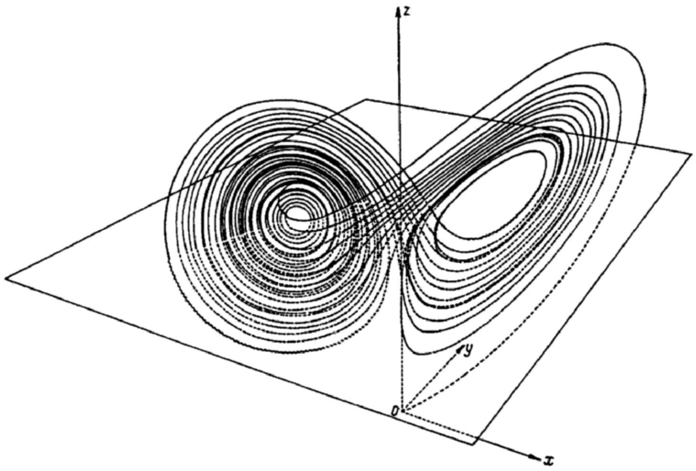

Figure 7. 7. Lorenz's attractor

Then one identifies the basin of attraction of each of these attractors; this can be numerically obtained, considering the solutions corresponding to a great number of initial data. It is a difficult task, that might be perhaps more efficiently carried over in the frame of the *theory of the invariant manifold*, initiated by Poincaré.
We thus obtain a portrait attractor –basin of attraction (AB) in the phase space. The whole procedure must be repeated if we modify the parameters of the system. In the new AB-portrait it is possible that some of the attractor dissapear and some others, of another type, replace them.
At the points of bifurcation, we observe qualitative changes of the topological structure of the portrait AB; the state changes are sometimes called *catastrophes*.
The theory of catastrophes, of the central manifold, of the bifurcations are all of them modern theories, with numerous applications in phenomenological studies.
One can conclude that the qualitative study of the solutions of the ODS depending on parameters represents a key to clarify and foresee a great number of physical phenomena, so far unexplained and, because of this fact, sometimes classified as "experimental errors".

3. Applications

Application 7.1

Problem. Study the stability of the position of equilibrium of a free or constraint particle P in the presence of a field of conservative forces.

Mathematical model. Let us consider first of all the case of a free particle subjected to the action of a conservative force of the form $\mathbf{F} = \operatorname{grad} U(\mathbf{r})$, where $\mathbf{r}$ is the position vector of the particle P ; the Newtonian equation of motion is written in the form

$$m\ddot{\mathbf{r}} = m\dot{\mathbf{v}} = \operatorname{grad} U , \tag{a}$$

where m is the mass of the particle, while $\mathbf{v}$ is the velocity. A scalar product by $\mathrm{d}\,\mathbf{r}$ leads to

$$m\ddot{\mathbf{r}} \cdot \mathrm{d}\mathbf{r} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathrm{d}\mathbf{r} = m\mathbf{v} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathrm{d}\left(\frac{1}{2}m\mathbf{v}^2\right) = \operatorname{grad} U \cdot \mathrm{d}\mathbf{r} = \mathrm{d}U ,$$

whence

$$\frac{1}{2}mv^2 = U + \frac{1}{2}mv_0^2 - U_0 ; \tag{b}$$

we found thus a first integral (the first integral of mechanical energy) of the differential equation (a).

Solution. We assume that the potential $U(\mathbf{r})$ has an isolated minimum at the origin O; noting that the potential is determinate abstraction of an arbitrary constant the gradient of which vanishes, we can take $U(\mathbf{0}) = 0$. Let be a convex closed surface S which contains the point O (e.g. a sphere of centre O), of arbitrary small dimensions, so that in the interior of the surface and on it the function $U(\mathbf{r})$ be negative, vanishing only at the point O. We may assume that there exists $p > 0$ sufficiently small so that on the surface S to have $-U > p$, hence $U + p < 0$. Let be P_0 an initial position of the particle P in the interior of the surface S, the corresponding velocity being $\mathbf{v_0}$; we may thus use the first integral (b) with $U_0 < 0$. We determine the position and the magnitude of the velocity at the initial moment by the condition $mv_0^2/2 - U_0 < p$; for this it is sufficient to take, for instance, $mv_0^2/2 < p/2$, $-U_0 < p/2$. The first relation shows that $v_0 < \eta' = \sqrt{p/m}$. As well, the function U is continuous and vanishes at the origin; there exists thus $\eta > 0$, such that $\overline{OP_0} < \eta$, corresponding to $-U_0 < p/2$. Hence, if – in the interior of the surface S – we give to the particle an initial position at a distance to O less than η, with an initial velocity less than η', then the theorem of energy leads to the inequality $mv^2/2 < U + p$; thus, the particle cannot come out from the interior of the surface S. Indeed, if the particle P would reach S, then the sum $U + p$ would become negative, which is not possible if we take into account the previous relation. Hence, we may state that it corresponds $\varepsilon > 0$ so that $\overline{OP} < \varepsilon$, $P = P(t)$. As well, $mv^2/2 < p$, because $U < 0$; it results $v(t) = \sqrt{2p/m} = \varepsilon' > 0$. The conditions for the point O to have a stable position of equilibrium are fulfillEd. We may thus state for a point $P^0 \equiv O$:

The position of equilibrium P^0 of a free particle P, in the presence of a field of conservative forces, the potential U having an *isolated maximum* at the point P^0, is a position of stable equilibrium (*the Lagrange-Dirichlet theorem*).

For instance, the origin of the co-ordinate axis is a stable position of equilibrium for a free particle subjected to the action of an elastic force of attraction $\mathbf{F}(\mathbf{r}) = -k\mathbf{r}$, $k > 0$, which derives from the potential $U(\mathbf{r}) = -k\mathbf{r}^2 / 2$.

In the case of a particle constrained to stay on a fixed smooth surface S there are introduced the generalized forces $Q_\alpha(u, v)$, $\alpha = 1, 2$, where u and v are co-ordinates along the co-ordinate lines on the respective surface. If $Q_1 \mathrm{d}u + Q_2 \mathrm{d}v = \mathrm{d}\overline{U}(u, v)$ is a total differential, then we are led to the study of the extrema of the potential $\overline{U} = \overline{U}(u, v)$, where u and v are generalized co-ordinates, the holonomic (geometric, integrable) and scleronomic (i.e.,which do not depend explicitly on time) constraints being eliminatEd. We may also obtain for $\overline{U}$ a maximum equal to zero at the point P^0, coinciding with the origin $(\overline{U}(0,0) = 0)$. We draw on the surface S a closed curve C around the point P^0, so that to have on the curve $\overline{U} < 0$; there exists thus $p > 0$ so that $\overline{U} + p < 0$ on C. Displacing the particle from P^0 at a neighbourhood point, interior to the curve C, we may follow the preceding demonstration. In general, we can state that the Lagrange-Dirichlet theorem may be applied in case of holonomic and scleronomic constraints too.

If the potential U has an *isolated minimum* at the point P^0, then that one represents a *labile position of equilibrium*.

Introducing the potential energy $V = -U$, we may affirm that, for a stable position of equilibrium, the potential energy has an isolated minimum, while, for a labile position of equilibrium, it has an isolated maximum.

In particular, let be the case of a gravitational field for which $V = mgz$ (the Oz-axis is along the ascendent vertical), where g is the gravitational acceleration; we obtain *the Torricelli's theorem*, which states that the stable position of equilibrium corresponds to the lowest position on a fixed smooth curve or surface. We may also state that a labile position of equilibrium corresponds to the highest such position.

Application 7.2

Problem. Study the motion of a particle with a single degree of freedom, subjected to scleronomic constraints in a conservative field.

Mathematical model. In the case of a particle (or of a mechanical system) with only one degree of freedom, for which the equation of motion is of the form

$$\ddot{q} = f(q), \tag{a}$$

where q is the generalized co-ordinate, we may set up a first integral of the energy in the form

$$\dot{q}^2 - \dot{q}_0^2 = 2[U(q) - U(q_0)], \quad U(q) = \int f(q)\mathrm{d}q. \tag{b}$$

We have introduced a simple potential U (or a scalar potential U_0 of a generalized potential); hence, the corresponding mechanical system is a *conservative system*. As well, one can show that a unidimensional conservative mechanical system (with only one degree of freedom) or a pluridimensional one (if we succeed to eliminate, by means of first integrals, the corresponding parameters, transforming it in an unidimensional system) leads to an equation of motion of the form (a).

Solution. We notice that the equation (a) corresponds to non-linear, non-damped free oscillations; in this case, the function $f(q)$ corresponds to a calling force. Integrating the equation (b), we get

$$t - t_0 = \pm\int_{q_0}^{q} \frac{\mathrm{d}\eta}{\sqrt{\varphi(\eta)}}, \tag{c}$$

where we have introduced the notation

$$\varphi(q) = \dot{q}_0^2 + 2[U(q) - U(q_0)]. \tag{d}$$

The sign + or – in (c) is taken as the function $q(t)$ is monotone increasing or decreasing, respectively. It is necessary to have $\varphi(q) \geq 0$ so that the motion be real. Noting that $\varphi(q_0) = \dot{q}_0^2 \geq 0$, we may assume that the function q begins to increase together with t (corresponding to the direction of the initial velocity); so that one chooses the sign +. A study of the variation of the function $f(q)$ and of its zeros allows to obtain interesting conclusions about the motion of the particle (or of the mechanical system).
Denoting $\dot{q} = p$, we may replace the equation (a) by the system

$$\frac{\mathrm{d}q}{\mathrm{d}t} = p, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = f(q), \tag{e}$$

which leads to

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{f(q)}{p}; \tag{f}$$

the motion of the particle is the equivalent to the motion of a representative point P in the phase space of co-ordinates q, p. The trajectory C in this space pierces the axis Oq under a right angle, a tangent to it being parallel to the same axis for $f(q) = 0$, $p \neq 0$; if we have $p = 0$ too, one obtains a singular point, corresponding to a position of equilibrium, as it results from the system (e).

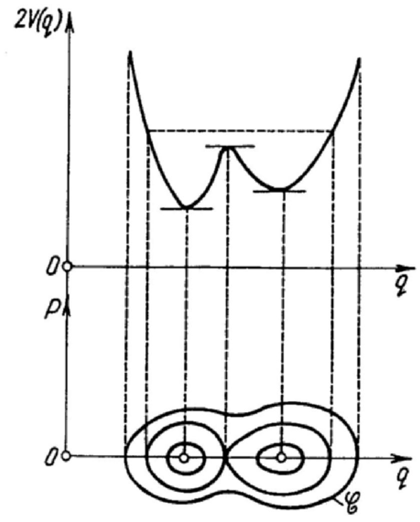

Figure 7. 8. Motion of a particle with one degree of freedom in a conservative case

Expressing the first integral (b) in the form

$$p^2 + 2V(q) = h\,,\; h = \dot{q}_0^2 + 2V(q_0)\,,\; V(q) = -U(q)\,, \tag{g}$$

where h is the energy constant; we notice that the trajectory C is symmetric with respect to the Oq-axis, being situated in the domain $2V(q) \le h$. Corresponding to the Lagrange-Dirichlet theorem (see Appl.7.1), to the points of minimum of the potential energy $V(q)$ correspond positions of stable equilibrium, while to the points of maximum correspond positions of labile equilibrium (Fig.7.8). From the first equation (e) it results that, for $p > 0$, q increases with the time t, which allows to specify the direction of the trajectory. The period of the motion is given by this equation in the form

$$T = \oint \frac{\mathrm{d}q}{p}\,, \tag{h}$$

the integral taking place along a closed curve.

Application 7.3

Problem. Study the topological structure of the phase trajectories in the motion of a particle with a single degree of freedom, subjected to scleronomic constraints in a field of conservative forces.

Mathematical model. In connection to the preceding application, the equation of motion in the generalized co-ordinate q, corresponding to a single degree of freedom, is of the form

$$\ddot{q} = f(q, \lambda)\,, \tag{a}$$

where λ is a parameter the values of which contribute to the variation of the topological structure of the phase trajectories. In a field of conservative forces, we have

$$f(q,\lambda) = -V_q'(q,\lambda) = -\frac{\partial V(q,\lambda)}{\partial q}, \tag{b}$$

the position of equilibrium being situated along the curve C of equation (Fig.7.9)

$$f(q,\lambda) = 0 . \tag{c}$$

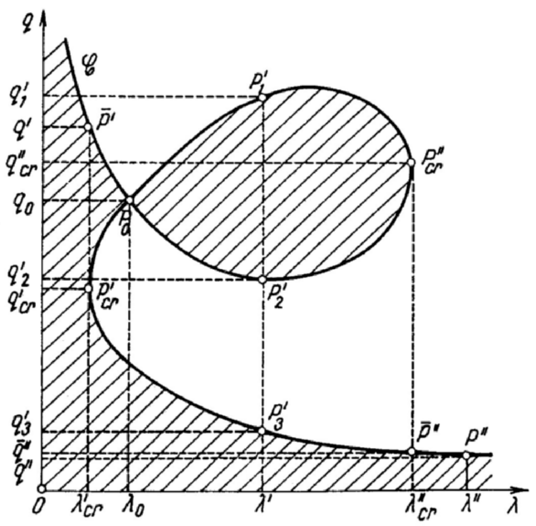

Figure 7. 9. Topological structure of the phase trajectories in the motion of a particle with a single degree of freedom, subjected to scleronomic constraints, in a field of conservative forces

Solution. For different values of the parameter λ one obtains three positions of equilibrium (for $\lambda = \lambda'$ correspond the points P_1', P_2', P_3' of ordinates q_1', q_2', q_3') or a single position of equilibrium (for $\lambda = \lambda''$ corresponds the point P'' of ordinate q''); one passes from three positions to only one position by critical values of the parameter λ $(\lambda = \lambda_{cr}', \lambda_{cr}'')$, to which correspond the points P_{cr}', P_{cr}'' of ordinates q_{cr}', q_{cr}'' and points $\overline{P'}$, $\overline{P''}$ of ordinates $\overline{q'}$, $\overline{q''}$. Noting that $\mathrm{d}q/\mathrm{d}\lambda = -f_\lambda'(q,\lambda)/f_q'(q,\lambda)$, it results that the critical points correspond to the solution of the equation $f_q'(q,\lambda) = 0$ (for which the tangent to the curve $f_q(q,\lambda) = 0$ is parallel to the Oq-axis) assuming that $f_\lambda'(q,\lambda) \neq 0$. We may conclude that the points of equilibrium appear and disappear two by two. We assume that C is a Jordan curve, which divides the plane in two regions. We observe that a straight line $\lambda = \lambda'$ pierces the curve C, e.g., at the point P_3'; if $f(q,\lambda') > 0$, hence $V_q'(q,\lambda') < 0$ under the curve C, then for q increasing $V'(q_3',\lambda') = 0$ on C and $V_q'(q,\lambda') > 0$ over the curve C. It results that $V_q(q_3',\lambda')$ represents an isolated minimum of the potential energy and the Lagrange-Dirichlet theorem (see Appl.7.1) allows to state that:

The positions of equilibrium of a particle which moves after the law $\ddot{q} = f(q, \lambda)$ in a conservative field are stable if the domain $f(q, \lambda) > 0$ is under the curve $f(q, \lambda) = 0$, $q > 0$, $\lambda > 0$, and labile if this domain is over the curve (*Poincaré's theorem*).

The hatched domain corresponds to $f(q, \lambda) > 0$ in Fig.7.9.

Application 7.4

Problem. Let be a surface S which passes through the origin O, so that the plane tangent at this point is horizontal; in the neighbourhood of this point, the surface is over this point. Study the small oscillations around the point O of a heavy particle staying on this surface.

Mathematical model. Corresponding to *Torricelli's theorem* (see Appl.7.1) the point O is a stable position of equilibrium for a heavy particle P of mass m. Taking the Oz-axis along the ascendent local vertical, the surface S may be represented in the vicinity of the point O by a Maclaurin series in the form

$$z = \frac{1}{2}\left(\frac{x^2}{R_1} + \frac{y^2}{R_2}\right) + \varphi(x, y), \tag{a}$$

where R_1, R_2 are the principal curvature radii of the surface at O, while $\varphi(x, y)$ contains terms at least of the third degree with respect to the co-ordinates x, y.

Solution. The simple potential corresponding to the gravitational field is $U(z) = -mgz$, where g is the gravitational acceleration; eliminating the constraint relation (a) and neglecting the terms of higher order, we get

$$U(x, y) = -\frac{mg}{2}\left(\frac{x^2}{R_1} + \frac{y^2}{R_2}\right),$$

the force which acts upon the particle being given by

$$\mathbf{F} = \operatorname{grad} U = -mg\left(\frac{x}{R_1}\mathbf{i} + \frac{y}{R_2}\mathbf{j}\right).$$

We obtain thus the equations of motion

$$\ddot{x} = -\omega_1^2 x, \ \ddot{y} = -\omega_2^2 y, \ \omega_1^2 = \frac{g}{R_1}, \ \omega_2^2 = \frac{g}{R_2}. \tag{b}$$

By integration, it results

$$x = a_1 \cos(\omega_1 t - \varphi_1), \ y = a_2 \cos(\omega_2 t - \varphi_2), \tag{c}$$

where the amplitudes a_1, a_2 and the phase differences φ_1, φ_2 are determined by initial conditions. In particular, if $R_1 = R_2 = R$, then one obtains the small motions corresponding to the spherical pendulum (see Appl.4.5 too).

Application 7.5

Problem. Study the small oscillations of a discrete system S of n particles subjected to holonomic and scleronomic constraints in a field of conservative forces, in the space E_3, around a stable position of equilibrium.

Mathematical model. We consider a system S of n particles P_i, $i = \overline{1,n}$, subjected to m holonomic (geometric) and scleronomic constraints; in this case, the system has $s = 3n - m$ degrees of freedom, hence there are necessary s generalized co-ordinates q_j, $j = \overline{1,s}$, to specify its position. Let be $P(q_1, q_2, \ldots, q_s)$ the representative point in the configuration space Λ_s with s dimensions of Lagrange and let be $V = V(q_1, q_2, \ldots, q_s)$ the potential energy corresponding to the given field of forces. The representative point specifies the position of the system S in the space Λ_s by the functions $q_j = q_j(t)$.

Because the potential energy is determined making abstraction of an arbitrary constant, we may choose this constant as to have $V(0,0,\ldots,0) = 0$ at the point $O(0,0,\ldots,0)$. The Lagrange-Dirichlet theorem (see Appl.7.1) shows that for the position of stable equilibrium P^0 (let that one be the origin of generalized co-ordinates) the potential energy has an isolated minimum. Let $P^0 \equiv O$ be the respective point; thus, in a neighbourhood of P^0 we have $V(P) > 0$. We assume that V may be developed into a power series in the form

$$V = V_0 + V_1 + V_2 + \ldots + V_n + \ldots, \tag{a}$$

where V_n is a polynomial of nth degree in generalized co-ordinates. We observe that $V(0,0,\ldots,0) = V_0 = 0$; then $V_1 = 0$ is a hyperplane which passes through P^0, hence $V_1(P)$ has not a constant sign in the neighbourhood of P^0. Having to do with small oscillations, the polynomials V_3, V_4, … may be neglected with respect to V_2. In this case $V = V_2$, hence a positive definite quadratic form ($V_2 > 0$ in the neighbourhood of P^0, vanishing only at P^0, hence if all generalized co-ordinates are zero); we may write

$$V = \frac{1}{2} \sum_{i=1}^{s} \sum_{j=1}^{s} a_{ij} q_i q_j \ , \ a_{ij} = \text{const} . \tag{b}$$

In the case of scleronomic constraints, the kinetic energy T is also a positive definite quadratic form in the generalized velocities $\dot{q}_j = \dot{q}_j(t)$. We may thus write

$$T = \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{s} b_{ij}\dot{q}_i\dot{q}_j \ , \tag{c}$$

where, in general, $b_{ij} = b_{ij}(q_1, q_2, \ldots, q_s)$; assuming a development into power series of those coefficients and taking into account that the oscillations are small, we may take $b_{ij} = \text{const}$.

Solution. As it is known, one can make always a linear transformation of generalized co-ordinates so that, in the new co-ordinates $\eta_k = \eta_k(t)$, $k = \overline{1, s}$ (called *normal co-ordinates*), the two quadratic forms be expressed simultaneously in the form of sums of squares (it is sufficient that only one of the quadratic forms be positive definite, while the other may be only positive)

$$T = \frac{1}{2}\sum_{k=1}^{s}\dot{\eta}_k^2 \ , \ V = \frac{1}{2}\sum_{k=1}^{s}\omega_k^2\eta_k^2 \ ; \tag{d}$$

here ω_k^2 are the s real and positive roots of the algebraic equation of s th degree

$$\det\left[a_{ij} - \omega^2 b_{ij}\right] = 0 \, . \tag{e}$$

Lagrange's kinetic potential being $\mathrm{L} = T - V$, one may write the Lagrange equations of motion in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathrm{L}}{\partial \dot{\eta}_k}\right) - \frac{\partial \mathrm{L}}{\partial \eta_k} = \ddot{\eta}_k + \omega_k^2\eta_k = 0 \, , \ k = \overline{1, s} \tag{f}$$

for the representative point. By integration, we obtain

$$\eta_k(t) = a_k \cos(\omega_k t - \varphi_k), \ k = \overline{1, s} \, , \tag{g}$$

where a_k and φ_k are the amplitudes and the phase differences, respectively. One may thus state that any permanent oscillatory phenomenon (scleronomic constraints) may be analysed by a superposition of independent harmonic oscillations (*D. Bernoulli's theorem*).
The *mechanical oscillations* are called also *vibrations*.

Application 7.6

Problem. Study the influence of a holonomic constraint which intervenes in the frame of a permanent oscillatory phenomenon.

Mathematical model. Consider a holonomic (geometric) constraint expressed in the configurations space Λ_s in the form

$$f(\eta_1, \eta_2, \ldots, \eta_s) = 0 \, , \tag{a}$$

where η_j, $j = \overline{1, s}$, are normal generalized co-ordinates (see Appl.7.5). Developing into a power series, we remain to the linear form, corresponding to small oscillations; thus, we have

$$\sum_{j=1}^{s} C_j \eta_j = 0 , \; C_j = \text{const} . \tag{b}$$

Solution. Eliminating these constraints, Lagrange's equations (the equations (f) in the mentioned application) become

$$\ddot{\eta}_k + \omega_k^2 \eta_k + \lambda C_k = 0 , \; k = \overline{1, s} , \tag{c}$$

where $\lambda = \lambda(t)$ is a Lagrange's multiplier. Let us assume that

$$\eta_j = \alpha_j \cos \omega t , \; \lambda = \mu \cos \omega t , \; \alpha_j, \omega, \mu = \text{const.} \tag{d}$$

By the condition of verifying the equation (c), we find

$$\alpha_j \left(\omega_j^2 - \omega^2 \right) + \mu C_j = 0 . \tag{e}$$

Taking into account (d), the condition (b) becomes

$$\sum_{j=1}^{s} C_j \alpha_j = 0 . \tag{f}$$

Replacing α_j given by (e), it results the algebraic equation

$$\sum_{j=1}^{s} \frac{C_j}{\omega_j^2 - \omega^2} = 0 , \tag{g}$$

which gives the values of ω^2, hence of ω, for which the equation (c) is verified; this equation is of $(s-1)$-degree and has $s-1$ real roots contained between ω_1^2, ω_2^2; ω_2^2, ω_3^2; …; ω_{s-1}^2, ω_s^2, assuming that $\omega_1 < \omega_2 < \ldots < \omega_s$.
We may thus state that, in a holonomic and scleronomic, discrete mechanical system, with s degrees of freedom, subjected to small oscillations around a stable position of equilibrium, the intervention of a holonomic constraint cannot bring down the fundamental note (the minimal frequency in acoustics) or cannot raise over the value of the frequency of the harmonic of s-th order (*Rayleigh's theorem*).

Application 7.7

Problem. Study the motion of the mathematical pendulum in the phase space.

Mathematical model. We use the results given in Appl.7.3, taking into consideration an equation of the form (a) in the phase space of co-ordinates q and p. With the notations in Appl.4.33, corresponding to a mathematical pendulum, and noting that $q = \theta$, we obtain

$$p^2 = 2\omega^2 \cos\theta + h\,,\; p = \dot{\theta}\,,\; V(\theta) = -\omega^2 \cos\theta\,, \tag{a}$$

with

$$\omega^2 = \frac{g}{l}\,,\; h = 2\omega^2 \overline{h}\,,\; \overline{h} = -\cos\alpha\,, \tag{b}$$

$\overline{h}$ being a *non-dimensional constant*.

Solution. Representing $2V(\theta)$ vs. θ (Fig.7.10), we see that the motion can take place only for $\overline{h} \in [-1,1]$; we may have also $\overline{h} > 1$, but it does not correspond to a real angle, the motion being – in this case – circular. The condition $2V(\theta) \le h$ allows to draw the curves $p = p(\theta)$, symmetric with respect to the $O\theta$-axis, as function of various values of $\overline{h}$ in the phase space. For $\overline{h} \in (-1,1)$ the motion is oscillatory (we have a simple pendulum), e.g. for $\overline{h} = 0$. If $\overline{h} = 1$, then the motion is asymptotic, obtaining the separation lines (drawn with a thicker line) in the phase space; it corresponds a labile position of equilibrium for $\alpha = \pi$. For $\overline{h} = -1$ we obtain a stable position of equilibrium (a point in the phase space), corresponding $\alpha = 0$. Noting that, for $p > 0$, q increases at the same time as t, we have indicated by an arrow the direction of motion in the phase space.

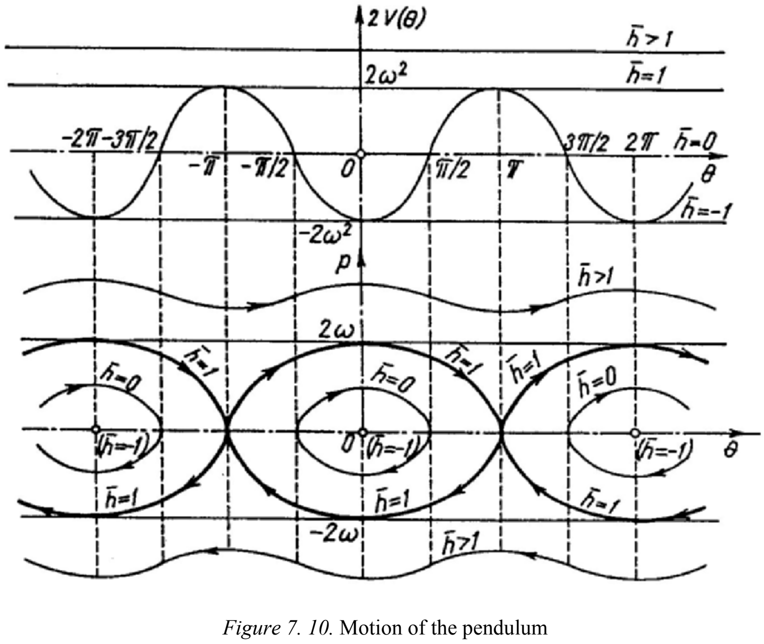

Figure 7. 10. Motion of the pendulum

We observe that the separation lines are phase trajectories of the representative point in the phase plane; they allow to pass from one type of motion to another one. We have seen that a singular point is specified by the equations $f(q) = 0$, $p = 0$, any other point being an ordinary one; it results that an ordinary point is characterized by a well definite direction of the tangent to the phase trajectory which passes through this point. We may thus state that:

Through any ordinary point in the phase space passes a phase trajectory and only one (*Cauchy's theorem*).

We notice that the equation $p\mathrm{d}p = f(q)\mathrm{d}q$ defines a field of vectors of components q, p, hence a field of velocities in the phase plane; the singular point represents the point in which the velocity in the phase plane vanishes.

The topological methods allow the study of the general topological properties of the phase trajectories in the neighbourhood of the stable points of equilibrium $(\overline{h} = -1)$. Such a singular point is called *centre*; analogous considerations lead to the denomination *saddle point* for a singular point of labile equilibrium $(\overline{h} = 1)$.

Application 7.8

Problem. Study the topological structure of the phase trajectories of a simple pendulum in a motion of rotation around a vertical axis.

Mathematical model. We use the results in Appl.4.26 to the study of the topological structure of the phase trajectories of a simple (mathematical) pendulum for which the vertical circle on which the heavy particle moves is rotating with a constant angular velocity ω about its vertical diameter. The results in this application lead to the differential equation $(q = \theta)$

$$\ddot{\theta} = (\cos\theta - \lambda)\sin\theta\,, \tag{a}$$

where $\lambda = g / l\omega^2 > 0$ is a parameter with respect to which is effected the study.

Solution. The curves C are given by the straight lines $\theta = 0$ and $\theta = \pm\pi$ and by the curve $\theta = \arccos\lambda$. Applying *Poincaré's theorem* (see Appl.7.3), we find stable branches of the curve C (the points of equilibrium of *centre type* being denoted by full circlets , i.e. $\theta = \arccos\lambda$ and $\theta = 0$, $\lambda > 1$ and $\theta = \pm\pi$, $\lambda < -1$) as well as *labile branches* (the points of equilibrium of *saddle type* being denoted by hollow circlets, i.e. $\theta = 0$, $\lambda < 1$ and $\theta = \pm\pi$, $\lambda > -1$) (Fig.7.11). The points $\theta = 0$, $\lambda = 1$ and $\theta = \pm\pi$, $\lambda = -1$ are points of branching of equilibrium, while the values $\lambda_{cr} = \pm 1$ are *critical values* (of *bifurcation*) of the parameter λ, corresponding to those points. Taking into account (a), it results that $\lambda > 0$, the domains of the figure being thus restraint; as well, to have $\lambda < 1$ the angular velocity ω must be sufficient great. If we put the condition that a separation line passes trough the singular point $\dot{\theta} = 0$, $\theta = 0$, then we find the first integral

$$\dot{\theta}^2 = \sin^2\theta - 2\lambda(1 - \cos\theta), \tag{b}$$

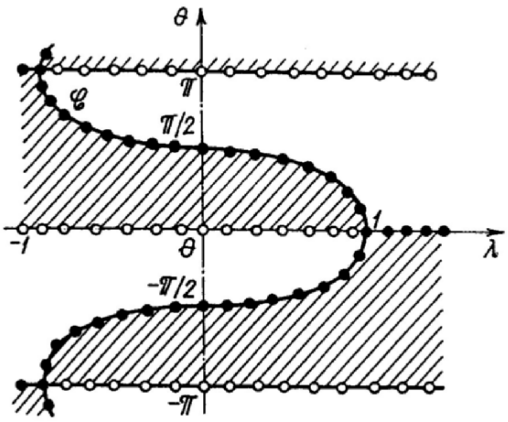

Figure 7. 11. Topological structure of the phase trajectories of a simple pendulum in motion of rotation about a vertical axis

whence

$$\dot{\theta} = \pm 2\sin\frac{\theta}{2}\sqrt{\cos^2\frac{\theta}{2} - \lambda}\ ; \tag{c}$$

if such a line passes through the singular points $\dot{\theta} = 0$, $\theta = \pm\pi$, the respective first integral becomes

$$\dot{\theta}^2 = \sin^2\theta + 2\lambda(1 + \cos\theta) \tag{d}$$

whence

$$\dot{\theta} = \pm 2\cos\frac{\theta}{2}\sqrt{\sin^2\frac{\theta}{2} + \lambda}\ . \tag{e}$$

For $\lambda < -1$, the singular points of saddle type $\dot{\theta} = 0$, $\theta = \pm\pi$ become singular points of centre type (Fig.7.12, a); passing through $\lambda'_{cr} = -1$, for $-1 < \lambda < 0$ two separation lines, C_1 and C_2, appear the first of those ones surrounding two centres, while the point O becomes a singular point of saddle type (Fig.7.12, b).

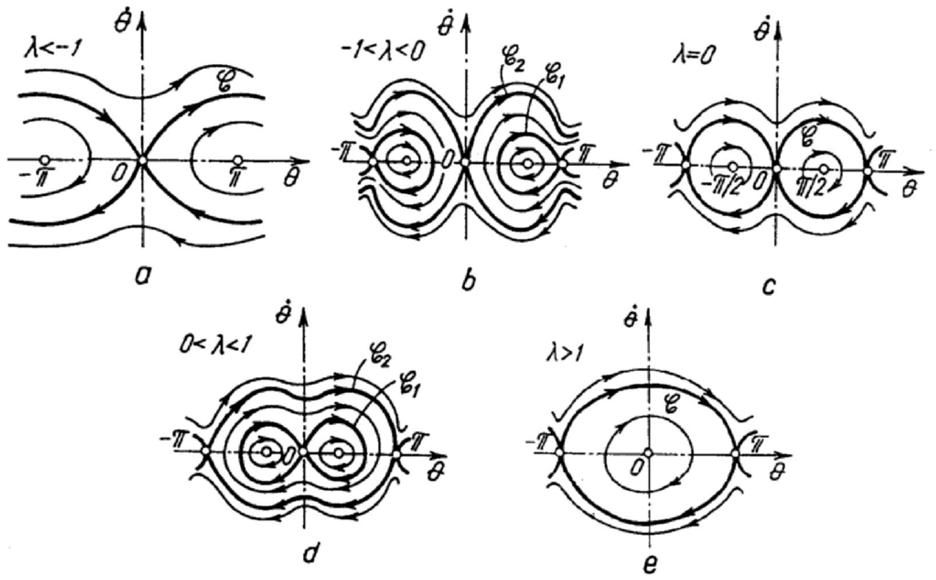

Figure 7. 12. Phase trajectories of a simple pendulum in motion of rotation about a vertical axis for: $\lambda < -1$ (a); $-1 < \lambda < 0$ (b); $\lambda = 0$ (c); $0 < \lambda < 1$ (d); $\lambda > 1$ (e)

If $\lambda = 0$, hence if $\omega \to \infty$, then the curves C_1 and C_2 coincide with the curve C and form only one line of separation; in this case, the centres are of abscissae $\theta = \pm\pi/2$ (Fig.7.12, c). For $0 < \lambda < 1$ one obtains two separation lines C_1 and C_2, corresponding to the equations (c), which pass through the singular points of saddle type $\dot{\theta} = 0$, $\theta = 0$, and $\dot{\theta} = 0$, $\theta = \pm\pi$, respectively; in the interior of the loops of the curve C_1 there exist two other singular points of centre type, having the abscissae $\theta = \pm 2 \arccos\sqrt{\lambda}$ (Fig.7.12, d). If $\lambda = \lambda''_{cr} = 1$, then the curve C_1 coincides with the singular point O, which becomes a point of centre type; for $\lambda > 1$, remains only one separation line C (Fig.7.12, e). We observe thus that the separation lines correspond to phase trajectories with different topological aspects.

The above considerations allow to state, without demonstration:

The closed phase trajectories of a particle which is moving after the law $\ddot{q} = f(q, \lambda)$ in a conservative field may surround only an odd number of singular points, the number of centres being greater than the number of singular points of saddle type (*Poincaré's theorem*).

Application 7.9

Problem. Study the topological structure of the phase trajectories of a simple pendulum in a resistant medium.

Mathematical model. In the case of motion of the simple pendulum in a resistant medium the field of forces is non-conservative (see Appl.1.10). If we put the condition $\dot{\theta} = \dot{\theta}_0$ greater or smaller than 0 for $\theta = 0$ in the formula (h), in the mentioned application, then we obtain

$$\dot{\theta}^2 = \left(\dot{\theta}_0^2 - \frac{2\omega^2}{4k^2+1} \right) e^{\mp 2k^2\theta} + \frac{2\omega^2}{4k^2+1} \left(\cos\theta \mp 2k^2 \sin\theta \right), \tag{a}$$

where the sign $\pm$ corresponds to $\dot{\theta}_0$ greater or smaller than 0 respectively.

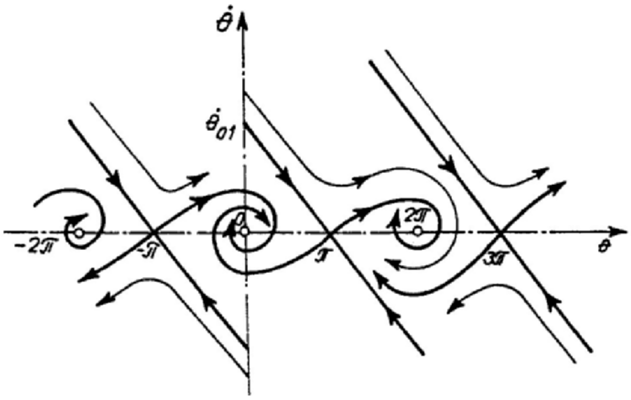

Figure 7. 13. Topological structure of the phase trajectories of a simple pendulum in a resistent medium

Solution. The points $\dot{\theta} = 0$, $\theta = n\pi$, $n \in \mathbb{Z}$, correspond to positions of equilibrium; the equilibrium is *stable* for n even (the corresponding singular points are of *focus type*), while for n odd the equilibrium is *labile* (there correspond singular points of *saddle type*) (Fig.7.13). If

$$\dot{\theta}_{0n}^2 = \frac{2\omega^2}{4k^2+1} \left(1 + \mathrm{e}^{2k^2 n\pi} \right), \ n \text{ odd}, \tag{b}$$

then we notice that for $\dot{\theta}_0 < \dot{\theta}_{01}$ the particle oscillates, the motion being damped around the stable position of equilibrium $\dot{\theta} = 0$, $\theta = 0$; if $\dot{\theta}_0 = \dot{\theta}_{01}$, then one obtains the asymptotic motion of the particle. For $\dot{\theta}_{01} < \dot{\theta}_0 < \dot{\theta}_{03}$ the particle effects a complete rotation and then its oscillatory motion is damped; in general, if $\dot{\theta}_{0n} < \dot{\theta}_0 < \dot{\theta}_{0,n+2}$, n odd, the particle effects $(n+1)/2$ complete rotations, passing then in a regime of damped oscillations around a stable position of equilibrium

Application 7.10

Problem. Study the small oscillations of a small sphere M of mass m, linked to a fixed point O by a spring of elastic constant k and of negligible weight, if it may rotate around the point O in a vertical plane (Fig.7.14). The length of the spring in non-deformed state is l_0.

Mathematical model. The position of the particle M may be specified by the angle φ and by the length l of the spring at a given moment, having two degrees of freedom.

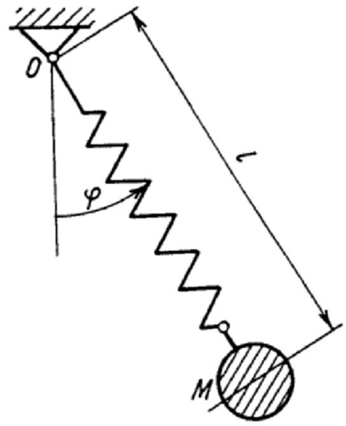

Figure 7. 14. Small oscillations of a particle linked by an elastic spring around a fixed point, in a vertical plane

The kinetic energy and the potential one are expressed in the form

$$T = \frac{1}{2}m\left(l^2\dot{\varphi}^2 + \dot{l}^2\right),\ V = \frac{1}{2}k(l - l_0)^2 + mg(l_0 - l\cos\varphi), \tag{a}$$

respectively. We may write Lagrange's equations (see Appl.6.2, formula (e)) in the form

$$\begin{aligned} &m\ddot{l} - ml\dot{\varphi}^2 + k(l - l_0) - mg\cos\varphi = 0\,, \\ &l\ddot{\varphi} + 2\dot{l}\dot{\varphi} + g\sin\varphi = 0\,. \end{aligned} \tag{b}$$

By the notations $l_1 = l_0 + m/k$, $p_1^2 = k/m$, $p_2^2 = g/l_1$ and by the change of variable $x = l - l_1$, the equations (b) become

$$\begin{aligned} &\ddot{x} + p_1^2 x = (l_1 + x)\dot{\varphi}^2 - g(1 - \cos\varphi), \\ &\ddot{\varphi} + p_2^2\sin\varphi = -\frac{x}{l_1}\ddot{\varphi} - \frac{2}{l_1}\dot{x}\dot{\varphi}\,, \end{aligned} \tag{c}$$

thus obtaining a new system of non-linear differential equations.
We observe that one may find a particular solution for $\varphi = 0$, the pendulum oscillating in this case only along the vertical, after the law

$$x = x_1 = x_1^0\cos(p_1 t - \psi),\ \varphi = \varphi_1 = 0\,. \tag{d}$$

This motion with the pulsation p_1 represents a natural mode of oscillation.

Assuming that φ is very small and neglecting the powers of higher order of this argument, the differential equations (c) become

$$\begin{aligned} &\ddot{x} + p_1^2 x = 0, \\ &\ddot{\varphi} + p_2^2 \varphi = 0 ; \end{aligned} \tag{e}$$

in this case x and φ become normal co-ordinates, the two modes of vibrations being no more couplEd. Although for a very small φ we obtain a system of non-coupled ODEs, a reciprocal influence of the two oscillations is still possible. This appears in the form of *the instability of the basic oscillation* $x_1 = x_1^0 \cos(p_1 t - \psi)$ and $\varphi = 0$.

To study this phenomenon, we consider the motions in the neighbourhood of the fundamental oscillation. Let thus be $x = x_1 + y$ and φ; we suppose that y and φ are sufficiently small to may linearize the terms which appear. Replacing in the equations (e), we get

$$\ddot{y} + p_1^2 y = 0 , \ \ddot{\varphi}\left(1 + \frac{x_1}{l_1}\right) + 2\frac{\dot{x}_1}{l_1}\dot{\varphi} + p_2^2 \varphi = 0 . \tag{f}$$

Although the vibrations are non-coupled with respect to the parameters y and φ, one may see that φ depends on the fundamental motion $x_1 = x_1^0 \cos(p_1 t - \psi)$. Because the equation in φ has coefficients variable with the pulsation p_1, by convenient changes of function it may be brought to the form of a differential equation of Hill's type with variable coefficients, with the same pulsation p_1. One may deduce that the solution of the equation in φ may have also domains of instability for some ratios between the pulsations p_1 and p_2; this instability puts in evidence also the instability of the fundamental oscillations x_1, the reciprocal influence of the two oscillations being thus proved.

Application 7.11

Problem. A *Watt centrifugal regulator* is composed of two rods OA and OB of the same length l, articulated at the point O of a vertical axle tree; at the ends of the rods are two balls of equal masses m. Other two rods CD and CE are articulated to the first ones at the points D and E and by a collar C, which slides along the axle tree; one assumes that the quadrangle $ODCE$ is a rhomb of side a. One considers a modelling of particle for the balls A and B (Fig.7.15). If the angular velocity of the axle tree increases, then the rods move away, while the masses raise; at the same time, the collar raises too, acting by a force P a system of levels which diminishes the admission of the vapour in a motor. Neglecting the masses of the rods and of the collar, study the stability of motion of the regulator.

Mathematical model. At a given moment, the position of the regulator is determined by the rotation angle θ of the plane of the regulator around the axle OC and by the angle

φ made by the rods OA and OB with the axle tree in the plane of the regulator; the mechanical system has thus two degrees of freedom. The moment of inertia of the parts in rotation, without the balls A and B, with respect to the axis of the tree is I_0; the moment of bringing back due to the variation $\Delta\varphi = \varphi - \varphi_0$ of the angle φ made by the rod OA with the axis of the tree with respect to an angle φ_0, in case of a constant angular velocity ω_0 of the axle tree, is $-k\Delta\varphi = -k(\varphi - \varphi_0)$, where k is a constant coefficient.

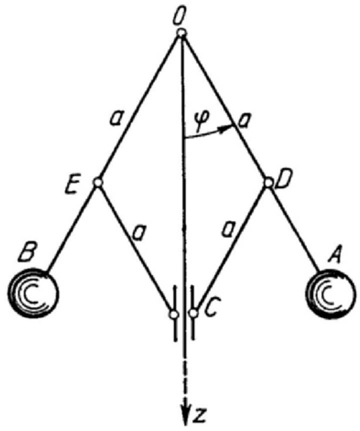

Figure 7. 15. Watt's centrifugal regulator

The motion of the regulator is composed from a rotation in its plane around an axis normal to the plane around the OC-axis, with an angular velocity $\dot{\varphi}$. The two axes are principal axes of inertia, so that the kinetic energy reads

$$T = \frac{1}{2}\left(I_1\dot{\theta}^2 + I_2\dot{\theta}^2\right), \tag{a}$$

with

$$I_1 = I_0 + 2ml^2 \sin^2 \varphi \,,\; I_2 = 2ml^2 \,; \tag{b}$$

finally, we get

$$T = \frac{1}{2}\left[\left(I_0 + 2ml^2 \sin^2 \varphi\right)\dot{\theta}^2 + 2ml^2\dot{\varphi}^2\right]. \tag{c}$$

Upon the regulator act the weights mg of the balls, the force P in the collar, the moment $-k(\varphi - \varphi_0)$ and the reactions at O and C, which give a zero virtual work. Assuming that only a virtual displacement $\delta\theta$ takes place, we obtain $\delta W = -k(\varphi - \varphi_0)\delta\theta$; hence it results the generalized force

$$Q_\theta = -k(\varphi - \varphi_0). \tag{d}$$

As well, the virtual displacement $\delta\varphi$ leads to

$$\delta W = P\delta z_C + mg\delta z_A + mg\delta z_B = P\delta z_C + 2mg\delta z_A;$$

but $z_C = 2a\cos\varphi$, $z_A = l\cos\varphi$, so that $\delta W = -2(aP + mgl)\sin\varphi\delta\varphi$, and the corresponding generalized force is given by

$$Q_\varphi = -2(aP + mgl)\sin\varphi. \tag{e}$$

We obtain thus Lagrange's equations (see formula (e) in Appl.6.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(I_1\dot{\theta}\right) = -k(\varphi - \varphi_0),\ I_2\ddot{\varphi} - \frac{1}{2}\dot{\theta}^2\frac{\partial I_1}{\partial\varphi} = -2(aP + mgl)\sin\varphi. \tag{f}$$

Solution. We search firstly the position of *relative equilibrium* of the regulator in its plane, corresponding to the rotation with a constant velocity $\dot{\theta} = \omega_0$ about the axis of the tree; let be φ_0 the angle corresponding to this position. Noting that $\ddot{\varphi} = 0$, the second equation (f) leads to $\sin\varphi_0\left(m\omega_0^2 l^2\cos\varphi_0 - aP - mgl\right) = 0$; one obtains thus two positions of relative equilibrium for $\varphi_0 = 0$ and for $\cos\varphi_0 = (aP + mgl)/m\omega_0^2 l^2$. The motion with a constant angular velocity ω_0 given by the second relation, for which we assume that $aP + mgl < m\omega_0^2 l^2$, is called *motion of régime* of the regulator.

We use now the equation (f) to study the small oscillations around this *motion of régime*. We denote $\varphi = \varphi_0 + \psi$, $\dot{\theta} = \omega_0 + \gamma$. The first equation (f) is written in the form

$$4ml^2\sin\varphi\cos\varphi\dot{\varphi}\dot{\theta} + I_1\ddot{\theta} = -k(\varphi - \varphi_0),$$

whence

$$2ml^2\sin 2(\varphi_0 + \psi)\dot{\psi}(\omega_0 + \gamma) + \left[I_0 + 2ml^2\sin^2(\varphi_0 + \psi)\right]\dot{\gamma} = -k\psi;$$

neglecting the powers of higher order ($\sin\psi \cong \psi$, $\cos\psi \cong 1$), we obtain

$$\left(I_0 + 2ml^2\sin^2\varphi_0\right)\dot{\gamma} + 2ml^2\omega_0^2\sin 2\varphi_0\dot{\psi} + k\psi = 0.$$

The second equation (f) becomes

$$I_2\ddot{\varphi} - 2ml^2\dot{\theta}^2\sin\varphi\cos\varphi = -2(aP + mgl)\sin\varphi \tag{g}$$

or

$$I_2\ddot{\psi} - ml^2(\omega_0 + \gamma)^2\sin^2(\varphi_0 + \psi) = -2(aP + mgl)\sin(\varphi_0 + \psi).$$

In the frame of the same approximations, we get

$$\ddot{\psi} - \omega_0 \sin 2\varphi_0 \gamma + \omega_0^2 \sin^2 \varphi_0 \psi = 0 . \tag{h}$$

The solutions of the system of equations (g), (h) are of the form $\psi = A_1 e^{\lambda t}$, $\gamma = A_2 e^{\lambda t}$ and lead to the characteristic equation (the necessary and sufficient condition to have non-zero A_1 and A_2)

$$a_0 \lambda^3 + a_2 \lambda + a_3 = 0 , \tag{i}$$

where

$$a_0 = \frac{I_0}{2ml^2} + \sin^2 \varphi_0 ,\ a_1 = 0 ,\ a_2 = \omega_0^2 \sin^2 \varphi_0 \left(1 + 3\cos^2 \varphi_0 + \frac{I_0}{2ml^2} \right),$$
$$a_3 = \frac{k}{2ml^2} \omega_0 \sin^2 \varphi_0 . \tag{j}$$

To have a stable motion, the real parts of the roots λ must be negative (so that the exponential does tend to zero for $t \to \infty$). In conformity to Hurwitz's criterion, this condition takes place if

$$a_1 > 0 ,\ \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = a_1 a_2 - a_0 a_3 > 0 ,\ \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3 \left(a_1 a_2 - a_0 a_3 \right) > 0 ; \tag{k}$$

one may see that, in the considered case, these conditions are not verified and the motion of régime is not stable. This fact, which is established experimentally too, imposes the introduction of new elements in the regulator system.

Application 7.12

Problem. Study the motion of *the centrifugal regulator* in Fig.7.16. Each ball has the mass m_1, the collar has the mass m_2, the spring is of elastic constant k, while the four rods are each one of length l; the weights of the rods and of the spring are negligible. The moment of inertia of the collar with respect to the axis of rotation is I. Upon the axis of the regulator acts a moment M. The regulator rotates with an angular velocity ω, the variation of which leads to a change of the distance of the balls to the rotation axis, to a displacement of the collar and to a deformation of the spring; by a fitment acts a valve which regulates the alimentation with fuel of the engine, so as to obtain a certain angular velocity. We assume also that the collar is linked to a hydraulic damper which yields a viscous force of resistance, the damping coefficient being c.

Mathematical model. We choose as generalized co-ordinates the angle φ of rotation about the vertical axis and the angle α indicated in the figure, the system having thus two degrees of freedom. We assume that the regulator is built up so as for $\alpha = 0$ the spring be non-deformed; measured from this position of the collar, the distances s_1 and s_2 indicated on the figure are given by $s_1 = l(2 - \cos\alpha)$, $s_2 = 2l(1 - \cos\alpha)$.

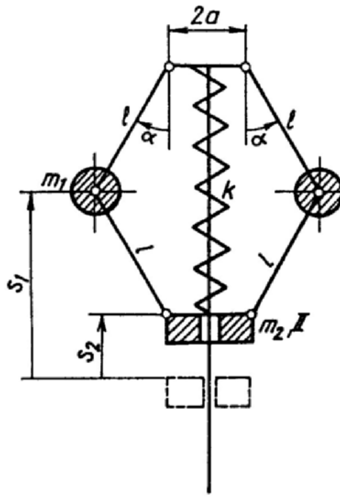

Figure 7. 16. Centrifugal regulator

The kinetic energy of the balls and of the collar, respectively, are

$$T_1 = 2(m_1/2)v_1^2,\ T_2 = (m_2/2)v_2^2 + (I/2)\dot{\varphi}^2 .$$

Noting that the relative and the transportation velocities are orthogonal and that $v_r = l\dot{\alpha}$, $v_t = (a + l\sin\alpha)\dot{\varphi}$, we get $v_1^2 = (a + l\sin\alpha)^2\dot{\varphi}^2 + l^2\dot{\alpha}^2$. The collar has a motion of rotation about the vertical axis with the angular velocity $\dot{\varphi}$ and a motion of translation with the velocity $v_2 = \mathrm{d}s_2/\mathrm{d}t = 2l\dot{\alpha}\sin\alpha$. Finally, the kinetic energy of the mechanical system is given by

$$T = T_1 + T_2 = \left[2m_1(a + l\sin\alpha)^2 + I\right]\frac{\dot{\varphi}^2}{2} + \left[2m_2 l^2 + 4m_2 l^2\sin^2\alpha\right]\frac{\dot{\alpha}^2}{2}. \tag{a}$$

The virtual work for a displacement compatible with the constraints is

$$\delta L = Q_\varphi\delta\varphi + Q_\alpha\delta\alpha = M\delta\varphi - 2m_1 g\delta s_1 - m_2 g\delta s_2 - ks_2\delta s_2 - c\dot{s}_2\delta s_2 ,$$

where ks_2 is the elastic force in the spring, while $c\dot{s}_2$ is the viscous resistant force. Calculating δs_1, δs_2 and $\dot{s}_2$ and replacing in the above relation, we may write

$$\delta W = M\delta\varphi + \left[-2m_1 gl\sin\alpha - 2m_2 gl\sin\alpha - 4l^2 k(1 - \cos\alpha)\sin\alpha - 4l^2 c\dot{\alpha}\sin^2\alpha\right]\delta\alpha ,$$

so that the generalized forces are

$$Q_\varphi = M ,\ Q_\alpha = -2l\sin\alpha\left[(m_1 + m_2)g + 2lk(1 - \cos\alpha) + 2lc\dot{\alpha}\sin\alpha\right]. \tag{b}$$

Lagrange's equations (formula (e), Appl.6.2) read

$$\left[2m_1(a+l\sin\alpha)^2+I\right]\ddot{\varphi}+4m_1(a+l\sin\alpha)\dot{\varphi}\dot{\alpha}l\cos\alpha=M,$$
$$\left(m_1+2m_2\sin^2\alpha\right)l^2\ddot{\alpha}+2m_2l^2\dot{\alpha}^2\sin\alpha\cos\alpha-m_1(a+l\sin\alpha)\dot{\varphi}^2l\cos\alpha \tag{c}$$
$$=-l\sin\alpha\left[(m_1+m_2)g+2lk(1-\cos\alpha)+2lc\dot{\alpha}\sin\alpha\right]$$

and form a system of non-linear differential equations.

Solution. The goal of the regulator is, obviously, to maintain a constant angular velocity ω_0 of the axis. First of all, we determine the position of *relative equilibrium* of the regulator corresponding to this angular velocity; thus the motion of the regulator will be a *motion of régime*. Let be, in this case, α_0 the value of α and $\dot{\varphi}=\omega_0=\text{const}$. As $\ddot{\varphi}=0$ and $\dot{\alpha}=0$, it results $M=0$ and

$$m_1(a+l\sin\alpha_0)\omega_0^2\cos\alpha_0-2kl(1-\cos\alpha_0)\sin\alpha_0-(m_1+m_2)g\sin\alpha_0=0, \tag{d}$$

thus obtaining the link between the régime angular velocity ω_0, the position α_0 of the regulator and the position of the collar $s_2=2l(1-\cos\alpha_0)$, a relation important in design. To put in evidence *the stability of the motion of régime*, we assume that that one is characterized by ω_0 and may be perturbed by the variation of the moment M. We may write $\dot{\varphi}=\omega=\omega_0+\omega_1$ and $\alpha=\alpha_0+\alpha_1$, where ω_1 and α_1 are small, so that we may consider

$$\sin(\alpha_0+\alpha_1)=\sin\alpha_0+\alpha_1\cos\alpha_0,\ \cos(\alpha_0+\alpha_1)=\cos\alpha_0-\alpha_1\sin\alpha_0,$$
$$M(\alpha_0+\alpha_1)=M(\alpha_0)+\alpha_1M'(\alpha_0)+\ldots\cong\alpha_1M'(\alpha_0),$$

because $M(\alpha_0)=0$.

Replacing in the equations (c) and taking into account the equation (d), we obtain the system

$$f\dot{\omega}_1+p\dot{\alpha}_1-M'\alpha_1=0,$$
$$h\ddot{\alpha}_1+b\dot{\alpha}_1+d\alpha_1-p\omega_1=0, \tag{e}$$

where

$$f=2m_1(a+l\sin\alpha_0)^2+I,\ p=4m_1(a+l\sin\alpha_0)\omega_0\cos\alpha_0,\ b=4cl^2\sin^2\alpha_0,$$
$$h=2\left(m_1+2m_2\sin^2\alpha_0\right)l^2,$$
$$d=2lm_1\omega_0^2\left[a\sin\alpha_0+l\left(2\sin^2\alpha_0-1\right)\right] \tag{f}$$
$$+4kl^2\left(\cos\alpha_0+2\sin^2\alpha_0-1\right)+2(m_1+m_2)gl\cos\alpha_0;$$

this system of linear ODEs determines the oscillations of the regulator about the motion of régime. Searching the solutions of this system in the form $\alpha_1=A_1e^{\lambda t}$, $\omega_1=A_2e^{\lambda t}$ we obtain the characteristic equation, which gives the pulsations

$$h\lambda^3 + b\lambda^2 + \left(d + \frac{p^2}{f} \right)\lambda - M'\frac{p}{f} = 0 ; \tag{g}$$

the oscillations are damped if the exponential decreases in time, hence if the real part of those equations is negative. According to *Hurwitz's theorem*, this condition is fulfilled if one verifies the conditions

$$h > 0 , \ b > 0 , \ b\left(d + \frac{p^2}{f} \right) - h\left(-M'\frac{p}{f} \right) > 0 , \ -M'\frac{p}{f}\left[b\left(d + \frac{p^2}{f} \right) + hM'\frac{p}{f} \right] > 0 ; \tag{h}$$

as $h > 0$, these conditions may be written in the form

$$b > 0 , \ M' < 0 , \ b\left(d + \frac{p^2}{f} \right) > -hM'\frac{p}{f} .$$

The condition $b > 0$ is fulfilled if there is a damping which satisfies the last condition (i). The condition $M' < 0$ is satisfied if, by a growth of the angle α, the regulator provokes a decreasing of the driving moment.

Application 7.13

Problem. The axle tree of a rotor rests in a spherical hinge with the centre at the point O (Fig.7.17). The weight of the system rotor-axle tree is P, the centre of gravity C being situated above the point O, at the distance $\overline{OC} = l_1$. The rotor is rotating with a constant angular velocity ω about the vertical axis of symmetry of the system. The inferior extremity of the axle tree is at the distance $\overline{OM} = l$ of the fixed point O. Study the stability of the motion of rotation of the system, knowing that the moment of inertia with respect to the symmetry axis is I_1, and with respect to any other axis normal to the first one at the point O is I_2.

Mathematical model. Let us consider the fixed frame of reference $Oxyz$, the axis Oz being vertical. The position of the rotation axis at a given moment is specified by the position M' of the point M along the axis of the tree; at a certain moment, it coincides with the axis OC, while the functions of time by means of which we may study the vibrations are the co-ordinates x and y of the point M.

Because the elastic forces at the point M' vanish, the differential equations of the vibrations are written in the form

$$\begin{aligned} I_2\ddot{x} + I_1\omega\dot{y} - l_1 Px = 0 , \\ I_2\ddot{y} - I_1\omega\dot{x} - l_1 Py = 0 ; \end{aligned} \tag{a}$$

multiplying the second equation by $\mathrm{i} = \sqrt{-1}$ and introducing the complex variable $u = x + \mathrm{i}y$, we get the differential equation in u

$$I_2\ddot{u} - \mathrm{i}I_1\omega\dot{u} - l_1 P u = 0\,. \tag{b}$$

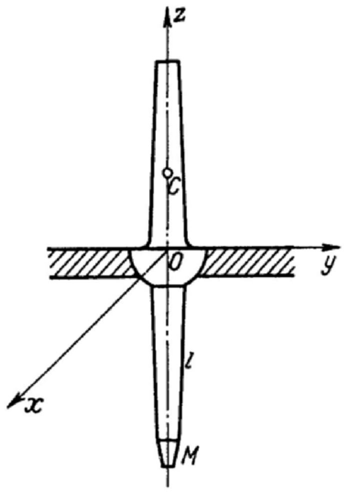

Figure 7. 17. Motion of a rotor-axle treee system

Solution. The characteristic equation

$$I_2 s^2 - \mathrm{i}I_1\omega s - l_1 P = 0 \tag{c}$$

has the roots

$$s_{1,2} = \frac{1}{2I_2}\left(\mathrm{i}I_1\omega \mp \sqrt{4I_2 l_1 P - I_1^2\omega^2}\right), \tag{d}$$

while the general solution of the differential equation (b) is

$$u = A_1 \mathrm{e}^{s_1 t} + A_2 \mathrm{e}^{s_2 t}\,; \tag{e}$$

if the complex integration constants are of the form $A_1 = C_1 e^{\mathrm{i}\alpha_1}$, $A_2 = C_2 e^{\mathrm{i}\alpha_2}$, where C_1, C_2, α_1, α_2 are real integration constants, the solution is written in the form

$$u = C_1 \mathrm{e}^{s_1 t + \mathrm{i}\alpha_1} + C_2 \mathrm{e}^{s_2 t + \mathrm{i}\alpha_2}\,. \tag{f}$$

If $I_1^2\omega^2 < 4I_2 l_1 P$, hence if $\omega < \omega_0 = (2/I_1)\sqrt{I_2 l_1 P}$, then the roots (d) are of the form $s_{1,2} = \mp a + \mathrm{i}b$, where $a = (1/2I_2)\sqrt{4I_2 l_1 P - I_1^2\omega^2} > 0$, $b = I_1\omega/2I_2 > 0$. In this case, the general solution is written in the form

$$u = C_1 \mathrm{e}^{-at + \mathrm{i}(bt + \alpha_1)} + C_2 \mathrm{e}^{at + \mathrm{i}(bt + \alpha_2)}\,. \tag{g}$$

One obtains thus the equations of motion

$$x = C_1 e^{-at} \cos(bt + \alpha_1) + C_2 e^{at} \cos(bt + \alpha_2),$$
$$y = C_1 e^{-at} \sin(bt + \alpha_1) + C_2 e^{at} \sin(bt + \alpha_2). \tag{h}$$

Noting that in the second term of the vibrations a factor increasing with time $(a > 0)$ appears, the corresponding component has an increasing amplitude, so that the motion of rotation of the system is labile. If $I_1^2 \omega^2 = 4 I_2 l_1 P$, hence if $\omega = \omega_0 = (2 / I_1)\sqrt{I_2 l_1 P}$, then the roots are equal, while the general solution is given by

$$u = (A_1 + A_2 t) e^{s_1 t} = C_1 e^{i(bt + \alpha_1)} + C_2 t e^{i(bt + \alpha_2)}, \tag{i}$$

so that the equations of motion are

$$x = C_1 \cos(bt + \alpha_1) + C_2 t \cos(bt + \alpha_2),$$
$$y = C_1 \sin(bt + \alpha_1) + C_2 t \sin(bt + \alpha_2). \tag{j}$$

In this case too, the amplitudes of the second component are increasing, and the motion of rotation of the system is unstable too.
If $I_1^2 \omega^2 > 4 I_2 l_1 P$, hence if $\omega > \omega_0 = (2 / I_1)\sqrt{I_2 l_1 P}$, then the roots $s_{1,2} = i p_{1,2}$, with $p_{1,2} = (2 / I_1)\left(I_1 \omega \mp \sqrt{I_1^2 \omega^2 - 4 I_2 l_1 P} \right)$ are purely imaginary. The general solution is given by

$$u = C_1 e^{i(p_1 t + \alpha_1)} + C_2 e^{i(p_2 t + \alpha_2)} \tag{k}$$

and the equations of motion are of the form

$$x = C_1 \cos(p_1 t + \alpha_1) + C_2 \cos(p_2 t + \alpha_2),$$
$$y = C_1 \sin(p_1 t + \alpha_1) + C_2 \sin(p_2 t + \alpha_2). \tag{l}$$

Hence, the two natural modes of vibration are harmonic. The amplitudes of the vibrations remain finite; that is, if $\omega > \omega_0$ the motion of rotation is stable. As we have seen, the vertical position of equilibrium of the mechanical system is unstable; it remains unstable for $0 \le \omega \le \omega_0$, but becomes stable for $\omega > \omega_0$.

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