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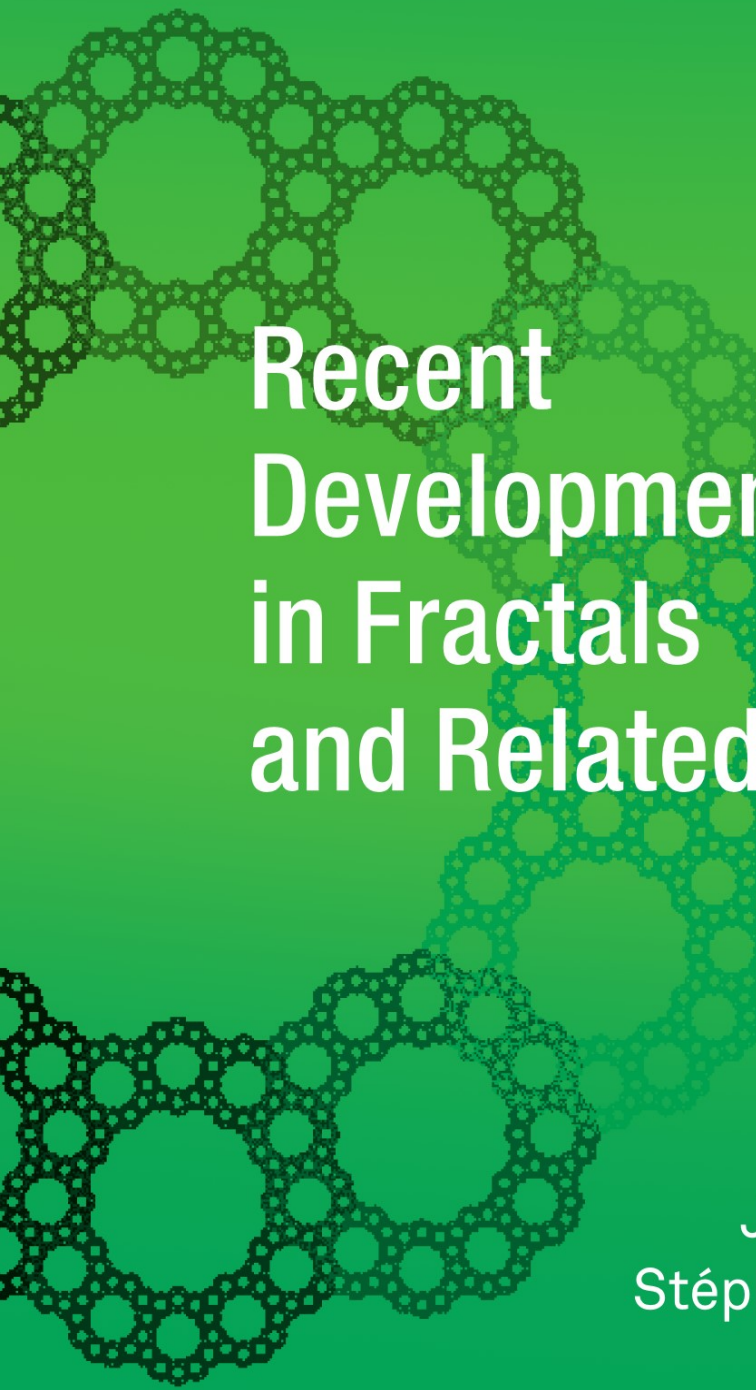
Applied and Numerical Harmonic Analysis



Recent Developments in Fractals and Related Fields

Julien Barral
Stéphane Seuret
Editors

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Jean - *Tiens, vous avez sorti le vitriol !*
Paul Volfoni - *Pourquoi vous dites ça ?*
Il a l'air pourtant honnête !
Fernand Naudin - *Sans être franchement malhonnête,*
au premier abord, comme ça,
il a l'air assez curieux.
Maître Folace - *Il date du mexicain, du temps des grandes heures,*
seulement on a dû arrêter la fabrication:
Il y a des clients qui devenaient aveugles,
alors ça faisait des histoires...
Raoul Volfoni - *Faut reconnaître, c'est du brutal !*

Michel Audiard

ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the

broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory
Biomedical signal processing
Digital signal processing
Fast algorithms
Gabor theory and applications
Image processing
Numerical partial differential equations
Prediction theory
Radar applications
Sampling theory
Spectral estimation
Speech processing
Time-frequency and time-scale analysis
Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

University of Maryland,
College Park

John J. Benedetto
Series Editor

Preface

This book originates from the conference "Fractals and Related Fields", held in September 2007 in Monastir (Tunisia) in honor of Jacques Peyrière. The purpose of this conference was to gather mathematicians sharing scientific interests with Jacques. It offered the opportunity to produce a state of the art in various active mathematical fields, which we list below. Moreover, this conference represented an important scientific event in the longstanding cooperation between the French and Tunisian mathematical communities.

The success of the conference and the enthusiasm demonstrated by many participants led us to the idea of gathering and editing editing these proceedings. Two years later, the process is now completed, and this book serves as witness of the great scientific moments we enjoyed in Monastir.

The book *Recent Developments in Fractals and Related Fields* provides the reader with a large overview and many recent developments in the mathematical fields related to fractals. It is thus intended for mathematicians working in the covered subjects, as well as for mathematicians and other scientists interested in discovering the fractal domain. The book gathers refereed original papers, as well as some surveys. We are very pleased that many young mathematicians contributed to this volume.

The following topics, related to fractals, are covered:

- Geometric measure theory and multifractals
- Harmonic and functional analysis and signal processing
- Dynamical systems and analysis on fractals
- Stochastic processes and random fractals
- Combinatorics on words

The content of each of the five parts is too rich and various to be shortly detailed here. Throughout this volume, the reader will discover interesting and motivating results and, we hope, sources of further research.

This preface gives us the opportunity to thank our Tunisian colleagues and friends Fathi Ben Nasr and Mounir Mensi for the local organization of the conference in Monastir, and Stéphane Jaffard for all of his of wise advice.

Paris *Julien Barral*
September 2009 *Stéphane Seuret*

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Part I

Geometric Measure Theory and Multifractals

Occupation Measure and Level Sets of the Weierstrass–Cellerier Function

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Summary. We show that the occupation measure of the Weierstrass–Cellerier function $\mathcal{W}(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x)$ is purely singular. Using our earlier results, we can deduce from this that almost every level set of $\mathcal{W}(x)$ is finite. These previous results and Besicovitch's projection theorem imply that for almost every c the occupation measure of $\mathcal{W}(x,c) = \mathcal{W}(x) + cx$ is purely singular. In this chapter we verify that this result holds for *all* $c \in \mathbb{R}$, especially for $c = 0$. As happens quite often, it is not that easy to obtain from an almost everywhere true statement one that holds everywhere.

1 Introduction

I proposed for the annual Miklós Schweitzer Mathematical Competition of the János Bolyai Mathematical Society in 2006 the following problem:

Suppose that $f(x) = \sum_{n=0}^{\infty} 2^{-n}||2^n x||$, *where* $||x||$ *is the distance of* x *from the closest integer* (*that is,* f *is Takagi's function*). *What can we say for Lebesgue almost every* $y \in f(\mathbb{R})$ *about the cardinality of the level set*

$$L_y = \{x \in [0,1] : f(x) = y\}?$$

The somewhat surprising answer is that it is finite.

It is natural to study the same question for functions defined similarly to Takagi's function. In this chapter we show that if in the above problem one uses the Weierstrass–Cellerier function

$$\mathcal{W}(x) = \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x), \tag{1}$$

then the same result holds, that is, almost every level set of $\mathcal{W}(x)$ is finite. The level sets are sets of the form $\{x : \mathcal{W}(x) = y\}$ for a $y \in \mathbb{R}$.

To answer the question about the cardinality of the level sets we have to study properties of occupation measures. If λ denotes the Lebesgue measure; and μ denotes the occupation measure of $\mathcal{W}$, then

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$$\mu(A) = \lambda\{x \in [0,1] : \mathcal{W}(x) \in A\} = \lambda(\mathcal{W}^{-1}(A) \cap [0,1])$$

for a (Borel) measurable set $A \subset \mathbb{R}$. Occupation measures are studied in the theory of stochastic processes. For example, with probability one, the occupation measure of the Brownian motion is absolutely continuous with respect to the Lebesgue measure; it satisfies the local time condition. One can find lots of interesting facts about occupation measures in the survey paper [12] by D. Geman and J. Horowitz. Results for stochastic processes are almost everywhere results. It is mentioned in [12] that it is difficult to apply the methods valid for random functions (with probability one) for nonrandom functions.

Level sets and occupation measures of some self-affine functions were studied by J. Bertoin in [3] and [4]. These occupation measures either satisfy the local time condition, or are singular with respect to the Lebesgue measure.

Theorem 19 of our paper [7] implies that from the singularity of the occupation measure of $\mathcal{W}(x)$ it follows that almost every level set of $\mathcal{W}(x)$ is finite.

Using Theorem 13 of [7] one can decompose the graph of $\mathcal{W}(x)$ over $[0,1]$ into two sets S^*_{irr} and S^*_{reg}. The set S^*_{reg} can be covered by the union of the graphs of countably many strictly monotone functions. The set S^*_{irr} is an irregular (or purely unrectifiable) 1-set. This means that $\mathcal{H}^1(S^*_{irr})$, the one-dimensional Hausdorff measure of S^*_{irr}, is positive and finite, moreover, S^*_{irr} intersects every continuously differentiable curve in a set of $\mathcal{H}^1$-measure zero. By Besicovitch's projection theorem (see [5] or [9]) the projection of S^*_{irr} in almost all directions is of zero Lebesgue measure.

It also turns out in [7] that the projection of S^*_{irr} onto the x-axis is of measure one, while S^*_{reg} projects onto a set of measure zero.

Hence the "occupation measure behavior" of $\mathcal{W}(x)$ over $[0,1]$ is completely determined by the part of its graph belonging to S^*_{irr}. As explained in Sect. 5.1 of [7], Besicovitch's projection theorem implies that for almost every $c \in \mathbb{R}$ the occupation measure of the function $\mathcal{W}(x,c) = \mathcal{W}(x) + cx$ is purely singular. For these c if $S^*_{irr,c}$ denotes the irregular 1-set on the graph of $\mathcal{W}(x,c)$ over $[0,1]$, the projection of $S^*_{irr,c}$ onto the y-axis is of zero Lebesgue measure, while its projection onto the x-axis is of measure one.

It is quite often rather difficult to obtain from an almost everywhere result, a result concerning a specific parameter value. My favorite example of this phenomenon is that it is known that for almost every $\theta > 1$ the sequence of the fractional part of θ^n is uniformly distributed in $[0,1]$ but no explicit example of a real number θ is known for which this sequence is uniformly distributed. For example, the case $\theta = 3/2$ is a famous open problem.

In this chapter we show that for all $c \in \mathbb{R}$ (with no exceptional values) the occupation measure of $\mathcal{W}(x,c)$ is purely singular. To deduce this result we have to consider a larger class of functions. This class was introduced in [7], and the main reason for the introduction of these functions was the fact that we need them to prove the singularity of the occupation measure of $\mathcal{W}(x)$.

We denote by $\mathcal{F}_{\mathcal{W}}$ the set of those twice continuously differentiable functions f_{-1} on $[0,1]$ for which for any trigonometric polynomial P the function $f_{-1}+P$ is piecewise strictly monotone, or constant. If a function is analytic on an open set $G \supset [0,1]$ then its restriction to $[0,1]$ belongs to $\mathcal{F}_{\mathcal{W}}$. Given $f_{-1} \in \mathcal{F}_{\mathcal{W}}$ we studied in Sect. 5 of [7] functions

$$f(x) = f_{-1}(x) + \sum_{n=0}^{\infty} 2^{-n} \sin(2\pi 2^n x) = f_{-1}(x) + \mathcal{W}(x). \tag{2}$$

If $f_{-1}(x) = 0$ identically then we obtain $\mathcal{W}(x)$, if $f_{-1}(x) = cx$ we obtain $\mathcal{W}(x,c)$. We also need in this chapter the special case when

$$f_{-1}(x) = \gamma_0 x + S_{-\infty}(x) \stackrel{\text{def}}{=} \gamma_0 x + \sum_{n=-\infty}^{-1} (2^{-n} \sin(2\pi 2^n x) - 2\pi x) \tag{3}$$

with a $\gamma_0 \in \mathbb{R}$. In Sect. 2 we show that this $f_{-1} \in \mathcal{F}_{\mathcal{W}}$. Theorem 18 of [7] implies that the occupation measure of f is nonatomic if f is defined in (2) with an $f_{-1} \in \mathcal{F}_{\mathcal{W}}$. Arguing as in Sect. 5.1 of [7] one can see that for almost every $\gamma_0 \in \mathbb{R}$ the occupation measure is singular for f defined in (2) by using f_{-1} from (3).

This paper is organized in the following way. In Sect. 2 some preliminary results are given. In Sect. 3 the main result on the singularity of the occupation measure of $\mathcal{W}(x,c)$ for all $c \in \mathbb{R}$ is proved based on two lemmas. The proof of one of these lemmas is quite technical, and the details of the proof of Lemma 3 are given in Sect. 4.

2 Notation and Preliminary Results

By $\lambda(A)$ we denote the Lebesgue measure of the set $A \subset \mathbb{R}$. Set $\mathcal{W}_N(x) = \sum_{n=0}^{N} 2^{-n} \sin(2\pi 2^n x)$. We will use the following estimate:

$$|\mathcal{W}_N''(x)| = \left| \sum_{n=0}^{N} -4\pi^2 2^n \sin(2\pi 2^n x) \right| < 8\pi^2 2^N. \tag{4}$$

For $N_1 \le -1$ and $x \in [-2,2]$ we set

$$S_{N_1}(x) = \sum_{n=N_1}^{-1} (2^{-n} \sin(2\pi 2^n x) - 2\pi x) \text{ and } S_{-\infty}(x) = \lim_{N_1 \to -\infty} S_{N_1}(x). \tag{5}$$

Next we need some elementary estimates of the derivatives of the terms of S_{N_1}. If $n \le -2$

$$\begin{aligned} \max_{x\in[-2,2]} |2\pi \cos(2\pi 2^n x) - 2\pi| &= 2\pi \max_{x\in[-2,2]} |\cos(2\pi 2^n x) - 1| \\ &= 2\pi |\cos(4\pi 2^n) - 1| \le 2\pi \max_{c\in[0,4\pi 2^n]} |4\pi 2^n \sin(c)| \le 8\pi^2 2^n. \end{aligned} \tag{6}$$

This implies that

$$S'_{N_1}(x) = \sum_{n=N_1}^{-1} (2\pi \cos(2\pi 2^n x) - 2\pi) \text{ converges uniformly on } [-2,2]. \quad (7)$$

Moreover, for $k \geq 2$ with $\phi(x) = \pm\sin(x)$ or $\pm\cos(x)$ we have

$$\left|S^{(k)}_{N_1}(x)\right| = \left|\sum_{n=N_1}^{-1} 2^{-n}(2\pi)^k (2^n)^k \phi(2\pi 2^n x)\right| \leq (2\pi)^k \sum_{n=N_1}^{-1} 2^n \leq (2\pi)^k. \quad (8)$$

From (7) and (8) it follows that $S_{-\infty}$ is a (real) analytic function on $[-2,2]$.

For $N \geq 0$ and $\gamma_0 \in \mathbb{R}$ set

$$\mathcal{W}_{-\infty}(x) = S_{-\infty}(x) + \mathcal{W}(x), \quad \mathcal{W}_{-\infty,N}(x) = S_{-\infty}(x) + \mathcal{W}_N(x), \quad \text{and}$$
$$\mathcal{W}_{-\infty}(x,\gamma_0) = \mathcal{W}_{-\infty}(x) + \gamma_0 \cdot x. \quad (9)$$

For $N \leq -1$ set $\mathcal{W}_{N,\infty}(x) \stackrel{\text{def}}{=} S_N(x) + \mathcal{W}(x)$.

Since $S_{-\infty}$ is analytic on an open interval containing $[0,1]$ it is in the class of functions $\mathcal{F}_{\mathcal{W}}$. By an argument analogous to that of Sect. 5.1 in [7] one can see by Besicovitch's projection theorem that for almost every $\gamma_0 \in \mathbb{R}$ the function $\mathcal{W}_{-\infty}(x,\gamma_0)$ has singular occupation measure. By Theorem 18 of [7] this occupation measure is nonatomic. These properties imply the following lemma.

Lemma 1. *For almost every $\gamma_0 \in \mathbb{R}$, given $\psi > 0$ there exist y_1 and h_1 such that*

$$\lambda\{x \in [0,1] : \mathcal{W}_{-\infty}(x,\gamma_0) \in (y_1, y_1 + h_1)\} > \psi h_1. \quad (10)$$

Moreover, there exists $\epsilon_{0,\psi} > 0$ such that if $g \in C[0,1]$ and

$$|\mathcal{W}_{-\infty}(x,\gamma_0) - g(x)| < \epsilon_{0,\psi} \text{ for all } x \in [0,1] \quad (11)$$

then

$$\lambda\{x \in [0,1] : g(x) \in (y_1, y_1 + h_1)\} > \psi h_1. \quad (12)$$

In the sequel we suppose that γ_0 is chosen so that the occupation measure of $\mathcal{W}_{-\infty}(x,\gamma_0)$ is singular (with respect to the Lebesgue measure) and Lemma 1 holds for this γ_0.

The next lemma is a simple consequence of Lebesgue's density theorem and is related to Vitali's covering theorem, see 2.8.17 in [11].

Lemma 2. *Suppose that $H \subset [0,1]$ is Lebesgue measurable and there exists $\rho > 0$ satisfying the following property. For every $x \in H$ there are arbitrarily*

small intervals J_x containing a measurable subset S_x such that $x \in J_x$ and $\lambda(S_x) > \rho\lambda(J_x)$. Then for almost every $x_0 \in H$ there exist sets S_x $(x \in H)$ of arbitrarily small diameter such that $x_0 \in S_x$.

Proof. Suppose there is $\epsilon > 0$ and a set $H' \subseteq H$ such that H' does not intersect the union of the sets of the form S_x with $x \in H$, $\operatorname{diam}(S_x) < \epsilon$. Suppose that x_0 is a Lebesgue density point of H'. Then by our assumptions there exists a sufficiently short J_{x_0} such that $\lambda(J_{x_0}) < \epsilon$, $\lambda(J_{x_0} \cap H') > (1-\rho)\lambda(J_{x_0})$, but this contradicts that $\lambda(S_{x_0}) > \rho\lambda(J_{x_0})$, $\operatorname{diam}(S_{x_0}) < \lambda(J_{x_0}) < \epsilon$, and $S_{x_0} \cap H' = \emptyset$. Therefore, H' has no Lebesgue density points and this implies that it is of measure zero. □

3 Main Results

Definition 1. *For $N = 0, 1, \ldots$ and $x \in [0,1)$ denote by $P(x,N)$ the point $\frac{k}{2^N}$ satisfying $x \in [\frac{k}{2^N}, \frac{k+1}{2^N})$.*

Lemma 3. *For almost every $x_0 \in [0,1)$, $\{\mathcal{W}'_N(P(x_0,N)) : N \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

We postpone the proof of this lemma to Sect. 4.

Remark 1. The main difficulty in Lemma 3 is that we need density of $\{\mathcal{W}'_N(P(x_0,N))\}$ and not of $\{\mathcal{W}'_N(x_0)\}$. The referee of this paper suggested that we mention that as an alternative to our direct "elementary" approach one could also use results from ergodic and probability theory to deal with properties of $\{\mathcal{W}'_N(x_0)\}$. Indeed, $\mathcal{W}'_N(x) = \sum_{n=0}^{N} 2\pi\cos(2\pi 2^n x)$ is the Birkhoff sum, $\sum_{n=0}^{N} \overline{\phi}(T^n x)$ of $\overline{\phi} \overset{\text{def}}{=} 2\pi\cos(2\pi x)$ with respect to the ergodic transformation $Tx = \{2x\}$ defined on $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Since $\int_X \overline{\phi} = 0$ by recurrence results concerning cocycles k and random walks [1, Chaps. 8, 8.1.2, 8.1.5], [2], the cocycle $\sum_{n=0}^{N} \overline{\phi}(T^n x)$ is recurrent, that is $\liminf_{N\to\infty} \left|\sum_{n=0}^{N} \overline{\phi}(T^n x)\right| = 0$. Using Lusin's theorem, a Lebesgue density argument, and the continuous differentiability of $\overline{\phi}$ one can also verify that $\overline{\phi}$ is not a coboundary, that is, there is no Borel measurable map $h : X \to \mathbb{R}$ such that $\overline{\phi}(x) = h(x) - h(Tx)$. By using the argument of [1, Corollary 8.3.4] one can see that if $\overline{\phi}$ is not a coboundary then its group of persistencies $\Pi(\overline{\phi}) = \{a \in \mathbb{R} : \forall A \in \mathcal{B}, \lambda(A) > 0, \forall \epsilon > 0, \exists N \geq 1, \lambda(A \cap T^{-N} A \cap \{x : |\sum_{n=0}^{N-1} \overline{\phi}(T^n x) - a| < \epsilon\}) > 0\}$ is noncompact. By [1, Proposition 8.2.1] it is a closed subgroup of $(\mathbb{R}, +)$. Therefore, for almost every $x_0 \in X$, $\liminf_{N\to\infty} \mathcal{W}'_N(x_0) = \liminf_{N\to\infty} \sum_{n=0}^{N} \overline{\phi}(T^n x_0) = -\infty$ and $\limsup_{N\to\infty} \mathcal{W}'_N(x_0) = \limsup_{N\to\infty} \sum_{n=0}^{N} \overline{\phi}(T^n x_0) = +\infty$.

We also remark that in [10] by using ergodic theory the authors estimate the Hausdorff dimension of those points where the growth rate of $\max_{N\le K}|\sum_{n=0}^{N}\overline{\phi}(T^n x)|$ is slower than the one expected from the law of the iterated logarithm theorem or from the central limit theorem. These results are also applied in [10] to the Weierstrass–Cellerier function (under the name of Hardy function) to estimate the Hausdorff dimension of those points where the Weierstrass–Cellerier–(Hardy) function exhibits various local behaviors.

Lemma 4. *For any $c\in\mathbb{R}$, $\psi>0$ there exist N_1 and h_1 for which the following holds. For almost every $x_0\in[0,1]$ there exist infinitely many N_0's such that if $x_0\in[\frac{k_0}{2^{N_0}},\frac{k_0+1}{2^{N_0}})$ then there exists y_1 such that*

$$\lambda\left\{x\in\left[\tfrac{k_0}{2^{N_0}},\tfrac{k_0}{2^{N_0}}+\frac{1}{2^{N_0+N_1+1}}\right):\mathcal{W}(x,c)\in\left[y_1,y_1+\frac{h_1}{2^{N_0+N_1+1}}\right)\right\}>\psi\frac{h_1}{2^{N_0+N_1+1}}. \tag{13}$$

Proof. Choose γ_0 such that $\mathcal{W}_{-\infty}(x)+\gamma_0 x$ has singular occupation measure. This implies that the conclusion of Lemma 1 holds for γ_0. By Lemma 1 choose y_1', h_1 and $\epsilon_{0,\psi}$, such that (10)–(12) hold. Suppose $x_0\in[0,1]$ is arbitrary and $x_0\in[\frac{k_0}{2^{N_0}},\frac{k_0+1}{2^{N_0}})$. Choose N_1' such that for any N_0 and $N_1\ge N_1'$ we have for any $x\in[\frac{k_0}{2^{N_0}},\frac{k_0}{2^{N_0}}+\frac{1}{2^{N_0+N_1}})\subset[0,1)$

$$\left|\mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right)-\mathcal{W}'_{N_0}(x)\right|\le\max_{x'\in[0,1)}|\mathcal{W}''_{N_0}(x')|\frac{1}{2^{N_0+N_1}} \tag{14}$$

(using (4))

$$\le 8\pi^2 2^{N_0}\frac{1}{2^{N_0+N_1}}\le\frac{8\pi^2}{2^{N_1'}}<\frac{\epsilon_{0,\psi}}{3}.$$

We also choose and fix $N_1\ge N_1'$ such that for $x\in[0,1]$

$$|\mathcal{W}_{-N_1,\infty}(x)-\mathcal{W}_{-\infty}(x)|\le\left|\sum_{n=-\infty}^{-N_1-1}(2^{-n}\sin(2\pi 2^n x)-2\pi x)\right|\le\frac{\epsilon_{0,\psi}}{3}. \tag{15}$$

Set

$$\gamma=\gamma_0-N_1 2\pi. \tag{16}$$

By Lemma 3 for almost every x_0 there exist infinitely many N_0 such that if $P(x_0,N_0)=\frac{k_0}{2^{N_0}}$ then $x_0\in[\frac{k_0}{2^{N_0}},\frac{k_0+1}{2^{N_0}})$ and

$$\left|\mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right)-\gamma+c\right|<\frac{\epsilon_{0,\psi}}{3}. \tag{17}$$

Choose and fix one N_0 satisfying the above assumption.

Clearly, for $x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$

$$\begin{aligned}\mathcal{W}(x,c) &= \left(\mathcal{W}_{N_0}(x) + cx - \gamma\left(x - \frac{k_0}{2^{N_0}}\right)\right) \\ &\quad + \left(\sum_{n=-N_1}^{\infty} 2^{-n}2^{-N_0-N_1-1}\sin\left(2\pi 2^n 2^{N_0+N_1+1}\left(x - \frac{k_0}{2^{N_0}}\right)\right)\right. \\ &\quad \left. + \gamma\left(x - \frac{k_0}{2^{N_0}}\right)\right) \\ &= \mathcal{V}_1(x) + \mathcal{V}_2(x).\end{aligned}$$

By (17), $|\mathcal{V}_1'(\frac{k_0}{2^{N_0}})| < \frac{\epsilon_{0,\psi}}{3}$ and by (14), $|\mathcal{V}_1'(\frac{k_0}{2^{N_0}}) - \mathcal{V}_1'(x)| < \frac{\epsilon_{0,\psi}}{3}$ for all $x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$. Therefore, by the Lagrange mean value theorem

$$\left|\mathcal{V}_1\left(\frac{k_0}{2^{N_0}}\right) - \mathcal{V}_1(x)\right| < \frac{2\epsilon_{0,\psi}}{3}\frac{1}{2^{N_0+N_1+1}} \tag{18}$$

holds for all $x \in [\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$. On the other hand, using (16)

$$\begin{aligned}\mathcal{V}_2(x) &= 2^{-N_0-N_1-1}\left(\sum_{n=-N_1}^{-1}\left(2^{-n}\sin\left(2\pi 2^n 2^{N_0+N_1+1}\left(x - \frac{k_0}{2^{N_0}}\right)\right)\right.\right. \\ &\qquad \left. - 2\pi 2^{N_0+N_1+1}\left(x - \frac{k_0}{2^{N_0}}\right)\right) \\ &\qquad + \sum_{n=0}^{\infty} 2^{-n}\sin\left(2\pi 2^n 2^{N_0+N_1+1}\left(x - \frac{k_0}{2^{N_0}}\right)\right) \\ &\qquad \left. + \gamma_0 2^{N_0+N_1+1}\left(x - \frac{k_0}{2^{N_0}}\right)\right) \\ &= 2^{-N_0-N_1-1}\left(\mathcal{W}_{-N_1,\infty}\left(2^{N_0+N_1+1}\left(x - \frac{k_0}{2^{N_0}}\right)\right)\right. \\ &\quad \left. + \gamma_0 2^{N_0+N_1+1}\left(x - \frac{k_0}{2^{N_0}}\right)\right).\end{aligned} \tag{19}$$

Let $u = 2^{N_0+N_1+1}(x - \frac{k_0}{2^{N_0}})$, $u \in [0,1)$. Then we have

$$\begin{aligned}&\mathcal{W}\left(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u, c\right) \\ &= \mathcal{V}_1\left(\frac{k_0}{2^{N_0}}\right) + \mathcal{V}_1\left(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u\right) - \mathcal{V}_1\left(\frac{k_0}{2^{N_0}}\right) \\ &\quad + 2^{-N_0-N_1-1}\left(\mathcal{W}_{-N_1,\infty}(u) + \gamma_0 u\right)\end{aligned} \tag{20}$$

$$= \mathcal{V}_1\left(\frac{k_0}{2^{N_0}}\right) + 2^{-N_0-N_1-1}\left(2^{N_0+N_1+1}\left(\mathcal{V}_1\left(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u\right)\right.\right.$$
$$\left.\left. -\mathcal{V}_1\left(\frac{k_0}{2^{N_0}}\right)\right) + \mathcal{W}_{-N_1,\infty}(u) + \gamma_0 u\right)$$
$$= \mathcal{V}_1\left(\frac{k_0}{2^{N_0}}\right) + 2^{-N_0-N_1-1}g(u). \tag{21}$$

By (18) we have for all $u \in [0,1)$

$$\left|2^{N_0+N_1+1}\left(\mathcal{V}_1\left(\frac{k_0}{2^{N_0}}\right) - \mathcal{V}_1\left(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u\right)\right)\right| < \frac{2\epsilon_{0,\psi}}{3}.$$

Hence (15) implies that

$$|g(u) - \mathcal{W}_{-\infty}(u,\gamma_0)| < \epsilon_{0,\psi}. \tag{22}$$

This can be interpreted as the approximate repeated similarity property. The function $\mathcal{W}(x,c)$ infinitely often (for infinitely many N_0's) approximates very well on intervals of the form $[\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}})$ a rescaled and translated copy of $\mathcal{W}_{-\infty}(x,\gamma_0)$, see (20)–(22). This explains why we need the functions $\mathcal{W}_{-\infty}(x,\gamma_0)$ and the class of functions $\mathcal{F}_{\mathcal{W}}$. This property is not a self-similarity property since not $\mathcal{W}(x,c)$, but $\mathcal{W}_{-\infty}(x,\gamma_0)$ is approximated by these rescaled and translated copies. (This property can also be interpreted by using the concept of micro tangent sets from [6] and [8]. As we zoom in the graph of $\mathcal{W}(x,c)$ at the point $(x_0,\mathcal{W}(x_0,c))$ at certain scales we see something very close to a translated copy of $\mathcal{W}_{-\infty}(x,\gamma_0)$.)

By (22) and by our choice of y_1' and h_1 we have the following version of (12):

$$\lambda\{u \in [0,1) : g(u) \in (y_1', y_1' + h_1)\} > \psi h_1. \tag{23}$$

Set $y_1 = \mathcal{V}_1(\frac{k_0}{2^{N_0}}) + 2^{-N_0-N_1-1}y_1'$. By (20)–(20) and (23) we have

$$\lambda\left\{u \in [0,1) : \mathcal{W}\left(\frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u, c\right) \in [y_1, y_1 + 2^{-N_0-N_1-1}h_1)\right\} > \psi h_1. \tag{24}$$

After the substitution $x = \frac{k_0}{2^{N_0}} + 2^{-N_0-N_1-1}u$ we obtain (13) and this proves the lemma. □

Based on Lemma 4 it is easy to verify the main result of this paper.

Theorem 1. *For all $c \in \mathbb{R}$ the occupation measure of the function $\mathcal{W}(x,c)$ is purely singular.*

Remark 2. When $c = 0$ this means that the occupation measure of the Weierstrass–Cellerier function is purely singular. By results of [7] this also implies that almost every level set of this function is finite.

Proof (Theorem 1). Suppose $c \in \mathbb{R}$ is fixed. For any $\psi > 0$ denote by X'_ψ the set of those x_0 for which Lemma 4 holds. Hence $\lambda(X'_\psi) = 1$ and we can choose N_1, h_1 such that for all $x_0 \in X'_\psi$ there are infinitely many N_0's for which if the interval $J_{x_0} = [\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}})$ contains x_0 then there is y_1 such that if

$$S_{x_0} = \left\{ x \in \left[\frac{k_0}{2^{N_0}}, \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0+N_1+1}} \right) : \mathcal{W}(x,c) \in \left[y_1, y_1 + \frac{h_1}{2^{N_0+N_1+1}} \right) \right\}$$

then we have (13), that is,

$$\lambda(S_{x_0}) > \psi \frac{h_1}{2^{N_1+1}} \lambda(J_{x_0}) = \psi \frac{h_1}{2^{N_0+N_1+1}}.$$

Using Lemma 2 with $\rho = \psi h_1 2^{-(N_1+1)}$ we can select a subset $X_\psi \subset X'_\psi$ such that $\lambda(X_\psi) = 1$ and for any $x \in X_\psi$ there are infinitely many N_0's and y_1's such that x belongs to a set of the form S_{x_0} defined above for an $x_0 \in X_\psi$. This implies that if $x_1 \in X_\psi$ then there exist infinitely many N's and y_1's such that $\mathcal{W}(x_1, c) \in [y_1, y_1 + \frac{h_1}{2^N})$ and

$$\mu\left(\left[y_1, y_1 + \frac{h_1}{2^N} \right) \right) = \lambda \left\{ x \in [0,1) : \mathcal{W}(x,c) \in \left[y_1, y_1 + \frac{h_1}{2^N} \right) \right\} > \psi \frac{h_1}{2^N}. \tag{25}$$

Set $X_\infty = \cap_{K=1}^{\infty} X_K$. Then $\lambda(X_\infty) = 1$ and hence $\mu(\mathcal{W}(X_\infty, c)) = 1$. On the other hand, for any $y \in \mathcal{W}(X_\infty, c)$ we can choose $x \in X_\infty$ such that $\mathcal{W}(x,c) = y$ and for any $\psi = K$ there are infinitely many N's and y_1's such that

$$y \in \left[y_1, y_1 + \frac{h_1}{2^N} \right) \text{ and } \mu\left(\left[y_1, y_1 + \frac{h_1}{2^N} \right) \right) > K \frac{h_1}{2^N}.$$

This implies that $\lambda(\mathcal{W}(X_\infty, c)) = 0$ and hence μ is singular with respect to λ. □

4 Proof of Lemma 3

Before proving Lemma 3 we need some auxiliary results.

Lemma 5. *For any $\epsilon_1 > 0$ there are integers N', k_1 and k_2 such that*

$$0 < \mathcal{W}'_{-\infty,N'}\left(\frac{k_1}{2^{N'}} \right) - \mathcal{W}'_{-\infty,N'}\left(\frac{k_2}{2^{N'}} \right) < \epsilon_1, \quad 0 < k_1 < k_2 < 2^{N'}. \tag{26}$$

Proof. For any $n \le 0$ and $N' > 2$

$$0 < \cos\left(2\pi 2^n \left(\frac{1}{4} - \frac{1}{2^{N'}} \right) \right) - \cos\left(2\pi 2^n \left(\frac{1}{4} + \frac{1}{2^{N'}} \right) \right) < 2\pi 2^n \frac{2}{2^{N'}}. \tag{27}$$

For $n > 0$

$$\cos\left(2\pi 2^n\left(\frac{1}{4}-\frac{1}{2^{N'}}\right)\right)-\cos\left(2\pi 2^n\left(\frac{1}{4}+\frac{1}{2^{N'}}\right)\right)=0. \tag{28}$$

Choose k_1, k_2 such that $\frac{k_1}{2^{N'}}=\frac{1}{4}-\frac{1}{2^{N'}}$ and $\frac{k_2}{2^{N'}}=\frac{1}{4}+\frac{1}{2^{N'}}$. Then by (27) and (28)

$$\begin{aligned}
0 &< \mathcal{W}'_{-\infty,N'}\left(\frac{k_1}{2^{N'}}\right)-\mathcal{W}'_{-\infty,N'}\left(\frac{k_2}{2^{N'}}\right)\\
&= \sum_{n=-\infty}^{N'} 2\pi\left(\cos\left(2\pi 2^n\left(\frac{1}{4}-\frac{1}{2^{N'}}\right)\right)-\cos\left(2\pi 2^n\left(\frac{1}{4}+\frac{1}{2^{N'}}\right)\right)\right)\\
&= \sum_{n=-\infty}^{0} 2\pi\left(\cos\left(2\pi 2^n\left(\frac{1}{4}-\frac{1}{2^{N'}}\right)\right)-\cos\left(2\pi 2^n\left(\frac{1}{4}+\frac{1}{2^{N'}}\right)\right)\right)\\
&< \sum_{n=-\infty}^{0} 4\pi^2\frac{2}{2^{N'}}2^n=\frac{16\pi^2}{2^{N'}}.
\end{aligned}$$

Therefore, if $16\pi^2/2^{N'}<\epsilon_1$ then we have (26). □

Lemma 6. *Given $\epsilon_0>0$ there exist N', q' and $0\le k'<2^{N'}$ such that for any $\gamma'\in[0,2\pi)$ there exists $1\le t'\le q'$, $t'\in\mathbb{N}$ such that*

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}\left(\frac{k'}{2^{N'}}\right)\right\}-\gamma'\right|<\epsilon_0. \tag{29}$$

(Here $\{.\}$ denotes the fractional part.)

Proof. If there exists N', k' such that $\theta\overset{\text{def}}{=}\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\notin\mathbb{Q}$ then $\{t'\theta\}$, $t'\in\mathbb{N}$ is dense in $[0,1)$ and hence we can choose a suitable q'. Suppose that for all N' and k', $\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\in\mathbb{Q}$. By Lemma 5 used with $\epsilon_1=\epsilon_0^2/4\pi^2$ we choose N', k_1 and k_2 such that if $\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k_1}{2^{N'}})=\frac{p_1}{q_1}\in\mathbb{Q}$ and $\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k_2}{2^{N'}})=\frac{p_2}{q_2}\in\mathbb{Q}$ with $0<q_1$, $0<q_2$, $(p_1,q_1)=1$, and $(p_2,q_2)=1$ then

$$0<\frac{p_1}{q_1}-\frac{p_2}{q_2}<\frac{\epsilon_0^2}{4\pi^2}.$$

This implies that $\frac{1}{q_1q_2}<\frac{\epsilon_0^2}{4\pi^2}$. Hence, either $\frac{1}{q_1}<\frac{\epsilon_0}{2\pi}$, or $\frac{1}{q_2}<\frac{\epsilon_0}{2\pi}$. If $\frac{1}{q_1}<\frac{\epsilon_0}{2\pi}$ then we set $q'=q_1$ and $k'=k_1$, otherwise we put $q'=q_2$ and $k'=k_2$. Therefore, we have $\{\frac{1}{2\pi}\mathcal{W}'_{-\infty,N'}(\frac{k'}{2^{N'}})\}=\frac{p'}{q'}$ with a $p'<q'$, $(p',q')=1$ and $\frac{1}{q'}<\frac{\epsilon_0}{2\pi}$. Using the fact that p' and q' are relatively prime we obtain that the set consisting of the numbers $\{t'\frac{p'}{q'}\}$, $t'=1,...,q'$ equals the set consisting of the numbers l/q', $l=0,...,q'-1$ and hence we have (29) for any $\gamma'\in[0,2\pi)$. □

Proof (Lemma 3). Suppose $c \in \mathbb{R}$ and $0 < \epsilon_1 < 1$ are given. We want to show that for almost every $x_0 \in [0,1)$ there exist infinitely many N such that

$$|\mathcal{W}'_N(P(x_0,N)) - c| < \epsilon_1. \tag{30}$$

Set $c_0 = 2\pi\{\frac{c}{2\pi}\}$ and $D_c = \lfloor \frac{c}{2\pi} \rfloor$. Denote by $X_{\pm\infty}$ the set of those x_0's for which $\liminf_{N\to\infty} \mathcal{W}'_N(x_0) = -\infty$ and $\limsup_{N\to\infty} \mathcal{W}'_N(x_0) = +\infty$. It follows from results of [6], or from the results mentioned in Remark 1, that almost every $x_0 \in [0,1)$ belongs to $X_{\pm\infty}$. Apply Lemma 6 with $\epsilon_0 = \frac{\epsilon_1}{32}$ to obtain N', q' and $0 \le k' < 2^{N'}$. Without limiting generality we can also suppose that

$$q' > \max\{|D_c| + 1, 4\pi\}. \tag{31}$$

Choose N'_1 such that for $N_1 \ge N'_1 \ge 4$

$$\frac{8\pi^2}{2^{N_1}} < \frac{\epsilon_1}{8q'}. \tag{32}$$

Set

$$\mathcal{W}_{N_1,N'}(x) = S_{-N_1}(x) + \mathcal{W}_{N'}(x), \text{ and } N'' = N' + N_1 + 1.$$

Then $\mathcal{W}_{-\infty,N'}(x) = \lim_{N\to\infty} \mathcal{W}_{N_1,N'}(x)$ and $q'N'' > 4\pi > 1$. We also suppose that for any $N_1 > N'_1$ and $N' > 0$ we have

$$\begin{aligned} |\mathcal{W}'_{-\infty,N'}(x) - \mathcal{W}'_{N_1,N'}(x)| &= |S'_{-\infty}(x) - S'_{-N_1}(x)| \\ &= \left| \sum_{n=-\infty}^{-N_1-1} (2\pi \cos(2\pi 2^n x) - 2\pi) \right| < \frac{\epsilon_1}{64\pi q'}. \end{aligned} \tag{33}$$

Since $|\mathcal{W}'_{N+1}(x_0) - \mathcal{W}'_N(x_0)| = |2\pi \cos(2\pi 2^{N+1} x_0)| \le 2\pi$ for any N, and $x_0 \in X_{\pm\infty}$ there are infinitely many N_0's such that

$$-101q'N'' < \frac{1}{2\pi}\mathcal{W}'_{N_0}(x_0) < -100q'N''. \tag{34}$$

Fix one such N_0 and choose k_0 such that $\frac{k_0}{2^{N_0}} = P(x_0, N_0)$. Then by (4), $q'N'' > 4\pi$, and the Lagrange mean value theorem

$$-102q'N'' < \frac{1}{2\pi}\mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right) < -99q'N''. \tag{35}$$

If $2\pi\{\frac{1}{2\pi}\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}})\} \le c_0$ then set $D_0 = \lfloor \frac{1}{2\pi}\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}}) \rfloor$ and

$$c'_{\text{corr}} = \mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right) - 2\pi D_0 = 2\pi\left\{\frac{1}{2\pi}\mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right)\right\}, \tag{36}$$

otherwise set $D_0 = \lceil \frac{1}{2\pi}\mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}})\rceil$ and

$$c'_{\rm corr} = \mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right) - 2\pi D_0 = 2\pi\left(\left\{\frac{1}{2\pi}\mathcal{W}'_{N_0}\left(\frac{k_0}{2^{N_0}}\right)\right\} - 1\right). \tag{37}$$

By (35) we have

$$-103q'N'' < D_0 < -98q'N''. \tag{38}$$

We also have $0 \le c_{\rm corr} \overset{\rm def}{=} c_0 - c'_{\rm corr} < 2\pi$. If $\frac{\epsilon_1}{16\pi} \le \frac{c_{\rm corr}}{2\pi} \le 1 - \frac{\epsilon_1}{16\pi}$ then set $\overline{c}_{\rm corr} = c_{\rm corr}$, if $0 \le \frac{c_{\rm corr}}{2\pi} < \frac{\epsilon_1}{16\pi}$ then set $\overline{c}_{\rm corr} = \frac{\epsilon_1}{16\pi}\cdot 2\pi = \frac{\epsilon_1}{8}$. Finally, if $1 - \frac{\epsilon_1}{16\pi} \le \frac{c_{\rm corr}}{2\pi} < 1$ then set $\overline{c}_{\rm corr} = 2\pi(1 - \frac{\epsilon_1}{16\pi}) = 2\pi - \frac{\epsilon_1}{8}$. For any of the previous cases we have $|c_{\rm corr} - \overline{c}_{\rm corr}| \le \frac{\epsilon_1}{8}$. To obtain N', q' and k' Lemma 6 was used with $\epsilon_0 = \frac{\epsilon_1}{32}$, by (29) we can choose $t' \le q'$, $t' \in \mathbb{N}$ such that

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}\left(\frac{k'}{2^{N'}}\right)\right\} - \overline{c}_{\rm corr}\right| < \frac{\epsilon_1}{32}. \tag{39}$$

By $|c_{\rm corr} - \overline{c}_{\rm corr}| \le \frac{\epsilon_1}{8}$ and $\frac{\epsilon_1}{8} \le \overline{c}_{\rm corr} \le 2\pi - \frac{\epsilon_1}{8}$ we also have

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}\left(\frac{k'}{2^{N'}}\right)\right\} - c_{\rm corr}\right| < \frac{\epsilon_1}{4} \text{ and} \tag{40}$$

$$\frac{\epsilon_1}{16} < 2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}\left(\frac{k'}{2^{N'}}\right)\right\} < 2\pi - \frac{\epsilon_1}{16},$$

that is,

$$\frac{\epsilon_1}{32\pi} < \left\{\frac{t'}{2\pi}\mathcal{W}'_{-\infty,N'}\left(\frac{k'}{2^{N'}}\right)\right\} < 1 - \frac{\epsilon_1}{32\pi}. \tag{41}$$

This, $t' \le q'$, (33), and (40) imply that

$$\left|2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}\left(\frac{k'}{2^{N'}}\right)\right\} - c_{\rm corr}\right| < \frac{\epsilon_1}{3}. \tag{42}$$

We will fix an integer $D_1 > q'N''$ later.

Set $N'_0 = N_0 + D_1 + t'(N'+1)$, $x'_0 = \frac{k_0}{2^{N_0}}$, and

$$x'_l = \frac{k_0}{2^{N_0}} + \sum_{t=1}^{l}\frac{k'}{2^{N_0+tN''}} \text{ for } l = 1, ..., t'. \tag{43}$$

Then

$$\mathcal{W}'_{N'_0}(x'_{t'}) = \sum_{n=0}^{N_0} 2\pi\cos(2\pi 2^n x'_{t'}) + \sum_{t=1}^{t'}\sum_{n=N_0+(t-1)N''+1}^{N_0+tN''} 2\pi\cos(2\pi 2^n x'_{t'}) \tag{44}$$

$$+ \sum_{n=N_0+t'N''+1}^{N'_0} 2\pi\cos(2\pi 2^n x'_{t'}) = \sum_{n=0}^{N_0} 2\pi\cos(2\pi 2^n x'_{t'})$$

(recall that $N'' = N' + 1 + N_1$)

$$+\left(\sum_{t=1}^{t'}\sum_{n=N_0+(t-1)N''+1}^{N_0+tN''} 2\pi\cos(2\pi 2^n(x'_{t'}-x'_{t-1}))\right)+2\pi(D_1-t'N_1)$$
$$\stackrel{\text{def}}{=}\mathcal{S}_0+\sum_{t=1}^{t'}\mathcal{S}_t+2\pi(D_1-t'N_1).$$

Recalling $0 \le k' < 2^{N'}$ and $N'' = N' + N_1 + 1$, by (4), (32), and (43) we obtain

$$|\mathcal{S}_0 - \mathcal{W}'_{N_0}(x'_0)| = |\mathcal{W}'_{N_0}(x'_{t'}) - \mathcal{W}'_{N_0}(x'_0)| \le 8\pi^2 2^{N_0}(x'_{t'}-x'_0)$$
$$< 8\pi^2 2^{N_0}\frac{1}{2^{N_0+N_1}} = 8\pi^2\frac{1}{2^{N_1}} < \frac{\epsilon_1}{8q'}. \tag{45}$$

Set $N''_t \stackrel{\text{def}}{=} N_0 + (t-1)N'' + N_1 + 1$. We have

$$\left|\mathcal{S}_t - \left(\mathcal{W}'_{N_1,N'}\left(\frac{k'}{2^{N'}}\right)+2\pi N_1\right)\right| \tag{46}$$
$$\le \sum_{n=-N_1}^{N'} 2\pi\left|\cos\left(2^{N''_t}2^n 2\pi(x'_{t'}-x'_{t-1})\right)-\cos\left(2^n 2\pi\left(\frac{k'}{2^{N'}}\right)\right)\right|$$
$$\le \sum_{n=-N_1}^{N'} 2^n 4\pi^2\left|2^{N''_t}\left(x'_{t'}-x'_{t-1}\right)-\frac{k'}{2^{N'}}\right|. \tag{47}$$

We rewrite

$$2^{N''_t}\left(x'_{t'}-x'_{t-1}\right) = 2^{N_0+(t-1)N''+N_1+1}\left(\sum_{l=t}^{t'}\frac{k'}{2^{N_0+lN''}}\right)$$
$$= 2^{N_0+(t-1)N''+N_1+1}\left(\frac{k'}{2^{N_0+tN''}}+\sum_{l=t+1}^{t'}\frac{k'}{2^{N_0+lN''}}\right)$$
$$= 2^{N_1+1}\frac{k'}{2^{N''}}\left(1+\sum_{l=1}^{t'-t}\frac{1}{2^{lN''}}\right)$$
$$= 2^{N_1+1}\frac{k'}{2^{N'+N_1+1}}\left(1+\sum_{l=1}^{t'-t}\frac{1}{2^{lN''}}\right) = \frac{k'}{2^{N'}}+\frac{k'}{2^{N'}}\sum_{l=1}^{t'-t}\frac{1}{2^{lN'}}.$$

Therefore, recalling that $k' < 2^{N'}$ and using (32) we infer that

$$\left|2^{N''_t}(x'_{t'} - x'_{t-1}) - \frac{k'}{2^{N'}}\right| \le \frac{k'}{2^{N'}} \sum_{l=1}^{t'-t} \frac{1}{2^{lN''}} < \frac{1}{2^{N''}} \sum_{l=0}^{\infty} \frac{1}{2^{lN''}}$$
$$\le \frac{2}{2^{N''}} = \frac{2}{2^{N'+N_1+1}} = \frac{1}{2^{N'+N_1}} < \frac{1}{2^{N'}} \cdot \frac{\epsilon_1}{8q'8\pi^2}. \tag{48}$$

Now we can continue (47) to have for all $t = 1, ..., t'$

$$\left|\mathcal{S}_t - \left(\mathcal{W}'_{N_1,N'}\left(\frac{k'}{2^{N'}}\right) + 2\pi N_1\right)\right| \le 4\pi^2 \frac{\epsilon_1}{2^{N'}8q'8\pi^2} \sum_{n=-N_1}^{N'} 2^n < \frac{\epsilon_1}{8q'}. \tag{49}$$

Using (44), (45), and (49) for all $t \in \{1, ..., t'\}$ we obtain

$$\left|\mathcal{W}'_{N'_0}(x'_{t'}) - \mathcal{W}'_{N_0}(x'_0) - \left(t'\mathcal{W}'_{N_1,N'}\left(\frac{k'}{2^{N'}}\right) + 2\pi t' N_1\right) - 2\pi(D_1 - t'N_1)\right|$$
$$< \frac{\epsilon_1}{8q'} + t'\frac{\epsilon_1}{8q'} < \frac{\epsilon_1}{4}. \tag{50}$$

Recalling that $c'_{\text{corr}} = \mathcal{W}'_{N_0}(\frac{k_0}{2^{N_0}}) - 2\pi D_0 = \mathcal{W}'_{N_0}(x'_0) - 2\pi D_0$ and simplifying we infer from (50) that

$$\left|\mathcal{W}'_{N'_0}(x'_{t'}) - t'\mathcal{W}'_{N_1,N'}\left(\frac{k'}{2^{N'}}\right) - c'_{\text{corr}} - 2\pi(D_1 + D_0)\right| < \frac{\epsilon_1}{4}. \tag{51}$$

Recall that $c'_{\text{corr}} = c_0 - c_{\text{corr}}$. We can rewrite (51) in the following way:

$$\frac{\epsilon_1}{4} > \left|\mathcal{W}'_{N'_0}(x'_{t'}) - 2\pi\left\{\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}\left(\frac{k'}{2^{N'}}\right)\right\}\right.$$
$$\left. + c_{\text{corr}} - c_0 - 2\pi\left\lfloor\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}\left(\frac{k'}{2^{N'}}\right)\right\rfloor - 2\pi(D_1 + D_0)\right|. \tag{52}$$

Set $D_2 = \lfloor\frac{t'}{2\pi}\mathcal{W}'_{N_1,N'}(\frac{k'}{2^{N'}})\rfloor$. Then using (42) and (52) we infer that

$$\frac{\epsilon_1}{4} + \frac{\epsilon_1}{3} > \left|\mathcal{W}'_{N'_0}(x'_{t'}) - c_0 - 2\pi(D_0 + D_1 + D_2)\right|. \tag{53}$$

Now,

$$|D_2| < 1 + \frac{t'}{2\pi}\left(\left(\sum_{n=-N_1}^{N'} 2\pi\left|\cos\left(2\pi 2^n \frac{k'}{2^{N'}}\right)\right|\right) + 2\pi N_1\right)$$
$$< 1 + t'(N' + 2N_1 + 1) < 3q'(N' + N_1 + 1) = 3q'N''.$$

Using (31) and (38) we can choose $D_1 > q'N''$, so that $D_1 < 110 \cdot q'N''$ and

$$D_0 + D_1 + D_2 = D_c. \tag{54}$$

By (53) we obtain that $|\mathcal{W}'_{N'_0}(x'_{t'}) - c| < \epsilon_1$. We have

$$N_0 + t'N'' \le N_0 + q'N'' < N'_0 = N_0 + D_1 + t'(N'+1) \le N_0 + 111q'N''. \tag{55}$$

Hence, there is $k'' \in \mathbb{Z}$ such that $x'_{t'} = \frac{k''}{2^{N'_0}}$. Now, recalling $0 \le k' < 2^{N'}$ and $N_1 \ge 4$ we infer that

$$\begin{aligned}\frac{k''}{2^{N'_0}} = x'_{t'} &= \frac{k_0}{2^{N_0}} + \sum_{t=1}^{t'} \frac{k'}{2^{N_0+tN''}} < \frac{k_0}{2^{N_0}} + \frac{1}{2^{N_0}} \sum_{t=1}^{\infty} \frac{k'}{2^{t(N'+N_1+1)}} \\ &< \frac{k_0}{2^{N_0}} + \frac{1}{2 \cdot 2^{N_0}} \sum_{t=1}^{\infty} \frac{1}{2^{tN_1}} < \frac{k_0}{2^{N_0}} + \frac{1}{2} \cdot \frac{1}{2^{N_0}}.\end{aligned}$$

Therefore, we have $[\frac{k''}{2^{N'_0}}, \frac{k''+1}{2^{N'_0}}) \subset [\frac{k_0}{2^{N_0}}, \frac{k_0+1}{2^{N_0}})$. It is also clear that, for $x \in [\frac{k''}{2^{N'_0}}, \frac{k''+1}{2^{N'_0}})$,

$$P(x, N'_0) = \frac{k''}{2^{N'_0}} \text{ and } |\mathcal{W}'_{N'_0}(P(x, N'_0)) - c| < \epsilon_1. \tag{56}$$

Hence, by (55) for any $x_0 \in X_{\pm\infty}$ there exist infinitely many N_0's such that in $(x_0 - \frac{1}{2^{N_0}}, x_0 + \frac{1}{2^{N_0}})$ there exists an interval of length longer than $\frac{1}{2^{111q'N''}} \cdot \frac{1}{2^{N_0}}$ such that for any x in this interval (56) holds.

By Lemma 2 this implies that for almost every $x \in X_{\pm\infty}$, that is, for almost every $x \in [0,1]$, there exist infinitely many N'_0's such that (56) holds.

Repeating the above procedure for all rational $c \in \mathbb{R}$ and $\epsilon_1 = \frac{1}{n}$ we obtain the statement of Lemma 3. □

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Space-Filling Functions and Davenport Series

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Summary. In this paper, we study the pointwise Hölder regularity of some space-filling functions. In particular, we give some general results concerning the pointwise regularity of the Davenport series.

1 Introduction

In the seminal paper [1′], Cantor showed that there exists a one-to-one mapping between the unit interval and the unit square. A few years later, Peano discovered a continuous map from the unit interval onto the unit square (see [13]). Such "Peano curves" have been used in connection with different branches of mathematical analysis and are still used in data transmission and mathematical programming, where one looks for functions from $\mathbb{R}$ to $\mathbb{R}^2$ which are onto and preserve some neighborhood relationship. In other words, one searches for curves that go through all the elements of an array in a "regular way." Such considerations naturally lead to the use of Peano functions, satisfying some "regularity" conditions (see, for instance, [12, 14]).

We will study the pointwise Hölder regularity of some space-filling functions; since several of the historical ones were given by Davenport series, we will also give some general results concerning the pointwise regularity of such series. This work improves some previous results of [6, 7].

2 Definitions

In this section, we recall the definitions related to the space-filling functions and the pointwise Hölder regularity. Let $\mathcal{L}^d$ denote the d-dimensional Lebesgue measure.

Definition 1. *A function $f : [0,1] \to \mathbb{R}^d$ $(d \geq 2)$ is space-filling if $\mathcal{L}^d(f([0,1])) > 0$; a Peano function is a continuous space-filling function.*

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Morayne proved that there is no everywhere differentiable space-filling function [11]. We will be interested in obtaining the precise Hölder regularity of such functions.

2.1 Hölder Spaces and Hölder Exponents

The Hölder exponent refines the notion of a continuous, non-differentiable function.

Definition 2. *Let $\alpha > 0$ and $x \in \mathbb{R}$; a locally bounded function f belongs to the Hölder space $C^\alpha(x)$ if there exist $C, R > 0$ and a polynomial P such that*

$$|r| < R \Rightarrow |f(x+r) - P(r)| \leq C|r|^\alpha. \tag{1}$$

A regularity index of f at each point x is given by the following definition.

Definition 3. *The Hölder exponent of f at x is*

$$h(x) = h(x; f) = \sup\{\alpha : f \in C^\alpha(x)\}. \tag{2}$$

Obviously, $h(x) < 1$ implies that f is not differentiable at x. This exponent is sometimes called the *lower Hölder exponent* in order to emphasize the difference from the *upper Hölder exponent*, which is a counterpart of the Hölder exponent, and is a way to measure the irregularity of a function at a point.

Definition 4. *Let $0 < \alpha \leq 1$; a function f belongs to $I^\alpha(x)$ if there exist $C, R > 0$ such that*

$$r < R \Rightarrow \sup_{|r'|<r} |f(x+r') - f(x)| \geq Cr^\alpha. \tag{3}$$

The upper Hölder exponent of f at x is

$$\overline{h}(x) = \overline{h}(x; f) = \inf\{\alpha : f \in I^\alpha(x)\}. \tag{4}$$

The spaces $I^\alpha(x)$ can be generalized for $\alpha > 1$ (see, e.g., [2]).

Definition 5. *The r-oscillation of a function f at x is*

$$\mathrm{osc}_r(x) = \mathrm{osc}_r(x; f) = \mathrm{diam} f(B(x, r)).$$

An equivalent definition of the spaces $C^\alpha(x)$ and $I^\alpha(x)$ can be given in terms of r-oscillation, which sheds light on the duality between these two notions; indeed one immediately checks that:

- A function f belongs to $C^\alpha(x)$ if and only if there exist $C, R > 0$ such that

$$r < R \Rightarrow \mathrm{osc}_r(x) \leq Cr^\alpha. \tag{5}$$

- A function f belongs to $I^\alpha(x)$ if and only if there exist $C, R > 0$ such that

$$r < R \Rightarrow \mathrm{osc}_r(x) \geq Cr^\alpha. \tag{6}$$

2.2 Uniform Hölder Spaces and Strongly Monohölder Functions

We give here the uniform versions of the pointwise Hölder spaces and introduce the important notion of *strongly monohölder functions* which formalizes the idea of a function which has everywhere the same regularity, in a way as uniform as possible.

The previous definitions concerning the pointwise regularity have a uniform counterpart.

Definition 6. *Let* $0 < \alpha < 1$*; a function* f *belongs to* C^α *if there exist* $C, R > 0$ *such that, for any* x*,*

$$|r| < R \Rightarrow |f(x+r) - f(x)| \leq C|r|^\alpha,$$

or equivalently,

$$r < R \Rightarrow \mathrm{osc}_r(x) \leq Cr^\alpha.$$

In the same way, $f \in I^\alpha$ *if there exist* $C, R > 0$ *such that, for any* x*,*

$$r < R \Rightarrow \sup_{|r'| \leq r} |f(x+r') - f(x)| \geq Cr^\alpha,$$

which can be rewritten as

$$r < R \Rightarrow \mathrm{osc}_r(x) \geq Cr^\alpha.$$

The regularity of most of the "historical Peano functions" is the same at every point; we will make an intensive use of the following notation.

Definition 7. *Let* $0 < \alpha < 1$*; a function* f *is strongly monohölder of exponent* α *(*$f \in SM^\alpha$*) if* $f \in C^\alpha \cap I^\alpha$*, i.e., if there exist* $C, R > 0$ *such that, for any* x*,*

$$r < R \Rightarrow \frac{1}{C} r^\alpha \leq \sup_{y \in B(x,r)} |f(y) - f(x)| \leq Cr^\alpha, \tag{7}$$

or equivalently,

$$r < R \Rightarrow \frac{1}{C} r^\alpha \leq \mathrm{osc}_r(x) \leq Cr^\alpha.$$

Strongly monohölder functions share the following property (see [3]): Let $\dim_B$ denote the box-counting dimension; if $f : [0,1] \to \mathbb{R}$ is continuous, then

$$f \in SM^{2-h} \Rightarrow \dim_B(\mathrm{graph}(f)) = h.$$

Since the associated box-counting dimension is larger than one, the graph of such a function is usually qualified as a "fractal set."

3 The Peano Function

The Peano function [13] is the first continuous space-filling function ever exhibited.

Let K be the function defined by $K(j) = 2 - j$ $(0 \leq j \leq 2)$; we denote by K^j the jth iterate of K, and set by convention $K^0(j) = j$. The Peano function is defined in [13] as follows (Figs. 1 and 2):

$$P : [0,1] \to [0,1]^2 \quad x \mapsto (p_1(x), p_2(x)),$$

where, if

$$x = \sum_{k=1}^{\infty} \frac{x_k}{3^k}, \tag{8}$$

with $x_k \in \{0,1,2\}$ $(\forall k)$,

$$p_1(x) = \sum_{k=1}^{\infty} \frac{K^{\sum_{l=1}^{k-1} x_{2l}}(x_{2k-1})}{3^k}$$

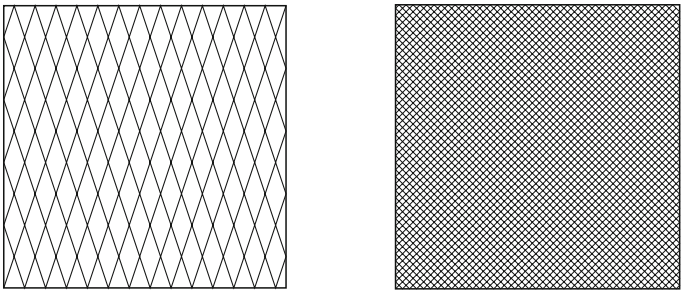

Fig. 1. Approximations of the Peano curve by polygonal curves: one sets $t_0^{(j)} = 0$ and $t_k^{(j)} = t_0 + k/j$; the polygonal curve parameterized by γ_j approximating the curve parameterized by γ is the piecewise linear curve made of the $j+1$ segments of extremities $\gamma(t_i)$, $\gamma(t_{i+1})$ $(0 \leq i < j)$. Here are represented γ_{3^5} and γ_{3^8}

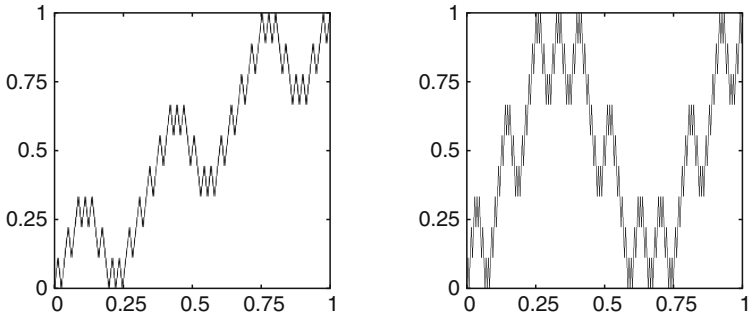

Fig. 2. The Peano functions p_1 and p_2

and

$$p_2(x) = \sum_{k=1}^{\infty} \frac{K^{\sum_{l=0}^{k-1} x_{2l+1}}(x_{2k})}{3^k}.$$

It is easy to check that this function is well defined, i.e.,

$$P\left(\sum_{k=1}^{k_0} \frac{x_k}{3^k}\right) = P\left(\sum_{k=1}^{k_0-1} \frac{x_k}{3^k} + \frac{x_{k_0}-1}{3^{k_0}} + \sum_{k=k_0+1}^{\infty} \frac{2}{3^k}\right)$$

for any $k_0 > 1$ and any sequence $(x_k)_{k\in\mathbb{N}}$ defined on $\{0,1,2\}$ such that $x_{k_0} \neq 0$, thanks to the operator K (in other words, $P(x)$ does not depend on the sequence chosen to represent x).

Peano proved the following result in [13].

Proposition 1. *The Peano function is onto* $[0,1]^2$.

Proof. If x has (8) as an expansion, let us denote $x_k^{(1)} = K^{\sum_{l=1}^{k-1} x_{2k}}(x_{2k-1})$ and $x_k^{(2)} = K^{\sum_{l=1}^{k-1} x_{2l-1}}(x_{2k})$. We have $K^j(l) = l$ if j is odd and $K^j(l) = 2-l$ otherwise. Since the operator K leaves the parity unchanged, $x_k^{(1)} = K^{\sum_{l=1}^{k-1} x_k^{(2)}}(x_{2k-1})$ and $x_k^{(2)} = K^{\sum_{l=1}^{k-1} x_k^{(1)}}(x_{2k})$. Therefore,

$$x_{2k-1} = K^{\sum_{l=1}^{k-1} x_k^{(2)}}(x_k^{(1)}), \qquad x_{2k} = K^{\sum_{l=1}^{k-1} x_k^{(1)}}(x_k^{(2)}) \tag{9}$$

and any element $(x^{(1)}, x^{(2)}) = (\sum_{k=1}^{\infty} x_k^{(1)} 3^{-k}, \sum_{k=1}^{\infty} x_k^{(2)} 3^{-k})$ of $[0,1]^2$ gives rise to an element $x = \sum_{k=1}^{\infty} x_k 3^{-k}$ of $[0,1]$, using relation (9). □

Proposition 2. *The Peano function belongs to* $SM^{1/2}$.

Proof. Let us work with p_1. The case of p_2 can be treated in the same way. Let $x \in [0,1]$ and $r > 0$. Define $k_0 \in \mathbb{N}$ such that

$$\frac{1}{3^{k_0}} \leq r < \frac{1}{3^{k_0-1}}.$$

If $y = \sum_{k=1}^{\infty} y_k 3^{-k}$ belongs to $B(x,r)$, we have

$$|x-y| = \sum_{k=k_0}^{\infty} \frac{\delta_k}{3^k},$$

for a sequence $(\delta_k)_{k\in\mathbb{N}}$ defined on $\{0,1,2\}$. One immediately gets

$$p_1(x) - p_1(y) = \sum_{\lceil k_0/2\rceil+1}^{\infty} \frac{K^{\sum_{l=1}^{k-1} x_{2l}}(x_{2k-1})}{3^k} - \sum_{\lceil k_0/2\rceil+1}^{\infty} \frac{K^{\sum_{l=1}^{k-1} y_{2l}}(y_{2k-1})}{3^k}.$$

Therefore, $|p_1(x) - p_1(y)| \leq C\sqrt{|x-y|} \leq C' 3^{-k_0/2}$ and $\mathrm{osc}_r(x) \leq C''\sqrt{r}$. Moreover, if $\delta_{k_0} \neq 0$, setting $\beta = \sum_{l=1}^{\lceil k_0/2\rceil} x_{2l}$, one has $K^\beta(x_{2\lceil k_0/2\rceil+1}) \neq K^\beta(y_{2\lceil k_0/2\rceil+1})$. This implies $\mathrm{osc}_r(x) \geq C' 3^{-k_0/2} \geq C''\sqrt{r}$. □

4 A Strong Monohölderianity Criterion

We now prove a general strong monohölderianity criterion. This criterion extends a less general one proved in [7]. It immediately implies that most of the "historical Peano functions" (e.g., the functions of Peano, Wunderlich, Hilbert, Moore, and Sierpinski (see [15]) and also the time-changed Polya function introduced in [10]) are strongly monohölder with Hölder exponent 1/2.

A tree T of subintervals of $[0,1)$ will be called regular if it satisfies the following requirements. There exists $C > 0$ such that:

1. The root of T is $[0,1)$.
2. Each element of T has at most C children, which form a subdivision of their parent.
3. For each generation G_j, $\forall e, f \in G_j$, $|e| \leq C|f|$.

Proposition 3. *Let $f : [0,1) \to \mathbb{R}^d$ be a continuous function satisfying the following conditions: There exists a constant $C' > 0$ and $\alpha \in (0,1)$ such that*

$$\forall e \in T; \qquad \frac{1}{C'}(\mathrm{diam}(e))^\alpha \leq \mathrm{diam}(f(e)) \leq C'(\mathrm{diam}(e))^\alpha, \tag{10}$$

where T is a regular tree. Then, the function f belongs to SM^α.

Proof. Let $x, y \in [0,1)$; there exists $e \in T$ such that

$$x \in e \qquad \text{and} \qquad |e| \leq |x - y| \leq C|e|.$$

The points x and y are separated by at most C intervals $e_1 = e, e_2, \ldots, e_k$ of the same generation, with endpoints $x_1, \ldots, x_k$. It follows that

$$|f(x) - f(y)| \leq |f(x) - f(x_1)| + |f(x_1) - f(x_2)| + \cdots + |f(x_k) - f(y)|$$

$$\leq (C+1)C'|e|^\alpha \leq (C+1)C'|f(x) - f(y)|^\alpha.$$

Let us now show the uniform irregularity. Because of the lower bound in (10) there exists two points u and v in the interval e_1 such that

$$|f(u) - f(v)| \geq C'|e_1|^\alpha.$$

Since the interval e_1 can be chosen including any point x, and at any scale, the uniform irregularity follows. □

The Peano functions introduced by Peano, Wunderlich, Hilbert, Moore, Sierpinski, and the time-changed Polya function are defined in a recursive way, so that the images of p-adic intervals are exactly triangles or squares; therefore one immediately checks in these examples that the assumptions of Proposition 3 are satisfied with $\alpha = 1/2$.

5 The Lebesgue Function

The Lebesgue function [9] is a classical example of an almost everywhere differentiable continuous space-filling function.

Let us first recall the definition of the triadic Cantor set.

Definition 8. *The triadic Cantor set K is the subset of $[0,1]$ such that*

$$x \in K \Leftrightarrow x = \sum_{k=1}^{\infty} \frac{2x_k}{3^k},$$

for a binary sequence $(x_k)_{k\in\mathbb{N}}$.

The Lebesgue function is defined on K as follows:

$$L_{|K} : K \to [0,1]^2 \quad x \mapsto (l_1(x), l_2(x)),$$

where, if

$$x = \sum_{k=1}^{\infty} \frac{2x_k}{3^k},$$

for a binary sequence $(x_k)_{k\in\mathbb{N}}$,

$$l_1(x) = \sum_{k=1}^{\infty} \frac{x_{2k-1}}{2^k}, \qquad \text{and} \qquad l_2(x) = \sum_{k=1}^{\infty} \frac{x_{2k}}{2^k}.$$

The Lebesgue function can be continuously extended to $[0,1]$ as follows. If $x \notin K$, let I_x denote the largest (open) interval of K^c containing x. The Lebesgue function L is defined as the continuous function satisfying (Figs. 3 and 4)

$$L : [0,1] \to [0,1]^2 \quad x \mapsto \begin{cases} L_{|K}(x) & \text{if } x \in K \\ L \text{ is linear on } \bar{I}_x & \text{if } x \in K^c \end{cases}.$$

Lebesgue showed in [9] that L is onto the unit square. We show here a slightly stronger result [7].

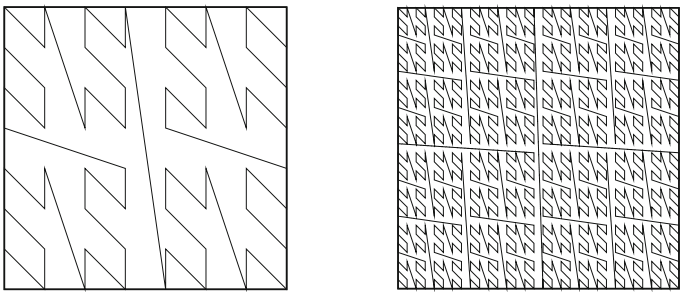

Fig. 3. Approximations of the Lebesgue curve by polygonal curves (see Fig. 1). Here are represented γ_{3^6} and $\gamma_{3^{10}}$

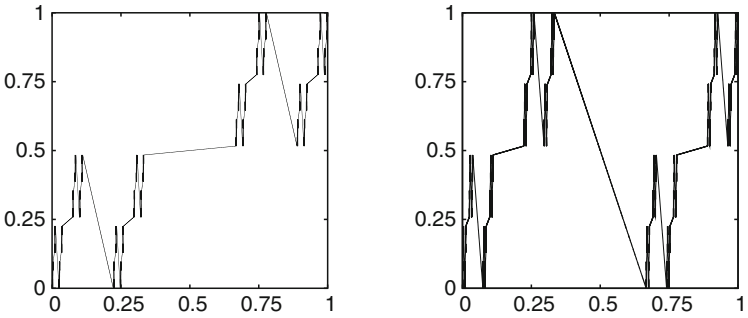

Fig. 4. The Lebesgue functions l_1 and l_2

Proposition 4. *Let K^* be the subset of $[0,1]$ such that*

$$x \in K^* \Leftrightarrow x = \sum_{k=1}^{\infty} \frac{2x_k}{3^k},$$

where $(x_k)_{k\in\mathbb{N}}$ is a binary sequence for which there is no k_0 such that $k > k_0 \Rightarrow x_k = 1$. The restriction of the Lebesgue function to K^* is onto $[0,1]^2 - \{(1,1)\}$, but is not a one-to-one function.

Proof. Let $(y,z) \in [0,1]^2 - \{(1,1)\}$. If $y = \sum_{k=1}^{\infty} y_k 2^{-k}$ and $z = \sum_{k=1}^{\infty} z_k 2^{-k}$, for two binary sequences $(y_k)_{k\in\mathbb{N}}$ and $(z_k)_{k\in\mathbb{N}}$, (y,z) defines a number $x \in K$,

$$x = \sum_{k=1}^{\infty} \frac{2y_k}{3^{2k-1}} + \sum_{k=1}^{\infty} \frac{2z_k}{3^{2k}}.$$

Moreover, since $(y,z) \neq (1,1)$, one of the associated binary sequences ($(y_k)_k$, say) can be chosen to be not ultimately equal to 1 (i.e., such that there is no k_0 such that $k > k_0 \Rightarrow y_k = 1$). As a consequence, $x \in K^*$ and, by construction, $L(x) = (y,z)$. If y or z is a dyadic number, it will have two different binary representations, which give rise to two different pre-images in K^*. □

The regularity of the Lebesgue function is given by the following result, which was obtained in a different way in [7].

Proposition 5. *The Lebesgue function L belongs to C^h, with $h = \log 2/2\log 3$. If $x \notin K$, $L \in C^\infty(x)$; if $x \in K$, the function belongs to $C^h(x) \cap I^h(x)$.*

Proof. The uniform regularity was obtained in [7], using a generic result (Proposition 17). Let us show the pointwise regularity. We will work with l_1 (defined on $[0,1]$); the case of l_2 is similar. We can suppose that $x \in K$; let $\alpha = \log 2/2\log 3$. If $y \in K$ is such that

$$|x - y| = \sum_{k=k_0}^{\infty} \frac{2\delta_k}{3^k},$$

for a binary sequence $(\delta_k)_{k\in\mathbb{N}}$, one has

$$|l_1(x) - l_1(y)| = \sum_{k=\lceil k_0/2\rceil+1}^{\infty} \frac{\delta_k}{2^k} \leq C|x-y|^\alpha. \tag{11}$$

Now, if $y \notin K$, let $I_y = (a,b)$ and set $c = a$ if $y > x$, $c = b$ otherwise. One has

$$|l_1(x)-l_1(y)| \leq |l_1(x)-l_1(c)|+|l_1(c)-l_1(y)| \leq C(|x-c|^\alpha+|c-y|^\alpha) \leq C|x-y|^\alpha,$$

where we have used either the relation (11) or the linearity of l_1. Let us now show the pointwise irregularity of l_1. Let $x \in K$, $r > 0$ and let $k_0 \in \mathbb{N}$ defined by the relations

$$\frac{1}{3^{k_0-1}} \leq r < \frac{1}{3^{k_0-2}}.$$

Let also $y \in B(x,r) \cap K$ such that

$$|x-y| = \sum_{k=k_0}^{\infty} \frac{2\delta_k}{3^k},$$

for a binary sequence $(\delta_k)_{k\in\mathbb{N}}$, with $\delta_{k_0} \neq 0$ (this can be done by choosing a point y in a different triadic interval of generation k_0 coming up in the construction of K). For such a number,

$$|l_1(x) - l_1(y)| = \sum_{k=\lceil k_0/2\rceil+1}^{\infty} \frac{\delta_k}{2^k} \geq C|x-y|^\alpha \geq C'3^{-k_0\alpha} \geq C''r^\alpha.$$

6 The Schoenberg Function

Let Λ be the 2-periodic even function such that

$$\Lambda(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/3 \\ 3x-1 & \text{if } 1/3 \leq x \leq 2/3 \\ 1 & \text{if } 2/3 \leq x \leq 1. \end{cases}$$

The Schoenberg function [16] is defined by (Figs. 5 and 6)

$$S : [0,1] \to [0,1]^2 \quad x \mapsto (s_1(x), s_2(x)),$$

where

$$s_1(x) = \sum_{k=1}^{\infty} \frac{\Lambda(3^{2(k-1)}x)}{2^k}$$

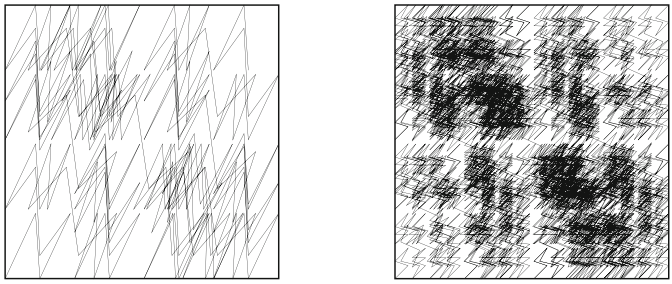

Fig. 5. Approximations of the Schoenberg curve by polygonal curves (see Fig. 1). Here are represented γ_{3^8} and $\gamma_{3^{10}}$

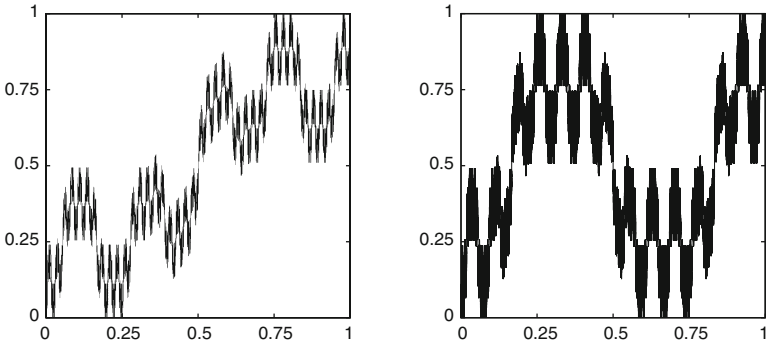

Fig. 6. The Schoenberg functions s_1 and s_2

and

$$s_2(x) = \sum_{k=1}^{\infty} \frac{\Lambda(3^{2k-1}x)}{2^k}.$$

The fact that S is onto the unit square can be deduced from Proposition 4 and the following result, which was obtained by Schoenberg in [16].

Proposition 6. *For any $x \in K$, $S(x) = L(x)$ (where L denotes the Lebesgue function).*

Proof. Let $x \in K$; if $x = \sum_{k=1}^{\infty} 2x_k 3^{-k}$ for a binary sequence $(x_k)_{k\in\mathbb{N}}$, we have

$$\Lambda(3^{k_0}x) = \Lambda\left(\sum_{k=k_0+1}^{\infty} \frac{2x_k}{3^k}\right) = x_{k_0+1},$$

by definition of Λ. Therefore,

$$s_1(x) = \sum_{k=1}^{\infty} \frac{\Lambda(3^{2(k-1)}x)}{2^k} = \sum_{k=1}^{\infty} \frac{x_{2k-1}}{2^k} = l_1(x),$$

by definition of l_1. In the same way, $s_2(x) = l_2(x)$. □

Concerning the regularity, S is strongly monohölder. The following result was stated in [7], but can be obtained in many different ways, see e.g., [1,5,8].

Proposition 7. *The Schoenberg function belongs to* $SM^{\log 2/2 \log 3}$.

7 The Cantor Function

In a letter to Dedekind, Cantor proposed the construction of a new function as a candidate to be the first one-to-one correspondence between the unit interval and the unit square [4]. Dedekind pointed out that this function actually is not one to one (as showed by Proposition 8).

In this section, any number $x \in [0,1)$ will be implicitly associated with the sequence $(x_k)_{k\in\mathbb{N}}$ of its proper expansion in the decimal base; i.e., it takes values in $\{0,\ldots,9\}$, satisfies

$$x = \sum_{k=1}^{\infty} \frac{x_k}{10^k}, \tag{12}$$

and there is no k_0 such that $k > k_0 \Rightarrow x_k = 9$.

The Cantor function $\mathcal{C} : [0,1) \to [0,1]^2$ is defined by $\mathcal{C}(x) = (c_1(x), c_2(x))$, where

$$c_1(x) = \sum_{k=1}^{\infty} \frac{x_{2k-1}}{10^k} \qquad \text{and} \qquad c_2(x) = \sum_{k=1}^{\infty} \frac{x_{2k}}{10^k}.$$

One extends $\mathcal{C}$ on $[0,1]$ by picking $\mathcal{C}(1) = (1,1)$ (Fig. 7).

Proposition 8. *The Cantor function is onto* $[0,1]^2$ *but is not a one-to-one function.*

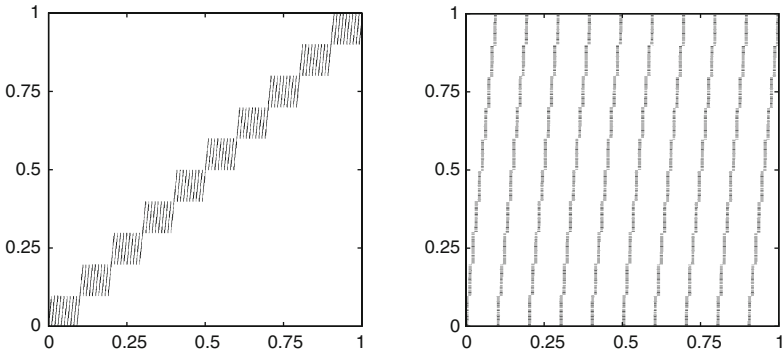

Fig. 7. The Cantor functions c_1 and c_2

Proof. Let $(y,z) \in [0,1]^2$. We can suppose that $(y,z) \neq (1,1)$. The expression

$$x = \sum_{k=1}^{\infty} \frac{y_k}{10^{2k-1}} + \sum_{k=1}^{\infty} \frac{z_k}{10^{2k}}$$

is the proper expansion of a number $x \in [0,1)$ such that $\mathcal{C}(x) = (y,z)$.

Let now $x \in [0,1)$ be a number defined by a sequence $(x_k)_{k\in\mathbb{N}}$ such that there exists an index $k_0 > 0$ for which $x_{k_0} < 9$ and $x_{k_0+2k} = 9$, $\forall k \in \mathbb{N}$. It is easy to check that the number y defined by the following sequence $(y_k)_{k\in\mathbb{N}}$,

$$y_k = \begin{cases} x_k & \text{if } k < k_0 \\ x_{k_0} + 1 & \text{if } k = k_0 \\ x_k & \text{if } k = k_0 + 2l + 1, \text{ with } l \in \mathbb{N} \\ 0 & \text{if } k = k_0 + 2l, \text{ with } l \in \mathbb{N} \end{cases}$$

is such that $x \neq y$ and $\mathcal{C}(x) = \mathcal{C}(y)$. □

Proposition 8 provides an injective map from $[0,1]^2$ to $[0,1]$. Since it is trivial to find an injective map from $[0,1]$ to $[0,1]^2$, the Schröder–Bernstein theorem implies that there exists a one-to-one mapping from $[0,1]$ to $[0,1]^2$.

The regularity of the Cantor function at a given point x depends on the order of the approximation of the number x by numbers of the form $k/10^l$ $(k,l \in \mathbb{N})$.

Proposition 9. *If x is not of the form $k/10^l$ $(k,l \in \mathbb{N}_0)$, let $\phi(x)$ be the supremum of the exponents ϕ such that the equation*

$$\left| x - \frac{k}{10^l} \right| \leq 10^{-l\phi} \quad (k < 10^l)$$

has infinitely many solutions. If $x = k/10^l$ for two $k,l \in \mathbb{N}_0$, one sets $\phi(x) = \infty$. The Hölder exponent of $\mathcal{C}$ at x is

$$h(x) = \frac{1}{2\phi(x)}.$$

The upper Hölder exponent of $\mathcal{C}$ at x is

$$\overline{h}(x) = \begin{cases} 1/2 \text{ if } x \neq k/10^l \\ 0 \quad \text{otherwise} \end{cases}.$$

Proof. Let $D = \{x : x = k/10^l,\ k,l \in \mathbb{N}\}$. Let us first show that the Cantor function is not left-continuous at numbers of the form $k/10^l$ and is continuous elsewhere. If $x \notin D$, let $N_x(y) = \inf\{j : x_j \neq y_j\} - 1$. The continuity of the Cantor function at x follows from the fact that for any sequence $(z_j)_{j\in\mathbb{N}}$,

$$\lim_j z_j = x \Leftrightarrow \lim_j N_x(z_j) \to \infty,$$

and from the inequality $N_{c_1(x)}(c_1(y)) \geq [N_x(y)/2]$ (the same relation holds for c_2). Suppose now that $x \in D$; there exists k_0 such that $x_{k_0} > 0$ and $x_k = 0$, $\forall k > k_0$. Let $x^{(l)}$ be the number satisfying

$$x_k^{(l)} = \begin{cases} x_k & \text{if } k < k_0 - 1 \\ x_{k_0} - 1 & \text{if } k = k_0 \\ 9 & \text{if } k_0 < k \leq k_0 + l \\ 0 & \text{if } k > k_0 + l \end{cases}.$$

One has

$$c_1(x^{(l)}) - c_1(x) = \sum_{k=k_0/2+1}^{k_0/2+\lceil l/2 \rceil} \frac{9}{10^k} \to 10^{-k_0/2}$$

if k_0 is even and $c_2(x^{(l)}) - c_2(x) \to 10^{-(k_0-1)/2}$ if k_0 is odd.

From now on, we can suppose that $x \notin D$. Since the Cantor function is continuous on $[0,1] - D$ but not on $[0,1]$, it is sufficient to look at $|\mathcal{C}(x) - \mathcal{C}(y)|$ where $y \in D$. For any $l \in \mathbb{N}_0$, let $k(l) \in \mathbb{N}_0$ be such that

$$\left| x - \frac{k(l)}{10^l} \right| = \min_{k<10^l} \left| x - \frac{k}{10^l} \right| \leq 10^{-\psi(l)},$$

where $\psi(l)$ is the largest integer such that the inequality holds. Let $y^{(l)} = k(l)/10^l$ if $k(l)/10^l > x$ and $y^{(l)} = k(l)/10^l - 10^{-\psi(l)}$ otherwise. Suppose that l is even (if l is odd, one can consider c_2 instead of c_1); it is easy to check that, as l goes to infinity,

$$|c_1(x) - c_1(y^{(l)})| \leq C10^{-l/2} \leq C(10^{-l\phi(x)})^{1/2\phi(x)} \leq C|x - y^{(l)}|^{1/2\phi(x)},$$

and $h(x) \leq 1/2\phi(x)$. Now, if $y = k/10^l$, with $k < 10^l$ and $k \neq k(l)$, one has, by definition of $k(l)$, $|c_1(x) - c_1(y)| \leq C|x - y|^{1/2\phi(x)}$, for y sufficiently close to x. Therefore $h(x) = 1/2\phi(x)$. The upper Hölder exponent is easy to get, since, by definition of $\mathcal{C}$, for any $l \in 2\mathbb{N}_0$, it is always possible to find a number y such that $10^{-l-1} \leq |x - y| < 10^{-l}$ and $|c_1(x) - c_1(y)| \geq 10^{(-l-1)/2}$. For any $r > 0$,

$$\sup_{y \in B(x,r)} |c_1(x) - c_1(y)| \geq C\sqrt{r}.$$

The case of c_2 is similar. □

Let us now show that the Cantor function is an example of a *Davenport series*. Such series are odd 1-periodic functions defined as follows: if $\{x\}$ denotes the "sawtooth function" $\{x\} = x - [x] - \frac{1}{2}$, then Davenport series are of the form

$$\sum_{n=1}^{\infty} a_n \{nx\} \qquad \text{with} \qquad (a_n) \in l^1, \tag{13}$$

see [6]; *p-adic Davenport series* correspond to the case where $a_n = 0$ except if $n = p^k$ for a p larger than 2, i.e., are of the form

$$f(x) = \sum_{j=1}^{\infty} a_j \{p^j x\}, \tag{14}$$

Recent results on Davenport series can be found in [6].

Let $\omega(x)$ be the 1-periodic function such that $\omega(x) = j$ if $x \in [j/10, (j+1)/10)$ $(0 \le j \le 9)$. Clearly, $x_n = \omega(10^{n-1}x)$, so that

$$c_1(x) = \sum_{k=1}^{\infty} \frac{\omega(10^{2k-2}x)}{10^k} \qquad \text{and} \qquad c_2(x) = \sum_{k=1}^{\infty} \frac{\omega(10^{2k-1}x)}{10^k}.$$

One easily checks that

$$\omega(x) = 10\{x\} - \{10x\} + \frac{9}{2}.$$

Therefore,

$$c_1(x) = \frac{1}{2} + \sum_{k=1}^{\infty} -\frac{\{(10^{2k-1}x)\}}{10^k} + \frac{\{(10^{2k-2}x)\}}{10^{k-1}}$$

$$c_2(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{\{(10^{2k-1}x)\}}{10^{k-1}} - \frac{\{(10^{2k}x)\}}{10^k}.$$

Thus, the coordinates of C are examples of 10-adic Davenport series. Another remarkable space-filling function that also turned out to be a p-adic Davenport series was the Lebesgue–Davenport function studied in [7]. We will now prove a general result which yields the pointwise Hölder regularity of any p-adic Davenport series (and therefore applies to these two space-filling functions). This results extends a previous one of [7], in which a regularity condition on the sequence of jumps was imposed, which turns out to be unnecessary.

8 Hölder Exponent of *p*-adic Davenport Series

Since $(a_j) \in l^1$, the function f defined by (14) is the sum of a normally convergent series; it follows that it is continuous at every non p-adic real number, and has a right and a left limit at every p-adic rational $k \cdot p^{-l}$ (where $\gcd(k,p) = 1$, which will be denoted $k \wedge p = 1$), with a jump of amplitude

$$b_l = a_l + a_{l+1} + \cdots .$$

Let $x_0 \in [0,1)$. We denote by ω_n the sequence of p-adic approximants of x_0, i.e., for each n, ω_n is the point of the form $k \cdot p^{-n}$ with $k \wedge p = 1$ which is closest to x_0.

Theorem 1. *Let f be given by (14), with $a_j \in l^1$. Let $x_0 \in \mathbb{R}$; if x_0 is not a p-adic rational, then*

$$h_f(x_0) = \liminf_{j\to\infty} \left(\frac{\log(|b_j|)}{\log(|x_0 - \omega_j|)} \right). \tag{15}$$

Assume now that $x_0 = k \cdot p^{-l}$ with $k \wedge p = 1$. If $b_l \neq 0$ then $h_f(t_0) = 0$, else (15) holds (and, in this case, $|x_0 - \omega_j| = p^{-j}$).

Proof. Denote by α the right-hand side of (15). First, note that f has a jump of amplitude b_j at ω_j; therefore, it follows from a classical lemma (see [6] for instance) that $h_f(x_0) \leq \alpha$. Therefore, we only have to prove the regularity at x_0. Let $\varepsilon > 0$. For j large enough,

$$|b_j| \leq (p^{-j})^{\alpha-\varepsilon}.$$

Let x be given; let J be defined by

$$p^{-J-1} < |x - x_0| \leq p^{-J},$$

and let l be the first integer such that x and x_0 are not in the same p-adic interval of length p^{-l}. We have $l \leq J$ and

$$f(x) - f(x_0) = \left(\sum_{j\leq J} a_j p^j \right) (x - x_0) + \sum_{j=l}^{J} a_j + \sum_{j>J} a_j (\{p^j x\} - \{p^j x_0\}).$$

Since $a_j = b_j - b_{j+1}$, the last term is bounded by

$$4 \sum_{j>J} |b_j| \leq C (p^{-J})^{\alpha-\varepsilon} \leq C |x - x_0|^{\alpha-\varepsilon}.$$

As regards the second term, since $|x_0 - \omega_l| \leq |x - x_0|$,

$$\left| \sum_{j=l}^{J} a_j \right| = |b_l - b_{J+1}| \leq |x_0 - \omega_l|^{\alpha-\varepsilon} + (p^{-J-1})^{\alpha-\varepsilon} \leq C |x - x_0|^{\alpha-\varepsilon}.$$

As regards the first term, we separate two cases; if $\alpha \leq 1$, then

$$\left| \sum_{j\leq J} a_j p^j \right| = \left| \sum_{j\leq J} (b_j - b_{j+1}) p^j \right| \leq C \cdot p^{(1-\alpha+\varepsilon)J} \leq C |x - x_0|^{\alpha-\varepsilon-1},$$

which yields the required bound for $|f(x) - f(x_0)|$.

If $\alpha > 1$, then the series $\sum a_j p^j = \sum (b_j - b_{j+1}) p^j$ is convergent; therefore, we can write

$$\sum_{j \leq J} a_j p^j = \sum_{j \in I\!N} a_j p^j - \sum_{j > J} a_j p^j$$

and

$$\left| \sum_{j>J} a_j p^j \right| = \left| \sum_{j>J} (b_j - b_{j+1}) p^j \right| \leq C \cdot p^{(1-\alpha+\varepsilon)J} \leq C |x - x_0|^{\alpha-\varepsilon-1},$$

which yields the required bound for $|f(x) - f(x_0) - (\sum a_j p^j)(x - x_0)|$. □

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Dimensions and Porosities

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Summary. We give a short overview of dimensional properties of porous sets and measures. A special emphasis is given to the heuristic ideas behind a recent result giving the best possible upper bound for the packing dimension of mean porous measures. This chapter is based on the paper Beliaev et al. (J Lond Math Soc, 2009), which is a joint work with D. Beliaev, M. Järvenpää, A. Käenmäki, T. Rajala, S. Smirnov, and V. Suomala.

1 Porous Sets

Intuitively, it seems obvious that if a set contains a lot of large holes, then it cannot be very big. Furthermore, the bigger holes there are, the smaller the set should be, and if the number of holes increases the size of the set should decrease. Can one transform this heuristic statement to a mathematical theorem? In order to do that, one has to specify what is meant by the size of a set and by the size and abundance of holes. Throughout this paper the size of sets or measures is measured either by the Hausdorff dimension $\dim_H$, by the packing dimension $\dim_p$, or by the Minkowski dimension $\dim_M$, which is also known as the box counting dimension. The size and abundance of holes are measured by porosity.

Definition 1. *Let $A \subset \mathbf{R}^n$. For all $x \in \mathbf{R}^n$ and $r > 0$, define*

$$\mathrm{por}(\mathrm{A}, \mathrm{x}, \mathrm{r}) = \sup\{\alpha \geq 0 \mid \mathrm{B}(\mathrm{y}, \alpha\mathrm{r}) \subset \mathrm{B}(\mathrm{x}, \mathrm{r}) \setminus \mathrm{A} \text{ for some } \mathrm{y} \in \mathbf{R}^{\mathrm{n}}\}.$$

Here $B(x, r)$ is the closed ball with center at x and radius r. The porosity of A at x is

$$\mathrm{por}(\mathrm{A}, \mathrm{x}) = \liminf_{\mathrm{r} \to 0} \mathrm{por}(\mathrm{A}, \mathrm{x}, \mathrm{r}),$$

and the porosity of A is

$$\mathrm{por}(\mathrm{A}) = \inf_{\mathrm{x} \in \mathrm{A}} \mathrm{por}(\mathrm{A}, \mathrm{x}).$$

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The definition can be easily extended to metric spaces (see [9]). The porosity of a set gives for all small scales the relative radius of the largest ball which fits into a reference ball centered at the set and which does not intersect the set. Clearly, $0 \le \mathrm{por}(A, x, r) \le \frac{1}{2}$ for all $x \in A$ implying that $0 \le \mathrm{por}(A, x) \le \frac{1}{2}$ and $0 \le \mathrm{por}(A) \le \frac{1}{2}$.

In Definition 1 we had to choose between the upper and the lower limit and between infimum and supremum. Since we want to find non-trivial upper bounds for the dimension of porous sets, it is obvious that it is not enough to find one point where the set is porous. Thus, we have inf instead of sup in the definition of por(A). The choice between the lower and the upper limit in the definition of por(A, x) is more subtle. In fact, the porosity was first defined by Denjoy [4] in the 1920s (although he called it index and not porosity), and he used the upper limit in the definition. However, it turns out that it is not a good choice from the point of dimension estimates, although it is very useful in many other connections. Indeed, for any $0 \le s \le n$ there exists a compact set $A \subset \mathbf{R}^n$ such that $\dim_H(A) = s$ and $\limsup_{r \to 0} \mathrm{por}(A, x, r) = \frac{1}{2}$ for all $x \in A$ (see [13, p. 64]). The concept defined in this way is called upper porosity to distinguish it from the (lower) porosity.

It is easy to see that if a set $A \subset \mathbf{R}^n$ is uniformly porous in the sense that $\mathrm{por}(A, x, r) \ge \alpha > 0$ for all $x \in A$ and for all $0 < r < r_0$, then the upper Minkowski dimension of A is less than n. Indeed, let $k \in \mathbf{N}$ be such that $k^{-1}\sqrt{n} < \alpha$. Then every k-adic cube of side length less than r_0 contains a k-adic cube of the next generation which does not intersect the set A. This immediately gives that $\dim_M(A) < n$. If $\mathrm{por}(A) \ge \alpha > 0$ one can represent A as a countable union of uniformly α-porous sets. Thus, one obtains a non-trivial upper bound for $\dim_p(A)$ from the above argument. Although this is a very crude estimate, it is amazingly good when α is close to zero. It gives the best possible asymptotic behavior in the limit as α tends to zero modulo a logarithmic term. For more discussion on this issue, see [9]. From now on, I concentrate on the asymptotic behavior as α tends to its maximum value $\frac{1}{2}$.

It is not difficult to see that the above brute force method cannot give very good estimates if the porosity is large. Therefore, something else has to be developed. The first result in this direction is due to Mattila [12]. Using upper conical densities of sets, he proved that there exists a decreasing function $b : (0, \frac{1}{2}) \to (n-1, n)$ such that $\dim_H(A) \le b(\mathrm{por}(A))$ and $\lim_{\alpha \to \frac{1}{2}} b(\alpha) = n - 1$. Later, Salli [14] extended this result to packing dimension using different, more direct methods. He even found the best possible asymptotic behavior of the function b. Indeed, he proved that there exists a constant $C > 0$ such that

$$\dim_p(A) \le n - 1 + \frac{C}{\log(1/(1-2\alpha))}$$

if $\mathrm{por}(A) \ge \alpha$. The bound is best possible in the sense that there exists $c \le C$ such that for all $\alpha < \frac{1}{2}$ one can construct a compact set A_α with $\mathrm{por}(A_\alpha) = \alpha$ and

$$\dim_p(A_\alpha) = n - 1 + \frac{c}{\log(1/(1-2\alpha))}.$$

Hence, Salli's results give an almost complete description of the dimensional behavior of porous sets.

I mentioned earlier that the upper porosity is not strong enough to help us conclude anything about the dimension, that is, there are sets with full dimension which contain large holes around every point at arbitrarily small scales. The definition of porosity guarantees that there are holes around every point at all small scales. One may ask if it is really necessary to have holes at all small scales or if it would be enough to have holes at many scales in order to obtain non-trivial upper bounds for the dimension. This is indeed the case. To make this precise, the following definition is needed.

Definition 2. *Let* $0 \le \alpha \le \frac{1}{2}$ *and* $0 \le p \le 1$. *A set* $A \subset \mathbf{R}^n$ *is mean* (α, p)*-porous if*

$$\liminf_{i\to\infty} \frac{\#\{1 \le j \le i \mid \mathrm{por(A, x, 2^{-j})} \ge \alpha\}}{i} \ge p$$

for all $x \in A$. *Here the number of the elements in a set* B *is denoted by* $\#B$.

According to Definition 2, in a mean porous set a certain percentage of scales is porous. The dimensional behavior of mean porous sets has been studied by Koskela and Rohde [11] (with a slightly different definition) in the case of small mean porosity and by Beliaev and Smirnov for both small and large porosities. In [2] Beliaev and Smirnov proved that there exists $C > 0$ such that for all mean (α, p)-porous sets $A \subset \mathbf{R}^n$

$$\dim_p(A) \le n - p + \frac{C}{\log(1/(1-2\alpha))}.$$

Since a hyperplane is $\frac{1}{2}$-porous, it is clear that one cannot obtain anything better than $n-1$ as the upper bound. To get something smaller, one may assume that a set contains holes in several different directions. This leads to the concept of k-porosity. In [10] an asymptotically sharp upper bound

$$\dim_p(A) \le n - k + \frac{C}{\log(1/(1-2\alpha))}$$

was established for k-porous sets $A \subset \mathbf{R}^n$.

2 Porous Measures

In the previous section I described the dimensional behavior of porous sets in $\mathbf{R}^n$. In many applications it is more convenient to study measures than sets. For example, in the measurements concerning the dimension of the galaxy distribution of the universe, it is more practical to interpret the data as a measure.

Thus, one may omit the irrelevant dust between galaxies, measurement errors, and unreliable measurements. For more information on this issue, see [6]. The discussion related to this problem led to the following definition of the porosity of a measure.

Definition 3. *Let μ be a finite Radon measure on $\mathbf{R}^n$. Define*

$$\mathrm{por}(\mu, \mathrm{x}, \mathrm{r}, \varepsilon) = \sup \left\{ \alpha \geq 0 : \exists\ y \quad \begin{array}{l} \textit{such that} \quad B(y, \alpha r) \subset B(x, r) \\ \textit{and} \quad \mu(B(y, \alpha r)) \leq \varepsilon \mu(B(x, r)) \end{array} \right\}.$$

The porosity of μ at x is

$$\mathrm{por}(\mu, \mathrm{x}) = \lim_{\varepsilon \to 0} \liminf_{\mathrm{r} \to 0} \mathrm{por}(\mu, \mathrm{x}, \mathrm{r}, \varepsilon)$$

and the porosity of μ is

$$\mathrm{por}(\mu) = \mu\text{-ess sup } \mathrm{por}(\mu, \mathrm{x}).$$

As mentioned earlier, in order to find non-trivial upper bounds for dimensions, it is not enough to have holes only at arbitrarily small scales. Thus, we have the lower limit in the definition of $\mathrm{por}(\mu, \mathrm{x})$. The parameter ε is the threshold to ignore the irrelevant "dust." In the definition of $\mathrm{por}(\mu, \mathrm{x})$, the order of the limits is crucial. If we first let the threshold go to zero and after that let r tend to zero, we will obtain the porosity of the support of μ. Unlike the definition of the porosity of a set, we have essential supremum instead of essential infimum in the definition of $\mathrm{por}(\mu)$. The reason for this is that the (lower) Hausdorff or packing dimension of a measure is defined using sets of positive measure, that is,

$$\dim(\mu) = \inf\{\dim(A) \mid A \text{ is a Borel set with } \mu(A) > 0\},$$

where dim is either $\dim_H$ or $\dim_p$. If one wants to estimate the upper dimensions of μ where the condition $\mu(A) > 0$ is replaced by $\mu(\mathbf{R}^n \setminus A) = 0$, then one has to replace ess sup by ess inf in the definition of $\mathrm{por}(\mu)$.

For sets it is obvious that $0 \leq \mathrm{por}(\mathrm{A}, \mathrm{x}) \leq \frac{1}{2}$ for all $x \in A$. In [5] it is shown that for measures one also has $0 \leq \mathrm{por}(\mu, \mathrm{x}) \leq \frac{1}{2}$ for μ-almost all $x \in \mathbf{R}^n$. Thus, it is reasonable to ask whether similar dimension bounds as for porous sets are valid for porous measures. This is indeed the case. In [5] it is shown that there exists a constant $C > 0$ such that for all doubling Radon measures μ with $\mathrm{por}(\mu) \geq \alpha$ we have

$$\dim_p(\mu) \leq n - 1 + \frac{C}{\log(1/(1 - 2\alpha))}.$$

Here a measure μ is doubling if

$$\limsup_{r \to 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} < \infty$$

for μ-almost all $x \in \mathbf{R}^n$. In [7] this result is extended for non-doubling measures. The argument in that paper works only for the Hausdorff dimension and not for the packing dimension as stated. This point is clarified in [8].

In terms of porosities, there is an essential difference between doubling and non-doubling measures. Hausdorff and packing dimensions of measures may be defined using local dimensions, that is,

$$\dim_H(\mu) = \mu\text{-ess inf } \underline{\dim}_{loc}(\mu, x) \text{ and } \dim_p(\mu) = \mu\text{-ess inf } \overline{\dim}_{loc}(\mu, x).$$

Here the lower and upper local dimensions are

$$\underline{\dim}_{loc}(\mu, x) = \liminf_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r}$$

and

$$\overline{\dim}_{loc}(\mu, x) = \limsup_{r \to 0} \frac{\log(\mu(B(x, r)))}{\log r}.$$

These definitions are equivalent to the ones given in terms of dimensions of sets. It is natural to ask whether the porosity of μ could be defined using porosities of sets, that is, is it true that

$$\text{por}(\mu) = \sup\{\text{por(A)} \mid \mu\text{(A)} > 0\}?$$

For doubling measures this is the case, as proven in [5]. However, there are non-doubling measures for which this is not true (see [7]).

3 Mean Porous Measures

It is well known that for a non-doubling measure most scales are doubling (for the exact statement see [3]). Since in the example mentioned at the end of the previous section the difference of porosities of sets and measures is due to the non-doubling scales, it is reasonable to expect that this problem may be circumvented by studying mean porous measures defined as follows.

Definition 4. *Let $0 \le \alpha \le \frac{1}{2}$ and $0 \le p \le 1$. A finite Radon measure μ on $\mathbf{R}^n$ is mean (α, p)-porous at $x \in \mathbf{R}^n$ if*

$$\lim_{\varepsilon \to 0} \liminf_{i \to \infty} \frac{\#\{1 \le j \le i \mid \text{por}(\mu, \text{x}, 2^{-\text{j}}, \varepsilon) \ge \alpha\}}{i} \ge p.$$

The measure μ is mean (α, p)-porous if there exists $A \subset \mathbf{R}^n$ with $\mu(A) > 0$ such that μ is mean (α, p)-porous at all $x \in A$.

With this definition one may ask the following question: if μ is mean (α, p)-porous, is there for all $\alpha' < \alpha$ and $p' < p$ a mean (α', p')-porous set A with $\mu(A) > 0$? This natural question was one of the central ideas in [2] when

estimating dimensions of porous measures. Quite surprisingly, the answer to this question is negative. Indeed, in [1, Theorem 4.1] we construct an example of a Radon probability measure μ on $\mathbf{R}^n$ such that μ is mean $(\alpha, 1)$-porous for all $0 \leq \alpha < \frac{1}{2}$ but

$$\sup\{\alpha \geq 0 \mid A \text{ mean } (\alpha, p)\text{-porous}, \mu(A) > 0\} = 0$$

for all $0 < p \leq 1$. This example indicates that one cannot estimate mean porous measures by mean porous sets. However, this does not imply that dimensions of mean porous measures cannot be estimated from above. It only implies that if there exists a non-trivial upper bound the proof must be based purely on measures. This is indeed the case. Our main theorem is as follows.

Theorem 1. *Let μ be a finite Radon measure on $\mathbf{R}^n$, $0 < \alpha \leq \frac{1}{2}$ and $0 < p \leq 1$. There exists $C > 0$ such that if μ is mean (α, p)-porous, then*

$$\dim_p(\mu) \leq n - p + \frac{C}{\log(1/(1-2\alpha))}.$$

For each fixed $p > 0$ the bound is asymptotically sharp as α tends to $\frac{1}{2}$.

The complete proof of Theorem 1 is given in [1, Theorem 3.1]. I will only sketch the main ideas. To obtain some understanding of why the above bound is correct, I consider first a simplified situation for a mean porous set $A \subset \mathbf{R}^n$. I need the following elementary lemma, which is proven in [1, Lemma 3.2].

Lemma 1. *Let $k \in \mathbf{N}$ and let $Q \subset \mathbf{R}^n$ be a dyadic cube. If $B_1, B_2, \ldots, B_m$ are closed balls with radii at least $\sqrt{n} r_Q$, then $\partial(Q \setminus \bigcup_{i=1}^m B_i)$ may be covered by $c2^{k(n-1)}$ dyadic cubes of side length $2^{-k} r_Q$. Here c is a positive and finite constant depending only on n, and r_Q is the side length of Q.*

If all the points of A inside Q are $\frac{1}{2}$-porous at a scale comparable to r_Q, then they are on the boundary of some hole which is represented by the balls $B_1, B_2, \ldots, B_m$ in Lemma 1. Thus, $Q \cap A$ may be covered by $2^{k(n-1)}$ cubes with side length $2^{-k} r_Q$. If $Q \cap A$ is not porous at scale r_Q, one needs 2^{kn} cubes with side length $2^{-k} r_Q$ to cover $Q \cap A$. Iterating this argument l times, one finds pl porous scales and $(1-p)l$ non-porous scales, which implies that one needs $2^{plk(n-1)+(1-p)lkn}$ cubes with side length 2^{-kl} to cover A. This in turn gives the following upper bound for the Minkowski dimension of A:

$$\dim_M(A) \leq \frac{\log 2^{plk(n-1)+(1-p)lkn}}{\log 2^{kl}} = n - p.$$

Hence, the term $n-p$ comes from the fact that the percentage of porous scales is p.

To understand the correction term, assume that $p = 1$. Since the balls in Lemma 1 are the holes given by porosity, the thickness of the boundary layer

around them where all the porous points are is $(1-2\alpha)r$, where r is the scale where the porosity is used. Thus, one needs $C2^{k(n-1)}\frac{(1-2\alpha)r}{2^{-k}}$ cubes of side length 2^{-k} to cover A, resulting in

$$\dim_M(A) \leq \frac{\log\left(C2^{k(n-1)}\frac{(1-2\alpha)r}{2^{-k}}\right)}{\log 2^k}.$$

Note that even though the boundary layer could be arbitrarily thin, one needs at least one cube to cover each point on the boundary. Thus, in the above estimate we have to assume that $\frac{(1-2\alpha)r}{2^{-k}} \geq 1$, resulting in $r \approx \frac{2^{-k}}{(1-2\alpha)}$. Furthermore, since $p = 1$ we may look at the situation at the starting scale $r \approx 1$. This implies that the optimal k is given by the formula $\log 2^k = \log \frac{1}{1-2\alpha}$, and therefore,

$$\dim_M(A) \leq n - 1 + \frac{\log C}{\log(1/(1-2\alpha)}.$$

Let us now consider the actual case of porous measures. The starting point is the following lemma [1, Lemma 3.5].

Lemma 2. *Let μ be a finite Radon measure on $\mathbf{R}^n$. Given $\varepsilon > 0$, there exists $C > 0$ such that any cube $Q \subset \mathbf{R}^n$ can be divided into three parts,*

$$Q = P \cup E \cup J,$$

where P is approximately $(n-1)$-dimensional, $\mu(E) \leq \varepsilon\mu(CQ)$, and J contains those points where μ is not porous. Here CQ is the cube with the same center as Q, but the side length is multiplied by C.

In Lemma 2 the set P is the boundary layer of the holes and E is the union of these holes. The proof is not too difficult, but one has to be careful with some issues related to constants. The next lemma [1, Lemma 3.2] is another important tool. The proof is a straightforward application of the definition of the packing dimension via local dimensions.

Lemma 3. *Let $m \in \mathbf{N}$ and $D > 0$. Let μ be a finite Radon measure on $\mathbf{R}^n$. Assume that for all $Q \in \mathcal{Q}$ there is $0 < \tau(Q) < D$ such that*

$$S := \sum_{Q \in \mathcal{Q}} r_Q^{\tau(Q)} \mu(Q)^{1-\frac{\tau(Q)}{D}} < \mu(\mathbf{R}^n)$$

for all collections $\mathcal{Q}$ of 2^m-adic cubes. Then

$$\dim_p(\mu) \leq D.$$

To prove Theorem 1 it is enough to show that the assumption of Lemma 3 is valid for all $D > n - p + \frac{C}{\log(1/(1-2\alpha))}$. This is the essential part of the proof.

There are two main problems. The first problem is that $\mu(E)$ in Lemma 2 is small compared to the measure of a neighborhood of Q and not compared to the measure of Q and the cubes in $Q \in \mathcal{Q}$ need not be of the same size. The other problem is that the measure of J in Lemma 2 is not necessarily small.

The solutions of these problems are quite technical, but I will try to explain the simplified ideas.

The neighborhood problem may be solved with the following procedure. Divide Q into the inner part and the boundary part. The inner part will not be problematic, since the neighborhood of a small cube in the inner part is still inside Q. In the boundary part one iterates this argument going into smaller and smaller scales. This heuristic is formalized in [1, Lemma 3.6].

The latter problem is handled with the following trick. We attach a weight for each Q in a way such that the weight is large when Q is porous. Then we multiply S by a large number and divide the multiplying factor into weights which are distributed in a suitable manner among the Q's. Finally, we show that the weighted sum converges, implying that S is small. This is done formally in [1, Lemmas 3.6 and 3.7]. The argument shows that when summing over porous cubes we obtain a small factor which decreases exponentially under iteration.

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On Upper Conical Density Results

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Summary. We report a recent development on the theory of upper conical densities. More precisely, we look at what can be said in this respect for other measures than just the Hausdorff measure. We illustrate the methods involved by proving a result for the packing measure and for a purely unrectifiable doubling measure.

1 Introduction

Conical density theorems are used in geometric measure theory to derive geometric information from given metric information. Classically, they deal with the distribution of the Hausdorff measure. The main applications of upper conical density results concern rectifiability and porosity. The extensive study of upper conical densities was pioneered by Besicovitch [1], who studied the conical density properties of purely 1-unrectifiable sets in the plane. Besides Besicovitch, the theory of upper conical densities has been developed by Morse and Randolph [16], Marstrand [13], and Federer [7].

2 Notation and Preliminaries

Let $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, and $G(n, n-m)$ denote the space of all $(n-m)$-dimensional linear subspaces of $\mathbf{R}^n$. The unit sphere of $\mathbf{R}^n$ is denoted by S^{n-1}. For $x \in \mathbf{R}^n$, $\theta \in S^{n-1}$, $0 < \alpha \leq 1$, $r > 0$, and $V \in G(n, n-m)$, we set

$$\begin{aligned} H(x, \theta, \alpha) &= \{y \in \mathbf{R}^n : (y-x) \cdot \theta > \alpha |y-x|\}, \\ X(x, V, \alpha) &= \{y \in \mathbf{R}^n : \operatorname{dist}(y-x, V) < \alpha |y-x|\}, \\ X(x, r, V, \alpha) &= B(x, r) \cap X(x, V, \alpha), \end{aligned}$$

where $B(x, r)$ is the closed ball centered at x with radius r. We also denote $B_V(x, r) = \operatorname{proj}_V\big(B(x, r)\big)$, where proj_V is the orthogonal projection onto V.

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By a measure we will always mean a locally finite nontrivial Borel regular (outer) measure defined on all subsets of $\mathbf{R}^n$. We use the notation $\mathcal{H}^s$ and $\mathcal{P}^s$ to denote the s-dimensional Hausdorff and packing measure, respectively. Consult [15, Sects. 4 and 5.10]. We follow the convention according to which $c = c(\cdots) > 0$ denotes a constant depending only on the parameters listed inside the parentheses.

If $A \subset \mathbf{R}^n$ is a Borel set with $0 < \mathcal{H}^s(A) < \infty$, then [15, Theorem 6.2(1)] implies that for $\mathcal{H}^s$-almost every $x \in A$ there are arbitrary small radii r so that $\mathcal{H}^s\big(A \cap B(x,r)\big)$ is proportional to r^s. So we know roughly how much of A there is in such small balls $B(x,r)$. But we would also like to know how the set A is distributed there. The following three upper conical density theorems give information on how much A there is near $(n-m)$-planes in the sense of the Hausdorff measure.

Theorem 1 ([18, Theorem 3.1]). *If $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, $m < s \leq n$, and $0 < \alpha \leq 1$, then there is a constant $c = c(n,m,s,\alpha) > 0$ satisfying the following: For every $A \subset \mathbf{R}^n$ with $\mathcal{H}^s(A) < \infty$ and for each $V \in G(n, n-m)$ it holds that*

$$\limsup_{r \downarrow 0} \frac{\mathcal{H}^s\big(A \cap X(x,r,V,\alpha)\big)}{(2r)^s} \geq c$$

for $\mathcal{H}^s$-almost every $x \in A$.

Remark 1. The role of the assumption $s > m$ in the above theorem is to guarantee that the set A is scattered enough. When $s \leq m$, it might happen that $A \in V^{\perp}$ for some $V \in G(n, n-m)$. Furthermore, if the set A is m-rectifiable with $\mathcal{H}^m(A) < \infty$, then it follows from [15, Theorem 15.19] that the result of Theorem 1 cannot hold. On the other hand, if the set A is purely m-unrectifiable with $\mathcal{H}^m(A) < \infty$, then the result of Theorem 1 holds, see [15, Corollary 15.16]. We refer the reader to [15, Sect. 15] for the basic properties of rectifiable and purely unrectifiable sets. See also Sect. 5.

Theorem 2 ([14, Theorem 3.3]). *If $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, $m < s \leq n$, and $0 < \alpha \leq 1$, then there is a constant $c = c(n,m,s,\alpha) > 0$ satisfying the following: For every $A \subset \mathbf{R}^n$ with $\mathcal{H}^s(A) < \infty$ it holds that*

$$\limsup_{r \downarrow 0} \inf_{V \in G(n,n-m)} \frac{\mathcal{H}^s\big(A \cap X(x,r,V,\alpha)\big)}{(2r)^s} \geq c$$

for $\mathcal{H}^s$-almost every $x \in A$.

The above theorem is a significant improvement of Theorem 1. It shows that in the sense of the Hausdorff measure, there are arbitrary small scales so that almost all points of A are well surrounded by A. Theorem 2 is actually applicable for more general symmetric cones. More precisely, the set $X(x,r,V,\alpha)$ can be replaced by $C_x \cap B(x,r)$, where $C_x = \bigcup_{V \in C}(V + x)$,

$C \subset G(n, n-m)$ is a Borel set with $\gamma(C) > \delta > 0$, and γ is the natural-isometry-invariant measure on $G(n, n-m)$. The infimum is then taken over all such sets C. The proof of Theorem 2 (and its more general formulation) is nontrivial, and it is based on Fubini-type arguments and an elegant use of what are called the sliced measures. The following theorem gives Theorem 2 a more elementary proof. The technique used there does not require the cones to be symmetric.

Theorem 3 ([11, Theorem 2.5]). *If $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, $m < s \leq n$, and $0 < \alpha \leq 1$, then there is a constant $c = c(n, m, s, \alpha) > 0$ satisfying the following: For every $A \subset \mathbf{R}^n$ with $\mathcal{H}^s(A) < \infty$ it holds that*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mathcal{H}^s\big(A \cap X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)\big)}{(2r)^s} \geq c$$

for $\mathcal{H}^s$-almost every $x \in A$.

The main application of this theorem is porosity. By porous sets we mean sets which have holes on every small scale. For a precise definition of a porous set and the connection between porosity and upper conical densities, the reader is referred to [15, Theorem 11.14], [11, Theorem 3.2], and [12, Sect. 3]. See [2–4, 6, 8–10, 17] for other related results.

We will now look at what kind of upper conical density results can be proven for other measures.

3 Packing Type Measures

The following result is Theorem 3 for the packing measure. A more general formulation can be found in [12]. To our knowledge, it is the first upper conical density result for other measures than the Hausdorff measure.

Theorem 4. *If $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, $m < s \leq n$, and $0 < \alpha \leq 1$, then there is a constant $c = c(n, m, s, \alpha) > 0$ satisfying the following: For every $A \subset \mathbf{R}^n$ with $\mathcal{P}^s(A) < \infty$ it holds that*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n, n-m)}} \frac{\mathcal{P}^s\big(A \cap X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)\big)}{(2r)^s} \geq c$$

for $\mathcal{P}^s$-almost every $x \in A$.

Proof. Fix $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, and $0 < \alpha \leq 1$. Observe that $G(n, n-m)$ endowed with the metric $d(V, W) = \sup_{x \in V \cap S^{n-1}} \operatorname{dist}(x, W)$ is a compact metric space and

$$\bigcup_{d(V,W) < \alpha} \{x : x \in W\} = X(0, V, \alpha) \tag{1}$$

for all $V \in G(n, n-m)$, see [18, Lemma 2.2]. Using the compactness, we may thus choose $K = K(n, m, \alpha) \in \mathbb{N}$ and $(n-m)$-planes $V_1, \ldots, V_K$ so that for each $V \in G(n, n-m)$ there is $j \in \{1, \ldots, K\}$ with

$$X(x, V, \alpha) \supset X(x, V_j, \alpha/2).$$

Let $t = \max\{t(\alpha/2), 1 + 6/\alpha\} \geq 1$, where $t(\alpha/2)$ is as in [5, Lemma 4.3] and take $q = q(n, \alpha/(2t))$ from [5, Lemma 4.2]. If $\lambda > 0$ and $\{B(y_i, t\lambda)\}_{i=1}^q$ is a collection of pairwise disjoint balls centered at $B(0,1)$, then it follows from the above-mentioned lemmas that there exists $i_0 \in \{1, \ldots, q\}$ such that for every $\theta \in S^{n-1}$ there is $k \in \{1, \ldots, q\}$ for which

$$B(y_k, \lambda) \subset B(y_{i_0}, 3) \setminus H(y_{i_0}, \theta, \alpha). \tag{2}$$

In fact, there are three center points that form a large angle. The choice of t also implies that for every $y, y_0 \in \operatorname{proj}_{V^\perp}^{-1}\big(B_{V^\perp}(0,\lambda)\big)$ with $|y - y_0| \geq t\lambda$ we have

$$B(y, \lambda) \subset X(y_0, V, \alpha/2). \tag{3}$$

Let $c_1 = c_1(m)$ and $c_2 = c_2(n)$ be such that the set $B_{V^\perp}(0,1)$ may be covered by $c_1\lambda^{-m}$ and the set $\operatorname{proj}_{V^\perp}^{-1}\big(B_{V^\perp}(0,\lambda)\big) \cap B(0,1)$ may be covered by $c_2\lambda^{m-n}$ balls of radius λ for all $V \in G(n, n-m)$. Finally, fix $m < s \leq n$ and set $\lambda = \lambda(n, m, s, \alpha) = \min\{2^{-1}t^{s/(m-s)}d^{1/(s-m)}, (3t)^{-1}\} > 0$, where $d = d(n, m, \alpha) = 1/(2c_1K(q-1))$.

It suffices to show that if $c = \lambda^n/(3^s 4 c_1 c_2 K) > 0$ and $A \subset \mathbf{R}^n$ with $\mathcal{P}^s(A) < \infty$, then

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ j \in \{1, \ldots, K\}}} \frac{\mathcal{P}^s\big(A \cap X(x, r, V_j, \alpha/2) \setminus H(x, \theta, \alpha)\big)}{(2r)^s} \geq c$$

for $\mathcal{P}^s$-almost every $x \in A$. Assume to the contrary that there are $r_0 > 0$ and a closed set $A \subset \mathbf{R}^n$ with $0 < \mathcal{P}^s(A) < \infty$ so that for every $x \in A$ and $0 < r < r_0$ there exist $j \in \{1, \ldots, K\}$ and $\theta \in S^{n-1}$ such that

$$\mathcal{P}^s\big(A \cap X(x, r, V_j, \alpha/2) \setminus H(x, \theta, \alpha)\big) < c(2r)^s.$$

Recalling [15, Theorem 6.10], we may further assume that

$$\liminf_{r \downarrow 0} \frac{\mathcal{P}^s\big(A \cap B(x, r)\big)}{(2r)^s} = 1 \tag{4}$$

for all $x \in A$. Pick $x_0 \in A$ and choose $0 < r' < r_0$ so that $\mathcal{P}^s\big(A \cap B(x_0, r')\big) \geq (2r')^s/2$. For notational simplicity, we assume that $r' = 1$ and $r_0 > 3$. Since $A = \bigcup_{j=1}^K A_j$, where

$$A_j = \{x \in A : \mathcal{P}^s\big(A \cap X(x, 3, V_j, \alpha/2) \setminus H(x, \theta, \alpha)\big) < c6^s \text{ for some } \theta\}, \tag{5}$$

we find $j \in \{1, \dots, K\}$ for which $\mathcal{P}^s\big(A_j \cap B(x_0, 1)\big) \geq 2^s/(2K)$. Going into a subset, if necessary, we may assume that $A_j \cap B(x_0, 1)$ is compact. Moreover, we may cover the set $B_{V_j^\perp}(x_0, 1)$ by $c_1\lambda^{-m}$ balls of radius λ. Hence,

$$\mathcal{P}^s\big(A_j \cap \mathrm{proj}_{V_j^\perp}^{-1}\big(B_{V_j^\perp}(y', \lambda)\big) \cap B(x_0, 1)\big) \geq \lambda^m 2^s/(2c_1K) \tag{6}$$

for some $y' \in V_j^\perp$. Next we choose q pairwise disjoint balls $\{B(y_i, t\lambda)\}_{i=1}^q$ centered at $A_j' = A_j \cap \mathrm{proj}_{V_j^\perp}^{-1}\big(B_{V_j^\perp}(y', \lambda)\big) \cap B(x_0, 1)$ so that for each $i \in \{1, \dots, q\}$ it holds that $\mathcal{P}^s\big(A \cap B(y_i, \lambda)\big) \geq \mathcal{P}^s\big(A \cap B(y, \lambda)\big)$ for all $y \in A_j' \setminus \bigcup_{k=1}^{i-1} U(y_k, t\lambda)$, where $U(x, r)$ denotes the open ball. This can be done since the set A_j' is compact and the function $y \mapsto \mathcal{P}^s\big(A \cap B(y, \lambda)\big)$ is upper semicontinuous. The set A_j' can be covered by $c_2\lambda^{m-n}$ balls of radius λ, whence

$$c_2\lambda^{m-n}\mathcal{P}^s\big(A \cap B(y_q, \lambda)\big) \geq \lambda^m 2^s/(2c_1K) - \sum_{i=1}^{q-1} \mathcal{P}^s\big(A \cap B(y_i, t\lambda)\big) \tag{7}$$

by recalling (6). Now (2), (3), and (5) give

$$\mathcal{P}^s\big(A \cap B(y_q, \lambda)\big) \leq \mathcal{P}^s\big(A \cap X(y_{i_0}, 3, V_j, \alpha/2) \setminus H(y_{i_0}, \theta, \alpha)\big) < c6^s,$$

and consequently,

$$\sum_{i=1}^{q-1} \mathcal{P}^s\big(A \cap B(y_i, t\lambda)\big) \geq \lambda^m 2^s/(2c_1K) - c6^s c_2\lambda^{m-n} = (q-1)2^s d\lambda^m/2$$

by (7) and the choices of c and d. Hence, $\mathcal{P}^s\big(A \cap B(x_1, t\lambda)\big) \geq 2^s d\lambda^m/2$ for some $x_1 \in \{y_1, \dots, y_{q-1}\} \subset A \cap B\big(x_0, 1\big)$. Recall that $\mathcal{P}^s\big(A \cap B(x_0, 1)\big) \geq 2^s/2$. Repeating now the above argument in the ball $B(x_1, t\lambda)$, we find a point $x_2 \in A \cap B(x_1, t\lambda)$ so that $\mathcal{P}^s\big(A \cap B(x_2, (t\lambda)^2)\big) \geq 2^s d^2\lambda^{2m}/2$. Continuing in this manner, we find for each $k \in \mathbb{N}$ a ball $B(x_k, (t\lambda)^k)$ centered at $A \cap B(x_{k-1}, (t\lambda)^{k-1})$ so that $\mathcal{P}^s\big(A \cap B(x_k, (t\lambda)^k)\big) \geq 2^s d^k\lambda^{km}/2$.

Now for the point $z \in A$ determined by $\{z\} = \bigcap_{k=0}^\infty B\big(x_k, (t\lambda)^k\big)$, we have

$$\begin{aligned}
\liminf_{r \downarrow 0} \frac{\mathcal{P}^s\big(A \cap B(z, r)\big)}{(2r)^s} &\geq \liminf_{k \to \infty} \frac{\mathcal{P}^s\big(A \cap B(x_{k+1}, (t\lambda)^{k+1})\big)}{2^s(t\lambda)^{(k-1)s}} \\
&\geq \liminf_{k \to \infty} \frac{d^{k+1}\lambda^{(k+1)m}}{2^s(t\lambda)^{(k-1)s}} \\
&= \liminf_{k \to \infty} 2^{-s} d^2\lambda^{2m}(d\lambda^{m-s}t^{-s})^{k-1} \\
&\geq \liminf_{k \to \infty} 2^{-s} d^2\lambda^{2m} 2^{(s-m)(k-1)} = \infty,
\end{aligned}$$

since $t\lambda \leq 1/3$, $s > m$ and $\lambda^{m-s} \geq 2^{s-m}t^s d^{-1}$. This contradicts (4). The proof is finished. □

The above result is a special case of the following more general result.

Theorem 5 ([12, Theorem 2.4]). *If $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, $0 < \alpha \leq 1$, and a nondecreasing function $h\colon (0,\infty) \to (0,\infty)$ satisfies*

$$\limsup_{r \downarrow 0} \frac{h(\gamma r)}{h(r)} < \gamma^m \tag{8}$$

for some $0 < \gamma < 1$, then there is a constant $c = c(n, m, h, \alpha) > 0$ satisfying the following: For every measure μ on $\mathbf{R}^n$ with

$$\liminf_{r \downarrow 0} \frac{\mu\big(B(x,r)\big)}{h(2r)} < \infty \quad \text{for } \mu\text{-almost all } x \in \mathbf{R}^n$$

it holds that

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mu\big(X(x,r,V,\alpha) \setminus H(x,\theta,\alpha)\big)}{h(2r)} \geq c \limsup_{r \downarrow 0} \frac{\mu\big(B(x,r)\big)}{h(2r)}$$

for μ-almost every $x \in \mathbf{R}^n$.

Remark 2. If instead of (8), the function $h\colon (0,\infty) \to (0,\infty)$ satisfies

$$\liminf_{r \downarrow 0} \frac{h(\gamma r)}{h(r)} \geq \gamma^m$$

for all $0 < \gamma < 1$, then [12, Proposition 3.3] implies that the result of Theorem 5 cannot hold. This shows that Theorem 5 fails for gauge functions such as $h(r) = r^m/\log(1/r)$ when $m > 0$.

4 Measures with Positive Dimension

When working with a Hausdorff or packing type measure μ, it is useful to study densities such as

$$\limsup_{r \downarrow 0} \frac{\mu\big(X(x,r,V,\alpha)\big)}{h(2r)},$$

where h is the gauge function used to construct the measure μ. However, most measures are so unevenly distributed that there are no gauge functions that could be used to approximate the measure in small balls. To obtain conical density results for general measures, it seems natural to replace the value of the gauge h in the denominator by the measure of the ball $B(x,r)$.

The following result is valid for all measures on $\mathbf{R}^n$.

Theorem 6 ([5, Theorem 3.1]). *If $n \in \mathbb{N}$ and $0 < \alpha \leq 1$, then there is a constant $c = c(n, \alpha) > 0$ satisfying the following: For every measure μ on $\mathbf{R}^n$ it holds that*

$$\limsup_{r \downarrow 0} \inf_{\theta \in S^{n-1}} \frac{\mu\big(B(x,r) \setminus H(x,\theta,\alpha)\big)}{\mu\big(B(x,r)\big)} \geq c$$

for μ-almost every $x \in \mathbf{R}^n$.

By assuming a lower bound for the Hausdorff dimension of the measure, the measure will be scattered enough so that we are able to prove a result similar to Theorem 3 for general measures. The *(lower) Hausdorff and packing dimensions of a measure μ* are defined by

$$\dim_{\mathrm{H}}(\mu) = \inf\{\dim_{\mathrm{H}}(\mathrm{A}) : \mathrm{A} \text{ is a Borel set with } \mu(\mathrm{A}) > 0\},$$
$$\dim_{\mathrm{p}}(\mu) = \inf\{\dim_{\mathrm{p}}(A) : A \text{ is a Borel set with } \mu(A) > 0\},$$

where $\dim_{\mathrm{H}}(\mathrm{A})$ and $\dim_{\mathrm{p}}(A)$ denote the Hausdorff and packing dimensions of the set $A \subset \mathbf{R}^n$, respectively. The reader is referred to [15, Sects. 4 and 5.9].

Theorem 7 ([5, Theorem 4.1]). *If $n \in \mathbb{N}$, $m \in \{0, \ldots, n-1\}$, $m < s \leq n$, and $0 < \alpha \leq 1$, then there is a constant $c = c(n, m, s, \alpha) > 0$ satisfying the following: For every measure μ on $\mathbf{R}^n$ with $\dim_H(\mu) \geq \mathrm{s}$ it holds that*

$$\limsup_{r \downarrow 0} \inf_{\substack{\theta \in S^{n-1} \\ V \in G(n,n-m)}} \frac{\mu\big(X(x,r,V,\alpha) \setminus H(x,\theta,\alpha)\big)}{\mu\big(B(x,r)\big)} \geq c$$

for μ-almost every $x \in \mathbf{R}^n$.

Question 1. Does Theorem 7 hold if we just assume $\dim_{\mathrm{p}}(\mu) \geq s$ instead of $\dim_{\mathrm{H}}(\mu) \geq \mathrm{s}$?

5 Purely Unrectifiable Measures

Another condition to guarantee the measure to be scattered enough is unrectifiability. A measure on $\mathbf{R}^n$ is called *purely m-unrectifiable* if $\mu(A) = 0$ for all m-rectifiable sets $A \subset \mathbf{R}^n$. We refer the reader to [15, Sect. 15] for the basic properties of rectifiable sets. Applying the ideas of [15, Lemma 15.14], we are able to prove the following theorem.

Theorem 8. *If $d > 0$ and $0 < \alpha \leq 1$, then there is a constant $c = c(d, \alpha) > 0$ satisfying the following: For every $n \in \mathbb{N}$, $m \in \{1, \ldots, n-1\}$, $V \in G(n, n-m)$, and purely m-unrectifiable measure μ on $\mathbf{R}^n$ with*

$$\limsup_{r \downarrow 0} \frac{\mu\big(B(x,2r)\big)}{\mu\big(B(x,r)\big)} < d \quad \text{for } \mu\text{-almost all } x \in \mathbf{R}^n \tag{9}$$

it holds that

$$\limsup_{r\downarrow 0} \frac{\mu\big(X(x,r,V,\alpha)\big)}{\mu\big(B(x,r)\big)} \geq c \tag{10}$$

for μ-almost every $x \in \mathbf{R}^n$.

Proof. Fix $d > 0$ and $0 < \alpha \leq 1$. Observe that there exists a constant $b = b(d,\alpha) > 0$ such that any measure μ satisfying (9) fulfills

$$\limsup_{r\downarrow 0} \frac{\mu\big(B(x,3r)\big)}{\mu\big(B(x,\alpha r/20)\big)} < b$$

for μ-almost all $x \in \mathbf{R}^n$.

We will prove that (10) holds with $c = (4bd)^{-1}$. Assume to the contrary that for some $n \in \mathbb{N}$, $m \in \{1,\ldots,n-1\}$, $V \in G(n,n-m)$, and purely m-unrectifiable measure μ satisfying (9) there exist $r_0 > 0$ and a Borel set $A \subset \mathbf{R}^n$ with $\mu(A) > 0$ so that

$$\mu\big(X(x,2r,V,\alpha)\big) < c\mu\big(B(x,2r)\big) \tag{11}$$

for all $0 < r < r_0$ and for every $x \in A$. We may further assume that

$$\mu\big(B(x,2r)\big) \leq d\mu\big(B(x,r)\big), \tag{12}$$

$$\mu\big(B(x,3r)\big) \leq b\mu\big(B(x,\alpha r/20)\big) \tag{13}$$

for all $0 < r < r_0$ and for every $x \in A$.

Recalling [15, Corollary 2.14(1)], we fix $x_0 \in A$ and $0 < r < r_0$ so that

$$\mu\big(A \cap B(x_0,r)\big) > \mu\big(B(x_0,r)\big)/2. \tag{14}$$

For each $x \in A \cap B(x_0,r)$ we define $h(x) = \sup\{|y-x| : y \in A \cap X(x,r,V,\alpha/4)\}$. Since μ is purely m-unrectifiable, it follows from [15, Lemma 15.13] that $h(x) > 0$ for μ-almost all $x \in A \cap B(x_0,r)$. For each $x \in A \cap B(x_0,r)$ with $h(x) > 0$ we choose $y_x \in A \cap X(x,r,V,\alpha/4)$ such that $|y_x - x| > 3h(x)/4$. Inspecting the proof of [15, Lemma 15.14], we see that

$$A \cap \operatorname{proj}_{V^\perp}^{-1}\big(B_{V^\perp}(x,\alpha h(x)/4)\big) \subset X(x,2h(x),V,\alpha) \cup X(y_x,2h(x),V,\alpha) \tag{15}$$

for all $x \in A \cap B(x_0,r)$. Applying the $5r$-covering theorem ([15, Theorem 2.1]) to the collection $\big\{B_{V^\perp}\big(x,\alpha h(x)/4\big) : x \in A \cap B(x_0,r) \text{ with } h(x) > 0\big\}$, we find a countable collection of pairwise disjoint balls $\big\{B_{V^\perp}\big(x_i,\alpha h(x_i)/20\big)\big\}_i$ so that

$$\bigcup_{h(x)>0} \operatorname{proj}_{V^\perp}^{-1}\big(B_{V^\perp}(x,\alpha h(x)/4)\big) \subset \bigcup_i \operatorname{proj}_{V^\perp}^{-1}\big(B_{V^\perp}(x_i,\alpha h(x_i)/4)\big). \tag{16}$$

Now (16), (15), (11), (13), and (12) imply that

$$\begin{aligned}\mu\big(A\cap B(x_0,r)\big) &\le \sum_i \mu\big(A\cap B(x_0,r)\cap \mathrm{proj}_{V^\perp}^{-1}\big(B_{V^\perp}(x_i,\alpha h(x_i)/4)\big)\big)\\ &\le c\sum_i \mu\big(B(x_i,2h(x_i))\big)+c\sum_i \mu\big(B(y_{x_i},2h(x_i))\big)\\ &\le 2c\sum_i \mu\big(B(x_i,3h(x_i))\big)\le 2cb\sum_i \mu\big(B(x_i,\alpha h(x_i)/20)\big)\\ &\le 2cb\mu\big(B(x_0,2r)\big)\le 2cbd\mu\big(B(x_0,r)\big)=\mu\big(B(x_0,r)\big)/2,\end{aligned}$$

that is, a contradiction with (14). The proof is finished. □

Remark 3. Theorem 8 does not hold without the assumption (9), see [5, Example 5.5] for a counterexample. Recall also Remark 1. Observe that one cannot hope to generalize the result by taking the infimum over all $V \in G(n, n-m)$ before taking the lim sup as in Theorem 3. A counterexample follows immediately from [5, Example 5.4] by noting that the set constructed in the example supports a 1-regular measure, that is, a measure giving for each small ball measure proportional to the radius. See also [12, Proposition 3.3 and Remark 3.4] and [12, Question 4.2] for related discussion.

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On the Dimension of Iterated Sumsets

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Summary. Let A be a subset of the real line. We study the fractal dimensions of the k-fold iterated sumsets kA, defined as

$$kA = \{a_1 + \cdots + a_k : a_i \in A\}.$$

We show that for any nondecreasing sequence $\{\alpha_k\}_{k=1}^{\infty}$ taking values in $[0, 1]$, there exists a compact set A such that kA has Hausdorff dimension α_k for all $k \geq 1$. We also show how to control various kinds of dimensions simultaneously for families of iterated sumsets.

These results are in stark contrast to the Plünnecke–Ruzsa inequalities in additive combinatorics. However, for lower box-counting dimensions, the analog of the Plünnecke–Ruzsa inequalities does hold.

1 Introduction and Statement of Results

Given sets A, B in some ambient group, let $A + B = \{a + b : a \in A, b \in B\}$ and $kA = \{a_1 + \cdots + a_k : a_i \in A\}$ be the arithmetic sum of A and B and the k-iterated sum of A, respectively. A general principle in additive combinatorics is that if the arithmetic sum of a finite set A with itself is "small," then the set A itself has "additive structure," and in particular iterated sums and differences such as $A + A + A$ and $A - A$ are also "small." One precise formulation of this principle are the Plünnecke–Ruzsa inequalities, which say that if A, B are two finite subsets of an abelian group and $|A + B| \leq K|A|$, then

$$|nB - mB| \leq K^{n+m}|A|.$$

In particular, taking $B = A$, this result gives a quantitative version of the above principle. The reader is referred to [2] for the precise definitions and statements, as well as general background in additive combinatorics.

In this chapter we investigate whether similar statements can be made when A is a subset of the real numbers (rather than the integers or a discrete

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group), and size is measured by some fractal dimension instead of cardinality. We were motivated in particular by the following question: if the Hausdorff dimension of $A + A$ is equal to the Hausdorff dimension of A, does it follow that the Hausdorff dimension of $A + A + A$ is also equal to the Hausdorff dimension of A? We prove that the answer is negative, in rather dramatic fashion: the sequence $\{\alpha_\ell\}_{\ell=1}^{\infty}$ of Hausdorff dimensions of the iterated sumsets ℓA can be completely arbitrary, subject to the obvious restrictions of being nondecreasing and taking values in $[0, 1]$. In particular, information about the Hausdorff dimension of the sumsets $A, 2A, \ldots, \ell A$ gives no information whatsoever about the Hausdorff dimension of $(\ell + 1)A$, other than the trivial fact that Hausdorff dimension is monotone. Thus, the fractal world exhibits very different behavior than the discrete world.

More generally, we investigate the possible simultaneous values of Hausdorff, lower box-counting, and upper box-counting dimensions of iterated sumsets (the reader is referred to [1] for the definitions and basic properties of Hausdorff and box-counting dimensions). This turns out to be a delicate problem — controlling various dimensions at once is substantially harder than controlling just one of them.

We will denote Hausdorff dimension by $\dim_H$, and lower and upper box dimensions by $\underline{\dim}_B$ and $\overline{\dim}_B$, respectively. The following is our main result.

Theorem 1. *Let $\{\alpha_i\}_{i=1}^{\infty}$, $\{\beta_i\}_{i=1}^{\infty}$, and $\{\gamma_i\}_{i=1}^{\infty}$ be nondecreasing sequences with $0 \le \alpha_i \le \beta_i \le \gamma_i \le 1$,*

$$\beta_\ell \le \beta_{\ell-1} + \beta_1 - \sum_{k=2}^{\ell-1} (\ell - k)(\beta_{k-1} + \beta_1 - \beta_k),$$

and

$$\gamma_\ell \le \gamma_{\ell-1} + \gamma_1 - \sum_{k=2}^{\ell-1} (\ell - k)(\gamma_{k-1} + \gamma_1 - \gamma_k)$$

for all $\ell \ge 2$.

There exists a compact set $A \subset [0, 1]$ such that

$$\dim_H(\ell A) = \alpha_\ell, \quad \underline{\dim}_B(\ell A) = \beta_\ell, \quad \text{and} \quad \overline{\dim}_B(\ell A) = \gamma_\ell$$

for $\ell = 1, 2, \ldots$. Additionally, if $\alpha_\ell = 1$ for some ℓ, we can also require that ℓA contain an interval.

Notice from the above result that, even if a set A has coinciding Hausdorff, lower box, and upper box dimensions, it is possible that for the sumset $A + A$ all three concepts of dimension differ.

Since the construction of the set A in Theorem 1 is rather complicated, rather than giving a full proof, we will present several examples of increasing complexity illustrating different features of the general construction. After these examples we indicate how to put them together to yield Theorem 1. This will be done in Sect. 2.

Theorem 1 does not negate a result in the spirit of Plünnecke–Ruzsa for upper or lower box-counting dimension. In Sect. 3 we will show that there is indeed a natural extension of the Plünnecke–Ruzsa estimates for the lower box dimension (but not for the upper box dimension); see Proposition 1 for the precise quantitative estimates.

2 Examples and Proof of the Main Result

2.1 Basic Facts

Before proving Theorem 1, we will present some simpler but significant examples illustrating the main features of the construction. The construction itself is quite technical and will be sketched at the end of the section.

We will consider the numbers in the unit interval in their base 2 expansion, i.e., to a real number $x \in [0,1]$ we associate a binary infinite sequence $\underline{x} = x_1x_2x_3\cdots$ such that $x_i \in \{0,1\}$ and $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$. This sequence is unique unless x is a dyadic rational, in which case we have exactly two representations. It will be apparent from the constructions that this will not affect the dimension calculations (in the case of Hausdorff dimension this is clear since countable sets have zero Hausdorff dimension).

We will use the following notation. If for a given set of sequences $x_1x_2\cdots$ the ith symbol is not specified–i.e., it can be chosen to be either 0 or 1–we will write $x_i = a$ (a stands for "arbitrary"). In all our constructions, the basic pieces of the set will be defined in terms of sequences which have 0 at some positions and a at the rest of the positions.

Before starting the constructions, we recall some basic properties of dimensions. The Hausdorff dimension and upper box dimension are stable under finite unions, i.e.,

$$\dim\left(\bigcup_{i=1}^{m} A_i\right) = \max_{i=1}^{m} \dim(A_i),$$

where dim stands for either $\dim_H$ or $\overline{\dim}_B$. However, the lower box dimension $\underline{\dim}_B$ is *not* stable under finite unions. These facts will be exploited repeatedly in our constructions.

Let $A \subset [0,1]$. For $\underline{x} \in A$, we define

$$\#_{off}(n, \underline{x}, A) := \begin{cases} 1 & \text{if } [x_1\cdots x_n a] \cap A \neq \emptyset \\ 0 & \text{otherwise} \end{cases},$$

and

$$\mathcal{OFF}_n(A) := \min_{\underline{x}\in A} \frac{1}{n} \sum_{i=1}^{n} \#_{off}(i, \underline{x}, A).$$

Lemma 1. *For any set $A \subset [0,1]$, we have*

$$\liminf_n \mathcal{OFF}_n(A) \leq \dim_H A.$$

Proof. Let log denote the logarithm to base 2. Then for any measure μ on the space of binary sequences,

$$d_\mu(\underline{x}) := \liminf_{\varepsilon \to 0} \frac{\log \mu(B(\underline{x}, \varepsilon))}{\log \varepsilon} = \liminf_{n \to \infty} -\frac{1}{n} \log \mu(C_n(\underline{x})),$$

where $C_n(\underline{x})$ denotes the set of infinite binary sequences starting with $x_1 \cdots x_n$. Now let μ be the measure on A that gives equal weight to any offspring of a given cylinder, i.e., it gives half of the measure if $\#_{off}(i, \underline{x}, A) = 1$ and full measure otherwise. Then for any $\underline{x} \in A$ we have

$$\liminf_{n \to \infty} -\frac{1}{n} \log \mu(C_n(\underline{x})) \geq \liminf_n \mathcal{OFF}_n(A).$$

An application of the mass distribution principle (see, e.g., [1, Chap. 4]) concludes the proof. □

2.2 Examples for Hausdorff Dimension

The first example shows how one can control the Hausdorff dimension of simple sumsets.

Example 1. For $0 \leq \alpha_1 \leq \alpha_2 \leq 1$, we construct a compact set $A \subset [0,1]$ such that

$$\dim_H A = \alpha_1 \quad \dim_H (A + A) = \alpha_2.$$

Proof (Construction). First we fix a sufficiently rapidly increasing sequence of natural numbers, say $n_k = 2^{2^k}$. The set A will be constructed as an (almost disjoint) union of two sets. Let

$$A_1 := \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{3k}, \left[\frac{n_{3k}}{\alpha_1} \right]_* - 1 \right] \\ 0 & i \in \left[n_{3k+2}, \left[\frac{n_{3k+2}}{\alpha_2} \right]_* - 1 \right] \\ a & \text{otherwise} \end{cases} \right\},$$

where $\left[\frac{n_i}{\alpha_j} \right]_*$ denotes the minimum of the integer part of $\frac{n_i}{\alpha_j}$ and $n_{i+1} - n_i$ if $\alpha_j \neq 0$, and in_i otherwise (we will use this notation for the rest of this construction and the next one). Note that, if $\alpha_j \neq 0$, then $\left[\frac{n_i}{\alpha_j} \right]_*$ equals the integer part of $\frac{n_i}{\alpha_j}$ for all but finitely many values of i. The second set is defined (only if $\alpha_2 \neq 0$, otherwise it is empty) as

$$A_2 := \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{3k+1}, \left[\frac{n_{3k+1}}{\alpha_1} \right]_* - 1 \right] \\ 0 & i \in \left[n_{3k+2}, \left[\frac{n_{3k+2}}{\alpha_2} \right]_* - 1 \right] \\ a & \text{otherwise} \end{cases} \right\}.$$

If we consider cylinders of length $\left[\frac{n_{3k}}{\alpha_1} \right]_*$ we see that there are at most $2^{n_{3k}}$ intersecting A_1, and if we consider cylinders of length $\left[\frac{n_{3k+1}}{\alpha_1} \right]_*$ there are at most $2^{n_{3k+1}}$ intersecting A_2. Hence, $\alpha_1 \geq \underline{\dim}_B A_i \geq \dim_H A_i$. On the other hand, $\liminf_n \mathcal{OFF}_n(A_i) = \alpha_1$ since $\alpha_2 \geq \alpha_1$. Thus, Lemma 1 gives the lower bound for $\dim_H A$.

For the simple sumset we can argue as follows. Firstly, we have that

$$2A_1 \cup 2A_2 \cup (A_1 + A_2) \subset \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{3k+2}, \left[\frac{n_{3k+2}}{\alpha_2} \right]_* - 2 \right] \\ a & \text{otherwise} \end{cases} \right\} =: B_1,$$

where the "-2" accounts for a possible carry. Secondly,

$$A_1 + A_2 \supset \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{3k+2}, \left[\frac{n_{3k+2}}{\alpha_2} \right]_* - 1 \right] \\ a & \text{otherwise} \end{cases} \right\} =: B_2.$$

For both sets on the right-hand side we have that

$$\alpha_2 \leq \liminf_n \mathcal{OFF}_n(B_i) \leq \dim_H B_i \leq \underline{\dim}_B B_i \leq \alpha_2.$$

This shows that the set $A = A_1 \cup A_2$ has the desired properties. □

The second example shows how one can control the Hausdorff dimension of triple sumsets. This gives an idea about the general induction process.

Example 2. For $0 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq 1$ we construct a compact set $A \subset [0,1]$ such that

$$\dim_H A = \alpha_1 \quad \dim_H (A + A) = \alpha_2 \quad \dim_H (A + A + A) = \alpha_3.$$

Proof (Construction). This example is a modification of the previous one. We just need to add a third component to control the triple sums.

Again we fix a sufficiently rapidly increasing sequence of natural numbers, say $n_k = 2^{2^k}$. The set A will be constructed as an (almost disjoint) union of three sets. Let

$$A_1 := \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{6k}, \left[\frac{n_{6k}}{\alpha_1} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+2}, \left[\frac{n_{6k+2}}{\alpha_2} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+4}, \left[\frac{n_{6k+4}}{\alpha_2} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+5}, \left[\frac{n_{6k+5}}{\alpha_3} \right]_* - 1 \right] \\ a & \text{otherwise} \end{cases} \right\},$$

$$A_2 := \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{6k+1}, \left[\frac{n_{6k+1}}{\alpha_1} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+2}, \left[\frac{n_{6k+2}}{\alpha_2} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+3}, \left[\frac{n_{6k+3}}{\alpha_2} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+5}, \left[\frac{n_{6k+5}}{\alpha_3} \right]_* - 1 \right] \\ a & \text{otherwise} \end{cases} \right\},$$

$$A_3 := \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{6k}, \left[\frac{n_{6k}}{\alpha_1} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+3}, \left[\frac{n_{6k+3}}{\alpha_2} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+4}, \left[\frac{n_{6k+4}}{\alpha_2} \right]_* - 1 \right] \\ 0 & i \in \left[n_{6k+5}, \left[\frac{n_{6k+5}}{\alpha_3} \right]_* - 1 \right] \\ a & \text{otherwise} \end{cases} \right\}.$$

By a similar reasoning as in the previous example, we have that $\dim_H A_1 = \dim_H A_2 = \dim_H A_3 = \dim_H A = \alpha_1$, where $A = A_1 \cup A_2 \cup A_3$. Also for the simple sumsets we have that

$$\begin{aligned} \dim_H(A_i + A_i) &\le \dim_H(A_1 + A_2) = \dim_H(A_2 + A_3) = \dim_H(A_1 + A_3) \\ &= \dim_H(A + A) = \alpha_2. \end{aligned}$$

For the triple sumset, we remark that

$$\bigcup_{i,j,l=1,2,3} (A_i + A_j + A_l) \subset \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{6k+5}, \left[\frac{n_{6k+5}}{\alpha_3} \right]_* - 3 \right] \\ a & \text{otherwise} \end{cases} \right\} =: B_1,$$

where the "-3" accounts for carryovers. Secondly,

$$A_1 + A_2 + A_3 \supset \left\{ \underline{x} : x_i = \begin{cases} 0 & i \in \left[n_{6k+5}, \left[\frac{n_{6k+5}}{\alpha_3} \right]_* - 1 \right] \\ a & \text{otherwise} \end{cases} \right\} =: B_2.$$

Again for both sets on the right-hand side we have that

$$\alpha_3 \le \liminf_n \mathcal{OFF}_n(B_i) \le \dim_H B_i \le \underline{\dim}_B B_i \le \alpha_3.$$

This shows that the set $A = A_1 \cup A_2 \cup A_3$ has the desired properties. □

The previous example can clearly be generalized to control the Hausdorff dimension of any finite sumsets, i.e., of sets A, $2A, \dots, \ell A$ for any $\ell \in \mathbb{N}$. Moreover, it is easy to see that, in Example 2, if some $\alpha_i = 1$ for $i = 1, 2, 3$, then $iA \supset [0, 1]$, since the choice of digits becomes completely arbitrary. Next, we show how to control an infinite number of sumsets.

Example 3. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a nondecreasing sequence taking values in $[0,1]$. Then there exists a compact set $A \subset [0,1]$ such that

$$\dim_H(\ell A) = \alpha_\ell$$

for all $\ell \in \mathbb{N}$. Furthermore, if $\alpha_\ell = 1$ for some ℓ, we can also require that ℓA contain an interval.

Proof (Construction). In short, the construction consists in pasting together along the dyadic structure all the sets obtained for finite sequences $\alpha_1, \ldots, \alpha_\ell$. By proceeding as in Example 2, we see that for every $\ell \geq 2$ there exists a compact set $A_\ell \subset [0,1]$ such that

$$\dim_H(iA_\ell) = \alpha_i, \quad i = 1, \ldots, \ell.$$

Let $\{M_\ell\}_{\ell=1}^{\infty}$ be a rapidly increasing sequence. Set $S_1 = 0$ and $S_\ell = \sum_{i=1}^{\ell-1} M_\ell$ for $\ell > 1$, and let

$$A := \left\{ \underline{x} : x_{S_\ell+1} \cdots x_{S_{\ell+1}} = y_1 \ldots y_{M_\ell} \text{ for some } \underline{y} \in A_\ell, \text{ for each } \ell \in I\!N \right\}.$$

Roughly speaking, A is defined by following the construction of A_1 for the first M_1 binary digits, then the construction of A_2 for the following M_2 binary digits, and so on. Note that A is a countable intersection of compact sets, so it is compact.

We claim that

$$\dim_H(A) = \liminf_i \dim_H(A_i) = \alpha_1,$$

provided $\{M_\ell\}$ grows fast enough. Indeed, by taking M_ℓ large enough, we can cover each A_ℓ by dyadic intervals $\{I_\ell^{(r)}\}$ of length at least 2^{-M_ℓ}, satisfying

$$\sum_r \left| I_\ell^{(r)} \right|^{\alpha_\ell + 1/\ell} < 1.$$

Then we can cover A by at most $2^{S_i \ell}$ translated and scaled-down (by a factor of 2^{-S_ℓ}) copies of the family $\{I_\ell^{(r)}\}$. Since $M_\ell \gg S_\ell$, this yields the upper bound for $\dim_H(A)$. For the lower bound, we use Frostman's lemma: there are measures μ_ℓ supported on A_ℓ, such that

$$\mu_\ell(I) \leq |I|^{\alpha_\ell - 1/\ell}, \tag{1}$$

for all dyadic intervals of length $|I| \leq 2^{-M'_\ell}$, where $M'_\ell \ll M_{\ell-1}$ (making $M_{\ell-1}$ larger if necessary). We can paste all these measures together dyadically in a similar way to the construction of A. More precisely,

$$\mu(C(x_1 \cdots x_{S_\ell})) := \mu_1(C(x_1 \cdots x_{M_1})) \cdots \mu_i(C(x_{S_{\ell-1}+1} \cdots x_{S_\ell})), \tag{2}$$

where $C(y_1 \cdots y_j)$ denotes the set of all dyadic sequences starting with $y_1 \cdots y_j$. Combining (1) and (2) and applying the mass distribution principle yields the lower bound, completing the proof of the claim.

More generally, since addition preserves the dyadic structure except for carryovers, and these are negligible due to the presence of blocks of zeros in the construction of each A_ℓ (unless $\alpha_\ell = 1$), we see that

$$\dim_H(\ell A) = \liminf_i \dim_H(\ell A_i) = \alpha_\ell.$$

Finally, if $\alpha_\ell = 1$ for some ℓ, then there is no restriction on the dyadic digits of ℓA_i for any $i \geq \ell$, thus, there is no restriction on the dyadic digits of ℓA except for finitely many of them. Hence, ℓA contains an interval, as desired. □

2.3 Examples for Hausdorff and Box-Counting Dimensions

Next, we start controlling various notions of dimension simultaneously. In the first example of this kind we show how to control the Hausdorff and lower box-counting dimension for simple sumsets.

Example 4. Given $0 \leq \alpha_i \leq b_i \leq 1$, $i = 1, 2$ with $\alpha_1 \leq a_2$ and $\beta_1 \leq \beta_2 \leq 2\beta_1$, we construct a compact set $A \subset [0, 1]$ with

$$\dim_H A = \alpha_1 \quad \underline{\dim}_B A = \beta_1,$$

and

$$\dim_H(A + A) = \alpha_2 \quad \underline{\dim}_B(A + A) = \beta_2.$$

Proof (Construction). Again we fix a fast-increasing sequence

$$n_k = \min\left\{n \in \mathbb{N} \,:\, n \geq 2^{2^k} \text{ and } (k-1)|(n - n_{k-1})\right\}. \tag{3}$$

For each $k \in \mathbb{N}$ we define four numbers:

$$l_k := [k\beta_1], \quad m_k := [k\beta_2],$$

$$d_i(k) := \begin{cases} k\left[\frac{1}{k} n_k \left(\frac{\beta_i}{\alpha_i} - 1\right)\right] & \text{if } \alpha_i \neq 0 \\ kn_k & \text{if } \alpha_i = 0 \end{cases}, \quad i = 1, 2.$$

Given a word u, we let u^r denote the word consisting of r consecutive copies of u; in particular, a^r is the word consisting of r consecutive "arbitrary symbols" a. If $r = 0$, then u^r is the empty word. We will define four types of blocks:

$$t_{\alpha_1}(k) := \Big\{x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} = \cdots = x_{n_k + d_1(k) - 1} = 0,$$
$$x_{n_k + d_1(k)} \cdots x_{n_{k+1}-1} = \left(a^{l_k} 0^{k - l_k}\right)^{(n_{k+1} - n_k - d_1(k))/k}\Big\},$$

$$t_{\alpha_2}(k) := \Big\{ x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} = \cdots = x_{n_k+d_2(k)-1} = 0,$$
$$x_{n_k+d_2(k)} \cdots x_{n_{k+1}-1} = \left(0^{m_k-l_k} a^{l_k} 0^{k-m_k}\right)^{(n_{k+1}-n_k-d_2(k))/k} \Big\},$$

$$t_{\beta_1}(k) := \left\{ x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} \cdots x_{n_{k+1}-1} = \left(a^{l_k} 0^{k-l_k}\right)^{(n_{k+1}-n_k)/k} \right\},$$

$$t_{\beta_2}(k) := \left\{ x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} \cdots x_{n_{k+1}-1} = \left(0^{m_k-l_k} a^{l_k} 0^{k-m_k}\right)^{(n_{k+1}-n_k)/k} \right\}.$$

The set A will be defined as the union of six components A_i, which are defined as follows:

$$A_1 := \left\{ \underline{x} \,:\, \begin{cases} x_{n_{3k}} \cdots x_{n_{3k+1}-1} \in t_{\alpha_1}(3k) \\ x_{n_{3k+1}} \cdots x_{n_{3k+2}-1} \in t_{\alpha_2}(3k+1) \\ x_{n_{3k+2}} \cdots x_{n_{3(k+1)}-1} \in t_{\beta_1}(3k+2) \end{cases} \right\},$$

$$A_2 := \left\{ \underline{x} \,:\, \begin{cases} x_{n_{3k}} \cdots x_{n_{3k+1}-1} \in t_{\beta_1}(3k) \\ x_{n_{3k+1}} \cdots x_{n_{3k+2}-1} \in t_{\alpha_1}(3k+1) \\ x_{n_{3k+2}} \cdots x_{n_{3(k+1)}-1} \in t_{\alpha_2}(3k+2) \end{cases} \right\},$$

$$A_3 := \left\{ \underline{x} \,:\, \begin{cases} x_{n_{3k}} \cdots x_{n_{3k+1}-1} \in t_{\alpha_2}(3k) \\ x_{n_{3k+1}} \cdots x_{n_{3k+2}-1} \in t_{\beta_1}(3k+1) \\ x_{n_{3k+2}} \cdots x_{n_{3(k+1)}-1} \in t_{\alpha_1}(3k+2) \end{cases} \right\},$$

$$A_4 := \left\{ \underline{x} \,:\, \begin{cases} x_{n_{3k}} \cdots x_{n_{3k+1}-1} \in t_{\alpha_2}(3k) \\ x_{n_{3k+1}} \cdots x_{n_{3k+2}-1} \in t_{\alpha_1}(3k+1) \\ x_{n_{3k+2}} \cdots x_{n_{3(k+1)}-1} \in t_{\beta_2}(3k+2) \end{cases} \right\},$$

$$A_5 := \left\{ \underline{x} \,:\, \begin{cases} x_{n_{3k}} \cdots x_{n_{3k+1}-1} \in t_{\beta_2}(3k) \\ x_{n_{3k+1}} \cdots x_{n_{3k+2}-1} \in t_{\alpha_2}(3k+1) \\ x_{n_{3k+2}} \cdots x_{n_{3(k+1)}-1} \in t_{\alpha_1}(3k+2) \end{cases} \right\},$$

$$A_6 := \left\{ \underline{x} \,:\, \begin{cases} x_{n_{3k}} \cdots x_{n_{3k+1}-1} \in t_{\alpha_1}(3k) \\ x_{n_{3k+1}} \cdots x_{n_{3k+2}-1} \in t_{\beta_2}(3k+1) \\ x_{n_{3k+2}} \cdots x_{n_{3(k+1)}-1} \in t_{\alpha_2}(3k+2) \end{cases} \right\}.$$

Note that the block structure of A_1, A_2, and A_3 follows a cyclic pattern, and that the block structure of A_{3+i} is specular to that of A_i, in the sense that blocks $t_{*_i}(k)$ are replaced by $t_{*_{2-i}}(k)$, for $* = \alpha, \beta$.

We will use the following notation. Given a finite word u, by $\mathrm{Fr}_a(u)$ we denote the frequency of symbols "a" in u, i.e.,

$$\mathrm{Fr}_a(u) = \frac{|\{i : u_i = a\}|}{|u|},$$

where $|u|$ is the length of u. A quantity that goes to zero as $k \to \infty$ will be denoted by $o(1)$.

Notice that all types of blocks have a frequency $\beta_1 + o(1)$ of "a" symbols. Since the blocks $t_{\alpha_1}(k)$ start with $d_1(k)$ zeros, we see that if $k = 3l$, then

$$\mathcal{OFF}_{n_k+d_1(k)}(A_1) = \alpha_1 + o(1),$$

and $\mathcal{OFF}_n(A_1) \geq \alpha_1$ for all n, so by Lemma 1 we get

$$\alpha_1 \leq \dim_H A_1 \leq \underline{\dim}_B A_1 \leq \alpha_1.$$

The same argument holds for the other A_i, so we obtain $\dim_H A = \alpha_1$. Next, notice that for each k there is at least one component (in fact two) A_i for which the block $x_{n_k} \cdots x_{n_{k+1}-1}$ is neither of type $t_{\alpha_1}(k)$ nor $t_{\alpha_2}(k)$. This shows that $\underline{\dim}_B A = \beta_1$.

Let us write

$$t_*(k) + t_{**}(k) := \{x_{n_k} \cdots x_{n_{k+1}-1} = v + w \,:\, v \in t_*(k),\ w \in t_{**}(k)\},$$

where $a+a = a+0 = a$ (thus, carryovers are ignored, but due to the structure of the blocks they are negligible in the estimates). Note that

$$\mathrm{Fr}_a(t_*(k) + t_{**}(k)) \leq \beta_2 + o(1) \tag{4}$$

for any choice of $*, **$, and therefore $\overline{\dim}_B(A_i) \leq \beta_2$. On the other hand, each $A_i + A_j$ contains blocks of either type $t_{\alpha_1}(k)+t_{\alpha_2}(k)$ or $t_{\alpha_2}(k)+t_{\alpha_2}(k)$. Taking into account the definition of $d_2(k)$, we see that $\dim_H(A_i + A_j) \leq \alpha_2$ for each i, j. In the opposite direction, note that since $\beta_2 \leq 2\beta_1$, we have

$$\mathrm{Fr}_a(t_{\beta_1}(k) + t_{\beta_2}(k)) = \beta_2 + o(1).$$

Since $A_1 + A_4$ has infinitely many blocks $t_{\alpha_1}(k) + t_{\alpha_2}(k)$ preceded by $t_{\beta_1}(k-1) + t_{\beta_2}(k-1)$, and it has no blocks of the form $t_{\alpha_1}(k) + t_{\alpha_1}(k)$ or blocks of the form $t_{\beta_1} + t_{\beta_1}$, we see that $\liminf_n \mathcal{OFF}_n(A_1 + A_4) \geq \alpha_2$. Hence, we have shown that

$$\dim_H(A + A) = \max_{i,j} \dim_H(A_i + A_j) = \dim_H(A_1 + A_4) = \alpha_2.$$

By (4), $\overline{\dim}_B(A_i + A_j) \leq \beta_2$, so that

$$\underline{\dim}_B(A + A) \leq \overline{\dim}_B(A + A) = \max_{i,j} \overline{\dim}_B(A_i + A_j) \leq \beta_2.$$

On the other hand, for each k, there is $i \in \{1, 2, 3\}$ such that the block $x_{n_k} \cdots x_{n_{k+1}-1}$ in $A_i + A_{3+i}$ is of type $t_{\beta_1}(k) + t_{\beta_2}(k)$; all of these have frequency $\beta_2 + o(1)$ of "a" distributed in small (relative to n_k) chunks of length k, so $\underline{\dim}_B(A + A) \geq \beta_2$, as desired. □

The next example shows how to control Hausdorff, lower box-counting, and upper box-counting dimensions at once in simple sumsets.

Example 5. Given $0 \le \alpha_i \le \beta_i \le \gamma_i \le 1$, $i = 1, 2$ with $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2$, $\gamma_1 \le \gamma_2$, $\beta_2 \le 2\beta_1$, and $\gamma_2 \le 2\gamma_1$, we construct a compact set $A \subset [0, 1]$ with

$$\dim_H A = \alpha_1 \quad \underline{\dim}_B A = \beta_1 \quad \overline{\dim}_B A = \gamma_1,$$

and

$$\dim_H (A + A) = \alpha_2 \quad \underline{\dim}_B (A + A) = \beta_2 \quad \overline{\dim}_B (A + A) = \gamma_2.$$

In particular, if $\alpha_1 = \beta_1 = \gamma_1$ the regularity of the set A does not imply the regularity of the sumset $A + A$.

Proof (Construction). This example will be a modification of the previous one (one can check that in Example 4, $\underline{\dim}_B(A) = \overline{\dim}_B(A)$ and $\underline{\dim}_B(A + A) = \overline{\dim}_B(A+A)$). All we need to do is to add in the construction of each A_i blocks of type $t_{\gamma_1}(k)$ or $t_{\gamma_2}(k)$, at the same positions in each A_i, and preceding any blocks of the form $t_{\alpha_1}(k)$ or $t_{\alpha_2}(k)$ (to prevent the Hausdorff dimension from dropping too much).

We use the same fast-increasing sequence defined in (3). For each $k \in \mathbb{N}$ we define six numbers. The numbers l_k, m_k are defined exactly as in Example 4, while the numbers $d_i(k)$ are redefined as

$$d_i(k) := \begin{cases} k\left[\frac{1}{k} n_k \left(\frac{\gamma_i}{\alpha_i} - 1\right)\right] & \text{if } \alpha_i \neq 0 \\ k n_k & \text{if } \alpha_i = 0 \end{cases}.$$

Additionally, we define

$$p_k := [k\gamma_1], \quad q_k := [k\gamma_2].$$

We will use six types of blocks; the blocks $t_{\alpha_i}(k), t_{\beta_i}(k)$, $i = 1, 2$ are defined just as in the previous example. We additionally define

$$t_{\gamma_1}(k) := \left\{ x_{n_k} \cdots x_{n_{k+1}-1} : x_{n_k} \cdots x_{n_{k+1}-1} = \left(a^{p_k} 0^{k-p_k}\right)^{(n_{k+1}-n_k)/k} \right\},$$

$$t_{\gamma_2}(k) := \left\{ x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} \cdots x_{n_{k+1}-1} = \left(0^{q_k - p_k} a^{p_k} 0^{k-q_k}\right)^{(n_{k+1}-n_k)/k} \right\}.$$

The set A will have six components A_i. The first three are defined as

$$A_1 := \left\{ \underline{x} : \begin{cases} x_{n_{6k}} \cdots x_{n_{6k+1}-1} \in t_{\gamma_1}(6k) \\ x_{n_{6k+1}} \cdots x_{n_{6k+2}-1} \in t_{\alpha_1}(6k+1) \\ x_{n_{6k+2}} \cdots x_{n_{6k+3}-1} \in t_{\gamma_2}(6k+2) \\ x_{n_{6k+3}} \cdots x_{n_{6k+4}-1} \in t_{\alpha_2}(6k+3) \\ x_{n_{6k+4}} \cdots x_{n_{6k+5}-1} \in t_{\gamma_1}(6k+4) \\ x_{n_{6k+5}} \cdots x_{n_{6(k+1)}-1} \in t_{\beta_1}(6k+5) \end{cases} \right\},$$

$$A_2 := \left\{ \underline{x} : \begin{cases} x_{n_{6k}} \cdots x_{n_{6k+1}-1} \in t_{\gamma_1}(6k) \\ x_{n_{6k+1}} \cdots x_{n_{6k+2}-1} \in t_{\beta_1}(6k+1) \\ x_{n_{6k+2}} \cdots x_{n_{6k+3}-1} \in t_{\gamma_1}(6k+2) \\ x_{n_{6k+3}} \cdots x_{n_{6k+4}-1} \in t_{\alpha_1}(6k+3) \\ x_{n_{6k+4}} \cdots x_{n_{6k+5}-1} \in t_{\gamma_2}(6k+4) \\ x_{n_{6k+5}} \cdots x_{n_{6(k+1)}-1} \in t_{\alpha_2}(6k+5) \end{cases} \right\},$$

$$A_3 := \left\{ \underline{x} : \begin{cases} x_{n_{6k}} \cdots x_{n_{6k+1}-1} \in t_{\gamma_2}(6k) \\ x_{n_{6k+1}} \cdots x_{n_{6k+2}-1} \in t_{\alpha_2}(6k+1) \\ x_{n_{6k+2}} \cdots x_{n_{6k+3}-1} \in t_{\gamma_1}(6k+2) \\ x_{n_{6k+3}} \cdots x_{n_{6k+4}-1} \in t_{\beta_1}(6k+3) \\ x_{n_{6k+4}} \cdots x_{n_{6k+5}-1} \in t_{\gamma_1}(6k+4) \\ x_{n_{6k+5}} \cdots x_{n_{6(k+1)}-1} \in t_{\alpha_1}(6k+5) \end{cases} \right\}.$$

Note that the only difference with the sets in Example 4 is the addition of blocks corresponding to the upper box dimension. Likewise, for $i \in \{1,2,3\}$, the sets A_{i+3} are defined in a specular way to A_i as in the previous example; namely, blocks of type $t_{*_i}(k)$ are replaced by blocks of type $t_{*_{2-i}}(k)$ for $* = \alpha, \beta$ and γ.

One can check that $\dim_H A = \alpha_1$, $\underline{\dim}_B A = \beta_1$ just as in Example 4 (for the Hausdorff dimension, it is useful to note that blocks of type $t_{\alpha_1}(k)$ are always preceded by blocks of type $t_{\gamma_1}(k-1)$). Since

$$\mathrm{Fr}_a(t_{\gamma_i}(k)) = \gamma_1 + o(1),$$

for $i = 1, 2$, it follows that $\overline{\dim}_B(A) = \gamma_1$.

For the sumset we argue just as in Example 4. For the upper box dimension, all we need to observe is that

$$\mathrm{Fr}_a(t_{\gamma_1}(k) + t_{\gamma_2}(k)) = \gamma_2 + o(1),$$

and such blocks occur infinitely often in $A_1 + A_4$; any other block $t_*(k) + t_{**}(k)$ has a lower frequency of "a". Thus,

$$\dim_H(A+A) = \alpha_2 \quad \underline{\dim}_B(A+A) = \beta_2 \quad \overline{\dim}_B(A+A) = \gamma_2. \qquad \square$$

Example 6. Suppose $0 \le \alpha_i \le \beta_i \le \gamma_i \le 1$, $i = 1, 2, 3$, and

$$\beta_2 \le 2\beta_1 \quad \beta_3 \le 2\beta_2 - \beta_1,$$

$$\gamma_2 \le 2\gamma_1 \quad \gamma_3 \le 2\gamma_2 - \gamma_1.$$

We construct a compact set $A \subset [0,1]$ with

$$\dim_H A = \alpha_1 \quad \underline{\dim}_B A = \beta_1 \quad \overline{\dim}_B A = \gamma_1,$$

$$\dim_H (A+A) = \alpha_2 \quad \underline{\dim}_B (A+A) = \beta_2 \quad \overline{\dim}_B (A+A) = \gamma_2,$$

and

$$\dim_H (A+A+A) = \alpha_3 \quad \underline{\dim}_B (A+A+A) = \beta_3 \quad \overline{\dim}_B (A+A+A) = \gamma_3.$$

Proof (Construction). We fix again the fast-increasing sequence n_k given by (3). For each $k \in \mathbb{N}$ we will define nine numbers. The numbers l_k, m_k, p_k, and q_k are defined exactly as in Example 5. The numbers $d_i(k)$ are also defined in the same way, except that now we also allow the index $i = 3$. We also define new numbers:

$$s_k := [k\beta_3], \quad v_k := [k\gamma_3].$$

We will define nine types of blocks. The blocks $t_{\alpha_i}(k), t_{\beta_i}(k)$, and $t_{\gamma_i}(k)$, $i = 1, 2$, are the same as in Example 5. We further define

$$t_{\alpha_3}(k) := \Big\{ x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} = \cdots = x_{n_k + d_3(k) - 1} = 0,$$
$$x_{n_k + d_3(k)} \cdots x_{n_{k+1}-1} = \left(0^{m_k - l_k} a^{l_k + m_k - s_k} 0^{s_k - m_k} a^{s_k - m_k} 0^{k - s_k}\right)^{(n_{k+1} - n_k - d_3(k))/k} \Big\},$$

$$t_{\beta_3}(k) := \Big\{ x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} \cdots x_{n_{k+1}-1} = \left(0^{m_k - l_k} a^{l_k + m_k - s_k} 0^{s_k - m_k} a^{s_k - m_k} 0^{k - s_k}\right)^{(n_{k+1} - n_k)/k} \Big\},$$

$$t_{\gamma_3}(k) := \Big\{ x_{n_k} \cdots x_{n_{k+1}-1} \,:\, x_{n_k} \cdots x_{n_{k+1}-1} = \left(0^{q_k - p_k} a^{p_k + q_k - v_k} 0^{v_k - q_k} a^{v_k - q_k} 0^{k - v_k}\right)^{(n_{k+1} - n_k)/k} \Big\}.$$

It is easy to verify that $t_{\beta_3}(k)$ and $t_{\gamma_3}(k)$ are well defined due to the assumptions made on the β_i and γ_i.

The set A will have 18 components A_i, which are defined by specifying the types of blocks

$$x_{n_k} \cdots x_{n_{k+1}-1} = t_*(k)$$

as follows:

$12k+$	0	1	2	3	4	5	6	7	8	9	10	11
A_1	γ_1	α_1	γ_2	α_2	γ_3	α_3	γ_1	α_1	γ_2	α_2	γ_3	β_1
A_2	γ_1	α_1	γ_2	α_2	γ_3	α_3	γ_1	α_1	γ_2	β_1	γ_3	α_3
A_3	γ_1	α_1	γ_2	α_2	γ_3	α_3	γ_1	β_1	γ_2	α_2	γ_3	α_3
A_4	γ_1	α_1	γ_2	α_2	γ_3	β_1	γ_1	α_1	γ_2	α_2	γ_3	α_3
A_5	γ_1	α_1	γ_2	β_1	γ_3	α_3	γ_1	α_1	γ_2	α_2	γ_3	α_3
A_6	γ_1	β_1	γ_2	α_2	γ_3	α_3	γ_1	α_1	γ_2	α_2	γ_3	α_3
A_7	γ_2	α_2	γ_3	α_3	γ_1	α_1	γ_3	α_3	γ_1	α_1	γ_2	β_2
A_8	γ_2	α_2	γ_3	α_3	γ_1	α_1	γ_3	α_3	γ_1	β_2	γ_2	α_2
A_9	γ_2	α_2	γ_3	α_3	γ_1	α_1	γ_3	β_2	γ_1	α_1	γ_2	α_2
A_{10}	γ_2	α_2	γ_3	α_3	γ_1	β_2	γ_3	α_3	γ_1	α_1	γ_2	α_2
A_{11}	γ_2	α_2	γ_3	β_2	γ_1	α_1	γ_3	α_3	γ_1	α_1	γ_2	α_2
A_{12}	γ_2	β_2	γ_3	α_3	γ_1	α_1	γ_3	α_3	γ_1	α_1	γ_2	α_2
A_{13}	γ_3	α_3	γ_1	α_1	γ_2	α_2	γ_2	α_2	γ_3	α_3	γ_1	β_3
A_{14}	γ_3	α_3	γ_1	α_1	γ_2	α_2	γ_2	α_2	γ_3	β_3	γ_1	α_1
A_{15}	γ_3	α_3	γ_1	α_1	γ_2	α_2	γ_2	β_3	γ_3	α_3	γ_1	α_1
A_{16}	γ_3	α_3	γ_1	α_1	γ_2	β_3	γ_2	α_2	γ_3	α_3	γ_1	α_1
A_{17}	γ_3	α_3	γ_1	β_3	γ_2	α_2	γ_2	α_2	γ_3	α_3	γ_1	α_1
A_{18}	γ_3	β_3	γ_1	α_1	γ_2	α_2	γ_2	α_2	γ_3	α_3	γ_1	α_1

Arguing as in the previous examples we see that

$$\dim_H A = \alpha_1 \quad \underline{\dim}_B A = \beta_1 \quad \overline{\dim}_B A = \gamma_1.$$

For the sumset $A + A$, notice that each pair $A_i + A_j$ contains infinitely many blocks of the form $t_{\alpha_*}(k) + t_{\alpha_{**}}(k)$, where $*$ and $**$ are either 1 or 2 (possibly $* = **$). Hence, again arguing as in the previous constructions, $\dim_H(A) \le \alpha_2$. On the other hand, for $A_1 + A_7$ all blocks of the form $t_{\alpha_1}(k) + t_{\alpha_2}(k)$ are preceded by blocks of the form $t_{\gamma_1}(k-1) + t_{\gamma_2}(k-1)$, and there are infinitely many such blocks. Also, there are no blocks of the form $t_{*_1} + t_{*_1}$ for $* = \alpha, \beta$ or γ. From this we deduce that

$$\dim_H(A + A) \ge \dim_H(A_1 + A_7) \ge \alpha_2.$$

For the lower box dimension, we note that for each k there is an $i \in \{1, \dots, 7\}$ such that $A_i + A_{i+6}$ contains a block of type either $t_{\beta_1}(k) + t_{\beta_2}(k)$ or $t_{\gamma_1}(k) + t_{\gamma_2}(k)$, so that $\underline{\dim}_B(A + A) \ge \beta_2$. On the other hand, if $k = 12l + 1$, then

$$\mathrm{Fr}_a(t_*(k) + t_{**}(k)) \le \beta_2,$$

for all possible occurrences of $*$ and $**$ (to see this in the case $* = \beta_1$ or β_2 and $** = \beta_3$, one needs to make use of the assumption $\beta_3 \le 2\beta_2 - \beta_1$). Hence, $\underline{\dim}_B(A + A) \le \beta_2$. It is easy to check that $\overline{\dim}_B(A + A) = \gamma_2$.

Finally, if we consider $A + A + A$, we see that for each i, j, k there are infinitely many blocks of the form $t_{\alpha_*}(k) + t_{\alpha_{**}}(k) + t_{\alpha_{***}}(k)$ in $A_i + A_j + A_k$, where $*, **, *** \in \{1, 2, 3\}$. This implies that $\dim_H(A + A + A) \le \alpha_3$.

On the other hand, $A_1 + A_7 + A_{13}$ contains infinitely many blocks of the form $t_{\alpha_1}(k) + t_{\alpha_2}(k) + t_{\alpha_3}(k)$, all of them preceded by blocks of the form $t_{\gamma_1}(k-1) + t_{\gamma_2}(k-1) + t_{\gamma_3}(k-1)$; moreover, any block in $A_1 + A_7 + A_{13}$ is of the form $t_{*_1}(k) + t_{**_2}(k) + t_{***_3}(k)$. It follows that $\dim_H(A + A + A) \geq \alpha_3$.

The arguments for $\underline{\dim}_B(A + A + A)$ and $\overline{\dim}_B(A + A + A)$ are just like those for the sums $A + A$. □

2.4 Proof of the Main Result

Proof (of Theorem 1). The result is proved by combining the examples above, in particular, Examples 3 and 6. Since the actual construction is quite complicated, we sketch the main ideas, leaving the details to the interested reader.

Example 6 can be generalized to ℓ-sumsets in a straightforward way if ℓ is a prime number (and there is no loss of generality in assuming this). We need $(\ell-1)\ell^2$ components A_i and 3ℓ different types of blocks, and we have to control the sequences for $2(\ell-1)\ell$ consecutive n_k's. The restrictions on the dimensions that arise are precisely those stated in the theorem.

Given $\ell \geq 2$, let $A_\ell \subset [0,1]$ be a compact set such that

$$\dim_H(iA_\ell) = \alpha_i, \quad \underline{\dim}_B(iA_\ell) = \beta_i, \quad \overline{\dim}_B(iA_\ell) = \gamma_i,$$

for $i = 1, \ldots, \ell$.

We are going to combine the sets A_ℓ exactly as in Example 3. Let $\{M_\ell\}_{\ell=1}^{\infty}$ be a rapidly increasing sequence. Write $S_\ell = \sum_{i=1}^{\ell} M_i$, with $S_1 = 0$, and define

$$A := \left\{ \underline{x} : x_{S_\ell+1} \cdots x_{S_{\ell+1}} = y_1 \ldots y_{M_\ell} \text{ for some } \underline{y} \in A_\ell, \text{ for each } \ell \in I\!N \right\}.$$

The set A is clearly compact, and its structure translates to any finite sumset ℓA, apart from carryovers, which can be ignored. Therefore, we get

$$\begin{aligned}
\dim_H(\ell A) &= \liminf_{i\to\infty} \dim_H(\ell A_i) = \alpha_\ell, \\
\underline{\dim}_B(\ell A) &= \liminf_{i\to\infty} \underline{\dim}_B(\ell A_i) = \beta_\ell, \\
\overline{\dim}_B(\ell A) &= \limsup_{i\to\infty} \overline{\dim}_B(\ell A_i) = \gamma_\ell,
\end{aligned}$$

provided M_ℓ grows quickly enough (this was proved for the Hausdorff dimension in Example 3; the proof for box dimensions is similar but easier). If $\alpha_\ell = 1$ for some ℓ, then obviously $\beta_\ell = \gamma_\ell = 1$ as well, and one can check that there is no restriction on the digits of ℓA_i for all $i \geq \ell$. Thus, A has no restriction on all but finitely many of its binary digits, and therefore it contains an interval. This concludes the sketch of the proof. □

Remark 1. Recall that for any two bounded sets $A, B \subset \mathbb{R}$ we have

$$\overline{\dim}(A + B) \leq \overline{\dim}(A \times B) \leq \overline{\dim}(A) + \overline{\dim}(B),$$

but no such inequality holds for the lower box dimension in general. In contrast, for sumsets $A + A$ we do have a bound

$$\underline{\dim}_B(A + A) \le \underline{\dim}_B(A \times A) \le 2\underline{\dim}_B A,$$

since we can cover the product by squares coming from a cover of the components approximating the lower box dimension. Besides these "product" bounds, there are "Plünnecke" bounds between the different β_ℓ; see Proposition 1 in Sect. 3. Thus, finding the most general possible relations between the sequences α_n, β_n, and γ_n appears to be rather difficult.

3 Plünnecke Estimates for Box-Counting Dimensions

We begin by observing that in Theorem 1, if $\gamma_2 = \gamma_1$, then necessarily $\gamma_\ell{=}\gamma_1$ for all ℓ, so this theorem does not directly negate the possibility of a "Plünnecke estimate" for the upper box dimensions. However, it is possible to modify the construction to obtain counterexamples. We indicate how to show that for any $0 < \gamma < 1$ there exists a compact set $A \subset [0, 1]$ such that

$$\overline{\dim}_B A = \overline{\dim}_B(A + A) = \gamma,$$

but

$$\overline{\dim}_B(A + A + A) = \min(1, 3\gamma/2) > \gamma.$$

Recall from Example 6 that there exist compact sets $A', A'' \subset [0, 1]$ such that

$$\overline{\dim}_B(A') = \gamma/2, \quad \overline{\dim}_B(A' + A') = \gamma, \quad \overline{\dim}_B(A' + A' + A') = \min(1, 3\gamma/2),$$

$$\overline{\dim}_B(A'') = \gamma, \quad \overline{\dim}_B(A'' + A'') = \gamma, \quad \overline{\dim}_B(A'' + A'' + A'') = \gamma.$$

Moreover, these sets are constructed by specifying types of blocks for finite sequences of binary digits $x_{n_k} \cdots x_{n_{k+1}-1}$, where $\{n_k\}$ is a rapidly increasing sequence. Now let M_r be another rapidly increasing sequence, say $M_r = 2^{2^r}$. The set A is defined by using the blocks corresponding to A' for all $k \in [M_{2r-1}, M_{2r})$ for some r, and the blocks corresponding to A'' for $k \in [M_{2r}, M_{2r+1})$ for some r. It is then easy to check that

$$\overline{\dim}_B(iA) = \max(\overline{\dim}_B(iA'), \overline{\dim}_B(iA'')) \quad i = 1, 2, 3,$$

which yields the claim.

We finish the paper with the positive result mentioned in the introduction, which bounds the lower box dimension of iterated sumsets ℓB in terms of the lower box dimensions of A and $A + B$. The proof is a straightforward discretization argument using the Plünnecke–Ruzsa inequalities.

Proposition 1. *Let $A, B \subset \mathbb{R}$ be bounded sets. Then for all $\ell \geq 2$,*

$$\underline{\dim}_B(\ell B) \leq \ell \underline{\dim}_B(A+B) - (\ell-1)\underline{\dim}_B A.$$

In particular, if $\underline{\dim}_B(A+A) = \underline{\dim}_B A$, then

$$\underline{\dim}_B(\ell A) = \underline{\dim}_B A$$

for all $\ell \in \mathbb{N}$.

Proof. Let $\mathcal{D}_{j,1}$ be the family $\{[i2^{-j}, (i+1)2^{-j}] : i \in \mathbb{Z}\}$ of dyadic intervals of length 2^{-j}, and for $\ell \geq 2$ let

$$\mathcal{D}_{j,\ell} = \{[i2^{-j}, (i+\ell)2^{-j}] : i \in \mathbb{Z}\}.$$

Given a set $A \subset \mathbb{R}$, write $\mathcal{D}_{j,\ell}(A)$ for the intervals in $\mathcal{D}_{j,\ell}$ having nonempty intersection with A. Note that for any sets $A_1, \ldots, A_\ell \subset \mathbb{R}$ and any $j \geq 1$, if $b \in A_1 + \cdots + A_\ell$, then there are $I_i \in \mathcal{D}_{j,1}(A_i)$ such that $b \in I_1 + \cdots + I_\ell \in \mathcal{D}_{j,\ell}(A_1 + \cdots + A_k)$. Since b belongs to at most $\ell+1$ elements of $\mathcal{D}_{j,\ell}$, we have

$$|\mathcal{D}_{j,\ell}(A_1 + \cdots + A_\ell)| \leq (\ell+1)|\mathcal{D}_{j,1}(A_1) + \cdots + \mathcal{D}_{j,1}(A_\ell)|. \tag{5}$$

Moreover, it is easy to see that for a fixed ℓ we have

$$\underline{\dim}_B(A) = \liminf_{j\to\infty} \frac{\log|\mathcal{D}_{j,\ell}(A)|}{j}. \tag{6}$$

(Recall that log is the base 2 logarithm.)

The Plünnecke–Ruzsa theorem says that if E, F are finite subsets of $\mathbb{Z}$ with $|E+F| < K|E|$, then $|\ell F| < K^\ell |E|$; see [2, Sect. 6.5] for the proof and further background. We apply this result to $E = \mathcal{D}_{j,1}(A), F = \mathcal{D}_{j,1}(B)$ and use (5) to obtain

$$|\mathcal{D}_{j,\ell}(B)| \leq (\ell+1)\left(\frac{|\mathcal{D}_{j,2}(A+B)|}{|\mathcal{D}_j(A)|}\right)^\ell |\mathcal{D}_j(A)|. \tag{7}$$

Let $j_r \to \infty$ be a sequence such that

$$\lim_{r\to\infty} \frac{\log|\mathcal{D}_{j_r,2}(A+B)|}{j_r} = \underline{\dim}_B(A+B).$$

Using (7), we conclude that

$$\begin{aligned}
\underline{\dim}_B(\ell B) &\leq \liminf_{r\to\infty} \frac{\log|\mathcal{D}_{j_r,\ell}(B)|}{j_r} \\
&\leq \ell \liminf_{r\to\infty} \frac{\log|\mathcal{D}_{j_r,2}(A+B)|}{j_r} - (\ell-1)\limsup_{r\to\infty} \frac{\log|\mathcal{D}_{j_r}(A)|}{j_r} \\
&\leq \ell\underline{\dim}_B(A+B) - (\ell-1)\underline{\dim}_B B.
\end{aligned}$$

□

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Geometric Measures for Fractals

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Summary. In this survey article, we review the concept of fractal curvatures and fractal curvature measures and discuss some of the results known for self-similar sets. We emphasize in particular the close relations to the Minkowski content.

1 Introduction

Curvature measures are an important tool in geometry. Introduced for sets with positive reach by Federer [2], they have been extended to various classes of sets in $\mathbb{R}^d$ using methods like additive extension, approximation by parallel sets, or axiomatic approaches. A central role is played by Steiner type formulas, which describe the volume of the parallel sets and in which curvature measures appear as "coefficients."

The attempt to introduce some notion of curvature for fractal sets by means of approximation with parallel sets in [17] led to the definition of *fractal curvatures* and *fractal curvature measures*. In this survey chapter, we recall this concept and discuss the main results obtained so far for the class of self-similar sets. There are close relations to the well-known Minkowski content, which we want to emphasize here, since the analogies may be helpful for understanding the new concept. But the relation goes beyond analogy. Viewing Minkowski content as a curvature measure leads to a localization of this notion.

Our aim is to illustrate the main ideas in [17] in a concise and introductory way, to make the results more accessible. Most statements are presented without proofs; occasionally we try to sketch the main ideas. For details the reader is referred to [17]. Only the proof of the upper bound for the scaling exponents is discussed in more detail. It illustrates very well the kind of arguments required in this theory and yet is not too long to be included here.

We start by recalling Minkowski content and explaining its "localization." Curvature measures are in general signed measures. In Sect. 3, we discuss the difficulties arising when passing on from measures to signed measures.

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In Sect. 4, we recall curvature measures for polyconvex sets and define fractal curvatures. Then we are ready to discuss the results for self-similar sets in Sects. 5 and 6. Finally, in Sect. 7 we give some references to very recent advances and discuss possible generalizations and extensions.

2 Minkowski Content

For $A \subset \mathbb{R}^d$ and $\varepsilon > 0$, the *ε-parallel* set A_ε of A is given by

$$A_\varepsilon := \{x \in \mathbb{R}^d : \inf_{y \in A} \|x - y\| \leq \varepsilon\}.$$

We write $V(A_\varepsilon)$ for the volume or Lebesgue measure of A_ε. The *s-dimensional upper and lower Minkowski content* of A are defined by

$$\overline{\mathcal{M}}^s(A) := \limsup_{\varepsilon \to 0} \varepsilon^{s-d} V(A_\varepsilon) \qquad \text{and} \qquad \underline{\mathcal{M}}^s(A) := \liminf_{\varepsilon \to 0} \varepsilon^{s-d} V(A_\varepsilon).$$

If $\underline{\mathcal{M}}^s(A) = \overline{\mathcal{M}}^s(A)$, then the common value $\mathcal{M}^s(A)$ is the s-dimensional Minkowski content of A. The Minkowski content gives rise to the (upper and lower) *Minkowski* or *box dimension*, which characterizes the "optimal" scaling exponents s for A in the above definition:

$$\begin{aligned}\overline{D}(A) &= \inf\{t \geq 0 : \overline{\mathcal{M}}^t(A) = 0\} = \sup\{t \geq 0 : \overline{\mathcal{M}}^t(A) = \infty\}\\ \underline{D}(A) &= \inf\{t \geq 0 : \underline{\mathcal{M}}^t(A) = 0\} = \sup\{t \geq 0 : \underline{\mathcal{M}}^t(A) = \infty\}.\end{aligned}$$

If $\underline{D}(A) = \overline{D}(A)$, the common value $D = D(A)$ is called the Minkowski dimension of A. We will omit the dimension index D and write $\mathcal{M}(A)$ for the Minkowski content $\mathcal{M}^D(A)$ of dimension D. In many situations the Minkowski dimension is known, while the computation of the Minkowski content is a difficult problem (similar to the case of the Hausdorff dimension and the exact value of the corresponding Hausdorff measure). Often it is not even known if the limit $\mathcal{M}(A)$ exists. A set is called *Minkowski measurable* if and only if $\mathcal{M}(A)$ exists and is positive and finite.

Due to applications in spectral theory and as a "lacunarity" parameter, see [5, 9], the question of Minkowski measurability aroused considerable interest. However, even for self-similar sets $F \subset \mathbb{R}^d$ satisfying the open set condition (OSC; see definitions in Sect. 5), the question remained open for some time. For subsets of $\mathbb{R}$, the first results were obtained by Lapidus [5] and Falconer [1]. The following general characterization for self-similar sets in $\mathbb{R}^d$ was given by Gatzouras [3, Theorems 2.3 and 2.4]: Non-arithmetic sets are Minkowski measurable, while for arithmetic sets, only an averaged limit $\widetilde{\mathcal{M}}(F)$ can be shown to exist. $\widetilde{\mathcal{M}}(F)$ is called the *average Minkowski content* and is defined by

$$\widetilde{\mathcal{M}}(F) := \lim_{\delta \searrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-d} V(F_\varepsilon) \frac{d\varepsilon}{\varepsilon}.$$

Theorem 2.1 (Gatzouras's theorem) *Let F be a self-similar set satisfying the OSC. Then the average Minkowski content $\widetilde{\mathcal{M}}(F)$ of F exists and is positive and finite. If F is non-arithmetic, then also the Minkowski content $\mathcal{M}(F)$ of F exists and $0 < \mathcal{M}(F) < \infty$, i.e., F is Minkowski measurable.*

The result is based on the renewal theorem. From Gatzouras's result explicit formulas for the computation of $\mathcal{M}(F)$ and $\widetilde{\mathcal{M}}(F)$ can be derived (see Theorem 5.4 below).

It is well known that the Minkowski content $\mathcal{M}(\,\cdot\,)$ is finitely additive but not σ-additive, i.e., it is a content but not a measure. However, changing the point of view slightly, the Minkowski content of F can be interpreted as a uniform mass distribution on F, hence a measure. The key to this is to "localize" the approximation by parallel sets. Let $C_d(F_\varepsilon, \cdot\,) := V(F_\varepsilon \cap \cdot\,)$. $C_d(F_\varepsilon, \cdot\,)$ is a measure on F_ε and it is natural to ask for the limiting behavior of these measures as $\varepsilon \to 0$. If they are appropriately rescaled, they do indeed converge in the weak sense to a limit measure on F, provided their total masses converge, i.e., provided the Minkowski content of F exists. Otherwise, only some average versions of these measures can be shown to converge. Let μ_F be the normalized s-dimensional Hausdorff measure on F (where s is the dimension of F) i.e., $\mu_F = (\mathcal{H}^s(F))^{-1}\mathcal{H}^s|_F(\,\cdot\,)$.

Theorem 2.2 *Let F be a self-similar set satisfying the OSC. If F is non-arithmetic, then the measures $\varepsilon^{D-d}C_d(F_\varepsilon, \cdot\,)$ converge weakly to $\mathcal{M}(F)\mu_F =: \mathcal{C}_d(F, \cdot\,)$ as $\varepsilon \searrow 0$. The measures*

$$\widetilde{C}_d(F_\delta, \cdot\,) := \frac{1}{|\ln \delta|}\int_\delta^1 \varepsilon^{D-d}C_d(F_\varepsilon, \cdot\,)\frac{d\varepsilon}{\varepsilon} \tag{1}$$

always converge weakly to $\widetilde{\mathcal{C}}_d(F, \cdot\,) = \widetilde{\mathcal{M}}(F)\mu_F$ as $\delta \searrow 0$.

Theorem 2.2 localizes Gatzouras's theorem. Not only does the total (average) ε-parallel volume of self-similar sets converge, as $\varepsilon \searrow 0$. The convergence even takes place locally for "nice" subsets of $\mathbb{R}^d$. More precisely, if $B \subset \mathbb{R}^d$ is a μ_F*-continuity set*, i.e., if $\mu_F(\partial B) = 0$, then $\varepsilon^{D-d}C_d(F_\varepsilon, B) \to \mathcal{M}(F)\mu_F(B)$ as $\varepsilon \searrow 0$ for F non-arithmetic (and $\widetilde{C}_d(F_\varepsilon, \cdot\,) \to \widetilde{\mathcal{M}}(F)\mu_F(B)$ for general F).

The approximation with parallel sets induces a measure on F. The uniformity of the measures $C_d(F_\varepsilon, \cdot\,)$ carries over to the limit measure (which is by no means obvious). However, any uniform measure on F is necessarily a multiple of the Hausdorff measure on F. The Minkowski content comes in naturally as the total mass of the limit measure. The theorem parallels known results on the continuity of curvature measures and, in particular, the volume (see, e.g., Proposition 4.2(6)) and extends them to the fractal setting. (The classical results only give $C_d(F_\varepsilon, \cdot\,) \xrightarrow{\text{w}} C_d(F, \cdot\,) = 0$ as $\varepsilon \searrow 0$ for fractal sets F of dimension less than d.)

In the light of the above results for Minkowski content and its localization, it is natural to ask for the limiting behavior of other geometric measures

associated to the parallel sets, for instance, the surface area $\mathcal{H}^{d-1}(\partial A_\varepsilon)$ of the boundary of A_ε, its Euler characteristic $\chi(A_\varepsilon)$, or, more generally, curvature measures, as $\varepsilon \searrow 0$. In fact, it was the study of curvature measures that suggested the local interpretation of Minkowski content.

3 Sequences of Signed Measures

Curvature measures are signed measures in general, and this is one of the main reasons why things become more difficult compared to the situation for the Minkowski content. In order to see the difficulties arising from the non-positivity of the measures, we briefly discuss some of the phenomena arising in the study of sequences of signed measures.

Let $(\mu_\varepsilon)_{\varepsilon>0}$ be a sequence of finite signed measures (i.e., with finite total variation) on some metric space X. Denote by $\mu_\varepsilon^+, \mu_\varepsilon^-$, and $\mu_\varepsilon^{\mathrm{var}}$ the positive, negative, and total variation measure of μ_ε, respectively, and let $M(\varepsilon)$ and $M^{\mathrm{var}}(\varepsilon)$ be the total masses of μ_ε and $\mu_\varepsilon^{\mathrm{var}}$.

Recall that the sequence (μ_ε) is said to converge weakly to a limit measure μ as $\varepsilon \searrow 0$, in symbols $\mu_\varepsilon \xrightarrow{\mathrm{w}} \mu$, iff for each bounded continuous function f the integrals $\mu_\varepsilon(f) := \int_X f d\mu_\varepsilon$ converge to the value of the integral $\mu(f) := \int_X f d\mu$. (The integral with respect to a signed measure μ is defined by $\int_X f d\mu = \int_X f d\mu^+ - \int_X f d\mu^-$.)

In general, the weak convergence $\mu_\varepsilon \xrightarrow{\mathrm{w}} \mu$ does not imply the convergence of the variation measures.

Example 3.1 *For $\varepsilon > 0$, let the measures μ_ε on $\mathbb{R}$ be defined by $\mu_\varepsilon := \delta_0 - \delta_\varepsilon$ for $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, and $\mu_\varepsilon := 0$ otherwise. (Here δ_x denotes the Dirac measure at x.) Then $\mu_{1/n}^{\mathrm{var}} = \delta_0 + \delta_{1/n}$ and so $M^{\mathrm{var}}(1/n) = 2$, while $M^{\mathrm{var}}(\varepsilon) = 0$ otherwise. Hence $\mu_\varepsilon \xrightarrow{\mathrm{w}} 0$, while $\mu_\varepsilon^{\mathrm{var}}$ does not converge as $\varepsilon \searrow 0$.*

If additionally the convergence of $M^{\mathrm{var}}(\varepsilon)$ is assumed, then the variation measures converge weakly to some limit measures $\mu^{(+)}, \mu^{(-)}$, and $\mu^{(\mathrm{var})}$ and the relations $\mu = \mu^{(+)} - \mu^{(-)}$ and $\mu^{(\mathrm{var})} = \mu^{(+)} + \mu^{(-)}$ carry over to the limits. However, in general, $\mu^{(+)}, \mu^{(-)}$, and $\mu^{(\mathrm{var})}$ are not the variation measures of μ. The limit $\mu^{(\mathrm{var})}$ of the total variation measures is only an upper bound for μ^{var}, the total variation of the limit measure μ (and similarly for μ^+, μ^-); hence the notation $\mu^{(\mathrm{var})}$ with parentheses. The reason is that the measures μ_ε^+ and μ_ε^- have essentially disjoint supports, a property which is not necessarily preserved in the limit. $\mu^{(+)}$ and $\mu^{(-)}$ may even have equal support.

Example 3.2 *For $\varepsilon > 0$, let the measures μ_ε on $\mathbb{R}$ be defined by $\mu_\varepsilon = \delta_\varepsilon - \delta_1$. Then $\mu_\varepsilon^{\mathrm{var}} = \mu_\varepsilon^+ + \mu_\varepsilon^- = \delta_\varepsilon + \delta_1$ and so $M^{\mathrm{var}}(\varepsilon) = 2$, while $M(\varepsilon) = 0$ for each $\varepsilon > 0$. The total mass zero gives no hint on the real "size" of the underlying measures μ_ε. For $\varepsilon \searrow 0$, one has $\mu_\varepsilon \xrightarrow{\mathrm{w}} \mu = \delta_0 - \delta_1$ and $\mu_\varepsilon^{\mathrm{var}} \xrightarrow{\mathrm{w}} \mu^{(\mathrm{var})} = \delta_0 + \delta_1$. In this case, $\mu^{(\mathrm{var})}$ is the variation of the limit measure, i.e., $\mu^{(\mathrm{var})} = \mu^{\mathrm{var}}$.*

Now let $\tilde{\mu}_\varepsilon = \delta_\varepsilon - \delta_0$. *Then* $\tilde{\mu}_\varepsilon^{\mathrm{var}} = \delta_\varepsilon + \delta_0$, $\widetilde{M}(\varepsilon) = 0$, *and* $\widetilde{M}^{\mathrm{var}}(\varepsilon) = 2$ *as before. But now, as* $\varepsilon \searrow 0$, *the situation is different:* $\tilde{\mu}_\varepsilon \xrightarrow{\mathrm{w}} \tilde{\mu} = \delta_0 - \delta_0 = 0$ *and* $\tilde{\mu}_\varepsilon^{\mathrm{var}} \xrightarrow{\mathrm{w}} \tilde{\mu}^{(\mathrm{var})} = 2\delta_0$. *The limit* $\tilde{\mu}$ *is the zero measure (with* $\tilde{\mu}^{\mathrm{var}} = 0$*) and so* $\tilde{\mu}^{(\mathrm{var})}$ *is not the total variation measure of* $\tilde{\mu}$.

The above examples show that for the limiting behavior of the measures μ_ε, the behavior of the mass $M^{\mathrm{var}}(\varepsilon)$ is essential. In case, $M^{\mathrm{var}}(\varepsilon) \to \infty$ or $M^{\mathrm{var}}(\varepsilon) \to 0$, rescaling with a factor ε^t for some $t \in \mathbb{R}$ may help to obtain non-trivial limits. We define the *(upper) mass scaling exponent* m of the sequence $(\mu_\varepsilon)_{\varepsilon>0}$ by

$$m := \inf\{t \in \mathbb{R} : \lim_{\varepsilon \searrow 0} \varepsilon^t M^{\mathrm{var}}(\varepsilon) = 0\}. \tag{2}$$

Remark 3.3 *There are several other scaling exponents associated to the sequence* (μ_ε) *which are sometimes useful. The numbers* m', m^+, *and* m^- *are defined by replacing* $M^{\mathrm{var}}(\varepsilon)$ *in the above definition with* $|M(\varepsilon)|$, $M^+(\varepsilon)$, *and* $M^-(\varepsilon)$, *respectively (where* $M^+(\varepsilon)$ $(M^-(\varepsilon))$ *is the total mass of* μ_ε^+ (μ_ε^-)*). In general,* $m' \le m = \max\{m^+, m^-\}$. *Similarly, lower scaling exponents* $\underline{m}, \underline{m}', \underline{m}^+$, *and* $\underline{m}^-$ *can be introduced by replacing the limits with* lim inf *'s (which in general may differ from the corresponding upper exponents). Often the interrelations of these eight exponents allow conclusions on the existence of limits and their properties (see below).*

Let

$$M := \lim_{\varepsilon \searrow 0} \varepsilon^m M(\varepsilon) \quad \text{and} \quad M^{\mathrm{var}} := \lim_{\varepsilon \searrow 0} \varepsilon^m M^{\mathrm{var}}(\varepsilon)$$

be the rescaled limits of the total masses, provided they exist. Of course, in general, these limits do not exist and then lim inf's and lim sup's can be considered, but in this case the corresponding (rescaled) measures cannot converge, either. Provided that they exist, the weak limit of the rescaled sequence $(\varepsilon^m \mu_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \searrow 0$ will be denoted by μ and the limit of $(\varepsilon^m \mu_\varepsilon^{\mathrm{var}})_{\varepsilon>0}$ by $\mu^{(\mathrm{var})}$,

$$\varepsilon^m \mu_\varepsilon \xrightarrow{\mathrm{w}} \mu \qquad \text{and} \qquad \varepsilon^m \mu_\varepsilon^{\mathrm{var}} \xrightarrow{\mathrm{w}} \mu^{(\mathrm{var})}.$$

If positive and negative parts scale with different exponents, i.e., if $m^+ \neq m^-$, then the limit measure is purely positive or purely negative, depending on which of these exponents is larger. For instance, if $m^+ > m^-$, then $\mu = \mu^+ = \mu^{\mathrm{var}}$ and $\mu^- = 0$. The most interesting case is when positive and negative parts scale with the same exponent. Then the limit measure can have positive and negative part. However, it can also happen in this case that positive and negative variation measures cancel each other out in the limit.

Example 3.4 *For* $\varepsilon > 0$ *and* $s, t \in \mathbb{R}$, *let* $\mu_\varepsilon^{s,t} := \varepsilon^{-s}\delta_\varepsilon - \varepsilon^{-t}\delta_1$. *If* $s > t$, *then* $m = s$ *is the correct scaling exponent and* $\varepsilon^m \mu_\varepsilon^{s,t} = \delta_\varepsilon - \varepsilon^{s-t}\delta_1 \to \delta_0$ *as* $\varepsilon \searrow 0$. *Hence, the limit measure is purely positive. If* $s < t$, *then* $m = t$ *and* $\varepsilon^m \mu_\varepsilon^{s,t} = \varepsilon^{t-s}\delta_\varepsilon - \delta_1 \to -\delta_1 =: \mu$ *as* $\varepsilon \searrow 0$, *i.e., the limit measure is*

purely negative. The most interesting case is $s = t$. Then $m = s = t$ and $\varepsilon^m \mu_\varepsilon^{s,t} = \delta_\varepsilon - \delta_1 \to \delta_0 - \delta_1$, i.e., the limit measure has positive and negative parts. If the measures $\hat{\mu}_\varepsilon^{s,t} := \varepsilon^{-s}\delta_\varepsilon - \varepsilon^{-t}\delta_0$ are considered instead, then, for $s = t$, still $m = s$ but now $\varepsilon^m \hat{\mu}_\varepsilon^{s,t} \to \delta_0 - \delta_0 = 0$ as $\varepsilon \searrow 0$, i.e., the limit measure is the zero measure, although s is the optimal scaling exponent.

If the total masses of the measures μ_ε do not converge (even if appropriately rescaled), then (Cesàro) averaging may improve the convergence behavior, in particular, if self-similar sets are considered (compare, e.g., the results for the averaged Minkowski content). Let

$$\widetilde{M} := \lim_{\delta \searrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^m M(\varepsilon) \frac{d\varepsilon}{\varepsilon}$$

and if $\widetilde{M}$ exists, one can ask for the weak convergence of the sequence $(\tilde{\mu}_\varepsilon)_{\varepsilon>0}$ defined by

$$\tilde{\mu}_\delta(\,\cdot\,) := \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^m \mu_\varepsilon(\,\cdot\,) \frac{d\varepsilon}{\varepsilon}$$

for $\delta > 0$. The limit measure will be denoted by $\tilde{\mu}$. If M exists, then $\widetilde{M} = M$ and if the weak limit μ exists, then $\tilde{\mu} = \mu$. Thus, the average limits are a reasonable extension of the corresponding limits.

Remark 3.5 *If the measure μ_ε is positive (as for instance the volume of the ε-parallel set), then $\mu_\varepsilon^{\mathrm{var}} = \mu_\varepsilon$ and $M^{\mathrm{var}}(\varepsilon) = M(\varepsilon)$, thus $m' = m$, $M = M^{\mathrm{var}}$ and the limit measure of the sequence $(\varepsilon^m \mu_\varepsilon)$ is non-negative (if it exists). Hence the situation simplifies and the number of exponents and limits to look at reduces. In particular, if μ_ε is the volume of the ε-parallel set F_ε (of a compact set $F \subset \mathbb{R}^d$), then $m = \overline{D} - d$ (where $\overline{D}$ was the (upper) Minkowski dimension) and M (if it exists) is the Minkowski content of F.*

4 Curvature Measures and Fractal Curvatures

First we recall the notion of curvature measure and discuss some properties. For simplicity, we restrict considerations to polyconvex sets. For more details see, for instance, Schneider [14].

4.1 Curvature Measures

Recall that a set $K \subseteq \mathbb{R}^d$ is *convex* iff for any two points $x, y \in K$ the line segment connecting them is contained in K. We write $\mathcal{K}^d$ for the family of all *convex bodies*, i.e., of all non-empty compact convex sets in $\mathbb{R}^d$. A set K is called *polyconvex* if it has a representation as a finite union of convex bodies. The convex ring $\mathcal{R}^d$ is the family of all polyconvex sets in $\mathbb{R}^d$. It is called a *ring* because of its stability with respect to finite unions and intersections.

For each set $K \in \mathcal{R}^d$, *curvature measures* $C_0(K, \cdot)$, $C_1(K, \cdot)$, ..., $C_d(K, \cdot)$ can be defined. For convex bodies K, they are characterized by the *local Steiner formula*. Let π_K denote the *metric projection* onto the convex set $K \in \mathcal{K}^d$, mapping a point $x \in \mathbb{R}^d$ to its (unique) nearest point in K. For $\varepsilon > 0$, the set $K_\varepsilon \cap \pi_K^{-1}(B)$ is the *local ε-parallel set* of K with respect to the Borel set B.

Theorem 4.1 *For each $K \in \mathcal{K}^d$, there are uniquely determined finite Borel measures $C_0(K, \cdot)$, ..., $C_d(K, \cdot)$ on $\mathbb{R}^d$, such that*

$$V(K_\varepsilon \cap \pi_K^{-1}(B)) = \sum_{k=0}^{d} \varepsilon^{d-k} \kappa_{d-k} C_k(K, B)$$

for each Borel set $B \subseteq \mathbb{R}^d$ and $\varepsilon > 0$.

Here κ_i is the i-dimensional volume of the unit ball in $\mathbb{R}^i$.

Curvature measures of convex bodies are measures in the second argument and they are *additive* in the first. If $K, L, K \cup L \in \mathcal{K}^d$, then

$$C_k(K \cup L, B) = C_k(K, B) + C_k(L, B) - C_k(K \cap L, B). \tag{3}$$

The additivity allows us to extend curvature measures to sets $K \in \mathcal{R}^d$, by using representations with convex sets. Curvature measures of polyconvex sets are in general signed measures. However, for $k = d$ and $d-1$, $C_k(K, \cdot)$ is always non-negative. $C_d(K, \cdot) = V(K \cap \cdot)$ is the volume restricted to K and $C_{d-1}(K, \cdot)$ is half the surface area of K, provided K is the closure of its interior. Except for $k = d$, the measures $C_k(K, \cdot)$ are concentrated on the boundary ∂K of K. If the boundary of K is sufficiently smooth, curvature measures have a representation as integrals of the symmetric functions of principal curvatures. The total mass $C_k(K) := C_k(K, \mathbb{R}^d)$ of the measure $C_k(K, \cdot)$ is called the *kth total curvature* of K. Total curvatures are also known as *intrinsic volumes* or *Minkowski functionals*. $C_0(K)$ coincides with the *Euler characteristic* of K, by the Gauss–Bonnet theorem. For $K \in \mathcal{K}^d$, $C_0(K) = 1$. We collect some important properties of curvature measures and total curvatures.

Proposition 4.2 *Let $K, L \in \mathcal{R}^d$ and $B \subseteq \mathbb{R}^d$ be an arbitrary Borel set. For each $k \in \{0, \ldots, d\}$ we have:*

1. Additivity: $C_k(K \cup L, B) = C_k(K, B) + C_k(L, B) - C_k(K \cap L, B)$.
2. Motion invariance: *If g is a rigid motion, then* $C_k(gK, gB) = C_k(K, B)$.
3. Homogeneity: *For $\lambda > 0$,* $C_k(\lambda K, \lambda B) = \lambda^k C_k(K, B)$.
4. Locality: *If $K \cap A = L \cap A$ for some open set $A \subseteq \mathbb{R}^d$, then* $C_k(K, B) = C_k(L, B)$ *for all Borel sets $B \subseteq A$.*
5. Continuity: *If $K, K^1, K^2, \ldots \in \mathcal{K}^d$ with $K^i \to K$ as $i \to \infty$ (w.r.t. the Hausdorff metric) then the measures $C_k(K^i, \cdot)$ converge weakly to $C_k(K, \cdot)$,* $C_k(K^i, \cdot) \xrightarrow{\text{w}} C_k(K, \cdot)$. *In particular,* $C_k(K^i) \to C_k(K)$.

6. Continuity II: *If* $K \in \mathcal{R}^d$, *then* $C_k(K_\varepsilon, \cdot) \xrightarrow{\text{w}} C_k(K, \cdot)$ *as* $\varepsilon \searrow 0$. *In particular,* $\lim_{\varepsilon \searrow 0} C_k(K_\varepsilon) = C_k(K)$.
7. Monotonicity of the total curvatures: *If* $K, L \in \mathcal{K}^d$ *and* $K \subseteq L$, *then* $C_k(K) \leq C_k(L)$.

The additivity leads to the following useful formula called the *inclusion-exclusion principle*. If $K^1, \ldots, K^n \in \mathcal{R}^d$ and $K := \bigcup_{i=1}^n K^i$, then for all Borel sets $B \subseteq \mathbb{R}^d$

$$C_k(K, B) = \sum_{I \in N_n} (-1)^{\#I-1} C_k\big(\bigcap_{i \in I} K^i, B\big). \tag{4}$$

Here N_n is the family of all non-empty subsets I of $\{1, \ldots, n\}$. Hence, the sum is over all intersections of the K^i. $\#I$ denotes the cardinality of the set I.

Remark 4.3 *The properties of motion invariance, homogeneity, and locality in Proposition 4.2 carry over to the variation measures, the continuity II does so only if* K *itself is already a parallel set (of some other set). Unfortunately, additivity does not hold for the variation measures.*

4.2 Fractal Curvature Measures

Let $A \subset \mathbb{R}^d$ be a compact set. To define fractal curvatures for A, we need the curvature measures $C_k(A_\varepsilon, \cdot)$ of the parallel sets A_ε to be well defined for all $\varepsilon > 0$ (or at least for small ε). This can, for instance, be ensured by requiring that all the parallel sets are polyconvex and, for simplicity, we will assume this in the sequel. Keep in mind that more general notions of curvature exist to which the following concepts similarly apply.

If the $C_k(A_\varepsilon, \cdot)$ are well defined, then they form a sequence of signed measures (μ_ε) as in the previous section. Hence, the notions of scaling exponents, total mass limits, and weak limits specialize to the situation here. We first discuss the appropriate scaling exponents.

Definition 4.4 *Let* $A \subseteq \mathbb{R}^d$ *be compact with* $A_\varepsilon \in \mathcal{R}^d$ *for* $\varepsilon > 0$ *and let* $k \in \{0, 1, \ldots, d\}$. *The* (upper) kth curvature scaling exponent *of* A *is the number*

$$s_k = s_k(A) := \inf\left\{t : \varepsilon^t C_k^{\text{var}}(A_\varepsilon) \to 0 \text{ as } \varepsilon \searrow 0\right\}.$$

Hence, s_k is the exponent m of Sect. 3 for the kth curvature measures. The total masses $M(\varepsilon)$ specialize to the total curvatures $C_k(A_\varepsilon)$ (and $M^{\text{var}}(\varepsilon)$ to $C_k^{\text{var}}(A_\varepsilon)$). Their limits are the *fractal curvatures* of the set A.

Definition 4.5 *For* $k \in \{0, \ldots, d\}$, *let the* kth fractal curvature *of* A *be defined by*

$$\mathcal{C}_k(A) := \lim_{\varepsilon \searrow 0} \varepsilon^{s_k} C_k(A_\varepsilon)$$

and let the corresponding average kth fractal curvature *be the number*

$$\widetilde{\mathcal{C}}_k(A) := \lim_{\delta \searrow 0} \frac{1}{|\ln \delta|} \int_\delta^1 , \varepsilon^{s_k} C_k(A_\varepsilon) \frac{d\varepsilon}{\varepsilon},$$

provided these limits exist.

Note that if $\mathcal{C}_k(A)$ exists, then $\widetilde{\mathcal{C}}_k(A) = \mathcal{C}_k(A)$. Finally, we introduce the *fractal curvature measures* A as weak limits of the curvature measures of A_ε.

Definition 4.6 *For $k \in \{0, \dots, d\}$, we call a (signed) measure $\mathcal{C}_k(A, \cdot)$ the* kth fractal curvature measure *of A, iff*

$$\varepsilon^{s_k} C_k(A_\varepsilon, \cdot) \xrightarrow{\mathrm{w}} \mathcal{C}_k(A, \cdot),$$

and we call $\widetilde{\mathcal{C}}_k(A, \cdot)$ the average kth fractal curvature measure *of A, iff it is the weak limit of the measures*

$$\widetilde{C}_k(A_\delta, \cdot) := \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{s_k} C_k(A_\varepsilon, \cdot) \frac{d\varepsilon}{\varepsilon}$$

as $\delta \searrow 0$.

We state some general properties of fractal curvatures and their scaling exponents. $\mathcal{C}_k(A)$ is the total mass of the fractal curvature measure, i.e., $\mathcal{C}_k(A, \mathbb{R}^d) = \mathcal{C}_k(A)$. The support of $\mathcal{C}_k(A, \cdot)$ is contained in ∂A, if $k < d$. For $k = d$, $s_d = \overline{D} - d$ and $\mathcal{C}_d(F) = \mathcal{M}(F)$, i.e., it is just the Minkowski content. The motion invariance and homogeneity of $C_k^{\mathrm{var}}(A_\varepsilon)$ (see Proposition 4.2) imply that $s_k(A)$ is motion and scaling invariant, i.e., we have $s_k(gA) = s_k(A)$ for rigid motions g and $s_k(rA) = s_k(A)$ for $r > 0$. Similarly, the fractal curvatures are motion invariant and homogeneous: $\mathcal{C}_k(gA) = \mathcal{C}_k(A)$ and $\mathcal{C}_k(rA) = r^{k+s_k} \mathcal{C}_k(rA)$. Note that the order of homogeneity is $k + s_k$. If $s_k = D - k$, where D is the Minkowski dimension of A (this is what one would expect for s_k; also see the results for self-similar sets below), then $k + s_k = D$, i.e., the fractal curvatures are homgeneous of order D. Finally, by the continuity of curvature measures, fractal curvatures and fractal curvature measures are consistent with their classical counterparts. For polyconvex sets K with $C_k^{\mathrm{var}}(K) \neq 0$, one has $s_k(K) = 0$ and thus $\mathcal{C}_k(K, \cdot) = C_k(K, \cdot)$ (see Proposition 2.2.10 in [17]).

5 Curvature Measures for Self-Similar Sets

Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, \dots, N$, be contracting similarities. Denote the contraction ratio of S_i by $r_i \in (0, 1)$. It is well known that there is a unique non-empty and compact *invariant set* $F \subset \mathbb{R}^d$ for the system $\{S_1, \dots, S_N\}$, i.e., a set F satisfying the equation

$$F = \bigcup_{i=1}^N S_i F.$$

F is called the *self-similar set* generated by $\{S_1, \dots, S_N\}$. Throughout we will assume that F (or, more precisely, the system $\{S_1, \dots, S_N\}$) satisfies the *open set condition* (OSC), i.e., there exists an open, non-empty, bounded

subset $O \subset \mathbb{R}^d$ such that $\bigcup_i S_i O \subseteq O$ and $S_i O \cap S_j O = \varnothing$ for all $i \neq j$. If the OSC holds, then the set O is not unique and, by [13], it is always possible to choose O such that $O \cap F \neq \varnothing$. (Then F is said to satisfy the *strong open set condition* (SOSC) for O.) The unique solution s of $\sum_{i=1}^N r_i^s = 1$ is called the *similarity dimension* of F. It is well known that under the OSC s coincides with the Hausdorff and Minkowski dimensions of F, i.e., $s = D$.

Let $h > 0$. A finite set of positive real numbers $\{y_1, ..., y_N\}$ is called *h-arithmetic* if h is the largest number such that $y_i \in h\mathbb{Z}$ for $i = 1, \ldots, N$. If no such number h exists for $\{y_1, ..., y_N\}$, the set is called *non-arithmetic*. We call a self-similar set F (non-)arithmetic if the set $\{-\ln r_1, \ldots, -\ln r_N\}$ is.

For a self-similar set $F \in \mathbb{R}^d$, it is sufficient to assume that there exists some parallel set $F_\varepsilon, \varepsilon > 0$ which is polyconvex, since this implies already that all the parallel sets of F have this property (see [17, Proposition 2.3.1]). So for self-similar sets we have the dichotomy that either all its parallel sets are polyconvex or none are. A set with all parallel sets polyconvex is, for instance, the Sierpinski gasket, and an example with no polyconvex parallel sets is the Koch curve. For subsets of $\mathbb{R}$, all parallel sets are polyconvex.

For self-similar sets F, an upper bound for the (upper) scaling exponents $s_k = s_k(F)$ is given by the following result.

Theorem 5.1 [17, Theorem 2.3.2] *Let F be a self-similar set satisfying OSC and $F_\varepsilon \in \mathcal{R}^d$, and let $k \in \{0, \ldots, d\}$. The expression $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$ is uniformly bounded in $(0, 1]$, i.e. there is a constant M such that for all $\varepsilon \in (0, 1]$, $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon) \leq M$.*

Corollary 5.2 *$s_k \leq s - k$.*

Another immediate consequence is that the expression $\varepsilon^{s-k} |C_k(F_\varepsilon)|$ is also bounded by M for $\varepsilon \in (0, 1]$. A proof of Theorem 5.1 is given in Sect. 6.

Lower bounds for s_k are considerably harder to establish. For most self-similar sets, the equality $s_k = s - k$ holds. However, equality is not true in general, which is easily seen from the following example.

Example 5.3 *The unit cube $Q = [0, 1]^d \subset \mathbb{R}^d$ (considered as a self-similar set generated by 2^d similarities, each with contraction ratio $\frac{1}{2}$) has similarity dimension $s = d$. For the curvature measures of its parallel sets no rescaling is necessary. Q is convex and so are its parallel sets Q_ε. The continuity implies that, for $k = 0, \ldots, d$, $C_k(Q_\varepsilon, \cdot) \to C_k(Q, \cdot)$ as $\varepsilon \searrow 0$. Therefore, $s_k(Q) = 0$, which, for $k < d$, is certainly different from $d - k$.*

Theorem 5.4 [17, Theorem 2.3.6] *Let F be a self-similar set satisfying the OSC and let $F_\varepsilon \in \mathcal{R}^d$. Let $k \in \{0, \ldots, d\}$ and assume that $s_k = s - k$. Then the following holds:*

(i) The average kth fractal curvature $\widetilde{\mathcal{C}}_k(F)$ exists and equals

$$X_k := \frac{1}{\eta} \int_0^1 \varepsilon^{s-k-1} R_k(\varepsilon)\, d\varepsilon, \tag{5}$$

where $\eta = -\sum_{i=1}^{N} r_i^s \ln r_i$ *and the function* $R_k : (0,1] \to \mathbb{R}$ *is given by*

$$R_k(\varepsilon) = C_k(F_\varepsilon) - \sum_{i=1}^{N} \mathbf{1}_{(0,r_i]}(\varepsilon) C_k((S_i F)_\varepsilon). \tag{6}$$

(ii) If F *is non-arithmetic, then the* k*th fractal curvature* $\mathcal{C}_k(A)$ *exists and equals* X_k*.*

The formula (5) allows explicit calculations of fractal curvatures. Many examples are considered in [17, Sect. 2.4].

Remark 5.5 *The result extends to the case* $s_k < s - k$*, in the sense that* $\lim_{\varepsilon \searrow 0} \varepsilon^{s-k} C_k(F_\varepsilon)$ *(or its averaged counterpart, respectively) equals* X_k*. But in this case, we have* $X_k = 0$ *and obtain no information on the existence of fractal curvatures, since we are looking at the wrong scaling exponent* $(s - k$ *instead of* $s_k)$*. However, it is very useful to know that the formula holds in general. Typically, the scaling exponent is not a priori known. Computing* X_k *with the given formula allows us to verify that* $s_k = s - k$*. Namely, if* $X_k \neq 0$*, then necessarily* $s_k = s - k$*. If* $X_k = 0$*, then both cases are possible for* s_k*, either* $s_k = s - k$ *or* $s_k < s - k$ *(see Example 2.4.6 in [17]). In this situation, the computation of* X_k^{var} *or the study of the local behavior of the curvature measures helps (see Theorem 2.3.8 in [17]).*

For the proof we used a renewal theorem; a version suitable for taking limits as $\varepsilon \searrow 0$ (rather than to ∞) is stated in [17, Theorem. 4.1.4]. The main observation is that the function $f(\varepsilon) := C_k(F_\varepsilon)$ satisfies a renewal equation with error term $R_k(\varepsilon)$:

$$f(\varepsilon) = \sum_{i=1}^{N} r_i^k \mathbf{1}_{(0,r_i]}(\varepsilon) f(\varepsilon / r_i) + R_k(\varepsilon),$$

which is due to the equality $C_k((S_i F)_\varepsilon) = r_i^k C_k(F_{\varepsilon / r_i})$. The difficulty is to verify that the hypotheses of the renewal theorem are satisfied. We require some bound on the growth of $R_k(\varepsilon)$ as $\varepsilon \searrow 0$ and the continuity of $C_k(F_\varepsilon)$ and thus of $R_k(\varepsilon)$ in ε up to a discrete set of exceptions. The latter is easily derived from the properties of the curvature measures, while for the bounds on R_k some considerable effort is required. The following lemma is the key to most of the results on fractal curvatures and fractal curvature measures obtained so far.

Let $\Sigma^* = \bigcup_{n \in \mathbb{N} \cup \{0\}} \{1, \ldots, N\}^n$ and, for $0 < r \leq 1$, let $\Sigma(r)$ be the family of all finite words $w = w_1 \ldots w_n \in \Sigma^*$ such that

$$r_w < r \leq r_w r_{w_n}^{-1}. \tag{7}$$

Choose a set O such that F satisfies the SOSC for O, i.e., $F \cap O \neq \varnothing$. For $0 < r \leq 1$, we define the set $O(r)$ by

$$O(r) := \bigcup_{v \in \Sigma(r)} S_v O \tag{8}$$

and, for $r > 1$, by $O(r) := O$. In particular, $O(1) = \mathbf{S}O = \bigcup_i S_i O$. Note that F satisfies the OSC with the open set $O(r)$, $r > 0$. For the complement $O(r)^c$ of these sets the following estimate holds.

Lemma 5.6 [17, Lemma 5.2.1] *For each $r > 0$, there exist constants $c, \gamma, \rho > 0$ such that for all $\varepsilon \leq \delta \leq \rho r$*

$$C_k^{\mathrm{var}}(F_\varepsilon, (O(r)^c)_\delta) \leq c\varepsilon^{k-s}\delta^\gamma.$$

In fact, only the constant c depends on r, while ρ and γ merely depend on the choice of O. The estimate roughly means that, as ε approaches 0, the mass of $C_k^{\mathrm{var}}(F_\varepsilon, \cdot\,)$ close to the boundary of $O(r)$ is small compared to its total mass. The bound is obtained by careful decomposition of the parallel sets into convex pieces of approximately equal size using the family $\Sigma(r)$, and by estimating the number of mutual intersections of these pieces and the total number of pieces involved as $\varepsilon \searrow 0$.

If for some self-similar set F, $s_k = s - k$, then by the above result the existence of the fractal curvatures of F is only ensured in the non-arithmetic case. So the best one can hope for is the existence of fractal curvature measures in this case. Indeed, if $\mathcal{C}_k(F)$ exists, then the corresponding fractal curvature measure $\mathcal{C}_k(F, \cdot\,)$ exists.

Theorem 5.7 *Let F be a self-similar set satisfying the OSC and $F_\varepsilon \in \mathcal{R}^d$. Let $k \in \{0, \ldots, d\}$ and assume $s_k = s - k$. Then the average fractal curvature measures of F exist and $\widetilde{\mathcal{C}}_k(F, \cdot\,) = \widetilde{\mathcal{C}}_k(F)\mu_F$. If additionally F is non-arithmetic, then the fractal curvature measures exist and $\mathcal{C}_k(F, \cdot\,) = \mathcal{C}_k(F)\mu_F$.*

The idea of the proof is as follows. Since the families $(C_k(F_\varepsilon, \cdot\,))_{\varepsilon \in (0,1]}$ (and $(\widetilde{C}_k(F_\varepsilon, \cdot\,))_{\varepsilon \in (0,1]}$) are tight, by Prohorov's theorem, there exist converging subsequences, and the task is to show that the limit measures of all these subsequences are the same. This is done by proving that the limit measure μ_k of each fixed converging subsequence coincides with the measure $\mathcal{C}_k(F)\mu_F$. For the equivalence of two measures, it is sufficient that they coincide on an intersection stable generating class of the Borel σ-algebra. Since the computation of the limit $\lim_{\varepsilon \searrow 0} \varepsilon^{s-k} C_k(F_\varepsilon, B)$ is difficult for arbitrary sets B, the generator has to be adapted to the structure of F to include only sets B for which the limit can be determined. The generator $\mathcal{A}$ used in the proofs consists of copies of the open set O, i.e., the family $\{S_w O : w \in \Sigma^*\}$, and of all subsets C of the complement of some $O(r)$ (see (8)). For these sets, μ_F is known ($\mu_F(S_w O) = r_w^s$ and $\mu_F(C) = 0$) and the values for μ_k can be computed by approximation of the sets with continuous functions and using the weak convergence of the chosen (sub-)sequence. Here the estimate of Lemma 5.6 is used.

Remark 5.8 *Results analogous to those in Theorems 5.4 and 5.7 hold for the limiting behavior of the corresponding variation measures* $C_k^{\mathrm{var}}(F_\varepsilon, \cdot)$ *and their total masses* $C_k^{\mathrm{var}}(F_\varepsilon)$.

6 Proof of Theorem 5.1

We give a proof of the boundedness of the expression $\varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$, which is self-contained except for an application of Lemma 5.6. The proof shows the kind of arguments required to obtain results on fractal curvatures. The first step is the following general observation regarding parallel sets. If for a compact set $A \subset \mathbb{R}^d$, $A_\varepsilon \in \mathcal{R}^d$ for some $\varepsilon > 0$, then $A_{\varepsilon+r} \in \mathcal{R}^d$ for all $r > 0$.

Lemma 6.1 *Let* $A \subset \mathbb{R}^d$ *be compact and* $0 < a < b$. *Assume that* $A_a \in \mathcal{R}^d$. *Then there is a constant* $c > 0$ *such that* $C_k^{\mathrm{var}}(A_\varepsilon) \le c$ *for* $\varepsilon \in [a, b]$.

Proof. Let $A_a = \bigcup_{j=1}^n K^j$ be a representation of A_a by convex sets K^j. Then $A_\varepsilon = \bigcup_{j=1}^n (K^j)_{\varepsilon-a}$ is a representation of A_ε by convex sets for $\varepsilon > a$. Let $\mathbb{R}^d = H^+ \cup H^-$ be a Hahn decomposition of $\mathbb{R}^d$ for the measure $C_k(A_\varepsilon, \cdot)$. Then, by the inclusion-exclusion formula (4), we have for $a \le \varepsilon \le b$

$$\begin{aligned} C_k^{\mathrm{var}}(A_\varepsilon) &= C_k(A_\varepsilon, H^+) + C_k(A_\varepsilon, H^-) \\ &= \sum_{I \in N_n} (-1)^{\#I-1} \left(C_k\Big(\bigcap_{i\in I} K^i_{\varepsilon-a}, H^+\Big) + C_k\Big(\bigcap_{i\in I} K^i_{\varepsilon-a}, H^-\Big) \right) \\ &\le \sum_{I \in N_n} C_k \left(\bigcap_{i\in I} K^i_{\varepsilon-a} \right) \le \#N_n \cdot C_k(\mathrm{conv}(F_b)) =: c \end{aligned}$$

Here the last inequality is due to the fact that $K^i_{\varepsilon-a} \subset \mathrm{conv}(F_b)$ and the monotonicity of the total curvatures for convex sets. □

Setting $r = 1$ and $\varepsilon = \delta$, we have $O(1) = \bigcup_i S_i O$, and Lemma 5.6 specializes as follows.

Corollary 6.2 *There are constants* $c, \gamma > 0$ *such that for all* $\varepsilon \le 1$

$$C_k^{\mathrm{var}}(F_\varepsilon, (O(1)^c)_\varepsilon) \le c\varepsilon^{k-s+\gamma}.$$

Proof. Setting $r = 1$ and $\varepsilon = \delta$ in Lemma 5.6, we get the assertion for $0 < \varepsilon \le \rho$. By Lemma 6.1, $C_k^{\mathrm{var}}(F_\varepsilon) \ge C_k^{\mathrm{var}}(F_\varepsilon, (O(1)^c)_\varepsilon)$ is bounded by some constant for $\varepsilon \in [\rho, 1]$. Hence, if the constant c is suitably enlarged, the estimate holds for $\varepsilon \in (0, 1]$. □

The next step towards the proof of Theorem 5.1 is the following inequality.

Lemma 6.3 *For $\varepsilon > 0$ and $0 < r \leq 1$, we have*

$$C_k^{\mathrm{var}}(F_\varepsilon) \leq \sum_{w \in \Sigma(r)} C_k^{\mathrm{var}}((S_w F)_\varepsilon) + C_k^{\mathrm{var}}(F_\varepsilon, (O(r)^c)_\varepsilon)\,.$$

Proof. Fix $\varepsilon > 0$. Let $U = \bigcup_{v,w \in \Sigma(r)} (S_v F)_\varepsilon \cap (S_w F)_\varepsilon$ and $B_w = (S_w F)_\varepsilon \setminus U$ for $w \in \Sigma(r)$. Then $F_\varepsilon = U \cup \bigcup_{w \in \Sigma(r)} B^w$ and this union is disjoint. Thus,

$$C_k^{\mathrm{var}}(F_\varepsilon) = \sum_{w \in \Sigma(r)} C_k^{\mathrm{var}}(F_\varepsilon, B^w) + C_k^{\mathrm{var}}(F_\varepsilon, U)\,.$$

The set $A^w := \left(\bigcup_{v \in \Sigma(r)\setminus\{w\}} (S_v F)_\varepsilon\right)^c$ is open (the complement is a finite union of closed sets). Moreover, $B^w \subseteq A^w$ and $F_\varepsilon \cap A^w = (S_w F)_\varepsilon \cap A^w$. Hence, by locality (see proposition 4.2), we have $C_k^{\mathrm{var}}(F_\varepsilon, B^w) = C_k^{\mathrm{var}}((S_w F)_\varepsilon, B^w) \leq C_k^{\mathrm{var}}((S_w F)_\varepsilon)$. It remains to show that $U \subset (O(r)^c)_\varepsilon$. Let $x \in U$. We show that $d(x, O(r)^c) \leq \varepsilon$ and thus $x \in (O(r)^c)_\varepsilon$. Assume $d(x, O(r)^c) > \varepsilon$. Since the union $O(r) = \bigcup_{w \in \Sigma(r)} S_w O$ is disjoint, there is a unique $v \in \Sigma(r)$ such that $x \in S_v O$. Moreover, $d(x, \partial S_v O) > \varepsilon$. Since $x \in U$, there is at least one word $w \in \Sigma(r), w \neq v$ such that $x \in (S_w F)_\varepsilon$ and thus a point $y \in S_w F$ with $d(x, y) \leq \varepsilon$. But then $y \in S_w F \cap S_v O$, a contradiction to the OSC. Hence, $d(x, O(r)^c) \leq \varepsilon$. □

Combining the above statements, the upper bound for s_k is now easily derived.

Proof of Theorem 5.1. Set $g(\varepsilon) := \varepsilon^{s-k} C_k^{\mathrm{var}}(F_\varepsilon)$. We have to show that $\sup\{g(\varepsilon) : \varepsilon \in (0, 1]\}$ is bounded by some constant $M > 0$. Observe that $C_k^{\mathrm{var}}((S_i F)_\varepsilon) = r_i^k C_k^{\mathrm{var}}(F_{\varepsilon/r_i}) = \varepsilon^{k-s} r_i^s g(\varepsilon/r_i)$. Combining Lemma 6.3 and Corollary 6.2, there exist $c, \gamma > 0$ such that, for $0 < \varepsilon \leq 1$,

$$g(\varepsilon) \leq \sum_{i=1}^{N} r_i^s g(\varepsilon/r_i) + c\varepsilon^\gamma. \tag{9}$$

Let $r_{\max} := \max\{r_i | i = 1, \ldots, N\}$. For $n \in \mathbb{N}$ set $I_n := (r_{\max}^n, 1]$ and $M_1 := \max\{\sup_{\varepsilon \in I_1} g(\varepsilon), c\}$. Note that $M_1 < \infty$ is ensured by Lemma 6.1. We claim that for $n \in \mathbb{N}$,

$$\sup_{\varepsilon \in I_n} g(\varepsilon) \leq M_n := M_1 \sum_{j=0}^{n-1} (r_{\max}^\gamma)^j, \tag{10}$$

which we show by induction. For $n = 1$, the statement is obvious. So assume that (10) holds for $n = k$. Then for $\varepsilon \in I_k$, we have $g(\varepsilon) \leq M_k \leq M_{k+1}$ and for $\varepsilon \in I_{k+1} \setminus I_k$ we have $\varepsilon/r_i \geq \varepsilon/r_{\max} \geq r_{\max}^k$, i.e., $\varepsilon/r_i \in I_k$ for all i. Hence, by (9),

$$g(\varepsilon) \leq \sum_{i=1}^{N} r_i^s g(\varepsilon/r_i) + c\varepsilon^\gamma \leq \sum_{i=1}^{N} r_i^s M_k + M_1 r_{\max}^{\gamma k} = M_{k+1},$$

proving (10) for $n = k + 1$ and hence for all $n \in \mathbb{N}$. Now observe that the sequence $(M_n)_{n\in\mathbb{N}}$ is bounded ($M_n = M_1 \sum_{j=0}^{n} (r_{\max}^{\gamma})^j \to (1 - r_{\max}^{\gamma})^{-1}$ as $n \to \infty$). Hence, $g(\varepsilon)$ is bounded in $(0, 1]$, completing the proof of Theorem 5.1.

7 Generalizations

The results obtained in [17] and discussed here should be seen as the outset of a larger project to find geometric measures and characteristics suitable to describe the geometry of fractal sets. The current research is aimed in several directions.

Generalization of the results to other classes of self-similar sets. In particular, the assumption of polyconvexity for the parallel sets is a serious restriction of the applicability of the results in [17]. Some progress in eliminating this assumption has been made by Zähle [18]. Here self-similar sets are considered, for which the closed complements of almost all parallel sets are sets with positive reach (for the cost of replacing limits by essential limits). Up to now it is not clear whether all self-similar sets have this property. Moreover, in this paper the concepts are extended to self-similar random sets and results on *random fractal curvatures* are obtained.

In [11], the limiting behavior of the surface area of parallel sets is shown to be closely tied to the Minkowski content. The relations are derived from the fact that for most parallel sets the surface area is the derivative of the volume. Some results apply to arbitrary compact sets and thus go beyond the self-similar setting.

Curvature-direction measures. For self-similar sets, the fractal curvature measures turn out to be multiples of the Hausdorff measure, and so only the total curvatures may give new insights into the geometry of the sets. One possible approach to obtain more detailed information is to work with generalized curvature measures (or curvature-direction measures) of the parallel sets. They live on the normal bundle and take into account, in addition to the boundary points, also the normal directions. The First results have been obtained by Rothe in his diploma thesis [12]. He showed the existence of *fractal curvature direction measures* for self-similar sets with polyconvex parallel sets and showed that they have a product structure. Examples suggest that the directional components of the limit measures carry non-trivial geometric information.

Other classes of fractals. In [4], Kombrink studied the fractal curvatures of self-conformal sets. Some bounds for the scaling exponents have been obtained. The approach is slightly different from ours. The parallel sets of the fractals are not used for the approximation, but instead the parallel sets of certain covers by convex sets.

Steiner-type formulas. For certain self-similar sets (and other sets that can be described by fractal sprays) tube formulas have been obtained, which

describe the volume of the parallel sets and relate it to the complex dimensions of the set, see [6–8, 10]. The classical Steiner formulas suggest that the coefficients should be interpreted as curvatures in some way. Up to now the relations to fractal curvatures are not clear.

Estimation of fractal dimension and fractal curvatures from digital images. Fractal curvatures and the associated scaling exponents provide a whole set of characteristics which may be used to distinguish and classify fractal sets. They can easily be estimated from digital images; some methods have been implemented by Straka [16] and tested for self-similar sets. Using several geometric characteristics instead of just one (usually the volume) may also improve estimates of fractal dimension, see [15].

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Part II

Harmonic and Functional Analysis and, Signal Processing

A Walk from Multifractal Analysis to Functional Analysis with $\mathcal{S}^\nu$ Spaces, and Back

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Summary. With the $\mathcal{S}^\nu$ spaces introduced by Jaffard in the context of multifractal analysis to extend the Besov spaces environment, functional analysis received a gift from concrete applications. These spaces led to new results in multifractal analysis, but also brought concrete objects to study as new examples by the typical various tools and aspects of functional analysis, in the hope of providing some new points of view from which to consider multifractal analysis questions.

1 Introduction

1.1 Where do the Spaces $\mathcal{S}^\nu$ Come from?

The representation of a signal by means of its wavelet coefficients is a widely used tool. From a functional analysis point of view, a property of signals which has an expression in terms of wavelet coefficients independent of the chosen wavelet basis can intrinsically be studied using sequence spaces. In particular, the pointwise Hölder regularity can be studied in such a way since the Hölder exponent of a function can be characterized by means of its wavelet coefficients, under some mild regularity hypothesis. Then, multifractal analysis and multifractal formalisms related to this concept of regularity can be studied.

In various domains, Besov spaces constitute a natural mathematical setting in which to study signals, as they have a convenient wavelet characterization and fit naturally to approximation problems. Nevertheless, it appears that this setting is not sufficient in general to handle all the accurate information contained in the distribution of the wavelet coefficients, see Jaffard [13]. In particular, what is called the thermodynamic multifractal formalism, related to Besov spaces, deals with Legendre transforms, and hence with concave spectra. The introduction of spaces $\mathcal{S}^\nu$ provides a suitable tool to obtain multifractal results in the non-concave case.

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1.2 Heuristic Description

Originally, the spaces of type $\mathcal{S}^\nu$ were introduced as spaces of functions or distributions on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, whose histograms of wavelet coefficients have a certain controlled behavior. The wavelet coefficients of a 1-periodic function, in an L^∞-normalized infinitely regular wavelet basis

$$\{\psi_{j,k} : (j,k) \in \Lambda\},$$

form a sequence indexed by the set

$$\Lambda := \bigcup_{j\in\mathbb{N}} \{j\} \times \{0, \dots, 2^j - 1\}.$$

Let $\nu : \mathbb{R} \to \{-\infty\} \cup [0,1]$ be a non-decreasing and right-continuous function; roughly speaking, a sequence $x \in \mathbb{C}^\Lambda$ belongs to $\mathcal{S}^\nu$ if and only if, for each scale j and for every number α, there are less than $2^{\nu(\alpha)j}$ coefficients larger than $2^{-\alpha j}$.

To ensure that such a sequence of coefficients indeed represents the wavelet coefficients of a finite order distribution, we assume that there exists an $\alpha_{\min} \in \mathbb{R}$ such that $\nu(\alpha) = -\infty$ for $\alpha < \alpha_{\min}$ and $\nu(\alpha) \geq 0$ for $\alpha \geq \alpha_{\min}$. No other assumption is made when we study the topological and functional analysis aspects of the sequence spaces $\mathcal{S}^\nu$; for instance, $\alpha_{\max} := \inf\{\alpha : \nu(\alpha) = 1\}$ could be equal to $+\infty$.

When we want to study the multifractal properties (spectrum of singularities) of the corresponding functions, then it is required that $\alpha_{\min} > 0$ in order to be able to use the now well-known correspondence between pointwise regularity and wavelet coefficients. Let us also note here that the condition $\alpha_{\min} > 0$ means that the scaling function

$$\eta(p) := \inf_{\alpha \geq \alpha_{\min}} \{\alpha p - \nu(\alpha) + 1\}$$

is strongly admissible and that the critical exponent p_c (see Jaffard [12]) is given by $p_c = 1/h_{\max}$ if $h_{\max} := \inf_{\alpha\geq\alpha_{\min}} \frac{\alpha}{\nu(\alpha)}$; this value $h_{\max}$ means that the information about the spectrum is available for $h \leq h_{\max}$ (see Sects. 3.1 and 3.2).

1.3 Outline of this Chapter

In Sect. 2, we first review the results that were proved in [4] and presented at the Fractals and Related Fields 2007 conference in Monastir. Newer findings, of a more technical nature and proved in [1, 2], are also mentioned in this survey. During our trip in the forest of functional analysis with $\mathcal{S}^\nu$ as luggage, we discover the richness of the topological structure of these spaces; they are born in concrete applications, and it is very exciting to realize how well they illustrate several of the most exotic aspects of functional analysis. Then in

Sect. 3, we present ideas that go in the other direction: what can functional analysis tell us about multifractals? Some results were already established in [3], while others are new and proved here.

2 $\mathcal{S}^\nu$ Metrizable Topology

The definition and basic topology of $\mathcal{S}^\nu$ are rather straightforward (Sect. 2.1). Then, from the functional analysis point of view, natural questions have appeared: are these spaces locally p-convex (see Sect. 2.2) for some $p > 0$? In particular, when $p = 1$, this would mean that they are locally convex, hence Fréchet spaces. We characterize the p-convexity by means of a condition on ν, and we also give a complete description of p-seminorms defining the topology in that case (Sect. 2.2). After this, it turns out that a slight modification of the proof leading to this result about the p-seminorms allows us to claim that in any case, the topological vector space $\mathcal{S}^\nu$ is locally pseudoconvex. Using a concrete characterization of the compact sets, it has also appeared that in the locally convex case, these spaces are Fréchet–Montel. Pursuing our investigation in this setting, we recently studied the nuclear property and the diametral dimension of these spaces, classical aspects of the functional analysis tool (Sect. 2.3). Finally, we have also obtained a complete characterization of the dual as a union of $\mathcal{S}^{\nu'}$ spaces. In the locally convex case, the strong dual is in fact a countable inductive limit of Fréchet–Montel spaces of type $\mathcal{S}^{\nu'}$ (Sect. 2.4).

2.1 Definitions and Notation

The spaces $\mathcal{S}^\nu$ we are dealing with are sequence spaces precisely defined as follows: if $\nu : \mathbb{R} \to \{-\infty\} \cup [0, 1]$ is a non-decreasing and right-continuous function such that $\alpha_{\min} := \inf\{\alpha : \nu(\alpha) \geq 0\} > -\infty$, then $\mathcal{S}^\nu$ is defined as the set of sequences $x = \left(x_{j,k}\right)_{(j,k)\in\Lambda}$ such that

$$\forall \alpha \in \mathbb{R}, \forall C, \varepsilon > 0, \exists J : \#E_j(C, \alpha)(x) \leq 2^{(\nu(\alpha)+\varepsilon)j}, \forall j \geq J,$$

where

$$E_j(C, \alpha)(x) = \left\{ k \ : \ |x_{j,k}| \geq C 2^{-\alpha j} \right\}$$

and using the convention that $2^{-\infty j} = 0$ for any $j \geq 0$. If $\Omega := \mathbb{C}^\Lambda$ denotes the set of all sequences and if the wavelet profile of x is defined as

$$\nu_x(\alpha) := \lim_{\varepsilon \to 0^+} \limsup_j \left(\frac{\log(\#E_j(C, \alpha + \varepsilon))}{\log 2^j} \right),$$

then

$$\mathcal{S}^\nu \;=\; \left\{ x \in \Omega : \nu_x(\alpha) \leq \nu(\alpha),\ \forall \alpha \in \mathbb{R} \right\}.$$

These spaces are linear spaces and they can be endowed with a unique metrizable topology τ that is stronger than pointwise convergence and makes them complete, separable topological vector spaces. In fact, we define, for $\alpha \in \mathbb{R}$ and $\beta \in \{-\infty\} \cup [0, +\infty)$, the distance

$$d_{\alpha,\beta}(x, y) := \inf \left\{ C \geq 0 : \forall j, \#\left\{ |x_{j,k} - y_{j,k}| \geq C 2^{-\alpha j} \right\} \leq C 2^{\beta j} \right\}$$

on the ancillary metric space

$$E(\alpha, \beta) := \{ x \in \Omega : d_{\alpha,\beta}(x, 0) < \infty \}.$$

Then for any sequence α_n dense in $\mathbb{R}$ and any sequence $\varepsilon_m \searrow 0$ we have

$$\mathcal{S}^\nu = \bigcap_{n,m} E(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$$

and τ coincides with the projective limit topology (the coarsest topology which makes each inclusion $\mathcal{S}^\nu \hookrightarrow E(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ continuous).

For future reference, we also recall that given $p, q \in (0, \infty]$ and $s \in \mathbb{R}$, the Besov sequence space $b^s_{p,q}$ is the quasi-Banach space defined by the quasi-norm

$$\|x\|_{b^s_{p,q}} := \left(\sum_{j \in \mathbb{N}} 2^{j(s-1/p)q} \left(\sum_{k=0}^{2^j - 1} |x_{j,k}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

with the obvious modifications in the case p or $q = \infty$.

2.2 More Functional Analysis: p-Convexity

For $0 < p \leq 1$, a topological vector space E is called *locally p-convex* if its topology is induced by a family of p-seminorms, i.e., applications $\|\cdot\| : E \to \mathbb{R}^+$ satisfying $\|\lambda x\| = |\lambda| \|x\|$ and $\|x + y\|^p \leq \|x\|^p + \|y\|^p$. As an introductory example, let us consider the elementary case $(0 < a \leq 1)$

$$\nu(\alpha) := \begin{cases} -\infty \text{ if } \alpha < 0 \\ a\alpha \ \text{ if } \alpha \in [0, 1/a]. \\ 1 \ \text{ if } \alpha \geq 1/a \end{cases}$$

In that case we have

$$\mathcal{S}^\nu = \bigcap_{\varepsilon > 0} b^{\frac{1}{a} - \varepsilon}_{a,\infty}.$$

This space is locally a-convex and we note that $a = \frac{\partial}{\partial \alpha} \nu(\alpha)$ when $\alpha \in (0, 1/a)$.

More generally, given ν as usual, we define

$$\underline{\partial}^+\nu(\alpha) := \liminf_{h\to 0^+} \frac{\nu(\alpha+h)-\nu(\alpha)}{h}$$

and

$$p_0 := \min\left\{1, \inf_{\alpha:0\le\nu(\alpha)<1} \underline{\partial}^+\nu(\alpha)\right\}.$$

In [1], the following result is shown.

Proposition 1 *If $p_0 > 0$, then the topological vector space $\mathcal{S}^\nu$ is locally p_0-convex. Conversely, if the space is locally p-convex for some $p > 0$, then $p_0 \ge p$.*

A complete description of a set of p_0-seminorms (actually, p_0-norms) defining the topology was also obtained. It involves interpolating Besov and Hölder norms for $\alpha \in [\alpha_{\min}, \alpha_{\max})$, where $\alpha_{\max} := \inf\{\alpha : \nu(\alpha) = 1\}$.

Proposition 2 *A desired set of such p_0-seminorms is given by*

$$\begin{aligned} p_{\alpha,\varepsilon}(x) &= \inf\left\{\|x'\|_{b^{\alpha-\varepsilon+\frac{1-\nu(\alpha)}{p_0}}_{p_0,\infty}} + \|x''\|_{b^\alpha_{\infty,\infty}} : x = x' + x''\right\} \text{ if } 0 \le \nu(\alpha) < 1 \\ &= \|x\|_{b^\alpha_{\infty,\infty}} \text{ if } \nu(\alpha) = -\infty, \end{aligned}$$

where $\varepsilon > 0$ and $\alpha < \alpha_{\max}$.

A slight adaptation of the arguments of the proof of this proposition even leads to the property of pseudoconvexity of the spaces $\mathcal{S}^\nu$. Let us recall the following.

Definition 3 *A topological linear space is said to be locally pseudoconvex if there exists a family of r-seminorms ($0 < r \le 1$) defining the topology of the space.*

Indeed, such is the case for $\mathcal{S}^\nu$:

Proposition 4 *Assume that $\alpha_{\min} > -\infty$. Then for any sequence $p_m \in (0,1]$ ($m \in \mathbb{N}$) converging to 0, the topology of the metrizable linear topological space $\mathcal{S}^\nu$ can be defined by a sequence of p_m-seminorms $\|\cdot\|_m$.*

2.3 More Functional Analysis: Typical Properties

Case $p_0 = 1$

In this case, $\mathcal{S}^\nu$ is in fact a Fréchet space. Many questions arise in this context: what kind of typical functional analysis properties does this space carry? We first have to recall some definitions (see, for instance, Jarchow [14]) of the

properties that we will handle in what follows. Given U an absolutely convex 0-neighborhood in a locally convex space E, we denote by

$$E(U) := E/\{x : \forall \rho > 0, x \in \rho U\}$$

the space normed by $\|x\|_U := \inf\{\rho : x \in \rho U\}$.

Definition 5 *A Fréchet space E is called a* Montel space *if every bounded set is relatively compact in E.*

A locally convex space E is called a Schwartz space *if, for any absolutely convex 0-neighborhood U, there exists a 0-neighborhood V absorbed by U such that the canonical inclusion $E(V) \hookrightarrow E(U)$ is compact.*

A locally convex space E is called a nuclear space *if, for any absolutely convex 0-neighborhood U, there exists a 0-neighborhood V absorbed by U such that the canonical inclusion $E(V) \hookrightarrow E(U)$ is nuclear.*

In our first paper [4], we obtained a complete description of the compact sets.

Proposition 6 *In $\mathcal{S}^\nu$, the compact sets are the closed bounded sets.*

In the locally convex case, this means that the Fréchet space $\mathcal{S}^\nu$ is also Montel. The fact that it carries the Schwartz property[1] is already contained within the lines of the results of [4]. It is also explicitly contained (as a consequence) in [2], where the nuclearity as well as the diametral dimension of such spaces is studied. More precisely, we can conclude the following.

Proposition 7 *$\mathcal{S}^\nu$ is a Schwartz space, but not nuclear.*

These rather technical topological properties are not at all gratuitous. For instance, nuclear spaces have nice tensor product qualities, related to the theory of Fredholm kernels and partial differential equations (and one may see the multifractal spectrum of singularities associated to a physical phenomenon, such as turbulence, as an indication about the function space in which to look for solutions to the corresponding equations); diametral dimensions are also related to approximation properties which are of foremost importance in signal processing.

Case $p_0 < 1$

In this case, $\mathcal{S}^\nu$ is no longer locally convex and the above definitions need to be adapted; again, see [16] and again [2] for a discussion. We shall just mention here that $\mathcal{S}^\nu$ is still a Schwartz space, even when it is only locally pseudoconvex. This gives us the occasion to revisit an example of Ligaud: in [17], he gives an explicit (but tricky and somewhat artificial) construction

[1] A Fréchet–Schwartz space is always a Montel space, but the converse is not true.

of a metrizable topological vector space which is a locally pseudoconvex non p-convex Schwartz space. Actually, his example is a particular case of $\mathcal{S}^\nu$.

Ligaud starts with a decreasing sequence $1 \geq p_n \to 0^+$ and another sequence $\varepsilon_n > 0$, which we shall assume for our convenience to be summable. Then he constructs a decreasing sequence of spaces E_n such that (for $n \geq 2$) the following diagram commutes (horizontal "=" means isomorphism)

$$\begin{array}{ccc} E_n & = & l_{p_n} \\ \uparrow{\scriptstyle i_n} & & \uparrow{\scriptstyle u_n} \\ E_{n+1} & = & l_{p_{n+1}} \end{array}$$

where i_n is the canonical inclusion, l_{p_n} the standard p_n-normed space of sequences indexed by $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$, and

$$u_n : (\xi_l)_{l\in\mathbb{N}^*} \mapsto \left(\frac{\xi_l}{l^{\varepsilon_n}}\right)_{l\in\mathbb{N}^*}.$$

Note that u_n, therefore i_n, is compact, whence the Schwartz property holds for the projective limit $E := \bigcap E_n$, which is indeed pseudoconvex but not p-convex.

If we now define $s_n := -\sum_{l=n}^{\infty} \varepsilon_l$ and the Besov space $\mathfrak{l}_n := b^{s_n+1/p_n}_{p_n,p_n}$, then we have an isomorphism φ_n from l_{p_n} to $\mathfrak{l}_n$ which is explicitly given by

$$\varphi_n : (\xi_l)_{l\in\mathbb{N}^*} \mapsto \left(x_{j,k} := \frac{\xi_{2^j+k}}{(2^j+k)^{s_n}}\right)_{(j,k)\in\Lambda}.$$

It is an isomorphism because

$$\|\xi\|_{l^{p_n}} \leq \|\varphi_n(\xi)\|_{\mathfrak{l}_n} = \left(\sum_{j,k} 2^{js_np_n}\left|\frac{\xi_{2^j+k}}{(2^j+k)^{s_n}}\right|^{p_n}\right)^{\frac{1}{p_n}} \leq 2^{-s_n}\|\xi\|_{l^{p_n}}.$$

Now if $\mathfrak{i}_n$ represents the canonical inclusion $\mathfrak{l}_{n+1} \hookrightarrow \mathfrak{l}_n$, the following diagram also commutes:

$$\begin{array}{ccc} l_{p_n} & = & \mathfrak{l}_n \\ \uparrow{\scriptstyle u_n} & & \uparrow{\scriptstyle \mathfrak{i}_n} \\ l_{p_{n+1}} & = & \mathfrak{l}_{n+1} \end{array}$$

So we have for the projective limit

$$E \simeq \bigcap_{n\geq 2} \mathfrak{l}_n = \bigcap_{n\geq 2} b^{s_n+1/p_n}_{p_n,\infty} = \mathcal{S}^\nu$$

with $\nu(\alpha) := -\infty$ if $\alpha < 0$ and $\nu(\alpha) = 0$ if $\alpha \geq 0$. The first equality above holds because of standard Besov injections, the second one stems from the link between $\mathcal{S}^\nu$ and Besov spaces, see [4].

2.4 More Functional Analysis: Properties of the Dual

We employ the usual scalar product

$$\langle x, y\rangle := \sum_{(j,k)\in\Lambda} x_{j,k}\overline{y_{j,k}}$$

to identify the topological dual $(\mathcal{S}^\nu)'$ of $\mathcal{S}^\nu$ to some sequence space. This identification is made as follows: for all $(j,k) \in \Lambda$, let $e^{j,k}$ be the sequence whose only non-zero component is $e^{j,k}_{j,k} = 1$. Given $u \in (\mathcal{S}^\nu)'$, let us define

$$y := \sum_{j,k\in\Lambda} \overline{u(e^{j,k})}e^{j,k}.$$

This sequence y indeed satisfies $u(x) = \langle x, y\rangle$ because, for all $x \in \mathcal{S}^\nu$, the sum $\sum_{j,k\in\Lambda} x_{j,k}e^{j,k}$ converges to x in $\mathcal{S}^\nu$.

Which sequences can be in the dual of $\mathcal{S}^\nu$? Suppose that x has at scale j a number $2^{\nu(\alpha)j}$ of coefficients equal to $2^{-\alpha j}$ and that $\langle x, y\rangle$ converges: at worst these coefficients have to be multiplied by as many as $(2^{\nu(\alpha)j} = 2^{\nu'(\alpha')j})$ coefficients of y of size $2^{-\alpha' j}$. We are on the verge of convergence when $\nu(\alpha) = \nu'(\alpha') = \alpha + \alpha'$. Making this reasoning rigorous leads to the following.

Theorem 8 *The topological dual of* $\mathcal{S}^\nu$ *is*

$$(\mathcal{S}^\nu)' = \bigcup_{\varepsilon>0} \mathcal{S}^{\nu'_\varepsilon}$$

with $\nu'_\varepsilon(\alpha') = \nu'(\alpha' - \varepsilon)$ *and*

$$\nu'(\alpha') := [[\alpha' + \inf\{\alpha : \nu(\alpha) - \alpha > \alpha'\}]]$$

using the notation

$$[[\beta]] := \begin{cases} -\infty & \text{if } \beta < 0 \\ \beta & \text{if } 0 \le \beta \le 1 \\ 1 & \text{if } \beta \ge 1. \end{cases}$$

It should be noted that the slope of ν' is always ≥ 1; therefore, the spaces $\mathcal{S}^{\nu'_\varepsilon}$ are automatically Fréchet–Montel.

As a union of increasing topological vector spaces, the dual of $\mathcal{S}^\nu$ can be endowed with the countable inductive limit topology. On the other hand, as a dual, it carries naturally the strong topology (uniform convergence on the bounded sets of $\mathcal{S}^\nu$). We can summarize the situation so far as follows.

Proposition 9 *The strong dual* $(\mathcal{S}^\nu)'_b$ *and the inductive limit* $\mathrm{ind}_m\, \mathcal{S}^{\nu'_m}$ *have the following properties.*

(i) *A subset of* $\mathrm{ind}_m \mathcal{S}^{\nu'_m}$ *is bounded if and only if it is bounded in one of the* $\mathcal{S}^{\nu'_m}$ *(i.e., the inductive limit is regular).*
(ii) *The strong dual* $(\mathcal{S}^\nu)'_b$ *and the inductive limit* $\mathrm{ind}_m \mathcal{S}^{\nu'_m}$ *have the same bounded sets.*
(iii) *The inductive limit and the strong dual are sequentially complete spaces.*
(iv) *At least in the locally convex case, we have* $(\mathcal{S}^\nu)'_b = \mathrm{ind}_m \mathcal{S}^{\nu'_m}$.

L. Frerick and J. Wengenroth recently proved (private communication) that the strong dual of a complete metrizable Schwartz space is an (LB)-space. This implies in particular that (iv) holds in all cases.

3 Contributions to Multifractal Analysis

In return, the functional analysis point of view sheds a new light on the problems of multifractal analysis. This had already been noticed by Jaffard [10] when he proved that the thermodynamic multifractal formalism yields an upper bound on the spectrum of singularities. Then in [12] he proved that this formalism is generically valid (in the sense of Baire categories), and in [7] with A. Fraysse the same result was established in a prevalent setting. Comparable but more precise results hold in the $\mathcal{S}^\nu$ framework for prevalence (Sect. 3.1) and Baire categories (Sect. 3.2). But considerations not limited to this framework also hold: under mild hypotheses, we can show that the local convexity index of a function space shapes the maximal spectrum of singularities in that space (Sect. 3.3).

3.1 Prevalent Properties

Let us first recall the upper bound on the spectrum of singularities proved in [5]. Given an admissible profile ν, we set $h_{\max} := \inf_{h \geq \alpha_{\min}} \frac{h}{\nu(h)}$ and

$$d_\nu(\alpha) := \begin{cases} \alpha \sup_{\alpha \geq \alpha'} \frac{\nu(\alpha')}{\alpha'} \text{ if } \alpha \leq h_{\max}. \\ -\infty \text{ if } \alpha > h_{\max} \end{cases}$$

Theorem 10 *Suppose* $\alpha_{\min} > 0$. *Then for any* $x \in \mathcal{S}^\nu$ *and* $f := \sum x_{j,k}\psi_{j,k}$.

$$d_f(\alpha) \leq d_\nu(\alpha), \forall \alpha \leq h_{\max}.$$

A lower bound holding in all generality on $\mathcal{S}^\nu$ does not exist, but the topological properties of $\mathcal{S}^\nu$ make it a suitable space in which to have prevalent subsets. Let us recall the terminology employed by Hunt et al. [8, 9].

Definition 11 *Let* X *be a complete metric vector space. A Borel set* $A \subset X$ *is called* shy *(*Haar-null *in [6]) if there exists a Borel measure* μ, *strictly positive on some compact set* $K \subset X$, *such that* $\forall x \in X, \mu(A+x) = 0$. *Such a measure is called* transverse *to* A. *A subset* A *of* E *is shy if it is included in a shy Borel set. A set is* prevalent *if its complement is shy.*

A prevalent set is "big" in the sense that it is always dense and this property is stable by translation, non-zero dilations, and countable intersections. A prevalent property (i.e., valid on a prevalent subset of X) is also called *almost sure* in X. Then the following can be proved (see [3]).

Theorem 12 *In $\mathcal{S}^\nu$, almost surely $\nu_x = \nu$. If furthermore $\alpha_{\min} > 0$, then for $f := \sum x_{j,k}\psi_{j,k}$, almost surely $d_f = d_\nu$ and the almost-everywhere regularity of f is $h_{\max}$.*

3.2 Quasi-Sure Properties

A property is said to hold *quasi-surely* on a Baire space X if it is true on (at least) a countable intersection of dense sets. Historically, this notion of genericity predates the notion of prevalence and was first considered to describe regularity properties by many authors, including Jaffard [12]. Most (but not all, see Kahane [15]) results concerning pointwise regularity are valid both in the prevalence and quasi-sure sense; this is also true in $\mathcal{S}^\nu$.

Theorem 13 *In $\mathcal{S}^\nu$, quasi-surely $\nu_x = \nu$. If furthermore $\alpha_{\min} > 0$, then for $f := \sum x_{j,k}\psi_{j,k}$, quasi-surely $d_f = d_\nu$ and the almost-everywhere regularity of f is $h_{\max}$.*

Proof. The key to this kind of proof is to build a "saturating sequence" (using the terminology of [12]) $z \in \mathcal{S}^\nu$ having the properties that we want to show to be quasi-sure, and to prove that they remain true in a countable intersection of open dense sets built upon z.

For instance, using a random wavelet series (RWS) adapted to ν as in [3,5], we can build a sequence z having for arbitrary $\varepsilon > 0$, for each $\alpha \in \mathbb{R}$, and at each scale j large enough, a number $2^{(\nu(\alpha)-\varepsilon)j}$ of wavelet coefficients $2^{-\alpha j} \leq |z_{j,k}| \leq 2^{-(\alpha-\varepsilon)j}$; furthermore, these coefficients are "well spread" in the sense that the intervals of size $2^{-(\nu(\alpha)-2\varepsilon)j}$ centered on them cover $\mathbb{T}$. In fact, almost all trajectories of an adapted RWS have this property.

A deterministic construction of such a sequence is also feasible. Let us define a partition of the set of the natural numbers using particular finite sets A_n ($n \geq 1$): for each n, A_n contains n consecutive elements, the smallest of which is $n(n-1)/2$. Let also α_n ($n \geq 1$) be a dense sequence of $[\alpha_{min}, +\infty[$ (with $\alpha_n \neq \alpha_N$ if $n \neq N$). Fix $n \in \mathbb{N}$; for the sequence of natural numbers $(j_m)_{m \geq n}$ defined as j_m is the nth elements of the set A_m, let $z_{j_m,k} := 2^{-\alpha_n j_m}$ for $\left\lceil r 2^{\nu(\alpha_n) j_m} \right\rceil$ values of k uniformly distributed in $\{0, \ldots, 2^j - 1\}$ and let $z_{j_m,k} := 0$ for the other values of k. If this construction is applied for every n, then a sequence $z = (z_{j,k})_{(j,k)\in\Lambda}$ is built, and it can be shown that it satisfies all the desired properties.

We pick sequences α_n and ε_m as in Sect. 2.1 and write $d_{m,n} := d_{\alpha_n, \nu(\alpha_n)+\varepsilon_m}$. Whatever the construction process used for the saturating sequence z, we retain that for each $m, n \in \mathbb{N}$ there exists an infinite set $J_{m,n}$

such that, for all $j \in J_{m,n}$, there are at least $2^{(\nu(\alpha_n)-\varepsilon_m)j}$ coefficients $z_{j,k}$ bracketed by $2^{-\alpha_n j} \leq |z_{j,k}| \leq 2^{-(\alpha_n-\varepsilon_m)j}$.

As we mentioned earlier, $\mathcal{S}^\nu$ is separable and, more precisely, there exists a dense sequence of finite rational sequences $(x^l)_{l\in\mathbb{N}}$: we note $j_l \in \mathbb{N}$ such that $j \geq j_l$ implies $x^l_{j,k} = 0$. Let us now build $y^l := x^l + \epsilon_l z$, where $\epsilon_l \leq 1$ is an arbitrary sequence with limit 0. The sequence y^l is again dense in $\mathcal{S}^\nu$ and so is, for every $m, n, L \in \mathbb{N}$, the open set

$$U_{m,n,L} := \bigcup_{l\geq L} y^l + B_{m,n,l},$$

where

$$B_{m,n,l} := \left\{x : d_{m,n}(x,0) < \frac{\epsilon_l}{2} 2^{-2\varepsilon_m j_{m,n,l}}\right\},$$

where $j_{m,n,l}$ is the smallest element of $J_{m,n}$ to be larger than j_l. The radius of $B_{m,n,l}$ has been chosen small enough so that, at scale $j := j_{m,n,l}$, an element of this ball has less than $\frac{1}{2}2^{(\nu(\alpha_n)-\varepsilon_m)j}$ coefficients larger than $\frac{\epsilon_l}{2}2^{-\alpha_n j}$. Therefore, an element $x \in W := \bigcap_{m,n,L} U_{m,n,L}$ has for each m, n, for infinitely many scales j, a set K_j, of cardinal at least $\frac{1}{2}2^{(\nu(\alpha_n)-\varepsilon_m)j}$, of coefficients larger than $\frac{\epsilon_l}{2}2^{-\alpha_n j}$ (but smaller than $2\epsilon_l 2^{-(\alpha_n-\varepsilon_m)j}$). It follows directly that $\nu_x = \nu$.

By the cardinal of K_j and the fact that its elements are still well spread in $\{0, \ldots, 2^j - 1\}$, an element $x \in W$ verifies all the properties that almost surely the trajectories of RWS enjoy, that are used in the proof of [5, Theorem 2], relying in particular on ubiquity techniques that were developed in [11]. If $\alpha_{\min} > 0$ and $f := \sum x_{j,k}\psi_{j,k}$, it follows that $d_f = d_\nu$ and almost everywhere $h_f(t) = h_{\max}$. □

3.3 On the Maximal Spectrum in Locally p-Convex Spaces

Let us now consider a more general situation where we have some functional analysis-type information on a space X of functions and try, with minimal hypotheses, to deduce information of the multifractal kind. Given a space X of locally bounded functions, let us define its *maximal spectrum* of singularities

$$d_X : h \mapsto \sup_{f\in X} d_f(h).$$

In the particular case $X = \mathcal{S}^\nu$, we have already noticed a relationship between the local convexity index p_0 and the slopes of the maximal spectrum (which was $d_\nu(h)$ in Sect. 3.1, for $h \leq h_{\max}$). This leads us to formulate a more general statement.

We shall use an infinitely regular periodized wavelet basis

$$\{\psi_{j,k} : (j,k) \in \Lambda\}$$

with L^∞ normalization. We say that f is *strongly irregular* with exponent h at a point $t \in \mathbb{T}$ if there exist $j_0 < \infty$ and $C > 0$ such that $j \geq j_0$ implies that there exists (j,k),

$$C\left|2^{-j}k - t\right| \leq 2^{-j} \text{ and } |x_{j,k}| \geq C2^{-hj}.$$

In this context, uniformly strongly irregular will mean that the constants j_0 and C do not depend on the point t.

Theorem 14 *Let X be a complete topological vector space of functions on $\mathbb{T}$ satisfying the following:*

(i) X is locally p-convex for some $0 < p \leq 1$.
(ii) The translations $\tau_t : f(\cdot) \mapsto f(\cdot - t)$ are continuous on X, uniformly in t.
(iii) For all h such that $d_X(h) > 0$, for all $0 < \varepsilon < \frac{1}{3}d_X(h)$, there exists a uniformly Hölder function $f \in X$, which is uniformly strongly irregular with exponent h on some subset $A(h)$ of $\mathbb{T}$ having an upper box dimension $\geq d_X(h) - \varepsilon$.
(iv) If $d_X(h) = 0$, there exists a uniformly Hölder function $f \in X$ having at least one point t at which f is strongly irregular with exponent h.

Then for all h such that $0 \leq d_X(h) < 1$, one has $\underline{\partial}^+ d_X(h) \geq p$.

Proof. Let us first consider an h for which $d_X(h) > 0$ and we have f as in (iii). From (ii), there exists a 0-neighborhood basis $\mathcal{U}$ of X whose elements are balanced, p-convex, and uniformly absorb f and all its translations. More precisely, for an arbirary $V \in \mathcal{U}$, there exists $\gamma_V > 0$ such that $|\gamma| \leq \gamma_V$ and $t \in \mathbb{T}$ imply that V contains $\gamma\tau_t f$, as well as finite p-convex combinations of such terms.

Because of the strong irregularity, there exists $C > 0$ such that each point of $A(h)$ is within distance $\leq \frac{1}{C}2^{-j}$ of a wavelet coefficient of f that is $\geq C2^{-jh}$, and therefore, by definition of the upper box dimension, there exists an infinite set $J \subset \mathbb{N}$ such that if $j \in J$, then f has at least $2^{(d_X(h)-2\varepsilon)j}$ such wavelet coefficients.

Let $\delta > 0$ be fixed. For each $j \in J$ we consider $2^{p\delta j}$ random variables $t_{j,l}$ uniformly distributed on $\left\{k2^{-j} : 0 \leq k < 2^j\right\}$ and the same number of random variables $\theta_{j,l}$ uniformly distributed on $[0, 2\pi]$, all independent. Then

$$f_j := \sum_{l=1}^{2^{p\delta j}} 2^{-\delta j} e^{i\theta_{j,l}} \tau_{t_{j,l}} f$$

is a p-convex combination such that $\gamma_V f_j$ belongs to V. So, when we choose m_0 large enough so that $\sum_{j=m_0}^{\infty} j^{-2} \leq \gamma_V^p$, then for any $m_1, m_2 \geq m_0$ the finite Cauchy sum

$$\sum_{j=m_1}^{j=m_2} j^{-2/p} f_j = \sum_{j=m_1}^{j=m_2} \frac{j^{-2/p}}{\gamma_V} \gamma_V f_j$$

is another p-convex combination that remains in V. Since X is complete, we have proved that

$$g := \sum_{j \in J} j^{-2/p} f_j$$

converges in X. We claim that, almost surely, $d_g(h+\delta) \geq d_X(h) + p\delta - 4\varepsilon$, which implies the announced result.

Indeed, using the Borel–Cantelli lemma, there exists $j_1 < \infty$ such that with probability one, g has at each scale $j \in J \cap [j_1, \infty)$ a number $2^{(d_X(h)+p\delta-3\varepsilon)j}$ wavelet coefficients of size $\geq \frac{C}{2} 2^{-(h+\delta)j} j^{-\frac{p+1}{p}}$ which are "well spread" in the sense that the intervals of size $2^{-(d_X(h)+p\delta-4\varepsilon)j}$ centered on them cover $\mathbb{T}$. By the usual ubiquity technique (again we refer to [5, Theorem 2]), this gives the lower bound on the spectrum.

The same idea may be directly adapted to the case $d_X(h) = 0$, using (iv). □

Some comments on this theorem: (i) is the functional analysis-type hypothesis we previously alluded to; (ii) is very common and basically serves to avoid trivial cases; (iii) is a relatively strong hypothesis, but it could probably be weakened; on the other hand, it is easily seen that nothing meaningful can be said in this direction if one allows the maximal spectrum to be approached by functions having only chirp-like singularities and/or no uniformity; finally (iv) is really a variant of (iii) for the dimension 0.

It may also seem strange that from some hypothesis on the box dimension of $A(h)$ one gets a lower bound on the Hausdorff dimension of some other set. This is so because randomized translations of infinitely many copies of the original set are used to build a set of larger dimension.

3.4 Open Questions

And then? Several questions occur. For example:

- What are the minimal hypotheses on X to make Theorem 14 work? A great limitation seems to come from the use of wavelets for the proof; could we do without them?
- Under what hypotheses on X is the maximal spectrum also the prevalent and/or the quasi-sure spectrum?
- How could one give a sense to a spectrum of singularities when $\alpha_{\min} < 0$? In that case, in general, elements of $\mathcal{S}^\nu$ are sequences of wavelet coefficients of distributions. Remark that in that case the dual $(\mathcal{S}^\nu)'$ is a union of Hölder regular function spaces.
- In the case where ν is concave, the space $\mathcal{S}^\nu$ amounts to an intersection of Besov spaces. Does this situation bring something more for the functional analysis properties?

- The minimum slope of ν gives the convexity index. Can other features, such as the concavity of ν, be detected on purely topological properties of the space?
- It is clear that $\mathcal{S}^{\nu_1}$ and $\mathcal{S}^{\nu_2}$ are isomorphic if $\nu_2(\cdot) = \nu_1(\cdot - r)$ for some $r \in \mathbb{R}$. Is the converse true?

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Concentration of the Integral Norm of Idempotents

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Summary. This chapter is a companion of a recent paper, entitled *Integral concentration of idempotent trigonometric polynomials with gaps*. New results of the present work concern L^1 concentration, while the above-mentioned paper deals with L^p concentration.

Our aim here is twofold. First we try to explain methods and results, and give further straightforward corollaries. On the other hand, we also push forward the methods to obtain a better constant for the possible concentration (in the L^1 norm) of an idempotent on an arbitrary symmetric measurable set of positive measure. We prove a rather high level $\gamma_1 > 0.96$, which contradicts strongly the conjecture of Anderson et al. that there is no positive concentration in the L^1 norm.

The same problem is considered on the group $\mathbb{Z}/q\mathbb{Z}$, with q, say, a prime number. There, the property of absolute integral concentration of idempotent polynomials fails, which is in a way a positive answer to the conjecture mentioned above. Our proof uses recent results of B. Green and S. Konyagin on the Littlewood problem.

1 Introduction and Statement of Results

The problem of p-concentration on the torus for idempotent polynomials was considered first in [1,3,6]. We use the notation $\mathbb{T} := I\!R/\mathbb{Z}$ for the torus. Then $e(t) := e^{2\pi it}$ is the usual exponential function adjusted to interval length 1, and we denote by e_h the function $e(ht)$. For obvious reasons of being convolution idempotents, the set

$$\mathbf{P} := \left\{ \sum_{h\in H} e_h \ : \ H \subset I\!N, \ \sharp H < \infty \right\} \tag{1}$$

is called the set of *(convolution-)idempotent exponential (or trigonometric) polynomials*, or just *idempotents* for short. The p-concentration problem comes from the following definition.

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Definition 1. *Let $p > 0$. We say that there is p-concentration if there exists a constant $\gamma > 0$ so that for any symmetric (with respect to 0) measurable set E of positive measure one can find an idempotent $f \in \mathbf{P}$ with*

$$\int_E |f|^p \geq \gamma \int_{\mathbb{T}} |f|^p. \tag{2}$$

The supremum of all such constants γ will be denoted as γ_p, and called the level of p-concentration.

The main theorem of [2] can be stated as follows.

Theorem 1 (Anderson, Ash, Jones, Rider, Saffari). *There is p-concentration for all $p > 1$.*

We prove in our recent paper [4] that there is p-concentration for all $p > 1/2$, while these authors conjectured that idempotent concentration fails already for $p = 1$. Moreover, we prove that the constant γ_p is equal to 1 when $p > 1$ and p is not an even integer. This is in line with the fact that L^p norms behave differently depending on whether p is an even integer or not in a certain number of problems, such as the Hardy–Littlewood majorant problem (*does an inequality on absolute values of Fourier coefficients imply an inequality on L^p norms?*) or the Wiener property for periodic positive definite functions (*does a positive definite function (with large gaps in its Fourier series) belong to L^p when it is the case on a small interval?*). The fact that one can find idempotents among counter-examples to the Hardy–Littlewood majorant problem had been conjectured by Montgomery [10] and was recently proved by Mockenhaupt and Schlag [9], and we rely on their construction in [4]. At the same time, we were able to revisit the Wiener property in order to construct counter-examples among idempotents [5].

Even if we disproved the conjecture of [2] for $p = 1$, the situation is not yet entirely clear. Indeed, the constant γ can be taken arbitrarily close to 1 when we restrict the class of symmetric measurable sets to symmetric open sets or enlarge the class of trigonometric polynomials to all positive definite ones, that is, allow all nonnegative coefficients and not only 0 or 1. So one may conjecture that $\gamma_1 = 1$ (even if we understand that one should be cautious with such conjectures). By pushing forward our techniques, we improve our previous constant and prove the following.

Theorem 2. *For $p = 1$ there is concentration at the level $\gamma_1 > 0.96$. Moreover, for arbitrarily large given N the corresponding concentrating idempotent can be chosen with gaps at least N between consecutive frequencies.*

In order to prove this theorem, we will describe the main steps of our proofs in [4] before focusing on the improvements. When doing this, we also give a relatively simple proof of the fact that the best constant γ_2 for symmetric measurable sets is the same as for open sets. This is proved in [2], as it is

a particular case of their general result, but their proof is not easy to read. We describe it here so that a simpler, explanatory proof will be available. The constant for open sets has been obtained by Déchamps-Gondim, Lust-Piquard and Queffélec [6,7], so that

$$\gamma_2 = \sup_{0\le x} \frac{2\sin^2 x}{\pi x} = 0.4613\cdots . \tag{3}$$

In all proofs, the same kind of estimate as (2), but with finite sums on a grid of points replacing integrals, plays a central role in the proofs. So it was natural to become interested in best constants on these finite structures. This led us to the same problem, but taken on finite groups, which we describe now.

Let us consider $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$, which identifies with the grid (or subgroup) $\mathbb{G}_q := \{k/q; k = 0, 1, \ldots, q-1\}$ contained in the torus. We do not assume that q is a prime number at this point. We still denote by $e(x) := e^{2\pi i x/q}$ the exponential function adapted to the group $\mathbb{Z}_q$ and by e_h the function $e(hx)$. Again the set

$$\mathbf{P}_q := \left\{ \sum_{h\in H} e_h \ : \ H \subset \{0,\ldots,q-1\} \right\} \tag{4}$$

is called the set of *idempotents* on $\mathbb{Z}_q$. In this context, the set of idempotents has 2^q elements.

We then adapt the definition of p-concentration to the setting of $\mathbb{Z}_q$.

Definition 2. *Let $p > 0$. We say that there is uniform (in q) p-concentration for $\mathbb{Z}_q$ if there exists a constant $\gamma > 0$ so that for each prime number q one can find an idempotent $f \in \mathbf{P}_q$ with*

$$2|f(1)|^p \ge \gamma \sum_{k=0}^{q-1} |f(k)|^p. \tag{5}$$

Moreover, writing $\gamma_p^\sharp(q)$ for the maximum of all such constants γ, we set

$$\gamma_p^\sharp := \liminf_{q\to\infty} \gamma_p^\sharp(q).$$

Then $\gamma_p^\sharp$ is called the uniform level of p-concentration.

Here we can formulate a discrete analog of the problem in [1,2]. *Does q-uniform concentration fail for $p = 1$?*

The reader may note that in order to define p-concentration in the setting of $\mathbb{Z}_q$, one should also look for f that satisfies (5), but with $f(a)$, for some arbitrary $a \in \mathbb{Z}_q$, on the left-hand side. This is easy when q is prime. Indeed, for $a = 0$ the Dirac mass at 0, which is an idempotent, has the required property with constant 1. Otherwise, if $a \neq 0$ and f satisfies (5), then the function $g(x) := f(a^{-1}x)$ satisfies the same inequality, but with $g(a)$ on the

left-hand side. Here a^{-1} is the unique inverse for the multiplication in $\mathbb{Z}_q$. Clearly, $g(a) = f(1)$, and all other values taken by f are taken by g since multiplication is one to one in $\mathbb{Z}_q$ for q prime, so that the right-hand side is the same for f and g.

Remark 1. One can also replace 1 by a on the left-hand side of (5) when q is any integer, but a and q are coprime.

As we said, p-concentration on $\mathbb{Z}_q$ plays a role in proofs for p-concentration on the torus. In order to solve the 2-concentration problem on the torus, Déchamps-Gondim, Lust-Piquard, and Queffélec [6, 7] have considered the concentration problem on $\mathbb{Z}_q$, proving the precise value that we already mentioned,

$$\gamma_2^\sharp = \sup_{0\le x} \frac{2\sin^2 x}{\pi x} = 0.4613\cdots . \tag{6}$$

Moreover, they obtained $\gamma_p^\sharp \ge 2(\gamma_2^\sharp/2)^{p/2}$ for all $p > 2$. The last assertion is an easy consequence of the decrease of ℓ^p norms with p, and we have, in general,

$$\gamma_p^\sharp \ge 2(\gamma_{p'}^\sharp/2)^{p/p'} \tag{7}$$

for $p > p'$.

Let us also mention that they considered the same problem for the class of positive definite polynomials, that is,

$$\mathbf{P}_q^+ := \left\{ \sum_{h\in H} a_h e_h \ : \ a_h \ge 0, h \in \{0,\dots,q-1\} \right\}. \tag{8}$$

We say that there is uniform p-concentration on $\mathbb{Z}_q$ for the class of positive definite polynomials if there exists some constant γ such that (5) holds for some $f \in \mathbf{P}_q^+$. We denote by c_p^+ the level of p-concentration for the class of positive definite polynomials, which is defined as the maximum of all admissible constants in (5) (similarly to the class of idempotents).

With this notation, it has been proved in [6] that $c_2^+ = 1/2$. Since the class of positive definite polynomials is stable by taking products, it follows that, for all even integers $2k$,

$$\gamma_{2k}^\sharp \le c_{2k}^+ \le 1/2.$$

It is easy to see that there is uniform p-concentration on $\mathbb{Z}_q$ for all $p > 1$, using Dirichlet kernels. This has been used in our paper [4], where the discrete problem under consideration here has been largely studied, at least for p an even integer.

On the other hand, coming back to our main point, i.e., to the case of $p = 1$, and using the recent results of B. Green and S. Konyagin [8], we answer negatively in this case, which gives an affirmative answer to the conjecture of [2] for finite groups $\mathbb{Z}_q$.

All the results on $\mathbb{Z}_q$ summarize in the following theorem, which gives an almost complete answer to the p-concentration problem under consideration, except for the best constants, which are not known for $p \neq 2$.

Theorem 3. *For all $1 < p < \infty$ there is uniform p-concentration on $\mathbb{Z}_q$. The constant γ_2 is given by (3), and $0.495 < \gamma_4 \leq 1/2$. For all $p > 2$, we have $\gamma_p > 0.483$. On the other hand, for $p \leq 1$ there is not uniform p-concentration.*

Positive results are implicitly contained in [4], where they are used as tools for the problem of concentration on the torus. As far as necessary upper bounds for $\gamma_p^\sharp$ are considered, since the polynomials f with positive coefficients have their maximum at 0, we have the trivial upper bound $\gamma_p^\sharp \leq 2/3$. Moreover, for p an even integer, we have seen that $\gamma_p^\sharp \leq 1/2$. Let us remark that (7) provides an improvement on the bound $2/3$ between two even integers. Indeed, for $p \leq 2k$, we have

$$\gamma_p^\sharp \leq 2^{1-p/k}.$$

In the next two sections, we will consider the case of $\mathbb{Z}_q$, first for $p > 1$, then for $p = 1$. Then, in Sect. 4, we will come back to the case $p = 2$ on the torus and exploit the proof for giving concentration results by means of the use of the grid $\mathbb{G}_q$. In the last section, we prove Theorem 2.

We tried to keep the notation for the constants the same as in [4], since we refer to the proofs there, and apologize for the fact that sometimes this notation seems more complicated than it should be.

2 Uniform p-Concentration

In this section, we will recall the situation on the group $\mathbb{Z}_q$ by transferring the results that have been obtained for the grid

$$\mathbb{G}_q := \{k/q; k = 0, 1, \ldots, q-1\}$$

contained in $\mathbb{T}$. By a slight abuse of notation, let us still denote by

$$\mathbf{P}_q := \left\{ \sum_{h \in H} e_h \ : \ H \subset \{0, \ldots, q-1\} \right\} \tag{9}$$

the set of trigonometric idempotents of degree less than q on $\mathbb{T}$, with e_h denoting the exponential $e_h(x) := e^{2\pi i h x}$ adapted to $\mathbb{T}$. When restricted to $\mathbb{G}_q$ identified with $\frac{1}{q}\mathbb{Z}_q$, it coincides with the corresponding idempotent on $\mathbb{Z}_q$ (the coefficients are the same, but the exponential is now adapted to $\mathbb{Z}_q$). This is a one-to-one correspondence between idempotents of $\mathbb{Z}_q$ and idempotents of degree less than q, since these last ones are determined by their values on q points, and, in particular, on $\mathbb{G}_q$. We will prefer to deal with ordinary trigonometric polynomials, and see $\mathbb{Z}_q$ as the grid $\mathbb{G}_q$.

Unless explicitly mentioned, we will only consider Taylor polynomials, that is, trigonometric polynomials with only nonnegative frequencies.

We consider the following quantities, written in this new notation, and identify them with the quantities defined for $\mathbb{Z}_q$ in the Introduction.

$$\gamma_p := \liminf_{q\to\infty} \gamma_p(q), \qquad \gamma_p(q) := \sup_{R\in\mathbf{P}_q} \frac{2\left|R\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p}. \tag{10}$$

One can obtain a lower bound of $\gamma_p^\sharp$, with $p>1$, by the only consideration of the Dirichlet kernels

$$D_n(x) := \sum_{\nu=0}^{n-1} e(\nu x) = e^{\pi i(n-1)x}\frac{\sin(\pi n x)}{\sin(\pi x)}. \tag{11}$$

Here the constraint on the degree restricts us to $n<q$. Having n and q tend to infinity with n/q tending to t, we proved the following in [4] (see Lemma 35).

Lemma 1. *For $p>1$, we have the inequality*

$$2(\gamma_p^\sharp)^{-1} \le \inf_{0<t<1/2} B(p,t), \tag{12}$$

where, for $\lambda>1$,

$$B(\lambda,t) := \left(\frac{\pi t}{\sin \pi t}\right)^\lambda \left(1+2\sum_{k=1}^{\infty}\left|\frac{\sin(k\pi t)}{k\pi t}\right|^\lambda\right). \tag{13}$$

It is clear that $B(\lambda,t)$ is bounded for $\lambda>1$, so that $\gamma_p^\sharp>0$ and there is uniform p-concentration: just take as a bound the value for $t=1/4$. Let us try to get more precise estimates. The computation of $\inf_{0<t<1/2} B(\lambda,t)$ can be executed explicitly for $\lambda=2$ and $\lambda=4$. In the first case we recognize in the sum the Fourier coefficients of $\chi_{[-t/2,t/2]}$, whose L^2 norm is $\sqrt{t}$. So (12) leads to the minimization of the function $\frac{2\sin^2 t}{\pi t}$, and to the estimate $\gamma_2 \ge \sup_{0\le t}\frac{2\sin^2 t}{\pi t} = 0.4613\cdots$. This is the formula given by Déchamps-Gondim, Lust-Piquard, and Queffélec in [6]. We refer to them for the necessity of the condition, for which they give a smart proof. For $\lambda=4$, we recognize in the sum of (13) the Fourier coefficients of the convolution product $\chi_{[-t/2,t/2]} * \chi_{[-t/2,t/2]}$, whose L^2 norm is equal to $(2t^3/3)^{1/2}$. Using the Plancherel formula we obtain

$$\gamma_4^\sharp \ge \max_{0<t<1/2} \frac{3\left(\sin^4(\pi t)\right)}{\pi^4 t^3} > 0.495. \tag{14}$$

For larger integer values of λ, the computations do not seem to be easily handled. But we can prove that there exists a uniform lower bound for $\gamma_p^\sharp$

when $p > 2$. To see this, we need another lemma that can be found in [4]. Let us first give new definitions, relative to positive definite polynomials.

As for idempotents, by the same slight abuse of notation, let us still denote by

$$\mathbf{P}_q^+ := \left\{ \sum_{h\in H} a_h e_h \ : \ a_h \geq 0,\ h \in \{0,\dots,q-1\} \right\} \tag{15}$$

the set of trigonometric polynomials with nonnegative coefficients of degree less than q on $\mathbb{T}$, with e_h denoting the exponential adapted to $\mathbb{T}$. Again, when restricted to $\mathbb{G}_q$, it coincides with the corresponding positive definite polynomial with nonnegative coefficients on $\mathbb{Z}_q$, and this defines a one-to-one correspondence between positive definite polynomials of $\mathbb{Z}_q$ and positive definite polynomials on $\mathbb{T}$ of degree less than q. The constant c_p^+ can then be defined by

$$c_p^+ := \liminf_{q\to\infty} c_p^+(q), \qquad c_p^+(q) := \sup_{R\in\mathbf{P}_q^+} \frac{2\left|R\left(\frac{1}{q}\right)\right|^p}{\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p}. \tag{16}$$

It is much easier to find positive definite polynomials in $\mathbf{P}_q^+$ than idempotents. In particular, whenever P is in $\mathbf{P}_q$, then, for each positive integer L the polynomial Q, which has degree less than q and has the same values on $\mathbb{G}_q$ as P^L, is in $\mathbf{P}_q^+$. So we can also take powers of Dirichlet kernels as polynomials R on the right-hand side of (16). This leads to the following bounds, using Lemma 1:

$$\begin{aligned} 2(c_p^+)^{-1} &\leq \inf_{L\geq 1}\ \inf_{0<t<1/2} B(Lp,t) \leq \inf_{\kappa>0} \limsup_{\lambda\mapsto\infty} B\left(\lambda, \kappa\sqrt{6/\lambda}\right) \\ &\leq 4.13273. \end{aligned} \tag{17}$$

The two last estimates may be found in [4], see (55), and lead to

$$c_p^+ > 0.483. \tag{18}$$

The first one gives a nonexplicit bound for a fixed p:

$$c_p^+ \geq 2 \sup_{L\geq 1}\ \sup_{0<t<1/2} B(Lp,t)^{-1}. \tag{19}$$

We prove now that we have the same estimates for γ_p when $p > 2$.

Theorem 4. *We have* $\gamma_p > 0.483$ *uniformly for all* $p > 2$.

This is a consequence of the following proposition, which is more general than the corresponding results in [4].

Proposition 1. *Let $p > 2$ and $c > 0$, $\varepsilon > 0$. Then there exists $q_0 := q_0(c, \varepsilon)$ such that, if $q > q_0$ and $P := \sum_0^{q-1} a_h e_h$ is a polynomial of degree less than q that satisfies the two conditions*

$$cq \max_h |a_h| \le \sum |a_h| \le c^{-1} |P(1/q)|, \tag{20}$$

$$|P(1/q)| \ge c \left(\sum_{k=0}^{q-1} |P(k/q)|^p \right)^{1/p}, \tag{21}$$

then there exists a polynomial Q of degree less than q, whose coefficients are either $a_h/|a_h|$ or 0, such that

$$|Q(1/q)| \ge (1 - \varepsilon) |P(1/q)|, \tag{22}$$

$$\left(\sum_{k=0}^{q-1} |Q(k/q) - P(k/q)|^p \right)^{1/p} \le \varepsilon |P(1/q)|. \tag{23}$$

Observe that, for P positive definite, Q is an idempotent. In this case, the first condition can be reduced to $P(0) \ge cq \max_h |a_h|$. Indeed, the fact that $|P(1/q)| \ge cP(0)$ follows from the second one.

Let us take the proposition for granted, and use it in our context.

We claim that this allows us to conclude for the bound below

$$\gamma_p^\sharp > 0.483. \tag{24}$$

Proof (of Theorem 4). Let us take for P a positive definite polynomial of degree less than q for which

$$\frac{2 \left| P\left(\frac{1}{q}\right) \right|^p}{\sum_{k=0}^{q-1} \left| P\left(\frac{k}{q}\right) \right|^p} \ge c_0 > 0.483.$$

We claim that there exists an idempotent Q for which the same ratio is bounded below by $c_0 C(\varepsilon)$, with $C(\varepsilon)$ tending to 1 when ε tends to 0. Indeed, we can apply the proposition as soon as we have proved that P satisfies the condition (20) (uniformly for q large). We have seen that P can be taken as the polynomial of degree less than q, which coincides with D_n^L on the grid $\mathbb{G}_q$, for n chosen in such a way that $n/q \approx t = \kappa\sqrt{6/\lambda}$ is small enough so that we approach the extremum in (17). Next, it is easy to see that $P(0) = n^L$, while $|\hat{P}(k)| \le Ln^{L-1}$. So we have (20) with a very small constant c, but what is important is that it does not depend on q tending to ∞ (for fixed ε). To conclude the proof, we use the fact that, by Minkowski's inequality, and using the assumption on P, we have

$$\left(\sum_{k=0}^{q-1}\left|Q\left(\frac{k}{q}\right)\right|^p\right)^{1/p} \le \left(\sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p\right)^{1/p} + \varepsilon|P(1/q)|$$
$$\le ((2/c_0)^{1/p}+\varepsilon)|P(1/q)|$$
$$\le (1-\varepsilon)((2/c_0)^{1/p}+\varepsilon)|Q(1/q)|.$$

The constant tends to $(2/c_0)^{1/p}$ when ε tends to 0, which concludes the proof. □

The same method leads to

$$\gamma_p^\sharp \ge 2 \sup_{L\ge 1} \sup_{0<t<1/2} B(Lp,t)^{-1}. \tag{25}$$

This finishes the proof of the part of Theorem 3 concerning $p > 1$, except for the proof of Proposition 1, which we do now. It relies on the construction of random polynomials, which may have an independent interest.

Proof (of Proposition 1). Without loss of generality we may assume that $\max_h |a_h| = 1$. We set $\alpha_k := |a_k|$ and $\sigma := \sum \alpha_k$, so that $0 \le \alpha_k \le 1$ and $cq \le \sigma \le c^{-1}|P(1/q)|$. We take a sequence of independent random variables $X_0, X_1, \ldots, X_{q-1}$ that follow the Bernoulli law with parameters $\alpha_0, \alpha_1, \ldots, \alpha_{q-1}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and set

$$P_\omega := \sum_0^{q-1} b_h X_h(\omega) e_h$$

with $b_h := a_h/|a_h|$ for $a_h \ne 0$; otherwise $b_h = 0$. Then the expectation of P_ω is equal to P. We will prove that $Q = P_\omega$ satisfies (22) and (23) with positive probability. Let us first consider (22), and prove that the converse inequality holds with probability less than 1/3 for q large enough. Indeed, one has the inclusions

$$\{\omega; |P_\omega(1/q)| \le (1-\varepsilon)|P(1/q)|\} \subset \{\omega; |P_\omega(1/q) - P(1/q)| > \varepsilon|P(1/q)|\},$$

so that, by the Markov inequality, using the fact that the variance of $P_\omega(1/q)$ is $\sum \alpha_k(1-\alpha_k) \le \sigma$, we have

$$\mathbb{P}\left(\left|\frac{P_\omega(1/q)}{P(1/q)}\right| \le 1-\varepsilon\right) \le c^{-2}\varepsilon^{-2}\sigma^{-1}.$$

By (20) we know that this quantity is small for q large.

Next, to show (23), in view of (20) it is sufficient to prove that with probability 2/3,

$$\sum_{k=0}^{q-1} |P_\omega(k/q) - P(k/q)|^p \le c^p\varepsilon^p\sigma^p.$$

We claim that there exists some uniform constant C_p, for $p > 2$, such that, for each k,

$$\mathbb{E}(|P_\omega(k/q) - P(k/q)|^p) \le C_p \sigma^{p/2}. \tag{26}$$

Let us take this for granted and finish the proof. By simple estimation,

$$\mathbb{P}\left(\sum |P_\omega(k/q) - P(k/q)|^p \ge (c\varepsilon\sigma)^p\right) \le c^{-p}\varepsilon^{-p} C_p\, q\, \sigma^{-p/2}.$$

From this we conclude easily, using the fact that $\sigma \ge cq$, so that the right-hand side tends to 0 when q tends to infinity. Finally, (26) is a well-known property of independent sums of Bernoulli variables, e.g., in [4] (Lemma 54) a proof of the following lemma can be found.

Lemma 2. *For $p > 2$ there exists some constant C_p with the following property. Let $\alpha_k \in [0,1]$ and $b_k \in \mathbb{C}$ be arbitrary for $k = 0, 1, \dots, N$. For X_k a sequence of independent Bernoulli random variables with parameter α_k, we have*

$$\mathbb{E}\left(\left|\sum_{k=0}^{N} b_k(X_k - \alpha_k)\right|^p\right) \le C_p \cdot \max_{k=1,\dots,N} |b_k|^p \cdot \left(1 + \sum_{k=0}^{N} \alpha_k\right)^{p/2}.$$

□

Of course, one would like to know whether constants are the same for classes $\mathbf{P}_q$ and $\mathbf{P}_q^+$. We know that it is not the case for $p = 2$ from the work of Déchamps-Gondim, Lust-Piquard, and Queffélec, but the last proposition induces us to conjecture that they are the same for $p > 2$. Note that Proposition 1 holds when (20) is replaced by the weaker assumption $\sigma \ge A(q) q^{2/p} \max |a_h|$, with A tending to infinity with q.

3 Failure of Uniform 1-Concentration on $\mathbb{Z}_q$

We prove here the negative result of Theorem 3. It will be more convenient, in this section, to work directly on $\mathbb{Z}_q$, and not on the grid $\mathbb{G}_q$. We now restrict to q prime, which is sufficient to conclude negatively.

Assume that there exist some constant c and some idempotent $f = \sum_{h\in H} e_h$ such that

$$|f(1)| \ge c \sum_{k=0}^{q-1} |f(k)|. \tag{27}$$

We claim that H may be assumed to have cardinality $\le q/2$. Indeed, H is certainly not the whole set $\{0, \dots, q-1\}$, since the corresponding idempotent is q times the Dirac mass at 0. Assume that f satisfies (27) with $\#H > q/2$. Then the idempotent $\widetilde{f} := \sum_{h\in {}^cH} e_h$ takes the same absolute values as f outside 0, while its value at 0 is $q - \#H < f(0)$. So it also satisfies (27).

From now on, let $r := \#H \leq q/2$. We have by assumption (27) $\sum_{k=0}^{q-1} |f(k)| \leq |f(1)|/c \leq f(0)/c = r/c$. So the function

$$g := r^{-1}\,(f - r\delta_0)$$

is 0 at 0 and has ℓ^1 norm bounded by $\frac{1}{c} + 1$, while its Fourier coefficients are equal to $1/r - 1/q$ (r of them), or $-1/q$, since the delta function has all Fourier coefficients equal to $1/q$. But, according to Theorem 1.3 of [8], we should have $q \min_k |\hat{g}(k)|$ tending to 0 when q tends to ∞ (note that the Fourier transform here is replaced by the inverse Fourier transform in [8], which is the reason for multiplication by q compared to the statement given there). This gives a contradiction, and allows us to conclude that there is no uniform 1-concentration. This finishes the proof.

We leave the following as an open question.

Problem 1. In line with Definition 2, for given fixed q denote

$$\gamma_1^\sharp(q) := \max_{f \in \mathbf{P}_q} 2|f(1)|/\sum_{k=0}^{q-1} |f(k)|.$$

Determine $\beta := \liminf_{q\to\infty} \log(1/\gamma_1^\sharp(q))/\log\log q$.

Using the full strength of the result of [8], the constant c in (27) may be chosen uniformly bounded from below in q by $\log^{-\alpha} q$, with α less than $1/3$ (that is, the proof by contradiction shows that $c > \log^{-\alpha} q$ is not possible, hence $\beta \geq 1/3$). On the other hand, the Dirichlet kernel exhibits $\gamma_1(q) \geq C/\log q$, i.e., $\beta \leq 1$. This leaves open the question if β achieves 1, i.e., $\log(1/\gamma_1^\sharp(q))/\log\log q$ can be taken as anything less than 1. The problem is in relation with the Littlewood conjecture on groups $\mathbb{Z}_q$, for which there have been new improvements by Sanders [11].

4 The 2-Concentration on Measurable Sets

We prove in this section that $\gamma_2 \geq \gamma_2^\sharp$. The converse inequality follows from the fact that the constant for measurable sets is smaller than the one when restricted to open sets, which is $\gamma_2^\sharp$, whose explicit value is given by (6). In this paragraph we shall basically use the method of Anderson et al. [2]. Our improvements are mainly expository. The method is valid for all $p > 1$, and we will write it in this context, even if better results can be obtained for $p \neq 2$. Indeed, it will be easier, later on, to explain how to improve the method starting from this first one.

So we are going to prove the following proposition.

Proposition 2. *For $p > 1$, we have*

$$\gamma_p \geq \gamma_p^\sharp.$$

Proof. We are given an arbitrary symmetric measurable set, with $|E| > 0$. We want to find some idempotent f that concentrates on E. We will use a variant of Khintchine's theorem in Diophantine approximation, which we summarize in the next lemma (Proposition 36 in [4]).

Lemma 3. *Let E be a measurable set of positive measure in $\mathbb{T}$. For all $\theta > 0$, $\eta > 0$, and $q_0 \in \mathbb{N}$, there exists an irreducible fraction a/q such that $q > q_0$ and*

$$\left| \left(\frac{a}{q} - \frac{\theta}{q^2}, \frac{a}{q} + \frac{\theta}{q^2} \right) \cap E \right| \geq (1-\eta) \frac{2\theta}{q^2}. \tag{28}$$

Moreover, given a positive integer ν, it is possible to choose q such that $(\nu, q) = 1$.

The parameter θ will play no role at the moment, so we can set it as 1. It will appear as necessary for generalizations only later. We consider the grid $\mathbb{G}_q := \{k/q; k = 0, 1, \ldots, q-1\}$ contained in the torus, for a and q given by Lemma 3, for given values of η and q_0 to be fixed later on. We assume that q is sufficiently large so that we can find $R \in \mathbf{P}_q$ with the property that

$$2|R(a/q)|^p \geq c \sum_{k=0}^{q-1} |R(k/q)|^p, \tag{29}$$

with $\varepsilon > 0$ chosen arbitrarily small and where $c > \gamma_p - \varepsilon$. When $a = 1$, the existence of such a P follows from the definition of $\gamma_p^\sharp$. See Remark 1 for the fact that we can replace 1 by a whenever a and q are co-prime. We then claim that the polynomial $Q(t) := R(t)D_n(qt)$, which is an idempotent, is such that

$$\int_E |Q|^p \geq c\kappa(\varepsilon) \int_{\mathbb{T}} |Q|^p,$$

with $\kappa(\varepsilon) < 1$ tending to 1 when ε tends to 0, and where parameters η and n are chosen suitably depending on ε.

The idea of the proof goes as follows: since D_n concentrates the L^p norm near 0 (it can be concentrated in any subset F of the interval $\left(-\frac{1}{q}, +\frac{1}{q}\right)$, with $|F| > 2(1-\eta)/q$), then $D_n(qt)$ concentrates equally on the q subsets around the points of the grid $\mathbb{G}_q$. We take F such that as qt belongs to F when t belongs to $\left(\frac{a}{q} - \frac{\theta}{q^2}, \frac{a}{q} + \frac{\theta}{q^2}\right) \cap E$. Now multiplication by R will concentrate the integral on the subset around a/q, which we wanted. We need to know that the polynomial R is almost constant on each of these subsets, which is given by Bernstein's theorem.

Let us now enter into details. We have the following lemma on Dirichlet kernels.

Lemma 4. *Let $p > 1$. For ε given, one can find $\eta > 0$ and $\delta_0 > 0$ such that, for all $0 < \delta < \delta_0$, if F is a measurable subset of $(-\delta, +\delta) \subset \mathbb{T}$ of measure larger than $2\delta(1-\eta)$, we can find some suitable $n \in \mathbb{N}$ so that*

$$\int_F |D_n|^p \geq (1-\varepsilon) \int_{\mathbb{T}} |D_n|^p.$$

Proof. It is well known that $\int_{\mathbb{T}} |D_n|^p \geq \kappa_p n^{p-1}$ (see [2], for instance, for precise estimates). So it is sufficient to prove that we can obtain

$$\int_{{}^c F} |D_n|^p \leq \varepsilon n^{p-1}.$$

This is a consequence of the fact that

$$\int_{(-\delta,+\delta)\setminus F} |D_n|^p \leq 2n^p \eta\delta,$$

while

$$\int_{\mathbb{T}\setminus(-\delta,+\delta)} |D_n|^p \leq \left(\frac{\pi}{2}\right)^p \int_{|t|>\delta} t^{-p} dt = \kappa'_p \delta^{1-p}.$$

We choose for n the smallest integer larger than $(2\kappa'_p/\varepsilon)^{1/(p-1)}\delta^{-1}$ and η such that $8(2\kappa'_p/\varepsilon)^{1/(p-1)}\eta = \varepsilon$.

We remark that here we did not need the flexibility linked to the parameter δ_0. It is here for further generalizations. □

Next we recall classical Bernstein and Marcinkiewicz–Zygmund type inequalities, in the forms tailored to our needs and proved in [4], Lemma 41. Recall that here polynomials are Taylor polynomials, that is, trigonometric polynomials with only nonnegative frequencies, which is the case for the polynomial R.

Lemma 5. *For $1 < p < \infty$ there exists a constant K_p such that, for P a polynomial of degree less than q and for $|t| < 1/2$, we have the two inequalities*

$$\sum_{k=0}^{q-1} |P(t + k/q)|^p \leq K_p \sum_{k=0}^{q-1} |P(k/q)|^p, \tag{30}$$

$$\sum_{k=0}^{q-1} ||P(t + k/q)|^p - |P(k/q)|^p| \leq K_p |qt| \sum_{k=0}^{q-1} |P(k/q)|^p. \tag{31}$$

For our polynomial R, this gives the inequality

$$||R(t + a/q)|^p - |R(a/q)|^p| \leq 2c^{-1} K_p qt |R(a/q)|^p. \tag{32}$$

This implies that, for $|t - \frac{a}{q}| < \frac{\theta}{q^2}$ with q large enough,

$$|R(t)|^p \geq (1 - \varepsilon)|R(a/q)|^p. \tag{33}$$

We have also, for $|t| < \frac{\theta}{q^2}$,

$$\sum_{k=0}^{q-1} |R(t + k/q)|^p \leq \sum_{k=0}^{q-1} |R(k/q)|^p + 2K_p \frac{\theta}{q} c^{-1} |R(a/q)|^p,$$

which leads to the inequality, valid for $|t| < \frac{\theta}{q^2}$ for q large enough,

$$\sum_{k=0}^{q-1} |R(t+k/q)|^p \le 2c^{-1}(1+\varepsilon)|R(a/q)|^p. \tag{34}$$

Let us finally remark that (30) leads to the following, valid for all t:

$$\sum_{k=0}^{q-1} |R(t+k/q)|^p \le 2c^{-1}K_p|R(a/q)|. \tag{35}$$

We can now proceed to the proof of the required inequality for R. We have fixed ε and chosen q_0 large enough so that estimates (33) and (34) hold (recall that for the moment $\theta = 1$). Then we use Lemma 3, which fixes some a/q, and find D_n, which is assumed to be adapted to $\delta := \frac{\theta}{q}$. We denote $\tau^p := \int_{\mathbb{T}} |D_n|^p$ and $I := \left(\frac{a}{q} - \frac{\theta}{q^2}, \frac{a}{q} + \frac{\theta}{q^2}\right)$.

$$\begin{aligned}\frac{1}{2}\int_E |Q|^p \ge \int_{I\cap E} |R|^p|D_n|^p &\ge (1-\varepsilon)|R(a/q)|^p \int_{I\cap E} |D_n(qt)|^p dt \\ &\ge \frac{1}{q}(1-\varepsilon)|R(a/q)|^p \int_{F\cap(-\delta,+\delta)} |D_n|^p \\ &\ge \frac{(1-\varepsilon)^2\tau^p}{q}|R(a/q)|^p. \end{aligned}\tag{36}$$

Here F is the pre-image by $t \mapsto qt$ of $I \cap E$, which has measure at least $2(1-\eta)\delta$, and so concentrates the integral of $|D_n|^p$.

Let us now look for a bound of the whole integral. We write

$$\int_{\mathbb{T}} |Q|^p = \int_{-1/q}^{1/q} \left(\sum_k |R(t+\frac{k}{q})|^p\right) |D_n(qt)|^p dt$$

and cut the integral into two parts, depending on whether $|t| \le \frac{\theta}{q^2}$ or not. For the first part we use (34), for the second one (35). We recall that the integral of D_n outside the interval $(-\theta/q, \theta/q)$ is bounded by $\varepsilon\tau^p$. Finally,

$$\begin{aligned}\int_{\mathbb{T}} |Q|^p &\le 2c^{-1}\,\frac{1+\varepsilon}{q}|R(a/q)|^p \int_{\mathbb{T}} |D_n|^p + 2c^{-1}K_p \cdot \frac{\varepsilon}{q}|R(a/q)|^p \int_{\mathbb{T}} |D_n|^p \\ &\le 2c^{-1}\,\frac{(1+C\varepsilon)\tau^p}{q}|R(a/q)|^p.\end{aligned}$$

We conclude by comparison with (36). □

As said above, we have obtained optimal results for $p = 2$. At this point, we can see how results can be improved for $p \neq 2$. The main point is the possibility to replace the Dirichlet kernel D_n by an idempotent T, which satisfies nearly

the same properties as the Dirichlet kernel that are summarized in Lemma 4, but has the additional property to have arbitrarily large gaps. More precisely, we say that T *has gaps larger than* N if $|k-k'| \le N$ implies that one of the two Fourier coefficients $\hat{T}(k)$ and $\hat{T}(k')$ is zero. We state the existence of such idempotents T as a lemma, and we refer to [4] for their construction.

Lemma 6. *Let $p > 0$ be different from 2. Then for $\varepsilon > 0$ there exist $\delta_0 > 0$ and $\eta > 0$ such that, for all $\delta < \delta_0$ and $N \in \mathbb{N}$, if E is a measurable set that satisfies, for $\alpha = 0$, the assumption $|E \cap [\alpha-\delta, \alpha+\delta]| > 2(1-\eta)\delta$, then there exists an idempotent T with gaps larger than N such that*

$$\int_{E\cap[\alpha-\delta,\alpha+\delta]} |T|^p > (1-\varepsilon)\int_0^1 |T|^p.$$

Moreover, if p is not an even integer, this is also valid for $\alpha = 1/2$.

For the moment we use this lemma with $\alpha = 0$. We are no longer restricted to consider polynomials of degree less than q in order that $R(t)T(qt)$ be an idempotent. It is sufficient that the degree of R be less than Nq, and, since N is arbitrary, this gives essentially no constraint. The fact that R has degree less than q was also used for (33) and (34). It is where the flexibility given by the parameter θ can be used: if R has degree less than q^2, then roughly speaking we can also use the Bernstein inequality, but θ/q has to be replaced by θ in (32). This is of no inconvenience, since θ can be chosen arbitrarily small.

At this point, we could proceed with a polynomial of degree less than q^2 for (33), but certainly not for Lemma 5, since such a polynomial can be identically 0 on the grid $\mathbb{G}_q$. To develop such inequalities for polynomials S of degree larger than q, we will restrict to those that can be written as $S(t) := R(t)R((q+1)t)$, with R an idempotent that satisfies (29), but for $2p$ instead of p (so that the condition on p is now $p > 1/2$). The important point is that S is also an idempotent, and so is ST if T has sufficiently large gaps. Also, $|S(k/q)|^p = |R(k/q)|^{2p}$ at each point of the grid, and, in particular, at a/q. Moreover, it is easy to see that, for θ small enough, one still has the inequalities (33), (34), and (35) with $2p$ in place of p, both for the polynomials $R(t)$ and $R((q+1)t)$ (for this last one we have to choose θ small enough, as we mentioned earlier). The fact that (33), (34), and (35) are valid for S follows from the Cauchy–Schwarz inequality. The rest of the proof goes the same way as the previous one and leads to the following, for which we leave the details to the reader.

Proposition 3. *One has p-concentration for $p > 1/2$, and, for $p \neq 2$, one has the inequality $\gamma_p \ge \gamma_{2p}^\sharp$. In particular, $\gamma_1 \ge \gamma_2^\sharp$.*

We could as well have taken $S(t) = R_1(t)R_2((q+1)t)$ and used Hölder's inequality, taking R_1 approaching the maximum concentration on the grid for the exponent r and R_2 approaching the maximum concentration on the grid for the exponent s, with $\frac{p}{r} + \frac{p}{s} = 1$. This leads to the following generalization of the last proposition.

Proposition 4. *One has p-concentration for $p > 1/2$, and, for $p \neq 2$, one has the inequality $\gamma_p \geq \left(\gamma_r^\sharp\right)^{p/r} \left(\gamma_s^\sharp\right)^{p/s}$ for all $r > p$ and $s > p$ such that $\frac{p}{r} + \frac{p}{s} = 1$.*

Before concluding this section, let us make a last observation. Once we use an idempotent T with arbitrarily large gaps, it is not difficult to build idempotents with arbitrarily large gaps. It is sufficient to start from the polynomial $R(\nu t)$, with ν arbitrarily large. Recall that, when using Lemma 3, we can take q such that $(\nu, q) = 1$. This means that there exists $b \pmod q$ such that $\nu a = b \pmod q$, and we choose R to satisfy (29), but with b/q in place of a/q. The rest of the proof can be adapted. We state it as a proposition.

Proposition 5. *In Proposition 2 and Proposition 4, when $p \neq 2$, we can have arbitrarily large gaps. That is, when $1/2 < p \neq 2$, given a symmetric measurable set E of positive measure, and any constant $c < \gamma_p^\sharp$ (resp. $\left(\gamma_r^\sharp\right)^{p/r} \left(\gamma_s^\sharp\right)^{p/s}$), there exists an idempotent P with arbitrarily large gaps such that*

$$\int_E |P|^p > c \int_{\mathbb{T}} |P|^p.$$

5 Improvement of Constants for p not an Even Integer

We proved in [4] that $\gamma_p = 1$ for $p > 1$ and p not an even integer. Let us give the main lines of the proof, which will be used again for the improvement of the constant when $p = 1$. As we shall see, it has been slightly simplified compared to the proof in [4]. The main ingredient is the fact that there are idempotents that concentrate as the Dirichlet kernels, but with arbitrarily large gaps, and at $1/2$ instead of 0. We have already stated this in Lemma 6.

If we take such a peaking function T, then $T(qx)$ concentrates around the points of the translated grid

$$\mathbb{G}_q^\star := \frac{1}{2q} + \mathbb{G}_q = \left\{ \frac{2k+1}{2q} \quad ; \quad k = 0, \dots, q-1 \right\}. \tag{37}$$

We have considerably gained with this new grid compared to $\mathbb{G}_q$ because 0 – where, by positive definiteness, we always must have a maximal value of any idempotent – does not belong to the grid any longer. Thus, we will even be able to find idempotents P such that the maximal value of $|P|$ (over the grid) will be attained at the points $\pm 1/2q$; moreover, the sum of the values $|P|^p$ on $\mathbb{G}_q^\star$ is just slightly larger than $2|P(1/2q)|^p$.

Let us interpret the new constants that we will introduce in terms of another concentration problem on a finite group. More precisely, we view $\mathbb{G}_q^\star$ as $\mathbb{G}_{2q} \setminus \mathbb{G}_q$, and identify $\mathbb{G}_{2q}$ with $\mathbb{Z}_{2q}$, while $\mathbb{G}_q^*$ identifies with a coset. Recall that the idempotents on $\mathbb{Z}_{2q}$ are identified with polynomials in $\mathbf{P}_{2q}$. We are interested in relative concentration inside the coset, and we give the following definition.

Definition 3. *We define*

$$\Gamma_p^\star := \sup_{K<\infty} \liminf_{q\to\infty} \Gamma_p^\star(q,K), \tag{38}$$

where $\Gamma_p^\star(q,K)$ *is the maximum of all constants* γ *for which there exists* $R \in \mathbf{P}_{2q}$ *satisfying*

$$2\left|R\left(\frac{1}{2q}\right)\right|^p \geq \gamma \sum_{k=0}^{q-1} \left|R\left(\frac{2k+1}{2q}\right)\right|^p \tag{39}$$

$$2\left|R\left(\frac{1}{2q}\right)\right|^p \geq \gamma K^{-1} \sum_{k=0}^{q-1} \left|R\left(\frac{k}{q}\right)\right|^p. \tag{40}$$

In other words, $\Gamma_p^\star$ is positive when there is uniform concentration at $1/2q$ (which is the case for $p > 1$), but the grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$ do not play the same role; the constant $\Gamma_p^\star$ is only the relative concentration on $\mathbb{G}_q^\star$, which we try to maximize.

Remark 2. We can also replace 1 by $2a+1$ on the left-hand side of (39) when q is any integer, but $2a+1$ and $2q$ are coprime.

This is the equivalent of Remark 1. Multiplication by b, such that $b(2a+1) \equiv 1$ modulo $2q$, will send 1 to $2a+1$ and define a bijection on $\mathbb{G}_q^\star$ (resp. $\mathbb{G}_q$).

Lower bounds for $\Gamma_p^\star$ are given in the following lemma, which is a slight modification of Lemma 34 in [4].

Lemma 7. *For* $p > 1$*, we have the inequality*

$$\frac{1}{\Gamma_p^\star} \leq \inf_{0<t<1/2} A(p,t), \tag{41}$$

where, for $\lambda > 1$*,*

$$A(\lambda,t) := \frac{1}{(\sin(\pi t))^\lambda} \sum_{k=0}^{\infty} \left|\frac{\sin((2k+1)\pi t)}{2k+1}\right|^\lambda. \tag{42}$$

The inequality is obtained by taking Dirichlet kernels D_n, with $n/2q$ tending to t, a point that will be used later on. Observe that $A(\lambda,t)$ tends to ∞ when t tends to 0, so that the infimum is obtained away from 0. The uniformity in the second inequality (40) is given by a bound of (a small modification of) $B(\lambda,t)$ defined in (13), for which we have the inequality

$$B(\lambda,t) \leq \left(\frac{\pi}{2}\right)^\lambda + 2\left(\sum_k k^{-\lambda}\right) t^{-\lambda}. \tag{43}$$

Observe that (for fixed t) $A(\lambda,t)$, and hence also $\inf_{0<t<1/2} A(\lambda,t)$ are decreasing functions of λ. In [4], recognizing the Fourier coefficients (at k

and $-k$) of the function $\frac{\pi}{2}\left(\chi_{[-t/2,t/2]}(x) - \chi_{[-t/2,t/2]}(x-1/2)\right)$, we used the Plancherel formula to calculate

$$A(2,t) = \frac{\pi^2 t}{4\sin^2(\pi t)}. \tag{44}$$

Substituting $x = \pi t$ and recalling (3) we find that

$$\Gamma_2^\star \geq 2\gamma_2 \approx 0.9226.$$

Moreover, it is easy to see that $\inf_{0<t<1/2} A(\lambda, t)$ is left continuous in λ at 2, so that

$$\liminf_{p\to 2-0} \Gamma_p^\star \geq 2\gamma_2. \tag{45}$$

Our main estimate for $\Gamma_p^\star$ is the following.

Proposition 6. *For $p > 2$ we have $\Gamma_p^\star = 1$.*

We postpone the proof of this proposition and show how to use it. We need an adaptation of the Khintchine-type theorem that we used in Sect. 4. The next lemma uses the inhomogeneous extension of Khintchine's Diophantine approximation theorem, first proved by Szüsz [13] and later generalized by Schmidt [12]. This is Proposition 37 of [4].

Lemma 8. *Let E be a measurable set of positive measure in $\mathbb{T}$. For all $\theta > 0$, $\eta > 0$, and $q_0 \in \mathbb{N}$, there exists an irreducible fraction $\frac{2a+1}{2q}$ such that $q > q_0$ and*

$$\left|\left[\frac{2a+1}{2q} - \frac{\theta}{q^2}, \frac{2a+1}{2q} + \frac{\theta}{q^2}\right] \cap E\right| \geq (1-\eta)\frac{2\theta}{q^2}. \tag{46}$$

Moreover, given a positive integer ν, it is possible to choose q such that $(\nu, q) = 1$.

Our main result is the following.

Theorem 5. *For p not an even integer, one has the inequalities $\gamma_p \geq \Gamma_p^\star$ and $\gamma_p \geq (\Gamma_r^\star)^{p/r}\,(\Gamma_s^\star)^{p/s}$ for all $r > p$ and $s > p$ such that $\frac{p}{r} + \frac{p}{s} = 1$. Moreover, given a symmetric measurable set E of positive measure, and any constant $c < \Gamma_p^\star$ (resp. $(\Gamma_r^\star)^{p/r}\,(\Gamma_s^\star)^{p/s}$), there exists an idempotent P with arbitrarily large gaps such that*

$$\int_E |P|^p > c \int_{\mathbb{T}} |P|^p.$$

Proof. We shall first prove the inequality $\gamma_p \geq \Gamma_p^\star$. We will then show how to modify the proof for the other statements.

We are given a symmetric measurable set E. We consider the grid $\mathbb{G}_q^\star = \mathbb{G}_{2q} \setminus \mathbb{G}_q$ contained in the torus, with a and q given by Lemma 8. At this point

we have already fixed some $\varepsilon > 0$. The values of q_0, η, and θ are also fixed, but we will say how to choose them later on. We assume that q is sufficiently large so that we can find $R \in \mathbf{P}_{2q}$ with the property that

$$2\left|R\left(\frac{1}{2q}+\frac{a}{q}\right)\right|^p \geq c\sum_{k=0}^{q-1}\left|R\left(\frac{1}{2q}+\frac{k}{q}\right)\right|^p, \tag{47}$$

with $c > (1-\varepsilon)\Gamma_p^\star$. Moreover, we can assume that

$$\sum_{k=0}^{q-1}\left|R\left(\frac{k}{q}\right)\right|^p \leq 2Kc^{-1}\left|R\left(\frac{1}{2q}+\frac{a}{q}\right)\right|^p \tag{48}$$

for some uniform constant K. The existence of such an R is given by Definition 3 and by Remark 2. Once we have chosen R, we choose a peaking function T at $1/2$ for the value ε. We assume now that η has been chosen sufficiently small for the existence of such a function T, built for $\delta := \theta/q$, which is possible if $\theta/q_0 \leq \delta_0$.

We choose the idempotent $Q(t) := R(t)T(qt)$ (indeed it is an idempotent if T has sufficiently large gaps) and fix $I := \left(\frac{2a+1}{2q} - \frac{\theta}{q^2}, \frac{2a+1}{2q} + \frac{\theta}{q^2}\right)$. We also set $\tau^p := \int_{\mathbb{T}} |T|^p$. From this point on, the proof follows the same lines as the proof of Proposition 2. We have the inequality

$$\begin{aligned}\frac{1}{2}\int_E |Q|^p \geq \int_{I\cap E}|R|^p|T|^p &\geq (1-\varepsilon)\left|R\left(\frac{2a+1}{2q}\right)\right|^p \int_{I\cap E}|T(qt)|^p dt\\ &\geq \frac{1}{q}(1-\varepsilon)\left|R\left(\frac{2a+1}{2q}\right)\right|^p \int_{F\cap(-\delta,+\delta)}|T|^p\\ &\geq \frac{(1-\varepsilon)^2\tau^p}{q}\left|R\left(\frac{2a+1}{2q}\right)\right|^p.\end{aligned}$$

We have used the fact that the image F of $I \cap E$ by $t \mapsto qt$ has measure at least $2(1-\eta)\delta$, and concentrates the integral of $|T|^p$ at $1/2$. We have also used the inequality

$$|R(t)|^p \geq (1-\varepsilon)\left|R\left(\frac{2a+1}{2q}\right)\right|^p, \tag{49}$$

valid for $|t - \frac{2a+1}{2q}| < \frac{\theta}{q}$ with θ small enough. This is an easy consequence of Lemma 6 for polynomials of degree $2q$, since the sum of values of $|R|^p$ on the whole grid $\mathbb{G}_{2q}$ is bounded by $2c^{-1}(K+1)$ times its value at $\frac{2a+1}{2q}$. Just take θ small enough (we fix θ in such a way that this is valid).

Before going on, let us remark that the other two inequalities can be deduced from Lemma 6. First, for $|t - \frac{2a+1}{2q}| < \frac{\theta}{q}$ with θ small enough, we have also

$$\sum_{k=0}^{q-1} |R\left(t+\frac{k}{q}\right)|^p \le 2c^{-1}(1+\varepsilon)\left|R\left(\frac{2a+1}{2q}\right)\right|^p. \tag{50}$$

Finally, for all t, we have, for some constant κ,

$$\sum_{k=0}^{q-1}\left|R\left(t+\frac{k}{q}\right)\right|^p \le \kappa\left|R\left(\frac{2a+1}{2q}\right)\right|^p. \tag{51}$$

Here we can take $\kappa := 2c^{-1}K_p(K+1)$. Next we look for a bound of the whole integral

$$\int_{\mathbb{T}} |Q|^p = \int_0^{1/q}\left(\sum_k \left|R\left(t+\frac{k}{q}\right)\right|^p\right)|T(qt)|^p dt$$

and cut the integral into two parts, depending on the fact that $|t-\frac{1}{2q}| \le \frac{\theta}{q}$ or not. For the first part we use (50), for the second one (51). We recall that the integral of T outside the interval $(\frac{1}{2}-\frac{\theta}{q}, \frac{1}{2}+\frac{\theta}{q})$ is bounded by $\varepsilon\tau^p$.

$$\begin{aligned}\int_{\mathbb{T}} |Q|^p &\le 2c^{-1}\,\frac{1+\varepsilon}{q}\left|R\left(\frac{2a+1}{2q}\right)\right|^p \quad \tau^p + \kappa\frac{\varepsilon}{q}\left|R\left(\frac{2a+1}{2q}\right)\right|^p \tau^p \\ &\le 2c^{-1}\,\frac{(1+C\varepsilon)\tau^p}{q}\left|R\left(\frac{2a+1}{2q}\right)\right|^p.\end{aligned}$$

We conclude by comparison with the integral on E. This allows us to conclude for the first case, $\gamma_p \ge \Gamma_p^\star$.

Let us now indicate the necessary modification for finding

$$\gamma_p \ge (\Gamma_r^\star)^{p/r}\,(\Gamma_s^\star)^{p/s}.$$

In the following we denote $r_1 := r$ and $r_2 := s$: the index j will always cover the two values $j=1$ and $j=2$. Instead of starting from one polynomial, we start from two polynomials R_1 and R_2 in $\mathbf{P}_{2q}$, which satisfy the following inequalities, for $j=1,2$:

$$2\left|R_j\left(\frac{2a+1}{2q}\right)\right|^{r_j} \ge c_j\sum_{k=0}^{q-1}\left|R_j\left(\frac{2a+1}{2q}\right)\right|^{r_j}, \tag{52}$$

with $c_j > (1-\varepsilon)\Gamma_{r_j}^\star$. Moreover, we assume that

$$\sum_{k=0}^{q-1}\left|R_j\left(\frac{k}{q}\right)\right|^{r_j} \le 2Kc^{-1}\left|R_j\left(\frac{2a+1}{2q}\right)\right|^{r_j} \tag{53}$$

for some uniform constant K. We then set $R(t) := R_1(t)R_2((2q+1)t)$. We remark that, on $\mathbb{G}_{2q}$, the values of R coincide with the values of the product R_1R_2. We will prove that we still have inequalities (49) and (50) for

$|t - \frac{2a+1}{2q}| < \frac{\theta}{q^2}$, and (51) for all t. Let us first prove that (51) holds for some constant κ. Indeed, by Hölder's inequality with conjugate exponents r_1/p and r_2/p and a periodicity of R_2, we have

$$\sum_{k=0}^{q-1} \left| R\left(t + \frac{k}{q}\right)\right|^p \le \left(\sum_{k=0}^{q-1} \left| R_1\left(t + \frac{k}{q}\right)\right|^{r_1}\right)^{\frac{p}{r_1}} \times \left(\sum_{k=0}^{q-1} \left| R_2\left((2q+1)t + \frac{k}{q}\right)\right|^{r_2}\right)^{\frac{p}{r_2}}.$$

Both factors are bounded, up to a constant, respectively by $|R_1(\frac{2a+1}{2q})|^p$ and $|R_2(\frac{2a+1}{2q})|^p$, which allows us to conclude.

In view of (49) and (50), we remark that, when t differs from $\frac{2a+1}{2q}$ by less than $\frac{\theta}{q^2}$, then $(2q+1)t$ differs from $\frac{2a+1}{2q}$ (modulo 1) by less than $\frac{3\theta}{q}$. So we still have, for $|t - \frac{2a+1}{2q}| < \frac{\theta}{q^2}$ with θ small enough,

$$|R(t)|^p \ge (1-\varepsilon)\left|R\left(\frac{2a+1}{2q}\right)\right|^p. \tag{54}$$

For inequality (50), we first use Hölder's inequality with conjugate exponents r_1/p and r_2/p as before, then the same kind of estimate for each factor.

From this point, the proof is the same.

It remains to indicate how to modify the proof to get peaking idempotents with arbitrarily large gaps. So we fix ν as a large odd integer, and we will prove that we can replace the polynomial R used above by some

$$S(x) := R_1(\nu x) R_2((2q+1)\nu x),$$

which has gaps larger than ν. Recall first that we can take arbitrarily large q satisfying $(\nu, q) = 1$, and get an idempotent by multiplication by $T(qx)$ for T having sufficiently large gaps. The value taken by the polynomial S at $\frac{2a+1}{2q}$ is the value of $R_1 R_2$ at $\frac{2b+1}{2q}$, with $\nu(2a+1) \equiv 2b+1 \bmod 2q$. So we choose R_1 and R_2 as before, but with b in place of a.

From this point the proof is identical, apart from an additional factor ν, which modifies the value of θ. We know that $S(\nu x)$ and $R(x)$ take globally the same values on both grids $\mathbb{G}_q$ and $\mathbb{G}_q^\star$, because in each case we multiply by an odd integer that is co-prime with $2q$. □

Now Theorem 2 is an easy consequence of Proposition 6 and Theorem 5: take $r < 2$ and $s > 2$. For s the constant may be taken equal to 1, so that $\gamma_1 \ge (\Gamma_r^\star)^{1/r}$. Then take the limit of Γ_r for $r \to 2 - 0$ using (45).

Proof (of Proposition 6). The proof is in the same spirit as the proof of the inequality $\gamma_p^\sharp > 0.483$. Let us first fix $\gamma < 1$ and prove that we can find a positive definite polynomial of degree less than $2q$ such that

$$2\left|P\left(\frac{1}{2q} + \frac{a}{q}\right)\right|^p \ge \gamma \sum_{k=0}^{q-1} \left|P\left(\frac{1}{2q} + \frac{k}{q}\right)\right|^p,$$

while

$$2\left|P\left(\frac{1}{2q}+\frac{a}{q}\right)\right|^p \geq \gamma \sum_{k=0}^{q-1}\left|P\left(\frac{k}{q}\right)\right|^p .$$

Indeed, it is proved in [4] (and is elementary) that $A(Lp, 1/4)$ has limit $1/2$ when L tends to ∞, which means that we can take for P a polynomial that coincides with D_n^L on the grid $\mathbb{G}_{2q}$. We fix L large enough, and choose n to be approximately $q/4$. The second inequality follows from (43).

At this point one can use Proposition 1, with q replaced by $2q$, to find the idempotent Q. □

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Le calcul symbolique dans certaines algèbres de type Sobolev

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Summary. On cherche à décrire le calcul symbolique de l'algèbre de Sobolev $\mathcal{H}^s(\mathbb{R}^n) = H^s \cap L_\infty(\mathbb{R}^n)$ $(s > 0)$, autrement dit: à caractériser simplement les fonctions définies sur $\mathbb{R}$ qui opèrent sur $\mathcal{H}^s$ par composition à gauche. Pour $s > 3/2$, il est raisonnable de conjecturer que ce sont précisément les fonctions appartenant localement à H^s et s'annulant à l'origine. Cette conjecture est maintenant un théorème dans le cas $n = 1$. Elle reste ouverte pour $n > 1$, avec des résultats partiels significatifs. Des résultats similaires ont été obtenus dans les espaces Besov $B^s_{p,q}(\mathbb{R})$ et de Lizorkin-Triebel $F^s_{p,q}(\mathbb{R})$ pour $s > 1 + (1/p)$. Salah-Eddine Allaoui, Massimo Lanza de Cristoforis, Madani Moussai et Winfried Sickel ont participé à ce programme de recherche.

1 Généralités sur le calcul symbolique

Le calcul symbolique (ou calcul fonctionnel) dans une algèbre de Banach commutative A consiste à donner un sens à $f(a)$ pour certaines fonctions f de la variable complexe définie sur le spectre de $a \in A$. Si par exemple

$$f(z) = \sum_{k=1}^{\infty} c_k z^k$$

est une fonction entière définie sur $\mathbb{C}$ et s'annulant en 0, il est naturel de poser

$$f(a) = \sum_{k=1}^{\infty} c_k a^k \,.$$

puisque cette série converge normalement dans A. Le problème se pose d'étendre le calcul symbolique à des fonctions f plus générales. Ainsi l'existence d'un calcul fonctionnel continu dans les C^*-algèbres constitue le fondement de la théorie spectrale classique.

Dans cette introduction, nécessairement brève, nous nous limiterons à des algèbres de Banach de fonctions sur un espace compact. Ce point de vue est

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en fait très proche du cas général, puisque le calcul symbolique s'introduit et s'étudie essentiellement par le truchement de la transformation de Gelfand. Nous suivrons de près le chapitre VIII du livre de Katznelson [13].

1.1 Le cas des algèbres de fonctions

Soit donc X un espace topologique compact et A une partie de $C(X)$, sur laquelle on fera les hypothèses suivantes:

(i) A une sous-algèbre unifère de $C(X)$, pour la multiplication usuelle des fonctions.
(ii) A est munie d'une norme qui en fait une algèbre de Banach.
(iii) A est *pleine*, au sens où tout élément de A est inversible dans A dès qu'il l'est dans $C(X)$.
(iv) A est auto-adjointe (Si $f \in A$, on a aussi $\overline{f} \in A$).

Les conditions (i)-(iv) entraînent que la transformation de Gelfand de A n'est autre que l'injection canonique $A \hookrightarrow C(X)$ (voir [3, sec. 7.1, prop.1]). Étudier le calcul symbolique dans ce contexte consiste donc à rechercher quelles sont les fonctions f, définies sur une partie Ω de $\mathbb{C}$, pour lesquelles $f \circ g \in A$ pour toute fonction $g \in A$ telle que $g(X) \subset \Omega$. On dispose *a minima* du calcul symbolique holomorphe:

Théorème 1.1 *Sous les hypothèses (i)-(iv), toute fonction holomorphe sur un ouvert de* $\mathbb{C}$ *opère sur* A.

La preuve classique du théorème consiste à écrire

$$f(g(x)) = \frac{1}{2\pi i} \int_\gamma f(z)(z - g(x))^{-1} dz \,,$$

où γ est un chemin fermé d'indice 1 par rapport à tout point de $g(X)$ et d'indice 0 par rapport à tout point extérieur au domaine de f.

1.2 Le cas des algèbres régulières

On dit qu'une classe de fonctions $A \subset C(X)$ est *régulière* si, pour tout compact K de X et tout $a \in X \setminus K$, il existe une fonction $f \in A$ telle que $f(x) = 1$, pour tout $x \in K$, et $f(a) = 0$.

Jusqu'à la fin de la section 1, A désignera une algèbre de fonctions sur l'espace compact X, satisfaisant les conditions (i)-(iv) ainsi que la régularité.

Définition 1 *Soit $E \subset C(X)$. On dit qu'une fonction $g \in C(X)$ appartient localement à E si, pour tout $x \in X$, il existe un voisinage V de x et un élément h de E tels que $g|_V = h|_V$.*

Proposition 1. *Pour tout recouvrement de X par des ouverts $U_1, \ldots, U_N$, il existe des fonctions $\varphi_j \in A$ telles que $\operatorname{supp} \varphi_j \subset U_j$, pour tout j, et $\sum_{j=1}^{N} \varphi_j = 1$.*

Preuve: Voir [13, sec. 5].

Corollaire 2 *Toute fonction $g \in C(X)$ appartenant localement à A est un élément de A.*

Le corollaire permet de retrouver le calcul symbolique holomorphe sans passer par la formule de Cauchy. Soit en effet $g \in A$ et f une fonction holomorphe sur un ouvert de $\mathbb{C}$ contenant $g(X)$. Étant donné $x_0 \in X$, on va vérifier que $f \circ g$ coincide avec un élément de A au voisinage de x_0. Sans perte de généralité, on peut supposer que $g(x_0) = 0$. Par hypothèse, f s'écrit

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

pour $|z| < R$, pour un certain $R > 0$. Alors $V = \{x : |g(x)| \leq R/2\}$ est un voisinage de x_0. Par la régularité de A, il existe $\varphi \in A$ tel que $\varphi(x) = 1$ au voisinage de x_0, $0 \leq \varphi(x) \leq 1$, et $\varphi(x) = 0$ en dehors de V – voir [13], lemme 8.1. L'hypothèse (iii) a pour conséquence que le rayon spectral de φg dans A est majoré par $R/2$. Ceci implique que la série

$$\sum_{k=0}^{\infty} a_k (\varphi g)^k$$

converge dans A. Sa somme est un élément de A qui coincide avec $f \circ g$ au voisinage de x_0.

Outre d'être élémentaire, la preuve ci-dessus a l'intérêt de se généraliser aisément aux fonctions analytiques réelles:

Théorème 1.2 *Toute fonction analytique réelle, définie sur un ouvert de $\mathbb{C}$ ou un ouvert de $\mathbb{R}$, opère sur A.*

Il existe des algèbres sur lesquelles seules les fonctions analytiques opèrent. C'est ce qu'exprime le célèbre théorème de Kahane et Katznelson (voir [13, 8.6]):

Théorème 1.3 *Soit $A(\mathbb{T})$ l'algèbre du cercle, à savoir l'ensemble des fonctions 2π-périodiques g telles que $\sum_{k\in\mathbb{Z}} |\widehat{g}(k)| < +\infty$, où $\widehat{g}$ désigne la suite des coefficients de Fourier de g. Si f est une fonction définie sur un intervalle I de $\mathbb{R}$ et opérant sur $A(\mathbb{T})$, alors f est analytique réelle en tout point de I.*

1.3 Vers un calcul symbolique maximal?

Le théorème 1.2 établit que le calcul analytique est en sens minimal, ce minimum étant atteint d'après le théorème 1.3. Y-a-t'il à l'inverse un calcul symbolique maximal?

On serait tenté de répondre que le calcul symbolique est maximal si *toutes les fonctions continues opèrent*. Mais, si c'est le cas, l'algèbre A est soit triviale (égale à $C(X)$) soit, en un sens, pathologique.

Pour établir ce fait, il sera commode d'introduire l'opérateur de composition $T_f(g) = f \circ g$, ainsi que l'ensemble $\Re(A)$ des fonctions à valeurs réelles appartenant à A. Si la fonction $f : \mathbb{R} \to \mathbb{R}$ opère sur A, l'opérateur T_f envoie l'ensemble $\Re(A)$ dans lui-même.

De façon générale, si E et F sont deux espaces métriques, une application $T : E \to F$ est dite *bornée* si, pour toute partie bornée B de E, $T(B)$ est borné dans F. L'opérateur T_f n'étant pas linéaire, il n'y aucune raison *a priori* pour que T_f soit continu, ou borné.

Nous noterons $\mathcal{G}(A)$ l'ensemble des homéomorphismes γ de X sur lui-même pour lesquels l'application $g \mapsto g \circ \gamma$ est un automorphisme de A.

Théorème 1.4 *Supposons que l'ensemble* $\{\gamma(x) : \gamma \in \mathcal{G}(A)\}$ *soit infini, quel que soit* $x \in X$. *Alors, pour toute fonction* $f : \mathbb{R} \to \mathbb{R}$ *opérant sur* A, *l'opérateur* $T_f : \Re(A) \to \Re(A)$ *est borné au voisinage de* 0.

Preuve: Voir [13], pp. 275-276.

Théorème 1.5 *Si la fonction* $f(x) = \sqrt{|x|}$ *opère sur* A *et si l'opérateur* $T_f : \Re(A) \to \Re(A)$ *est borné au voisinage de* 0, *on a* $A = C(X)$.

Preuve: Voir [13], exercice 8.4, p. 281.

1.4 Le cas des algèbres sur le cercle

Le cercle (ou tore de dimension 1) est l'espace quotient $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Nous identifions $C(\mathbb{T})$ à l'algèbre des fonctions continues, 2π-périodiques, de $\mathbb{R}$ dans $\mathbb{C}$.

Dans cette sous-section, nous supposerons que A est une sous-algèbre de $C(\mathbb{T})$ vérifiant les conditions (i)-(iv). Nous supposerons de plus que $C^\infty(\mathbb{T}) \subset A$, condition qui implique *a fortiori* la régularité de A. Dans ce contexte, on dispose d'une condition nécessaire d'opérance qui s'avère réaliste, au sens où elle est également suffisante dans bien des cas.

Définition 3 *Soit* $E \subset C(\mathbb{T})$. *On dit qu'une fonction* $g : \mathbb{R} \to \mathbb{C}$ *appartient localement à* E, *ce qu'on notera* $g \in E_{loc}$, *si, pour tout* $x \in \mathbb{R}$, *il existe un voisinage* V *de* x *dans* $\mathbb{R}$ *et un élément* h *de* E *tels que* $g|_V = h|_V$.

Notons qu'une fonction appartenant à E_{loc} n'a aucune raison d'être périodique, contrairement aux fonctions considérées dans la définition 1.

Théorème 1.6 *Toute fonction $f : \mathbb{R} \to \mathbb{C}$ qui opère sur A appartient à A_{loc}.*

Preuve: Soit I un intervalle ouvert de longueur inférieure à 2π. On construit sans difficulté une fonction $\varphi \in C^\infty(\mathbb{T})$ telle que $\varphi(x) = x$ sur I. On a alors $f \circ \varphi \in A$ et f coincide avec $f \circ \varphi$ sur I.

Définition 4 *On dit que l'algèbre A a un calcul symbolique maximal si toute fonction appartenant à A_{loc} opère sur A.*

Voici une première série d'exemples et de contre-exemples. Les algèbres $C^k(\mathbb{T})$, $k \in \mathbb{N}$, ont un calcul symbolique maximal. Par contre, si $0 < s < 1$, l'algèbre $C^s(\mathbb{T})$, des fonctions 2π-périodiques höldériennes d'exposant s, n'admet pas de calcul symbolique maximal, car les fonctions qui opèrent sont nécessairement localement lipschitziennes, voir par exemple [7].

2 Calcul symbolique dans les espaces de Sobolev

2.1 Définition et théorème principal

Pour $s \in \mathbb{R}$, l'espace $H^s(\mathbb{T})$ est l'ensemble des distributions 2π-périodiques f telles que

$$\|f\|_{H^s} = \left(\sum_{k \in \mathbb{Z}} (1 + k^2)^s |\widehat{f}(k)|^2 \right)^{1/2} < +\infty \,.$$

Théorème 2.1 *$H^s(\mathbb{T})$ est inclus dans $L_\infty(\mathbb{T})$ si et seulement si $s > 1/2$ et, sous cette condition, c'est une sous-algèbre vérifiant les hypothèses de la sous-section 1.4.*

Preuve: Supposons $s > 1/2$. Si $f \in H^s(\mathbb{T})$, on a

$$\widehat{f}(k) = (1 + k^2)^{-s/2} \left((1 + k^2)^{s/2} \widehat{f}(k) \right) .$$

Ceci montre que la suite $\widehat{f}$ est le produit de deux suites appartenant à $l^2(\mathbb{Z})$, autrement dit un élément de $l^1(\mathbb{Z})$. On a ainsi établi le plongement $H^s(\mathbb{T}) \subset A(\mathbb{T})$. Soit maintenant $f, g \in H^s(\mathbb{T})$. En partant de l'inégalité

$$(1 + k^2)^s \leq 4^s \left((1 + (k - k')^2)^s + (1 + k'^2)^s \right) , \quad \forall k, k' \in \mathbb{Z} \,,$$

on obtient

$$\begin{aligned} (1 + k^2)^{s/2} |\widehat{fg}(k)| \leq c \sum_{k' \in \mathbb{Z}} & \left((1 + (k - k')^2)^{s/2} |\widehat{f}(k - k')| \right) |\widehat{g}(k')| \\ & + c \sum_{k' \in \mathbb{Z}} \left((1 + k'^2)^{s/2} |\widehat{g}(k')| \right) |\widehat{f}(k - k')| \,. \end{aligned}$$

Dans chacun des termes du membre de droite, on reconnait la convolution d'une suite de $l^2(\mathbb{Z})$ avec une suite de $l^1(\mathbb{Z})$. On conclut que la suite

$$k \mapsto (1+k^2)^{s/2}\widehat{fg}(k)$$

appartient à $l^2(\mathbb{Z})$, ce qui signifie que $fg \in H^s$, la même preuve donnant l'inégalité $\|fg\|_{H^s} \leq c\|f\|_{H^s}\|g\|_{H^s}$. Pour établir que $H^s(\mathbb{T})$ est une sous-algèbre pleine, il suffit, d'après [3, sec. 7.1, prop.1], d'établir que les caractères de $H^s(\mathbb{T})$ sont les évaluations $f \mapsto f(t)$, pour $t \in \mathbb{T}$. Soit donc χ un caractère de $H^s(\mathbb{T})$. La restriction de χ à $C^N(\mathbb{T})$, avec N entier $> s$, est un caractère de l'algèbre $C^N(\mathbb{T})$. Il existe donc $t_0 \in \mathbb{T}$ tel que $\chi(f) = f(t_0)$ pour tout $f \in C^N(\mathbb{T})$. Par densité de $C^N(\mathbb{T})$ dans $H^s(\mathbb{T})$, la même égalité est vérifiée pour tout $f \in H^s(\mathbb{T})$, ce qu'il fallait démontrer. On verra plus loin (preuve du théorème 3.1) qu'il existe une fonction $f \in H^{1/2}(\mathbb{R})$ portée par $[-1,1]$ et n'appartenant pas à $L_\infty(\mathbb{R})$. La périodisée de f appartient à $H^{1/2}(\mathbb{T})$ mais n'est pas essentiellement bornée.

Théorème 2.2 *Soit $s > 1/2$. L'algèbre $H^s(\mathbb{T})$ a un calcul symbolique maximal si et seulement si $s > 3/2$.*

La partie difficile de la preuve du théorème est d'établir la maximalité dans le cas $s > 3/2$. Ce sera l'objet des sous-sections 2.3 et 2.4. La non-maximalité pour $s \leq 3/2$ s'obtient aisément comme suit. On sait que toute fonction qui opère est localement lipschitzienne (voir [7]). On sait aussi que $H^{3/2}(\mathbb{T})_{loc}$ contient des fonctions qui ne sont pas localement lipschitziennes (voir la remarque 9). Pour $s \leq 3/2$, il y a donc des fonctions de $H^s(\mathbb{T})_{loc}$ qui n'opèrent pas sur $H^s(\mathbb{T})$.

Remarque 5 *Que dire du calcul symbolique pour $1/2 < s \leq 3/2$? Il est bien connu (et non-maximal!) pour $1/2 < s \leq 1$: on sait qu'alors les fonctions localement lipschitziennes opèrent, et seulement elles. Il reste mystérieux pour $1 < s \leq 3/2$, car on connait alors des fonctions lipschitziennes qui n'opèrent pas.*

Remarque 6 *Pour $0 \leq s \leq 1/2$, $H^s(\mathbb{T})$ n'est plus une algèbre. On peut cependant déterminer les fonctions continues $f : \mathbb{C} \to \mathbb{C}$ telles que T_f envoie $H^s(\mathbb{T})$ dans lui-même. Pour $0 < s \leq 1/2$, ce sont les fonctions globalement lipschitziennes (voir [7]) et, pour $s = 0$, les fonctions telles que $f(z) = O(|z|)$ pour $|z| \to +\infty$ (voir [2]).*

2.2 Reformulation du théorème principal

Le théorème 2.2 est valable aussi bien pour les espaces de Sobolev sur la droite réelle, et c'est d'ailleurs comme cela qu'on l'établit. Rappelons que $H^s(\mathbb{R}^n)$ est l'ensemble des distributions tempérées f sur $\mathbb{R}^n$ telles que

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} < +\infty\,,$$

où $\widehat{f}$ désigne la transformée de Fourier de f. L'espace $H^s(\mathbb{R}^n)_{loc}$ s'obtient suivant la définition 3. En dimension 1, les relations entre les divers H^s sont décrites par l'énoncé suivant, dont la preuve est élémentaire:

Proposition 2. *Pour tout $s \geq 0$, $H^s(\mathbb{T})$ n'est autre que l'ensemble des fonctions 2π-périodiques appartenant à $H^s(\mathbb{R})_{loc}$, et les espaces locaux $H^s(\mathbb{R})_{loc}$ et $H^s(\mathbb{T})_{loc}$ coïncident.*

Voici maintenant les énoncés concernant le calcul symbolique:

Théorème 2.3 *Si $s > 3/2$, toute fonction f telle que $f(0) = 0$ et que $f' \in H^{s-1}(\mathbb{R})$ opère sur $H^s(\mathbb{R})$.*

Corollaire 7 *Si $s > 3/2$, toute fonction f appartenant à $H^s(\mathbb{R})_{loc}$ et s'annulant en 0 opère sur $H^s(\mathbb{R})$.*

Corollaire 8 *Si $s > 3/2$, toute fonction $f \in H^s(\mathbb{R})_{loc}$ opère sur $H^s(\mathbb{R})_{loc}$.*

Nous reviendrons à la preuve du théorème 2.3. Mais voyons d'abord comment les autres énoncés en découlent.

2.3 Preuve des corollaires 7 et 8

Soit $f \in H^s(\mathbb{R})_{loc}$, s'annulant en 0, et $g \in H^s(\mathbb{R})$. Si $\varphi \in \mathcal{D}(\mathbb{R})$ vérifie $\varphi(x) = 1$ sur $g(\mathbb{R})$, on a $f \circ g = f\varphi \circ g$ et la fonction $f\varphi$ appartient à $H^s(\mathbb{R})$, et s'annule en 0. Du théorème 2.3, on déduit que $f \circ g \in H^s(\mathbb{R})$.

Soit maintenant f, g des fonctions à valeurs réelles appartenant à $H^s(\mathbb{R})_{loc}$. En retranchant à f la constante $f(0)$, on se ramène au cas $f(0) = 0$. On considère ensuite une fonction $\psi_1 \in \mathcal{D}(\mathbb{R})$, puis $\psi_2 \in \mathcal{D}(\mathbb{R})$ telle que $\psi_1\psi_2 = \psi_1$. Il vient $\psi_1 \cdot (f \circ g) = \psi_1 \cdot (f \circ (\psi_2 g))$. Du corollaire 7, il découle que $\psi_1 \cdot (f \circ g) \in H^s(\mathbb{R})$. Puisque ψ_1 est arbitraire, on conclut que $f \circ g \in H^s(\mathbb{R})_{loc}$.

Enfin en combinant le corollaire 8 et la proposition 2, on obtient aussitôt la partie difficile du théorème 2.2.

2.4 Preuve du théorème 2.3

Préliminaires

On se limite ici à $3/2 < s < 2$ (le cas $s = 2$ est classique, le cas $2 < s \leq 5/2$ plus technique, l'extension à $s > 5/2$ est une récurrence facile; les preuves détaillées figurent dans les articles [5, 11, 12]). Nous établirons, plus précisément, l'existence d'une constante $c = c(s) > 0$ telle que

$$\|f \circ g\|_{H^s(\mathbb{R})} \leq c\, \|f'\|_{H^{s-1}(\mathbb{R})} \left(1 + \|g\|_{H^s(\mathbb{R})}\right)^{s-(1/2)}, \tag{1}$$

pour toute fonction f telle que $f(0) = 0$ et $f' \in H^{s-1}(\mathbb{R})$, et toute fonction $g \in H^s(\mathbb{R})$, à valeurs réelles.

Tout d'abord la condition $f(0) = 0$ et le plongement de $H^s(\mathbb{R})$ dans $Lip(\mathbb{R})$, pour $s > 3/2$, nous donnent

$$\|f \circ g\|_2 \leq c\|f'\|_{H^{s-1}(\mathbb{R})}\|g\|_2\,,$$

de sorte qu'il nous reste à montrer que $\|(f \circ g)'\|_{H^{s-1}(\mathbb{R})}$ est estimé par le second membre de (1).

Nous allons mettre en jeu *trois idées*. La première consiste à se limiter à des fonctions g très régulières, en l'occurrence analytiques réelles. C'est un moyen commode de donner un sens raisonnable à l'inévitable changement de variable $y = g(x)$. Une fois l'estimation (1) acquise pour de telles fonctions g, le cas général résultera d'un argument standard d'approximation. Les deux autres idées — l'utilisation de normes équivalentes convenables dans H^{s-1} et le plongement dans l'espace BV_2 — font l'objet des paragraphes suivants.

Normes équivalentes dans l'espace de Sobolev

Si $0 < \sigma < 1$, on montre classiquement que l'expression suivante est équivalente à la norme usuelle de $H^\sigma(\mathbb{R})$:

$$\|f\|_2 + \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{|h|^{2\sigma+1}}|f(x+h) - f(x)|^2 dh\,dx\right)^{1/2} \tag{2}$$

Si $1/2 < \sigma < 1$, on dispose (moins classiquement: voir [16, thm. 3.5.3, p. 194]) d'une version équivalente "maximale" de la norme ci-dessus:

$$\|f\|_2 + \left(\int_{\mathbb{R}}\int_0^\infty \frac{1}{t^{2\sigma+1}} \sup_{|v|\leq t} |f(x+v) - f(x)|^2 dt\,dx\right)^{1/2} \tag{3}$$

La condition $\sigma > 1/2$ y est optimale, car la finitude de (3) implique que f est localement bornée, ce qui est faux en général pour les fonctions de $H^{1/2}$.

La 2-variation d'une fonction

Pour toute fonction g définie sur $\mathbb{R}$, on note $\nu_2(g)$ la borne supérieure des nombres

$$\left(\sum_{k=1}^N |g(t_k) - g(t_{k-1})|^2\right)^{1/2},$$

prise sur toutes les suites finies $t_0 < t_1 < \cdots < t_N$. Une fonction g est à 2-*variation bornée* si $\nu_2(g) < +\infty$. On note $BV_2(\mathbb{R})$ l'espace des telles fonctions (introduit par Wiener [17]). On dispose du *plongement de Peetre* $H^\sigma(\mathbb{R}) \hookrightarrow BV_2(\mathbb{R})$ si $\sigma > 1/2$. Le théorème de Peetre est un énoncé plus précis, qui nécessite l'introduction des espaces de Besov, voir [14, thm. 7, p. 112] ou [10]. L'idée, c'est que l'espace BV_2 est très voisin de $H^{1/2}$.

Les détails de la preuve

Compte tenu de la norme équivalente (2), il nous suffit d'établir que

$$A = \int_{\mathbb{R}} \int_0^\infty \frac{(f'(g(x+h))g'(x+h) - f'(g(x))g'(x))^2}{h^{2s-1}}\, dh\, dx$$

est estimé par le carré du second membre de (1). L'intégrale portant sur les $h < 0$ ferait l'objet d'un traitement similaire. On note d'abord que $\sqrt{A} \leq \sqrt{A_1} + \sqrt{A_2}$, où

$$A_1 = \int_{\mathbb{R}} \int_0^\infty \frac{(f'(g(x+h)) - f'(g(x)))^2\, g'(x)^2}{h^{2s-1}}\, dh\, dx\,,$$

$$A_2 = \int_{\mathbb{R}} \int_0^\infty \frac{f'(g(x+h))^2\, (g'(x+h) - g'(x))^2}{h^{2s-1}}\, dh\, dx\,.$$

On a aussitôt $\sqrt{A_2} \leq \|f'\|_\infty \|g'\|_{H^{s-1}}$. Le plongement de H^{s-1} dans L_∞ permet d'en finir avec A_2. Pour estimer A_1, on commence par écrire le complémentaire de l'ensemble des zéros de g' comme une réunion d'intervalles ouverts disjoints $\{I_l\}$ et, pour $x \in I_l$, on considère la distance $\eta_l(x)$ de x à l'extrémité droite de I_l. Alors $\sqrt{A_1} \leq \sqrt{V_1} + \sqrt{V_2}$, où

$$V_1 = \sum_l \int_{I_l} \int_0^{\eta_l(x)} \frac{(f'(g(x+h)) - f'(g(x)))^2\, g'(x)^2}{h^{2s-1}}\, dh\, dx$$

et V_2 est défini de même, en intégrant pour $h > \eta_l(x)$.

Estimation de V_1. On effectue le changement de variable $y = g(x)$ sur chaque intervalle I_l. En posant $a_l = \sup_{I_l} |g'|$ et en utilisant la norme (3), on obtient

$$\begin{aligned} V_1 &\leq \sum_l a_l \int_{\mathbb{R}} \int_0^\infty \frac{1}{h^{2s-1}} \sup_{|v| \leq a_l h} (f'(y+v) - f'(y))^2\, dh\, dy \\ &= \sum_l a_l^{2s-1} \int_{\mathbb{R}} \int_0^\infty \frac{1}{t^{2s-1}} \sup_{|v| \leq t} (f'(y+v) - f'(y))^2\, dt\, dy \\ &\leq c\, \|f'\|^2_{H^{s-1}(\mathbb{R})} \sum_l a_l^{2s-1}\,. \end{aligned}$$

La condition $2s - 1 > 2$ et le fait que g' s'annule aux extrémités de I_l nous donnent

$$\sum_l a_l^{2s-1} \leq \left(\sum_l a_l^2 \right)^{s-(1/2)} \leq \nu_2(g')^{2s-1}\,.$$

Le plongement de Peetre implique

$$V_1 \le c\,\|f'\|^2_{H^{s-1}(\mathbb{R})}\|g'\|^{2s-1}_{H^{s-1}(\mathbb{R})}\,.$$

Estimation de V_2. On utilise l'inégalité triviale $|f'(g(x+h)) - f'(g(x))| \le 2\|f'\|_\infty$ et le fait que g' s'annule aux extrémités de I_l, pour écrire

$$\begin{aligned}
V_2 &\le c_1\|f'\|^2_\infty \sum_l \int_{I_l} \left(\int_{\eta_l(x)}^\infty \frac{dh}{h^{2s-1}} \right) g'(x)^2 dx \\
&= c_2\|f'\|^2_\infty \sum_l \int_{I_l} \left(\frac{g'(x)}{\eta_l(x)^{s-1}} \right)^2 dx \\
&\le c_2\|f'\|^2_\infty \sum_l \int_{I_l} \left(\sup_{h\in\mathbb{R}} \frac{|g'(x+h) - g'(x)|}{|h|^{s-1}} \right)^2 dx \\
&\le c_2\|f'\|^2_\infty \int_{\mathbb{R}} \left(\sup_{h\in\mathbb{R}} \frac{|g'(x+h) - g'(x)|}{|h|^{s-1}} \right)^2 dx \le c_3\|f'\|^2_\infty \|g'\|^2_{H^{s-1}(\mathbb{R})}\,.
\end{aligned}$$

3 Diverses Extensions

Il est naturel de chercher à étendre le théorème 2.3

- aux espaces de Besov $B^s_{p,q}(\mathbb{R})$ et de Lizorkin-Triebel $F^s_{p,q}(\mathbb{R})$, dont $H^s(\mathbb{R})$ est un cas particulier puisque $H^s = B^s_{2,2} = F^s_{2,2}$,
- aux espaces de Sobolev sur $\mathbb{R}^n$,
- aux espaces de fonctions à valeurs dans $\mathbb{R}^m$, la fonction opérante étant elle-même définie sur $\mathbb{R}^m$.

Dans ces diverses directions, on dispose de résultats partiels et de conjectures.

3.1 Le cas des espaces de Besov et de Lizorkin-Triebel

On peut penser que le théorème 2.3 s'étend *mutatis mutandis* aux espaces $B^s_{p,q}(\mathbb{R})$ et $F^s_{p,q}(\mathbb{R})$, pour $p, q \in [1, +\infty]$ et $s > 1 + (1/p)$. On sait le prouver pour $p > 4/3$ *ou* pour $s - [s] > 1/p$ (avec de plus $q \ge p$ dans le cas Besov) [12].

3.2 Le cas des espaces définis sur $\mathbb{R}^n$

Tout d'abord, voici la version n-dimensionnelle du théorème 2.1:

Théorème 3.1 *$H^s(\mathbb{R}^n)$ est inclus dans $L_\infty(\mathbb{R}^n)$ si et seulement si $s > n/2$ et, sous cette condition, c'est une sous-algèbre de $C_0(\mathbb{R}^n)$.*

Preuve: Les propriétés de $H^s(\mathbb{R}^n)$ pour $s > n/2$ se démontrent comme dans le théorème 2.1. Pour prouver que $H^{n/2}(\mathbb{R}^n)$ ne se plonge pas dans L_∞, on part d'une fonction $\varphi \in \mathcal{D}(\mathbb{R}^n)$, positive, valant 1 sur la boule unité. On pose

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j} \varphi(2^j x)\,.$$

On voit que f est à support compact, de classe C^∞ en dehors de 0, et qu'il existe $c > 0$ tel que $f(x) \geq c \ln(-\ln|x|)$ pour $0 < |x| < 1/e$. On a donc $f \notin L_\infty(\mathbb{R}^n)$. De la relation

$$\widehat{f}(x) = \sum_{j=1}^{\infty} \frac{1}{j} 2^{-jn} \widehat{\varphi}(2^{-j} x)$$

et de l'estimation $|\widehat{\varphi}(x)| \leq c \min(1, |x|^{-n-1})$, on déduit que

$$\widehat{f}(x) = O\left(|x|^{-n} \ln^{-1}|x|\right) \quad (|x| \to +\infty)\,,$$

ce qui donne $f \in H^{n/2}(\mathbb{R}^n)$.

Remarque 9 *Soit $\psi \in \mathcal{D}(\mathbb{R})$, telle que $\psi(x) = x$ sur l'intervalle $[-1, 1]$ et $\psi(x) \geq 0$ pour $x \geq 0$, et soit*

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j} 2^{-j} \psi(2^j x)\,.$$

En raisonnant comme ci-dessus, on voit qu'il existe $c > 0$ tel que $f(x) \geq cx \ln(-\ln x)$ pour $0 < x < 1/e$. On obtient ainsi une fonction à support compact appartenant à $H^{3/2}(\mathbb{R})$, mais non localement lipschitzienne.

Pour $3/2 \leq s < n/2$, le calcul symbolique est trivial: les seules fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que T_f envoie $H^s(\mathbb{R}^n)$ dans lui-même sont les fonctions linéaires [4, 6]. Par contre l'espace modifié $\mathcal{H}^s(\mathbb{R}^n) = H^s \cap L_\infty(\mathbb{R}^n)$ est une algèbre de Banach quel que soit $s \geq 0$, voir [15, thm. 4.6.4(1)].

Conjecture 1. Soit $s > 3/2$. Une fonction $f : \mathbb{R} \to \mathbb{R}$ opère sur $\mathcal{H}^s(\mathbb{R}^n)$ si et seulement si elle appartient localement à $H^s(\mathbb{R})$ et s'annule en zéro.

Rappelons que le calcul symbolique sur $\mathcal{H}^s(\mathbb{R}^n)$ ou $H^s(\mathbb{R}^n)$ est entièrement déterminé pour $0 \leq s \leq 1$ (voir [7]), mais qu'il reste mystérieux pour $1 < s \leq 3/2$. Le cas critique $s = n/2$ devrait se résoudre comme suit:

Conjecture 2. Soit $s = n/2 > 3/2$. Une fonction $f : \mathbb{R} \to \mathbb{R}$ opère sur $H^s(\mathbb{R}^n)$ si et seulement si f' appartient à $H^{s-1}(\mathbb{R})$ localement-uniformément, et $f(0) = 0$.

Rappelons qu'une fonction g appartient à $H^s(\mathbb{R})$ *localement-uniformément* si on a

$$\sup_{a\in\mathbb{R}} \|g\,\varphi(\cdot - a)\|_{H^s} < +\infty$$

pour une (et donc toute) fonction non nulle $\varphi \in \mathcal{D}(\mathbb{R})$.

Les conditions apparaissant dans les conjectures 1 et 2 sont connues pour être nécessaires. En dimension $n > 1$, on a pu établir les deux conjectures pour les valeurs entières de s. La preuve se trouve, pour l'essentiel, dans notre travail de 1991 [5].

L'énoncé suivant peut se voir comme une bonne approximation de la conjecture 1. Sa preuve [8] est un sous-produit de celle du théorème 2.3.

Théorème 3.2 *Si $m + (1/2) < s < s' < m + 1$, avec $m \in \mathbb{N}^*$, toute fonction $f : \mathbb{R} \to \mathbb{R}$ vérifiant $f(0) = 0$ et $f' \in H^{s'-1}(\mathbb{R})$, opère sur l'espace $\mathcal{H}^s(\mathbb{R}^n)$.*

3.3 Le cas des espaces de Sobolev à valeurs vectorielles

Dans les théorèmes 2.2 et 2.3, ainsi d'ailleurs que dans la définition 4, on s'est limité à des fonctions opérantes définies sur $\mathbb{R}$. Cette restriction n'est pas une simple commodité d'exposition. En fait, si on excepte le cas $0 \le s \le 1$, on ne sait pas grand-chose des fonctions $f : \mathbb{C} \to \mathbb{C}$ qui opèrent sur H^s ou sur $\mathcal{H}^s$. Cependant il est vraisemblable que la conjecture 1 se généralise aux espaces de fonctions de $\mathbb{R}^n$ dans $\mathbb{R}^m$, *à condition que $m \le n$.*

Conjecture 3. Soient $s > 3/2$ et $m \le n$. Une fonction $f : \mathbb{R}^m \to \mathbb{R}$ opère de $\mathcal{H}^s(\mathbb{R}^n, \mathbb{R}^m)$ dans $\mathcal{H}^s(\mathbb{R}^n)$ si et seulement si f vérifie les trois conditions suivantes:

1- $f(0) = 0$;
2- f est localement lipschitzienne;
3- f appartient à $H^s(\mathbb{R}^m)_{loc}$.

Salah Eddine Allaoui a établi la nécessité des conditions ci-dessus [1].

4 Régularité du calcul symbolique

À partir du moment où une fonction $f : \mathbb{R} \to \mathbb{R}$ opère sur un espace normé de fonctions E, il est naturel d'étudier la régularité de T_f en tant qu'application de E dans lui-même. Il s'agira de rechercher des conditions sur f — si possible nécessaires et suffisantes — qui assurent que T_f est bornée, continue, localement ou globalement lipschitzienne, de classe C^r, etc.

4.1 Calcul symbolique borné

En prouvant le théorème 2.3, nous avons non seulement trouvé toutes les fonctions qui opèrent, mais encore montré que l'opérateur de composition T_f est toujours borné sur H^s. Dans le cas qui nous intéresse ici, on peut établir que l'opérateur T_f est automatiquement borné, au moins en un sens affaibli:

Proposition 3. *Si une fonction $f : \mathbb{R} \to \mathbb{R}$ opère sur $E = \mathcal{H}^s(\mathbb{R}^n)$ (ou sur $E = H^s(\mathbb{R}^n)$), alors, pour tout compact K de $\mathbb{R}^n$, il existe une boule B de $\Re(E)$ telle que la restriction de T_f à l'ensemble des fonctions de B portées par K soit une application bornée.*

La preuve de la proposition 3 est essentiellement la même que celle du théorème 1.4. Elle repose notamment sur l'invariance de nos espaces par translation.

4.2 Calcul symbolique de classe C^r

Soit Φ_s l'ensemble des fonctions $f : \mathbb{R} \to \mathbb{R}$ pour lesquelles T_f est une application bornée de $\mathcal{H}^s(\mathbb{R}^n)$ dans lui-même. On munit Φ_s de la famille de seminormes:

$$\rho_k(f) = \sup\{\|f \circ g\|_{\mathcal{H}^s} \ : \ \|g\|_{\mathcal{H}^s} \leq k\}\,, \qquad \forall k \in \mathbb{N}^*\,.$$

Pour tout $r \in \mathbb{N}$, on note $W^r(\Phi_s)$ l'espace de Sobolev des fonctions $f : \mathbb{R} \to \mathbb{R}$ telles que $f^{(j)} \in \Phi_s$ pour tout $j = 0, \ldots, r$.

Théorème 4.1 *Soient $r \in \mathbb{N}$ et $s > 0$. Si $f : \mathbb{R} \to \mathbb{R}$ est une fonction continue appartenant à la fermeture de $C^\infty(\mathbb{R}) \cap W^r(\Phi_s)$ dans $W^r(\Phi_s)$, alors T_f est une application de classe C^r de $\mathcal{H}^s(\mathbb{R}^n)$ dans lui-même.*

Théorème 4.2 *Soient $r \in \mathbb{N}$ et $s > 0$. Si $f : \mathbb{R} \to \mathbb{R}$ est une fonction continue telle que T_f soit une application de classe C^r de $\mathcal{H}^s(\mathbb{R}^n)$ dans lui-même, alors $f \in H^{s+r}(\mathbb{R})_{loc}$.*

Dans les cas où la conjecture 1 est un théorème, donc notamment pour $n = 1$ et $s > 3/2$, les deux théorèmes ci-dessus débouchent sur une condition nécessaire et suffisante: *T_f est une application de classe C^r de $\mathcal{H}^s(\mathbb{R}^n)$ dans lui-même si et seulement si $f(0) = 0$ et $f \in H^{s+r}(\mathbb{R})_{loc}$.*

Le lecteur trouvera dans l'article [9] la preuve de ce résultat, ainsi que celles des deux théorèmes précédents. Le même article traite le cas plus général des espaces de Besov et de Lizorkin-Triebel, ainsi que la continuité lipschitzienne de T_f. L'extension aux espaces à valeurs vectorielles est abordé par Allaoui [1].

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L^p-Norms and Fractal Dimensions of Continuous Function Graphs

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Summary. We study the fractal dimensions of continuous function graphs and more general fractal parameters. They are all obtained from the L^p-norms of some well-built operators. We give general results about these norms in the continuous and the discrete cases. For a function that is uniformly Hölderian, they allow us to estimate in a very easy way a large family of dimensional indices, like the box dimension and regularization dimension.

1 Introduction

Let $D \geq 1$. Let $f : \mathbb{R}^D \to \mathbb{R}$ be a continuous function and let

$$\Gamma_f = \{ (t, f(t)) \, : \, t \in [0,1]^D \} \subset \mathbb{R}^{D+1}$$

be the graph of f restricted to $[0,1]^D$. We are interested in indices that characterize the irregularity of Γ_f. Some of them are called "fractal dimensions" and yield different definitions for $\dim(\Gamma_f)$. They are of two kinds: those related to the measure theory (Hausdorff dimension, packing dimension) and those obtained as the growth order of a quantity $Q(\varepsilon)$ that vanishes when ε decreases towards 0, i.e., they can be written as

$$\dim(\Gamma_f) = \lim_{\varepsilon \to 0} \frac{\log Q(\varepsilon)}{\log \varepsilon} \tag{1}$$

for a certain function Q. Since it is possible that this limit does not exist, we shall consider (1) with lower or upper limits. Such quantities $Q(\varepsilon)$ are obtained by counting methods or integration methods. Let us give here some examples.

1. The (*upper*) *box dimension* $\dim_B$ is obtained by a counting method (see [15]). Let $N(\varepsilon)$ be the minimal number of balls of diameter ε, centered in Γ_f, which cover Γ_f. Then

$$\dim_B(\Gamma_f) = \overline{\lim_{\varepsilon \to 0}} \frac{\log(1/N(\varepsilon))}{\log \varepsilon}. \tag{2}$$

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2. The *regularization dimension* $\dim_R$ is obtained by integration methods (see [13]). Here we have to suppose that $D = 1$. The box dimension of Γ_f is obtained by considering coverings of Γ_f, made with unions of balls of increasingly smaller diameters, and evaluating the speed at which these sets approach Γ_f. Here we want to approach Γ_f with continuous function graphs instead of ball unions. To do this we consider "regular versions" of f, obtained by convolution with a very smooth kernel. Specifically, we choose a function K in the Schwartz class, even, and vanishing outside $[-1, 1]$. For $\varepsilon > 0$ we write $K_\varepsilon(x) = \varepsilon^{-1} K(\varepsilon^{-1}x)$ and we consider the regular version $f_\varepsilon = f * K_\varepsilon$ of f built with K_ε:

$$f_\varepsilon(x) = \int_{\mathbb{R}} f(t)K_\varepsilon(x-t)\,dt = \int_{\mathbb{R}} f(x-t)K_\varepsilon(t)\,dt.$$

The graph of f_ε is of finite length given by $\mathcal{L}_\varepsilon = \int_0^1 \sqrt{1 + f'_\varepsilon(x)^2}\,dx$. We evaluate the speed at which this length converges to the length of Γ_f. From this point of view, one defines the regularization dimension of Γ_f by

$$\dim_R(\Gamma_f) = 1 - \varliminf_{\varepsilon\to 0} \frac{\log \mathcal{L}_\varepsilon}{\log \varepsilon},$$

proving that this limit does not depend on the kernel K.

3. Wavelets are also common tools used to study the irregularity of a function. Let us suppose again that $D = 1$. For $\varepsilon > 0$ and $x \in \mathbb{R}$ we write $C(\varepsilon, x) = \varepsilon^{-1} \int_0^1 f(t)\Psi(\varepsilon^{-1}(t-x))\,dt$, where Ψ is an even wavelet, vanishing outside $[-1, 1]$. Then we define for $q > 0$:

$$Z_q(\varepsilon) = \left(\int_{\mathbb{R}} |C(\varepsilon, x)|^q\,dx\right)^{1/q} \quad \text{and} \quad \omega(q) = \varliminf_{\varepsilon\to 0} \frac{\log Z_q(\varepsilon)}{\log \varepsilon}. \tag{3}$$

These indices do not depend on the wavelet Ψ. They are used to investigate the fractal and multifractal properties of f (see [11]).

4. For $q > 0$, the *q-structure function* of f is the function S_q defined by

$$\forall\, \varepsilon > 0 \qquad S_q(\varepsilon) = \left(\int_0^1 |f(x+\varepsilon) - f(x)|^q\,dx\right)^{1/q}. \tag{4}$$

These functions are also a classical tool in signal analysis. They lead to another approach of the multifractal spectrum of f (see [11]), and in a generalized version they give a family of dimensions $\Delta^{(\alpha,\beta)}$ (see [14]).

We often study (1) as

$$\dim(\Gamma_f) = D + 1 - \varliminf_{\varepsilon\to 0} \frac{\log d(\varepsilon)}{\log \varepsilon},$$

where $d(\varepsilon) = \varepsilon^{D+1}/Q(\varepsilon)$. We call d a "determining function." Some of these functions (for example, $\varepsilon^{D+1}N(\varepsilon)$) allow us to analyze any compact set of the plan, whereas others are specific to graphs of functions. Moreover, one will take advantage of a good choice for d. For example, when f is very irregular, the length of Γ_f is not finite. In this case one can prove that

$$\dim_R(\Gamma_f) = 2 - \underline{\lim}_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \int_0^1 |\varepsilon f'_\varepsilon(x)|\, dx, \tag{5}$$

i.e., $d(\varepsilon) = \int_0^1 |\varepsilon f'_\varepsilon(x)|\, dx$ is a determining function for $\dim_R$.

One can observe a common point among all the previous indices. In each case the determining function may be a derivative from the norm of an operator in a certain L^p-space which depends on both f and ε. We are precisely interested in these dimensional indices. The integrals which define the norms are generally difficult to calculate explicitly. The main reason is that we integrate with respect to continuous measures. Our main purpose is to replace these integrals by discrete ones in order to work with finite sums, which easier to handle. This will be possible if f has a minimal regularity. Therefore, we obtain results when f is uniformly Hölderian on A with exponent $H \in (0, 1]$. Let us recall that this means that there exists a constant $C > 0$ such that

$$\forall\, x, y \in A \qquad |f(x) - f(y)| \leq C \|x - y\|^H .$$

In Sect. 2 we define precisely the general context of these L^p-spaces. We give several general lemmas and state the first main result (Theorem 1). It will be possible to obtain lower and upper bounds for general indices using L^p-spaces with discrete measures.

In Sect. 3 we state the second main result (Proposition 1). We obtain estimates for $\dim_B(\Gamma_f)$ using the q-discrete structure functions of f. They can be seen as other generalizations of the quadratic variations

$$2^{-K} \sum_{k=0}^{2^K-1} \left| f((k+1)2^{-K}) - f(k2^{-K}) \right|^2 ,$$

often used to estimate the Hölder exponent of f (see, for example, [9]). Proposition 1 will allow us to consider multivariate functions.

In Sect. 4 we suppose that $D = 1$. We obtain methods to estimate different classical indices (like $\omega(q)$) using Theorem 1 in special cases. We specify some relations between several fractal indices. We also study the regularization dimension and provide two ways to determine it (Corollary 1 and Proposition 3). Other results on the box dimension are stated.

The last section is devoted to examples. First we present two well-known deterministic functions. We apply results of Sect. 3. We give estimates for $\dim_B(\Gamma_f)$ when f is a Riemann function and determine the exact value when f is a Weierstrass function. Next we study random functions called fractal

sums of pulses (see [2]). They provide an example of functions defined on $\mathbb{R}^D$ for which we can determine $\dim_B(\Gamma_f)$. When $D = 1$ we find $\dim_R(\Gamma_f)$ and several dimensional indices using the results of Sect. 4.

Notations
We end this section with some notation used throughout the next sections.

Let $\{e_1, \ldots, e_D\}$ be the canonical basis of $\mathbb{R}^D$ ($D \geq 1$). It is endowed with the maximum norm $\|\cdot\| : \forall\, t = (t_1, \ldots, t_D) \quad \|t\| = \max\{|t_i|\}$. We write $B(t, \varepsilon)$ for the closed ball of center t and radius $\varepsilon > 0$ with respect to $\|\cdot\|$.

Let $a, b : (0, 1] \to (0, 1]$ be two functions such that $\lim_{\varepsilon \to 0} a(\varepsilon) = \lim_{\varepsilon \to 0} b(\varepsilon) = 0$. A comparison of a and b is made by using the following rules: $a \preccurlyeq b$ means that the ratio a/b is bounded from above, $a \succcurlyeq b$ that it has a strictly positive lower bound, and $a \simeq b$ means that both properties are verified. In this case there exist two constants $C_i > 0$ such that $C_2 \leq a(\varepsilon)/b(\varepsilon) \leq C_1$. Finally, the same notation will be used for two sequences $a = (a_n)$ and $b = (b_n)$.

2 Operator Norms

In the sequel E denotes a non-empty Borel set of $\mathbb{R}^D$ (or $\mathbb{R}^D \times \mathbb{R}^D$) endowed with its Borel algebra $\mathcal{B}$. We consider a family $\xi = (\Phi_\varepsilon, \mu_\varepsilon)_{\varepsilon > 0}$ where, for all $\varepsilon > 0$, Φ_ε is a bounded continuous function defined on E, with values in $\mathbb{R}$, and μ_ε is a probability on $(E, \mathcal{B})$. Under these conditions we define

$$N_p(\xi, \varepsilon) = \left(\int_E |\Phi_\varepsilon(x)|^p \, d\mu_\varepsilon(x) \right)^{1/p} \tag{6}$$

and $N_\infty(\xi, \varepsilon) = \sup\{|\Phi_\varepsilon(x)| \ : \ x \in E \cap Supp(\mu_\varepsilon)\}$ where $Supp(\mu_\varepsilon)$ is the support of μ_ε.

Finally we define the corresponding indices

$$\overline{\sigma}(\xi, p) = \overline{\lim_{\varepsilon \to 0}} \, \frac{\log N_p(\xi, \varepsilon)}{\log \varepsilon} \quad \text{and} \quad \underline{\sigma}(\xi, p) = \underline{\lim_{\varepsilon \to 0}} \, \frac{\log N_p(\xi, \varepsilon)}{\log \varepsilon}.$$

It follows from Jensen's lemma that for all fixed $\varepsilon > 0$ the function $p \mapsto N_p(\xi, \varepsilon)$ is increasing. One can check that it is bounded by $N_\infty(\xi, \varepsilon)$. We will assume that the function $\varepsilon \mapsto N_p(\xi, \varepsilon)$ is continuous and vanishes when ε tends to 0. We want to estimate this speed of convergence. In this sense the following result is essential.

Lemma 1. *Let p, H_1, H_2 be positive real numbers such that*

$$\varepsilon^{H_2} \preccurlyeq N_p(\xi, \varepsilon) \leq N_\infty(\xi, \varepsilon) \preccurlyeq \varepsilon^{H_1}.$$

Then for all $q > 0$,

$$\varepsilon^{H_1 + (H_2 - H_1) \max\{1, p/q\}} \preccurlyeq N_q(\xi, \varepsilon) \preccurlyeq \varepsilon^{H_1}.$$

In particular, $N_q(\xi, \varepsilon) \simeq \varepsilon^H$ if $H_1 = H_2 = H$.

Proof. The right bound is clear. For the other side we have to distinguish two cases. If $q > p$ the inequality is a consequence of the fact that $p \mapsto N_p(\xi, \varepsilon)$ is increasing. If $q \leq p$ we write

$$\begin{aligned}\varepsilon^{pH_2} \preccurlyeq N_p^p(\xi,\varepsilon) &= \int_E |\Phi_\varepsilon(x)|^p \, d\mu_\varepsilon(x) \leq \int_E N_\infty^{p-q}(\xi,\varepsilon)\, |\Phi_\varepsilon(x)|^q \, d\mu_\varepsilon(x) \\ &\leq N_\infty^{p-q}(\xi,\varepsilon) \int_E |\Phi_\varepsilon(x)|^q \, d\mu_\varepsilon(x) \preccurlyeq \varepsilon^{(p-q)H_1} N_q^q(\xi,\varepsilon).\end{aligned}$$

So $N_q(\xi, \varepsilon) \succcurlyeq \varepsilon^{H_1+(H_2-H_1)(p/q)}$. □

An ideal situation is when we can compare $N_p(\xi, \varepsilon)$ to a power of ε: if $N_p(\xi, \varepsilon) \simeq \varepsilon^H$ we get $\underline{\sigma}(\xi, p) = \overline{\sigma}(\xi, p) = H$. However, it remains difficult to estimate these indices from above. With a weaker hypothesis we get a first useful result.

Lemma 2. *Assume that there exists $H > 0$ such that $N_\infty(\xi, \varepsilon) \preccurlyeq \varepsilon^H$. Then*

$$\forall\, p > 0 \qquad \underline{\sigma}(\xi, p) \geq H.$$

Proof. There exists $C > 0$ such that for all $\varepsilon > 0$ and all $p > 0$,

$$N_p(\xi, \varepsilon) \leq N_\infty(\xi, \varepsilon) \leq C\varepsilon^H.$$

One can assume that $\log \varepsilon < 0$, thus

$$\frac{\log N_p(\xi,\varepsilon)}{\log \varepsilon} \geq \frac{\log N_\infty(\xi,\varepsilon)}{\log \varepsilon} \geq \frac{\log C}{\log \varepsilon} + H.$$

The result follows by taking the upper limit as ε tends to 0. □

When E is a parallelepiped and μ_ε is the uniform measure on E, the operator $\Phi_\varepsilon(x)$ depends on the two continuous variables ε and x. That makes the computation of $N_p(\xi, \varepsilon)$ difficult. Our aim is to obtain good bounds for $\underline{\sigma}(\xi, p)$ knowing only estimates about discrete versions of $N_p(\xi, \varepsilon)$. To get an upper bound it is sufficient to consider a discrete sequence of values (ε_K) decreasing to 0. This allows us to discretize $N_p(\xi, \varepsilon)$ with respect to the variable ε. A way to solve the problem with respect to x is to use only discrete measures for which integrals (6) are easier to calculate. Thus, for all $\varepsilon \in (0, 1]$, we consider

$$D_\varepsilon = \{(k_1\varepsilon, \ldots, k_D\varepsilon) \,:\, k_i \in \mathbb{N}\,,\, 0 \leq k_i \leq 1/\varepsilon\} \subset [0,1]^D.$$

This is a finite subset with cardinal $m_\varepsilon = (1 + [1/\varepsilon])^D \simeq \varepsilon^{-D}$, where the notation $[x]$ stands for $[x] = \max\{j \in \mathbb{Z} : j \leq x\}$. For example, when $D = 1$ and $\varepsilon = 2^{-K}$, $D_\varepsilon = \{k2^{-K} \,:\, k \in \{0, \ldots, 2^K\}\}$ is the usual set of dyadic numbers of order K. Let us note that one can choose another grid D_ε provided that its gap will be equivalent to ε and its cardinal will be equivalent to ε^{-D}.

Then it is possible to discretize both parameter ε and variable x using different grids of type $u + D_\varepsilon$. The following lemma shows these two levels of discretization. We restrict ourselves to the dimension 1.

Lemma 3. *Let $\Phi : [0,1] \to \mathbb{R}$ be a continuous function. Let us fix $K \in \mathbb{N}^*$, $s \geq 1$ and write $u_M = 2^{(s-1)K+M}$. Then Φ is integrable and*

$$\int_0^1 \Phi(x)\,dx = \lim_{M\to+\infty} \frac{1}{u_M} \sum_{i=0}^{[u_M]-1} \frac{1}{2^K} \sum_{k=0}^{2^K-1} \Phi\left(\frac{k}{2^K} + \frac{i}{2^{sK+M}}\right).$$

Moreover if $\Phi \geq 0$ then

$$\int_0^1 \Phi(x)\,dx \geq \lim_{M\to+\infty} \frac{1}{u_M} \sum_{i=0}^{2^M-1} \frac{1}{2^K} \sum_{k=0}^{2^K-1} \Phi\left(\frac{k}{2^K} + \frac{i}{2^{sK+M}}\right).$$

Proof. (a) Since Φ is continuous we will calculate its integral with Riemann's sums. Let $k \in \{0,\dots,2^K-1\}$. We consider the subdivision $\{x_i\}_{i=0}^{[u_M]+1}$ of $[k2^{-K},(k+1)2^{-K}]$ defined by

$$x_i = \begin{cases} \dfrac{k}{2^K} + \dfrac{i}{2^{sK+M}} & \text{for} \quad i = 0,\dots,[u_M] \\ \\ \dfrac{k+1}{2^K} & \text{for} \quad i = [u_M]+1. \end{cases}$$

We have

$$\int_{k2^{-K}}^{(k+1)2^{-K}} \Phi(x)\,dx = \lim_{M\to+\infty} \sum_{i=0}^{[u_M]} (x_{i+1}-x_i)\,\Phi(x_i)$$

and one checks that

$$\sum_{i=0}^{[u_M]} (x_{i+1}-x_i)\,\Phi(x_i) = \frac{1}{2^K}\frac{1}{u_M} \sum_{i=0}^{[u_M]-1} \Phi(x_i) \;+\; \frac{1}{2^K}\left(1 - \frac{[u_M]}{u_M}\right)\Phi(x_{[u_M]}).$$

Consequently,

$$\begin{aligned} \int_0^1 \Phi(x)\,dx &= \sum_{k=0}^{2^K-1} \int_{k2^{-K}}^{(k+1)2^{-K}} \Phi(x)\,dx \\ &= \lim_{M\to+\infty} \left(\frac{1}{u_M} \sum_{i=0}^{[u_M]-1} \frac{1}{2^K} \sum_{k=0}^{2^K-1} \Phi(x_i) \right) \\ &\quad + \lim_{M\to+\infty} \left(\frac{1}{2^K} \sum_{k=0}^{2^K-1} \left(1 - \frac{[u_M]}{u_M}\right) \Phi\left(x_{[u_M]}\right) \right). \end{aligned}$$

It just remains to prove that the second limit is 0. Let us recall that K and s are fixed. The sequence $(u_M)_{M\geq 1}$ is increasing and tends to $+\infty$ so

$(1 - [u_M]/u_M) \to 0$. Since Φ is bounded, we have the required limit and the result.

(b) Since $s \geq 1$ we have $[u_M] \geq 2^M$. We split here the sum on i in the limit. The second part is positive if $\Phi \geq 0$. □

Here we suppose that $E = [0, 1]$ and μ_ε is the Lebesgue measure. For $u \in E$ and $\varepsilon > 0$ we consider the discrete measure

$$\lambda_\varepsilon^u = \frac{1}{m_\varepsilon} \sum_{x \in u + D_\varepsilon} \delta_x,$$

where δ_x denotes the Dirac measure at the point x. Notice that λ_ε^u is simply the uniform probability on the translated grid $u + D_\varepsilon$. We note $s_p(u, \varepsilon) = N_p(\xi_u, \varepsilon)$ with $\xi_u = (\Phi_\varepsilon, \lambda_\varepsilon^u)_{\varepsilon > 0}$. We will simply write $s_p(\varepsilon)$ when $u = 0$. The following theorem will be crucial in the sequel: it will be enough to specialize the family ξ to work with different dimensional indices.

Theorem 1. *Assume that there exist α, β, $\gamma \in (0, 1]$ such that $\alpha \geq \beta$ and*

(i) $N_\infty(\xi, \varepsilon) \preccurlyeq \varepsilon^\gamma$,

(ii) The family $(\Phi_\varepsilon)_{\varepsilon > 0}$ is equi-Hölderian on E with exponent β:

$$\forall\, \varepsilon > 0 \quad \forall\, x, y \in E \quad |\Phi_\varepsilon(x) - \Phi_\varepsilon(y)| \leq C_1 \, |x - y|^\beta$$

where $C_1 > 0$ is a constant which depends neither on x, nor on y, nor on ε,

(iii) There exists $p \geq 1$ such that for all $M \in \mathbb{N}$, there exists an integer K_M such that

$$\forall\, i \in \{0, \ldots, 2^M - 1\} \quad \forall\, K \geq K_M \quad s_p(u_i, 2^{-K}) \geq C_2 \, 2^{-K\alpha}$$

with $u_i = \dfrac{i}{2^{sK+M}}$, $s = \alpha/\beta$, and $C_2 > 0$ a constant which depends neither on M, nor on i, nor on K.

Then for all $q > 0$,

$$\gamma \leq \underline{\sigma}(\xi, q) \leq \gamma + (\alpha - \gamma) \max\{1, p/q\} + \frac{\alpha - \beta}{q\beta}.$$

In particular, $\underline{\sigma}(\xi, q) = \gamma$ if $\alpha = \beta = \gamma$.

Proof. (a) Let us fix $i \in \{0, \ldots, 2^M - 1\}$. Let $u \in [u_i, u_i + 1)$. We will show that for a certain constant $C_3 > 0 : s_p(u, 2^{-K}) \geq C_3 \, 2^{-K\alpha}$. Indeed, for $\varepsilon > 0$ and $t \in D_\varepsilon$, the triangular inequality yields

$$|\Phi_\varepsilon(u_i + t)| \leq |\Phi_\varepsilon(u_i + t) - \Phi_\varepsilon(u + t)| + |\Phi_\varepsilon(u + t)| \, .$$

Thus, hypothesis (ii) implies that

$$|\Phi_\varepsilon(u_i+t)| \le C_1\,|u-u_i|^\beta + |\Phi_\varepsilon(u+t)| \le C_1\,2^{-(sK+M)\beta} + |\Phi_\varepsilon(u+t)|\,.$$

Consequently, since $p \ge 1$, Minkowski's inequality yields

$$s_p(u_i,\varepsilon) \le C_1\,2^{-(sK+M)\beta} + s_p(u,\varepsilon).$$

For $\varepsilon = 2^{-K}$ hypothesis (iii) gives

$$s_p(u,2^{-K}) \ge C_2\,2^{-K\alpha} - C_1\,2^{-(sK+M)\beta},$$

and because of the choice of s,

$$C_2\,2^{-K\alpha} - C_1\,2^{-(sK+M)\beta} = 2^{-K\alpha}(C_2 - C_1\,2^{-M\beta}).$$

So let $C_3(M) = C_2 - C_1\,2^{-M\beta}$. We fix $M_0 \ge 0$ such that $C_1 2^{-M_0\beta} \le C_2/2$ and in the sequel we suppose that $M \ge M_0$. The result holds with $C_3 = C_3(M_0)$.

(b) For all $u \in E$ hypothesis (i) leads to $s_\infty(u,2^{-K}) \preccurlyeq 2^{-K\gamma}$ and (a) gives $s_p(u,2^{-K}) \succcurlyeq 2^{-K\alpha}$. Thus, we can apply Lemma 1 with $H_1 = \gamma$ and $H_2 = \alpha$. We have, in particular,

$$s_q(u,2^{-K}) \succcurlyeq 2^{-K(\gamma+(\alpha-\gamma)\max\{1,p/q\})},$$

which amounts to $s_q(u,2^{-K}) \ge C'\,2^{-K\theta}$ with $\theta = \gamma + (\alpha-\gamma)\max\{1,p/q\}$ and $C' > 0$ a certain constant.

(c) Since $\alpha \ge \beta$ we have $s \ge 1$, so according to Lemma 3,

$$N_q^q(\xi,2^{-K}) \ge 2^{(1-s)K} \lim_{M\to+\infty} \frac{1}{2^M} \sum_{i=0}^{2^M-1} s_q^q(u_i,2^{-K}).$$

But according to the previous points, for all indices i considered,

$$s_q^q(u_i,2^{-K}) \ge C'\,2^{-K\theta q}.$$

Hence, $2^{-K(\theta+(s-1)/q)} \preccurlyeq N_q(\xi,2^{-K})$, so

$$\varliminf_{K\to\infty} \frac{\log N_q(\xi,2^{-K})}{\log 2^{-K}} \le \gamma + (\alpha-\gamma)\max\{1,p/q\} + \frac{\alpha-\beta}{q\beta}.$$

By definition of the lower limit, since $N_q(\xi,\varepsilon)$ is continuous, we always have

$$\underline{\sigma}(\xi,q) \le \varliminf_{K\to\infty} \frac{\log N_q(\xi,2^{-K})}{\log 2^{-K}},$$

which gives the upper bound. According to hypothesis (i), the lower bound is given by Lemma 2. □

3 Application to Box Dimension

To calculate $\dim_B(\Gamma_f)$, we cannot directly use the original definition given by (2) because it is too general. We have to find a better determining function written as a certain $N_p(\xi, \varepsilon)$ and more adapted to the graph of a function.

For every non-empty set $B \subset [0,1]^D$ let us define $osc(f, B)$ as the oscillation of f over B:

$$osc(f, B) = \sup\{|f(t) - f(s)| \,:\, s, t \in B\}.$$

It is possible to consider the oscillation of f over all the balls $B(x, \varepsilon)$ centered in $[0,1]^D$. The arithmetic mean of these oscillations gives the variation of f:

$$V(f, \varepsilon) = \int_{[0,1]^D} osc(f, B(x, \varepsilon))\, dx.$$

If f is constant on $[0,1]^D$, its variation is zero for all $\varepsilon > 0$. Otherwise, $V(f, \varepsilon)$ is a strictly positive quantity which vanishes as ε tends to 0. The rate of decrease of $V(f, \varepsilon)$ is related to the regularity of f and to $\dim_B(\Gamma_f)$. One can prove (see [5, 6]) that if f is not constant, then $V(f, \varepsilon)$ determines the box dimension:

$$\dim_B(\Gamma_f) = D + 1 - \underline{\lim}_{\varepsilon \to 0} \frac{\log V(f, \varepsilon)}{\log \varepsilon}.$$

If one can control the increments of f, then one can estimate its variation. For example, if f is differentiable, then $V(f, \varepsilon) \simeq \varepsilon$ and the limit above is D. More generally, if f is uniformly Hölderian with exponent β, then one easily gets

$$\dim_B(\Gamma_f) \leq D + 1 - \beta. \tag{7}$$

It is more difficult to get a lower bound. It is possible if conversely f is anti-Hölderian, but such a property is difficult to check (see [15]).

Here we suppose that $E = [0,1]^D \times [0,1]^D$ and μ_ε is the Lebesgue measure. We introduce a set of edges of the grid D_ε:

$$D_\varepsilon^* = \{(s, t) \in D_\varepsilon \times D_\varepsilon \,:\, t = s + \varepsilon e_i \text{ for a certain } i \in \{1, \ldots, D\}\}.$$

This is again a finite set with cardinal $n_\varepsilon \simeq \varepsilon^{-D}$. For $u \in E$ and $\varepsilon > 0$ we consider the uniform probability on D_ε^*:

$$\lambda_\varepsilon^u = \frac{1}{n_\varepsilon} \sum_{x \in u + D_\varepsilon^*} \delta_x.$$

We choose for Φ_ε the map defined by

$$\forall\, x = (s, t) \in E \qquad \Phi_\varepsilon(x) = f(t) - f(s).$$

For a fixed $u \in E$ we write again $s_p(u,\varepsilon) = N_p(\xi_u,\varepsilon)$ with $\xi_u = (\Phi_\varepsilon, \lambda_\varepsilon^u)_{\varepsilon>0}$ and we will simply write $s_p(\varepsilon)$ if $u = 0$. This last function is called the *discrete p-structure function* of f (see [4]) and so it is defined by

$$s_p(\varepsilon) = \left(\frac{1}{n_\varepsilon} \sum_{(s,t)\in D_\varepsilon^*} |f(t) - f(s)|^p \right)^{1/p}.$$

One important fact is that for $p = 1$ one always obtains a lower approximation for the variation.

Lemma 4. *For all $\varepsilon > 0$ the inequality $s_1(\varepsilon) \preccurlyeq V(f,\varepsilon)$ holds.*

We refer to [4] for the proof. However, due to technical reasons, calculations with $s_1(\varepsilon)$ are often difficult. It will be very useful to have results knowing estimates on $s_p(\varepsilon)$ for some $p > 0$.

Proposition 1. *Assume that there exist α, $\beta \in (0,1]$ such that*

(i) f is uniformly Hölderian on $[0,1]^D$ with exponent β,
(ii) There exists $p > 0$ such that $s_p(2^{-K}) \succcurlyeq 2^{-\alpha K}$.

Then the following bounds hold:

$$D + 1 - (\beta + (\alpha - \beta)\max\{1,p\}) \le \dim_B(\Gamma_f) \le D + 1 - \beta.$$

In particular, $\dim_B(\Gamma_f) = D + 1 - \beta$ if $\alpha = \beta$.

Proof. (a) (i) implies $osc(f, B(x,\varepsilon)) \preccurlyeq \varepsilon^\beta$, so $V(f,\varepsilon) \preccurlyeq \varepsilon^\beta$ and $\dim_B(\Gamma_f) \le D + 1 - \beta$.

(b) The support of λ_ε^0 is D_ε^*, so (i) yields $s_\infty(\varepsilon) \preccurlyeq \varepsilon^\beta$. Hypothesis (ii) implies $s_p(2^{-K}) \succcurlyeq 2^{-K\alpha}$. We can apply Lemma 1 with $H_1 = \beta$, $H_2 = \alpha$, and $q = 1$, and we obtain in particular

$$s_1(2^{-K}) \succcurlyeq 2^{-K(\beta + (\alpha-\beta)\max\{1,p\})}.$$

Thus, according to Lemma 4, $V(f,2^{-K}) \succcurlyeq 2^{-K(\beta+(\alpha-\beta)\max\{1,p\})}$. Consequently,

$$\underline{\lim}_{\varepsilon\to 0} \frac{\log V(f,\varepsilon)}{\log \varepsilon} \le \underline{\lim}_{K\to\infty} \frac{\log V(f,2^{-K})}{\log 2^{-K}} \le \beta + (\alpha - \beta)\max\{1,p\},$$

which gives the required lower bound. □

Let us remark that if only the hypothesis (i) is used to prove the right inequality, we have to use it again to prove the left inequality.

4 Application to Several Fractal Indices

In this section we suppose that $D=1$. We will show how Theorem 1 can be applied to study classical fractal indices. We will give three examples. In Sect. 1 we have defined $\omega(q)$ for $q>0$. Here let us add:

1. The indices associated to the q-structure functions (4) and defined by

$$\sigma(q) = \underline{\lim}_{\varepsilon\to 0} \frac{\log S_q(\varepsilon)}{\log \varepsilon}. \tag{8}$$

2. The indices associated to the regularization dimension and defined, according to Sect. 1, by

$$\rho(q) = \underline{\lim}_{\varepsilon\to 0} \frac{\log R_q(\varepsilon)}{\log \varepsilon}, \quad \text{where} \quad \forall\, \varepsilon>0 \quad R_q(\varepsilon) = \left(\int_0^1 |\varepsilon f'_\varepsilon(x)|^q\, dx\right)^{1/q}. \tag{9}$$

We may express each function Z_q, S_q and R_q as an appropriate operator norm $N_p(\xi,\varepsilon)$. Thus, each function ω, σ, and ρ will be the corresponding $\underline{\sigma}(\xi,q)$.

Proposition 2. *Assume that there exists $C>0$, $p\geq 1$, α, $\beta\in(0,1]$ such that:*

(i) f is uniformly Hölderian on $[0,1]$ with exponent β.
(ii) For all $M\in\mathbb{N}$ there exists an integer K_M such that

$$\forall\, i\in\{0,\dots,2^M-1\},\ \forall\, K\geq K_M,\ \frac{1}{2^K}\sum_{k=0}^{2^K-1}\left|U\left(\frac{k}{2^K},\frac{i}{2^{M+\alpha K/\beta}}\right)\right|^p \geq C\,2^{-Kp\,\alpha},$$

with $U(k\,2^{-K},u) = f((k+1)2^{-K}+u) - f(k\,2^{-K}+u)$ (resp. $2^{-K}f'_{2^{-K}}(k\,2^{-K}+u)$, $C(2^{-K},2(k\,2^{-K}+u)-\frac{1}{2})$).

Then for all $q>0$ and $\delta=\sigma$ (resp. ρ, ω),

$$\beta\leq\delta(q)\leq\beta+(\alpha-\beta)\left(\max\{1,p/q\}+\frac{1}{q\beta}\right).$$

In particular, $\delta(q)=\beta$ if $\alpha=\beta$.

Proof. We consider $E=[0,1]$ and for all $\varepsilon>0$, μ_ε is the Lebesgue measure on E. We want to apply Theorem 1. To do this we have to correctly choose the operator $\Phi_\varepsilon(x)$ and check in each case the three hypotheses. Actually, hypotheses (i) and (ii) are a consequence of the fact that f is uniformly Hölderian. Hypothesis (iii) will be clear as soon as we check that the expression of $s_p(u_i,2^{-K})$ is the one given in (ii) with the corresponding U.

(a) First we take $\Phi_\varepsilon(x) = f(x+\varepsilon) - f(x)$, $x \in E$. Then $N_q(\xi,\varepsilon) = S_q(\varepsilon)$ and $\underline{\sigma}(\xi,q) = \sigma(q)$. Using (i):

$$\exists\, C_1 > 0 \quad \forall\, x \in E \qquad |\Phi_\varepsilon(x)| = |f(x+\varepsilon) - f(x)| \leq C_1\, \varepsilon^\beta,$$

thus $N_\infty(\xi,\varepsilon) \preccurlyeq \varepsilon^\beta$.

The triangular inequality gives

$$\forall\, \varepsilon > 0 \quad \forall\, x, y \in E \qquad |\Phi_\varepsilon(x) - \Phi_\varepsilon(y)| \leq 2\, C_1\, |x-y|^\beta\,,$$

so the family $(\Phi_\varepsilon)_{\varepsilon>0}$ is equi-Hölderian with exponent β.

Moreover,

$$s_p(u_i, 2^{-K}) = \left(\frac{1}{2^K} \sum_{k=0}^{2^K-1} \left| f\left(\frac{k+1}{2^K} + u_i\right) - f\left(\frac{k}{2^K} + u_i\right) \right|^p \right)^{1/p}.$$

(b) Here we take $\Phi_\varepsilon(x) = \varepsilon f'_\varepsilon(x)$ for $x \in E$. Then $N_q(\xi,\varepsilon) = R_q(\varepsilon)$ and $\underline{\sigma}(\xi,q) = \rho(q)$.

Let $\varepsilon > 0$ and $x \in E$. We keep the notation of the Introduction (see page 146). Since K' is odd, we can write

$$f'_\varepsilon(x) = \int_{B(x,\varepsilon)} f(t) K'_\varepsilon(x-t)\, dt = \int_{B(x,\varepsilon)} (f(t) - f(x)) K'_\varepsilon(x-t)\, dt.$$

Consequently,

$$\begin{aligned} |f'_\varepsilon(x)| &\leq \sup_{t \in B(x,\varepsilon)} |f(t) - f(x)| \int_{B(x,\varepsilon)} |K'_\varepsilon(x-t)|\, dt \\ &\leq \sup_{t \in B(x,\varepsilon)} |f(t) - f(x)| \left(\frac{1}{\varepsilon} \int_{\mathbb{R}} |K'(u)|\, du \right). \end{aligned}$$

Thus, $|\Phi_\varepsilon(x)| \preccurlyeq osc(f, B(x,\varepsilon))$ and (i) implies $N_\infty(\xi,\varepsilon) \preccurlyeq \varepsilon^\beta$.
Let $\varepsilon > 0$ and $x, y \in [0,1]$. We can write

$$f'_\varepsilon(x) - f'_\varepsilon(y) = \int_{\mathbb{R}} (f(x-t) - f(y-t)) K'_\varepsilon(t)\, dt\,;$$

thus,

$$\begin{aligned} |f'_\varepsilon(x) - f'_\varepsilon(y)| &\leq \int_{\mathbb{R}} |f(x-t) - f(y-t)|\, |K'_\varepsilon(t)|\, dt \leq C\, |x-y|^\beta \int_{\mathbb{R}} |K'_\varepsilon(t)|\, dt \\ &\leq C\, |x-y|^\beta \left(\frac{1}{\varepsilon} \int_{\mathbb{R}} |K'(u)|\, du \right). \end{aligned}$$

We obtain $|\Phi_\varepsilon(x) - \Phi_\varepsilon(y)| \preccurlyeq |x-y|^\beta$, so the family $(\Phi_\varepsilon)_{\varepsilon>0}$ is equi-Hölderian with exponent β.

Moreover,

$$s_p(u_i, 2^{-K}) = \left(\frac{1}{2^K} \sum_{k=0}^{2^K-1} \left| \frac{1}{2^K} f'_{2^{-K}} \left(\frac{k}{2^K} + u_i \right) \right|^p \right)^{1/p}.$$

c) We keep the notation of the Introduction (see page 146). The calculation of the upper limit in ω allows us to assume that $\varepsilon < 1/2$ and the assumption on the support of Ψ implies $C(\varepsilon, x) = 0$ if $[x - \varepsilon, x + \varepsilon] \cap [0, 1] = \varnothing$. So we have

$$\forall\, \varepsilon \in (0, 1/2), \qquad Z_q(\varepsilon) = \left(\int_{-1/2}^{3/2} |C(\varepsilon, x)|^q \, dx \right)^{1/q}.$$

Thus, we write for all $\varepsilon < 1/2$,

$$\widetilde{Z}_q(\varepsilon) = \left(\frac{1}{2} \int_{-1/2}^{3/2} |C(\varepsilon, x)|^q \, dx \right)^{1/q} = \left(\int_0^1 \left| C\left(\varepsilon, 2x - \frac{1}{2} \right) \right|^q dx \right)^{1/q},$$

and we notice that

$$\omega(q) = \varliminf_{\varepsilon \to 0} \frac{\log \widetilde{Z}_q(\varepsilon)}{\log \varepsilon}.$$

Thus, we finally take $\Phi_\varepsilon(x) = C(\varepsilon, 2x - 1/2)$, $x \in E$. Then $N_q(\xi, \varepsilon) = \widetilde{Z}_q(\varepsilon)$ and $\underline{\sigma}(\xi, q) = \omega(q)$. We proceed in the same way as in (b), changing K in Ψ. We obtain again $N_\infty(\xi, \varepsilon) \preccurlyeq \varepsilon^\beta$ and an equi-Hölderian family $(\Phi_\varepsilon)_{\varepsilon > 0}$ with exponent β.

Moreover,

$$s_p(u_i, 2^{-K}) = \left(\frac{1}{2^K} \sum_{k=0}^{2^K-1} \left| C\left(\frac{1}{2^K}, 2\left(\frac{k}{2^K} + u_i \right) - \frac{1}{2} \right) \right|^p \right)^{1/p}.$$

Finally, it is clear that necessarily $\alpha \geq \beta$. Therefore, in each case one can apply Theorem 1 with $\gamma = \beta$ and the selected family $(\Phi_\varepsilon)_{\varepsilon > 0}$. □

Now we show precisely the relationship between these indices. We can deduce estimates for the box dimension and the regularization dimension of Γ_f.

Corollary 1. *Let* dim *denote here either* $\dim_B$ *or* $\dim_R$. *Then, under the same assumptions as in Proposition 2, we have*

$$2 - \beta - (\alpha - \beta)(p + \frac{1}{\beta}) \leq \dim(\Gamma_f) \leq 2 - \beta.$$

In particular, $\dim(\Gamma_f) = 2 - \beta$ *if* $\alpha = \beta$.

Proof. (a) First let us suppose that dim = $\dim_B$. Since f is uniformly Hölderian with exponent β, the upper bound holds (see (7)). For the lower one we have $S_1(\varepsilon) \preccurlyeq V(f,\varepsilon)$ so $2-\sigma(1) \leq \dim_B(\Gamma_f)$. We apply Proposition 2 with $\delta = \sigma$ and $q = 1$.
(b) Now let us suppose that dim = $\dim_R$. Since $\dim_R(\Gamma_f) = 2-\rho(1)$ the result is a consequence of Proposition 2 with $\delta = \rho$ and $q = 1$. □

The estimate obtained for $\dim_B(\Gamma_f)$ is less precise than the one given by Proposition 1 and the hypotheses are more restrictive. If $D = 1$ and $p \geq 1$, then Proposition 1 gives

$$\dim_B(\Gamma_f) \geq 2-\beta-(\alpha-\beta)p > 2-\beta-(\alpha-\beta)(p+\frac{1}{\beta}).$$

However, the indices $\sigma(q)$ have their own interest. We will show, for example, how to use them to calculate the regularization dimension of Γ_f. The dimensions $\dim_B(\Gamma_f)$ and $\dim_R(\Gamma_f)$ are closely related. We always have the inequalities

$$1 \leq \dim_R(\Gamma_f) \leq \dim_B(\Gamma_f) \leq 2.$$

For many mathematical models the central inequality is in fact an equality. We can describe a general situation where these two dimensions coincide. We use the indices $\omega(q)$. Let us recall that the following facts hold (see [11] and [13]):

1. If $q > 1$ and if $\sigma(q) < 1$ then $\sigma(q) = \omega(q)$
2. If $0 < \omega(1) < 1$ then $\dim_B(\Gamma_f) = 2-\omega(1)$
3. $\dim_R(\Gamma_f) = \max\{1, 2-\omega(1)\}$

It is clearly possible to relate $\dim_R(\Gamma_f)$ to $\dim_B(\Gamma_f)$ using the indices $\sigma(q)$ and $\omega(q)$.

Proposition 3. *Let $\alpha \in (0,1)$ be such that:*

(i) f is uniformly Hölderian with exponent β for all $\beta < \alpha$.
(ii) There exists $p > 1$ such that $\sigma(p) = \alpha$.

Then $\dim_B(\Gamma_f) = \dim_R(\Gamma) = 2-\alpha$.

Let us remark that (i) is true if f is uniformly Hölderian with exponent α but allows us to deal with functions for which there exists only a "critical" exponent α. For example, the standard Brownian motion is uniformly Hölderian with exponent β for all $\beta < 1/2$, but not uniformly Hölderian with exponent $\alpha = 1/2$.

Proof. Given the facts above, it is enough to show that $\omega(1) = \alpha$, that is, $\omega(1) = \sigma(p)$. To this end we will relate $\omega(1)$ to $\omega(p)$. We keep the notation of part (c) of the proof of Proposition 2. So we have $\widetilde{Z}_q(\varepsilon) = N_q(\xi,\varepsilon)$.

Let $\eta > 0$. With $\beta = \alpha - \eta$ (i) implies that $N_\infty(\xi,\varepsilon) \leq \varepsilon^{\alpha-\eta}$. Hypothesis (ii) gives, with the well-known fact 1, $\omega(p) = \sigma(p) = \alpha$. With the definition

of $\omega(p)$ we get, for all ε small enough, $N_p(\xi,\varepsilon) \succcurlyeq \varepsilon^{\alpha+\eta}$. We apply Lemma 1 with $H_1 = \alpha - \eta$, $H_2 = \alpha + \eta$, and $q = 1$:

$$\varepsilon^{\alpha+(2p-1)\eta} \preccurlyeq N_1(\xi,\varepsilon) \preccurlyeq \varepsilon^{\alpha-\eta}.$$

Therefore, $\alpha - \eta \leq \omega(1) \leq \alpha + (2p-1)\,\eta$. Since this is true for all $\eta > 0$, we have $\omega(1) = \alpha$ and the result follows. □

This proposition is useful because, among the indices σ, ρ, and ω, the easiest to handle is σ since it is directly built with the increments of f. Notice that in every case one has to evaluate them to determine as Hölder exponent for f. Proposition 2 may be used to obtain $\sigma(p)$.

5 Examples

Since the second half of the nineteenth century, many mathematicians have proposed examples of continuous nowhere differentiable functions. Nowadays they are still studied as models of fractal and multifractal functions. Moreover, since their graphs are very irregular, these functions are used as models for many physical phenomena such as rough profiles or stock market prices. We expose, here three examples of such functions.

1. The Riemann functions: we only obtain inequalities for $\dim_B$.
2. The Weierstrass functions: we can prove again a known result about $\dim_B$ (see [15]).
3. The fractal sums of pulses: we collect some results stated in [2] and [4].

5.1 Riemann Functions

Let $r > 1$ be an integer, $H \in (1, r]$, and $(X_n)_{n\geq 0}$ be a sequence of real numbers. We define on $\mathbb{R}$ the function

$$R_H(x) = \sum_{n=1}^{\infty} n^{-H} \sin(2\pi(n^r x + X_n)).$$

Since $H > 1$ the series converges and defines a continuous function. With $r = H = 2$ and $(X_n) = 0$ one obtains the original function introduced by Riemann in 1861 (see [7,16]).

Proposition 4. *The following properties of R_H hold:*

(i) R_H is uniformly Hölderian with exponent $(H-1)/r$.
(ii) $S_1(\varepsilon) \succcurlyeq \varepsilon^{H/r}$.

Thus, $2 - \dfrac{H}{r} \leq \dim_B(\Gamma_{R_H}) \leq 2 - \dfrac{H-1}{r}$.

Proof. Let us write $S_n(x) = \sin(\pi n^r x)$ and $C_n(x) = \cos(\pi n^r x + 2\pi X_n)$. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. We have $S_n(x+\varepsilon) - S_n(x) = 2S_n(\varepsilon)C_n(2x+\varepsilon)$.
(i) Using the triangular inequality and the relation above,

$$|F(x+\varepsilon) - F(x)| \leq 2\sum_{n=1}^{\infty} n^{-H} |S_n(\varepsilon)| .$$

We use a well-known technique consisting in splitting the sum. We choose $N \geq 1$ such that $(N+1)^{-r} < h \leq N^{-r}$ and using $\sin x \leq \min\{x, 1\}$ we get

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-H} |S_n(\varepsilon)| &\leq \sum_{n=1}^{N} n^{-H} |\pi n^r \varepsilon| + \sum_{n=N+1}^{\infty} n^{-H} \\ &\preccurlyeq \varepsilon\, N^{1+r-H} + (N+1)^{1-H} \preccurlyeq \varepsilon^{(H-1)/r}.\end{aligned}$$

Hence assertion (i).

(ii) For any $N \geq 1$ we can write

$$\begin{aligned}|F(x+\varepsilon) - F(x)| &= \left| 2\sum_{n=1}^{\infty} n^{-H} S_n(\varepsilon) C_n(2x+\varepsilon) \right| \\ &\geq \left| 2\sum_{n=1}^{\infty} n^{-H} S_n(\varepsilon) C_n(2x+\varepsilon) \right| |C_N(2x+\varepsilon)| \\ &\geq \left| -2\sum_{n=1}^{\infty} n^{-H} S_n(\varepsilon) C_n(2x+\varepsilon) C_N(2x+\varepsilon) \right| .\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}S_1(\varepsilon) = \int_0^1 |F(x+\varepsilon) - F(x)|\, dx &\geq \int_0^1 \Big| -2\, N^{-H} S_N(\varepsilon) C_N^2(2x+\varepsilon) \\ &\quad - 2\sum_{n\neq N} n^{-H} S_n(\varepsilon) C_n(2x+\varepsilon) C_N(2x+\varepsilon) \Big| dx \\ &\geq \left| -2N^{-H} S_N(\varepsilon) \int_0^1 C_N^2(2x+\varepsilon) dx \right| \\ &\quad - \left| 2\sum_{n\neq N} n^{-H} S_n(\varepsilon) \int_0^1 C_n(2x+\varepsilon) C_N(2x+\varepsilon) dx \right| .\end{aligned}$$

Since $r \in \mathbb{N}^*$, one easily checks that $\int_0^1 C_N^2(2x+\varepsilon)dx = 1/2$ and $\int_0^1 C_n(2x+\varepsilon)$ $C_N(2x+\varepsilon)dx = 0$ if $n \neq N$.

Finally, we get $S_1(\varepsilon) \geq N^{-H} |S_N(\varepsilon)|$. Let us choose N such that $(N+1)^{-r} < \varepsilon \leq N^{-r}$. First we have $N^r \varepsilon \geq 2^{-r}$ so $|S_N(\varepsilon)| \geq \sin(\pi 2^{-r}) > 0$. But $N^{-H} \geq \varepsilon^{H/r}$ so $S_1(\varepsilon) \succcurlyeq \varepsilon^{H/r}$. Hence assertion (ii).

Then we can apply Proposition 1 with $\alpha = (H-1)/r$, $\beta = H/r$, and $p = 1$, which gives the last result. □

Let us state that certain more precise results are known (see [1,10,12]), but the proofs are rather technical. Our approach yields easily general inequalities.

5.2 Weierstrass Functions

Weierstrass, too, was interested in continuous functions which do not have a well-defined derivative. He thought that the example of Riemann was rather complicated, so in 1875 he proposed another example of such functions (see [16]).

Let $a > 1$ be an integer, $H > 0$, and $(X_n)_{n \geq 0}$ be a sequence of real numbers. We define on $\mathbb{R}$ the function

$$W_H(x) = \sum_{n=0}^{\infty} a^{-nH} \cos(2\pi(a^n x + X_n)).$$

The series converges and defines a continuous function. This function is close to R_H but its irregularity is more homogeneous. However, the behavior of its increments is very difficult to determine (see [8]). Here we can obtain the exact behavior of all its q-structure functions S_q and the exact value of $\dim_B(\Gamma_{W_H})$.

Proposition 5. *The function W_H satisfies the following:*

(i) W_H is uniformly Hölderian with exponent H.
(ii) $S_1(\varepsilon) \succcurlyeq \varepsilon^H$.

Thus, $S_q(\varepsilon) \simeq \varepsilon^H$ *for all* $q > 0$ *and* $\dim_B(\Gamma_{W_H}) = 2 - H$.

Proof. The proofs of (i) and (ii) are the same as those of Proposition 4. Thus, we can apply Lemma 1 with $N_p(\xi, \varepsilon) = S_p(\varepsilon)$, $p = 1$ and $H_1 = H_2 = H$. This gives the first part of the conclusion. Next we have $\dim_B(\Gamma_{W_H}) \leq 2 - H$, using (i) and (7). Finally, we have $\dim_B(\Gamma_{W_H}) \geq 2 - \sigma(1)$ and (ii) implies that $\sigma(q) = H$ for all $q > 0$. □

5.3 Fractal Sums of Pulses

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with the following properties: g is even, decreasing on $[0, 1]$, and vanishing on $[1, +\infty)$ and $g(0) = 1$. We define on $\mathbb{R}^D$ the "elementary pulse" G as the radial function $G(t) = g(\|t\|_G)$, where $\|\cdot\|_G$ is an arbitrary norm on $\mathbb{R}^D$ such that its unit ball $B_G(0, 1)$ is included in the cube $[-1, 1]^D$.

We study the function F_H defined on $\mathbb{R}^D$ by

$$F_H(x) = \sum_{n=1}^{\infty} n^{-H/D} G(n^{1/D}(x - X_n)) \tag{10}$$

with $H \in (0,1)$ and $(X_n)_{n\geq 1}$ a sequence of iid random variables with the same distribution. We assume that this distribution is the uniform probability on $[-1,2]^D$ (a more natural choice would be the uniform probability on $[0,1]^D$, but our assumption will prevent some boundary effects).

We show that the series (10) defines almost surely a continuous function on $[0,1]^D$ and we can show precisely given the regularity of its paths. A proof of the following is given in [4].

Proposition 6. *Assume that g is uniformly Hölderian on $\mathbb{R}$ with exponent γ. Then, for every $0 < \beta < \min\{H, \gamma\}$, F_H is almost surely uniformly Hölderian with exponent β. Consequently, we get almost surely* $\dim_B(\Gamma_{F_H}) \leq D + 1 - \min\{H, \gamma\}$.

Thus, the main hypothesis on F_H is satisfied. One can think that the inequality above is an equality. This is true if g is not too irregular. To state this result, we still use Proposition 1. We have to control the asymptotic behavior of the discrete structure functions s_p of F_H. Due to Lemma 4, the value $p = 1$ seems to be a natural choice (see the previous examples). Actually, when one studies a random process, the value $p = 2$ yields exact calculations whereas $p = 1$ often does not.

Proposition 7. *Assume that g is uniformly Hölderian on $\mathbb{R}$ with exponent $\gamma > H$. Then almost surely $s_2(2^{-K}) \simeq (2^{-K})^H$. Thus, we get almost surely* $\dim_B(\Gamma_{F_H}) = D + 1 - H$.

Sketch of proof. (See again [2, 4] for details.) It is the same to prove that $s_2^2(2^{-K}) \simeq 2^{-2KH}$. We proceed in two steps. First we show that this equivalence is true for the expectation of $s_2^2(2^{-K})$. Next we evaluate the distance between $s_2^2(2^{-K})$ and $\mathbb{E}(s_2^2(2^{-K}))$, calculating $Var(s_2^2(2^{-K}))$. This last point is rather complicated, so we do not state it here. □

From now on we study the particular case $D = 1$. We will show how to use Proposition 2. First we have to state a more precise result adapting the previous proof.

Proposition 8. *Assume that g is uniformly Hölderian on $\mathbb{R}$ with exponent $\gamma > H$. Then there exist constants $C_i > 0$ such that for all $u \in [0,1]$ and all $s \geq 1$, there exists an integer $K(u,s)$ such that almost surely, for all $K \geq K(u,s)$:*

$$C_2 2^{-KH} \leq \left(\frac{1}{2^K} \sum_{k=0}^{2^K - 1} \left| F_H\left(\frac{k+1}{2^K} + \frac{u}{2^{sK}} \right) - F_H\left(\frac{k}{2^K} + \frac{u}{2^{sK}} \right) \right|^2 \right)^{1/2} \leq C_1 2^{-KH}.$$

Corollary 2. *Assume that g is uniformly Hölderian on $\mathbb{R}$ with exponent $\gamma > H$. Then almost surely,*

(i) For all $q > 0$, $\sigma(q) = H$
(ii) $\dim_R(\Gamma_{F_H}) = \dim_B(\Gamma_{F_H}) = 2 - H$

Proof. (i) Let us fix $\eta > 0$ and take $s = H/(H - \eta)$. For all $u \in [0, 1]$ written as $u = i2^{-M}$ we call $K(i, M, s)$ the integer given by Proposition 8 and

$$K_M = \max\{K(i, M, s) : i = 0, \ldots, 2^M - 1\}.$$

With this notation we can apply Proposition 2 with $\beta = H - \eta$, $\alpha = H$, $p = 2$, and $\delta = \sigma$. Thus, for all $q > 0$ we have almost surely

$$H - \eta \leq \sigma(q) \leq H + \eta \left(\max\{1, 2/q\} + 1/(q(H - \eta)) - 1\right).$$

Since it is true for all $\eta > 0$ we have the result considering a sequence of values (η_k) tending to 0.

(ii) We get in particular $\sigma(2) = H$, thus we just apply Proposition 3 with $\alpha = H$ and $p = 2$. □

It is possible to use Proposition 2 with $\delta = \rho$ to calculate directly $\dim_R(\Gamma_{F_H})$. However, in this situation hypothesis (ii) is more difficult to check (see [2]).

The methods presented here provide lower bounds for $\dim_B(\Gamma_f)$ and $\dim_R(\Gamma_f)$. Often one directly looks for the Hausdorff dimension $\dim_H(\Gamma_f)$ of the graph. Indeed, the inequality $\dim_H(\Gamma_f) \leq \dim_B(\Gamma_f)$ always holds, so a lower bound for $\dim_H(\Gamma_f)$ is one for $\dim_B(\Gamma_f)$ too. However, this new problem is generally more difficult to solve. For the fractal sums of pulses F_H, the Hausdorff dimension of the graph is also equal to $D + 1 - H$ for a large class of pulses G (see [3]). But for certain cases we cannot prove that $\dim_H(\Gamma_{F_H}) = D + 1 - H$, whereas $\dim_B(\Gamma_{F_H})$ is known (see [2]). Moreover, when $D = 1$ we cannot deduce $\dim_R(\Gamma_f)$ from $\dim_H(\Gamma_f)$ since inequality $\dim_H(\Gamma_f) \leq \dim_R(\Gamma_f)$ is conjectured but not proved.

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Uncertainty Principles, Prolate Spheroidal Wave Functions, and Applications

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Summary. In the literature, the prolate spheroidal wave functions (PSWFs) are often regarded as mysterious set of functions of $L^2(\mathbb{R})$, with no explicit or standard representation and too difficult to compute numerically. Nonetheless, the PSWFs exhibit the unique properties to form an orthogonal basis of $L^2([-1,1])$, an orthonormal system of $L^2(\mathbf{R})$ and an orthonormal basis of B_c, the Paley-Wiener space of $c-$band-limited functions. Recently, there is a growing interest in the computational side of the PSWFs as well as in the applications of these laters in solving many problems from different scientific area, such as physics, signal processing and applied mathematics. In this work, we first give a brief description of the main properties of the PSWFs. Then, we give a detailed study of our two recent computational methods of these PSWFs and their associated eigenvalues. Also, we give a brief description for a composite quadrature based method for the approximation of the values and the eigenvalues of the high frequency PSWFs. In the applications part of this work, we study the quality of approximation by the PSWFs in the space of almost band-limited functions. Moreover, we study the contribution of the PSWFs in the reconstruction of band-limited functions with missing data sets. Finally, we provide the reader with some numerical examples that illustrate the results of this work.

1 Introduction

For a given real number $c > 0$, the prolate spheroidal wave functions (PSWFs) denoted by $(\psi_{n,c}(\cdot))_{n\geq 0}$, have been known since the early 1930s as the eigenfunctions of the Sturm–Liouville operators L_c, defined on $C^2([-1,1])$ by

$$L_c(\psi) = (1-x^2)\frac{d^2\psi}{d\,x^2} - 2x\frac{d\psi}{d\,x} - c^2x^2\psi. \tag{1}$$

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The above operator arises from solving the Helmholtz equation $\Delta\Phi + k^2\Phi = 0$ by the separation of variables method with the use of the prolate spheroidal coordinates given by

$$x = a\xi\eta, \quad y = a\sqrt{(1-\eta^2)(\xi^2-1)}\cos\theta, \quad z = a\sqrt{(1-\eta^2)(\xi^2-1)}\sin\theta,$$

where $|\eta| < 1,\ |\xi| > 1$. A solution $\Phi(\xi,\eta,\theta)$ is given by $\Phi(\xi,\eta,\theta) = S_m(c,\eta)\cdot e^{im\theta}$, where $c = ak/2$ and where $S_m(c,\eta)$ is the angular wave function given by the following Sturm–Liouville differential equation:

$$(1-\eta^2)\frac{d^2S_m(c,\eta)}{d\,\eta^2} - 2\eta\frac{dS_m(c,\eta)}{d\,\eta} + (\chi_n(c) - c^2\eta^2)S_m(c,\eta) = 0. \tag{2}$$

The eigenvalues $\chi_n(c)$ are fixed by the requirement that $S_m(c,\eta)$ is bounded as $|\eta| \to 1^-$. For more details, the reader is referred to [1]. Note that in the special case $m = 0$, the differential operators given by (1) and (2) coincide.

A breakthrough in the theory and the computation of the PSWFs was made in the early 1960s by D. Slepian, H. Pollak, and H. Landau [16,24]. More precisely, starting from a nonclassical version of the uncertainty principles, D. Slepian and his collaborators have discovered that the PSWFs are also the eigenfunctions of a self-adjoint compact integral operator F_c that commutes with L_c and is defined on $L^2([-1,1])$ by

$$F_c(\psi)(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)}\,\psi(y)\,dy. \tag{3}$$

If Q_c denotes the finite Fourier transform operator defined by

$$Q_c : L^2[-1,1] \to [-1,1],\ f \to \int_{-1}^{1} e^{i\,c\,x\,y} f(y)\,dy, \tag{4}$$

then

$$Q_c^*(Q_c f)(x) = \frac{2\pi}{c}\int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)}\,f(y)\,dy = \frac{2\pi}{c}F_c(f)(x).$$

Hence, the $\psi_{n,c}(\cdot)$ are also the eigenfunctions of Q_c, that is,

$$\int_{-1}^{1} e^{i\,c\,x\,y}\psi_{n,c}(y)\,dy = \mu_n(c)\psi_{n,c}(x), \quad \forall\, x \in \mathbf{R}. \tag{5}$$

Moreover, it is shown in [24] that the PSWFs exhibit unique properties to form an orthogonal basis of $L^2([-1,1])$, an orthonormal system of $L^2(\mathbf{R})$, and an orthonormal basis of B_c, the Paley–Wiener space of $c-$band-limited functions defined by $B_c = \left\{f \in L^2(\mathbf{R}),\ \text{Support}\ \widehat{f} \subset [-c,c]\right\}$. Here, $\widehat{f}$ denotes the Fourier transform of f, defined by

$$\widehat{f}(\xi) = \int_{\mathbf{R}} e^{-ix\xi} f(x)\,dx, \quad \xi \in \mathbf{R}.$$

Note that there is no other known system in mathematics that exhibits the previous three properties.

In the literature, the PSWFs are often regarded as a mysterious set of functions with no explicit or standard representation and as being too difficult to compute numerically. Recently, there is a growing interest in the computational side of the PSWFs as well as in their applications in solving many problems from different scientific areas. Some efficient and accurate new numerical methods have been developed for the computation of the PSWFs and their eigenvalues; see, for example, [3, 15, 22, 25, 27]. Also, it is expected that the popularity of the PSWFs will grow in the future, due to their promising applications in many fields. The PSWFs have already found applications in numerical analysis, signal processing, and physics; see, for example, [6–8, 18, 20]. The PSWFs have also found important applications in the theory of random matrices. It is shown in [19] that the eigenvalues of the PSWFs play a crucial role in the asymptotic behavior of the spectrum of the unitary Gaussian random matrices.

In this chapter, we give a brief description of D. Slepian's uncertainty principle and the derivation of the integral operator F_c given by (3). Also, we list the main properties of the PSWFs. Then, we briefly describe the classical Flammer's method for the computation of the PSWFs. Moreover, we give a detailed study of our two recent computational methods of these PSWFs and their associated eigenvalues. The first method is based on an exact matrix representation of the operator Q_c with respect to the basis of $L^2([-1,1])$, given by the set of orthonormal Legendre polynomials. This method can be described as an improved version of a similar method that we have developed in [14]. The second method, see [14], is based on an appropriate Gaussian quadrature method for the operator Q_c. Also, we give a brief description for a composite quadrature-based method for the approximation of the values and the eigenvalues of high frequency PSWFs. In the applications part of this work, we study the quality of approximation by the PSWFs in the space of almost band-limited functions. Moreover, we study the contribution of the PSWFs in the reconstruction of band-limited functions with missing data sets. Finally, we provide the reader with some numerical examples that illustrate the results of this work.

2 Uncertainty Principles and Prolate Spheroidal Wave Functions

2.1 Uncertainty Principles

Let $f \in L^2(\mathbf{R})$ be a function satisfying xf, $\xi\widehat{f} \in L^2(\mathbf{R})$. If

$$x_0 = \frac{1}{\|f\|_2^2}\int_{\mathbf{R}} t\,|f(t)|^2\,dt, \quad \xi_0 = \frac{1}{\|\widehat{f}\|_2^2}\int_{\mathbf{R}} \xi\,|f(\xi)|^2\,d\xi,$$

then the classical Heisenberg uncertainty principle says that $\sigma_f \sigma_{\widehat{f}} \geq \dfrac{\|f\|_2^2}{2}$, where $\sigma_f = \|(x - x_0)f\|_2$, $\sigma_{\widehat{f}} = \|(\xi - \xi_0)\widehat{f}\|_2$. This classical uncertainty principles says that one cannot arbitrarily localize a signal in both time and frequency, around x_0 and ξ_0, respectively. It is has been mentioned in [23] that the above uncertainty principle has little significance for signal processing engineers. In fact, due to the limitation of the measuring devices, the signals encountered in practice are rather band-limited signals belonging to the Paley–Wiener space B_c, for some $c > 0$. A nonclassical uncertainty principle given in [24] is described as follows. If $\tau > 0$, $\|f\|_{2,\tau}^2 = \displaystyle\int_{-\tau}^{\tau} |f(t)|^2\, dt$, and if one considers the practical measure of a signal concentration given by

$$\alpha^2(\tau) = \frac{\|f\|_{2,\tau}^2}{\|f\|_2^2} \quad \text{in the time interval } [-\tau, \tau],$$

then how large is the ratio $\alpha^2(T)$ for $f \in B_\omega$? Since for a given $f \in B_\omega$, we have $f(t) = \mathcal{F}^{-1}(\mathcal{F}f)(t)$, $\forall t \in [-\tau, \tau]$, where $\mathcal{F}$ denotes the Fourier transform operator, then

$$\alpha^2(\tau) = \frac{\int_{-\tau}^{\tau} \left| \int_{-\omega}^{\omega} e^{ity} \widehat{f}(y)\, dy \right|^2 dt}{\int_{-\omega}^{\omega} |\widehat{f}(y)|^2\, dy} = \frac{\int_{-\tau}^{\tau} \int_{-\omega}^{\omega} e^{ity} \widehat{f}(y)\, dy \cdot \int_{-\omega}^{\omega} e^{-itx} \overline{\widehat{f}(x)\, dx}\, dt}{\int_{-\omega}^{\omega} |\widehat{f}(y)|^2\, dy}$$
$$= \frac{\int_{-\omega}^{\omega} \left(\int_{-\omega}^{\omega} \frac{\sin 2\pi\tau(x-y)}{\pi(x-y)} \widehat{f}(y)\, dy \right) \overline{\widehat{f}(x)}\, dx}{\int_{-\omega}^{\omega} \widehat{f}(x) \overline{\widehat{f}(x)}\, dx}.$$

Hence, from the theory of the eigenvalues of integral operators with symmetric kernels, see [[9], pp. 122–125], $\alpha^2(\tau)$ attains its maximum value if and only if it is a solution of the following eigenvalue problem:

$$\int_{-\omega}^{\omega} \frac{\sin 2\pi\tau(x-y)}{\pi(x-y)} \widehat{f}(y)\, dy = \alpha^2(\tau) \widehat{f}(x),\ |x| \leq \omega. \tag{6}$$

With the appropriate change of variables and by using the fact that the solutions of the above equation have analytic extension to $\mathbf{R}$, one concludes that (6) can be rewritten as follows:

$$F_c(\psi)(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} \psi(y)\, dy = \lambda\ \psi(x)\ \ \forall x \in \mathbf{R}. \tag{7}$$

Finally, we briefly describe a third uncertainty principle which is due to D. Donoho and P. Stark, see [11]. This uncertainty principle will be needed in the applications part of this work. The statement of this principle requires the following definition.

Definition 1. *Let T and Ω be two measurable sets. A function pair $(f, \widehat{f})$ is said to be ϵ_T-concentrated in T and ϵ_Ω-concentrated in Ω if*

$$\int_{T^c} |f(t)|^2 \, dt \leq \epsilon_T^2, \qquad \int_{\Omega^c} |\widehat{f}(\omega)|^2 \, d\omega \leq \epsilon_\Omega^2.$$

Theorem 1. *[11]: If $\|f\|_2 = 1$ and $(f, \widehat{f})$ is ϵ_T-concentrated in T and ϵ_Ω-concentrated in Ω, then*

$$|\Omega||T| \geq 2\pi \left(1 - \epsilon_T - \frac{\epsilon_\Omega}{\sqrt{2\pi}}\right)^2.$$

2.2 Properties of the PSWFs and Their Eigenvalues

A breakthrough in the theory of PSWFs was made in the early 1960s by D. Slepian and his collaborators H. Pollack and H. Landau [17, 18, 25, 28]. In particular, D. Slepian has discovered that the integral operator F_c given by (3) commutes with the differential operator L_c given by (1). Hence, the PSWFs originally known as the different eigenfunctions of a Sturm–Liouville differential operator are also the eigenfunctions of the integral operator F_c. It is worthwhile to mention that this last property of the PSWFs was first proved by J. Morrison, see [21]. Moreover, since $F_c = Q_c^* \cdot Q_c$, where Q_c is the Fourier transform operator given by (4), then the PSWFs are also the eigenfunctions of Q_c. Since the kernel $K(x, y) = \dfrac{\sin c(x-y)}{\pi(x-y)}$ associated with the integral operator F_c is symmetric, nondegenerate, and belongs to $L^2([-1, 1])$, then F_c is a self-adjoint compact operator with a nondegenerate kernel. From the basic properties of this kind of operator; see, for example, [9], one can easily conclude that $\rho(F_c)$, the spectrum of F_c, is infinite and countable. Moreover, we have

$$\rho(F_c) = \{\lambda_n(c),\, n \in N;\; 1 > \lambda_0(c) > \lambda_1(c) > \cdots \lambda_n(c) > \cdots\},\ \lim_{n \to +\infty} \lambda_n(c) = 0.$$

If $[x]$ denotes the integer part of x, then it is well known that $(\lambda_n(c))_n$, the spectrum of F_c, has a critical region around $\left[\dfrac{2c}{\pi}\right]$, where the eigenvalues fall from nearly 1 to nearly 0. Moreover, the width of this region is approximately equal to $\log(c)$. More precisely, the behavior of the sequence $(\lambda_n(c))_n$ is given by the following theorem, which has been proven in [17].

Theorem 2. *For any positive real number c, and $0 < \alpha < 1$, the number $N(\alpha)$ of eigenvalues of the operator F_c that are greater than α is given by the following equation:*

$$N(\alpha) = \frac{2c}{\pi} + \left(\frac{1}{\pi^2} \log \frac{1-\alpha}{\alpha}\right) \log(c) + o\left(\log(c)\right).$$

The above behavior of the $(\lambda_n(c)_n$ is illustrated by Fig. 2 at the end of Section 3. Also, note that the sequence of the eigenvalues $(\lambda_n(c))_n$ satisfies the following equality (see [22]):

$$|\lambda_n(c)| = \frac{c^{2n+1}(n!)^4}{2((2n)!)^2\Gamma(3/2+n)^2}e^{G(c)}, \tag{8}$$

where

$$G(c) = 2\int_0^c \left(\frac{2(\psi_{n,t}(1))^2 - 1}{2t} - \frac{n}{t}\right) dt$$

and where $\Gamma(\cdot)$ denotes the gamma function. Numerical evidence shows that for a fixed $c > 0$, $G(c) < 0$. Moreover, by using the above equality together with Wallis's formula and the asymptotic behavior of $\Gamma(n)$ (See, for example, [2]), one gets the following exponential decay estimate of the sequence $(\lambda_n(c))_n$, for large enough n,:

$$\lambda_n(c) \le \beta_n(c), \quad \beta_n(c) \sim ec\, e^{-(2n+1)\log 4}\left(\frac{ce}{n+1/2}\right)^{2n}. \tag{9}$$

We should mention that the super-exponential decay of the $\lambda_n(c)$ has been shown by H. Widom [26]. On the other hand, since the PSWFs are the eigenfunctions of the Sturm–Liouville differential operator (1), then from the basic properties of this kind of operator (see [9]), one concludes that for any integer $n \ge 0$, $\psi_{n,c}$, the nth-order PSWF has the same parity as n. Moreover, $\psi_{n,c}$ has n distinct zeros inside $[-1, 1]$. Also, it has been shown in [24] that for any integers $n, m \ge 0$, we have

$$\int_{-1}^{1} \psi_{n,c}(x)\psi_{m,c}(x)\, dx = \lambda_n(c)\delta_{n,m}, \quad \int_R \psi_{n,c}(x)\psi_{m,c}(x)\, dx = \delta_{n,m}.$$

The main properties of the PSWFs are obtained because of the identity (5). These properties are given by the following theorem.

Theorem 3. *For any positive real number c and any integer $n \ge 0$, $\psi_{n,c}$, the nth order PSWF is a c-band-limited function and its Fourier transform is given by*

$$\widehat{\psi}_{n,c}(\xi) = (-i)^n\sqrt{\frac{2\pi}{c\,\lambda_n(c)}}\,\psi_{n,c}\left(\frac{\xi}{c}\right)\, 1_{[-c,c]}(\xi).$$

Moreover, the set $B = \{\psi_{n,c},\ n \in \mathbf{N}\}$ is an orthogonal basis of $L^2([-1,1])$, an orthonormal system of $L^2(\mathbf{R})$, and an orthonormal basis of B_c, the Paley–Wiener space of c-band-limited functions.

Note that the proof of the different results given in the previous theorem can be found in [24]. Also, we should mention that no other known system in mathematics has all the properties given by the previous theorem.

3 Computation of the PSWFs and Their Eigenvalues

To study the different computational methods of the eigenvalues and the values of the PSWFs, we need some mathematical preliminaries on orthogonal polynomials and Bessel functions. This is the subject of the following subsection. We mention that the material of this subsection is borrowed from [14].

3.1 Mathematical Preliminaries

We first describe a method for the construction of a set of orthogonal polynomials over a given interval $[a, b]$, where a, b are arbitrary real numbers. Then, we describe some properties of these polynomials as well as the Gaussian quadrature method based on them. This set of orthogonal polynomials $\{P_n(x),\ n \geq 0\}$ is given by the following Rodriguez formula:

$$P_n(x) = h_n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(a-x)^n (b-x)^n\right], \quad n \geq 0, \tag{10}$$

where h_n is a normalization constant to be fixed later on. For any integers $0 \leq m \leq n$, n integrations by parts applied to $\displaystyle\int_a^b P_n(x) P_m(x)\, dx$, give us

$$\begin{aligned}\int_a^b P_n(x) P_m(x)\, \mathrm{d}x &= h_n h_m \int_a^b (a-x)^n (b-x)^n \frac{\mathrm{d}^{n+m}}{\mathrm{d}x^{n+m}} \left[(a-x)^m (b-x)^m\right] dx \\ &= \begin{cases} 0 & \text{if } m < n, \\ (h_n)^2 (2n)! \int_a^b (a-x)^n (b-x)^n\, dx & \text{if } m = n. \end{cases}\end{aligned}$$

By using the substitution $t = (a-x)/(a-b)$ inside the last integral, one can easily check that

$$\int_a^b P_n(x)^2\, dx = (2n)!(b-a)^{2n+1} B(n+1, n+1) h_n^2.$$

Here, $B(x, y)$ is the beta function given by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x,\ y > -1. \tag{11}$$

Hence, the normalization constant is simply given by $h_n = \dfrac{\sqrt{2n+1}}{(b-a)^{n+1/2} n!}$ and the orthogonal polynomial over $[a, b]$ of degree n is given by

$$P_n(x) = \frac{\sqrt{2n+1}}{(b-a)^{n+1/2} n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(a-x)^n (b-x)^n\right], \quad n \geq 0. \tag{12}$$

If $a_n > 0$ denotes the highest coefficient of P_n, then $a_n = \dfrac{(2n)!h_n}{n!}$, $n \geq 0$. It is well known (see, for example, [2]) that such orthogonal polynomials satisfy the following three-term recursion relation:

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad n \geq 0. \tag{13}$$

By using (12) and evaluating the coefficients of x^{n+1}, x^n, x^{n+1} from both sides of (13), one can easily check that the coefficients A_n, B_n, C_n given in the previous equality are as follows.

$$A_n = \frac{2}{b-a}\frac{\sqrt{(2n+3)(2n+1)}}{n+1},\ B_n = -\frac{b+a}{b-a}\frac{\sqrt{(2n+3)(2n+1)}}{n+1},$$
$$C_n = \frac{n}{n+1}\sqrt{\frac{2n+3}{2n-1}}, \quad n \geq 1. \tag{14}$$

It is important to mention that $\forall\, n \geq 0$, $P_n(x)$ has n distinct zeros inside $[a, b]$, and these zeros are rapidly and accurately computed as the eigenvalues of the following nth-order tridiagonal symmetric matrix D:

$$D = [d_{i,j}]_{1\leq i,j\leq n}, \quad d_{j,j} = \frac{b+a}{2}, \quad d_{j,j+1} = d_{j+1,j} = -\frac{j(b-a)}{2\sqrt{2j+1}\sqrt{2j-1}}, \tag{15}$$

and $d_{i,j} = 0$ if $i \neq j-1, j, j+1$. Also, note that a Gaussian quadrature method of order $2n$, associated with the orthogonal polynomial $P_n(x)$, is given by

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} \omega_k f(x_k), \tag{16}$$

where $f \in C([a, b])$, and the nodes $(x_k)_{1\leq k\leq n}$ are the different zeros of $P_n(x)$. The weights $(\omega_k)_{1\leq k\leq n}$ are given by the use of the following practical formula:

$$\omega_k = -\frac{a_{n+1}}{a_n}\frac{1}{P_{n+1}(x_k)P_n'(x_k)}, \quad 1 \leq k \leq n. \tag{17}$$

It is well known (see, for example, [2]) that if $f \in C^{2K}([a, b], \mathbf{R})$, then we have the following desirable formula for estimating the error of the quadrature formula (16):

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} \omega_k f(x_k) + \frac{1}{a_n^2}\frac{f^{(2n)}(\eta)}{(2n)!}\int_a^b P_n^2(x)\,dx, \quad a \leq \eta \leq b. \tag{18}$$

Note that the set of the normalized Legendre polynomials $\overline{P_k}$ are special cases of the previous general orthogonal sets of polynomials with $b = 1 = -a$. Hence, the $\overline{P_k}$ are given by the following Rodriguez formula:

$$\overline{P_k}(x) = \sqrt{k+1/2}P_k(x) = \sqrt{k+1/2}\frac{(-1)^k}{2^k k!}\frac{d^k}{dx^k}\left[(1-x^2)^k\right].$$

Here $P_n(x)$ stands for the Legendre polynomials. They satisfy the differential equation

$$(1-x^2)\frac{d^2P_n(x)}{d\,x^2} - 2x\frac{dP_n(x)}{d\,x} = n(n+1)P_n(x), \quad n \geq 0, \quad \forall\, x \in [-1,1]. \tag{19}$$

Also, they satisfy the following identity:

$$\begin{aligned} x^2P_k(x) = {} & \frac{(k+1)(k+2)}{(2k+1)(2k+3)}P_{k+2}(x) + \left(\frac{2k(k+1)-1}{(2k-1)(2k+3)}\right)P_k(x) \\ & + \frac{k(k-1)}{(2k+1)(2k-1)}P_{k-2}(x)\; k \geq 1. \end{aligned} \tag{20}$$

Next, consider a real number $\alpha > -1$; then the Bessel function of the first type and order α denoted by $J_\alpha(\cdot)$ is given by

$$J_\alpha(y) = \left(\frac{y}{2}\right)^\alpha \sum_{j=0}^{\infty}(-1)^j \frac{(y)^{2j}}{4^j\, j!\, \Gamma(\alpha+j+1)}, \quad y \in \mathbf{R}. \tag{21}$$

A useful relation relating Bessel functions and Legendre polynomials is the following (see [10]):

$$\int_{-1}^{1} e^{ixy}\overline{P_n}(y)\, dy = i^n\, \sqrt{n+1/2}\sqrt{\frac{2\pi}{x}}J_{n+\frac{1}{2}}(x), \quad \forall\; x \neq 0. \tag{22}$$

3.2 A Classical Computational Method

In this subsection, we briefly describe the classical Flammer's method for the computation of the PSWFs. The $\psi_{n,c}$ are the bounded eigenfunctions of L_c, that is,

$$\begin{aligned} L_c(\psi_{n,c})(x) &= (1-x^2)\psi''_{n,c}(x) - 2x\psi'_{n,c}(x) - c^2x^2\psi_{n,c}(x) \\ &= \chi_n(c)\psi_{n,c}(x), \quad x \in [-1,1]. \end{aligned} \tag{23}$$

Let $\mathcal{B} = \left\{\overline{P_k},\ k \geq 0\right\}$ be the set of the normalized Legendre polynomials. Since $\psi_{n,c}(\cdot) \in L^2([-1,1])$ and since $\mathcal{B}$ is a Hilbert basis of $L^2([-1,1])$, then the following Legendre expansion of $\psi_{n,c}(x)$ holds:

$$\psi_{n,c}(x) = {\sum_{k=0,1}^{\infty}}' \beta_k^n \overline{P_k}(x). \tag{24}$$

Here, the sign ${\sum_{k=0,1}^{\infty}}'$ means that the sum is over even or odd integers depending on whether the order n is even or odd. By substituting (24) into (23) and by using (19) and (20), one concludes that the different expansion

coefficients $(\beta_k^n)_k$ as well as the corresponding eigenvalue $\chi_n(c)$ are solutions of the following eigensystem:

$$\frac{(k+1)(k+2)}{(2k+3)\sqrt{(2k+5)(2k+1)}}c^2\beta_{k+2}^n + \big(k(k+1) + \frac{2k(k+1)-1}{(2k+3)(2k-1)}\big)c^2\beta_k^n$$
$$\frac{k(k-1)}{(2k-1)\sqrt{(2k+1)(2k-3)}}c^2\beta_{k-2}^n = \chi_n(c)\beta_k^n, \quad k \geq 0.$$

The above computational method of the PSWFs inside $[-1,1]$ is due to C. J. Bouwkamp, [5]. A detailed description of this method can be found in [12]. By inserting (24) into (5) and by using the identity (22), it can be easily shown that the values of the PSWFs outside $[-1,1]$ are computed by the use of the following formula:

$$\psi_{n,c}(x) = \frac{\sqrt{2\pi}}{|\mu_n(c)|}\sum_{k\geq 0}(-1)^k\beta_k^n\frac{J_{k+1/2}(cx)}{\sqrt{cx}}, \quad \forall\, |x| > 1, \tag{25}$$

where

$$\mu_n(c) = \frac{2\pi}{c}\left[\frac{\sum'_{k\geq 0,1} i^k\sqrt{k+1/2}\;\beta_k^n\;J_{k+1/2}(c)}{\sum_{k\geq 0}\beta_k^n\sqrt{k+1/2}}\right]. \tag{26}$$

Next, we develop new practical methods for the computation of the PSWFs and their associated eigenvalues. Unlike the previous classical method, these methods do not depend on the differential operator L_c. Moreover, they are accurate and easy to use. The first method uses the exact matrix representation of the finite Fourier transform operator Q_c, and it is given in the following subsection.

3.3 Matrix Representation of Q_c and PSWFs

We first prove the following technical lemma.

Lemma 1. *Let $k, l \geq 0$ be two integers and let $c > 0$ be a positive real number. Let $a_{kl}(c) =< Q_c(\overline{P_k}), \overline{P_l} >$, where $< \cdot, \cdot >$ denotes the usual $L^2([-1,1])$ inner product. If $l + k$ is odd, then $a_{kl}(c) = 0$; otherwise, we have*

$$a_{kl}(c) = i^k\pi\frac{\sqrt{k+1/2}\sqrt{l+1/2}\,c^k}{2^{2k}}, \tag{27}$$
$$\sum_{n\geq 0}\frac{(\sqrt{c}/2)^{4n}(2n+k)!}{n!(n+(k-l)/2)!\Gamma(n+k+3/2)\Gamma(n+(k+l+3)/2)}.$$

Proof. Since $J_{k+1/2}(cx) = \sum_{n\geq 0}\dfrac{(-1)^n(cx/2)^{2n+k+1/2}}{n!\,\Gamma(n+k+3/2)}$, then by using (22), one gets

$$a_{kl}(c) = \sqrt{\pi}\sum_{n\geq 0}\frac{(-1)^n c^{2n+k}}{2^{2n+k}n!\,\Gamma(n+k+3/2)}\int_{-1}^{1}x^{2n+k}\overline{P_l}(x)\,dx. \tag{28}$$

Next, we check that

$$M_{mj} = \int_{-1}^{1} x^m \overline{P_j}(x)dx = \begin{cases} 0 & \text{if } m < j \text{ or } m-j \text{ is odd} \\ \frac{\sqrt{\pi}m!}{2^m\left(\frac{m-j}{2}\right)!\Gamma\left(\frac{m+j+3}{2}\right)} & \text{if } m-j \geq 0 \text{ and is even.} \end{cases} \tag{29}$$

In fact, since $P_j(x)$ is orthogonal to $\text{Span}\{1, x, \ldots, x^{j-1}\}$, then $M_{mj} = 0$ if $0 \leq m < j$. Moreover, if $m - j$ is a positive odd integer, then $x \to x^m P_j(x)$ is an odd function and, consequently, $M_{mj} = 0$. Next, if $m - j$ is an even positive integer, then we have

$$M_{mj} = \sqrt{j+1/2}\frac{(-1)^j}{2^j j!}\int_{-1}^{1} x^m \frac{d^j}{dx^j}\left[(1-x^2)^j\right]\, dx.$$

m successive integrations by parts applied to the above integral give us

$$\begin{aligned} M_{mj} &= \frac{2m!\sqrt{j+1/2}}{2^j(m-j)!j!}\int_0^1 (x^2)^{(m-j)/2}(1-x^2)^j\, dx, \quad y = x^2, \\ &= \frac{m!\sqrt{j+1/2}}{2^j(m-j)!j!}\int_0^1 y^{(m-j-1)/2}(1-y)^j\, dy \\ &= \frac{m!\sqrt{j+1/2}}{2^j(m-j)!j!}B\left(\frac{m-j+1}{2}, j+1\right). \end{aligned} \tag{30}$$

Here, $B(x, y), x, y > -1$ denotes the beta function given by (11). Since the gamma function satisfies the identity $\Gamma(n+1/2) = \sqrt{\pi}\frac{(2n)!}{2^{2n}n!}$, $n \geq 0$, then by using (11) and (30), together with the previous identity, one obtains (29). Finally, by combining (29) and (28), one gets the desired result (27). □

Next, define the infinite order matrix $A(c) = [a_{kl}(c)]_{k,l\geq 0}$. It can be easily checked that $A(c)$ is nothing but the matrix representation of Q_c with respect to the orthonormal Legendre basis $\mathcal{B}$. Moreover, it has been shown in [15] that the spectrum of Q_c coincides with the spectrum of $A(c)$. It is interesting to mention that for moderate values of the bandwidth c, the coefficients of $A(c)$ have a fast decay to zero. This is given by the following lemma, borrowed from [15].

Lemma 2. *For any $c > 0$ and any positive integers k, l with $\alpha = \max(k, l) \geq c - 1$, we have $|a_{kl}(c)| \leq \frac{c^\alpha}{\alpha!}\frac{1}{2\alpha+1}$.*

For the proof of the above lemma, the reader is referred to [15]. Note that $A(c)$ is a Hilbert–Schmidt matrix. If $(\mu_n(c))_{n\geq 0}$ denotes the infinite set of eigenvalues of $A(c)$, arranged in decreasing order, that is,

$$|\mu_0(c)| > |\mu_1(c)| > \cdots |\mu_n(c)| \cdots,$$

then the different components of the eigenvector $V_n = (d_k^n)_{k\geq 0}$ associated with $\mu_n(c)$ are the coefficients of the Legendre expansion of $\psi_{n,c}(c)$. It is important to mention that computing the complete spectrum of an infinite order matrix is impossible. In practice and since we already knew that the sequence of the eigenvalues decays exponentially to zero, we need only to compute accurate approximations of a finite number of these eigenvalues. For this purpose, we consider a positive integer $N > 1$ and we let $A_N(c)$ be the submatrix of $A(c)$ of order $N+1$ obtained from $A(c)$ by keeping only the first $N+1$ rows and $N+1$ columns from $A(c)$. The following theorem shows that one can get very highly accurate approximations to any finite number of the exact eigenvalues of Q_c.

Theorem 4. *Under the above notation, for any $\epsilon > 0$ and any integer $K > 0$, there exists an integer $N_\epsilon \geq K$, such that*

$$0 \leq |\eta(k)|^2 - |\mu(k)|^2 < \epsilon, \quad \forall\ 0 \leq k \leq K, \tag{31}$$

where $(\eta(k))_{0\leq k\leq N_\epsilon}$ and $(\mu(k))_{k\geq 0}$ are the decreasing sequences of the eigenvalues of $A_{N_\epsilon}(c)$ and Q_c, respectively.

Proof. Let $\mathcal{K}_c(x,y) = e^{icxy}$ denote the kernel of the operator Q_c. Since $\mathcal{K}_c \in L^2([-1,1]^2)$, then Q_c is a Hilbert–Schmidt operator. The infinite matrix $A(c) = [a_{kk'}(c)]_{k,k'\geq 0}$ is the matrix representation of the operator Q_c with respect to the orthonormal basis $\mathcal{B} = \{\overline{P}_k,\ k \geq 0\}$. If $\|Q_c\|_H$ denotes the Hilbert–Schmidt norm of Q_c, then it is well known that

$$\|Q_c\|_H^2 = \|\mathcal{K}_c\|_{[-1,1]^2}^2 = 4. \tag{32}$$

Moreover, it is well known (see [13]) that the Hilbert–Schmidt norm of an operator does not depend on a specific matrix representation of this later. Hence,

$$\|Q_c\|_H^2 = \sum_{k,k'\in\mathbf{N}} |a_{kk'}(c)|^2. \tag{33}$$

Combining (32) and (33), one concludes that $\sum_{k,k'\geq 0} |a_{kk'}(c)|^2 = 4$. Consequently, for any integer $N \in \mathbf{N}$, $\sum_{0\leq k,k'\leq N} |a_{kk'}(c)|^2 < 4$ and $A_N(c)$ is a matrix representation of a finite rank, Hilbert–Schmidt operator Q_c^N. Moreover, we have

$$\|Q_c - Q_c^N\|^2 \leq \|Q_c - Q_c^N\|_H^2 = \sum_{k,k'\geq N+1} |a_{kk'}(c)|^2 \to 0 \quad \text{as} \quad N \to +\infty. \tag{34}$$

Since $Q_c^* \cdot Q_c$, $(Q_c^N)^* \cdot Q_c^N$ are two Hermitian operators and since $\lambda(A(c)^* \cdot A(c)) = \lambda(Q_c^* \cdot Q_c)$, $\lambda((Q_c^N)^* \cdot Q_c^N) = \lambda(A_N^*(c) \cdot A_N(c))$, then Weyl's perturbation theorem gives us

$$\begin{aligned}\max_{0\le k\le N}\left||\eta(k)|^2-|\mu(k)|^2\right| &= \max_{0\le j\le N}|\lambda_j(A_N^*(c)\cdot A_N(c))-\lambda_j(A^*(c)\cdot A(c))|\\ &\le \|(Q_c^N)^*\cdot Q_c^N - Q_c^*\cdot Q_c\|\\ &\le \|Q_c^*\|_H\|Q_c-Q_c^N\|_H+\|Q_c^N\|_H\|Q_c^*-(Q_c^N)^*\|_H\\ &\le 2\|Q_c-Q_c^N\|_H+2\|Q_c^*-(Q_c^N)^*\|_H\\ &\le 4\left(\sum_{k,k'\ge N+1}|a_{kk'}(c)|^2\right)^{1/2}\to 0 \text{ as } N\to+\infty.\end{aligned} \tag{35}$$

Since $(Q_c^N)^*\cdot Q_c^N=\mathcal{P}_N Q_c^*\cdot Q_c\mathcal{P}_N$, where $\mathcal{P}_N$ is the projection operator over $\text{Span}\{P_n(x),\ 0\le n\le N\}$, then the Rayleigh–Ritz theorem gives us

$$0\le |\eta(k)|^2-|\mu(k)|^2,\quad \forall\ 0\le k\le N. \tag{36}$$

Finally, by combining (35) and (36), one obtains the desired result (31). □

Remark 1. For small values of the bandwidth c, the sequence of finite rank operators Q_c^N converges fast to Q_c. Hence, the matrix order N_ϵ given by the above theorem is not large in this case. For example, for $c=\pi$, $\epsilon=10^{-20}$, $K=39$, the matrix order N_ϵ that ensures the computation of the first 40 eigenvalues of Q_π within 10^{-20} precision is given by $N_\epsilon=49$.

3.4 A Quadrature Method for the Computation of the PSWFs

In this subsection, we describe a second method for computing the PSWFs and their eigenvalues. This method is based on a Gaussian quadrature formula applied to the eigenproblem (5). The following theorem, borrowed from [14], provides us with a discretization formula for the eigenproblem (5) as well as an interpolation formula for the eigenfunctions of this later.

Theorem 5. *Consider an integer $n\ge 0$ and an arbitrary real number $0<\epsilon<1$, and let*

$$K_\epsilon=\inf\left\{k\in\mathbf{N},\ \frac{\sqrt{2c}(k!)^4}{(2k+1)(2k)^{4k+3/2}\pi^2}<\epsilon|\mu_n(c)|\right\}. \tag{37}$$

Then for any integer $K\ge\max\left([2ce^3]+1,K_\epsilon\right)$, we have

$$\sup_{|x|\le b}\left|\psi_{n,c}(x)-\frac{1}{\mu_n(c)}\sum_{j=1}^{K}\omega_j e^{icxy_j}\psi_{n,c}(y_j)\right|<\epsilon.$$

Here, $[x]$ denotes the integer part of x; the y_j and the ω_j are the different nodes and weights associated with the Legendre polynomial $P_K(x)$.

Proof. Since for $x \in \mathbf{R}$, we have

$$\psi_{n,c}(x) = \frac{1}{\mu_n(c)} \int_{-1}^{1} e^{icxy} \psi_{n,c}(y)\, dy = \frac{1}{(\mu_n(c))^2} \int_{-1}^{1} \int_{-1}^{1} e^{ic(x+t)y} \psi_{n,c}(t)\, dt\, dy, \tag{38}$$

then, $\forall\ |x| \leq 1$, one obtains

$$\left| \psi_{n,c}(x) - \frac{1}{\mu_n(c)} \sum_{j=1}^{K} \omega_j e^{icxy_j} \psi_{n,c}(y_j) \right|$$
$$\leq \frac{1}{|\mu_n(c)|^2} \int_{-1}^{1} \left| \int_{-1}^{1} e^{ic(x+t)y}\, dy - \sum_{j=1}^{K} \omega_j e^{icy_j(x+t)} \right| |\psi_{n,c}(t)|\, dt. \tag{39}$$

By considering the real and the imaginary parts of $f(y) = e^{icy(x+t)}$, and by using the previous inequality together with formula (18), one gets

$$\begin{aligned}
\mathcal{E}_{K,c} &= \left| \psi_{n,c}(x) - \frac{1}{\mu_n(c)} \sum_{j=1}^{K} \omega_j e^{icxy_j} \psi_{n,c}(y_j) \right| \\
&\leq \frac{1}{|\mu_n(c)|^2} \frac{2(2c)^{2K}}{(2K)!} \frac{1}{a_K^2} \int_{-1}^{1} |\psi_{n,c}(t)|\, dt \\
&\leq \frac{1}{|\mu_n(c)|^2} \frac{2(2c)^{2K}}{(2K)!} \frac{1}{a_K^2} \sqrt{2}\, \|\psi_{n,c}\|_{2,[-1,1]} \\
&\leq \frac{1}{|\mu_n(c)|^2} \frac{2\sqrt{2}(2bc)^{2K}}{(2K)!} \frac{1}{a_K^2} \sqrt{\frac{c}{2\pi}} |\mu_n(c)| = \sqrt{\frac{2}{\pi}} \frac{(2c)^{2K+1/2}}{(2K)!} \frac{1}{a_K^2} \frac{1}{|\mu_n(c)|}.
\end{aligned}$$

By using (12) with $b = 1 = -a$, one can easily check that the coefficient a_K is given by

$$a_K = \frac{\sqrt{K + 1/2}(2K)!}{2^K (K!)^2}. \tag{40}$$

By using (40) together with the previous inequality, one concludes that $\mathcal{E}_{K,c} < \epsilon$, whenever

$$\frac{(4c)^{2K} (\sqrt{2})^3 (K!)^4}{(2K+1)((2K)!)^3} \sqrt{\frac{2c}{\pi}} < \epsilon |\mu_n(c)|. \tag{41}$$

Since Stirling's formula (see [2]) we have

$$(2K)! \geq \sqrt{2\pi} (2K)^{2K+1/2} e^{-2K}, \quad K \geq 1, \tag{42}$$

then straightforward manipulations show that (41) holds for any integer K satisfying the following inequality:

$$\left(\frac{(4ce^3)}{2K} \right)^{2K} \frac{\sqrt{2c}(K!)^4}{(2K+1)(2K)^{4K+3/2}\pi^2} < \epsilon |\mu_n(c)|. \tag{43}$$

Consider the integer K_ϵ given by (37), then from (43), one concludes that, for any integer $K \geq \max([2ce^3] + 1, K_\epsilon)$, we have

$$\sup_{|x|\leq 1} \left| \psi_{n,c}(x) - \frac{1}{\mu_n(c)} \sum_{j=1}^{K} \omega_j e^{icxy_j} \psi_{n,c}(y_j) \right| < \epsilon.$$

The following Weyl's perturbation theorem (see, for example, [4]) will be used frequently in our error analysis studies. □

Theorem 6. *(Weyl's perturbation theorem): Let A and B be two Hermitian matrices of order n. Then*

$$\max_{0\leq j\leq n-1} |\lambda_j(A) - \lambda_j(B)| \leq \|A - B\|.$$

Here,

$$\{|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots |\lambda_{n-1}(A)|\}\,,\ \{|\lambda_0(B)| \geq |\lambda_1(B)| \geq \cdots |\lambda_{n-1}(B)|\}$$

are the rearrangements of the spectrums of A and B, respectively.

The error analysis of our quadrature method for computing the the eigenvalues of Q_c is given by the following theorem.

Theorem 7. *Let $K > 1$ be an integer and let $c > 0$ be a positive real number. Let $(\mu_i(c))_{0\leq i\leq K-1}$ denote the first K eigenvalues of Q_c arranged in decreasing order of their magnitudes. Let $x_1, \ldots, x_K \in [-1, 1]$ and $\omega_1, \ldots, \omega_K$ be the K different nodes and weights corresponding to the Kth-degree Legendre polynomial. Assume that*

$$\sup_{0\leq n\leq K-1} \sup_{1\leq l\leq K} \left| \psi_{n,c}(x_l) - \frac{1}{\mu_n(c)} \sum_{j=1}^{K} \omega_j e^{icx_l x_j} \psi_{n,c}(x_j) \right| \leq \epsilon. \tag{44}$$

Moreover, assume that the matrix $B = [\psi_{l-1}(x_j)]_{1\leq l,j\leq K}$ is nonsingular. Consider the matrix $\widetilde{A}_K = \left[\omega_j e^{icx_l x_j}\right]_{1\leq l,j\leq K}$; then we have

$$\max_{0\leq j\leq K-1} \left| \lambda_j(\widetilde{A}_K^* \widetilde{A}_K) - \lambda_j(Q_c^* Q_c) \right| \leq 4\epsilon(2\sqrt{K} + K\epsilon). \tag{45}$$

Also, let $A_{e,K}$, $A_{o,K}$ be the two Hermitian matrices of order K, defined by

$$\widetilde{A}_{e,K} = [\omega_j \cos(cx_l x_j)]_{1\leq l,j\leq K}\,, \quad \widetilde{A}_{o,K} = [\omega_j \sin(cx_l x_j)]_{1\leq l,j\leq K}\,,$$

where the ω_j, x_j are the weights and the nodes corresponding to a Kth-degree orthogonal polynomial over $[0, 1]$. Assume that

$$\sup_{0\leq n\leq K-1} \sup_{1\leq l\leq K} \left| \psi_{2n,c}(x_l) - \frac{2}{\mu_{2n}(c)} \sum_{j=1}^{K} \omega_j \cos(cx_l x_j) \psi_{2n,c}(x_j) \right| \leq \epsilon, \tag{46}$$

$$\sup_{0\le n\le K-1}\sup_{1\le l\le K}\left|\psi_{2n+1,c}(x_l)-\frac{2}{\mu_{2n+1}(c)}\sum_{j=1}^{K}\omega_j\sin(cx_lx_j)\psi_{2n+1,c}(x_j)\right|\le\epsilon; \tag{47}$$

then, we have

$$\max_{0\le j\le K-1}\left|\lambda_j\left(\widetilde{A}_{e,K}\right)-\lambda_j\left(\frac{1}{2}(Q_c^*+Q_c)\right)\right|\le\epsilon\sqrt{K}, \tag{48}$$

$$\max_{0\le j\le K-1}\left|\lambda_j\left(\widetilde{A}_{o,K}\right)-\lambda_j\left(\frac{1}{2i}(Q_c-Q_c^*)\right)\right|\le\epsilon\sqrt{K}. \tag{49}$$

Proof. To prove (45), we proceed as follows. We first define a Hilbert space E_K by $E_K=\text{Span}\,\{\psi_{0,c},\psi_{1,c},\ldots,\psi_{K-1,c}\}$. Let $Q_{K,c}$, $\widetilde{Q}_c:E_K\to\mathbf{R}^K$ be the two operators defined by

$$Q_{K,c}(f)=\left[\int_{-1}^{1}e^{icx_jy}f(y)\,dy\right]_{1\le j\le K}^{t},\ \widetilde{Q}_{K,c}(f)=\left[\sum_{k=1}^{K}\omega_ke^{icx_jy_k}f(y_k)\right]_{1\le j\le K}^{t}. \tag{50}$$

It is clear that $Q_{K,c}$, $\widetilde{Q}_c$ are Hilbert–Schmidt operators. Also, it is easy to see that

$$Q_{K,c}(\psi_i)=\mu_i(c)\Psi_i,\quad\forall\,0\le i\le K-1, \tag{51}$$

where $\Psi_i=[\psi_i(x_1),\ldots,\psi_i(x_K)]^t\in\mathbf{R}^K$. Note that, by assumption,

$$B=\{\Psi_0,\Psi_1,\ldots,\Psi_{K-1}\}$$

is a basis of $\mathbf{R}^K$. Moreover, since

$$\widetilde{Q}_{K,c}(\psi_l)=\left[\sum_{k=1}^{K}\omega_ke^{icx_jx_k}\psi_l(x_k)\right]_{1\le j\le K}^{t}=\widetilde{A}_K\Psi_l,\quad 0\le l\le K-1,$$

then $\widetilde{A}_K$ is nothing but the matrix representation of $\widetilde{Q}_{K,c}$ with respect to the usual basis of $\mathbf{R}^K$. Next, let's denote by $\|T\|_H$ the Hilbert–Schmidt norm of an operator T. Then, we have

$$\begin{aligned}\|Q_{K,c}-\widetilde{Q}_{K,c}\|^2&\le\|Q_{K,c}-\widetilde{Q}_{K,c}\|_H^2\\&=\sum_{l=0}^{K-1}|\mu_l(c)|^2\cdot\sum_{j=1}^{K}\left[\psi_l(x_j)-\frac{1}{\mu_l(c)}\sum_{k=1}^{K}\omega_ke^{icx_jx_k}\psi_l(x_k)\right]^2\\&\le\sum_{0\le l\le K-1}K\epsilon^2|\mu_l(c)|^2\le K\epsilon^2\sum_{l\ge0}|\mu_l(c)|^2.\end{aligned}$$

Moreover, since $4 = \int_{-1}^{1}\int_{-1}^{1} |e^{icxy}|^2\,dx\,dy = \|Q_c\|_H^2 = \sum_{l\geq 0} |\mu_l(c)|^2$, then the previous inequality implies

$$\|Q_{K,c} - \widetilde{Q}_{K,c}\|_H \leq 2\sqrt{K}\epsilon. \tag{52}$$

Next, let $Q_{K,c}^*, \widetilde{Q}_{K,c}^*$ denote the adjoint of the operators $Q_{K,c}, \widetilde{Q}_{K,c}$, respectively. Since $Q_{K,c}^* Q_{K,c}$, $\widetilde{Q}_{K,c}^* \widetilde{Q}_{K,c}$, are Hermitian operators and since $\lambda(\widetilde{Q}_{K,c}^* \widetilde{Q}_{K,c}) = \lambda(\widetilde{A}_K^* \widetilde{A}_K)$, and $\{\lambda_j(Q_{K,c}^* Q_{K,c}) = |\mu_j(c)|^2,\ j = 0, \ldots, K-1\}$, then by Weyl's perturbation theorem, one gets

$$\begin{aligned}
&\max_{1\leq j\leq K} |\lambda_j(Q_{K,c}^* Q_{K,c}) - \lambda_j(\widetilde{Q}_{K,c}^* \widetilde{Q}_{K,c})| = \max_{1\leq j\leq K} \left||\mu_j(c)|^2 - \lambda_j(\widetilde{A}_K^* \widetilde{A}_K)\right| \\
&\leq \|Q_{K,c}^* Q_{K,c} - \widetilde{Q}_{K,c}^* \widetilde{Q}_{K,c})\|_H \\
&\leq \|Q_{K,c}^*\|_H \cdot \|Q_{K,c} - \widetilde{Q}_{K,c}\|_H + \|\widetilde{Q}_{K,c}\|_H \cdot \|Q_{K,c}^* - Q_{K,c}\|_H \\
&\leq 4\sqrt{K}\epsilon + (2 + 2\sqrt{K}\epsilon)\cdot 2\sqrt{K}\epsilon = 8\sqrt{K}\epsilon + 4K\epsilon^2.
\end{aligned}$$

Next, to prove (48), we consider the $K-$dimensional Hilbert space $E_{e,K} = \mathrm{Span}\{\Psi_{2i},\ 0 \leq i \leq K-1\}$. Then, we consider two Hermitian operators $T_{K,c}, \widetilde{T}_{K,c} : E_{e,K} \to \mathbf{R}^K$, defined by

$$T_{K,c}(f) = \left[\int_0^1 \cos(cx_j y) f(y)\,dy\right]_{1\leq j\leq K}^t$$

and

$$\widetilde{T}_{K,c}(f) = \left[\sum_{k=1}^{K} \omega_k \cos(cx_j x_k) f(x_k)\right]_{1\leq j\leq K}^t.$$

It is clear that $T_{K,c}(\Psi_{2l}) = \dfrac{\mu_{2l}(c)}{2}\Psi_{2l},\ 0 \leq l \leq K-1$, and the matrix $A_{e,K}$ is the matrix representation of $\widetilde{T}_{K,c}$. Hence, we have

$$\begin{aligned}
\|T_{K,c} - \widetilde{T}_{K,c}\|^2 &\leq \|T_{K,c} - \widetilde{T}_{K,c}\|_H^2 \\
&= \sum_{l=0}^{K-1} \frac{|\mu_{2l}(c)|^2}{4} \cdot \sum_{j=1}^{K} \left[\psi_l(x_j) - \frac{2}{\mu_l(c)} \sum_{k=1}^{K} \omega_k \cos(cx_k x_j)\psi_l(x_k)\right]^2 \\
&\leq K\epsilon^2 \sum_{l\geq 0} \frac{|\mu_{2l}(c)|^2}{4} = K\epsilon^2 \int_0^b \int_0^b \cos^2(cxy)\,dx\,dy \leq K\epsilon^2.
\end{aligned}$$

By applying Weyl's perturbation theorem, one concludes that

$$\max_{0\leq j\leq K-1} |\mu_j(A_{e,K}) - \mu_j(T_{K,c})| \leq \sqrt{K}\epsilon.$$

Finally, to get (48), it suffices to remark that

$$\mu_j(T_{K,c}) = \mu_j\left(\frac{Q_c + Q_c^*}{2}\right),\ \forall 0 \le j \le K-1.$$

Similarly, one can easily prove (49).

From the previous theorem, one concludes that if A_K, B_K denote the square matrices of order K, defined by

$$A_K = [\omega_j \cos(cx_iy_j)]_{1\le i,j\le 2K}, B_K = [\omega_j \sin(cx_iy_j)]_{1\le i,j\le 2K+1}, \tag{53}$$

then the sets of the eigenvalues of A_K, B_K are approximations of a finite set of the eigenvalues of the operator Q_c, of even and odd order, respectively. Moreover, for any integer $0 \le n \le 2K+1$, the eigenvector U_n corresponding to the approximate eigenvalue $\mu_n(c)$ is given by $U_n = \left[\widetilde{\psi}_{n,c}(x_i)\right]_{1\le i\le K'}$. Here, $\widetilde{\psi}_{n,c}(\cdot)$ denotes the numerical approximation to the exact PSWFs $\psi_{n,c}(\cdot)$. Finally, to provide approximate values of $\psi_{n,c}(x)$ along the interval $[-1,1]$, we use the interpolation formulae given by (46), (47). □

3.5 Examples

In this subsection, we give two examples that illustrate the results of this section. In the first example, we have considered a bandwidth $c = 10\pi$, then we have applied the techniques of the matrix representation of Q_c and obtained accurate numerical approximations to the exact values of the $\psi_{n,10\pi}$ for various values of n. The graphs of the latter are given in Fig. 1.

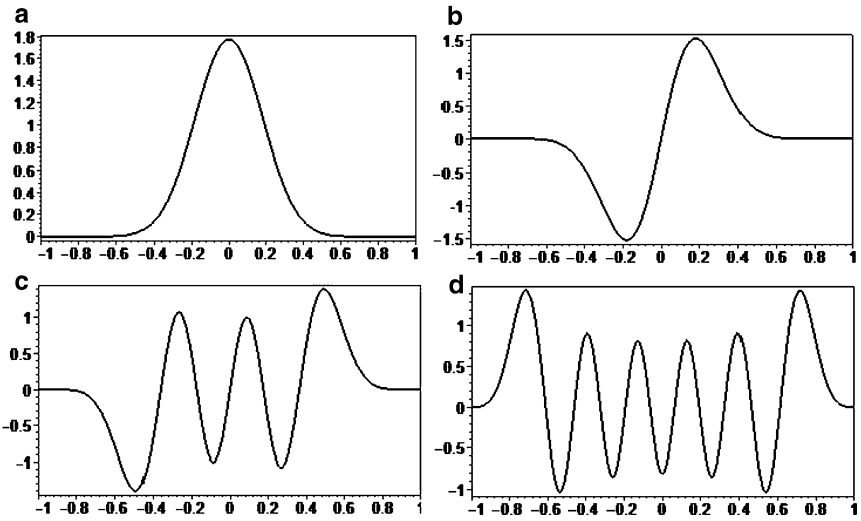

Fig. 1. Graphs of the PSWFs $\psi_{n,c}$, $c = 10\pi$ and with (**a**) $n = 0$, (**b**) $n = 1$, (**c**) $n = 5$, (**d**) $n = 10$

In the second example, we have applied a K-point quadrature-based method for the computation of the eigenvalues $\lambda_n(c)$ with $K = 80$, and computed approximate values of $\lambda_n(c)$, for various values of the bandwidth $20 \leq c \leq 200$ and for different values of the integer n. Some of the obtained numerical results are listed in Table 1. Moreover, the plots of the obtained approximate values of the $(\lambda_n(c))_n$ are given in Fig. 2.

Table 1. Eigenvalues $\lambda_n(c)$ obtained by method 2

$c = 20$		$c = 60$		$c = 100$	
n	$\lambda_n(c)$	n	$\lambda_n(c)$	n	$\lambda_n(c)$
0	1.00000000000000 E-01	0	1.00000000000000 E-00	0	1.00000000000000 E-00
6	9.99998807256537 E-01	24	9.99999999998610 E-01	15	1.00000000000000 E-00
9	9.97432513731077 E-01	32	9.99907810939978 E-01	30	1.00000000000000 E-00
12	5.88793382749218 E-01	35	9.80144818469178 E-01	45	9.99999999999999 E-01
15	7.42123382509520 E-03	40	3.68048271697460 E-02	60	9.86548363597656 E-01
21	2.04865146097706 E-09	48	7.04366904038692 E-09	75	2.15369364968569 E-09
27	1.27516304402105 E-17	59	1.07764362377985 E-20	90	3.29303151559032 E-24
36	3.58084843969160 E-32	64	9.29366143801287 E-27	105	1.26347622109739 E-41

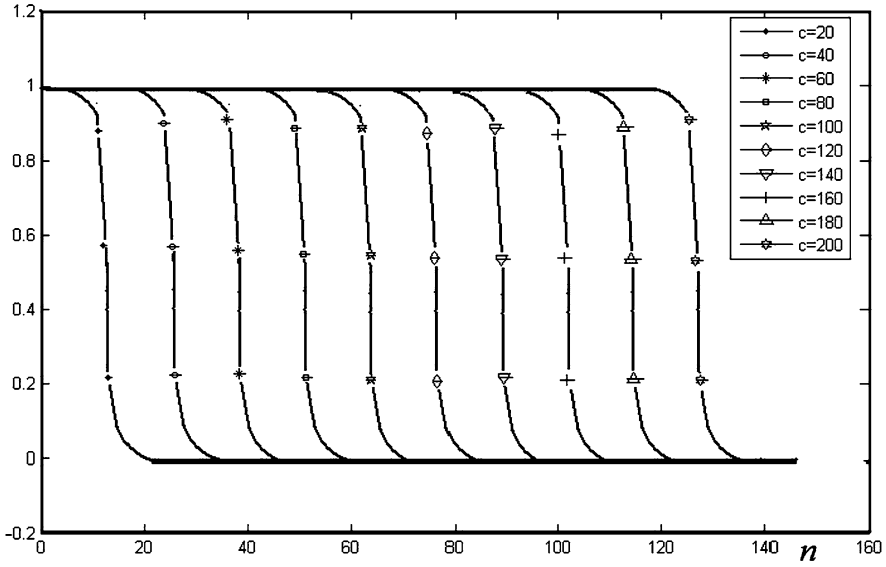

Fig. 2. Graphs of the $\lambda_n(c)$ for various values of c

4 Computation of the Spectrum of High Frequency PSWFs

We first note that the computational methods of the previous section for the approximation of the values and the spectra of the PSWFs are only valid for small or moderate values of the parameter c. For large values of c, the previous methods become impractical. In this section, we describe an efficient quadrature-based method for computing the eigenvalues and the values of the PSWFs $\psi_{n,c}$ with very large values of the bandwidth c. We should mention that most of the content of this part has been taken from [14]. The first step of our high frequency quadrature method consists in converting the eigenvalue problem (5) with large value of the parameter c, into an equivalent problem but with a much smaller bandwidth c_1. More precisely, given $c > 0$, consider a positive real number $b > 0$, then choose a large enough integer M and a small enough bandwidth c_1 such that $c = c_1(Mb)^2$. Denote by ψ_{n,c_1}, $\mu_{n,Mb}(c_1)$, the nth-order PSWFs concentrated in $[-MB, Mb]$ and its corresponding eigenvalue, respectively. Hence, we have

$$\int_{-Mb}^{Mb} e^{ic_1xy}\psi_{n,c_1}(y)\,dy = \mu_{n,Mb}(c_1)\psi_{n,c_1}(x), \quad x \in \mathbf{R}. \tag{54}$$

By considering the substitutions $Y = y/(Mb)$, $X = x/(Mb)$, the above equation is rewritten as follows:

$$\int_{-1}^{1} e^{ic_1(Mb)^2XY}\psi_{n,c_1}(MbY)\,dy = \frac{\mu_{n,Mb}(c_1)}{Mb}\psi_{n,c_1}(MbX), \quad X \in \mathbf{R}. \tag{55}$$

By comparing (5) with (55), one concludes that

$$\psi_{n,c}(x) = \psi_{n,c_1}(Mbx), \quad c = c_1(Mb)^2, \quad x \in \mathbf{R}. \tag{56}$$

Moreover, the eigenvalues of the operators F_c, Q_c, given by (3), (5) are, respectively, given by

$$\mu_n(c) = \frac{\mu_{n,Mb}(c_1)}{Mb}, \quad \lambda_n(c) = \frac{2c_1}{\pi}|\mu_{n,Mb}(c_1)|^2, \quad c = c_1(Mb)^2. \tag{57}$$

Hence, the problem of computing the classical PSWFs and their eigenvalues with large values of the bandwidth c, is reduced to the computation of the values of the low frequency PSWFs ψ_{n,c_1} and their corresponding eigenvalues $\mu_{n,Mb}(c_1)$. Since the ψ_{n,c_1} are either odd or even depending on whether n is odd or even, then we have

$$\int_{0}^{Mb} \cos(c_1xy)\psi_{2n,c_1}(y)\,dy = \frac{\mu_{2n,Mb}(c_1)}{2}\psi_{2n,c_1}(x), \quad \forall\, x \in \mathbf{R}, \quad \forall\, n \geq 0, \tag{58}$$

$$\int_0^{Mb} \sin(c_1 xy)\psi_{2n+1,c_1}(y)\,dy = \frac{\mu_{2n,Mb}(c_1)}{2}\psi_{2n+1,c_1}(x), \quad \forall x \in \mathbf{R}, \quad \forall n \geq 0. \tag{59}$$

We should mention that an important feature of the orthogonal polynomial-based quadrature methods is the possibility to associate a composite rule with these methods. More precisely, given the orthogonal polynomial-based quadrature method over $[0, b]$, given by (16) and (17) with $a = 0$, then for an integer $M \geq 1$, the associated composite quadrature formula over $[0, Mb]$ is given by the following formula:

$$\int_a^{Mb} f(x)\,dx \approx \sum_{k=0}^{M-1}\sum_{j=1}^{K} \omega_j f(x_j + kb), \tag{60}$$

where $(x_j)_{1\leq j\leq K}$ are the different zeros of the Kth-degree orthogonal polynomial $P_K(x)$, given by (12) with $a = 0$. The following theorem borrowed from [14], shows that the previous composite quadrature method is well adapted for the approximation of the ψ_{n,c_1} and consequently for the high frequency PSWFs.

Theorem 8. *Under the above notation, consider an integer $n \geq 0$ an arbitrary real number $0 < \epsilon < 1$, and let*

$$K_{1,\epsilon} = \inf\left\{k \in \mathbf{N},\ \frac{\sqrt{c_1 b^3}}{4\pi^2(2k)^{3/2}(2k+1)}\left(\frac{k!}{(2k)^k}\right)^4 < \frac{\epsilon}{M^{3/2}|\mu_{2n+1,Mb}(c_1)|}\right\}. \tag{61}$$

Then, for $K \geq \max([c_1 b^2 e^3] + 1, K_{1,\epsilon})$, we have

$$\sup_{|x|\leq b}\left|\psi_{2n,c_1}(x) - \frac{2}{\mu_{2n,Mb}(c_1)}\sum_{k=0}^{M-1}\sum_{j=1}^{K}\omega_j \cos(c_1 x(y_j + kb))\psi_{2n,c_1}(y_j + kb)\right| < \epsilon,$$

$$\sup_{|x|\leq b}\left|\psi_{2n+1,c_1}(x) - 2\frac{\sum_{k=0}^{M-1}\sum_{j=1}^{K}\omega_j \sin(c_1 x(y_j + kb))\psi_{2n+1,c_1}(y_j + kb)}{\mu_{2n+1,Mb}(c_1)}\right| < \epsilon. \tag{62}$$

To illustrate the efficiency of the composite quadrature-based method for the computation of the spectrum of high frequency PSWFs, we have considered different large values of the bandwidth c, with $5{,}000 \leq c \leq 8{,}424$. The composite quadrature method is generated by a polynomial $P_K(x)$ with $K = 100$ and orthogonal over $[0, b]$ with $b = 6$. For each value of c, we choose an appropriate positive integer M such that $c = c_1(Mb)^2$, for small enough c_1 and such that the composite quadrature method over M subintervals provides us with accurate approximations of the eigenvalues $\lambda_n(c), 0 \leq n \leq MK - 1 = 100M - 1$. Table 2 lists some values of these $\lambda_n(c)$ where n belongs to the critical region, where the eigenvalues fall from nearly 1 to nearly 0.

Table 2. Values of $\lambda_n(c)$, obtained by the composite quadrature method

$M=30,\ c=5{,}000$		$M=40,\ c=6{,}000$		$M=60,\ c=8{,}424$	
n	$\lambda_n(c)$	n	$\lambda_n(c)$	n	$\lambda_n(c)$
3,158	9.9999999306270E-01	3,798	9.9999999939782E-01	5,338	9.9997136507377E-01
3,165	9.9999997736575E-01	3,818	7.3180875636663E-01	5,358	9.7490851479382E-01
3,178	9.8274366116523E-01	3,825	6.1666389115765E-03	5,365	1.0644733713574E-01
3,185	1.1524774788029E-01	3,838	8.8973445595927E-09	5,378	4.6931509106707E-07
3,198	2.5600404945757E-07	3,845	3.5323962029152E-12	5,385	2.9536474399836E-10
3,205	1.0009200735176E-10	3,858	8.3443768247892E-19	5,398	1.6149072522293E-16
3,218	2.0477425005413E-17	3,865	1.6519672232251E-22	5,405	5.1909303029293E-20
3,225	3.6004818793843E-21	3,878	3.6297236681301E-27	5,418	1.1083806564304E-26

5 Applications of the PSWFs

In this section, we describe two applications of the PSWFs. The first application deals with the quality of approximation by the PSWFs. The second application is the contribution of the PSWFs in the reconstruction of band-limited signals with missing data.

5.1 PSWFs and Quality of Approximation

In this subsection, we study the quality of approximation of band-limited and almost band-limited functions by the PSWFs. Let f be a band-limited function, that is, $f \in B_c$, for some $c > 0$. From the properties of the PSWFs, one concludes that there exists $(a_n)_{n\geq 0} \in \mathbf{C}$ such that

$$f(t) = \sum_{n\geq 0} a_n \psi_{n,c}(t),\ \forall t \in \mathbf{R},\ a_n = \int_{\mathbf{R}} f(t)\psi_{n,c}(t)\,dt, \quad n \in \mathbf{N}.$$

Moreover, since $f \in L^2([-1,1])$, then a PSWF-based expansion formula of f over $[-1,1]$ is given by $f(t)=\sum_{n\geq 0} \alpha_n \psi_{n,c}(t)$. Since $\int_{-1}^{1} \psi_{n,c}\psi_{m,c}=\lambda_n(c)\delta_{nm}$, then

$$\alpha_n = \frac{1}{\sqrt{\lambda_n(c)}} \int_{-1}^{1} f(t)\psi_{n,c}(t)\,dt \leq \|f\|_{L^2(\mathbf{R})}\sqrt{\lambda_n(c)}.$$

Let f_N denote the N-term truncated PSWF series expansion of f, given by $f_N(t) = \sum_{n=0}^{N} \alpha_n \psi_{n,c}(t), \quad t \in [-1,1]$. Since $\lambda_n(c) = O\left(e^{-(2n+1)\log 4(\frac{ce}{n})^{2n}}\right)$, and since $\|\psi_{n,c}\|_\infty \leq \sqrt{c/\pi}$, then f_N converges rapidly to f. More importantly, if f is an almost time- and band-limited function according to Definition 1, then we show that f is well approximated by its truncated PSWFs series expansion. More precisely, Let $T = [-\tau, \tau]$ and $\Omega = [-c, c]$, and define the time-limiting operator P_T and the band-limiting operator P_Ω by

$$P_T(f)(x) = \chi_T(x)f(x), \qquad P_\Omega(f)(x) = \frac{1}{2\pi}\int_{\Omega} e^{ix\omega}\widehat{f}(\omega)\,d\omega.$$

If $\Psi_{n,c}(x) = \frac{1}{\sqrt{\tau}}\psi_{n,c}\left(\frac{x}{\tau}\right)$, $f_N(x) = \sum_{n=0}^{N} \alpha_n \Psi_{n,c}(x)$, then the following proposition borrowed from [14] provides us with the quality of approximation of almost time- and band-limited functions by the PSWFs.

Proposition 1. *If $(f, \widehat{f})$ is ϵ_T-concentrated in T and ϵ_Ω-concentrated in Ω, then for any positive integer N, we have*

$$\|f - P_T(P_T f)_N\|_2 \leq 2\left(\epsilon_T + \frac{\epsilon_\Omega}{\sqrt{2\pi}}\right) + \|f\|_2 \sqrt{\sum_{n \geq N+1} \lambda_n(c)}. \tag{63}$$

In particular, we have

$$\|f - (P_T f)_N\|_{2,T} \leq \left(3\epsilon_T + \sqrt{\frac{2}{\pi}}\epsilon_\Omega\right) + \|f\|_2 \sqrt{\sum_{n \geq N+1} \lambda_n(c)}. \tag{64}$$

5.2 Exact Reconstruction of Band-Limited Functions by the PSWFs

In [11], the authors have shown the uncertainty principle given by Theorem 1. As a consequence of this uncertainty principle, it is shown in [11] that the following band-limited reconstruction problem has a unique solution in B_Ω:

$$\text{Find } S \in B_\Omega \text{ such that } r(t) = \chi_T(t)S(t) \text{ is known.}$$

Here, Ω and T are two measurable sets of $\mathbf{R}$. More precisely, if $\|f\|_2 = 1$ and $(f, \widehat{f})$ is ϵ_T-concentrated in T and ϵ_Ω-concentrated in Ω, then

$$|\Omega||T| \geq 2\pi\left(1 - \epsilon_T - \frac{\epsilon_\Omega}{\sqrt{2\pi}}\right)^2.$$

By using Theorem 1, it is shown in [11] that if $|\Omega||T^c| < 2\pi$, then the previous reconstruction problem has a unique solution. This unique solution S is given by the following formula:

$$S(t) = \sum_{n \geq 0} \left(P_{T^c} P_\Omega\right)^n r(t), \quad t \in \mathbf{R}.$$

The above method has a serious limitation to require the strong condition $|\Omega||T^c| < 2\pi$. In the case where the sets T^c, Ω, are bounded, the PSWFs provide us with a more powerful method for solving the band-limited reconstruction problem. More precisely and without loss of generality, we may assume that $T^c = [-\tau, \tau]$, $\Omega = [-c, c]$, for some $\tau, \omega > 0$. Then, we have

$$P_\Omega P_{T^c}(f)(x) = \int_{-\tau}^{\tau} \frac{\sin c(x-y)}{\pi(x-y)} f(y)\, dy, \quad x \in \mathbf{R}.$$

Hence,

$$\|P_{T^c} P_\Omega\| \leq \lambda_0(c) < 1.$$

Consequently, the band-limited reconstruction problem has a band-limited solution S, no matter how large T^c and Ω are.

5.3 Examples

In the first example, we have considered a bandwidth $c = 100$ and the time-limited and almost band-limited and bad-behaved function f given by $f(x) = \chi_{[-3/4,3/4]}(x)$. Then, we have computed f_N, the truncated PSWF series expansion of f, by its first N even indexed terms with $N = 45$. Figure 3 shows the graphs of f and its relatively good approximation f_N.

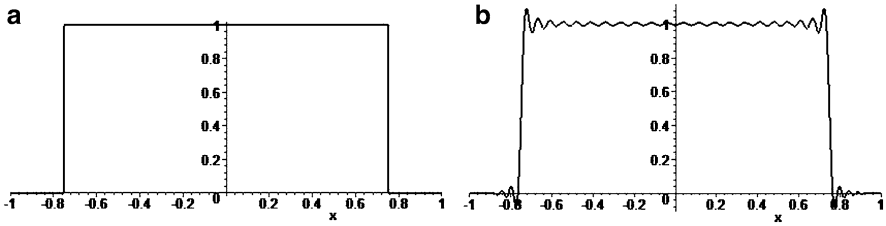

Fig. 3. (**a**) Graph of $f(x)$, (**b**) graph of $f_N(x)$ with $N = 45$

In the second example, we have considered a bandwidth $c = 16\pi$ and the almost band-limited function g given by

$$g(x) = (\cos(25x) + \sin(50x)) \exp(x^2).$$

Then, we have computed g_N, the truncated N-term PSWF series expansion of g, with $N = 52$. Figure 4 shows the graphs of g and its highly accurate approximation g_N. The maximum error norm is given numerically by $\max_{0 \le i \le 100} |g(t_i) - g_N(t_i)| \approx 8.0\ 10^{-5}$, $t_i = -1 + i/50$.

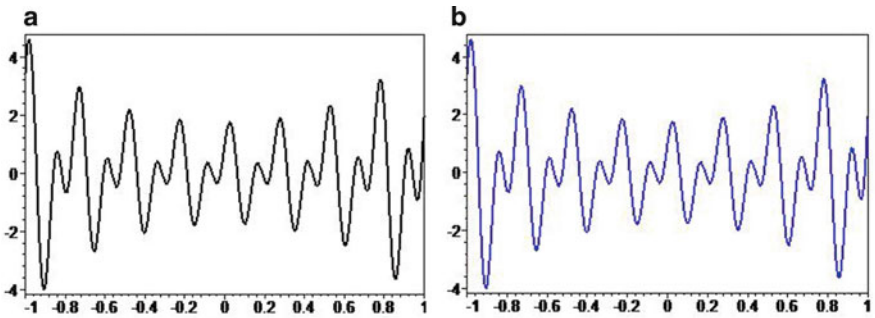

Fig. 4. (**a**) Graph of $g(x)$, (**b**) graph of $g_N(x)$ with $N = 52$

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2-Microlocal Besov Spaces

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Summary. We introduce 2-microlocal Besov spaces which generalize the 2-microlocal spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ by Bony. We give a unified Fourier-analytic approach to define generalized 2-microlocal Besov spaces and we present a wavelet characterization for them. Wavelets provide a powerful tool for studying global and local regularity properties of functions. Further, we prove a characterization with wavelets for the local version of the 2-microlocal Besov spaces and we give first connections and generalizations to local regularity theory.

1 Introduction and Preliminaries

In this paper we introduce 2-microlocal Besov spaces which generalize the 2-microlocal spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ introduced by Bony [4] and Jaffard [8] in two directions. For these spaces, which we call $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$, we give a characterization with wavelets and use this result to describe the local 2-microlocal Besov spaces.

2-Microlocal spaces initially appeared in the book of Peetre [21] and have been studied by Bony [4] in the context of non-linear hyperbolic equations and were widely elaborated by Jaffard and Meyer [9]. In [16] Lévy Véhel and Seuret developed the 2-microlocal formalism, which is similar to the multifractal formalism. It turned out that the 2-microlocal spaces are a useful tool to measure the local regularity of functions. The approach is Fourier analytic and the spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ are defined by size estimates of the Littlewood–Paley decomposition.

More precisely, let φ_0 be a positive function from the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of infinitely differentiable and rapidly decreasing functions with

$$\varphi_0(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2. \end{cases} \tag{1}$$

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We set $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and define $\varphi_j(x) = \varphi(2^{-j}x)$ for $j = 1, 2, \ldots$. Then we have $\sum_{j=0}^{\infty} \varphi_j(x) = 1$, and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is called a smooth dyadic resolution of unity.

The dual space of $\mathcal{S}(\mathbb{R}^n)$ is the space of tempered distributions which we denote by $\mathcal{S}'(\mathbb{R}^n)$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and its inverse on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. We will also use the symbols $\hat{f}$ and $f^\vee$ for $\mathcal{F}f$ and $\mathcal{F}^{-1}f$.

For $f \in \mathcal{S}'(\mathbb{R}^n)$ and a smooth resolution of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$ we have the fundamental decomposition

$$f = \sum_{j=0}^{\infty} (\varphi_j \hat{f})^\vee, \text{ convergence in } \mathcal{S}'(\mathbb{R}^n).$$

A distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ does belong to the space $C^{s,s'}_{x_0}(\mathbb{R}^n)$ if the estimates

$$|(\varphi_j \hat{f})^\vee(x)| \le c 2^{-js}(1 + 2^j|x - x_0|)^{-s'} \tag{2}$$

hold for all $x \in \mathbb{R}^n$ and all $j \in \mathbb{N}_0$. We can reformulate (2) as

$$\sup_{x\in\mathbb{R}^n} w_j(x)|(\varphi_j \hat{f})^\vee(x)| < c 2^{-js}, \tag{3}$$

with the weight sequence

$$w_j(x) = (1 + 2^j|x - x_0|)^{s'}. \tag{4}$$

With the same weight functions w_j from (4) the spaces $H^{s,s'}_{x_0}(\mathbb{R}^n)$ are defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ with

$$c_j^2 = \int_{\mathbb{R}^n} w_j^2(x)|(\varphi_j \hat{f})^\vee(x)|^2 dx \qquad \text{and} \sum_{j\in\mathbb{N}_0} 2^{2js} c_j^2 < \infty. \tag{5}$$

Spaces of this type have been introduced by Bony [4]. A characterization of $C^{s,s'}_{x_0}(\mathbb{R}^n)$ by wavelets has been given by Jaffard in [8]. Wavelets provide a powerful tool for studying the regularity properties of functions, as can be seen in Lévy Véhel and Seuret [16]. They used the wavelet characterization of $C^{s,s'}_{x_0}(\mathbb{R}^n)$ and developed the 2-microlocal formalism. It turned out that 2-microlocal spaces provide a fine way of measuring the local smoothness of distributions. Many regularity exponents, such as the local and pointwise Hölder exponents, the chirp exponent, the oscillating exponent, and the weak scaling exponent, can be derived just by calculating the 2-microlocal domain (see [17] and [16] for details). This 2-microlocal domain is the set

$$E(f, x_0) = \left\{(s, s') \in \mathbb{R}^2 : f \text{ belongs to } C^{s,s'}_{x_0}(\mathbb{R}^n) \text{ locally around } x_0 \right\}.$$

We will introduce a more general 2-microlocal domain in Sect. 3 based on the 2-microlocal Besov spaces $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$ which are defined in the following paragraphs.

Conditions (3) and (5) suggest to consider $C^{s,s'}_{x_0}(\mathbb{R}^n)$ and $H^{s,s'}_{x_0}(\mathbb{R}^n)$ as a kind of weighted Besov space. In general, a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n, w)$ for $s \in \mathbb{R}$ and $0 < p, q \le \infty$, if the (quasi-)norm of f satisfies

$$\left\| f \middle| B^s_{p,q}(\mathbb{R}^n, w) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| (\varphi_j \hat{f})^\vee \middle| L_p(\mathbb{R}^n, w) \right\|^q \right)^{1/q} < \infty, \tag{6}$$

where w is an admissible weight function (see [6]). Here, $L_p(\mathbb{R}^n)$ denotes the usual Lebesgue space, and its weighted version $L_p(\mathbb{R}^n, w)$ is normed by

$$\| f | L_p(\mathbb{R}^n, w) \| = \| wf | L_p(\mathbb{R}^n) \| = \left(\int_{\mathbb{R}^n} |w(x) f(x)|^p dx \right)^{1/p}. \tag{7}$$

Now, it becomes obvious how to modify the definition of the Besov space norm (6) to obtain generalized 2-microlocal Besov spaces. We replace w in (6) by the special weights w_j from (4), depending also on $j \in \mathbb{N}_0$.

We will deal with a further generalization with respect to the weight sequence. Instead of the weights from (4) we introduce the notion of admissible weight sequences.

Definition 1 (Admissible weight sequence). *Let $\alpha \ge 0$ and let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 \le \alpha_2$. A sequence of non-negative measurable functions $\boldsymbol{w} = \{w_j\}_{j=0}^{\infty}$ belongs to the class $\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if, and only if,*

(i) There exists a constant $\mathrm{C} > 0$ such that

$$0 < w_j(x) \le \mathrm{C} w_j(y) \left(1 + 2^j |x - y|\right)^{\alpha} \quad \text{for all } j \in \mathbb{N}_0 \text{ and all } x, y \in \mathbb{R}^n.$$

(ii) For all $j \in \mathbb{N}_0$ and all $x \in \mathbb{R}^n$ we have

$$2^{\alpha_1} w_j(x) \le w_{j+1}(x) \le 2^{\alpha_2} w_j(x).$$

Such a system $\{w_j\}_{j=0}^{\infty} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ is called an admissible weight sequence.

For $U \subset \mathbb{R}^n$ we denote $\operatorname{dist}(x, U) = \inf_{y \in U} |x - y|$ and we define for $s' \in \mathbb{R}$ the 2-microlocal weights by

$$w_j(x) = (1 + 2^j \operatorname{dist}(x, U))^{s'}. \tag{8}$$

These weights are an admissible weight sequence with $\alpha_1 = \min(0, s')$, $\alpha_2 = \max(0, s')$, and $\alpha = |s'|$. Note that for $U = \{x_0\}$ we get the 2-microlocal weights (4) from the beginning. Further examples of admissible weight sequences can be found in [11].

Now, we are able to give the definition of generalized 2-microlocal Besov spaces.

Definition 2. *Let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *and let* $\{\varphi_j\}_{j\in\mathbb{N}_0}$ *be a smooth resolution of unity. Further, let* $0 < p, q \leq \infty$ *and* $s \in \mathbb{R}$*, then*

$$B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\| f \middle| B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w}) \right\| < \infty \right\}, \qquad \textit{where}$$

$$\left\| f \middle| B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w}) \right\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| \left(\varphi_j \hat{f}\right)^{\vee} \middle| L_p(\mathbb{R}^n, w_j) \right\|^q \right)^{1/q}.$$

These spaces have been introduced in [11]. Using a Fourier multiplier theorem for weighted Lebesgue spaces of entire analytic functions ([22, Theorem 1.7.5]), it is easy to show that the definition is independent of the chosen resolution of unity (see Theorem 2.13 in [11]).

If $w_j(x) = 1$ for $j \in \mathbb{N}_0$, then we obtain the usual Besov spaces from (6), studied in detail by Triebel in [24] and [25]. If we set

$$w_j(x) = w_0(x)$$

for all $j \in \mathbb{N}_0$, then we derive weighted Besov spaces $B^s_{p,q}(\mathbb{R}^n, w_0)$, which were studied in [6, Chap. 4].

Regarding the 2-microlocal weight sequence

$$w_j(x) = (1 + 2^j |x - x_0|)^{s'},$$

we get for $p = q = \infty$ the spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ introduced by Jaffard [8] and for $p = q = 2$ we obtain the spaces $H^{s,s'}_{x_0}(\mathbb{R}^n)$ introduced by Bony [4]. With these weight functions, Xu studied in [28] 2-microlocal Besov spaces with $1 \leq p, q \leq \infty$, and in [18] Meyer and Xu used these spaces to characterize chirps by means of their wavelet transforms.

Using as admissible weight sequence the weights from (8) with open $U \subset \mathbb{R}^n$, Moritoh and Yamada introduced in [19] 2-microlocal Besov spaces of homogeneous type and studied local properties of functions.

Taking $w_j(x) = \sigma_j$ for $j \in \mathbb{N}_0$, where $\sigma_j \in \mathbb{R}$ satisfies $c_1\sigma_j \leq \sigma_{j+1} \leq c_2\sigma_j$ for some $c_1, c_2 > 0$, we derive the Besov spaces of generalized smoothness introduced by Kalyabin [10] and studied in [7] and [20]. More generally, we can set

$$w_j(x) = 2^{js(x)}$$

with suitable conditions on $s(x) : \mathbb{R}^n \to \mathbb{R}$ [14], and we obtain spaces of variable smoothness introduced by Underberger and Bokobza [26] and Beauzamy [2] with more recent results due to Leopold [15] and Besov [3].

The above definition with weights satisfying Definition 1 was given in [11] by Kempka, and characterizations by local means, atoms, and wavelets have been established [11, 13]. Moreover, there exists also a characterization by differences of $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$ proved by Besov in [3].

To this end, we define by $\Delta_h f(x) = f(x+h) - f(x)$ and $\Delta_h^M = \Delta_h^{M-1}\Delta_h$ the iterated differences for $x, h \in \mathbb{R}^n$ and $M \in \mathbb{N}$. Two norms $(\|\cdot\|_1, \|\cdot\|_2)$ are called equivalent on a space X if there exists a constant $c > 0$ such that

$$\frac{1}{c}\|x\|_1 \le \|x\|_2 \le c\|x\|_1 \quad \text{for all } x \in X.$$

Proposition 1 (Besov 2003). *Let $1 < p, q \le \infty$, $s > 0$ and $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. If $M > s + \alpha_2$, then*

$$\left(\sum_{k=1}^{\infty} 2^{ks} \sup_{|h|\le 1} \left\| w_k \Delta^M_{2^{-k}h} f \middle| L_p(\mathbb{R}^n)\right\|^q\right)^{1/q} + \| w_0 f | L_p(\mathbb{R}^n)\|$$

is an equivalent norm on $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$.

This corresponds to the time-domain characterization of the local version of $C^{s,s'}_{x_0}(\mathbb{R}^n)$ presented in [23] by Seuret and Lévy Véhel.

Another approach, which is not covered by Definition 2, is to generalize $H^{s,s'}_{x_0}(\mathbb{R}^n)$ as weighted Triebel–Lizorkin spaces. This has been done by Andersson in [1] for the 2-microlocal weights from (4). In a more general context these spaces have been studied with admissible weight sequences from Definition 1 in [14], and local means characterizations have been established.

In the next section we present an adapted wavelet characterization based on Daubechies wavelets for $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$ with weights from (8). Section 3 deals with the local version of these spaces, and we use the results from the previous section to describe them with wavelet decompositions as in [9, Proposition 1.4] and [16, Theorem 1] for $U = \{x_0\}$ and $p = q = 2$ or $p = q = \infty$. Although we do not develop a full regularity theory of functions as in [16], our results seem to be promising for further research.

2 Characterization with Wavelets

In this section we will present a wavelet characterization for $B^{s,mloc}_{p,q}(\mathbb{R}^n, \boldsymbol{w})$ with the weight sequence from (8). In comparison to $C^{s,s'}_{x_0}(\mathbb{R}^n)$ we will denote them by $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$.

The most important characterization of the local spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ is due to the wavelet characterization. To this end, we have to give a modified version of the wavelet characterization in Theorem 4 in [13]. We adopt the notation from [25, 4.2.1]. For sufficiently large $k \in \mathbb{N}_0$, let us assume that

$$\psi_M, \psi_F \in C^k(\mathbb{R}) \tag{9}$$

are real, compactly supported Daubechies wavelets (see [5,27]) with

$$\int_{\mathbb{R}} x^{\beta} \psi_M(x) dx = 0 \qquad \text{for } |\beta| < k \tag{10}$$

and $\mathrm{Supp}\psi_M, \mathrm{Supp}\psi_F \subset B_{2^J}(0)$, with $J \in \mathbb{N}$. Here, $B_r(x)$ denotes the open ball around $x \in \mathbb{R}^n$ with radius $r > 0$. Let $l \in \mathbb{N}_0$, then

$$G = G^{l,l} = \{F, M\}^n \qquad \text{and} \qquad G^{\nu,l} = \{F, M\}^{n*} \quad \text{for } \nu > l,$$

where the * indicates that at least one G_i of $G = (G_1, \ldots, G_n) \in \{F, M\}^{n*}$ must be an M. It is well known that $\{\Psi^{\nu,l}_{Gm} : \nu \geq l,\ G \in G^{\nu,l} \text{ and } m \in \mathbb{Z}^n\}$ is an orthonormal basis of $L_2(\mathbb{R}^n)$ for fixed $l \in \mathbb{N}_0$ with

$$\Psi^{\nu,l}_{Gm}(x) = 2^{\nu\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^\nu x_r - m_r) \quad \text{where } G = (G_1, \ldots, G_n) \in G^{\nu,l}.$$

We have to adapt our sequence spaces to the new situation. A sequence of complex-valued numbers $\{\lambda^{\nu,l}_{Gm}\}$ belongs to $b^{s,s'}_{p,q;l}(U)$ if, and only if,

$$\left\| \lambda | \, b^{s,s'}_{p,q;l}(U) \right\|$$

$$= \left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{\nu,l}_{Gm}|^p (1 + 2^\nu \mathrm{dist}(2^{-\nu} m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty.$$

We introduce the number $\sigma_p = \max(0, n(1/p - 1))$, which is zero if $p \geq 1$. By unconditional convergence of a sum we mean that each rearrangement of the sum converges to the same limit. The next corollary follows from Theorem 4 in [13].

Corollary 1. *Let $U \subset \mathbb{R}^n$ bounded, $s, s' \in \mathbb{R}$, and $l \in \mathbb{N}_0$. Further, let $0 < p, q \leq \infty$ and*

$$k > \max(\sigma_p - s - \min(0, s'), s + \max(0, s')) \tag{11}$$

in (9) and (10). Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ if, and only if, it can be represented as

$$f = \sum_{\nu=l}^{\infty} \sum_{G \in G^{\nu,l}} \sum_{m \in \mathbb{Z}^n} \lambda^{\nu,l}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu,l}_{Gm} \qquad \text{with } \lambda \in b^{s,s'}_{p,q;l}(U), \tag{12}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any $B^{t,t'}_{p,q}(\mathbb{R}^n, U)$ with $t < s$ and $t' < s'$. The representation (12) is unique,

$$\lambda^{\nu,l}_{Gm} = \lambda^{\nu,l}_{Gm}(f) = 2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle, \tag{13}$$

and

$$I : f \mapsto \left\{ 2^{\nu\frac{n}{2}} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle \right\}, \tag{14}$$

is an isomorphic map from $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ onto $b^{s,s'}_{p,q;l}(U)$. Moreover, if in addition $\max(p, q) < \infty$, then $\{\Psi^{\nu,l}_{Gm}\}$ is an unconditional basis in $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$.

The advantage of this representation with additional index $l \in \mathbb{N}_0$ is that the size of the support of the wavelets on the zero level $\nu = l$ is $\mathrm{Supp}\Psi^{l,l}_{Gm} \subset B_{2^{J-l}}(2^{-l}m)$ and can be minimized by taking large $l \in \mathbb{N}_0$.

Remark 1. We assume in the following that the Daubechies wavelets have enough regularity, which means $k > \max(\sigma_p - s - \min(0, s'), s + \max(0, s'))$. Note that in the case $p \geq 1$ this means $k > \max(|s|, |s + s'|)$.

3 The Local Spaces $B^{s,s'}_{p,q}(U)^{loc}$

This section is devoted to the study of the local spaces $B^{s,s'}_{p,q}(U)^{loc}$. They are an appropriate instrument for measuring the local regularity of functions, as has been done intensively by Jaffard and Meyer, Seuret and Lévy Véhel, and many others [9, 16, 18]. We would like to point out some connections to the known case, $p = q = \infty$ and $U = \{x_0\}$, and give first results.

For the rest of the chapter we fix $U \subset \mathbb{R}^n$ as a compact subset and $s, s' \in \mathbb{R}$ and $0 < p, q \leq \infty$ are arbitrary but fixed numbers.

3.1 Definition and Wavelet Characterization

In this subsection we define the local version of $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ for compact $U \subset \mathbb{R}^n$ and give a characterization by wavelets for them.

Definition 3. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then f belongs to the local space $B^{s,s'}_{p,q}(U)^{loc}$ if there exists an open neighborhood $V_0 \supset U$ and $g \in B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ globally such that $f = g$ on V_0.*

From a pointwise multiplier statement for the global spaces $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ (Theorem 4.10 in [11]) we obtain the following.

Lemma 1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in B^{s,s'}_{p,q}(U)^{loc}$ if, and only if, there exists an open neighborhood $V_0 \supset U$ and $\varphi \in C^\infty_0(\mathbb{R}^n)$ with $\varphi(x) = 1$ on V_0 and $\varphi f \in B^{s,s'}_{p,q}(\mathbb{R}^n, U)$.*

Now, we are able to characterize the local spaces $B^{s,s'}_{p,q}(U)^{loc}$ in terms of wavelets.

Theorem 1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then f belongs to $B^{s,s'}_{p,q}(U)^{loc}$ if, and only if, there exists an $l \in \mathbb{N}_0$ and an $A > 0$ with*

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \mathit{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty, \tag{15}$$

where

$$U_\nu = \{m \in \mathbb{Z}^n : \mathit{dist}(2^{-\nu}m, U) \leq A\} \quad \text{and} \quad \lambda^{\nu,l}_{Gm}(f) = 2^{\nu n/2} \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle.$$

Proof. First Step: We have $f \in B^{s,s'}_{p,q}(U)^{loc}$, which means that we can find open sets V_0, V such that $U \subset V_0 \subset V$ and $\varphi \in C^\infty_0(\mathbb{R}^n)$ with $\varphi(x) = 1$ on V_0, Supp$\varphi \subset V$ and $\varphi f \in B^{s,s'}_{p,q}(\mathbb{R}^n, U)$. We choose a number, $-h \in \mathbb{N}_0$ such that $U_{2^h} \subset V_0$, where $U_{2^h} = \{x \in \mathbb{R}^n : \text{dist}(x, U) \leq 2^h\}$. We would like to take these $\Psi^{\nu,l}_{Gm}$ which fulfill

$$\left\langle \varphi f, \Psi^{\nu,l}_{Gm} \right\rangle = \left\langle f, \Psi^{\nu,l}_{Gm} \right\rangle, \tag{16}$$

which means that Supp$\Psi^{\nu,l}_{Gm} \subset U_{2^h} \subset V_0$. This is fulfilled if $\text{dist}(2^{-j}m, U) \leq 2^h - 2^{J-\nu}$. To have a positive number on the right-hand side we have to demand $\nu > J - h$. Now, we fix $l = J - h + 1$ and $A > 0$ by $A = 2^h - 2^{J-l}$. From Corollary 1 we derive that

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{\nu,l}_{Gm}(\varphi f)|^p (1 + 2^\nu \text{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty$$

and that finally gives us with (16) and $U_\nu = \{m \in \mathbb{Z}^n : \text{dist}(2^{-\nu}m, U) \leq A\}$ with $A > 0$ as above

$$\left(\sum_{\nu=l}^{\infty} 2^{\nu(s-n/p)q} \sum_{G \in G^{\nu,l}} \left(\sum_{m \in U_\nu} |\lambda^{\nu,l}_{Gm}(f)|^p (1 + 2^\nu \text{dist}(2^{-\nu}m, U))^{s'p} \right)^{q/p} \right)^{1/q} < \infty.$$

Second step: If we have (15) for some $l \in \mathbb{N}_0$ and $A > 0$, then we can define

$$\tilde{\lambda}^{\nu,l}_{Gm} = \begin{cases} \lambda^{\nu,l}_{Gm}, & \text{for } m \in U_\nu \\ 0, & \text{otherwise.} \end{cases}$$

Then $f = \sum_{\nu,G,m} \tilde{\lambda}^{\nu,l}_{Gm} 2^{-\nu\frac{n}{2}} \Psi^{\nu,l}_{Gm}$ belongs to $B^{s,s'}_{p,q}(\mathbb{R}^n, U)$ by Corollary 1 and this implies $f \in B^{s,s'}_{p,q}(U)^{loc}$. □

Remark 2. Let us emphasize that this theorem is similar to [9, Proposition 1.4] and [16, Theorem 1] in the cases $p = q = \infty$, $p = q = 2$, and $U = \{x_0\}$.

3.2 Embeddings

The aim of this subsection is to present some embedding theorems for the local spaces. These embeddings are well known in the case $p = q = \infty$ and $U = \{x_0\}$.

Lemma 2. *Let $f \in B^{s,s'}_{p,q}(U)^{loc}$, then f belongs to $B^{s-\varepsilon,s'+\varepsilon}_{pq}(U)^{loc}$ for all $\varepsilon > 0$.*

The proof is a simple application of the theorem above. More generally, we can prove the following embedding.

Theorem 2.

$$B^{s,s'}_{p,q}(U)^{loc} \hookrightarrow B^{t,t'}_{p,q}(U)^{loc} \quad \textit{if, and only if,} \quad t \leq s \textit{ and } t + t' \leq s + s'.$$

Proof. The sufficiency of the conditions with respect to the parameters $s, t, s', t' \in \mathbb{R}$ is proved again using Theorem 1. To get the necessity we have to be more careful. The embedding is equivalent to the fact that we can find $l \in \mathbb{N}_0$, $A > 0$ and $c > 0$ such that

$$2^{(t-s)\nu} \leq c(1 + 2^{\nu}\text{dist}(2^{-\nu}m, U))^{s'-t'} \quad \text{holds for all } \nu \geq l \text{ and } m \in U_\nu. \quad (17)$$

We have to distinguish two cases. First, we assume that $s < t$, then for $\nu \geq l$ large enough, we can find $m_\nu \in U_\nu$ with $\text{dist}(2^{-\nu}m_\nu, U) \sim 2^{-\nu}$. This implies that the left-hand side of (17) is increasing in ν. But, the right-hand side is independent of ν, which is a contradiction to (17).

In the second case we assume that $t + t' > s + s'$. Then we take for all $\nu \geq l$ an $m_\nu \in U_\nu$ with $\text{dist}(2^{-\nu}m_\nu, U) \sim A$. We can estimate the right-hand side of (17) by

$$(1 + 2^{\nu}\text{dist}(2^{-\nu}m_\nu, U))^{s'-t'} \leq c2^{\nu(s'-t')} \quad \text{where } c > 0 \text{ is independent of } \nu.$$

This gives us a contradiction to (17), because there does not exist $c > 0$ with $2^{\nu(t-s)} \leq c2^{\nu(s'-t')}$ for all $\nu \geq l$. □

Remark 3. This embedding theorem is in contrast to the global spaces, where we have (Remark 2.3.4 in [12])

$$B^{s,s'}_{p,q}(\mathbb{R}^n, U) \hookrightarrow B^{t,t'}_{p,q}(\mathbb{R}^n, U) \quad \text{if, and only if,} \quad t \leq s \text{ and } t' \leq s'.$$

These results are well known in the case of the local spaces $C^{s,s'}_{x_0}(\mathbb{R}^n)$ ([17, Corollary III/3.4]). Moreover, this theorem is the starting point for the definition of the 2-microlocal frontier, see [17, III.5] and [16, Chap. 2].

3.3 The 2-Microlocal Domain

Similarly as in [17] we give in this subsection a generalized approach to define a 2-microlocal domain for a given function $f \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 4. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then for fixed $0 < p, q \leq \infty$,*

$$E_{p,q}(f, U) = \{(s, s') \in \mathbb{R}^2 : f \in B^{s,s'}_{p,q}(U)^{loc}\}$$

defines the 2-microlocal domain.

We have generalized the 2-microlocal domain from [17] and [16] where case $p = q = \infty$ has been considered. We get from the embedding Theorem 2 the following.

Lemma 3. *Let* $(s, s') \in E_{p,q}(f, U)$ *and let*

$$t \leq s \quad \textit{and} \quad t + t' \leq s + s',$$

then $(t, t') \in E_{p,q}(f, U)$.

Moreover, an easy application of Theorem 1 shows that this domain is convex.

Lemma 4. *The 2-microlocal domain is convex. This means if* $(s, s') \in E_{p,q}(f, U)$ *and* $(t, t') \in E_{p,q}(f, U)$, *then* $(\lambda s + (1 - \lambda)t, \lambda s' + (1 - \lambda)t') \in E_{p,q}(f, U)$ *for all* $\lambda \in [0, 1]$.

Remark 4. This 2-microlocal domain clearly gives us new information about the local regularity of functions (distributions). As a first example we take the delta distribution and $U \subset \mathbb{R}^n$ compact with $0 \in U$. Then we have for $0 < q < \infty$,

$$\delta \in B_{p,q}^{s,s'}(U)^{loc} \Leftrightarrow s < \frac{n}{p} - n,$$

and for $q = \infty$,

$$\delta \in B_{p,\infty}^{s,s'}(U)^{loc} \Leftrightarrow s \leq \frac{n}{p} - n.$$

Hence, one easily recognizes the role played by the parameter p and, less importantly, q.

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Refraction on Multilayers

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à Jacques Peyrière

Summary. Given an unknown number N of transparent plates stacked one onto the other of unknown thicknesses a_k and unknown indices of refraction n_k, it is possible to discover N, a_k, n_k by observing how incident beams of monochromatic light behave on the opposite side, at least if the measurements are very precise. But what if they are not?

1 Precise Measurement

Consider N plates with respective indices of refraction $n_1, n_2, \ldots, n_N$ and respective thicknesses $a_1, a_2, \ldots, a_N$. We assume $n_i \neq n_j$ for $i \neq j$. The plates are represented vertically on Fig. 1.

At some point O of the front face of the system, send a ray of monochromatic light making an angle $\theta \in [0, \pi/2[$, with the x-axis. The Snell–Descartes law of refraction teaches us that the refracted ray penetrates the first layer with angle θ_1,

$$\sin\theta = n_1 \sin\theta_1.$$

If the successive angles within the plates are $\theta_1, \theta_2, \ldots, \theta_n$, then

$$n_i \sin\theta_i = n_{i+1} \sin\theta_{i+1}$$

for $i = 1, 2, \ldots, N-1$. Simple geometrical considerations show that AM denoted $\phi(\theta)$ is equal to

$$\phi(\theta) = \sum_{\ell=1}^{N} a_\ell \frac{\sin\theta}{\sqrt{n_\ell^2 - \sin^2\theta}}.$$

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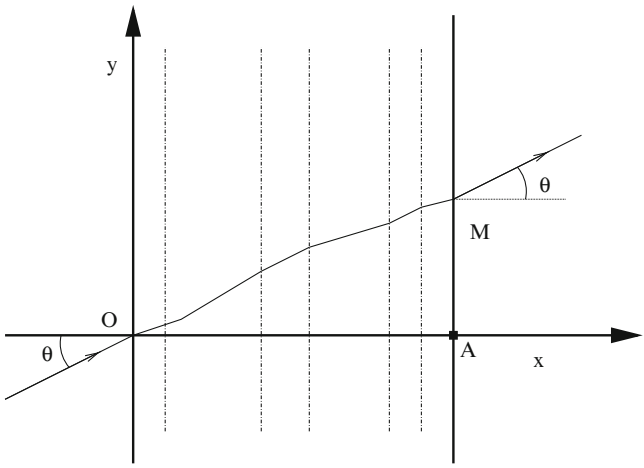

Fig. 1. Ray through plates

Set $\sin\theta = \xi \in [0,1[$, and $\phi(\theta) = \psi(\sin\theta)$. Then

$$\psi(\xi) = \sum_{\ell=1}^{N} a_\ell \frac{\xi}{\sqrt{n_\ell^2 - \xi^2}}.$$

Measuring $\phi(\theta)$ precisely for different θ, i.e., $\psi(\xi)$ for different ξ, allows one to obtain N, n_i, a_i. Indeed, if the n_i are ordered increasingly, then n_1 is the first singularity of ψ, and

$$a_1 = \lim_{\xi \to n_1} \frac{\Psi(\xi)}{\xi} \sqrt{n_1^2 - \xi^2}.$$

Then n_2 is the first singularity of

$$\psi_1(\xi) = \psi(\xi) - \frac{a_1 \xi}{\sqrt{n_1^2 - \xi^2}} = \sum_{\ell=2}^{N} a_\ell \frac{\xi}{\sqrt{n_\ell^2 - \xi^2}}.$$

Inductively, $n_2, n_3, \ldots$ and $a_2, a_3, \ldots$ are obtained, and finally N when there are no more singularities. This is discussed in detail in our paper [1].

2 Fuzzy Measurements and Algebra

Suppose that the measurement of $\phi(\theta)$ is not precise. In particular, suppose that we know the successive odd derivatives $\phi'(0), \phi'''(0), \ldots, \phi^{(2M-1)}(0)$ quite precisely, but that our knowledge of higher derivatives at the origin is somehow

questionable. Can we still get some information concerning the optical system? Note that ϕ is obviously an odd function so that all the derivatives of even order vanish at the origin.

M real numbers $\alpha_1, \alpha_2, \ldots, \alpha_M$ are said be k-algebraically independent if there exist no nonzero polynomials $P \in \mathbb{Z}[X_1, \ldots, X_M]$ of total degree at most k such that $P(\alpha_1, \alpha_2, \ldots, \alpha_M) = 0$.

Theorem 1. *If the M values $\phi'(0), \phi'''(0), \ldots, \phi^{(2M-1)}(0)$ are $M!$ algebraically independent, then the optical system contains at least $N > \frac{M}{2}$ layers.*

Proof. We first observe that the derivatives $\phi^{(\nu)}(0), \nu \leq \nu_0$ are $\mathbb{Q}$-linearly dependent of the derivatives $\psi^{(\nu)}(0), \nu \leq \nu_0$. Therefore, in the theorem we can replace the hypothesis by the independence of $\phi'(0), \phi'''(0), \ldots, \phi^{(2M-1)}(0)$.

Now

$$\psi(\xi) = \sum_{i=1}^{N} a_i \frac{\xi}{\sqrt{n_i^2 - \xi^2}} = \sum_{i=1}^{N} a_i \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{\xi^{2k+1}}{n_i^{2k+1}}$$
$$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \xi^{2k+1} \sum_{i=1}^{N} \frac{a_i}{n_i^{2k+1}} = \sum_{k=0}^{\infty} \frac{\psi^{(2k+1)}(0)}{(2k+1)!} \xi^{2k+1}.$$

Then

$$\frac{\psi^{(2k+1)}(0)}{(2k+1)!} = \binom{-\frac{1}{2}}{k} \sum_{i=1}^{N} \frac{a_i}{n_i^{2k+1}},$$

and finally

$$\sum_{i=1}^{N} \frac{a_i}{n_i^{2k+1}} = \frac{\psi^{(2k+1)}(0)}{(2k+1)!} \binom{-\frac{1}{2}}{k}^{-1}.$$

By hypothesis, the right-hand side is known for $k = 0, 1, \ldots, M-1$. To simplify the notation, set $g_k = \binom{-\frac{1}{2}}{k}^{-1} \frac{\psi^{(2k+1)}(0)}{(2k+1)!}$ and $\nu_i = n_i^{-2}, \alpha_i = \frac{a_i}{n_i}$. Then,

$$\sum_{i=1}^{N} \alpha_i \nu_i^k = g_k, \quad k = 0, 1, \ldots, M-1.$$

Considering the α_i and ν_i as unknowns, the above system consists of M equations with $2N$ unknowns α_i, ν_i. The degree of each one of the equations is respectively $1, 2, \ldots, M-1, M$. A classical theorem (see Perron [2], p. 129, satz 57) implies that the system is necessarily solvable as soon as $P(g_0, \ldots, g_{M-1}) = 0$ for a certain nonzero polynomial P of degree $M!$. Therefore, if no such polynomial exists, N must be larger than $\frac{M}{2}$. □

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Wavelet Shrinkage: From Sparsity and Robust Testing to Smooth Adaptation

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Summary. Wavelet transforms are said to be sparse in that they represent smooth and piecewise regular signals by coefficients that are mostly small except for a few that are significantly large. The WaveShrink or wavelet shrinkage estimators introduced by Donoho and Johnstone in their seminal work exploit this sparsity to estimate or reconstruct a deterministic function from the observation of its samples corrupted by independent and additive white Gaussian noise AWGN. After a brief survey on wavelet shrinkage, this chapter presents several theoretical results, from which we derive new adaptable WaveShrink estimators that overcome the limitations of standard ones, have an explicit close form and can apply to any wavelet transform (orthogonal, redundant, multi-wavelets, complex wavelets, among others) and to a large class of estimation problems. These WaveShrink estimators do not induce additional computational cost that could depend on the application and let the user choose freely the wavelet transform suitable for a given application, in contrast to all parametric methods and also in contrast to some non-parametric methods. In addition, our estimators can be adapted to each decomposition level thanks to known properties of the wavelet transform. The two theoretical results on which our WaveShrink estimators rely are, fist, a new measure of sparsity for sequences of noisy random signals and, second, the construction of a family of smooth shrinkage functions, the so-called Smooth Sigmoid Based Shrinkage (SSBS) functions. The measure of sparsity is based on recent results in non-parametric statistics for the detection of signals with unknown distributions and unknown probabilities of presence in independent AWGN. The SSBS functions allow for a flexible control of the shrinkage thanks to parameters that directly relate to the attenuation wanted for the small, median and large coefficients. The relevance of the approach is illustrated in image denoising, a typical application field for WaveShrink estimators.

1 Introduction

Wavelet transforms are sparse for smooth and piecewise regular signals in the sense that they represent these signals by coefficients whose amplitudes are mostly small except for a few that are significantly large [8, 13]. The

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WaveShrink (wavelet shrinkage) estimator proposed in [8] exploits this sparsity to estimate or reconstruct a deterministic function from the observation of its samples corrupted by independent and additive white Gaussian noise (AWGN). Standard WaveShrink basically involves three steps. The first one consists in transforming the noisy signal via the wavelet transform. The second step is the shrinkage of the noisy signal wavelet coefficients: the small coefficients, that is, those whose amplitudes are less than or equal to a certain threshold height, are forced to zero because they are considered to contain too little information about the signal to recover; the large coefficients, that is, those whose amplitudes are above the threshold height, are considered to pertain to the signal to estimate and, as such, they are processed so as to reduce the influence of noise or are even kept at their values. The third step in WaveShrink is the inverse transform of the shrunk coefficients to obtain the estimate of the signal.

The hard and the soft thresholding functions are the basic shrinkages proposed in [8]. These shrinkage functions are usually adjusted by either the universal or the minimax threshold proposed by Donoho and Johnstone in [8]. In practice, the hard and soft thresholding functions adjusted with these thresholds present severe drawbacks for practical applications: the hard thresholding function induces an important variance and the soft thresholding function, a large bias [5]. Several suggestions have been made to improve the performance of the WaveShrink estimators. On the one hand, in [10] and [2], the authors propose thresholds other than the universal and the minimax ones for WaveShrink by soft shrinkage. On the other hand, several authors have addressed the design of the shrinkage itself: among others, parametric shrinkages (when a priori information is used to parameterize the distribution of the signal) are proposed in [7, 15, 27, 28, 31, 34], whereas non-parametric ones are propounded in [1, 3, 11, 20]. Such contributions have led to numerous wavelet-based methods for image denoising, a typical application field for WaveShrink estimators. Many of these methods use intrascale or interscale predictors and exploit redundancy in the wavelet domain to improve the performance of the shrinkage. However, for denoising applications where the processing of large databases or images requires fast and robust techniques, WaveShrink estimators that are easily portable without extra computational load or specific adaptation are desirable. In this respect, this chapter addresses the design of adaptable WaveShrink estimators that overcome the limitations of standard WaveShrinks, have an explicit close form, and can apply to any wavelet transform (orthogonal, redundant, multi-wavelet, and complex wavelet, among others) and to a large class of estimation problems. The WaveShrink estimators proposed here do not induce additional computational cost that could depend on the application. Moreover, they let the user freely choose the wavelet transform suitable for its application, in contrast to all parametric methods and also in contrast to some non-parametric methods. For instance, albeit

non-parametric, the SURELET[1] of [20] is non-adaptable with respect to the desired denoising level since SURELET parameters are computed via a SURE optimization. In practical applications, this lack of flexibility does not allow for a tuning of the denoising if the SURE optimal parameters are not satisfactory, or if we wish to reduce noise without significantly impacting the signal. In addition, the SURELET cannot be transposed from one wavelet transform to another without additional computation because of the use of interscale predictors for detecting the significant wavelet coefficients.

Our WaveShrink estimators rely on two major points. The first is a theoretical background involving a new way to quantitatively characterize or measure the sparsity of a sequence of noisy random signals. This measure of sparsity is based on recent results in non-parametric statistics for the detection of signals with unknown distributions and unknown probabilities of presence in independent AWGN. As such, this characterization of sparsity leads to a whole family of thresholds, called the *detection thresholds* (DeTs), aimed at distinguishing large from small coefficients. The second point on which our WaveShrink estimator relies concerns a family of smooth shrinkage functions, called the smooth sigmoid-based shrinkage (SSBS) functions. Originally introduced in [3], these shrinkage functions overcome the limitations of standard ones. The main feature of the SSBS functions is that they allow for a flexible control of the shrinkage by using parameters that directly relate to the attenuation wanted for the small, median, and large coefficients. Any SSBS function is adjusted by a threshold whose role, as usual, is to separate large from small coefficients. As such, this threshold height can be a DeT. However, as an extension of [14], it is possible to take into account some specific properties of the wavelet transform that suggest adapting the threshold height to each decomposition level. DeTs can then be calculated at each decomposition level with respect to these properties of the wavelet transform. The resulting thresholds are called adapted detection thresholds (ADeTs). By performing the shrinkage via an SSBS function whose threshold height is an ADeT at each decomposition level, the shrinkage becomes adapted to each decomposition level and overcomes the limitations of standard WaveShrink estimators. Hence, this new type of shrinkage is called smooth adapted WaveShrink with adapted detection thresholds (SAW-ADeTs). Even though the relevance of SAW-ADeTs will hereafter be illustrated in image denoising, note that SAW-ADeTs could apply to other fields of interest such as speech processing, ECG analysis, and so forth.

With respect to its purpose, this chapter is organized as follows. In Sect. 2, we begin with some theoretical background on WaveShrink and the standard soft thresholding. Section 3 focuses on sparsity: on the basis of some results recalled in Sect. 3.2, Sect. 3.3 presents our model for noisy sparse sequences and the definition of DeTs. An application to WaveShrink by soft thresholding

[1]The SURELET uses a SURE [33] optimization on a shrinkage function described as a linear expansion of thresholds (LET). See [20] for details.

is then given in Sect. 3.4. Section 4 begins by recalling the main features of the SSBS functions (see Sect. 4.1) in comparison to standard shrinkage functions that are briefly described. ADeTs are then introduced in Sect. 4.2. We then combine the standard shrinkage functions with the different thresholds (universal, minimax, UniDeT, and ADeTs). The experimental results presented and commented in Sect. 4.3 highlight that the best combination is, as could be expected, SAW-ADeTs, that is, the combination of SSBS with ADeTs. Section 5 concludes this chapter with some perspectives.

2 WaveShrink: Estimation by Sparse Transform and Thresholding

2.1 Background

The material and assumptions introduced in this section will always have the same meaning throughout. Let $\boldsymbol{Z} = \{Z_i\}_{1\leqslant i\leqslant N}$ stand for a sequence of random variables such that $Z_i = f(t_i) + e_i$, $i = 1, 2, \ldots, N$, where f is an unknown deterministic function to be estimated, and the random variables $\{e_i\}_{1\leqslant i\leqslant N}$ are independent and identically distributed (i.i.d), Gaussian with zero mean and variance σ^2: for every $i = 1, 2, \ldots, N$, $e_i \sim \mathcal{N}(0, \sigma^2)$. The problem addressed is the non-parametric estimation of $\boldsymbol{f} = \{f(t_i)\}_{1\leqslant i\leqslant N}$ on the basis of $\boldsymbol{Z}$. The following description of the WaveShrink procedure slightly extends [8] and this extension is clarified below.

[Sparse transform:] A sparse transform, represented by an orthonormal matrix $\mathcal{W}$, is applied to $\boldsymbol{Z}$. This linear transform returns the coefficients

$$Y_i = \theta_i + X_i, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\boldsymbol{Y} = \{Y_i\}_{1\leqslant i\leqslant N} = \mathcal{W}\boldsymbol{Z}$, $\boldsymbol{\theta} = \{\theta_i\}_{1\leqslant i\leqslant N} = \mathcal{W}\boldsymbol{f}$, and $\boldsymbol{X} = \{X_i\}_{1\leqslant i\leqslant N} = \mathcal{W}\boldsymbol{e}$ with $\boldsymbol{e} = \{e_i\}_{1\leqslant i\leqslant N}$. The random variables $\{X_i\}_{1\leqslant i\leqslant N}$ are i.i.d and $X_i \sim \mathcal{N}(0, \sigma^2)$. The transform must be sparse in that only the amplitudes of a few coefficients θ_i are large in comparison to those of the noise coefficients X_i. The discrete wavelet transform (DWT) is a typical example of a sparse transform for smooth or piecewise regular signals [8, 13].

[Shrinkage by thresholding function:] Large coefficients θ_i are less affected by noise than small ones, which may even become negligible in noise, especially, of course, when the noise standard deviation is large. Therefore, it is reasonable to consider that large coefficients θ_i contain the most information about the signal and that small coefficients can be forced to zero without significant loss of the signal. The shrinkage is then performed by a thresholding function $\delta_\lambda(\cdot)$ with threshold height λ: the coefficients Y_i with amplitude less than or equal to λ are considered to be derived from too small, or even null, signal components and, as such, are forced to zero; the coefficients Y_i with amplitude above λ are kept or processed so as to reduce the noise influence.

[Computation of the estimate:] Denoting by $\widehat{\boldsymbol{\theta}} = \{\delta_\lambda(Y_i)\}_{1\leqslant i\leqslant N}$ the outcome of the shrinkage of the coefficients $\{Y_i\}_{1\leqslant i\leqslant N}$ by the thresholding function $\delta_\lambda(\cdot)$, the estimate of $\boldsymbol{f}$ is then $\widehat{\boldsymbol{f}} = \mathcal{W}^{\mathsf{T}}\widehat{\boldsymbol{\theta}}$, where $\mathcal{W}^{\mathsf{T}}$ is the transpose, and thus, the inverse, of $\mathcal{W}$.

The risk function or cost used to measure the accuracy of this estimation procedure is the standard mean-squared error (MSE) $r_{\delta_\lambda}(\boldsymbol{f}, \widehat{\boldsymbol{f}}) = (1/N)\mathbb{E}\|\boldsymbol{f} - \widehat{\boldsymbol{f}}\|^2$ between $\boldsymbol{f}$ and its estimate $\widehat{\boldsymbol{f}}$. In this equality and throughout the rest of the text, given any natural number d and any element $x = (x_1, \ldots, x_d)$ of $\mathbb{R}^d$, $\|x\| = \sqrt{\sum_{k=1}^d x_k^2}$ stands for the standard Euclidean norm of x. In image denoising, instead of the MSE, practitioners prefer using the peak signal-to-noise ratio (PSNR), in decibels, defined by

$$\mathrm{PSNR}_{\delta_\lambda}(\boldsymbol{f}, \widehat{\boldsymbol{f}}) = 10\log_{10}\left(255^2/r_{\delta_\lambda}(\boldsymbol{f}, \widehat{\boldsymbol{f}})\right). \tag{2}$$

Since $\mathcal{W}$ is assumed to be orthonormal, $r_{\delta_\lambda}(\boldsymbol{f}, \widehat{\boldsymbol{f}}) = r_{\delta_\lambda}(\boldsymbol{\theta}, \widehat{\boldsymbol{\theta}})$, where $r_{\delta_\lambda}(\boldsymbol{\theta}, \widehat{\boldsymbol{\theta}}) = \frac{1}{N}\mathbb{E}\|\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}\|^2 = \frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big(\theta_i - \delta_\lambda(Y_i)\Big)^2$.

2.2 WaveShrink by Soft Thresholding

One standard shrinkage function is called the soft thresholding function with threshold height $\lambda \geqslant 0$ defined by

$$\delta_\lambda^S(x) \triangleq \begin{cases} x - \mathrm{sgn}(x)\lambda & \text{if } |x| \geqslant \lambda, \\ 0 & \text{elsewhere,} \end{cases} \tag{3}$$

where $\mathrm{sgn}(x) = 1$ (resp. -1) if $x \geqslant 0$ (resp. $x < 0$). The soft thresholding function is very popular for its appreciable properties of smoothness and adaptation (see [9, 10]). The following lemma gives an upper bound for the estimation risk yielded by this shrinkage function.

Lemma 1. *Given the model of (1), the estimation risk $r_{\delta_\lambda^S}$ for the estimation of $\boldsymbol{\theta}$ by soft thresholding with threshold height $\lambda \geqslant 0$ is such that*

$$r_{\delta_\lambda^S}(\boldsymbol{\theta}, \widehat{\boldsymbol{\theta}}) \leqslant (1 + \lambda^2/\sigma^2) \times \left(\sigma^2 e^{-\lambda^2/2\sigma^2} + r_0(\boldsymbol{\theta})\right),$$

where

$$r_0(\boldsymbol{\theta}) = \frac{1}{N}\sum_{i=1}^N \min\left(\theta_i^2, \sigma^2\right)$$

is called the oracle risk [8].

Proof. See [2]. □

For adjusting the soft thresholding function, two thresholds have been used extensively. On the one hand, the universal threshold is $\lambda_u(N) \stackrel{\Delta}{=} \sigma\sqrt{2\ln N}$. The relevance of this threshold relies on the following fact: since $X_i \overset{\text{iid}}{\sim} \mathcal{N}(0,\sigma^2)$, it follows from [4, (9.2.1), (9.2.2), Sect. 9.2, p. 187] or [21, p. 454], [29, Sect. 2.4.4, p. 91] that

$$\lim_{N\to+\infty} \mathrm{P}\left[\lambda_u(N) - \frac{\sigma\ln\ln N}{\ln N} \leqslant \max\Big\{|X_i|, 1\leqslant i\leqslant N\Big\} \leqslant \lambda_u(N)\right] = 1. \quad (4)$$

The maximum amplitude of $\{X_i\}_{1\leqslant i\leqslant N}$ has, thus, a strong probability of being close to $\lambda_u(N)$ when N is large. According to [8, Theorem 1], or as a straightforward consequence of Lemma 1 above, the risk $r_{\delta^S_{\lambda_u(N)}}(\boldsymbol{\theta},\widehat{\boldsymbol{\theta}})$ for the estimation of $\boldsymbol{\theta}$ by soft thresholding with threshold $\lambda_u(N)$ is such that

$$r_{\delta^S_{\lambda_u(N)}}(\boldsymbol{\theta},\widehat{\boldsymbol{\theta}}) \leqslant (1+2\ln N)\left(N^{-1}\sigma^2 + r_0(\boldsymbol{\theta})\right). \quad (5)$$

On the other hand, the *minimax threshold* $\lambda_m(N)$ is defined as the largest threshold height among those attaining the minimax risk bound $\Upsilon(N) = \inf_{\lambda>0}\sup_{\mu\in\mathbb{R}} \frac{r_{\delta^S_\lambda}(\mu,\widehat{\mu})}{N^{-1}+r_0(\mu)}$. Since we have no close form for the minimax threshold, the risk $r_{\delta^S_{\lambda_m(N)}}(\boldsymbol{\theta},\widehat{\boldsymbol{\theta}})$ is bounded by using result other than Lemma 1: according to [8, Theorem 2], the risk for the estimation of $\boldsymbol{\theta}$ by soft thresholding with threshold $\lambda_m(N)$ is such that

$$r_{\delta^S_{\lambda_m(N)}}(\boldsymbol{\theta},\widehat{\boldsymbol{\theta}}) \leqslant \Upsilon(N)\left(N^{-1}\sigma^2 + r_0(\boldsymbol{\theta})\right). \quad (6)$$

Since $\Upsilon(N) \leqslant 1+2\ln N$ and $\Upsilon(N) \overset{N\to\infty}{\sim} 2\ln N$, it follows from (5) and (6) that the upper bound on $r_{\delta^S_\lambda}(\boldsymbol{\theta},\widehat{\boldsymbol{\theta}})$ is of the same order as $2r_0(\boldsymbol{\theta})\ln N$ when N tends to ∞ and λ is either the universal or the minimax threshold.

3 From Non-Parametric Statistical Decision to Sparsity

3.1 Motivation

Let $\mathcal{T}_\lambda(\cdot)$ be *the thesholding test with threshold height* $\lambda \in [0,\infty)$ defined for every $x \in \mathbb{R}^d$ by

$$\mathcal{T}_\lambda(x) = \begin{cases} 1 & \text{if } |\,x\,| \geqslant \lambda, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Given $\lambda \in [0,\infty)$, the reader must not confuse the thresholding test $\mathcal{T}_\lambda(\cdot)$ with threshold height λ with the soft thresholding function δ^S_λ of Sect. 2.2. However, these two measurable maps relate with each other in $\mathbb{R}$ since $\delta^S_\lambda(x) = \mathcal{T}_\lambda(x)(x-\mathrm{sgn}(x)\lambda)$ for any $x \in \mathbb{R}$. This equation highlights that,

for the estimation by the sparse transform and thresholding function, the primary role of the threshold λ is to decide which coefficients must be processed–because they can reasonably be expected to contain significant information about the signal–and which coefficients must be forced to zero–because they are assumed to contain no or too little information about this same signal. Therefore, the choice of the threshold can be regarded as a statistical decision problem where it is to be decided whether a given coefficient contains significant information about the signal or not. This is the approach initiated in [2] and developed through Sects. 3.2 and 3.4 below. More specifically, Sect. 3.2 begins with a general result, namely, Theorem 1, concerning the statistical detection of a d-dimensional real random vector with unknown distribution and prior in additive and independent standard Gaussian noise with standard deviation σ. By standard Gaussian noise with standard deviation σ, we mean a centered d-dimensional real random vector whose distribution is Gaussian with covariance matrix $\sigma^2\mathbf{I}_d$, where $\mathbf{I}_d$ is the $d\times d$ identity matrix. In Theorem 1, the signal is constrained by two conditions only: its norm or amplitude must be greater than a given bound ϱ and its probability of presence, or prior, must be less than or equal to some bound $\mathsf{p} \leqslant 1/2$. Theorem 1 states two results: first, the existence of a sharp upper bound $V(\varrho, \mathsf{p})$ for the probability of error of the *minimum-probability-of-error* (MPE) test [26, Sect. II.B], that is, the test with the smallest possible probability of error among all possible tests; second, the existence of a thresholding test whose threshold height does not depend on the signal distribution and whose probability of error has the same sharp upper bound $V(\varrho, \mathsf{p})$ as the MPE test. The assumptions of Theorem 1 about the amplitude and the prior of the signal can be regarded as sparsity assumptions. They actually mean that the signal is relatively big and less present than absent. This simple remark is exploited in Sect. 3.3, where we propose a sparsity measure and introduce the notion of detection thresholds (DeTs). The results of [2], briefly summarized in Sect. 3.4, derive from the theoretical framework of Sects. 3.2 and 3.3 when WaveShrink by soft thresholding is adjusted by the *universal detection threshold* (UniDeT), a particular case of DeT.

3.2 Detection of Random Signals with Unknown Distribution and Prior in White Gaussian Noise

We consider the problem of deciding, on the basis of a measurement or observation Y, whether a d-dimensional real random signal Θ is present or absent in a background of additive and independent standard Gaussian noise with standard deviation σ. Binary hypothesis testing is a suitable framework for the description of such problems: the null hypothesis $\mathcal{H}_0$ is that only noise X is present; the alternative hypothesis $\mathcal{H}_1$ is that the observation is the sum $\Theta + X$. For this type of problem, non-randomized tests are sufficient: in what follows, a test will be a measurable map of $\mathbb{R}^d$ into $\{0, 1\}$, the set of the two possible hypotheses. In many practical applications, very little is known about

the signals or most of their describing parameters [16, Sect. I, p. 2232], and the probability distributions of the signals are partially or definitely unknown. This issue is encountered in electronic (warfare) support measures (ESM) systems that must cope with non-cooperative communications. This is also the case with passive sensors such as sonar systems aimed at detecting motor noise or hull vibrations transmitted through a fluctuating environment. For such situations when our lack of prior knowledge prevents the use of the standard Bayes, minimax, and Neyman–Pearson criteria ([17, 19, 26, 35], among others), non-parametric statistical tests are aimed at keeping invariant some *performance characteristic* over a wide range of possible distributions [26, Sect. III.E.1]. Constant false alarm rate (CFAR) detectors [22], commonly used in radar processing, are typical examples of such tests since their purpose is to keep the probability of false alarm at a predefined value regardless of the environment in which the radar is operating.

In contrast with the Neyman–Pearson approach, we assume the existence of the a priori probability of occurrence of hypothesis $\mathcal{H}_1$. If $\mathrm{P}(\mathcal{H}_0)$ and $\mathrm{P}(\mathcal{H}_1)$ stand for the probability of occurrence of hypothesis $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, our performance criterion for the quality of a test $\mathcal{T}$ is the probability of error $\mathrm{P}_e(\mathcal{T}) = \mathrm{P}(\mathcal{H}_0)\mathrm{P}[\mathcal{T}(X) = 1] + \mathrm{P}(\mathcal{H}_1)\mathrm{P}[\mathcal{T}(\Theta + X) = 0]$. In Theorem 1, the importance of these probabilities of occurrence will be significantly reduced by imposing two constraints on the signal: the signal norm must be larger than or equal to a given bound ϱ, and the probability of occurrence must be less than $\mathsf{p} \leqslant 1/2$. In the following statement, $V(\rho, p)$ stands for the function defined for every $\rho \in [0, \infty)$ and every $p \in (0, 1/2]$ by

$$V(\rho, p) = p\mathcal{R}(\rho, \xi(\rho, p)) + (1 - p)\big[1 - \mathcal{R}(0, \xi(\rho, p))\big], \tag{8}$$

where, for any $\lambda \in [0, \infty)$,

$$\mathcal{R}(\rho, \lambda) = \frac{e^{-\rho^2/2}}{2^{d/2-1}\Gamma(d/2)} \int_0^\lambda e^{-t^2/2} t^{d-1} {}_0\mathrm{F}_1(d/2\,;\, \rho^2 t^2/4)\mathrm{d}t, \tag{9}$$

$\xi(\rho, p)$ is the unique solution to x in the equation

$${}_0\mathrm{F}_1(d/2\,;\, \rho^2 x^2/4) = e^{\rho^2/2}\,(1 - p)/p, \tag{10}$$

${}_0\mathrm{F}_1$ is the generalized hypergeometric function [18, p. 275], and Γ is the standard gamma function.

Theorem 1. *Consider the binary hypothesis testing problem*

$$\begin{cases} \mathcal{H}_0 : Y \sim \mathcal{N}(0, \sigma^2 I_d) \\ \mathcal{H}_1 : Y = \Theta + X,\ \Theta \neq 0 \text{ (a.s.)}, \|\Theta\| \geqslant \varrho \geqslant 0 \text{ (a.s.)}, X \sim \mathcal{N}(0, \sigma^2 I_d), \end{cases}$$

where Θ and X are independent and $\sigma > 0$. Assume that the probability of occurrence of hypothesis $\mathcal{H}_1$ is less than or equal to some $\mathsf{p} \leqslant 1/2$.

Then, $V(\varrho/\sigma, \mathsf{p})$ is an upper bound for the probability of error of both the likelihood ratio test with the least probability of error among all possible tests and the threshold test $\mathcal{T}_{\sigma\xi(\varrho/\sigma,\mathsf{p})}$. This bound is reached by both tests when the probability of occurrence of hypothesis $\mathcal{H}_1$ equals p and the distribution of Θ is uniform on the sphere centered at the origin with radius in $\mathbb{R}^d$.

Proof. This result is derived from [24, Theorem VII.1] and [2, Proposition 1]. □

When $d = 1$, the expressions of ξ and V simplify significantly.

Lemma 2. *Let* F *be the cumulative distribution function of the standard normal distribution $\mathcal{N}(0,1)$. When $d = 1$,*

$$\xi(\rho, p) = \frac{\rho}{2} + \frac{1}{\rho}\left[\ln\frac{1-p}{p} + \ln\left(1 + \sqrt{1 - \frac{p^2}{(1-p)^2}e^{-\rho^2}}\right)\right], \tag{11}$$

and

$$V(\rho, p) = p\left[\mathrm{F}\left(\rho + \xi(\rho, p)\right) - \mathrm{F}\left(\rho - \xi(\rho, p)\right)\right] + 2(1-p)\left[1 - \mathrm{F}(\xi(\rho, p))\right] \tag{12}$$

for every $\rho \in [0, \infty)$ and every $p \in [0, 1/2]$.

Proof. The proof is a simple application of the fact that ${}_0\mathrm{F}_1(1/2\,; x^2/4) = \cosh(x)$. □

In the one-dimensional case, the sphere with radius ϱ and centered at the origin is the discrete set $\{-\varrho, \varrho\}$ so that $V(\varrho/\sigma, \mathsf{p})$ is attained by both the Bayes test with the least probability of error and $\mathcal{T}_{\sigma\xi(\varrho/\sigma,\mathsf{p})}$ if $\mathrm{P}[\Theta = \varrho] = \mathrm{P}[\Theta = -\varrho] = 1/2$ and the probability of occurrence is p.

3.3 Sparse Sequences and Detection Thresholds

The assumptions made by Theorem 1 on the norm and the probability of presence of the signal Θ are aimed at bounding our lack of prior knowledge for the statistical decision problem under consideration. In fact, by assuming that Θ has norm equal to or above ϱ and that the probability of occurrence of Θ does not exceed $\mathsf{p} \in [0, 1/2]$, Theorem 1 basically applies to random vectors with arbitrarily large amplitudes and arbitrarily small probabilities of presence. The assumptions of Theorem 1 can then be used to characterize the sparsity of a sequence of random vectors. More specifically, let $\mathcal{I}$ be some countable index and suppose that $\{Y_i\}_{i\in\mathcal{I}}$ is a sequence of d-dimensional real random vectors such that, for every $i \in \mathcal{I}$, Y_i obeys the binary hypothesis model:

$$\begin{cases} \mathcal{H}_{0,i} : Y_i = X_i, \\ \mathcal{H}_{1,i} : Y_i = \Theta_i + X_i, \end{cases} \tag{13}$$

where $\{X_i\}_{i\in\mathcal{I}}$ is a sequence of independent standard random vectors with the same standard deviation $\sigma > 0$, and $\{\Theta_i\}_{i\in\mathcal{I}}$ is a sequence of d-dimensional real random vectors such that, for each $i \in \mathcal{I}$, Θ_i and X_i are independent and $\|\Theta_i\| \neq 0$ (a.s.). In addition, let us make the following assumptions, which extend those of Theorem 1 to every element of the sequence $\{Y_i\}_{i\in\mathcal{I}}$.

[Amp(ϱ)]: There exists $\varrho \geqslant 0$ such that $\|\Theta_i\| \geqslant \varrho$ (a.s.) for every $i \in \mathcal{I}$.

[Occ(p)]: The probability of occurrence of each alternative hypothesis $\mathcal{H}_{1,i}$ is less than or equal to some p in $(0, 1/2]$.

Parameters ϱ and p in assumptions **[Amp(ϱ)]** and **[Occ(p)]** basically specify the sparsity degree of the sequence $\{Y_i\}_{i\in\mathcal{I}}$. In particular, when ϱ is large and p is small in assumptions **[Amp(ϱ)]** and **[Occ(p)]**, vectors with large norms are few in number, which corresponds to the standard notion of sparsity. However, in contrast to the standard notion of sparsity, the model described by (13) and assumptions **[Amp(ϱ)]** and **[Occ(p)]** does not require ϱ to be large and the number of large signal coefficients to be small. Also, this model involves the case where the signal is represented by random vectors.

With this model, the problem of distinguishing the vectors that contain significant information about the signal from those due to noise alone amounts to accepting or rejecting the null hypothesis $\mathcal{H}_{i,0}$ when we observe Y_i, $i \in \mathcal{I}$. According to Theorem 1, the thresholding test $\mathcal{T}_{\sigma\xi(\varrho/\sigma,\mathsf{p})}$ with threshold height $\sigma\xi(\varrho/\sigma, \mathsf{p})$ is appropriate: applied to every Y_i, this test makes it possible to decide which hypothesis, $\mathcal{H}_{i,0}$ or $\mathcal{H}_{i,1}$, is true with a probability of error less than or equal to $V(\varrho/\sigma\,,\,\mathsf{p})$ for any i; if ϱ is large enough, this probability of error becomes very small. For any $\varrho \in [0, \infty)$ and any $\mathsf{p} \in (0, 1/2]$, we then define the detection threshold $\lambda_D(\varrho, \mathsf{p})$ by setting

$$\lambda_D(\varrho, \mathsf{p}) \stackrel{\Delta}{=} \sigma\xi(\varrho/\sigma, \mathsf{p}). \tag{14}$$

3.4 Application to WaveShrink by Soft Thresholding and the Universal Detection Threshold

We now apply the foregoing results to WaveShrink by soft thresholding. The transform $\mathcal{W}$ intervening in WaveShrink is said to be sparse because most of the coefficients it returns for the signal are small, except for a few that are large. These small coefficients may become negligible in comparison with noise. Only a few coefficients are supposed to contain significant information about the signal. Moreover, $\lambda_u(N)$ is regarded as the maximum noise amplitude when N is large enough (see (4)). Therefore, the model adopted in [2] for carrying out the selection of the large coefficients is that the outcome of the sparse transform is a sequence of coefficients obeying the binary hypothesis model of (13) with assumptions **[Amp($\lambda_u(N)$)]** and **[Occ(1/2)]** for $\Theta_i = \theta_i$ for $i \in \mathcal{I} = \{1, \ldots, N\}$. These assumptions are acceptable for modeling the statistical behavior of the wavelet coefficients for smooth or piecewise regular signals. With this sparse model, the decision about the presence or the

absence of significant information about the signal in any given coefficient Y_i amounts to testing the null hypothesis $Y_i \sim \mathcal{N}(0, \sigma^2)$ against the alternative hypothesis $Y_i \sim \mathcal{N}(\theta_i, \sigma^2)$ with $|\theta_i| \geqslant \lambda_u(N)$ when the probability of occurrence of the alternative hypothesis is less than or equal to one half. The DeT suitable for detecting the large coefficients of the sparse sequence satisfying assumptions **[Amp(**$\lambda_u(N)$**)]** and **[Occ(**1/2**)]** is then the universal detection threshold (UniDeT) defined by

$$\lambda_D^*(N) \stackrel{\Delta}{=} \lambda_D(\lambda_u(N), 1/2), \tag{15}$$

which follows from (14). The probability of error of the thresholding test $\mathcal{T}_{\lambda_D^*(N)}$ with threshold height $\lambda_D^*(N)$ is then less than or equal to $V(\sqrt{2 \ln N}, 1/2)$, computed according to (12). This probability of error decreases with N (see Table 1 in [2]). For small values of N, the value of $\lambda_D^*(N)$ is close to that of the minimax threshold; for large values of N (above or equal to 2,048, see Table 2 in [2]), it is about $\lambda_u(N)/2$ and smaller than the minimax threshold. In practice, **[Amp(**$\lambda_u(N)$**)]** and **[Occ(**1/2**)]** are not fulfilled when the signal is not smooth enough or not sufficiently regular. In this respect, the following result [2] gives a bound, which does not depend on assumptions **[Amp(**$\lambda_u(N)$**)]** and **[Occ(**1/2**)]**, on the risk for the estimation of $\boldsymbol{\theta}$ by WaveShrink with soft thresholding $\delta^S_{\lambda_D^*(N)}(\cdot)$.

Proposition 1. *With respect to the model of* (1)*, assume that* $N \geqslant 2$*. The risk* $r_{\delta^S_{\lambda_D^*(N)}}$ *for the estimation of* $\boldsymbol{\theta}$ *by soft thresholding with UniDeT is such that* $r_{\delta^S_{\lambda_D^*(N)}}(\boldsymbol{\theta}, \widehat{\boldsymbol{\theta}}) \leqslant (\ln N/2 + \eta(N)) \left(\sigma^2 \zeta(N) + r_0(\boldsymbol{\theta})\right)$*, with* $\eta(N) = 1 + \ln\left(1 + \sqrt{1 - 1/N^2}\right) + \ln^2\left(1 + \sqrt{1 - 1/N^2}\right)/2 \ln N$*, and* $\zeta(N) = N^{-1/4}(1 + \sqrt{1 - 1/N^2})^{-1/2} \exp\left(-\ln^2(1 + \sqrt{1 - 1/N^2})/4 \ln N\right)$.

Proof. The proof is a application of Lemma 1 when the threshold height is $\lambda_D^*(N)$. □

According to [8, Theorem 3], $2r_0(\boldsymbol{\theta}) \ln N$ is the optimal order for the upper bound on the estimation risk when diagonal estimators such as the soft thresholding are used and $\theta \in \mathbb{R}^N$. However, by focusing on a subclass of signals, a subclass sufficiently large to contain many or even most of the signals met in practice, the upper bound on the estimation risk when soft thresholding is adjusted with UniDeT is proved to be from about twice to four times smaller than $2r_0(\boldsymbol{\theta}) \ln N$. This value is the order of the risk upper bounds obtained by using either the universal or the minimax threshold. In this respect, UniDeT performs better than these standard thresholds on the subclass under consideration. This theoretical result is experimentally verified: for a large class of synthetic signals and standard images pertaining to the subclass of interest, UniDeT actually achieves smaller risks for the

estimation by soft thresholding than the universal and the minimax thresholds. The reader is asked to refer to [2] for further details. In this section, we summarize them as follows.

The minimax and the universal thresholds are suitable for recovering smooth signals. In contrast, UniDeT is suitable for estimating less smooth signals, including piecewise regular signals, which are known to be over-smoothed by the minimax and the universal thresholds. In fact, wavelet representations of smooth signals are very sparse in the sense given by [8]: for such signals, large coefficients are indeed very few in number. On the other hand, large coefficients are not very few for natural images, which are piecewise regular rather than smooth. In such cases, assumption **[Occ(1/2)]** is appropriate and UniDeT performs better than the standard minimax and universal thresholds. The interested reader may refer to the many experimental results given in [2]. To illustrate the discussion above, an example of image denoising is given in the first row of Fig. 6 (Sect. 4.3). The noise standard deviation is $\sigma = 35$. As can be seen, the image denoised by soft thresholding with UniDeT is sharper than that obtained by soft thresholding with minimax or universal thresholds. Moreover, the contours of the original image are better restored in the image returned by soft thresholding with UniDeT than in the other two.

4 Smooth Adapted WaveShrink with Adapted Detection Thresholds

For applications in image denoising, the zero-forcing performed by the standard WaveShrink estimators is detrimental because it induces visual artifacts. In addition, the WaveShrink discussed above does not take into account that the proportion of significant coefficients of smooth and piecewise regular signals increases [21, Sect. 10.2.4, p. 460] and that the amplitudes of the signal detail coefficients tend to decrease, when the decomposition level increases (see [21, Theorem 6.4, p. 171], for instance).

These limitations are now addressed in order to significantly improve the WaveShrink estimator. Specifically, in the next section, after discussing the zero-forcing detrimental effects, we describe a family of shrinkage functions, namely the smooth sigmoid-based shrinkage (SSBS) functions. The main feature of these functions is that they do not force small coefficients to zero but attenuate them less than large ones. In Sect. 4.2, we present how WaveShrink, either based on standard shrinkage functions or on SSBS functions, can be adapted very easily to the aforementioned properties of the wavelet transforms. In fact, since we know that DeTs perform better than the standard universal and minimax thresholds, we adapt the DeTs to the properties of the wavelet decompositions and introduce the adapted detection thresholds (ADeTs). At each decomposition level, the ADeT is a DeT adapted to this decomposition level and enables as to distinguish large from small coefficients

at this specific decomposition level. The shrinkage is thus performed differently with respect to the decomposition level. Among the several WaveShrink estimators that can be constructed by adjusting a shrinkage function with ADeTs, the smooth adapted WaveShrink with ADeTs (SAW-ADeTs) is obtained by combining an SSBS function with ADeTs. The experimental results presented to conclude this section highlight the relevance of combining SSBS Functions with DeTs, UniDeTs, or ADeTs.

4.1 SSBS Functions

The soft and hard thresholding functions [8] involve forcing to zero the coefficients with amplitudes lower than their specified threshold, while any coefficient with amplitude above this threshold is kept by hard thresholding and shrunk by soft thresholding. The discontinuities of the hard thresholding function induce an important variance of the estimate. On the other hand, the soft thresholding function is continuous, but attenuates large coefficients, which results in an over-smoothing and an important bias for the estimated signal [11]. The non-negative garrote (NNG) function [11] and the smoothly clipped absolute deviation (SCAD) function [1] have been proposed to limit this drawback. The graphs of these functions are given in Fig. 1. Basically, the NNG and SCAD functions achieve a certain trade-off between the hard and the soft thresholding functions.

However, the zero-forcing performed by each of these functions generates a significant variance of the estimation because of the sensitivity of the inverse wavelet transform. Moreover, natural images tend to be piecewise regular rather than smooth. As a consequence, the wavelet representation of these images fails to be sparse enough: large coefficients are not very few, whereas textures and contours are characterized by many small coefficients. Therefore, it can be useful to process the small coefficients as well because these small coefficients can contain relevant information about the signal. More specifically, according to the foregoing discussion and the one in [3], a suitable shrinkage function should satisfy the following properties:

(P1) Smoothness: the shrinkage function should reduce the shrinkage variability among the data with close values;

(P2) Penalized shrinkage: a strong (resp. a weak) attenuation should be imposed to small (resp. large) data.

The SSBS functions, originally introduced in [3], basically satisfy properties **(P1–2)**. Every SSBS function depends on a threshold aimed at distinguishing small from large coefficients so as to process differently these two kinds of data.

The SSBS functions are the family of real-valued functions defined by [3]

$$\delta_{t,\tau,\lambda}(x) = \frac{\operatorname{sgn}(x)(|x| - t)_+}{1 + e^{-\tau(|x|-\lambda)}}, \tag{16}$$

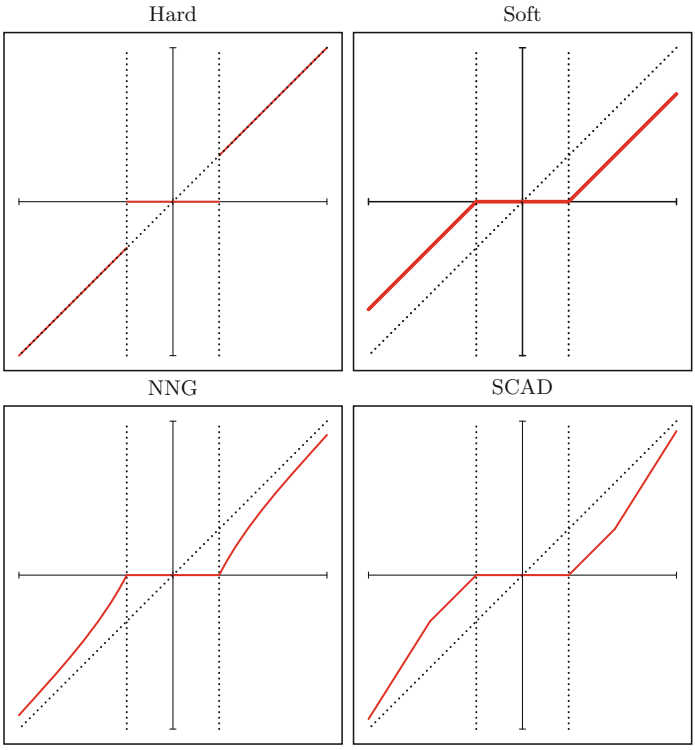

Fig. 1. Standard non-parametric WaveShrink functions with explicit close forms

for $x \in \mathbb{R}$, $(t, \tau, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+$, where $\mathrm{sgn}(x) = 1$ (resp. -1) if $x \geqslant 0$ (resp. $x < 0$), and $(x)_+ = x$ (resp. 0) if $x \geqslant 0$ (resp. $x < 0$). Each $\delta_{t,\tau,\lambda}$ is the product of the soft thresholding function with a sigmoid-like function. As such, the function $\delta_{t,\tau,\lambda}$ is called a smooth sigmoid-based shrinkage (SSBS) function.

Given t and λ, the function $\delta_{t,\tau,\lambda}(x)$ tends to the soft thresholding function $\mathrm{sgn}(x)(|x| - T)_+$ with $T = \max(t, \lambda)$ when τ tends to $+\infty$. On the other hand, when τ tends to infinity, $\delta_{0,\tau,\lambda}$ tends simply to $\delta_{0,\infty,\lambda}$, which is a hard thresholding function defined by

$$\delta_{0,\infty,\lambda}(x) = \begin{cases} x\mathbb{1}_{\{|x|>\lambda\}} & \text{if } x \in \mathbb{R} \setminus \{-\lambda, \lambda\}, \\ \pm\lambda/2 & \text{if } x = \pm\lambda, \end{cases} \tag{17}$$

where $\mathbb{1}_\Delta$ is the indicator function of a given set $\Delta \subset \mathbb{R}$: $\mathbb{1}_\Delta(x) = 1$ if $x \in \Delta$; $\mathbb{1}_\Delta(x) = 0$ if $x \in \mathbb{R} \setminus \Delta$. Note that $\delta_{0,\infty,\lambda}$ sets a coefficient with amplitude λ to half of its value, and so minimizes the local variation around λ, since $\lim_{x\to\lambda^+} \delta_{0,\infty,\lambda}(x) - 2\delta_{0,\infty,\lambda}(\lambda) + \lim_{x\to\lambda^-} \delta_{0,\infty,\lambda}(x) = 0$. According to the foregoing remarks, the soft and hard thresholding functions are degenerate SSBS functions.

The following addresses the role of the SSBS parameters. First, note that t controls the attenuation imposed to data with large amplitudes (see Fig. 2). Thus, it will be called the *asymptotic attenuation* parameter. In what follows, we assume that $\lambda \geqslant t$. In fact, if $\lambda < t$, then $\delta_{t,\tau,\lambda}$ behaves as the soft thresholding function $\mathrm{sgn}(x)(|x| - t)_+$, which is known to over-smooth the estimate. Second, λ is called the *threshold* or *threshold height* of the SSBS function since it acts as a threshold: $\delta_{0,\infty,\lambda}$ is a hard thresholding function with threshold height λ.

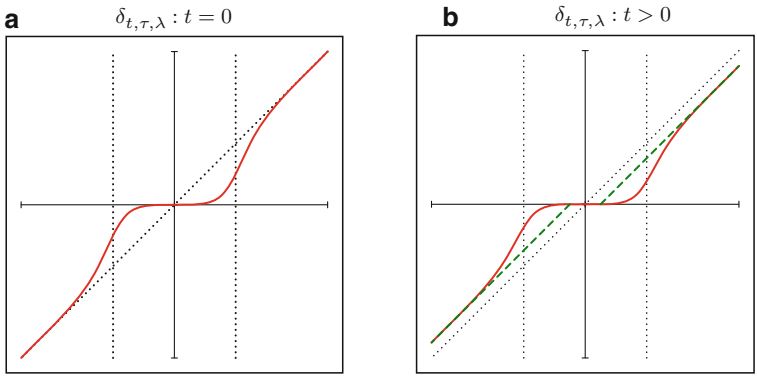

Fig. 2. Graphs of $\delta_{t,\tau,\lambda}$ for $t = 0$ and $t \neq 0$. The vertical dotted lines cut the x-axis at the value of $\pm\lambda$ where λ is the SSBS threshold height chosen to draw these figures. The intersection between this vertical line and the curve of an SSBS function is a fixed point of this SSBS curve. Therefore, below this threshold height, amplitudes are attenuated more strongly than above this threshold height. This is one of the main features of SSBS functions

Now, parameter τ has a geometric interpretation according to the choice of t and λ. Let $t \geqslant 0$ and $\lambda > t$. When τ varies, the SSBS functions thus obtained admit two fixed points: in Cartesian coordinates, $A = (\lambda, (\lambda - t)/2)$ and $A' = (-\lambda, -(\lambda - t)/2)$ belong to the graph of $\delta_{t,\tau,\lambda}$ for every $\tau > 0$ since, according to (20), $\delta_{t,\tau,\lambda}(\pm\lambda) = \pm(\lambda - t)/2$ for any $\tau > 0$. It follows that τ parameterizes the curvature of the arc of the SSBS function in the interval (t, λ). This curvature directly relates to the attenuation degree we want to apply to the data whose amplitudes belong to the interval (t, λ).

Let C be the intersection between the abscissa axis and the tangent at point A to the curve of the SSBS function. The equation of this tangent is $y = 0.25(2 + \tau(\lambda - t))(x - \lambda) + 0.5(\lambda - t)$. The coordinates of point C are $C = ((2t + \tau\lambda(\lambda - t))/(2 + \tau(\lambda - t)), 0)$. We can easily control the arc $\widehat{OA}$ curvature via the angle, denoted by ϕ, between $\overrightarrow{OA}$, which is fixed, and $\overrightarrow{CA}$, which is carried by the tangent to the curve of $\delta_{t,\tau,\lambda}$ at point A. The larger

ϕ, the stronger the attenuation of the coefficients with amplitudes in (t, λ). Given t and λ, the relation between ϕ and τ is

$$\cos\phi = \frac{10\lambda - 2t + \tau(\lambda - t)^2}{\sqrt{4\lambda^2 + (\lambda - t)^2}\sqrt{20 + 4\tau(\lambda - t) + \tau^2(\lambda - t)^2}}. \tag{18}$$

Thereby,

$$0 < \phi < \arccos\left(\lambda - t\right)/\sqrt{4\lambda^2 + (\lambda - t)^2}).$$

When $\phi = \arccos\left((\lambda - t)/\sqrt{4\lambda^2 + (\lambda - t)^2}\right)$, then $\tau = +\infty$, and $\delta_{t,\tau,\lambda}$ is the hard thresholding function of (17). Henceforth, ϕ is called the *attenuation degree*. According to (18), τ is a function of t, ϕ, and λ:

$$\tau(t, \phi, \lambda) = \frac{1}{\lambda - t}\left(-\frac{\Psi(t, \phi, \lambda)}{\Lambda(t, \phi, \lambda)} + \sqrt{\frac{\Psi^2(t, \phi, \lambda)}{\Lambda^2(t, \phi, \lambda)} - \frac{20\Phi(t, \phi, \lambda) - 4(5\lambda - t)^2}{\Lambda(t, \phi, \lambda)}}\right), \tag{19}$$

where $\Phi(t, \phi, \lambda) = \left(\lambda^2 + (\lambda - t)^2\right)\cos^2\phi$, $\Lambda(t, \phi, \lambda) = \Phi(t, \phi, \lambda) - (\lambda - t)^2$, and $\Psi(t, \phi, \lambda) = 2\left(\Phi(t, \phi, \lambda) - (\lambda - t)(5\lambda - t)\right)$.

In practice, we can then control the attenuation degree we want to impose to the data in the interval (t, λ) by choosing ϕ, a rather natural parameter, and calculating τ according to (19). For this reason, we henceforth set $\delta_{t,\phi,\lambda} = \delta_{t,\tau(\phi,\lambda),\lambda}$, where $\tau(t, \phi, \lambda)$ is given by (19). The considerations above make it easier to select convenient parameters for practical applications. Summarizing, the estimation procedure is performed in three steps:

1. Fix the asymptotic attenuation t, the threshold λ, and the attenuation degree ϕ of the SSBS function.
2. Compute the corresponding value of τ from (19).
3. Shrink the data according to the SSBS function $\delta_{t,\tau,\lambda}$.

When no attenuation is required for large data, we are concerned by the particular case $t = 0$ (the SSBS shape is that of Fig. 3), and if we set $\delta_{\tau,\lambda} = \delta_{0,\tau,\lambda}$, (16), (18), and (19) simplify to:

$$\delta_{\tau,\lambda}(x) = \frac{x}{1 + e^{-\tau(|x| - \lambda)}}, \tag{20}$$

$$\cos\phi = \frac{10 + \tau\lambda}{\sqrt{5(20 + 4\tau\lambda + \tau^2\lambda^2)}} \text{ and } \tau(\phi, \lambda) = \frac{10}{\lambda}\,\frac{\sin^2\phi + 2\sin\phi\cos\phi}{5\cos^2\phi - 1}, \tag{21}$$

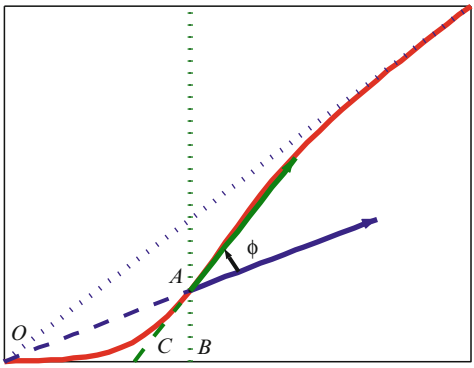

Fig. 3. Graph of $\delta_{t,\tau,\lambda}$ in the positive half plane ($t = 0$)

with $0 < \phi < \arccos(\sqrt{5}/5)$. As above, we set $\delta_{\phi,\lambda} = \delta_{\tau(\phi,\lambda),\lambda}$ with $\tau(\phi, \lambda)$ given by (21). Some SSBS graphs are plotted in Fig. 4 for different values of ϕ, a fixed λ, and $t = 0$.

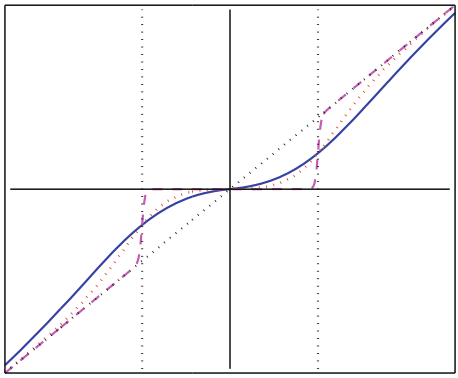

Fig. 4. Shapes of SSBS functions $\delta_{\phi,\lambda}$ for different values of the attenuation degree ϕ: $\phi = \pi/6$ for the continuous curve, $\phi = \pi/4$ for the dotted curve, and $\phi = \pi/3$ for the dashed curve

4.2 Adapted Detection Thresholds

Section 3.3 has highlighted that DeTs are suitable for the detection of significant coefficients of a sparse sequence such as that returned by the wavelet transform. As such, they are well adapted for WaveShrink. In fact, DeTs can be adapted to the wavelet transform by adjusting them at each decomposition level so as to take into account some specific features of the wavelet transform. These features relate to the sparsity of the transform. More specifically, for smooth and piecewise regular signals, the proportion of significant

coefficients, which plays a role similar to p, increases [21, Sect. 10.2.4, p. 460]. Moreover, for smooth and piecewise regular signals again, the amplitudes of the signal detail coefficients tend to decrease (see [21, Theorem 6.4, p. 171], for instance), when the decomposition level increases. If J is the decomposition level, the foregoing suggests giving upper bounds $(\mathsf{p}_j)_{j=1,2,\cdots,J}$ where $\mathsf{p}_j \leqslant 1/2$ for every $j \in \{1, 2, \ldots, J\}$ and lower bounds $(\varrho_j)_{j=1,2,\cdots,J}$ for the amplitudes of the significant wavelet coefficients so as to derive level-dependent DeTs for performing soft thresholding adapted to each resolution level. The minimum amplitudes ϱ_j must decrease and the maximum probabilities of occurrence p_j must increase.

Since significant information tends to be absent among the first resolution level detail wavelet coefficients, it is reasonable to set $\varrho_1 = \sigma\sqrt{2\ln N}$, that is, the universal threshold. Now, when the resolution level increases, it follows from [21, Theorem 6.4] that a convenient choice for $\varrho_j, j > 1$ is $\varrho_j = \varrho_1/\sqrt{2^{j-1}}$ when the signal of interest is smooth or piecewise regular.

As far as the upper bounds p_j are concerned, we must stop the shrinkage at a resolution level J for which p_J is less than or equal to $1/2$ since DeTs are defined for $\mathsf{p}_j \leqslant 1/2$. We propose the use of exponentially or geometrically increasing sequences for the values $(\mathsf{p}_j)_{j=1,2,\cdots,J}$ since p_1 must be a very small value (significant information tends to be absent among the first resolution level detail wavelet coefficients), and the presence of significant information increases significantly as the resolution level increases. In the following, we consider a sequence $(\mathsf{p}_j)_{j=1,2,\cdots,J}$ such that $\mathsf{p}_{j+1} = (\mathsf{p}_j)^{1/\mu}$ with $\mu > 1$. Summarizing, we consider the adapted detection thresholds (ADeTs) defined by

$$\lambda_D(\varrho_j, \mathsf{p}_j) \stackrel{\Delta}{=} \sigma\xi(\varrho_j/\sigma, \mathsf{p}_j), \tag{22}$$

where

$$\varrho_j \stackrel{\Delta}{=} \sigma\sqrt{\ln N}/2^{j/2-1} \text{ and } \mathsf{p}_j \stackrel{\Delta}{=} 1/2^{\mu^{J-j}}. \tag{23}$$

4.3 Experimental Results

In order to assess the performance improvement provided by SSBS, DeTs, and ADeTs, experimental tests were carried out. These experiments concerned image denoising. This application field is a very common one for WaveShrink estimators and quite illustrative. Our purpose was then to compare the several WaveShrink estimators that can be obtained by combining the different WaveShrink functions (soft thresholding, hard thresholding, NNG, SCAD, SSBS) with the different thresholds we have considered (universal, minimax, UniDeT, and ADeTs). In particular, SAW-ADeTs corresponds to the pair (SSBS, ADeTs). Standard images were then corrupted by AWGN with different noise standard deviations.

Each noisy image was then decomposed via the stationary wavelet transform (SWT) proposed in [6]. We chose the Haar wavelet with maximum decomposition level equal to $J = 4$. Similar results would have been obtained

by using other SWT algorithms such as the " à trous" algorithm [12, 30]. The SWT is recommended by practitioners for its appreciable properties in denoising [6, 21]. Actually, since the SWT can be seen as an average version of DWTs adjusted with different decimation steps, it is translation-invariant [21]. Moreover, its redundancy makes it possible to reduce residual noise and possible artifacts incurred by the translation sensitivity of the orthonormal wavelet transform. The support of the Haar wavelet is suitable for dealing with images that are often piecewise regular signals. The SWT coefficients of each noisy image were then shrunk according to the different possible shrinkages previously mentioned and described by a pair (WaveShrink function, threshold). As far as the parameters t and ϕ of the SSBS are concerned, they were set on the basis of preliminary tests, which suggested using small (resp. large) asymptotic attenuation and attenuation degrees for small (resp. large) noise levels. Specifically, we chose the following values:

- $t = 0$ and $\phi = \pi/10$ when the noise standard deviation σ is less than or equal to 5.
- $t = \sigma/5$ and $\phi = \pi/6$ when $5 < \sigma \leqslant 15$.
- $t = \sigma/3$ and $\phi = \pi/5$ when σ is larger than 15.

Regarding ADeTs, they were calculated at each decomposition level according to (22), where $(\varrho_j, \mathsf{p}_j)$, for $j = 1, 2, \ldots, J$ are given by (23) with $\mathsf{p}_J = 1/2$. In (23), we set $\mu = 2.35$ on the basis of preliminary tests. The PSNR (see (2)) was then used to assess the quality of every denoised image.

For a given noise standard deviation, we generated 40 noisy observations of every image. Each possible WaveShrink estimator defined by a pair (WaveShrink function, threshold) was then applied to denoise each of these noisy observations. For a given noise standard deviation, a given image, and a given WaveShrink estimator, we therefore obtained 40 PSNRs, whose average value is reported in Tables 1 and 2 for comparison to the PSNR of the original input image. It is worth mentioning that, in Tables 1 and 2, the 20 average PSNRs obtained for every given noise standard deviation and every given image are statistically different in the ANOVA (analysis of variance) sense. Indeed, for every given noise standard deviation and every given image, we performed a one-way ANOVA of the 20 average PSNRs obtained by the 20 available WaveShrink estimators and all the p-values returned by the 24 ANOVAs thus performed were significantly smaller than the standard level 0.05.

The experimental results of Tables 1 and 2 emphasize the relevance of ADeTs, UniDeT, and DeTs and that of the SSBS functions. More specifically:

- *For a given shrinkage function*, the results yielded by "global" shrinkages, that is, shrinkages performed by using one single threshold–the universal threshold, the minimax threshold, or UniDeT–without any adaptation, show that UniDeT outperforms the minimax threshold, which, in turn,

Table 1. Average values of the PSNRs computed over 40 noise realizations, when denoising is performed on test images by the soft, hard, NNG, SCAD, and SSBS functions

Image		House	Peppers	Barbara	Lena	Finger	Boat
		$\boldsymbol{\sigma = 5}$ ($\Longrightarrow$ Input PSNR = 34.15)					
	Universal	32.32	30.67	28.03	31.55	26.33	29.64
Soft	Minimax	33.89	32.57	29.78	32.94	28.21	31.06
	UniDeT	35.14	34.04	31.73	34.48	30.26	32.67
	ADeTs	34.13	32.78	29.78	33.76	29.25	31.70
	Universal	35.83	35.40	33.21	35.33	31.92	33.58
Hard	Minimax	36.84	36.34	34.67	36.29	33.14	34.69
	UniDeT	37.18	36.46	35.49	36.75	33.94	35.44
	ADeTs	36.66	36.17	34.35	36.43	32.90	34.67
	Universal	34.68	33.78	30.59	33.72	29.94	31.79
NNG	Minimax	35.99	35.49	32.70	35.19	31.91	33.28
	UniDeT	36.99	36.56	34.67	36.47	33.61	34.81
	ADeTs	35.89	35.30	32.38	35.52	32.20	33.57
	Universal	34.14	32.00	29.34	33.00	29.14	31.01
SCAD	Minimax	35.46	34.85	31.56	34.45	31.19	32.54
	UniDeT	36.46	36.07	33.82	35.92	33.02	35.16
	ADeTs	35.41	34.69	31.25	34.99	31.65	32.97
	Universal	37.25	36.54	34.09	36.57	33.45	34.90
SSBS	Minimax	37.84	37.25	35.47	37.34	34.47	35.89
	UniDeT	37.89	37.27	36.17	37.59	34.92	36.43
	ADeTs	37.85	37.26	35.26	37.55	34.68	36.01
		$\boldsymbol{\sigma = 15}$ ($\Longrightarrow$ Input PSNR = 24.61)					
	Universal	27.09	24.81	23.29	27.01	20.47	25.11
Soft	Minimax	28.74	26.59	24.35	28.17	21.94	26.26
	UniDeT	30.05	28.04	25.79	29.55	23.68	27.66
	ADeTs	29.58	27.30	24.86	29.63	23.76	27.44
	Universal	31.35	29.58	25.89	30.37	25.54	28.34
Hard	Minimax	32.23	30.75	27.50	31.46	26.78	29.57
	UniDeT	31.54	30.33	28.38	31.34	27.02	29.90
	ADeTs	32.36	30.92	27.43	32.08	27.71	30.11
	Universal	29.31	27.19	24.24	28.54	22.57	26.47
NNG	Minimax	31.08	29.19	25.65	29.89	24.47	27.87
	UniDeT	32.18	30.52	27.43	31.31	26.39	29.41
	ADeTs	31.38	29.52	25.90	31.07	26.10	28.86
	Universal	28.49	26.19	23.67	27.84	21.48	25.73
SCAD	Minimax	30.40	28.38	24.83	29.18	23.49	27.09
	UniDeT	31.75	29.92	26.47	30.71	25.70	28.70
	ADeTs	30.87	28.77	25.24	30.57	25.44	28.28
	Universal	31.86	30.05	26.71	30.89	25.86	28.93
SSBS	Minimax	32.57	31.08	28.12	31.83	27.25	30.06
	UniDeT	32.02	30.83	28.96	31.77	27.74	30.41
	ADeTs	32.74	31.27	28.09	32.37	28.03	30.54

The tested images are corrupted by independent AWGN with standard deviation $\sigma = 5$, 15. The SWT is computed by using the Haar wavelet.

Table 2. Average values of the PSNRs computed over 40 noise realizations, when denoising is performed on test images by the soft, hard, NNG, SCAD, and SSBS functions

Image		House	Peppers	Barbara	Lena	Finger	Boat
		$\sigma = 25$ ($\Longrightarrow$ Input PSNR = 20.17)					
	Universal	24.89	22.44	22.11	25.31	18.39	23.54
Soft	Minimax	26.38	24.05	22.81	26.31	19.62	24.47
	UniDeT	27.62	25.41	23.88	27.53	21.17	25.68
	ADeTs	27.61	24.96	23.58	27.97	21.61	25.83
	Universal	28.86	26.64	23.70	28.14	22.30	26.05
Hard	Minimax	29.90	28.03	24.88	29.25	24.01	27.28
	UniDeT	28.62	27.34	25.54	28.75	24.44	27.37
	ADeTs	30.18	28.27	25.11	30.07	25.35	28.10
	Universal	26.58	24.17	22.73	26.46	19.67	24.46
NNG	Minimax	28.39	26.18	23.62	27.68	21.39	25.66
	UniDeT	29.59	27.60	24.96	29.01	23.33	27.09
	ADeTs	29.20	26.76	24.21	29.19	23.53	26.98
	Universal	25.74	23.13	22.37	25.84	18.71	23.87
SCAD	Minimax	27.58	25.15	23.15	27.01	20.31	24.96
	UniDeT	29.04	26.86	24.30	28.41	22.38	26.37
	ADeTs	28.66	25.85	23.87	28.72	22.86	26.45
	Universal	29.02	26.83	24.20	28.37	22.37	26.34
SSBS	Minimax	30.00	28.16	25.33	29.39	24.00	27.50
	UniDeT	29.28	27.92	26.07	29.23	24.79	27.84
	ADeTs	30.28	28.39	25.51	30.13	25.17	28.22
		$\sigma = 35$ ($\Longrightarrow$ Input PSNR = 17.25)					
	Universal	23.64	21.10	21.54	24.36	17.35	22.71
Soft	Minimax	24.99	22.54	22.09	25.24	18.34	23.48
	UniDeT	26.16	23.80	22.93	26.34	19.70	24.55
	ADeTs	26.46	23.56	22.99	26.98	20.34	24.89
	Universal	27.13	24.62	22.84	26.78	20.22	24.69
Hard	Minimax	28.29	26.18	23.64	27.83	22.01	25.86
	UniDeT	26.60	25.33	23.89	26.94	22.66	25.67
	ADeTs	28.74	26.48	24.05	28.72	23.65	26.80
	Universal	24.97	22.34	22.05	25.27	18.14	23.38
NNG	Minimax	26.69	24.25	22.72	26.37	19.59	24.40
	UniDeT	27.88	25.67	23.74	27.59	21.41	25.68
	ADeTs	27.86	24.96	23.50	28.03	21.94	25.86
	Universal	24.15	21.53	21.73	24.70	17.44	22.90
SCAD	Minimax	25.85	23.25	22.35	25.75	18.63	23.80
	UniDeT	27.27	24.80	23.27	27.02	20.40	25.02
	ADeTs	27.38	24.25	23.27	27.62	21.27	25.40
	Universal	27.25	24.79	23.24	27.00	20.39	24.99
SSBS	Minimax	28.26	26.22	24.09	27.91	22.00	26.05
	UniDeT	27.25	25.86	24.47	27.41	22.90	26.13
	ADeTs	28.73	26.51	24.40	28.73	23.50	26.88

The tested images are corrupted by AWGN with standard deviation $\sigma = 25, 35$. The SWT is computed by using the Haar wavelet.

outperforms the universal threshold. Shrinkages with ADeTs are generally better than "global" shrinkages, even when these global shrinkages are adjusted with UniDeT.

- *For a given threshold*, the SSBS function is more efficient than the standard soft, hard, NNG, and SCAD thresholding functions. Among the standard shrinkage functions, the hard thresholding function gives the best PSNRs.
- With the criterion that, for a given standard deviation and a given image, a WaveShrink estimator ranks first each time it achieves a PSNR average above the others, SAW-ADeTs ranks first 12 times, SSBS adjusted with UniDeT ranks first 9 times, and (Hard, ADeTs) ranks first 3 times.

In order to illustrate the foregoing comments and the quantitative performance measurements of Tables 1 and 2, some denoising examples are given in Fig. 6 for the case where the original image is that of Fig. 5. The reader will appreciate the quality of the denoised images and especially that achieved by SAW-ADeTs and, more generally, by the WaveShrink estimators involving either SSBS functions or detection thresholds (DeT, UniDeT, ADeTs).

Fig. 5. Noisy "Lena" image. The noise is AWGN with standard deviation 35

5 Conclusion

In this chapter, non-parametric WaveShrink estimators are designed on the basis of an original theoretical approach on sparsity and the design of smooth shrinkage functions. More specifically, we have derived from our characterization of sparsity a family of detection thresholds, called the DeTs. In addition, smooth sigmoid-based shrinkage (SSBS) functions have been discussed as an alternative to standard shrinkage functions. Like standard shrinkages, SSBS depends on a threshold height for the distinction between large and small coefficients. It is recommended to choose a DeT for this threshold height. In contrast to standard shrinkages, SSBS does not systematically force to zero

Fig. 6. Denoising of Lena image. The noisy image considered is that of Fig. 5

every coefficient below the threshold height, but attenuates some of them. As a consequence, SSBS reduces the estimation variance. The relevance of our theoretical approach on sparsity and smooth shrinkage is then highlighted by constructing a new WaveShrink estimator, SAW-ADeTs. Basically, SAW-ADeTs

combines SSBS with the adapted detection thresholds (ADeTs), which are DeTs adjusted to the decomposition levels with respect to specific properties of the wavelet transform. SAW-ADeTs can apply to any wavelet transform (orthogonal, redundant, multi-wavelet, complex wavelet, among others) without extra computation. In image denoising, experimental results have emphasized that DeTs, UniDeT, and ADeTs perform better than the standard minimax and universal thresholds and that SSBS performs better than the standard soft, hard, NNG, and SCAD thresholding functions. As a consequence, SAW-ADeTs mostly outperforms the standard WaveShrink estimators. In fact, especially for application to image processing, SAW-ADeTs could be completed by interscale predictors such as those used in [27] or [20]. A perspective of interest is then the comparison of SAW-ADeTs to the non-parametric SURELET approach of [20] and the parametric (computationally expensive) method BLS-GSM of [27]: the former is non-parametric and is shown to be as effective as the latter, which is considered the best up-to-date parametric method in image denoising. In this respect, a SURE [33] optimization of the SSBS parameters, especially the asymptotic attenuation and the attenuation degree (see Sect. 4.1), is thinkable. It could also be interesting to compare the value of the threshold height returned by a SURE optimization of the SSBS parameters to DeT, UniDeT, and ADeTs.

From a general theoretical point of view, the theoretical approach on sparsity proposed in Sect. 3 seems promising. In fact, on the basis of the theoretical background presented in this chapter, applications to speech processing [25], as a continuation of [23], and to orthogonal frequency division multiple access [32]–a promising multiple access technology for new generation wireless networks–have been proposed in [25]. Where application to communication electronic support is also suggested for gathering information intercepted from radio-frequency emissions of non-cooperative communication systems. From a purely theoretical point of view, we aim at refining the model described by assumptions $[\mathbf{Amp}(\varrho)]$ and $[\mathbf{Occ}(\mathsf{p})]$. In fact, such assumptions characterize sparsity in a strong sense since, in this model, a signal is either null or has an amplitude bounded away from 0. A problem of theoretical interest, even for applications other than WaveShrink, is then the study of a more realistic model where discriminating small from large amplitudes would amount to testing whether a random signal observed in noise has an amplitude above or below a specified value.

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Part III

Dynamical Systems and Analysis on Fractals

Simple Infinitely Ramified Self-Similar Sets

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Summary. Self-similar sets form a mathematically tractable class of fractals. Any family of contractive mappings $f_1, ..., f_m$ generates a unique corresponding fractal set A. It is more difficult to find conditions for the f_k which ensure that A has a nice structure. We describe a technique which allows us to determine self-similar sets with a particularly simple structure. Some of the resulting examples are known, like the Sierpiński gasket and carpet, some others seem to be new.

A simple structure is necessary when we want to do classical analysis on A; for instance, define harmonic functions and a Laplace operator. So far, much analysis has been realized on a very small class of fractal spaces–essentially the relatives of the Sierpiński gasket. In this chapter, we discuss two classes of infinitely ramified fractals which seem to be more realistic from the point of view of physical modeling, and we give examples for which fractal analysis seems to be possible. A property of the boundaries for these fractal classes is verified.

1 Introduction

Fractals naturally arise in many mathematical fields which work with recursive procedures, for example, in dynamical systems and stochastic processes. They often have a very intricate structure, and even their dimension may be hard to calculate. On the other hand, there are artificial constructions of more simple fractals for which there is some hope of understanding the geometry and developing some analysis on the space. The best-known class of such fractals is that of the plane self-similar sets, but even these can still be very complicated. Here we discuss how to construct particularly simple self-similar sets.

First, let us briefly discuss our main motivation, the development of fractal analysis on infinitely ramified spaces. The talk in Monastir actually contained some estimates, but this work is still in progress and cannot be presented here.

1.1 A Few Remarks on Fractal Analysis

When fractals are taken as models of porous materials, it is natural to ask for a mathematical theory of heat and electricity flow on such spaces, and

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for a description of vibrations and other physical phenomena. In 1982–1983, two papers by French physicists on the Sierpiński gasket marked the starting point of fractal analysis. Since 1990, mathematicians have developed an analytical theory, dealing with harmonic functions, the resistance metric, and the Laplace operator [13, 18], as well as a probalistic theory of Brownian motion on fractals [6]. Both have become united by Dirichlet forms [14] and heat kernel estimates [11, 12].

As far as concrete spaces were concerned, work was focused on a very special class of self-similar sets, called *finitely ramified* by Mandelbrot. The standard example is the Sierpiński gasket, shown in Fig. 1 together with a four-piece gasket. The pieces, which are similar to the whole set, intersect at a single point. These points are taken as control stations for the transport. It is possible to approximate the space by graphs, where the vertices are control points, and the edges model the pieces of a certain level. From the viewpoint of mathematical techniques, this is very convenient. From the viewpoint of physics, however, it is somewhat unrealistic to assume that a connected space will fall into pieces if a finite set of points is removed.

There is only one type of example outside this class for which fractal analysis was developed. For the *Sierpiński carpet* shown on the left of Fig. 2, Barlow and Bass [7] defined in 1989 a Brownian motion as a weak limit point of reflecting Brownian motions on approximating open sets. Only very recently, four experts joined efforts to finally prove the uniqueness of this process [8]. There are tremendous technical difficulties since the process was not constructed using self-similarity.

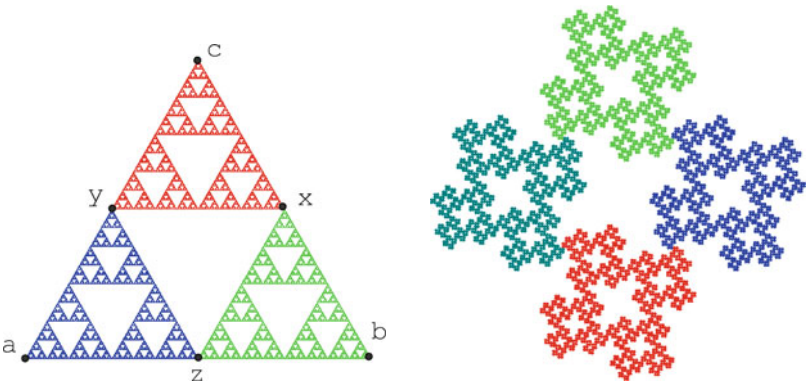

Fig. 1. Sierpiński gasket and four-piece gasket–finitely ramified fractals

1.2 The Advantage of Self-Similarity

Why is it helpful to have self-similarity of a space in order to develop analysis? The answer is that self-similarity is a symmetry property; it means that certain

structures in the space repeat on different scales. More symmetry means fewer conditions, and fewer equations to solve for obtaining parameters of the structure. Classical analysis is possible since Euclidean spaces have extremely rich symmetry! All points have similar neighborhoods, which are transformed into each other by translations. Moreover, we have similarity maps which transform small neighborhoods to big ones, and rotational symmetries of circular neighborhoods.

Self-similar fractals, like those in Fig. 1, retain at least a part of this rich symmetry structure. Let us discuss how this helps us to define harmonic functions. In any metric space, we can say that a real function g is harmonic if it fulfills the mean-value property on spheres:

$$g(x) = \frac{1}{\sigma(S)} \int_S g(y)\, \mathrm{d}\sigma(y). \tag{1}$$

Here S denotes a small sphere around x and σ the surface measure. Such a definition makes sense if

- There are many balls or ball-like sets with similar structure.
- There is a concept of surface measure.

On self-similar sets, pieces or unions of pieces can play the part of balls. In the finitely ramified case, the boundary (= surface) of a piece is finite, so the counting measure can be taken as σ.

This provides an explicit construction of harmonic functions on the Sierpiński gasket S (Fig. 1). Consider the point x and assume that the side length of S is 1. The ball around x with radius $\frac{1}{2}$ consists of two pieces of S, and its boundary consists of the points b, z, y, c. The Euclidean ball would also intersect the third piece, but here we refer to the interior metric, given by the length of the shortest polygonal path within S. A harmonic function g on S thus must fulfill (1) in its explicit form

$$g(x) = \frac{1}{4}(g(b) + g(c) + g(y) + g(z)).$$

Similar equations hold for y and z. Thus, if we assume that the values of g at the boundary points a, b, c of S are given–this is the classical Dirichlet problem–we can use the three equations to determine g at the points x, y, z. We get what Strichartz [18] calls the $\frac{1}{5}$-$\frac{2}{5}$-rule:

$$g(x) = \frac{1}{5}(g(a) + 2g(b) + 2g(c)).$$

Once we have these values, we can determine the values of the intersection points of the pieces of the second level in the same way, and so on by induction. The set of intersection points of all levels is dense in S. By a continuity argument, we can uniquely extend g to a continuous function on S.

Here we used (1) only for countably many, rather special points x, and for the particular values $r = 2^{-k}, k = 1, 2, ...$ of the radius, and we got a unique solution of the Dirichlet problem. It seems difficult and not so useful to check (1) for other values of x and r. For the point x above and $r = \frac{1}{3}$, for instance, the boundary is a Cantor set of dimension $\frac{1}{2}$, so the $\frac{1}{2}$-dimensional Hausdorff measure should be taken as σ. The construction of Brownian motion on S [6] has shown, however, that the given definition of harmonic functions is the unique mathematically correct approach.

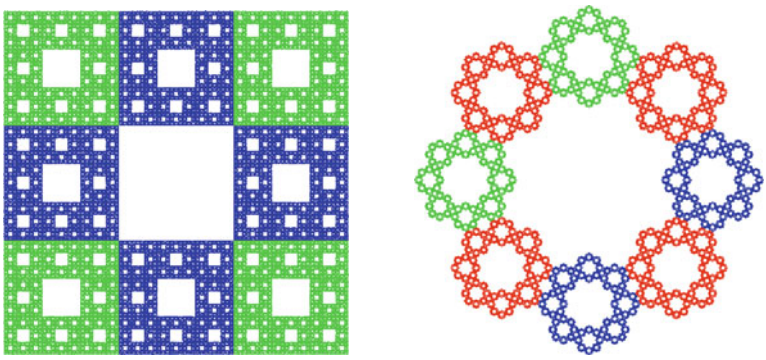

Fig. 2. Sierpiński carpet and octagasket–infinitely ramified fractals

1.3 Contents of the Chapter

The above calculation was easy because S admits self-similarity as well as rotational symmetry, like $\mathbb{R}^n$. The essential difference is that similarity maps and rotations on S act on a discrete scale, while on $\mathbb{R}^n$ they vary continuously. This problem cannot be avoided; it is inherent in all deterministic fractals. The only thing we can do is to choose examples as nice as S. In Sect. 2 we shall construct self-similar sets with a sufficiently strong symmetry structure like S. Strong symmetry means low complexity! In Sects. 3 and 4 we discuss new infinitely ramified examples and prove a minimality property of their boundary. Finally, in Sect. 5 we present some examples of self-similar sets with exact overlap.

2 Self-Similar Sets of Low Complexity

2.1 Basic Definitions

Self-similar sets form a mathematically tractable class of fractals. We review some basic concepts and refer to [10] for details. A self-similar set is a compact subset A of $\mathbb{R}^d$ which is a union of subsets $A_1, ..., A_m$ which are similar to A.

So there are similitudes $f_1, ..., f_m$ on $\mathbb{R}^d$ with $A_k = f_k(A)$. A basic theorem of Hutchinson says that for given contracting similitudes f_k there is a unique compact nonempty A with

$$A = f_1(A) \cup ... \cup f_m(A)\,. \tag{2}$$

This equation implies that each piece $A_k = f_k(A)$ is a union of smaller pieces $A_{kj} = f_k f_j(A)$, and so on. For each word $u = u_1...u_n$ from the alphabet $I = \{1, ..., m\}$ we write the corresponding piece of level n as $A_u = f_u(A) = f_{u_1} f_{u_2} ... f_{u_n}(A)$. Thus, the set I^* of words from I addresses all the pieces of A.

While the theory is treated in this general context, all examples in our paper belong to a very special subclass: we focus on dimension two and orientation-preserving similitudes with equal contraction factors which can be written as complex functions

$$f_k(z) = \lambda_k z + c_k \text{ with } r = |\lambda_k| < 1. \tag{3}$$

The existence theorem does not guarantee that we get nice fractals, like those in our figures. For small contraction factors r we get a Cantor set A, and for large r the pieces A_k overlap too much. The *open set condition* ensures that the pieces can overlap only at their boundaries. It says that there exists an open set U with $f_k(U) \subset U$ and $f_j(U) \cap f_k(U) = \varnothing$ for $j, k \in I, j \neq k$. Since it is not always obvious how to obtain such an open set, an algebraic equivalent of the open set condition was given in [2]:

The identity map is not a limit of maps $f_u^{-1} f_v$ with $u, v \in I^*$.

The idea here is that $f_u^{-1} f_v$ describes the overlap of A_u and A_v, and this overlap should never be complete.

2.2 Counting Neighbor Types to Measure Complexity

For the purpose of analysis, it is appropriate to strengthen this condition. It is not enough to require that overlaps be small. *We want to have only very few types of overlaps of pieces, up to similarity.*

In the Sierpiński carpet, two neighboring pieces of the same size can have a point or an edge in common; otherwise, they are disjoint. Moreover, if two pieces have an edge in common, there is a similitude s which maps the first piece into the unit square and the second piece into its right neighbor square, so that the edge is at $x = 1$. So there is only one type of overlap at an edge, and similarly only one type of overlap at a point. Thus, the Sierpiński carpet has two neighbor types. We say it is type 2. The Sierpiński gasket is type 1; the examples in Fig. 3 are type 2.

In general, we define a neighbor type as a pair of copies $(g(A), h(A))$ of A which intersect each other. Here g, h are similitudes with the same factor. (Or

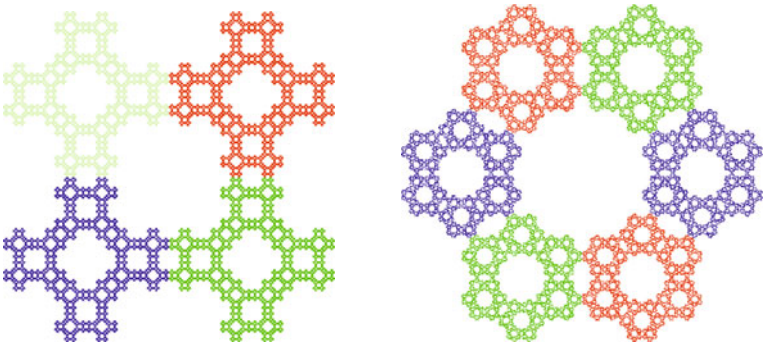

Fig. 3. Self-similar sets with two neighbor types which are Cantor sets

almost the same factor, if the f_k have different contraction factors. To keep our discussion simple, we assume that the contraction factors of all f_k coincide.) Moreover, when s is a similitude, then $(sg(A), sh(A))$ should represent the same type.

We get a canonical representation of the neighbor type by taking g as an identity map (this would be obtained from $s = g^{-1}$). So we have a representing pair $(A, h(A))$. Thus, a neighbor type is given by an isometry map h which we call a *neighbor map* since it maps A to a possible neighboring position.

The neighbor type $(f_u(A), f_v(A))$ of two intersecting pieces of A has the canonical representation $(A, f_u^{-1} f_v(A))$ and the neighbor map $h = f_u^{-1} f_v$. We say that A is of finite type if there are only finitely many different maps of the form $h = f_u^{-1} f_v$ for which the pieces A_u, A_v intersect each other.

This definition can be transformed into an algorithm which from the data of the f_k decides whether A is of finite type [1,5,15]. This algorithm was used to produce most of our new examples. In the case where the linear parts of the f_k are equal, all $f_u^{-1} f_v$ are translations $t(x) = x + c$, and the condition that A_u, A_v intersect can be replaced by $|c| < C$ for some constant C.

There is a slight complication in the case that A has symmetries. When s, t are isometries with $s(A) = A = t(A)$, then $(A, h(A))$ and $(s(A), ht(A))$ represent the same neighbor type. Thus, the neighbor maps h and $s^{-1}ht$ are equivalent, and a neighbor type is defined as a *conjugacy class* of neighbor maps under the symmetry group of A.

The number of neighbor types is only one measure of complexity of the geometry of A. One could also take the number of *neighborhood* types–a piece together with *all* its neighbors. A related concept was used by Ngai and Wang [16]. For the calculation of resistances as in [13], we need a small number of *pairs of intersection sets* of neighbors with a piece, up to similarity, and also a small number m of first-level pieces, which influences the renormalization step.

3 Symmetric Examples

3.1 Carpet and Gasket Constructions

It is easy to construct examples of type 1 if m can be large [15]. For example, any carpet construction on an $n \times n$ checkerboard will be of type 1 if the scheme is invariant under the symmetries of the square (rotations and reflections), and the corner pieces are left out. One example with $n = 5$ is given in Fig. 4. The contraction factor is $r = \lambda = \frac{1}{n} = 0.2$.

Let us mention one property of the left-hand examples in Figs. 1, 2, and 4 which is tightly connected to the chain condition of Grigor'yan and Kumagai [12]. *These examples contain line segments.* Self-similarity implies that whenever there exists a tiny line segment in A, then there must be a line segment in A which intersects at least two first-level pieces, extending as far as those pieces do, and certainly passing through a point of their intersection. This property makes it possible to verify that the right-hand examples in the figures, and both examples of Fig. 3 do not contain line segments. The proof requires some lengthy arguments and is not given here.

From a physical point of view, there is no reason to expect that porous materials contain line segments. For the Sierpiński gasket and carpet, this property seems just a consequence of the special nature of the mappings.

The right-hand example of Fig. 4 is a gasket construction based on a subdivision of the equilateral triangle into 4×4 triangles. Thus, $\lambda = \pm\frac{1}{4}$ in (3). The three corner triangles and the central triangle were removed. It is easy to see that such a construction will be type 1 whenever the omitted triangles include the corners, and form a set which is invariant under the symmetries of the big triangle (rotations and reflections). So we can construct many similar examples.

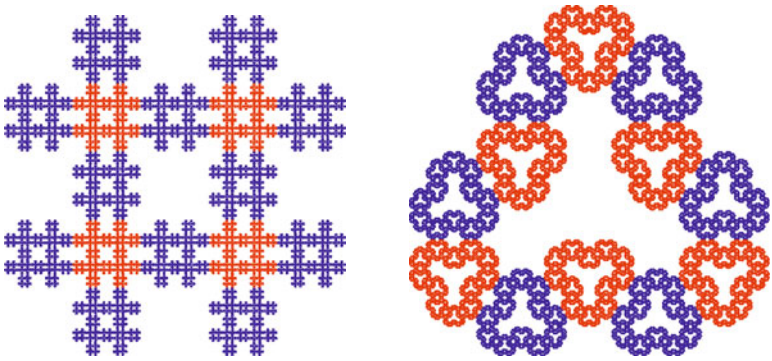

Fig. 4. Type 1 carpet and gasket constructions

3.2 Fractal m-gons

Now let us concentrate on small m. We take symmetric examples where the λ_k in (3) are all equal, and the c_k are the mth roots of unity:

$$f_k(z) = \lambda z + b^k,\ k = 1, ..., m \ \text{ with } r = |\lambda| < 1 \text{ and } b = e^{2\pi i/m}. \qquad (4)$$

These spaces were termed *fractal m-gons* in [4], and they are discussed here for two reasons. On one hand, symmetry makes it much easier to get examples with small neighbor type. When A_1 and A_m are disjoint, then no pieces will intersect each other [4], so A is type 0 by definition. And when A_1 intersects A_m, the space will be automatically connected. The neighbor map $h = f_1^{-1} f_m$ will coincide (up to conjugacy by multiplication with b^q) with all $f_{k+1}^{-1} f_k$, just by symmetry. On the other hand, for given m we have only one complex parameter λ, and we can write $A = A(\lambda)$. We now look for parameters λ for which $A(\lambda)$ is type 1.

Proposition 1. *(a) If the fractal m-gon $A(\lambda)$ is type 1, then*

$$\frac{1}{\lambda} = b^q + \frac{b^j - b^k}{b-1} \quad \text{for some} \quad j, k, q \in I. \qquad (5)$$

(b) If $A_1 \cap A_m \neq \varnothing$, the open set condition holds, and (5) is fulfilled with $q = q(j,k)$ for all pairs (j,k) for which $A_{1j} \cap A_{mk} \neq \varnothing$, then A is type 1.

Proof. (a) From (4) it follows that $h(z) = f_1^{-1} f_m(z) = z + \frac{b-1}{\lambda}$. Assume A is type 1, and $A_{1j} \cap A_{mk} \neq \varnothing$. Then the map

$$f_{1j}^{-1} f_{mk}(z) = f_j^{-1} f_1^{-1} f_m(z) f_k(z) = f_j^{-1}\left(\lambda z + b^k + \frac{b-1}{\lambda}\right) = z + \frac{b^k - b^j}{\lambda} + \frac{b-1}{\lambda^2}$$

must be conjugate to h, that is, equal to $b^q h(b^p z)$ for some $p, q \in I$. The equation

$$b^q \cdot \left(b^p z + \frac{b-1}{\lambda}\right) = z + \frac{b^k - b^j}{\lambda} + \frac{b-1}{\lambda^2}$$

implies $p = -q$ and $\frac{b-1}{\lambda} = b^q(b-1) + b^j - b^k$. This proves (a).

(b) The open set condition implies that $A_j \cap A_k = \varnothing$ for $|j-k| > 1$ (modulo m) [4]. Thus, by assumption, there is up to conjugacy only one neighbor map $h = f_1^{-1} f_m$ among pieces of the first and of the second level. Note that two neighbors A_{uv}, A_{uw} have the same type as A_v, A_w, for arbitrary $u, v, w \in I^*$. What remains is to show that neighbors A_v, A_w of level three or higher have the type h if they are in different first-level pieces, that is, $v_1 \neq w_1$. We show how to go from level 2 to level 3. The induction step to higher levels is exactly the same argument.

Let $J = \{(j,k) \in I^2 | A_{1j} \cap A_{mk} \neq \varnothing\}$. We assumed that J has $\ell \geq 1$ elements, and

$$ {}^q h = f_j^{-1} h f_k \quad \text{for} \quad (j,k) \in J \quad \text{and} \quad q = q(j,k)\,, \tag{6} $$

where ${}^q h$ denotes conjugation by b^q, that is, ${}^q h(z) = b^q h(b^{-q} z)$. Note that ${}^q(f \cdot g) = {}^q f \cdot {}^q g$ and ${}^p({}^q f) = {}^{p+q} f$ with $+$ modulo m. Moreover, ${}^q f_k = f_{k+q}$ and ${}^q f_j^{-1} = f_{j+q}^{-1}$ for arbitrary $j,k,q \in I$, as one can easily check. We now take (6) as it stands, conjugate with some $q' = q(j',k')$ where $(j',k') \in J$, and apply (6) once more to j', k', q'.

$$ {}^{q'+q} h = {}^{q'} f_j^{-1} \left({}^{q'} h \right) {}^{q'} f_k = f_{j+q'}^{-1} \left(f_{j'}^{-1} h f_{k'} \right) f_{k+q'}\,. $$

That is, for every $(j',k') \in J$ we find ℓ different pairs $(j+q', k+q')$ (all $(j,k) \in J$ are possible) such that $A_{1j'(j+q')}$ and $A_{mk'(k+q')}$ are neighbors of type h. However, since $A_{1j'}, A_{mk'}$ has the same neighbor type as A_1, A_m, and ℓ was the number of pairs of level 2 pieces in which A_1, A_m intersect each other, there are exactly ℓ pairs of level 3 pieces in which $A_{1j'}, A_{mk'}$ intersect. So they all have neighbor type h. This completes the proof. □

With the help of this proposition and a computer, we can easily find all fractal m-gons of type 1. The parameter q in (5) can be dropped since λ and $b^q\lambda$ generate the same A. There are only m^2 pairs $(j,k) \in I^2$, and hence only m^2 possible values λ fulfilling (5). Most of these values can be excluded since $A_{1j'} \cap A_{mk'} \neq \varnothing$ for some $(j',k') \neq (j,k)$ for which (5) will not hold.

However, we can also use the proposition to check type 1 directly for our figures, when we believe that these computer pictures do correctly show which first-and second-level pieces intersect and which do not. Since $\ell \leq 2$, and h is given above, there is little calculation.

Examples. For the four-piece gasket in Fig. 1 we have $\frac{1}{\lambda} = 2 + i$ and $b = i$, so the only solution of (5) is obtained for $j = 2, k = 4$, and $q = 1$. For the octagasket we have $\frac{1}{\lambda} = 2 + \sqrt{2}$ and $b = \frac{1+i}{\sqrt{2}}$. In (5) we must have $|\frac{b^j - b^k}{b-1}| \geq 1 + \sqrt{2}$, and in the case of equality, the fraction must be positive and real, and $q = m$. There are two solutions $j = 2, k = 7$ and $j = 3, k = 6$ which also denote the intersecting subpieces of A_1, A_m in Fig. 2. Again, we have type 1.

Examples of type 2 can be studied with a little more effort: one has to also consider level 3 pieces. In Fig. 3 and the left-hand part of Fig. 5 this is easy since the first type goes into the second type on the next level, and conversely. The values of λ in Fig. 3 are $\frac{\tau}{\sqrt{2}} e^{\pi i/4}$ and $\frac{\tau}{\sqrt{3}} e^{\pi i/6}$, where $\tau = \frac{\sqrt{5}-1}{2}$ denotes the golden mean.

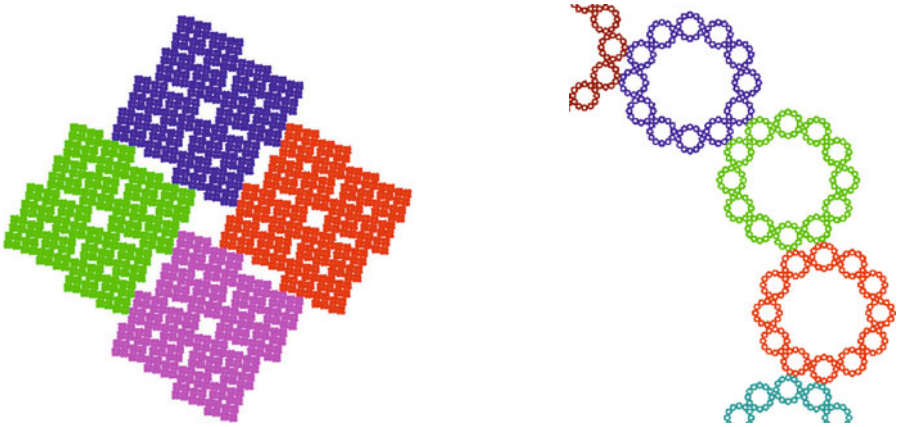

Fig. 5. Type 2 and type 1 fractal m-gons with Cantor set intersection

4 The Boundary Structure

4.1 Self-Similarity of the Intersection Sets

Three types of neighbor intersection sets may appear in the plane:

- Finite sets, which means singleton intersections for subpieces
- Cantor sets, intervals, or fractal arcs
- Complete subpieces of the self-similar set

When we assume type 1, there are of course no mixed cases, like the Sierpiński carpet. The first, finitely ramified case is well known (Fig. 1). For type 1 fractal m-gons, it corresponds to the case $\ell = 1$ in the preceding proof.

Cantor set intersections, as shown in Figs. 2, 3, 4, and 5, seem to be more frequent even if the open set condition is assumed [3]. In the case of type 1, they correspond to $\ell \geq 2$. It is well known that such Cantor intersection sets have a self-similar structure themselves. In general, a system of equations of the form (2) can be established for these sets [5]. For the type 1 case (left Fig. 2, right Fig. 5) the situation is much simpler. (A similar statement holds for our type 2 examples, but is more difficult to prove.)

Proposition 2. *For a type 1 fractal m-gon, the set $A_1 \cap A_m$ is the self-similar set with respect to the ℓ mappings $g = f_1 f_j \, {}^q f_1^{-1} = f_m f_k \, {}^q f_m^{-1}$ where (j, k) runs through the pairs with $A_{1j} \cap A_{mk} \neq \varnothing$, and $q = q(j, k)$ in the proof of Proposition 1. The open set condition is fulfilled.*

Proof. The equality of the mappings was proved in part (a) above. It implies that $g(A_1) \subset A_1$ and $g(A_m) \subset A_m$. Thus, the self-similar set with respect to the ℓ mappings g must be a subset of A_1 and of A_m, and thus of $A_1 \cap A_m$.

Moreover, it was shown in part (b) above that for each n, the image of A_1 under the possible ℓ^n compositions of n mappings g consists of the ℓ^n pieces of level $n+1$ inside A_1 which intersect A_m. This shows that the self-similar set of the ℓ maps g is the whole set $A_1 \cap A_m$. Moreover, there is an open set U for $f_1, ..., f_m$, and $f_1(U)$ (or $f_m(U)$ or the union of both) can be taken as an open set for the g, due to the properties just discussed. □

Corollary 1. *For a type 1 fractal m-gon, the set $C = A_1 \cap A_m$ has Hausdorff dimension $d_C = \frac{\log \ell}{-\log r}$ while the dimension of A is $d_A = \frac{\log m}{-\log r}$.*

This follows from the dimension formula for self-similar sets [10] since all mappings g have the factor r. Moreover, the d_C-dimensional Hausdorff measure on intersection sets may serve as the surface measure σ, as discussed in the Introduction, even though the resulting harmonic measure need not be absolutely continuous with respect to σ.

4.2 Intersection Sets as Minimal Cuts

Now we point out an essential difference between our m-gon examples and the Sierpiński carpet (cf. Fig. 2). The intersection sets of neighboring pieces of the carpet are intervals, while the intersection of the carpet with other line segments usually has dimension smaller than 1. A well-known theorem of Marstrand says that, for almost all lines, the intersection with the carpet has dimension $d_A - 1 \approx 0.89$. For the diagonals of the carpet, the intersection is a middle-third Cantor set with dimension $\frac{\log 2}{\log 3} \approx 0.63$, and this is the minimum value, as can be shown by the method pointed out below.

When for control of a transport process on A we consider the flow through the intersection of A with a curve, it seems reasonable to require that this intersection be rather small, a "bottleneck" of the flow. For finitely ramified sets like the gasket we used intersection sets C, which are singletons. They are minimal, while in the carpet the intersections have maximal dimension. This may be a reason why self-similarity could not help in the study of Brownian motion on the carpet.

We now show that some infinitely ramified m-gon constructions have the same property as the gasket: intersection sets have minimal dimension, they are "bottlenecks." Consider a fractal m-gon with symmetry center 0, and a continuous curve $\varphi : [0,1] \to \mathbb{R}^2 \cup \{\infty\}$. The set $D = \varphi([0,1]) \cap A$ will be called a cut of A.

Theorem 1. *For a type 1 fractal m-gon, where neighbors meet in $\ell = 2$ subpieces, the set $C = A_1 \cap A_m$ has the minimal Hausdorff dimension among all cuts D of A.*

Examples are the octagasket (Fig. 2) and the dodecagasket (Fig. 5). However, the proof below applies to the examples of Fig. 4 as well, and, with small

modifications, also to Fig. 3 and the left-hand example of Fig. 5. In all these examples, it is clear that C is a cut. In the general case, this must be proved by showing that the set $\{x \in \mathbb{R}^2 | d(x, A_1) = d(x, A_m)\}$ (together with a line from ∞ to 0) separates A_1 from A_m, and thus contains the image of a curve φ from 0 to ∞ which separates A_1 and A_m and thus must pass through all points of C.

Proof. A sequence $A_{v_1}, , ..., A_{v_k}$ of k small subpieces of A of the same level n is called a ring if only the neighboring pieces intersect: $A_{v_j} \cap A_{v_{j+1}} \neq \varnothing$, with $+$ mod k, and if 0 is separated from ∞ by the union of the A_v. It is obvious that every cut D must intersect at least one set from every ring.

Moreover, it follows from type 1 and $\ell = 2$ that each ring splits into two disjoint rings on the next level, an outer one and an inner one (i.e., the new sets A_v of both rings are different, not necessarily disjoint).

Now starting with the ring $A_1, ..., A_m$ and using induction, we see that any cut D must intersect at least 2^{n-1} subpieces of A of level n. This lower bound is just the number of subpieces of the cut C, provided we count only on one side (i.e., on A_1 and not on A_m).

To prove $\dim D \geq d_C$, let $d = d_C = \frac{\log 2}{-\log r}$. It suffices to show that the d-dimensional Hausdorff measure $\mu^d(D)$ is positive (see [10]). Since D is compact, we can work with finite coverings by open sets U. First, let us note that for any finite family $\mathcal{A}$ of pieces A_v of A which cover D,

$$\sum_{A_v \in \mathcal{A}} |A_v|^d \geq (r|A|)^d, \tag{7}$$

where $|A|$ denotes the diameter of A and hence $r|A|$ the diameter of A_i. This follows from the fact that when we cover with pieces of the next level, at least two smaller pieces have to be used to replace an A_v.

The open set condition of A implies that there is a constant c such that for all n, a set U of diameter $\leq r^n|A|$ can intersect at most c sets A_v of level n [17]. If $r^{n+1}|A| < |U| \leq r^n|A|$, let $\mathcal{A}_U$ denote the family of all nth-level sets A_{i_j} which intersect U. Then

$$c|U|^d \geq r^d \sum_{A_v \in \mathcal{A}_U} |A_v|^d.$$

Now we take an arbitrary finite cover $\mathcal{U}$ of D by open sets. Then $\bigcup_{U \in \mathcal{U}} \mathcal{A}_U$ is a cover of D by pieces A_v which fulfills (7), and hence

$$\sum_{U \in \mathcal{U}} |U|^d \geq \frac{r^d}{c} (r|A|)^d.$$

Since $\mathcal{U}$ was arbitrary, $\mu^d(D)$ is positive. This completes the proof. □

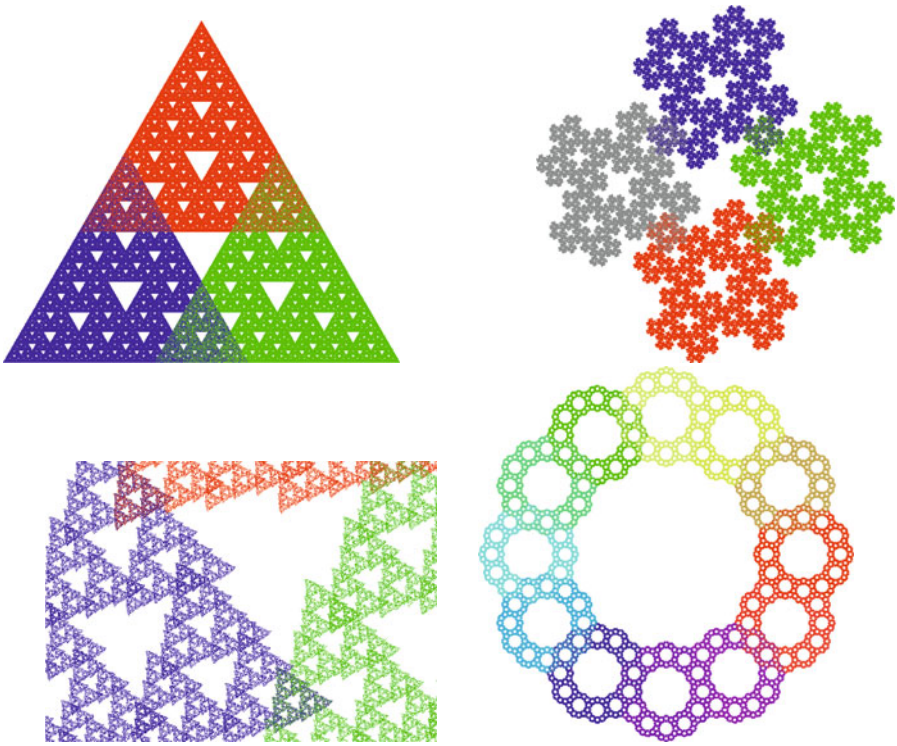

Fig. 6. The golden gasket and some new overlap examples

5 Examples with Exact Overlap of Pieces

There are numerous papers on the Hausdorff dimension of self-similar sets with overlap [16], but few examples. The golden gasket in Fig. 6 was studied by Broomhead, Montaldi, and Sidorov [9]; it has type 4. We give some new examples in Fig. 6. The example with four maps has type 2, and the ring with 12 pieces has type 1. For the example with three pieces we show a magnification since the structure is more complicated. It is not yet clear how to develop an analysis on spaces with such overlaps.

In these examples, two pieces which coincide are not considered as neighbors. In other words, the identity map is not considered as a neighbor map, although $f_u^{-1} f_v = id$ for $A_u = A_v$. For given words $u = u_1 ... u_n$ and $v = v_1 ... v_n$, this equation will be a polynomial equation in λ, see [2]:

$$b^{v_1} - b^{u_1} + \lambda(b^{v_2} - b^{u_2}) + \cdots + \lambda^{n-1}(b^{v_n} - b^{u_n}) = 0. \tag{8}$$

Example. In the upper-right picture of Fig. 6, $A_{423} = A_{141}$. With $b = i$, the equation for λ is $i - 1 + 2\lambda + 2i\lambda^2 = 0$, resulting in $\lambda \approx 0.275 - 0.409i$. The two types are given by "overlap of a second-level piece" and "overlap of

a first-level piece." In the lower-right example, the unique type is "overlap of two second-level pieces plus smaller pieces."

An equation of the form (8) will rarely lead to such simple examples. Usually, many other neighbor types will occur. There seem to be topological restrictions concerning possible overlap sets. While the golden gasket immediately generalizes to higher dimensions, it is not clear whether there are many other examples in dimension ≥ 3. It is also not clear how an analysis can be developed on these spaces.

Acknowledgments

I would like to thank the organizers of the conference on Fractals and Related Fields for the invitation and for the very inspiring atmosphere in Monastir. Jacques Peyrière is a famous expert in random fractals and multifractals. I am convinced that in the end only random fractals can model physical phenomena like transport in porous media, or turbulence. Our deterministic fractals are toy models that we need on the way to develop our geometric understanding.

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Quantitative Uniform Hitting in Exponentially Mixing Systems

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Summary. Consider an exponentially mixing metric measure preserving system $(X, \mathscr{B}, \mu, T, d)$. Let $\alpha_{\max}$ be the maximal local dimension of μ. It is proved that if $\tau < 1/\alpha_{\max}$, then for μ-almost all x and for every $y \in X$ we have $\liminf_{n\to\infty} n^{\tau} d(T^n x, y) = 0$. The critical value $1/\alpha_{\max}$ is optimal in many cases.

1 Introduction

Let $(X, \mathscr{B}, \mu, T, d)$ be a metric measure preserving system (m.m.p.s.), by which we mean that (X, d) is a metric space, $\mathscr{B}$ is a σ-field containing the Borel σ field of X, and $(X, \mathscr{B}, \mu, T)$ is a measure preserving system. Under the assumption that (X, d) has a countable base, the Poincaré recurrence theorem implies that μ-almost all $x \in X$ is recurrent in the sense that

$$\liminf_{n\to\infty} d(T^n x, x) = 0 \tag{1}$$

(for example, see [11]). If, furthermore, μ is ergodic, then μ-almost all $x \in X$ hit every fixed point $y \in X$ in the sense that

$$\liminf_{n\to\infty} d(T^n x, y) = 0. \tag{2}$$

In 1993, Boshernitzan [5] obtained the following quantitative improvement of the above recurrence (1): Let $(X, \mathscr{B}, \mu, T, d)$ be an m.m.p.s. Assume that, for some $\alpha > 0$, the Hausdorff α-measure H_α is σ-finite on X. Then for μ-almost all $x \in X$, we have

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) < \infty. \tag{3}$$

If, moveover, $H_\alpha(X) = 0$, then for μ-almost all $x \in X$, $\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) = 0$.

There were some subsequent works on quantitative recurrence, sometimes referred to as the dynamical Borel–Cantelli lemma or the shrinking target

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problem (see, for example, [1, 3, 4, 6, 12, 16–18, 22]). In the present work, we would like to quantify the hitting property (2) for all y simultaneously. Let us first introduce some notation before stating our results. Define the local lower and upper dimensions of μ at $x \in X$ as follows:

$$\underline{\alpha}(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}, \qquad \overline{\alpha}(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ denotes the ball centered at x of radius r. Let

$$\alpha^* = \sup_{x\in X} \underline{\alpha}(x) \qquad \text{and} \qquad \alpha_{\max} = \limsup_{r\to 0} \sup_{x\in X} \frac{\log \mu(B(x,r))}{\log r}.$$

It is evident that $\alpha^* \le \alpha_{\max}$ so that $\frac{1}{\alpha_{\max}} \le \frac{1}{\alpha^*}$.

We say that an m.m.p.s. $(X, \mathscr{B}, \mu, T, d)$ is *exponentially mixing* if there exist two constants $c > 0$ and $0 < \gamma < 1$ such that

$$|\mu(E|T^{-n}F) - \mu(E)| \le c\gamma^n \qquad (\forall n \ge 1) \tag{4}$$

holds for any ball E and any measurable set $F \in \mathscr{B}$ with $\mu(F) > 0$. Here $\mu(A|B)$ denotes the conditional probability $\frac{\mu(A\cap B)}{\mu(B)}$. Sometimes we say that μ is exponentially mixing.

Theorem 1.1 *Let $(X, \mathscr{B}, \mu, T, d)$ be an m.m.p.s. Suppose that the system is exponentially mixing and $\alpha_{\max} < \infty$. If $\tau < 1/\alpha_{\max}$, then for μ-almost all $x \in X$, we have*

$$\liminf_{n\to\infty} n^\tau d(T^n x, y) = 0 \qquad (\forall y \in X). \tag{5}$$

If $\tau > 1/\alpha^*$, then there exists $y \in X$ such that

$$\lim_{n\to\infty} n^\tau d(T^n x, y) = \infty \qquad (a.e.\ x \in X). \tag{6}$$

Theorem 1.1 was proved, among others, in [10] for the doubling map $Tx = 2x \pmod 1$ on the interval $[0,1)$ and for Gibbs measures μ associated to Hölder potentials. The method presented, inspired [8], is different from that in [10] and is applicable to a large class of dynamical systems. The following are some systems to which Theorem 1.1 applies. Philipp [19] showed that for the β-shift $T_\beta x = \beta x \pmod 1$ on the interval $[0,1)$, the Parry measure is exponentially mixing and that for the Gauss map $Sx = \{\frac{1}{x}\} \pmod 1$ on the interval $[0,1)$, the Gauss measure is exponentially mixing. Pommerenke [20] showed that for the boundary map of an inner function in the unit disk, the invariant harmonic measure is exponentially mixing. In all these cases, we have $\alpha^* = \alpha_{\max} = 1$.

The crucial point of Theorem 1.1 is the first assertion (5) on the uniformity on y. The second assertion of Theorem 1.1 is just a consequence of the dynamical Borel–Cantelli lemma (see Proposition 4.1). Although the Borel–Cantelli lemma together with a Fubini argument shows that the lim inf in (5) is finite for almost all y, the uniformity is far from evident.

Theorem 1.1 will be proved as a consequence of Theorem 2.1, which is valid in a probabilistic setting. This probabilistic setting will be described in Sect. 2. In Sect. 3, we will discuss the exponentially mixing property, which will be first used to prove a weighted Borel–Cantelli lemma in Sect. 4. Sections 5 and 6 are devoted to the proof of the main result (Theorem 2.1). In Sect. 7, we will give an application to subshifits of finite type.

2 Probabilistic Setting

Let $(\xi_n)_{n\geq 0}$ be a stationary process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, taking values in a metric space (X, d). Let μ be the initial probability measure defined by

$$\mu(B) = \mathbb{P}(\xi_0 \in B)$$

for any Borel set B in X. We always assume that X is the support of μ.

For $n \geq 1$, let $\mathcal{A}^n$ be the sub-σ-field generated by $(\xi_{n+j})_{j\geq 0}$. We say that the process (ξ_n) is *exponentially mixing* if there exist two constants $c > 0$ and $0 < \gamma < 1$ such that

$$|\mathbb{P}(\xi_0 \in E|D) - \mathbb{P}(\xi_0 \in E)| \leq c\gamma^n \tag{7}$$

holds for any $n \geq 1$, for any ball E in X and any $D \in \mathcal{A}^n$.

An independent and identically distributed (i.i.d.) sequence (ξ_n) is exponentially mixing. If $(X, \mathcal{B}, \mu, T, d)$ is an exponentially mixing m.m.p.s., then the process (ξ_n) defined by

$$\xi_n(x) = T^n x$$

is an exponentially mixing process defined on the probability space $(X, \mathcal{B}, \mu)$ taking values in X.

Let us go back to a general stationary process (ξ_n). We always make the following assumptions on the initial probability measure μ:

$$\varphi_1(r) \leq \mu(B(x, r)) \leq \varphi_2(r) \qquad (\forall x \in X, \forall r > 0) \tag{8}$$

holds for two increasing functions φ_1 and φ_2 satisfying $\lim_{r\to 0} \varphi_i(r) = 0$ and $\varphi_i(2r) \leq K\varphi_i(r)$ for $i = 1, 2$ $(\exists K > 0, \forall r > 0)$. For a sequence of positive numbers $\{r_n\}$, we will also need the following condition:

$$\lim_{n\to\infty} \frac{\log^2 n}{n\varphi_1(r_n)} = 0. \tag{9}$$

We will use χ_E to denote the indicator function of a set E.

Theorem 2.1 *Let (ξ_n) be an exponentially mixing stationary process whose initial probability satisfies the assumption (8). Let (r_n) be a decreasing sequence of positive numbers satisfying (9). Then*

$$\text{a.s.} \quad \liminf_{N\to\infty} \frac{\min_{y\in X} \sum_{n=1}^{N} \chi_{B(y,r_n)}(\xi_n(\omega))}{\sum_{n=1}^{N} \varphi_1(r_n)} > 0. \tag{10}$$

$$\text{a.s.} \quad \limsup_{N\to\infty} \frac{\max_{y\in X} \sum_{n=1}^{N} \chi_{B(y,r_n)}(\xi_n(\omega))}{\sum_{n=1}^{N} \varphi_2(r_n)} < +\infty. \tag{11}$$

Assume $\varphi_1(r) \geq Ar^s$ and $r_n = B/n^\tau$ ($A > 0, B > 0$ being constant). If $\tau s < 1$, we can apply Theorem 2.1 and we get Theorem 1.1.

When (ξ_n) is an i.i.d. sequence of random variables uniformly distributed on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, the problem we study here was initiated by Dvoretzky in 1956. The question was: Under what condition do we have

$$\text{a.s.} \quad \mathbb{T} = \limsup_{n\to\infty} B(\xi_n, r_n)?$$

In 1972, L. Shepp [23] gave a complete solution by finding a necessary and sufficient condition

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e^{2(r_1+\cdots+r_n)} = \infty.$$

See [2,7,8,13–15] for more information on further development of the subject.

3 On the Exponentially Mixing Property

We shall use the exponentially mixing property in the following form.

Proposition 3.1 *Let $(\xi_n)_{n\geq 0}$ be an exponentially mixing stationary process with initial probability measure μ, taking values in a metric space X. Let $n \geq 1$ be an integer and B_i $(1 \leq i \leq n)$ balls in X. Consider $g_i(x) = a_i\chi_{B_i}(x) + b_i$ with $a_i, b_i \in \mathbb{R}$. Then for any integer $d \geq 1$, we have*

$$\mathbb{E}\prod_{i=1}^{n} g_i(\xi_{(i-1)d}) = \left[\prod_{i=1}^{n} \int g_i(x)d\mu(x)\right] \left[\prod_{i=1}^{n-1} \left(1 + \frac{O(a_i\gamma^d)}{\int g_i(x)d\mu(x)}\right)\right],$$

where $0 < \gamma < 1$ and the constants involved in O are absolute constants ($f = O(g)$ means that there exists a constant C such that $|f| \leq C|g|$).

Proof. The mixing property (7) may be stated as

$$\mathbb{P}(\{\xi_0 \in E\} \cap D) = \mathbb{P}(\{\xi_0 \in E\})\mathbb{P}(D) + O(\gamma^d)\mathbb{P}(D),$$

where E is a ball in X and $D \in \mathcal{A}^d$. In other words,

$$\mathbb{E}(\chi_E(\xi_0)h) = \left(\int \chi_E(x)d\mu(x) + O(\gamma^d)\right)\mathbb{E}(h),$$

where h is the characteristic function of D and $\mu = \mathbb{P} \circ \xi_0^{-1}$. This equality is actually valid for any bounded or non-negative $\mathcal{A}^d$-measurable variable h. Then, by the linearity of the integral, for any function of the form $g = a\chi_E + b$ $(a, b \in \mathbb{R})$ we get

$$\mathbb{E}(g(\xi_0)h) = \left(\int g(x)d\mu(x) + O(a\gamma^d) \right) \mathbb{E}(h). \tag{12}$$

Apply the (12) to $g = g_1$ and $h(\omega) := \prod_{i=2}^{n} g_i(\xi_{(i-1)d}(\omega))$, which is $\mathcal{A}^d$-measurable. Using the stationarity of the process, we get

$$\mathbb{E} \prod_{i=1}^{n} g_i(\xi_{(i-1)d}) = \left(\int g_1(x)d\mu(x) + O(a_1\gamma^d) \right) \mathbb{E} \prod_{i=1}^{n-1} g_{i+1}(\xi_{(i-1)d}).$$

In this way, by induction on n, we get the result. □

Proposition 3.2 *Let $(X, \mathscr{B}, \mu, T, d)$ be an m.m.p.s.*

(i) *The system is exponentially mixing if there exist $0 < \delta < 1$ and an integer $w \geq 1$ such that for any $n \geq 1$, any ball E, and any measurable set F, we have*

$$\mu(E \cap T^{-wn}F) = (\mu(E) + O(\delta^n))\, \mu(F).$$

(ii) *If the system is exponentially mixing, so is the process $(T^n x)_{n\geq 0}$.*

Proof. For any integer $n \geq 1$ write $n = wq + i$ (i, q integers and $0 \leq i < w$). Then

$$\mu(E \cap T^{-n}F) = \mu(E)\mu(F) + \mu(F)O(\delta^q) = \mu(E)\mu(F) + \mu(F)O(\gamma^n),$$

where $\gamma = \delta^{\frac{1}{2w}}$. This proves (i).

The exponential mixing property of the process $\xi_n = T^n x$ is stated as

$$\mu(E \cap D) = \mu(E)\mu(D) + O(\gamma^n)\mu(D),$$

where E is a ball in X and $D \in \mathcal{A}^n$. It suffices to notice that $\mathcal{A}^n = T^{-n}\mathscr{B}$. □

4 Weighted Borel–Cantelli Lemma

The following is a weighted Borel–Cantelli Lemma.

Proposition 4.1 *Suppose that $(\xi_n)_{n\geq 0}$ is an exponentially mixing stationary process. Let (r_n) be a decreasing sequence and (a_n) be a non-negative bounded sequence. For any point $y \in X$,*

(i) $\sum_{n=1}^{\infty} a_n \mu(B(y, r_n)) < +\infty \Rightarrow$ a.s. $\sum_{n=1}^{\infty} a_n \chi_{B(y,r_n)}(\xi_n) < +\infty$.

(ii) $\sum_{n=1}^{\infty} a_n \mu(B(y, r_n)) = +\infty \Rightarrow$ a.s. $\sum_{n=1}^{\infty} a_n \chi_{B(y,r_n)}(\xi_n) = +\infty$.

Proof. The conclusion (i) follows immediately from

$$\mathbb{E}\left(\sum_{n=1}^{\infty} a_n \chi_{B(y,r_n)}(\xi_n)\right) = \sum_{n=1}^{\infty} a_n \mu(B(y,r_n)) < +\infty.$$

Let $Z_N = \sum_{n=1}^{N} a_n \chi_{B(y,r_n)}(\xi_n)$. Then

$$\mathbb{E}(Z_N^2) = \sum_{n=1}^{N} a_n^2 \mu(B(y,r_n)) + 2 \sum_{1\le m<n\le N} a_m a_n \mathbb{E}\left(\chi_{B(y,r_m)}(\xi_m)\chi_{B(y,r_n)}(\xi_n)\right). \tag{13}$$

By Proposition 3.1, we have

$$\mathbb{E}\big(\chi_{B(y,r_m)}(\xi_m)\chi_{B(y,r_n)}(\xi_n)\big) = \mu(B(y,r_m))\mu(B(y,r_n)) + O(\gamma^{n-m})\mu(B(y,r_n)). \tag{14}$$

Assume that the sequence (a_n) is bounded by above by M and the sequence $(\sum_{m=1}^{n-1} a_m O(\gamma^{n-m}))$ is bounded by M'. Combining (13) and (14) gives

$$\mathbb{E}(Z_N^2) = I_1 + I_2 + I_3,$$

where

$$I_1 = \sum_{n=1}^{N} a_n^2 \mu(B(y,r_n)) \le M\mathbb{E}(Z_N)$$

$$I_2 = 2 \sum_{1\le m<n\le N} a_m a_n \mu(B(y,r_m))\mu(B(y,r_n)) \le \mathbb{E}(Z_N)^2$$

$$I_3 = 2 \sum_{1\le m<n\le N} a_m a_n O(\gamma^{n-m})\mu(B(y,r_n)) \le 2M'\mathbb{E}(Z_N).$$

So, $\mathbb{E}(Z_N^2) \le (M+2M')\mathbb{E}(Z_N) + \mathbb{E}(Z_N)^2$. Then, using the Paley–Zygmund inequality, for any $\lambda > 0$, we obtain

$$\mu\left\{Z_N > \lambda\mathbb{E}(Z_N)\right\} \ge (1-\lambda)^2 \frac{\mathbb{E}(Z_N)^2}{\mathbb{E}(Z_N^2)} \ge (1-\lambda)^2 \frac{\mathbb{E}(Z_N)}{M+2M'+\mathbb{E}(Z_N)}.$$

Let $N \to +\infty$ and $\lambda \to 0$, and we get (ii). □

5 Fundamental Inequalities

5.1 Basic Inequalities

For two positive integers $p \le q$, define the truncated sum

$$S_{p,q}(\omega, y) = \sum_{n=p}^{q} \chi_{B(y,r_n)}(\xi_n(\omega))$$

and its maximal and minimal values on y,

$$S^M_{p,q}(\omega) = \max_{y\in X} S_{p,q}(\omega, y), \qquad S^m_{p,q}(\omega) = \min_{y\in X} S_{p,q}(\omega, y).$$

Let $0 < \beta < 1 < \alpha$. Define

$$\overline{S}_{p,q}(\omega, y) = \sum_{n=p}^{q} \chi_{B(y,\alpha r_n)}(\xi_n(\omega)), \quad \underline{S}_{p,q}(\omega, y) = \sum_{n=p}^{q} \chi_{B(y,\beta r_n)}(\xi_n(\omega)).$$

We will often write $S(\omega, y), S^M(\omega), S^m(\omega), \overline{S}(\omega, y), \underline{S}(\omega, y)$, respectively, instead of $S_{p,q}(\omega, y)$, $S^M_{p,q}(\omega)$, $S^m_{p,q}(\omega)$, $\overline{S}_{p,q}(\omega, y)$, $\underline{S}_{p,q}(\omega, y)$.

One key point in proving Theorem 2.1 is the following lemma. Its proof is inspired from [8].

Lemma 5.1 *For any fixed $\omega \in \Omega$ and any integers $p \le q$, we have*

$$\mu\left\{y \in X : \overline{S}_{p,q}(\omega, y) \ge S^M(\omega)\right\} \ge \varphi_1\left((\alpha - 1)r_q\right) \tag{15}$$

$$\mu\left\{y \in X : \underline{S}_{p,q}(\omega, y) \le S^m(\omega)\right\} \ge \varphi_1\big((1 - \beta)r_q\big). \tag{16}$$

Proof. Let y_0 be a maximal point such that $S(\omega, y_0) = S^M(\omega)$. Then there are $S^M(\omega)$ integers n in the interval $[p, q]$ such that $y_0 \in B(\xi_n, r_n)$. Thus, for such an n, $y \in B(y_0, (\alpha - 1)r_q)$ implies $y \in B(\xi_n, \alpha r_n)$ just by the triangular inequality:

$$d(y, \xi_n) \le d(y, y_0) + d(y_0, \xi_n) < (\alpha - 1)r_n + r_n = \alpha r_n.$$

In other words, $\overline{S}(\omega, y) \ge S^M(\omega)$ for $y \in B(y_0, (\alpha - 1)r_q)$. Then

$$B(y_0, (\alpha - 1)r_q) \subset \left\{y \in X : \overline{S}(\omega, y) \ge S^M(\omega)\right\},$$

so we have proved (15). Similarly, we prove (16) by showing that

$$B(y_0', (1 - \beta)r_q) \subset \left\{y \in X : \underline{S}(\omega, y) \le S^m(\omega)\right\},$$

where y_0' is a minimal point such that $S(\omega, y_0') = S^m(\omega)$. □

Lemma 5.2 *There exist positive constants M, B_1, B_2 such that for any $y \in X$ and any integers $p \le q$,*

$$\mathbb{E}\exp\left\{(\log q)^{-1}\overline{S}_{p,q}(\omega, y)\right\} \le M \exp\left\{B_1(\log q)^{-1}\sum_{n=p}^{q}\varphi_2(r_n)\right\} \tag{17}$$

and

$$\mathbb{E}\exp\left\{-(\log q)^{-1}\underline{S}_{p,q}(\omega, y)\right\} \le M \exp\left\{-B_2(\log q)^{-1}\sum_{n=p}^{q}\varphi_1(r_n)\right\}. \tag{18}$$

Proof. Let $F = \sum_{n=p}^{q} \varphi_1(r_n)$ and $G = \sum_{n=p}^{q} \varphi_2(r_n)$. Let $\lambda > 0$ be a real number which will be chosen later. Write

$$\exp\{\lambda \overline{S}(\omega, y)\} = \prod_{n=p}^{q} \Big[(e^{\lambda} - 1)\chi_{B(y,\alpha r_n)}(\xi_n) + 1\Big].$$

Let $d \geq 1$ be an integer, which will be determined later. We divide the integers between p and q into

$$\Lambda_i = \{p \leq n \leq q : n \equiv i \pmod{d}\} \quad (0 \leq i \leq d-1).$$

Notice that

$$[(e^{\lambda} - 1)\chi_{B(y,\alpha r_n)}(\xi_n) + 1]^d = (e^{d\lambda} - 1)\chi_{B(y,\alpha r_n)}(\xi_n) + 1.$$

Applying Hölder's inequality to $\mathbb{E}\prod_{i=0}^{d-1} Z_i$, where Z_i is defined by $Z_i = \prod_{n\in\Lambda_i} \big[(e^{\lambda} - 1)\chi_{B(y,\alpha r_n)}(\xi_n) + 1\big]$, we get

$$\mathbb{E}\exp\{\lambda \overline{S}(\omega, y)\} \leq \left[\prod_{i=0}^{d-1} \mathbb{E}Z_i\right]^{\frac{1}{d}}.$$

Now choose $\lambda = (\log q)^{-1}$ and $d = \lfloor k_0 \log q \rfloor$ with $k_0 > 1$ large enough such that $e\gamma^{k_0} < 1$, where $\lfloor x \rfloor$ is the integer part of x. Applying Proposition 3.1 to each expectation in the above product, we obtain

$$\mathbb{E}Z_i = \prod_{n\in\Lambda_i} [C(d\lambda)\mu(B(y, \alpha r_n)) + 1] \prod_{n\in\Lambda_i} \left(1 + \frac{C(d\lambda)O\left(\gamma^d\right)}{C(d\lambda)\mu\left(B(y, \alpha r_n)\right) + 1}\right),$$

where $0 < C(d\lambda) = e^{d\lambda} - 1 \leq e^{k_0} - 1 := C(k_0)$. So

$$\mathbb{E}\exp\{\lambda \overline{S}(\omega, y)\} \leq M_{p,q} \left[\prod_{n=p}^{q} \left[C(k_0)\mu\left(B(y, \alpha r_n)\right) + 1\right]\right]^{\frac{1}{d}}, \tag{19}$$

where

$$M_{p,q} = \left[\prod_{n=p}^{q} \left(1 + \frac{C(d\lambda)O\left(\gamma^d\right)}{C(d\lambda)\mu\left(B(y, \alpha r_n)\right) + 1}\right)\right]^{\frac{1}{d}} \leq \left[\prod_{n=p}^{q} \left(1 + C(k_0)C\gamma^d\right)\right]^{\frac{1}{d}},$$

C being the constant involved by O. Notice that $q\gamma^d$ is bounded and so $M_{p,q}$ is bounded, say by a constant M independent of p and q. In fact, the boundedness of $q\gamma^d$ is implied by the fact that $e\gamma^{k_0} < 1$. Since $\mu(B(y, \alpha r_n)) \leq \varphi_2(\alpha r_n) \leq K_\alpha \varphi_2(r_n)$ for some constant K_α, from (19) we get

$$\mathbb{E}\exp\{\lambda \overline{S}(\omega, y)\} \leq M \exp\left\{d^{-1}C(k_0)K_\alpha G\right\}.$$

Thus, we have proved (17) with $B_1 = \frac{C(k_0)K_\alpha}{k_0-1}$. Similarly, we write

$$\mathbb{E}\exp\{-\lambda \underline{S}(\omega,y)\} = \mathbb{E}\prod_{n=p}^{q}\Big[(e^{-\lambda}-1)\chi_{B(y,\beta r_n)}(\xi_n)+1\Big].$$

Using similar skills to treat $\exp\big\{-\lambda\overline{S}(\omega,y)\big\}$, we can obtain

$$\mathbb{E}\exp\{-\lambda \underline{S}(\omega,y)\} \le M'_{p,q}\exp\left\{\frac{e^{-d\lambda}-1}{d}\sum_{n=p}^{q}\mu(B(y,\beta r_n))\right\},$$

where

$$M'_{p,q} = \prod_{n=p}^{q}\left(1+\frac{(e^{-d\lambda}-1)O(\gamma^d)}{(e^{-d\lambda}-1)\mu(B(y,\beta r_n))+1}\right)^{\frac{1}{d}} \le \prod_{n=p}^{q}\left(1+(1-e^{-d\lambda})C\gamma^d\right)^{\frac{1}{d}}.$$

We now choose $\lambda = (\log q)^{-1}$ and $d = \lfloor k_0 \log q\rfloor$ with $k_0 > 1$ large enough such that $e\gamma^{k_0} < 1$. As $M_{p,q}$, $M'_{p,q}$ is bounded, say by M. Since $\mu(B(y,\beta r_n)) \ge \varphi_1(\beta r_n) \ge K_\beta\varphi_1(r_n)$ and $e^{-d\lambda}-1$ is negative,

$$\mathbb{E}\exp\{-\lambda \underline{S}(\omega,y)\} \le M\exp\left\{d^{-1}(e^{-d\lambda}-1)K_\beta F\right\}. \tag{20}$$

Thus (18) is proved with $B_2 = (1-e^{-k_0})K_\beta/k_0$. □

Proposition 5.3 *Suppose that there exists a constant $C_1 > 0$ such that*

$$\sum_{n=p}^{q}\varphi_2(r_n) \ge C_1(\log q)\max\{-\log\varphi_1(r_q), \log q\}. \tag{21}$$

Then there exist absolute constants $C > 0, M_0 > 0$, and $C_0 > 1$ such that

$$\mathbb{P}\left\{\omega : S^M(\omega) \ge C\sum_{n=p}^{q}\varphi_2(r_n)\right\} < M_0 q^{-C_0}.$$

Proof. Let $G = \sum_{n=p}^{q}\varphi_2(r_n)$. For any $\lambda > 0$, by Markov's inequality, we have

$$\mathbb{P}\left\{\omega : S^M(\omega) \ge CG\right\} \le \exp\{-\lambda CG\}\,\mathbb{E}(e^{\lambda S^M(\omega)}). \tag{22}$$

We are thus led to estimate the Laplace transform of $S^M(\omega)$. Fixing ω and using Markov's inequality with respect to y, we get

$$\mu\left\{y \in X : \overline{S}(\omega,y) \ge S^M(\omega)\right\} \le \frac{\int e^{\lambda\overline{S}(\omega,y)}d\mu(y)}{e^{\lambda S^M(\omega)}}.$$

Combining this with the inequality (15) in Lemma 5.1, we get

$$e^{\lambda S^M(\omega)} \leq \frac{\int e^{\lambda \overline{S}(\omega,y)} d\mu(y)}{\varphi_1\left((\alpha-1)r_q\right)} \leq \frac{\int e^{\lambda \overline{S}(\omega,y)} d\mu(y)}{K_\alpha \varphi_1(r_q)}$$

for $\varphi_1((\alpha-1)r_q) \geq K_\alpha \varphi_1(r_q)$. Then

$$\mathbb{E} e^{\lambda S^M(\omega)} \leq \frac{\mathbb{E}\int e^{\lambda \overline{S}(\omega,y)} d\mu(y)}{K_\alpha \varphi_1(r_q)}. \tag{23}$$

The Laplace transform of $\bar{S}(\omega, y)$ with respect to the product measure $\mathbb{P} \times \mu$ is easier to estimate. In fact, when $\lambda = (\log q)^{-1}$, by Fubini's theorem and the inequality (17) of Lemma 5.2, we have

$$\mathbb{E}\int e^{\lambda \overline{S}(\omega,y)} d\mu(y) = \int \mathbb{E} e^{\lambda \overline{S}(\omega,y)} d\mu(y) \leq M \exp\left\{B_1 (\log q)^{-1} G\right\}.$$

Combining (23) and (22), we obtain

$$\mathbb{P}\left\{\omega : S^M(\omega) \geq CG\right\} \leq M K_\alpha^{-1} \cdot \exp\left\{(B_1 - C)(\log q)^{-1} G - \log \varphi_1(r_q)\right\}.$$

Using the assumption (21), we can choose $C > 0$ (independent of p, q) large enough such that $-C_0 = (B_1 - C + C_1^{-1})C_1 < -1$. Let $M_0 = M K_\alpha^{-1}$, we have

$$\mathbb{P}\left\{\omega : S^M(\omega) \geq CG\right\} \leq M K_\alpha^{-1} \cdot \exp\left\{\left(B_1 - C + C_1^{-1}\right)(\log q)^{-1} G\right\} \leq \frac{M_0}{q^{C_0}}.$$

□

Proposition 5.4 *Suppose that there exists a constant $C_2 > 0$ such that $C_2 > 2B_2^{-1}$ and*

$$\sum_{n=p}^{q} \varphi_1(r_n) \geq C_2 (\log q) \max\left\{-\log \varphi_1(r_q), \log q\right\}. \tag{24}$$

Then there exist absolute constants $C' > 0$, $K_\beta > 0$, and $C_0 > 1$ such that

$$\mathbb{P}\left\{\omega : S^m(\omega) \leq C' \sum_{n=p}^{q} \varphi_1(r_n)\right\} < K_\beta^{-1} q^{-C_0}. \tag{25}$$

Remark that the condition (24) implies the condition (21) since $\varphi_2 \geq \varphi_1$.

Proof. Let $F = \sum_{n=p}^{q} \varphi_1(r_n)$. For $\lambda > 0$, by Markov's inequality, we have

$$\mathbb{P}\left\{S^m(\omega) \leq C'F\right\} \leq \exp(\lambda C' F) \mathbb{E}(e^{-\lambda S^m(\omega)}). \tag{26}$$

By Markov's inequality and the inequality (16) in Lemma 5.1, we obtain

$$e^{-\lambda S^m(\omega)} \le \frac{\int e^{-\lambda \underline{S}(\omega,y)} d\mu(y)}{\varphi_1\big((1-\beta)r_q\big)} \le \frac{\int e^{-\lambda \underline{S}(\omega,y)} d\mu(y)}{K_\beta \varphi_1(r_q)}$$

for $\varphi_1((1-\beta)r_q) \ge K_\beta \varphi_1(r_q)$. Integrating with respect to ω, we get

$$\mathbb{E}(e^{-\lambda S^m(\omega)}) \le \frac{\mathbb{E}\int e^{-\lambda \underline{S}(\omega,y)} d\mu(y)}{K_\beta \varphi_1(r_q)}. \tag{27}$$

Choose $\lambda = (\log q)^{-1}$. Combining Fubini's theorem and the inequalities (26), (27), and (18), we obtain

$$\mathbb{P}\left\{S^m(\omega) \le C'F\right\} \le K_\beta^{-1} \exp\left\{(-B_2 + C')\,(\log q)^{-1} F - \log \varphi_1(r_q)\right\}.$$

By the condition (24), we can choose $C' > 0$ such that $C_0 := (B_2 - C_2^{-1} - C')C_2 > 1$, and we have

$$\mathbb{P}\left\{S^m(\omega) \le C'F\right\} \le K_\beta^{-1} \exp\left\{-(B_2 - C' - C_2^{-1})(\log q)^{-1} F\right\} \le K_\beta^{-1} q^{-C_0}.$$

□

6 Proofs of Theorems

Let $m > 1$ be an integer. Let $N_j = m^j$. We write $S_j^M(\omega)$ and $S_j^m(\omega)$ instead of $S_{N_j,N_{j+1}}^M(\omega)$ and $S_{N_j,N_{j+1}}^m(\omega)$.

6.1 Proof of Theorem 2.1

First we notice that, by the monotonicity of φ_2 and of r_n, since $\varphi_2 \ge \varphi_1$, we have

$$\sum_{N_j+1}^{N_{j+1}} \varphi_2(r_n) \ge (N_{j+1} - N_j)\varphi_2(r_{N_{j+1}}) \ge (1 - m^{-1})N_{j+1}\varphi_1(r_{N_{j+1}}).$$

Hence the condition (9) implies that for large j we have

$$\sum_{N_j+1}^{N_{j+1}} \varphi_2(r_n) \ge (\log N_{j+1}) \max\left\{-\log \varphi_1(r_{N_{j+1}}), \log N_{j+1}\right\},$$

which is just the inequality (21) with $p = N_j + 1$ and $q = N_{j+1}$ when j is large enough. By Proposition 5.3 and the Borel–Cantelli lemma, we get $S_j^M(\omega) \le C \sum_{N_j+1}^{N_{j+1}} \varphi_2(r_n)$ almost surely. For any $N \ge 1$, there exists $k \in \mathbb{N}$ such that $N_k \le N < N_{k+1}$. Then we have

$$\max_{y\in X}\sum_{n=1}^{N}\chi_{B(y,r_n)}(\xi_n)\le\sum_{j=0}^{k}S_j^M\le C\sum_{j=0}^{k}\sum_{N_j+1}^{N_{j+1}}\varphi_2(r_n)$$
$$\le C\left(\sum_{n=1}^{N_k}\varphi_2(r_n)+\sum_{N_k+1}^{N_{k+1}}\varphi_2(r_n)\right).$$

Since $N_{k+1}-N_k=(m-1)m^k$ and $\sum_{n=1}^{N_k}\varphi_2(r_n)\ge m^k\varphi_2(r_{N_k+1})$,

$$\sum_{N_k+1}^{N_{k+1}}\varphi_2(r_n)\le(m-1)m^k\varphi_2(r_{N_k+1})\le(m-1)\sum_{n=1}^{N_k}\varphi_2(r_n).$$

We have thus proved (11).

The condition (9) also implies the inequality (24) with $p=N_j+1$ and $q=N_{j+1}$ when j is large enough. So, we can use Proposition 5.4 to prove (10). □

6.2 Proof of Theorem 1.1

The second assertion can be obtained by applying Proposition 4.1 to a point y with $\underline{\alpha}(y)\tau>1$, $r_n=bn^{-\tau}$ with $b\gg1$, and $a_n=1$. Let us prove the first assertion. By the definition of $\alpha_{\max}$, we have $\mu(B(x,r))\ge r^{\alpha_{max}+\varepsilon}$ for any $x\in X$, any $0<\varepsilon<1/\tau-\alpha_{\max}$, and small r. We take $\varphi_1(r)=r^{\alpha_{max}+\varepsilon}$ and $r_n=1/n^\tau$. Then it is easy to check the condition (9) and the divergence of the series $\sum_{n=1}^{+\infty}\varphi_1(r_n)$. The result follows from Theorem 2.1. □

7 Gibbs Measures on Subshifts of Finite Type

Let $m\ge2$ be an integer and A be an $m\times m$ matrix of 0's and 1's. Denote

$$\Sigma_A=\left\{x\in\{0,1,\dots,m-1\}^{\mathbb{N}}:A_{x_ix_{i+1}}=1\right\}$$

and define $\sigma:\Sigma_A\to\Sigma_A$ by $\sigma(x)_i=x_{i+1}$. Suppose A to be primitive. Then Σ_A is topology mixing. For $y=(y_i)_{i\ge0}\in\Sigma_A$, let $C_n(y):=[y_0,y_1,\dots,y_{n-1}]$ denote the cylinder containing y of $n:=|C_n(y)|$. Let $d(x,y)$ be the distance of $x,y\in\Sigma_A$ defined by $d(x,y)=1/m^{n-1}$, where n is the least integer such that $x_n\ne y_n$. Let $\phi:\Sigma_A\to\mathbb{R}$ be an α-Hölder potential, i.e.,

$$[\phi]_\alpha:=\sup_{x,y\in\sum_A}\frac{|\phi(x)-\phi(y)|}{d(x,y)^\alpha}<+\infty.$$

The transfer operator associated to ϕ is defined as follows:

$$L_\phi f(x)=\sum_{\sigma y=x}e^{\phi(y)}f(y).$$

We define the norm $\|f\|_\alpha = \|f\|_\infty + [f]_\alpha$ on the space of α-Hölder continuous functions $\mathscr{H}_\alpha$. The well-known Ruelle theorem [21] asserts that the spectral radius $\lambda > 0$ of $L_\phi : \mathscr{H}_\alpha \to \mathscr{H}_\alpha$ is an eigenvalue with a strictly positive eigenfunction h and there is a probability eigenmeasure ν for the adjoint operator L_ϕ^*, i.e., $L_\phi^* \nu = \lambda \nu$.

Choose h such that $\int h d\nu = 1$. Let $P(\phi) = \log \lambda$ (called the pressure of ϕ). The measure $\mu := h\nu$ is called the Gibbs measure associated to ϕ. The function $\phi + (\log h \circ \sigma - \log h) - P(\phi)$ is the normalization of ϕ. We note again this normalization ϕ, we have $\lambda = 1$, $P(\phi) = 0$, and $h = 1$. The Gibbs measure has the Gibbs property: there exists a constant $\gamma > 1$ such that

$$\frac{1}{\gamma} e^{S_n \phi(x)} \le \mu(C_n(x)) \le \gamma e^{S_n \phi(x)}$$

holds for all $x \in \Sigma_A$ and all $n \ge 1$ where $S_n f(y) := \sum_{j=0}^{n-1} f(\sigma^j y)$. Denote

$$e_{\min} = \min_{m \in M(\Sigma_A, \sigma)} \int -\phi dm \quad \text{and} \quad e_{\max} = \max_{m \in M(\Sigma_A, \sigma)} \int -\phi dm,$$

where $M(\Sigma_A, \sigma)$; is the set of invariant measures over $\sum_A$; the minimum and maximum exist by the weak-$*$ compactness of $M(\Sigma_A, \sigma)$. Note that $e_{\min} > 0$. In fact, we can deduce $S_N \phi(x) < 0$ for some $N \ge 1$ and for all $x \in \sum_A$ from the topological mixing property and the fact that $L_\phi \chi = \chi$, where χ denotes the function identically equal to 1 on $\sum_A$. So, for any invariant measure m, we have

$$\int -\phi dm = -\int \frac{S_N \phi}{N} dm > 0.$$

Proposition 7.1 *For any $0 < \delta < e_{\min}$, there exists $N(\delta) > 0$, such that for any cylinder C_n $(n \ge N(\delta))$*

$$\gamma^{-1} e^{-n(e_{\max} + \delta)} \le \mu(C_n) \le \gamma e^{-n(e_{\min} - \delta)}. \tag{28}$$

Proof. Firstly, we prove

$$\liminf_{n \to \infty} \min_{x \in \Sigma_A} \left(-\frac{S_n \phi(x)}{n} \right) \ge e_{\min}. \tag{29}$$

In fact, for any $\varepsilon > 0$, we have $\liminf_{n \to \infty} \min_{x \in \Sigma_A} (-\frac{S_n \phi(x)}{n}) \ge e_{\min} - \varepsilon$. Otherwise, there exist infinitely many x_n such that $-\frac{S_n \phi(x_n)}{n} < e_{\min} - \varepsilon$. Consider the measures $\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j x_n}$. Let μ_∞ be the limit of some subsequence $\{\mu_{n_i}\}$. Since

$$\int -\phi d\mu_{n_i} = \frac{1}{n_i} \sum_{j=0}^{n_i - 1} -\phi(T^j x_{n_i}) = -\frac{S_{n_i} \phi(x_{n_i})}{n_i} < e_{\min} - \varepsilon,$$

we get $\int -\phi d\mu_\infty \leq e_{\min} - \varepsilon$, which contradicts the definition of $e_{\min}$. Letting $\varepsilon \to 0^+$, we obtain (29). Therefore, for any $0 < \delta < e_{\min}$, when n is large enough, $-\frac{S_n\phi(x)}{n} \geq e_{\min} - \delta$ for all $x \in \Sigma_A$. For any cylinder C_n, using the Gibbs property, when n is large enough, for any $x \in C_n$,

$$\mu(C_n) \leq \gamma e^{S_n\phi(x)} \leq \gamma e^{-n(e_{\min}-\delta)}.$$

Similarly, we can prove $\limsup_{n\to\infty} \max_{x\in\Sigma_A}(-\frac{S_n\phi(x)}{n}) \leq e_{\max}$. Then using the Gibbs property, we obtain $\mu(C_n(x)) \geq \gamma^{-1}e^{-n(e_{\max}+\delta)}$. □

Proposition 7.2 *The system (Σ_A, σ, μ) is exponentially mixing, i.e., there exists $0 < \delta < 1$ such that for any cylinder E and measurable set F we have*

$$\mu(E \cap T^{-n}F) = \mu(E)\mu(F) + \mu(F)O(\delta^n)$$

for all $n \geq 1$, where the constant involved by the O is absolute.

Proof. For any $f \in L^1(\Sigma_A, \mu)$ and $g \in \mathscr{H}_\alpha$,

$$\left|\int f \circ \sigma^n g d\mu\right| \leq \mathbb{E}(f)\theta^n \|g\|_\alpha, \tag{30}$$

where $0 < \theta < 1$ is a constant (see, for example, [9]). First consider the case $E = C_n$ (C_n being an n-cylinder). Let $f = \chi_F$ and $g = \chi_{C_n} - \mu(C_n)$. Applying (30) to f, g, we obtain

$$|\mu(C_n \cap \sigma^{-\omega n}F) - \mu(C_n)\mu(F)| \leq \mu(F)\theta^{\omega n}\left(\|\chi_{C_n} - \mu(C_n)\|_\infty + [\chi_{C_n}]_\alpha\right).$$

We choose large enough ω such that $\delta_0 := \theta^\omega m^\alpha < \frac{1}{m}$. Since $\|\chi_{C_n} - \mu(C_n)\|_\infty \leq 1$ and $[\chi_{C_n}]_\alpha \leq m^{n\alpha}$, for any $n \geq 0$ we have

$$|\mu(C_n \cap \sigma^{-\omega n}F) - \mu(C_n)\mu(F)| \leq 2\mu(F)\delta_0^n. \tag{31}$$

For an arbitrary cylinder E, we distinguish two cases.

Case 1. There exists a cylinder C_n such that $E \subset C_n$. By Proposition 7.1, we have $\mu(E) \leq \mu(C_n) \leq \gamma\delta_1^n$, where δ_1 is a constant involved in (28). So

$$\begin{aligned}|\mu(E \cap T^{-\omega n}F) - \mu(E)\mu(F)| &\leq |\mu(C_n \cap T^{-\omega n}F) - \mu(C_n)\mu(F)| \\ &\quad + \mu(C_n)\mu(F) + \mu(E)\mu(F) \\ &\leq (2{\delta_0}^n + 2\gamma{\delta_1}^n)\mu(F).\end{aligned}$$

Case 2. E is the union of at most m^n n-cylinders $C_n^{(i)}$. Then

$$\begin{aligned}|\mu(E \cap \sigma^{-\omega n}F) - \mu(E)\mu(F)| &\leq \sum_i |\mu(C_n^{(i)} \cap \sigma^{-\omega n}F) - \mu(C_n^{(i)})\mu(F)| \\ &\leq 2\mu(F)(m\delta_0)^n.\end{aligned}$$

Let $\delta = \max\{\delta_1, m\delta_0\}$. We conclude that $(\sum_A, \sigma, \mu)$ is exponentially mixing by Proposition 3.2. □

Let

$$D^- = (\log m)^{-1}(e_{\min} - \delta) \quad \text{and} \quad D^+ = (\log m)^{-1}(e_{\max} + \delta),$$

where $0 < \delta < e_{\min}$ is an arbitrary number. Take $\varphi_1(r) = \gamma^{-1} r^{D^+}$, $\varphi_2(r) = \gamma r^{D^-}$. Combining Proposition 7.1 and Proposition 7.2, we can apply Theorem 2.1 to obtain the following

Theorem 7.3 *If* $\lim_{n\to\infty} r_n^{-D^+} \log^2 n/n = 0$, *then for* μ*-almost all* $x \in \Sigma_A$, *there exist* $0 < A(x), B(x) < \infty$, *such that*

$$A(x) \sum_{n=1}^{N} r_n^{D^+} \le \min_{y\in X} \sum_{n=1}^{N} \chi_{B(y,r_n)}(\sigma^n x) \le \max_{y\in X} \sum_{n=1}^{N} \chi_{B(y,r_n)}(\sigma^n x) \le B(x) \sum_{n=1}^{N} r_n^{D^-},$$

when N *is large enough.*

For the Bernoulli measure with the probability vector $(p, 1-p)$ $(0 < p \le \frac{1}{2})$, we can choose $\varphi_1(r) = r^{-\log p}$ and $\varphi_2(r) = r^{-\log(1-p)}$. Notice that $\alpha_{\max} = \alpha^* = -\log(1-p)/\log 2$.

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Some Remarks on the Hausdorff and Spectral Dimension of V-Variable Nested Fractals

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Summary. New families of random fractals, referred to as *V-variable fractals* (developed by Barnsley, Hutchinson, and Stenflo), are presented. In order to define and investigate a Laplacian and a Brownian motion–or, equivalently– a Dirichlet form on them, we introduce the class of *V-variable nested fractals.* These are nested fractals over families of iterated function systems with nested attractors and a uniform set of essential fixed points. So, the underlying graphs in the construction steps form sequences of comparable resistance networks. Dirichlet forms are defined ω-wise in a canonical way. In a survey style, we explain how to get Hausdorff and spectral dimension of such fractals by applying results of Furstenberg and Kesten on limits of products of random matrices.

1 Introduction

Fractal sets often arise as attractors of deterministic or random *iterated function systems*. However, for many "real world applications" the model of a deterministic as well as of a random homogeneous fractal is too "rigid," while the model of a standard random fractal "allows too much local inhomogeneity." Here, new families of random fractals, referred to as V-variable (developed by Barnsley, Hutchinson, and Stenflo), are regarded, which are intermediate between the notions of random homogeneous and standard random fractals. The parameter V describes the degree of "variability": At each magnification level any V-variable fractal has at most V key "shapes" or "patterns."

Several attempts in modeling physical phenomena on porous sets which carry a recursive structure–but not a strictly deterministic self-similar one– led to the development of an analysis on certain random models. Hereby, models from the theory of random fractals are modified in such a manner that one still can define a reasonable analysis on them. So, the range of possible random sets is restricted to realizations which are still connected and finitely ramified. The assumption of a finite ramification is crucial here, as we follow the classical theory developed by Kusuoka [21]; see also the monograph [19] and the references therein.

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The simplest examples of such models consist of a "random mixture" of a finite number of iterated function systems which can be "combined" in a suitable way. In this chapter, we regard V-variable nested fractals. For a better illustration, we restrict ourselves to the model case of fractals obtained by a V-variable mixing of two iterated function systems only, both having Sierpinski gasket-type attractors.

2 The Model: Two Shapes and Their Random Mixing

2.1 Classical Models

Let us be given two different triangle-shaped self-similar fractals, namely, the Sierpinski gasket $SG(2)$ and a modified Sierpinski gasket $SG(3)$ (see Fig. 1).

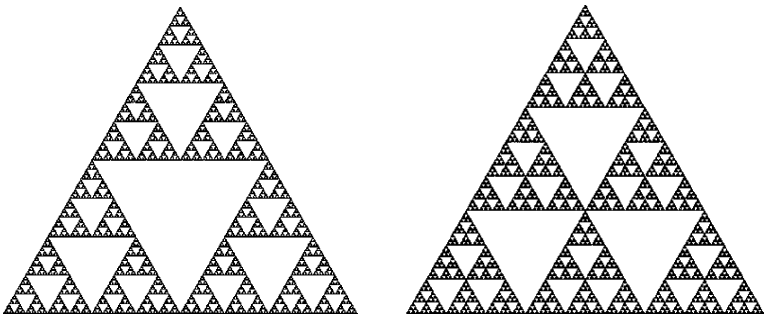

Fig. 1. The Sierpinski gasket $SG(2)$ (*left*) and the modified Sierpinski gasket $SG(3)$ (*right*)

Note that the first figure $SG(2)$ is self-similar with respect to an iterated function system $\mathcal{F} = \{f_1, f_2, f_3\}$ consisting of three contractive similitudes acting on the real plane $\mathbb{R}^2$, while the latter one can be obtained as the attractor of an iterated function system $\mathcal{G} = \{g_1, g_2, g_3, g_4, g_5, g_6\}$ containing six similitudes.

In order to get a wider class of possible shapes, we are now interested in "random mixing" of these two key patterns. Historically, this has been done in two different manners: Firstly, we have the "homogenous random" (see Fig. 2) model introduced by, e.g., Kifer [17]. The main feature of this class of random fractals is that in each construction step only *one* of the possible key patterns applies. The randomness can be coded with the help of the sequence of the iterated function systems which are applied. In the example shown in Fig. 2 this would be $\mathcal{G}, \mathcal{F}, \mathcal{G}, \mathcal{F}, \ldots$, if we code "from outside in".

Fundamental analytical and stochastic objects on such fractals have been introduced and studied mainly by Hambly; see [13] for the construction of a

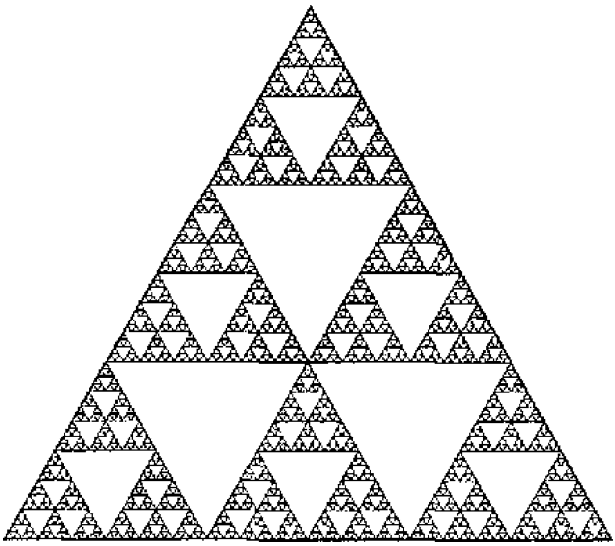

Fig. 2. Realization of a homogenous random fractal

Brownian motion, [14] for heat kernel estimates, and [15] for the definition and fundamental properties of a Laplacian.

On the other hand, in 1986, Mauldin and Williams (see [23]) and Falconer (see [5]) introduced independently the model of "random recursive" or "standard random" fractals. Hereby, in each construction step, each "cell" in the pre-fractals is replaced according to one of the iterated function systems $\mathcal{F}$ or $\mathcal{G}$, independently of the past of the process and independently of whatever happens in all the other cells. Here, the natural coding can be done with the help of a labeled tree where the number of offsprings from a node depends on the label at this node.

In the example shown in Fig. 3, the root has label $\mathcal{F}$ and, hence, it has three offsprings labeled $\mathcal{G}$, $\mathcal{G}$, and $\mathcal{F}$, respectively. Then the nodes with label $\mathcal{G}$ have six offsprings, while the node with label $\mathcal{F}$ has three offsprings, and so on. Properties of the natural Brownian motion on such kind of random fractals have been investigated by Hambly [12], and the corresponding Laplacian has been treated by Hambly and Barlow in [1].

In this chapter, we introduce new families of random fractals, called V-variable fractals, which "interpolate" somehow between the two models mentioned above. Thus, the case $V = 1$ will lead to the class of homogenous random fractals, while the case $V = \infty$ corresponds to the model of a random recursive fractal. In particular, we will point out how the parameter V–the degree of variability in the random machinery–affects the Hausdorff and spectral dimensions of a typical sample in the setting.

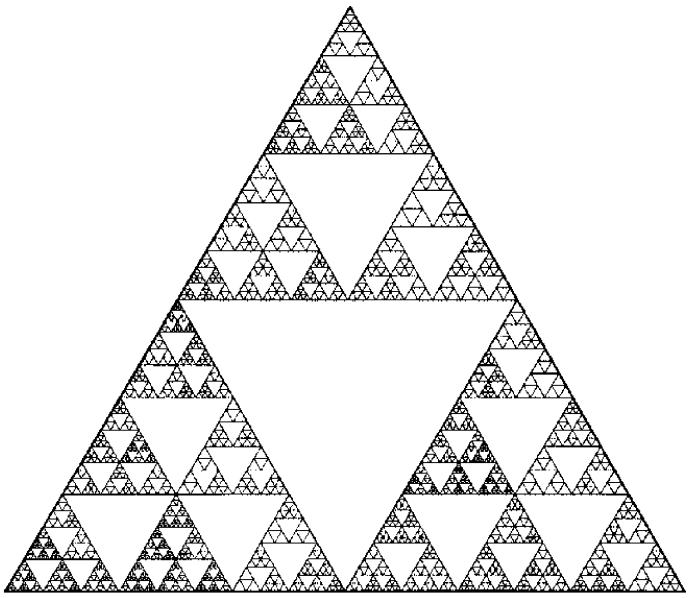

Fig. 3. Realization of a random recursive fractal

2.2 The V-Variable Model

In this section, we outline the definition of a V-variable fractal. This model has been developed by Barnsley, Hutchinson, and Stenflo. For details we refer the reader to [2–4].

Assume that V is an arbitrary positive integer $V \in \mathbb{N} \cup \{\infty\}$, and let us be given a collection of iterated function systems $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$ (not necessarily countably many) with a certain probability distribution $p_1, p_2, p_3, \ldots$. We discuss the construction with the help of the model case $V = 5$ and $(\mathcal{F}_1, \mathcal{F}_2) = (SG(2), SG(3)) =: (\mathcal{F}, \mathcal{G})$, where the iterated function systems $\mathcal{F}$ and $\mathcal{G}$ are those used in the Introduction. Moreover, fix a number $p_F \in (0, 1)$ and set $p_G := 1 - p_F$.

The construction of the V-variable fractal is done by constructing V-tuples of sets. We explain how to get the V members of the $(k+1)$th generation from the V members of the kth generation (see Fig. 4):

1° Choose one of the iterated function systems $\mathcal{F}$ or $\mathcal{G}$ according to (p_F, p_G).

2° Choose 3 (or 6, resp.) "parents" from generation k for "child" i of generation $k + 1$.

Run this loop V times.

We can code this machinery with the help of a matrix a_k (here: a 5×7-matrix) as shown in Fig. 4. In each row, in the first column, we store theinfor-

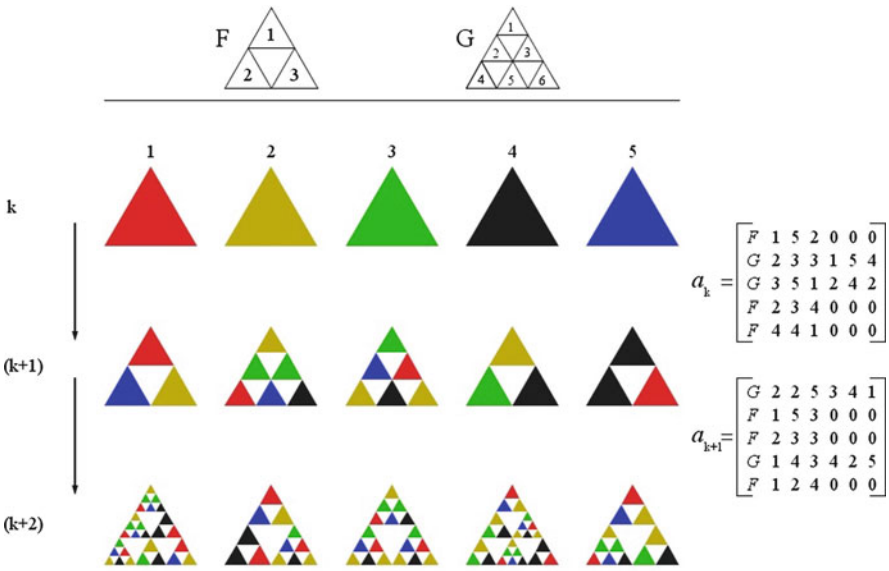

Fig. 4. The V-variable machinery and its coding by matrices

mation on which "genetic rule" applies for the corresponding child; in the rest of the row the (ordinal) numbers of the parents are listed. If rule $\mathcal{F}$ applies, we have three parents only, and we fill up the row with zeros.

3 Hausdorff Dimension

In determining the Hausdorff dimension of random fractals, which fit into the two classical approaches mentioned in Sect. 2.1, we distinguish two main techniques: Homogeneous random fractals ($V = 1$) are treated with the law of large numbers as we have to investigate limits of products of random numbers coding up the randomness. For finer results, typically the law of iterated logarithm is employed. In contrast, methods from generalized branching processes apply in the recursive ($V = \infty$) case. So, the challenge in considering the V-variable case is somehow to find the "structural interpolation" between sequences and trees. It turns out that regarding products of random $V \times V$ matrices which code up information in the construction process is a good tool.

For a better illustration we sketch here the Hausdorff dimension result obtained in [4]. Associated with the transformation from level k to level $k+1$ of V-tuples of triangles (referring to the example shown in Fig. 4, hence

$V = 5$), we define a $V \times V$-matrix $M^{(k)}(\alpha)$ as follows (hereby, $\alpha > 0$ is a free parameter):

$$M^{(k)}(\alpha) = \begin{pmatrix} \frac{1}{2^\alpha} & \frac{1}{2^\alpha} & 0 & 0 & \frac{1}{2^\alpha} \\ \frac{1}{3^\alpha} & \frac{1}{3^\alpha} & \frac{2}{3^\alpha} & \frac{1}{3^\alpha} & \frac{1}{3^\alpha} \\ \frac{1}{3^\alpha} & \frac{2}{3^\alpha} & \frac{1}{3^\alpha} & \frac{1}{3^\alpha} & \frac{1}{3^\alpha}. \\ 0 & \frac{1}{2^\alpha} & \frac{1}{2^\alpha} & \frac{1}{2^\alpha} & 0 \\ \frac{1}{2^\alpha} & 0 & 0 & \frac{2}{2^\alpha} & 0 \end{pmatrix}.$$

For example, as the first "child" of generation $k+1$ is built according to rule $\mathcal{F}$, we "distribute" three entries $(1/2)^\alpha$ over the first row of the matrix $M^{(k)}(\alpha)$, one for each "ancestor". Into the second row, we write six entries $(1/3)^\alpha$, the third "ancestor" appears twice in building the second "child", and hence the element $(2,3)$ of $M^{(k)}(\alpha)$ equals $2(1/3)^\alpha$. Note that $1/2$ and $1/3$ are the contraction ratios of the iterated function systems $\mathcal{F}$ and $\mathcal{G}$, respectively.

We define the *(Hausdorff) pressure function* by

$$\gamma_V^H(\alpha) := \lim_{k\to\infty} \frac{1}{k} \log\left(\frac{1}{V}\left\|M^{(1)}(\alpha)...M^{(k)}(\alpha)\right\|\right),$$

where $\|A\|$ is the norm given by the sum of all entries of the matrix A. Furstenberg–Kesten techniques (see [11]) ensure that $\gamma_V^H(\alpha)$ exists as a function of α and is independent of the realization of the experiment.

Moreover, $\gamma_V^H(.)$ is monotone decreasing and $\exists!\ d_H : \gamma_V^H(d_H) = 0$.

In [4], the following result is proved.

Theorem 1. *This zero of $\gamma_V(.)$ gives the Hausdorff (as well as the box counting) dimension of the V-variable fractal with probability one.*

Proof. We give only a very short sketch of the proof. Firstly, we observe that

$$\sum_{|i|=k} (\operatorname{diam} T_i)^\alpha = \left\|M^{(1)}(\alpha)...M^{(k)}(\alpha)\right\|,$$

where $T_i = T_{i_1...i_k}$ is the k-cell with address $i_1...i_k$. Hence, $\left\|M^{(1)}(\alpha)...M^{(k)}(\alpha)\right\|$ is proportional to the α-dimensional Hausdorff measure of the union of the V sets in the V-tuple of sets. Then, from the definition of the pressure function, we have the following trichotomy. If $\alpha < d$, then $\gamma_V^H(\alpha)$ blows up exponentially fast. If $\alpha > d$, then $\gamma_V^H(\alpha)$ decays to zero exponentially fast. If $\alpha = d$, then $\gamma_V^H(\alpha)$ blows up or decays at most subexponentially.

In view of the definition of the Hausdorff dimension as the jumping point of Hausdorff measures from $+\infty$ down to zero which are defined on their part by means of optimal δ-coverings, it remains to show that small triangles are "effective" coverings. The proof is quite technical and can be found in [4]. □

In [4], the following is obtained with the help of Monte Carlo simulations.

Proposition 1. *The Hausdorff dimension of the V-variable fractal is increasing in V.*

Remark 1. In the special cases $V = 1$ and $V = \infty$, these results agree with those obtained in [12,15] (for the case $V = 1$) and [5,23] (for the case $V = \infty$), respectively.

4 Spectral Dimension

4.1 The Deterministic Case

In order to determine the spectral dimension of a V-variable fractal, we first need to define a Laplacian, or equivalently, a Dirichlet form on it. To this end, we recall the corresponding construction in the deterministic case, outlined in, e.g., [6]; for a rigorous representation of the theory see [19].

Let us be given an iterated function system $\Psi = \{\psi_1, \ldots, \psi_M\}$ such that $K = \bigcup_{i=1}^{M} \psi_i(K)$ is a nested fractal (see Definition 2 in Sect. 5, or [22]). Denote by V_0 the set of essential fixed points (these are the three vertices of the starting triangle in both the cases $SG(2)$ and $SG(3)$) and set $V_n := \Psi(V_{n-1}), n \geq 1$. Then it holds that $(V_n)_{n\geq 1}$ forms an increasing sequence of finite sets of points, and the fractal K can be recovered from its supremum, i.e., $\overline{V_*} = K$, where $V_* := \bigcup_n V_n$.

Now a sequence of discrete Dirichlet forms is introduced as follows. For a function $u : V_* \longrightarrow \mathbb{R}$ define

$$\mathcal{E}_n[u] := \varrho^n \sum_{p\in V_n} \sum_{q\sim_n p} (u(p) - u(q))^2. \tag{1}$$

By the "right choice" of ϱ, the sequence $(\mathcal{E}_n[u])_n$ is nondecreasing, and constant if and only if u is harmonic. Note that ϱ is obtained by the *Gaussian principle* (see, e.g., [6]). In our model cases $\mathcal{F} = SG(2)$ and $\mathcal{G} = SG(3)$, these energy scaling factors are given by $\varrho_F = 5/3$ and $\varrho_G = 15/7$, respectively. Setting

$$\mathcal{D}(\mathcal{E}) := \{u : V_* \longrightarrow \mathbb{R} : \lim_{n\to\infty} \mathcal{E}_n[u] < \infty\},$$

it holds that $\mathcal{D}(\mathcal{E}) \hookrightarrow C(K)$. The limit quadratic form is defined by

$$\mathcal{E}[u] := \lim_{n\to\infty} \mathcal{E}_n[u] \quad \text{on} \quad \mathcal{D}(\mathcal{E}),$$

and the corresponding bilinear form is obtained by polarization:

$$\mathcal{E}(u, v) := \frac{1}{2} \left(\mathcal{E}[u + v] - \mathcal{E}[u] - \mathcal{E}[v]\right).$$

A real number λ is called the *eigenvalue* of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, if $\mathcal{E}(u,v) = \lambda\langle u, v\rangle$, $\forall v \in \mathcal{D}(\mathcal{E})$. Then the *eigenvalue counting function* $N(x) := \sharp\{\lambda_k \leq x\}$, $x > 0$ –counting according to multiplicities– is well defined (see, e.g., [20]). The spectral dimension d_S (of K) is the number describing the order of the asymptotic behavior of the eigenvalues and is the unique positive number such that

$$N(x) \asymp x^{d_S/2}, \quad x \to \infty.$$

Here, the symbol $\asymp$ means that there exist positive numbers C_1, C_2 and x_0 such that

$$C_1 x^{d_S/2} \leq N(x) \leq C_2 x^{d_S/2}, \quad x \geq x_0. \tag{2}$$

In the special case of nested fractals, d_S is given by $d_S = \frac{2\ln M}{\ln(\varrho M)}$, where ϱ is the energy scaling factor introduced in (1) while M denotes the number of similitudes in the iterated function system (see [20]). Thus, for our examples we have

$$d_S(SG(2)) = \frac{\ln 9}{\ln 5} \approx 1.365,$$

and

$$d_S(SG(3)) = \frac{\ln 36}{\ln 90 - \ln 7} \approx 1.403.$$

4.2 The V-Variable Case

4.3 Construction of the Energy Form

In the V-variable case, the construction of the energy form is done hierarchically and ω-wise. Following the outlines above, we define an increasing set of "nodes" approaching the fractal, which is done individually for any realization ω in the V-variable setting. The sets $V_0, V_1, V_2, \ldots$ are defined in a canonical way, as shown in Fig. 5.

Given the values of a function u on V_0, we are now seeking the harmonic extension on $V_1 \setminus V_0$ (i.e., we look for the value of the function in 3 (or 7) more points). Then we calculate the harmonic extension to $V_2 \setminus V_1$ on *any* of the 3 (or 6) subtriangles of V_1 according to an $\mathcal{F}$ (or $\mathcal{G}$) rule and independently on whatever happens in the neighbor cells. Note that hereby the finite ramification of the fractal triangles is crucial. Proceeding so defines a sequence of energies

$$\mathcal{E}_n^{(\omega)}[f] = \sum_{\overline{\imath}\in\omega, |\overline{\imath}|=n} R(\overline{\imath})\mathcal{E}_0[f \circ \psi_{\overline{\imath}}], \tag{3}$$

where

$$R(\overline{\imath}) = \prod_{j=1}^{|\overline{\imath}|} \varrho_j, \qquad \varrho_j \in \{\varrho_F = 5/3, \varrho_G = 15/7\},$$

and the summation in (3) is intended over all rooted limbs of length n in the tree ω.

By construction we have

$$\mathcal{E}_n^{(\omega)}[f_{|V_n}] = \inf\{\mathcal{E}_{n+1}^{(\omega)}[g] : g_{|V_n} = f_{|V_n}\}.$$

The limit form $\mathcal{E}^{(\omega)}$ with domain $\mathcal{D}(\mathcal{E}^{(\omega)})$ is introduced ω-wise, as described above for the deterministic case. Denote $K(\omega)$ the realization of the set and define $\mu(\omega)$ to be the Monge–Kantorovich limit measure by applying compositions of the corresponding Markov operators $\mathcal{M}_{\mathcal{F}}$ or $\mathcal{M}_{\mathcal{G}}$ according to the tree ω (see [16]). Then the following holds.

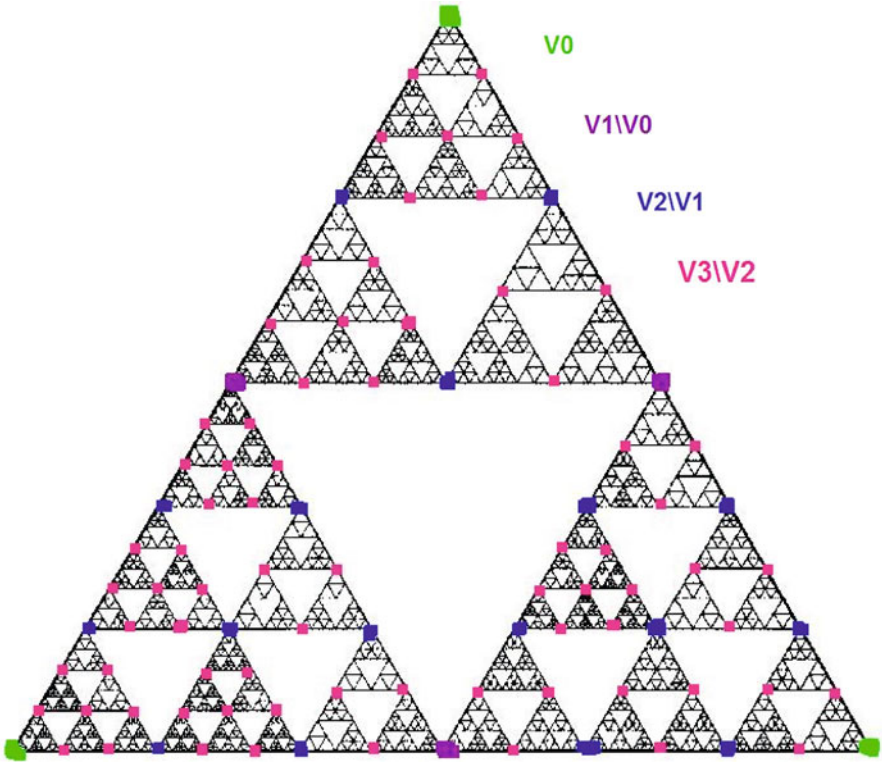

Fig. 5. The sets V_0, V_1, V_2, and V_3 are defined ω-wise

Proposition 2. [*See, e.g., [15]*]*For any $\omega \in \Omega$, the limit form $(\mathcal{E}^{(\omega)}, \mathcal{D}(\mathcal{E}^{(\omega)}))$ is a Dirichlet form on $L_2(K(\omega), \mu(\omega))$. The eigenvalues of $(\mathcal{E}^{(\omega)}, \mathcal{D}(\mathcal{E}^{(\omega)}))$ form a countable sequence with no accumulation point except $+\infty$. In particular, the eigenvalue counting function*

$$N^{(\omega)}(x) := \sharp\{\lambda_k \le x : \lambda_k \text{ is an eigenvalue of } (\mathcal{E}^{(\omega)}, \mathcal{D}(\mathcal{E}^{(\omega)}))\}, \quad x > 0,$$

is well defined for any ω.

4.4 Spectral Asymptotics

Now we explain how to get d_S in the V-variable case. In order to get the right resistance scalings, we need products and sums of the parameters ϱ_F and ϱ_G according to the V-variable (tree) setting which can be obtained as was done in determining the Hausdorff dimension by multiplying random matrices.

Define $T_F := \varrho_F M_F$ and $T_G := \varrho_G M_G$, hence $T_F = 5$ and $T_G = 90/7$. Note that these numbers are the mean crossing times of a random walk through the generating graph. (This holds in view of Einstein's relation, see, e.g., [7].) Associated with the transformation from level k to level $k+1$ of V-tuples of triangles we define a $V \times V$-matrix $M^{(k)}(\alpha)$ as follows (referring to the example shown in Fig. 4, see page 271):

$$M^{(k)}(\alpha) = \begin{pmatrix} \frac{1}{T_F^\alpha} & \frac{1}{T_F^\alpha} & 0 & 0 & \frac{1}{T_F^\alpha} \\ \frac{1}{T_G^\alpha} & \frac{1}{T_G^\alpha} & \frac{2}{T_G^\alpha} & \frac{1}{T_G^\alpha} & \frac{1}{T_G^\alpha} \\ \frac{1}{T_G^\alpha} & \frac{2}{T_G^\alpha} & \frac{1}{T_G^\alpha} & \frac{1}{T_G^\alpha} & \frac{1}{T_G^\alpha} \\ 0 & \frac{1}{T_F^\alpha} & \frac{1}{T_F^\alpha} & \frac{1}{T_F^\alpha} & 0 \\ \frac{1}{T_F^\alpha} & 0 & 0 & \frac{2}{T_F^\alpha} & 0 \end{pmatrix}.$$

Similarly as was done in Sect. 3, we define a (spectral) pressure function γ_V^S:

$$\gamma_V^S(\alpha) := \lim_{k\to\infty} \frac{1}{k} \log\left(\frac{1}{V}\left\|M^{(k)}(\alpha)...M^{(1)}(\alpha)\right\|\right),$$

where, as before, $\|A\|$ is the norm given by the sum of all entries of the matrix A.

Again, $\gamma_V^S(\alpha)$ exists as a function of α, and is independent of the realization of the experiment. Moreover, γ_V^S is monotone decreasing and has a unique zero $d_S/2 = d_S(V)/2$. It is worth pointing out that the corresponding result on the asymptotic behavior of the eigenvalue counting function is weaker than the Kigami–Lapidus result for the deterministic case stated in inequality (2). The proof of the following theorem can be found in [9].

Theorem 2. *Denote by $d_S/2 = d_S(V)/2$ the zero of the pressure function $\gamma_V^S(d)$. Then it holds that*

$$N^{(\omega)}(x)x^{-\alpha} \longrightarrow 0 \qquad \text{for a.e. } \omega \quad \text{if } \alpha > d_S/2,$$

and

$$N^{(\omega)}(x)x^{-\alpha} \longrightarrow \infty \qquad \text{for a.e. } \omega \quad \text{if } \alpha < d_S/2,$$

where $N^{(\omega)}(.)$ is the eigenvalue counting function introduced in Proposition 2.

Hence, we can call this value $d_S = d_S(V)$ the *spectral dimension* of the V-variable fractal.

Proof. Here, we give only a very rough sketch of the proof. For details we refer to [9].

1° Assume that there is a number $d > 0$ such that a.s. it holds that

$$0 < \liminf_{x\to\infty} N(x)x^{-d} < \limsup_{x\to\infty} N(x)x^{-d} < \infty.$$

Denote by $N_i^{(k)}(.)$ the eigenvalue counting function of the Dirichlet form of the ith component at generation $k, i = 1, ..., V$.

Denote

$$h_i^{(k)}(x) := x^{-d}N_i^{(k)}(x), \qquad h^{(k)}(x) := \left(h_1^{(k)}(x), ..., h_V^{(k)}(x)\right)^T.$$

As the space of harmonic functions is finite dimensional, we have that

$$N_i^{(k+1)}(x) \sim \sum_{j=1}^{M} N_j^{(k)}(x), \qquad M \in \{M_F, M_G\},\ j \in \{1, ..., V\}.$$

Moreover, the quantities in the last formula satisfy–with an error of smaller order–a scaling property of the type

$$N_j^{(k)}(x) \sim N_i^{(k+1)}(x/\varrho M),$$

where $\varrho = \varrho_F$ if $M = M_F$ and $\varrho = \varrho_G$ if $M = M_G$.

Then we get from the Dirichlet–Neumann bracketing that

$$h^{(k+1)}(x) \sim M^{(k)}(\alpha)...M^{(1)}(\alpha)h_1,$$

where h_1 can be chosen to be a vector with positive and finite components.

The assumption that each component of $h^{(k)}(x)$ is bounded for sufficiently large x leads to the assertion.

2° If the assumption made in 1° would be true, then we could get d as the solution of $\gamma_V(d) = 0$. Unfortunately, the assumption is not true. But one can prove

$$N(x)x^{-\alpha} \longrightarrow 0 \qquad \text{a.s.} \quad \text{if } \alpha > d,$$

and

$$N(x)x^{-\alpha} \longrightarrow \infty \qquad \text{a.s.} \quad \text{if } \alpha < d. \qquad \square$$

Similarly as for the Hausdorff dimension, we have the following.

Proposition 3. *The spectral dimension $d_S = d_S(V)$ is increasing in V.*

Note that, in addition, the value of $d_S = d_S(V)$ of course depends on the choice of the vector of probabilities (p_F, p_G). More precisely, the following holds.

Proposition 4. *The spectral dimension $d_S = d_S(V)$ is a continuous function of the parameter p_F.*

The last observation can be employed in the construction of V-variable models with prescribed geometric and analytic properties (see [8]).

5 Generalization: V-Variable Nested Fractals

In the last section, we just give some guidelines as to which classes of families of iterated function systems the $SG(2)$–$SG(3)$-model could be extended.

Mathematically precisely, the model we have in mind is a *V-variable nested fractal* (see Definition 3 below). This is defined to be a V-variable fractal (hence, a collection of iterated function systems together with a probability distribution), where

1. Each of the iterated function systems $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, ...$ is an iterated function system generating a *nested fractal* .
2. All of the iterated function systems $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, ...$ have the same set of *essential fixed points.*

These assumptions are crucial: Nested fractals (see Definition 2 below for a rigorous definition) satisfy certain symmetry and ramification properties which are fundamental in constructing an energy form or a diffusion process in the sense of Kusuoka (see [21]). The second assumption ensures that the attractors of the iterated function systems in the random mixing "match each other" in an appropriate way so that we can construct sequences of compatible resistance networks in the sense of Kigami (see [18]).

For the convenience of the reader, we recall here the definition of essential fixed point as well as of nested fractal. Let $\Psi = \{\psi_1, \dots, \psi_N\}$ be an iterated function system consisting of contractive similitudes acting on the real plane. Note that all the definitions and results extend in a natural way to any complete metric space.

As each of the maps ψ_i, $i = 1, \dots, N$, is a contraction, it has a unique fixed point in $\mathbb{R}^2$. Denote by V the set of fixed points of the maps $\psi_1, \dots, \psi_N$, which are assumed to be pairwise different, i.e., $\sharp V = N$.

Definition 1. *A point $P \in V$ is called an* essential fixed point of the family Ψ, *if there exist $i, j \in \{1, \dots, N\}$, $i \neq j$ and a point $Q \in V$ such that $\psi_i(P) = \psi_j(Q)$.*

Example 1. The standard Sierpinski gasket is the attractor of the iterated function system $\mathcal{F} = \{f_1, f_2, f_3\}$ where the similitudes $f_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are given by

$$f_i(x) := \frac{1}{2}(x - P_i) + P_i, \quad i = 1, 2, 3.$$

Here the point P_i is the fixed point of the mapping f_i, hence the points P_1, P_2, P_3 are just the vertices of the outer triangle (see Fig. 1), which we, without loss of generality, assume to be $P_1 = (0, 0)$, $P_2 = (1, 0)$, and $P_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Each of the fixed points is an essential fixed point, because it holds that $\psi_2(P_1) = \psi_1(P_2)$, $\psi_3(P_1) = \psi_1(P_3)$, and $\psi_3(P_2) = \psi_2(P_3)$.

In contrast, the set $SG(3)$ is the attractor of the iterated function system $\mathcal{G} = \{g_1, \ldots, g_6\}$ with similitudes $g_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$g_i(x) := \frac{1}{3}(x - Q_i) + Q_i, \quad i = 1, \ldots, 6,$$

where $Q_1 = P_1$, $Q_3 = P_2$, and $Q_6 = P_3$, while the points Q_2, Q_4, and Q_5 are the midpoints of the line segments P_1P_2, P_1P_3, and P_2P_3, respectively. It is easy to see that the fixed points Q_1, Q_3, Q_6 are essential, but not so the fixed points Q_2, Q_4, Q_5.

Denote $V_0 := \{P_1, \ldots, P_M\}, M \leq N$, the set of the essential fixed points of the family Ψ. We define for any $(i_1, \ldots, i_n) \in \{1, \ldots, N\}^n$ the mapping

$$\psi_{i_1,\ldots,i_n}(A) := \psi_{i_1} \circ \ldots \circ \psi_{i_n}(A), \qquad A \subseteq \mathbb{R}^2.$$

We call a set

$$V_{i_1,\ldots,i_n} := \psi_{i_1,\ldots,i_n}(V_0)$$

an *n-cell* , and a set

$$K_{i_1,\ldots,i_n} := \psi_{i_1,\ldots,i_n}(K)$$

an *n-complex.*

Definition 2. *A set K is a* nested fractal *if it is the attractor of an iterated function system $\Psi = \{\psi_1, \ldots, \psi_N\}$ with uniform ratio satisfying*

1. Connectivity: *For each pair of 1-cells C and $\tilde{C}$, there is a sequence $\{C_i, i = 0, \ldots, k\}$ of 1-cells, such that $C_0 = C$, $C_k = \tilde{C}$, and $C_{i-1} \bigcap C_i \neq \varnothing$, $i = 1, \ldots, k$.*
2. Symmetry: *For any $P, Q \in V_0$, the reflection at the hyperplane*

$$H_{PQ} := \{x \in \mathbb{R}^2 : |x - P| = |x - Q|\}$$

maps K to itself.
3. Nesting: *For any pair of distinct n-tuples $(i_1, \ldots, i_n), (j_1, \ldots, j_n) \in \{1, \ldots, N\}^n$ the following* nesting condition *is satisfied:*

$$\psi_{i_1,\ldots,i_n}(K) \cap \psi_{j_1,\ldots,j_n}(K) = \psi_{i_1,\ldots,i_n}(V_0) \cap \psi_{j_1,\ldots,j_n}(V_0).$$

4. Open set condition: *There exists a nonempty, bounded, open set O such that the sets $\psi_i(O), i = 1, \ldots, N$, are pairwise disjoint and $\bigcup_{i=1}^{N} \psi_i(O) \subseteq O$.*

From the nesting axiom it follows that nested fractals are finitely ramified; well-known examples are the Sierpinski gasket and the von Koch snowflake. "Prototypes" of self-similar fractals which are not nested are the classical middle third Cantor set (connectivity fails) and the Sierpinski carpet (nesting fails).

Definition 3. *A* V-variable nested fractal *is a V-variable fractal over a family $\mathbf{F} = \{F^\lambda : \lambda \in \Lambda\}$ of iterated function systems such that*

1. *Each iterated function system F^λ, $\lambda \in \Lambda$, is a nested fractal.*
2. *Each of the iterated function systems $\mathbf{F} = \{F^\lambda : \lambda \in \Lambda\}$ has the same set of essential fixed points.*

In view of Example 1 it follows that the iterated function systems $\mathcal{F} = \{f_1, f_2, f_3\}$ and $\mathcal{G} = \{g_1, \ldots, g_6\}$ generating the fractals $SG(2)$ and $SG(3)$ form a family in the sense of the latter definition.

Another example could be the V-variable mixing of a filled and a perforated snowflake, as pictured in Fig. 6.

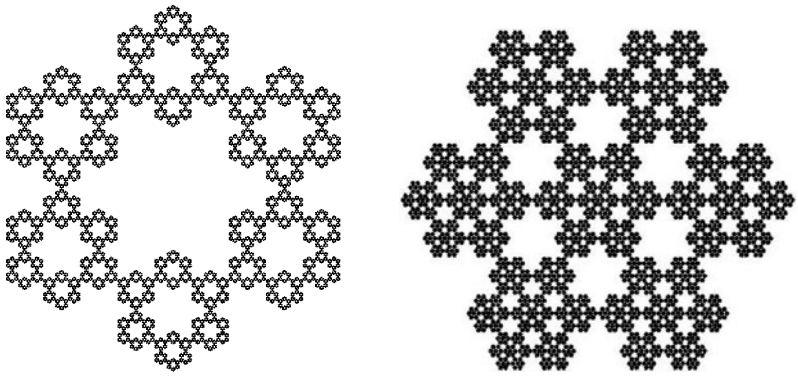

Fig. 6. Nested snowflakes, generated by 6 and 7 similitudes, respectively. The set of essential fixed points consists, for both fractals, of the 6 vertices of the outer hexagon. The snowflake shown on the right possesses an additional (nonessential) fixed point, namely its central point

Moreover, one could think about fractals built by combining in a V-variable manner intervals and Koch curves, two nested fractals with the same set of essential fixed points. This might be a useful and appealing model in approaching variational problems on domains with fractal boundaries or transmission problems across a fractal layer whenever the assumption of strict self-similarity is too rigid (but the randomness is not "too random").

Finally, we want to emphasize that one could also combine more than two iterated function systems. Look again at Example 1 and Fig. 1. For any $\lambda \in \mathbb{N}$ there is a corresponding fractal $SG(\lambda)$, obtained again iteratively by subdividing each triangle of construction step K_n into λ^2 smaller triangles and removing all downward-pointing ones. Every step $\lambda(\lambda+1)/2$ of the λ^2 smaller triangles is kept; hence, the Hausdorff dimension of $SG(\lambda)$ is given by

$$d_H SG(\lambda) = \frac{\ln\left[\lambda(\lambda+1)/2\right]}{\ln \lambda} \longrightarrow 2, \text{ as } \lambda \to \infty.$$

See, e.g., [10] for details of this family of fractals. Each of these sets $SG(\lambda)$ has the same set of essential fixed points, namely, P_1, P_2, P_3 (see Example 1). Hence, arbitrarily many of sets $SG(\lambda)$ can be combined in the sense of Definition 3, leading to a possibility of modeling "everything between string and drum" by a random combination of finer and finer Sierpinski-like gaskets.

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Cantor Boundary Behavior of Analytic Functions

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Summary. Let $\mathbb{D}$ be the open unit disc and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For $f(z)$ analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, it follows from the open mapping theorem that $\partial f(\mathbb{D}) \subset f(\partial\mathbb{D})$. These two sets have very rich and intrigue geometric properties. When $f(z)$ is univalent, then they are equal and there is a large literature to study their boundary behaviors. Our interest is on the class of analytic functions $f(z)$ for which the image curves $f(\partial\mathbb{D})$ form infinitely many loops everywhere, they are not univalent of course. We formulate this as the *Cantor boundary behavior*. We give sufficient conditions for such property, making use of the distribution of the zeros of f' and the mean growth rate of f'. Examples includes the complex Weierstrass functions, and the Cauchy transform of the canonical Hausdorff measure on the Sierpiski gasket.

1 Introduction

Let $\mathbb{D}$ be the open unit disk and let $\partial\mathbb{D}$ be the boundary of $\mathbb{D}$. For f analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, it follows from the open mapping theorem that $\partial f(\mathbb{D}) \subset f(\partial\mathbb{D})$. These two sets have very rich and intriguing geometric properties. In fact, when f is conformal, then they are equal and there is a large literature on the study of their boundary behaviors; the reader can refer to Pommerenke [19] and Duren [8] for the classical developments, and to Lawler [13] for the more recent development in connection with the Brownian motion. Also, the well-known conjecture that the Mandelbrot set M is locally connected can be treated as a problem of boundary behavior of conformal maps, because the complement of M in $\mathbb{C}_\infty (= \mathbb{C} \cup \{\infty\})$ is the image of a conformal map f on $\mathbb{D}$ [2,7]. Hence, the problem is equivalent to whether the f can be extended continuously to the boundary of $\mathbb{D}$ [19].

Our interest is in the class of analytic functions f for which the image curve $f(\partial\mathbb{D})$ forms infinitely many loops everywhere; they are not univalent of course. Intuitively, for any open arc I on $\partial\mathbb{D}$, $f(I)$ contains at least one loop (which is inside $f(\mathbb{D})$). If we let $C = f^{-1}(\partial f(\mathbb{D}))$, then $C = \partial\mathbb{D} \setminus \bigcup_{i=1}^{\infty} I_i$,

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where I_i are open arcs of $\partial\mathbb{D}$, $f(I_i) \subset f(\mathbb{D})$, and $\overline{\bigcup_{i=1}^{\infty} I_i} = \partial\mathbb{D}$. The condition of loops everywhere implies that C is a nowhere dense close set (a Cantor-type set) and the image stretches out to be $f(C) = \partial f(\mathbb{D})$. This boundary behavior was first observed by Strichartz et al. [14] through some computer graphics of the Cauchy transform on the Sierpinski gasket (see Fig. 3).

We formulate this property as the *Cantor boundary behavior* on $\mathbb{D}$ and carry out an investigation via (a) the distribution of zeros of f' and (b) the fast mean growth rate of $|f'|$ for z near the boundary (faster than the well-known rate for univalent functions [18]). Our theorems allow us to use the infinite Blaschke product to construct examples with the Cantor boundary behavior. We show that the complex Weierstrass functions will have this property (see Fig. 1). In the fractal case, we show that the Cauchy transform of the canonical Hausdorff measure on the Sierpinski gasket also possesses this property, which answers the Cantor set conjecture in [14].

The detail of proofs will appear elsewhere.

2 The Basic Setup

The geometry of the curve $f(\partial\mathbb{D})$ can be very complicated, and there are difficulties in obtaining a precise meaning of "infinitely many loops" from the intuitive idea. Our approach is to use a weaker topological concept of the connected components determined by $f(\partial\mathbb{D})$.

By a component of a set E in a topological space, we mean a maximal connected subset of E. Let $K \subset \mathbb{C}$ be a compact subset, then $\mathbb{C}_\infty \setminus K$ has at most countably many components, they are simply connected if E is connected. Furthermore, if K is locally connected, then each component will have a locally connected boundary [21].

For Ω a bounded domain in $\mathbb{C}$, we will consider the components of $\mathbb{C}_\infty \setminus f(\partial\Omega)$ and $\Omega \setminus f^{-1}\big(f(\partial\Omega)\big)$. The former is used as a rigorous setup for the loose concept of loops of $f(\partial\Omega)$, and the second one divides Ω into connected subregions that map onto components of $\mathbb{C}_\infty \setminus f(\partial\Omega)$. These two classes of components play a key role in our consideration. In view of the facts stated in the last paragraph, we have the following.

Proposition 1. *Let Ω be a bounded simply connected domain. Let f be a non-constant analytic function in Ω and continuous on $\overline{\Omega}$. Suppose $\mathbb{C}_\infty \setminus f(\partial\Omega) = \bigcup_{j\geq 0} \mathcal{W}_j$ is the unique decomposition into components. Then*

(i) *Each $\mathcal{W}_j$ is a simply connected domain.*
(ii) *$f^{-1}(f(\partial\Omega))$ is connected and each component of $\Omega \setminus f^{-1}(f(\partial\Omega))$ is a simply connected domain.*

Let $n_f(w; K)$ denote the number of roots $z \in K$ for the equation $f(z) = w$, counting according to multiplicity. The more precise relationship of the components is as follows.

Proposition 2. *With the above assumption, suppose that* $\mathcal{W}_j \cap f(\Omega) \neq \emptyset$. *Let* $f^{-1}(\mathcal{W}_j) = \bigcup_{k=1}^{q_j} O_j^k$ *be the decomposition of the open set* $f^{-1}(\mathcal{W}_j)$ *into components. Then* $1 \leq q_j < +\infty$; *each* O_j^k *is a simply connected component of* $\Omega \setminus f^{-1}(f(\partial\Omega))$ *and*

$$f(O_j^k) = \mathcal{W}_j, \qquad f(\partial O_j^k) = \partial \mathcal{W}_j. \tag{1}$$

Moreover, for each $w \in \mathcal{W}_j$, $n_f(w; O_j^k) \equiv n_{j,k}$ *and* $\sum_{k=1}^{q_j} n_{j,k} \equiv n_f(w, \Omega)$.

If, in addition, $\partial\Omega$ *is locally connected, then all the* $\partial\mathcal{W}_j$ *and* ∂O_j^k *are locally connected.*

The above O_j^k has a close relationship with the zeros of f'.

Proposition 3. *With the above assumption and notation,* f' *has* $n_{j,k}-1$ *zeros in* O_j^k.

The proof depends on the following lemma and the Riemann mapping theorem.

Lemma 1. *Let* f *be analytic in* $\mathbb{D}$ *with* $f(\mathbb{D}) = \mathbb{D}$. *Suppose* $n_f(w; \mathbb{D}) \equiv k$ *for all* $w \in \mathbb{D}$; *then* f *is a finite Blaschke product of degree* k, *and* $f'(z)$ *has* $k-1$ *zeros in* $\mathbb{D}$.

We need a special result on the finite Blaschke product f, which provides a way to cut up the domain $\mathbb{D}$ into simply connected subregions so that f is univalent in each of the subregions. It will be applied to f from O_j^k onto $\mathcal{W}_j$ (Lemma 2). For clarity, we use $\mathbb{D}_z$ and $\mathbb{D}_w$ to denote the unit disk $\mathbb{D}$ as domain and range.

Proposition 4. *Let* f *be a Blaschke product of degree* k *and let* $\mathcal{Z}$ *be the set of zeros of* f' *in* $\mathbb{D}_z$. *Suppose* $f(\mathcal{Z}) \subset L$ *where* L *is a Jordan curve in* $\mathbb{D}_w$ *except for an end point* $\xi_0 \in \partial\mathbb{D}_w$. *Let* $G = \mathbb{D}_w \setminus L$ *(it is simply connected), and let* $f^{-1}(G) = \bigcup_{j=1}^{d} O_j$ *be the connected component decomposition as in Proposition 2. Then* $d = k$, *and* f *is univalent in* O_j *with* $f(O_j) = G$.

3 The Cantor Boundary Behavior

With the preceding notation, we can define the Cantor boundary behavior for f.

Definition 1. *Let* f *be analytic in* $\mathbb{D}$ *and continuous on* $\overline{\mathbb{D}}$. *We say that* f *has the* Cantor boundary behavior *if* $f^{-1}(\partial f(\mathbb{D}))$ *and* $\partial O \cap \partial\mathbb{D}$ *are Cantor type sets in* $\partial\mathbb{D}$ *(whenever it is non-empty) where* O *is any simply connected component of* $\mathbb{D} \setminus f^{-1}(f(\partial\mathbb{D}))$ *(as in Proposition 1).*

The geometric meaning of the definition is as follows: for $C := f^{-1}(\partial f(\mathbb{D})) \subset \partial\mathbb{D}$ to be a Cantor type set, $C = \partial\mathbb{D} \setminus \bigcup_{k=1}^{\infty} I_k$ where I_k are disjoint open arcs of $\partial\mathbb{D}$, with $\overline{\bigcup_k I_k} = \partial\mathbb{D}$ and $f(I_k) \subset f(\mathbb{D})$. Intuitively, the curve $f(I_k)$ forms a loop (closed curves) inside the image $f(\mathbb{D})$, and the outer boundary of the image $f(\mathbb{D})$ comes from the nowhere dense closed set C in $\partial\mathbb{D}$. The same explanation applies for $O_j^k \cap \partial\mathbb{D}$ with its image in the boundary of $f(O_j^k) = \mathcal{W}_j$ (as in Proposition 2). Putting these together, we can perceive that for each loop $f(I_k)$, there is another family of loops inside $f(I_k)$ with the Cantor boundary behavior, and inductively we can see that for $f(\partial\mathbb{D})$ there is an infinite family of loops inside loops.

Also, it is clear that the definition implies the following: for any subarc $I \subset \partial\mathbb{D}$, $f(I) \not\subseteq \partial\mathcal{W}$ for any component $\mathcal{W}$ of $\mathbb{C}_\infty \setminus f(\partial\mathbb{D})$.

Our main lemma is as follows.

Lemma 2. *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. If there is a non-degenerated arc $J \subset \partial\mathbb{D}$ such that $f(J) \subset \partial f(\mathbb{D})$, then there exists a non-degenerated subarc $I \subset J$ and a bounded simply connected domain $D \subset \mathbb{D}$ such that $I \subset \partial D$, ∂D is locally connected, and f is univalent in D.*

Sketch of proof. Let $J = \{e^{i\theta} : 0 \le \theta_1 \le \theta \le \theta_2 < 2\pi\}$. We choose a Jordan curve γ such that $\gamma^o \subset \mathbb{D}$ and has two end points $e^{i\theta_1}, e^{i\theta_2}$. Let Ω be the closed region enclosed by the simple closed curve $J \cup \gamma$ and let $\widetilde{f} = f|\overline{\Omega}$. Then, by assumption, we have $\widetilde{f}(J) \subset \partial\widetilde{f}(\Omega)$. Let $\Gamma = \widetilde{f}(J \cup \gamma)$; then, by applying Propositions 1 and 2, we have the decompositions

$$C_\infty \setminus \Gamma = \bigcup_{j\ge 1} \mathcal{W}_j \quad \text{and} \quad \widetilde{f}^{-1}(\mathcal{W}_j) = \bigcup_{k=1}^{q_j} O_j^k.$$

As $\widetilde{f}(J) \subset \partial\widetilde{f}(\Omega)$, we can show that one of the O_j^k will contain a subarc $\ell \subset J$. We denote this simply connected domain by O^* and the corresponding $\mathcal{W}_j$ by $\mathcal{W}^*$.

Now consider $f : O^* \to \mathcal{W}^*$. By Proposition 2, $f(O^*) = \mathcal{W}^*$, $f(\partial O^*) = \partial\mathcal{W}^*$, and each $w \in \mathcal{W}^*$ has multiplicity, say, q. Let $\mathcal{Z}$ denote the $q-1$ zeros of f' in O^* and let L be a Jordan curve in $\mathcal{W}^*$ with one end point at $\partial\mathcal{W}^*$. We can apply Proposition 4 (through the Riemann mapping theorem) to divide O^* into simply connected regions $D_i, i = 1, \ldots, q$ and f is univalent on each of the regions. We select the one D_i such that $\ell \cap D_i$ is a non-degenerated arc of $\partial\mathbb{D}$. we denote this D_i by D, and the arc $\ell \cap D_i$ by I. □

We also need a similar lemma on the components.

Lemma 3. *Lemma 2 still holds if we replace the assumption $f(J) \subset \partial f(\mathbb{D})$ by $f(J) \subset \partial\mathcal{W}$ for some component $\mathcal{W}$ of $f(\mathbb{D}) \setminus f(\partial\mathbb{D})$.*

Now we can state our first theorem for the Cantor boundary behavior.

Theorem 1. *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose the set of limit points of $\mathcal{Z} = \{z \in \mathbb{D} : \ f'(z) = 0\}$ is $\partial\mathbb{D}$. Then f has the Cantor boundary behavior.*

The proof is simple by the two lemmas. We show that $C = f^{-1}(\partial f(\mathbb{D}))$ does not contain any subarc of $\partial\mathbb{D}$; this will imply that C is a Cantor-type set. Suppose otherwise, then there exists a circular arc $J = \{e^{i\theta} : \theta_1 \le \theta \le \theta_2\} \subset f^{-1}(\partial f(\mathbb{D}))$. It follows that $f(J) \subset \partial f(\mathbb{D})$. By Lemma 2, there exists a simply connected domain $D \subset \mathbb{D}$ and a non-degenerated subarc $I \subset J$ such that $I \subset \partial D$ and f is univalent in D. Hence, $f'(z) \neq 0$ in D, i.e., $\mathcal{Z} \cap D = \emptyset$ and $\mathcal{Z}$ does not have a limit point in I^o. This is a contradiction, and therefore C is a Cantor set. The case for the components $\mathcal{W}_j$ follows from the same proof.

We can construct analytic functions with the Cantor boundary behavior explicitly using the theorem and the infinite Blaschke product. For example, we let $\theta_{k,m} = m/k,\ m = 1, 2, \ldots, k-1,\ k = 2, 3, \ldots$, and let $z_{k,m} = (1 - k^{-s})e^{i2\pi\theta_{k,m}}$. Since $\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}(1 - |z_{k,m}|) = \sum_{k=2}^{\infty}(k-1)k^{-s} < \infty$ if $s > 2$, then the Blaschke product

$$p_s(z) = \prod_{k=2}^{\infty}\prod_{m=1}^{k-1}\frac{|z_{k,m}|}{z_{k,m}}\frac{z_{k,m} - z}{1 - \overline{z}_{k,m}z}$$

converges uniformly for $|z| \le r < 1$ and $|p_s(z)| \le 1$ for $z \in \mathbb{D}$. For $s > 2$, we define $f(z) = \int_0^z p_s(\xi)d\xi$. Then f satisfies the assumptions in Theorem 1 and hence has the desired property.

In general, the zeros of f' are not easy to locate. We will give another sufficient condition of different nature for f to have the Cantor boundary behavior. It is related to the growth rate of the integral mean of $|f'|$.

Let $\mathcal{S}$ denote the class of all analytic functions f with $f(0) = 0, f'(0) = 1$, that are univalent in $\mathbb{D}$. For $\lambda > 0$, we define

$$\beta(\lambda) = \sup_{f\in\mathcal{S}}\left(\limsup_{r\to 1^-}\frac{\log\left(\int_0^{2\pi}|f'(re^{i\theta})|^{\lambda}d\theta\right)}{-\log(1-r)}\right) \tag{1}$$

and call it the *integral mean spectrum* of $\mathcal{S}$ [18,19]. A nice survey of this and related topics can be found in [3]. It follows easily that for any $f \in \mathcal{S}$ and for any fixed $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) > 0$ such that

$$\frac{1}{2\pi}\int_0^{2\pi}|f'(re^{i\theta})|^{\lambda}d\theta \le \frac{C}{(1-r)^{\beta(\lambda)+\varepsilon}}, \quad \frac{1}{2} < r < 1. \tag{2}$$

The estimate of $\beta(\lambda)$ is a difficult problem. Up to now, the best upper bound estimate was given by Pommerenke:

$$\beta(\lambda) \le \lambda - \frac{1}{2} + \left(4\lambda^2 - \lambda + \frac{1}{4}\right)^{1/2} < 3\lambda^2 + 7\lambda^3, \qquad \lambda > 0. \tag{3}$$

The lower bound was considered by Makarov [15], and a sharper estimate was given by Kayumov [12] more recently: $\beta(\lambda) \ge \frac{1}{5}\lambda^2$ for $0 < \lambda \le \frac{2}{5}$.

Theorem 2. *Let f be analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose, for any non-degenerated interval $I \subset [0, 2\pi]$, there exist $\kappa > 0, C > 0$, and $0 < r_0 < 1$ such that, for sufficiently small $\lambda > 0$,*

$$\int_I |f'(re^{i\theta})|^\lambda d\theta \ \geq \ \frac{C}{(1-r)^{\lambda\kappa}}, \qquad r_0 < r < 1. \tag{4}$$

Then f has the Cantor boundary behavior.

Note that, by assumption, when $\lambda > 0$ is small, the mean growth rate of $|f'|$ is greater than the rate for all the univalent functions in $\mathcal{S}$ (i.e., $\lambda\kappa > 3\lambda^2 + 7\lambda^3$ in (3)). This, together with Lemmas 2 and 3 and a contrapositive argument (using the Riemann mapping theorem on D), yields the theorem.

4 The Complex Weierstrass Functions

In the following we consider the class of complex Weierstrass functions:

$$f(z) := f_{q,\beta}(z) = \sum_{n=1}^{\infty} q^{-\beta n} z^{q^n}, \ z \in \overline{\mathbb{D}},$$

where $0 < \beta < 1$ and $q \geq 2$ is an integer. It is well known that f is a Lipschitz function of order β and the Hausdorff dimension of $f(\partial\mathbb{D})$ is $1 < 1/\beta < 2$ for $\beta > \frac{1}{2}$ [9].

For $0 \leq \theta < 2\pi$, $0 < \alpha < \pi/2$, $\tau > 0$, we let

$$S_\alpha(\theta, \tau) = \{z : |z - e^{i\theta}| \leq \tau, \ |\arg(1 - e^{-i\theta} z)| \leq \alpha\}$$

to denote the *Stolz angle* at $e^{i\theta}$. By some rather delicate estimations, we show that the class of $f_{q,\beta}$ satisfies the following lemma.

Lemma 4. *For $\theta_{k,m} := 2\pi m q^{-k}$ with $m = 0, \ldots, q^k - 1$, $k = 1, 2, \ldots$, there exist $C > 0$, $0 < \alpha < 1$, and $0 < \tau_k < \delta q^{-k}$ such that*

$$\mathrm{Re}\big(e^{i\theta_{k,m}} f'(z)\big) \geq \frac{C}{(1-|z|)^{1-\beta}}, \qquad z \in S_\alpha(\theta_{k,m}, \tau_j) \setminus \{e^{i\theta_{k,m}}\}.$$

In order to apply Theorem 2, it is more convenient to modify the integral mean growth condition to be a discretized growth condition of $|f'|$.

Lemma 5. *For $\theta_{k,m} := 2\pi m q^{-k}$ with $m = 0, \ldots, q^k - 1$, $j = 1, 2, \ldots$, suppose there exist $\kappa > 0$, $\delta > 0$, and $\eta \in (0, \pi/2)$ such that*

$$|f'(z)| \ \geq \ c(1-|z|)^{-\kappa} \tag{1}$$

for $z \in S_\eta(\theta_{k,m}, \delta/2^k)$ and $\delta/2^{k+1} \leq 1 - |z| < \delta/2^k$. Then the integral mean condition in (4) of Theorem 2 is satisfied.

By using the two lemmas and Theorem 2, we prove the following.

Theorem 3. *For* $0<\beta<1,\ q\geq 2$ *an integer, the complex Weierstrass function* $f_{q,\beta}$ *has the Cantor boundary behavior.*

In Fig. 1, we display some graphics of the complex Weierstrass functions $f(z)=\sum_{n=1}^{\infty} q^{-\beta n} z^{q^n}$ for different values of q and β. It is seen that the

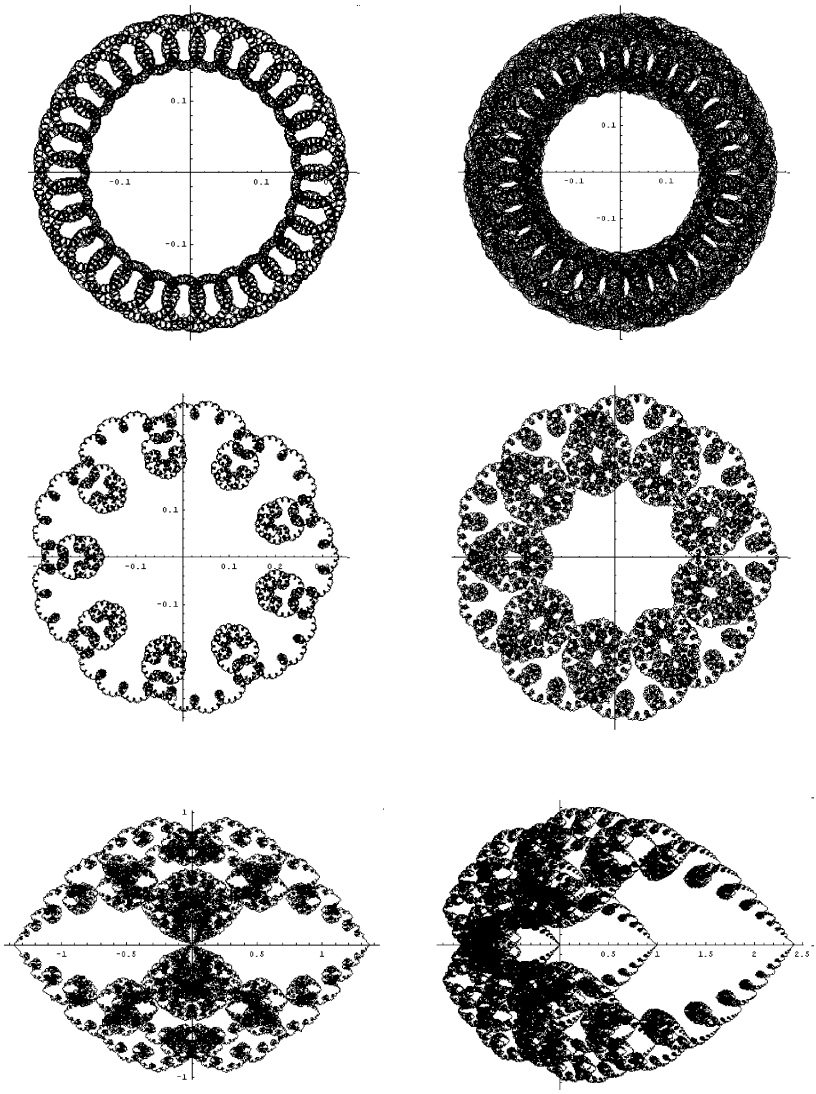

Fig. 1. The images $f_{q,\beta}(\partial\mathbb{D})$. The first two are $q=30, \beta=0.5, 0.4$; the second two are $q=10, \beta=0.6,\ 0.5$; the last two are $q=3, 2,\ \beta=0.5$

number of pedals depends on q. Indeed, there are $q-1$ symmetric pedals due to $q^n \equiv 1 \pmod{q-1}$. The curve $f(\partial\mathbb{D})$ can be space-filling on some regions inside $f(\mathbb{D})$, as was first observed by Salem and Zygmund. In [1], Barański proved this further for $q \geq 2$ and for β sufficiently close to 0. It is seen from the picture that as β is closer to 0, the mean growth rate of $|f'|$ is larger and the curve $f(\mathbb{D})$ loops more violently.

In our theorem, we make use of the fact that the gap ratio q of the series is an integer and the coefficients are a geometric progression. We do not know whether the more general lacunary series still have the Cantor boundary behavior. Also, it is well known that for $\mathrm{Re}f$ and $\mathrm{Im}f$, the box dimension of the graph is known to be $2-\beta$ [9]; however, the question for the Hausdorff dimension is still open (see [10,11,17]). It is seen that $f(\partial\mathbb{D})$ is a fractal curve, and it will be interesting to find the dimension in connection with the results in [20] and [1], and in particular for the dimension or Hausdorff measure of the Cantor set C and the outside boundary of the image $f(C)(=\partial f(\mathbb{D}))$.

5 Cauchy Transform on Sierpinski Gasket

Let $S_k z = \varepsilon_k + (z-\varepsilon_k)/2,\ k = 0,1,2$, where $\varepsilon_k = e^{2k\pi i/3}$. The attractor of this iterated function system $\{S_k\}_{k=0}^2$ is the Sierpinski gasket K (see Fig. 2). Recall that the α-Hausdorff measures satisfies $\mathcal{H}^\alpha(2E) = 2^\alpha \mathcal{H}^\alpha(E)$. For $\mu = \mathcal{H}^\alpha|_K$, where μ is a self-similar measure and normalized to 1, it satisfies $\mu = 3^{-1}\sum_{j=0}^2 \mu \circ S_i^{-1}$ [11,16]. The Cauchy transform of $\mu = \mathcal{H}^\alpha|_K$ is defined by

$$F(z) = \int_K \frac{d\mathcal{H}^\alpha(w)}{z-w}.$$

It is clear that F is analytic away from K and $F(\infty) = 0$. In [14], Strichartz et al. showed that F has a unique extension to be a Hölder continuous function over K of order $\log 3/\log 2 - 1$ (see also [4,5]). Let Δ_0 be the unbounded connected component of $\mathbb{C}\setminus K$, then $F(\Delta_0)$ is a bounded domain. In [14] they

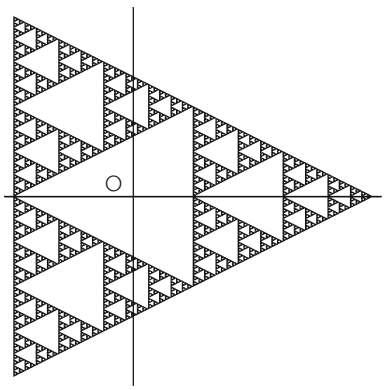

Fig. 2. The Sierpinski gasket K with vertices 1, $e^{2\pi i/3}$, $e^{4\pi i/3}$

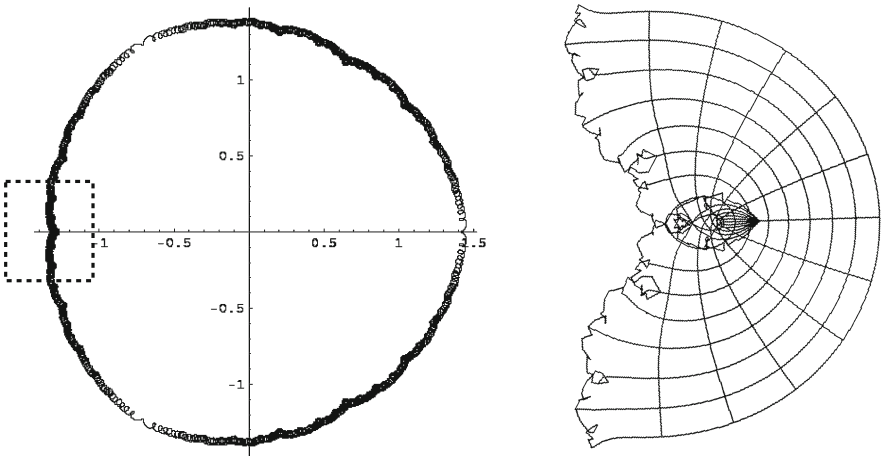

Fig. 3. The image of the outer triangle of K under $F(z)$; the picture on the right is the magnification around $F(-\frac{1}{2})$

also observed from computer graphics that the image $F(\partial\Delta_0)$ is a curve consisting of infinitely many fractal-looking loops (see Fig. 3), and they proposed the *Cantor set conjecture: there exists a Cantor-type set* $C \subset \partial\Delta_0$ *such that* $F(C) = \partial F(\Delta_0)$. This is actually the motivation of our investigation of the Cantor boundary behavior.

By symmetry we only consider the vertical line segment $\partial\Delta_0$; the dyadic points $z_{k,m}$ (not including the two end points) are of the form: for $1 \le m \le 2^k - 1$ and $k \ge 1$,

$$z_{k,m} = \frac{m}{2^k}\varepsilon_1 + \left(1 - \frac{m}{2^k}\right)\varepsilon_2 = -\frac{1}{2} + \frac{m - 2^{k-1}}{2^k}\sqrt{3}i. \tag{1}$$

For $\theta \in (0, \pi/2]$ and $r > 0$, we use the notation

$$\Omega(\theta) = \{z : \ |\arg z - \pi| < \theta\} \quad \text{and} \quad \Omega(\theta; r) = \{z \in \Omega(\theta) : |z| < r\}$$

to replace the Stolz angle on $\mathbb{D}$.

Theorem 4. *There exists a function* $\mathcal{G}$ *such that, for any* $z_{m,k}$,

$$F(z + z_{k,m}) = F(z_{k,m}) + \mathcal{G}(z)z^{\alpha-1} + zp_{k,m}(z), \quad 0 < \arg z < 2\pi,$$

where

(i) $\mathcal{G}$ *is continuous on* $\mathbb{C}\backslash\{0\}$, *analytic in* $\Omega(\pi/2)$, *and* $\mathcal{G}(2z) = \mathcal{G}(z)$ *in* $0 \le \arg z < 2\pi$.

(ii) $p_{k,m}(z)$ *is bounded continuous on* $\mathbb{C}$, *and analytic in* $\Omega(\pi/2) \cup \{z : \ |z| < 3/2^{k+1}\}$.

From this we can draw the following conclusion on the growth rate of F near $\partial\Delta_0$.

Proposition 5. *There exists $C > 0$ such that*

$$\max_{dist\ (z,K)\geq t} |F'(z)| \leq Ct^{\alpha-2};$$

and the order is attained at the dyadic points of $\partial\Delta_0$, in the sense that there exists $0 < \eta < \pi/2, \delta > 0$ and $c > 0$ such that for any $z \in \Omega(\eta; 2^{-k}\delta)$,

$$|F'(z + z_{k,m})| \geq c|z|^{\alpha-2}.$$

Let φ be the Riemann mapping that transforms the closed unit disk $\overline{\mathbb{D}}$ onto $\overline{\Delta_0} \cup \{\infty\}$ conformally. We can use Proposition 5 to show that Lemma 5 (with a slight modification on the $\theta_{k,m}$) is satisfied. Hence, $f(z) = F(\varphi(z))$ satisfies the growth rate condition in Theorem 2. Therefore, we have the following theorem which answers the Cantor set conjecture proposed by Strichartz et al. in [14].

Theorem 5. *The Cauchy transform F has the Cantor boundary behavior.*

The main idea in the proof of Theorem 4 and Proposition 5 is to make use of the following auxiliary functions:

$$g_k(z) = \int_{A_k} \frac{d\mathcal{H}^\alpha(w)}{w(z-w)}, \qquad H_k(z) = \int_{A_k} \frac{d\mathcal{H}^\alpha(w)}{(z-w)^2}$$

with $0 \leq k \leq 5$, where $A_k = e^{k\pi i/3} A_0$ and A_0 is the "Sierpinski cone" generated by the relocated gasket $T = 1 - K$ with vertex at 0 (see Fig. 4).

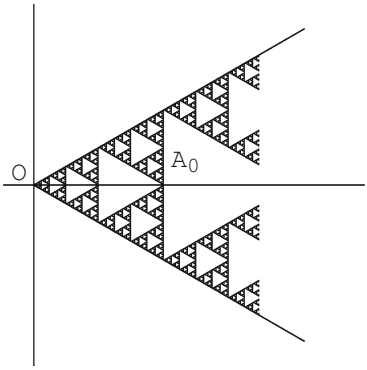

Fig. 4. The Sierpinski cones

These functions have the multiplicative periodic property (period 2). Formally $(zg_k(z))' = -H_k(z)$. The bounded function $\mathcal{G}$ in Theorem 4 is given by

$$\mathcal{G}(z) = z^{2-\alpha}\big(g_1(z) + g_5(z)\big).$$

The H_k's are used in the derivative F' in Proposition 5:

$$F'(z+z_{k,m}) = -(H_1(z)+H_5(z)) + O(1)$$

as $z \in \Omega(\pi/2)$ and $z \to 0$.

From the self-similar property of F, we see that there are "loops inside loops" in the image $F(\partial\Delta_0)$ (Fig. 2). The image points in these loops have multiplicity (from Δ_0) at least 2 and can be any large number. It is natural to ask whether the area of the Riemann region $F(\Delta_0)$ (counting according to multiplicity) is finite. We prove the following.

Theorem 6. *The area of the Riemann region $F(\Delta_0)$ is finite, but it is infinite for $F(\mathbb{C} \setminus K)$.*

The Cantor boundary behavior suggests that $F(\partial\Delta_0)$ is a fractal curve. Indeed, observe that $F(z)$ is Hölder continuous of order $\alpha - 1$ on K. We have immediately (by [9, p. 29]) the following proposition.

Proposition 6. $\dim_{\mathcal{H}} F(\partial\Delta_0) \le (\alpha-1)^{-1} (\approx 1.70951)$.

On the other hand, by using Theorem 4,

$$F(z+z_{m,k}) = F(z_{m,k}) + \mathcal{G}(z) z^{\alpha-1} + O(z), \tag{2}$$

we see that the order $\alpha - 1$ is attained on a dense subset of $\partial\Delta_0$. It is natural to make the following conjecture:

The box dimension and the Hausdorff dimension of $F(\partial\Delta_0)$ are $(\alpha-1)^{-1}$.

Let $\mathrm{Gr}(f;I) = \{(t, f(t)) : t \in I\}$ denote the graph of f on an interval I. It is known that if f is Hölder continuous of order $0 < s \le 1$, then the upper box dimension $\overline{\dim}_B \mathrm{Gr}(f;I) \le 2 - s$. It is easy to show ([9, p. 146]) that if there exists $c > 0$ such that for any dyadic subinterval $I_{k,m} \subset I,\ m = 0, \ldots, 2^k - 1,\ k > 0$,

$$\mathrm{Osc}_f(I_{m,k}) \ge c 2^{-sk},$$

then $\underline{\dim}_B \mathrm{Gr}(f;I) \ge 2 - s$. Based on this and the estimation on the oscillation of $\mathrm{Re}F(z)$ and $\mathrm{Im}F(z)$, we have the following.

Proposition 7. $\dim_B \mathrm{Gr}(\mathrm{Re}F; \partial\Delta_0)$ *and* $\dim_B \mathrm{Gr}(\mathrm{Im}F; \partial\Delta_0)$ *are* $3 - \alpha$.

We do not know if the Hausdorff dimension of the graphs of $\mathrm{Re}F$ and $\mathrm{Im}F$ is $3 - \alpha$. This question may be difficult, as the approximating function $\mathcal{G}$ in (2) has a series expression $\sum_{n\in\mathbb{Z}} 2^{(\alpha-2)}\phi(2^{-n}z)$ [6]. It is analogous to the well-known Weierstrass function, and as already mentioned, the Hausdorff dimension of its graph is still unsolved.

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Measures of Full Dimension on Self-Affine Graphs

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Summary. For a compact subset of the 2-torus that is left invariant by an expanding diagonal endomorphism, the Hausdorff and the Minkowski dimensions may not coincide: this *dimensional hiatus* is possible whenever the **x**-axis and **y**-axis expansion rates differ. The variational principle for dimension ensures that the Hausdorff dimension of the invariant compact set is obtained as the Hausdorff dimension of invariant probability measures called the measures of full dimension: we shall investigate such measures on examples related to classical self-affine graphs.

1 Introduction

Let $\mathbf{K}$ be a compact subset of the torus $\mathbb{R}/\mathbb{Z}$ that is left invariant by the endomorphism $x \mapsto \mathbf{b}x \pmod 1$, for $\mathbf{b} \geq 2$ an integer; Furstenberg [5] proved that the Hausdorff dimension $\dim_H \mathbf{K}$ equals the Minkowski dimension $\dim_M \mathbf{K}$. Now, consider $\mathbb{T}^2 = \mathbb{T}_{\mathbf{x}} \times \mathbb{T}_{\mathbf{y}}$, where $\mathbb{T}_{\mathbf{x}}$ and $\mathbb{T}_{\mathbf{y}}$ are two copies of $\mathbb{R}/\mathbb{Z}$; given $2 \leq \mathbf{b} \leq \mathbf{d}$ integers, $T : \mathbb{T}^2 \to \mathbb{T}^2$ is the expanding diagonal endomorphism such that $T(x, y) = (\mathbf{d}x, \mathbf{b}y) \pmod 1$. In this chapter, we are mainly interested in $\mathbf{K}$ being a T-invariant compact subset of $\mathbb{T}^2$. The case $\mathbf{b} = \mathbf{d}$ is close to the one-dimensional situation considered by Furstenberg; actually, the common value of Hausdorff and Minkowski dimensions is $\mathrm{h}_T(\mathbf{K})/\log \mathbf{b}$, where $\mathrm{h}_T(\mathbf{K})$ is the topological entropy of $T : \mathbf{K} \to \mathbf{K}$. This does not hold anymore when $\mathbf{b} < \mathbf{d}$. In particular, McMullen and Bedford [1,13] simultaneously gave a formula for the Hausdorff dimension of the *general Sierpiński carpets* and proposed examples of *dimensional hiatus* (i.e., when the Minkowski dimension is a strict upper bound for the Hausdorff dimension; see Sect. 3.1). This question has been developed by several authors in the context of the *variational principle for dimension* [6,7,10,11]. By Young's definition [25], the Hausdorff dimension of a measure,[1] η is the infimum of the Hausdorff dimension of the

[1] By *measure* we mean a *Borel probability measure*.

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Borel sets B with $\eta(B) = 1$. Of special importance are the η which belong to the Choquet simplex $\mathcal{M}_T(\mathbf{K})$ of the T-invariant measures with support in $\mathbf{K}$. It is clear that $\dim_H \mathbf{K}$ is bounded from below by the supremum of the Hausdorff dimension $\dim_H \eta$, for $\eta \in \mathcal{M}_T(\mathbf{K})$ and η is *of full dimension*, whenever $\dim_H \eta = \dim_H \mathbf{K}$. The existence of measures of full dimension has been established by Kenyon & Peres [10]. Their approach rests on (a simple version of) the Ledrappier–Young formula [12],[2] giving the Hausdorff dimension of any T-ergodic η, say

$$\dim_H \eta := \alpha \frac{\mathrm{h}_T(\eta)}{\log \mathbf{b}} + (1-\alpha)\frac{\mathrm{h}_{T_{\mathbf{y}}}(\eta_{\mathbf{y}})}{\log \mathbf{b}}. \tag{1}$$

Here, $\eta_{\mathbf{y}}$ is the $\mathbf{y}$-axis projection of η and $T_{\mathbf{y}} : \mathbb{T}_{\mathbf{y}} \to \mathbb{T}_{\mathbf{y}}$ the $\mathbf{y}$-axis marginal dynamic of T such that $T_{\mathbf{y}}(y) = \mathbf{b}y \pmod 1$. The parameter $\alpha = \log \mathbf{b}/\log \mathbf{d}$ plays a crucial role since it indicates the degree of nonconformality of the system; when $\mathbf{b} = \mathbf{d}$, one has $\alpha = 1$ and the $\mathbf{y}$-axis entropy term in (1) disappears.

Before going further, we shall give the basic notions of the *thermodynamic formalism* [2, 17, 22]. For $S : X \to X$ a continuous transformation of a compact metric space X, the topological entropy $\mathrm{h}_S(X)$ is defined as the supremum of the metric entropy $\mathrm{h}_S(\mu)$, for μ in the weak-$*$ compact Choquet simplex $\mathcal{M}_S(X)$ of the S-invariant measures. We shall always be concerned with $S : X \to X$ satisfying expansiveness (see [4, 21]), so that the map $\mu \mapsto \mathrm{h}_S(\mu)$ is affine and upper semicontinuous (u.s.c.); in that case, $\mathrm{h}_S(X)$ is a maximum reached on a nonempty compact face of $\mathcal{M}_S(X)$. When this face is reduced to a single measure (necessarily ergodic), we call it the Parry measure. The variational principle for entropy takes a more general form known as the *variational principle for pressure*. Given $F : X \to \mathbb{R}$ a continuous function, the metric pressure map $\mu \mapsto \mathrm{h}_S(\mu) + \mu(F) =: \mathrm{P}(F, \mu)$ is affine and u.s.c. with respect to $\mu \in \mathcal{M}_S(X)$ and reaches its maximum $\mathrm{P}(F) := \max_\mu\{\mathrm{P}(F, \mu)\}$, called the (topological) pressure, for μ in a nonempty compact face of $\mathcal{M}_S(X)$: these measures are called the *equilibrium states* of F.

Assuming that $T : \mathbf{K} \to \mathbf{K}$ satisfies a mixing property called specification (see [4, Sect. 2.1]), the Haydn–Ruelle RPF theorem [8, 18] ensures that the Parry measure $\bar{\eta}$ on $\mathbf{K}$ has the Gibbs property. More precisely, $\bar{\eta}$ is the unique equilibrium state of $\Theta : \mathbf{K} \to \mathbb{R}$ such that $\Theta(z) = 0$ and for $C_{i,j} = [i/\mathbf{d}^k\,; (i+1)/\mathbf{d}^k[\times[j/\mathbf{b}^k\,; (j+1)/\mathbf{b}^k[$ (with $0 \le i < \mathbf{d}^k$ and $0 \le j < \mathbf{b}^k$) one has either $\bar{\eta}(C_{i,j}) = 0$ or $1/c \le \bar{\eta}(C_{i,j})/e^{-k\mathrm{h}_T(\mathbf{K})} \le c$ for c a constant independent of k, i, j. Hence, the number of the $C_{i,j}$ which intersect $\mathbf{K}$ is approximately

[2] Actually the Kenyon–Peres argument requires the Ledrappier–Young formula for the Bernoulli measures, which can be proved by elementary means.

$e^{k\mathrm{h}_T(\mathbf{K})}$, each of them being tiled by $\mathbf{d}^k/\mathbf{b}^k$ *quasi-squares* of diameter $1/\mathbf{d}^k$. If, in addition, the $\mathbf{y}$-axis projection of $\mathbf{K}$ is assumed to coincide with $\mathbb{T}_\mathbf{y}$, then the number of quasi-squares of diameter $1/\mathbf{d}^k$ needed to tile $\mathbf{K}$ is approximately equal to $e^{\mathrm{h}_T(\mathbf{K})}\mathbf{d}^k/\mathbf{b}^k$; this provides a sketch of the classical computation giving the Minkowski dimension

$$\dim_M \mathbf{K} = \frac{\mathrm{h}_T(\mathbf{K})}{\log \mathbf{d}} + (1-\alpha). \tag{2}$$

The cases for which no dimensional hiatus arise have a simple characterization given in [10]. To see this, notice that $\dim_M \mathbf{K} \geq \dim_H \mathbf{K} \geq \dim_H \mu$, for any ergodic $\mu \in \mathcal{M}_T(\mathbf{K})$; then, using (1) and (2) simultaneously gives

$$\dim_M \mathbf{K} - \dim_H \mu = \frac{\mathrm{h}_T(\mathbf{K}) - \mathrm{h}_T(\mu)}{\log \mathbf{d}} + (1-\alpha)\frac{\log \mathbf{b} - \mathrm{h}_{T_\mathbf{y}}(\mu_\mathbf{y})}{\log \mathbf{b}} \geq 0. \tag{3}$$

Consider that μ^* is an ergodic measure in $\mathcal{M}_T(\mathbf{K})$ of full dimension; according to (3), the equality $\dim_H \mathbf{K} = \dim_M \mathbf{K}$ implies (use the variational principle for entropy twice) that $\mathrm{h}_T(\mu^*) = \mathrm{h}_T(\mathbf{K})$ and $\mathrm{h}_{T_\mathbf{y}}(\mu^*_\mathbf{y}) = \log \mathbf{b} = \mathrm{h}_{T_\mathbf{y}}(\mathbb{T}_\mathbf{y})$. This means that $\mu^* = \bar{\eta}$ and that $\mu^*_\mathbf{y} = \bar{\eta}_\mathbf{y}$ is the Lebesgue measure (on $\mathbb{T}_\mathbf{y}$). Conversely, suppose that $\bar{\eta}_\mathbf{y}$ coincides with the Lebesgue measure; then, (1) gives $\dim_H \bar{\eta} = \mathrm{h}_T(\mathbf{K})/\log \mathbf{d} + (1-\alpha) = \dim_M \mathbf{K}$.

Theorem 1. *[10] Suppose the Parry measure $\bar{\eta} \in \mathcal{M}_T(\mathbf{K})$ with $\mathrm{h}_T(\bar{\eta}) = \mathrm{h}_T(\mathbf{K})$ is well defined and the $\mathbf{y}$-axis projection of $\mathbf{K}$ coincides with $\mathbb{T}_\mathbf{y}$. Then, (i) : $\Longleftrightarrow$ (ii) $\Rightarrow$ (iii), with (i) : $\dim_H \mathbf{K} = \dim_M \mathbf{K}$, (ii) : the $\mathbf{y}$-axis projection $\bar{\eta}_\mathbf{y}$ of $\bar{\eta}$ is the Lebesgue measure, and (iii) : $\bar{\eta}$ is the unique measure of full dimension on $\mathbf{K}$.*

The second part of the variational principle for dimension is concerned with the uniqueness of the measure of full dimension; we shall see that, under a rather natural assumption, depending on the fractal nature of $\mathbf{K}$ (see condition (**H**) in Theorem 2 below), it may be handled by means of the variational principle for pressure. The Ledrappier–Young formula allows a transposition of the problem into symbolic dynamics. Consider the full shifts $\sigma_\mathbf{x} : \mathbf{X} \to \mathbf{X}$ with $\mathbf{X} := \{\mathbf{0}, \ldots, \mathbf{d}-1\}^\mathbb{N}$, $\sigma_\mathbf{y} : \mathbf{Y} \to \mathbf{Y}$ with $\mathbf{Y} := \{\mathbf{0}, \ldots, \mathbf{b}-1\}^\mathbb{N}$, and $\sigma : \mathcal{Z}^\mathbb{N} \to \mathcal{Z}^\mathbb{N}$ with $\mathcal{Z} = \{\mathbf{0}, \ldots, \mathbf{d}-1\} \times \{\mathbf{0}, \ldots, \mathbf{b}-1\}$ (we shall identify $\mathcal{Z}^n$ with $\{\mathbf{0}, \ldots, \mathbf{d}-1\}^n \times \{\mathbf{0}, \ldots, \mathbf{b}-1\}^n$ and $\mathcal{Z}^\mathbb{N}$ with $\mathbf{X} \times \mathbf{Y}$). The affine contraction $R_{(i,j)} : \mathbb{R}^2 \to \mathbb{R}^2$ is such that

$$R_{(i,j)}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i/\mathbf{d} \\ j/\mathbf{b} \end{pmatrix} + \begin{pmatrix} 1/\mathbf{d} & 0 \\ 0 & 1/\mathbf{b} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and the *representation* map $I\!R : \mathcal{Z}^\mathbb{N} \to \mathbb{T}^2$ is defined for $z = (x, y) \in \mathcal{Z}^\mathbb{N}$ by

$$I\!R(z) = \lim_{n \to +\infty} R_{z_0} \circ \cdots \circ R_{z_{n-1}}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} x_k/\mathbf{d}^{k+1} \\ y_k/\mathbf{b}^{k+1} \end{pmatrix} \pmod 1. \tag{4}$$

The symbolic model of $\mathbf{K}$ is the subshift $\mathbf{Z} \subset \mathcal{Z}^\mathbb{N}$ (i.e., $\mathbf{Z}$ compact with $\sigma(\mathbf{Z}) = \mathbf{Z}$) such that $\mathbf{K} = I\!R(\mathbf{Z})$. Given η any measure on $\mathcal{Z}^\mathbb{N}$, let $\eta_\mathbf{y} := \eta \circ \pi_\mathbf{y}^{-1}$,

where $\pi_{\mathbf{y}} : \mathcal{Z}^{\mathbb{N}} \to \mathbf{Y}$ is the projection such that $\pi_{\mathbf{y}}(x, y) = y$. According to (1) it is natural to define the LY-dimension of any $\eta \in \mathcal{M}_\sigma(\mathcal{Z}^{\mathbb{N}})$, say:

$$\dim_{LY} \eta := \alpha \frac{\mathrm{h}_\sigma(\eta)}{\log \mathbf{b}} + (1 - \alpha) \frac{\mathrm{h}_{\sigma_{\mathbf{y}}}(\eta_{\mathbf{y}})}{\log \mathbf{b}}. \tag{5}$$

The mapping $\eta \mapsto \dim_{LY} \eta$ being (affine and) u.s.c. on $\mathcal{M}_\sigma(\mathbf{Z})$, we define the LY-dimension of $\mathbf{Z}$ as the maximum $\dim_{LY} \mathbf{Z}$ of the $\dim_{LY} \eta$ for η in $\mathcal{M}_\sigma(\mathbf{Z})$. A measure $\eta \in \mathcal{M}_\sigma(\mathbf{Z})$ such that $\dim_{LY} \eta = \dim_{LY} \mathbf{Z}$ is said to be of full LY-dimension; moreover, for any $\eta \in \mathcal{M}_\sigma(\mathbf{Z})$, the measure $\mu = \eta \circ I\!R^{-1}$ ($\in \mathcal{M}_T(\mathbf{K})$) is of full dimension on $\mathbf{K}$ if and only if η is of full LY-dimension on $\mathbf{Z}$ and $\dim_H \mathbf{K} = \dim_{LY} \mathbf{Z}$ (see [10]).

Theorem 2. [*14, TheoremA*] *Suppose that $\sigma : \mathbf{Z} \to \mathbf{Z}$ satisfies specification together with condition* (**H**) *ensuring that the projection $\bar{\eta}_{\mathbf{y}}$ of the Parry measure $\bar{\eta}$ on $\mathbf{Z}$ is an e^ϕ-conformal measure for $\phi : \mathbf{Y} \to \mathbb{R}$ continuous* (*see below for definitions*). *Then,*

$$\dim_{LY} \mathbf{Z} = \frac{\mathrm{h}_\sigma(\mathbf{Z})}{\log \mathbf{d}} + \frac{\mathrm{P}(\alpha\phi)}{\log \mathbf{b}} \quad (= \dim_H \mathbf{K}) \; ; \tag{6}$$

moreover, $\eta \in \mathcal{M}_\sigma(\mathbf{Z})$ is of full LY-dimension if and only if it is an equilibrium state of the texture potential $\Phi : \mathbf{Z} \to \mathbb{R}$ such that $\Phi(x, y) = (\alpha - 1)\phi(y)$.

An analog of Theorem 2 is proved in a recent paper by Yayama [24] in the case where the symbolic model of $\mathbf{K}$ is a subshift of finite type. In Theorem 2, the assumption that $\sigma : \mathbf{Z} \to \mathbf{Z}$ satisfies specification strongly depends on Haydn & Ruelle's work [8,18]. As previously mentioned, it ensures the existence of the Parry measure, characterized as a Gibbs measure: this plays a crucial role in the proof of Theorem A in [14].[3] The other point is condition (**H**), which may be difficult to establish; a case where it fails to hold is given in [14, Example 3].

The aim of this chapter is to give examples of application of (6) for some classical self-affine graphs studied in [1,9,13,16,20] and to study the uniqueness of the measure of full dimension.

2 Notation and Background

Let f and g be two real-valued functions defined on a space $\mathcal{X}$; we use Xiangfan notation (see Peyrière in [15]), writing $f(x) \bowtie g(x)$ when there exists a constant K, such that $g(x)/K \leq f(x) \leq K\, g(x)$ for any $x \in \mathcal{X}$.

Let $\mathcal{S}$ be a finite alphabet and $n \geq 1$; an element in $\mathcal{S}^n$ is written as a string of digits in $\mathcal{S}$ called a word and $\mathcal{S}^* := \bigcup_{n=0}^{\infty} \mathcal{S}^n$ denotes the set of words on $\mathcal{S}$ (by convention $\mathcal{S}^0$ is reduced to the empty word ϕ). The sequence space $\mathcal{S}^{\mathbb{N}}$ is compact w.r.t. the product topology and its elements are written as one-sided

[3] The papers [24] and [14] have been issued independently.

infinite words; given $w \in \mathcal{S}^*$, let C_w be the cylinder set of the $\omega \in \mathcal{S}^{\mathbb{N}}$ whose prefix is w. Now, consider that Ω is a subshift of $\mathcal{S}^{\mathbb{N}}$, that is, a compact subset left invariant by the shift map $\sigma : \omega_0\omega_1\omega_2\cdots \mapsto \omega_1\omega_2\cdots$ (i.e., $\sigma(\Omega) = \Omega$). A word $w \in \mathcal{S}$ is said to be Ω-admissible whenever $C_w \cap \Omega$ is nonempty and $\Omega^{(n)}$ stands for the set of the Ω-admissible words in $\mathcal{S}^n$. Suppose that $F : \Omega \to \mathbb{R}$ is Hölder continuous; with the condition that $\sigma : \Omega \to \Omega$ satisfies specification, it follows from the Haydn–Ruelle RPF theorem that F has a unique equilibrium state μ_F which is a Gibbs measure in the sense that for any $\omega \in \Omega$,

$$\mu_F(C_{\omega_0\cdots\omega_{n-1}}) \bowtie e^{\sum_{k=0}^{n-1} F(\sigma^k\omega)}/e^{n\mathrm{P}(F)}, \tag{7}$$

where $\mathrm{P}(F)$ is the pressure of F as already defined. For $F = \Theta$ the identically zero function, $\mathrm{P}(\Theta) = \mathrm{h}_\sigma(\Omega)$ and the unique equilibrium state of Θ is the Parry measure $\bar{\eta}$. The probability measure μ on Ω is said to be e^G-conformal, for $G : \Omega \to \mathbb{R}$ continuous, whenever for any $w \in \Omega^{(n)}$ and any $A \subset C_w$,

$$\mu(\sigma^n(A)) = \int \mu(d\omega)\mathbf{1}_A(\omega)e^{-\sum_{k=0}^{n-1} G(\sigma^k\omega)}.$$

If, in addition to of being e^G-conformal, the measure μ is σ-invariant, then $\mathrm{P}(G) = 0$ and $\mathrm{h}_\sigma(\mu) + \mu(G) = 0$, meaning that μ is an equilibrium state of G. In practice we shall use the following *folklore* proposition.

Proposition 1. *Let $G : \Omega \to \mathbb{R}$ be a continuous map; the σ-invariant measure μ, supposed to have full support on Ω, is e^G-conformal if and only if the n-step potential $\phi_n : \Omega \to \mathbb{R}$ such that $\phi_n(\omega) = \log \mu(C_{\omega_0\cdots\omega_{n-1}})/\mu(C_{\omega_1\cdots\omega_{n-1}})$ converges to G μ-a.e.*

Let $\mathbf{Z}$ be a subshift of $\mathcal{Z}^{\mathbb{N}}$ (i.e., $\mathbf{Z}$ is a compact subset of $\mathcal{Z}^{\mathbb{N}}$ such that $\sigma(\mathbf{Z}) = \mathbf{Z}$). The compact set $\mathbf{K} = I\!R(\mathbf{Z})$ is called a *sofic affine-invariant set* [11] when $\mathbf{Z}$ is a sofic system [23]. Here, it means that there exist $\mathcal{L} \subset \mathcal{Z}$ and a finite $\mathcal{V}$ together with an *adjacency scheme* $\{(\Lambda_\zeta(u,v))_{u,v\in\mathcal{V}}\}_{\zeta\in\mathcal{L}}$, where the square matrix Λ_ζ has entries $\Lambda_\zeta(u,v) \in \{0,1\}$; the adjacency scheme determines an adjacency graph (i.e., an oriented labeled graph) with vertices in $\mathcal{V}$ and an admissible edge from vertex u to vertex v labeled by ζ whenever $\Lambda_\zeta(u,v) = 1$. The sofic system $\mathbf{Z}$ is the collection of the sequences $(z_n)_{n=0}^\infty \in \mathcal{L}^{\mathbb{N}}$ for which there exists a sequence $(v_n)_{n=0}^\infty$ of vertices such that $\Lambda_{z_n}(v_n, v_{n+1}) = 1$ for any $n \geq 0$ (moreover, any sofic system $\mathbf{Z} \subset \mathcal{Z}^{\mathbb{N}}$ is of this form [3]). The adjacency scheme is always assumed to be resolving in the sense that the edges starting from a vertex carry different labels. Under this condition, the coefficient $\Lambda_{z_0\cdots z_{n-1}}(u,v)$ is either 0 or 1 and $\Lambda_{z_0\cdots z_{n-1}}(u,v) = 1$ if and only if there exists $u = v_0, v_1, \ldots, v_n = v \in \mathcal{V}$ such that $\Lambda_{z_0}(v_0,v_1)\Lambda_{z_1}(v_1,v_2)\cdots\Lambda_{z_{n-1}}(v_{n-1},v_n) = 1$; hence, $z_0\cdots z_{n-1} \in \mathbf{Z}^*$ if and only if $\Lambda_{z_0\cdots z_{n-1}}(u,v) = 1$ for some $u, v \in \mathcal{V}$. For

$$\|\Lambda_{z_0\cdots z_{n-1}}\| = \begin{pmatrix}1\cdots 1\end{pmatrix}\Lambda_{z_0\cdots z_{n-1}}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

one has either $\|\Lambda_{z_0\cdots z_{n-1}}\| = 0$ if $z_0\cdots z_{n-1} \notin \mathbf{Z}^*$ or $1 \leq \|\Lambda_{z_0\cdots z_{n-1}}\| \leq \#\mathcal{V}$ if $z_0\cdots z_{n-1} \in \mathbf{Z}^*$. Define $A = \sum_{\zeta\in\mathcal{L}} \Lambda_\zeta$, the *adjacency matrix* associated with the sofic system $\mathbf{Z}$. Since $\|A^n\| = \sum_{w\in\mathcal{L}^n} \|\Lambda_w\| \bowtie \#\mathbf{Z}^{(n)}$, it follows from the definition of the topological entropy of $\mathbf{Z}$ (exponential growth rate of $\#\mathbf{Z}^{(n)}$) that $\mathrm{h}_\sigma(\mathbf{Z})$ is $\log \rho_A$, where ρ_A is the spectral radius of A. We now assume the adjacency matrix A to be irreducible, so that $\sigma : \mathbf{Z} \to \mathbf{Z}$ satisfies specification; then, the Parry measure $\bar\eta \in \mathcal{M}_\sigma(\mathbf{Z})$ is Gibbs, i.e., $\bar\eta(C_{z_0\cdots z_{n-1}}) \bowtie \rho_A^{-n}\mathbf{1}_{\mathbf{Z}}(z)$. Let $W_{(i,j)} := \rho_A^{-1}\Lambda_{(i,j)}$. The matrix $W_+ := \sum_{(i,j)} W_{(i,j)} = \rho_A^{-1}A$ is irreducible with spectral radius equal to 1 and (Perron–Frobenius theorem) there exist two vectors L and R, with positive entries such that $L^*W_+ = L^*$ and $W_+R = R$; with the additional condition that $L^*R = 1$, the map $\mathcal{Z}^* \ni z_0\cdots z_{n-1} \mapsto L^*W_{z_0\cdots z_{n-1}}R$ extends (Kolmogorov consistency theorem) to a probability measure in $\mathcal{M}_\sigma(\mathcal{Z}^{\mathbb{N}})$ supported by $\mathbf{Z}$. Moreover, as we saw, $\|\Lambda_{z_0\cdots z_{n-1}}\| \bowtie \mathbf{1}_{\mathbf{Z}^*}(z_0\cdots z_{n-1})$, so that $L^*W_{z_0\cdots z_{n-1}}R \bowtie \rho_A^{-n}\mathbf{1}_{\mathbf{Z}^*}(z_0\cdots z_{n-1})$, which means that $\bar\eta(C_{z_0\cdots z_{n-1}}) = L^*W_{z_0\cdots z_{n-1}}R$. In what follows, $M_j := \sum_i W_{(i,j)}$, for any $j \in \{\mathbf{0},\dots,\mathbf{b}-1\}$.

Proposition 2. *$\bar\eta_{\mathbf{y}}(J_{y_0\cdots y_{n-1}}) = L^*M_{y_0\cdots y_{n-1}}R$, where $J_{y_0\cdots y_{n-1}}$ is the cylinder set of the $y' \in \mathbf{Y}$ such that $y'_0\cdots y'_{n-1} = y_0\cdots y_{n-1}$.*

3 Classical Self-Affine Graphs and Their Sofic Coding

3.1 McMullen–Bedford (1984)

Take $\mathbf{d} = 3$ and $\mathbf{b} = 2$: a general Sierpiński carpet in the sense of McMullen–Bedford [1, 13] is of the form $\mathbf{K} = \mathbb{R}(\mathbf{Z})$, where $\mathbf{Z} = \mathcal{S}^{\mathbb{N}}$, for $\mathcal{S} \subset \{\mathbf{0},\mathbf{1},\mathbf{2}\} \times \{\mathbf{0},\mathbf{1}\}$; we shall consider the celebrated example when $\mathcal{S} = \{(\mathbf{0},\mathbf{0}),(\mathbf{1},\mathbf{1}),(\mathbf{2},0)\}$ (see Fig. 1).[4] Here the application of Theorem 2 is straightforward. The Parry measure $\bar\eta$ on $\mathbf{Z}$ is the Bernoulli measure with $\bar\eta\{z_0 = (\mathbf{0},\mathbf{0})\} = \bar\eta\{z_0 = (\mathbf{1},\mathbf{1})\} = \bar\eta\{z_0 = (\mathbf{2},\mathbf{0})\} = 1/3$, and its projection $\bar\eta_{\mathbf{y}}$ is the Bernoulli measure (on $\mathbf{Y} = \{\mathbf{0},\mathbf{1}\}^{\mathbb{N}}$) with $\bar\eta_{\mathbf{y}}\{y_0 = \mathbf{0}\} = 2/3$ and $\bar\eta_{\mathbf{y}}\{y_0 = \mathbf{1}\} = 1/3$; condition (**H**) is satisfied since $\bar\eta_{\mathbf{y}}$ is the e^ϕ-conformal measure with $\phi(y) = \mathbf{1}_{\{\mathbf{0}\}}(y_0)\log 2/3 + \mathbf{1}_{\{\mathbf{1}\}}(y_0)\log 1/3$. The pressure of $\alpha\phi$ is classically obtained as the logarithm of the partition function $(2/3)^\alpha + (1/3)^\alpha = (2^\alpha+1)/3^\alpha$, so that

$$\dim_H \mathbf{K} = \frac{\mathrm{h}_\sigma(\mathbf{Z})}{\log 3} + \frac{\mathrm{P}(\alpha\phi)}{\log 2} = 1 + \frac{\log(2^\alpha+1) - \alpha\log 3}{\log 2} = \frac{\log(2^\alpha+1)}{\log 2}.$$

Therefore, the unique measure of full dimension on $\mathbf{K}$ is $\mu^* \circ \mathbb{R}^{-1}$, where μ^* is the equilibrium state of the texture potential $\Phi : \mathbf{Z} \to \mathbb{R}$ such that

[4] Given $\mathcal{S}$ an alphabet, $(X,\star)$ a monoid with unit element e, and a map $\mathcal{S} \ni s \mapsto x_s \in X$, we note $x_{s_1\cdots s_n} := x_{s_1}\star\cdots\star x_{s_n}$, for any word $s_1\cdots s_n \in \mathcal{S}^*$ and the convention that $x_\emptyset = e$.

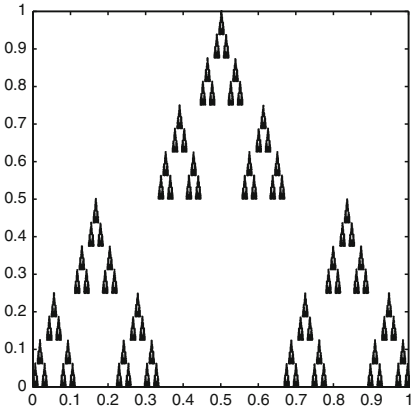

Fig. 1. A typical general Sierpiński carpet (with $\mathbf{d} = 3$ and $\mathbf{b} = 2$) studied by McMullen and Bedford [1, 13]

$$\Phi(x, y) = (\alpha - 1)\Big\{\mathbf{1}_{\{\mathbf{0}\}}(y_0)\log 2/3 + \mathbf{1}_{\{\mathbf{1}\}}(y_0)\log 1/3\Big\},$$

which is the Bernoulli measure identified by McMullen in [13].

3.2 Przytycki and Urbański (1989) and Urbański (1990)

Take $\mathbf{d} = 4$ and $\mathbf{b} = 2$; in Figs. 2 and 3, are represented the substitution rules together with the corresponding adjacency graph associated with a sofic affine invariant set $\mathbf{K} = I\!R(\mathbf{Z})$, where $\mathbf{Z}$ is a sofic system. The labels are taken in

$$\mathcal{L} = \{(\mathbf{0}, \mathbf{0}), (\mathbf{1}, \mathbf{0}), (\mathbf{2}, \mathbf{0}), (\mathbf{0}, \mathbf{3}), (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{1}), (\mathbf{1}, \mathbf{2}), (\mathbf{3}, \mathbf{1})\},$$

the vertices in $\mathcal{V} = \{\mathbf{u}, \mathbf{v}\}$, and the adjacency matrix is

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

The spectral radius of A is $\rho_A = 4$ so that $\mathrm{h}_\sigma(\mathbf{Z}) = \log 4$; according to Proposition 2, the $\mathbf{y}$-axis projection $\bar{\eta}_{\mathbf{y}}$ of the Parry measure is the $\sigma_{\mathbf{y}}$-ergodic measure such that $\bar{\eta}_{\mathbf{y}}(J_m) = L^* M_m R$, for any word $m \in \{\mathbf{0}, \mathbf{1}\}^*$, where

$$L = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad M_{\mathbf{0}} = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix} \quad M_{\mathbf{1}} = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \quad R = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Here, the condition (**H**) is very simple to check, since for $y \in \mathbf{Y}$ and $n \geq 1$,

$$\log \frac{L^* M_{y_0 \cdots y_{n-1}} R}{L^* M_{y_1 \cdots y_{n-1}} R} = \mathbf{1}_{\{\mathbf{0}\}}(y_0)\log(3/4) + \mathbf{1}_{\{\mathbf{1}\}}(y_0)\log(1/4) =: \phi(y).$$

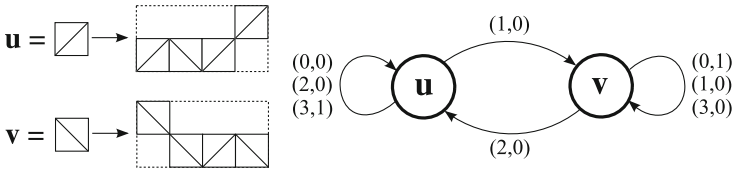

Fig. 2. The substitution rules and corresponding adjacency graph defining the sofic system related to a self-affine graph studied in [16] (see Fig. 3)

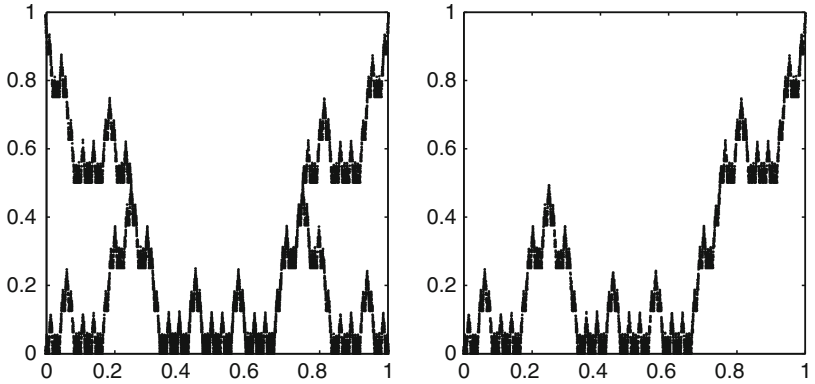

Fig. 3. The sofic affine-invariant set $\mathbf{K} = I\!R(\mathbf{Z})$ represented here (*left*) displays a dimensional hiatus, that is, $\dim_H \mathbf{K} < \dim_M \mathbf{K}$; $I\!R(\mathbf{Z_u})$ (*right*) coincides with a self-affine graph studied in [16] where it is proved that $\dim_H I\!R(\mathbf{Z_u}) < \dim_M I\!R(\mathbf{Z_u})$

This means that $\bar{\eta}_{\mathbf{y}}$ is the ergodic e^{ϕ}-conformal measure, that is, the Bernoulli measure with $\bar{\eta}_{\mathbf{y}}\{y_0 = \mathbf{0}\} = 3/4$ and $\bar{\eta}_{\mathbf{y}}\{y_0 = \mathbf{1}\} = 1/4$. The texture potential $\Phi : \mathbf{Z} \to \mathbb{R}$ such that $\Phi(x, y) = (\alpha - 1)\phi(y)$ being clearly Hölder continuous, it has a unique equilibrium state, say μ^*. The unique measure of full dimension on $\mathbf{K}$ is $\mu^* \circ I\!R^{-1}$, and it is quite simple to compute the Hausdorff dimension of $\mathbf{K}$ by means of (6): here, $\alpha = \log 2/\log 4 = 1/2$ so that $\mathrm{P}(\alpha\phi) = \log(\sqrt{3/4} + \sqrt{1/4}) = \log((1 + \sqrt{3})/2)$ and (recall that $\mathrm{h}_\sigma(\mathbf{Z}) = \log 4$)

$$\dim_H \mathbf{K} = \frac{\mathrm{h}_\sigma(\mathbf{Z})}{\log 4} + \frac{\log((1+\sqrt{3})/2)}{\log 2} = \frac{\log(1+\sqrt{3})}{\log 2}.$$

For a given vertex v, let us denote by $\mathbf{Z}_v$ the set of the sequences $z \in \mathbf{Z}$ for which there exists a sequence of vertices $v = v_0, v_1, \ldots$ such that $\Lambda_{z_0}(v_0, v_1)\Lambda_{z_1}(v_1, v_2) \cdots \Lambda_{z_{n-1}}(v_{n-1}, v_n) = 1$ for any $n \geq 0$. Then, $I\!R(\mathbf{Z_u})$ is a self-affine graph studied by Przytycki and Urbański in [16]; here one recovers that $\dim_H I\!R(\mathbf{Z_u}) < \dim_M I\!R(\mathbf{Z_u})$. Concerning this question, we also mention the application of the multifractal analysis of the weak quasi-Bernoulli measures developed by Testud in [19].

Now let us consider the sofic affine-invariant set $\mathbf{K} = I\!R(\mathbf{Z})$ in Figs. 4 and 5 which is related to a graph studied by Urbańsi in [20]; the correspond-

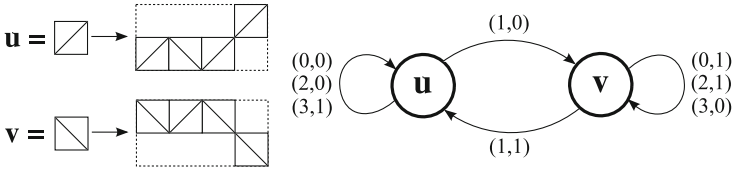

Fig. 4. The substitution rules and corresponding adjacency graph defining the sofic system related to a self-affine graph studied in [20] (see Fig. 5)

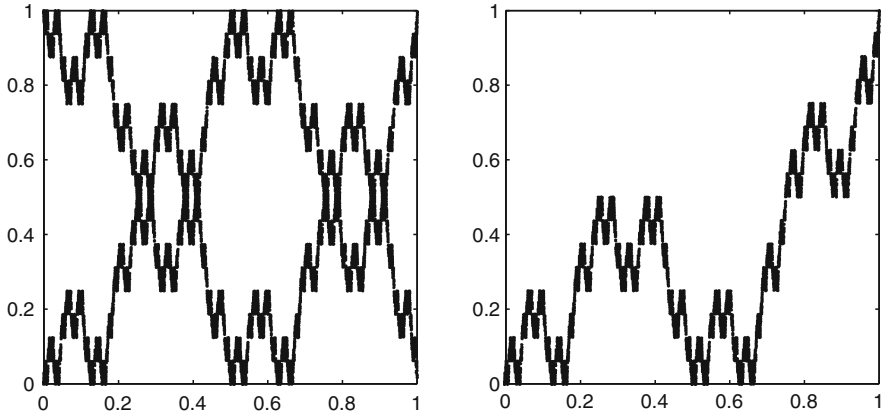

Fig. 5. The sofic affine-invariant set $\mathbf{K} = I\!R(\mathbf{Z})$ represented here (*left*) displays no dimensional hiatus, that is, $\dim_H \mathbf{K} = \dim_M \mathbf{K}$; $I\!R(\mathbf{Z_u})$ (*right*) coincides with a self-affine graph studied in [20] where it is proved that $\dim_H I\!R(\mathbf{Z_u}) = \dim_M I\!R(\mathbf{Z_u})$

ing adjacency scheme is very close to the previous one, where $\mathcal{L}$ and $\mathcal{V}$ are unchanged while

$$L = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad M_{\mathbf{0}} = \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/4 \end{pmatrix} \quad M_{\mathbf{1}} = \begin{pmatrix} 1/4 & 0 \\ 1/4 & 1/2 \end{pmatrix} \quad R = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}.$$

Since $L^* M_{\mathbf{0}} = L^* M_{\mathbf{1}} = 1/2 L^*$, one has $\bar{\eta}_{\mathbf{y}}(J_m) = 1/2^n$, for any $m \in \{\mathbf{0}, \mathbf{1}\}^n$, meaning that $\bar{\eta}_{\mathbf{y}}$ is the uniform Bernoulli measure on $\mathbf{Y} = \{\mathbf{0}, \mathbf{1}\}^{\mathbb{N}}$ (in particular, condition (**H**) is trivially satisfied for $\phi : \mathbf{Y} \to \mathbb{R}$ with $\phi(y) = \log 2$). By Theorem 1, there is no dimensional hiatus for $\mathbf{K}$, that is,

$$\dim_H \mathbf{K} = \dim_M \mathbf{K} = \frac{\mathrm{h}_\sigma(\mathbf{Z})}{\log 3} + \left(1 - \frac{\log 2}{\log 4}\right) = \frac{3}{2}.$$

Moreover, $\bar{\eta} \circ I\!R^{-1}$ is the unique T-ergodic measure of full dimension. Remark that $I\!R(\mathbf{Z_u})$ is a self-affine graph (left in Fig. 5) which has been studied by Urbański in [20]: one recovers that $\dim_H I\!R(\mathbf{Z_u}) = \dim_M I\!R(\mathbf{Z_u})$ (see also [19]).

3.3 Kamae (1986)

Take $\mathbf{d} = 3$ and $\mathbf{b} = 2$; the sofic affine-invariant set $\mathbf{K} = I\!R(\mathbf{Z})$ in Figs. 6 and 7 is related to the self-affine graph $\mathbf{G} = I\!R(\mathbf{Z}_{\mathbf{u}})$ studied by Kamae in [9]. The sofic system $\mathbf{Z}$ has labels in

$$\mathcal{L} = \{(\mathbf{0},\mathbf{0}), (\mathbf{1},\mathbf{0}), (\mathbf{2},\mathbf{0}), (\mathbf{0},\mathbf{1}), (\mathbf{1},\mathbf{1}), (\mathbf{2},\mathbf{1})\},$$

vertices in $\mathcal{V} = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, and adjacency matrix

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

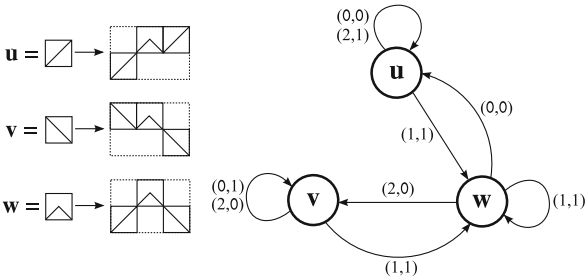

Fig. 6. The substitution rules and corresponding adjacency graph defining the sofic system related to a self-affine graph studied in [9] (see Fig. 7)

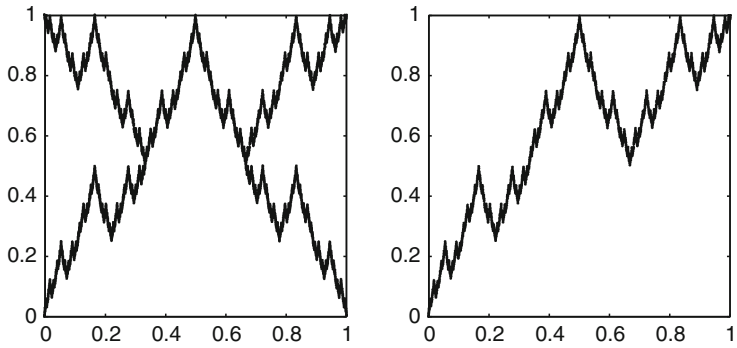

Fig. 7. The sofic affine-invariant set $\mathbf{K} = I\!R(\mathbf{Z})$ represented here (*left*) displays a dimensional hiatus, that is, $\dim_H \mathbf{K} < \dim_M \mathbf{K}$; the self-affine graph $I\!R(\mathbf{Z}_{\mathbf{u}})$ (*right*) was studied by Kamae in [9]

The spectral radius of A is $\rho_A = 3$ and $\mathrm{h}_\sigma(\mathbf{Z}) = \log 3$; according to Proposition 2, $\bar{\eta}_{\mathbf{y}}$ is the $\sigma_{\mathbf{y}}$-ergodic measure such that $\bar{\eta}_{\mathbf{y}}(J_m) = L^* M_m R$, for any word $m \in \{\mathbf{0}, \mathbf{1}\}^*$, where

$$L = 3R = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$M_{\mathbf{0}} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 1/3 & 1/3 & 0 \end{pmatrix} \quad M_{\mathbf{1}} = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

Theorem 3. K *has a unique T-ergodic measure of full dimension.*

By Theorem 2, the measures of full dimension on **K** are the equilibrium states of the texture potential; hence, Theorem 3 is a consequence of the following proposition together with the fact that (classical RPF theorem) Hölder continuous functions have a unique equilibrium state (a representation of the potential ϕ, associated with the texture potential, is given in Fig. 8).

Proposition 3. *The projection $\bar{\eta}_{\mathbf{y}}$ of the Parry measure on $\mathbf{Z}$ is e^{ϕ}-conformal for $\phi : \mathbf{Y} \to \mathbb{R}$ Hölder continuous and such that, for any sequence $a_1, a_2, \ldots$ of positive integers,*

$$\phi(\mathbf{0}^i \mathbf{1}^{a_1} \mathbf{0}^* \mathbf{1}^{a_2} \mathbf{0}^* \cdots) = \begin{cases} \log \frac{1}{3} & \text{if } i \geq 2\ , \\ \log \frac{1}{3} + \log\left(\frac{2a_1+1}{a_1+1}\right) & \text{if } i = 1, \\ \log \frac{1}{3} + \log\left(\frac{a_1+1}{a_1}\right) & \text{if } i = 0. \end{cases}$$

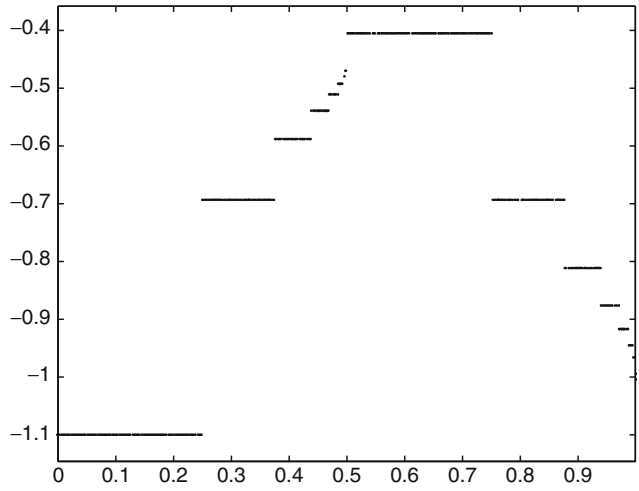

Fig. 8. Graph plot of $[0\,;1] \ni y \mapsto \phi(y_0 y_1 \cdots)$, where $0.y_0 y_1 \cdots$ is a dyadic expansion of y

Sketched Proof. Let $y = \mathbf{0}^i\mathbf{1}^{a_1}\mathbf{0}^*\mathbf{1}^{a_2}\mathbf{0}^*\cdots$ for $i \geq 0$, a sequence $a_1, a_2, \ldots$ of positive integers, and $\mathbf{0}^* = \mathbf{0}\cdots\mathbf{0}$ with an arbitrary positive length. For $\epsilon = \mathbf{0}$ or $\mathbf{1}$, set $A_\epsilon := 3M_\epsilon$; in order to use Proposition 1, we shall prove that

$$\lim_{n\to+\infty} \frac{L^* A_{y_0\cdots y_{n-1}} R}{L^* A_{y_1\cdots y_{n-1}} R} = 3e^{\phi(y)}. \tag{8}$$

To begin with, notice that for any integers $p, q \geq 1$,

$$A_{\mathbf{0}}^p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_{\mathbf{1}}^q = \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that (8) holds when $i \geq 2$. Now, consider that $i = 0$ (a similar argument applies when $i = 1$). According to the matricial identity

$$A_{\mathbf{1}}^q A_{\mathbf{0}}^p = A_{\mathbf{1}}^q A_{\mathbf{0}} = \begin{pmatrix} (q+1) & q & 0 \\ q & (q+1) & 0 \\ 1 & 1 & 0 \end{pmatrix},$$

it is suitable to introduce

$$B^{(q)} := \begin{pmatrix} (q+1) & q & 0 \\ q & (q+1) & 0 \\ (2q+1) & (2q+1) & 0 \end{pmatrix}.$$

Given $a_1, a_2, \ldots$ a sequence of positive integers, define $\varphi(a_1) = a_1$ and

$$\varphi(a_1, \ldots, a_n) := \frac{1}{2} \sum_{k=1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} 2^k a_{i_1} \cdots a_{i_k},$$

for any $n \geq 2$, so that, by a simple induction,

$$B^{(a_1)}(A_{\mathbf{1}}^{a_2} A_{\mathbf{0}}) \ldots (A_{\mathbf{1}}^{a_n} A_{\mathbf{0}}) = B^{(\varphi(a_1,\ldots,a_n))} =: B^{(a_1,\ldots,a_n)}.$$

For any $p \geq 0$, the matrix identities

$$B^{(a_1,\ldots,a_n)} R = B^{(a_1,\ldots,a_n)} A_{\mathbf{0}}^p R = \Big(2\varphi(a_1, \ldots, a_n) + 1\Big) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

together with

$$L^* A_{\mathbf{0}} A_{\mathbf{1}}^p = L^* A_{\mathbf{0}} B^{(p)} = \begin{pmatrix} 2 & 2 & 0 \end{pmatrix} B^{(p)}$$

and

$$L^* A_{\mathbf{1}}^p = \begin{pmatrix} 1 & 1 & (2a_1 + 1)^{-1} \end{pmatrix} B^{(p)}$$

allow us to write:

$$\lim_{n\to+\infty}\frac{L^*A_{y_0\cdots y_{n-1}}R}{L^*A_{y_1\cdots y_{n-1}}R}=\lim_{n\to+\infty}\frac{\begin{pmatrix}2&2&0\end{pmatrix}B^{(a_1,\cdots,a_n)}R}{\begin{pmatrix}1&1&(2a_1+1)^{-1}\end{pmatrix}B^{(a_1,\cdots,a_n)}R}$$

$$=\lim_{n\to+\infty}\frac{\begin{pmatrix}2&2&0\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}}{\begin{pmatrix}1&1&(2a_1+1)^{-1}\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}}=\frac{2a_1+1}{a_1+1}.$$

Finally, since $L^*A_{\mathbf{1}}^pA_{\mathbf{0}}=\begin{pmatrix}2p+2&2p+2&0\end{pmatrix}$, one concludes the following:

$$\lim_{n\to+\infty}\frac{L^*A_{y_0\cdots y_{n-1}}R}{L^*A_{y_1\cdots y_{n-1}}R}=\lim_{n\to+\infty}\frac{a_1+1}{a_1}\frac{\begin{pmatrix}1&1&0\end{pmatrix}B^{(a_2,\cdots,a_n)}R}{\begin{pmatrix}1&1&0\end{pmatrix}B^{(a_2,\cdots,a_n)}R}=\frac{a_1+1}{a_1}.$$

■

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Part IV

Stochastic Processes and Random Fractals

A Process Very Similar to Multifractional Brownian Motion

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Summary. Multifractional Brownian motion (mBm), denoted here by X, is one of the paradigmatic examples of a continuous Gaussian process whose pointwise Hölder exponent depends on the location. Recall that X can be obtained (see, e.g., Ayache and Taqqu Publ Mat 49:459–486,2005; Benassi et al. Rev Mat Iberoam 13:19–81, 1997) by replacing the constant Hurst parameter H in the standard wavelet series representation of fractional Brownian motion (fBm) by a smooth function $H(\cdot)$ depending on the time variable t. Another natural idea (see Benassi et al. Stat Infer Stoch Proc 3:101–111, 2000) which allows us to construct a continuous Gaussian process, denoted by Z, whose pointwise Hölder exponent does not remain constant all along its trajectory, consists in substituting $H(k/2^j)$ to H in each term of index (j,k) of the standard wavelet series representation of fBm. The main goal of our chapter is to show that, under some assumption on the bounds of $H(\cdot)$, X and Z only differ by a process R which is smoother than they are; this means that they are very similar from a fractal geometry point of view.

1 Introduction and Statement of the Main Results

Throughout this chapter we denote by $H(\cdot)$ an arbitrary function defined on the real line and with values in an arbitrary fixed compact interval $[a,b] \subset (0,1)$. We will always assume that on each compact $\mathcal{K} \subset \mathbb{R}$, $H(\cdot)$ satisfies a uniform Hölder condition of order $\beta > b$, i.e., there is a constant $c_1 > 0$ (which a priori depends on $\mathcal{K}$) such that for every $t_1, t_2 \in \mathcal{K}$ one has

$$|H(t_1) - H(t_2)| \leq c_1 |t_1 - t_2|^{\beta}; \tag{1}$$

typically $H(\cdot)$ is a Lipschitz function over $\mathbb{R}$. We will also assume that $a = \inf\{H(t) : t \in \mathbb{R}\}$ and $b = \sup\{H(t) : t \in \mathbb{R}\}$. Recall that multifractional Brownian motion (mBm) of functional parameter $H(\cdot)$, which we denote by $X = \{X(t) : t \in \mathbb{R}\}$, is the continuous and nowhere differentiable Gaussian process obtained by replacing the Hurst parameter in the harmonizable representation of fractional Brownian motion (fBm) by the function

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$H(\cdot)$. That is, the process X can be represented for each $t \in \mathbb{R}$ as the following stochastic integral:

$$X(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} \, d\widehat{W}(\xi), \tag{2}$$

where $d\widehat{W}$ is "the Fourier transform" of the real-valued white-noise dW in the sense that for any function $f \in L^2(\mathbb{R})$ one has a.s.

$$\int_{\mathbb{R}} f(x) \, dW(x) = \int_{\mathbb{R}} \widehat{f}(\xi) \, d\widehat{W}(\xi). \tag{3}$$

Observe that (3) implies that (see [10, 17]) the following equality holds a.s. for every t, to within a deterministic smooth bounded and nonvanishing deterministic function:

$$\int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} \, d\widehat{W}(\xi) = \int_{\mathbb{R}} \left\{ |t-s|^{H(t)-1/2} - |s|^{H(t)-1/2} \right\} dW(s).$$

Therefore, X is a real-valued process. MBm was introduced independently in [16] and [8] and since then there has been increasing interest in the study of multifractional processes. We refer, for instance, to [13, 18] for two excellent quite recent articles on this topic. The main three features of mBm are the following:

(a) X reduces to an fBm when the function $H(\cdot)$ is constant.
(b) Unlike fBm, $\alpha_X = \{\alpha_X(t) : \ t \in \mathbb{R}\}$, the pointwise Hölder exponent of X, may depend on the location and can be prescribed via the functional parameter $H(\cdot)$; in fact, one has (see [3, 5, 8, 16]) a.s. for each t,

$$\alpha_X(t) = H(t). \tag{4}$$

Recall that α_X, the pointwise Hölder exponent of an arbitrary continuous and nowhere differentiable process X, is defined, for each $t \in \mathbb{R}$, as

$$\alpha_X(t) = \sup \left\{ \alpha \in \mathbb{R}_+ : \ \limsup_{h \to 0} \frac{|X(t+h) - X(t)|}{|h|^\alpha} = 0 \right\}. \tag{5}$$

(c) At any point $t \in \mathbb{R}$, there is an fBm of Hurst parameter $H(t)$, which is tangent to mBm [8, 11, 12], i.e., for each sequence (ρ_n) of positive real numbers converging to 0, one has

$$\lim_{n \to \infty} \text{law} \left\{ \frac{X(t + \rho_n u) - X(t)}{\rho_n^{H(t)}} : \ u \in \mathbb{R} \right\} = \text{law}\{B_{H(t)}(u) : \ u \in \mathbb{R}\}, \tag{6}$$

where the convergence holds in distribution for the topology of uniform convergence on compact sets.

The main goal of our chapter is to give a natural wavelet construction of a continuous and nowhere differentiable Gaussian process $Z = \{Z(t)\}_{t\in\mathbb{R}}$ which has the same features (a), (b), and (c) as mBm X and which differs from it by a smoother stochastic process $R = \{R(t) : t \in \mathbb{R}\}$ (see Theorem 1).

In order to be able to construct Z, first we need to introduce some notation. In what follows we denote by $\{2^{j/2}\psi(2^j x - k) : (j,k) \in \mathbb{Z}^2\}$ a Lemarié–Meyer wavelet basis of $L^2(\mathbb{R})$ [14] and we define Ψ to be the function, for each $(x,\theta) \in \mathbb{R} \times \mathbb{R}$,

$$\Psi(x,\theta) = \int_{\mathbb{R}} e^{ix\xi} \frac{\widehat{\psi}(\xi)}{|\xi|^{\theta+1/2}}\, d\xi. \tag{7}$$

By using the fact that $\widehat{\psi}$ is a compactly supported C^∞ function vanishing on a neighborhood of the origin, it follows that Ψ is a well-defined C^∞ function satisfying for any $(l,m,n) \in \mathbb{N}^3$ with $l \geq 2$, the following localization property (see [5] for a proof):

$$c_2 = \sup_{\theta\in[a,b],\, x\in\mathbb{R}} (2+|x|)^\ell |(\partial_x^m \partial_\theta^n \Psi)(x,\theta)| < \infty, \tag{8}$$

where $\partial_x^m \partial_\theta^n \Psi$ denotes the function obtained by differentiating the function Ψ, n times with respect to the variable θ and m times with respect to the variable x. For convenience, let us introduce the Gaussian field $B = \{B(t,\theta) : (t,\theta) \in \mathbb{R} \times (0,1)\}$ defined for each $(t,\theta) \in \mathbb{R} \times (0,1)$ as

$$B(t,\theta) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{\theta+1/2}}\, d\widehat{W}(\xi). \tag{9}$$

Observe that for every fixed θ, the Gaussian process $B(\cdot,\theta)$ is an fBm of Hurst parameter θ on the real line. Also observe that mBm X satisfies, for each $t \in \mathbb{R}$,

$$X(t) = B(t, H(t)). \tag{10}$$

By expanding for every fixed (t,θ), the kernel function $\xi \mapsto \dfrac{e^{it\xi} - 1}{|\xi|^{\theta+1/2}}$ in the orthonormal basis of $L^2(\mathbb{R})$, $\{2^{-j/2}(2\pi)^{1/2} e^{i2^{-j}k\xi} \widehat{\psi}(-2^{-j}\xi) : (j,k) \in \mathbb{Z}^2\}$, and by using the isometry property of the stochastic integral in (9), it follows that

$$B(t,\theta) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-j\theta} \varepsilon_{j,k} \Big\{ \Psi(2^j t - k, \theta) - \Psi(-k,\theta) \Big\}, \tag{11}$$

where $\{\varepsilon_{j,k} : (j,k) \in \mathbb{Z}^2\}$ is a sequence of independent $\mathcal{N}(0,1)$ Gaussian random variables and where the series is, for every fixed (t,θ), convergent in $L^2(\Omega)$; throughout this article Ω denotes the underlying probability space. In fact, this series is also convergent in a much stronger sense; see part (i) of the following remark.

Remark 1. The field B has already been introduced and studied in [5]; we recall some of its useful properties.

(i) The series in (11) is a.s. uniformly convergent in (t,θ) on each compact subset of $\mathbb{R}\times(0,1)$, so B is a continuous Gaussian field. Moreover, combining (10) and (11), we deduce the following wavelet expansion of mBm:

$$X(t)=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}2^{-jH(t)}\varepsilon_{j,k}\left\{\Psi(2^jt-k,H(t))-\Psi(-k,H(t))\right\}. \quad (12)$$

(ii) The low frequency component of B, namely the field $\dot{B}=\{\dot{B}(t,\theta):(t,\theta)\in\mathbb{R}\times(0,1)\}$ defined for all $(t,\theta)\in\mathbb{R}\times(0,1)$ as

$$\dot{B}(t,\theta)=\sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{\infty}2^{-j\theta}\varepsilon_{j,k}\left\{\Psi(2^jt-k,\theta)-\Psi(-k,\theta)\right\}, \quad (13)$$

is a C^∞ Gaussian field. Therefore, (1) and (10) imply that the low frequency component of the mBm X, namely the Gaussian process $\dot{X}=\{\dot{X}(t)\}_{t\in\mathbb{R}}$ defined for each $t\in\mathbb{R}$ as

$$\dot{X}(t)=\sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{\infty}2^{-jH(t)}\varepsilon_{j,k}\left\{\Psi(2^jt-k,H(t))-\Psi(-k,H(t))\right\}, \quad (14)$$

satisfies a uniform Hölder condition of order β on each compact subset $\mathcal{K}$ of $\mathbb{R}$. Thus, in view of (b) and the assumption $b<\beta$, the pointwise Hölder exponent of X is only determined by its high frequency component, namely the continuous Gaussian process $\ddot{X}=\{\ddot{X}(t)\}_{t\in\mathbb{R}}$ defined for each $t\in\mathbb{R}$ as

$$\ddot{X}(t)=\sum_{j=0}^{+\infty}\sum_{k=-\infty}^{\infty}2^{-jH(t)}\varepsilon_{j,k}\left\{\Psi(2^jt-k,H(t))-\Psi(-k,H(t))\right\}. \quad (15)$$

Definition 1. *The process $Z=\{Z(t):t\in\mathbb{R}\}$ is defined for each $t\in\mathbb{R}$ as*

$$Z(t)=\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}2^{-jH(k/2^j)}\varepsilon_{j,k}\left\{\Psi(2^jt-k,H(k/2^j))-\Psi(-k,H(k/2^j))\right\}. \quad (16)$$

In view of (11) it is clear that the process Z reduces to an fBm when the function $H(\cdot)$ is constant; this means that the process Z has the same feature (a) as an mBm.

Remark 2. Using the same techniques as in [5], one can show that:

(i) The series in (16) is a.s. uniformly convergent in t on each compact interval of $\mathbb{R}$; therefore, Z is a well-defined continuous Gaussian process.

(ii) The low frequency component of the process Z, namely the process $\dot{Z} = \{\dot{Z}(t) : t \in \mathbb{R}\}$ defined for all $t \in \mathbb{R}$ as

$$\dot{Z}(t) = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2^{-jH(k/2^j)} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H(k/2^j)) - \Psi(-k, H(k/2^j)) \right\}, \tag{17}$$

is a C^∞ Gaussian process. The pointwise Hölder exponent of Z is therefore only determined by its high frequency component, namely the continuous Gaussian process $\ddot{Z} = \{\ddot{Z}(t) : t \in \mathbb{R}\}$ defined for all $t \in \mathbb{R}$ as

$$\ddot{Z}(t) = \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{\infty} 2^{-jH(k/2^j)} \varepsilon_{j,k} \left\{ \Psi(2^j t - k, H(k/2^j)) - \Psi(-k, H(k/2^j)) \right\}. \tag{18}$$

It is worth noticing that if one replaces in (18) the Hölder function $H(\cdot)$ by a step function, then one recovers the step fractional Brownian motion which has been studied in [2,6].

Let us now state our main result.

Theorem 1. *Let $R = \{R(t) : t \in \mathbb{R}\}$ be the process defined for any $t \in \mathbb{R}$ as*

$$R(t) = Z(t) - X(t). \tag{19}$$

Let $\mathcal{K}$ be a compact interval included in $\mathbb{R}$. Then, if a and b satisfy the following condition:

$$1 - b > (1 - a)(1 - ab^{-1}), \tag{20}$$

there exists an exponent $d \in (b, 1]$, such that the process R satisfies a uniform Hölder condition of order d on $\mathcal{K}$. More precisely, there is Ω^* an event of probability 1, such that, for all $\omega \in \Omega^*$ and for each $(t_0, t_1) \in \mathcal{K}^2$, one has

$$|R(t_1, \omega) - R(t_0, \omega)| \leq C_1(\omega) |t_1 - t_0|^d, \tag{21}$$

where C_1 is a nonnegative random variable of finite moment of every order only depending on Ω^* and $\mathcal{K}$.

Remark 3. We do not know whether Theorem 1 remains valid when condition (20) does not hold. Figure 1 indicates the region $\mathcal{D}$ in the unit cube satisfying (20).

From Theorem 1 we can obtain the following result, which shows that Z and X are very similar from a fractal geometry point of view.

Corollary 1. *Assume that a and b satisfy (20). Then the process Z has the same features (a), (b), and (c) as an mBm.*

Throughout this chapter, we use $[x]$ to denote the integer part of a real number x. Positive deterministic constants will be numbered as $c_1, c_2, \ldots$ while positive random constants will be numbered as $C_1, C_2, \ldots$.

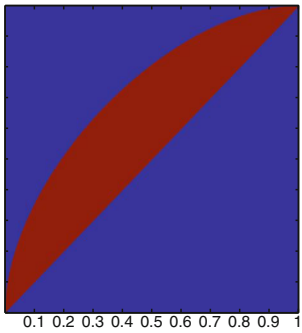

Fig. 1. The region $\mathcal{D}$ in the unit cube satisfying (20)

2 The Main Ideas of the Proofs

In the reminder of our chapter we always assume that condition (20) is satisfied and that $\mathrm{diam}(\mathcal{K}) := \sup\{|u-v| : (u,v) \in \mathcal{K}\} \leq 1/4$. Also notice that we will frequently make use of the inequality

$$\log(3+x+y) \leq \log(3+x) \times \log(3+y) \qquad \text{for all } (x,y) \in I\!R_+^2. \tag{22}$$

Let us now present the main ideas behind the proof of Theorem 1. First we need to state the following lemma, which allows us to conveniently bound the random variables $\varepsilon_{j,k}$. It is a classical result; we refer, for example, to [15] or [4] for its proof.

Lemma 1. *[4, 15] There are an event Ω^* of probability 1 and a nonnegative random variable C_2 of finite moment of every order such that the inequality*

$$|\varepsilon_{j,k}(\omega)| \leq C_2(\omega)\sqrt{\log(3+|j|+|k|)} \tag{23}$$

holds for all $\omega \in \Omega^*$ and $j,k \in Z\!\!Z$.

Proof of Theorem 1. In view of Remark 1 (ii) and Remark 2 (ii), it is sufficient to prove that Theorem 1 holds when the process R is replaced by its high frequency component, namely, the process $\ddot{R} = \{\ddot{R}(t) : t \in I\!R\}$ defined for each $t \in I\!R$ as

$$\ddot{R}(t) = \ddot{Z}(t) - \ddot{X}(t). \tag{24}$$

Let $g_{j,k}$ be the function defined on $I\!R \times I\!R$ by

$$g_{j,k}(t,\theta) = 2^{-j\theta}\left\{\Psi(2^j t - k, \theta) - \Psi(-k,\theta)\right\}. \tag{25}$$

It follows from (24), (15), (18), (25), and (23), that, for any $\omega \in \Omega^*$,

$$|\ddot{R}(t_1,\omega)-\ddot{R}(t_0,\omega)| \le C_2(\omega)\sum_{j=0}^{+\infty}\sum_{k=-\infty}^{+\infty}\sqrt{\log(3+j+|k|)}\times\Big|g_{j,k}(t_1,H(k/2^j)) - g_{j,k}(t_0,H(k/2^j)) - g_{j,k}(t_1,H(t_1)) + g_{j,k}(t_0,H(t_0))\Big|. \tag{26}$$

Next, we expand the term $g_{j,k}\big(t_i,H(\tau)\big)$ with $i=0$ or 1 and $\tau=t_1$ or $k/2^j$ with respect to the second variable in the neighborhood of $H(t_0)$. Indeed, since the function Ψ is $\mathcal{C}^\infty$, the functions $g_{j,k}$ are also $\mathcal{C}^\infty$. Thus, we can use a Taylor–Lagrange formula of order 1 with an integral remainder and we get

$$g_{j,k}(t_1,H(t_1)) = g_{j,k}(t_1,H(t_0)) + (H(t_1)-H(t_0))(\partial_\theta g_{j,k})(t_1,H(t_0)) + (H(t_1) - H(t_0))^2\int_0^1(1-\tau)(\partial_\theta^2 g_{j,k})(t_1,H(t_0)+\tau(H(t_1)-H(t_0)))\,d\tau, \tag{27}$$

$$g_{j,k}(t_0,H(k/2^j)) = g_{j,k}(t_0,H(t_0))+(H(k/2^j)-H(t_0))(\partial_\theta g_{j,k})(t_0,H(t_0)) +(H(k/2^j)-H(t_0))^2\int_0^1(1-\tau)(\partial_\theta^2 g_{j,k})(t_0,H(t_0)+\tau(H(k/2^j)-H(t_0)))\,d\tau, \tag{28}$$

and

$$g_{j,k}(t_1,H(k/2^j)) = g_{j,k}(t_1,H(t_0))+(H(k/2^j)-H(t_0))(\partial_\theta g_{j,k})(t_1,H(t_0)) +(H(k/2^j)-H(t_0))^2\int_0^1(1-\tau)(\partial_\theta^2 g_{j,k})(t_1,H(t_0)+\tau(H(k/2^j)-H(t_0)))\,d\tau. \tag{29}$$

By adding or subtracting relations (27), (28), and (29), the constant terms disappear and we get the following upper bound:

$$\begin{aligned}&\Big|g_{j,k}(t_1,H(k/2^j)) - g_{j,k}(t_0,H(k/2^j)) - g_{j,k}(t_1,H(t_1)) + g_{j,k}(t_0,H(t_0))\Big|\\ &\le |H(t_1)-H(t_0)|\Big|(\partial_\theta g_{j,k})(t_1,H(t_0))\Big|\\ &+\big|H(t_1)-H(t_0)\big|^2\int_0^1(1-\tau)\Big|(\partial_\theta^2 g_{j,k})(t_1,H(t_0)+\tau(H(t_1)-H(t_0)))\Big|\,d\tau\\ &+|H(k/2^j)-H(t_0)|\Big|(\partial_\theta g_{j,k})(t_1,H(t_0)) - (\partial_\theta g_{j,k})(t_0,H(t_0))\Big|\\ &+\big|H(k/2^j)-H(t_0)\big|^2\int_0^1(1-\tau)\Big|(\partial_\theta^2 g_{j,k})(t_1,H(t_0)+\tau(H(k/2^j)-H(t_0))) -(\partial_\theta^2 g_{j,k})(t_0,H(t_0)+\tau(H(k/2^j)-H(t_0)))\Big|\,d\tau.\end{aligned} \tag{30}$$

Then, we substitute the previous bound (30) into the inequality (26). We stress that the quantities $|H(t_1) - H(t_0)|$ and $\left|H(t_1) - H(t_0)\right|^2$ can be factorized outside the sum, whereas the quantities $|H(k/2^j) - H(t_0)|$ and $\left|H(k/2^j) - H(t_0)\right|^2$ remain inside the sum. We obtain

$$
\begin{aligned}
|\ddot{R}(t_1,\omega) - \ddot{R}(t_0,\omega)| &\leq C_2(\omega)|H(t_1) - H(t_0)| \\
&\quad \times \left\{ \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} \sqrt{\log(3+j+|k|)} \times \left| (\partial_\theta g_{j,k})(t_1, H(t_0)) \right| \right\} \\
&\quad + C_2(\omega)|H(t_1) - H(t_0)|^2 \times \left\{ \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} \sqrt{\log(3+j+|k|)} \right. \\
&\quad \left. \times \int_0^1 (1-\tau) \left| (\partial_\theta^2 g_{j,k})(t_1, H(t_0) + \tau(H(t_1) - H(t_0))) \right| d\tau \right\} \\
&\quad + C_2(\omega) \times \left\{ \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} \sqrt{\log(3+j+|k|)} |H(k/2^j) - H(t_0)| \right. \\
&\quad \left. \times \left| (\partial_\theta g_{j,k})(t_1, H(t_0)) - (\partial_\theta g_{j,k})(t_0, H(t_0)) \right| \right\} \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
&+ C_2(\omega) \times \left\{ \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} \sqrt{\log(3+j+|k|)} |H(k/2^j) - H(t_0)|^2 \right. \\
&\quad \times \int_0^1 (1-\tau) \left| (\partial_\theta^2 g_{j,k})(t_1, H(t_0) + \tau(H(k/2^j) - H(t_0))) \right. \\
&\quad \left. \left. - (\partial_\theta^2 g_{j,k})(t_0, H(t_0) + \tau(H(k/2^j) - H(t_0))) \right| d\tau \right\}.
\end{aligned}
$$

Then using the following two lemmas whose proofs will be given soon, we get

$$
\begin{aligned}
|\ddot{R}(t_1,\omega) - \ddot{R}(t_0,\omega)| &\leq C_2(\omega) \Big(|H(t_1) - H(t_0)| \mathcal{A}_1(\mathcal{K}; a, b) \\
&\quad + \left|H(t_1) - H(t_0)\right|^2 \mathcal{A}_2(\mathcal{K}; a, b) + |t_1 - t_0|^{d_1} \mathcal{G}_1(\mathcal{K}; a, b, d_1) \\
&\quad + |t_1 - t_0|^{d_2} \mathcal{G}_2(\mathcal{K}; a, b, d_2) \Big). \qquad (32)
\end{aligned}
$$

Finally, in view of (1) the latter inequality implies that Theorem 1 holds. ■

Lemma 2. *For every integer $n \geq 0$ and $(t, \theta) \in \mathbb{R} \times (0, +\infty)$ one sets*

$$
A_n(t,\theta) := \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{+\infty} |(\partial_\theta^n g_{j,k}(t,\theta)| \sqrt{\log(3+j+|k|)}. \qquad (33)
$$

Then one has

$$\mathcal{A}_n(\mathcal{K};a,b) := \sup\Big\{A_n(t,\theta):\ (t,\theta)\in\mathcal{K}\times[a,b]\Big\} < \infty. \tag{34}$$

Lemma 3. *For every integer $n \geq 1$ and $(t_0,t_1,\theta) \in \mathbb{R}^2 \times (0,+\infty)$ one sets*

$$G_n(t_0,t_1,\theta) := \sum_{j=0}^{+\infty}\sum_{k=-\infty}^{+\infty} |H(k/2^j)-H(t_0)|^n \times \sqrt{\log(3+j+|k|)} \times\Big|(\partial_\theta^n g_{j,k})(2^j t_1-k,\theta)-(\partial_\theta^n g_{j,k})(2^j t_0-k,\theta)\Big|.$$

Then, for every integer $n \geq 1$, there is an exponent $d_n \in (b,1]$ such that

$$\mathcal{G}_n(\mathcal{K};a,d_n) := \sup_{(t_0,t_1,\theta)\in\mathcal{K}^2\times[a,b]} |t_1-t_0|^{-d_n} G_n(t_0,t_1,\theta) < \infty. \tag{35}$$

Proof of Lemma 2. From Lemma 4 given in the next section, one can deduce

$$A_n(t,\theta) \leq \sum_{p=0}^{n} C_n^p |\log 2|^p \sum_{j=0}^{+\infty}\sum_{k=-\infty}^{+\infty} j^p 2^{-j\theta}\sqrt{\log(3+j+|k|)} \times\Big\{\big|(\partial_\theta^{n-p}\Psi)(2^j t-k,\theta)\big| + \big|(\partial_\theta^{n-p}\Psi)(-k,\theta)\big|\Big\}. \tag{36}$$

Note that the deepest bracket $\Big\{\big|(\partial_\theta^{n-p}\Psi)(2^j t-k,\theta)\big| + \big|(\partial_\theta^{n-p}\Psi)(-k,\theta)\big|\Big\}$ contains two terms: the first $\big|(\partial_\theta^{n-p}\Psi)(2^j t-k,\theta)\big|$ depends on $t\in\mathcal{K}$, while the second $\big|(\partial_\theta^{n-p}\Psi)(-k,\theta)\big|$ no longer depends on t. Therefore, it suffices to obtain a bound of the supremum for $t\in\mathcal{K}$ of the sum corresponding to the first term, then to use it in the special case $\mathcal{K}=\{0\}$ to bound the sum corresponding to the second term. Let us remark that there exists a real $K>0$ such that $\mathcal{K}\subset[-K,K]$. Therefore, without any restriction, we can suppose that $\mathcal{K}=[-K,K]$. Next, using (8), the convention that $0^0=1$, the change of variable $k=k'+[2^j t]$, the fact that $|t|\leq K$, (22), and the fact that $z=2^j t-[2^j t]\in[0,1]$, one has the following estimates for each $p\in\{0,\dots,n\}$ and $(t,\theta)\in[-K,K]\times[a,b]$:

$$\begin{aligned}
&\sum_{j=0}^{+\infty}\sum_{k=-\infty}^{+\infty} j^p 2^{-j\theta}\sqrt{\log(3+j+|k|)}\,\big|(\partial_\theta^{n-p}\Psi)(2^j t-k,\theta)\big| \\
&\quad\leq c_2\sum_{j=0}^{+\infty}\sum_{k=-\infty}^{+\infty} j^p 2^{-ja}\sqrt{\log(3+j+|k|)}\cdot(2+|2^j t-k|)^{-\ell} \\
&\quad\leq c_2\sum_{j=0}^{+\infty}\sum_{k'=-\infty}^{+\infty} j^p 2^{-ja}\sqrt{\log(3+j+|k'|+2^j K)}\cdot(2+|2^j t-[2^j t]-k'|)^{-\ell} \\
&\quad\leq c_2 c_3\sum_{j=0}^{+\infty} j^p 2^{-ja}\sqrt{\log(3+j+2^j K)} < \infty,
\end{aligned} \tag{37}$$

where

$$c_3 = \sup\left\{\sum_{k=-\infty}^{+\infty}(2+|z-k|)^{-l}\sqrt{\log(3+|k|)}:\ z\in[0,1]\right\}<\infty. \tag{38}$$

Clearly, (37) combined with (36) implies that (34) holds. ■

Proof of Lemma 3. The proof is very technical, so let us first explain the main ideas behind it. For simplicity, we make the change of notation $t_1 = t_0 + h$. Then we split the set of indices $\{(j,k) \in \mathbb{N} \times \mathbb{Z}\}$ into three disjoint subsets: $\mathcal{V}$ a neighborhood of radius r about t_0, a subset $\mathcal{W}$ corresponding to the low frequency ($j \leq j_1$) outside the neighborhood $\mathcal{V}$, and a subset $\mathcal{W}^c$ corresponding to the high frequency ($j > j_1$) outside the neighborhood $\mathcal{V}$ (the "good" choices of the radius r and of the cutting frequency j_1 will be clarified soon). Thus, the sum through which $G_n(t_0, t_1, \theta)$ is defined (see the statement of Lemma 3) can be decomposed into three parts: a sum over $\mathcal{V}$, a sum over $\mathcal{W}$, and a sum over $\mathcal{W}^c$; they can respectively be denoted $B_{1,n}(t_0, h, \theta)$, $B_{2,n}(t_0, h, \theta)$, and $B_{3,n}(t_0, h, \theta)$. In order to be able to show that, to within a constant, each of these three quantities is upper bounded by $|h|^{d_n}$ for some exponent $d_n > b$, we need to conveniently choose the radius r of the neighborhood $\mathcal{V}$ as well as the cutting frequency j_1. The most natural choice is to take $r = |h|$ and $2^{-j_1} \simeq |h|$. However, a careful inspection of the proof of Lemma 7 shows that this does not work, basically because $2^{j_1}|h|$ does not go to infinity when $|h|$ tends 0. Roughly speaking, to overcome this difficulty, we have taken $r = |h|^\eta$ and $2^{-j_1} \simeq |h|^\gamma$, where $0 < \eta < \gamma < 1$ are two parameters (the "good" choices of these parameters will be clarified soon) as shown by Fig. 2.

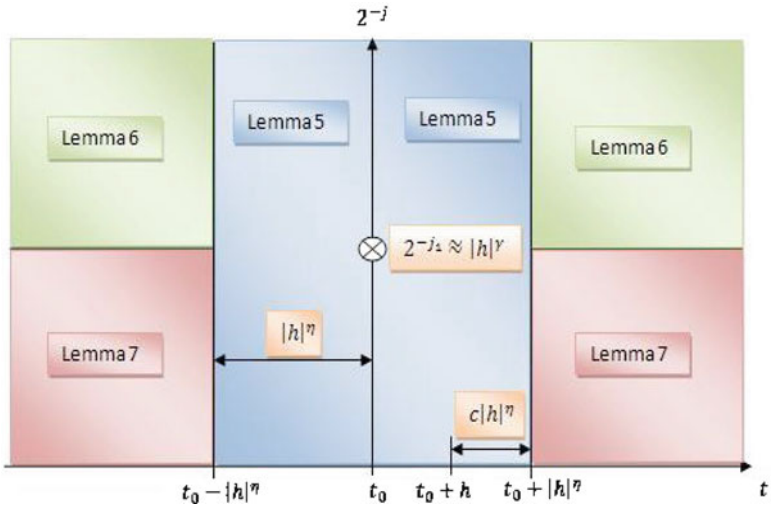

Fig. 2. Representation of the sets of indices used in Lemmas 5 through 7

More precisely, j_1 is the unique nonnegative integer satisfying

$$2^{-j_1-1} < |h|^\gamma \leq 2^{-j_1}, \tag{39}$$

and the sets $\mathcal{V}$, $\mathcal{V}^c$, $\mathcal{W}$ and $\mathcal{W}^c$ are defined by

$$\mathcal{V}(t_0,h,\eta) = \{(j,k) \in \mathbb{N} \times \mathbb{Z} :\ |k/2^j - t_0| \leq |h|^\eta\}, \tag{40}$$

$$\mathcal{V}^c(t_0,h,\eta) = \{(j,k) \in \mathbb{N} \times \mathbb{Z} :\ |k/2^j - t_0| > |h|^\eta\}, \tag{41}$$

$$\mathcal{W}(t_0,h,\eta,\gamma) = \{(j,k) \in \mathcal{V}^c(t_0,h,\eta) :\ 0 \leq j \leq j_1\} \tag{42}$$

and

$$\mathcal{W}^c(t_0,h,\eta,\gamma) = \{(j,k) \in \mathcal{V}^c(t_0,h,\eta) :\ j \geq j_1 + 1\}. \tag{43}$$

It follows from Lemmas 4 through 7 that

$$\begin{aligned} &G_n(t_0, t_0+h, \theta) \\ &\quad = \sum_{m=1}^{3} B_{m,n}(t_0,h,\theta) \leq \sum_{p=0}^{n} \sum_{m=1}^{3} C_n^p\, (\log 2)^p B_{m,n,p}(t_0,h,\theta) \\ &\quad \leq c_4 \Big(|h|^{a+\eta\beta} + |h|^{(1-\gamma)+\gamma a} + |h|^{(\gamma-\eta)(\ell-1-\varepsilon)+\gamma a} \Big) \log^{n+1/2}(1/|h|), \end{aligned} \tag{44}$$

where the constant $c_4 = \max\{c_5, c_8, c_{10}\} \sum_{p=0}^{n} C_n^p\, (\log 2)^p$ does not depend on (t_0,h,θ). In view of (44) and the inequality $\beta > b$ as well as the fact that ε is arbitrarily small, for proving that (35) holds, it is sufficient to show that there exist two reals $0 < \eta < \gamma < 1$ and an integer $\ell \geq 2$ satisfying the following inequalities:

$$\begin{cases} a + \eta b \geq b \\ (1-\gamma) + \gamma a > b \\ (\gamma - \eta)(\ell - 1) + \gamma a > b. \end{cases}$$

This is clearly the case. In fact, from (20) we can show that the first two inequalities have common solutions; moreover each of their common solutions is also a solution of the third inequality provided that ℓ is big enough. ■

Before ending this section let us prove that Corollary 1 holds.

Proof of Corollary 1. Let us first show that Z has the same feature (b) as an mBm. In view of Theorem 1 and Remark 3, it is clear that α_R, the pointwise Hölder exponent of R, satisfies a.s. for all $t \in \mathbb{R}$,

$$\alpha_R(t) \geq d. \tag{45}$$

Next combining (45), the fact that $d > b$, (19), and (4), it follows that a.s. for all $t \in \mathbb{R}$,

$$\alpha_Z(t) = H(t).$$

Let us now show that Z has the same feature (c) as mBm. Let (ρ_n) be an arbitrary sequence of positive real numbers converging to 0. In view of (19) and (6), to prove that for each $t \in \mathbb{R}$ one has

$$\lim_{n\to\infty} \text{law}\left\{\frac{Z(t+\rho_n u) - Z(t)}{\rho_n^{H(t)}} : \ u \in \mathbb{R}\right\} = \text{law}\{B_{H(t)}(u) : \ u \in \mathbb{R}\}, \quad (46)$$

in the sense of a finite dimensional distribution, it is sufficient to prove that for any $u \in \mathbb{R}$ one has

$$\lim_{n\to+\infty} E\left\{\left(\frac{R(t+\rho_n u) - R(t)}{\rho_n^{H(t)}}\right)^2\right\} = 0. \quad (47)$$

Observe that for all n big enough one has $\rho_n|u| \leq 1$. Therefore, taking $\mathcal{K} = [t-1, t+1]$ in Theorem 1, it follows that for n big enough,

$$E\left\{\left(\frac{R(t+\rho_n u) - R(t)}{\rho_n^{H(t)}}\right)^2\right\} \leq \rho_n^{2(d-H(t))} E(C_1^2), \quad (48)$$

and the latter inequality clearly implies that (47) holds. To have in (46) the convergence in distribution for the topology of the uniform convergence on compact sets, it is sufficient to show that, for any positive real L, the sequence of continuous Gaussian processes,

$$\left\{\frac{Z(t+\rho_n u) - Z(t)}{\rho_n^{H(t)}} : \ u \in [-L, L]\right\}, \ n \in \mathbb{N},$$

is tight. This tightness result can be obtained (see [9]) by proving that there exists a constant $c_{17} > 0$ only depending on L and t such that for all $n \in \mathbb{N}$ and each $u_1, u_2 \in [-L, L]$ one has

$$E\left\{\left(\frac{Z(t+\rho_n u_1) - Z(t)}{\rho_n^{H(t)}} - \frac{Z(t+\rho_n u_2) - Z(t)}{\rho_n^{H(t)}}\right)^2\right\} \leq c_{17}|u_1 - u_2|^{2H(t)}. \quad (49)$$

Without loss of generality, we may assume that, for every $n \in \mathbb{N}$, $\rho_n \in (0, 1]$. Then by using the fact that (49) is satisfied when Z is replaced by X (see [7] Proposition 2), as well as the fact that it is also satisfied when Z is replaced by R (this can be done similarly to (48)), one can establish that this inequality holds. ■

3 Some Technical Lemmas

Lemma 4. *For every integer $n \geq 0$ and any $(t, \theta) \in \mathbb{R} \times \mathbb{R}$, one has*

$$(\partial_\theta^n g_{j,k})(t,\theta) = \sum_{p=0}^{n} C_n^p (-j \log 2)^p 2^{-j\theta} \Big\{ (\partial_\theta^{n-p})\Psi(2^j t - k, \theta) - (\partial_\theta^{n-p}\Psi)(-k,\theta) \Big\}. \quad (50)$$

Proof of Lemma 4. The lemma can easily be obtained by applying the Leibniz formula for the nth derivative of a product of two functions. ■

Lemma 5. *For each integer $n \geq 1$ and $(t_0, h, \theta) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty)$, set*

$$B_{1,n,p}(t_0,h,\theta) := \sum_{(j,k)\in\mathcal{V}(t_0,h,\eta)} j^p 2^{-j\theta} |H(t_0) - H(k/2^j)|^n \times \sqrt{\log(3+j+|k|)} \times \Big| (\partial_\theta^{n-p}\Psi)(2^j(t_0+h)-k,\theta) - (\partial_\theta^{n-p}\Psi)(2^j t_0 - k,\theta) \Big|,$$

where $\mathcal{V}(t_0, h, \eta)$ is the set defined by (40). Then, for all real $K > 0$ and every integer $n \geq 1$ and $0 \leq p \leq n$, one has

$$c_5 := \sup_{(t_0,\theta)\in[-K,K]\times[a,b],|h|<1/4} |h|^{-a-n\eta\beta} \log^{-p-1/2}(1/|h|) B_{1,n,p}(t_0,h,\theta) < \infty. \quad (51)$$

Proof of Lemma 5. The method is standard. A detailed proof is given in the long version [1]. We set $j_0 \geq 2$ to be the unique integer such that $2^{-j_0-1} < |h| \leq 2^{-j_0}$ and use two different bounds at high and low frequencies. On one hand, by using (8), the high frequency terms are bounded by $c_6 |h|^a \log^{p+1/2}(1/|h|)$, where c_6 is a constant that does not depend on (t_0, h, η).

On the other hand, by using the mean-value theorem applied to the function $\partial_\theta^{n-p} g_{j,k}$ with respect to the first variable, (8), the fact that for all $2^j|h| \leq 1$ for all $j \in \{0, \ldots, j_0\}$, (22), (38), and the inequality $|t_0| \leq K$, we can bound the low frequency term by $c_7 |h|^a \log^{p+1/2}(1/|h|)$. Finally, one can deduce (51). ■

Lemma 6. *For all $(t, h, \theta) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty)$ and for all integers $n \geq 1$ and $0 \leq p \leq n$, set*

$$B_{2,n,p}(t_0,h,\theta) := \sum_{(j,k)\in\mathcal{W}(t_0,h,\eta,\gamma)} j^p 2^{-j\theta} |H(t_0) - H(k/2^j)|^n \times \sqrt{\log(3+j+|k|)} \times \Big| (\partial_\theta^{n-p}\Psi)(2^j(t_0+h)-k,\theta) - (\partial_\theta^{n-p}\Psi)(2^j t_0 - k,\theta) \Big|,$$

where $\mathcal{W}(t_0, h, \eta, \gamma)$ is the set defined by (42). Then, for any real $K > 0$, one has that

$$c_8 = \sup_{(t_0,\theta)\in[-K,K]\times[a,b],|h|<1/4} |h|^{-(1-\gamma)-\gamma a} \log^{-p-1/2}(1/|h|) B_{2,n,p}(t_0,h,\theta) < \infty.$$

Proof of Lemma 6. A detailed proof is given in the long version located in on Hal Arxiv [1]. We just stress the main ideas. First, note that

$$\text{for all } (j,k) \in I\!N \times Z\!\!Z, \qquad |H(t_0) - H(k/2^j)|^n < 1. \tag{52}$$

Second, by using the mean-value theorem applied to the function $\partial_\theta^{n-p}\Psi$ with respect to the first variable combined with (8), we get, for all $t_0 \in \mathcal{K}$ and $h \in I\!R$,

$$\begin{aligned}&\left|(\partial_\theta^{n-p}\Psi)(2^j(t_0+h)-k,\theta) - (\partial_\theta^{n-p}\Psi)(2^j t_0 - k,\theta)\right| \\ &\quad \le c_2\, 2^j|h|\left(2+|2^j t_0 - k + 2^j u h|\right)^{-\ell}.\end{aligned}$$

Third, it follows from the inequality $2^j|h| \le 1$ for all $j \in \{0,\dots,j_1\}$ and from the triangle inequality, that $|2^j t_0 - k + 2^j u h| \ge |2^j t_0 - k| - 1$. Next, by using standard calculations, we get

$$B_{2,n,p}(t_0,h,\theta) \le c_9 |h|^{(1-\gamma)+\gamma a} \log^{p+1/2}(1/|h|), \tag{53}$$

where the constant c_9 does not depend on (t_0,h,θ). ■

Lemma 7. *For all $(t,h,\theta) \in I\!R \times I\!R \times [a,b]$ and all integers $n \ge 1$ and $0 \le p \le n$, set*

$$\begin{aligned}B_{3,n,p}(t_0,h,\theta) := &\sum_{(j,k)\in\mathcal{W}^c(t_0,h,\eta,\gamma)} j^p 2^{-j\theta} |H(t_0) - H(k/2^j)|^n \times \sqrt{\log(3+j+|k|)} \\ &\times \left|(\partial_\theta^{n-p}\Psi)(2^j(t_0+h)-k,\theta) - (\partial_\theta^{n-p}\Psi)(2^j t_0 - k,\theta)\right|,\end{aligned}$$

where $\mathcal{W}^c(t_0,h,\eta,\gamma)$ is the set defined by (43). Then, for every real $K > 0$, for each arbitrarily small real $\varepsilon > 0$, and all integers $l \ge 2$, one has $c_{10} < \infty$, where

$$c_{10} := \sup_{(t_0,\theta)\in[-K,K]\times[a,b], |h|<1/4} |h|^{-(\gamma-\eta)(l-1-\varepsilon)-\gamma a} \log^{-p-1/2}(|h|^{-1}) B_{2,n,p}(t_0,h,\theta).$$

Proof of Lemma 7. A detailed proof is given in the long version [1]. The only interesting point is that, for every fixed j, the set of indices is divided into two subsets $\mathcal{T}_j^+$ and $\mathcal{T}_j^-$. Indeed, by using the triangle inequality combined with (43) and (41), one gets, for all $(j,k) \in \mathcal{W}^c(t_0,h,\eta,\gamma)$,

$$|(t_0+h) - k2^{-j}| \ge |t_0 - k2^{-j} + h| \ge |h|^\eta - |h| \ge \left(1 - 4^{\eta-1}\right)|h|^\eta. \tag{54}$$

In view of (54), let us consider the sets $\mathcal{T}_j^+$ and $\mathcal{T}_j^-$ defined by

$$\mathcal{T}_j^+ = \{|2^j(t_0+h) - k| : \quad k \in Z\!\!Z \text{ and } 2^j(t_0+h) - k \ge c_{11} 2^j |h|^\eta\}$$

and

$$\mathcal{T}_j^- = \{|2^j(t_0+h)-k| : \quad k \in \mathbb{Z} \text{ and } k - 2^j(t_0+h) \geq c_{11}2^j|h|^\eta)\}.$$

For every fixed j, each of the sets $\mathcal{T}_j^+$ and $\mathcal{T}_j^-$ can be viewed as a strictly increasing sequence. Next, by using standard calculations, for all $|h| \leq 1/4$, we get

$$\sum_{(j,k)\in\mathcal{W}^c(t_0,h,\eta,\gamma)} j^p 2^{-j\theta} \left|(\partial_\theta^{n-p}\Psi)(2^j(t_0+h)-k,\theta)\right| \sqrt{\log(3+j+|k|)}$$
$$\leq c_{15}\,|h|^{(\gamma-\eta)(\ell-1-\varepsilon)+\gamma a} \log^{p+1/2}(1/|h|),$$

where c_{15} is a constant which does not depend on (t_0, h, θ). ■

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Gaussian Fields Satisfying Simultaneous Operator Scaling Relations

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Summary. In this chapter we define a special class of group of self-similar Gaussian fields. We present a harmonizable representation of m-parameter group self-similar Gaussian fields by utilizing the Haar measure of this group. These fields also have stationary rectangular increments according to special directions linked to coreduction of matrices of the considered m-parameter group.

1 Introduction

Random fields are a useful tool for modeling spatial phenomena like those found in environmental fields, including, for example, hydrology, geology, oceanography, and medicine (medical images). Many times the chosen model has to include some statistical dependence structure that might be present across the scales. Thus, a usual assumption is self-similarity (see [12]), defined for a random field $\{X(x)\}_{x\in\mathbb{R}^d}$ on $\mathbb{R}^d$ by

$$\{X(ax)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d}$$

for some $H \in \mathbb{R}$ (called the Hurst index). As usual, $\overset{(f.d.)}{=}$ denotes equality of all finite-dimensional marginal distributions. The most famous example of self-similar processes is fractional Brownian motion (FBM) $\{B_H(x)\}_{x\in\mathbb{R}^d}$, introduced in 1940 by Kolmogorov (see [10]) and first studied in the famous paper of Mandelbrot and Van Ness (see [14]).

Moreover, in many cases, random fields have an anisotropic nature in the sense that they have different geometric characteristics along different directions (see, for example, Davies and Hall [6], Bonami and Estrade [4], and Benson et al. [2]). The classical notion of self-similarity–by construction isotropic–then has to be changed in order to fit anisotropic situations. For this reason, there has been increasing interest in defining a suitable concept for anisotropic self-similarity. Many authors have developed techniques to handle anisotropy in scaling: In [8] Hudson and Mason introduced operator

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self-similar processes $\{X(t)\}_{t\in\mathbb{R}}$ with values in $\mathbb{R}^d$. Moreover, in [17, 18] Schertzer and Lovejoy introduced a general concept of scaling with respect to a one-parameter group (which may be a matricial one) for random fields. These authors discussed in particular the linear case and its application to the study of atmospheric stratification.

In [9], A. Kamont introduced a first example of an anisotropic self-similar Gaussian field: the fractional Brownian sheet (FBS). For any $(H_1,\dots,H_d)$ in $(0,1)^d$, the FBS with Hurst indices $(H_1,\dots,H_d)$–denoted $\{B_{H_1,\dots,H_d}(x)\}_{x\in\mathbb{R}^d}$–can be defined through its harmonizable representation (see [1]):

$$B_{H_1,\dots,H_d}(x)=\int_{\mathbb{R}^d}\frac{(e^{i<x_1,\xi_1>}-1)\cdots(e^{i<x_d,\xi_d>}-1)}{|\xi_1|^{H_1+\frac{1}{2}}\cdots|\xi_d|^{H_d+\frac{1}{2}}}d\widehat{W}_{\xi_1,\dots,\xi_d},\qquad(1)$$

where $dW_{x_1,\dots,x_d}$ is a Brownian measure on $\mathbb{R}^d$ and $d\widehat{W}_{\xi_1,\dots,\xi_d}$ its Fourier transform. This definition implies that the FBS satisfies simultaneous scaling properties: For any $(a_1,\dots,a_d)\in(\mathbb{R}_+^*)^d$,

$$\begin{cases}\{B_{H_1,\dots,H_d}(a_1x_1,\dots,x_d)\}_{x\in\mathbb{R}^d}\overset{(f.d.)}{=}\{a_1^{H_1}B_{H_1,\dots,H_d}(x)\}_{x\in\mathbb{R}^d}\\ \qquad\qquad\vdots\\ \{B_{H_1,\dots,H_d}(x_1,\dots,a_dx_d)\}_{x\in\mathbb{R}^d}\overset{(f.d.)}{=}\{a_d^{H_d}B_{H_1,\dots,H_d}(x)\}_{x\in\mathbb{R}^d}\end{cases}.\qquad(2)$$

Moreover, it follows from definition (1) that the FBS admits stationary rectangular increments according to the coordinate axes. For example, in the bidimensional case ($d=2$) if we denote

$$\begin{aligned}\Delta_{h_1,h_2}B_{H_1,H_2}(x_1,x_2)=&B_{H_1,H_2}(x_1+h_1,x_2+h_2)-B_{H_1,H_2}(x_1+h_1,x_2)\\&-B_{H_1,H_2}(x_1,x_2+h_2)+B_{H_1,H_2}(x_1,x_2),\end{aligned}$$

then (Proposition 2 of [1]) for any $(x_1,x_2)\in\mathbb{R}^2$:

$$\{\Delta_{h_1,h_2}B_{H_1,H_2}(x_1,x_2)\}_{(h_1,h_2)\in\mathbb{R}^2}\overset{(f.d.)}{=}\{\Delta_{h_1,h_2}B_{H_1,H_2}(0,0)\}_{(h_1,h_2)\in\mathbb{R}^2}.$$

Conversely, any Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ of the form

$$X(x)=\int_{\mathbb{R}^d}(e^{i<x_1,\xi_1>}-1)\cdots(e^{i<x_d,\xi_d>}-1)\phi(\xi)d\widehat{W}_{\xi_1,\dots,\xi_d},\qquad(3)$$

with $\int\min(1,|\xi_1|^2)\cdots\min(1,|\xi_d|^2)|\phi(\xi)|^2d\xi<+\infty$, admits stationary rectangular increments according to the coordinate axes (see Sect. 3). Furthermore, if, as in the case of the FBS,

$$\forall(a_1,\dots,a_d)\in(\mathbb{R}_+^*)^d,\ |\phi(a_1\cdots a_d\xi)|^2=a_1^{-2H_1-1}\cdots a_d^{-2H_d-1}|\phi(\xi)|^2,\qquad(4)$$

then the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ defined by (3) satisfies properties (2).

Another model of an anisotropic self-similar random field is the class of operator scaling random fields (OSRF) introduced by H. Biermé, M. Meerschaert, and H.P. Scheffler in [3]. These fields satisfy the following scaling: relation:

$$\forall a > 0,\ \{X(a^E x)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a^H X(x)\}_{x\in\mathbb{R}^d} \tag{5}$$

for some matrix E (called an anisotropy of the field) whose eigenvalues have a positive real part and some $H > 0$ (called a Hurst index of the field). Recall that for any real $a > 0$, a^E denotes the matrix $a^E = \exp(E\log(a)) = \sum_{k\geq 0} \frac{E^k \log^k(a)}{k!}$. Moreover, Biermé, Meerschaert, and Scheffler defined a special class of OSRF with stationary increments: For any matrix E with positive real parts of the eigenvalues, any $H \in (0, \rho_{\min}(E))$–where $\rho_{\min}(E) = \min_{\lambda\in Sp(E)} (Re(\lambda))$–Gaussian fields with stationary increments satisfying (5) can be defined in the following way:

$$X(x) = \int_{\mathbb{R}^d} \left(e^{i<x,\xi>} - 1\right) \rho(\xi)^{-(H+\frac{Tr(E)}{2})} d\widehat{W}_\xi,$$

where ρ is a $(\mathbb{R}^d, E^t)$ pseudo-norm (see [13]) that is a continuous function defined on $\mathbb{R}^d \setminus \{0\}$ with positives values satisfying

$$\forall \xi \in \mathbb{R}^d \setminus \{0\},\ \rho(a^{E^t}\xi) = a\rho(\xi).$$

Then, the main difficulty to overcome is to define a suitable $(\mathbb{R}^d, E^t)$ pseudo-norm. In [3], for any matrix E whose eigenvalues have positive real parts, it is proved that

$$\rho(\xi) = \left(\int_{S_0}\int_0^\infty \left(1 - \cos\left(<\xi, r^E\theta>\right)\right) \frac{dr}{r^2} d\mu(\theta)\right)$$

is an $(\mathbb{R}^d, E^t)$ pseudo-norm (S_0 denotes the unit sphere of $\mathbb{R}^d$ for a well-chosen norm, and μ a finite measure on S_0). We will generalize this result using another approach based on the Haar measure of an m-parameter group.

Here, our purpose is to introduce another class of anisotropic Gaussian field satisfying given simultaneous operator scaling relations. For any $(a_1, \ldots, a_m) \in (\mathbb{R}_+^*)^m$,

$$\begin{cases} \{X(a_1^{E_1}x)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_1^{H_1}X(x)\}_{x\in\mathbb{R}^d}, \\ \qquad\qquad\vdots \\ \{X(a_m^{E_m}x)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_m^{H_m}X(x)\}_{x\in\mathbb{R}^d}, \end{cases} \tag{6}$$

where $E_1, \ldots, E_m$ are m given pairwise commuting matrices. We now illustrate through an example the potential usefulness of this model.

Example 1. Let us consider the two commuting matrices

$$E_1 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any $\theta \in \mathbb{R}$, denote $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ and note that for any positive numbers a_1, a_2,

$$a_1^{E_1} = \begin{pmatrix} a_1 R_{\log(a_1)} & 0 \\ 0 & 1 \end{pmatrix}, \; a_2^{E_2} = \begin{pmatrix} R_{\log(a_2)} & 0 \\ 0 & a_2 \end{pmatrix}.$$

As a consequence of Theorem 1 (see Sect. 4), for all $(H_1, H_2) \in (0,1)^2$ we can define an anisotropic field $\{X(x)\}_{x\in\mathbb{R}^3} = \{X(x_1, x_2, t)\}_{(x_1,x_2,t)\in\mathbb{R}^3}$ such that

$$\begin{cases} \forall a_1 \in \mathbb{R}_+^*, \{X(a_1^{E_1} x)\}_{x\in\mathbb{R}^3} \overset{(f.d.)}{=} \{a_1^{H_1} X(x)\}_{x\in\mathbb{R}^3}, \\ \forall a_2 \in \mathbb{R}_+^*, \{X(a_2^{E_2} x)\}_{x\in\mathbb{R}^3} \overset{(f.d.)}{=} \{a_2^{H_2} X(x)\}_{x\in\mathbb{R}^3}. \end{cases} \tag{7}$$

Let us make some comments about these two scaling properties of field $\{X(x)\}_{x\in\mathbb{R}^3}$. Assume that x_1, x_2 denote two space variables whereas t denotes time. Then, the first scaling property means that at fixed time $\{X(x)\}_{x\in\mathbb{R}^3}$ is a (maybe anisotropic) operator scaling Gaussian field. The second scaling property means that the anisotropy of the field evolves throughout time. In fact, we defined a fixed time anisotropic Gaussian field whose anisotropy rotates with time.

Our objective is now to define such Gaussian fields. In the following two sections, we present our approach. We first consider the special case of Gaussian fields satisfying simultaneous uncoupled relations.

2 A First Example of a Field Satisfying Simultaneous Operator Scaling Properties

We first consider a particular case. Let $(d_1, \ldots, d_m) \in \mathbb{N}^m$ such that $d_1 + \ldots + d_m = d$, $(E_1, \ldots, E_m)$ be m given matrices in $(M_{d_1}(\mathbb{R}), \ldots, M_{d_m}(\mathbb{R}))$ whose eigenvalues have positive real parts, and $(H_1, \ldots, H_m)$ in $\prod_{\ell=1}^{m} (0, \rho_{min}(E_\ell))$.

Combining the model of FBS and this one of the OSRF, one can easily define a Gaussian field $\{X(x_1, \ldots, x_m)\}_{(x_1,\ldots,x_m)\in\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_m}}$ satisfying the following simultaneous operator scaling relations: For any $(a_1, \ldots, a_m) \in (\mathbb{R}_+^*)^m$,

$$\begin{cases} \{X(a_1^{E_1} x_1, \ldots, x_m)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_1^{H_1} X(x_1, \ldots, x_m)\}_{x\in\mathbb{R}^d} \\ \qquad\qquad \vdots \\ \{X(x_1, \ldots, a_m^{E_m} x_m)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a_m^{H_m} X(x_1, \ldots, x_m)\}_{x\in\mathbb{R}^d} \end{cases}. \tag{8}$$

Indeed, the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ can be defined through its harmonizable representation

$$X(x) = Re\left(\int_{\mathbb{R}^{d_1}\times\cdots\times\mathbb{R}^{d_m}} \prod_{\ell=1}^{m}\left((e^{i<x_\ell,\xi_\ell>}-1)(e^{-i<y_\ell,\xi_\ell>}-1)|\phi_\ell(\xi_\ell)|^2\right) d\widehat{W}_\xi\right), \tag{9}$$

where for any ℓ, ϕ_ℓ denotes an $(\mathbb{R}^{d_\ell}\setminus\{0\}, E_\ell^t)$ pseudo-norm. By construction, the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ satisfies simultaneously the m operator scaling relationships (8).

As FBS, this field does not admit stationary increments but satisfies a weaker stationarity property: It admits rectangular stationary increments according to some special directions.

Before giving a precise statement about stationarity properties of the field $\{X(x)\}_{x\in\mathbb{R}^d}$, we first recall the notion of a Gaussian field with rectangular stationary increments. Let us begin by defining the notion of rectangular increments of a function.

Definition 1. *Let $M_1,\ldots,M_m$ be m subspaces of $\mathbb{R}^d$ in direct sum, and let f be a function defined on $\mathbb{R}^d$. For any $x\in\mathbb{R}^d$ and any $(h_1,\cdots,h_m)\in M_1\times\cdots\times M_m$, define*

$$\Delta_{h_1,\ldots,h_m} f(x) = \sum_{\ell=0}^{m}\sum_{1\le i_1<\cdots<i_\ell\le m} (-1)^{m-\ell} f(x+h_{i_1}+\cdots+h_{i_\ell}),$$

where, for $\ell=0$, $f(x+h_{i_1}+\cdots+h_{i_\ell})=f(x)$.

Example 2. In the case $m=1$, $M_1=\mathbb{R}^d$ $\Delta_h f(x)=f(x+h)-f(x)$.
In the case $m=d=2$, $M_1=\mathbb{R}\times\{0\}$, $M_2=\{0\}\times\mathbb{R}$, if $(h,k)\in M_1\times M_2$,

$$\Delta_{h,k} f(x_1,x_2) = f(x_1+h,x_2+k)-f(x_1+h,x_2)-f(x_1,x_2+k)+f(x_1,x_2).$$

Definition 2. *Let $M_1,\ldots,M_m$ be m subspaces of $\mathbb{R}^d$ in direct sum. A Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ admits stationary rectangular increments according to the directions $M_1,\ldots,M_m$ if, for any $x\in\mathbb{R}^d$,*

$$\{\Delta_{h_1,\ldots,h_m}X(x)\}_{(h_1,\ldots,h_m)\in M_1\times\cdots\times M_m} \overset{(\mathcal{L})}{=} \{\Delta_{h_1,\ldots,h_m}X(0)\}_{(h_1,\ldots,h_m)\in M_1\times\cdots\times M_m}.$$

Example 3. In the case $m=1$ we recover the classical notion of a random field with stationary increments. A bidimensional FBS admits stationary rectangular increments according to the directions $M_1=\mathbb{R}\times\{0\}$ and $M_2=\{0\}\times\mathbb{R}$.

Here, in the example above, the Gaussian field defined by (9) admits stationary rectangular increments according to

$$M_1=\mathbb{R}^{d_1}\times\ldots\times\{0_{\mathbb{R}^{d_m}}\},\ldots,M_m=\{0_{\mathbb{R}^{d_1}}\}\times\cdots\times\mathbb{R}^{d_m}.$$

We now use the special case introduced in this section in order to present a general approach to define Gaussian fields satisfying the m given scaling properties (6).

3 A General Approach

In the general case, we will formulate the problem as in Sect. 2 and define a Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ which admits stationary rectangular increments according to some special directions $(M_1,\dots,M_{m'})$ in direct sum with $m'\geq m$. These special directions follow from the simultaneous reduction of matrices $E_1,\dots,E_m$ (see Sect. 2 above) and then are invariant through these matrices. That is, for any j in $\{1,\dots,m\}$, for any ℓ in $\{1,\dots,m'\}$ $E_jM_\ell\subset M_\ell$. Subspaces $(M_1,\dots,M_{m'})$ will be called the renormalization directions of Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ and will have to be defined.

After, the definition of these renormalization directions, the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ will be defined as follows. We are given a function ϕ with positive values satisfying the m following simultaneous properties: For any $a_1,\dots,a_m$, for almost any ξ in $\mathbb{R}^d$,

$$\begin{cases}\phi(a_1^{-E_1}\xi) &= a_1^{H_1+\frac{Tr(E_1)}{2}}\phi(\xi)\\ &\vdots\\ \phi(a_m^{-E_m}\xi) &= a_m^{H_m+\frac{Tr(E_m)}{2}}\phi(\xi)\end{cases}\tag{10}$$

Furthermore, in order for integral (12) to exist, we assume that

$$\forall x\in\mathbb{R}^d,\ \int_{\mathbb{R}^d}|\phi(\xi)|^2\prod_\ell(\min(1,|<\xi,x_\ell>|^2))d\xi<+\infty.\tag{11}$$

Function $|\phi(\xi)|^2$ is then called a spectral density of Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$. Thereafter, for any $x=(x_1,\dots,x_{m'})\in M_1\times\cdots\times M_{m'}$, we set

$$X(x)=Re\left(\int_{\mathbb{R}^d}\prod_{\ell=1}^{m'}(e^{i<x_\ell,\xi>}-1)\phi(\xi)d\widehat{W}_\xi\right).\tag{12}$$

The following proposition proves that the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ satisfies the required properties.

Proposition 1. *Let $(E_1,\dots,E_m)$ be m pairwise commuting matrices. Assume that there exist $(M_1,\dots,M_{m'})$ m' subspaces of $\mathbb{R}^d$ in direct sum, invariant through the action of matrices $(E_1,\dots,E_m)$ and let ϕ be a function with positive values satisfying properties (10) and condition (11). Then $\{X(x)\}_{x\in\mathbb{R}^d}$ defined by (12) is with stationary rectangular increments according to directions $M_1,\dots,M_{m'}$ and satisfies the m simultaneous operator scaling relations (6).*

Proof. Indeed, since condition (11) is satisfied, $\{X(x)\}_{x\in\mathbb{R}^d}$ defined by (12) exists for any $x\in\mathbb{R}^d$. Denote

$$X_1 = Re\left(\int_{\mathbb{R}^d} h_1(\xi)d\widehat{W}_\xi\right),\ X_2 = Re\left(\int_{\mathbb{R}^d} h_2(\xi)d\widehat{W}_\xi\right),$$

and remark that using corollary 6.3.2 of [16], in the special case of Gaussian random variables,

$$\mathbb{E}(X_1X_2) = Re\left(\int_{\mathbb{R}^d} h_1(\xi)\overline{h_2(\xi)}d\xi\right).$$

It is then sufficient to prove that $\{Y(x)\}_{x\in\mathbb{R}^d}$, defined as

$$Y(x) = \int_{\mathbb{R}^d}\prod_{\ell=1}^{m'}(e^{i<x_\ell,\xi>}-1)\phi(\xi)d\widehat{W}_\xi,$$

satisfies the required properties.

Then note that the homogeneity properties (10) satisfied by ϕ imply that $\{Y(x)\}_{x\in\mathbb{R}^d}$ satisfies (6). Moreover, $\{Y(x)\}_{x\in\mathbb{R}^d}$ admits stationary rectangular increments according to directions $M_1,\ldots,M_{m'}$. Indeed, for any $x\in\mathbb{R}^d$, $(h_1,\ldots,h_{m'})\in M_1\times\cdots\times M_{m'}$, $(k_1,\ldots,k_{m'})\in M_1\times\cdots\times M_{m'}$

$$\mathbb{E}(\Delta_{h_1,\ldots,h_{m'}}Y(x)\overline{\Delta_{k_1,\ldots,k_{m'}}Y(x)}) = \int\prod_{\ell=1}^{m'}(e^{i<h_\ell,\xi>}-1)(e^{-i<k_\ell,\xi>}-1)|\phi(\xi)|^2d\xi.$$

This expression does not depend on x, and then the result follows. ■

Let us illustrate this proposition by giving an explicit construction of a Gaussian field satisfying the two simultaneous operator scaling properties introduced in Example 1.

Example 4. The notation is that of Example 1. Our objective is to define a Gaussian field satisfying the two simultaneous scaling properties (7). Keeping Proposition 1 in mind, we consider

$$X(x_1,x_2,t) = \int_{\mathbb{R}^3}\left(e^{i(x_1\xi^1_{space}+x_2\xi^2_{space})}-1\right)\left(e^{it\xi_{time}}-1\right)\phi\left(\xi^1_{space},\xi^2_{space},\xi_{time}\right)d\widehat{W}_\xi.$$

It will be required that function ϕ satisfies (10) and (11). Let us set

$$\phi(\xi^1_{space},\xi^2_{space},\xi_{time}) = \frac{\left|\xi^1_{space}\cos(\log(|\xi_{space}|\cdot|\xi_{time}|))-\xi^2_{space}\cos(\log(|\xi_{space}|\cdot|\xi_{time}|))\right|}{|\xi_{space}|^{H_1+1}|\xi_{time}|^{H_2+\frac{1}{2}}}.$$

One can easily check that ϕ fullfills the assumptions (10) and (11) of Proposition 1 and thus that the Gaussian field $\{X(x)\}_{x\in\mathbb{R}^3}$ satisfies the two simultaneous scaling properties (7).

We now state the existence results proved in this paper.

4 Existence Results

In order to state our existence results, some hypotheses are needed on matrices $(E_1, \dots, E_m)$. We first review the real diagonalizable part of a matrix.

4.1 Real Diagonalizable Part of a Matrix

As usual (see [8] or [15]), our main tool will be the complete additive Jordan decomposition of a matrix. We refer to lemma 7.1, Chap. 9 of [7].

Proposition 2. *Any matrix M of $M_d(\mathbb{R})$ can be decomposed into a sum of commuting real matrices $M = D + S + N$, where D is a diagonalizable matrix in $M_d(\mathbb{R})$, S is a diagonalizable matrix in $M_d(\mathbb{C})$ with zero or imaginary complex eigenvalues, and N is a nilpotent matrix. Matrix D is called the real diagonalizable part of M.*

Below, we give examples of the real diagonalizable part of a matrix E.

1. If $E = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ or $E = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$, then E admits λ as a unique eigenvalue and as real diagonalizable part $D = \lambda Id$, where Id denotes the identity matrix.
2. If $E = \begin{pmatrix} \Delta & & 0 \\ & \ddots & \\ 0 & & \Delta \end{pmatrix}$ or $E = \begin{pmatrix} \Delta & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & \Delta \end{pmatrix}$ with $\Delta = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, then E admits exactly two conjugate complex eigenvalues $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$. This implies that $D = \alpha Id$.

Recall that, in general, a square real matrix is similar to a block diagonal matrix where each block is a square matrix of the form above. Thus, from the previous examples we can deduce the complete additive Jordan decomposition of any square real matrix. We now state our assumptions on matrices $E_1, \dots, E_m$.

4.2 Assumptions on Matrices $E_1, \dots, E_m$

Denote by $D_1, \dots, D_m$ the real diagonalizable parts of matrices $E_1, \dots, E_m$.

Hypotheses 4.1 *We assume that*

1. *Matrices $E_1, \dots, E_m$ are pairwise commuting matrices.*

2. *Matrices* $(D_1, \ldots, D_m)$ *are linearly independent matrices in* $M_d(\mathbb{R})$.

Under these assumptions we now state our main result.

Theorem 1. *Assume that Hypotheses 4.1 are satisfied. Then for some* $C_0 \in (0,1)$ *depending only on matrices* $E_1, \ldots, E_m$, *for any* $\ell \in \{1, \ldots, m\}$ *and any* $0 < H_\ell < C_0 \rho_{\min}(E_\ell)$ *there exists* $m' \geq m$, m' *subspaces* $M_1, \ldots, M_{m'}$ *of* $\mathbb{R}^d$ *in direct sum and a Gaussian field* $\{X(x)\}_{x \in \mathbb{R}^d}$ *with rectangular stationary increments according to the directions* $M_1, \ldots, M_{m'}$ *satisfying the* m *simultaneous operator scaling properties:*

$$\forall (a_1, \ldots, a_m) \in (\mathbb{R}_+^*)^m, \begin{cases} \{X(a_1^{E_1} x)\}_{x \in \mathbb{R}^d} \overset{(f.d.)}{=} \{a_1^{H_1} X(x)\}_{x \in \mathbb{R}^d} \\ \vdots \\ \{X(a_m^{E_m} x)\}_{x \in \mathbb{R}^d} \overset{(f.d.)}{=} \{a_m^{H_m} X(x)\}_{x \in \mathbb{R}^d} \end{cases}. \tag{13}$$

Before obtaining any proof of this result, we need first to reformulate our problem in terms of group self-similarity. It will then allow us to use the concept of Haar measure in order to define a spectral density of Gaussian field $\{X(x)\}_{x \in \mathbb{R}^d}$.

5 Reformulation of the Problem in Terms of Group Self-Similarity

In [11], S. Kolodynski and J. Rosinski defined the notion of a group self-similar random field. We adapt this definition to our setting

Definition 3. *Let* $\mathcal{A}$ *be a subgroup of* $Gl_d(\mathbb{R})$ *and* χ *be a continuous mapping from* $\mathcal{A}$ *into* $\mathbb{R}_+^*$. *The random field* $\{X(x)\}_{x \in \mathbb{R}^d}$ *is* $\mathcal{A}$*-self-similar with coefficient* χ *if*

$$\forall A \in \mathcal{A}, \{X(Ax)\}_{x \in \mathbb{R}^d} \overset{(f.d.)}{=} \{\chi(A) X(x)\}_{x \in \mathbb{R}^d}. \tag{14}$$

Remark 1. Remark that (see [11]) the self-similarity coefficient of a Gaussian field is necessarily a homomorphism from $\mathcal{A}$ into $\mathbb{R}_+^*$.

Our problem can now be reformulated in terms of group self-similarity. Let us define the following m-parameter group:

$$\mathcal{A} = \{a_1^{E_1} \cdots a_m^{E_m}, (a_1, \ldots, a_m) \in (\mathbb{R}_+^*)^m\}. \tag{15}$$

Proposition 3. *The two problems are equivalent:*

1. *Find sufficient conditions on* $H_1, \ldots, H_m$ *for the existence of a Gaussian field* $\{X(x)\}_{x \in \mathbb{R}^d}$ *satisfying the simultaneous scaling properties (6).*

2. *Find sufficient conditions on homomorphism χ for the existence of a Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ $\mathcal{A}$-self-similar with coefficient χ where $\mathcal{A}$ is the m-parameter group defined by (15).*

The proof of Proposition 3 is based on the complete description of the continuous homomorphisms from $\mathcal{A}$ into $\mathbb{R}_+^*$. More precisely, one can easily check the following proposition (a more detailed proof may be found in [5]).

Proposition 4. *If Hypotheses (4.1) are satisfied, the mapping $(a_1,\dots,a_m)\mapsto a_1^{E_1}\dots a_m^{E_m}$ is a bicontinous isomorphism from $(\mathbb{R}_+^*)^m$ into $\mathcal{A}$.*

Since the homomorphisms from $(\mathbb{R}_+^*)^m$ into $\mathbb{R}_+^*$ are well known, if Hypotheses 4.1 are satisfied, Proposition 4 directly implies Proposition 5 (a more detailed proof may be found in [5]).

Proposition 5. *Let $\mathcal{A}$ be a subgroup of $Gl_d(\mathbb{R})$ of the form (15). Assume that Hypotheses 4.1 are satisfied. Then for any continuous homomorphism χ from $\mathcal{A}$ into $\mathbb{R}_+^*$, there exists a unique $(H_1,\dots,H_m)\in(\mathbb{R}_+^*)^m$ such that*

$$\forall (a_1,\dots,a_m)\in(\mathbb{R}_+^*)^m,\ \chi(a_1^{E_1}\cdots a_m^{E_m})=a_1^{H_1}\cdots a_m^{H_m}.$$

Then any $\mathcal{A}$-self-similar Gaussian field with coefficient χ, where χ is a continuous mapping from $\mathcal{A}$ into $\mathbb{R}_+^*$, necessarily satisfies

$$\forall (a_1,\dots,a_m)\in(\mathbb{R}_+^*)^m,\ \{X(a_1^{E_1}\cdots a_m^{E_m}x)\}_{x\in\mathbb{R}^d}\overset{(\mathcal{L})}{=}\{a_1^{H_1}\cdots a_m^{H_m}X(x)\}_{x\in\mathbb{R}^d}$$

for some $(H_1,\dots,H_m)\in(\mathbb{R}_+^*)^m$. We have then proved Proposition 3.

Example 5. We now give examples of $\mathcal{A}$-group self-similar Gaussian fields with coefficient χ.

- FBM is group self-similar with $\mathcal{A}=\{aId\}_{a\in\mathbb{R}_+^*}$ and $\chi(aId)=a^H$.
- FBS is group self-similar where $\mathcal{A}$ is the group of diagonal matrices with positive coefficients and $\chi\left(\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\right)=\lambda_1^{H_1}\cdots\lambda_n^{H_n}$.
- Any OSRF is group self-similar with $\mathcal{A}=\{a^E, a>0\}$, $\chi(a^E)=a^H$.
- The Gaussian field defined in Sect. 2 is also group self-similar with $\mathcal{A}=\left\{\begin{pmatrix}a_1^{E_1} & & 0\\ & \ddots & \\ 0 & & a_m^{E_m}\end{pmatrix},\ \forall i\ a_i>0\right\}$, $\chi\left(\begin{pmatrix}a_1^{E_1} & & 0\\ & \ddots & \\ 0 & & a_m^{E_m}\end{pmatrix}\right)=\prod_\ell a_\ell^{H_\ell}$.

6 Definition of the Desired Gaussian Field

As stated in Sect. 3, we now define the desired Gaussian field in two steps. We first define a suitable spectral density (see Sect. 6.1). Then, using simultaneous reduction of matrices $E_1, \ldots, E_m$, we define renormalization directions $M_1, \ldots, M_{m'}$ (see Sect. 6.2). Finally, in Sect. 6.3, we will use Proposition 1 to prove that the Gaussian field we just defined satisfies the required properties.

6.1 Definition of a Suitable Spectral Density

We reformulate the properties required on spectral density using group $\mathcal{A}$ defined by (15).

Proposition 6. *Let ϕ be a function defined on $\mathbb{R}^d$ with positive values. The two properties are equivalent:*

1. *Function $|\phi(\xi)|^2$ satisfies the m simultaneous relations (10).*
2. *Function $|\phi(\xi)|^2$ is $\mathcal{A}$ homogeneous with coefficient $\chi^2(\cdot)|\det(\cdot)|$, that is,*

$$\forall A \in \mathcal{A}, \text{ a.e. } \xi \in \mathbb{R}^n, \ |\phi((A^{-1})^t \xi)|^2 = |\chi(A)|^2 |\det(A)| |\phi(\xi)|^2, \tag{16}$$

with $\chi(a_1^{E_1} \cdots a_m^{E_m}) = a_1^{H_1} \cdots a_m^{H_m}$.

We now define a suitable spectral density using a Haar measure $\mu_{\mathcal{A}}$ of group $\mathcal{A}$. Let

$$|\phi(\xi)|^2 = \int_{\mathcal{A}} \chi(A)^2 |\det(A)| |\widehat{\psi}(-A^t \xi)|^2 d\mu_{\mathcal{A}}(A), \tag{17}$$

where $\psi \in H^{m+1}(\mathbb{R}^d)$. As usual,

$$H^{m+1}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d), |\xi|^{m+1} \widehat{f} \in L^2(\mathbb{R}^d)\}.$$

Then, the invariance property of any Haar measure of group $\mathcal{A}$ implies that the following.

Proposition 7. *Function $|\phi(\xi)|^2$ defined by (17) is an $\mathcal{A}^t$ homogeneous function with coefficient $\chi^2(\cdot)|\det(\cdot)|$.*

Remark 2. Proposition 7 does not prove that function $|\phi(\xi)|^2$ defined by (17) is finite almost everywhere. The finiteness of function $|\phi(\xi)|^2$ comes from the existence of the covariance function of the required Gaussian field.

Function $|\phi(\xi)|^2$ defined by (17) then satisfies the required properties of a spectral density. We now give some details about the definition of renormalization directions.

6.2 Definition of Renormalization Directions

In this section, our purpose is to define m' special subspaces of $\mathbb{R}^d$ $M_1, \dots, M_{m'}$ in direct sum, invariant through matrices $E_1, \dots, E_m$ such that hypotheses of Proposition 1 are fullfilled. These subspaces will be called renormalization directions and are invariant through group $\mathcal{A}$. Proposition 1 then ensures the existence of a Gaussian field satisfying the required properties. Moreover (see Sect. 2), if

$$\mathcal{A} = \{diag(a_1^{E_1}, \dots, a_m^{E_m}), (a_1, \dots, a_m) \in (\mathbb{R}_+^*)^m\}, \tag{18}$$

with $(E_1, \dots, E_m) \in M_{d_1}(\mathbb{R}) \times \cdots \times M_{d_m}(\mathbb{R})$, whose eigenvalues have positive real parts (see Sect. 2), we can choose as renormalization directions

$$M_1 = \mathbb{R}^{d_1} \times \ldots \times \{0\}, \dots, M_m = \{0\} \times \cdots \times \mathbb{R}^{d_m}.$$

Now we want to extend the approach of Sect. 2 to the general case. In order to define renormalization directions, we simultaneously diagonalize matrices $D_1, \dots, D_m$ using the following proposition.

Proposition 8. *Let $E_1, \dots, E_m$ be m pairwise commuting square matrices. Denote by $D_1, \dots, D_m$ their real diagonalizable parts. Then*

1. *Matrices $D_1, \dots, D_m$ are pairwise commuting and then simultaneously diagonalizable.*
2. *Matrices $E_1, \dots, E_m$ are all commuting with matrices $D_1, \dots, D_m$.*

Definition of renormalization directions will follow from this simultaneous reduction of matrices $D_1, \dots, D_m$. The following notation will be needed.

Notation 1 *Let $\mathcal{A}$ be the group defined by (15). Then denote*

$$\mathcal{A}_D = \{a_1^{D_1} \cdots a_m^{D_m}, (a_1, \dots, a_m) \in (\mathbb{R}_+^*)^m\}.$$

We can reduce simultaneously matrices of group $\mathcal{A}_D$.

Proposition 9. *Assume that Hypotheses 4.1 are fullfilled. There exists an invertible matrix P such that*

$$\mathcal{A}_D = \{P \times diag(a_1^{\Delta_1}, \dots, a_m^{\Delta_m}, a_1^{D_1^{m+1}} .. a_m^{D_m^{m+1}}) \times P^{-1}, (a_1, \dots, a_m) \in (\mathbb{R}_+^*)^m\},$$

where

1. *For any $k \in \{1, \dots, m\}$, $\Delta_k = \begin{pmatrix} \Delta_k^+ & 0 \\ 0 & \Delta_k^- \end{pmatrix}$.*
2. *For any $k \in \{1, \dots, m\}$, matrices Δ_k^+ (resp. Δ_k^-, D_ℓ^{m+1}) are diagonal matrices with positive coefficients (resp. negative, unspecified).*

3. Matrices Δ_k^+ always exist for any $k \in \{1, \ldots, m\}$ whereas matrices Δ_k^-, D_ℓ^{m+1} can possibly not exist for some values of k or ℓ.

Proof. The proof is detailed in [5].
Let us illustrate Proposition 9 through an example. ■

Example 6. Let us consider the following group:

$$\mathcal{A} = \{a_1^{E_1} a_2^{E_2}, (a_1, a_2) \in (\mathbb{R}_+^*)^2\},$$

with $E_1 = D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}, E_2 = D_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Here $\mathcal{A} = \mathcal{A}_D$, matrices D_1 and D_2 are diagonal, and

$$Vect < D_1, D_2 >= Vect < \begin{pmatrix} \Delta & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 2\Delta & 0 \\ 0 & 1 \end{pmatrix} >, \text{ with } \Delta = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

To prove that matrices (D_1, D_2) are linearly independent matrices, note that $\left(\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$ are linearly independent vectors. Since

$$Vect < \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} >= Vect < \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} >,$$

it implies that

$$Vect < \begin{pmatrix} \Delta & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 2\Delta & 0 \\ 0 & 1 \end{pmatrix} >= Vect < \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} > .$$

Hence we deduce that $\mathcal{A} = \left\{ \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1^{-2} & 0 \\ 0 & 0 & a_2 \end{pmatrix}, (a_1, a_2) \in (\mathbb{R}_+^*)^2 \right\}$. Thus, we recover the result of Proposition 9 with $\Delta_1^+ = (1), \Delta_1^- = (-2), \Delta_2^+ = (2)$.

Proposition 9 implies the following description of group $\mathcal{A}$.

Proposition 10. *We use the notation of Proposition 9. Group $\mathcal{A}$ is of the form*

$$\mathcal{A} = \{P a_1^{F_1} \cdots a_m^{F_m} P^{-1}, (a_1, \ldots, a_m) \in (\mathbb{R}_+^*)^m\},$$

where for any ℓ, matrix F_ℓ admits as the real diagonalizable part the diagonal matrix $diag(0, \ldots, 0, \Delta_\ell, 0, \ldots, D_\ell^{m+1})$.

Proof. The proof is detailed in [5]. ■

Proposition 10 will allow us to define renormalization directions. Let us define for any $\ell \in \{1,\dots,m\}$, $W_\ell^+ = \mathbb{R}^{d_1^+}$, $W_\ell^- = \mathbb{R}^{d_1^-}, W_{m+1} = \mathbb{R}^{d_{m+1}}$. We then choose as renormalization directions

$$\forall \ell \in \{1,\dots,m\},\ V_\ell^+ = P^{-1}W_\ell^+,\ V_\ell^- = P^{-1}W_\ell^-, V_{m+1} = P^{-1}W_{m+1}. \quad (19)$$

These subspaces are all invariant through matrices $E_1,\dots,E_m$. The sets V_l^-, V_{m+1} can be possibly equal to $\{0\}$.

Notation 2 *Denote by $M_1,\dots,M_{m'}$ the m' nonzero sets within the spaces $V_1^+,\dots,V_m^+, V_1^-,\dots,V_m^-, V_{m+1}$. Subspaces $M_1,\dots,M_{m'}$ are called nontrivial renormalization directions.*

Example 7. In Example 6 above, the renormalization directions are

$$M_1^+ = \mathbb{R}\times\{0\}\times\{0\},\ M_1^- = \{0\}\times\mathbb{R}\times\{0\},\ M_2^+ = \{0\}\times\{0\}\times\mathbb{R}.$$

In the following section, we prove that this construction method is effective.

6.3 Proof of Theorem 1

Consider function $|\phi(\xi)|^2$ defined by (17) and the renormalization directions $M_1,\dots,M_{m'}$ defined in Sect. 6.2. We want to find sufficient conditions on χ in order for condition (11) of Proposition 1 to be fullfilled. Let us first note the following.

Lemma 1. *Let $\mathcal{A}_P$ be the following m-parameter group*

$$\mathcal{A}_P = P^{-1}\mathcal{A}P = \{a_1^{F_1}\cdots a_m^{F_m}, (a_1,\dots,a_m)\in(\mathbb{R}_+^*)^m\},$$

and let χ_P be defined as $\chi_P(A_P) = \chi(PA_PP^{-1})$ for any $A_P\in\mathcal{A}_P$. Condition (11) is satisfied iff for any $x\in\mathbb{R}^d$ the following integral:

$$\int_{\mathbb{R}^d}\prod_\ell(\min(1,|<x_{W_l^+},\zeta>|^2)\min(1,|<x_{W_l^-},\zeta>|^2))|\phi_{P^t}(\zeta)|^2d\zeta, \quad (20)$$

is finite with $|\phi_{P^t}(\cdot)|^2 = |\phi(P^t\cdot)|^2 = \int\chi_P^2(A_P)|\det(A_P)||\widehat{\psi_P}(-A_P^t\cdot)|^2d\mu_{\mathcal{A}_P}(A_P)$ and $\psi_P(\cdot)=\psi(P\cdot)$.

Remark 3. In the proof of the existence of the desired Gaussian field, one can then replace $\mathcal{A}$ by $\mathcal{A}_P$, χ by χ_P, and ψ by ψ_P.

Proof. To prove this result, we perform the changes of variable $\xi = P^t\zeta$, $A_P = PAP^{-1}$.
Lemma 1 leads us to consider a special case. ■

Proposition 11. *The notation is that of Proposition 10. Assume that for any $i\in\{1,\dots,m\}$, $-\rho_{\min}(\Delta_i^-) < H_i' < \rho_{\min}(\Delta_i^+)$, with $\rho_{min}(\Delta_i^-) = 0$ if matrix Δ_i^- does not exist. Then condition (20) is satisfied for χ_P defined from $\mathcal{A}_P$ into $\mathbb{R}_+^*$ as $\chi(a_1^{F_1}\cdots a_m^{F_m}) = a_1^{H_1'}\cdots a_m^{H_m'}$.*

Proof. The proof is detailed in [5]. ■

Lemma 1 and Proposition 11 then imply Theorem 1 (see [5] for more details).

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On Randomly Placed Arcs on the Circle

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Summary. We completely describe in terms of Hausdorff measures the size of the set of points of the circle that are covered infinitely often by a sequence of random arcs with given lengths. We also show that this set is a set with large intersection.

1 Introduction and Statement of the Results

Let us consider a nonincreasing sequence $\ell = (\ell_n)_{n\geq 1}$ of positive reals converging to zero. In 1956, A. Dvoretzky [11] raised the question of finding a necessary and sufficient condition on the sequence ℓ to ensure that the whole circle $\mathcal{T} = \mathbb{R}/\mathbb{Z}$ is covered almost surely by a sequence of random arcs with lengths ℓ_1, ℓ_2, etc. To be specific, let $A(x, l)$ denote the open arc with center $x \in \mathcal{T}$ and length $l > 0$, that is, the set of all $y \in \mathcal{T}$ such that $\mathrm{dist}(y, x) < l/2$, where dist denotes the usual quotient distance on $\mathcal{T}$. Then, let $(X_n)_{n\geq 1}$ be a sequence of random points independently and uniformly distributed on $\mathcal{T}$ and let

$$E_\ell = \limsup_{n\to\infty} A(X_n, \ell_n).$$

Dvoretzky's problem amounts to finding a necessary and sufficient condition to ensure that

$$\text{a.s.} \qquad E_\ell = \mathcal{T}. \tag{1}$$

This longstanding problem, along with several of its extensions, has raised the interest of many mathematicians. In fact, Dvoretzky himself gave in [11] a simple sufficient condition ensuring that (1) holds. Subsequently, J.-P. Kahane heard of the problem from P. Lévy and established in [18] that (1) holds for $\ell_n = a/n$ when $a > 1$ but not when a is too small. Moreover, P. Erdős [12] announced without publishing a proof that (1) also holds when $a = 1$, and P. Billard showed in his thesis that (1) does not hold when $a < 1$. In addition, Dvoretzky's problem has found various applications, notably in the study of multiplicative processes and that of random series of functions. We refer to Kahane's book [19] and his survey paper [20] for details.

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Finally, Dvoretzky's problem was completely solved in 1972 by L. Shepp, who showed in [23] that (1) holds if, and only if,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(\ell_1 + \cdots + \ell_n) = \infty. \tag{2}$$

Still, several related questions remained open and were considered afterwards, such as that of the covering of a fixed part K of the circle by more general random subsets; see [19, 20] and the references therein.

Furthermore, in the case where the series in (2) diverges, the whole circle is covered with probability one, so that the series $\sum_n \mathbf{1}_{\{x \in A(X_n, \ell_n)\}}$ diverges everywhere. As L. Carleson did in a private communication to Kahane, it is then natural to ask for the study of the asymptotic behavior of the partial sums

$$C_n(x) = \sum_{m=1}^{n} \mathbf{1}_{\{x \in A(X_m, \ell_m)\}},$$

which tell how many random arcs among the first n ones cover each point x of the circle. Significant results on that topic were obtained by J. Barral, A.-H. Fan, and Kahane, and we refer to [2, 15, 16] for precise statements.

Now, if the series in (2) converges, one may wonder which proportion of the circle is actually covered infinitely often by the random arcs. In other words, what is the size of the set E_ℓ? A first answer may be given by computing the value of its Lebesgue measure $\mathcal{L}(E_\ell)$. This is, in fact, trivial, since Fubini's theorem and the Borel–Cantelli lemma directly imply that

$$\text{a.s.} \qquad \mathcal{L}(E_\ell) = \begin{cases} 0 & \text{if} \quad \sum_n \ell_n < \infty \\ 1 & \text{if} \quad \sum_n \ell_n = \infty. \end{cases} \tag{3}$$

A standard way of refining the description of the size of the set E_ℓ is then to compute the value of its Hausdorff dimension. This has been done recently by Fan and J. Wu [17] in the particular case where $\ell_n = a/n^\alpha$, for some $a > 0$ and $\alpha > 1$. Their result states that

$$\forall a > 0 \quad \forall \alpha > 1 \quad \text{a.s.} \qquad \dim_{\mathrm{H}} E_{(a/n^\alpha)} = \frac{1}{\alpha},$$

where $\dim_{\mathrm{H}}$ stands for Hausdorff dimension. Corollary 1 in Sect. 1.1 ensures that, for any nonincreasing sequence $\ell = (\ell_n)_{n \geq 1}$ converging to zero, the Hausdorff dimension of the set E_ℓ is almost surely equal to

$$s_\ell = \sup\bigl\{ s \in (0,1) \bigm| \textstyle\sum_n {\ell_n}^s = \infty \bigr\} = \inf\bigl\{ s \in (0,1) \bigm| \textstyle\sum_n {\ell_n}^s < \infty \bigr\}, \tag{4}$$

with the convention that $\sup \emptyset = 0$ and $\inf \emptyset = 1$. Corollary 1 will follow from Theorem 1, which, in fact, gives the value of the Hausdorff g-measure of the set E_ℓ for any gauge function g and not only the monomial functions

used to define the Hausdorff dimension (see Sect. 1.1). Therefore, this theorem provides a complete description of the size of the set E_ℓ in terms of Hausdorff measures, for any sequence ℓ.

We also establish in this chapter that the set E_ℓ enjoys a remarkable property originally introduced by K. Falconer [13], namely, it is a set with large intersection. Roughly speaking, this means that E_ℓ is "large and omnipresent" in the circle in a strong measure theoretic sense; see Sect. 1.2. As shown by several authors, numerous examples of sets with large intersection appear in the theory of Diophantine approximation and that of dynamical systems; see [3, 4, 7–9, 13]. Their first occurrence in probability theory was pointed out by J.-M. Aubry and S. Jaffard [1] when performing the multifractal analysis of a specific model of random wavelet series. Since then, sets with large intersection have been shown to arise in the multifractal analysis of other stochastic processes; see [6, 10].

1.1 Size Properties of the Set E_ℓ

A typical way of precisely describing the size of a subset of the circle is to determine the value of its Hausdorff g-measure for any gauge function g; see [14, 22]. We call a gauge function any function g defined on $[0, \infty)$ that is nondecreasing near zero, enjoys $\lim_{0+} g = g(0) = 0$, and is such that $r \mapsto g(r)/r$ is nonincreasing and positive near zero. For any gauge function g, the Hausdorff g-measure of a set $F \subseteq \mathcal{T}$ is then defined by

$$\mathcal{H}^g(F) = \lim_{\delta \downarrow 0} \uparrow \mathcal{H}^g_\delta(F) \quad \text{with} \quad \mathcal{H}^g_\delta(F) = \inf_{F \subseteq \bigcup_p U_p \text{ and } |U_p| < \delta} \sum_{p=1}^{\infty} g(|U_p|).$$

The infimum is taken over all sequences $(U_p)_{p \geq 1}$ of sets with $F \subseteq \bigcup_p U_p$ and $|U_p| < \delta$ for all $p \geq 1$, where $|\cdot|$ denotes diameter. Note that if $g(r)/r$ goes to infinity at zero, every nonempty open subset of $\mathcal{T}$ has infinite Hausdorff g-measure and that, otherwise, $\mathcal{H}^g$ coincides up to a multiplicative constant with the Lebesgue measure on the Borel subsets of $\mathcal{T}$.

The size properties of E_ℓ are then described in terms of the Hausdorff measures by the following result. Note that the value of the packing dimension of that set is given by Corollary 2 below.

Theorem 1. *Let $\ell = (\ell_n)_{n \geq 1}$ be a nonincreasing sequence of positive reals converging to zero and let g be a gauge function. Then, with probability one, for any open subset V of $\mathcal{T}$,*

$$\mathcal{H}^g(E_\ell \cap V) = \begin{cases} \mathcal{H}^g(V) & \text{if} \quad \sum_n g(\ell_n) = \infty \\ 0 & \text{if} \quad \sum_n g(\ell_n) < \infty. \end{cases}$$

Let Id denote the identity function on the nonnegative reals. Then, the Hausdorff dimension of a nonempty set $F \subseteq \mathcal{T}$ is defined with the help of the monomial functions Id^s by

$$\dim_{\mathrm{H}} F = \sup\{s \in (0,1) \mid \mathcal{H}^{\mathrm{Id}^s}(F) = \infty\} = \inf\{s \in (0,1) \mid \mathcal{H}^{\mathrm{Id}^s}(F) = 0\},$$

with the same convention regarding the infimum and the supremum of the empty set as in (4). Also, it is customary to let $\dim_{\mathrm{H}} \emptyset = -\infty$. Using Theorem 1, it is then possible to determine the value of the Hausdorff dimension of the set E_ℓ, thereby generalizing the result of [17].

Corollary 1. *For every nonincreasing sequence $\ell = (\ell_n)_{n\geq 1}$ of positive reals converging to zero,*

$$a.s. \qquad \dim_{\mathrm{H}} E_\ell = s_\ell,$$

where s_ℓ is defined by (4).

Theorem 1 and Corollary 1 are proven in Sects. 2 and 3, respectively.

1.2 Large Intersection Properties of the Set E_ℓ

Rigorously, the fact that E_ℓ is a set with large intersection means that it belongs to some classes $\mathrm{G}^g(V)$ of subsets of the circle that we defined in [10, Sect. 5]. These classes were introduced with the aim of both transposing to the periodic setting and refining the classes $\mathcal{G}^s$ of sets with large intersection introduced by Falconer in [13]. Recall that, for any real $s \in (0,1]$, the class $\mathcal{G}^s$ is the maximal class of G_δ-subsets of $\mathbb{R}$ with Hausdorff dimension at least s that is closed under countable intersections and similarities.

We refer to [7, 10] for a precise definition of our generalized classes, and we content ourselves with stressing the fact that, for any gauge function g and any nonempty open subset V of the circle, the class $\mathrm{G}^g(V)$ of *sets with large intersection in V with respect to g* enjoys, among other properties, the following.

Proposition 1. *For any gauge function g and any nonempty open $V \subseteq \mathcal{T}$,*

(a) The class $\mathrm{G}^g(V)$ is closed under countable intersections.
(b) Every set $F \in \mathrm{G}^g(V)$ enjoys $\mathcal{H}^{\overline{g}}(F) = \infty$ for any gauge $\overline{g}$ with $\overline{g} \prec g$.
(c) $\mathrm{G}^g(V) = \bigcap_{\overline{g}} \mathrm{G}^{\overline{g}}(V)$ where $\overline{g}$ is a gauge function satisfying $\overline{g} \prec g$.
(d) $\mathrm{G}^g(V) = \bigcap_U \mathrm{G}^g(U)$ where U is a nonempty open subset of V.
(d) Every G_δ-set with full Lebesgue measure in V belongs to the class $\mathrm{G}^g(V)$.

The notation $\overline{g} \prec g$ means that $\overline{g}/g$ monotonically tends to infinity at zero. In view of Proposition 1, every set in the class $\mathrm{G}^g(V)$ has infinite Hausdorff $\overline{g}$-measure in every nonempty open subset of V for any gauge function $\overline{g} \prec g$, and any countable intersection of such sets enjoys the same property. Therefore, the classes $\mathrm{G}^g(V)$ provide a rigorous way of stating that a set is large and omnipresent in V in a strong measure theoretic sense.

In order to describe the large intersection properties of the set E_ℓ, we shall make use of the following result, which gives a simple sufficient condition for a lim sup of arcs to be a set with large intersection in the circle. It may be seen as the analog for the periodic setting of the *ubiquity* result established in [7].

Proposition 2. *Let $(y_n)_{n\geq 1}$ be a sequence in $\mathcal{T}$ and let $(r_n)_{n\geq 1}$ be a sequence of positive reals converging to zero. Then, for any gauge function g,*

$$\mathcal{L}\left(\limsup_{n\to\infty} A(y_n, 2g(r_n))\right) = 1 \qquad \Longrightarrow \qquad \limsup_{n\to\infty} A(y_n, 2r_n) \in \mathrm{G}^g(\mathcal{T}).$$

Proposition 2 may be interpreted as follows. Given that any gauge function g is bounded below by the identity function (up to a multiplicative constant), the lim sup of the arcs $A(y_n, 2r_n)$ may be seen as a "reduced" version of the lim sup of the arcs $A(y_n, 2g(r_n))$. If the latter lim sup is large and omnipresent enough to contain Lebesgue-almost every point of the circle, then its reduced version is also large and omnipresent, in the weaker sense that it belongs to the class $\mathrm{G}^g(\mathcal{T})$. We refer to Sect. 5 for a proof of Proposition 2.

The large intersection properties of the set E_ℓ are then completely described by the following result.

Theorem 2. *Let $\ell = (\ell_n)_{n\geq 1}$ be a nonincreasing sequence of positive reals converging to zero and let g be a gauge function. Then, almost surely, for any nonempty open subset V of $\mathcal{T}$,*

$$E_\ell \in \mathrm{G}^g(V) \qquad \Longleftrightarrow \qquad \sum\nolimits_n g(\ell_n) = \infty.$$

In addition, the fact that the set E_ℓ belongs to the class $\mathrm{G}^g(\mathcal{T})$ of sets with large intersection in the whole circle $\mathcal{T}$ with respect to some gauge function g enables us to determine the value of its packing dimension (see [14] for the definition), as shown by the following result.

Corollary 2. *Let $\ell = (\ell_n)_{n\geq 1}$ be a nonincreasing sequence of positive reals converging to zero such that the real s_ℓ defined by (4) is positive. Then,*

$$a.s. \qquad \dim_{\mathrm{P}} E_\ell = 1,$$

where $\dim_{\mathrm{P}}$ *denotes packing dimension.*

The remainder of this chapter is organized as follows: Theorems 1 and 2 are established in Sect. 2, Corollaries 1 and 2 are proven in Sects. 3 and 4, respectively, and the proof of Proposition 2 is given in Sect. 5. Before detailing the proofs, let us mention that we shall basically only make use of the main properties of the classes $\mathrm{G}^g(V)$ given by Proposition 1, the ubiquity result given by Proposition 2, and the value (3) of the Lebesgue measure of the set E_ℓ. In particular, unlike the authors of [17], we do not need to call upon any specific result on the spacings between the random centers X_n of the arcs. This also means that the method developed here can effortlessly be extended to the case of balls randomly placed on the d-dimensional torus for any $d \geq 2$.

2 Proof of Theorems 1 and 2

Theorems 1 and 2 follow from four lemmas which we now state and prove. Throughout the section, $\ell = (\ell_n)_{n\geq 1}$ is a nonincreasing sequence of positive reals converging to zero.

Lemma 1. *For any gauge function g,*

$$\sum\nolimits_n g(\ell_n) < \infty \qquad \Longrightarrow \qquad \forall V \text{ open} \quad \mathcal{H}^g(E_\ell \cap V) = 0.$$

Proof. For any $\delta > 0$, there is an integer $n_0 \geq 1$ such that $0 < \ell_n < \delta$ for any $n \geq n_0$. Moreover, the set E_ℓ is covered by the arcs $A(X_n, \ell_n)$ for $n \geq n_0$, so that $\mathcal{H}^g_\delta(E_\ell) \leq \sum_{n=n_0}^{\infty} g(\ell_n)$. If the series $\sum_n g(\ell_n)$ converges, then letting n_0 tend to infinity and δ go to zero yields $\mathcal{H}^g(E_\ell) = 0$. ■

Lemma 2. *For any gauge function g,*

$$\sum\nolimits_n g(\ell_n) < \infty \qquad \Longrightarrow \qquad \forall V \neq \emptyset \text{ open} \quad E_\ell \notin \mathrm{G}^g(V).$$

Proof. Let us assume that the series $\sum_n g(\ell_n)$ converges. Then, one may build a gauge function $\overline{g}$ such that $\overline{g} \prec g$ and the series $\sum_n \overline{g}(\ell_n)$ converges too, for example, by adapting a construction given in the proof of [5, Theorem 3.5]. By Lemma 1, the set E_ℓ has Hausdorff $\overline{g}$-measure zero in V and thus cannot belong to the class $\mathrm{G}^g(V)$, due to Proposition 1.b. ■

Lemma 3. *For any gauge function g,*

$$\sum\nolimits_n g(\ell_n) = \infty \qquad \Longrightarrow \qquad \text{a.s.} \quad \forall V \neq \emptyset \text{ open} \quad E_\ell \in \mathrm{G}^g(V).$$

Proof. If the series $\sum_n g(\ell_n)$ diverges, then $\sum_n g(\ell_n/2)$ diverges as well (because $r \mapsto g(r)/r$ is nonincreasing near zero). Hence, from (3), the limsup of the arcs $A(X_n, 2g(\ell_n/2))$ has Lebesgue measure one with probability one. We conclude using Propositions 2 and 11. ■

Lemma 4. *For any gauge function g,*

$$\sum\nolimits_n g(\ell_n) = \infty \qquad \Longrightarrow \qquad \text{a.s.} \quad \forall V \text{ open} \quad \mathcal{H}^g(E_\ell \cap V) = \mathcal{H}^g(V).$$

Proof. We may obviously assume that V is nonempty. Let us suppose that the series $\sum_n g(\ell_n)$ diverges. Then, again by following a construction given in the proof of [5, Theorem 3.5], it is possible to build a gauge function $\underline{g}$ such that $g \prec \underline{g}$ and the series $\sum_n \underline{g}(\ell_n)$ diverges too, provided that $g \prec \mathrm{Id}$. Therefore, from Lemma 3, the set E_ℓ belongs to the class $\mathrm{G}^{\underline{g}}(V)$. Hence, $\mathcal{H}^g(E_\ell \cap V) = \infty = \mathcal{H}^g(V)$, from Proposition 1. In the case where $g \not\prec \mathrm{Id}$, the Hausdorff g-measure coincides, up to a multiplicative constant, with the Lebesgue measure on the Borel subsets of the circle, and the result follows from (3). ■

3 Proof of Corollary 1

Let us consider a nonincreasing sequence $\ell = (\ell_n)_{n\geq 1}$ of positive reals converging to zero. Theorem 1, along with the definition (4) of the real s_ℓ, implies that for any real $s \in (0,1)$, with probability one,

$$\mathcal{H}^{\mathrm{Id}^s}(E_\ell) = \begin{cases} \infty \text{ if } s < s_\ell \\ 0 \quad \text{if } s > s_\ell. \end{cases}$$

Let us assume that $s_\ell \in (0,1]$. Then, for all m large enough, with probability one, the set E_ℓ has infinite Hausdorff $\mathrm{Id}^{s_\ell - 1/m}$-measure, so that its Hausdorff dimension is at least $s_\ell - 1/m$. Therefore, the dimension of E_ℓ is almost surely at least s_ℓ. Likewise, if $s_\ell \in [0,1)$, then E_ℓ has $\mathrm{Id}^{s_\ell+1/m}$-measure zero with probability one for all m large enough, so that its Hausdorff dimension is almost surely at most s_ℓ. As a result, with probability one, $\dim_{\mathrm{H}} E_\ell = s_\ell$ if $s_\ell \in (0,1]$ and $\dim_{\mathrm{H}} E_\ell \leq 0$ if $s_\ell = 0$.

It remains to establish that E_ℓ is almost surely nonempty when $s_\ell = 0$. Note that the set $E_{(1/n)}$, obtained by picking $\ell_n = 1/n$, has Lebesgue measure one with probability one, by virtue of (3). Let us assume that this property holds. Furthermore, note that $\ell_n = \mathrm{o}(1/n)$ as n goes to infinity, by Olivier's theorem [21]. In particular, $\ell_n \leq 1/n$ for any integer n greater than or equal to some $n_1 \geq 1$. Let $I_1 = A(X_{n_1}, \ell_{n_1}/2)$. The union over $n > \max\{n_1, 8/\ell_{n_1}\}$ of the arcs $A(X_n, 1/n)$ has full Lebesgue measure in the circle, so its intersection with the arc $A(X_{n_1}, \ell_{n_1}/4)$ is nonempty. Therefore, there is an integer $n_2 > \max\{n_1, 8/\ell_{n_1}\}$ such that $A(X_{n_2}, 1/n_2) \subseteq I_1$. Then, let $I_2 = A(X_{n_2}, \ell_{n_2}/2)$. Repeating this procedure, one may obtain a nested sequence of open arcs I_n, and the intersection of their closures yields a point that belongs to the set E_ℓ.

4 Proof of Corollary 2

Let us consider a nonincreasing sequence $\ell = (\ell_n)_{n\geq 1}$ of positive reals converging to zero and assume that the real s_ℓ defined by (4) is positive. Then, Theorem 2 ensures that with probability one, the set E_ℓ belongs to the class $\mathrm{G}^{\mathrm{Id}^{s_\ell/2}}(\mathcal{T})$ of sets with large intersection in the whole circle $\mathcal{T}$ with respect to the gauge function $\mathrm{Id}^{s_\ell/2}$. Letting ϕ denote the canonical surjection from $\mathbb{R}$ onto $\mathcal{T}$, we deduce that the set $\phi^{-1}(E_\ell)$ belongs to the class $\mathrm{G}^{\mathrm{Id}^{s_\ell/2}}(\mathbb{R})$ of sets with large intersection in $\mathbb{R}$ with respect to the same gauge function. This last class is defined in [7] and is included in the class $\mathcal{G}^{s_\ell/2}$ of Falconer; see [7,10]. Theorem D in [13] finally implies that the packing dimension of $\phi^{-1}(E_\ell) \cap [0,1)$ is almost surely equal to one, which yields the desired statement.

5 Proof of Proposition 2

Let g be a gauge function such that the lim sup of the arcs $A(y_n, 2g(r_n))$ has Lebesgue measure one. Thus, following the terminology of [7], the family $(k + \dot{y}_n, g(r_n))_{(k,n)\in\mathbb{Z}\times\mathbb{N}}$ is a homogeneous ubiquitous system in $\mathbb{R}$. Here, each $\dot{y}_n$ is the only real in $[0, 1)$ such that $\phi(\dot{y}_n) = y_n$. From [7, Theorem 2], the set of all reals x such that $|x - k - \dot{y}_n| < r_n$ for infinitely many $(k, n) \in \mathbb{Z} \times \mathbb{N}$ belongs to the class $\mathrm{G}^g(\mathbb{R})$ of sets with large intersection in $\mathbb{R}$ with respect to the gauge function g. Equivalently, the inverse image under ϕ of the lim sup of the arcs $A(y_n, 2r_n)$ belongs to $\mathrm{G}^g(\mathbb{R})$, which ensures that this lim sup belongs to the class $\mathrm{G}^g(\mathcal{T})$, see [10, Sect. 5].

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T-Martingales, Size Biasing, and Tree Polymer Cascades

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Summary. Polymer models generally refer to random paths having probabilities induced by a random potential. In the case of tree polymers, the paths are defined by connecting vertices to a root of a binary tree, with probabilities given by a random multiplicative cascade normalized to a probability. The basic theory then concerns almost sure probability laws governing (asymptotically) long polymer paths. Weak and strong disorder refer to events in which the (non-normalized) cascade lives or dies, respectively. An almost sure central limit theorem (clt) is established in the full range of weak disorder, extending early results of Bolthausen. Also, almost sure Laplace large deviation rates are obtained under both disorder types. Open problems are included along the way.

1 Introduction

Directed lattice polymers on the $d+1$ dimensional integer lattice are modeled by (random) distributions of graphs of polygonal paths in $\mathbf{N} \times \mathbf{Z}^d$ for which the horizontal coordinate serves to direct the path as a self-avoiding chain of connected monomers.

Tree polymer models were considered early on by Bolthausen [8] as a special framework in which to illustrate certain L^2-martingale methods introduced to analyze directed lattice polymers. In this chapter we will use the term **lattice polymer** in reference to the directed polygonal paths on the $d+1$ dimensional integer lattice, and **tree polymer** for the case of polygonal paths of a binary tree.

While the primary focus of polymer research is aimed at low-dimensional lattice polymer models, where sharp results are rare, the tree polymer is important for testing lattice methods because sharp results are often possible to obtain for tree paths. In fact, one can demand sharp results and precise cutoffs of tree polymer theory, whereas this seems a less realistic requirement of lattice polymer theory.

Although focused on lattice polymer theory, the work of Comets and Yoshida [12] is likely state of the art for the research frontiers on dispersion

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problems for the case of lattice polymers. However, less actually appears to be available in the literature explicitly focused on the case of tree polymers in terms of precise bounds and results (see remarks of the next section). Since the importance of tree polymers is precisely that of furnishing sharp results, part of the purpose of this chapter is to provide a complete and self-contained treatment of best possible bounds and results for the basic dispersion problem of tree polymer theory.

This leads to a number of additional interesting open mathematical problems for tree polymers from the perspective of Kahane's **T**-martingale theory, where formerly most of the focus has been on describing the fine-scale structure of a.s. surviving cascades (weak disorder). In fact, it motivates a number of entirely new questions for the continued development of **T**-martingale theory for both weak and strong disorder types. For example, as will be seen, even in the analysis of weak disorder the approach of this paper involves *differentiation* of a certain class of **T**-martingales along lines introduced by Barral [2]. This naturally leads to a new notion of **signed** or **complex** **T**-martingales for which the authors know of no general theory. The strong disorder problems involve very new phenomena that escape direct application of existing theory.

In the next section, the tree polymer model is introduced. Bolthausen's notions of **weak** and **strong disorder** environments, respectively, are precisely defined and some basic polymer problems are identified. Section 3 contains a brief summary of Kahane's **T**-martingale theory appropriate to this application, as well as an overview of the extension of Peyrière's mean size-bias probability to the strong disorder environments introduced in Waymire and Williams [30]. As something of a warm-up, Sect. 4 opens with a simple result demonstrating that, regardless of disorder type, the tree polymer paths are **nonballistic** in the sense of an almost sure law of large number convergence to zero. This is followed by a discussion of a polymer diffusion problem and includes asymptotic polymer **path free energy**-type calculations in cases of both weak and strong disorder. These are in contrast to the free energy calculations provided by Buffet et al. [10]. The latter are made simpler by restricting their considerations to normalization constants. Section 5 contains the complete proof of a.s. long-chain Gaussian fluctuations under $n^{\frac{1}{2}}$ diffusive scaling within the *full range* of weak disorder; i.e., a tree polymer path central limit theorem (CLT). In Sect. 6 *vector cascades* are introduced as a class of **T**-martingales for which one can obtain an alternative representation of path free energy under strong disorder. Related directions and open questions for extensions of **T**-martingale theory are briefly discussed in Sect. 7.

2 Background and Notation

Let $T := \cup_{n=0}^{\infty}\{-1,1\}^n$ denote the set of **vertices** of the complete binary tree rooted at 0, with the convention that $\{-1,1\}^0 := \{0\}$. Equivalently, each vertex $v \neq 0$ defines a unique *edge* adjacent to this vertex on the unique

connected path joining the vertex v to the root 0. Such an edge is unambiguously also denoted by v. The **tree path space** is defined by the Cartesian product $\partial T := \{-1, 1\}^{\mathbf{N}}$, where each $s = (s_1, s_2, \dots) \in \partial T$ defines a possible *polymer path*. It is convenient to denote the vertex (or edge) at the jth level of the path s, read "s restricted to j", by $s|j := (s_1, \dots, s_j)$, for each $j = 1, 2, \dots$, and $s|0 := 0$. The same notation applies to a **finite path segment** $t = (t_1, \dots, t_n) \in \{-1, 1\}^n$ for $j \leq n$. In this case, $|t| = n$ denotes the **length** of the finite path segment. The **polymer path position** of $s = (s_1, s_2, \dots) \in \partial T$ at the nth link is defined by the special notation $(s)_0 = 0$, $(s)_n = \sum_{j=1}^n s_j$, for $n \geq 1$. The **normalized Haar measure** $\lambda(ds)$ on ∂T, regarded as a compact Abelian group under coordinatewise multiplication and the Cartesian product topology for the discrete topology on each factor $\{-1, 1\}$, defines the uniform distribution on polymer path space ∂T.

Next, the **environment** is defined by a collection $\{X(v) : v \in T\}$ of i.i.d. (strictly) positive random variables on a probability space $(\Omega, \mathcal{F}, P)$ indexed by the vertices (or edges) of T. Define a sequence of random probability measures $prob_n(ds, \omega) << \lambda(ds)$, $n \geq 1, \omega \in \Omega$, by the corresponding sequence of Radon–Nikodym derivatives

$$\frac{dprob_n}{d\lambda}(s, \omega) = Z_n^{-1}(\omega) \prod_{j=1}^n X(s|j), \quad n = 1, 2, \dots,$$

where $Z_n(\omega)$ denotes the normalization constant (or **partition function**) given by

$$Z_n(\omega) := \int_{\partial T} \prod_{j=1}^n X(s|j) \lambda(ds) = \sum_{|t|=n} \prod_{j=1}^n X(t|j)(\omega) 2^{-n},$$

and the sum is over all finite path segments t of length n. In particular, on the **finite dimensional cylinder sets** $\Delta_n(t) := \{s \in \partial T : s|n = t\}, t \in \{-1, 1\}^n$, $n \geq 1$ of the Borel σ-field of ∂T, one has

$$prob_n(\Delta_n(t), \omega) = Z_n^{-1}(\omega) \prod_{j=1}^n X(t|j)(\omega) 2^{-n}.$$

The factor 2^{-n} cancels in the ratio, but is convenient to display as it makes the sequence $\{Z_n/(E_P X)^n : n \geq 1\}$ a positive martingale. Observe, also, that for each finite dimensional cylinder set $\Delta_n(t)$, one has sample pointwise on Ω

$$prob_{n+m}(\Delta_n(t)) = \frac{Z_n \sum_{|s|=m} \prod_{j=1}^m X(t * (s|j)) 2^{-m}}{Z_{n+m}} prob_n(\Delta_n(t)),$$

where $*$ denotes concatenation of word strings defining vertices.

Definition 1. *Given an environment* $\{X(v) : v \in T\}$ *of i.i.d. (strictly) positive random variables on a probability space* $(\Omega, \mathcal{F}, P)$ *indexed by the vertices (or edges) of* T*, the* ***tree polymer*** *is defined by the sequence* $prob_n(dt),\ n \geq 1$*, of (random) probabilities defined on the Borel sigma-field of* ∂T*.*

2.1 Some Special Notation and Assumptions

The explicit dependence of random variables on $\omega \in \Omega$ will generally be suppressed as per standard probability convention. Also, a number of different probabilities will appear throughout this paper, e.g., P, $prob_n$, $\mathcal{Q}$, etc., whose role in expected value computations will be indicated by a subscript to the expectation symbol E. It will be assumed throughout that there is a number $p > 1$ such that

$$E_P X^p < \infty. \tag{1}$$

This condition is easily satisfied by the following basic examples of polymer theories, namely, (i) $X = e^{\beta Z}$, for standard normal Z, and (ii) $X = \begin{cases} a & \text{with probability } p \\ b & \text{with probability } q = 1 - p, \end{cases}$ for some $a, b > 0$, $0 < p < 1$. Without loss of generality, one may take $E_P X = 1$, since replacing X by $X/E_P X$ is canceled by the respective factors $(E_P X)^n$ of the new normalization constants. This normalization will also be assumed throughout, modifying the form of these examples accordingly.

In view of the martingale convergence theorem, $Z_\infty = \lim_{n\to\infty} Z_n$ exists P-a.s. Also by Kolmogorov's zero-one law and sure positivity of the environmental weights, the event $[Z_\infty = 0]$ has P-probability zero or one. The environment $\{X(v) : v \in T\}$ is referred to as one of **weak disorder** if and only if P-a.s. $Z_\infty > 0$; otherwise, the environment is that of **strong disorder**. In the case of weak disorder one has the existence of an a.s. unique tree polymer limit probability $prob_\infty(dt)$ on ∂T defined by the a.s. weak limit of the tree polymer $prob_n(dt), n \geq 1$. In Waymire and Williams [33] the existence of a unique weak limit probability was proven under strong disorder as a Dirac point mass concentrated on a random path $\tau \in \partial T$ with respect to the mean size-biasing change of P-measure described in the next section. Moreover the mean size-biasing change of measure and P are mutually singular under strong disorder.

Remark 1. Yuval Peres (personal communication) suggested that under strong disorder the set of limit points of $prob_n(dt), n \geq 1$, might a.s. consist of Dirac point masses on paths.

Remark 2. The sharp criticality condition for transitions between weak and strong disorder is known precisely for tree polymer models as a result of the seminal paper of Kahane and Peyrière [24]. In addition, Bolthausen's

weak/strong disorder criticality condition was improved by Birkner [6] for the case of lattice polymers using a size-bias change of measure. Birkner's criticality condition indeed coincides with the sharp determination that one obtains using the Kahane and Peyrière [24] theory for the case of tree polymers. This illustrates a benchmark role for tree polymer theory for evaluating the sharpness of lattice polymer methods mentioned at the outset.

In the context of tree polymers, the basic theory concerns the P-a.s. asymptotic behavior of segments of *random* polygonal paths $S \in \partial T$ of length n distributed, respectively, according to the sequence $prob_n(ds, \omega)$. For example, a.s. strong laws governing averages $\frac{(S)_n}{n}$, and a.s. limit distributions governing fluctuations $\frac{(S)_n - c_n}{a_n}$ for suitable centering c_n and scaling constants $a_n > 0$, as $n \to \infty$ are desired. While these are only a few of the problems of interest here, they do play a central role.

Remark 3. The L^2-martingale methods developed in Bolthausen [7,8] for the lattice polymer do indeed provide the CLT for tree polymers with $c_n = 0, a_n = \sqrt{n}$, but in a *strict* subregion of weak disorder. Comets and Yoshida [12] note that subsequent L^2-martingale methods developed for lattice polymers extend the range of weak disorder in sufficiently high dimensions d. Specifically, Albeverio and Zhou [1], Imbrie and Spencer [20], Song and Zhou [29], and Birkner [6] are noted in this regard. In these results, however, the asymptotic diffusion coefficient in $d+1$ dimensions is $\frac{1}{d}$. The extension of these results and methods to the context of tree polymers does not seem obvious, although it appears that they are presumed to hold.

The chapter, will provide an explicit, self-contained, and complete proof for the CLT problem in the case of tree polymers of weak disorder type based on differentiated cascades. The sharpness obtained for tree polymers suggests that corresponding methods might also prove useful for lattice polymers. It is also shown that the same diffusive scaling limit is not possible under strong disorder. More generally, another important motivation for this chapter is to uncover the extent of applicability of existing **T**-martingale theory and identify new directions under strong disorder.

3 T-Martingales and Size-Bias Theory

For a complete metric space $(\mathbf{T}, d)$, Kahane's **T-martingale** refers to a sequence of nonnegative random functions $Q_n, n = 1, 2, \ldots$ on $\mathbf{T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ adapted to a filtration $\mathcal{F}_n, n = 1, 2 \ldots$, such that for each $t \in \mathbf{T}$, $Q_n(t)$ is a mean-one martingale with respect to this filtration. Given a Radon measure σ on the Borel sigma-field $\mathcal{B}(\mathbf{T})$, the **T**-martingale induces a sequence of random measures $Q_n\sigma(dt)$ defined by

$$\int_{\mathbf{T}} f(t) Q_n \sigma(dt) = \int_{\mathbf{T}} f(t) Q_n(t) \sigma(dt)$$

for all continuous bounded functions f on $\mathbf{T}$, i.e., $\frac{dQ_n\sigma}{d\sigma}(t) = Q_n(t), t \in \mathbf{T}$. As such, using martingale convergence theory and Kahane's $\mathbf{T}$-martingale decomposition, e.g., see Kahane [22] and Waymire and Williams [30], one may obtain a (possibly degenerate) random measure $Q_n\sigma \Rightarrow \sigma_\infty$ as an a.s. vague limit.

For the case of tree polymers, consider $\mathbf{T} = \partial T$ introduced in the previous section, with

$$Q_n(s) = \prod_{j=1}^{n} X(s|j), \quad s \in \partial T,$$

and for example, $\sigma = \lambda$, the Haar measure on ∂T. As noted earlier, in the context of tree polymers $X(v), v \in t$ is a strictly positive, mean-one random variable. One may write

$$prob_n(ds) = \frac{Q_n\lambda(ds)}{Q_n\lambda(\partial T)}, \qquad n \geq 1.$$

and, in the case of weak disorder, one has

$$prob_\infty(ds) = \frac{\lambda_\infty(ds)}{\lambda_\infty(\partial T)}.$$

Peyrière's **mean size bias** was introduced to compute the fine-scale structure of surviving cascades, i.e., weak disorder in the context of polymers. The consideration of such transformations is naturally motivated by more basic elements of Cramèr–Chernoff exponential size biasing in the computation of large deviation rates; e.g., see Bhattacharya and Waymire [3]. Specifically, since the product of i.i.d. mean-one nondegenerate random variables along any one path is a.s. zero, the survival of cascades requires deviations from this average behavior made possible by the uncountably many paths of ∂T. Moreover, exponential size biasing of the logarithm of a random variable is precisely mean size biasing. We summarize here the basic framework developed in Waymire and Williams [30–33] to use size biasing to determine the asymptotic total mass in cases of both weak and strong disorder.

By restricting the formulation to the sigma-fields generated, respectively, by the first finitely many levels of the environment $\mathcal{F}_n := \sigma(X(v) : |v| \leq n)$, and the finite dimensional cylinder sets of tree paths $\mathcal{R}_n := \sigma(\Delta_n(t) : |t| = n)$, for $n \geq 1$, with the aid of the Kolmogorov consistency theorem, one may define a joint probability $\mathcal{Q}(d\omega \times dt)$ (on $\Omega \times \partial T$) of the environment and paths that, for a given path s, size-biases the environment along this path. Namely, one has

$$\mathcal{Q}(d\omega \times ds) = \prod_{j=1}^{n} X(s|j)(\omega)P(d\omega)\lambda(ds) = P_s(d\omega)\lambda(ds), \tag{2}$$

where

$$P_s << P \quad \text{on} \quad \mathcal{F}_{\mathrm{n}} = \sigma(\mathrm{X(v)} : |\mathrm{v}| \leq \mathrm{n}).$$

In other words, the measures $\prod_{j=1}^{n} X(s|j)(\omega)P(d\omega)\lambda(ds), n \geq 1$, provide a consistent specification of the finite dimensional distributions of $(\{X(v) : v \in T\}, S)$ under $\mathcal{Q}(d\omega \times ds)$ on $\mathcal{F} \otimes \mathcal{B}$. Accordingly, under $\mathcal{Q}(d\omega \times ds)$, for a given path $S = s$, the environment variable $X(v)$ has distribution $P \circ X^{-1}(dx)$ if v is not on s, while it is $xP \circ X^{-1}(dx)$ if v is along the path s.

Letting π_Ω $\pi_{\partial T}$ denote the coordinate projection maps of $\Omega \times \partial T$ onto Ω and ∂T, respectively, one obtains by integrating out the coordinates that

$$(i) \quad \mathcal{Q} \circ \pi_\Omega^{-1}(d\omega) = Z_n(\omega)P(d\omega), \qquad (ii) \quad \mathcal{Q} \circ \pi_{\partial T}^{-1}(ds) = \lambda(ds). \tag{3}$$

From here one readily obtains the following variant on the *Bayes formula*:

$$\mathcal{Q}(d\omega \times ds) = prob_n(ds, \omega)\mathcal{Q} \circ \pi_\Omega^{-1}(d\omega). \tag{4}$$

In particular, the polymer path distribution $prob_n(ds, \omega)$ is the conditional path probability given the environment.

Next one has the *Lebesgue decomposition*

$$\mathcal{Q}(d\omega \times ds) = Q_\infty \lambda(ds, \omega)P(d\omega) + \mathbf{1}[Q_\infty \lambda(\partial T, \omega) = \infty]p_\infty(ds, \omega)\mathcal{Q} \circ \pi_\Omega^{-1}(d\omega), \tag{5}$$

where $p_\infty(ds, \omega)$ denotes the $\mathcal{Q} \circ \pi_\Omega^{-1}$-a.s. weak limit of $p_n(ds)$ as $n \to \infty$. The structure of $p_\infty(ds, \omega)$ in the case of strong disorder is described in Proposition 2 of Sect. 4. Accordingly, with regard to weak and strong disorder, the event $[Q_\infty \lambda(\partial T) = 0]$ is a zero-one event under P if and only if $[Q_\infty \lambda(\partial T) = \infty]$ is a zero-one event under $\mathcal{Q} \circ \pi_\Omega^{-1}(d\omega)$.

Next we record the *(weighted) first departure bounds* developed in Waymire and Williams [33] for the special case of the product probabilities $\sigma = \mu \times \mu \times \cdots$ of a (generic) Bernoulli probability μ on $\{-1, 1\}$ that will naturally appear in forthcoming tree polymer applications. Namely, for an arbitrary path $s \in \partial T$, and positive constants C_n, one has

$$\begin{aligned} &\prod_{j=1}^{n} X(s|j)\mu(+)^{\#(s|n)}\mu(-)^{\#(s|n)} \\ &\quad \leq Q_n\sigma(\partial T) \leq \prod_{j=1}^{n} X(s|j)\mu(+)^{\#^+(s|n)}\mu(-)^{\#^-(s|n)} + C_n A_n, \end{aligned} \tag{6}$$

where $C_n > 0$ and $A_n, n \geq 1$ is a positive submartingale (dependent on the choice of C_n). The symbols $\#^\pm(s|n)$ count the respective number of ± 1 coordinates of the path segment $s|n$, and $\mu(\pm) = \mu(\{\pm 1\})$, respectively. The lower bound is obvious since a sum of positive terms is larger than any single term. The upper bound is obtained by splitting off the term corresponding to the product along the s-path and decomposing the remaining sum with respect to the level of first departure from the s-path.

This summarizes the essential elements of the theory which will be needed for this paper.

4 Asymptotic Polymer Path Free Energy-Type Calculations for Weak and Strong Disorder

In addressing the asymptotic structure of tree polymers without regard to disorder type, one is forced to consider weak limits; i.e., limits with respect to the sequence $prob_n(dt), n \geq 1$. The following simple lemma is somewhat surprising on first glance in view of the random normalization.

Lemma 1. *On $\mathcal{R}_n$ one has*

$$E_P prob_n(B) = \lambda(B) = E_P Q_n \lambda(B), \quad B \in \mathcal{R}_n.$$

Proof. Simply observe that $E_P prob_n(\Delta_n(t)) = 2^{-n}$ since the expression is independent of $t \in \partial T$ and sums to one. ■

As an application of this lemma, one can readily obtain an expression of the nonballistic character of polymers regardless of disorder type.

Proposition 1. *Regardless of the disorder strength one has*

$$\lim_{n\to\infty} E_{prob_n} |\frac{(S)_n}{n}| = 0 \quad P - a.s.$$

Proof. Let $A_n = E_{prob_n} |\frac{(S)_n}{n}|$. Then for $h > 1$, applying Jensen's inequality to the integral with respect to $prob_n(ds)$, one has

$$E_P A_n^h = n^{-h} E_P \left(\int_{\partial T} |(s)_n| prob_n(ds) \right)^h \leq n^{-h} \int_{\partial T} |(s)_n|^h E_P prob_n(ds) \leq C n^{-\frac{h}{2}}$$

with $C > 0$. Now take $h = 4$ and apply Borel–Cantelli to obtain the assertion. ■

The result quoted in the previous section that identifies $prob_\infty(ds)$ in the case of strong disorder as concentrated on a single random path $\mathcal{Q} \circ \pi_\Omega^{-1}(d\omega)$-a.s. is repeated here for ease of reference and to correct some typographical errors in the proof in Waymire and Williams [33].

Proposition 2 ([33]). *In the case of strong disorder there is a random path $\tau = \tau(\omega) \in \partial T, \omega \in \Omega$, such that $\mathcal{Q} \circ \pi_\Omega^{-1}(d\omega)$-a.s. as $n \to \infty$,*

$$prob_n(ds) \Rightarrow \delta_\tau(ds).$$

Proof. Fix a path s. If, for example, $s_1 = +1$, then the total mass on the "left side" of the tree, $Z_n(-) = \sum_{|t|=n, t_1=-1} \prod_{j=1}^n X(t|j) 2^{-(n-1)}$, is a positive martingale under P_s since the environment off the path s is i.i.d. distributed under P. In particular, therefore, P_s-a.s. one has

$$Z_\infty(-) = \lim_{n\to\infty} Z_n(-) < \infty.$$

A similar observation holds if $s_1 = -1$, and so on down the tree off the path s. But under strong disorder, for any path s, since $Z_\infty = Q_\infty \lambda(\partial T) = 0 \;\; P$-a.s., from the Lebesgue decomposition one observes that

$$P_s(Z_\infty = \infty) = 1.$$

Let $\omega \in [Z_\infty = \infty]$. Then, removing an event of $\mathcal{Q} \circ \pi_\Omega^{-1}$-probability zero if necessary, one has either $Z_\infty(+, \omega) = \infty$ or $Z_\infty(-, \omega) = \infty$, but not both. Define $\tau_1(\omega) = \pm 1$ according to $Z_\infty(\pm, \omega) = \infty$. Now iterate this procedure down the tree accordingly. □

For the a.s. distributional limits of interest in the next two sections, it will be convenient to have calculations of the a.s. asymptotic behavior of **polymer path free energies** (or *cumulant generating functions*) of the form $F(r) = \lim_{n\to\infty} \frac{\ln M_n(r)}{n}$, where

$$M_n(r) = E_{prob_n} e^{r(S)_n}.$$

Remark 4. In Buffet et al. [10] the authors consider a different type of free energy density calculations which, in the present notation, may be defined for environments $X = e^{-\beta V}$ (for a particular class of real-valued random variables V), as

$$\psi = \lim_{n\to\infty} \frac{\ln Z_n}{n},$$

where Z_n is the corresponding normalizing constant (partition function). Such considerations will follow as a special case of path free energy results presented here.

Lemma 2. *Let*

$$p_r(\pm 1) = \frac{e^{\pm r}}{e^r + e^{-r}}, \quad \lambda_r = p_r \times p_r \times \cdots \times p_r \times \cdots.$$

Then

$$M_n(r) = \cosh^n(r) \frac{Q_n \lambda_r(\partial T)}{Q_n \lambda_0(\partial T)}, \quad -\infty < r < \infty.$$

Proof. One has

$$\begin{aligned} M_n(r) &= Z_n^{-1} \sum_{|s|=n} \prod_{j=1}^n e^{r s_j} \prod_{j=1}^n X(s|j) 2^{-n} \\ &= \cosh^n r Z_n^{-1} \sum_{|s|=n} \prod_{j=1}^n p_r(s_j) \prod_{j=1}^n X(s|j) \\ &= \cosh^n r \frac{\sum_{|s|=n} \prod_{j=1}^n X(s|j) \prod_{j=1}^n p_r(s_j)}{\sum_{|s|=n} \prod_{j=1}^n X(s|j) 2^{-n}}. \end{aligned} \tag{7}$$

This completes the proof. ■

The following formula is well known by various methods starting with Borel normal numbers and its extensions by Eggleston [14], Billingsley [5], Kifer [25], Fan [17], and Peyrière [28]. We write $supp\sigma$ for the maximal Borel support of a probability σ on ∂T. That is, $supp\sigma = \inf\{\dim(A) : \sigma(A^c) = 0\}$, where $\dim A$ denotes the Hausdorff dimension of Borel $A \subseteq \partial T$ (for the metric $\rho(s,t) = 2^{-|s\wedge t|}$, $s,t \in \partial T$, where $s \wedge t$ denotes the common part of the paths s,t emanating from the root, until first departure). With this notation and terminology, one has $dim(supp\lambda_r) = h_2(r)$, where

$$h_2(r) := -\frac{e^r}{e^r+e^{-r}}\log_2\left(\frac{e^r}{e^r+e^{-r}}\right) - \frac{e^{-r}}{e^r+e^{-r}}\log_2\left(\frac{e^{-r}}{e^r+e^{-r}}\right). \tag{8}$$

We also refer to $h_2(r)$ as the *base 2-entropy* of λ_r. In particular, note that the Haar measure (uniform distribution) λ_0 has full support of dimension one, i.e., *maximal entropy* among λ_r, $-\infty < r < \infty$.

The following is a special case of more general theorems of Kahane [21] on conditions for survival of multiplicative cascades with respect to initial measures σ on ∂T using potential theoretic/capacity methods. In the case of product measures such as λ_r, this also follows from the weighted size-bias theory developed in Waymire and Williams [33]. It may also be obtained from necessary and sufficient conditions obtained by Fan [18] for Markov measures. In essence, the support must be sufficiently large relative to the variability in the environment for the cascade to survive. In the case of Haar measure λ_0, this may be equivalently interpreted as the condition that the branching number 2 must be large enough relative to variability of the environment. Namely, we have the following.

Proposition 3. *For arbitrary $r \in \mathbf{R}$ one has*

$$Q_\infty \lambda_r(\partial T) > 0 \quad \text{a.s.}$$

if and only if

$$E_P X \log_2 X < h_2(r).$$

Proof. The proof follows precisely the lines of Waymire and Williams [30], using the weighted first departure bounds. For necessity, suppose that $E_P X \log_2 X \geq h_2(r)$. Then, for any fixed path $s \in supp(\lambda_r)$, one has, using the lower bound,

$$\begin{aligned} Q_n\lambda_r(\partial T) &\geq \prod_{j=1}^n X(s|j) p_r^{\#^+(s|n)}(+) p_r^{\#^-(s|n)}(-) \\ &= \exp\left\{ n\left(\frac{1}{n}\sum_{j=1}^n \ln X(s|j) + \frac{\#^+(s|n)}{n}\ln p_r(+) + \frac{\#^-(s|n)}{n}\ln p_r(-)\right)\right\}. \end{aligned}$$

By two applications of the strong law of large numbers, one has, respectively, that P_s-a.s. $\frac{1}{n}\sum_{j=1}^n \ln X(s|j) \to E_P X \ln X$, and λ_r- a.e. $\frac{\#^\pm(s|n)}{n} \to p_r^\pm$ as $n \to \infty$. It follows from this that

$$\int_{\partial T}\int_{\Omega} \mathbf{1}[Q_\infty(\partial T)=\infty]P_s(d\omega)\lambda_r(ds)=1$$

in the case $E_P X\log_2 X > h_2(r)$. The same can be seen to hold when $E_P X\log_2 X = h_2(r)$ using the Chung–Fuchs theorem in place of the strong law of large numbers. The converse is proved similarly using the upper first departure bound. □

Remark 5. In the case of Haar measure $\lambda=\lambda_0$, $h_2(0)=\ln 2$ and Proposition 3 provides the usual condition on the variability in the environment with respect to the branching number for weak and strong disorder.

Corollary 1. *Under weak disorder one has P-a.s. that there is a $\delta>0$ such that*

$$F(r)=\lim\frac{\ln M_n(r)}{n}=\ln\cosh(r)\quad |r|\le\delta.$$

Proof. Since weak disorder is equivalent to $E_P X\log_2 X < h_2(0)$, and $h_2(0)$ is maximal entropy, using continuity of $h_2(r)$, there is a $\delta>0$ such that $E_P X\log_2 X < h_2(r)$ for $|r|\le\delta$. The result follows immediately from Proposition 3 taking logarithms in Lemma 2. ■

Remark 6. Observe that in the case of simple symmetric random walk paths obtained by taking deterministic $X\equiv 1$, one has the sure identity

$$\frac{\ln M_n(r)}{n}\equiv\ln\cosh(r),\quad n=1,2,\ldots.$$

Moreover,

$$\ln\cosh^n\left(\frac{r}{\sqrt{n}}\right)\sim\left(1+\frac{r^2}{2n}+o(1)\right)^n\sim e^{\frac{r^2}{2}}\quad \text{as}\ \ \mathrm{n}\to\infty.$$

So formally, at least, one expects the diffusive (CLT) limit to hold almost surely from Corollary 1.

The computation of the path free energy under strong disorder is a little more delicate than the case of Corollary 1. We will make a size-bias calculation for an upper bound on lim sup. However, the lower bound on lim inf obtained by the corresponding approach is too small. Nonetheless we will see that the lim sup bound is indeed the asserted a.s. limit. Also see Corollary. 2 in Sect. 6.

Proposition 4. *Under strong disorder there is a $\delta>0$ such that*

$$F(r)=\lim\frac{\ln M_n(r)}{n}=\ln\cosh(r)+\frac{\ln E_P X^{h(r)}+\ln\left(p_r^{h(r)}(+)+p_r^{h(r)}(-)\right)}{h(r)},$$

where $h=h(r)$ is a uniquely determined positive solution to

$$E_P\left\{\frac{X^h}{E_P X^h}\ln\frac{X^h}{E_P X^h}\right\}=\epsilon(\overline{p}_{r,h}(+),\overline{p}_{r,h}(-)),$$

for

$$\overline{p}_{r,h}(\pm) := \frac{p_r^h(\pm)}{p_r^h(+) + p_r^h(-)}$$

and $\epsilon(a,b) = -a\ln a - b\ln b$.

Proof. We begin by using size biasing to compute an upper bound on the quantity $\limsup_{n\to\infty} \frac{\ln Q_n\lambda_r(\partial T)}{n}$. Fix $c > 0$, $0 < h < 1$. The size-bias change of measure in this context may be obtained by the modification denoted

$$\mathcal{Q}^{(r)}(d\omega \times ds) = P_s(d\omega)\lambda_r(ds).$$

In particular, on $\mathcal{F}_n$,

$$\begin{aligned}\mathcal{Q}^{(r)} \circ \pi_\Omega^{-1}(d\omega) &= \int_{\partial T} P_s(d\omega)\lambda_r(ds) = \sum_{t\in\{-1,1\}^n} \int_{\Delta_n(t)} \prod_{j=1}^n X(t|j)P(d\omega)\lambda_r(dt)\\ &= m_n(r)P(d\omega),\end{aligned}$$

where

$$m_n(r) := \frac{M_n(r)}{\cosh^n(r)}.$$

Now,

$$\begin{aligned}&P(Q_n\lambda_r(\partial T) > e^{nc})\\ &= E_P\mathbf{1}[Q_n\lambda_r(\partial T) > e^{nc}]\\ &= E_{\mathcal{Q}^{(r)}\circ\pi_\Omega^{-1}} m_n(r)^{-1}\mathbf{1}[m_n(r) > e^{nc}]\\ &\le \int_{\partial T}\int_\Omega \frac{m_n^h(r)e^{-nch}}{m_n(r)} P_s(d\omega)\lambda_r(ds)\\ &\le e^{-nhc}\int_{\partial T}\int_\Omega \frac{1}{\prod_{j=1}^n X^{1-h}(s|j)p_r^{1-h}(s_j)} P_s(d\omega)\lambda_r(ds)\\ &= e^{-nhc}\int_{\partial T}\int_\Omega \prod_{j=1}^n X^h(s|j)p_r(s_j)^{h-1}\frac{1}{\prod_{j=1}^n X(s|j)} P_s(d\omega)\lambda_r(ds)\\ &= e^{-nhc}(E_P X^h)^n \int_{\partial T}\prod_{j=1}^n p_r^{h-1}(s_j)\lambda_r(ds) \qquad (9)\\ &= e^{-nhc}(E_P X^h)^n \left(\int_{\partial T} p_r^{h-1}(s_1)\lambda_r(ds)\right)^n\\ &= \exp\{-n\left[hc - (\ln E_P X^h + \ln(p_r^h(+) + p_r^h(-)))\right]\}.\end{aligned}$$

Thus, the probability is summable for

$$c > \sup_{0<h<1} \frac{\ln E_P X^h + \ln(p_r^h(+) + p_r^h(-))}{h}.$$

Using Borel–Cantelli one therefore obtains P-a.s. that

$$\limsup_{n\to\infty} \frac{\ln M_n(r)}{n} \le \sup_{0<h<1} \frac{\ln E_P X^h + \ln(p_r^h(+) + p_r^h(-))}{h}.$$

Next we verify that this upper bound on the lim sup is also a lower bound on the lim inf, and therefore is the desired limit P-almost surely. Define for fixed r,

$$Z_n(r,h) := \sum_{|s|=n} \prod_{j=1}^{n} X^h(s|j) p_r^h(s_j), \quad h \in \mathbf{R}.$$

Then

$$Q_n \lambda_r(\partial T) = Z_n(r,1).$$

Also note that

$$E_P Z_n(r,h) = (E_P X^h)^n \left(p_r^h(+) + p_r^h(-)\right)^n.$$

Viewing X^h as a new polymer environment, and normalizing $p_r^h(\pm)$ to a probability distribution given by

$$\overline{p}_{r,h}(\pm) = \frac{p_r^h(\pm)}{p_r^h(+) + p_r^h(-)},$$

one sees from Proposition 3 that for each fixed r there is a unique $h(r)$ defined by

$$E_P \frac{X^h}{E_P X^h} \ln \frac{X^h}{E_P X^h} = \epsilon(r,h),$$

where $\epsilon(r,h) = -\overline{p}_{r,h}(+) \ln \overline{p}_{r,h}(+) - \overline{p}_{r,h}(-) \ln \overline{p}_{r,h}(-)$, such that

$$\lim_{n\to\infty} \frac{Z_n(r,h)}{E_P Z_n(r,h)} > 0 \quad P\text{-a.s.}$$

if and only if $h < h(r)$. Thus, for $h < h(r)$, one has

$$\lim_{n\to\infty} \frac{\ln Z_n(r,h)}{n} = \ln E_P X^h + \ln\left(p_r^h(+) + p_r^h(-)\right).$$

The uniqueness of $h = h(r)$ follows by checking that for fixed r, $h \to E_P \frac{X^h}{E_P X^h} \ln \frac{X^h}{E_P X^h} - \epsilon(r,h)$ is monotone increasing on $0 < h < 1$. Define

$$g(r,h) := \ln E_P X^h + \ln\left(p_r^h(+) + p_r^h(-)\right).$$

Now, for $\epsilon > 0$, rewrite a bit, and apply Jensen's inequality to obtain

$$\begin{aligned}\frac{Z_n(r,1)}{Z_n(r,h)} &= Z_n(r,h)^{-1} \sum_{|s|=n} \prod_{j=1}^{n} X(s|j) p_r(s_j) \\ &= \sum_{|s|=n} \prod_{j=1}^{n} X^{1-h}(s|j) p_r^{1-h}(s_j) Z_n(r,h)^{-1} \prod_{j=1}^{n} X^h(s|j) p_r^h(s_j)\end{aligned}$$

$$= \sum_{|s|=n} \left(\prod_{j=1}^{n} X^{\frac{1-h}{1+\epsilon}}(s|j) p_r^{\frac{1-h}{1+\epsilon}}(s_j) \right)^{1+\epsilon} Z_n(r,h)^{-1} \prod_{j=1}^{n} X^h(s|j) p_r^h(s_j)$$

$$\geq \frac{\left(\sum_{|s|=n} \prod_{j=1}^{n} X^{\frac{1-h}{1+\epsilon}+h}(s|j) p_r^{\frac{1-h}{1+\epsilon}+h}(s_j) \right)^{1+\epsilon}}{Z_n(r,h)^{1+\epsilon}}.$$

Thus,

$$Z_n(r,1) \geq \frac{Z_n^{1+\epsilon}(r, h + \frac{1-h}{1+\epsilon})}{Z_n^{\epsilon}(r,h)}.$$

In particular, therefore,

$$\frac{\ln Z_n(r,1)}{n} \geq (1+\epsilon) \frac{\ln Z_n(r, \frac{1+\epsilon h}{1+\epsilon})}{n} - \epsilon \frac{\ln Z_n(r,h)}{n}$$

$$= \frac{\ln Z_n(r, \frac{1+\epsilon h}{1+\epsilon})}{n} + \epsilon \left[\frac{\ln Z_n(r, \frac{1+\epsilon h}{1+\epsilon})}{n} - \frac{\ln Z_n(r,h)}{n} \right].$$

Now, $\frac{1+\epsilon h}{1+\epsilon} < h(r)$ for $\epsilon > \frac{1-h(r)}{h(r)-r} > 0$. Thus, taking lim inf as $n \to \infty$, followed by letting $\epsilon \downarrow \frac{1-h(r)}{h(r)-h}$, yields

$$\liminf_{n\to\infty} \frac{\ln Z_n(r,1)}{n} \geq g(r,h(r)) + \frac{1-h(r)}{h(r)-h} \left[g(r,h(r)) - g(r,h) \right].$$

Finally, let $h \uparrow h(r)$ to obtain

$$\liminf_{n\to\infty} \frac{\ln Z_n(r,1)}{n} \geq g(r,h(r)) + (1-h(r)) \frac{\partial g}{\partial h}(r,h(r)).$$

With a bit of tedious algebra, one may check that the size-bias bound on the lim sup coincides with this lower bound on the lim inf and, therefore, is the a.s. limit. Also see Corollary 2, Sect. 6 for an alternative formula. ■

5 Diffusive Limits Under Full Range of Weak Disorder

Taking the deterministic environment $X \equiv 1$ for which the tree polymer paths are then distributed as simple symmetric random walks, one clearly has

$$\frac{(S)_n}{\sqrt{n}} \Rightarrow Z \quad n \to \infty,$$

where Z has the standard normal law. The objective here is to show that this law a.s. persists throughout the entire range of weak disorder.

Remark 7. As remarked earlier, from the a.s. calculation $F(r) = \ln\cosh(r)$, one expects a diffusive scaling limit to hold. A theorem of Ellis [15] is known to lead from asymptotic calculations of the form $F(r) = \lim_n \ln M_n(r)/n$ under sufficient convexity conditions of such functions and their derivatives; e.g., see Cox and Griffeath [13] and Maxwell [27] for indications of successful applications to certain particle systems and to certain asymptotic enumerations, respectively. It has not been possible to verify the convexity conditions for the polymer model. However, as will be seen, it is fruitful to consider derivatives of $M_n(r)$ nonethless.

The following lemma follows from straightforward calculations that are left to the reader to verify.

Lemma 3. *Let $\delta > 0$ be arbitrary. (i) $\widetilde{m}_n(r) := \frac{z_n M_n(r)}{\cosh^n(r)}$, $-\delta \le r \le \delta$, is a continuously differentiable $\boldsymbol{T}$-martingale on $\boldsymbol{T} = [-\delta, \delta]$ with the usual Euclidean metric. Also, the corresponding derived processes (ii) $\widetilde{m}_n'(r) \equiv \frac{d\widetilde{m}_n(r)}{dr}$, $-\delta \le r \le \delta$, is a (signed) $\boldsymbol{T}$-martingale. Moreover,*

$$\widetilde{m}_n'(r) = \sum_{j=1}^{n} m_{n,j}(r) = \frac{1}{\cosh^n(r)} \sum_{j=1}^{n} \int_{\partial T} \{s_j - \tanh(r)\} e^{r(s)_n} Q_n \lambda(ds),$$

where $m_{n,j}(r), 1 \le j \le n$, are defined by the indicated terms of the second sum.

Remark 8. As noted earlier, this lemma illustrates a natural role for the extended notions of signed (or more generally complex) **T**-martingales, as well as **T**-martingale difference sequences. The following lemma makes explicit use of the assumption (1).

To take advantage of the symmetries of the binary tree and environment, we say a permutation (i.e., bijection) $\pi : T \to T$ of $T := \cup_{n=0}^{\infty}\{-1, 1\}^n$ is **lattice preserving** if, for each $v \in T$, both (i) $|\pi(v)| = |v|$, and (ii) $\pi(v|j) = (\pi(v)|j)$, for $j \le |v|$. Let $\mathcal{P}_{\le n}$ denote the collection of lattice-preserving permutations which also satisfy $\pi(u * v) = \pi(u) * v$, if $|u| = n$, for $u, v \in T$, where $*$ is concatenation of the two sequences. Now, for $A \in \mathcal{F} = \sigma(X(v) : v \in T)$, say $A = [X(v_1) \in B_1, \ldots, X(v_k) \in B_k]$, write $\pi(A) = [X(\pi(v_1)) \in B_1, \ldots, X(\pi(v_k)) \in B_k]$, $B_i \in \mathcal{B}(0, \infty)$. Define

$$\mathcal{S}_n := \{A \in \mathcal{F} : A = \pi(A) \ \forall\ \pi \in \mathcal{P}_n\}.$$

Lemma 4. *Under weak disorder, equivalently $E_P X \ln X < \ln 2$, there is a number $1 < q < 2$ and a positive number δ such that*

$$\lim_{n\to\infty} \sum_{j=1}^{n} \sup_{|r|\le\delta} ||m_{n,j}(r)||_{L^q(\Omega,\mathcal{F},P)} < \infty.$$

Proof. For $1 < q < 2$, $\frac{q}{2} < 1$. Note that

$$||m_{n,j}(r)||_{L^q}^q = E_P|m_{n,j}(r)|^q = E_P\left(|m_{n,j}(r)|^2\right)^{\frac{q}{2}} \le E_P\left(E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\}\right)^{\frac{q}{2}}.$$

Here $E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\}$ is the essentially unique positive $\mathcal{S}_n$-measurable random variable guaranteed by the Radon–Nikodym theorem. However, $|m_{n,j}(r)|^2$ need not be integrable (with respect to P), so that the usual L^1-expectation needs to be replaced by the L^+-version. Let $\mathcal{P}_n$ denote the collection of permutations on $T_n := \cup_{k=0}^n\{-1,1\}^k$ "depending on at most the first n levels," i.e., $\pi \in \mathcal{P}_n$ if and only if there is a $\hat{\pi} \in \mathcal{P}_{\le n}$ such that $\pi = \hat{\pi}|_{T_n}$. With this notation one may compute

$$E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\} = \frac{1}{\cosh^{2n} r}\frac{1}{\#\mathcal{P}_n^2}\int_{\partial T}\int_{\partial T}\sum_{\pi\in\mathcal{P}_n}\Big\{(\pi(s)(j)-\tanh(r))\times$$

$$(\pi(t)(j)-\tanh(r))\,e^{r(\pi(t))_n}e^{r(\pi(s))_n}\Big\}\left(\sum_{\gamma\in\mathcal{P}_n}\prod_{i=0}^n X(\gamma(s|i))X(\gamma(t|i))\right)\lambda(ds)\lambda(dt).$$

Moreover,

$$\frac{1}{\cosh^{2n} r}\frac{1}{\#\mathcal{P}_n}\sum_{\pi\in\mathcal{P}_n}\{(\pi(s)(j)-\tanh(r))\,(\pi(t)(j)-\tanh(r))\times e^{r(\pi(t))_n}e^{r(\pi(s))_n}\}$$

$$= \begin{cases} 0 \ \text{ if } j > |s\wedge t| \\ \frac{1}{2^j\cosh^{2j}(r)}\sum_{|s|=j}\{(s_j-\tanh(r))^2\,e^{2r(s)_j} \ \text{ if } j \le |s\wedge t| \end{cases}$$

$$= \begin{cases} 0 \ \text{ if } j > |s\wedge t| \\ \frac{(1-\tanh r)^2e^2+(-1-\tanh r)^2e^{-2}}{2\cosh^2(r)}\frac{\cosh^{j-1}(2r)}{\cosh^{2j-2}(r)} \ \text{ if } j \le |s\wedge t|. \end{cases}$$

Thus, one may write

$$E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\} = \mu_j(r)\int_{\partial T}\int_{\partial T}\mathbf{1}[|s\wedge t|\ge j]$$

$$\times\frac{1}{\#\mathcal{P}_n}\left(\sum_{\pi\in\mathcal{S}_n}\prod_{i=0}^n X(\pi(s|i))X(\pi(t|i))\right)\lambda(ds)\lambda(dt)$$

$$= \mu_j(r)\sum_{k=j}^n\sum_{|s|=k}2^{-2k}\prod_{i=0}^k X^2(s|k)\int_{\partial T}\prod_{i=0}^{n-k-1}X(s*(1)*t|i)\lambda(dt)$$

$$\times\int_{\partial T}\prod_{i=0}^{n-k-1}X(s*(-1)*t|i)\lambda(dt),$$

where

$$\mu_j(r) = \frac{(1-\tanh r)^2e^2+(-1-\tanh r)^2e^{-2}}{2\cosh^2(r)}\frac{\cosh^{j-1}(2r)}{\cosh^{2j-2}(r)},$$

and one makes the convention that

$$\int_{\partial T} \prod_{i=0}^{-1} X(s*(1)*t|i)\lambda(dt) \int_{\partial T} \prod_{i=0}^{-1} X(s*(-1)*t|i)\lambda(dt) \equiv 1.$$

Recall that for the models considered here there is a $p > 1$ such that $E_P X^p < \infty$. Under weak disorder, therefore, there is a $1 < \hat{q} < 2$ such that

$$\frac{E_P X^q}{2^{q-1}} < 1 \quad \text{for any } 1 < \mathrm{q} < \hat{\mathrm{q}}.$$

So, for $q \in (1, \hat{q})$, it follows that

$$\begin{aligned}
&||m_{n,j}(r)||_q^q \\
&\quad \le E_P \Bigg\{ \mu_j(r) \sum_{k=j}^{n} \sum_{|s|=k} 2^{-2k} \prod_{i=0}^{k} X^2(s|k) \\
&\quad \times \int_{\partial T} \prod_{i=0}^{n-k-1} X(s*(1)*t|i)\lambda(dt) \int_{\partial T} \prod_{i=0}^{n-k-1} X(s*(-1)*t|i)\lambda(dt) \Bigg\}^{\frac{q}{2}} \\
&\quad \le E_P \Bigg\{ \mu_j^{\frac{q}{2}}(r) \sum_{k=j}^{n} \sum_{|s|=k} 2^{-2k} \prod_{i=0}^{k} X^q(s|k) \\
&\quad \times \left(\int_{\partial T} \prod_{i=0}^{n-k-1} X(s*(1)*t|i)\lambda(dt) \int_{\partial T} \prod_{i=0}^{n-k-1} X(s*(-1)*t|i)\lambda(dt) \right)^{\frac{q}{2}} \Bigg\} \\
&\quad \le \mu_j^{\frac{q}{2}}(r) \sum_{k=j}^{n} \frac{(E_P X^q)^k}{2^{(q-1)k}} \\
&\quad \le \mu_j^{\frac{q}{2}}(r) \frac{(E_P X^q)^j}{2^{(q-1)j}} \left(1 - \frac{E_P X^q}{2^{q-1}} \right)^{-\frac{1}{q}}.
\end{aligned}$$

Next choose $\delta > 0$ sufficiently small that for $|r| \le \delta$

$$\frac{\cosh(2r)}{\cosh^2(r)} \frac{E_P X^q}{2^{q-1}} < \frac{1}{2} \left(\frac{E_P X^q}{2^{q-1}} + 1 \right).$$

Then, letting $C = \max_{|r| \le \delta} \sqrt{\frac{(1-\tanh r)^2 e^2 + (-1-\tanh r)^2 e^{-2}}{2\cosh^2(r)}}$, it follows that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \sup_{|r|\le\delta} ||m_{n,j}(r)||_{L^q(\Omega,\mathcal{F},P)} \le C \left(1 - \frac{E_P X^q}{2^{q-1}} \right)^{-\frac{1}{q}} \sum_{j=1}^{\infty} \left\{ \frac{1}{2} \left(\frac{E_P X^q}{2^{q-1}} + 1 \right) \right\}^j < \infty$$

as asserted. ■

We conclude this section with a main result of this chapter for tree polymers under the full range of weak disorder.

Theorem 1. *Assume weak disorder. Then P-a.s. there is a $\delta > 0$ and an absolutely continuous random function G on $-\delta \le r \le \delta$ such that uniformly on $|r| \le \delta$ one has*

$$\lim_{n\to\infty} \widetilde{m}'_n(r) \to G(r), \quad -\delta \le r \le \delta.$$

In particular, P-a.s.

$$\frac{(S)_n}{\sqrt{n}} \Rightarrow Z,$$

where Z has a standard normal distribution.

Proof. From Lemma 4 there are numbers $\delta > 0$ and $1 < q < 2$ such that

$$M := \lim_{n\to\infty} \sum_{j=1}^{n} \sup_{|r|\le\delta} ||m_{n,j}(r)||_{L^q(\Omega,\mathcal{F},P)} < \infty.$$

Then

$$\begin{aligned}\left(E_P \int_{-\delta}^{\delta} |\widetilde{m}'_n(r)|^q dr\right)^{\frac{1}{q}} &\le \sum_{j=1}^{n} \left(E_P \int_{-\delta}^{\delta} |m_{n,j}(r)|^q dr\right)^{\frac{1}{q}} \\ &\le \sum_{j=1}^{n} \left(2\delta \sup_{|r|\le\delta} E_P |m_{n,j}(r)|^q\right)^{\frac{1}{q}} \\ &\le (2\delta)^{\frac{1}{q}} M.\end{aligned}$$

Thus, $\widetilde{m}'_n(r)$ converges P-a.s. and for almost every $r \in [-\delta, \delta]$, to some $G(r)$. In fact, with $C_q := (\frac{q}{q-1})^q$, one has by the L^q-maximal inequality that

$$\begin{aligned}\int_{-\delta}^{\delta} E_P \sup_{n\le N} |\widetilde{m}'_n(r)|^q dr &\le C_q \int_{-\delta}^{\delta} E_P |\widetilde{m}'_N(r)|^q dr \\ &\le 2\delta C_q M^q.\end{aligned}$$

Thus, P-a.s. $\widetilde{m}'_n(r) \to G(r)$ in $L^q([-\delta,\delta], dr)$. In particular, $m_\infty(r) = \lim_{n\to\infty} \widetilde{m}_n(r)$, $-\delta \le r \le \delta$ is uniform, and $\widetilde{m}_\infty(r)$ is absolutely continuous with derivative $G(r)$. Note that $\widetilde{m}_\infty(0) = z_\infty$ since $\widetilde{m}_n(0) \equiv z_n$ for each n. Since $\widetilde{m}_\infty(r)$ is P-a.s. continuous in a neighborhood of $r = 0$, one has P-a.s. for $-\delta \le r \le \delta$,

$$M_n\left(\frac{r}{\sqrt{n}}\right) = \frac{M_n(\frac{r}{\sqrt{n}})}{\cosh^n(\frac{r}{\sqrt{n}})} \cosh^n\left(\frac{r}{\sqrt{n}}\right) \to \frac{\widetilde{m}_\infty(0)}{z_\infty} e^{\frac{r^2}{2}} \equiv e^{\frac{r^2}{2}} \quad \text{as} \quad n \to \infty.$$

■

Remark 9. One may, in fact, show with only a little more effort that the limits G and m_∞ are both a.s. analytic functions.

6 Vector Cascades

This section provides an extension of i.i.d. scalar cascades within the framework of **T**-martingales. As an application an alternative approach to asymptotic path free energy calculations is given. For this, suppose that $\mathbf{W} = (W_1, W_2)$ is a symmetric random vector with a.s. positive components defined on a probability space $(\Omega, \mathcal{F}, P)$. We will denote a (scalar) random variable with the common marginal distribution of the (possibly correlated) components W_1, W_2 by W. Let

$$g(h) = E_P W^h, \quad h \in H = \{h \geq 0 : E_P W^h < \infty\}.$$

Then g is continuous on H and we restrict our attention to distributions for which H is a nondegenerate subinterval of $[0, \infty)$. For $h \in H$, define

$$W_h = \frac{W^h}{g(h)} \qquad \mathbf{W}_h = (W_{h,1}, W_{h,2}) = \left(\frac{W_1^h}{g(h)}, \frac{W_2^h}{g(h)} \right).$$

For $h \in H^0$, the interior of H, one has that $\ln W$ has a finite moment generating function and, therefore,

$$E_P \left(W_h (\ln W)^n \right) < \infty, \quad \forall n = 1, 2, \ldots.$$

Moreover, from the dominated convergence theorem, one has for $h \in H^0$

$$\frac{d}{dh} \ln g(h) = E_P \left(W_h \ln W \right)$$

and

$$\frac{d^2}{dh^2} \ln g(h) = E_P \left(W_h (\ln W)^2 \right) - \left(E_P (W_h \ln W) \right)^2 = \mathrm{var}_h (\ln W) \geq 0,$$

where var_h denotes variance computed with respect to the size-biased probability $dQ_h = W_h dP$. In particular, $\ln g(h)$ is convex on H. In fact, the function $h \to E_P (W_h \ln W_h), h \in H^0$, is increasing since

$$\frac{d}{dh} E_P (W_h \ln W_h) = h \mathrm{var}_h (\ln W) \geq 0.$$

Thus, if W is not an a.s. constant then $h \to E_P (W_h \ln W_h), h \in H^0$ is strictly increasing.

Now suppose that $\{\mathbf{W}_v = (W_{v,1}, W_{v,2}) : v \in T\}$ is an i.i.d. tree-indexed collection of random vectors defined on the probability space $(\Omega, \mathcal{F}, P)$ distributed as $\mathbf{W}$. Let

$$W_{v,(h,i)} = \frac{W_{v,i}^h}{g(h)}, \quad v \in T, i = 1, 2, h \in H,$$

and define

$$Q_n^{(h)}(t) = \prod_{j=1}^{n} W_{t|(j-1),(h,t_j)}, \quad t \in \partial T.$$

Then $\{Q_n^{(h)} : n \geq 1\}$ defines a positive $\mathbf{T}$-martingale in the sense of Kahane.

We will require a few lemmas based on the size-biasing theory of Sect. 3. Let us denote the size-bias probabilities corresponding to the $\mathbf{T}$-martingale $\{Q_n^{(h)} : n \geq 1\}$ by $P_{h,t}$ and $\mathcal{Q}_h$, accordingly. The first is a law of large numbers under the size-bias change of measures.

Proposition 5. *Let $h, h' \in H$ and $t \in \partial T$.*

1. *$P_{h,t}$-a.s., $\frac{1}{n} \ln Q_n^{(h')}(t) \to E_P(W_h \ln W_{h'})$. Moreover, if there is an $h'' \in H$ such that $h < h''$, then*

$$\sum_{n=1}^{\infty} E_{P_{h,t}} \left(\frac{1}{n} \ln Q_n^{(h')}(t) - E_P(W_h \ln W_{h'}) \right)^4 < \infty.$$

2. *$\mathcal{Q}_h$-a.s., $\frac{1}{n} \ln Q_n^{(h')} \circ \pi_{\partial T}(t) \to E_P(W_h \ln W_{h'})$. Moreover,*

$$\sum_{n=1}^{\infty} E_{\mathcal{Q}_h} \left(\frac{1}{n} \ln Q_n^{(h')} \circ \pi_{\partial T}(t) - E_P(W_h \ln W_{h'}) \right)^4 < \infty.$$

Proof. From the definitions,

$$\frac{1}{n} \ln Q_n^{(h')}(t) = \frac{1}{n} \sum_{j=1}^{n} \ln W_{t|j-1,(h',t_j)}$$

is a sample average of i.i.d. terms under $P_{h,t}$ with mean $E_P(W_h \ln W_{h'})$. Thus, the first assertion of the first statement is merely a version of the strong law of large numbers, and the second assertion of the first statement is the fourth-moment Borel–Cantelli condition for the strong law of large numbers. Specifically, under the condition $h < h'' \in H$, one has

$$E\left(W_h (\ln W_{h'})^4\right) = E\left(W_h (h' \ln W - \ln g(h'))\right)^4 < \infty.$$

For the second statement, observe that the first statement is true for λ-a.e. $t \in \partial T$. Also, by symmetry,

$$E_{\mathcal{Q}_h} \left(\frac{1}{n} \ln Q_n^{(h')} \circ \pi_{\partial T} - E(W_h \ln W_{h'}) \right)^4 = E_{P_{h,t}} \left(\frac{1}{n} \ln Q_n^{(h')}(t) - E_P(W_h \ln W_{h'}) \right)^4,$$

and is therefore also summable in n. □

Lemma 5. *If $E_P(W_h \ln W_h) < \ln 2$ and $h' \in H$, then a.s.*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \geq E_P\left(W_h \ln \frac{W_{h'}}{W_h}\right).$$

Moreover, if h_c exists such that $E_P(W_{h_c} \ln W_{h_c}) = \ln 2$, then a.s. one has

$$\liminf_{n\to\infty} \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \geq E_P\left(W_{h_c} \ln \frac{W_{h'}}{W_{h_c}}\right).$$

Proof. Using the size-bias change of measure, one has

$$\begin{aligned}\frac{\int_{\partial T} Q_n^{(h')}(t)\lambda(dt)}{\int_{\partial T} Q^{(h)}(t)\lambda(dt)} &= \frac{\int_{\partial T} \frac{Q_n^{(h')}(t)}{Q_n^{(h)}(t)} Q_n^{(h)}(t)\lambda(dt)}{\int_{\partial T} Q^{(h)}(t)\lambda(dt)} \\ &= E_{\mathcal{Q}}\left(E_{\mathcal{Q}}\left(\frac{Q_n^{(h')}}{Q_n^{(h)}} \circ \pi_{\partial T} | \mathcal{F}_n\right)\right). \end{aligned} \tag{10}$$

Using the convexity of $x \to -\ln x$, one has

$$\begin{aligned}&\frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \\ &\quad \geq E_{\mathcal{Q}}\left(\frac{1}{n} \frac{Q_n^{(h')}}{Q_n^{(h)}} \circ \pi_{\partial T} | \mathcal{F}_n\right) + \frac{1}{n} \ln \int_{\partial T} Q_n^{(h)}(t)\lambda(dt) \\ &\quad = E_{\mathcal{Q}}\left(\frac{1}{n} Q_n^{(h')} \circ \pi_{\partial T} - \frac{1}{n} \ln Q_n^{(h)} \circ \pi_{\partial T} | \mathcal{F}_n\right). \end{aligned} \tag{11}$$

Using the fourth-moment summability of the second part of Proposition 5, it follows that

$$E_{\mathcal{Q}}\left(\frac{1}{n} Q_n^{(h')} \circ \pi_{\partial T} - \frac{1}{n} \ln Q_n^{(h)} \circ \pi_{\partial T} | \mathcal{F}_n\right) \to E_P(W_h \ln W_{h'}) - E_P(W_h \ln W_h).$$

Thus, the asserted lower bound holds $\mathcal{Q}$-a.s. But, $E_P(W_h \ln W_h) < \ln 2$ implies that $\mathcal{Q} \circ \pi_{\partial T}^{-1} << P$. This proves the first assertion of the lemma. The second assertion follows from the continuity of $h \to E_P(W_h \ln \frac{W_{h'}}{W_h})$. ■

Lemma 6. *If $E_P(W_h \ln W_h) < \ln 2$ and $h' \in H$, then a.s.*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \leq E_P\left(W_h \ln \frac{W_{h'}}{W_h}\right).$$

Moreover, if h_c exists such that $E_P(W_{h_c} \ln W_{h_c}) = \ln 2$, then a.s. one has

$$\liminf_{n\to\infty} \frac{1}{n} \ln \int_{\partial T} Q_n^{(h)}(t)\lambda(dt) \leq E_P\left(W_{h_c} \ln \frac{W_h}{W_{h_c}}\right).$$

Proof. For any $t \in \partial T$ one has the Chebyshev bound

$$P\left(\prod_{j=1}^{n} W_{t|j-1,t_j} \geq c^n\right) \leq E_P\left(\frac{\prod_{j=1}^{n} W_{t|j-1,(h,t_j)}}{c^{hn}}\right) = \left(\frac{g(h)}{c^h}\right)^n.$$

The right side is minimized at the Legendre transform value $\frac{d}{dh}(\ln g(h) - h \ln c) = 0$. In other words, $E_P(W_h \ln W) = \ln c$ optimizes to the extent that

$$P\left(\prod_{j=1}^{n} W_{t|j-1,t_j} \geq e^{nE_P(W_h \ln W)}\right) \leq e^{-nE_P(W_h \ln W_h)}.$$

Thus, for $h, h' \in H$, one has

$$P\left(\prod_{j=1}^{n} W_{t|j-1,(h',t_j)} \geq e^{nE_P(W_h \ln W_{h'})}\right) \leq e^{-nE_P(W_h \ln W_h)}.$$

In particular, for h_c defined by

$$E_P(W_{h_c} \ln W_{h_c}) = \ln 2,$$

one has by monotonicity that

$$E_P(W_h \ln W_h) > \ln 2 \quad \text{for} \quad h > h_c.$$

Thus,

$$\sum_{n=1}^{\infty} 2^n P\left(\prod_{j=1}^{n} W_{t|j-1,(h',t_j)} \geq e^{nE_P(W_h \ln W_{h'})}\right) < \infty,$$

and therefore,

$$P\left(\cup_{N=1}^{\infty} \cap_{n \geq N}^{\infty} [\prod_{j=1}^{n} W_{t|j-1,(h',t_j)} < e^{nE_P(W_h \ln W_{h'})} \forall t \in \partial T]\right) = 1.$$

Now consider $h' > h_c$. Then

$$\begin{aligned} &\limsup \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t) \lambda(dt) \\ &\quad = \limsup \frac{1}{n} \ln \int_{[Q_n^{(h')}(t) < e^{nE_P(W_h \ln W_{h'})}]} Q_n^{(h')}(t) \lambda(dt). \end{aligned}$$

Although $E_P \ln W < 0$ may not be finite, $\lim_{h \downarrow 0} E_P(W_h \ln W) = E_P(\ln W)$. Also, $h \to E_P(W_h \ln W)$ is continuous and increasing on H^0. Let $0 < h_1 < h_2 < \cdots < h_{m-1} < h_c < h_m < h'$, and $c_i = \exp E_P(W_{h_i} \ln W)$. Consider the random set

$$A_{n,i} = \left\{ t \in \partial T : c_{i-1}^n < \prod_{j=1}^{n} W_{t|j-1,t_j} \le c_j^n \right\}.$$

For all n large, one-as a.s. that

$$\frac{c_1^m}{g^n(h')} + \sum_{i=2}^{n} \frac{c_i^m}{g^n(h')} \lambda(A_{n,i}) \ge \int_{\partial T} Q_n^{(h')}(t)\lambda(dt).$$

In any case, for $i \ge 2$ one has

$$\frac{c_{i-1}^m}{g^n(h_{i-1})} \lambda(A_{n,i}) \le \int_{\partial T} Q_n^{(h_{i-1})}(t)\lambda(dt).$$

Thus, P-a.s.,

$$\begin{aligned} &\liminf \frac{1}{n} \ln \left(\frac{c_1^m}{g^n(h')} + \sum_{i=2}^{n} \frac{c_i^{nh'} g^n(h_{i-1})}{c_{i-1}^{nh_{i-1}} g^n(h')} \int_{\partial t} Q_n^{(h_{i-1})}(t)\lambda(dt) \right) \\ &\quad \ge \liminf \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt). \end{aligned}$$

Since for $0 < h < h_c$, $\lim_n \int_{\partial T} Q_n^{(h)}(t)\lambda(dt)$ exists and is positive, one has

$$\begin{aligned} &\liminf \frac{1}{n} \ln \left(\frac{c_1^m}{g^n(h')} + \sum_{i=2}^{n} \frac{c_i^{nh'} g^n(h_{i-1})}{c_{i-1}^{nh_{i-1}} g^n(h')} \int_{\partial t} Q_n^{(h_{i-1})}(t)\lambda(dt) \right) \\ &= \ln \max \left\{ \frac{c_1^{h'}}{g(h')}, \frac{c_i^{nh'} g^n(h_{i-1})}{c_{i-1}^{nh_{i-1}} g^n(h')} \right\} \\ &= \max \left\{ E_P(W_{h_1} \ln W_{h'}), E_P(W_{h_{i-1}} \ln W_{h'}) - E_P(W_{h_{i-1}} \ln W_{h_{i-1}}), i \ge 2 \right\}. \end{aligned}$$

Thus, one has P-a.s. that

$$\begin{aligned} &\liminf \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \\ &\quad \le \inf_{0<h_1<\cdots<h_{m-1}<h_c<h_m<h'} \max\{E_P(W_{h_1} \ln W_{h'}), \\ &\quad E_P(W_{h_{i-1}} \ln W_{h'}) - E_P(W_{h_{i-1}} \ln W_{h_{i-1}}), i \ge 2\}. \end{aligned}$$

Now use the uniform continuity of $E_P(W_x \ln W_{h'}) - E_P(W_y \ln W_y)$ for $x, y \in [h_1, h']$ together with the fact that $E_P(\ln W_{h'}) = E_P(W_0 \ln \frac{W_{h'}}{W_0})$ to proceed as follows:

$$\begin{aligned} &\inf_{0<h_1<\cdots<h_{m-1}<h_c<h_m<h'} \max\{E_P(W_{h_1} \ln W_{h'}), \\ &E_P(W_{h_{i-1}} \ln W_{h'}) - E_P(W_{h_{i-1}} \ln W_{h_{i-1}}), i \ge 2\} \\ &\quad \le \inf_{0<h_1<h_c} \max\{E_P(W_{h_1} \ln W_{h'}), \sup_{h_1<h<h_c} [E_P(W_h \ln W_{h'}) - E_P(W_h \ln W_h)]\} \end{aligned}$$

$$\leq \max\left\{E_P(\ln W_{h'}), E_P\left(W_{h_c}\ln\frac{W_{h'}}{W_{h_c}}\right)\right\} = E_P\left(W_{h_c}\ln\frac{W_{h'}}{W_{h_c}}\right).$$

Finally, if $h' \in H$ and $h' > h_c$, then P-a.s.,

$$\liminf \frac{1}{n}\ln\int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \leq E_P\left(W_{h_c}\ln\frac{W_{h'}}{W_{h_c}}\right).$$

This completes the derivation of the upper bound. □

Combining these lemmas one arrives at the following result.

Theorem 2. *For $h \in H$ and $h > h_c$ one has P-a.s.*

$$\liminf_{n\to\infty} \frac{1}{n}\ln\int_{\partial T} Q_n^{(h)}(t)\lambda(dt) = -E_P\left(W_{h_c}\ln\frac{W_{h_c}}{W_h}\right).$$

Remark 10. Notice that if $h > h_c$ then it follows from the previously noted monotonicity that $E_P(W_h \ln W_h) > \ln 2$.

To apply this to the polymer model let $X_{v,j}, v \in T, j = 1, 2$, be i.i.d. positive random variables distributed as X. Assume that H is a nondegenerate interval for X defined by

$$g(h) = E_P X^h < \infty, \quad h \in H \subseteq [0, \infty).$$

Also, suppose that Y is a symmetric Bernoulli ± 1-valued random variable, independent of X, and define

$$g(r, h) = E_P(e^{rY}X^h) = g(h)\cosh r, \quad h \in H, r \geq 0,$$

and consider the vector cascade weights

$$\mathbf{W}_{(r,h)} = \left(\frac{e^{rY}X_1^h}{g(r,h)}, \frac{e^{-rY}X_2^h}{g(r,h)}\right) = \left(W_{(r,h),1}, W_{(r,h),2}\right).$$

Note that defining

$$Y_r = \frac{e^{rY}}{\cosh r},$$

one has $E_P(Y_r \ln Y_r) = r\tanh r - \ln\cosh r$. Moreover, one has

$$\left\{(\cosh r)^{-n}\int_{\partial T} Q_n^{(r,h)}(t)\lambda(dt) : n \geq 1\right\} =^{\text{dist}} \left\{\int_{\partial T} e^{r(t)_n} Q_n^{(h)}(t)\lambda(dt) : n \geq 1\right\}.$$

With this one may obtain the following equivalent representation of the asymptotic path free energy under strong disorder; cf. Proposition 4 Sect. 4.

Theorem 3. *Suppose that for $r \geq 0, h \in H, h > h_c$, there is a unique pair $(r,h)^* \equiv (r^*,h^*)$ such that $(r^*,h^*) = \gamma(r,h)$ for some $0 < \gamma < 1$, and*

$$E_P\left(W_{(r^*,h^*)} \ln W_{(r^*,h^*)}\right) \equiv E_P\left(X_{h^*} \ln X_{h^*}\right) + E_P\left(Y_{r^*} \ln Y_{r^*}\right) = \ln 2,$$

where $Y_r = \frac{e^{rY}}{\cosh r}$. Then P-a.s. one has

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{\int_{\partial T} e^{r(t)_n} Q_n^{(h)}(t)\lambda(dt)}{\int_{\partial T} Q_n^{(h)}(t)\lambda(dt)} = \ln\cosh r + E_P\left(W_{(r,h)^*} \ln W_{(r,h)}\right) - E_P\left(W_{(0,h)^*} \ln W_{(0,h)}\right).$$

Proof. The proof is essentially an application of Theorem 2, using the fact that the limit has already been shown to exist. More specifically, one has P-a.s. that

$$\begin{aligned}\lim \frac{1}{n} \ln \int_{\partial T} Q_n^{(r,h)}(t)\lambda(dt) &= -E_P(W_{(r,h)^*} \ln W_{(r,h)^*} + E_P(W_{(r,h)^*} \ln W_{(r,h)}) \\ &= -\ln 2 + E_P(W_{(r,h)^*} \ln W_{(r,h)}).\end{aligned}$$

Thus,

$$\lim \frac{1}{n} \ln \frac{\int_{\partial T} Q_n^{(r,h)}(t)\lambda(dt)}{\int_{\partial T} Q_n^{(0,h)}(t)\lambda(dt)} = -E_P(W_{(r,h)^*} \ln W_{(r,h)}) + E_P(W_{(0,h)^*} \ln W_{(0,h)}).$$

Now

$$\lim \frac{1}{n} \ln \frac{\int_{\partial T} e^{r(t)_n} Q_n^{(r,h)}(t)\lambda(dt)}{\int_{\partial T} Q_n^{(0,h)}(t)\lambda(dt)} = \ln\cosh r + E_P(W_{(r,h)^*} \ln W_{(r,h)} - E_P(W_{(0,h)^*} \ln W_{(0,h)})$$

as asserted. ■

In the strong disorder case $E_P X \ln X > \ln 2$, normalized to $E_P X = 1$, one has $h_c < 1$ by the monotonicity of $h \to E_P X^h \ln X^h$. Taking $h = 1$ in this theorem gives the alternative path free energy formula. Namely, we have the following.

Corollary 2. *Assume X is a positive random variable normalized to $E_P X = 1$ such that $E_P X \ln X > \ln 2$; i.e., strong disorder. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_{\partial T} e^{r(t)_n} prob_n(dt) = \ln\cosh r + E_P\left(W_{(r,1)^*} \ln W_{(r,1)}\right) - E_P\left(W_{(0,1)^*} \ln W_{(0,1)}\right).$$

7 Related Directions in T-Martingale Theory

T-martingale theory and size-bias methods occupy a central role in determining sharp results for the existence, and in the analysis, of the fine-scale structure of diverse models; see Kahane [23] for a review of general theory and other applications. Most of the theory, however, is devoted to the analysis of fine-scale structure in the weak disorder regime. Tree polymers present entirely new challenges to the theory in the case of strong disorder, and naturally motivate new directions. On the purely mathematical side, the contemplation of a companion theory for **complex T-martingales** on manifolds suggests a number of new and interesting challenges.

In the context of tree polymer models, sharp determination of the a.s. probability laws governing polymer paths under weak and strong disorder should eventually evolve. One may not expect surprises under weak disorder but, as illustrated here for the a.s. CLT, the techniques and estimates may be delicate in the full range of weak disorder. The limits on Bolthausen's L^2 approach to a CLT for tree polymers can only be asserted when such a CLT has been established as has been achieved here. It seems to be generally accepted that the lattice polymer approach of Comets and Yoshida [12] would also provide the CLT for tree polymers in the full range of weak disorder, but such a proof has not been available in the literature.

As is evidenced here, there is a huge amount of symmetry present both in the tree and in the environment. In general, there seems to be much to understand about how and when symmetry breaking may occur. A loosely related phenomenon illustrating symmetries was observed in Waymire and Williams [31,32] in the consideration of a Markovian environment (along tree paths); such results were also considered by Fan [18]. In Waymire and Williams [31, 32], the authors demonstrate that for finite-state time-reversible ergodic Markov environments, the structure of the multiplicative cascade coincides with that of i.i.d. environments distributed according to the unique invariant probability. However, examples are provided to show that this is no longer true for nonreversible Markov chains.

While the emphasis of this chapter is that of the theory of **T**-martingales, it is widely recognized that results obtained for branching random walks originating in Kingman [26] and Biggins [4] closely parallel this development. So it is not surprising that both theoretical frameworks can be applicable to tree polymers. The very recent paper by Hu and Shi [19] illustrates many aspects of the continued development of this companion framework. In particular, Hu and Shi [19] analyze the free energy-type calculations for polymers on Galton–Watson trees within the branching random walk framework. It seems natural, by extension, to consider the polymer path laws in the Galton–Watson environment within either framework; e.g., see Peyrière [28] and Burd and Waymire [11] for some multiplicative cascade theory on Galton–Watson trees.

In addition to providing a complete and self-contained diffusive limit for tree polymers in the full range of weak disorder, the goal of this chapter has

been to suggest new directions for extensions of the multiplicative cascade theory. The extension of Proposition 2 to corresponding P-a.s. weak limit points under strong disorder aptly illustrates such a need.

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Part V

Combinatorics on Words

Univoque Numbers and Automatic Sequences

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Summary. A set of binary sequences related to the iteration of unimodal continuous functions of the interval $[0,1]$ appears in a 1982–1983 work of Cosnard and the first author. An almost identical set of binary sequences occurs in a 1990 paper by Erdős, Joó, and Komornik; it consists of expansions of 1 in *univoque* bases β in $(1,2)$ (the base β is univoque if 1 admits a unique β-expansion). We generalize a result of the second author and Niu by proving, using the 1982–1983 results, that a large class of Thue-Morse-like sequences belong to these sets of binary sequences. The case of alphabets of size larger than 2 yields similar results.

1 Introduction

A set of binary sequences related to the iteration of continuous unimodal functions of the interval $[0,1]$ was introduced at the beginning of the 1980s by M. Cosnard and the first author [1, 4, 13]. This set is obtained by looking at the kneading sequences of the point 1 under continuous unimodal maps from $[0,1]$ into itself, then replacing R, L in the kneading sequences by $0, 1$, and finally replacing each binary sequence $(a_n)_{n\geq 0}$ obtained in that way by the sequence $(\sum_{0\leq j\leq n} a_j \bmod 2)_{n\geq 0}$. This set is fractal (actually self-similar in some sense, see [1,4]). An almost identical set was introduced independently in 1990 by Erdős, Joó, and Komornik [20] to characterize univoque real numbers in the interval $(1,2)$. Recall that a real number $\beta > 1$ is called *univoque* if there is only one expansion of the number 1 as $1 = \sum_{n\geq 1} a_n/\beta^n$, with $a_n \in \{0, 1, \ldots, \lceil \beta \rceil - 1\}$. The set studied in [20] is the set of binary sequences $(a_n)_{n\geq 0}$ such that the (unique) real number $\beta > 1$ satisfying $1 = \sum_{n\geq 1} a_n/\beta^n$ is univoque.

Those two sets of binary sequences are, respectively, the sets Γ and Γ_{strict} defined by

$$\begin{aligned} \Gamma &:= \{A = (a_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}},\ \forall k \geq 0,\ \overline{A} \leq \sigma^{(k)}A \leq A\} \\ \Gamma_{strict} &:= \{A = (a_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}},\ \forall k \geq 1,\ \overline{A} < \sigma^{(k)}A < A\}, \end{aligned}$$

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where σ is the *shift* on sequences and the *bar* operation switches 0's and 1's, i.e., if $A = (a_n)_{n\geq 0}$, then $\sigma A := (a_{n+1})_{n\geq 0}$, and $\overline{A} := (1-a_n)_{n\geq 0}$; the symbol $\leq$ denotes the lexicographical order on sequences induced by $0 < 1$, and the notation $A < B$ means as usual that $A \leq B$ and $A \neq B$.

Note that the original set studied in [1, 4, 13] is equal, with our notation here, to $\Gamma \setminus \{(10)^\infty\}$. Also note that the original definition of the set that we call here Γ_{strict} is slightly different from, but equivalent to, the definition above, see [6, Remark 4, p. 328]. Finally, note that the sets Γ and Γ_{strict} only differ by a set of (purely) periodic sequences.

In 1998, Komornik and Loreti [22] proved that there exists a smallest univoque number β_0, and that the expansion of 1 in base β_0 is the shifted Thue–Morse sequence: 1 1 0 1 0 0 1 1 .... (For more about the Thue–Morse sequence, see, e.g., [10].) J. Shallit indicated to the first author that this result appeared in the 1982–1983 work of Allouche–Cosnard: namely, the smallest nonperiodic element of Γ (see [1, 4]) is clearly the smallest element of Γ_{strict} (see [22], see also [5, 6]).

This unexpected occurrence of the Thue–Morse sequence $(t_n)_{n\geq 0}$ is not isolated. Other variations or avatars of this sequence also belong to the sets Γ and Γ_{strict} or to their generalizations to alphabets of size > 2: the q-mirror sequences (see [1, 4]), and the sequences $(d + t_{n+1})_{n\geq 0}$ and $(d + t_{n+1} - t_n)_{n\geq 0}$ for a fixed integer d (see [23]), and the fixed point beginning with 3 of the morphism $3 \to 31$, $2 \to 30$, $1 \to 03$, $0 \to 02$ that governs several sequences in (generalizations of) the set Γ_{strict}, including the sequences $(d + t_{n+1})_{n\geq 0}$ and $(d + t_{n+1} - t_n)_{n\geq 0}$ above (see [7]).

In [25] M. Niu and the second author exhibited a class of generalized Thue–Morse sequences $(\varepsilon_n)_{n\geq 1}$, called the "$m$-tuplings Morse sequences," which among other properties belong to the set Γ_{strict}. The m-tuplings Morse sequence $(\varepsilon_n)_{n\geq 1}$ will be called the *m-fold Morse sequence* here. It is defined as the fixed point beginning with 0 of the morphism $0 \to 01^{m-1}$, $1 \to 10^{m-1}$. The purpose of this chapter is to prove that a result more general than the one in [25] can be easily deduced from several lemmas proved by the first author in [1].

Remark 1. Note that m-fold Morse sequences are particular cases of the generalized Thue–Morse sequences defined by Doche in [19], and of the symmetric D0L words defined by Frid in [21] (see also the paper of Astudillo [12]).

2 A Class of Sequences Belonging to Γ_{strict}

Before stating the main theorem of this chapter, we need to introduce some notation. Recall that the length of a (finite) word u is denoted by $|u|$.

Definition 1. *For any integer $r \geq 2$ we define the map Φ_r on periodic sequences of the form $(u0)^\infty$ with minimal period $|u| + 1$ by*

$$\Phi_r((u0)^\infty) := (u1(\overline{u}1)^{r-2}\overline{u}0)^\infty.$$

Theorem 1. *Let u be a finite word on the alphabet $\{0,1\}$, such that the sequence $(u0)^\infty$ belongs to Γ and has minimal period $|u|+1$. Then*

- *The sequence $\Phi_r((u0)^\infty)$ belongs to Γ and has minimal period $(r|u|+r)$.*
- *The limit $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ exists, and it belongs to Γ_{strict}.*

This theorem will be proved in the next section. We deduce from Theorem 1 the following corollaries. The first one is essentially the first part of [1, théorème fondamental de structure, p. 24] for $(|u|+1)$-*mirror sequences*, and the second one is Theorem 1 in [25].

Corollary 1 ([1]). *If the sequence $(u0)^\infty$ belongs to Γ and has minimal period $|u|+1$, then the sequence $\lim_{k\to\infty} \Phi_2^{(k)}((u0)^\infty)$ belongs to Γ_{strict}.*

Corollary 2 ([25]). *Let m be an integer ≥ 2, and let $(\varepsilon_n)_{n\geq 0}$ be the fixed point beginning with 0 of the morphism $0 \to 01^{m-1}$, $1 \to 10^{m-1}$. Then the sequence $(\varepsilon_n)_{n\geq 1}$ belongs to Γ_{strict}.*

Proof. The sequence $(1^{m-1}0)^\infty$ clearly belongs to Γ and it has minimal period m. Hence, applying Theorem 1 with $u := 1^{m-1}$ and $r = m$, we see that the sequence $\lim_{k\to\infty} \Phi_m^{(k)}((1^{m-1}0)^\infty)$ belongs to Γ_{strict}. It thus suffices to prove that the fixed point of the morphism λ_m defined by $\lambda_m(0) := 01^{m-1}$, $\lambda_m(1) := 10^{m-1}$, i.e., the sequence $\lambda_m^{(\infty)}(0)$, satisfies: $\lambda_m^{(\infty)}(0) = 0 \lim_{k\to\infty} \Phi_m^{(k)}((1^{m-1}0)^\infty)$.

Since for each letter x in $\{0,1\}$ we have $\lambda_m(\overline{x}) = \overline{\lambda_m(x)}$, we have for any word w on $\{0,1\}$ the equality $\lambda_m(\overline{w}) = \overline{\lambda_m(w)}$. Now define x_k by $\lambda_m^{(k)}(0) = 0x_k$. We thus have $\lambda_m^{(k)}(1) = 1\overline{x_k}$. Furthermore,

$$0x_{k+1} = \lambda_m^{(k+1)}(0) = \lambda_m^{(k)}(\lambda_m(0)) = \lambda_m^{(k)}(01^{m-1}) = \lambda_m^{(k)}(0)(\lambda_m^{(k)}(1))^{m-1}.$$

Hence, $0x_{k+1} = 0x_k(1\overline{x_k})^{m-1}$, which shows that

$$x_{k+1} = x_k(1\overline{x_k})^{m-1}.$$

Now, since the sequence $\lambda_m^{(\infty)}(0)$ is the limit of the sequence of words $\lambda_m^{(k)}(0)$ when k goes to infinity, it is also the limit of the sequence of periodic sequences $(\lambda_m^{(k)}(0))^\infty = (0x_k)^\infty = 0(x_k0)^\infty$ when k goes to infinity. Hence, $\lambda_m^{(\infty)}(0) = 0 \lim_{k\to\infty}(x_k0)^\infty = 0 \lim_{k\to\infty}(x_{k+1}0)^\infty$. But

$$\Phi_m((x_k0)^\infty) = (x_k1(\overline{x_k}1)^{m-2}\overline{x_k}0)^\infty = (x_k(1\overline{x_k})^{m-1}0)^\infty = (x_{k+1}0)^\infty.$$

An immediate induction on k implies that

$$\Phi_m^{(k)}((1^{m-1}0)^\infty) = \Phi_m^{(k)}((x_10)^\infty) = (x_{k+1}0)^\infty.$$

Hence,

$$\lambda_m^{(\infty)}(0) = 0 \lim_{k\to\infty} (x_{k+1}0)^\infty = 0 \lim_{k\to\infty} \Phi_m^{(k)}((1^{m-1}0)^\infty). \quad \blacksquare$$

3 Proof of Theorem 1

We first note that the limit in the second assertion of the theorem exists: namely, $\Phi_r^{(k)}((u0)^\infty)$ and $\Phi_r^{(k+1)}((u0)^\infty)$ coincide on their prefixes of length $r^k|u|+r^k-1$, and this quantity tends to infinity. Since a limit of sequences belonging to Γ clearly belongs to Γ, what we have to prove is the first assertion, and the fact that $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ is not periodic.

3.1 Proof of the First Assertion

We will make use of two results proved in [1] and that we recall below (up to notation).

Lemma 1 ([1]). *If a binary sequence belongs to Γ and begins with $u\overline{u}$ for some word u, then it is equal to $(u\overline{u})^\infty$.*

Lemma 2 ([1]). *Let $(u0)^\infty$ be a sequence in Γ, of smallest period $|u|+1$. Suppose $u = xy$ where x, y are two binary words, and x is not empty. Then $\overline{x}\ \overline{y}\ 0 < y\ 1\ \overline{x} < x\ y\ 1$, and $\overline{x}\ \overline{y}\ 0 < \overline{y}\ 1\ \overline{x} < x\ y\ 1$.*

Remark 2. Lemma 1 is [1, Lemme 2, pp. 26–27]. The first double inequality in Lemma 2 is [1, Lemme 5, p. 30], while the second can be easily obtained by combining the second assertion of [1, Proposition 2, p. 34] and [1, Lemme 3, p. 27].

Now suppose that $(u0)^\infty$ is a sequence in Γ, of smallest period $|u|+1$. We want to prove that $\Phi_r((u0)^\infty)$ belongs to Γ and has minimal period $r|u|+r$, i.e., that we have

$$\begin{cases} (\overline{u}0(u0)^{r-2}u0)^\infty \le (u1(\overline{u}1)^{r-2}\overline{u}0)^\infty \\ (\overline{u}0(u0)^{r-2}u0)^\infty \le \sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty) < (u1(\overline{u}1)^{r-2}\overline{u}0)^\infty, \quad \forall k \ge 1 \end{cases} \qquad (*)$$

Note that, if u is reduced to 1, or if u begins with 10, then, from Lemma 1, the sequence $(u0)^\infty$ must be equal to $(10)^\infty$; hence, the sequence $(u1(\overline{u}1)^{r-2}\overline{u}0)^\infty$ must be equal to $(11(01)^{r-2}00)^\infty$ and inequalities $(*)$ clearly hold. We thus may suppose that u begins with 11.

- First case: $k \equiv 0 \bmod (|u|+1)$.
  If $k \equiv 0 \bmod (r|u|+r)$ inequalities $(*)$ are clearly true (note that u must begin with 1 since $(u0)^\infty$ belongs to Γ, hence $(\overline{u}1)^\infty \le (u0)^\infty$).
  If $k \equiv j(|u|+1) \bmod (r|u|+r)$, with $j \in [1, r-2]$, the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins with $\overline{u}1$, and the inequalities are clear.
  If $k \equiv (r-1)(|u|+1) \bmod (r|u|+r)$, then the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins with $\overline{u}0u1$, and the inequalities are clear again.
- Second case: $k \equiv j(|u|+1)-1 \bmod (r|u|+r)$, with $j \in [1, r]$, then the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins with $1\overline{u}$ or $0u$. It thus suffices to check that $\overline{u}0 < 1\overline{u} < u1$ and that $\overline{u}0 < 0u < u1$, which are easy consequences of the fact that u begins with 11.

- Third case: $k \not\equiv 0, -1 \bmod (|u|+1)$. There exist two words x and y with $x \neq \emptyset$ and $u = xy$ such that the sequence $\sigma^k((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty)$ begins either with $y1\overline{x}$, or with $\overline{y}1\overline{x}$, or with $\overline{y}0x$, and Lemma 2 permits us to conclude.

3.2 The Sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ is Not Periodic

As will be proved in Theorem 2 in Sect. 4, if the sequence $(u0)^\infty$ is periodic with smallest period $|u|+1$, and if we set $(x_n)_{n\geq 0} := \lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$, then the sequence $(x_{qn})_{n\geq 1}$ is the shifted sequence of the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$. It thus suffices to prove that the r-fold Morse sequence is not eventually periodic. Being the fixed point of the morphism λ_r, $0 \to 01^{r-1}$, $1 \to 10^{r-1}$, the sequence $(\varepsilon_n)_{n\geq 0}$ satisfies

$$\begin{array}{ll} \varepsilon_{rn} = \varepsilon_n & \text{for all } n \geq 0, \\ \varepsilon_{rn+j} = 1 - \varepsilon_n & \text{for all } n \geq 0, \text{ for all } j \in [1, r-1]. \end{array}$$

Now suppose that $(\varepsilon_n)_{n\geq 0}$ is eventually periodic with smallest period T. We will show that this is impossible by looking at T modulo r. If $T = r\ell$ for some integer ℓ, then, for n large enough, we have $\varepsilon_{n+\ell} = \varepsilon_{rn+r\ell} = \varepsilon_{rn+T} = \varepsilon_{rn} = \varepsilon_n$, which would imply that $(\varepsilon_n)_{n\geq 0}$ is eventually periodic with period $\ell < T$. If $T = r\ell + j$ for some integer ℓ, and some integer $j \in [1, r-1]$, then, for n large enough, we have $\varepsilon_{n+\ell} = 1 - \varepsilon_{rn+r\ell+j} = 1 - \varepsilon_{rn+T} = 1 - \varepsilon_{rn} = 1 - \varepsilon_n$. This implies that $\varepsilon_{n+2\ell} = 1 - \varepsilon_{n+\ell} = \varepsilon_n$ for n large enough. Hence, the sequence $(\varepsilon_n)_{n\geq 0}$ is eventually periodic with period 2ℓ. But $2\ell \leq r\ell < r\ell + j = T$.

Remark 3. Since the morphism λ_r is primitive, the sequence $(\varepsilon_n)_{n\geq 0}$ is minimal. Hence, it cannot be eventually periodic without being periodic. A general criterion for periodicity of fixed points of constant length morphisms is given in [16, II.9 (iii), p. 226].

4 Automatic Sequences and the Sets Γ and Γ_{strict}

The sequences in Γ_{strict} given by Theorem 1 are generalizations of the m-fold Morse sequences of [25]. We will prove that they can be obtained by shuffling m-fold Morse sequences. In particular, they are automatic (for more about automatic sequences, see, e.g., [11]). We start with a definition and a lemma.

Definition 2. *If r is an integer ≥ 2, we define the map Ψ_r on binary words by*

$$\Psi_r(w) := w(\overline{w})^{r-1}.$$

Lemma 3. *Let u be a binary word such that the sequence $(u0)^\infty$ has minimal period $|u|+1$. Then, for all $k \geq 0$,*

$$0\ \Phi_r^{(k)}((u0)^\infty) = (\Psi_r^{(k)}(0u))^\infty \quad \text{and} \quad 0\ \lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty) = \lim_{k\to\infty} \Psi_r^{(k)}(0u).$$

Proof. The first assertion is trivial for $k = 0$, and it is an immediate consequence of the definitions of Φ_r and Ψ_r for $k = 1$. Suppose it is true for some k, then, using the induction hypothesis for k, and the case $k = 1$ of the induction hypothesis for the sequence $(u1(\overline{u}1)^{r-2}\overline{u}0)^\infty$, we have

$$\begin{aligned} 0\ \Phi_r^{(k+1)}((u0)^\infty) &= 0\ \Phi_r^{(k)}(\Phi_r((u0)^\infty)) = 0\ \Phi_r^{(k)}((u1(\overline{u}1)^{r-2}\overline{u}0)^\infty) \\ &= (\Psi_r^{(k)}(0u1(\overline{u}1)^{r-2}\overline{u}))^\infty = (\Psi_r^{(k)}(0u(1\overline{u})^{r-1}))^\infty \\ &= (\Psi_r^{(k)}(\Psi_r(u0)))^\infty = (\Psi_r^{(k+1)}(0u))^\infty. \end{aligned}$$

The second assertion is a consequence of the first one and of the two remarks that $\Psi_r^{(k+1)}(w)$ begins with $\Psi_r^{(k)}(w)$ for all binary words w, and that $|\Psi_r^{(k)}(w)|$ tends to infinity with k. ■

Theorem 2. *Let u be a finite word on the alphabet $\{0,1\}$, such that the sequence $(u0)^\infty$ belongs to Γ and has minimal period $q := |u| + 1$. Let $(x_n)_{n\geq 1}$ be the binary sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$. Then each of the sequences $(x_{qn+j})_{n\geq 0}$ for $j = 1, 2, \ldots, q-1$ is either the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$ or the sequence $(\overline{\varepsilon_n})_{n\geq 0}$. Furthermore, the sequence $(x_{qn})_{n\geq 1}$ is the shifted sequence of the r-fold Morse sequence. In other words, we have*

$$\begin{aligned} x_{qn+j} &= \varepsilon_n + x_j \quad \text{for all } n \geq 0 \text{ and for all } j = 1, 2, \ldots, q-1, \\ x_{qn} &= \varepsilon_n \qquad\quad \text{for all } n \geq 1. \end{aligned}$$

In particular, the sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ is r-automatic.

Proof. Recall that the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$ is the fixed point beginning with 0 of the morphism λ_r defined by $\lambda_r(0) = 01^{r-1}$ and $\lambda_r(1) = 10^{r-1}$. We first note that the sequence $(\varepsilon_n)_{n\geq 0}$ (resp. $(\overline{\varepsilon_n})_{n\geq 0}$) satisfies $(\varepsilon_n)_{n\geq 0} = \lim_{k\to\infty} \Psi_r^k(0)$ (resp. $(\overline{\varepsilon_n})_{n\geq 0} = \lim_{k\to\infty} \Psi_r^k(1)$). Namely, it was stated during the proof of Corollary 2, that $\lambda_r^{(\infty)}(0) = 0 \lim_{k\to\infty} \Phi_r^k((1^{r-1}0)^\infty)$. Hence, using Lemma 3, we have $\lambda_r^{(\infty)}(0) = \lim_{k\to\infty} \Psi_r^{(k)}(01^{r-1})$. Since $|\Psi_r^{(k)}(0)|$ tends to infinity with k, we have $\lim_{k\to\infty} \Psi_r^{(k)}(01^{r-1}) = \lim_{k\to\infty} \Psi_r^{(k)}(0)$. Hence, $(\varepsilon_n)_{n\geq 0} = \lambda_r^{(\infty)}(0) = \lim_{k\to\infty} \Psi_r^{(k)}(0)$, and $(\overline{\varepsilon_n})_{n\geq 0} = \lambda_r^{(\infty)}(1) = \lim_{k\to\infty} \Psi_r^{(k)}(1)$.

Now an induction on k shows that

$$\Psi_r^{(k)}(0u) = 0v_1b_1v_2b_2 \ldots b_{r^k-1}v_{r^k},$$

where $v_1 = u$, the v_j's are equal to u or $\overline{u}$ and b_j is 0 or 1. Furthermore,

$$0v_1b_1v_2b_2 \ldots b_{r^{k+1}-1}v_{r^{k+1}} = 0v_1b_1v_2b_2 \ldots b_{r^k-1}v_{r^k}(\overline{0v_1b_1v_2b_2 \ldots b_{r^k-1}v_{r^k}})^{r-1},$$

which implies that, for all $k \geq 0$,

- The words $B_k := 0b_1b_2 \ldots b_{r^k-1}$ satisfy the relation $B_{k+1} = B_k(\overline{B_k})^{r-1} = \Psi_r(B_k)$.

- The words $V_k := v_1 v_2 \dots v_{r^k}$ satisfy the relation $V_{k+1} = V_k(\overline{V_k})^{r-1} = \Psi_r(V_k)$.
- If $c_{j,i}$ is the jth letter of v_i (with $j \in [1, |u|]$), then the words $C_{j,k} := c_{j,1} c_{j,2} \dots c_{j,r^k}$ satisfy the relation $C_{j,k+1} = C_{j,k}(\overline{C_{j,k}})^{r-1} = \Psi_r(C_{j,k})$.

All these relations imply that the sequence $(b_n)_{n\geq 1}$ is the shifted sequence of the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$, and that each of the sequences $(c_{j,n})_{n\geq 0}$ for $j \in [1, |u|]$ is either the sequence $(\varepsilon_n)_{n\geq 0}$ or the sequence $(\overline{\varepsilon_n})_{n\geq 0}$.

The sequence $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ is thus r-automatic: this is a classical result, given that all subsequences on arithmetic progressions of length q are r-automatic (see, e.g., [11]). ■

Remark 4.

- Generalizing the definition introduced in [1], we will call the preceding sequences $\lim_{k\to\infty} \Phi_r^{(k)}((u0)^\infty)$ "(q, r)-mirror sequences" (where $q = |u| + 1$). The q-mirror sequences in [1] are thus $(q, 2)$-mirror sequences.
- It was proved in [25] that the r-fold Morse sequence $(\varepsilon_n)_{n\geq 0}$ has the property that $\varepsilon_n = 0$ if and only if the base r expansion of n has an even number of nonzero digits. The penultimate and last assertions of Theorem 2 are thus generalizations to (q, r)-mirror sequences of properties of q-mirror sequences (see [1, Propriétés, p. 21]).
- Relations like the ones satisfied by the sequences of words B_k, V_k, and $C_{j,k}$ above are called *locally catenative formulas*. For a systematic study of locally catenative formulas and morphic sequences, we refer the reader to [26].

Corollary 3. *For any $r \geq 2$, the set Γ_{strict} contains infinitely many r-automatic sequences.*

Proof. The sequences $(1^\ell 0)^\infty$ (with $\ell \geq 1$) clearly belong to Γ. They have minimal period $\ell + 1$. Using Theorems 1 and 2 we deduce that the (distinct) sequences $\lim_{k\to\infty} \Phi_r^{(k)}((1^\ell 0)^\infty)$ belong to Γ_{strict} and are r-automatic. ■

Remark 5. Another proof of Corollary 3 can be deduced from a result in [9]. Namely, if B is any nonperiodic minimal r-automatic sequence, then the sequence $A := \max\{\sup\{\sigma^k B,\ k \geq 0\},\ \sup\{\sigma^k \overline{B},\ k \geq 0\}\}$ is r-automatic (see [9]), and it clearly belongs to Γ_{strict} (it belongs a priori to Γ and cannot be periodic since B is minimal and nonperiodic). Now the sequences $1^\ell A$ (with $\ell \geq 1$) belong to Γ_{strict} and they are r-automatic.

5 Alphabets with More Than Two Letters

All the preceding results extend to (finite) alphabets with more than two letters. As noted in [1], the set Γ is the set of binary expansions of real

numbers $x \in [0,1]$ such that, for all $k \geq 0$, we have $1 - x \leq \{2^k x\} \leq x$, where $\{y\}$ is the fractional part of the real number y. It is natural to ask about the real numbers for which the inequalities $1 - x \leq \{b^k x\} \leq x$ hold (where b is some fixed integer ≥ 3). The base b expansions of these real numbers form the generalized Γ set

$$\Gamma := \{A = (a_n)_{n\geq 0} \in [0, b-1]^{\mathbb{N}},\ \forall k \geq 0,\ \overline{A} \leq \sigma^{(k)} A \leq A\},$$

whose combinatorial properties were studied in [1, Troisième partie, pp. 63–90]. (Here σ is again the *shift* on sequences and the *bar* operation replaces $x \in [0, b-1]$ by $b-1-x$, i.e., if $A = (a_n)_{n\geq 0}$, then $\sigma A := (a_{n+1})_{n\geq 0}$, and $\overline{A} := (b-1-a_n)_{n\geq 0}$; the symbol $\leq$ denotes the lexicographical order on sequences induced by $0 < 1$, and the notation $A < B$ means as usual that $A \leq B$ and $A \neq B$.) The corresponding set

$$\Gamma_{strict} := \{A = (a_n)_{n\geq 0} \in [0, b-1]^{\mathbb{N}},\ \forall k \geq 0,\ \overline{A} < \sigma^{(k)} A < A\}$$

is the set of *admissible sequences* introduced in [23]. (Note that the definition given in [23] is slightly different, but it can be proved equivalent, see, e.g., [7, Proposition 1].)

Results quite similar to the case of a binary alphabet–in particular, generalizations of Theorems 1 and 2 above–can be proved by using the results of [1, Troisième partie]. We will not enter details for brevity, but no really serious difficulty occurs.

6 Conclusion

Combinatorial properties of the sets Γ and Γ_{strict} are crucial in the study of univoque numbers but also in the study of iterations of unimodal functions. An attempt to explain why these almost identical sets appear in seemingly disjoint fields can be found in [3]. Other papers on univoque numbers make use of combinatorial properties of Γ and Γ_{strict} (to cite a few, see [8,14,15,17,18,24]). One may also ask whether the sets Γ and Γ_{strict} contain other "classical" infinite sequences from combinatorics on words. For example, it is not hard to prove that, defining the binary morphism τ by $\tau(1) := 10$, $\tau(0) := 1$ and denoting by F the Fibonacci sequence $\tau^{(\infty)}(1) = 10110101101\ldots$, the sequence $1F = 110110101101\ldots$ belongs to Γ_{strict} (see, e.g., [2]). This provides us with an open question: Which morphic sequences belong to Γ or Γ_{strict}?

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A Crash Look into Applications of Aperiodic Substitutive Sequences

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Summary. Applications of the theory of finite automata, automatic and substitutive sequences to the description of physical situations intermediate between crystallographic order and random disorder in theoretical and experimental Condensed Matter Physics are described. Particular reference is made to Trace Mapping techniques, experimental applications include the investigation of multilayer heterostructures designed after such automatic or substitutive sequences and the use of their specific properties.

1 Into the Past: The "Gang of Five"

A research group composed of Dr. Jean-Paul Allouche and Pr. Michel Mendès France (Laboratoire de Théorie analytique des Nombres, Department of Mathematics, University of Bordeaux), Pr. Jacques Peyrière (Laboratoire d'Analyse Harmonique, Department of Mathematics, UPS, Orsay), Dr. Maurice Kléman, and myself (Laboratoire de Physique des Solides, UPS, Orsay) held regular work meetings in Paris from 1984 to 1987 supported by DRET.

This "gang of five" had chosen for its center of interest the possible role of the theory of finite automata, automatic and substitutive sequences, and their applications [13] in the description of physical situations intermediate between crystallographic order and disorder [7, 41] around the following questions:

> Which are the types of geometric order that can be conceived in solids?
> Which are the theoretical and experimental signatures of long distance order?
> Which are the types of geometrical order that can be built by nature?

which laid the ground for a novel and fertile research field in condensed matter physics.

After the workshop "Aperiodic Crystals" co-organized in les Houches by Denis Gratias (CECM-CNRS, Vitry) and Louis Michel (March 11–20, 1986) [18], the Winter School "Beyond Quasicrystals" (March 7–18, 1994),

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co-organized by Denis Gratias and Françoise Axel [7], then the Winter School "Order, Chance and Risk, from Solid State to Finance (February 23–March 6, 1998) co-organized by Jean-Pierre Gazeau and Françoise Axel [6] both had their roots in these "gang of five" meetings.

Other authors, independently, started theoretical [16, 17, 22–24, 27, 32, 33, 48] as well as experimental [12, 28, 29, 42, 44] studies, mainly on quasi-periodic structures using a wide variety of techniques, often basing them on Fibonacci and Fibonacci-like sequences, with interest focused on localization properties.

In the following, examples of topics where the use of substitutive sequences has led to creative advances in theoretical and experimental condensed matter physics have been selected. A number of mathematicians and physicists over the years have worked in this field: I apologize in advance because, in such a short outline, I shall be able to quote neither all relevant works nor all authors.

2 Non-Bragg Diffraction in Systems with Aperiodic Order and Lebesgue Classification

Measures are to be decomposed into the three classes of Lebesgue's theorem, which states that any measure μ has a unique decomposition into "absolutely continuous" (AC), "singular continuous" (SC), and "atomic" (AT) components:

$$\mu = \mu_{\mathrm{AC}} + \mu_{\mathrm{SC}} + \mu_{\mathrm{AT}}.$$

For example, the measures associated with the Fourier Transform of the Rudin–Shapiro abstract sequence is purely AC, with the Thue–Morse abstract sequence purely SC, with the Fibonacci, period doubling, paper-folding abstract sequences purely AT.

Substitutive sequences can be defined either by the action of a substitution σ on an alphabet $\mathcal{A}$, often a two-letter alphabet e.g. $(0, 1)$, in the Thue–Morse case [31, 37, 43]:

$$\sigma(0) = 01$$
$$\sigma(1) = 10,$$

which yields, for the first few iterations starting with word 0

$$0$$
$$01$$
$$0110$$
$$01101001$$
$$0110100110010110 \text{ etc.}$$

or in many cases recursively, in the Thue–Morse case, for all $n > 0$:

$$\varepsilon_{2n} = \varepsilon_n$$
$$\varepsilon_{2n+1} = 1 - \varepsilon_n.$$

Certain substitutive sequences can also be generated using an automaton. The reader is referred to [13] and [7] for details.

In 1986, E. Bombieri and J. Taylor had raised their question: "Which distributions of matter diffract?" [11], and at that time, the doxa was that to be "sufficiently ordered to diffract ", a non periodic system was to have some tiling with an inflation rule with a characteristic equation having a Pisot number solution. In other words, loosely speaking, the capability to diffract could be extended from the crystalline systems, to quasi-periodic systems only, which then exhibited Bragg peak diffraction spectra.

Three-dimensional planar multilayers of two types made of GaAs and AlAs had been studied experimentally since 1985 by various groups in France, the USA and Japan, using either X-ray and neutron diffraction or Raman diffusion [12, 28, 29, 42, 44].

High-resolution X-ray diffraction spectra of GaAs-AlAs multilayers assembled according to the Thue–Morse sequence obtained in Pr. H. Terauchi's group were an excellent quality highly accurate body of data.

The analysis of these high-resolution X-ray diffraction spectra experiments on GaAs/AlAs Thue–Morse multilayers made by Terauchi showed that, contrary to this doxa, a diffraction spectrum keeping the essential characteristics of the Thue–Morse singular measure, different from those currently studied in classical crystallography could exist and be thoroughly interpreted, which laid the ground to a novel description of certain disordered systems [10].

The analysis of this data showed that most of the spectrum peaks are labeled, in convenient units, by $2k+1/3.2^p$, with p and k integers, therefore by very specific *rationals* with a precision of at least $1/200$, instead of the *integers* found in the case of perfect crystals or incommensurate crystals. The evolution of lineshapes as well as their height as a function of sample size and wave vector carries an information that can be specific of the generating aperiodic disorder and is described in particular by the Hölder exponent $\alpha_n(q) < 2$. (The value 2 is the Bragg peak value, for an atomic measure situation.)

For the first time an X-ray diffraction spectrum exhibiting the properties of a singular continuous measure–in the limit of infinite sample length–had been observed and analyzed. It could be shown that the X-ray spectrum of the finite size system keeps the essential characteristics of the underlying measure, which in this case are different from the "usual" Bragg peaks currently studied in "classical" crystallography and in the most frequently found quasi-crystalline systems.

One can then identify, by a complete and rigorous analysis [35] of the properties of such high-resolution experimental spectra, the respective contributions of the various spectra ingredients. In the Thue–Morse Terauchi samples case, the ingredients are:

- "Singular continuous" non-Bragg part
- "Dirac-type" Bragg part
- Contribution from layer substrate

– Effect of detector width
– effect of intrinsic surface roughness due to molecular beam epitaxy (MBE) fabrication method, which can be approached by a random variation of layer thickness.

In the case of the Rudin–Shapiro and generalized Rudin–Shapiro sequences, where the Fourier transform has an associated AC measure with bounded density–analogous in this to a random sequence–a similar analysis can be conducted [5].

The next step would be to ask whether and how, from the X-ray diffraction spectrum of a sample with aperiodic unknown 1D deterministic order, one can retrieve the generating deterministic sequence. This problem has been addressed in a preliminary fashion using genetic algorithm techniques [25].

3 Substitutive Sequences as Sources for Discrete Laplacians: Spectrum, Band Structure, and Eigenstates in Localization Studies

The differential operator for the study of the vibrations of a finite classical one-dimensional mass and spring system composed of N identical springs and masses is well known; it is essentially a discrete Laplacian.

When the N masses are of two kinds, m_0 and m_1 and distributed according to a deterministic aperiodic sequence on a two-letter alphabet, this sequence, say $m(j)$, plays the role of a source, and when $m(j)$ is the non periodic, non quasi-periodic Thue–Morse sequence, it is found that [4,8,9]:

– The infinite length limit vibration spectrum is a Cantor set with integrated density of states $2k + 1/3.2^p$ (k and p integers) when the energy value is in a gap, numerically proven to be of zero Lebesgue measure.
– To a countable number of elements of the spectrum correspond Born von Karman extended states.

Such results with deterministic aperiodic substitutive sequences as sources, when they were obtained, particularly [8,9], went against the usual habits of thinking for localization problems.

Discrete Laplacians with a source are also found with a variety of other situations, in discrete tight binding or diffusion equations, and also in Maxwell's equations in layered systems [49].

Transfer matrices naturally come in all such calculations. First introduced by Lord Rayleigh [38], and applied more recently to Optics by F. Abeles [1], they entail the use of the "trace mapping technique" first described by J.-P. Allouche and J. Peyrière [2]. The use of this potent method, which has its origins in Mathematics [15, 19, 20] and has a research field of its own [3,34,36,46,47], in particular results in shorter and more accurate calculations.

Over the years, abundant work has been performed to calculate and measure optical transmission coefficients and many other properties of multilayer heterostructures designed after a variety of automatic and substitutive sequences [14, 21, 26, 30, 39, 40, 45], with the hope new devices with promising properties can be discovered and built.

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Invertible Substitutions with a Common Periodic Point

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Summary. We characterize invertible substitutions over a two-letter alphabet which share a common periodic point (or fixed point). The argument is geometrical.

1 Introduction

Let $A = \{1, 2\}$ be an alphabet. Let $A^* = \cup_{n\geq 0} A^n$ denote the free monoid over $\{1, 2\}$ endowed with the concatenation operation. A non-erasing homomorphism σ of the free monoid A^* is called a *substitution*.

A substitution can be extended to infinite sequences in an obvious way. An infinite word $s \in A^{\mathbb{N}}$ is a *fixed point* of the substitution σ if $\sigma(s) = s$; it is called a *periodic point* of σ if $\sigma^k(s) = s$ for some $k \geq 1$.

For a substitution σ, let $M_\sigma = (m_{ij})$ be its *incidence matrix*, where m_{ij} counts the number of occurrences of the letter i in $\sigma(j)$. We say that σ is *unimodular* if $\det M_\sigma = \pm 1$, and is *primitive* if M_σ is primitive; i.e., M_σ^n has only positive entries for some $n \geq 1$.

A substitution is *invertible* if it is an automorphism of the free group $\mathcal{F}$ generated by the alphabet A. An invertible substitution is necessarily unimodular. There are numerous works on invertible substitutions, especially on invertible substitutions over a two-letter alphabet. See, for example, [3, 4, 12, 18, 20, 21]. An excellent survey can be found in Chap. 2 of [11].

We write $\sigma \sim \tau$ if substitutions σ and τ share a common periodic point. In general, it is difficult to tell when two substitutions share common periodic points or fixed points. In this chapter, we give an answer to the question for invertible substitutions over a two-letter alphabet.

It is easy to show that an invertible substitution σ over $\{1, 2\}$ is non-primitive if and only if M_σ has the form

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \ \text{or} \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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In these cases, the fixed points of the substitutions can only be 1^∞, 12^∞, 2^∞, or 12^∞, which is not interesting. So in the following, we study the primitive invertible substitutions. Let us denote by $\mathcal{I}_p$ the set of invertible substitutions over $\{1, 2\}$. Our main result is the following.

Theorem 1 *Let σ and τ belong to $\mathcal{I}_p$. Then $\sigma \sim \tau$ if and only if there is $\gamma \in \mathcal{I}_p$ such that $\sigma = \gamma^n$, $\tau = \gamma^m$, $m, n \geq 1$.*

As a corollary, we have the following

Corollary 2 *Let σ and τ belong to $\mathcal{I}_p$. If σ and τ share a fixed point ω, then there is $\gamma \in \mathcal{I}_p$ with fixed point ω such that $\sigma = \gamma^n$, $\tau = \gamma^m$, $m, n \geq 1$.*

Proof. By Theorem 1, $\sigma = \gamma^m$ and $\tau = \gamma^n$. If γ does not possess a fixed point, then the word $\gamma(1)$ is initialed by 2 and $\gamma(2)$ is initialed by 1. To guarantee σ and τ having fixed points, m and n must be even numbers. Hence, the corollary holds if we replace γ by γ^2. ■

We will use a geometrical method to prove the preceding results, where the notion of *Rauzy fractal* plays a central role. We have sought for a combinatorial proof but did not succeed. It would be interesting to know a combinatorial proof.

Rauzy fractals have many applications in number theory (see, for instance, [5, 9, 16, 17]), and this chapter gives a new one. For a general theory of Rauzy fractals, we refer to [2, 10, 14].

2 Some Known Results Concerning Invertible Substitutions

2.1 Frequency of a Substitution, the Generating Matrix

Let σ be a primitive unimodular substitution over $\{1, 2\}$. Let β be the maximal eigenvalue of the incidence matrix M_σ; then its algebraic conjugate β' is also an eigenvalue of M_σ. By the Perron–Frobenius theorem, we have $\beta > 1$. Now $\beta\beta' = \det M = \pm 1$ implies $|\beta'| < 1$. Therefore, β is a Pisot number, and the substitution σ is said to be of *Pisot type*.

It is well known that the frequencies of occurrences of letters exist in periodic points of primitive substitutions (see [15]). Let $1 - \alpha$ and α, $0 \leq \alpha \leq 1$, be the frequencies of the letters 1 and 2, respectively. We shall call α the *frequency* of σ. It is obvious that $\sigma \sim \tau$ implies that σ and τ have the same frequency. We shall denote $\mathcal{I}_p(\alpha)$ the collection of primitive invertible substitutions with frequency α.

It is clear that $(1-\alpha, \alpha)$ is an *expanding eigenvector*, that is, an eigenvector of M associated with the expanding eigenvalue β. The number α is quadratic; the vector $(1 - \alpha', \alpha')$ is an eigenvector of the eigenvalue β'. Still by the

Perron–Frobenius theorem, the coordinates $1-\alpha'$, α' cannot both be positive; otherwise, it is a Perron–Frobenius eigenvector. Hence, $\alpha'(1-\alpha') \leq 0$, which is equivalent to $\alpha' \notin [0,1]$. A quadratic number α with $0 < \alpha < 1$ and $\alpha' \notin [0,1]$ is called a *Sturm number* according to [1]. Hence, $\mathcal{I}_p(\alpha)$ is an empty set if α is not a Sturm number. For a Sturm number α, the set $\mathcal{I}_p(\alpha)$ is well understood.

Proposition 3 *Let α be a Sturm number. Then there is a non-negative primitive unimodular matrix $M(\alpha)$ such that $\sigma \in \mathcal{I}_p(\alpha)$ if and only if $\sigma \in \mathcal{I}_p$ and $M_\sigma = (M(\alpha))^k$ for some $k \geq 1$.*

We shall call $M(\alpha)$ the generating matrix of α. The preceding proposition is essentially contained in Wen et al. [21]. Berthé and Rao [6] give an explicit construction of the generating matrix $M(\alpha)$ according to α.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a primitive unimodular matrix. Séébold [18] proved the following.

Proposition 4 *Let M be a unimodular non-negative integral matrix. The number of invertible substitutions with incidence matrix M is equal to $a+b+c+d-1$.*

Various characterizations of these $a+b+c+d-1$ substitutions can be found in the book [11].

2.2 Sturmian Words

Sturmian words are infinite words over a binary alphabet, say $\{1,2\}$, that have exactly $n+1$ factors of length n for each $n \geq 1$. Sturmian words can be defined constructively in terms of rotations.

Let $\alpha \in (0,1)$. Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ denote the one-dimensional torus. The rotation of angle α of $\mathbb{T}^1$ is defined by $R_\alpha(x) := x + \alpha$. We introduce two partitions of $\mathbb{T}^1$ as follows:

$$\underline{I}_1 = [0, 1-\alpha), \quad \underline{I}_2 = [1-\alpha, 1); \quad \overline{I}_1 = (0, 1-\alpha], \quad \overline{I}_2 = (1-\alpha, 1].$$

For a real number ρ, tracing the orbit of $R_\alpha^n(\rho)$, we define two infinite words as follows:

$$\underline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \texttt{if } R_\alpha^n(\rho) \in \underline{I}_1, \\ 2 & \texttt{if } R_\alpha^n(\rho) \in \underline{I}_2, \end{cases}$$

$$\overline{s}_{\alpha,\rho}(n) = \begin{cases} 1 & \texttt{if } R_\alpha^n(\rho) \in \overline{I}_1, \\ 2 & \texttt{if } R_\alpha^n(\rho) \in \overline{I}_2. \end{cases}$$

It is proved in [8,13] that an infinite word is a Sturmian word if and only if it is in the form $\overline{s}_{\alpha,\rho}$ or $\underline{s}_{\alpha,\rho}$ and α is an irrational number. We shall call the word $\underline{s}_{\alpha,\rho}$ the *lower Sturmian word* and the word $\overline{s}_{\alpha,\rho}$ the *upper Sturmian word*.

It is well known that the periodic points of a primitive invertible substitution are Sturmian words (see [11]).

3 Rauzy Fractals of Invertible Substitutions

Let us first give an alternative definition of the Rauzy fractals of substitutions over two letters. For the original definition, see, for example, [2,10].

Let σ be a primitive and unimodular substitution over $\{1,2\}$. Let $s = s_0 s_1 s_2 \dots$ be a periodic point of σ. Let α be the frequency of σ.

We define an oriented walk on the real line as follows. Starting from the origin, in the nth step, if $s_{n-1} = 1$, we move to the right side with length α; if $s_{n-1} = 2$, we move to the left side of length $1-\alpha$. Taking the closure of the orbit, we obtain

$$X = \mathtt{cl}\ \{|s_0 s_1 \dots s_{n-1}|_1 \cdot \alpha + |s_0 s_1 \dots s_{n-1}|_2 \cdot (\alpha - 1);\ n \geq 0\},$$

where $|s_0 s_1 \dots s_n|_j$ denotes the occurrences of letter j in the word $s_0 s_1 \dots s_n$. Furthermore, we define

$$\begin{aligned} X_1 &= \mathtt{cl.}\ \{\alpha|s_0 s_1 \dots s_{n-1}|_1 + (\alpha-1)|s_0 s_1 \dots s_{n-1}|_2;\ s_n = 1,\ n \geq 0\} \\ X_2 &= \mathtt{cl.}\ \{\alpha|s_0 s_1 \dots s_{n-1}|_1 + (\alpha-1)|s_0 s_1 \dots s_{n-1}|_2;\ s_n = 2,\ n \geq 0\} \end{aligned} \tag{1}$$

We shall call X the *Rauzy fractal* of σ, and we call X_1, X_2 in formula (1) the *partial Rauzy fractals* of σ. In particular, formula (1) is well defined for any Sturmian word s, and we also call X_1, X_2 the Rauzy fractals of the Sturmian word s.

The Rauzy fractals defined above are affine images of the original Rauzy fractals [5]. Also, the definition of the Rauzy fractals does not depend on the choice of the particular periodic point [10].

Obviously the Rauzy fractals of a Sturmian word are intervals, and the length of X is 1. The periodic points of a primitive invertible substitution are Sturmian, and hence the associated Rauzy fractals are intervals, and the length of X is 1. (Actually, it is shown that [5,7], if σ is a primitive unimodular substitution over $\{1,2\}$, then the Rauzy fractals are intervals if and only if σ is invertible.)

So the Rauzy fractals X_1,X_2 are intervals with length $1-\alpha$ and α, respectively. Let us denote by $h = h_\sigma$ the intersection $X_1 \cap X_2$; then X_1, X_2 have the form

$$X_1 = [-1 + \alpha + h, h], \quad X_2 = [h, \alpha + h].$$

Lemma 5. *Let $\sigma, \tau \in \mathcal{I}_p$. Then $\sigma \sim \tau$ if and only if they have the same Rauzy fractals; in other words, if and only if $h_\sigma = h_\tau$.*

Proof. It is shown in [5] that, if the Rauzy fractals of σ are intervals, then they can be obtained from at most two Sturmian sequences by the above oriented walk, and they are all periodic points of σ. The lemma is proved. ■

From this lemma, it is seen that the Rauzy fractal is a suitable tool to handle the problem of when two substitutions share a common periodic point.

4 Characterization of Rauzy Fractals Stepped Surface

4.1 The Stepped Surface

Denote by V the expanding eigenspace of the matrix M_σ corresponding to the eigenvalue β, and by V' the contractive eigenspace corresponding to β'. Then V and V' are generated by the vectors $\mathbf{v} = (1-\alpha, \alpha)$ and $\mathbf{v}' = (1-\alpha', \alpha')$, respectively. According to the direct sum $V \oplus V' = \mathbb{R}^2$, two natural projections are defined:

$$\pi : \mathbb{R}^2 \to V' \quad \texttt{and} \quad \pi' : \mathbb{R}^2 \to V.$$

We denote the right side of V' (including V') by $(V')^+$, that is,

$$(V')^+ = \{x \in \mathbb{R}^2;\ \pi'(x) \geq 0\}.$$

Let us consider the unit segment connecting two integer points in $\mathbb{R}^2$. Two such segments are neighbors if they belong to one line and they share an endpoint. So a segment has two neighbors.

Let S be the collection of unit segments in $(V')^+$ which has a neighbor intersecting V'. Let $\bar{S}$ be the union of segments in S. Then $\bar{S}$ is a broken line approximating V'. We shall call $\bar{S}$ the *stepped surface* of V'. See Fig. 1.

The notion of stepped surface was introduced by Arnoux and Ito [2], to handle the set equations of the Rauzy fractals.

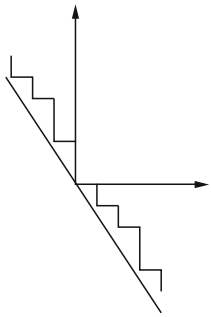

Fig. 1. Stepped surface

4.2 A Tiling Associated with the Stepped Surface

Projecting the segments of the stepped surface S to V', we obtain a tiling $\mathcal{J}'$ of V':

$$\mathcal{J}' = \{\pi(\mathbf{s});\ \mathbf{s} \in S\}.$$

The prototiles of $\mathcal{J}'$ consist of two kinds of segments with length $|\pi(\mathbf{e}_1)|$ and $|\pi(\mathbf{e}_2)|$, respectively, where $\mathbf{e}_1, \mathbf{e}_2$ is the canonical basis of $\mathbb{R}^2$.

Let ϕ be the linear transformation which maps V' to the real line $\mathbb{R}$ such that $\phi\circ\pi(\mathbf{e}_1)>0$, and $(|\phi\circ\pi(\mathbf{e}_1)|, |\phi\circ\pi(\mathbf{e}_2)|)=(1-\alpha,\alpha)$. Then $\mathcal{J}=\phi(\mathcal{J}')$ is a tiling of the real line consisting of two kinds of segments with length $1-\alpha$ and α, respectively.

Let $G=\{g_k;\ k\in\mathbb{Z}\}$ be the endpoints of tiles in $\mathcal{J}$, where g_k is increasing.

Projecting the integer points on $\bar{S}$ to V by π', one can regard the projection points as the orbit of a rotation on the torus V ($\mathtt{mod}\ \pi'(\mathbf{e_1}+\mathbf{e_2})$) with angle $\pi'(\mathbf{e}_2)$. Hence, it is not difficult to show the following [5].

Theorem 6 *If α is a Sturm number, then*

$$G=\{g\in\mathbb{Z}[\alpha];\ 0\le g'<2\alpha'-1\}\ \textit{when}\ \alpha'>1,$$
$$G=\{g\in\mathbb{Z}[\alpha];\ 2\alpha'-1<g'\le 0\}\ \textit{when}\ \alpha'<0,$$

where $\mathbb{Z}[\alpha]:=\{m\alpha+n;\ m,n\in\mathbb{Z}\}$.

4.3 Invertible Substitutions with a Given Incidence Matrix

Let $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ be a primitive unimodular matrix, then there are $a+b+c+d-1$ invertible substitutions with incidence matrix M (Proposition 4). Let us denote them by σ_k and denote by h_k the intersection point of the partial Rauzy fractals of σ_k, $1\le k\le a+b+c+d-1$. We arrange σ_k in the order such that h_k is increasing.

By the connectedness and the self-similarity of the Rauzy fractals of invertible substitutions, [5] determined the intersections h_k in terms of the set G.

Theorem 7 *Let σ_k, $1\le k\le a+b+c+d-1$, be the invertible substitutions with the incidence matrix M, let β be the maximal eigenvalue of M. Then*

(i) The values h_k are given by

$$h_k=\begin{cases}\frac{g_{-k+a+b}}{\beta-1}, & \textit{if}\ \det M=1\\ -\frac{g_{-k+c+d}}{\beta+1}, & \textit{if}\ \det M=-1.\end{cases}$$

(ii) Another characterization of h_k is

$$\{h_k;\ 1\le k\le a+b+c+d-1\}=\begin{cases}\frac{G}{\beta-1}\cap[-\alpha,1-\alpha], & \textit{if}\ \det M=1\\ -\frac{G}{\beta+1}\cap[-\alpha,1-\alpha], & \textit{if}\ \det M=-1.\end{cases}$$

As a direct consequence of item (i), if two primitive invertible substitutions have the same incidence matrix and share a periodic point, then they must coincide.

Tan and Wen [19] also give an alternative method to determine the value h for a given invertible substitution. In [5], Theorem 7 is designed to prove a theorem of Yasutomi [22] concerning substitution invariant Sturmian sequences.

5 Proof of Theorem 1

Now we are in a position to prove our main theorem.

Proof. Let σ, τ be two primitive invertible substitutions over $\{1, 2\}$ such that $\sigma \sim \tau$. Let α be their common frequency, which must be a Sturm number. Let $M = M(\alpha)$ be the generating matrix of α, let β be the maximal eigenvalue of M. The assumption $\sigma \sim \tau$ implies that they have the same Rauzy fractal (Lemma 5).

First, by Proposition 3, there are integers $m, n \geq 1$ such that $M_\sigma = M^m, M_\tau = M^n$. If $m = n$, then by Theorem 7 (i), $\sigma = \tau$. The theorem is true.

Hence, in what follows we assume that $m > n$ without loss of generality. We shall show that there is a primitive invertible substitution γ such that $\gamma \sim \sigma \sim \tau$ and $M_\gamma = M^{m-n}$.

Let us first deal with the case $\det M_\sigma = \det M_\tau = 1$. By Theorem 7 (ii), we have

$$h_\sigma = \frac{g_1}{\beta^m - 1}, \ h_\tau = \frac{g_2}{\beta^n - 1}, \tag{2}$$

for some $g_1, g_2 \in G$. Our assumption $\sigma \sim \tau$ implies that $h_\sigma = h_\tau$.

From $h'_\sigma = h'_\tau$ we infer that

$$g'_1 = \frac{(\beta')^m - 1}{(\beta')^n - 1} g'_2,$$

which implies that $|g'_1| > |g'_2|$. (Remember that $|\beta'| < 1$.) By (2) we have

$$h_\sigma = \frac{g_1 - \beta^{m-n} g_2}{\beta^{m-n} - 1}.$$

We claim that $g_1 - g_2\beta^{m-n} \in G$.

If $\alpha' > 1$, then $0 \leq g'_2 < g'_1 < 2\alpha' - 1$ by Theorem 6. Hence $0 \leq (\beta')^{m-n} g'_2 \leq g'_2$. For in the case that $\beta' < 0$, $\det M_\sigma = (\beta\beta')^m = 1$ implies that $(\beta')^m > 0$ and so that m is an even number; likewise, n is also an even number.

So $0 \leq g'_1 - g'_2(\beta')^{m-n} \leq 2\alpha' - 1$, and it follows that $g_1 - g_2\beta^{m-n} \in G$ by Theorem 6.

If $\alpha' < 0$, then $2\alpha' - 1 < g'_1 < g'_2 \leq 0$. Hence $g'_2 \leq (\beta')^{m-n} g'_2 \leq 0$ since $m - n$ is an even number in case that $\beta' < 0$. So $2\alpha' - 1 < g'_1 - g'_2(\beta')^{m-n} \leq 0$, and it follows that $g_1 - g_2\beta^{m-n} \in G$. Our claim is proved.

On the other hand, $-\alpha \leq h \leq 1 - \alpha$, since $\{h\} = X_1 \cap X_2$ is the intersection of the partial Rauzy fractals of σ and $0 \in X_1 \cup X_2$. By Theorem 7 (ii), there is a primitive invertible substitution γ with incidence matrix M^{m-n} such that $h_\gamma = h_\sigma = h_\tau$.

The cases $(\det M_\sigma, \det M_\tau) = (1, -1), (-1, 1), (-1, -1)$ can be proved in the same manner.

Repeating the above argument, we conclude that there is a primitive invertible substitution θ with incidence matrix M^l, $l = \gcd\{m, n\}$, such that $h_\theta = h_\sigma = h_\tau$.

Let $p = m/l$, $q = n/l$. Then θ^p and σ have the same incidence matrix and share a common periodic point. Hence $\sigma = \theta^p$. Likewise $\tau = \theta^q$. The theorem is proved. □

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Some Studies on Markov-Type Equations

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Summary. A concept of prime solution is introduced in solving Markoff-type equations and the structure of solutions is discussed.

1 Introduction

In this chapter, we will discuss the structure of solutions of the following Markov-type Diophantine equation:

$$x^2 + y^2 + z^2 - axyz - b = 0 \quad (a, b \in \mathbb{Z}). \tag{1}$$

We denote by $\mathbb{S}_{a,b}$ the set of the solutions of the (1). The case $a = 3$, $b = 0$ was studied first by Markov (Markoff) [4]; he proved that there are infinitely many solutions of (1) which are given by a simply algorithm, called the Markov chain. The general cases were also considered by Hurwitz [3], Mordell [5], Schwartz and Muhly [6], and A. Baragar [1], and A. Hone[2]. The following ideas and main results are due to them.

Let $(u, v, w) \in \mathbb{S}_{a,b}$ be a solution of the (1). Then we can get other solutions in the following three ways:

(a) Permuting u, v, w

(b) Changing the signs of any two of u, v, w

(c) Replacing one of u, v, w by $avw - u$, $auw - v$, $auv - w$

The operations (a), (b), and (c) are called *elementary operations*. A solution (u, v, w) is called *positive* if all u, v, w are positive, and is called *degenerate* if two of u, v, w are zero. A positive solution (u, v, w) is called *fundamental* if $0 < u \leq v \leq w$ and if no elementary operation could reduce the sum $u+v+w$ without destroying its positive character.

Proposition 1. *[6] With the proceeding notions, we have*

A. If a non-degenerate solution exists, we can find a fundamental solution from which the given non-degenerate solutions can be obtained by a finite number of elementary operations.

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B. Suppose $b \geq 0$. If (u, v, w) is a fundamental solution, then $u^2 + v^2 \leq b$ except for the cases $a = 1,\ b = 4$ or $a = 2,\ b = 1$.

C. Suppose $a = 1$ and $b = 4$. Then for any integer $v \geq 2$, $(2, v, v)$ is a fundamental solution; if $v = 1$, this is a permutation of the fundamental solution $(1, 1, 2)$. All fundamental solutions can be obtained in the above ways.

D. Suppose $a = 2$ and $b = 1$. Then for any integer $v \leq 1$, $(1, v, v)$ is a fundamental solution. Moreover, all fundamental solutions are of the above forms.

Corollary 1. *Except for the cases C and D of Proposition 1, there are at most a finite number of fundamental solutions.*

Remark 1. The case $b < 0$ was discussed elsewhere. According to the preceding results, all the fundamental solutions of the (1) can be obtained by finite steps of computation for given a and b, so we can get any other solutions by an algorithm.

Now a question arises naturally: whether and how a fundamental solution may change into another different one; furthermore, whether and how by a finite number of elementary operations, a fundamental solution changes into a bigger one. We will give complete answers to these questions in this chapter, and we can understand the structure of the (1).

We define a partial order "$\succeq$" on $\mathbb{Z}^3$ as follows:
$(u, v, w) \succeq (u_1, v_1, w_1) \Longleftrightarrow u \geq u_1, v \geq v_1, w \geq w_1$;
$(u, v, w) \succ (u_1, v_1, w_1) \Longleftrightarrow (u, v, w) \succeq (u_1, v_1, w_1)$ and $u + v + w > u_1 + v_1 + w_1$.

Let $id := (u, v, w)$. Let G denote the group generated by elementary operations under composition of operations. An integer triple (u, v, w) with $0 < u \leq v \leq w$ is called a *fundamental a-triple* if no elementary operation will reduce the sum $u + v + w$ without destroying its positive character; and it is called a *primitive a-triple* if for any element (u_1, v_1, w_1) in the orbit $G(u, v, w)$ with $0 < u_1 \leq v_1 \leq w_1$, we have $(u, v, w) \preceq (u_1, v_1, w_1)$; a solution of (1) is called a *primitive solution* if it is a primitive a-triple.

By the definition of fundamental a-triple, we immediately get the following.

Proposition 2. *An integer triple (u, v, w) with $0 < u \leq v \leq w$ is a fundamental a–triple if and only if either $auv \geq 2w$, or $auv \leq w$.*

Theorem 1. *An integer triple (u, v, w) with $0 < u \leq v \leq w$ is a primitive a-triple if and only if one of the following assertions holds:*

(i) $auv \geq 2w$

(ii) $auv = w$

(iii) $auv < w$, and there exists a triple which is a permutation of $(u, u(w - auv) + v, w - auv)$ and $\succeq (u, v, w)$.

We will prove the theorem in Sect. 2.

Corollary 2. *(1) A primitive a-triple is always fundamental.*
(2) There exist fundamental a-triples which are not primitive.

Theorem 2. *For any positive integer triple (u, v, w), there is a unique primitive a-triple in $G(u, v, w)$.*

Proof. By the definition, it is impossible that $G(u, v, w)$ contains two different primitive a-triples. So we only need to show that there exists one primitive a-triple in $G(u, v, w)$. Suppose that it is not true. We can just let (u, v, w) be minimal (e.g., for any positive triple (u_1, v_1, w_1) with $(u_1, v_1, w_1) \prec (u, v, w)$, there exists one primitive a-triple in $G(u_1, v_1, w_1)$.) By Theorem 1,

(a) $0 < auv - w < w$, or

(b) $auv - w < 0$ and there exists $(\overline{u}, \overline{v}, \overline{w})$ which is a permutation of $(u, au(w - auv) + v, w - auv)$ and less than (u, v, w).

In the first case, we let $(u_1, v_1, w_1) = (u, v, auv - w)$.

In the second case, let $(u_1, v_1, w_1) = (\overline{u}, \overline{v}, \overline{w})$. It is obvious that $(u_1, v_1, w_1) \in G(u, v, w)$ and $(u_1, v_1, w_1) \prec (u, v, w)$, which contradicts our assumption. ■

Lemma 1. *The following four operations are generators of group G:*
(i) $\Phi_\alpha = (v, u, w)$ (e.g., $\Phi_\alpha(u, v, w) = (v, u, w)$, the following notions are similar).
(ii) $\Phi_\beta = (u, w, v)$.
(iii) $\Phi_\gamma = (u, v, auv - w)$.
(iv) $\Phi_\rho = (-u, -v, w)$.

Proof. By the definition of G. ■

If $A \subset G$ and $B \subset G$, we define $AB = \{ab;\ a \in A,\ b \in B\}$.

Lemma 2. *(i) $(\Phi_\alpha)^2 = (\Phi_\beta)^2 = (\Phi_\gamma)^2 = (\Phi_\rho)^2 = id$*
(ii) $\langle \Phi_\alpha, \Phi_\beta \rangle \cong S_3$ (symmetric group on three letters)
(iii) $\langle \Phi_\alpha, \Phi_\beta, \Phi_\rho \rangle \cong S_4$ (symmetric group on four letters)

By Lemma 2, we let S_3 and S_4 denote $\langle \Phi_\alpha, \Phi_\beta \rangle$ and $\langle \Phi_\alpha, \Phi_\beta, \Phi_\rho \rangle$ respectively.

Lemma 3. *(i)* $(S_3\Phi_\rho S_3)(S_3\Phi_\rho S_3) = (S_3\Phi_\rho S_3) \bigcup S_3$

(ii) $(S_3\Phi_\rho S_3)(S_3\Phi_\gamma S_3) = (S_3\Phi_\gamma S_3)(S_3\Phi_\rho S_3)$

(iii) $$G = \left((S_3\Phi_\rho S_3) \bigcup_{m=1}^{+\infty} (S_3\Phi_\gamma S_3)^m\right) \bigcup \left(\bigcup_{m=1}^{+\infty} (S_3\Phi_\gamma S_3)^m\right) \bigcup S_4$$

Proof. Since the number of elements in $S_3\Phi_\rho S_3$ and $S_3\Phi_\gamma S_3$ are finite, we can show (i) and (ii) directly. By Lemma 1, any element in G can be written as the product of some $\Phi_i, i = 1, 2, \ldots$, where $\Phi_i \in S_3\Phi_\rho S_3, S_3\Phi_\gamma S_3$, or S_4. So by (i) and (ii), we obtain (iii). ■

Lemma 4. *For all* $(u,v,w) \in \mathbb{Z}^3$ *and* $\Phi \in S_3 \bigcup (S_3\Phi_\gamma S_3)$*, there exists* $\Theta \in S_3$*, such that* $\Theta\Phi(u,v,w) \succeq (u,v,w)$*, or* $\Theta\Phi(u,v,w) \preceq (u,v,w)$*.*

Proof. By direct check. ■

Lemma 5. *If an integer triple* (u,v,w) *satisfies* $1 \le u \le v \le w$ *and* $auv < w$*, then* $(u,v,w) \succeq (u, w-auv, au(w-auv)+v)$*, or* $(u,v,w) \prec (u, w-auv, au(w-auv)+v)$*.*

Proof. $au(w-auv)+v-w = (au-1)(w-(au+1)v)$. So when $w-auv \ge v$, we have $au(w-auv)+v \ge w$. When $w-auv < v$, we have $au(w-auv-w)+v \le w$. ■

Lemma 6. *Suppose that integer triple* (u,v,w) *satisfies the following two conditions:*

(c1) $2 \le u \le v \le w$*, or* $1 = u \le v \le w$ *and* $a \ge 2$

(c2) There exists $\Phi \in S_3\Phi_\gamma S_3$*, such that* $\Phi(u,v,w) \prec (u,v,w)$

Then we have

(i) $\Phi(u,v,w) \in S_3\Phi_\gamma(u,v,w)$

(ii) $auv < 2w$

Lemma 7. *Let* (u,v,w)*,* (u_1,v_1,w_1) *be two integer triples with* $(u,v,w) \notin S_4(u_1,v_1,w_1)$*, and there exists* $\Phi \in G$*, such that* $(u,v,w) = \Phi(u_1,v_1,w_1)$*. Then there exist* $\Theta \in (S_3\Phi_\rho S_3) \bigcup id$*, and* $\Phi_i \in S_3\Phi_\gamma S_3$ *(*$1 \le i \le m$*), such that*

$$\Theta\Phi_m \cdots \Phi_1(u,v,w) = (u_1,v_1,w_1).$$

Let m *be the least integer such that the above equality holds; then there do not exist* i,j *satisfying* $1 \le i < j \le m$*, and*

$$\Phi_j\Phi_{j-1}\cdots\Phi_1(u,v,w) \in S_4(\Phi_i\Phi_{i-1}(u,v,w))$$

or

$$\Phi_j\Phi_{j-1}\cdots\Phi_1(u,v,w) \in S_4(u,v,w).$$

Proof. By Lemma 3 and the minimal character of m, we can obtain the first part and last part of the results, respectively. ■

Lemma 8. *Under the conditions of Lemma 7, if we also know*

(1) $2 \le u \le v \le w$ *(or* $1 = u \le v \le w$*,* $a \ge 2$*), and* $1 \le u_1 \le v_1 \le w_1$

(2) $(u_1,v_1,w_1) \not\succeq (u,v,w)$

we then have

(i) $\Phi_1(u,v,w) \in S_3\Phi_\gamma(u,v,w)$

(ii) $auv < 2w$

Proof. (1) Suppose that there exists $\Theta \in S_3$, such that $\Phi_1(u,v,w) < \Theta(u,v,w,)$. Then we get the above results immediately, by Lemma 6.

(2) If the assumption in (1) does not hold, then by Lemma 4, there exists $\Theta \in S_3$ such that $\Phi_1(u,v,w) \succ \Theta(u,v,w)$. We know that $\Phi_2\Phi_1(u,v,w) \notin$

$S_3(u,v,w)$ by Lemma 7. Thus $\Phi_2\Phi_1(u,v,w) \succ \Theta_1\Phi_1(u,v,w)$ for some $\Theta_1 \in S_3$ (if not, then by Lemma 4, there exists $\Theta_2 \in S_3$ with $\Phi_2\Phi_1(u,v,w) \prec \Theta_2\Phi_1(u,v,w)$. By Lemma 6, we know that for some $\Theta_3 \in S_3$ $\Phi_2\Phi_1(u,v,w) = \Theta_3{\Phi_1}^{-1}\Phi_1(u,v,w) = \Theta_3(u,v,w)$, this leads to a contradiction by Lemma 7). In this way, we can get $\Phi_m\Phi_{m-1}\cdots\Phi_1(u,v,w) \succ \cdots \succ \overline{\Theta}(u,v,w)$ for some $\overline{\Theta} \in S_3$, which contradicts the conditions of the lemma. ■

Some results (Theorem 3, Theorem 4, etc.) on the structure of solutions of (1) will be given in Sect. 3.

2 Proof of Theorem 1

Now we begin to prove Theorem 1.

First of all, we prove the necessity of conditions. If an integer triple (u,v,w) does not satisfy any of the three conditions, then

(a) $w < auv < 2w$, or

(b) $auv < w$, and there exists $\Theta \in S_3$ such that $\Theta(u, au(w-auv)+v, w-auv) \prec (u,v,w)$ (by Lemma 5).

When (a) occurs, $\Phi_\gamma(u,v,w) = (u,v,auv-w) \prec (u,v,w)$.

When (b) occurs, let $\Gamma = \Theta\Phi_\beta\Phi_\alpha\Phi_\beta\ \Phi_\rho\Phi_\beta\Phi_\alpha\Phi_\gamma\Phi_\beta\Phi_\gamma$, then $\Gamma(u,v,w) = \Gamma(u, au(w-auv)+v, w-auv) \succeq (u,v,w)$. All the above show that (u,v,w) is not a primitive a-triple.

Now we are going to prove the sufficiency by considering the following three cases.

Case 1. $auv \geq 2w$.

Assumption (u,v,w) is not a primitive triple; then there exist (u_1,v_1,w_1), and $\Gamma \in G$, such that $(u_1,v_1,w_1) = \Gamma(u,v,w)$, $0 \leq u_1 \leq v_1 \leq w_1$, and $(u_1,v_1,w_1) \not\succeq (u,v,w)$. By Lemmas 7 and 8, we have $auv < 2w$, which leads to a contradiction.

Case 2. $auv = w$.

We give the same assumptions as in case 1, then by Lemma 7, there exist $\Theta \in (S_3\Phi_\rho S_3)\bigcup id$, and $\Phi_1,\ldots,\Phi_m \in S_3\Phi_\gamma S_3$, such that

$$\Theta\Phi_m\cdots\Phi_1(u,v,w) = (u_1,v_1,w_1),$$

where m is minimal integer.

When (A) $u \geq 2$, or $u = 1$ and $a \geq 2$, then, by Lemma 8,

$$\Phi_1(u,v,w) \in S_3\Phi_\gamma(u,v,w) = S_3(u,v,0).$$

So $\Phi_2\Phi_1(u,v,w) \in S_3(u,0,-v)\bigcup S_3(-u,0,v)\bigcup S_3(u,v,w)$, which contradicts the minimum property of m by Lemma 7.

When (B) $u = 1$, $a = 1$, we will treat it with the following two subcases:

(b1) $v = 1 = w$. In this subcase, we know there that are only finite elements in the set $G(u, v, w)$, so it can be easily shown that u is a primitive triple.

(b2) $v = w \geq 2$. We know that $\Phi_1(u, v, w) \in S_3(1, v, 0) \bigcup S_3(v, v, v^2 - 1)$.

If $\Phi_1(u, v, w) \in S_3(1, v, 0)$, then $\Phi_2\Phi_1(u, v, w) \in S_4(1, v, 0) \bigcup S_3(u, v, w)$, e.g.,

$$\Phi_2\Phi_1(u, v, w) \in S_4\Phi_1(u, v, w) \bigcup S_3(u, v, w),$$

which leads to a contradiction by Lemma 7. If $\Phi_1(u, v, w) \in S_3(v, v, v^2 - 1)$, then by Lemma 8,

$$\Phi_2\Phi_1(u, v, w) \in S_3{\Phi_1}^{-1}\Phi_1(u, v, w) = S_3(u, v, w),$$

which also leads to a contradiction.

Case 3. $auv \leq w$, and there exists $\Omega \in S_3$, such that

$$\Omega\,(u, au(w - auv) + v, w - auv) \succeq (u, v, w).$$

We give the same assumption as in case 2; then we have the following

(a) When $u \geq 2$, or $u = 1$ and $a \geq 2$, by Lemma 8, $\Phi_1(u, v, w) \in S_3(u, v, auv - w)$. Thus,

$$\Phi_2\Phi_1(u,v,w) \in S_3(u, auv-w, au(auv-w)-v) \bigcup S_3(v, auv-w, av(auv-w)-u),$$

so there exists $\Omega_1 \in S_4$, such that

$$\Omega\Phi_2\Phi_1(u, v, w) = (u, w - auv, au(w - auv) + v)$$

or

$$(v, w - auv, av(w - auv) + u).$$

So, by Lemma 3, there exist $\Theta_1 \in S_4$, and $\overline{\Phi}_i \in S_3\Phi_\gamma S_3$ $(3 \leq i \leq m)$, such that

$$\Theta_1\overline{\Phi}_m \cdots \overline{\Phi}_3\Omega\Phi_2\Phi_1(u, v, w) = \Theta\Phi_m \cdots \Phi_1(u, v, w).$$

Since $\Omega(u, au(w-auv)+v, w-auv) \succeq (u, v, w)$, we know that $w-auv \geq u \geq 2$. Thus, by Lemma 8, we can deduce that $\overline{\Phi}_3\Omega\Phi_2\Phi_1(u, v, w) \in S_3(u, v, w)$, which leads to a contradiction.

(b) When $u = 1$, $a = 1$, the proof is just tedious and simple, so we omit it.

Corollary 3. *A solution $(u, v, w) \in \mathbb{S}_{a,b}$ with $0 < u \leq v \leq w$ is primitive if and only if one of the following three conditions holds:*

(i) $auv^2 - u^2 - 2v^2 + b \leq 0$ and $w = \frac{1}{2}\left(auv - \sqrt{(auv)^2 - 4(u^2 + v^2 - b)}\right)$

(ii) $u^2 + v^2 = b$, $w = auv$

(iii) $u^2 + v^2 < b$, and $w = \frac{1}{2}\left(auv + \sqrt{(auv)^2 - 4(u^2 + v^2 - b)}\right)$, and there exists $\Theta \in S_3$, such that $\Theta(u, w - auv, au(w - auv) + v) \preceq (u, v, w)$

Proof. Since $(u, v, w) \in \mathbb{S}_{a,b}$,

$$w = \frac{1}{2}\left(auv \pm \sqrt{(auv)^2 - a(u^2 + v^2 - b)}\right),$$

then by the condition $0 \leq u \leq v \leq w$ and Theorem 1, we obtain the above results.■

Corollary 4. *For any fixed $n \in \mathbb{N}, a \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that the number of primitive solutions of (1) is bigger than n.*

Proof. For an arbitrary $n \in \mathbb{N}$, there exists $b \in \mathbb{N}$ such that the following Diophantine equation has more than n different solutions:

$$u^2 + v^2 = b.$$

So we can deduce that (1) has more than n different primitive solutions by Corollary 3. ■

Example. All the primitive solutions of

$$x^2 + y^2 + z^2 - 2xyz = 1$$

are given by $(1, n, n)$, where $n \in \mathbb{N}$. By induction, all the positive solutions can be written as $(U_l(n), U_m(n), U_{l+m}(n))$ or its permutation, where $l, m \geq 0$, $n \geq 1$, and for $k \geq 0$, $U_k(x)$ is a polynomial induced by $U_0(x) = 0$, $U_1(x) = 1$, $U_{i+1}(x) = xU_i(x) - U_{i-1}(x)$ for $i \geq 1$. All the primitive solutions of

$$x^2 + y^2 + z^2 - xyz = 4$$

are given by $(2, n, n)$. All the positive solutions can be written as $(V_l(n), V_m(n), V_{l+m}(n))$, where $V_k(x)$ is induced by $V_0(x) = 2$, $V_1(x) = x$, $V_{i+1}(x) = xV_i(x) - V_{i-1}(x)$.

Remark 2. In [5], Mordell showed that all positive solutions of

$$x^2 + y^2 + z^2 - 2xyz = 1$$

are given as follows:

$$2x = \alpha^{r+s} + \beta^{r+s}, \quad 2y = \alpha^r + \beta^r, \quad 2z = \alpha^s + \beta^s,$$

where r, s are arbitrary integers, and $\alpha = A + B\sqrt{\lambda}, \quad \beta = A - B\sqrt{\lambda}$, with $A^2 - \lambda B^2 = 1$, $\lambda \in \mathbb{Z}$.

3 The Structure of Solutions of Equation (1)

Markoff [4] showed that for every solution (u, v, w) with $0 < u \le v \le w$ of (1), (u, v, w) or (v, u, w) can be obtained from $(1, 1, 1)$ by a finite sequence of the following two basic operations:

(1) $\Psi_1(x, y, z) = (x, z, xz - y)$

(2) $\Psi_2(x, y, z) = (z, y, zy - x)$

To generalize this result, we note that

$$\Psi_{a,1} := \Phi_\gamma \Phi_\beta = (x, z, axz - y),$$
$$\Psi_{a,2} := \Phi_\gamma \Phi_\alpha \Phi_\beta \Phi_\alpha = (z, y, azy - x).$$

Theorem 3. *Every non-fundamental solution (u, v, w) of (1) with $0 < u \le v \le w$ can be obtained from a fundamental solution $(\overline{u}, \overline{v}, \overline{w})$ or its permutation $(\overline{u}, \overline{w}, \overline{v})$ by a finite sequence of basic operations $\Psi_{a,1}$ and $\Psi_{a,2}$.*

Proof. If the above results cannot hold for one non-fundamental solution (u, v, w) with $0 < u \le v \le w$, let $(u_1, v_1, w_1) = \Psi_{a,1}^{-1}(u, v, w) = (u, auv - w, v)$. We know $v_1 > 0$ for (u, v, w) is not fundamental (e.g., $0 < auv - w < w$), and $v_1 < w_1$ (otherwise, $auv - w \ge v$, and $(u, v, auv - w)$ is fundamental for $auv \ge 2(auv - w)$). It is obvious that $u_1 + v_1 + w_1 < u + v + w$.

Now, let

$$(u_2, v_2, w_2) = \begin{cases} \Psi_{a,1}^{-1}(u_1, v_1, w_1), & \text{if } u_1 \le v_1, \\ \Psi_{a,2}^{-1}(u_1, v_1, w_1), & \text{otherwise.} \end{cases}$$

We also know $0 < u_2, v_2 < w_2$, and $u_2 + v_2 + w_2 < u_1 + v_1 + w_1$. So we can take the above operations forever, which contradicts the fact that $u + v + w$ is finite. ■

Suppose Σ_a is a semigroup generated by $\Psi_{a,1}$ and $\Psi_{a,2}$, e.g.,

$$\Sigma_a = \{\Phi_m \cdots \Phi_1;\ \Phi_i = \Psi_{a,1} \text{ or } \Psi_{a,2},\ 1 \le i \le m,\ m \in \mathbb{N}\}.$$

Theorem 4. *For an arbitrary integer triple (u, v, w) with $2 \le u, v \le w$, and $u \ne v$, and two different elements Λ_1, Λ_2 in Σ_a, we have*

$$\Lambda_1(u, v, w) \notin S_3 \Lambda_2(u, v, w).$$

Proof. Let $\Lambda_1 = \Phi_m \cdots \Phi_1$ and $\Lambda_2 = \overline{\Phi}_n \cdots \overline{\Phi}_1$, where Φ_i, $\overline{\Phi}_j = \Psi_{a,1}$ or $\Psi_{a,2}$, $1 \le i \le m$, $1 \le j \le n$, $m \le n$. It is obvious that

$$\Lambda_1(u, v, w) \prec (u, v, w), \qquad \Lambda_2(u, v, w) \prec (u, v, w).$$

Let $(u_1, v_1, w_1) = \Lambda_1(u, v, w)$, $(u_2, v_2, w_2) = \Lambda_2(u, v, w)$.

Suppose $(u_1, v_1, w_1) \in S_3(u_2, v_2, w_2)$, and let m be the minimum integer such that the above assumption occurs. Since $0 < u_1, v_1 < w_1, 0 < u_2, v_2 < w_2$, we have either $(u_1, v_1, w_1) = (u_2, v_2, w_2)$ or $(u_1, v_1, w_1) = \Phi_\alpha(u_2, v_2, w_2) = (v_2, u_2, w_2)$. If $(u_1, v_1, w_1) = (u_2, v_2, w_2)$, then

$$\Phi_m = \overline{\Phi}_n \quad \text{and} \quad \Phi_{m-1} \cdots \Phi_1(u, v, w) = \overline{\Phi}_{n-1} \cdots \overline{\Phi}_1(u, v, w),$$

which contradicts the minimum property of m. If $(u_1, v_1, w_1) = (v_2, u_2, w_2)$, then

$$\Phi_m \neq \overline{\Phi}_n \quad \text{and} \quad \Phi_{m-1} \cdots \Phi_1(u, v, w) = \Phi_\alpha \overline{\Phi}_{n-1} \cdots \overline{\Phi}_1(u, v, w).$$

In this case, $\Phi_{m-1} \cdots \Phi_1 \neq \overline{\Phi}_{n-1} \cdots \overline{\Phi}_1$ (otherwise, we can deduce that $(u_1, v_1, w_1) \neq (v_2, u_2, w_2)$ at once), and it contradicts the minimum property of m, too. ■

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